Some Problems Related to Plethysm

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It's still magic, even if you know how it's done.

Terry Pratchett

Abstract

This thesis is concerned with plethysm. It investigates certain plethysm coefficients and also studies a diagram algebra whose representation theory is related to plethysm.

We use techniques involving plethystic semistandard Young tableaux in order to provide information about near-maximal constituents of the plethysm $s_\nu \circ s_\mu$ when $\mu = (m), (1^2)$ or $(2,1)$. We study further the case where $\mu = (1^2)$ by the means of a recursive formula of Law and Okitani.

We study the ramified partition algebra, proving some new results about its representation theory. We show that the ramified partition algebra is a cellular algebra and investigate its cell modules. We show that the cell modules of the ramified partition algebra form a stratifying system, and hence prove an analogue of the Hemmer-Nakano theorem for this algebra. We give partial results on the semisimplicity of the ramified partition algebra over $\mathbb{C}$, making a conjecture for the general case.

Finally, we study the restriction of the cell modules for the ramified partition algebra to the partition algebra, investigating two filtrations and making progress on the decomposition of such modules.

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David Brown, July 2023.

Contents

Chapter 1

Introduction

The key theme of this thesis is the *plethysm coefficients* appearing in the plethysm of two Schur functions. Our initial setting is the ring Λ^{sym} of symmetric functions with its famous basis of *Schur functions* s_λ. The plethysm $s_\nu \circ s_\mu$ of two Schur functions is roughly given by inputting the monomials occurring in s_μ into s_ν, giving some flavour of composition of functions. It is known that $s_\nu \circ s_\mu$ is a symmetric function, and thus can be expanded in terms of Schur functions s_λ. The coefficients $a^\lambda_{\nu,\mu} = \langle s_\nu \circ s_\mu, s_\lambda \rangle$ in this expansion are the famous *plethysm coefficients*.

The calculation of plethysm coefficients is a complex central problem in algebraic combinatorics. Complete knowledge of the plethysm $s_{(n)} \circ s_{(m)}$ was identified by Stanley as a key problem in algebraic combinatorics, being Problem 9 in [46]. This particular family of plethysms was already of real interest. For example, Foulkes' famous conjecture of 1950 [15], which states that $\langle s_{(m)} \circ s_{(n)}, s_\lambda \rangle \leq \langle s_{(n)} \circ s_{(m)}, s_\lambda \rangle$ when $m \leq n$, has been an important question in the subject since it was conceived, with much study being undertaken in pursuit of special cases without a method for general proof being identified. The study of general plethysms $s_\nu \circ s_\mu$ is yet more difficult.

The study of plethysm is not restricted to the ring of symmetric functions. Indeed, plethysm coefficients describe the answers to many problems in representation theory. In particular, the plethysm coefficients are the composition multiplicities of certain modules for the wreath product $S_m \wr S_n$ of two symmetric groups, induced up to the symmetric group S_{mn}. In the setting of the general linear group $GL(E)$, the plethysm coefficients encode information about the decomposition of $\nabla^\nu(\nabla^\mu(E))$ where ∇^ν is the famous Schur functor associated to the partition ν. These two descriptions in particular tell us that all plethysm coefficients are non-negative integers; a remarkable fact that is not obvious from the symmetric function context. A result of Kennedy [30] says that the ramified partition algebra lies in Schur-Weyl duality with $k(S_m \wr S_n)$ in certain circumstances, suggesting that plethysm coefficients are also related to the representation theory of the ramified partition algebra. This is an avenue of current research by Bowman, Paget and Wildon. Plethysm coefficients are to be found in many places in representation theory.

The study of plethysm coefficients can be divided into several subthemes. There are some combinatorial results for certain families of partitions. The general expansions $s_{(n)} \circ s_{(m)}$ are known when n is equal to $2, 3, 4$ (see [35], [49], and [25] respectively), but in general the problem is not solved. The decomposition of $s_{(2)} \circ s_\mu$ is entirely known, but the question of the plethysm $s_\nu \circ s_{(2)}$ is still open. Other study has been focused on inequalities which hold for various plethysm coefficients. We have already mentioned Foulkes' conjecture, for which there is progress from the calculations of $s_{(n)} \circ s_{(m)}$ for various small values of m and n as mentioned above, as well as for when n is much larger than m (see [3]) and a proof when $n = 5$ in [5]. Foulkes' second conjecture, which states that $\langle s_{(n)} \circ s_{(m)}, s_\lambda \rangle \leq \langle s_{(n)} \circ s_{(m+1)}, s_{\lambda+(n)} \rangle$, was proved by Brion in [3] by means of considering a more general problem. Brion showed that these coefficients are eventually stable. Study of the stability of plethysm coefficients has also been of interest, for example in the work of Colmenarejo [8], Law and Okitani [33], and the recent work of Bowman and Paget [2].

There has been other recent interest in plethysm coefficients. Recent use has been made of *plethystic Young tableaux*. For example, in [1], Bessenrodt, Bowman and Paget study multiplicity-free plethysms. A step in their proof relies on establishing bijections between sets of these plethystic tableaux and sets of semistandard Young tableaux in order to establish the coefficients $\langle s_\nu \circ s_{(2)}, s_\lambda \rangle$ when $\lambda_1 = |\nu| + \nu_1$ or $|\nu| + \nu_1 - 1$. These coefficients are then used to say more about other families of coefficients not being multiplicity-free. These calculations rely on a recursive formula implicit in [10], a paper studying plethysm by considering the general linear group and using plethystic Young tableaux. From a different angle, Law and Okitani [34] presented a recursive formula for $\langle s_{\nu'} \circ s_{(m)}, s_\lambda \rangle = \langle s_{\nu'} \circ s_{(1^m)}, s_{\lambda'} \rangle$ which is complicated, but becomes manageable when m is small and $\ell(\lambda)$ is close to $|\nu|$. This is another situation where *near-maximal constituents of plethysms* can be calculated more readily than others. This is far from a full account of all of the recent progress on plethysm, of course.

Recently, a diagram algebra has been seen to be related to plethysm. In [30], Kennedy showed that the wreath product $S_m \wr S_n$ of two symmetric groups lies in Schur-Weyl duality with what he refers to as the class partition algebra with parameters m and n, but which is in fact identical to the ramified partition algebra of Martin and Elgamal [39]. The symmetric group S_{mn} is already known to be in Schur-Weyl duality with the partition algebra, so this leads us to suggest that there is some hope in deriving information about plethysm coefficients from studying the ramified partition algebra - this is currently being researched by Bowman, Paget and Wildon. Martin and Elgamal were the first to define the ramified partition algebra, with their motivation coming from statistical mechanics. They give some structural results and examples, but leave many avenues open for further study.

In this thesis, we will examine these different strategies for finding information about plethysm coefficients in more detail.

In **Chapter 2**, we establish necessary background on the combinatorics and representation theory of the symmetric group.

In **Chapter 3**, we establish combinatorial results about near-maximal coefficients of the plethysms $s_\nu \circ s_{(m)}, s_\nu \circ s_{(1^2)}$ and $s_\nu \circ s_{(2,1)}$ by generalising the results of Bessenrodt, Bowman and Paget in [1]. We use the recursive formula of Law and Okitani [34] to verify our results for $s_\nu \circ s_{(1^2)}$ and dig a little deeper.

In **Chapter 4**, we recall essential background on the partition algebra $P_r(\delta)$ and its representation theory. We give a proof that $P_r(\delta)$ is a cellular algebra, both because we will require this warmup to prove that $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is cellular in the next chapter, and because the original proof by Xi [50] uses a Lemma containing a minor flaw. We remind the reader of its cell modules, which play the same role as the Specht modules for the symmetric group, before reviewing results on the semisimplicity of $P_r(\delta)$ due to Martin [37]. We recall the notion of a stratifying system and Hartmann, Henke, König and Paget's result that the cell modules of the partition algebra form a stratifying system provided that the characteristic of the ground field is not 2 or 3 [41]. This implies an analogue of the celebrated Hemmer-Nakano theorem for the partition algebra.

In **Chapter 5**, we recall Martin and Elgamal's definition [39] of the ramified partition algebra $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ and some important results about its basic structure. Our first result is the proof that $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is a cellular algebra. We see that its representation theory is intimately linked with that of direct products of wreath products of symmetric groups, and describe its cell modules. We study these cell modules and prove that they form a stratifying system for $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ provided that the characteristic of the ground field is not 2 or 3. Hartmann, Henke, König and Paget [41] proved that various diagram algebras, such as the Brauer and partition algebra, have a stratifying system. Their proof uses the relationship between the diagram algebra and the symmetric group, but we will make crucial use of Green's work [20] showing that the group algebra of a wreath product of symmetric groups has a stratifying system. We then prove some results about values of δ^{in} and δ^{out} where $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is not semisimple over $\mathbb{C}$, giving a conjecture about the full situation.

Finally, in **Chapter 6**, we examine the problem of restricting cell modules for the ramified partition algebra to the partition algebra, ending with a partial result which shows us that the decomposition numbers of these modules in terms of cell modules for the partition algebra may be expressed in terms of plethysm coefficients. This could be an exciting direction for further study, perhaps finding explicit formulae and applying knowledge about the plethysm coefficients occurring.

We will illustrate with examples throughout, and we check calculations with the computer algebra system SAGE [48] in **Appendix A**.

Chapter 2

Symmetric Groups and Representation Theory

2.1 Algebras and their Representation Theory

We assume the following list of basic concepts surrounding algebras and their representation theory:

- Associative algebras [12, §1.1] and their modules [12, §1.3].
- Tensor product of algebras [12, §4.2].
- Submodules and quotient modules [12, §1.4].
- Direct sums [12, §1.6] and tensor products of modules [29, §19].
- Restriction of modules to a subalgebra.
- Simple [12, §1.5], semisimple [12, §2.2] and indecomposable [12, §1.6] modules.
- Projective modules [12, §3.3].
- Group algebras [29, §6].

- Maschke's Theorem [29, §8].
- Inflation of modules for group algebras [29, §17].
- Basic theory of characters of groups [29, §13]
- Induction [29, §21] and restriction [29, §20] of modules for group algebras.
- Endomorphism algebras [12, §1.7].
- The Hom [43, p.17] and Ext [43, p.365] functors.
- The behaviour of the Ext functor where projective modules are concerned [43, Thm 6.64].
- Short exact sequences [43, p.47].
- The long exact sequence of Hom and Ext modules [43, Cor 6.46].

We also set up notation for the *outer tensor product* of two modules. Explicitly, suppose that M is an A-module and N is a B-module. Then the outer tensor product $M \boxtimes N$ is the $(A \otimes B)$-module with vector space $M \otimes N$ and action $(a \otimes b)(m \otimes n) = (am) \otimes (bn)$.

2.2 The Symmetric Group and its Representation Theory

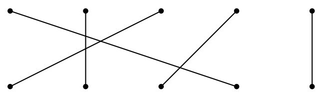
2.2.1 The Symmetric Group and Associated Combinatorial Definitions

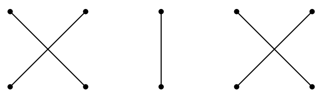
We will recall some important combinatorial objects associated with the representation theory of the symmetric group S_n. There are many sources for this information such as [28] and [44], but we will draw mainly on James' excellent introductory book [27].

Recall that the symmetric group S_n is the group of permutations of $\{1, \ldots, n\}$. It has standard set of generators $\{s_i | i = 1, \ldots, n-1\}$ where s_i is the transposition $(i, i+1)$. We multiply from right to left, so that $(1,2)(2,3) = (1,2,3)$. This lends itself naturally to left actions, with the action of $\tau \in S_n$ on $i \in \{1, \ldots, n\}$ written as τi or $\tau(i)$.

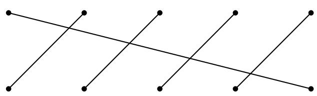
One may represent symmetric group elements as diagrams. To represent $\tau \in S_r$, draw two rows of r dots and join the ith dot on the bottom row to the $\tau(i)$th dot on the top row. One can then multiply by concatenation, where for example $\tau\rho$ means putting τ on top of ρ.

Example 2.1. *Consider the elements* $(1,3,4)$ *and* $(1,2)(4,5)$ *represented by diagrams.*

$(1,3,4) =$ 

$(1,2)(4,5) =$ 

Concatenating the diagram of $(1,3,4)$ *on top of the diagram for* $(1,2)(4,5)$ *and following through connections, one sees that*

$(1,3,4)(1,2)(4,5) =$  $= (1,2,3,4,5).$

Definition 2.2. [28, p.46] [27, Def 2.2] A composition λ of n is a tuple $(\lambda_1, \ldots, \lambda_t)$ of non-negative integers such that $\lambda_1 + \ldots + \lambda_t = n$. By convention, we do not allow trailing zeroes. We write $\lambda \vDash n$ when λ is a composition of n and write $\ell(\lambda) = t$, the length of λ.

If $\lambda \vDash n$ and $\lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0$, then we say λ is a *partition* of n and write $\lambda \vdash n$.

Notation. We abbreviate repetitions of the same number in a partition using index notation. For example, we may write $(2^2, 1^5)$ instead of $(2,2,1,1,1,1,1)$. We write $\emptyset$ for the empty partition (of 0).

Definition 2.3. Partitions of the form (a^b) are called rectangles and partitions of the form $(a, 1^b)$ are known as hooks.

Example 2.4. *The partitions of* 5 *are* $(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1)$ *and* $(1,1,1,1,1)$. *The partitions* $(5), (4,1), (3,1,1), (2,1,1,1)$ *and* $(1,1,1,1,1)$ *are all hooks. The partition* $(3^4) \vdash 12$ *is a rectangular partition of* 12. *The tuple* $(1,1,3,2,5)$ *is a composition of* 12 *which is not a partition.*

We will occasionally make use of the well known Young subgroups of S_n.

Definition 2.5. [27, p.13] Given a partition $\lambda \vdash n$, recall that the Young subgroup S_λ associated to $\lambda = (\lambda_1, \dots, \lambda_t)$ is defined to be the group

$$S_{\lambda_1} \times \cdots \times S_{\lambda_t}.$$

The standard embedding of S_λ inside S_n is

$$S_{\{1,\cdots,\lambda_1\}} \times S_{\{\lambda_1+1,\cdots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{\lambda_1+\cdots+\lambda_{t-1}+1,\dots,n\}}.$$

Example 2.6. *The Young subgroup* $S_{(3,2,1)}$ *associated to the partition* $\lambda = (3,2,1) \vdash 6$ *is* $S_3 \times S_2 \times S_1$, *which can be embedded into* S_6 *in standard fashion as* $S_{\{1,2,3\}} \times S_{\{4,5\}} \times S_{\{6\}}$.

Definition 2.7. [27, Def 3.1] Suppose that $\lambda \vdash n$. The Young diagram $[\lambda]$ of shape λ is the left aligned collection of boxes such that there are λ_1 boxes in the first row, λ_2 boxes in the row below it, and so on, with $\lambda_{\ell(\lambda)}$ boxes in the final row. The box in the ith row and the jth column is labelled by the coordinate (i, j).

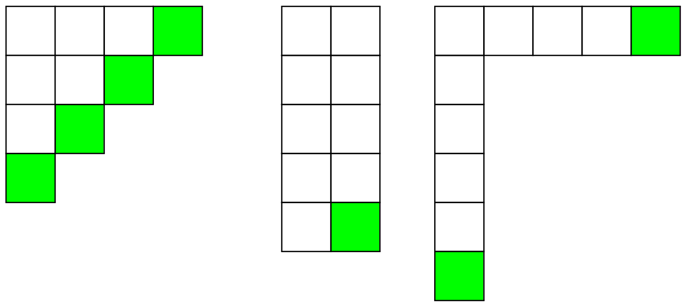
Definition 2.8. A box $(i, j) \in [\lambda]$ is a *removable box* if the spaces $(i + 1, j)$ and $(i, j + 1)$ immediately to the right and below are not occupied by boxes. Equivalently, box (i, j) is removable if $[\lambda] \setminus \{(i, j)\}$ is still a valid Young diagram of some partition.

A coordinate (i, j) is an *addable box* if (i, j) is not in $[\lambda]$ but the spaces immediately to the left and above are either boxes in $[\lambda]$ or outside of the region $i, j \geq 1$. That is, (i, j) is addable if $[\lambda] \sqcup \{(i, j)\}$ is the Young diagram of a partition.

Remark 2.9. From now on we will often just write λ instead of $[\lambda]$ and identify a partition with its Young diagram.

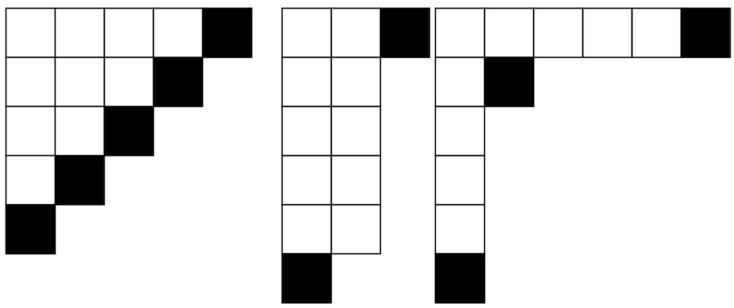
Example 2.10. *Below are the Young diagrams of some partitions of* 10 *with their removable boxes highlighted in green.*

$(4, 3, 2, 1)$ $(2, 2, 2, 2, 2)$ $(5, 1, 1, 1, 1, 1)$



We also show the same partitions with all addable boxes displayed in black.

$(4, 3, 2, 1)$ $(2, 2, 2, 2, 2)$ $(5, 1, 1, 1, 1, 1)$



Definition 2.11. [27, Def 3.5] Given a partition $\lambda \vdash n$, define the *conjugate partition* λ' of λ to be the partition whose Young diagram is the reflection of $[\lambda]$ in the line $i = j$. Explicitly, $\lambda'_i = |\{j | \lambda_j \geq i\}|$.

Remark 2.12. The partition λ' has λ as its list of column lengths, and clearly $(\lambda')' = \lambda$.

Example 2.13. *Referring to the pictures above,* $(4, 3, 2, 1)$ *is self-conjugate,* $(2^5)' = (5^2)$*, and* $(5, 1^5)' = (6, 1^4)$*. More generally,* $(a^b)' = (b^a)$ *and* $(a, 1^b)' = (b+1, 1^{a-1})$.

Definition 2.14. The *lexicographic order* on partitions is the total order where $\lambda > \mu$ if there is some $d \geq 1$ such that $\lambda_i = \mu_i$ for each $i = 1, \ldots, d-1$ but $\lambda_d > \mu_d$.

Example 2.15. *The partitions of* 5 *in Example 2.4 are listed in lexicographic order. Among partitions of* n*,* (n) *is always the maximal and* (1^n) *always the minimal partition of* n *in the lexicographic order.*

Definition 2.16. [27, Def 3.2] The dominance order $\triangleright$ is a partial order on partitions of n, and is defined as follows. Suppose $\lambda, \mu \vdash n$. If $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for every $i \in \{1, \ldots, \ell(\lambda)\}$, then we say λ dominates μ and write $\lambda \trianglerighteq \mu$.

Remark 2.17. In terms of Young diagrams, $\lambda \trianglerighteq \mu$ if and only if for every i, there are at least as many boxes in the first i rows of λ as in the first i rows of μ.

Example 2.18. *The partition* $\lambda = (5, 4, 3, 2, 1)$ *dominates* $(4, 3^3, 2)$*, and is dominated by* $(9, 5, 1)$*. Compare* λ *to* $\mu = (6, 3, 2^3)$*. There are more boxes in the first row of* μ *than* λ*, but more boxes in the first three rows of* λ *than in the first three rows of* μ*. Therefore neither dominates the other and the two are incomparable in the dominance order.*

Lemma 2.19. *The dominance ordering implies the lexicographic ordering: if* $\lambda \trianglerighteq \mu$ *then* $\lambda \geq \mu$ *lexicographically.*

Proof. Suppose $\lambda \trianglerighteq \mu$ and pick l maximal such that $\lambda_i = \mu_i$ for each $i < l$. By dominance, $\lambda_1 + \cdots + \lambda_l \geq \mu_1 + \cdots + \mu_l$, and thus $\lambda_l \geq \mu_l$, meaning $\lambda \geq \mu$ lexicographically. $\square$

Example 2.20. *The lexicographic and dominance orders are the same for* $n \leq 5$*. For* $n \geq 6$ *we have* $(n-3, 1^3) > (n-4, 2^2)$ *lexicographically, but they are incomparable in the dominance order.*

Definition 2.21. A *Young tableau of shape* $\lambda \vdash n$ *and weight* $\mu \vDash n$ is a filling of the boxes of the Young diagram $[\lambda]$ with μ_1 ones, μ_2 twos and so on.

A Young tableau is called semistandard if the entries increase weakly left to right along rows and strictly down columns. We write $\mathrm{SStd}(\lambda, \mu)$ for the set of semistandard Young tableaux of shape λ and weight μ.

A Young tableau is called standard if $\mu = (1^n)$, and entries strictly increase left to right across rows and down columns. Clearly, standard tableaux are always semistandard. We write $\mathrm{Std}(\lambda)$ for the set of standard tableaux of shape λ.

Example 2.22. *Consider the following tableaux.*

$$t_1 = \begin{ytableau} 2 & 3 & 1 & 1 \\ 5 & 4 & 6 \\ 3 \\ 1 \\ 1 \end{ytableau} \qquad t_2 = \begin{ytableau} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 \\ 3 & 3 \\ 4 & 5 \\ 6 \end{ytableau} \qquad t_3 = \begin{ytableau} 1 & 2 & 3 & 5 \\ 4 & 7 \\ 6 & 8 \\ 9 \\ 10 \end{ytableau}$$

The tableau t_1 is a tableau of shape $(4,3,1^3)$ and weight $(4,1,2,1^3)$ which is not semistandard.

The tableau t_2 is a semistandard tableau of shape $(5,3,2^2,1)$ and weight $(4^2,2,1^3)$.

The tableau t_3 is a standard tableaux of shape $(4,2^2,1^2)$.

It is clear to see that S_n acts on tableaux of shape $\lambda \vdash n$ and weight (1^n) by acting individually on the number in each box. More explicitly, write $t_{i,j}$ for the entry of t in the (i,j)th box. Suppose $\pi \in S_n$, then $(\pi t)_{i,j} = \pi t_{i,j}$.

For example,

$$(1,2,3) \; \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} = \begin{ytableau} 2 & 3 \\ 1 \end{ytableau}$$

Lemma 2.23. *[27, Proof of Cor 13.17] When $\lambda \ntrianglerighteq \mu$, $|\,\mathrm{SStd}(\lambda,\mu)| = 0$. When the shape and weight are the same, there is only one tableau, that is $|\,\mathrm{SStd}(\lambda,\lambda)| = 1$ for all $\lambda \vdash n$.*

Proof. Suppose that $\lambda \ntrianglerighteq \mu$ and $t \in \mathrm{SStd}(\lambda,\mu)$. Then for some i, there are more instances of the numbers $1, 2, \ldots, i$ to fill into boxes of t than there are boxes of t in the first i rows. This forces there to be some place at which the numbers in t do not increase down columns.

For the second result, since numbers must increase down columns, the only possible semistandard tableau of shape and weight λ is the one with first row consisting of λ_1 ones, second row consisting of λ_2 twos, and so on. $\square$

Definition 2.24. Place an order on semistandard tableaux of a given shape as follows. Given tableaux s and t of the same shape, examine the leftmost column where the two differ. If the least entry not occurring in both columns occurs only in s, then declare $s \prec t$.

Example 2.25. *In the case of semistandard* (2)*-tableaux,* $\boxed{a\,|\,b} \prec \boxed{c\,|\,d}$ *if and only if* $a < c$ *or* $a = c$ *and* $b < d$*. In the case of* (1^2)*-tableaux,* $\begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \prec \begin{array}{|c|}\hline c \\ \hline d \\ \hline\end{array}$ *if and only if* $a < c$ *or* $a = c$ *and* $b < d$*. In a similar fashion, this order boils down to a lexicographic-style order on* (n)*- and* (1^n)*-tableaux.*

For (2^2)*-tableaux, for example*

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array} \prec \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array} \prec \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \prec \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}.$$

Partitions and Young tableaux are essential to understanding the basic representation theory of the symmetric group, which we will now review.

2.2.2 Young Permutation and Specht Modules

Let k be an arbitrary field. We will describe kS_n modules. We recall two well known families of modules for the symmetric group S_n: the Young permutation modules M^λ and the Specht modules S^λ, both indexed by the partitions of n. We again draw mainly from [27], but the material can be reviewed in [28], [44] and indeed many other places.

Definition 2.26. [27, Def 3.9] Suppose that t is a Young tableau of shape and weight $\lambda \vdash n$. The row stabiliser R_t of t is the subgroup of S_n whose elements fix the entries of each row of t setwise. Given another Young tableau s of shape and weight λ, write $s \sim t$ if and only if $s = \pi t$ for some $\pi \in R_t$. Write $\{t\}$ for the equivalence class of t under this relation; this is a *tabloid*.

A tabloid $\{t\}$ is represented by removing vertical lines from the tableau t. We will soon see an example of this notation in Example 2.28

Remark 2.27. A tabloid $\{t\}$ is best thought of as the tableau t with the order of entries within a row ignored. Thus, we can always write tabloids with entries increasing along rows.

Example 2.28. *The tabloid associated to the Young tableau*

$$t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & 4 \\ \cline{1-2} 8 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$$

is

$$\{t\} = \begin{array}{ccc}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 8 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}.$$

The tabloid $\{t\}$ *is fixed for example by* $(123)(68)$ *and* $(23)(45)$*.*

Definition 2.29. [27, Def 4.1] Given $\lambda \vdash n$, the Young permutation module M^λ is the kS_n-module whose basis is the set of tabloids $\{t\}$ of shape λ under the action $\pi\{t\} = \{\pi t\}$.

The action $\pi\{t\} = \{\pi t\}$ is well defined by the following argument. Suppose that $\{s\} = \{t\}$, so $s = \rho t$ for some $\rho \in R_t$. Thus $\pi s = \pi\rho t$, but $\pi\rho \in R_{\pi t}$ and thus $\{\pi s\} = \{\pi t\}$.

The Young permutation module M^λ is generated by any λ-tabloid [27, Lem 4.2], and is thus a cyclic module. The Young permutation modules contain the famous Specht modules, which we will now define, as submodules.

Remark 2.30. Suppose $\lambda \vdash n$. Note that the row stabiliser R_t for a λ-tableau t, and thus the group $\mathrm{stab}_{S_n}(\{t\})$, is isomorphic to the Young subgroup S_λ of S_n. This implies the following well-known isomorphism:

$$M^\lambda \cong \mathbb{1}\uparrow^{S_n}_{\mathrm{stab}\{t\}} = \mathbb{1}\uparrow^{S_n}_{S_\lambda}.$$

Definition 2.31. [27, Def 4.3] Suppose that t is a tableau of shape and weight λ. Then C_t, the column stabiliser of t, is defined to be the subgroup of S_n whose elements fix the columns of t setwise.

The polytabloid e_t associated to t is the signed sum

$$e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi\{t\}$$

of tabloids.

The Specht module S^λ associated to $\lambda \vdash n$ is the kS_n-submodule of M^λ spanned by all polytabloids e_t.

Example 2.32. *Let t be the (2^2)-tableau*

$$s = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}.$$

Then $C_s = \{1, (1,3)(2,4), (1,3), (2,4)\}$ and

$$e_s = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} + \begin{array}{cc}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} - \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} - \begin{array}{cc}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array}.$$

The Specht module S^λ is also cyclic, generated by any chosen polytabloid [27, Lem 4.5]. Note that although e_t is a sum of tabloids, it depends on the tableau t and not just the tabloid $\{t\}$. We resolve this inconvenience by observing that there is a standard basis of S^λ allowing us to choose a specific tableau for each tabloid.

Theorem 2.33. *[27, Thm 8.4] The set*

$$\{e_t | t \textit{ is a standard tableau of shape } \lambda\}$$

is a basis for S^λ.

Example 2.34. *The Specht module $S^{(2^2)}$ has basis $\{e_s, e_t\}$ where*

$$s = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \textit{ and } t = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}.$$

We already observed that

$$e_s = \begin{array}{c}\hline 1\ \ 2 \\ \hline 3\ \ 4 \\ \hline\end{array} + \begin{array}{c}\hline 3\ \ 4 \\ \hline 1\ \ 2 \\ \hline\end{array} - \begin{array}{c}\hline 2\ \ 3 \\ \hline 1\ \ 4 \\ \hline\end{array} - \begin{array}{c}\hline 1\ \ 4 \\ \hline 2\ \ 3 \\ \hline\end{array},$$

and we can calculate that

$$e_t = \begin{array}{c}\hline 1\ \ 3 \\ \hline 2\ \ 4 \\ \hline\end{array} + \begin{array}{c}\hline 2\ \ 4 \\ \hline 1\ \ 3 \\ \hline\end{array} - \begin{array}{c}\hline 2\ \ 3 \\ \hline 1\ \ 4 \\ \hline\end{array} - \begin{array}{c}\hline 1\ \ 4 \\ \hline 2\ \ 3 \\ \hline\end{array}.$$

Now, for example

$$(1,3)e_s = \begin{array}{c}\hline 2\ \ 3 \\ \hline 1\ \ 4 \\ \hline\end{array} + \begin{array}{c}\hline 1\ \ 4 \\ \hline 2\ \ 3 \\ \hline\end{array} - \begin{array}{c}\hline 1\ \ 2 \\ \hline 3\ \ 4 \\ \hline\end{array} - \begin{array}{c}\hline 3\ \ 4 \\ \hline 1\ \ 2 \\ \hline\end{array} = -e_s,$$

and

$$(1,2)e_s = \begin{array}{c}\hline 1\ \ 2 \\ \hline 3\ \ 4 \\ \hline\end{array} + \begin{array}{c}\hline 3\ \ 4 \\ \hline 1\ \ 2 \\ \hline\end{array} - \begin{array}{c}\hline 1\ \ 3 \\ \hline 2\ \ 4 \\ \hline\end{array} - \begin{array}{c}\hline 2\ \ 4 \\ \hline 1\ \ 3 \\ \hline\end{array} = e_s - e_t.$$

The multiplicity over $\mathbb{C}$ of the Specht modules as composition factors of Young permutation modules is known. This result is one of several linked results known as *Young's Rule.*

Lemma 2.35 (Young's Rule). *[27, 14.1] Over a field of characteristic* 0*, the multiplicity of* S^λ *as a composition factor of* M^μ *is equal to* $|\operatorname{SStd}(\lambda,\mu)|$.

The Specht modules are the key to obtaining simple modules for kS_n. Over a field of most characteristics, they form a full set of simple modules themselves.

Theorem 2.36. *[27, Thm 4.12] When the characteristic of* k *is* 0*,* $\{S^\lambda | \lambda \vdash n\}$ *is a full set of simple modules for* kS_n *without repetition.*

In small characteristic, we must look within the Specht modules indexed by p-regular partitions, which we are about to define, for simple modules.

Definition 2.37. [27, Def 10.1] Given $p \in \mathbb{N}$, a partition λ is p-regular if each entry occurs at most $p-1$ times. Otherwise, λ is p-singular.

Example 2.38. *The partition* (1^n) *is* p*-singular if and only if* $n \geq p$*. The partition* $(3, 2^2, 1)$ *is* 2*-singular but* 3*-regular. A* 2*-regular partition is simply a partition with no repeated parts.*

Theorem 2.39. *[27, Thm 11.5] Suppose that* k *is a field of characteristic* $p > 0$*. Then when* λ *is* p*-regular,* S^λ *has as a unique maximal submodule whose quotient module is denoted* D^λ*. The set*

$$\{D^\lambda | \lambda \vdash n \textit{ is } p\textit{-regular}\}$$

is a complete set of simple modules for kS_n *without repetition.*

It is easy to observe that when $p > n$, all partitions of n are p-regular. In this situation, $D^\lambda = S^\lambda$ as we would expect in the semisimple case. More is known about the simple modules D^λ, and there are many interesting and challenging open questions, but we will not delve further into this matter in this thesis.

2.2.3 Row and Column Insertion

The Robinson-Schensted row insertion algorithm is a powerful combinatorial tool which describes a method of inserting new numbers into an existing semistandard tableau. The following exposition is based on [45] and [16].

The algorithm is reliant on inserted numbers displacing those in existing boxes, a process which is called bumping. If x displaces a number y already inhabiting a box, then x *bumps* y and y has *been bumped by* x.

Definition 2.40 (Row Insertion Algorithm)**.** Let t be a semistandard tableau of any shape and weight, with $x \in \mathbb{N}$ to be inserted. Define $t \leftarrow x$ to be the tableau obtained by applying the following algorithm:

1. Insert x into the first row of t by

    - adding x to the end of the row if it is greater than or equal to all entries in the row.

   - bumping the leftmost occurrence of the smallest number strictly greater than x otherwise.

2. If y is bumped from the first row, insert y into the second row in the same fashion and so on (y may be placed at the end of this row, or bump some z in that row).

3. Terminate when some number is added to the end of a row. This row may be the empty row just below the tableau, in which case the empty row becomes a row consisting of a single entry.

We will sometimes begin the row insertion process in the second row. If insertion starts in the 2nd row we will write $t \leftarrow_2 x$ for the tableau obtained.

Example 2.41. *Consider row inserting* 2 *into the tableau*

$$\begin{array}{|c|c|c|c|}
\hline
2 & 2 & 2 & 3 \\
\hline
3 & 4 & 5 \\
\cline{1-3}
5 \\
\cline{1-1}
6 \\
\cline{1-1}
\end{array}.$$

The 2 *bumps the* 3 *in the first row, leaving us to insert* 3 *into the second row of*

$$\begin{array}{|c|c|c|c|}
\hline
2 & 2 & 2 & 2 \\
\hline
3 & 4 & 5 \\
\cline{1-3}
5 \\
\cline{1-1}
6 \\
\cline{1-1}
\end{array}.$$

This must bump the 4 *in the second row:*

$$\begin{array}{|c|c|c|c|}
\hline
2 & 2 & 2 & 2 \\
\hline
3 & 3 & 5 \\
\cline{1-3}
5 \\
\cline{1-1}
6 \\
\cline{1-1}
\end{array}.$$

We must insert 4 *into the third row, which bumps* 5 *into the fourth row, where* 6 *is bumped:*

$$\begin{ytableau} 2 & 2 & 2 & *(green) 2 \\ 3 & *(green) 3 & 5 \\ *(green) 4 \\ *(green) 5 \end{ytableau}.$$

Finally, the 6 *is inserted on the end of the empty fifth row, leaving us with a final result of*

$$\begin{ytableau} 2 & 2 & 2 & *(green) 2 \\ 3 & *(green) 3 & 5 \\ *(green) 4 \\ *(green) 5 \\ *(green) 6 \end{ytableau}.$$

Definition 2.42 (Column Insertion Algorithm)**.** With the same setup, use the notation $x \to t$ for the tableau obtained by column inserting the number x into the tableau t. That is, applying the row insertion algorithm above but with the word "rows" interchanged with the word "columns", and insertion governed by:

- add x to the bottom of the column if it is strictly greater than all entries in the column.
- otherwise displace the smallest number greater than or equal to x.

Remark 2.43. It is important to stress that when defining column insertion in analogy to row insertion, all weak inequalities are exchanged for strict inequalities and vice versa.

Example 2.44. *We exhibit some further examples of row and column insertion. Consider the following tableau.*

$$t = \begin{ytableau} 1 & 1 & 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 \end{ytableau}$$

We show the results of some insertions below, with the path of inserted numbers shown in green

$t \leftarrow 4$

1	1	1	1	4
2	2			
3	3			
4				

$3 \rightarrow t$

1	1	1	1
2	2	3	
3	3		
4			

$t \leftarrow_2 2$

1	1	1	1
2	2	2	
3	3		
4			

Definition 2.45. Given a partition λ, write $\lambda + \epsilon_i$ for the composition obtained from λ by adding a box to the end of the ith row of the Young diagram. Similarly, write $\lambda + \zeta_j$ for the composition obtained by adding a box to the bottom of the jth column. Write $-\epsilon_i$ and $-\zeta_j$ as notation for removing from rows and columns respectively.

Remark 2.46. If $(i, \lambda_i + 1)$ is an addable box of λ then $\lambda + \epsilon_i$ is a partition, with a similar observation being true for $\lambda + \zeta_j$ when $(\lambda'_j + 1, j)$ is an addable box of λ.

Remark 2.47. If t is of shape λ, then $t \leftarrow x$ and $x \rightarrow t$ are of shapes $\lambda + \epsilon_i$ and $\lambda + \zeta_j$ for some i, j.

Lemma 2.48. *[45, Theorem 1] Suppose t is a semistandard tableau and $x \in \mathbb{N}$. Then the tableaux $t \leftarrow x$ and $x \rightarrow t$ are also semistandard.*

Remark 2.49. Suppose t is a semistandard tableau whose first row is at least two boxes longer than those below. Suppose further that this first row is filled entirely with ones, apart from the box at the end which is filled with some $y \geq 1$. Then the tableau $t \leftarrow_2 x$ is still a semistandard tableaux when $x \geq 2$. We will use this fact often when discussing and generalising the work of Bowman, Bessenrodt and Paget [1] later.

This insertion algorithm is powerful, since it is reversible as long as we know which box was added to the tableau. The following Lemma paraphrases the exposition from [16].

Lemma 2.50 (Reversibility of Row Insertion)**.** *[16, §1.1] Given a tableau t and nominated row i containing a removable box, we may calculate a unique tableau $\tilde{t}$ with shape* $\text{shape}(t) - \epsilon_i$ *and a number x such that $t = \tilde{t} \leftarrow x$.*

Proof. If $i = 1$ then set $\tilde{t}$ to be t with the final box of the first row removed, and x to be the entry which was in this box.

Otherwise $i \geq 2$. Remove the box of t at the end of the row i to obtain $\tilde{t}_i$ and denote the entry in this removed box by x_i. To obtain $\tilde{t}_{i-1}$, replace the rightmost entry in row $i-1$ of $\tilde{t}_i$ which is strictly less than x_i by x_i, and denote the replaced entry by x_{i-1}. Follow this algorithm until we reach the first row, and set $\tilde{t} = \tilde{t}_1$ and $x = x_1$. □

Remark 2.51. We may also find $\tilde{t}$ and x such that $t = \tilde{t} \leftarrow_2 x$ by stopping the algorithm at row 2 instead of row 1. Column insertion is also reversible by a similar argument.

We now understand well the process of inserting numbers into rows and columns, together with how to reverse this process. It remains to ask what happens when one inserts several elements in sequence into a tableau using this method. The following exposition is based on [16, §1.1] and [16, §A.2], with some results specialised to our context for conciseness.

Lemma 2.52. *[16, p.11] Consider the row insertion $t \leftarrow x$ followed by the row insertion $(t \leftarrow x) \leftarrow x'$ with $x \leq x'$. Suppose box B is added by the first insertion and box B' by the second. Then B is strictly to the left of and weakly below B'.*

Corollary 2.53. *Suppose that $t = (\tilde{t} \leftarrow x) \leftarrow x'$ with $x \leq x'$. Then the two boxes added in the insertion process are in different columns.*

If a number is inserted into a tableau, followed by a number at least as big, we can reverse this double row insertion using the following result.

Corollary 2.54. *[16, §1.1] Suppose one is given a tableau t of shape λ with two designated boxes identified at the end of rows i and j respectively, with the boxes* *in different columns. Then there are unique x, x' such that $x \leq x'$ and a unique tableau $\tilde{t}$ of shape $\lambda - \epsilon_i - \epsilon_j$ such that $t = (\tilde{t} \leftarrow x) \leftarrow x'$.*

Proof. Since one knows that the first number inserted was less than or equal to the second, we know that of the two added boxes, the rightmost one is added last, say at the end of row i. By the reverse algorithm, we may obtain a unique tableau $\tilde{t}_0$ of shape $\lambda - \epsilon_i$ and a unique x' such that $t = \tilde{t}_0 \leftarrow x'$.

Now, given $\tilde{t}_0$, we know a single box was added at the end of row, say, j, and therefore again by the reverse algorithm we can find a unique tableau $\tilde{t}$ of shape $\lambda - \epsilon_i - \epsilon_j$ and a unique x such that $\tilde{t}_0 = \tilde{t} \leftarrow x$.

Then $\tilde{t}, x$ and x' have all been uniquely chosen, and $t = (\tilde{t} \leftarrow x) \leftarrow x'$ as required.

□

Remark 2.55. The constraint that $x \leq x'$ is very important here, we must know that a value x was inserted, followed by some x' of greater or equal value.

Remark 2.56. We could prove similar results for the case when $x' > x$ for row insertion, and for the case when $x \geq x'$ for column insertion, but we will not need such results in this thesis.

A corresponding pair of results apply to column insertion.

Lemma 2.57. *[16, §A.2 Exercise 3] Consider the tableau $x' \to (x \to t)$ with the first insertion adding a box B and the second a box B'. If $x < x'$, then B' is strictly below and weakly to the left of B.*

Corollary 2.58. *Suppose that $t = x' \to (x \to \tilde{t})$ with $x < x'$. Then the two boxes added in the insertion process are in different rows.*

Corollary 2.59. *Suppose one is given a tableau t of shape λ with two designated boxes identified at the end of columns i and j respectively, with the boxes in different rows. There are unique x, x' and a unique tableau $\tilde{t}$ of shape $\lambda - \zeta_i - \zeta_j$ such that $x < x'$ and $t = x' \to (x \to \tilde{t})$.*

2.3 The Wreath Product $S_m \wr S_n$

We recall the wreath product of two symmetric groups and examine its modules. Wreath products are intimately related to plethysm coefficients and the ramified partition algebra, both of which will be discussed at length in this thesis. We will mostly follow [6], but [28] is another good source using slightly different notation. Although [6] is phrased in terms of general wreath products of algebras, we will only consider the wreath product of two symmetric groups, or subgroups thereof. We use slightly different notation so as to work initially with groups rather than group algebras.

Definition 2.60. [28, §4.1] Let S_m and S_n be symmetric groups. As a set, the wreath product $S_m \wr S_n$ is the Cartesian product of S_n with n copies of S_m, with a typical element written as

$$(\alpha_1, \dots, \alpha_n; \tau),$$

where τ lies in S_n and $\alpha_i \in S_m$ for each i.

The multiplication is given by

$$(\alpha_1, \dots, \alpha_n; \tau)(\beta_1, \dots, \beta_n; \pi) = (\alpha_1\beta_{\tau^{-1}(1)}, \dots, \alpha_n\beta_{\tau^{-1}(n)}; \tau\pi).$$

It is clear that we may replace S_n with one of its Young subgroups S_γ in order to obtain a sensible group, which we will refer to as the wreath product $S_m \wr S_\gamma$.

Loosely speaking, we have a copy of S_n which permutes n copies of S_m, which interact with each other according to that matching. It is clear that $S_m \wr S_n$ contains the subgroup $\{(\alpha_1, \dots, \alpha_n; \mathrm{Id}_{S_n}) | \alpha_i \in S_m\} \cong S_m \times \cdots \times S_m$ with n copies of S_m, called the base group, and thus contains many subgroups isomorphic to S_m. This subgroup is normal, and its quotient, the top group, is isomorphic to S_n. This quotient is explicitly given by the map $(\alpha_1, \dots, \alpha_n; \tau) \mapsto \tau$.

The group $S_m \wr S_n$ acts naturally on mn letters. To see this, divide the set $\{1, \ldots, mn\}$ into n subsets $\{1, \ldots, m\}, \ldots, \{(n-1)m+1, \ldots, nm\}$. Each copy of S_m acts on one of these sets and S_n permutes the sets around.

More precisely, the element $(\alpha_1, \ldots, \alpha_n; \tau)$ acts as the permutation [28, 4.1.18]

$$\begin{pmatrix} (j-1)m+i \\ (\tau(j)-1)m+\alpha_{\tau(j)}(i) \end{pmatrix}_{1 \le i \le m, 1 \le j \le n}.$$

In particular, it is easy to see that $S_m \wr S_n$ is a subgroup of S_{mn}.

Example 2.61. *Consider the element* $((1,2), \mathrm{Id}, (2,3); (1,2,3)) \in S_3 \wr S_3$. *We represent this as an element of* S_9 *as follows. First, divide* $\{1, \ldots, 9\}$ *into subsets* $\{1,2,3\}, \{4,5,6\}, \{7,8,9\}$. *The element* $(1,2,3)$ *acts by permuting these subsets, with the other elements permuting within some individual subset. The resulting permutation is*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 9 & 8 & 2 & 1 & 3 \end{pmatrix} = (1,4,7,2,5,9,3,6,8),$$

since $(1,2,3)$ *dictates that the* $1st$ *subset be replaced by the* $2nd$ *and so on, with* $(2,3)$ *swapping the last two elements of the subset* $\{7,8,9\}$ *and* $(1,2)$ *swapping the first two elements of the set* $\{1,2,3\}$.

Let us consider the representation theory of the group algebra $k(S_m \wr S_n)$. We will consider left modules, constructing the "Specht" modules for this group. We cite [6, §3] and follow the notation of [20, §4.3] for the appropriate ideas.

First, suppose that X is a left kS_m-module. Define $X^{\tilde{\boxtimes} n}$ to be the $k(S_m \wr S_n)$-module which is the vector space $X^{\otimes n}$ with action

$$(\alpha_1, \ldots, \alpha_n; \tau)(x_1 \otimes \ldots \otimes x_n) = (\alpha_1 x_{\tau^{-1}1}) \otimes \cdots \otimes (\alpha_n x_{\tau^{-1}n}).$$

In a more general fashion, let $X_1, \ldots, X_t$ be a collection of kS_m-modules, with $\gamma \vdash n$ such that $\ell(\gamma) = t$. Define $(X_1, \ldots, X_t)^{\tilde{\boxtimes} \gamma}$ to be the vector space $(X_1^{\otimes \gamma_1}) \otimes \cdots \otimes (X_t^{\otimes \gamma_t})$ with an action of $S_m \wr S_\gamma$:

$$(\alpha_1, \ldots, \alpha_n; \tau)(x_1 \otimes \ldots \otimes x_n) = (\alpha_1 x_{\tau^{-1}1}) \otimes \cdots \otimes (\alpha_n x_{\tau^{-1}n})$$

where each x_i lies in the appropriate X_j. This action is well defined as the fact $\tau \in S_\gamma$ implies that when the ith coordinate corresponds to X_j, so does the $\tau^{-1}(i)$th coordinate.

Equivalently, view $X_i^{\tilde{\boxtimes}\gamma_i}$ as a $k(S_m \wr S_{\gamma_i})$-module as above. One may consider the outer tensor product of modules $X_1^{\gamma_1} \boxtimes \cdots \boxtimes X_t^{\gamma_t}$ for the group algebra $k(S_m \wr S_{\gamma_1}) \otimes \cdots \otimes k(S_m \wr S_{\gamma_t})$. This is exactly the module $(X_1, \ldots, X_t)^{\tilde{\boxtimes}\gamma}$ for the group algebra $k(S_m \wr S_\gamma) \cong k(S_m \wr S_{\gamma_1}) \otimes \cdots \otimes k(S_m \wr S_{\gamma_{\ell(\gamma)}})$.

Next, suppose that Y is a kS_n-module. Then Y can easily be seen as a $k(S_m \wr S_n)$-module by inflation,

$$(\alpha_1, \ldots, \alpha_n; \tau)y = \tau y.$$

Refer to this module as $\mathrm{Inf}_{S_n}^{S_m \wr S_n} Y$, since it is an inflation from the quotient group S_n of $S_m \wr S_n$.

Suppose that Y is a kS_n-module, with Z a $k(S_m \wr S_n)$-module. Define $Z \oslash Y$ to be the left $k(S_m \wr S_n)$ module $Z \otimes \mathrm{Inf}_{S_n}^{S_m \wr S_n} Y$. Similarly, if Z is a $k(S_m \wr S_\gamma)$-module and Y is a kS_γ-module, then $Z \oslash Y = Z \otimes \mathrm{Inf}_{S_\gamma}^{S_m \wr S_\gamma} Y$ is a module for $S_m \wr S_\gamma$. To be explicit, the action is given by

$$(\alpha_1, \cdots, \alpha_n; \tau)(z \otimes y) = (\alpha_1, \cdots, \alpha_n; \tau)z \otimes (\tau y).$$

We are now ready to define the Specht modules for the wreath product $S_m \wr S_n$. These are indexed by the *t-multipartitions* of n, where t is the number of partitions of m.

Definition 2.62. [21, Above Thm 6] A t-multipartition $\underline{\lambda}$ of n is a t-tuple $(\lambda_1, \ldots, \lambda_t)$ of (possibly empty) partitions such that $|\lambda_1| + \cdots + |\lambda_t| = n$.

Example 2.63. *We will take several choices of m and n and give some examples of multipartitions indexing Specht modules for $S_m \wr S_n$.*

- *Suppose we are looking for multipartitions for $S_2 \wr S_5$. There are 2 partitions of 2, therefore we seek 2-multipartitions of 5, that is, pairs of partitions (λ_1, λ_2) such that $|\lambda_1| + |\lambda_2| = 5$.*

  Examples are $((5), \emptyset), ((2,1), (1^2))$ and $((2), (1^3))$.

- *Suppose we are looking for multipartitions for $S_4 \wr S_7$. There are 5 partitions of 4, therefore we seek 5-multipartitions of 7.*

  Examples are

  $$((1), \emptyset, (3,2), (1), \emptyset), (\emptyset, \emptyset, (2,1^5), \emptyset, \emptyset)$$

  and

  $$((2,1), (1), (1), (1), (1)).$$

We will always represent multipartitions by underlined symbols such as $\underline{\lambda}$. Now, write $\mu^1 > \cdots > \mu^t$ for the list of all partitions of m in lexicographic order.

We define the *Specht module* [20, §4.4] for the wreath product corresponding to the multipartition $\underline{\lambda}$ to be

$$S^{\underline{\lambda}} = \left[\left(S^{\mu^1}, \ldots, S^{\mu^t} \right)^{\tilde{\boxtimes} |\underline{\lambda}|} \oslash \left(S^{\lambda_1} \boxtimes \cdots \boxtimes S^{\lambda_t} \right) \right] \Big\uparrow_{S_m \wr S_{|\underline{\lambda}|}}^{S_m \wr S_n}.$$

We will not need to do in-depth computations with these modules in this thesis, and therefore we include only some simpler examples where the induction is trivial.

Example 2.64. *Consider the wreath product $S_3 \wr S_2$ of symmetric groups. There are 3 partitions of 3. In lexicographic order they are $(3) > (2,1) > (1^3)$. Therefore the Specht modules for $k(S_3 \wr S_2)$ are indexed by 3-multipartitions of 2.*

An example of such a multipartition is $(\emptyset, (1^2), \emptyset)$, which corresponds to the module $M = (S^{(2,1)})^{\tilde{\boxtimes} 2} \oslash S^{(1^2)}$. A basis for $S^{(2,1)}$ is

$$\left\{ e_u = \frac{\overline{1\ 2}}{\underline{3}} - \frac{\overline{2\ 3}}{\underline{1}}, e_v = \frac{\overline{1\ 3}}{\underline{2}} - \frac{\overline{2\ 3}}{\underline{1}} \right\}$$

and a basis for $S^{(1^2)}$ is

$$\left\{ e_w = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} - \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \right\}.$$

One may choose the element $(e_u \otimes e_v \oslash e_w) \in M$ and consider the action of some elements of $S_3 \wr S_2$. For example,

$$\begin{aligned} ((2,3),(1,2,3);\mathrm{Id})(e_u \otimes e_v \oslash e_w) &= (2,3)e_u \otimes (1,2,3)e_v \oslash \mathrm{Id}_{S_3} e_w \\ &= e_v \otimes (e_u - e_v) \oslash e_w = e_v \otimes e_u \oslash e_w - e_v \otimes e_v \oslash e_w, \end{aligned}$$

whereas

$$\begin{aligned} ((1,2),(1,3);(1,2))(e_u \otimes e_v \oslash e_w) &= ((1,2)e_v \otimes (1,3)e_u \oslash (1,2)e_w) \\ &= (-e_v) \otimes (-e_u) \oslash (-e_w) = -(e_v \otimes e_u \oslash e_w). \end{aligned}$$

One may induce Specht modules $(S^\mu)^{\tilde{\boxtimes} n} \oslash S^\nu$ for $S_m \wr S_n$ to S_{mn}, and these modules are called *Foulkes modules*, if $\mu = (m)$ and $\nu = (n)$, or *generalised Foulkes modules.* For example, we write $H^{(m^n)}$ for the Foulkes module $\left[(S^{(m)})^{\tilde{\boxtimes} n} \oslash S^{(n)}\right] \uparrow_{S_m \wr S_n}^{S_{mn}}$.

We write set partitions of a set S by writing the contents in S in set brackets, with different parts of the set partition separated by a $|$ symbol. For example, the set partition of $\{a, b, c, d, e\}$ with parts $\{a, c\}, \{b, e\}$ and $\{d\}$ is written as $\{a, c|b, e|d\}$.

One may view this module $H^{(m^n)}$ as the permutation action of S_{mn} on the collection of set partitions of $\{1, \cdots, mn\}$ into n sets, each of which is of size m. To see this explicitly, notice that $(S^{(m)})^{\tilde{\boxtimes} n} \oslash S^{(n)}$ is the trivial module for $S_m \wr S_n$, and that the usual embedding of $S_m \wr S_n$ into S_{mn} is the stabiliser of the set partition $\{1 \dots m| \cdots |(mn - m + 1) \dots mn\}$.

Example 2.65. *We choose the module $H^{(3^2)}$ to illustrate this class of modules. A basis of $H^{(3^2)}$ is given by the collection of set partitions of $\{1, \dots, 6\}$ into two sets of equal size.*

For example, one basis element is $v = \{1, 2, 3|4, 5, 6\}$. The permutations $(14)(25)(36)$ *and* (12) *preserve v since they lie within the standard embedding of*

$S_3 \wr S_2$ into S_6. The permutation $(16)(25)$ does not lie within this embedding of the wreath product, and indeed $(16)(25)v = \{1, 2, 4|3, 5, 6\} \neq v$.

2.4 Symmetric Functions

Many questions in this thesis concern *plethysm coefficients*. These are readily defined in terms of *symmetric functions*. Here we will introduce symmetric functions and give an interpretation of them in terms of representations of symmetric groups. An excellent but expansive treatment of symmetric functions can be found in MacDonald's book [36], while Sagan's book [44] provides a condensed account which we will mainly follow.

A symmetric function is, broadly speaking, a polynomial in infinitely many variables which is invariant under the action of any symmetric group S_n permuting the first n variables. MacDonald makes this idea precise by using inverse limits in [36, §I.2], but we will instead give Sagan's definition with explicit generators.

Consider the formal power series ring $\mathbb{C}[[x_1, x_2, \ldots]]$. For $n \geq 1$, the symmetric group S_n acts on $\mathbb{C}[[x_1, x_2, \ldots]]$ by permuting the variables $x_1, \ldots, x_n$. One is interested in elements of $\mathbb{C}[[x_1, x_2, \ldots]]$ which are fixed by the action of S_n for every $n \geq 1$. A family of such fixed points is the *monomial symmetric functions*.

Definition 2.66. [44, Def 4.3.1] Let λ be a partition of length l. The monomial symmetric function m_λ is defined by

$$m_\lambda(x_1, x_2, \ldots) = \sum x_{i_1}^{\lambda_1} \cdots x_{i_l}^{\lambda_l}$$

where the sum runs over indices such that every monomial with exponents $\lambda_1, \ldots, \lambda_l$ appears exactly once.

Example 2.67. *Some monomial symmetric functions are*

$$m_{(1)} = x_1 + x_2 + x_3 + \cdots$$

$$m_{(2)} = x_1^2 + x_2^2 + x_3^2 + \cdots$$

$$m_{(3,2)} = x_1^3 x_2^2 + x_1^2 x_2^3 + x_1^3 x_3^2 + x_1^2 x_2^3 + \cdots$$

These are the simplest symmetric functions, and the ring of symmetric functions is generated by them as a vector space. We take this as our definition.

Definition 2.68. [44, Def 4.3.2] The ring $\Lambda^{\mathrm{sym}}(x_1, x_2, \ldots)$ of symmetric functions is the vector subspace of $\mathbb{C}[[x_1, x_2, \ldots]]$ spanned by $\{m_\lambda | \lambda \vdash n$ where $n = 1, 2, 3, \ldots\}$, viewed as a subring with the usual product.

Remark 2.69. [44, §4.3] The ring $\Lambda^{\mathrm{sym}}(x_1, x_2, \ldots)$ does not contain, for example, $\Pi_{i\geq 1}(1+x_i)$, which is a formal power series invariant under the permutation action of symmetric groups. This is because $\Pi_{i\geq 1}(1+x_i)$ cannot be expressed as a finite linear combination of m_λ where λ is some partition.

The ring Λ^{sym} is a graded ring with the nth graded layer Λ_n^{sym} consisting of those symmetric functions of homogeneous degree n. There are several bases of interest for Λ_n^{sym}. One can review Definition 4.3.4 and Theorem 4.3.7 from [44] in order to learn more, but we will immediately focus on the famous *Schur functions*.

Definition 2.70. Given a tableau T of shape $\lambda \vdash n$ and weight μ, one may define the monomial

$$\mathbf{x}^T := \prod_{(i,j)\in[\lambda]} x_{T_{i,j}},$$

a monomial of degree n whose nonzero exponents, when listed in decreasing order, give the partition μ.

Example 2.71. *Supppose that T is the tableau*

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

of weight $(1, 2, 2, 1, 1, 1)$*. Then* $\mathbf{x}^T = x_1 x_2^2 x_3^2 x_4 x_5 x_6$ *of degree* 8.

There are several explicit definitions of the Schur functions, but it is simplest to define them explicitly in terms of these monomials.

Definition 2.72. [44, Def 4.4.1] Suppose that $\lambda \vdash n$ is a partition. The *Schur function* associated to λ is the element of $\mathbb{C}[[x_1, x_2, \ldots]]$, homogeneous of degree n, defined by

$$s_\lambda(x_1, x_2, \ldots) = \sum_{T \in \mathrm{SStd}(\lambda, \bullet)} \mathbf{x}^T,$$

where $\mathrm{SStd}(\lambda, \bullet)$ is the set of semistandard tableaux of shape λ and any weight.

Example 2.73. *Some examples of Schur functions are:*

$$s_\emptyset = 1$$

$$s_{(1)} = x_1 + x_2 + \cdots$$

$$s_{(2)} = x_1^2 + x_2^2 + x_1 x_2 \cdots .$$

$$s_{(1^2)} = x_1 x_2 + x_1 x_3 + x_2 x_3 + \cdots$$

Theorem 2.74. *[44, Cor 4.4.4.] Each Schur function s_λ is a symmetric function of homogeneous degree $n = |\lambda|$, and the set*

$$\{s_\lambda | \lambda \vdash n\}$$

is a basis for the symmetric functions Λ_n^{sym} of homogeneous degree n.

The symmetric functions are intimately related to the representation theory of the symmetric group through the *characteristic map*, and the Schur functions have a particularly important role to play in this case. Let $\mathrm{Cl}(S_n)$ be the ring of class functions on S_n with the usual inner product and write χ^λ for the ordinary character of the Specht module S^λ. There is a natural Hermitian inner product on Λ_n^{sym} given by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.

Proposition 2.75. *[44, Def 4.7.1, Prop 4.7.2] There is an isometry*

$$\mathrm{ch}^n : \mathrm{Cl}(S_n) \to \Lambda_n^{\mathrm{sym}},$$

known as the characteristic map, such that

$$\mathrm{ch}^n(\chi^\lambda) = s_\lambda.$$

The fact that $\mathrm{Cl}(S_n)$ and Λ_n^{sym} are isometric means that we may study questions regarding the representation theory of S_n by studying the symmetric functions of degree n.

2.4.1 The Littlewood-Richardson Rule

We will briefly study the famous *Littlewood-Richardson coefficients* $c^\lambda_{\nu,\mu}$, as their calculation is used in later results on plethysm coefficients.

Definition 2.76. [36, §9] The Littlewood-Richardson coefficients $c^\lambda_{\nu,\mu}$ are defined to be the structure constants of the usual multiplication on Schur functions. That is, if $\mu \vdash m, \nu \vdash n$ and $\lambda \vdash (m+n)$, then

$$s_\nu s_\mu = \sum_\lambda c^\lambda_{\nu,\mu} s_\lambda.$$

Equivalently, $c^\lambda_{\nu,\mu} := \langle s_\nu s_\mu, s_\lambda \rangle$.

Remark 2.77. By the theory of symmetric functions found for example in [44, §4.9] or [36, §9], using the characteristic map these are also known to be the multiplicities of Specht modules in the induced outer tensor product of two Specht modules. Explicitly,

$$c^\lambda_{\nu,\mu} = [S^\lambda : (S^\nu \boxtimes S^\mu) \uparrow^{S_{n+m}}_{S_n \times S_m}].$$

We will not use this fact explicitly, but it is an important context in which Littlewood-Richardson coefficients are used, and it makes clear the fact that $c^\lambda_{\nu,\mu} = c^\lambda_{\mu,\nu} \geq 0$ without difficult argument.

Definition 2.78. [27, §15] Suppose that $\mu \vdash k$. A *word of shape* μ is a sequence of k integers in $\{1, \ldots, l\}$ such that there are μ_1 ones, μ_2 twos, and so on.

A *Littlewood-Richardson word of shape* μ is a word w of shape μ obeying the additional condition that for any i and j, the subsequence consisting of the first j terms of w always has at least as many occurrences of i as it does of $i+1$.

Note that the phrase "Littlewood-Richardson word" is not used in [27], though the objects in question are described there. Other sources, such as [36], use the phrase "lattice permutation".

Example 2.79. *The Littlewood-Richardson words of type* $(3,2)$ *are*

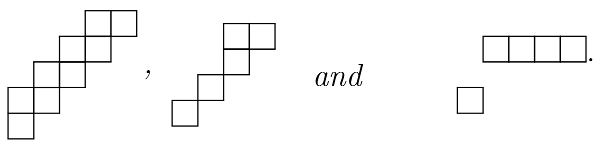
$$12121, 12112, 11221, 11212, 11122.$$

The sequence 21211 *is not a Littlewood-Richardson word as the subsequence consisting of the first element, or indeed the first three elements, has more instances of the number* 2 *than* 1.

There is a well-known theorem on the calculation of Littlewood-Richardson coefficients, known as the *Littlewood-Richardson rule.* This theorem uses fillings of skew diagrams, which we now recall.

Definition 2.80. Given two partitions λ and ν where $\nu \subset \lambda$, we call the pair of partitions represented by the symbol λ/ν a skew partition. The skew diagram of λ/ν is the diagram obtained from the Young diagram of λ by deleting the first ν_i nodes from row i for $1 \leq i \leq \ell(\nu)$.

Example 2.81. *The diagrams of the skew partitions* $(5,4,3,2,1)/(3,2,1)$, $(4,3,2,1)/(2,2,1)$ *and* $(5,1^2)/(1^2)$ *are respectively:*



Theorem 2.82 (Littlewood-Richardson Rule)**.** *[27, §16] [36, 9.2] Suppose* $\mu \vdash m, \nu \vdash n$ *and* $\lambda \vdash m+n$*. Then the Littlewood-Richardson coefficient* $c^{\lambda}_{\nu,\mu}$ *is equal to the number of fillings of the skew diagram* λ/ν *with a word of shape* μ *such that:*

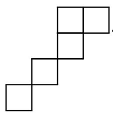
- *Filled numbers weakly increase along rows, left to right.*
- *Filled numbers strictly decrease down columns.*
- *Reading the filled numbers from right to left, and each row after the one above, one obtains a Littlewood-Richardson word.*

When $\nu \not\subseteq \lambda$, there is no such skew shape λ/ν to fill and thus the associated coefficient $c^{\lambda}_{\nu,\mu}$ is equal to 0*.*

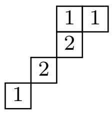
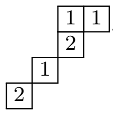
It is clear that since $c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$, we can just as easily exchange the roles of μ and ν in this rule to calculate $c^{\lambda}_{\mu,\nu}$, filling λ/μ with words of shape ν.

Remark 2.83. When $\mu = \emptyset$, $c^{\lambda}_{\mu,\nu} = 1 = c^{\lambda}_{\nu,\mu}$ if and only if $\lambda = \nu$. This mirrors the fact that $(S^{\nu} \boxtimes \{0\}) \uparrow^{n}_{n+0} = S^{\nu}$.

Example 2.84. *Let us calculate the Littlewood-Richardson coefficient $c^{(4,3,2,1)}_{(2,2,1),(3,2)}$. The appropriate skew diagram here is $(4,3,2,1)/(2,2,1)$:*

.

We already saw in Example 2.79 that the Littlewood-Richardson words of shape $(3,2)$ *are* 12121, 12112, 11221, 11212 *and* 11122*, so let us consider filling this skew tableaux from right to left and down the rows with these words. The first two words cannot work, as they both lead to a first row* 21 *which does not weakly increase. The last word also does not work as it places a* 1 *in the same column as another* 1*, which is not allowed. The other two words give valid fillings; in pictures these are*

 *and* .

Two families of Littlewood-Richardson coefficients are more easily described by Young's Rule and Pieri's Rule. Before we recall these two results, we establish some notation.

Notation. Suppose μ is a partition and n a positive integer. Write $\lambda \in S_n^c(\mu)$ if λ may be obtained from μ by adding n boxes to the end of rows, such that no two boxes are in the same column. Similarly, write $\lambda \in S_n^r(\mu)$ if λ may be obtained from μ by adding n boxes to the end of rows, such that no two boxes are in the same row. Write $\lambda = \mu + \square$ when $\lambda \in S_1^c(\mu) = S_1^r(\mu)$ as a shorthand.

Write $\lambda \in S_{-n}^c(\mu)$ when λ can be obtained from μ by deleting n boxes, no two of which are in the same column. That is, $\lambda \in S_{-n}^c(\mu) \iff \mu \in S_n^c(\lambda)$. Define $S_{-n}^r(\mu)$ similarly.

For the avoidance of doubt, define $S_0^c(\mu) = S_0^r(\mu) = \{\mu\}$.

Young's and Pieri's Rule are results that have been well-known in various forms for many years. We observe how they follow from the Littlewood-Richardson rule.

Lemma 2.85 (Young's Rule)**.** *[28, 2.8.2] Let $n \geq 0$, $\mu \vdash m$ and $\lambda \vdash m+n$. Then*

$$c_{\mu,(n)}^{\lambda} = \begin{cases} 1 & \text{if } \lambda \in S_n^c(\mu) \\ 0 & \text{else.} \end{cases}$$

Proof. This follows from the fact that the only Littlewood-Richardson word of weight (n) is $1 \cdots 1$, and that two ones must not be placed in the same column. $\square$

Lemma 2.86 (Pieri's Rule)**.** *Let $n \geq 0$, $\mu \vdash m$ and $\lambda \vdash m+n$. Then*

$$c_{\mu,(1^n)}^{\lambda} = \begin{cases} 1 & \text{if } \lambda \in S_n^r(\mu) \\ 0 & \text{else.} \end{cases}$$

Proof. This follows from the fact that the only Littlewood-Richardson word of weight (1^n) is $12 \cdots n$, and numbers may not decrease left to right along rows when placed in a skew tableau. $\square$

2.4.2 Plethysm Coefficients

We have already seen how to use the Littlewood-Richardson rule to calculate the product of two Schur functions in the ring of symmetric functions. We now recall a different product on symmetric functions called plethysm. Plethysm is somewhat like composition of functions. One can find a brief discussion of plethysm in [36, §8], but we draw from Stanley's account in [47].

Definition 2.87. [47, 7A2.6] Suppose that $g \in \Lambda^{\mathrm{sym}}$ is a symmetric function such that

$$g = \sum_{i \geq 1} x^{a^i}.$$

Given $f \in \Lambda^{\mathrm{sym}}$, define

$$f \circ g = f(x^{a^1}, x^{a^2}, ...).$$

We say $f \circ g$ is the plethysm of f and g.

Theorem 2.88. *[47, Thm 2.7] [36, 8.10] If μ and ν are partitions, then the plethysm $s_\nu \circ s_\mu$ is an integral linear combination of Schur functions with non-negative coefficients.*

Example 2.89. *Since the Schur function $s_\emptyset$ is just the polynomial 1, one may easily calculate:*

$$s_\emptyset \circ s_\mu = s_\emptyset$$

$$s_\nu \circ s_\emptyset = s_\emptyset.$$

The Schur polynomial $s_{(1)}$ is simply equal to $x_1 + x_2 + \cdots$, so

$$s_\nu \circ s_{(1)} = s_\nu(x_1, x_2, \ldots) = s_\nu.$$

Similarly, $s_{(1)} \circ s_\mu$ is just the sum of the monomials appearing in s_μ, being s_μ itself:

$$s_{(1)} \circ s_\mu = s_\mu.$$

Example 2.90. *We compute some slightly less trivial small examples which will be needed in the latter half of section 2. We will only compute the terms involving the first two variables. We start by computing*

$$\begin{aligned} s_{(2)} \circ s_{(2)} &= s_{(2)}(x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2, \ldots) \\ &= (x_1^2)^2 + (x_1)^2x_1x_2 + (x_1x_2)^2 + x_1^2x_2^2 + x_1x_2x_2^2 + (x_2^2)^2 + \cdots \\ &= (x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4 + \cdots) + (x_1^2x_2^2 + \cdots) \\ &= s_{(4)} + s_{(2,2)}. \end{aligned} \tag{2.1}$$

Throughout this thesis we will verify computations using SAGE [48], for this computation see SAGE Computation A.1.

We also compute

$$\begin{aligned} s_{(1^2)} \circ s_{(2)} &= s_{(1^2)}(x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2, \ldots) \\ &= x_1^3x_2 + x_1^2x_2^2 + x_1x_2x_2^2 + \cdots \\ &= s_{(31)} \end{aligned} \tag{2.2}$$

For this computation see SAGE Computation A.2.

The general calculation of $s_{(n)} \circ s_{(2)}$ is well known.

Lemma 2.91. *[36, §8 Ex6] One may calculate the plethysm*

$$s_{(n)} \circ s_{(2)} = \sum s_\mu$$

where μ runs over those partitions of $2n$ such that all parts are even.

There is a famous result on plethysm coefficients known as the *sign twist.* We state it here, without going into the mechanics of why it works.

Lemma 2.92 (Sign Twist). *[36, §1.8 Ex 1] There is an equality of plethysm coefficients*

$$\langle s_\nu \circ s_\mu, s_\lambda \rangle = \langle s_{\nu^*} \circ s_{\mu'}, s_{\lambda'} \rangle,$$

where $\nu^* = \nu$ when $|\mu|$ is even and $\nu^* = \nu'$ when $|\mu|$ is odd.

Corollary 2.93. *[36, §8 Ex6] One may calculate the plethysm*

$$s_{(n)} \circ s_{(1^2)} = \sum s_\lambda,$$

where λ runs over all partitions of $2n$ where each part is repeated an even number of times.

Proof. Although one may prove this in other methods, we may just apply a sign twist to Lemma 2.91 and observe that μ having all parts even is equivalent to μ' having each part repeated an even number of times. $\square$

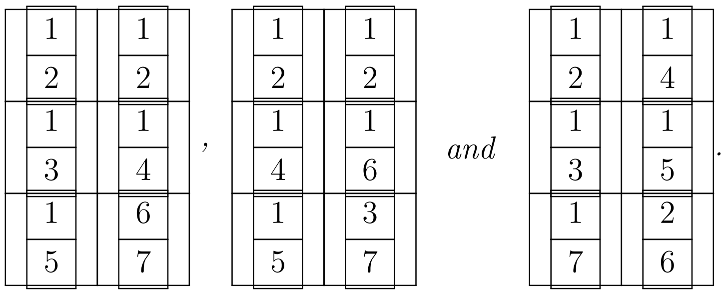
A key combinatorial object in the calculation of plethysm is the *plethystic semistandard Young tableau.* These have been used in many places, see for example [10].

Definition 2.94. Suppose that $\mu \vdash m, \nu \vdash n$ and $\lambda \vdash mn$. A plethystic semistandard Young tableau of shape μ^ν and weight λ is a Young diagram of shape ν where each box contains a semistandard Young tableau of shape μ, where the number of entries equal to i across all the boxes of all the μ-tableaux is equal to λ_i. The μ-tableaux in the boxes of the ν-tableau must weakly increase along rows from left to right and strictly increase down columns according to the $\prec$ order on tableaux we defined in Definition 2.24.

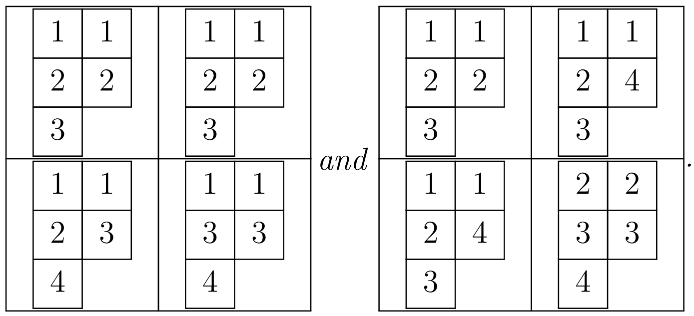
Write $\mathrm{PStd}(\mu^\nu, \lambda)$ for the set of all plethystic semistandard Young tableaux of shape μ^ν and weight λ. We will sometimes refer to the ν-tableau as the outer diagram, and the μ-tableaux contained in its boxes as inner tableaux.

Example 2.95. $\mathrm{PStd}((1)^\nu, \lambda) = \mathrm{SStd}(\nu, \lambda)$ *and* $\mathrm{PStd}(\mu^{(1)}, \lambda) = \mathrm{SStd}(\mu, \lambda)$ *by replacing single boxes by their filling.*

Some elements of $\mathrm{PStd}((1^2)^{(2^3)}, (5, 2, 1^5))$ *are*



Two plethystic semistandard Young tableaux of shape $(2^2, 1)^{(2^2)}$ *are*



These are of weight $(8, 5, 5, 2)$ *and* $(6, 6, 5, 3)$ *respectively.*

One might consider the plethysm of two Schur functions, expanded in terms of Schur functions. It would be natural to write these in lexicographic order. The lexicographically maximal partition in particular is known for any plethysm of Schur functions by the following theorem.

Theorem 2.96. *[42, Cor 9.1] [26, Thm 4.2] Let* $\mu \vdash m$ *and* $\nu \vdash n$ *be partitions. The maximal term of* $s_\nu \circ s_\mu$ *in the lexicographic order is*

$$\lambda_{\max} = (n\mu_1, \ldots, n\mu_{\ell(\mu)-1}, n\mu_{\ell(\mu)} - n + \nu_1, \nu_2, \ldots, \nu_{\ell(\nu)}),$$

and the corresponding plethysm coefficient $\langle s_\nu \circ s_\mu, s_{\lambda_{\max}} \rangle$ *is equal to* 1.

Notation. Some special cases of this result are used later, so we give them their own notation:

- When $\mu = (2)$, $\lambda_{\max}$ is equal to $\bar{\nu} = (n + \nu_1, \nu_2, \ldots, \nu_{\ell(\nu)})$.
- When $\mu = (1^2)$, $\lambda_{\max}$ is equal to $\tilde{\nu} = (n, \nu_1, \nu_2, \ldots, \nu_{\ell(\nu)})$.

We make extensive use of the following proposition in the next few sections; it will be used to prove results about plethysm coefficients inductively.

Proposition 2.97. *[1, Prop 2.9] Suppose that μ, ν and λ are partitions, then*

$$\langle s_\nu \circ s_\mu, s_\lambda \rangle = |\operatorname{PStd}(\mu^\nu, \lambda)| - \sum_{\substack{\beta \rhd \lambda \\ \beta \leq \lambda_{\max}}} \langle s_\nu \circ s_\mu, s_\beta \rangle |\operatorname{SStd}(\beta, \lambda)|.$$

Chapter 3

Calculating Plethysm Coefficients

3.1 Near-maximal constituents of $s_\nu \circ s_{(m)}$

As we saw in the background, the decomposition of $s_{(n)} \circ s_{(2)}$ is completely understood. Advancing from this, the general plethysm $s_\nu \circ s_{(2)}$ is still far from understood.

In their paper [1], Bessenrodt, Bowman and Paget give a combinatorial description for certain near-maximal constituents of the plethysm $s_\nu \circ s_{(2)}$. In particular, they calculate $\langle s_\nu \circ s_{(2)}, s_\lambda \rangle$ when $\lambda_1 = n + \nu_1$ with no restriction on ν, and also when $\lambda_1 = n + \nu_1 - 1$, provided that $\nu \neq (n)$.

Firstly, we present the important results from [1, §4], referring the reader to the paper to read the proofs. We then use a straightforward generalisation of the proofs in that paper to prove the same results but replacing the partition (2) with the partition (m) where $m \geq 2$. That is, we will calculate coefficients for some near-maximal constituents of the plethysm $s_\nu \circ s_{(m)}$. The methods used involve establishing bijections between the family of plethystic tableaux of shape $(m)^\nu$ with weight λ and families of semistandard tableaux, also of weight λ but of various shapes. The results are then proved by an application of Proposition 2.97, which links plethysm coefficients, plethystic tableaux and semistandard tableaux.

3.1.1 Summary of the Case $m = 2$

In this subsection we will summarise the results of Bessenrodt, Bowman and Paget [1] with some examples of our own added to demonstrate the content. Throughout, let $\nu \vdash n$ and $\lambda \vdash 2n$. Recall that we defined $\bar{\nu} = (n + \nu_1, \dots, \nu_{\ell(\nu)})$.

The first two lemmas are a summary of the initial discussion [1, §4] concerning the case where $\lambda_1 = n + \nu_1$. We will see their simple proofs when we generalise later, replacing (2) with (m).

Lemma 3.1. *[1, §4] There is an equality* $|\operatorname{PStd}((2)^\nu, \lambda)| = |\operatorname{SStd}(\bar{\nu}, \lambda)|$ *when* $\lambda_1 = n + \nu_1$.

Using this result together with Proposition 2.97 gives a simple proof of the following lemma.

Lemma 3.2. *[1, Eqn 4.2] Suppose that* $\nu \vdash n$ *and* $\lambda_1 = n + \nu_1$*, then*

$$\langle s_\nu \circ s_{(2)}, s_\lambda \rangle = \begin{cases} 1 & \lambda = \bar{\nu} \\ 0 & \textit{else.} \end{cases}$$

Now, let us turn our attention to the case where instead $\lambda_1 = n + \nu_1 - 1$. From this point on, we must assume that $\nu \neq (n)$, leaving out an easy known case. Indeed, the plethysm $s_{(n)} \circ s_{(2)}$ is understood in full as we observed in Lemma 2.91. We must recall some definitions and notation from [1] in order to state the value of the plethysm coefficients $\langle s_\nu \circ s_{(2)}, s_\lambda \rangle$ in combinatorial terms.

Definition 3.3. Let $\nu \vdash n$. Define $M(\nu)$ to be the set of all partitions $\beta \vdash 2n$ such that β is obtainable from $\bar{\nu}$ by removing one box from the first row, and one box from a row below the first row, before adding two boxes in rows $a \geq b \geq 2$ under the condition that $\beta_a \neq \beta_b$ when $a \neq b$. This condition $\beta_a \neq \beta_b$ means simply that the added boxes are not in the same column.

Given $\beta \in M(\nu)$, write $I_\nu(\beta)$ for the set of possible pairs (a, b) in the above description. If $\beta \notin M(\nu)$, let $I_\nu(\beta) = \emptyset$.

We give two examples. Later we will need to differentiate between the cases where $\nu_1 = \nu_2$ and when $\nu_1 \neq \nu_2$, so we give an example of each case.

Example 3.4. *Let $\nu = (3,2,2,1) \vdash 8$, so that $\bar{\nu} = (11,2,2,1)$. The possibilities obtainable by removing a box each from the first row and a lower row of $\bar{\nu}$ are $(10,2,1,1)$ and $(10,2,2)$. The values $\lambda \in M(\nu)$ together with the sets $I_\nu(\lambda)$ are summarised in the following table.*

λ	$I_\nu(\lambda)$
$(10,4,2)$	$\{(2,2)\}$
$(10,4,1,1)$	$\{(2,2)\}$
$(10,3,2,1)$	$\{(3,2),(4,2)\}$
$(10,3,1^3)$	$\{(5,2)\}$
$(10,2^3)$	$\{(4,4)\}$
$(10,2^2,1^2)$	$\{(5,3)\}$

Note that the partition $(10,2^2,1^2)$ is obtainable from $(10,2,2)$ by adding boxes in the 4th and 5th rows, but we do not count this since two boxes are added in the same column.

Example 3.5. *Now consider $\nu = (3,3,1) \vdash 7$, so that $\bar{\nu} = (10,3,1)$. The possibilities obtainable by removing a box each from the first row and a lower row of $\bar{\nu}$ are $(9,2,1)$ and $(9,3)$. The values $\lambda \in M(\nu)$ together with the sets $I_\nu(\lambda)$ are summarised in the following table.*

λ	$I_\nu(\lambda)$
$(9,5)$	$\{(2,2)\}$
$(9,4,1)$	$\{(2,2),(3,2)\}$
$(9,3,2)$	$\{(3,2),(3,3)\}$
$(9,3,1^2)$	$\{(4,2)\}$
$(9,2^2,1)$	$\{(4,3)\}$

In a situation similar to the previous example, we ignore the fact that $(9,3,1^2)$ can be obtained from $(9,3)$ by adding boxes in the 3rd and 4th rows, since these boxes would be added in the same column.

We will now recall two results we alluded to in the introduction of this section. These results of Bessenrodt, Bowman and Paget express the plethysm coefficients we mentioned in terms of $|I_\nu(\lambda)|$. We must distinguish between two cases, and we start with the case where $\nu_1 \neq \nu_2$.

Lemma 3.6. *[1, Cor. 4.6] Let $\nu \vdash n$ with $\nu_1 \neq \nu_2$ and $\lambda \vdash 2n$ with $\lambda_1 = n+\nu_1-1$. Then*

$$\langle s_\nu \circ s_{(2)}, s_\lambda \rangle = |I_\nu(\lambda)|.$$

In particular, $\langle s_\nu \circ s_{(2)}, s_\lambda \rangle = 0$ unless $\lambda \in M(\nu)$.

Example 3.7. *Refer back to Example 3.4 with $\nu = (3,2,2,1)$. We extend the table from that example with the associated plethysm coefficients included.*

λ	$I_\nu(\lambda)$	$\langle s_{(3,2,2,1)} \circ s_{(2)}, s_\lambda \rangle = \lvert I_\nu(\lambda)\rvert$
$(10,4,2)$	$\{(2,2)\}$	1
$(10,4,1,1)$	$\{(2,2)\}$	1
$(10,3,2,1)$	$\{(3,2),(4,2)\}$	2
$(10,3,1^3)$	$\{(5,2)\}$	1
$(10,2^3)$	$\{(4,4)\}$	1
$(10,2^2,1^2)$	$\{(5,3)\}$	1

These coefficients can be verified by computer algebra as in SAGE Computation A.3.

Some minor changes are required when treating the case in which $\nu_1 = \nu_2$.

Lemma 3.8. *[1, Cor. 4.8] Let $\nu \vdash n$ with $\nu_1 = \nu_2$ and $\lambda \vdash 2n$ with $\lambda_1 = n+\nu_1-1$. Then*

$$\langle s_\nu \circ s_{(2)}, s_\lambda \rangle = \begin{cases} |I_\nu(\lambda)| - 1 & \lambda = (n) + \nu - \epsilon_1 + \epsilon_2 \\ |I_\nu(\lambda)| & \textit{otherwise.} \end{cases}$$

Again, $\langle s_\nu \circ s_{(2)}, s_\lambda \rangle = 0$ unless $\lambda \in M(\nu)$.

Example 3.9. *Referring back to our Example 3.5 with $\nu = (3,3,1)$, we extend the table from that example with information to calculate the associated plethysm coefficients.*

λ	$I_\nu(\lambda)$	$\lvert I_\nu(\lambda)\rvert$	$\lambda = \bar{\nu} - \epsilon_1 + \epsilon_2$?	$\langle s_{(3,3,1)} \circ s_{(2)}, s_\lambda\rangle$
$(9,5)$	$\{(2,2)\}$	1	$\times$	1
$(9,4,1)$	$\{(2,2),(3,2)\}$	2	$\checkmark$	1
$(9,3,2)$	$\{(3,2),(3,3)\}$	2	$\times$	2
$(9,3,1^2)$	$\{(4,2)\}$	1	$\times$	1
$(9,2^2,1)$	$\{(4,3)\}$	1	$\times$	1

These coefficients can be verified by computer algebra as in SAGE Computation A.4.

3.1.2 Generalisation to the case $m \geq 2$

Let $m \geq 2$. We now calculate the coefficient $\langle s_\nu \circ s_{(m)}, s_\lambda\rangle$ provided that $\lambda_1 = (m-1)n + \nu_1$ or $(m-1)n + \nu_1 - 1$. The proofs in this section are straightforward generalisations of the proofs for the results above presented in [1, §4] by Bessenrodt, Bowman and Paget. Their work only required the case where $m = 2$, as different techniques were used to garner knowledge about other plethysm coefficients from this case.

From Theorem 2.96, the lexicographically maximal constituent of $\langle s_\nu \circ s_{(m)}, s_\lambda\rangle$ is $\bar{\nu} := ((m-1)n + \nu_1, \nu_2, \cdots, \nu_{\ell(\nu)})$. We begin with the case where λ has first part $((m-1)n + \nu_1)$, the same as the first part of $\bar{\nu}$.

Lemma 3.10. *Suppose $\nu \vdash n, \lambda \vdash mn$ and $\lambda_1 = (m-1)n + \nu_1$. Then there is an equality* $|\operatorname{PStd}((m)^\nu, \lambda)| = |\operatorname{SStd}(\bar{\nu}, \lambda)|$.

Proof. We establish a bijection

$$\Phi : \operatorname{PStd}((m)^\nu, \lambda) \mapsto \operatorname{SStd}(\bar{\nu}, \lambda)$$

as follows. Suppose $T \in \operatorname{PStd}((m)^\nu, \lambda)$ is a plethystic semistandard Young tableau. Since there are $(m-1)n+\nu_1$ ones in T and only boxes in the first row may contain the minimal (m)-tableau $\boxed{1}\boxed{\cdots}\boxed{1}$, we know by the pigeonhole principle that each

(m)-tableau in a box of T is of the form $\boxed{1}\cdots\boxed{1}\boxed{x}$ for some $x \geq 1$. Note that in the first row we must have $x = 1$.

Define $\Phi(T)$ to be the tableau obtained from T by replacing each (m)-tableau $\boxed{1}\cdots\boxed{1}\boxed{x}$ with the single entry x, and then appending $(m-1)n$ ones to the beginning of the first row of the resulting tableau. The shape of this tableau is $\bar{\nu}$, the weight is unchanged and semistandardness arises from the fact that $\boxed{1}\cdots\boxed{1}\boxed{x} \leq \boxed{1}\cdots\boxed{1}\boxed{y} \iff x \leq y$.

Suppose $t \in \mathrm{SStd}(\bar{\nu}, \lambda)$. By semistandardness, the $\lambda_1 = (m-1)n + \nu_1$ ones in t must occur in the initial boxes of the first row. Define $\Phi^{-1}(t)$ to be the tableau obtained by deleting the first $(m-1)n$ ones from the first row of t and then replacing each entry x of t with the (m)-tableau $\boxed{1}\cdots\boxed{1}\boxed{x}$. This shows that Φ is a bijection, completing the proof. $\square$

Example 3.11. *Suppose that $m = 3, \nu = (3, 2, 2)$ and $\lambda = (17, 2, 1, 1)$ with T being the plethystic tableau*

$$\begin{array}{|c|c|c|}\hline \boxed{1}\boxed{1}\boxed{1} & \boxed{1}\boxed{1}\boxed{1} & \boxed{1}\boxed{1}\boxed{1} \\ \hline \boxed{1}\boxed{1}\boxed{2} & \boxed{1}\boxed{1}\boxed{2} \\ \cline{1-2} \boxed{1}\boxed{1}\boxed{3} & \boxed{1}\boxed{1}\boxed{4} \\ \cline{1-2}\end{array}\ .$$

Then $\bar{\nu} = (17, 2, 2)$ and

$$\Phi(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ \hline 2&2 \\ \cline{1-2} 3&4 \\ \cline{1-2}\end{array}\ .$$

We can use this equality, together with Proposition 2.97, to give a complete description of $\langle s_\nu \circ s_{(m)}, s_\lambda \rangle$ when $\lambda_1 = (m-1)n + \nu_1$.

Lemma 3.12. *Suppose that $\nu \vdash n$ and $\lambda \vdash mn$ with $\lambda_1 = (m-1)n + \nu_1$, then*

$$\langle s_\nu \circ s_{(m)}, s_\lambda \rangle = \begin{cases} 1 & \lambda = \bar{\nu} \\ 0 & \textit{else.} \end{cases}$$

Proof. By Proposition 2.97,

$$\langle s_\nu \circ s_{(m)}, s_\lambda \rangle = |\operatorname{PStd}((m)^\nu, \lambda)| - \sum_{\beta \rhd \lambda} \langle s_\nu \circ s_{(m)}, s_\beta \rangle |\operatorname{SStd}(\beta, \lambda)|.$$

We already know that $\langle s_\nu \circ s_{(m)}, s_{\bar{\nu}} \rangle = 1$ and that $\langle s_\nu \circ s_{(m)}, s_\lambda \rangle = 0$ when $\lambda > \bar{\nu}$, so assume $\lambda < \bar{\nu}$. For sake of induction assume $\langle s_\nu \circ s_{(m)}, s_\beta \rangle = 0$ for all $\lambda < \beta < \bar{\nu}$. Then the formula becomes

$$\langle s_\nu \circ s_{(m)}, s_\lambda \rangle = |\operatorname{PStd}((m)^\nu, \lambda)| - |\operatorname{SStd}(\bar{\nu}, \lambda)| = 0.$$

$\square$

Now, let us turn our attention to the case where $\lambda_1 = (m-1)n + \nu_1 - 1$.

Let us first briefly deal with the case $\nu = (n)$. The only partition $\lambda \vdash mn$ with $\lambda_1 = (m-1)n + \nu_1 - 1$ is the partition $(mn-1, 1)$. There is exactly one plethystic standard tableau of shape $(m)^{(n)}$ and weight $(mn-1, 1)$, namely

$$\begin{array}{|c|c|c|c|}\hline \begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & 1 \\ \hline \end{array} & \cdots & \begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & 2 \\ \hline \end{array} \\ \hline \end{array}.$$

.

There is also only one tableau in $\operatorname{SStd}((mn), (mn-1, 1))$, which is

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & 2 \\ \hline \end{array}.$$

Therefore, by Proposition 2.97,

$$\langle s_{(n)} \circ s_{(m)}, s_{(mn-1,1)} \rangle = |\operatorname{PStd}((m)^{(n)}, (mn-1, 1))| - |\operatorname{SStd}((mn), (mn-1, 1))|,$$

which is equal to $1 - 1 = 0$.

From now on, assume that $\nu \neq (n)$.

Let us consider the possibilities for a tableau T of shape $(m)^\nu$ and weight λ where $\lambda_1 = (m-1)n + \nu_1 - 1$. The inner tableaux of T have exactly $(m-1)n + \nu_1 - 1$ ones in their boxes. This leaves one of two possibilities: either all boxes are filled with (m)-tableaux $\begin{array}{|c|c|c|c|}\hline 1&\cdots&1&x\\\hline\end{array}$ for $x \geq 1$ and the box at the end of the first row has $x \geq 2$ (with all others in the first row having $x = 1$); or every box in the first row is filled with $\begin{array}{|c|c|c|}\hline 1&\cdots&1\\\hline\end{array}$ and there is some box at the end of a lower row containing a tableau $\begin{array}{|c|c|c|c|c|}\hline 1&\cdots&1&y&z\\\hline\end{array}$ where $z \geq y \geq 2$, with all other boxes being of the form $\begin{array}{|c|c|c|c|}\hline 1&\cdots&1&x\\\hline\end{array}$ with $x \geq 1$.

Definition 3.13. Suppose $T \in \mathrm{PStd}((m)^\nu, \lambda)$ as above, where $\lambda_1 = (m-1)n + \nu_1 - 1$. Then

- If T has all boxes filled with tableaux $\begin{array}{|c|c|c|c|}\hline 1&\cdots&1&x\\\hline\end{array}$ where $x \geq 1$, then we say T is type one.
- If T has a single box filled with a tableau $\begin{array}{|c|c|c|c|c|}\hline 1&\cdots&1&y&z\\\hline\end{array}$ with $z \geq y \geq 2$, then we say T is type two.

Example 3.14. *Let us give an example of each type when* $m = 3, \nu = (3,3,2,1) \vdash 9$ *and* $\lambda = (20, 2^3, 1)$. *A tableau of type one is*

$$T_1 = \begin{array}{|c|c|c|}\hline \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} \\\hline \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&1&4\\\hline\end{array} \\\hline \begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&1&4\\\hline\end{array} \\\cline{1-2} \begin{array}{|c|c|c|}\hline 1&1&5\\\hline\end{array} \\\cline{1-1} \end{array},$$

whereas a tableau of type two is for example

$$T_2 = \begin{array}{|c|c|c|}\hline \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \\\hline \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&1&4\\\hline\end{array} \\\hline \begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&3&4\\\hline\end{array} \\\cline{1-2} \begin{array}{|c|c|c|}\hline 1&1&5\\\hline\end{array} \\\cline{1-1} \end{array}.$$

Recall that given a partition α, we write $\alpha + \epsilon_i$ for the partition obtained from α by adding a box to the end of row i, and we write $\alpha - \epsilon_i$ for the partition obtained from α by removing a box from the end of row i. If adding or removing a box would create a composition that is not a partition, then $\alpha \pm \epsilon_i$ is not defined as far as we are concerned.

Definition 3.15. We define a map

$$\Phi : \mathrm{PStd}((m)^\nu, \lambda) \to \mathrm{SStd}(\bar{\nu}, \lambda) \sqcup \bigsqcup_{\substack{\beta = \bar{\nu} - \epsilon_1 - \epsilon_x + \epsilon_a + \epsilon_b \\ x,a,b \geq 2}} \mathrm{SStd}(\beta, \lambda)$$

as follows. Suppose $T \in \mathrm{PStd}((m)^\nu, \lambda)$.

If T is of type one, every outer box in T is filled with an (m)-tableau $\boxed{1}\boxed{\cdots}\boxed{1}\boxed{x}$ with $x \geq 1$. Define $\Phi(T)$ to be the tableau obtained from T by replacing each tableau $\boxed{1}\boxed{\cdots}\boxed{1}\boxed{x}$ with the number x, and then appending $(m-1)n$ ones to the beginning of the first row. This is a tableau of weight λ and shape $((m-1)n) + \nu = \bar{\nu}$.

If T is of type two, all outer boxes are filled with the (m)-tableau $\boxed{1}\boxed{\cdots}\boxed{1}\boxed{x}$ with $x \geq 1$ except one removable box which contains an (m)-tableau $\boxed{1}\boxed{\cdots}\boxed{1}\boxed{y}\boxed{z}$ with $z \geq y \geq 2$. Delete this special box, replace each $\boxed{1}\boxed{\cdots}\boxed{1}\boxed{x}$ with x and append $(m-1)n - 1$ ones to the beginning of the first row to obtain a tableau $\tilde{t}$ of shape $\bar{\nu} - \epsilon_1 - \epsilon_e$ where e is the row number of the deleted box. Define $\Phi(T) = (\tilde{t} \leftarrow_2 y) \leftarrow_2 z$, the tableau obtained from $\tilde{t}$ by inserting y into the second row, followed by z. This gives a tableau of shape $\bar{\nu} - \epsilon_1 - \epsilon_e + \epsilon_a + \epsilon_b$ such that $e, a, b \geq 2$, where a and b are the row numbers at which boxes are added in the insertion process.

Example 3.16. *Referring back to Example 3.14, the tableau T_1 of type one is mapped to the $(21, 3, 2, 1)$-tableau*

$$\Phi(T_1) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \cdots & 1 & 2 \\ \hline 2 & 3 & 4 & \multicolumn{3}{c}{} \\ \cline{1-3} 3 & 4 & \multicolumn{4}{c}{} \\ \cline{1-2} 5 & \multicolumn{5}{c}{} \\ \cline{1-1} \end{array} .$$

For the tableau T_2 of type two, the box to delete is the one at the end of the third row containing $\ytableaushort{134}$. *We obtain the* $(20,3,1,1)$*-tableau* $\tilde{t}$ *in this case to be*

$$\begin{ytableau} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & 4 \\ 3 \\ 5 \end{ytableau}\ .$$

Inserting the deleted 3 *into this tableau at the second row, followed by* 4, *leaves us with the* $(20,4,2,1)$*-tableau*

$$\Phi(T_2) = \begin{ytableau} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & 3 & 4 \\ 3 & 4 \\ 5 \end{ytableau}\ .$$

Definition 3.17. Define $M(\nu)$ to be the set of all partitions $\beta \vdash mn$ such that β is obtainable from $\bar{\nu}$ by removing one box from the first row, and one box from below, before adding two boxes in rows $a \geq b \geq 2$ which are not in the same column. In other words,

$$M(\nu) = \{\beta \vdash mn | \beta = \bar{\nu} - \epsilon_1 - \epsilon_e + \epsilon_a + \epsilon_b \text{ where } a, b, e \geq 2 \text{ and } \beta_a \neq \beta_b\}.$$

Given $\beta \in M(\nu)$, write $I_\nu(\beta)$ for the set of possible pairs (a, b) in the above description. If $\beta \notin M(\nu)$, let $I_\nu(\beta) = \emptyset$.

Remark 3.18. Note that from the data (ν, β, a, b) we can recover the row number of the removed box, and therefore we do not need to record this information.

Example 3.19. *Let us consider an example where* $\nu = (3,2,1) \vdash 6$ *and* $m = 3$, *so that* $\bar{\nu} = (15,2,1)$. *The partitions of the form* $\bar{\nu} - \epsilon_1 - \epsilon_e$ *with* $e \geq 2$ *are* $(14,1^2)$ *and* $(14,2)$. *By adding boxes to* $(14,1^2)$ *according to our conditions, remembering that they may not be added in the same column, we can obtain* $(14,3,1)$ *and* $(14,2,1^2)$, *whereas adding boxes to* $(14,2)$ *gives us* $(14,4), (14,3,1)$ *and* $(14,2^2)$, *so*

$$M(\nu) = \{(14,4), (14,3,1), (14,2^2), (14,2,1^2)\}.$$

We may observe that

$$(14,3,1) = (14,1^2) + \epsilon_2 + \epsilon_2 = (14,2) + \epsilon_2 + \epsilon_3,$$

so $I_\nu(14,3,1) = \{(2,2),(3,2)\}$ *and* $|I_\nu(\lambda)| = 2$. *We summarise all the sets* $I_\nu(\lambda)$ *in the following table.*

λ	$I_\nu(\lambda)$
$(14,4)$	$\{(2,2)\}$
$(14,3,1)$	$\{(2,2),(3,2)\}$
$(14,2^2)$	$\{(3,3)\}$
$(14,2,1^2)$	$\{(4,2)\}$

The combinatorial descriptions of plethysm coefficients to be presented later in Propositions 3.22 and 3.28 will rely on the set $M(\nu)$ and the multiplicities $|I_\nu(\lambda)|$. Due to technical reasons concerning the restriction on entries in columns of tableaux, we must distinguish between the cases $\nu_1 = \nu_2$ and $\nu_1 \neq \nu_2$.

Lemma 3.20. *Suppose* $\nu \vdash n$ *with* $\nu_1 \neq \nu_2$*, and that* $\lambda \vdash mn$ *with* $\lambda_1 = (m-1)n + \nu_1 - 1$. *Then there is a bijection*

$$\hat{\Phi} : \mathrm{PStd}((m)^\nu, \lambda) \to \mathrm{SStd}(\bar{\nu}, \lambda) \sqcup \left(\bigsqcup_{\substack{\beta \in M(\nu) \\ \beta \trianglerighteq \lambda}} \mathrm{SStd}(\beta, \lambda) \times I_\nu(\beta) \right)$$

defined by $\hat{\Phi}(T) = \Phi(T)$ *when* T *is of type one, and* $\hat{\Phi}(T) = (\Phi(T), (a,b))$*, where* a *and* b *are the row numbers of added boxes, in the case that* T *is of type two. Here,* $\Phi(T)$ *is as defined in Definition 3.15.*

Proof. We show $\hat{\Phi}$ is invertible. Pick an element in the codomain. This is either $t \in \mathrm{SStd}(\bar{\nu}, \lambda)$ or $(t,(a,b)) \in \mathrm{SStd}(\beta, \lambda) \times I_\nu(\beta)$ for some $\beta \in M(\nu)$. In the first case, the preimage of t is formed by deleting $(m-1)n$ ones from the start of the first row of t and then replacing each remaining entry x with the (m)-tableau $\boxed{1}\boxed{\cdots}\boxed{1}\boxed{x}$. In the second case, knowing β, ν, a and b allows us to find the unique e so that we can write $\beta = \bar{\nu} - \epsilon_1 - \epsilon_e + \epsilon_a + \epsilon_b$. Applying Corollary 2.54 to t allows us to find some semistandard tableau s of shape $(\beta - \epsilon_a - \epsilon_b)$ such that $t = (s \leftarrow y) \leftarrow z$ with $y \leq z$. Thus s must be of weight $\lambda - \epsilon_y - \epsilon_z$. Now, replace each entry x of s with the (m)-tableau $\begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & x \\ \hline \end{array}$, and add an extra box to the end of row e containing the (m)-tableau $\begin{array}{|c|c|c|c|c|}\hline 1 & \cdots & 1 & y & z \\ \hline \end{array}$. The resulting tableau is the preimage of $(t, (a, b))$ under $\hat{\Phi}$. This tableau is semistandard since $\begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & w \\ \hline \end{array} < \begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & x \\ \hline \end{array} \iff w < x$. $\square$

Remark 3.21. The restriction that $\nu_1 \neq \nu_2$ is necessary for this bijection. For example, when $\nu = (2^2)$, $m = 3$ and $\lambda = (9, 1^3)$, the tableau

$$t = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$$

is contained in $\mathrm{SStd}(\bar{\nu}, \lambda)$, but it would have to be the image of

$$\begin{array}{|c|c|}\hline \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline \end{array} \\ \hline \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline \end{array} \\ \hline \end{array}$$

which is not a plethystic semistandard tableau.

Proposition 3.22. *Let $\nu \vdash n, \nu \neq (n)$ with $\nu_1 \neq \nu_2$ and $\lambda \vdash mn$ with $\lambda_1 = (m-1)n + \nu_1 - 1$. Then*

$$\langle s_\nu \circ s_{(m)}, s_\lambda \rangle = |I_\nu(\lambda)|.$$

In particular, $\langle s_\nu \circ s_{(m)}, s_\lambda \rangle = 0$ unless $\lambda \in M(\nu)$.

Proof. We proceed by induction on the lexicographic order. Recall from Proposition 2.97 that

$$\langle s_\nu \circ s_{(m)}, s_\lambda \rangle = \mathrm{PStd}((m)^\nu, \lambda) - \sum_{\beta \rhd \lambda} \langle s_\nu \circ s_{(m)}, s_\beta \rangle |\, \mathrm{SStd}(\beta, \lambda)|.$$

Using the bijection of the previous lemma, the right hand side becomes

$$|\operatorname{SStd}(\bar{\nu},\lambda)| + \sum_{\substack{\beta\in M(\nu)\\ \beta\trianglerighteq\lambda}} |I_\nu(\beta)||\operatorname{SStd}(\beta,\lambda)| - \sum_{\beta\triangleright\lambda}\langle s_\nu\circ s_{(m)}, s_\beta\rangle|\operatorname{SStd}(\beta,\lambda)|.$$

By the inductive hypothesis, $|I_\nu(\beta)| = \langle s_\nu \circ s_{(m)}, s_\beta\rangle$ when $\beta \triangleright \lambda$, and thus each term in the first sum cancels with a corresponding term in the second. From the first sum we are left with the term $|I_\nu(\lambda)||\operatorname{SStd}(\lambda,\lambda)| = |I_\nu(\lambda)|$.

If $\bar{\nu} \triangleright \lambda$, we are left with the term $|\operatorname{SStd}(\bar{\nu},\lambda)|$ corresponding to the index $\beta = \bar{\nu}$ from the second sum. Thus in this case

$$\langle s_\nu \circ s_{(m)}, s_\lambda\rangle = |\operatorname{SStd}(\bar{\nu},\lambda)| + |I_\nu(\lambda)| - |\operatorname{SStd}(\bar{\nu},\lambda)| = |I_\nu(\lambda)|.$$

If $\bar{\nu} \not\triangleright \lambda$ then $\operatorname{SStd}(\bar{\nu},\lambda) = \emptyset$ and there is also no $\beta = \bar{\nu}$ term in the second sum, so we are simply left with

$$\langle s_\nu \circ s_{(m)}, s_\lambda\rangle = |I_\nu(\lambda)|$$

as required. □

Example 3.23. *Referring back to Example 3.19, we may extend the table to calculate the coefficient $\langle s_{(3,2,1)} \circ s_{(3)}, s_\lambda\rangle$ when $\lambda_1 = 14$.*

λ	$I_\nu(\lambda)$	$\lvert I_\nu(\lambda)\rvert = \langle s_{(3,2,1)} \circ s_{(3)}, s_\lambda\rangle$
$(14,4)$	$\{(2,2)\}$	1
$(14,3,1)$	$\{(2,2),(3,2)\}$	2
$(14,2^2)$	$\{(3,3)\}$	1
$(14,2,1^2)$	$\{(4,2)\}$	1

These coefficients can be verified by computer algebra as in SAGE Computation A.5

Lemma 3.24. *Suppose $\nu \vdash n$ with $\nu_1 = \nu_2$, and that $\lambda \vdash mn$ with $\lambda_1 = (m-1)n + \nu_1 - 1$. Then there is a bijection*

$$\tilde{\Phi} : \mathrm{PStd}((m)^\nu, \lambda) \to \mathrm{SStd}(\nu, \lambda - ((m-1)n)) \sqcup \left(\bigsqcup_{\substack{\beta \in M(\nu) \\ \beta \trianglerighteq \lambda}} \mathrm{SStd}(\beta, \lambda) \times I_\nu(\beta) \right)$$

defined as follows. If T is of type one, $\tilde{\Phi}(T)$ is equal to the tableau resulting from deleting $(m-1)n$ ones from the first row of $\Phi(T)$, with Φ defined as in Definition 3.15. If T is of type two, then $\tilde{\Phi}(T)$ is defined in exactly the same way as $\hat{\Phi}$ is defined in the type two case of Lemma 3.20.

Proof. The proof is very similar to the proof of Lemma 3.20, with the only difference being when $t \in \mathrm{SStd}(\nu, \lambda - ((m-1)n)$. In this case the preimage is found by replacing each entry x of t with the (m)-tableau $\boxed{1}\boxed{\cdots}\boxed{1}\boxed{x}$. $\square$

Example 3.25. *We examine an example of the map above, but only in the type one case. Take $\nu = (3, 3, 1) \vdash 7$ and $\lambda = (16, 2, 2, 1)$. The $(3)^\nu$-tableau*

$$\begin{array}{|c|c|c|}\hline \boxed{1}\boxed{1}\boxed{1} & \boxed{1}\boxed{1}\boxed{1} & \boxed{1}\boxed{1}\boxed{3} \\ \hline \boxed{1}\boxed{1}\boxed{2} & \boxed{1}\boxed{1}\boxed{2} & \boxed{1}\boxed{1}\boxed{4} \\ \hline \boxed{1}\boxed{1}\boxed{3} \\ \cline{1-1} \end{array}$$

is mapped to the ν-tableau

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}.$$

Lemma 3.26. *Suppose that $\nu \vdash n$ with $\nu_1 = \nu_2$ and $\lambda \vdash mn$ such that $\lambda_1 = (m-1)n + \nu_1 - 1$. We have the equality:*

$$|\mathrm{SStd}(\bar{\nu}, \lambda)| = |\mathrm{SStd}(\nu, \lambda - ((m-1)n))| + |\mathrm{SStd}(\bar{\nu} - \epsilon_1 + \epsilon_2, \lambda)|.$$

Proof. We define a map

$$f : \mathrm{SStd}(\bar{\nu}, \lambda) \to \mathrm{SStd}(\nu, \lambda - ((m-1)n)) \cup \mathrm{SStd}(\bar{\nu} - \epsilon_1 + \epsilon_2, \lambda)$$

as follows. Suppose $t \in \mathrm{SStd}(\bar{\nu}, \lambda)$. If the last element in the first row of t is strictly less than the element at the end of the second row of t, delete a total of $(m-1)n$ boxes containing ones from the beginning of the first row to obtain $f(t) \in \mathrm{SStd}(\nu, \lambda - ((m-1)n)$. Otherwise, move the last box in the first row of t to the end of the second row to obtain $f(t) \in \mathrm{SStd}(\bar{\nu} - \epsilon_1 + \epsilon_2, \lambda)$. Now, let s be a tableau in the codomain of f. If $s \in \mathrm{SStd}(\nu, \lambda - ((m-1)n))$, append $(m-1)n$ ones to the beginning of the first row of s to obtain its preimage under f. Otherwise, $s \in \mathrm{SStd}(\bar{\nu} - \epsilon_1 + \epsilon_2, \lambda)$ and we may move the box at the end of the second row to the end of the first in order to obtain the preimage. Therefore f is bijective, yielding the result. $\square$

Example 3.27. *Suppose that* $m = 3, \nu = (2, 2, 1) \vdash 5$ *so* $\bar{\nu} = (12, 2, 1)$. *Choose* $\lambda = (11, 2, 2) \vdash 15$. *Define*

$$t_1 = \begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 2 & 3 \\ 3 \end{ytableau}$$

and

$$t_2 = \begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 \\ 2 & 2 \\ 3 \end{ytableau}.$$

Then

$$f(t_1) = \begin{ytableau} 1 & 2 \\ 2 & 3 \\ 3 \end{ytableau} \in \mathrm{SStd}(\nu, (1, 2, 2))$$

and

$$f(t_2) = \begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 \end{ytableau} \in \mathrm{SStd}(\bar{\nu} - \epsilon_1 + \epsilon_2, (10, 2, 2)).$$

Proposition 3.28. *Let* $\nu \vdash n$ *with* $\nu_1 = \nu_2$ *and* $\lambda \vdash mn$ *with* $\lambda_1 = (m-1)n + \nu_1 - 1$. *Then*

$$\langle s_\nu \circ s_{(m)}, s_\lambda \rangle = \begin{cases} |I_\nu(\lambda)| - 1 & \lambda = ((m-1)n) + \nu - \epsilon_1 + \epsilon_2 \\ |I_\nu(\lambda)| & \textit{otherwise.} \end{cases}$$

Again, $\langle s_\nu \circ s_{(m)}, s_\lambda \rangle = 0$ *unless* $\lambda \in M(\nu)$.

Proof. We again proceed by induction on the lexicographic order. As before, we have the equation

$$\langle s_\nu \circ s_{(m)}, s_\lambda\rangle = |\operatorname{PStd}((m)^\nu, \lambda)| - \sum_{\beta \rhd \lambda} \langle s_\nu \circ s_{(m)}, s_\beta\rangle |\operatorname{SStd}(\beta, \lambda)|.$$

Using our new bijection from Lemma 3.24 and the inductive hypothesis, we arrive at the expression

$$|\operatorname{SStd}(\nu, \lambda - ((m-1)n))| + \sum_{\substack{\beta \unrhd \lambda \\ \beta \in M(\nu)}} |I_\nu(\beta)||\operatorname{SStd}(\beta, \lambda)| - \sum_{\beta \rhd \lambda} \langle s_\nu \circ s_{(m)}, s_\beta\rangle |\operatorname{SStd}(\beta, \lambda)|$$

for the right hand side of this equation. One may extract terms from sums to arrive at the expression

$$|\operatorname{SStd}(\nu, \lambda - ((m-1)n))| + |I_\nu(\lambda)||\operatorname{SStd}(\lambda, \lambda)| - \langle s_\nu \circ s_{(2)}, s_{\bar\nu}\rangle |\operatorname{SStd}(\bar\nu, \lambda)|$$

$$+ \sum_{\substack{\beta \rhd \lambda \\ \beta \in M(\nu)}} |I_\nu(\beta)||\operatorname{SStd}(\beta, \lambda)| - \sum_{\substack{\beta \rhd \lambda \\ \beta \in M(\nu)}} \langle s_\nu \circ s_{(m)}, s_\beta\rangle |\operatorname{SStd}(\beta, \lambda)|,$$

which upon using the equality in Lemma 3.26, as well as the fact that $|\operatorname{SStd}(\lambda, \lambda)|$ and $\langle s_\nu \circ s_{(2)}, s_{\bar\nu}\rangle$ are equal to 1, we can simplify. We arrive at the equation

$$\langle s_\nu \circ s_{(m)}, s_\lambda\rangle = |I_\nu(\lambda)| - |\operatorname{SStd}(\bar\nu - \epsilon_1 + \epsilon_2, \lambda)|$$

$$+ \sum_{\substack{\beta \rhd \lambda \\ \beta \in M(\nu)}} \big(|I_\nu(\beta)| - \langle s_\nu \circ s_{(m)}, s_\beta\rangle\big) |\operatorname{SStd}(\beta, \lambda)|.$$

We now split into three cases to finish the proof.

- If λ is lexicographically greater than $\bar\nu - \epsilon_1 + \epsilon_2$, then $|I_\nu(\beta)| = \langle s_\nu \circ s_{(m)}, s_\beta\rangle$ for every $\beta \rhd \lambda$ by the inductive hypothesis and $|\operatorname{SStd}(\bar\nu - \epsilon_1 + \epsilon_2, \lambda)| = 0$, thus we are left with $\langle s_\nu \circ s_{(m)}, s_\lambda\rangle = |I_\nu(\lambda)|$.

- If $\lambda = \bar\nu - \epsilon_1 + \epsilon_2$, then still $|I_\nu(\beta)| = \langle s_\nu \circ s_{(m)}, s_\beta\rangle$ for every $\beta \rhd \lambda$, but $|\operatorname{SStd}(\bar\nu - \epsilon_1 + \epsilon_2, \lambda)| = 1$, thus we are left with $\langle s_\nu \circ s_{(m)}, s_\lambda\rangle = |I_\nu(\lambda)| - 1$.

- If λ is less than $\bar{\nu}-\epsilon_1+\epsilon_2$, then $|I_\nu(\beta)| = \langle s_\nu \circ s_{(m)}, s_\beta\rangle$ for every $\beta \rhd \lambda$ apart from $\beta = \bar{\nu}-\epsilon_1+\epsilon_2$, in which case $|I_\nu(\beta)| = \langle s_\nu \circ s_{(m)}, s_\beta\rangle + 1$. Thus the sum term reduces just to $|\operatorname{SStd}(\bar{\nu}-\epsilon_1+\epsilon_2,\lambda)|$, which cancels and leaves us with $\langle s_\nu \circ s_{(m)}, s_\lambda\rangle = |I_\nu(\lambda)|$.

$\square$

Example 3.29. *Let us consider the calculation of* $\langle s_{(3,3,2)} \circ s_{(3)}, s_\lambda\rangle$. *In this case* $\bar{\nu} = (19,3,2)$ *and we must consider the case where* $\lambda_1 = 18$. *The partitions of the form* $\bar{\nu}-\epsilon_1-\epsilon_e$ *are* $(18,2^2)$ *and* $(18,3,1)$. *We summarise the rest of the calculation in a table.*

λ	$I_\nu(\lambda)$	$\lvert I_\nu(\lambda)\rvert$	$\lambda = \bar{\nu}-\epsilon_1+\epsilon_2$?	$\langle s_{(3,3,2)} \circ s_{(3)}, s_\lambda\rangle$
$(18,5,1)$	$\{(2,2)\}$	1	$\times$	1
$(18,4,2)$	$\{(2,2),(3,2)\}$	2	$\checkmark$	1
$(18,4,1^2)$	$\{(4,2)\}$	1	$\times$	1
$(18,3^2)$	$\{(3,3)\}$	1	$\times$	1
$(18,3,2,1)$	$\{(4,2),(4,3)\}$	2	$\times$	2
$(18,2^3)$	$\{(4,4)\}$	1	$\times$	1

One can verify these coefficients by computer algebra as in SAGE Computation A.6.

3.2 Near maximal constituents of $s_\nu \circ s_{(1^2)}$

A sensible route for generalisation of the previous section would be to replace the partition (2) with the partition (1^2), or the partition (m) with $(m-1,1)$. One might be tempted to replace with (1^m) but this gets much harder as m increases, and does not represent a single case. In this section, we will describe the plethysm coefficient $\langle s_\nu \circ s_{(1^2)}, s_\lambda\rangle$ in combinatorial terms when λ_1 is n or $n-1$, again using Lemma 2.97, as well as bijections from plethystic to semistandard tableaux, as our tools.

Recall that $\nu \vdash n$. Theorem 2.96 tells us that the lexicographically maximal constituent of $s_\nu \circ s_{(1^2)}$ appearing with nonzero coefficient is $s_{\tilde{\nu}}$, where
$\tilde{\nu} = (n, \nu_1, \cdots, \nu_{\ell(\nu)})$. We already know that this constituent appears with coefficient 1, so we will progress to asking about the value of $\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle$ when $\lambda_1 = n$.

Lemma 3.30. *We have the equality* $|\,\mathrm{PStd}((1^2)^\nu, \lambda)| = |\,\mathrm{SStd}(\tilde{\nu}, \lambda)|$ *whenever* $\lambda_1 = n$*.*

Proof. We establish a bijection Ψ between these two sets. Let $T \in \mathrm{PStd}((1^2)^\nu, \lambda)$. Then T has exactly n entries equal to one, and each inner tableau must be of the form $\begin{array}{|c|}\hline 1\\\hline x\\\hline\end{array}$ for some $x \geq 2$. Define $\Psi(T)$ to be the semistandard tableau obtained from T by replacing each tableau $\begin{array}{|c|}\hline 1\\\hline x\\\hline\end{array}$ with its entry x, and then affixing a new row consisting of n ones to the top of this tableau.

Given a tableau $s \in \mathrm{SStd}(\tilde{\nu}, \lambda)$ with $\lambda_1 = n$, we may simply delete the first row, which must consist of n ones, and replace each remaining entry x with the (1^2)-tableau $\begin{array}{|c|}\hline 1\\\hline x\\\hline\end{array}$, which is semistandard since $x > 1$, in order to obtain $S \in \mathrm{PStd}((1^2)^\nu, \lambda)$ such that $\Psi(S) = s$. Thus Ψ is a bijection and the result follows. $\square$

Example 3.31. *Consider* $\nu = (4, 3, 2, 1^2) \vdash 11$ *with* $\lambda = (11, 4, 2^2, 1^3)$*. The* $(1^2)^\nu$*-tableau*

$$\begin{array}{|c|c|c|c|}\hline \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \\\hline \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} \\\cline{1-3} \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array} \\\cline{1-2} \begin{array}{|c|}\hline 1\\\hline 6\\\hline\end{array} \\\cline{1-1} \begin{array}{|c|}\hline 1\\\hline 7\\\hline\end{array} \\\cline{1-1} \end{array}$$

is mapped to the $(11, 4, 3, 2, 1^2)$*-tableau*

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&1&1&1\\\hline 2&2&2&2\\\cline{1-4} 3&3&4\\\cline{1-3} 4&5\\\cline{1-2} 6\\\cline{1-1} 7\\\cline{1-1}\end{array}$$

In a similar way to the (m) case, we are easily able to look at those constituents λ for which $\lambda_1 = n$.

Lemma 3.32. *If $\nu \vdash n$ and $\lambda \vdash 2n$ with $\lambda_1 = n$, then*

$$\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle = \begin{cases} 1 & \text{if } \lambda = \tilde{\nu} \\ 0 & \text{else.} \end{cases}$$

Proof. The cases where $\lambda \geq \tilde{\nu}$ are dealt with by Theorem 2.96 so it suffices to assume $\lambda < \tilde{\nu}$. Assume that the result holds for each $\beta > \lambda$. Proposition 2.97 tells us that

$$\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle = |\operatorname{PStd}((1^2)^\nu, \lambda)| - \sum_{\beta \rhd \lambda} \langle s_\nu \circ s_{(1^2)}, s_\beta \rangle |\operatorname{SStd}(\beta, \lambda)|.$$

We know that $\langle s_\nu \circ s_{(1^2)}, s_{\tilde{\nu}} \rangle = 1$, and by assumption $\langle s_\nu \circ s_{(1^2)}, s_\beta \rangle = 0$ for all $\lambda < \beta < \tilde{\nu}$. Then the formula reduces to

$$\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle = |\operatorname{PStd}((1^2)^\nu, \lambda)| - |\operatorname{SStd}(\tilde{\nu}, \lambda)| = 0$$

by Lemma 3.30. $\square$

We now fully understand the coefficients $\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle$ when $\lambda_1 = n$, so we will now assume that $\lambda_1 = n - 1$. We will exclude the cases where $\nu \in \{(n), (n-1, 1)\}$ from the main argument. We refer to existing work in order to resolve these cases now.

The case where $\nu = (n)$ was already discussed in Corollary 2.93.

We now turn to the case where $\nu = (n-1,1)$. The following result has been translated into the language of symmetric functions from De Boeck's thesis. If $\gamma = (\gamma_1, \ldots, \gamma_{\ell(\gamma)})$ is a partition, write 2γ for the partition whose ith part is $2\gamma_i$, and denote the number of distinct parts of 2γ by $\mathrm{dis}(2\gamma)$. Note that $\mathrm{dis}(2\gamma)$ is also exactly the number of distinct parts of γ.

Lemma 3.33. *[9, Cor 7.1.3] We have the formula*

$$s_{(n-1,1)} \circ s_{(2)} = \sum_{\gamma \vdash n} (\mathrm{dis}(2\gamma) - 1) s_{2\gamma} + \sum_{\mu} s_{\mu},$$

where the latter sum runs over those $\mu \vdash 2n$ with two distinct odd parts and the other parts even.

By Lemma 2.92, $\langle s_{(n-1,1)} \circ s_{(1^2)}, s_\lambda \rangle = \langle s_{(n-1,1)} \circ s_{(2)}, s_{\lambda'} \rangle$, so the lemma determines all constituents in this case. To compare with later results, we record those constituents of $s_{(n-1,1)} \circ s_{(1^2)}$ with first part n or $n-1$, and thus we must calculate $\langle s_{(n-1,1)} \circ s_{(2)}, s_{\lambda'} \rangle$ when $\ell(\lambda') \in \{n, n-1\}$.

The partitions $\gamma \vdash n$ of length n or $n-1$ are (1^n) and $(2, 1^{n-2})$, with 2γ being (2^n) and $(4, 2^{n-2})$ respectively. It is easy to see that $\mathrm{dis}(2^n) = 1$ and hence $s_{(2^n)}$ does not appear as a constituent of $s_{(n)} \circ s_{(2)}$, and that $\mathrm{dis}(4, 2^{n-2}) = 2$, so we deduce that $\langle s_{(n-1,1)} \circ s_{(1^2)}, s_{((n-1^2),1^2)} \rangle = \langle s_{(n-1,1)} \circ s_{(2)}, s_{(4,2^{n-2})} \rangle = 1$.

Now let us turn to those $\mu \vdash 2n$ where $\ell(\mu)$ is n or $n-1$ and μ has two distinct odd parts with all other parts being even. The only μ fitting this description are $(5, 2^{n-3}, 1), (4, 3, 2^{n-4}, 1)$ and $(3, 2^{n-2}, 1)$, and we conjugate them to obtain the remaining required constituents of $s_{(n-1,1)} \circ s_{(1^2)}$.

In summary, suppose that $\lambda_1 = n$ or $n-1$. Then $\langle s_{(n-1,1)} \circ s_{(1^2)}, s_\lambda \rangle = 1$ if $\lambda \in \{(n, n-1, 1), ((n-1)^2, 1^2), (n-1, n-2, 1^3), (n-1, n-2, 2, 1)\}$ and 0 otherwise. This is in fact valid only when $n \geq 4$ as some of the listed partitions no longer have distinct parts when n is small. When $n = 2$, the only constituent is $s_{(2,1,1)}$, and in the case $n = 3$, the consituents are $s_{(2^2,1^2)}$ and $s_{(2,1^4)}$.

Now assume ν is not (n) or $(n-1)$. We will calculate plethysm coefficients $\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle$ where $\lambda_1 = n-1$ in terms of membership of a set $R(\nu)$, as well as a quantity $|J_\nu(\lambda)|$. We define those concepts now.

Definition 3.34. Suppose that $\nu \vdash n$ with $\nu \neq (n), (n-1, 1)$.

- Define $R(\nu)$ to be the set

$$R(\nu) = \{\beta \vdash 2n | \beta = \tilde{\nu} - \epsilon_1 + \epsilon_f \text{ where } f \geq 2\},$$

  the set of partitions obtainable by moving the box at the end of the first row of $\tilde{\nu}$ to a subsequent row.

- Define $C(\nu)$ to be the set

$$C(\nu) = \{\beta \vdash 2n | \beta = \tilde{\nu} - \epsilon_1 - \epsilon_e + \epsilon_a + \epsilon_b \text{ where } e \geq 2 \text{ and } a > b > 1\}$$

  of partitions obtainable by removing boxes at the end of the first and eth rows of $\tilde{\nu}$ and adding boxes at the end of rows a and b such that the added boxes are in different rows, neither of which is the first row.

- Suppose $\beta \in C(\nu)$, define $J_\nu(\beta)$ to be the set of pairs (a, b) with $a > b$ that appear as row numbers where boxes are added to $\tilde{\nu} - \epsilon_1 - \epsilon_e$ to arrive at β.

Remark 3.35. Note that $|J_\nu(\lambda)|$ is simply the number of ways of writing

$$\lambda = \tilde{\nu} - \epsilon_1 - \epsilon_e + \epsilon_a + \epsilon_b \text{ where } e \geq 2 \text{ and } a > b > 1.$$

Example 3.36. *Let us consider the case where ν is the two row partition $(5, 2)$, in which case $n = 7$ and $\tilde{\nu} = (7, 5, 2)$. Then*

$$R(\nu) = \{(6, 6, 2), (6, 5, 3), (6, 5, 2, 1)\}.$$

We now determine $C(\nu)$. The partitions $\tilde{\nu} - \epsilon_1 - \epsilon_e$ for $e \geq 2$ are $(6, 4, 2)$ and $(6, 5, 1)$. We summarise the values $\lambda \in C(\nu)$ together with the associated set $J_\nu(\lambda)$ in the following table.

λ	$J_\nu(\lambda)$
$(6,6,2)$	$\{(3,2)\}$
$(6,6,1^2)$	$\{(4,2)\}$
$(6,5,3)$	$\{(3,2)\}$
$(6,5,2,1)$	$\{(4,3),(4,2)\}$
$(6,5,1^3)$	$\{(5,4)\}$
$(6,4,3,1)$	$\{(4,3)\}$
$(6,4,2,1^2)$	$\{(5,4)\}$

Note that the method of obtaining $(6,6,2)$ from $(6,4,2)$ by adding two boxes in row 2 is not included, since we are not permitted to add two boxes in the same row. A similar observation holds for $(6,5,3)$.

One notices that in this example $R(\nu) \subset C(\nu)$, and in fact it is important to our discussion that this is the case, so we briefly prove it.

Lemma 3.37. *For any partition $\nu \vdash n$ with $\nu \neq (n),(n-1,1)$, we have $R(\nu) \subset C(\nu)$.*

Proof. Suppose $\lambda = \tilde{\nu} - \epsilon_1 + \epsilon_f \in R(\nu)$. By definition $f \geq 2$. Choose $e > 1$ different from f. This is guaranteed to exist as ν has at least two rows, and so $\tilde{\nu}$ has at least three rows. Observe that $\lambda = \tilde{\nu} - \epsilon_1 - \epsilon_e + \epsilon_e + \epsilon_f \in C(\nu)$. □

Lemma 3.38. *Suppose $\nu \vdash n$ where $\nu \neq (n),(n-1,1)$ and let $\lambda \vdash 2n$ such that $\lambda_1 = n-1$. Then*

$$|\operatorname{SStd}(\tilde{\nu},\lambda)| = \sum_{\beta \in R(\nu)} |\operatorname{SStd}(\beta,\lambda)|.$$

Proof. Define a map

$$f : \operatorname{SStd}(\tilde{\nu},\lambda) \to \bigsqcup_{\beta \in R(\nu)} \operatorname{SStd}(\beta,\lambda)$$

as follows. Suppose $t \in \operatorname{SStd}(\tilde{\nu},\lambda)$. Then since $\lambda_1 = n-1$, the entry at the end of the first row of t must be some $x \geq 2$. Write t_0 for the tableau obtained from t be deleting the box at the right end of the first row, and then set $f(t) = t_0 \leftarrow_2 x$.

Given a tableau $s \in \mathrm{SStd}(\beta, \lambda)$ where $\beta \in R(\nu)$, we may identify the row number i which is longer in β than in $\tilde{\nu}$. By Lemma 2.50, we may find unique $y \in \mathbb{N}$ and semistandard s_0 of shape $\beta - \epsilon_i$ such that $s = s_0 \leftarrow_2 y$. Write s_1 for the semistandard tableau obtained from s_0 by appending a new box with entry y to the end of the first row. This is semistandard since we are adding an entry to a row of ones in a semistandard tableau. One see thats $f(s_1) = s$. □

Example 3.39. *Suppose* $\nu = (4, 3, 3, 1) \vdash 11$ *and* $\lambda = (10, 3^3, 2, 1)$. *Then*

$$t = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 6 \\ \hline 2 & 2 & 2 & 5 \\ \cline{1-4} 3 & 3 & 3 \\ \cline{1-3} 4 & 4 & 4 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array}$$

is a semistandard $\tilde{\nu}$*-tableau of weight* λ. *The partition* $(10, 5, 3^2, 1)$ *is in* $R(\nu)$ *and* $f(t)$ *is equal to the* $(10, 5, 3^2, 1)$*-tableau*

$$f(t) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 5 & 6 \\ \cline{1-5} 3 & 3 & 3 \\ \cline{1-3} 4 & 4 & 4 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array} .$$

We now define a map

$$\Psi_0 : \mathrm{PStd}((1^2)^\nu, \lambda) \to \bigsqcup_{\beta \in C(\nu)} \mathrm{SStd}(\beta, \lambda)$$

when $\nu \neq (n), (n-1, 1)$. We must first make some observations about plethystic $(1^2)^\nu$-tableaux of weight $\lambda \vdash 2n$ when $\lambda_1 = n - 1$. Suppose T is such a tableau. Then T contains $n-1$ ones, and these can only occur in the upper box of an inner tableau, therefore all of the (1^2)-tableaux inside T must be of the form $\begin{array}{|c|}\hline 1 \\ \hline x \\ \hline\end{array}$ with $x > 1$, apart from a single tableau $\begin{array}{|c|}\hline y \\ \hline z \\ \hline\end{array}$ with $z > y > 1$ at the end of some row $(e-1)$. Write t^- for the tableau obtained from T by deleting this box at the end of row $(e-1)$ and replacing the remaining entries $\begin{array}{|c|}\hline 1 \\ \hline x \\ \hline\end{array}$ with their entry x, before finally appending a new row of $n-1$ ones to the top. Define $\Psi_0(T) = z \to (y \to t^-)$, the tableau obtained by inserting y and then z into the first column of t^-. Define $\Psi(T)$ to be $(\Psi_0(T), (a,b))$ where a and b are the row numbers where boxes are added in the column insertion process, listed in decreasing order. The added boxes in the column insertion must be in different rows by Corollary 2.58, and thus $a > b$ and $(a,b) \in J_\nu(\beta)$.

Lemma 3.40. *Suppose that $\nu \vdash n$ where $\nu \neq (n), (n-1,1)$ and $\lambda \vdash 2n$ with $\lambda_1 = n-1$. Then the map*

$$\Psi : \mathrm{PStd}((1^2)^\nu, \lambda) \to \bigsqcup_{\beta \in C(\nu)} (\mathrm{SStd}(\beta, \lambda) \times J_\nu(\beta))$$

is a bijection.

Proof. Suppose that $s \in \mathrm{SStd}(\beta, \lambda)$ for some $\beta \in C(\nu)$ with $(c,d) \in J_\nu(\beta)$. We aim to find an $S \in \mathrm{PStd}((1^2)^\nu, \lambda)$ such that $\Psi(S) = (s, (c,d))$. Firstly, by Corollary 2.59, we may find s^- of shape $\beta - \epsilon_c - \epsilon_d$ such that $s = z \to (y \to s^-)$ and $z > y$. This tableau s^- is clearly of weight $\lambda - \epsilon_y - \epsilon_z$. Now, form a plethystic tableau S^- by deleting the first row of s^-, which must be a row of $n-1$ ones, and replacing each entry x with the (1^2)-tableau $\boxed{\begin{smallmatrix}1\\x\end{smallmatrix}}$. One can see that S^- is of shape $(1^2)^{\nu-\epsilon_e}$ after this process, and still weight $\lambda - \epsilon_y - \epsilon_z$. Thus to obtain an element of $\mathrm{PStd}((1^2)^\nu, \lambda)$, we have no choice but to affix a box containing $\boxed{\begin{smallmatrix}y\\z\end{smallmatrix}}$ to the end of the eth row. Define S to be this tableaux. One can verify that $\Psi(S) = (s, (c,d))$. Deleting the box at the end of row e leaves us again with S^-, and replacing each entry $\boxed{\begin{smallmatrix}1\\x\end{smallmatrix}}$ in S^- with x leaves us exactly with s^-. Finally, $s = z \to (y \to s^-)$, with the boxes added in rows c and d. We have just inverted the map Φ, and thus it is bijective. ∎

Example 3.41. *We give an example of the inversion process described in the proof of the previous Lemma. Let us take, for example, $\nu = (3,2,1) \vdash 6$ and*

$\lambda = (5,3,2,1,1) \vdash 12$. *Note that the tableau*

$$s = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\ \hline 2&2&2&3 \\ \cline{1-4} 3&4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

is of shape $(5,4,2,1) \in C(\nu)$, *so we may consider the preimage of* $(s,(2,3))$ *under* Ψ *by using the process described in the proof of the previous Lemma.*

We use reverse column insertion to remove boxes at the end of rows 2 *and* 3 *and end up with* $y = 2$ *and* $z = 3$ *removed leaving the tableau*

$$s^- = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\ \hline 2&2&3 \\ \cline{1-3} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} .$$

From this, we form the plethystic tableau

$$S^- = \begin{array}{|c|c|c|}\hline \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \\ \hline \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} \\ \cline{1-1} \begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array}. \\ \cline{1-1}\end{array}$$

One can see that S^- *is of shape* $(1^2)^{\nu-\epsilon_2}$, *and thus we must add a new box filled with* $\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$ *to the end of row* 2, *resulting in*

$$S = \begin{array}{|c|c|c|}\hline \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \\ \hline \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} & \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \\ \cline{1-2} \begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array}. \\ \cline{1-1}\end{array}$$

One can then verify that $\Psi(S) = s$.

Notation. Suppose λ and ν are partitions. Write

$$r(\lambda) = \begin{cases} 1 & \text{if } \lambda \in R(\nu) \\ 0 & \text{if } \lambda \notin R(\nu). \end{cases}$$

Theorem 3.42. *Suppose $\lambda \vdash 2n$ such that $\lambda_1 = n-1$, and $\nu \vdash n$ with $\nu \neq (n), (n-1,1)$. Then*

$$\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle = |J_\nu(\lambda)| - r(\lambda).$$

Proof. We know already that when $\lambda_1 = n$, $\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle = 0$ apart from when $\lambda = \tilde{\nu}$, in which case the coefficient is 1. Now, assume that $\lambda_1 = n-1$ and proceed by induction on the lexicographic order.

We use Proposition 2.97 to state that

$$\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle = |\operatorname{PStd}((1^2)^\nu, \lambda)| - \sum_{\beta \rhd \lambda} \langle s_\nu \circ s_{(1^2)}, s_\beta \rangle |\operatorname{SStd}(\beta, \lambda)|. \tag{3.1}$$

We may apply the inductive hypothesis to split up the right hand sum:

$$\begin{aligned} &\sum_{\beta \rhd \lambda} \langle s_\nu \circ s_{(1^2)}, s_\beta \rangle |\operatorname{SStd}(\beta, \lambda)| \\ &\qquad = |\operatorname{SStd}(\tilde{\nu}, \lambda)| + \sum_{\substack{\beta \rhd \lambda \\ \beta_1 = n-1}} (|J_\nu(\beta)| - r(\beta)) |\operatorname{SStd}(\beta, \lambda)| \\ &= |\operatorname{SStd}(\tilde{\nu}, \lambda)| + \sum_{\substack{\beta \rhd \lambda \\ \beta_1 = n-1}} |J_\nu(\beta)| |\operatorname{SStd}(\beta, \lambda)| - \sum_{\substack{\beta \rhd \lambda \\ \beta_1 = n-1}} r(\beta) |\operatorname{SStd}(\beta, \lambda)|. \end{aligned} \tag{3.2}$$

We may include $|\operatorname{SStd}(\tilde{\nu}, \lambda)|$ in this expression because either $\tilde{\nu} \rhd \lambda$ or $|\operatorname{SStd}(\tilde{\nu}, \lambda)| = 0$ and it is not possible that $\tilde{\nu} = \lambda$ since they have differing first part.

By Lemma 3.38, we know that

$$|\operatorname{SStd}(\tilde{\nu}, \lambda)| = \sum_{\beta \in R(\nu)} |\operatorname{SStd}(\beta, \lambda)| = r(\lambda) + \sum_{\substack{\beta \rhd \lambda \\ \beta_1 = n-1}} r(\beta) |\operatorname{SStd}(\beta, \lambda)|,$$

since $|\operatorname{SStd}(\lambda,\lambda)| = 1$, so Equation (3.2) boils down simply to

$$\sum_{\beta \rhd \lambda} \langle s_\nu \circ s_{(1^2)}, s_\beta \rangle |\operatorname{SStd}(\beta,\lambda)| = r(\lambda) + \sum_{\substack{\beta \rhd \lambda \\ \beta_1 = n-1}} |J_\nu(\beta)||\operatorname{SStd}(\beta,\lambda)|.$$

Substituting this into Equation (3.1), we obtain

$$\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle = |\operatorname{PStd}((1^2)^\nu,\lambda)| - r(\lambda) - \sum_{\substack{\beta \rhd \lambda \\ \beta_1 = n-1}} |J_\nu(\beta)||\operatorname{SStd}(\beta,\lambda)|. \quad (3.3)$$

Lemma 3.40 tells us that

$$\begin{aligned} |\operatorname{PStd}((1^2)^\nu,\lambda)| &= \sum_{\beta \in C(\nu)} |J_\nu(\beta)||\operatorname{SStd}(\beta,\lambda)| \\ &= \sum_{\substack{\beta \unrhd \lambda \\ \beta_1 = n-1}} |J_\nu(\beta)||\operatorname{SStd}(\beta,\lambda)| = |J_\nu(\lambda)| + \sum_{\substack{\beta \rhd \lambda \\ \beta_1 = n-1}} |J_\nu(\beta)||\operatorname{SStd}(\beta,\lambda)|, \quad (3.4) \end{aligned}$$

since $|J_\nu(\beta)| = 0$ when $\beta \notin C(\nu)$, $|\operatorname{SStd}(\beta,\lambda)| = 0$ if $\beta \not\trianglerighteq \lambda$ and $\beta_1 = n-1$ when $\beta \in C(\nu)$, with the final equality following from the fact that $|\operatorname{SStd}(\lambda,\lambda)| = 1$. Finally, substituting equation (3.4) into equation (3.3), we arrive at

$$\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle = |J_\nu(\lambda)| - r(\lambda)$$

as required. □

Remark 3.43. Note that this theorem tells us that $\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle = 0$ whenever $\lambda \notin C(\nu)$, as in this case $J_\nu(\lambda) = \emptyset$.

Example 3.44. *Let us continue from Example 3.36, where ν was equal to $(5,2)$ and we already calculated $R(\nu)$ and $C(\nu)$, as well as $J_\nu(\lambda)$ for every $\lambda \in C(\nu)$. We summarise the data in a table and show the associated plethysm coefficients.*

λ	$J_\nu(\lambda)$	$\|J_\nu(\lambda)\|$	$r(\lambda)$	$\langle s_{(5,2)} \circ s_{(1^2)}, s_\lambda \rangle = \|J_\nu(\lambda)\| - r(\lambda)$
$(6,6,2)$	$\{(3,2)\}$	1	1	0
$(6,6,1^2)$	$\{(4,2)\}$	1	0	1
$(6,5,3)$	$\{(3,2)\}$	1	1	0
$(6,5,2,1)$	$\{(4,3),(4,2)\}$	2	1	1
$(6,5,1^3)$	$\{(5,4)\}$	1	0	1
$(6,4,3,1)$	$\{(4,3)\}$	1	0	1
$(6,4,2,1^2)$	$\{(5,4)\}$	1	0	1

These coefficients can be verified by computer algebra as in SAGE Computation A.7.

Remark 3.45. Note that $\lambda \in C(\nu)$ does not imply that $\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle \neq 0$. This is because it is possible to have $r(\lambda) = |J_\nu(\lambda)| = 1$, as we have just seen when $\lambda = (6,5,3)$ and $\nu = (5,2)$.

Example 3.46. *We provide a final example looking at the plethysm $s_{(4,4,3,2,2)} \circ s_{(1^2)}$. Here $\nu = (4,4,3,2,2)$ and $\tilde{\nu} = (15,4,4,3,2,2)$. By moving a box from the end of the first row of $\tilde{\nu}$ to the end of subsequent rows, we may determine that*

$$R(\nu) = \{(14,5,4,3,2,2),(14,4,4,4,2,2),(14,4,4,3,3,2),(14,4,4,3,2,2,1)\}.$$

The partitions of the form $\tilde{\nu} - \epsilon_1 - \epsilon_i$ where $i \geq 2$ are $(14,4,3,3,2,2)$, $(14,4,4,2,2,2)$ and $(14,4,4,3,2,1)$. We summarise the remaining information in a table.

λ	$J_\nu(\lambda)$	$\lvert J_\nu(\lambda)\rvert$	$r(\lambda)$	$\langle s_{(4,4,3,2,2)} \circ s_{(1^2)}, s_\lambda \rangle$
$(14,5,5,3,2,1)$	$\{(3,2)\}$	1	0	1
$(14,5,5,2,2,2)$	$\{(3,2)\}$	1	0	1
$(14,5,4,4,2,1)$	$\{(4,2)\}$	1	0	1
$(14,5,4,3,3,1)$	$\{(5,2)\}$	1	0	1
$(14,5,4,3,2,2)$	$\{(3,2),(4,2),(5,2)\}$	3	1	2
$(14,5,4,3,2,1,1)$	$\{(7,2)\}$	1	0	1
$(14,5,4,2,2,2,1)$	$\{(7,2)\}$	1	0	1
$(14,5,3,3,3,2)$	$\{(5,2)\}$	1	0	1
$(14,5,3,3,2,2,1)$	$\{(7,2)\}$	1	0	1
$(14,4,4,4,3,1)$	$\{(5,4)\}$	1	0	1
$(14,4,4,4,2,2)$	$\{(4,3),(6,4))\}$	2	1	1
$(14,4,4,4,2,1,1)$	$\{(7,4)\}$	1	0	1
$(14,4,4,3,3,2)$	$\{(5,3),(5,4),(6,5)\}$	3	1	2
$(14,4,4,3,3,1,1)$	$\{(7,5)\}$	1	0	1
$(14,4,4,3,2,2,1)$	$\{(7,3),(7,4),(7,6)\}$	3	1	2
$(14,4,4,3,2,1,1,1)$	$\{(8,7)\}$	1	0	1
$(14,4,4,2,2,2,1,1)$	$\{(8,7)\}$	1	0	1
$(14,4,3,3,3,3)$	$\{(6,5)\}$	1	0	1
$(14,4,3,3,3,2,1)$	$\{(7,5)\}$	1	0	1
$(14,4,3,3,2,2,1,1)$	$\{(8,7)\}$	1	0	1

This completes our study of $s_{(4,4,3,2,2)} \circ s_{(1^2)}$.

3.2.1 Rectangles

We will apply the above results to rectangles. Take $\nu = (a^b)$ where $b \geq 3$, a rectangle with more than two rows, so $n = ab$. In this case, $\tilde{\nu} = (n, a^b)$. One can verify that

$$R(\nu) = \{(n-1, a+1, a^{b-1}), (n-1, a^b, 1)\}.$$

Now assume that $a \geq 2$. The only removable node of $\tilde{\nu}$ below the first row is the one in the lower right corner of the rectangle, so the only possibility for $\tilde{\nu} - \epsilon_1 - \epsilon_e$ is $(n-1, a^{b-1}, a-1)$, corresponding to $e = b+1$. We summarise the remaining information in a table.

λ	$J_\nu(\lambda)$	$\lvert J_\nu(\lambda)\rvert$	$r(\lambda)$	$\langle s_{(a^b)} \circ s_{(1^2)}, s_\lambda \rangle$
$(n-1, (a+1)^2, a^{b-3}, a-1)$	$\{(3,2)\}$	1	0	1
$(n-1, a+1, a^{b-1})$	$\{(b+1,2)\}$	1	1	0
$(n-1, a+1, a^{b-2}, a-1, 1)$	$\{(b+2,2)\}$	1	0	1
$(n-1, a^b, 1)$	$\{(b+2,b+1)\}$	1	1	0
$(n-1, a^{b-1}, a-1, 1^2)$	$\{(b+3,b+2)\}$	1	0	1

An example of these coefficients is calculated in SAGE Computation A.8.

In the case $a = 1$ we have $b = n$ and the table simplifies.

λ	$J_\nu(\lambda)$	$\lvert J_\nu(\lambda)\rvert$	$r(\lambda)$	$\langle s_{(1^b)} \circ s_{(1^2)}, s_\lambda \rangle$
$(n-1, 2^2, 1^{n-3})$	$\{(3,2)\}$	1	0	1
$(n-1, 2, 1^{n-1})$	$\{(n+1,2)\}$	1	1	0
$(n-1, 1^{n+1})$	$\{n+2, b+1)\}$	1	1	0

An example of these coefficients is calculated in SAGE Computation A.9.

3.2.2 Hooks

Suppose $\nu = (a, 1^b)$ with $a \geq 3, b \geq 3$ a genuine hook with arm and leg length at least 3. Then $n = a + b$ and $\tilde{\nu} = (n, a, 1^b)$. One can verify that

$$R(\nu) = \{(n-1, a+1, 1^b), (n-1, a, 2, 1^{b-1}), (n-1, a, 1^{b+1})\}.$$

There are two options for $\tilde{\nu} - \epsilon_1 - \epsilon_e$ corresponding to the cases $e = 2$ and $e = b+2$. These partitions are $(n-1, a-1, 1^b)$ and $(n-1, a, 1^{b-1})$ respectively. We summarise the remaining information in a table.

λ	$J_\nu(\lambda)$	$\lvert J_\nu(\lambda)\rvert$	$r(\lambda)$	$\langle s_{(a,1^b)} \circ s_{(1^2)}, s_\lambda \rangle$
$(n-1, a+1, 2, 1^{b-2})$	$\{(3,2)\}$	1	0	1
$(n-1, a+1, 1^b)$	$\{(b+2,2)\}$	1	1	0
$(n-1, a, 2^2, 1^{b-3})$	$\{(4,3)\}$	1	0	1
$(n-1, a, 2, 1^{b-1})$	$\{(b+2,3),(3,2)\}$	2	1	1
$(n-1, a, 1^{b+1})$	$\{(b+3,b+2),(b+3,2)\}$	2	1	1
$(n-1, a-1, 2^2, 1^{b-2})$	$\{(4,3)\}$	1	0	1
$(n-1, a-1, 2, 1^b)$	$\{(b+3,3)\}$	1	0	1
$(n-1, a-1, 1^{b+2})$	$\{(b+4,b+3)\}$	1	0	1

The cases $a = 1, a = 2$ and $b = 2$ are not excluded from our method. The case $a = 1$ overlaps with the rectangle case above with $a = 1$. If $a = 2$, we delete the entries $(n-1, a-1, 2^2, 1^{b-2})$ and $(n-1, a-1, 2, 1^b)$ from the table since these are no longer valid partitions. If $b = 2$, we delete the entry $(n-1, a, 2^2, 1^{b-3})$ from the table since this no longer makes sense as a partition.

Some examples are calculated in SAGE Computations A.10 and A.11.

3.2.3 Two Rows

Suppose that $\nu = (a, b)$ is a two row partition, and recall that in this circumstance $n = a+b$ and $\tilde{\nu} = (n, a, b)$. For now, we shall assume that $a \geq b+2$ and deal with the remaining cases later. One may verify that

$$R(\nu) = \{(n-1, a+1, b), (n-1, a, b+1), (n-1, a, b, 1)\}.$$

The possibilities for $\tilde{\nu} - \epsilon_1 - \epsilon_e$ are $(n-1, a-1, b)$, corresponding to $e = 2$, and $(n-1, a, b-1)$, corresponding to the case where $e = 3$. We summarise the remaining information in a table.

λ	$J_\nu(\lambda)$	$\lvert J_\nu(\lambda)\rvert$	$r(\lambda)$	$\langle s_{(a,1^b)} \circ s_{(1^2)}, s_\lambda \rangle$
$(n-1, a+1, b)$	$\{(3,2)\}$	1	1	0
$(n-1, a+1, b-1, 1)$	$\{(4,2)\}$	1	0	1
$(n-1, a, b+1)$	$\{(3,2)\}$	1	1	0
$(n-1, a, b, 1)$	$\{(4,3),(4,2)\}$	2	1	1
$(n-1, a, b-1, 1^2)$	$\{(5,4)\}$	1	0	1
$(n-1, a-1, b+1, 1)$	$\{(4,3)\}$	1	0	1
$(n-1, a-1, b, 1^2)$	$\{(5,4)\}$	1	0	1

When $a = b + 1$, the coefficients are the same as in the table above, apart from the row for $(n-1, a-1, b+1, 1)$ can be deleted entirely.

When $a = b$, $R(\nu) = \{(n-1, a+1, b), (n-1, a, b, 1)\}$ and the only option for $\tilde{\nu} = \epsilon_1 - \epsilon_e$ is $(n-1, a, b-1)$. The appropriate table is below.

λ	$J_\nu(\lambda)$	$\lvert J_\nu(\lambda)\rvert$	$r(\lambda)$	$\langle s_{(a,b)} \circ s_{(1^2)}, s_\lambda \rangle$
$(n-1, a+1, b)$	$\{(3,2)\}$	1	1	0
$(n-1, a+1, b-1, 1)$	$\{(4,2)\}$	1	0	1
$(n-1, a, b, 1)$	$\{(4,3),(4,2)\}$	1	1	0
$(n-1, a, b-1, 1^2)$	$\{(5,4)\}$	1	0	1

Some examples are calculated in SAGE Computations A.12, A.13 and A.14.

3.3 Near maximal coefficients of $s_\nu \circ s_{(2,1)}$

Finally, we consider the case where instead of (m) or (1^2), we use $\mu = (2, 1)$ for the inner partition in the plethysm $s_\nu \circ s_\mu$.

Throughout this subsection, we will need to use a different order on tableaux than the previous sections. This is because it is desirable for our purposes to have (21)-tableaux of the form $\begin{array}{|c|c|}\hline 1 & y \\ \hline z & \\ \cline{1-1}\end{array}$ with $y, z \geq 2$ at the end of rows of plethystic tableaux. To this end, in this section we declare a new order as follows. Suppose that s, t

are tableaux and examine the first row where the two differ. If the lowest entry not occurring in both rows occurs only in s, then declare $s \prec_r t$. For example

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \prec_r \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} \prec_r \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array},$$

and in general

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline x \\ \cline{1-1} \end{array} \prec_r \begin{array}{|c|c|}\hline 1 & y \\ \hline z \\ \cline{1-1} \end{array}$$

when $y \geq 2$.

Recall that $\nu \vdash n$. In this section we will use the notation $\tilde{\nu} = (2n, \nu_1, \cdots, \nu_{\ell(\nu)})$ and note by Theorem 2.96 that this labels the lexicographically maximal constituent of the plethysm $s_\nu \circ s_{(2,1)}$, and that $\langle s_\nu \circ s_{(2,1)}, s_{\tilde{\nu}} \rangle = 1$. It is easy to prove the following result in the same manner as Lemma 3.32.

Lemma 3.47. *Suppose that $\nu \vdash n, \lambda \vdash 3n$ and $\lambda_1 = 2n$. Then*

$$\langle s_\nu \circ s_{(2,1)}, s_\lambda \rangle = \begin{cases} 1 & \text{if } \lambda = \tilde{\nu} \\ 0 & \text{else.} \end{cases}$$

Now, suppose that $\nu \vdash n, \lambda \vdash 3n$ and $\lambda_1 = 2n - 1$. We will proceed to give a combinatorial description of $\langle s_\nu \circ s_{(2,1)}, s_\lambda \rangle$ in a fashion reminiscent of our previous results. We require definitions as follows.

Definition 3.48. Suppose $\nu \vdash n$.

- Define $R(\nu) = \{\beta \vdash 3n | \beta = \tilde{\nu} - \epsilon_1 + \epsilon_f \text{ where } f \geq 2\}$. Recall that $r(\lambda) = 1$ if $\lambda \in R(\nu)$ and 0 otherwise.
- Define $D(\nu) = \{\beta \vdash 3n | \beta = \tilde{\nu} - \epsilon_1 - \epsilon_e + \epsilon_a + \epsilon_b \text{ where } e, a, b \geq 2\}$
- Define $K_\nu(\beta)$ by the following set of rules. Suppose that $\beta = \tilde{\nu} - \epsilon_1 - \epsilon_e + \epsilon_a + \epsilon_b$ where $e, a, b \geq 2$, and assume $b \geq a$.
    - If $b = a$, then $(a, a) \in K_\nu(\beta)$.

– If $b > a$ and $\beta_a = \beta_b$, then $(a, b) \in K_\nu(\beta)$ and $(b, a) \notin K_\nu(\beta)$. This is the case where the boxes at the ends of rows a and b are in the same column.

– If $b > a$ and $\beta_a \neq \beta_b$, then $(a, b) \in K_\nu(\beta)$ and $(b, a) \in K_\nu(\beta)$.

Example 3.49. *Let us consider a partial example for the partition* $\nu = (4, 3, 1) \vdash 8$*, so that* $\tilde{\nu} = (16, 4, 3, 1)$ *and*

$$R(\nu) = \{(15, 5, 3, 1), (15, 4, 4, 1), (15, 4, 3, 2), (15, 4, 3, 1^2)\}.$$

The partitions of the form $\tilde{\nu} - \epsilon_1 - \epsilon_e$ *are* $(15, 3, 3, 1), (15, 4, 2, 1)$ *and* $(15, 4, 3)$.

Let us first set $\lambda^1 = (15, 4, 3, 2)$*. To obtain this from* $(15, 3, 3, 1)$ *we must add boxes to the ends of rows* 2 *and* 4*, and we may do this in either order, so* $(2, 4), (4, 2) \in K_\nu(\lambda^1)$*. To obtain this from* $(15, 4, 2, 1)$ *we must add boxes to the ends of rows* 3 *and* 4*, again in either order, so* $(3, 4), (4, 3) \in K_\nu(\lambda^1)$*. However, to obtain* λ *from* $(15, 4, 3)$ *we must add two boxes to the end of row* 4*, and thus only one value is added to* $K_\nu(\lambda^1)$*, being* $(4, 4)$*. In summary,*

$$K_\nu((15, 4, 3, 2)) = \{(2, 4), (3, 4), (4, 2), (4, 3), (4, 4))\}.$$

Consider instead the partition $\lambda^2 = (15, 3, 3, 1^3)$*. This is not obtainable from* $(15, 4, 2, 1)$ *or* $(15, 4, 3)$*, only from* $(15, 3, 3, 1)$*, to which we need to add boxes in rows* 5 *and* 6*. However, these added boxes are in the same column and can only be added in one order so only one value is added to* $K_\nu(\lambda^2)$*. In summary,*

$$K_\nu(15, 3, 3, 1^3) = \{(5, 6)\}.$$

Finally, consider the partition $\lambda^3 = (15, 6, 3)$*. This is only obtainable from* $(15, 4, 3)$*, and we must add two boxes in the second row. Therefore*

$$K_\nu(15, 6, 3) = \{(2, 2)\}.$$

Lemma 3.50. *For any partition $\nu \vdash n$, the set $R(\nu)$ is contained in the set $D(\nu)$.*

Lemma 3.51. *Suppose $\nu \vdash n$ and let $\lambda \vdash 3n$ such that $\lambda_1 = 2n-1$. Then*

$$|\operatorname{SStd}(\tilde{\nu}, \lambda)| = \sum_{\beta \in R(\nu)} |\operatorname{SStd}(\beta, \lambda)|$$

Proof. Identical to Lemma 3.38. □

Suppose $\nu \vdash n$ and $\lambda \vdash 3n$ with $\lambda_1 = 2n-1$. We define a map

$$\Omega_0 : \operatorname{PStd}((2,1)^\nu, \lambda) \to \bigsqcup_{\beta \in D(\nu)} \operatorname{SStd}(\beta, \lambda)$$

as follows. Suppose $T \in \operatorname{PStd}((2,1)^\nu, \lambda)$ is a plethystic semistandard tableau. This tableau must contain exactly $2n-1$ ones, and therefore each outer box must be filled with some tableau $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline x \\ \cline{1-1}\end{array}$ where $x \geq 2$, apart from a single box which is filled with a tableau $\begin{array}{|c|c|}\hline 1 & y \\ \hline z \\ \cline{1-1}\end{array}$ where $y, z \geq 2$. Note that it may be that $y > z$, $y = z$ or $y < z$, there is no restriction other than both being at least 2. Indeed, our new order guarantees that $\begin{array}{|c|c|}\hline 1 & y \\ \hline z \\ \cline{1-1}\end{array}$ fills some removable box at the end of a row. Say that this box occurs at the end of row $e-1$. Write t for the tableau obtained by deleting this box, replacing all the entries $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline x \\ \cline{1-1}\end{array}$ of the other boxes with the number x and appending a new row of $2n-1$ ones to the top of t. This tableau t is of shape $\tilde{\nu} - \epsilon_1 - \epsilon_e$. We define $\Omega_0(T)$ to be equal to $(z \to (y \to t))$, the tableau obtained from t by inserting y into the first column, followed by z.

Now, observe that the insertion of y in this process will add a box to the end of some row $a \geq 2$, and the subsequent insertion of z will add a box to the end of some row $b \geq 2$. Without knowing the relationship between y and z, we cannot comment on the relative positions of these two boxes. Define

$$\Omega : \operatorname{PStd}((21)^\nu, \lambda) \to \bigsqcup_{\beta \in D(\nu)} (\operatorname{SStd}(\beta, \lambda) \times K_\nu(\beta))$$

by

$$\Omega(T) = (\Omega_0(T), (a, b)),$$

where, to reiterate, a is the row number where a box is added from the insertion of y, and b is the row number where a box is added from the insertion of z. It may be that $a > b, a = b$ or $a < b$, and there are no restrictions on which added box is above the other.

Example 3.52. *We will exhibit the effect of Ω on some plethystic tableaux of shape $(2,1)^{(3,1,1)}$ and weight $(9,4,1,1)$. We consider in tandem the tableaux T_1 and T_2 which are identical apart from the transposition of two entries in a single inner tableau.*

$$T_1 = \begin{array}{|c|c|c|}\hline \begin{array}{|c|c|}\hline 1&1\\\hline 2& \\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 2& \\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 2& \\\cline{1-1}\end{array} \\\hline \begin{array}{|c|c|}\hline 1&1\\\hline 3& \\\cline{1-1}\end{array} \\\cline{1-1} \begin{array}{|c|c|}\hline 1&2\\\hline 4& \\\cline{1-1}\end{array} \\\cline{1-1}\end{array} \qquad T_2 = \begin{array}{|c|c|c|}\hline \begin{array}{|c|c|}\hline 1&1\\\hline 2& \\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 2& \\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 2& \\\cline{1-1}\end{array} \\\hline \begin{array}{|c|c|}\hline 1&1\\\hline 3& \\\cline{1-1}\end{array} \\\cline{1-1} \begin{array}{|c|c|}\hline 1&4\\\hline 2& \\\cline{1-1}\end{array} \\\cline{1-1}\end{array}$$

We can remove the box containing a single one in the last row, replace all inner $(2,1)$-tableaux $\begin{array}{|c|c|}\hline 1&1\\\hline x& \\\cline{1-1}\end{array}$ by their entry x and append a new row of ones to the top of each tableau to obtain

$$t = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&1\\\hline 2&2&2\\\cline{1-3} 3\\\cline{1-1}\end{array}.$$

Recall that $\Omega_0(T_1) = (4 \to (2 \to t))$ and $\Omega_0(T_2) = (2 \to (4 \to t))$. One follows this through to discover that

$$\Omega_0(T_1) = \Omega_0(T_2) = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&1\\\hline 2&2&2&2\\\cline{1-4} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}.$$

However, in the case of T_1 the box at the end of the second row is added, followed by the new fourth row box. In the case of T_2, the reverse is true. Therefore $\Omega(T_1) = (t,(2,4))$ whereas $\Omega(T_1) = (t,(4,2))$.

Now, consider a different tableau

$$T_3 = \begin{array}{|c|c|c|}\hline \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array} \\ \hline \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 & \\ \cline{1-1}\end{array} \\ \cline{1-1} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array} \\ \cline{1-1}\end{array}$$

which has no transposed counterpart in the same relationship as T_1 and T_2 are paired. One can calculate that

$$\Omega_0(T_3) = t_3 := \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 3 \\ \cline{1-5} 4 \\ \cline{1-1}\end{array}$$

with $\Omega(T_3) = (t_3, (2,2))$.

Lemma 3.53. *Suppose that $\nu \vdash n$ and $\lambda \vdash 3n$ with $\lambda_1 = 2n-1$. Then the map*

$$\Omega : \mathrm{PStd}((2,1)^\nu, \lambda) \to \bigsqcup_{\beta \in D(\nu)} \mathrm{SStd}(\beta, \lambda) \times K_\nu(\beta)$$

is a bijection.

Proof. Suppose that $(s, (c,d)) \in \mathrm{SStd}(\beta, \lambda) \times K_\nu(\beta)$. Apply reverse column insertion to obtain a tableau s_0 of shape $(\mathrm{shape}(s) - \epsilon_d)$ such that $s = (z \to s_0)$. Apply reverse column insertion again to obtain a tableau s_1 of shape $(\mathrm{shape}(s) - \epsilon_d - \epsilon_c)$ such that $s_0 = (y \to s_1)$. Delete the row of ones from the top of s_0 and replace each entry x with the tableau $\begin{smallmatrix}1 & 1 \\ x & \end{smallmatrix}$. Finally, append a box containing $\begin{smallmatrix}1 & y \\ z & \end{smallmatrix}$ to the end of the $(e-1)$st row to obtain $S \in \mathrm{PStd}((2,1)^\nu, \lambda)$ such that $\Omega(S) = s$. □

Theorem 3.54. *Suppose $\nu \vdash n$ and $\lambda \vdash 3n$ such that $\lambda_1 = 2n-1$. Then*

$$\langle s_\nu \circ s_{(2,1)}, s_\lambda \rangle = |K_\nu(\lambda)| - r(\lambda)$$

Proof. The proof follows the method of Theorem 3.42 closely, substituting $J_\nu(\lambda)$ for $K_\nu(\lambda)$, and thus we omit it. □

Example 3.55. *Let us apply our method to the plethysm* $s_{(3^3)} \circ s_{(2,1)}$. *Here* $\nu = (3^3) \vdash 9$ *and* $\tilde{\nu} = (18, 3^3)$ *and we are able to study constituents* λ *with first part* 17. *One can easily see that*

$$R(\nu) = \{(17,4,3,3),(17,3^3,1)\},$$

and observe that the only partition of the form $\tilde{\nu} - \epsilon_1 - \epsilon_e$ *for* $e \geq 2$ *is* $(17,3,3,2)$. *We summarise the remaining information in a table, the contents of which one can verify by computer algebra as in SAGE Computation A.15.*

λ	$K_\nu(\lambda)$	$\lvert K_\nu(\lambda)\rvert$	$r(\lambda)$	$\langle s_{(3^3)} \circ s_{(2,1)}, s_\lambda \rangle$
$(17,5,3,2)$	$\{(2,2)\}$	1	0	1
$(17,4,4,2)$	$\{(2,3)\}$	1	0	1
$(17,4,3,3)$	$\{(2,4),(4,2)\}$	2	1	1
$(17,4,3,2,1)$	$\{(2,5),(5,2)\}$	2	0	2
$(17,3^3,1)$	$\{(4,5),(5,4)\}$	2	1	1
$(17,3,3,2,2)$	$\{(5,5)\}$	1	0	1
$(17,3,3,2,1^2)$	$\{(5,6)\}$	1	0	1

Example 3.56. *Now we consider* $s_{(5,3)} \circ s_{(2,1)}$, *so that* $\nu = (5,3) \vdash 8, \tilde{\nu} = (16,5,3)$ *and*

$$R(\nu) = \{(15,6,3),(15,5,4),(15,5,3,1)\}.$$

The options for $\tilde{\nu} - \epsilon_1 - \epsilon_e$ *with* $e \geq 2$ *are* $(15,4,3)$ *and* $(15,5,2)$. *We summarise the remaining information in a table verifiable by computer algebra as in SAGE Computation A.16.*

λ	$K_\nu(\lambda)$	$\lvert K_\nu(\lambda)\rvert$	$r(\lambda)$	$\langle s_{(5,3)} \circ s_{(2,1)}, s_\lambda \rangle$
$(15,7,2)$	$\{(2,2)\}$	1	0	1
$(15,6,3)$	$\{(2,2),(2,3),(3,2)\}$	3	1	2
$(15,6,2,1)$	$\{(2,4),(4,2)\}$	2	0	2
$(15,5,4)$	$\{(2,3),(3,2),(3,3)\}$	3	1	2
$(15,5,3,1)$	$\{(2,4),(3,4),(4,3),(4,2)\}$	4	1	3
$(15,5,2,2)$	$\{(4,4)\}$	1	0	1
$(15,5,2,1,1)$	$\{(4,5)\}$	1	0	1
$(15,4,4,1)$	$\{(3,4),(4,3)\}$	2	0	2
$(15,4,3,2)$	$\{(4,4)\}$	1	0	1
$(15,4,3,1,1)$	$\{(4,5)\}$	1	0	1

Example 3.57. *Finally, we discuss* $s_{(3,1^2)} \circ s_{(21)}$*, so* $\nu = (3,1^2) \vdash 5$ *and* $\tilde{\nu} = (10,3,1^2)$*. Here*

$$R(\nu) = \{(9,4,1^2),(9,3,2,1),(9,3,1^3)\}$$

and the possibilities for $\tilde{\nu} - \epsilon_1 - \epsilon_e$ *with* $e \geq 2$ *are* $(9,2,1^2)$ *and* $(9,3,1)$*. The remaining information is summarised in a table and verifiable by computer algebra as in SAGE Example A.17.*

λ	$K_\nu(\lambda)$	$\lvert K_\nu(\lambda)\rvert$	$r(\lambda)$	$\langle s_{(3,1^2)} \circ s_{(2,1)}, s_\lambda \rangle$
$(9,5,1)$	$\{(2,2)\}$	1	0	1
$(9,4,2)$	$\{(2,3),(3,2)\}$	2	0	2
$(9,4,1,1)$	$\{(2,2),(2,4),(4,2)\}$	3	1	2
$(9,3,3)$	$\{(3,3)\}$	1	0	1
$(9,3,2,1)$	$\{(2,3),(3,2),(3,4),(4,3)\}$	4	1	3
$(9,3,1^3)$	$\{(2,5),(4,5),(5,2)\}$	3	1	2
$(9,2^3)$	$\{(3,4)\}$	1	0	1
$(9,2,2,1,1)$	$\{(2,4),(4,2)\}$	2	0	2
$(9,2,1^4)$	$\{(5,6)\}$	1	0	1

3.4 Computations with a Formula of Law and Okitani

In their paper [34], Law and Okitani derive a new recursive formula for some of the plethysm coefficients $\langle s_\nu \circ s_{(m)}, s_\lambda \rangle$.

Recall the Littlewood-Richardson coefficients $c^\gamma_{\alpha,\beta}$ defined in Definition 2.76 and calculated by the Littlewood-Richardson Rule stated in Theorem 2.82. Law and Okitani use these and plethysm coefficients to define the *skew plethysm coefficients*. For brevity of notation, we follow Law and Okitani and write $a^\lambda_{\nu,\mu}$ for the plethysm coefficient $\langle s_\nu \circ s_\mu, s_\lambda \rangle$.

Definition 3.58. [34, Eqn 2.4] The skew plethysm coefficient $a^{\alpha/\beta}_{\gamma/\delta,\mu}$ is defined by the formula

$$a^{\alpha/\beta}_{\gamma/\delta,\mu} = \sum_{\substack{\eta \vdash |\gamma|-|\delta| \\ \zeta \vdash |\alpha|-|\beta|}} c^\gamma_{\eta,\delta} c^\alpha_{\zeta,\beta} a^\zeta_{\eta,\mu}.$$

It is readily seen that the skew plethysm coefficient $a^{\alpha/\emptyset}_{\gamma/\emptyset,\mu}$ is equal to the usual plethysm coefficient $a^\alpha_{\gamma,\mu}$. This is simply because the Littlewood-Richardson coefficient $c^\alpha_{\zeta,\emptyset}$ is equal to 1 if $\alpha = \zeta$, and 0 otherwise. From now on, we will write α instead of $\alpha/\emptyset$ to condense notation, for example, writing $a^{\alpha/(k-i)}_{\beta',(m)}$ instead of $a^{\alpha/(k-i)}_{\beta'/\emptyset,(m)}$.

We recall the main theorem from [34] concerning the calculation of plethysm coefficients.

Theorem 3.59. *[34, Theorem A] Let $\nu \vdash n$ and $\lambda \vdash mn$ with $\ell(\lambda) \leq n$. Write $k := n - \ell(\lambda)$ and define $\hat{\lambda} = \lambda - (1^{n-k})$, the partition obtained from λ by removing its first column. Then*

$$a^\lambda_{\nu',(m)} = \sum_{i=0}^{k} (-1)^{k+i} \cdot \left(\sum_{\substack{\alpha \vdash k+(m-1)i \\ \beta \vdash i}} a^{\alpha/(k-i)}_{\beta',(m)} \cdot a^{\hat{\lambda}/\alpha}_{\nu/\beta,(m-1)} \right). \tag{3.5}$$

The equation in this theorem becomes more complicated as $k := n-\ell(\lambda)$ increases. It also increases in complexity when m increases, since in a calculation for $\mu = (m)$ we must already know the answer to a similar calculation for $\mu = (m-1)$. Using a sign twist (Lemma 2.92) allows us to use Theorem 3.59 to compute plethysm coefficients $a^{\lambda}_{\nu,(1^m)}$. Instead of the quantity $n-\ell(\lambda)$, the calculation will become more difficult as $n-\lambda_1$ increases. In the previous section, we calculated $a^{\lambda}_{\nu,(1^m)}$ when $m = 2$ and $\lambda_1 = n$ or $\lambda_1 = n-1$. We will confirm these results with alternative proofs relying on Theorem 3.59 before giving a description of the case where $m = 2$ and $\lambda_1 = n-2$.

3.5 Near-maximal constituents of $s_\nu \circ s_{(1^2)}$ revisited.

We observed from Theorem 2.96 that the lexicographically maximal constituent of $s_\nu \circ s_{(1^2)}$ is $s_{(n,\nu_1,\cdots,\nu_{\ell(\nu)})}$, that this constituent appears with multiplicity 1, and that all other partitions with first part n have multiplicity 0 in this plethysm. As a warmup, we will use Theorem 3.59 to rederive this result as a simple case. Firstly, note that to find the constituents of $s_\nu \circ s_{(1^2)}$ with first part n, we can use the sign twist (Lemma 2.92) and find consituents of $s_\nu \circ s_{(2)}$ with length n.

We can now calculate $a^{\theta}_{\nu,(2)}$ when $\ell(\theta) = n$ using Theorem 3.59. In this case, we have $k = 0$ and thus the outer sum has only one constituent corresponding to $i = 0$. Equation (3.5) with these parameters is

$$a^{\theta}_{\nu,(2)} = \sum_{\substack{\alpha \vdash 0+(1)0 \\ \beta \vdash 0}} a^{\alpha}_{\beta',(2)} \cdot a^{\hat{\theta}/\alpha}_{\nu'/\beta,(1)} = a^{\emptyset}_{\emptyset,(2)} \cdot a^{\hat{\theta}}_{\nu',(1)}.$$

Now, we know that $a^{\emptyset}_{\emptyset,(2)} = 1$, and thus

$$a^{\theta}_{\nu,(2)} = a^{\hat{\theta}}_{\nu',(1)} = \begin{cases} 1 & \text{if } \hat{\theta} = \nu' \\ 0 & \text{else.} \end{cases}$$

Applying the sign twist, taking $\theta' = \lambda$, we see that

$$a^{\lambda}_{\nu,(1^2)} = \begin{cases} 1 & \text{if } \widehat{(\lambda')} = \nu' \\ 0 & \text{else.} \end{cases}$$

Now,

$$\widehat{(\lambda')} = \nu' \iff \lambda' = (1^n) + \nu' \iff \lambda = (n, \nu_1, \ldots, \nu_{\ell(\nu)}) = \tilde{\nu},$$

so

$$a^{\lambda}_{\nu,(1^2)} = \begin{cases} 1 & \text{if } \lambda = \tilde{\nu} \\ 0 & \text{else.} \end{cases}$$

This gives an alternative derivation of Lemma 3.32 on the value of $a^{\lambda}_{\nu,(1^2)}$ when $\lambda_1 = n$.

We will now proceed to rederive Theorem 3.42 from Theorem 3.59. We want to determine $\langle s_\nu \circ s_{(1^2)}, s_\lambda \rangle$ when $\lambda_1 = n - 1$, which is exactly $\langle s_\nu \circ s_{(2)}, s_{\lambda'} \rangle$ where $\ell(\lambda') = n-1$. We write $\theta := \lambda'$ and for now keep m arbitrary. Suppose that $\nu \vdash n$, and note that $\theta \vdash mn$ with $\ell(\theta) = n-1$. In the setting of Lemma 3.59, this means $k = 1$ and $\hat{\theta} = \theta - (1^{n-1})$. Equation (3.5) becomes

$$a^{\theta}_{\nu,(m)} = \sum_{i=0}^{1} \left((-1)^{1+i} \cdot \left(\sum_{\substack{\alpha \vdash 1+(m-1)i \\ \beta \vdash i}} a^{\alpha/(1-i)}_{\beta',(m)} \cdot a^{\hat{\theta}/\alpha}_{\nu'/\beta,(m-1)} \right) \right).$$

In the $i = 1$ term, α will run over all partitions of m and β will be fixed at the partition (1), whereas in the $i = 0$ term, α must be (1) and β must be $\emptyset$. Thus the sum notation simplifies to

$$a^{\lambda}_{\nu,(m)} = \left(\sum_{\alpha \vdash m} a^{\alpha}_{(1),(m)} \cdot a^{\hat{\theta}/\alpha}_{\nu'/(1),(m-1)} \right) - a^{(1)/(1)}_{\emptyset,(m)} \cdot a^{\hat{\theta}/(1)}_{\nu',(m-1)}.$$

Recall that

$$a^{\alpha}_{(1),(m)} = \begin{cases} 1 & \text{if } \alpha = (m) \\ 0 & \text{else} \end{cases}$$

and note that $a^{(1)/(1)}_{\emptyset,(m)} = 1$.

Thus in the first sum we only need to consider the $\alpha = (m)$ term and our expression reduces to

$$a^{\theta}_{\nu,(m)} = a^{\hat{\theta}/(m)}_{\nu'/(1),(m-1)} - a^{\hat{\theta}/(1)}_{\nu',(m-1)}.$$

Referring to Definition 3.58, the first term can be expressed as

$$a^{\hat{\theta}/(m)}_{\nu'/(1),(m-1)} = \sum_{\substack{\eta \vdash n-1 \\ \zeta \vdash mn-m-n+1}} c^{\nu'}_{\eta,(1)} \cdot c^{\hat{\theta}}_{\zeta,(m)} \cdot a^{\zeta}_{\eta,(m-1)} \tag{3.6}$$

with the second becoming

$$a^{\hat{\theta}/(1)}_{\nu',(m-1)} = \sum_{\substack{\eta \vdash n \\ \zeta \vdash mn-n}} c^{\nu'}_{\eta,\emptyset} \cdot c^{\hat{\theta}}_{\zeta,(1)} \cdot a^{\zeta}_{\eta,(m-1)} = \sum_{\zeta \vdash mn-n} c^{\hat{\theta}}_{\zeta,(1)} \cdot a^{\zeta}_{\nu',(m-1)}. \tag{3.7}$$

Referring back to our background results in the Littlewood-Richardson Section, we recall Young's Rule (Lemma 2.85):

$$c^{\alpha}_{\beta,(m)} = \begin{cases} 1 & \text{if } \alpha \in S^c_m(\beta) \\ 0 & \text{else,} \end{cases}$$

where $S^c_m(\beta)$ is the set of all partitions obtainable from β by adding m boxes, no two of which are in the same column. We also established the notation that $S^c_{-m}(\alpha)$ is the set of all partitions obtainable from α by removing m boxes, no two of which are in the same column. Recall that we write $\alpha = \beta + \square$ when $\alpha \in S^c_1(\beta)$ and that $S^c_0(\beta) = \{\beta\}$.

From these known Littlewood-Richardson coefficients, one can simplify equations (3.6) and (3.7) to arrive at the equation

$$a^{\theta}_{\nu,(m)} = \sum_{\substack{\eta=\nu'-\square \\ \zeta\in S^c_{-m}(\hat{\theta})}} a^{\zeta}_{\eta,(m-1)} - \sum_{\zeta=\hat{\theta}-\square} a^{\zeta}_{\nu',(m-1)}$$

in the case where $\lambda'_1 = n-1$.

Let us specialise this equation again to the case where $m=2$ to reprove Theorem 3.42. In this case, the above expression becomes

$$a^{\theta}_{\nu,(2)} = \sum_{\substack{\eta=\nu'-\square \\ \zeta\in S^c_{-2}(\hat{\theta})}} a^{\zeta}_{\eta,(1)} - \sum_{\zeta=\hat{\theta}-\square} a^{\zeta}_{\nu',(1)},$$

which is simply

$$a^{\theta}_{\nu,(2)} = \sum_{\substack{\eta=\nu'-\square \\ \zeta\in S^c_{-2}(\hat{\theta}) \\ \eta=\zeta}} 1 - \sum_{\substack{\zeta=\hat{\theta}-\square \\ \nu'=\zeta}} 1.$$

Let us now compare this to Theorem 3.42 by means of a sign twist.

We know from Lemma 2.92 that $a^{\theta'}_{\nu,(1^2)} = a^{\theta}_{\nu,(2)}$. That is, setting $\lambda := \theta'$,

$$a^{\lambda}_{\nu,(1^2)} = \sum_{\substack{\eta=\nu'-\square \\ \zeta\in S^c_{-2}(\widehat{\lambda'}) \\ \eta=\zeta}} 1 - \sum_{\substack{\zeta=\widehat{\lambda'}-\square \\ \nu'=\zeta}} 1.$$

Let us consider the two sums separately. Write $\lambda_{\geq 2}$ for the partition $(\lambda_2,\ldots,\lambda_{\ell(\lambda)})$. Note that $\widehat{(\lambda')}$ is the same as $(\lambda_{\geq 2})'$, as conjugating and removing the first column is equivalent to removing the first row and then conjugating. Therefore, the first sum is equal to the number of ways of forming $(\lambda_{\geq 2})'$ by removing a box from ν', and adding two boxes in different columns. This is the same as forming $\lambda_{\geq 2}$ by removing a box from ν, and adding two boxes in different rows. Finally, this is the same as forming λ from $\tilde{\nu} = (n, \nu_1, \cdots, \nu_{\ell(\nu)})$ by removing a box from the first row and a lower row, before adding two boxes in rows below the first, no two of which are in the same row. Therefore, the first sum is equal to the number of distinct ways to write

$$\lambda = \tilde{\nu} - \epsilon_1 - \epsilon_e + \epsilon_a + \epsilon_b$$

with $e \geq 2$ and $a > b > 1$, which is precisely $|J_\nu(\lambda)|$.

The second sum is 1 if one can write $\widehat{\lambda'} = \nu' + \square$, and 0 otherwise. This is the same as saying the sum is 1 if $\lambda = \tilde{\nu} - \epsilon_1 + \epsilon_f$ where $f \geq 2$, and 0 otherwise. Thus the second sum is equal to $r(\lambda)$. Thus we see that

$$a^{\lambda}_{\nu,(1^2)} = |J_\nu(\lambda)| - r(\lambda),$$

which is precisely the content of Theorem 3.42.

3.6 Further near-maximal constituents of $s_\nu \circ s_{(1^2)}$.

We will compute $\langle s_{\nu'} \circ s_{(2)}, s_\theta \rangle$ when $\ell(\theta) = n - 2$. By writing $\theta := \lambda'$, we see this is the same as computing $\langle s_{\nu'} \circ s_{(1^2)}, s_\lambda \rangle$, and that we have $\lambda_1 = n - 2$.

We use Theorem 3.59, with $\nu \vdash n$ and $\theta \vdash 2n$ satisfying $\ell(\theta) = n - 2$. We recall that $\hat{\theta} := \theta - (1^{n-2})$, that is θ with its first column removed. We leave m generic for now and specialise later. Since $k = 2$, Equation (3.5) becomes

$$a^{\theta}_{\nu',(m)} = \sum_{i=0}^{2} (-1)^{2+i} \left(\sum_{\substack{\alpha \vdash 2+(m-1)i \\ \beta \vdash i}} a^{\alpha/(2-i)}_{\beta',(m)} a^{\hat{\theta}/\alpha}_{\nu/\beta,(m-1)} \right).$$

Making the first sum explicit, we may write this as

$$\sum_{\substack{\alpha \vdash 2 \\ \beta \vdash 0}} a^{\alpha/(2)}_{\beta',(m)} a^{\hat{\theta}/\alpha}_{\nu/\beta,(m-1)} - \sum_{\substack{\alpha \vdash m+1 \\ \beta \vdash 1}} a^{\alpha/(1)}_{\beta',(m)} a^{\hat{\theta}/\alpha}_{\nu/\beta,(m-1)} + \sum_{\substack{\alpha \vdash 2m \\ \beta \vdash 2}} a^{\alpha}_{\beta',(m)} a^{\hat{\theta}/\alpha}_{\nu/\beta,(m-1)}.$$

Replacing the placeholder β with these partitions, we can arrive at four sums each running over a single index α. We label each term for subsequent expansion.

$$\underbrace{\sum_{\alpha\vdash 2} a^{\alpha/(2)}_{\emptyset,(m)} a^{\hat{\theta}/\alpha}_{\nu,(m-1)}}_{\text{Term A}} - \underbrace{\sum_{\alpha\vdash m+1} a^{\alpha/(1)}_{(1),(m)} a^{\hat{\theta}/\alpha}_{\nu/(1),(m-1)}}_{\text{Term B}}$$

$$+ \underbrace{\sum_{\alpha\vdash 2m} a^{\alpha}_{(1^2),(m)} a^{\hat{\theta}/\alpha}_{\nu/(2),(m-1)}}_{\text{Term C}} + \underbrace{\sum_{\alpha\vdash 2m} a^{\alpha}_{(2),(m)} a^{\hat{\theta}/\alpha}_{\nu/(1^2),(m-1)}}_{\text{Term D}}$$

We will now examine each of Terms A, B, C and D in turn. From now on we will specialise to $m = 2$.

We begin with Term A $= \sum_{\alpha\vdash 2} a^{\alpha/(2)}_{\emptyset,(2)} a^{\hat{\theta}/\alpha}_{\nu,(1)}$. Note that Term A only makes sense if $\alpha \subseteq (2)$, i.e. when $\alpha = (2)$. Thus

$$\text{Term A } = a^{\hat{\theta}/(2)}_{\nu,(1)}.$$

Let us turn to Term B $= \sum_{\alpha\vdash 3} a^{\alpha/(1)}_{(1),(2)} a^{\hat{\theta}/\alpha}_{\nu/(1),(1)}$. Again using the definition, one sees that

$$a^{\alpha/(1)}_{(1),(2)} = \sum_{\substack{\eta\vdash 1\\ \zeta\vdash 2}} c^{(1)}_{\eta,\emptyset} c^{\alpha}_{\zeta,(1)} a^{\zeta}_{\eta,(2)} = \sum_{\zeta\vdash 2} c^{(1)}_{(1),\emptyset} c^{\alpha}_{\zeta,(1)} a^{\zeta}_{(1),(2)}.$$

As before, $c^{(1)}_{(1),\emptyset} = 1$ and $a^{\zeta}_{(1),(2)}$ is 1 if $\zeta = (2)$, and 0 otherwise. Thus

$$a^{\alpha/(1)}_{(1),(2)} = c^{\alpha}_{(2),(1)} = \begin{cases} 1 & \text{if } \alpha = (3) \text{ or } \alpha = (2,1) \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\text{Term B } = a^{\hat{\theta}/(3)}_{\nu/(1),(1)} + a^{\hat{\theta}/(2,1)}_{\nu/(1),(1)}.$$

Continuing with Term C $= \sum_{\alpha\vdash 4} a^{\alpha}_{(1^2),(2)} a^{\hat{\theta}/\alpha}_{\nu/(2),(1)}$, things become easier as $a^{\alpha}_{(1^2),(2)}$ is a nonskew plethysm coefficient. We know that $s_{(1^2)} \circ s_{(2)} = s_{(3,1)}$, and thus

$$\text{Term C} = a^{\hat{\theta}/(3,1)}_{\nu/(2),(1)}.$$

Finally, Term D $= \sum_{\alpha\vdash 4} a^{\alpha}_{(2),(2)} a^{\hat{\theta}/\alpha}_{\nu/(1^2),(1)}$. Again, a known plethysm coefficient appears, so we can use the fact that $s_{(2)} \circ s_{(2)} = s_{(4)} + s_{(2,2)}$ to conclude that

$$\text{Term D } = a^{\hat{\theta}/(4)}_{\nu/(1^2),(1)} + a^{\hat{\theta}/(2,2)}_{\nu/(1^2),(1)}.$$

Referring back to our initial calculation, we know that

$$a^{\theta}_{\nu',(2)} = \text{Term A } - \text{Term B } + \text{Term C } + \text{Term D}$$

$$= a^{\hat{\theta}/(2)}_{\nu,(1)} - a^{\hat{\theta}/(3)}_{\nu/(1),(1)} - a^{\hat{\theta}/(2,1)}_{\nu/(1),(1)} + a^{\hat{\theta}/(3,1)}_{\nu/(2),(1)} + a^{\hat{\theta}/(4)}_{\nu/(1^2),(1)} + a^{\hat{\theta}/(2,2)}_{\nu/(1^2),(1)}. \tag{3.8}$$

It remains to make sense of these particular skew plethysm coefficients combinatorially. The following lemmas shed light on these types of coefficients.

Lemma 3.60. *Suppose $\beta \vdash x+2$. We have*

$$a^{\hat{\theta}/\beta}_{\nu/(x),(1)} = \sum_{\zeta\in S^c_{-x}(\nu)} c^{\hat{\theta}}_{\zeta,\beta}.$$

This is equal to the number of fillings of $\hat{\theta}/\zeta$ with a Littlewood-Richardson word of type β, where ζ runs over all possibilities obtainable by removing x boxes from ν, no two of which are in the same column.

Proof. By definition

$$a^{\hat{\theta}/\beta}_{\nu/(x),(1)} = \sum_{\substack{\eta\vdash|\nu|-x\\ \zeta\vdash|\nu|-x}} c^{\nu}_{\eta,(x)} c^{\hat{\theta}}_{\zeta,\beta} a^{\zeta}_{\eta,(1)}.$$

Since $a^{\zeta}_{\eta,(1)}$ is 1 if $\zeta = \eta$ and 0 otherwise, this expression can be simplified to

$$\sum_{\zeta \vdash |\nu| - x} c^{\nu}_{\zeta,(x)} c^{\hat{\theta}}_{\zeta,\beta}.$$

By Young's Rule (Lemma 2.85)

$$c^{\nu}_{\zeta,(x)} = \begin{cases} 1 & \nu \in S^{c}_{x}(\zeta) \\ 0 & \text{else,} \end{cases}$$

so ultimately our sum becomes

$$\sum_{\zeta \in S^{c}_{-x}(\nu)} c^{\hat{\theta}}_{\zeta,\beta}$$

as required. The Littlewood-Richardson coefficient $c^{\hat{\theta}}_{\zeta,\beta}$ is equal to the number of fillings of $\hat{\theta}/\zeta$ with a Littlewood-Richardson word of type β, thus the second part of the lemma also holds. □

Lemma 3.61. *Let $\beta \vdash x + 2$. The skew plethysm coefficient $a^{\hat{\theta}/\beta}_{\nu/(1^x),(1)}$ is equal to the number of fillings of $\hat{\theta}/\zeta$ with a Littlewood-Richardson word of type β, where ζ runs over all possibilities in $S^{r}_{-x}(\nu)$.*

Proof. The proof is analogous to the previous lemma, relying on Pieri's Rule (Lemma 2.86) instead of Young's Rule. □

We can use these lemmas together with equation (3.8) to provide a combinatorial description of $a^{\theta}_{\nu',(2)}$ when θ has length $n-2$. In particular, this coefficient is equal to the *sum* of the following numbers:

- The number of ways of filling $\hat{\theta}/\nu$ with Littlewood-Richardson words of type (2),
- The number of ways of filling $\hat{\theta}/\zeta$ with Littlewood-Richardson words of type (3, 1) where ζ is obtained from ν by removing two boxes in different columns,

- The number of ways of filling $\hat{\theta}/\zeta$ with Littlewood-Richardson words of type $(2,2)$ or (4) where ζ is obtained from ν by removing two boxes in different rows,

minus

- The number of ways of filling $\hat{\theta}/\zeta$ with Littlewood-Richardson words of type $(2,1)$ or (3) where ζ is obtained from ν by removing a single box.

This description is unwieldy, so we organise it into a step-by-step algorithm which may be used to calculate the plethysm coefficient $a^{\theta}_{\nu',(2)}$.

Algorithm 3.62. We present a method of calculation for the plethysm coefficient $a^{\theta}_{\nu',(2)}$ in the case where $\ell(\theta) = n-2$.

- Assign $N = 0$ a placeholder variable.
- Form $\hat{\theta} = \theta - (1^{n-2})$, that is θ with the first column of the Young diagram removed.
- Compute the number of Littlewood-Richardson words of type (2) that can be validly entered in the skew shape $\hat{\theta}/\nu$. Add this number of words to N.
- Find all partitions ζ which can be obtained from ν by removing a single box. For each of these partitions ζ, compute the number of Littlewood-Richardson words of type (3) and $(2,1)$ that can be validly entered in the skew shape $\hat{\theta}/\zeta$. Subtract this number from N, possibly leaving a negative value.
- Find all partitions ζ which can be obtained from ν by removing two boxes, with attention paid to whether these boxes are in the same row or column. Count the number of words as follows.
  - If the two removed boxes are in the same column, compute the number of Littlewood-Richardson words of type $(2,2)$ or (4) which can be validly entered in the skew shape $\hat{\theta}/\zeta$. Add this number to N.

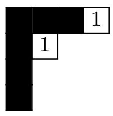
  - If the two removed boxes are in the same row, compute the number of Littlewood-Richardson words of type $(3,1)$ which can be validly entered in the skew shape $\hat{\theta}/\zeta$. Add this number to N.
  - If the two removed boxes are in the different rows and columns, compute the number of Littlewood-Richardson words of type $(2,2),(3,1)$ or (4) which can be validly entered in the skew shape $\hat{\theta}/\zeta$. Add this number to N.

- The final value of N is equal to $a^{\theta}_{\nu',(2)}$.

Suppose we want to compute the plethysm coefficient $a^{\lambda}_{\nu',(1^2)}$ where $\lambda_1 = n-2$. By the sign twist, this is as simple as setting $\theta := \lambda'$, noting that $\ell(\theta) = n-2$, and applying Algorithm 3.62 to compute $a^{\theta}_{\nu',(2)}$

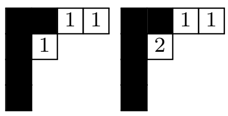
Example 3.63. *Let us calculate* $a^{(4^2,2,1^2)}_{(4,1^2),(1^2)} = a^{(5,3,2^2)}_{(4,1^2),(2)}$. *Here* $n = 6$ *and* $\lambda = (4^2,2,1^2)$*, so* $\theta = (5,3,2^2)$ *which has length* $4 = 6-2$. *We must label* $\nu = (4,1^2)' = (3,1^3)$ *and note* $\hat{\theta} = (4,2,1^2)$.

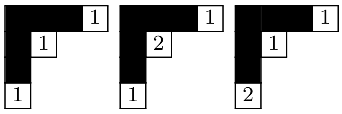
- *Set* $N = 0$.
- *There is only one way to fill* $\hat{\theta}/\nu$ *with a Littlewood-Richardson word of type* (2)*:*



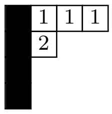
  Therefore we add one to get $N = 1$.

- *The two partitions which can be obtained by removing a box from* $\nu = (3,1^3)$ *are* $(2,1^3)$ *and* $(3,1^2)$. *We may fill with Littlewood-Richardson words of type* (3) *or* $(2,1)$. *The appropriate fillings of* $\hat{\theta}/(2,1^3)$ *are* 111 *and* 112. *The appropriate fillings of* $\hat{\theta}/(3,1^2)$ *are* 111, 121 *and* 112. *We have found* 5 *words, so* N *is now* $-4 = 1-5$. *We illustrate the fillings below.*

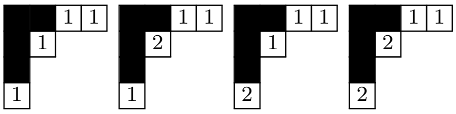




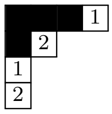
- *The partitions obtainable by removing two boxes from ν are $(1^4), (2, 1^2)$ and $(3, 1)$.*
  - *The partition (1^4) has both removed boxes in the same row, so we may only use type $(3, 1)$. The single allowable filling is* 1112.



  - *The partition $(2, 1^2)$ has removed boxes in different rows and columns, so we may use all of the types $(2, 2), (3, 1)$ and (4). The allowed fillings are* $1111, 1121, 1112$ *and* 1122.



  - *The partition $(3, 1)$ has both removed boxes in the same column, so we may use types $(2, 2)$ and (4). The only allowable filling is* 1212.



  We have found six new fillings to add to N, and thus finally $N = -4 + 6 = 2$.

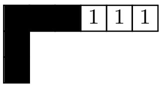
  We conclude that $a^{(4^2,2,1^2)}_{(4,1^2),(1^2)} = 2$. One may verify this calculation by means of a computer algebra system as in SAGE Computation A.18.

Example 3.64. *We calculate $a^{(4,3,1^5)}_{(4,1^2),(1^2)} = a^{(7,2^2,1)}_{(4,1^2),(2)}$. We have $\nu = (3, 1^3) = (4, 1^2)'$, so $n = 6$. Furthermore, $\lambda = (4, 3, 1^5)$ and so $\theta = (7, 2^2, 1)$ with $\hat{\theta} = (6, 1^2)$. One sees that $\ell(\theta) = 4 = 6 - 2$.*

- *Set $N = 0$.*

- *The partition ν is not contained within $\hat{\theta}$, so we can ignore the first step.*

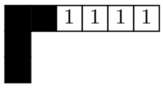
- *The partitions obtainable from ν by removing a box are $(2,1^3)$ and $(3,1^2)$. Only $(3,1^2)$ is contained within $\hat{\theta}$, and the only eligible filling is 111 of type (3):*



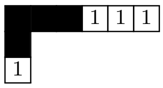
Thus $N = -1$.

- *The partitions obtainable from ν by removing two boxes are $(1^4), (2,1^2)$ and $(3,1)$. The partition (1^4) is not contained within $\hat{\theta}$ so we may ignore it.*

  - *The partition $(2,1^2)$ has removed boxes in different rows and columns, so all of the types $(2,2), (3,1)$ and (4) are allowed. The only allowed filling is 1111.*



  - *The partition $(3,1)$ has both removed boxes in the same column, so we may use types $(2,2)$ and (4). The only allowed filling is 1111.*



After this step we add 2 to N to conclude that the coefficient $a^{(4,3,1^5)}_{(4,1^2),(1^2)}$ is 1. Again, one may verify by computer algebra as in SAGE Computation A.18.

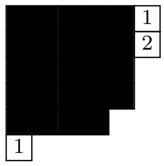
3.7 A Stability Result for Rectangles

We will use Algorithm 3.62 to prove that certain coefficients in the plethysm $s_{(b^a)} \circ s_{(1^2)}$ stabilise as the rectangle (b^a) gets large. We begin by calculating some coefficients for rectangles of different sizes. In particular, we will consider constituents λ that in some sense have the same shape "compared to" the rectangle each time - we will make this precise later.

Example 3.65. *We use Algorithm 3.62 to calculate* $a^{(23,6,5^3,4,2)}_{(5^5),(1^2)} = a^{(7^2,6^2,5,2,1^{17})}_{(5^5),(2)}$. *We label* $\lambda = (23, 6, 5^3, 4, 2)$ *so that* $\theta = (7^2, 6^2, 5, 2, 1^{17})$ *and* $\hat{\theta} = (6^2, 5^2, 4, 1)$ *with* $\nu' = \nu = (5^5)$. *Here* $n = 25$ *and* $\ell(\theta) = 23 = 25 - 2$.

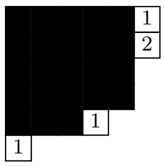
- *The partition* ν *is not contained within* $\hat{\theta}$, *so we may ignore the first step and leave* $N = 0$.

- *The only partition obtainable from* ν *by removing a single box is* $(5^4, 4)$, *and* $\hat{\theta}/(5^4, 4)$ *must be filled with Littlewood-Richardson words of type* (3) *or* $(2, 1)$. *The only valid filling is with the word* 121 *as pictured below, giving us* $N = -1$.

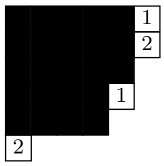


- *The partitions obtainable by removing two boxes from* ν *are* $(5^4, 3)$ *and* $(5^3, 4^2)$.

  – *The partition* $(5^4, 3)$ *has two boxes removed in the same row. Therefore, we may only use type* $(3, 1)$. *The only allowed filling is* 1211 *pictured below.*

  – *The partition* $(5^3, 4^2)$ *has two boxes removed in the same column. Therefore, we may use types* (4) *and* $(2, 2)$. *The only allowed filling is* 1212 *pictured below.*

  We have found two new valid fillings, and so now $N = 1$.

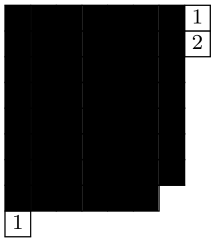
- *We conclude that* $a^{(7^2,6^2,5,2,1^{17})}_{(5^5),(2)} = 1$.

Example 3.66. *We use Algorithm 3.62 to calculate* $a^{(54,9,8^5,7,2)}_{(8^7),(1^2)} = a^{(9^2,8^5,7,2,1^{45})}_{(8^7),(2)}$. *We have* $\lambda = (54, 9, 8^5, 7, 2)$, *so we must label* $\theta = (9^2, 8^5, 7, 2, 1^{45})$, $\hat{\theta} = (8^2, 7^5, 6, 1)$

and $\nu' = (8^7)$ *so* $\nu = (7^8)$. *Note that* $n = 56$ *and* $\ell(\theta) = 54 = 56 - 2$ *so we may apply the algorithm.*

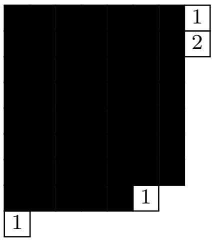
- *The partition* ν *is not contained within* $\hat{\theta}$*, so we may ignore the first step and leave* $N = 0$.

- *The only partition obtainable from* ν *by removing a single box is* $(7^7, 6)$*, and* $\hat{\theta}/(7^7, 6)$ *must be filled with Littlewood-Richardson words of type* (3) *or* $(2, 1)$. *The only valid filling is with the word* 121 *as pictured below, giving us* $N = -1$.

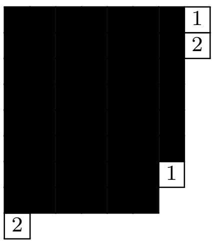


- *The partitions obtainable by removing two boxes from* ν *are* $(7^7, 5)$ *and* $(7^6, 6^2)$.

  - *The partition* $(7^7, 5)$ *has two boxes removed in the same row. Therefore, we may only use type* $(3, 1)$. *The only allowed filling is* 1211 *pictured below.*



  - *The partition* $(7^6, 6^2)$ *has two boxes removed in the same column. Therefore, we may use types* (4) *and* $(2, 2)$. *The only allowed filling is* 1212 *pictured below.*



  We have found two new valid fillings, and so now $N = 1$.

- *We conclude that* $a^{(9^2, 8^5, 7, 2, 1^{45})}_{(8^7),(2)} = 1$.

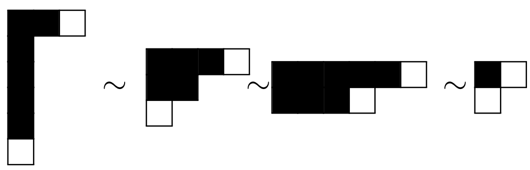
Note that the previous two examples calculated different plethysm coefficients, but were otherwise almost exactly identical. We saw that $a^{(7^2,6^2,5,2,1^{17})}_{(5^5),(2)} = a^{(9^2,8^5,7,2,1^{45})}_{(8^7),(2)} = 1$ from these two examples. In fact, provided a and b are big enough, the same calculation can be used to show that when we take $\theta \vdash 2ab$ such that $\hat{\theta} = ((a+1)^2, a^{b-3}, a-1, 1)$, we always have $a^{\theta}_{(b^a),(2)} = 1$. It is clear that the calculation depends only on filling a small number of skew shapes with Littlewood-Richardson words, and in fact that the size of the gaps between connected components of those skew shapes does not matter, only the connected components themselves.

We will state a result using this method soon, but we must first formalise this idea of considering skew shapes without worrying about the space between connected components via an equivalence relation.

Definition 3.67. Suppose that $\mu_1 \subset \lambda_1$, $\mu_2 \subset \lambda_2$ are pairs of partitions. Write $\lambda_1/\mu_1 \sim \lambda_2/\mu_2$ if the ordered list of shapes of their connected components is the same. For this purpose, boxes are deemed to be connected if they share an edge, but not if they share a corner. Write $[\lambda/\mu]_\sim$ for the equivalence class of λ/μ under this relation.

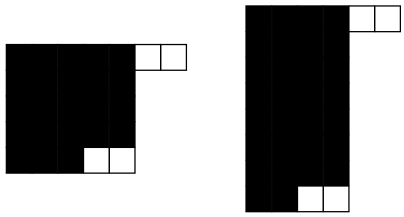
Remark 3.68. The equivalence class $[\lambda/\mu]_\sim$ can be viewed as the skew shape λ/μ with the size of the gaps between connected components ignored.

Example 3.69. *The skew shapes* $(3,1^5)/(2,1^4), (4,2,1)/(3,2)$, $(6,4)/(5,3)$ *and* $(21)/(1)$ *are all equivalent under* $\sim$. *One can mostly readily see this from the picture below.*



The equivalence class $[(2,1)/(1)]_\sim$ *can be viewed as the list* $((1),(1))$ *of connected components present.*

Looking at some larger skew shapes, $(7, 5^4)/(5^4, 3) \sim (6, 4^7)/(4^7, 2)$.



The equivalence class $[(7, 5^4)/(5^4, 3)]_\sim$ *can be viewed as the list* $((2), (2))$ *of connected components. Note that we could take* μ *to be an arbitrarily large rectangle, take two boxes from its bottom row and add two boxes to its top row to form* λ*, and still have* $\lambda/\mu \sim (7, 5^4)/(5^4, 3)$.

Considering our previous examples of different rectangle plethysm coefficients, one may observe further skew shapes that are equivalent under our relation.

Lemma 3.70. *Suppose that* $\nu = (a^b)$ *where* $a, b \geq 5$ *and write* $n = |\nu| = ab$*. Suppose that* $\theta \vdash 2n$ *with* $\ell(\theta) = n - 2$*, while* η *is a partition of* 50 *with* $\ell(\eta) = 23$ *such that* $\hat{\theta}/(a^b) \sim \hat{\eta}/(5^5)$*. Then* $a^{\theta}_{(b^a),(2)} = a^{\eta}_{(5^5),(2)}$.

Proof. Calculate the value of the coefficient $a^{\theta}_{\nu',(2)}$ using Algorithm 1. One may observe that the result of the algorithm depends only on the fillings of the skew shapes $\hat{\theta}/\zeta$ with Littlewood-Richardson words, where $\zeta \in \{\nu\} \cup S^c_{-1}(\nu) \cup S^c_{-2}(\nu) \cup S^r_{-2}(\nu)$. These skew shapes all depend on the skew shape $\hat{\theta}/(a^b)$, and in fact just on the equivalence class $[\hat{\theta}/(a^b)]_\sim$. The result follows by taking $\hat{\eta}/(5^5)$ as a representative of this equivalence class. $\square$

Remark 3.71. Although this proof does provide a method to calculate, it is mostly interesting because it shows a *stability property.* When ν is a rectangle and $\ell(\theta) = |\nu| - 2$, effectively the exact size of ν does not matter in calculating the associated plethysm coefficients $a^{\theta}_{\nu',(2)}$, as long as ν is big enough. The plethysm coefficients are eventually all the same.

Remark 3.72. This result is not restricted to rectangles, one could in theory extend it to other predefined shapes such as hooks or fat hooks, although the minimal shape from which the stability result applies would have to be deduced in each case.

Chapter 4

Preliminaries on the Partition Algebra

4.1 The Partition Algebra

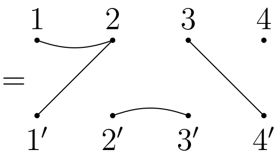
The following is standard material introducing the partition algebra. This exposition is formed from a combination of the papers of Martin [37], Halverson and Ram [23], and Doran and Wales [11].

Let k be any field. Let r be a positive integer and $\delta \in k$. The partition algebra $P_r(\delta)$ is the k-algebra whose basis consists of all set partitions of the set $\{1, \cdots, r, 1', \cdots, r'\}$ under a multiplication we will define momentarily.

Notation. We write a set partition by writing the underlying set in set brackets with the different parts of the set partition separated by a | symbol. For example, $\{1, 2, 1'|3, 4'|2', 3'|4'\}$ has four parts: $\{1, 2, 1'\}, \{3, 4'\}, \{2', 3'\}$ and $\{4'\}$.

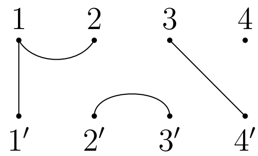
Multiplication is best understood in terms of partition diagrams consisting of two rows of dots $\{1, \cdots, r\}$ and $\{1', \cdots, r'\}$, where two dots are joined by some series of edges if and only if they appear in the same part of the set partition. In other words, the parts of the set partition are the connected components of the partition diagram. Since we are only concerned with the underlying set partition, there are many equivalent diagrams for a given set partition.

Example 4.1. *The partition* $\{1, 2, 1'|3, 4'|2', 3'|4\} \in P_4(\delta)$ *is represented by the diagram*

$d =$ 

,

but the diagram

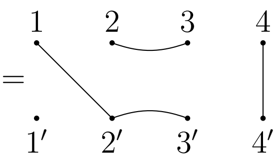


is equally valid.

We will usually use the diagram where each part has at most one line joining a primed and an unprimed dot, and this line occurs as far left as possible. The second diagram from the example is in this format.

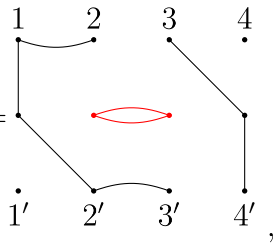
Suppose d and e are diagrams in $P_r(\delta)$. The diagram $d * e$ is obtained by placing d on top of e, identifying the bottom row of d with the top row of e, following through all connections and deleting any parts that do not intersect the new top or bottom row. This is called the *concatenation* of d and e. The integer $\gamma(d, e)$ is defined to be the number of connected components deleted in this process. The product of d and e in $P_r(\delta)$ is given by $de = \delta^{\gamma(d,e)} d * e$. We have defined multiplication of diagrams, and of course we extend to a full multiplication linearly.

Example 4.2. *Take* d *to be the diagram in Example 4.1 and write*

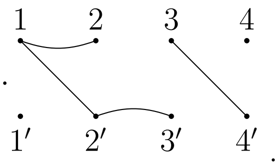
$e =$ 

.

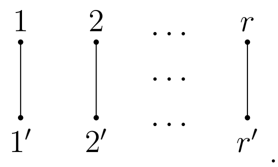
Concatenating the diagrams, we obtain

$d * e =$ 

so $\gamma(d,e) = 1$ *from deleting the single part containing the* 2*nd and* 3*rd dots of the middle row, and*

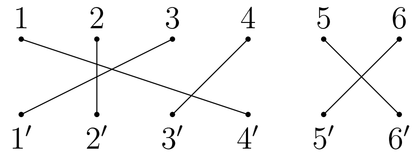
$de = \delta^1 \cdot$ 

The identity element of $P_r(\delta)$ is given by $\{1, 1' | \cdots | r, r'\}$, the diagram where each dot i is joined only to its primed counterpart i':

$\mathrm{Id}_{P_r(\delta)} =$ 

In general, diagrams in $P_r(\delta)$ are not invertible elements of the algebra. For example, the diagram d consisting entirely of singletons is not invertible: Suppose we multiply by some diagram e, then de cannot have connections between primed and unprimed dots, and therefore cannot be the identity diagram. We will expand upon this point further when we discuss propagating number.

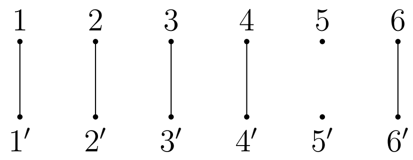
We now present a generating set for $P_r(\delta)$. There is a full copy of the symmetric group S_r inside $P_r(\delta)$, where $\tau \in S_r$ is represented by the diagram where i' is joined only to $\tau(i)$. Below we display the element $(134)(56) \in kS_r$ as an element of $P_r(\delta)$.



We introduce diagrams $\{A^i | i = 1, \cdots, r\}$ and $\{A^{i,j} | i, j = 1, \cdots, r \text{ with } i < j\}$ in order to complete a generating set.

Define A^i to be the diagram $\{1, 1' | \cdots | (i-1), (i-1)' | i | (i+1), (i+1)' | \cdots | r, r' | i'\}$ obtained from the identity diagram by removing the connection between i and i'.

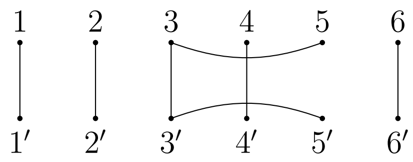
Example 4.3. *In $P_6(\delta)$, the diagram A^5 is*



.

Given a diagram d, the diagram $A^i d$ is obtained from d by isolating the point i. Similarly, dA^i is obtained from d by isolating the point i'.

Define $A^{i,j}$ to be the diagram $\{1, 1' | \cdots | i, j, i', j' | \cdots | (j-1), (j-1)' | (j+1), (j+1)' | \cdots | r, r'\}$ obtained from the identity by merging the parts $\{i, i'\}$ and $\{j, j'\}$. For example $A^{3,5}$ in $P_6(\delta)$ is given by



.

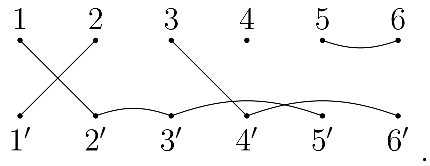
Given a diagram d, the diagram $A^{ij} d$ is obtained from d by merging the parts containing i and j. Similarly, dA^{ij} is obtained from d by merging the parts containing i' and j'.

Proposition 4.4. *[23, Cor of Thm 11.11] The partition algebra $P_r(\delta)$ is generated by a generating set of S_r, together with the sets $\{A^i | i = 1, \cdots, r\}$ and $\{A^{ij} | i, j = 1, \cdots, r$ with $i < j\}$.*

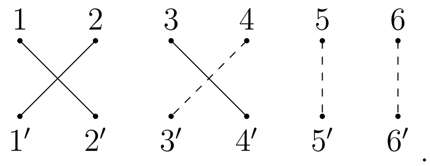
Rather than providing a proof, we briefly explain how to write a diagram as a product of generators. Let $d \in P_r(\delta)$ be a partition diagram. Without loss of generality we may assume that each part of d has at most one line connecting top to bottom, which is placed as far left as possible. Now, pick any element $\tau \in S_r$ which makes the same links from top to bottom as in d, disregarding any other strings. Note that there are usually several choices for τ. We may remove all strings of τ not in d by multiplying with elements A^i on the left. We may finally mimic connections solely on the top (resp. bottom) of d by multiplying on the left (resp. right) by some collection of $A^{i,j}$.

Example 4.5. *Take*

$d =$

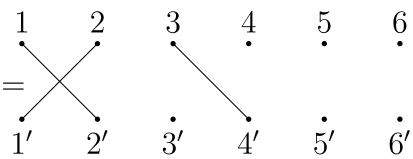


The strings connecting top and bottom have the same pattern as $\tau = (12)(34)$:



We want to isolate dots $4, 5$ and 6 at the top, so form

$A^4A^5A^6\tau =$

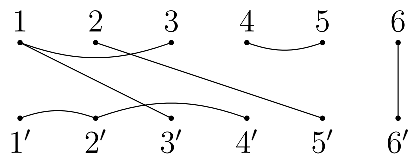


Finally, we want to connect 5 to 6, $2'$ to $3'$ to $5'$, and $4'$ to $6'$. Thus we achieve our result, $d = A^{5,6}A^4A^5A^6\tau A^{2,3}A^{3,5}A^{4,6}$ as required.

Remark 4.6. [23, Thm 11.11] We have chosen a large generating set, in fact it is sufficient to only use the set $\{s_1, \cdots, s_{r-1}, A^1, A^{12}\}$ since $s_1, \cdots, s_{r-1}$ generate the symmetric group and all other elements of our generating set can be obtained by conjugation with symmetric group elements.

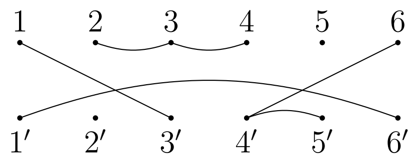
Partition diagrams can be organised by their *propagating number.* Let $d \in P_r(\delta)$ be a partition diagram. Refer to those parts of d which contain both primed and unprimed elements as *propagating parts*, and all other parts as non-propagating. Define the *propagating number* $p(d)$ to be the number of propagating parts of d.

Example 4.7. *An example of a diagram of propagating number* 3 *is*

$d =$ 

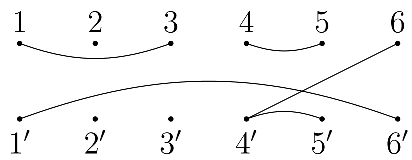
,

whereas a diagram of propagating number 2 *is*

$e =$ 

.

Multiplying these two together, we obtain

$de =$ 

,

which is of propagating number 1.

Example 4.8. *Every diagram in $S_r \subset P_r(\delta)$, including the identity diagram, is of propagating number r. The elements A^i and A^{ij} all have propagating number $r - 1$.*

One may note that in the above example, $p(de)$ did not exceed $p(d)$ or $p(e)$. This is an example of a general phenomenon.

Proposition 4.9. *[37, Prop. 2] Suppose that $d, e \in P_r(\delta)$ are partition diagrams, then*

$$p(de) \leq \min(p(d), p(e))$$

We provide a rough rationalisation for the proposition rather than a full proof. Observe the action of the generators on propagating number. Consider multiplying a diagram d on the left by a generator. The elements of S_r leave the propagating number unchanged as dots are only rearranged. The element A^i might make a propagating part non-propagating, and the element A^{ij} might merge two propagating parts. All other situations leave the propagating number unchanged. In particular, no generator can separate a propagating part into two or create new propagating parts. Multiplication on the right is completely analogous.

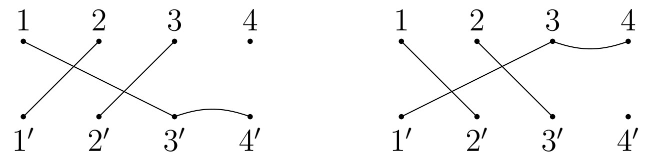
In particular, letting $P_r^i = \langle d : d \text{ is a diagram with } p(d) \leq i \rangle$, we obtain a filtration by ideals

$$0 \subset P_r^0 \subset \cdots \subset P_r^i \subset \cdots \subset P_r^r = P_r(\delta).$$

The partition algebra $P_r(\delta)$ has an additional piece of structure in the form of an anti-involution.

Definition 4.10. An *anti-involution* of a k-algebra A is a map $\sigma : A \to A$ such that $\sigma^2 = \mathrm{Id}_A$ and $\sigma(ab) = \sigma(b)\sigma(a)$ for any $a, b \in A$. We abbreviate anti-involutions simply to *involutions*.

The partition algebra has an involution σ, given by interchanging each unprimed element with its primed counterpart in the partition. Diagrammatically, this corresponds to performing the reflection which exchanges the two rows of the diagram. For example, the following two diagrams are images of each other under this involution σ.



Every non-permutation element of our chosen large generating set is preserved by σ. The involution preserves propagating number, that is $p(d) = p(\sigma(d))$ for every diagram $d \in P_r(\delta)$. When $\tau \in S_r$, $\sigma(\tau) = \tau^{-1}$.

We will soon investigate the representation theory of the partition algebra through the lens of cellularity.

4.2 Cellularity

We recall some background on cellular algebras, as we will examine algebras through this framework in this thesis. These were defined by Graham and Lehrer in [18], with extensive study and progress by König and Xi across many papers including [31, 32, 50].

Definition 4.11. [18, Def 1.1] A cellular algebra A is a k-algebra equipped with an involution σ together with *cell data* consisting of

- a poset Λ,
- a finite set $M(\lambda)$ for every $\lambda \in \Lambda$,
- a map $C : \bigsqcup_{\lambda\in\Lambda} M(\lambda) \times M(\lambda) \to A$.

These must satisfy the following axioms:

(C1) The map C is injective with image a basis of A. We write $C(T, B) = C^{\lambda}_{T,B}$ when $T, B \in M(\lambda)$.

(C2) For every $\lambda \in \Lambda$ and $T, B \in M(\lambda)$ we have $\sigma(C^{\lambda}_{T,B}) = C^{\lambda}_{B,T}$.

(C3) For any $a \in A, \lambda \in \Lambda$ and $T, B \in M(\lambda)$ we have that

$$aC^{\lambda}_{T,B} \equiv \sum_{T' \in M(\lambda)} r_a(T', T) C^{\lambda}_{T',B} \pmod{I_{<\lambda}}$$

for some scalars $r_a(T', T) \in k$, where $I_{<\lambda}$ is the submodule of A generated by all $C^{\mu}_{U,C}$ with $\mu < \lambda$. Here $r_a(T', T)$ does *not* depend on B.

Notation. We refer to the basis $\{C_{T,B} : T, B \in M(\lambda) \text{ for some } \lambda \in \Lambda\}$ as a *cellular basis* of A.

Remark 4.12. Immediately one may apply the involution σ to (C3) to see that multiplication of a basis element on the right by some $a \in A$ is equally strictly governed by the rule (C3R).

(C3R) For any $a \in A, \lambda \in \Lambda$ and $T, B \in M(\lambda)$ we have that

$$C^{\lambda}_{T,B} a \equiv \sum_{B' \in M(\lambda)} r_{\sigma(a)}(B', B) C^{\lambda}_{T,B'} \pmod{I_{<\lambda}}.$$

Together, (C3) and (C3R) imply that the submodule $I_{<\lambda}$ is in fact a two-sided ideal of A.

Example 4.13. *[32, Prop 3.4] The algebra $M_n(k)$ of $n \times n$ matrices is cellular. In this case, Λ is the trivial poset $\{1\}$, with $M(1)$ consisting simply of the numbers $\{1, \cdots, n\}$. The cellular basis elements are $C^1_{i,j} = E_{ij}$, the elementary matrices whose entries are 0 apart from a single 1 in the (i,j)th entry. The involution σ must then be matrix transposition. The axiom (C3) is easily verified when a is a basis element since $E_{ij}E_{kl} = \delta_{jk}E_{il}$, and therefore follows in general by linearity.*

Lemma 4.14. *[18, Ex 1.2] [40, (3.20)] The group algebra kS_r is a cellular algebra. The involution σ is given by $\sigma(g) = g^{-1}$ for $g \in S_r$, the poset Λ is the set of all partitions of r, and $M(\lambda)$ is the set of all standard Young tableaux of shape λ.*

Remark 4.15. One may specify a cellular basis for kS_r, for example the Murphy basis. However, we will not describe it here as the definition is quite involved and we do not need to utilise a precise cellular basis for kS_r in this thesis. A comprehensive exposition can be found in Mathas' book [40].

A cellular algebra has a special set of *cell modules* from which we can obtain all simple modules.

Definition 4.16. [18, Def 2.1] For each $\lambda \in \Lambda$ recall the left cell module Δ^λ to be the vector space with basis $\{C_T | T \in M(\lambda)\}$ of formal symbols and action

$$aC_T = \sum_{T' \in M(\lambda)} r_a(T', T) C_{T'},$$

where $r_a(T', T)$ is as in Axiom (C3).

The module action is defined by the multiplication in the cellular algebra A. All we need to do to calculate aC_T when $T \in M(\lambda)$ is choose any $B \in M(\lambda)$, do the multiplication $aC^\lambda_{T,B}$, reduce modulo $I_{<\lambda}$ and then ignore the B from the subscript of the answer. For example, if $aC^\lambda_{T,B} = pC^\lambda_{T,B} + qC^\lambda_{U,B} + rC^\lambda_{V,B} + C^\mu_{W,C}$ where $\mu < \lambda$, then $aC_T = pC_T + qC_U + rC_V$. The involution σ allows us to define a right cell module in a similar fashion, although we will not use it in this thesis.

Example 4.17. *In the case of the algebra $M_n(k)$, there is a single cell module Δ^1 with basis $\{E_1, \cdots, E_n\}$ and action $E_{ij}E_k = \delta_{jk}E_i$. This is of course isomorphic to the natural module k^n of column vectors under matrix multiplication.*

We recall from [18, Def 2.3] a bilinear form on Δ^λ. Consider multiplying two cellular basis elements in A. By considering (C3) and (C3R) we notice that

$$C^\lambda_{T',B} C^\lambda_{T,B'} \equiv r(B, T) C^\lambda_{T',B'} \pmod{I_{<\lambda}},$$

where $r(B, T)$ does not depend on B' or T'. Now, writing $\langle C_B, C_T \rangle_\lambda = r(B, T)$ and extending linearly, we obtain a bilinear form on Δ^λ.

This bilinear form reduces the search for simple modules to linear algebra in the following way.

Definition 4.18. [18, §3] For each $\lambda \in \Lambda$, define the submodule $\operatorname{rad}(\lambda)$ of Δ^λ by

$$\operatorname{rad}(\lambda) = \{x \in \Delta^\lambda : \langle x, y \rangle_\lambda = 0 \text{ for each } y \in \Delta^\lambda\}$$

and write L^λ for the quotient $\Delta^\lambda / \operatorname{rad}(\lambda)$. Write Λ_0 for the subset of Λ consisting of those λ for which the bilinear form $\langle \bullet, \bullet \rangle_\lambda$ is not identically 0.

Theorem 4.19. *[18, Section 3] Suppose A is a cellular algebra with cellular poset Λ and $\lambda \in \Lambda$. Then the module L^λ is simple when the bilinear form $\langle \bullet, \bullet \rangle_\lambda$ is not identically* 0. *Furthermore, the set $\{L^\lambda : \lambda \in \Lambda_0\}$ is a complete set of pairwise nonisomorphic simple modules for A.*

Corollary 4.20. *Suppose that A is a semisimple cellular algebra. Then the simple modules of A are exactly the cell modules of A.*

Lemma 4.21. *[40, Prop 1.22] With an appropriate choice of cellular basis, the cell module of the group algebra kS_r for the partition $\lambda \vdash r$ is the usual Specht module S^λ with the familiar action. Suppose that k has characteristic p. Then Λ_0 is the set of p-regular partitions and the modules L^λ are those described in §1.2.*

One may prove cellularity of an algebra directly from the definition of Graham and Lehrer, however proofs are often made significantly shorter and more intuitive by the use of *iterated inflations*. The method of iterated inflations was first described by König and Xi in [31]. Xi presented a lemma characterising them in [50], however this was shown to have an error in [22] and we instead state the following lemma of Green and Paget.

Lemma 4.22. *[22, Thm 1] Suppose that A is a k-algebra equipped with the anti-involution σ. Suppose that we have a finite poset I, and for each $i \in I$ a k-vector space V_i and cellular algebra B_i with involution σ_i such that*

$$A \cong \bigoplus_{i \in I} V_i \otimes B_i \otimes V_i$$

as vector spaces. Suppose that for each $i \in I$ we have (not necessarily cellular) bases $\mathcal{V}_i$ and $\mathcal{B}_i$ for V_i and B_i respectively, such that the following conditions hold.

- *For all $i \in I$, $t, b \in \mathcal{V}_i$ and $\tau \in \mathcal{B}_i$,*

$$\sigma(t \otimes \tau \otimes b) = b \otimes \sigma_i(\tau) \otimes t.$$

- *Let $\mathcal{A}$ be the basis of A consisting of all $(t \otimes \tau \otimes b)$ where $\tau \in \mathcal{B}_i, t, b \in \mathcal{V}_i$ for some $i \in I$. Then for any $i \in I$ there exist maps $\phi_i : \mathcal{A} \times \mathcal{V}_i \to V_i$ and*

$\theta_i : \mathcal{A} \times \mathcal{V}_i \to B_i$, such that for any $a \in \mathcal{A}, u, v \in \mathcal{V}_i, b \in \mathcal{B}_i$,

$$a \cdot (t \otimes \tau \otimes b) \equiv \phi_i(a,t) \otimes \theta_i(a,t)\tau \otimes b,$$

modulo the ideal $J(<i) = \bigoplus_{l<i} V_l \otimes B_l \otimes V_l$.

Then A is a cellular algebra with respect to the involution σ.

Suppose that the cellular data of B_i are (Λ_i, M_i, C_i). Then the cellular data of A are given by (Λ, M, C) where $\Lambda = \{(i,\lambda) : i \in I, \lambda \in \Lambda_i\}$ under the lexicographic order, $M(i,\lambda) = \mathcal{V}_i \times M_i(\lambda)$, and $C^{(i,\lambda)}_{(x,X),(y,Y)} = x \otimes C^{\lambda}_{X,Y} \otimes y$.

This iterated inflation structure leads to a precise description of the action of A on cell modules.

Lemma 4.23. *[22, Prop 2] Suppose A is an iterated inflation as in the setup above. Let $(i,\lambda) \in \Lambda$ and let Δ^λ be the cell module for B_i corresponding to the cell index λ. The cell module $\Delta^{(i,\lambda)}$ of A is obtained by equipping $V_i \otimes \Delta^\lambda$ with action*

$$a(x \otimes z) = \phi_i(a,x) \otimes \theta_i(a,x)z.$$

Notation. We will refer to the module $V_i \otimes \Delta^\lambda$ with the action described above as $V_i \oslash \Delta^\lambda$.

In his thesis [20] and resulting paper [21], Green studied the cellularity of the group algebra $k(S_m \wr S_n)$ of two symmetric groups via the method of iterated inflations. The result was proved earlier by Geetha and Goodman [17] without using the theory of iterated inflations.

Theorem 4.24. *[20, Thm 5.5.1] [21] [17] The group algebra $k(S_m \wr S_n)$ is a cellular algebra with involution given by inversion of group elements. The associated poset has underlying set Λ^n_r of r-multipartitions of n, where r is the number of partitions of m. The cell module associated to $\underline{\alpha} \in \Lambda^n_r$ is the usual Specht module $S^{\underline{\alpha}}$ defined in §2.3.*

Finally, we will need to consider the cellularity of the tensor product of two or more cellular algebras. We present exposition by Geetha and Goodman into one condensed lemma. We will not write down all of the details of the cell data, though those interested may find them in the reference.

Lemma 4.25. *[17, §3.2] Let $A_1, \cdots, A_s$ be cellular algebras. Suppose that the cellular poset of A_i is Γ_i, and that the involution on A_i is σ_i. Given $\gamma_i \in \Gamma_i$, write Δ^{γ_i} for the cell module of A_i associated to γ_i.*

Then $A_1 \otimes \cdots \otimes A_s$ is a cellular algebra with cellular poset $\Gamma = \Gamma_1 \times \cdots \times \Gamma_s$ under the product order, with involution σ given by $\sigma(a_1 \otimes \cdots \otimes a_s) = \sigma_1(a_1) \otimes \cdots \otimes \sigma_s(a_s)$. Furthermore, the cell module associated to $\gamma = (\gamma_1, \cdots, \gamma_s) \in \Gamma$ is the outer tensor product

$$\Delta^{\gamma_1} \boxtimes \cdots \boxtimes \Delta^{\gamma_s}$$

of cell modules for the tensor factors $A_1, \cdots, A_s$.

4.3 The Partition Algebra is Cellular

We take k to be an arbitrary field, and consider the partition algebra $P_r(\delta)$ over k. Xi's paper [50] developed theory to prove that $P_r(\delta)$ is cellular, however the lemma characterising iterated inflations in this paper was slightly incorrect, and thus we provide a proof of this well known result using Lemma 4.22 instead.

Definition 4.26. Define a *partition half-diagram of propagating number i on r dots* to be a set partition of $\{1, \cdots, r\}$ into at least i parts, with exactly i of these parts being distinguished as *propagating parts.* Let $\mathcal{W}_i$ be the set of all partition half-diagrams of propagating number i on r dots, and write W_i for the k-vector space with basis $\mathcal{W}_i$. If $t \in \mathcal{W}_i$, then write $p(t) = i$.

Notation. We may write a half diagram $t \in \mathcal{W}_i$ by writing its parts in a pair of braces, listing first propagating parts and then non-propagating parts, where the two lists are separated by a double pipe $||$.

Example 4.27. *Suppose t is the partition half-diagram of propagating number 3 on 6 dots whose propagating parts are $\{1,4\},\{3,5\},\{6\}$ and whose only non-propagating part is $\{2\}$. Then t may be represented by the notation $\{1,4|3,5|6||2\}$.*

For the remainder of this section we will refer to partition half-diagrams simply as half-diagrams and suppress the propagating index and number of dots. Half-diagrams have an obvious diagrammatic description as halves of partition diagrams - connect dots in the same part by lines, and leave small stubs on propagating parts.

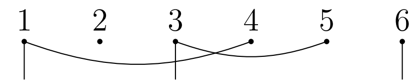
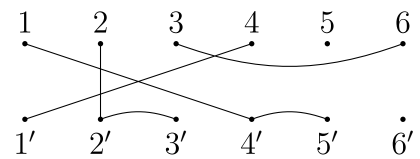


FIGURE 4.1: The half-diagram from Example 4.27.

Remark 4.28. Soon we will need to refer to, for example, the ith propagating part of some $t \in \mathcal{W}_i$. Therefore, we must order the propagating parts in any given half diagram. We order the propagating parts by their least element. Diagrammatically, speaking, this means a propagating part p is to the left of a propagating part q if the leftmost dot in p is to the left of the leftmost dot in q. In our running example $\{1,4|3,5|6||2\}$, the first propagating part is $\{1,4\}$, the second is $\{3,5\}$, and the third is $\{6\}$.

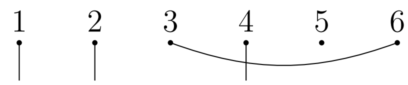
Given a partition diagram $d \in P_r(\delta)$, write $\mathrm{top}(d)$ for the half-diagram which is the restriction of d to $1,\cdots r$, with the propagating parts being those parts which are restrictions of propagating parts of d. We say that $\mathrm{top}(d)$ is the top of d. Similarly, we may define $\mathrm{bot}(d)$, the bottom of the diagram d, ensuring it is a half-diagram on $\{1,\cdots r\}$ by unpriming all elements.

Example 4.29. *Consider the diagram*

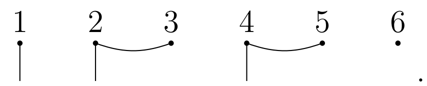


.

The top of the diagram is

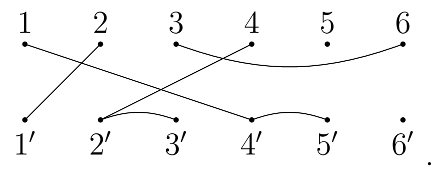


and the bottom is

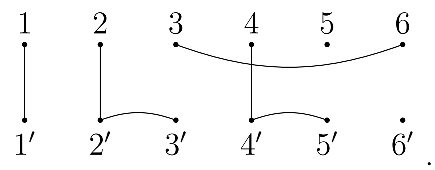


It is clear that several distinct diagrams may have the same top and bottom, thus we must also specify an additional piece of information describing how the top and bottom are connected in order to specify a diagram. Supplied with $t, b \in \mathcal{W}_i$, we connect them by specifying a permutation $\tau \in S_i$ and joining the jth propagating part of b to the $\tau(j)$th propagating part of t. If a diagram d of propagating number i has top and bottom connected by such a permutation τ, we write $\mathrm{con}(d) = \tau$,

Example 4.30. *Referring to the diagram in Example 4.29, one may see that* $\mathrm{con}(d) = (1, 3) \in S_3$*. If we take the same top and bottom, but replace the connecting permutation by* $(1, 2, 3) \in S_3$*, we would instead have the diagram*



If we chose the identity permutation of S_3 *in place of* (13)*, we would obtain the diagram*



It is now clear that a partition diagram $d \in P_r(\delta)$ of propagating index i can be uniquely specified by the triple $(\mathrm{top}(d), \mathrm{con}(d), \mathrm{bot}(d))$ where $\mathrm{top}(d), \mathrm{bot}(d) \in \mathcal{W}_i$

and $\mathrm{con}(d) \in S_i$, and thus

$$P_r(\delta) \cong \bigoplus_{i=1}^{r} W_i \otimes kS_i \otimes W_i$$

as vector spaces. When d is described by the triple (t, τ, b), we abuse notation slightly and write $d = t \otimes \tau \otimes b$.

We will now discuss how this structure interacts with multiplication. Let $a \in P_r(\delta)$ be any diagram and let $d = (t \otimes \tau \otimes b)$ be a diagram of propagating number i. Write t' for the half diagram on $\{1', \cdots, r'\}$ obtained from t by priming all of its parts. Write partition(a) and partition(t') for the underlying set partitions of a and t', respectively. Write $a \bigtriangledown t$ for the set partition of $\{1, \cdots, r, 1', \cdots, r'\}$ where two elements x, y are in the same part if there is a tuple $(x_0 = x, \cdots, x_s = y)$ of dots such that x_j and x_{j+1} lie in the same part of either partition(a) or partition(t') for each $j = 0, \cdots, s-1$.

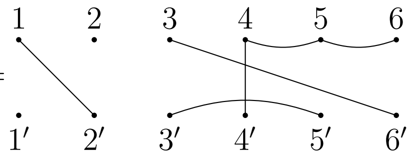
If in $a \bigtriangledown t$ each propagating part of t' meets exactly one propagating part of a, define a set partition $a \vee t = a \bigtriangledown t|_{\{1,\cdots,r\}}$. Turn this into a half diagram by specifying that the propagating parts of $a \vee t$ are those which are restrictions of parts of $a \bigtriangledown t$ that meet a propagating part of both a and t'. Otherwise, define $a \vee t = 0$. Write $\gamma(a, t)$ for the number of parts of $a \bigtriangledown t$ which consist entirely of primed elements.

Diagrammatically speaking, $a \bigtriangledown t$ is the diagram obtained from a by superimposing the half-diagram t over its second row, ignoring propagating parts of t. If $a \vee t$ is not zero, $a \vee t$ is obtained from $a \bigtriangledown t$ by only considering the top row of dots, how they are connected and whether they are propagating. The scalar $\gamma(a, t)$ is simply the number of parts which are confined to the bottom row of $a \bigtriangledown t$.

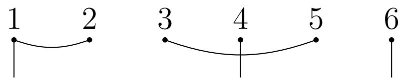
Define

$$\phi_i(a, t) = \delta^{\gamma(a,t)} a \vee t.$$

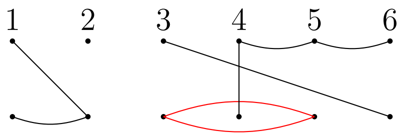
Example 4.31. *Suppose that*

$a =$ 

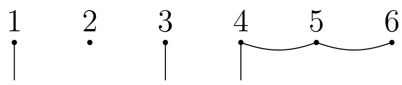
of propagating index 3 *and*

$t =$  ,

also of propagating index 3*. Then*

$a \bigtriangledown t =$ 

and one can observe that $\gamma(a,t) = 1$ *from the highlighted red part* $\{3', 5'\}$ *of* $a \bigtriangledown t$*. Finally,*

$a \vee t =$ 

and $\phi_3(a,t) = \delta(a \vee t)$*.*

We now define a map θ_i which will later be used to formalise multiplication. If $a \vee t = 0$, define $\theta_i(a,t) = 0$. Otherwise, define $\theta_i(a,t) \in S_i$ as follows. Suppose p is the lth propagating part of t'. Then p meets exactly one part of $a \bigtriangledown t$, whose restriction is some propagating part q of $a \vee t$. If q is the mth propagating part of $a \vee t$, then write $\theta_i(a,t)l = m$. Looking at all propagating parts together, this defines $\theta_i(a,t) \in S_i$.

Example 4.32. *Let us continue our running example. The propagating parts of* t'*, in order, are* $\{1', 2'\}, \{4'\}$ *and* $\{6'\}$*. These intersect the parts* $\{1, 1', 2'\}, \{4, 5, 6, 4'\}$ *and* $\{3, 6'\}$ *of* $a \bigtriangledown t$ *respectively. The restriction of* $\{1, 1', 2'\}$ *yields the first propagating part* $\{1\}$ *of* $a \vee t$*. The restriction of* $\{4, 5, 6, 4'\}$ *yields the third propagating* *part* $\{4,5,6\}$ *of* $a \vee t$*. Finally, the restriction of* $\{3,6'\}$ *yields the second propagating part* $\{3\}$ *of* $a \vee t$*. We conclude that* $\theta_3(a,t) = (2,3) \in S_3$*.*

Theorem 4.33. *[50, Thm 4.1] The partition algebra* $P_r(\delta)$ *is an iterated inflation with vector spaces* W_i *and cellular algebras* $B_i = kS_i$ *for* $i = 0, \cdots r$*, where* W_i *is the vector space of half-diagrams with* i *propagating parts as defined above. Hence* $P_r(\delta)$ *is cellular with cellular poset* $\{(i,\lambda)|i \in \{1,\cdots,r\}$ *and* $\lambda \vdash i\}$ *of propagating numbers paired with partitions. The cellular poset is ordered by* $(i,\lambda) < (j,\mu) \iff i < j$ *or* $i = j$ *and* $\lambda \lhd \mu$*.*

Proof. We will prove the theorem by using Lemma 4.22. In the context of the Lemma, let σ be the usual involution on $P_r(\delta)$ induced by reflection of partition diagrams. In the notation of the lemma, we will take $\mathcal{V}_i$ to be the set $\mathcal{W}_i$ of half-diagrams of propagating number i, and thus the vector space V_i in the lemma will be our W_i. This means that $\mathcal{A}$ from the lemma will be the usual diagram basis of $P_r(\delta)$. Let B_i be the group algebra kS_i and let $\mathcal{B}_i$ be the usual (non-cellular) basis of group elements. Let σ_i be the involution on kS_i induced by inversion of group elements.

We have already established the vector space isomorphism

$$P_r(\delta) \cong \bigoplus_{i=0}^{r} W_i \otimes kS_i \otimes W_i,$$

so let us examine how the involution σ acts on $d = t \otimes \tau \otimes b$ with propagating number i. Clearly $\mathrm{top}(\sigma(d)) = b$ and $\mathrm{bot}(\sigma(d)) = t$. The permutation τ still connects the jth propagating part of b to the $\tau(j)$th propagating part of t. Equivalently, the jth propagating part of $\mathrm{bot}(\sigma(d))$ is connected to the $\tau^{-1}(j)$th propagating part of $\mathrm{top}(\sigma(d))$, and therefore $\mathrm{con}(\sigma(d)) = \tau^{-1}$. We have determined that when $d = t \otimes \tau \otimes b$,

$$\sigma(d) = b \otimes \tau^{-1} \otimes t = b \otimes \sigma_i(\tau) \otimes t.$$

Multiplication obeys the rule $a \cdot (t \otimes \tau \otimes b) \equiv \phi_i(a,t) \otimes \theta_i(a,t)\tau \otimes b \ mod\ J(< i)$, with θ_i and ϕ_i as defined above. The case when $a \cdot (t \otimes \tau \otimes b)$ is in $J(< i)$ coincides exactly with the case where $\phi_i(a,t) = \theta_i(a,t) = 0$. Otherwise, one may discern from considering the diagram calculus that $a \vee t \in V_i$ is exactly the top of the diagram $a \cdot (t \otimes \tau \otimes b)$, and $\theta_i(a,t)\tau$ is exactly the map connecting b to $a \vee t$, which is $\mathrm{con}(a \cdot (t \otimes \tau \otimes b))$. □

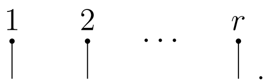
4.3.1 Cell Modules

We now describe the cell modules of the partition algebra. One may note that these coincide with the modules discussed in [11, 23, 37] for comparison and additional perspectives.

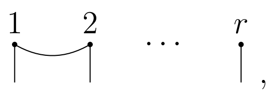
Following Lemma 4.23, we see that the cell modules of $P_r(\delta)$ must be of the form $W_i \oslash S^\lambda$ with $\lambda \vdash i$, equipped with the action relying on the maps ϕ and θ defined above.

Notation. When $\lambda = (n)$ for some $n \leq r$, S^λ is the trivial module of S_n and $W_i \oslash S^{(i)}$ is the same as W_i with the diagrammatic action. We shall refer to such modules W_i as *diagrammatic cell modules.*

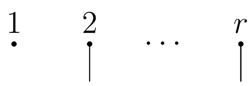
Example 4.34. *Suppose $\lambda \vdash r$ and S^λ has basis labelled by $v_1, \cdots v_s$, with t defined to be the half-diagram*

$t =$ 

Then the cell module $W_r \oslash S^\lambda$ for $P_r(\delta)$ has basis $\{t \oslash v_i | i = 1, \cdots, s\}$. The diagrammatic result of the multiplication $A^{12}t$ is

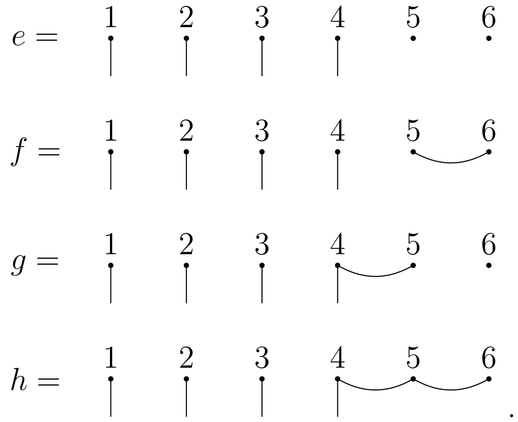


which is 0 in the module since the diagram in question is of propagating index $(r-1) < r$. Similarly, the multiplication $A^1 t$ gives the diagram



which is of propagating index $(r-1) < r$ and thus is equivalent to 0 in the module. In fact, each A^i and A^{ij} in $P_r(\delta)$ annihilates W_r. The diagram t is fixed by elements of S_r, so if $\tau \in S_r$, we have $\tau(t \oslash v_i) = t \oslash \tau v_i$. This tells us that $W_r \oslash S^\lambda$ is isomorphic to $\inf_{S_r}^{P_r(\delta)}(S^\lambda)$ when $\lambda \vdash r$.

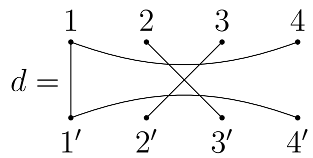
Example 4.35. *The cell module $W_4 \oslash S^{(4)}$ for $P_6(\delta)$ has several types of diagrams - we will exhibit one from each S_6-orbit:*



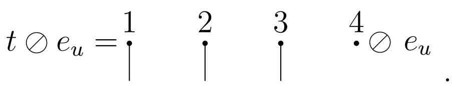
All the diagrams present are annihilated by $A^1, A^2, A^3, A^{12}, A^{13}, A^{23}$. We have $A^{45}e = g$ whereas $A^{56}g = h$. On the other hand, $A^5 g = A^5 A^6 h = e$.

Example 4.36. *In the cell module $W_3 \oslash S^{(2,1)}$ for $P_4(\delta)$, basis elements can be described by a half-diagram with propagating index 3, together with a polytabloid for one of the two standard $(2,1)$-tableaux $u = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $v = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$.*

An example multiplication would be to apply the diagram



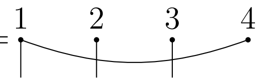
to the basis element



The diagrammatic part dt is easy to compute, however we must also calculate $\theta(d,t)$ in order to find out how to act correctly on the tableau. The diagram d acts on t by joining dots 1 and 4 up and swapping the second and third propagating parts with each other. For this reason, $\theta(d,t) = (2,3)$. Therefore the tableau component of the multiplication must be

$$(2,3)e_u = e_v$$

Therefore, we obtain the result:

$d \cdot (t \oslash e_u) =$  $\oslash e_v$.

4.3.2 Semisimplicity of the Partition Algebra over $\mathbb{C}$

The conditions under which the partition algebra is semisimple over $\mathbb{C}$ have been discussed in various papers over time. In particular, results on semisimplicity appear in [37] and [38]. Martin mentions semisimplicity in his introduction to [37] before giving a proposition that implies the semisimplicity result, which was previously proved in [38]. A weaker result on semisimplicity of $P_r(\delta)$ is proved in [23] in a more explicit fashion.

Theorem 4.37. *[38] [37, Prop 3] [23, Thm 3.27] Suppose that $k = \mathbb{C}$ with $\delta \in \mathbb{C}$. Then the partition algebra $P_r(\delta)$ is semisimple unless $\delta \in \{0, 1, \cdots, 2r-2\}$, in which case $P_r(\delta)$ is not semisimple.*

4.4 Stratifying Systems and Cellular Stratification

Stratifying systems were introduced by Cline, Parshall and Scott in order to study standardly stratified algebras [7]. We follow the paper of Erdmann and Sáenz [14] in order to introduce the definitions and useful results on stratifying systems.

Definition 4.38. A filtration $\mathcal{F}$ of an A-module M is a sequence

$$0 = \mathcal{F}_0 M \subset \mathcal{F}_1 M \subset \cdots \subset \mathcal{F}_s M = M$$

of submodules of M. If Θ is a set of A-modules, $\mathcal{F}$ is said to be a Θ-filtration of M if $(\mathcal{F}_i M/\mathcal{F}_{i-1} M) \in \Theta$ for each $i = 1, \cdots, s$. If M has a Θ-filtration, one may say that M is filtered by Θ.

Definition 4.39. [14, Def 1.1] Let A be an algebra and let $\Theta = \{\Theta(1), \cdots, \Theta(n)\}$ be a set of A-modules, with $Y = \{Y(1), \cdots, Y(n)\}$ a set of indecomposable A-modules. We say that (Θ, Y) is a stratifying system if:

1. $\mathrm{Hom}_A(\Theta(i), \Theta(j)) = 0$ whenever $i > j$.
2. For every i, there is an exact sequence

$$0 \longrightarrow \Theta(i) \longrightarrow Y(i) \longrightarrow Z(i) \longrightarrow 0$$

   where $Z(i)$ is filtered by $\Theta(j)$ with $j < i$.

3. $\mathrm{Ext}^1_A(X, Y) = 0$ where $Y = \bigoplus_{i=1}^n Y(i)$ and X is any module with a Θ-filtration.

The group algebra of the symmetric group has a stratifying system in most characteristics.

Theorem 4.40. *[13] Suppose k is a field of characteristic not 2 or 3. Then kS_r has a stratifying system where the set Θ consists of the Specht modules. Here, the indexing set is the set of partitions of r ordered lexicographically.*

A significant use of stratifying systems is that they guarantee a useful fact about filtration multiplicities as follows.

Theorem 4.41. *[14, Lemma 1.4] Suppose that (Θ, Y) is a stratifying system. Suppose that M has a filtration $\mathcal{F}$ by the modules $\Theta(i)$. Then the multiplicities occurring in any Θ-filtration $\mathcal{F}'$ of M are the same as those which occur in $\mathcal{F}$.*

Remark 4.42. Essentially, the result of the theorem means that Θ-filtration multiplicities for Θ-filtered modules are independent of the filtration. Therefore, the concept of a multiplicity $[M : \Theta(i)]$ is well defined.

Notably this can be used to show the celebrated *Hemmer-Nakano* property of kS_r, which states that modules with Specht filtrations have well defined Specht multiplicities provided that the characteristic of k is not 2 or 3. That is, any two Specht filtrations of a module have the same Specht module factors appearing the same amount of times.

We will show several algebras have stratifying systems. To simplify this process, it is useful to note that it is not necessary to specify the set Y. In fact, it is possible to get a stratifying system by just finding modules Θ with appropriate conditions on them.

Lemma 4.43. *[14, Thm 1.8] Suppose we have A-modules $\Theta(1), \cdots, \Theta(n)$ such that*

1. *Each $\Theta(i)$ is indecomposable.*
2. $\mathrm{Hom}_R(\Theta(j), \Theta(i)) = 0$ *when* $j > i$.
3. $\mathrm{Ext}^1_R(\Theta(j), \Theta(i)) = 0$ *when* $j \geq i$.

Then there exist modules $Y(i)$ such that (Θ, Y) forms a stratifying system.

In [20], Green proves the following results on wreath products, exhibiting a stratifying system for $k(S_m \wr S_n)$. We briefly recall the necessary background and results. Firstly, recall that Specht modules for $S_m \wr S_n$ are labelled by t-multipartions of n, where t is the number of distinct partitions of m. The Specht module $S^{\underline{\alpha}}$ is defined explicitly in §2.3.

In order to talk about stratifying systems, we must first impose an order on these multipartitions.

Definition 4.44. We recall [20, p.38] the *dominance order* $\triangleright$ *on multipartitions.* Suppose that $\underline{\alpha} = (\alpha^1, \cdots, \alpha^t)$ and $\underline{\beta} = (\beta^1, \cdots, \beta^t)$. Then $\underline{\alpha} \trianglerighteq \underline{\beta}$ if for every $p \in \{1, \cdots, t\}, q \geq 0$, we have

$$\sum_{i=1}^{p-1} |\alpha^i| + \sum_{i=1}^{q} \alpha_i^p \geq \sum_{i=1}^{p-1} |\beta^i| + \sum_{i=1}^{q} \beta_i^p, \tag{4.1}$$

taking any part that is undefined to be zero.

Example 4.45. *We have*

$$((3,2,1),(1),(2),\emptyset) \trianglerighteq ((2,2,2),(1),(1),(1))$$

in the dominance order.

Notation. We will for convenience abbreviate symmetric groups to their describing index in some situations. For example, Hom_n will mean Hom_{S_n} and $\mathrm{Ext}^1_{S_m \wr S_n}$ will be represented by $\mathrm{Ext}^1_{m \wr n}$.

Theorem 4.46. *[20, 10.1.1, 10.1.3] Suppose that $\underline{\alpha}, \underline{\beta}$ are t-multipartions of n. Then*

1. *When the characteristic of k is different from 2,*

$$\mathrm{Hom}_{m \wr n}(S^{\underline{\alpha}}, S^{\underline{\beta}}) \cong \begin{cases} k & \text{if } \underline{\alpha} = \underline{\beta}, \\ 0 & \text{if } \underline{\alpha} \ntrianglerighteq \underline{\beta} \end{cases}.$$

2. *When the characteristic of k is neither 2 or 3 and $\underline{\alpha} \ntriangleright \underline{\beta}$, then*

$$\mathrm{Ext}^1_{m \wr n}(S^{\underline{\alpha}}, S^{\underline{\beta}}) = 0.$$

By completing the dominance order on multipartitions to a total order, Green showed that the Specht modules form a stratifying system for $k(S_m \wr S_n)$ when the characteristic of k is not 2 or 3. In particular, the Hemmer-Nakano phenomenon for symmetric groups carries over to wreath products of symmetric groups. See [20, §10.2] for the full details of the stratifying system.

Corollary 4.47 (Hemmer-Nakano Property for Wreath Products). *[20, Thm 20.1] Over an algebraically closed field k with characteristic different from $2, 3$, the Specht factors appearing in a Specht-filtered $k(S_m \wr S_n)$-module's filtration are independent of the choice of that filtration.*

4.4.1 The Partition Algebra has a Stratifying System

We will review the proof that $P_r(\delta)$ has a stratifying system from the work of Hartmann, Henke, König and Paget on *cellularly stratified algebras* [41]. In many circumstances, this class of algebras have stratifying systems on cell modules. We will recall the argument that $P_r(\delta)$ is a cellularly stratified algebra and verify that it has the correct conditions.

Definition 4.48. [41, Def 2.1] Suppose that $A = \bigoplus_{l=1}^{n} V_l \otimes B_l \otimes V_l$ is an iterated inflation of cellular algebras B_l. Suppose that for each $l = 1, \cdots n$ there is some idempotent $e_l = t_l \otimes 1_{B_l} \otimes b_l$ such that $t_l, b_l \in V_l$. If $e_l e_m = e_m = e_m e_l$ whenever $l > m$, A is a *cellularly stratified algebra.*

The authors study in detail the homomorphisms and extensions between modules of cellularly stratified algebras, arriving at the following result. Note that the authors of [41] use the term "standard system" instead of "stratifying system".

Theorem 4.49. *[41, Thm 10.2] Let $A = \bigoplus_{l=1}^{n} V_l \otimes B_l \otimes V_l$ be a cellularly stratified algebra where the cell modules of B_l form a stratifying system. Then the cell modules of A form a stratifying system.*

It remains to recall the proof from [41] that the partition algebra $P_r(\delta)$ is a cellularly stratified algebra.

Lemma 4.50. *[41, Prop 2.6] The partition algebra $P_r(\delta)$ is cellularly stratified when $\delta \neq 0$.*

Proof. We have already verified that $P_r(\delta)$ is an iterated inflation of cellular algebras kS_i. We take directly from the exposition in §2.4 of [41] that an appropriate set of idempotents is given by

$$e_0 = \{1, 2, \cdots, r | 1', \cdots, r'\}$$

and

$$e_l = \{1, 1' | \cdots | l, l' | l+1, \cdots, r, (l+1)', \cdots, r'\}$$

for $l \geq 1$. □

Corollary 4.51. *[41, Cor 10.3] Suppose that $\delta \neq 0$ and that the characteristic of the ground field k is not equal to 2 or 3. Then the cell modules of $P_r(\delta)$ form a stratifying system.*

Proof. Combine results 4.49 and 4.40 and apply the result to the partition algebra $P_r(\delta)$. □

Chapter 5

The Structure of the Ramified Partition Algebra

5.1 The Ramified Partition Algebra

The ramified partiton algebra was defined by Martin and Elgamal in [39]. In fact, they introduced a wider family of ramified partition algebras of increasing complexity, however we focus (in their notation) on the 2-ramified partition algebra, referring to it simply as the ramified partition algebra. In his paper [30], Kennedy discusses an algebra known as the *class partition algebra*, which, upon closer inspection, can be seen to be identical to the ramified partition algebra.

Definition 5.1. Suppose that a and b are set partitions of some set S. Then a is a refinement of b if every part of a is contained entirely in some part of b. Equivalently, a is a refinement of b if every part of b is a union of parts of a.

Definition 5.2. [39, §2.1] [30, §3.1] Let $\delta^{\text{in}}, \delta^{\text{out}} \in k$ be parameters. We define the ramified partition algebra $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ over the field k to be the subalgebra of $P_r(\delta^{\text{in}}) \otimes P_r(\delta^{\text{out}})$ generated by the set

$$\{d^{\text{in}} \otimes d^{\text{out}} : d^{\text{in}} \text{ and } d^{\text{out}} \text{ are partition diagrams with } d^{\text{in}} \text{a refinement of } d^{\text{out}}\}.$$

We refer to the elements in this set as *ramified partition diagrams*. Given a ramified partition diagram $d^{\mathrm{in}} \otimes d^{\mathrm{out}}$, we refer to d^{in} as its inner partition and d^{out} as its outer partition.

Notation. We will henceforth write $(d^{\mathrm{in}}, d^{\mathrm{out}})$ for the diagram $d^{\mathrm{in}} \otimes d^{\mathrm{out}}$. We will use bold letters to refer to ramified partition diagrams, for example $\mathbf{d} = (d^{\mathrm{in}}, d^{\mathrm{out}})$.

We recall a diagram calculus as in [39, §2.2]. The diagram of $\mathbf{d}$ is given by taking the partition diagram of d^{in} from $P_r(\delta^{\mathrm{in}})$, and then grouping parts of d^{in} lying in the same part of d^{out} together by placing them into bubbles. Multiplication corresponds simply to concatenating inner and outer parts of the diagram separately, replacing inner parts that do not meet the top or bottom with a factor of δ^{in}, and outer bubbles that don't meet top or bottom with δ^{out}. For example, the ramified diagram of the identity element is given in figure 5.1.

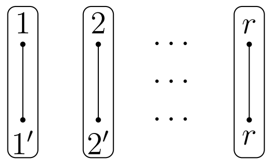


FIGURE 5.1: The identity element for $RP_r(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$.

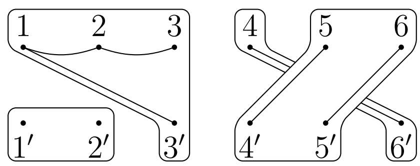
N.B. In the following example, some outer parts pass under others. This is for readability and aesthetic reasons, and does NOT represent over/under data.

Example 5.3. *The element given by the pair of partitions*

$$(\{1,2,3,3'|4,6'|5,4'|6,5'|1'|2'\}, \{1,2,3,3'|4,6'|5,6,4',5'|1',2'\})$$

can be represented by the diagram

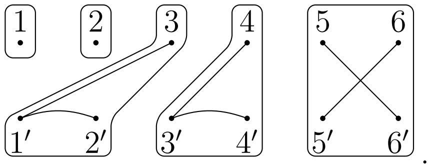
$\mathbf{d} =$



.

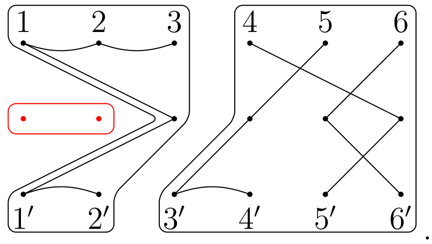
We multiply on the right by the diagram

$$\mathbf{e} =$$



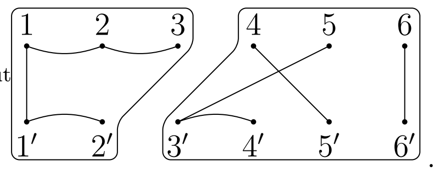
Concatenating, we obtain,

$$\mathbf{d} * \mathbf{e} =$$



Replacing the isolated outer part (bubble) by a factor of δ^{out}, and the two isolated inner parts by a factor of δ^{in}, we obtain the answer

$$\mathbf{de} = (\delta^{\text{in}})^2\delta^{\text{out}}$$



The diagonal map $d \mapsto (d, d)$ allows us to embed the partition algebra $P_r(\delta^{\text{in}}\delta^{\text{out}})$ into $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$. Diagrammatically, this inclusion simply corresponds to taking a partition diagram and drawing a bubble around each of its parts individually. In particular, this guarantees that the entire symmetric group S_r is contained within $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$.

Proposition 5.4. *[39, Cor of Prop 5] A generating set for $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is $\{(A^i, A^i), (A^i, 1) | i = 1, \cdots, r-1\} \cup \{(A^{ij}, A^{ij}), (1, A^{ij}) | i, j = 1, \cdots, r; i < j\} \cup S_r$.*

We do not present a full proof, however the idea is very similar to the one discussed below Lemma 4.4. Also analogous to the partition algebra case, a smaller generating set is $\{(A^1, A^1), (A^1, 1), (A^{12}, A^{12}), (1, A^{12}), (s_1, s_1), \cdots, (s_{r-1}, s_{r-1})\}$, but we will use the larger set of generators. We display the non-permutation elements of this smaller set of generators in figure 5.2.

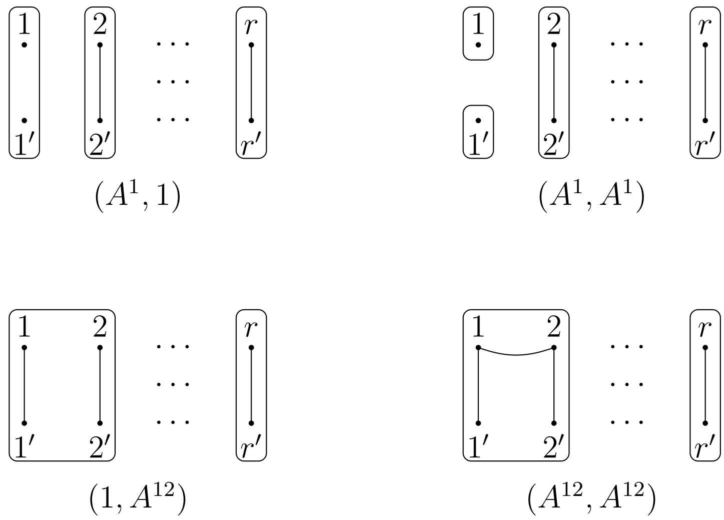


FIGURE 5.2: The non-permutation generators of $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$.

As is the case with the partition algebra, $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ has an involution σ given by interchanging each unprimed element with its primed counterpart. This corresponds diagrammatically to the reflection which exchanges the two rows of the diagram. Each of the generators $(A^i, A^i), (A^i, 1), (A^{ij}, A^{ij})$ and $(1, A^{ij})$ are preserved by σ.

In the previous section, we saw that partition diagrams could be organised by their propagating number. The diagrams of the ramified partition algebra can similarly be organised by their *propagating index*. We follow the definitions from [39, §3.1]. We first need some new terminology to describe the situation.

Definition 5.5. A *tail partition* λ is a tuple of non-negative integers $(\lambda_1, \cdots, \lambda_s)$ such that $\lambda_1 \geq \cdots \geq \lambda_s$. Unlike partitions, we do not suppress zeroes at the end of a tail partition, and the number of zeroes matters. Note that the empty partition $\emptyset$ is also a tail partition.

Given a tail partition λ, define $\text{env}(\lambda)$ to be the sum of the entries of λ, plus the number of zeroes in λ. We still write $|\lambda| = \lambda_1 + \ldots \lambda_{\ell(\lambda)}$.

Write Λ_n^{tail} for the set of those tail partitions λ for which $\text{env}(\lambda) \leq n$.

Example 5.6. *All partitions are also tail partitions. The tail partitions* $\lambda_1 = (3, 2, 1, 0)$ *and* $\lambda_2 = (3, 2, 1, 0^2)$ *are different tail partitions as they have different numbers of trailing zeroes.*

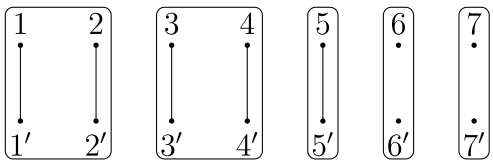
If $\lambda \vdash n$ *is a partition, then* $\mathrm{env}(\lambda) = n$*. We have* $\mathrm{env}(\lambda_1) = 7$*, whereas* $\mathrm{env}(\lambda_2) = 8$.

The propagating index $p(\mathbf{d})$ of a ramified diagram $\mathbf{d}$ is a tail partition which describes the pattern of the inner and outer propagating parts.

Definition 5.7. [39, §3] Suppose $\mathbf{d} \in RP_r(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$ is a ramified diagram. For each propagating part of d^{out}, write the number (possibly zero) of inner propagating parts contained within. The propagating index $p(\mathbf{d})$ of $\mathbf{d}$ is defined to be the list of these numbers in descending order.

Remark 5.8. It is possible that $p(\mathbf{d}) = \emptyset$. This occurs in the case that $\mathbf{d}$ has no inner or outer propagating parts.

Example 5.9. *The diagram*



has propagating index $(2^2, 1, 0^2)$.

Example 5.10. *The elements in Example 5.3 have propagating indices* $p(\mathbf{d}) = (2, 1^2), p(\mathbf{e}) = (2, 1^2)$ *and* $p(\mathbf{de}) = (3, 1)$ *respectively. Note that in the multiplication, an outer part with* 2 *inner propagating parts is merged with one containing* 1 *propagating part to make an outer part containing* 3 *inner propagating parts. This corresponds to merging the* 2 *and a* 1 *to make a* 3 *in the propagating index.*

The propagating index plays an important role in the structure of $RP_r(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$. In order to see this, we must first recall an appropriate order on propagating indices.

Definition 5.11. [39] Let $\lambda, \mu \in \Lambda_n^{\mathrm{tail}}$ for some n. Write $\lambda \prec \mu$ if

1. (Subtraction) λ can be obtained from μ by subtracting 1 from one of its nonzero parts, or

2. (Merging) λ can be obtained from μ by merging two of its parts.

It is necessary also to define $\emptyset \prec (0)$. Taking the transitive completion of these rules gives us the *propagating order* $\prec$ on propagating indices.

We will refer to the poset $(\Lambda_n^{\text{tail}}, \prec)$ of tail partitions under the propagating order simply by the notation Λ_n^{prop}.

Remark 5.12 (Deletion). Suppose λ is obtained from μ by deleting a part. Then $\lambda \prec \mu$.

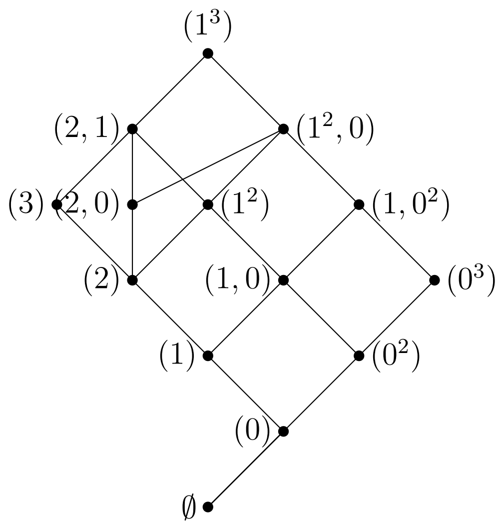


FIGURE 5.3: The Hasse diagram for Λ_3^{prop}.

In Λ_n^{prop}, the unique minimal element is the empty partition $\emptyset$ and (1^n) is the unique maximal element. This poset gets increasingly unwieldy, and is difficult to draw in a satisfying way even for $n = 5$. Figure 5.3 contains the Hasse diagram for Λ_3^{prop}.

The propagating index cannot be increased by multiplication. This is because multiplication can only ever merge parts or reduce the number of inner propagating parts, never split or increase, respectively.

Lemma 5.13. *[39, Prop 6] Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be ramified diagrams. Then* $p(\mathbf{ab}) \preccurlyeq \min(p(\mathbf{a}), p(\mathbf{b}))$*. Furthermore, the set of all ramified diagrams of propagating index at most* λ *generates an ideal* I_λ *of* $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$*, with* $I_\lambda \subset I_\mu \iff \lambda \prec \mu$*.*

We recall an idempotent $\mathbf{e}_\lambda$ for each propagating index $\lambda \in \Lambda_r^{\text{prop}}$, as defined in [39] with the notation $\mathcal{I}_\lambda$. We must first define an associated diagram $\mathbf{d}_\lambda$, built out of smaller diagrams. Given $a, b > 0$, define $\mathbf{b}_{(a^b)}$ to be the ramified diagram on ab dots with inner partition $1_{P_{ab}}$ and outer partition grouping the first a primed and unprimed dots together, then the next a and so on. Define $\mathbf{b}_{(0^b)}$ to be the ramified diagram on b dots obtained from $\mathbf{b}_{(1^b)}$ by making every dot singleton in the inner partition. Finally, define $\mathbf{b}_\emptyset$ to be the diagram (A^1, A^1) on 1 dot. Given diagrams $\mathbf{d}$ on r dots and $\mathbf{e}$ on s dots, define $\mathbf{d} \oplus \mathbf{e}$ to be the diagram on $r+s$ dots obtained by placing $\mathbf{e}$ directly to the right of $\mathbf{d}$ without making any new connections.

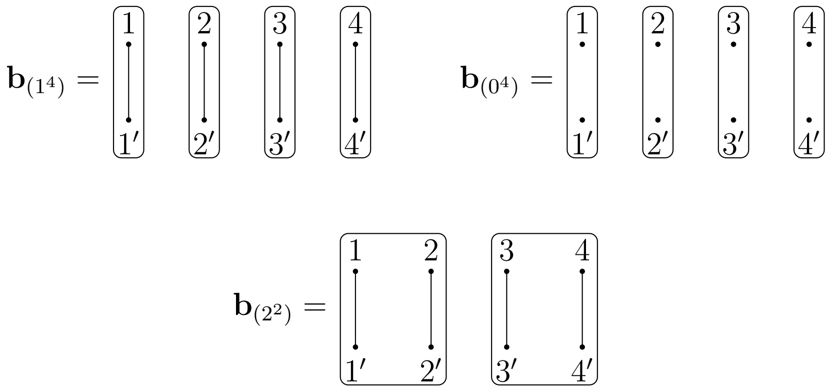


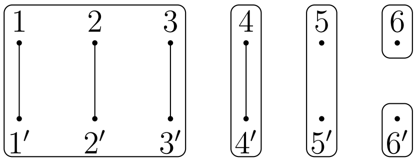
FIGURE 5.4: Pictures of the diagrams $\mathbf{b}_{(1^4)}$, $\mathbf{b}_{(0^4)}$ and $\mathbf{b}_{(2^2)}$.

Now, suppose that we are working on r dots and propagating index $\lambda \in \Lambda_r^{\text{prop}}$. Write $\lambda = (a_1^{b_1}, \cdots, a_s^{b_s})$ with $a_1 > \cdots > a_s \geq 0$ and $b_i > 0$. Define the diagram $\mathbf{d}_\lambda^{[r]}$ to be

$$\mathbf{d}_\lambda^{[r]} = \left(\bigoplus_{i=1}^{s} \mathbf{b}_{(a_i^{b_i})} \right) \oplus \left(\bigoplus_{j=1}^{r-\text{env}(\lambda)} \mathbf{b}_\emptyset \right).$$

Example 5.14. *If* $\lambda = (a^b)$ *and* $r = \text{env}((a^b)) = ab$*, then* $\mathbf{d}_\lambda^{[r]}$ *is simply the diagram* $\mathbf{b}_\lambda$ *defined above.*

The diagram $\mathbf{d}^{[6]}_{(3,1,0)}$ *on* 6 *dots is*



One further example is the diagram in example 5.9, which is in fact $\mathbf{d}^{[7]}_{(2^2,1,0^2)}$.

Now, one observes that upon squaring the diagram $\mathbf{d}^{[r]}_\lambda$, we obtain

$$(\mathbf{d}^{[r]}_\lambda)^2 = (\delta^{\text{in}})^p(\delta^{\text{out}})^q\mathbf{d}^{[r]}_\lambda,$$

where p is the number of inner singletons on the bottom row, and q is the number of dots which are both inner and outer singletons on the bottom row. In particular, $p = r - |\lambda|$ and $q = r - \text{env}(\lambda)$. Therefore, we must scale by this factor.

Definition 5.15. [39] Provided that $\delta^{\text{in}}, \delta^{\text{out}} \neq 0$, the idempotent $\mathbf{e}^{[r]}_\lambda$ is defined to be

$$\mathbf{e}^{[r]}_\lambda = \frac{1}{\left(\delta^{\text{in}}\right)^{(r-|\lambda|)}\left(\delta^{\text{out}}\right)^{(r-\text{env}(\lambda))}}\mathbf{d}^{[r]}_\lambda.$$

From now on, we will drop the superscript $[r]$ when the number of dots, and hence the algebra we are working with, is clear from context.

$$\frac{1}{(\delta^{\text{in}})^3(\delta^{\text{out}})^2}$$

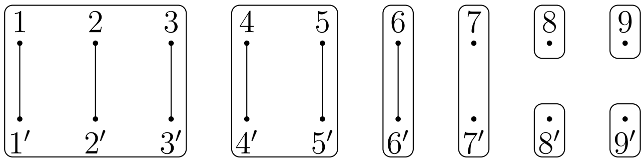


FIGURE 5.5: The element $\mathbf{e}_{(3,2,1,0)} \in RP_9(\delta^{\text{in}}, \delta^{\text{out}})$.

5.2 The Ramified Partition Algebra is Cellular

We show that the ramified partition algebra $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is cellular over a general field k. In particular, we will show that it is an iterated inflation of the group algebras of a direct products of wreath products of symmetric groups.

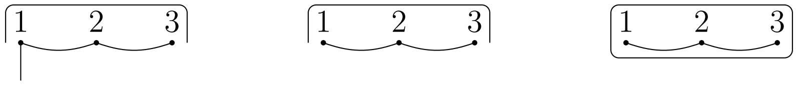


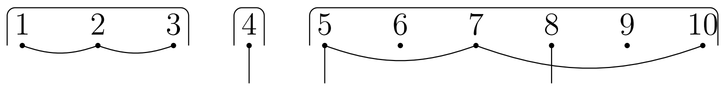
Figure 5.6: A propagating inner part vs. a propagating outer part with no inner propagating parts vs. a non propagating outer part.

Definition 5.16. Define a *ramified half-diagram on r dots* to be a pair $\mathbf{t} = (t^{\text{in}}, t^{\text{out}})$ of partition half-diagrams on r dots such that t^{in} is a refinement of t^{out} with all propagating parts of t^{in} being inside some propagating part of t^{out}.

List the number of inner propagating parts within each outer propagating part of $\mathbf{t}$ in descending order including zeroes. From this process, we obtain a tail-partition λ. In this case, we say that $\mathbf{t}$ has propagating index λ, and write $p(\mathbf{t}) = \lambda$. Define $\mathcal{V}_\lambda$ to be the set of ramified half-diagrams of propagating index λ on r dots, and write V_λ for the vector space with basis $\mathcal{V}_\lambda$.

Ramified half-diagrams have a diagrammatic representation as halves of ramified diagrams in the same way that partition half-diagrams are halves of partition diagrams. We have r dots, and draw lines between dots in order to signify that they are in the same inner part, leaving stubs on inner parts which are propagating. We join inner parts in the same outer part by drawing a bubble around them, leaving this bubble open at the bottom if the outer part in question is propagating. An example is shown in Figure 5.6.

For example a half-diagram of propagating index $(2, 1, 0)$ for $RP_{10}(\delta^{\text{in}}, \delta^{\text{out}})$ is

$\mathbf{t} =$ 

We discuss the group of permutations which describes how to connect together two ramified half-diagrams of propagating index λ. Martin and Elgamal discuss this group in [39] for different reasons and we use their notation $S[\lambda]$. Suppose that the tail partition λ is given in index notation by $(a_1^{b_1}, \cdots, a_s^{b_s})$ with $a_1 > \cdots > a_s$.

We define the permutation group

$$S[\lambda] := \prod_{i=1}^{s} S_{a_i} \wr S_{b_i},$$

a direct product of wreath products of symmetric groups. As algebras,

$$kS[\lambda] \cong \bigotimes_{i=1}^{s} (kS_{a_i} \wr S_{b_i}),$$

and therefore by results 4.24 and 4.25, $kS[\lambda]$ is cellular. Recall that S_0 and S_1 are both the trivial group, and that $S_0 \wr S_b \cong S_1 \wr S_b \cong S_b$. The involution σ_λ on $kS[\lambda]$ is simply the one induced by inverting group elements.

Recall that in Definition 2.62, we defined a t-multipartition of n to be a t-tuple $\underline{\lambda} = (\lambda_1, \ldots, \lambda_t)$ of partitions such that $|\lambda_1| + \cdots |\lambda_t| = n$. Recall further that the cell modules of $S_m \wr S_n$ are indexed by t-multipartitions of n, where t is the number of partitons of m.

The cell indices of $kS[\lambda]$ are *tuples of multipartitions*, where the ith entry is a multipartition labelling a cell module for $S_{a_i} \wr S_{b_i}$. We shall call these *tuples of multipartitions of shape* λ, and write them with double underlining, for example $\underline{\underline{\alpha}}, \underline{\underline{\beta}}$.

Example 5.17. *We exhibit a tuple of multipartitions which is of shape* $\lambda = (4^7, 3^2, 2^5)$. *This must be of the form* $(\underline{\alpha}^1, \underline{\alpha}^2, \underline{\alpha}^3)$ *where* $\underline{\alpha}^1$ *is a multipartition for* $S_4 \wr S_7$, $\underline{\alpha}^2$ *is a multipartition for* $S_3 \wr S_2$, *and* $\underline{\alpha}^3$ *is a multipartition for* $S_2 \wr S_5$.

Looking back to Example 2.63, we must have a 5*-multipartition of* 7 *(since there are* 5 *partitions of* 4*), a* 3*-multipartition of* 2*, and a* 2*-multipartition of* 5 *respectively. Drawing again from Example 2.63, we see that therefore some tuples of multipartitions of shape* λ *are*

$$(((2,1),(1),(1),(1),(1)),((1),\emptyset,(1)),((2),(1^3)))$$

and

$$(((1),\emptyset,(3,2),(1),\emptyset),(\emptyset,(1^2),\emptyset),((2,1),(1^2))).$$

We will mention the cell modules of $kS[\lambda]$ later.

Suppose that $\mathbf{t}, \mathbf{b} \in \mathcal{V}_\lambda$ are two ramified half-diagrams, and that $\tau \in S[\lambda]$ is the permutation chosen to connect them. Note that by definition $\tau = (\tau_1, \cdots, \tau_s)$ where $\tau_i \in S_{a_i} \wr S_{b_i}$.

Recall that we ordered propagating parts of a partition half-diagram by their leftmost dot (see Remark 4.28). One may extend this logic to order the inner and outer propagating parts of a ramified half-diagram, as well as the inner propagating parts in some outer part.

Look only at the outer propagating parts of $\mathbf{t}$ and $\mathbf{b}$ which contain exactly $a := a_i$ inner propagating parts. There are exactly $b := b_i$ of these in each half-diagram by assumption. We will connect these outer and inner propagating parts by the permutation $\tau_i = (\tau_{i,1}, \cdots, \tau_{i,b}; \tau_{i,0}) \in S_a \wr S_b$. For clarity, we remind the reader that this means $\tau_{i,0} \in S_b$, while $\tau_{i,j} \in S_a$ for $i = 1, \cdots, b$. Connect the lth outer propagating part of $\mathbf{b}$ to the $\tau_{i,0}(l)$th outer propagating part of $\mathbf{t}$. Connect the mth inner propagating part of the lth outer propagating part of $\mathbf{b}$ to the $\tau_{i,l}(m)$th inner propagating part of the $\tau_{i,0}(l)$th outer propagating part of $\mathbf{t}$. Running this process over all $i = 1, \cdots, s$, we have connected $\mathbf{b}$ to $\mathbf{t}$ by the permutation $\tau \in S[\lambda]$ - see Example 5.19 for an explicit example.

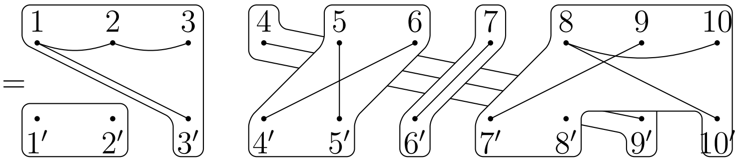
Analogously to the partition algebra case, this means that

$$RP_r(\delta^{\text{in}}, \delta^{\text{out}}) \cong \bigoplus_{\lambda \in \Lambda_r^{\text{prop}}} V_\lambda \otimes kS[\lambda] \otimes V_\lambda$$

as k-vector spaces, where V_λ is the vector space of ramified half-diagrams of propagating index λ, and Λ_r^{prop} is the poset of all propagating indices λ for $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$. When d is described by the triple $(\mathbf{t}, \tau, \mathbf{b})$, we abuse notation and write $d = \mathbf{t} \otimes \tau \otimes \mathbf{b}$. We write $\text{top}(\mathbf{d}) = \mathbf{t}$, $\text{bot}(\mathbf{d}) = \mathbf{b}$ and $\text{con}(\mathbf{d}) = \tau$.

Remark 5.18. For ease of reading, we will sometimes draw $\text{top}(\mathbf{d})$ and $\text{bot}(\mathbf{d})$ with opposite open ends, however they do indeed live in the same vector space and top half-diagrams are not different from bottom half-diagrams.

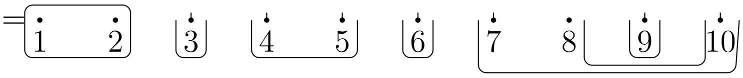
Example 5.19. *Consider the diagram*

$\mathbf{d} =$ 

.

This has top and bottom as follows:

$\text{top}(\mathbf{d}) =$ 

$\text{bot}(\mathbf{d}) =$ 

and $p(\mathbf{d}) = (2^2, 1^3)$. *This means that our connecting element should be an element of* $S[2^2, 1^3] = S_2 \wr S_2 \times S_1 \wr S_3 \cong S_2 \wr S_2 \times S_3$. *Firstly, consider those outer propagating parts which contain two inner propagating parts, which will be attached with an element* $(\alpha_1, \alpha_2; \alpha_0)$ *of* $S_2 \wr S_2$. *There are two of these each in* $\text{top}(d)$ *and* $\text{bot}(d)$*:* $\{5, 6\}$ *and* $\{8, 9, 10\}$ *on top and* $\{4', 5'\}$ *and* $\{7', 8', 10'\}$ *on the bottom.The first part on the bottom is matched to the first part on top, and the second to the second, so we must have* $\alpha_0 = \text{Id}$. *Within these connections, in both cases the two propagating inner parts are swapped over in the attaching map, so we must have* $\alpha_1 = \alpha_2 = (1, 2)$.

Those outer parts containing a single inner propagating part, namely $\{1, 2, 3\}, \{4\}$ *and* $\{7\}$ *on top, and* $\{3\}, \{6\}$ *and* $\{9\}$ *on the bottom, must be attached with an element of* $S_1 \wr S_3 \cong S_3$. *The first of the bottom inner propagating parts is attached to the first inner propagating part on top, and the other two are swapped in order, so the attaching map is* $(2, 3)$. *We can make this into an element of* $S_1 \wr S_3$ *notationally by instead writing it as* $(\text{Id}, \text{Id}, \text{Id}; (2, 3))$.

Therefore, we obtain our connecting map

$$\text{con}(d) = ((1, 2), (1, 2); \text{Id}) \times (\text{Id}, \text{Id}, \text{Id}; (2, 3)) \in S[(2^2, 1^3)].$$

We examine how the involution σ on ramified partition diagrams interacts with the tensor product structure. Suppose that $\mathbf{d} = \mathbf{t} \otimes \tau \otimes \mathbf{b}$ is of propagating index $\lambda = (a_1^{b_1}, \cdots, a_s^{b_s})$ with $a_1 > \cdots > a_s$. It is clear that $\sigma(\mathbf{d}) = \mathbf{b} \otimes \mathrm{con}(\sigma(\mathbf{d})) \otimes \mathbf{t}$, but we must identify $\mathrm{con}(\sigma(\mathbf{d}))$. Adopting our previous notation, we write $\tau = (\tau_1, \cdots, \tau_s)$ where $\tau_i = (\tau_{i,1}, \cdots, \tau_{i,b_i}; \tau_{i,0})$. We will determine the nature of $\mathrm{con}(\sigma(\mathbf{t}))$.

For this paragraph, let us look only at those outer propagating parts with exactly a_i inner propagating parts. In $\sigma(\mathbf{d})$, the lth outer propagating part of $\mathbf{b}$ is connected to the $\tau_{i,0}(l)$th outer propagating part of $\mathbf{t}$. That is to say, writing $p = \tau_{i,0}(l)$, the pth outer propagating part of $\mathrm{bot}(\sigma(\mathbf{d})) = \mathbf{t}$ is connected to the $\tau_{i,0}^{-1}(p)$th outer propagating part of $\mathrm{top}(\sigma(\mathbf{d})) = \mathbf{b}$. The mth inner propagating part of the lth outer propagating part of $\mathbf{b}$ is connected to the $\tau_{i,l}(m)$th inner propagating part of the $\tau_{i,0}(l)$th outer propagating part of $\mathbf{t}$. Again, considering this differently by writing $n = \tau_{i,l}(m)$, the nth inner propagating part of the pth outer propagating part of $\mathrm{bot}(\sigma(\mathbf{d})) = \mathbf{t}$ is connected to the $\tau^{-1}_{i,\tau_{i,0}^{-1}(p)}(n)$th inner propagating part of the $\tau_{i,0}^{-1}(p)$th outer part of $\mathrm{top}(\sigma(\mathbf{d})) = \mathbf{b}$. In summary, considering only the outer propagating parts which contain exactly a_i inner propagating parts, $\mathrm{bot}(\sigma(\mathbf{d}))$ is connected to $\mathrm{top}(\sigma(\mathbf{d}))$ by the permutation

$$(\tau_{i,\tau_{i,0}^{-1}(1)}, \cdots, \tau_{i,\tau_{i,0}^{-1}(a_i)}; \tau_{i,0}^{-1}) = \tau_i^{-1}.$$

Thus considering all outer propagating parts at once, we see that

$$\mathrm{con}(\sigma(\mathbf{d})) = (\tau_1^{-1}, \cdots, \tau_s^{-1}) = \tau^{-1}.$$

In conclusion, we have shown that

$$\sigma(\mathbf{t} \otimes \tau \otimes \mathbf{b}) = \mathbf{b} \otimes \tau^{-1} \otimes \mathbf{t}.$$

Finally, we must again turn our attention to finding the bilinear forms ϕ_λ and θ_λ from Theorem 4.22 governing multiplication. We will require the appropriate bilinear forms for the partition algebra $P_r(\delta)$ which can be found after Example 4.31. We will particularly recall that $\phi_i(a, t)$ is equal to either 0 or a scalar multiple of $a \vee t$, where partition($a \vee t$) is obtained by taking a, connecting dots i' and j' if i and j are connected in t to obtain $a \nabla t$, and then restricting to the top row $\{1, \cdots, r\}$ of dots. Fuller details are available in the earlier section, as mentioned.

Pick any ramified diagram $\mathbf{a} = (a^{\text{in}}, a^{\text{out}})$, as well as a ramified half-diagram $\mathbf{t} = (t^{\text{in}}, t^{\text{out}})$ of propagating index λ. Consider the (scalar multiple of a) ramified half-diagram given by $(\phi_i(a^{\text{in}}, t^{\text{in}}), \phi_j(a^{\text{out}}, t^{\text{out}}))$, where $i = p(t^{\text{in}})$ and $j = p(t^{\text{out}})$. If at least one entry here is equal to 0, then set $\phi_\lambda(a, t) = 0$. Otherwise, we obtain a (scalar multiple of a) ramified half-diagram, in which case set $\phi_\lambda(\mathbf{a}, \mathbf{t}) = (\phi_i(a^{\text{in}}, t^{\text{in}}), \phi_j(a^{\text{out}}, t^{\text{out}}))$. One must check that $\phi_i(a^{\text{in}}, t^{\text{in}})$ is a refinement of $\phi_j(a^{\text{out}}, t^{\text{out}})$ to verify that this is indeed a ramified half-diagram. It is enough to show that $a^{\text{in}} \nabla t^{\text{in}}$ is a refinement of $a^{\text{out}} \nabla t^{\text{out}}$. Suppose that x and y are connected in $a^{\text{in}} \nabla t^{\text{out}}$, then there is some sequence $\{x = p_0, p_1, \cdots p_s = y\}$ such that p_i and p_{i+1} are in the same part of a^{in} or t^{in}. Considering the same sequence in $a^{\text{out}} \nabla t^{\text{out}}$, we see that p_i is still connected to p_{i+1} in a^{out} or t^{out}, and hence x and y are also connected in $a^{\text{out}} \nabla t^{\text{out}}$, meaning that $a^{\text{in}} \nabla t^{\text{in}}$ is a refinement of $a^{\text{out}} \nabla t^{\text{out}}$.

We also need to produce an element $\theta_\lambda(\mathbf{a}, \mathbf{t}) \in S[\lambda]$. For the sake of simplicity, because of the direct product structure we describe the case $\lambda = (m^n)$ so that $S[\lambda] = S_m \wr S_n$. Consider the permutations $\theta_{mn}(a^{\text{in}}, t^{\text{in}}) \in S_{mn}$ and $\theta_n(a^{\text{out}}, t^{\text{out}}) \in S_n$. If either permutation is 0, set $\theta_\lambda(\mathbf{a}, \mathbf{t}) = 0$. Otherwise, proceed as follows.

The permutation $\theta_\lambda(\mathbf{a}, \mathbf{t})$ can be written as $(\tau_1, \cdots, \tau_n; \tau_0)$ with $\tau_0 \in S_n$ and $\tau_i \in S_m$ when $i \geq 1$. The permutation $\theta_n(a^{\text{out}}, t^{\text{out}})$ describes the effect of multiplication on the connection of outer propagating parts, so set $\tau_0 = \theta_n(a^{\text{out}}, t^{\text{out}}) \in S_n$. For $l > 0$, set τ_l to be the restriction of $\theta_{mn}(a^{\text{in}}, t^{\text{in}})$ to the m dots within the lth outer propagating part of $\mathbf{t}$, expressed as a permutation in S_m. The fact that $\theta_\lambda(\mathbf{a}, \mathbf{t})$ is in $S_m \wr S_n$ follows from the fact that inner parts are refinements of outer parts in all cases.

We have dealt with the case $S[\lambda] = S_m \wr S_n$, and the general case can be easily obtained from the direct product structure of $S[\lambda] = \prod S_{a_i} \wr S_{b_i}$. Explicitly, the component of $\theta_\lambda(\mathbf{a}, \mathbf{t})$ in $S_{a_i} \wr S_{b_i}$ can be obtained by only considering those outer propagating parts with a_i inner propagating parts and applying the simplified description in the last two paragraphs.

Theorem 5.20. *The ramified partition algebra $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is a cellular algebra with the cellular involution σ given by diagram reflection. Furthermore, the cellular poset is*

$$\{(\lambda, \underline{\underline{\alpha}}) | \lambda \in \Lambda_r^{\text{prop}} \text{ and } \underline{\underline{\alpha}} \text{ is a tuple of multipartitions of shape } \lambda\}.$$

Proof. As in the partition algebra case, we appeal to Lemma 4.22 to show that $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is an iterated inflation of cellular algebras. The poset I in the lemma will be our poset Λ_r^{prop} of propagating indices. The cellular algebra B_λ in the lemma will be the group algebra $kS[\lambda]$, with $\mathcal{B}_\lambda$ the (non-cellular) basis of group elements and involution σ_λ induced by inversion of group elements. Let $\mathcal{V}_\lambda$ in the lemma be the set of ramified half-diagrams of propagating index λ as in our current exposition, so that V_λ in the lemma is the same V_λ as discussed in this section. With these definitions taken, the set $\mathcal{A}$ is the usual basis of ramified diagrams for $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$.

We have already established that

$$RP_r(\delta^{\text{in}}, \delta^{\text{out}}) \cong \bigoplus_{\lambda \in \Lambda_r^{\text{prop}}} V_\lambda \otimes kS[\lambda] \otimes V_\lambda$$

as vector spaces, and that

$$\sigma(\mathbf{t} \otimes \tau \otimes \mathbf{b}) = \mathbf{b} \otimes \tau^{-1} \otimes \mathbf{t} = \mathbf{b} \otimes \sigma_\lambda(\tau) \otimes \mathbf{t}$$

for every $\mathbf{t}, \mathbf{b} \in \mathcal{V}_\lambda$ and $\tau \in kS[\lambda]$.

Suppose that $\mathbf{a} \in \mathcal{A}$ with $\mathbf{t}, \mathbf{b} \in \mathcal{V}_\lambda$ and $\tau \in kS[\lambda]$. It remains to show that

$$\mathbf{a}(\mathbf{t} \otimes \tau \otimes \mathbf{b}) \equiv \phi_\lambda(\mathbf{a}, \mathbf{t}) \otimes \theta_\lambda(\mathbf{a}, \mathbf{t}) \otimes \mathbf{b} \pmod{J(< i)},$$

where ϕ_λ and θ_λ are the maps defined immediately before this theorem. This follows from the fact that ϕ_i and θ_j are the corresponding set of maps for the partition algebra $P_r(\delta)$, and that multiplication in $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is induced from its containment in $P_r(\delta^{\text{in}}) \otimes P_r(\delta^{\text{out}})$. $\square$

We will describe the cell modules of $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$, but we first need to look at the cell modules for $kS[\lambda]$. We will write the cell module for $kS[\lambda]$ indexed by the tuple of multipartitions $\underline{\underline{\alpha}}$ as $S^{\underline{\underline{\alpha}}}$ instead of the usual Δ, and of course if $\underline{\underline{\alpha}} = (\underline{\alpha}^1, \cdots, \underline{\alpha}^t)$, where $\underline{\alpha}^i$ are multipartitions, then

$$S^{\underline{\underline{\alpha}}} = S^{\underline{\alpha}^1} \boxtimes \cdots \boxtimes S^{\underline{\alpha}^t}.$$

If $\lambda = (0^s), (1^s)$ or (s) then we can see tuples of multipartitions $\underline{\underline{\alpha}}$ of shape λ simply as partitions α of s, so that the Specht module $S^{\underline{\underline{\alpha}}}$ is simply the usual Specht module S^α.

We may thus use Lemma 4.23 to see that the cell modules of this algebra are indexed by pairs $(\lambda, \underline{\underline{\alpha}})$, where λ is a propagating index for r dots, and $\underline{\underline{\alpha}}$ is a tuple of multipartitions of shape λ.

The cell module for $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ corresponding to $(\lambda, \underline{\underline{\alpha}})$ is $V^\lambda \oslash S^{\underline{\underline{\alpha}}}$ with the action

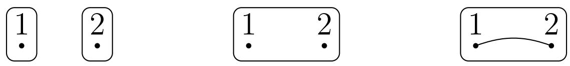
$$\mathbf{d}(\mathbf{t} \oslash x) = \phi_\lambda(\mathbf{d}, \mathbf{t}) \oslash \theta_\lambda(\mathbf{d}, \mathbf{t})x.$$

That is, a diagram $\mathbf{d}$ acts on half-diagrams of shape V_λ by the sensible diagrammatic action, and acts on modules for $kS[\lambda]$ by the element of $kS[\lambda]$ which connects its top and bottom, which is an element of $S[\lambda]$. We will give some examples of cell modules and some actions within.

We will mostly focus in this thesis on *diagrammatic cell modules* for the ramified partition algebra. That is, those cell modules whose cell index $(\lambda, \underline{\underline{\alpha}})$ has $\underline{\underline{\alpha}}$ labelling the trivial module for $S[\lambda]$. Consequently, we may ignore the presence of $\underline{\underline{\alpha}}$ and the module can be viewed solely in terms of diagrams. We will just write V_λ for such modules since the trivial module for $kS[\lambda]$ makes no difference to the multiplication.

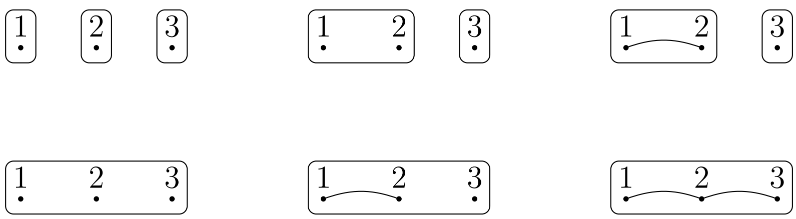
We give several practical examples, starting with specific, small numbers of dots.

Example 5.21. *Consider the module $V_\emptyset$ for $RP_2(\delta^{\text{in}}, \delta^{\text{out}})$. There are only three diagrams, as pictured below.*



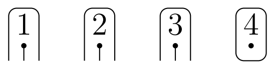
Note that these diagrams are not annihilated by any of our standard generators.

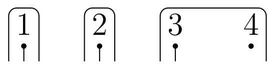
The same applies when we look at $V_\emptyset$ for $RP_3(\delta^{\text{in}}, \delta^{\text{out}})$. We draw one ramified half-diagram from each S_3-orbit in this module.

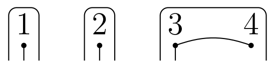


Indeed, none of our standard set of generators annihilate any elements of the ramified half-diagram basis of $V_\emptyset$ for $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ for general r.

Example 5.22. *Consider the module $V_{(1^3)}$ for $RP_4(\delta^{\text{in}}, \delta^{\text{out}})$. This module is of dimension 22, so we draw one ramified half-diagram from each S_4 orbit below.*

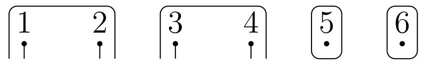
$e =$ 

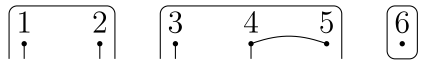
$f =$ 

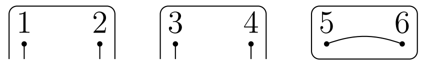
$g =$ 

The orbits of e, f and g are of sizes $4, 12$ and 6 respectively. One may confirm that $e = \frac{1}{\delta^{\text{in}}}(A^4, A^4)f = (A^4, A^4)g$, whereas $f = (A^4, 1)g = (1, A^{3,4})e$. Finally, $g = (A^{3,4}, A^{3,4})e = (A^{3,4}, A^{3,4})f$.

Example 5.23. *Consider the module $V_{(2^2)}$ for $RP_6(\delta^{\text{in}}, \delta^{\text{out}})$. We will look at only at 3 diagrams e, f, g as follows:*

$e =$ 

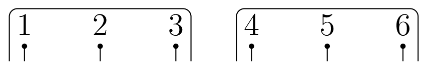
$f =$ 

$g =$ 

We have $(A^{4,5}, A^{4,5})e = f$ and $(A^{5,6}, A^{5,6})e = g$. In the other direction, $(A^5, A^5)f = (A^5, A^5)g = e$. It is not possible to transform f into g or vice versa by a single generator, but it is clear we can do so in two generators using e as an intermediate point.

To finish our study of modules with small fixed amounts of dots, we will briefly look at an example that is not purely diagrammatic.

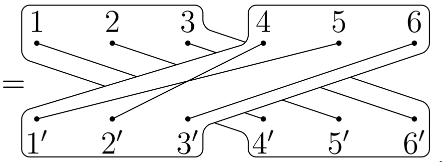
Example 5.24. *Consider the module $V_{(3^2)}$ for $RP_6(\delta^{\text{in}}, \delta^{\text{out}})$. The ramified half-diagram basis of $V_{(3^2)}$ is of size $\frac{1}{2}\binom{6}{3} = 10$. Due to complexity of drawing diagrams with overlapping parts, we display only one of these basis elements, being*

$\mathbf{t} =$ .

One sees that $(A^i, A^i)\mathbf{t} = (A^i, 1)\mathbf{t} = (A^{i,j}, A^{i,j})\mathbf{t} = 0$ for every $i, j \in \{1, \cdots, 6\}$. Considering the generator $(1, A^{i,j})$, this fixes $\mathbf{t}$ if i and j are either both in $\{1, 2, 3\}$ or both in $\{4, 5, 6\}$. However, for example, $(1, A^{3,4})\mathbf{t} = 0$ as two outer propagating parts are merged.

Let us consider the action of something a little more complicated. Consider the diagram $\mathbf{d} = (d^{\text{in}}, d^{\text{out}})$ given by $d^{\text{in}} = \{1, 4'|2, 5'|3, 6'|4, 2'|5, 1'|6, 3'\}$ and $d^{\text{out}} =$

$\{1, 2, 3, 4', 5', 6'|4, 5, 6, 1', 2', 3'\}$. *In a picture,*

$\mathbf{d} =$ 

It is clear from looking at the diagrammatic action that $\mathbf{dt} = \mathbf{t}$. *The element* $\theta(\mathbf{d}, \mathbf{t})$ *is given by* $\theta(\mathbf{d}, \mathbf{t}) = (\mathrm{Id}, (12); (12))$. *This corresponds to the element* $(1, 5, 2, 4)(3, 6)$ *when embedded into* S_6*, which is precisely the element of* S_6 *we can read off from the diagram by considering inner connections from bottom row dots to top row dots.*

Assume that we are now working in the module $V_{(3^2)} \oslash S^{\underline{\alpha}}$ *for* $RP_6(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$*, so that* $\underline{\alpha}$ *is a* 3*-multipartition of* 2 *and thus* $S^{\underline{\alpha}}$ *is a Specht module for* $S_3 \wr S_2$. *Suppose that* $v \in S^{\underline{\alpha}}$. *Then* $\mathbf{d} \cdot (\mathbf{t} \oslash v) = \mathbf{t} \oslash w$ *where* $w = (\mathrm{Id}, (12); (12)) \cdot v$.

Now, we examine some infinite families of cell modules where $\mathrm{env}(\lambda) = r$. Suppose that $\lambda = (1^r)$, note that in this case, $S[\lambda] = S_1 \wr S_r \cong S_r$. In this case, a tuple of multipartitions $\underline{\underline{\alpha}}$ of shape λ can simply be interpreted as a partition $\alpha \vdash r$. A similar fact is true when $\lambda = (r)$ or (0^r).

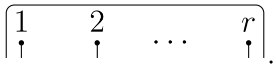
Example 5.25. *Let* S^α *be a Specht module for* S_r*, so that we may consider the cell module* $V_{(1^r)} \oslash S^\alpha$ *for* $RP_r(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$. *There is a single ramified half-diagram* $\mathbf{t}$ *of propagating index* (1^r)*, given by*

$\mathbf{t} =$ 

One may quickly verify by looking at the diagrammatic multiplication that this diagram is annihilated by each of the generators $(A^i, A^i), (A^i, 1), (A^{i,j}, A^{i,j})$ *and* $(1, A^{i,j})$ *of* $RP_r(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$. *Symmetric group elements act by fixing* $\mathbf{t}$ *and acting as normal on* S^α.

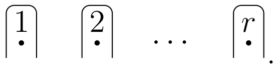
This is enough information for us to say that $V_{(1^r)} \oslash S^\alpha \cong \mathrm{inf}_{S_r}^{RP_r}(S^\alpha)$ *as* RP_r*-modules.*

Example 5.26. *Consider the module $V_{(r)}$ for $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$. This module is one-dimensional with basis element*

$\mathbf{t} =$ 

This basis element $\mathbf{t}$ *is annihilated by the generators* $(A^i, A^i), (A^i, 1)$ *and* $(A^{i,j}, A^{i,j})$. *It is preserved by* $(1, A^{i,j})$ *(and* S_r*), so the module* $V_{(r)} \oslash S^\alpha$ *is not isomorphic to* $V_{(1^r)} \oslash S^\alpha$*, even though they are of the same dimension.*

Example 5.27. *Considering the module $V_{(0^r)}$ for $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$, we see again the module is one dimensional with basis element*

$\mathbf{t} =$ 

The basis element $\mathbf{t}$ *is annihilated by* $(A^i, A^i), (A^{i,j}, A^{i,j})$ *and* $(1, A^{i,j})$*, while* $(A^i, 1)$ *acts as the scalar* δ^{in}. *Thus* $V_{(0^r)} \oslash S^\alpha$ *is isomorphic to neither* $V^{(1^r)} \oslash S^\alpha$ *nor* $V^{(r)} \oslash S^\alpha$.

5.3 Semisimplicity of the Ramified Partition Algebra over $\mathbb{C}$

We will discuss the semisimplicity of $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ over the field $\mathbb{C}$ of complex numbers when the values of the parameters δ^{in} and δ^{out} have different relationships to the number of dots r.

Firstly, we provide a wide range of values for which $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is semisimple. This relies on Kennedy's paper [30] which proves that, when δ^{in} and δ^{out} are in fact integers m and n, $RP_r(m, n)$ lies in Schur-Weyl duality with the algebra $\mathbb{C}(S_m \wr S_n)$. In particular, one may find the following definition and technical result in that paper, noting that Kennedy switches the order of the parameters when compared to our notation.

Definition 5.28. [30, §3.2] Define W to be the complex vector space $\mathbb{C}^{mn}$ with the natural action of $S_m \wr S_n \subset S_{mn}$ by permutation. We write $W^{\otimes r}$ for the r-fold tensor product of W with diagonal action of $S_m \wr S_n$ given by

$$\tau \cdot (w_1 \otimes \cdots \otimes w_r) := (\tau \cdot w_1) \otimes \cdots \otimes (\tau \cdot w_r).$$

While proving Schur-Weyl duality, Kennedy describes $RP_r(m,n)$ as an endomorphism algebra of $W^{\otimes r}$ when the parameters m and n are large in comparison to the number of dots r.

Lemma 5.29. *[30, Cor 3.3.3] Suppose that m and n are integers with $m, n \geq 2r$. Then*

$$RP_r(m,n) \cong \mathrm{End}_{\mathbb{C}(S_m \wr S_n)}(W^{\otimes r})$$

as algebras.

This allows us to establish a large family of parameters for which $RP_r(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$ is indeed semisimple.

Lemma 5.30. *Suppose that $\delta^{\mathrm{in}} = m \in \mathbb{Z}$ and $\delta^{\mathrm{out}} = n \in \mathbb{Z}$. If $m, n \geq 2r$, then $RP_r(m,n)$ is a semisimple algebra.*

Proof. In the circumstances of the Lemma, we have already seen that $RP_r(m,n) \cong \mathrm{End}_{\mathbb{C}(S_m \wr S_n)}(W^{\otimes r})$. By Maschke's Theorem, $\mathbb{C}(S_m \wr S_n)$ is a semisimple algebra. If M is a module over a semisimple algebra A, then $\mathrm{End}_A(M)$ is a semisimple algebra [12, Thm 2.6.4]. Therefore $\mathrm{End}_{\mathbb{C}(S_m \wr S_n)}(W^{\otimes r})$ is semisimple and this is precisely $RP_r(m,n)$ up to isomorphism. $\square$

We have recalled a large family of integer values of $(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$ for which $RP_r(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$ is semisimple. We will now use the theory of idempotent truncation detailed in [19, §6.2], as well as our established knowledge about the partition algebra, to exhibit some families of integer values of $(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$ for which $RP_r(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$ is not semisimple.

Given an idempotent $e \neq 0$ of a k-algebra A, the theory of idempotent truncation allows us to transition between the modules of A and eAe. If M is a module for A, eM is a module for eAe using the usual action. In particular, the following result on simple modules holds.

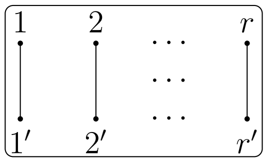
Lemma 5.31. *[19, 6.2b] Suppose that A is a k-algebra with e a nonzero idempotent. Suppose M is a simple module for A. Then eM is either zero or a simple eAe-module.*

Our strategy will be to exhibit the partition algebras $P_r(\delta^{\text{in}})$ and $P_r(\delta^{\text{out}})$ as subalgebras of $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ of the form $eRP_r(\delta^{\text{in}}, \delta^{\text{out}})e$, find some cell module M such that the $P_r(\delta^{\text{in}})$- or $P_r(\delta^{\text{out}})$-module eM is neither 0 nor simple, and deduce that M is not simple. As the cell modules of a semisimple algebra are simple, we will conclude that $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is not semisimple.

5.3.1 Values of the parameter δ^{in} for which $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is not semisimple.

We will do this first for the parameter δ^{in}, recalling that we are working over the field $\mathbb{C}$ of complex numbers. We will actually need to work on a quotient of $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ in order to use our knowledge of $P_r(\delta^{\text{in}})$. Write $I_\emptyset$ for the subalgebra of $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ generated by all ramified diagrams with propagating index $\emptyset$. By our results on propagating index, $I_\emptyset$ is an ideal and one may form the quotient $RP_r^{\text{prop}}(\delta^{\text{in}}, \delta^{\text{out}}) = RP_r(\delta^{\text{in}}, \delta^{\text{out}})/I_\emptyset$, which has basis consisting of the ramified diagrams with at least one propagating part. Since a quotient of a semisimple algebra is semisimple, it is enough to show that $RP_r^{\text{prop}}(\delta^{\text{in}}, \delta^{\text{out}})$ is not semisimple in order to show that $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is not semisimple.

Let $\mathbf{e}_{\text{inner}} \in RP_r^{\text{prop}}(\delta^{\text{in}}, \delta^{\text{out}})$ be the ramified diagram
$(\{1, 1' | \cdots | r, r'\}, \{1, 1', \cdots, r, r'\})$. Diagrammatically,

$\mathbf{e}_{\text{inner}} =$  .

Multiplying diagramatically, we see that $\mathbf{e}_{\text{inner}}^2 = \mathbf{e}_{\text{inner}}$ and thus we have chosen an idempotent in $RP_r^{\text{prop}}(\delta^{\text{in}}, \delta^{\text{out}})$.

For the rest of this section, we freely abbreviate $e = \mathbf{e}_{\text{inner}}$ and $A = RP_r^{\text{prop}}(\delta^{\text{in}}, \delta^{\text{out}})$. Write F_r for the set partition of $\{1, \cdots, r, 1', \cdots, r'\}$ consisting of the entire set so that $e = (1_{P_r(\delta^{\text{in}})}, F_r)$

Lemma 5.32. *The algebra eAe is isomorphic to the partition algebra $P_r(\delta^{\text{in}})$.*

Proof. The subalgebra eAe of A is spanned by the ramified diagrams (d^{in}, F_r). Define an isomorphism

$$\rho_{\text{inner}} : eAe \to P_r(\delta^{\text{in}})$$

by

$$\rho_{\text{inner}}((d^{\text{in}}, F_r)) = d^{\text{in}}.$$

This is a homomorphism since $(d_1^{\text{in}}, F_r)(d_2^{\text{in}}, F_r) = (d_1^{\text{in}} d_2^{\text{in}}, F_r)$, and bijectivity is clear. $\square$

We are interested in those cell modules of A which are not annihilated by e, which we now characterise. Recall that W_s is the diagrammatic cell module for the partition algebra whose basis consists of all the half-diagrams of propagating number s, and that the set of cell modules for the partition algebra $P_r(\delta)$ is $\{W_s \oslash S^{\alpha} | s = 0, 1, \cdots, r, \alpha \vdash s\}$.

Lemma 5.33. *Suppose that $\lambda \in \Lambda_r^{\text{prop}}$ is a propagating index which is not the empty propagating index $\emptyset$. Suppose further that $\underline{\underline{\alpha}}$ is a tuple of multipartitions of shape λ. Then*

$$e(V_\lambda \oslash S^{\underline{\underline{\alpha}}}) = 0$$

unless $\lambda = (s)$ for some $s = 0, 1, \cdots, r$.

In the case where $\lambda = (s)$, $\underline{\underline{\alpha}}$ *can be seen as a partition* $\alpha \vdash s$, *and* $S^{\underline{\underline{\alpha}}}$ *is just the usual Specht module* S^{α}. *Furthermore*

$$e(V_{(s)} \oslash S^{\alpha}) \cong W_s \oslash S^{\alpha}$$

as a module for $eAe \cong P_r(\delta^{\mathrm{in}})$.

Proof. Recall that V_λ is the module spanned by those ramified half-diagrams $\mathbf{d}$ of propagating index λ. If λ has more than one part, then $\mathbf{d}$ has more than one outer propagating part and $e\mathbf{d} = 0$ due to merging of propagating parts. Therefore $e(V_\lambda \oslash S^{\underline{\underline{\alpha}}}) = 0$ unless $\lambda = (s)$ for some $s = 0, 1, \cdots, r$.

Now, suppose that $\lambda = (s)$. Ramified half-diagrams in $eV_{(s)}$ are of the form $(t^{\mathrm{in}}, t^{\mathrm{out}})$ where $t^{\mathrm{in}} \in W_s$ and t^{out} is the half-diagram where all dots are in a single propagating part, and it is easy to see that there is a map

$$\rho : eV_{(s)} \to W_s$$

defined by $\rho(t^{\mathrm{in}}, t^{\mathrm{out}}) = t^{\mathrm{in}}$. One can see that this a homomorphism of $P_r(\delta^{\mathrm{in}})$-modules since $d \in P_r(\delta^{\mathrm{in}})$ acts on $eV_{(s)}$ as (d, F_r), effectively only acting on the inner parts of ramified half-diagrams, leaving the outer parts unchanged. It follows that $eV_s \oslash S^{\alpha} \cong W_s \oslash S^{\alpha}$ as an $eAe \cong P_r(\delta^{\mathrm{in}})$-module. □

Recall the following standard fact which can be deduced, for example, from the discussion below Theorem 1.4.8 in [12].

Lemma 5.34. *Suppose that* A *is an algebra,* I *is an ideal and* M *is a simple* A*-module. If* $IM = 0$ *then* M *is a simple* A/I*-module.*

Proof. The module M is an A/I-module since it is annihilated by I. Suppose that N is a proper submodule of M as an A/I-module. Then N can be viewed as an A-submodule of M through the quotient map, and so must be the zero module. □

Proposition 5.35. *Suppose that $\delta^{\text{in}} \in \{0, 1, \cdots, 2r-2\}$. Then the ramified partition algebra $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is not semisimple.*

Proof. Assume that $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is semisimple, so that its simple modules are exactly the cell modules by Corollarly 4.20. Consider the cell module $M = V_{(s)} \oslash S^{\alpha}$ for $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$. This module is annihilated by $I_{\emptyset}$ and is thus a simple $A = RP_r(\delta^{\text{in}}, \delta^{\text{out}})/I_{\emptyset}$-module.

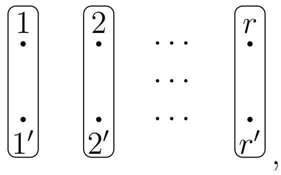
Since $\delta^{\text{in}} \in \{0, 1, \cdots, 2r-2\}$, the partition algebra $P_r(\delta^{\text{in}})$ is not semisimple and there is some $\alpha \vdash s$ such that $W_s \oslash S^{\alpha}$ is not simple. However, $W_s \oslash S^{\alpha} \cong e(V_{(s)} \oslash S^{\alpha})$. Therefore, there is a simple module M for A such that eM is neither zero nor simple as an eAe-module, contradicting Lemma 5.31. Hence our assumption that $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is semisimple was incorrect. $\square$

5.3.2 Values of the parameter δ^{out} for which $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is not semisimple.

Throughout this section, we assume $\delta^{\text{in}} \neq 0$ and recall that we are still working over the field $\mathbb{C}$ of complex numbers. Let $\mathbf{e}_{\text{outer}} \in RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ be the scalar multiple of a ramified diagram given by

$$\mathbf{e}_{\text{outer}} = (1/(\delta^{\text{in}})^r)(\{1|\ldots|r|1'|\ldots|r'\}, \{1, 1'|\ldots|r, r'\}).$$

Diagramatically,

$$\mathbf{e}_{\text{outer}} = \tfrac{1}{(\delta^{\text{in}})^r}$$ 

and observe that $\mathbf{e}_{\text{outer}}^2 = \mathbf{e}_{\text{outer}}$, meaning $\mathbf{e}_{\text{outer}}$ is idempotent. Of course we must note that δ^{in} must be nonzero for this to make sense. For the remainder of this section, we freely abbreviate $f = \mathbf{e}_{\text{outer}}$. Write D_r for the set partition of $\{1, \cdots, r, 1', \cdots, r'\}$ consisting entirely of singletons so that

$$f = \frac{1}{(\delta^{\text{in}})^r}(D_r, 1_{P_r(\delta^{\text{out}})}).$$

Lemma 5.36. *Suppose $\delta^{\text{in}} \neq 0$. Then the algebra $fRP_r(\delta^{\text{in}}, \delta^{\text{out}})f$ is isomorphic to the partition algebra $P_r(\delta^{\text{out}})$.*

Proof. The algebra $fRP_r(\delta^{\text{in}}, \delta^{\text{out}})f$ is spanned by diagrams of the form (D_r, d^{out}) for some $d^{\text{out}} \in P_r(\delta^{\text{out}})$. Define an isomorphism

$$\rho_{\text{outer}} : fRP_r f \to P_r(\delta^{\text{out}})$$

by

$$\rho_{\text{outer}}(D_r, d^{\text{out}}) \mapsto (\delta^{\text{in}})^r d^{\text{out}},$$

where the factor of $(\delta^{\text{in}})^r$ is to ensure that $\rho_{\text{outer}}(f) = 1_{P_r(\delta^{\text{out}})}$. This is a homomorphism since

$$(D_r, d_1^{\text{out}})(D_r, d_2^{\text{out}}) = (\delta^{\text{in}})^r (D_r, d_1^{\text{out}} d_2^{\text{out}}),$$

and bijectivity is again clear. □

We classify those cell modules M of RP_r such that $fM \neq 0$.

Lemma 5.37. *Suppose that $\lambda \in \Lambda_r^{\text{prop}}$ is a propagating index with $\underline{\underline{\alpha}}$ a tuple of multipartitions of shape λ. Then*

$$f(V_\lambda \oslash S^{\underline{\underline{\alpha}}}) = 0$$

unless $\lambda = \emptyset$ or $\lambda = (0^s)$ for some $s = 1, \cdots, r$. Recall that in the case $\lambda = (0^s)$, $\underline{\underline{\alpha}}$ can be interpreted as a Specht module S^α for S_s for $\alpha \vdash s$. Furthermore, as $fRP_r f \cong P_r(\delta^{\text{out}})$-modules,

$$fV_\emptyset \cong W_0$$

and

$$f(V_{(0^s)} \oslash S^\alpha) \cong W_s \oslash S^\alpha.$$

Proof. Suppose that λ has some nonzero part. Then ramified half-diagrams in V_λ have at least one propagating inner part which is cut when multiplying by f. Thus $f \cdot V_\lambda = 0$.

Now, suppose $\lambda = (0^s)$. Ramified half-diagrams in $fV_{(0^s)}$ are of the form $(t^{\text{in}}, t^{\text{out}})$ where t^{in} is the half-diagram consisting entirely of non-propagating singletons and $t^{\text{out}} \in W_s$, and thus there is a map

$$\rho_{\text{outer}} : fV_{(0^s)} \to W_s$$

given by $\rho_{\text{outer}}((\{1|\ldots|r||\}, t^{\text{out}})) = t^{\text{out}}$. This is an $fRP_r(\delta^{\text{in}}, \delta^{\text{out}})f$-isomorphism and $fV_{(0^s)} \cong W_s$, and similarly $f \cdot (V_{(0^s)} \oslash S^\alpha) \cong W_s \oslash S^\alpha$ as $fRP_rf \cong P_r(\delta^{\text{out}})$-modules. The proof that $fV_\emptyset \cong W_0$ as $P_r(\delta^{\text{out}})$-modules is entirely similar. □

Proposition 5.38. *Suppose that $\delta^{\text{out}} \in \{0, 1, \cdots, 2r-2\}$. Then the ramified partition algebra $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is not semisimple.*

Proof. The case where $\delta^{\text{in}} = 0$ is already proved, so assume $\delta^{\text{in}} \neq 0$. Assume that $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is semisimple, so that its simple modules are exactly the cell modules by Corollary 4.20. Since $\delta^{\text{out}} \in \{0, 1, \cdots, 2r-2\}$, the partition algebra $P_r(\delta^{\text{out}})$ is not semisimple and there is some $\alpha \vdash s$ such that $W_s \oslash S^\alpha$ is not simple. However, $W_s \oslash S^\alpha \cong f(V_{(0^s)} \oslash S^\alpha)$ when $s > 0$, or, if $s = 0, W_s \cong fV_\emptyset$. In either case, there is a simple module M for $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ such that fM is neither zero nor simple as an fAf-module, contradicting Lemma 5.31. Hence our assumption that $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is semisimple was incorrect. □

5.3.3 Other results on the semisimplicity of $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ and a conjecture.

We have just shown that $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is not semisimple over $\mathbb{C}$ whenever δ^{in} or δ^{out} is in the set $\{0, 1, \cdots, 2r-2\}$. We previously showed that if δ^{in} and δ^{out} are both integers greater than $2r-1$, $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is semisimple.

We will not make a determination either way about the situations where δ^{in} or δ^{out} are equal to $2r-1$, but we will recall some further results about the semisimplicity of the ramified partition algebra over $\mathbb{C}$ due to Martin and Elgamal [39]. Their paper contains other interesting discussion for example of the Gram matrix of $V_{\emptyset}$, but we will focus on just two results. The first result says that $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is "generically semisimple", which we rephrase in the following proposition.

Proposition 5.39. *[39, §3, Thm 1] The algebra $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is semisimple, unless δ^{in} or δ^{out} are one of a finite list of values.*

Martin and Elgamal also prove the following theorem that given any pair of natural number parameters, there is always a number of dots above which the ramified partition algebra is not semisimple.

Proposition 5.40. *[39, §4, Thm 1] For any $\delta^{\text{in}}, \delta^{\text{out}} \in \mathbb{N}$, there exists an $s \in \mathbb{N}$ such that $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is not semisimple for all $r > s$.*

The previous two sections have verified this result, and told us that it is enough to choose s to be $\min\{\lceil \frac{\delta^{\text{in}}+1}{2} \rceil, \lceil \frac{\delta^{\text{out}}+1}{2} \rceil\}$.

We conjecture that the parameters at which $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is semisimple mimic those for the partition algebra. In particular, we conjecture that we have already described all the values at which $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is not semisimple.

Conjecture 5.41. *The algebra $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ over the field $\mathbb{C}$ of complex numbers is semisimple if and only if $\delta^{\text{in}} \notin \{0, 1 \ldots, 2r-2\}$ and $\delta^{\text{out}} \notin \{0, 1 \ldots, 2r-2\}$.*

5.4 The Ramified Partition Algebra has a Stratifying System

Now, assume k to be a field of characteristic p, where p is 0 or a prime number and $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ is a k-algebra. We take $\delta^{\text{in}}\delta^{\text{out}} \neq 0$ throughout.

Recall that we abbreviate many groups in subscript. For example, Hom_n will be used for Hom_{kS_n} and $\mathrm{Ext}_{[\lambda]}$ will be used instead of $\mathrm{Ext}_{kS[\lambda]}$.

We will show that $RP_r(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$ has a stratifying system under certain conditions. To achieve this, we must investigate stratifying systems on the group algebra $kS[\lambda]$, which forms an essential part of the cellular structure of $RP_r(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$.

Let $\lambda \in \Lambda_r^{\mathrm{prop}}$ be a propagating index. Suppose $\lambda = (a_1^{b_1}, \cdots, a_s^{b_s})$ such that $a_1 > a_2 > \cdots > a_s$ and $b_i \geq 1$ for $i = 1, \cdots s$. We recall that the group $S[\lambda]$ is defined by

$$S[\lambda] = S_{a_1} \wr S_{b_1} \times \cdots \times S_{a_s} \wr S_{b_s},$$

with the convention that S_0 is the trivial group.

Recall that the cell modules for $kS[\lambda]$ are indexed by *tuples of multipartitions*, where the ith entry is a multipartition appropriate for $S_{a_i} \wr S_{b_i}$. Recall that we refer to such tuples as *being of shape* λ.

The following order is just the natural product order induced by the dominance order on multipartitions.

Definition 5.42. We define the *dominance order* $\rhd$ *on tuples of multipartitions*. To be explicit, let $\unrhd$ be the dominance order on *multipartitions*, and let $\underline{\underline{\alpha}} = (\underline{\alpha}^1, \cdots, \underline{\alpha}^s)$ and $\underline{\underline{\beta}} = (\underline{\beta}^1, \cdots, \underline{\beta}^s)$ be tuples of multipartitions of shape λ. Then

$$\underline{\underline{\alpha}} \unrhd \underline{\underline{\beta}} \iff (\underline{\alpha}^i \unrhd \underline{\beta}^i) \text{ for each } i = 1, \cdots s.$$

We will still denote this order by the dominance order symbol $\rhd$ and we shall call it the *dominance order on tuples of multipartitions.*

Recall that the cell module for $kS[\lambda]$ labelled by $\underline{\underline{\alpha}}$ is just $S^{\underline{\underline{\alpha}}} = S^{\underline{\alpha}^1} \boxtimes \cdots \boxtimes S^{\underline{\alpha}^s}$, an outer tensor product of modules, where $S^{\underline{\alpha}^i}$ is a cell module for $S_{a_i} \wr S_{b_i}$. We show some Hom and Ext results for these modules, resulting in an analogue of Theorem 4.46 for $kS[\lambda]$.

We require a preliminary homological lemma which will allow us to investigate Hom and Ext for outer tensor products in terms of their factors. The following lemma follows from an abstract result in Cartan and Eilenberg [4, Chapter XI, Theorem 3.1], however there is a less general result in Green's thesis which has a more straightforward proof.

Lemma 5.43. *[20, 2.1.3] Suppose $A = \bigotimes_{i=1}^{s} A_i$, where each tensor factor is a finite dimensional algebra over k. For each i, let M_i and N_i be A_i-modules. Then we have isomorphisms of vector spaces*

$$\mathrm{Hom}_A(M_1 \boxtimes \cdots \boxtimes M_s, N_1, \boxtimes \cdots \boxtimes N_s) \cong \bigotimes_{i=1}^{s} \mathrm{Hom}_{A_i}(M_i, N_i)$$

and

$$\mathrm{Ext}^1_A(M_1 \boxtimes \cdots \boxtimes M_s, N_1, \boxtimes \cdots \boxtimes N_s) \cong \bigoplus_{i=1}^{s} \left(\mathrm{Ext}^1_{A_i}(M_i, N_i) \otimes \bigotimes_{j \neq i} \mathrm{Hom}_{A_j}(M_j, N_j) \right)$$

The following Lemma is an easy consequence of Lemma 5.43 and the work of Green [20, 21] on homomorphisms and extensions between wreath product cell modules stated in Theorem 4.46.

Lemma 5.44. *Consider the algebra $kS[\lambda]$ over a field k, letting $\underline{\underline{\alpha}}, \underline{\underline{\beta}}$ be tuples of multipartitions of shape λ. Then*

1. *Provided that the characteristic of k is not 2, we have that*

$$\mathrm{Hom}_{[\lambda]}(S^{\underline{\underline{\alpha}}}, S^{\underline{\underline{\beta}}}) \cong \begin{cases} k & \text{if } \underline{\underline{\alpha}} = \underline{\underline{\beta}}, \\ 0 & \text{if } \underline{\underline{\alpha}} \not\trianglerighteq \underline{\underline{\beta}} \end{cases}.$$

2. When the characteristic of k is neither 2 nor 3, then

$$\mathrm{Ext}^1_{[\lambda]}(S^{\underline{\underline{\alpha}}}, S^{\underline{\underline{\beta}}}) = 0$$

whenever $\underline{\underline{\alpha}} \not\trianglerighteq \underline{\underline{\beta}}$.

Proof. Take the modules $M_i = S^{\underline{\alpha}^i}$, $N_i = S^{\underline{\beta}^i}$ in Lemma 5.43. For short, write $H_i = \mathrm{Hom}_{A_i}(S^{\underline{\alpha}^i}, S^{\underline{\beta}^i})$ and $E_i = \mathrm{Ext}_{A_i}(S^{\underline{\alpha}^i}, S^{\underline{\beta}^i})$, where A_i is the ith direct factor $S_{a_i} \wr S_{b_i}$ of $S[\lambda]$.

1. When $\underline{\underline{\alpha}} = \underline{\underline{\beta}}$, Lemma 5.43 and Theorem 4.46 give us that $\mathrm{Hom}_{[\lambda]}(S^{\underline{\underline{\alpha}}}, S^{\underline{\underline{\beta}}})$ is a tensor product of s copies of k, which is of course k. Now, suppose that $\underline{\underline{\alpha}} \ntrianglerighteq \underline{\underline{\beta}}$, so that there is some i such that $\underline{\alpha}^i \ntrianglerighteq \underline{\beta}^i$. Then $H_i = 0$, and therefore the whole tensor product will be 0, since one of the factors vanishes.

2. Now, consider the Ext result. Suppose that $\underline{\underline{\alpha}} \not\triangleright \underline{\underline{\beta}}$, so that either $\underline{\underline{\alpha}} = \underline{\underline{\beta}}$ or $\underline{\alpha}^i \ntrianglerighteq \underline{\beta}^i$ for some i. If $\underline{\underline{\alpha}} = \underline{\underline{\beta}}$, then $\underline{\alpha}^j = \underline{\beta}^j$ always, and hence $E_j = 0$ for every $j = 1, \cdots, s$ by Theorem 4.46. If there is some $\underline{\alpha}^i \ntrianglerighteq \underline{\beta}^i$, then $H_i = E_i = 0$, and by a similar argument the relevant Ext group becomes simply a direct sum of tensor products, each of which has 0 as a factor.

□

At this point, one may apply the discussion in [20, §10.2] to observe that $kS[\lambda]$ has a stratifying system unless the characteristic of k is 2 or 3.

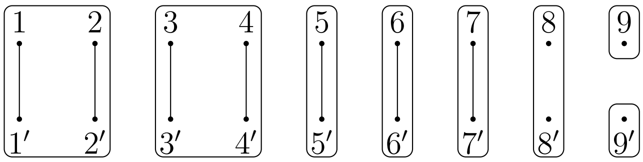
Note that the proof above is quite general. If we remove reference to $S[\lambda]$ and just take generic modules, we have the following result and its obvious generalisation to more factors, following easily from Lemma 5.43.

Lemma 5.45. *Suppose A, B are finite dimensional algebras with stratifying systems $\{M_i\}_{i\in I}$ and $\{N_j\}_{j\in J}$ respectively. Then $A \otimes B$ has a stratifying system $\{M_i \boxtimes N_i\}_{i\in I, j\in J}$ with (an order closely related to) the product order.*

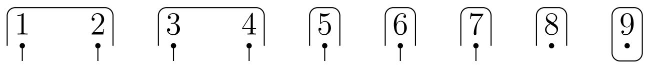
We will now prove Hom and Ext results for the ramified partition algebra, and hence show the existence of a stratifying system. We do this by exhibiting restrictions on when the Hom and Ext modules between cell modules can be nonzero in the sense of Lemma 4.43. The following results are inspired by work from [24], following their method of proof but with complications due to the more complex nature of the underlying permutation group $S[\lambda]$, and the fact that Λ_r^{prop} is not totally ordered. We assume that $\delta^{\text{in}}\delta^{\text{out}} \neq 0$ and abbreviate $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ simply to RP_r.

We will designate a special basis element of V_λ which we will use when proving homological results. Recall the idempotent $\mathbf{e}_\lambda$ from Definition 5.15. Write $\mathbf{t}_\lambda = \text{top}(\mathbf{e}_\lambda)$. Then it is clear that $\mathbf{e}_\lambda = \mathbf{t}_\lambda \otimes 1_{S[\lambda]} \otimes \mathbf{t}_\lambda$ up to multiplication by a scalar. We include a brief example to assist in recalling this definition.

Example 5.46. *Suppose that $\lambda = (2^2, 1^3, 0)$ over $r = 9$ dots. Then, up to multiplication by a nonzero scalar,*

$\mathbf{e}_\lambda =$ 

and

$\mathbf{t}_\lambda =$ .

We will investigate the effect of $\mathbf{e}_\lambda$ on half-diagram basis elements of V_λ. The following result follows from considering the diagram calculus and the fact that $\theta_\lambda(\mathbf{e}_\lambda, \mathbf{t}_\lambda) = \text{Id}_{S_\lambda}$.

Lemma 5.47. *Suppose $\delta^{\text{in}}\delta^{\text{out}} \neq 0$. Let $\lambda \in \Lambda_r^{\text{prop}}$ and let $\mathbf{t} \in V_\lambda$ be a ramified half-diagram.*

Then

- $\mathbf{e}_\lambda(\mathbf{t}_\lambda \otimes v) = (\mathbf{t}_\lambda \otimes v)$

- $\mathbf{e}_\lambda \mathbf{t} = \alpha \mathbf{t}_\lambda$ *for some* $\alpha \geq 0$.

Lemma 5.48. *Assume* $\delta^{\text{in}}\delta^{\text{out}} \neq 0$ *and suppose* $\lambda \in \Lambda_r^{prop}$. *Let* M, N *be* $kS[\lambda]$-*modules. Then there is an isomorphism*

$$\text{Hom}_{RP_r}(V_\lambda \oslash M, V_\lambda \oslash N) \cong \text{Hom}_{[\lambda]}(M, N).$$

Proof. We may write a basis of V_λ as $\mathbf{t}_1, \cdots, \mathbf{t}_s$ with $\mathbf{t}_1 = \mathbf{t}_\lambda$. Suppose that $f \in \text{Hom}_{RP_r}(V_\lambda \oslash M, V_\lambda \oslash N)$. Write $f(\mathbf{t}_\lambda \oslash m) = \sum_{i=1}^s \mathbf{t}_i \oslash n_i$ for some $n_i \in N$. Note that $f(\mathbf{t}_\lambda \oslash m) = f(\mathbf{e}_\lambda(\mathbf{t}_\lambda \oslash m)) = \mathbf{e}_\lambda f(\mathbf{t}_\lambda \oslash m)$. This is then equal to

$$\sum_{i=1}^{s} \mathbf{e}_\lambda(\mathbf{t}_i \oslash n_i) = \mathbf{t}_\lambda \oslash n'.$$

Therefore, every $f \in \text{Hom}_{RP_r}(V_\lambda \oslash M, V_\lambda \oslash N)$ induces an $\hat{f} \in \text{Hom}_{[\lambda]}(M, N)$ by setting $\hat{f}(m) = n'$ as in the above. This is a $kS[\lambda]$-homomorphism since $\tau \in kS[\lambda] \subset RP_r$ acts by multiplication by $\theta(\tau, \mathbf{t}_\lambda)$ on both M and N.

Now, suppose that $\hat{f} \in \text{Hom}_{[\lambda]}(M, N)$. Define

$$f((\mathbf{t}_i \oslash m)) := \mathbf{t}_i \oslash \hat{f}(m).$$

and let $\mathbf{d} \in RP_r$, then

$$\mathbf{d}f(\mathbf{t}_i \oslash m) = \mathbf{d}(\mathbf{t}_i \oslash \hat{f}(m)) = \phi_\lambda(\mathbf{d}, \mathbf{t}_i) \oslash \theta_\lambda(\mathbf{d}, \mathbf{t_i})\hat{f}(m),$$

which, since $\hat{f}$ is an $S[\lambda]$-homomorphism, is equal to

$$\phi_\lambda(\mathbf{d}, \mathbf{t}_i) \oslash \hat{f}(\theta_\lambda(\mathbf{d}, \mathbf{t}_i)m) = f(\phi_\lambda(\mathbf{d}, \mathbf{t}_i) \oslash \theta_\lambda(\mathbf{d}, \mathbf{t}_i)m) = f(\mathbf{d}(\mathbf{t}_i \oslash m)).$$

Thus $f \in \text{Hom}_{RP_r}(V_\lambda \oslash M, V_\lambda \oslash N)$ as required. □

Corollary 5.49. *Suppose* $\lambda \in \Lambda_r^{prop}$ *with the characteristic of* k *being different from* 2, *with* $\underline{\underline{\alpha}}, \underline{\underline{\beta}}$ *tuples of multipartitions of shape* λ. *Then*

$$\text{Hom}_{[\lambda]}(V_\lambda \oslash S^{\underline{\underline{\alpha}}}, V_\lambda \oslash S^{\underline{\underline{\beta}}}) = 0$$

whenever $\underline{\underline{\alpha}} \not\trianglerighteq \underline{\underline{\beta}}$*. Furthermore,*

$$\mathrm{Hom}_{[\lambda]}(V_\lambda \oslash S^{\underline{\underline{\alpha}}}, V_\lambda \oslash S^{\underline{\underline{\alpha}}}) \cong k.$$

Proof. This follows from Lemma 5.48 and Lemma 5.44. □

We now examine homomorphisms between cell modules of differing propagating indices. Recall that $\succ$ is the propagating order on propagating indices.

Lemma 5.50. *Suppose that M is an $S[\lambda]$-module and N is an $S[\mu]$-module where $\lambda, \mu \in \Lambda_r^{prop}$, and that $\delta^{\mathrm{in}}\delta^{\mathrm{out}} \neq 0$. Then* $\mathrm{Hom}_{RP_r}(V_\lambda \oslash M, V_\mu \oslash N) = 0$ *whenever $\lambda \not\preceq \mu$.*

Proof. Choose any $f \in \mathrm{Hom}_{RP_r}(V_\lambda \oslash M, V_\mu \oslash N)$. The map f is determined by the values of $f(\mathbf{t}_\lambda \oslash x)$ for each $x \in M$. The element $f(\mathbf{t}_\lambda \oslash x)$ must be equal to $\sum_{i=1}^{s}(a_i\mathbf{t}_i \oslash y_i)$ for some $a_i \in k, y_i \in N$, where $\mathbf{t}_i$ runs over the standard basis of V_μ consisting of half-diagrams. Repeating a previous technique, we know that $f(\mathbf{t}_\lambda \oslash x) = \mathbf{e}_\lambda f(\mathbf{t}_\lambda \oslash x)$.

Let $\mathbf{t}$ be a half-diagram of propagating index μ. Then $p(e_\lambda\mathbf{t})$ is less than or equal to both λ and μ in the propagating order. If $p(e_\lambda\mathbf{t})$ is equal to μ, then $p(\mathbf{e}_\lambda\mathbf{t}) = \mu \preccurlyeq \lambda$ which is a contradiction. Therefore $p(\mathbf{e}_\lambda\mathbf{t}) \prec \mu$ and hence $\mathbf{e}_\lambda\mathbf{t} = 0$ in V_μ. Putting it all together,

$$f(\mathbf{t}_\lambda \oslash x) = \sum_{i=1}^{s}(\mathbf{e}_\lambda(a_i\mathbf{t}_i \oslash y_i)) = 0,$$

since $\mathbf{e}_\lambda$ annihilates each $\mathbf{t}_i$. Thus f is the zero homomorphism. □

We will now investigate extensions between cell modules of different propagating indices. We will first prove an intermediate lemma.

Lemma 5.51. *Let $\lambda, \mu \in \Lambda_r^{\mathrm{prop}}$ with $\lambda \not\succ \mu$. Suppose $\underline{\underline{\beta}}$ is a tuple of multipartitions of shape μ so that $(\mu, \underline{\underline{\beta}})$ is a cell index for RP_r. Suppose that $\delta^{\mathrm{in}}\delta^{\mathrm{out}} \neq 0$. Then* $\mathrm{Ext}^1_{RP_r}(V_\lambda \oslash X, V_\mu \oslash S^{\underline{\underline{\beta}}}) = 0$ *for any projective $S[\lambda]$-module X.*

Proof. Consider the left module RP_re_λ for RP_r. This module is clearly projective since e_λ is idempotent. We may construct a map ψ from RP_re_λ to $V_\lambda \oslash kS[\lambda]$ by writing

$$\psi(\mathbf{d}) = \begin{cases} \mathrm{top}(\mathbf{d}) \oslash \mathrm{con}(\mathbf{d}) & p(\mathbf{d}) = \lambda \\ 0 & p(\mathbf{d}) \prec \lambda. \end{cases}$$

This map is surjective, since $\mathbf{t} \oslash \tau = \psi(\mathbf{t} \otimes \tau \otimes \mathbf{t}_\lambda)$. The kernel of this morphism is generated by all diagrams in $RP_r\mathbf{e}_\lambda$ of propagating index strictly less than λ in the propagating order. As usual, we have a short exact sequence

$$0 \longrightarrow \ker(\psi) \hookrightarrow RP_r\mathbf{e}_\lambda \twoheadrightarrow V_\lambda \oslash kS[\lambda] \longrightarrow 0,$$

which leads to a long exact sequence segment:

$$\cdots \longrightarrow \mathrm{Hom}_{RP_r}(\ker(\psi), V_\mu \oslash S^{\underline{\underline{\underline{\beta}}}}) \longrightarrow \mathrm{Ext}^1_{RP_r}(V_\lambda \oslash kS[\lambda], V_\mu \oslash S^{\underline{\underline{\underline{\beta}}}}) \longrightarrow \mathrm{Ext}^1_{RP_r}(RP_r\mathbf{e}_\lambda, V_\mu \oslash S^{\underline{\underline{\underline{\beta}}}}) \longrightarrow \cdots.$$

Since $RP_r\mathbf{e}_\lambda$ is projective,

$$\mathrm{Ext}^1_{RP_r}(RP_r\mathbf{e}_\lambda, V_\mu \oslash S^{\underline{\underline{\underline{\beta}}}}) = 0.$$

Suppose that $f \in \mathrm{Hom}_{RP_r}(\ker(\psi), V_\mu \oslash S^{\underline{\underline{\underline{\beta}}}})$ and let $\mathbf{d}$ be a diagram in $\ker(\psi)$ with $p(\mathbf{d}) =: \nu$. Multiplying by generators that do not change propagating index, we may assume that the top of $\mathbf{d}$ is $\mathbf{t}_\nu$. Now, $f(\mathbf{d})$ is of propagating index μ, and since $\nu \prec \lambda$ and $\mu \not\prec \lambda$, we have $\mu \not\preceq \nu$. Therefore $f(\mathbf{d}) = f(\mathbf{e}_\nu\mathbf{d}) = \mathbf{e}_\nu f(\mathbf{d}) = 0$, and

$$\mathrm{Hom}_{RP_r}(\ker(\psi), V_\mu \oslash S^{\underline{\underline{\underline{\beta}}}}) = 0.$$

Hence by exactness, the middle term vanishes and

$$\mathrm{Ext}^1_{RP_r}(V_\lambda \oslash S[\lambda], V_\mu \oslash S^{\underline{\underline{\underline{\beta}}}}) = 0.$$

Finally, using the direct sum property of the Ext functor, we can replace $S[\lambda]$ by any of its projective summands X, yielding the result. $\square$

Label the $S[\lambda]$-projective cover of $S^{\underline{\underline{\alpha}}}$ by $P(\underline{\underline{\alpha}})$ so that, tensoring the usual sequence with V_λ, we obtain a short exact sequence

$$0 \longrightarrow V_\lambda \oslash K(\underline{\underline{\alpha}}) \hookrightarrow V_\lambda \oslash P(\underline{\underline{\alpha}}) \twoheadrightarrow V_\lambda \oslash S^{\underline{\underline{\alpha}}} \longrightarrow 0, \tag{5.1}$$

where $K(\underline{\underline{\alpha}})$ is the kernel of the projective cover $P(\underline{\underline{\alpha}}) \twoheadrightarrow S^{\underline{\underline{\alpha}}}$. We now have the notation and results to discuss extensions between cell modules of different propagating indices.

Proposition 5.52. *Suppose that* $\delta^{\text{in}}\delta^{\text{out}} \neq 0$. *Suppose* $(\lambda, \underline{\underline{\alpha}}), (\mu, \underline{\underline{\beta}})$ *are cell indices for* RP_r *with* $\lambda \not\leq \mu$. *Then*

$$\mathrm{Ext}^1_{RP_r}(V_\lambda \oslash S^{\underline{\underline{\alpha}}}, V_\mu \oslash S^{\underline{\underline{\beta}}}) = 0.$$

Proof. Apply the functor $\mathrm{Hom}_{RP_r}(\bullet, V_\mu \oslash S^{\underline{\underline{\beta}}})$ to (5.1) and pass to the long exact sequence to obtain:

$$\cdots \longrightarrow \mathrm{Hom}_{RP_r}(V_\lambda \oslash K(\underline{\underline{\alpha}}), V_\mu \oslash S^{\underline{\underline{\beta}}}) \longrightarrow \mathrm{Ext}^1_{RP_r}(V_\lambda \oslash S^{\underline{\underline{\alpha}}}, V_\mu \oslash S^{\underline{\underline{\beta}}}) \longrightarrow \mathrm{Ext}^1_{RP_r}(V_\lambda \oslash P(\underline{\underline{\alpha}}), V_\mu \oslash S^{\underline{\underline{\beta}}}) \longrightarrow \cdots$$

By Lemma 5.50, the left term vanishes. By Lemma 5.51, the right term vanishes. Thus by exactness, we have

$$\mathrm{Ext}^1_{RP_r}(V_\lambda \oslash S^{\underline{\underline{\alpha}}}, V_\mu \oslash S^{\underline{\underline{\beta}}}) = 0$$

as required. $\square$

Finally, we consider extensions between cell modules of the same propagating index.

Proposition 5.53. *Suppose $\underline{\underline{\alpha}}, \underline{\underline{\beta}}$ are tuples of multipartitions of shape λ with $\underline{\underline{\alpha}} \not\triangleright \underline{\underline{\beta}}$, and that the characteristic of k is not 2 or 3. Suppose that $\delta^{\mathrm{in}}\delta^{\mathrm{out}} \neq 0$. Then*

$$\mathrm{Ext}^1_{RP_r}(V_\lambda \oslash S^{\underline{\underline{\alpha}}}, V_\lambda \oslash S^{\underline{\underline{\beta}}}) = 0.$$

Proof. Apply the functor $\mathrm{Hom}_{RP_r}(\bullet, V_\lambda \oslash S^{\underline{\underline{\beta}}})$ to (5.1) and pass to the long exact sequence to obtain:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{RP_r}(V_\lambda \oslash S^{\underline{\underline{\alpha}}}, V_\lambda \oslash S^{\underline{\underline{\beta}}}) \longrightarrow \mathrm{Hom}_{RP_r}(V_\lambda \oslash P(\underline{\underline{\alpha}}), V_\lambda \oslash S^{\underline{\underline{\beta}}}) \\ &\longrightarrow \mathrm{Hom}_{RP_r}(V_\lambda \oslash K(\underline{\underline{\alpha}}), V_\lambda \oslash S^{\underline{\underline{\beta}}}) \longrightarrow \mathrm{Ext}^1_{RP_r}(V_\lambda \oslash S^{\underline{\underline{\alpha}}}, V_\lambda \oslash S^{\underline{\underline{\beta}}}) \longrightarrow 0, \end{aligned} \tag{5.2}$$

where the rightmost 0 is the module $\mathrm{Ext}^1_{RP_r}(V_\lambda \oslash P(\underline{\underline{\alpha}}), V_\lambda \oslash S^{\underline{\underline{\beta}}})$ shown to be zero in Lemma 5.51.

Recall the short exact sequence

$$0 \longrightarrow K(\underline{\underline{\alpha}}) \hookrightarrow P(\underline{\underline{\alpha}}) \twoheadrightarrow S^{\underline{\underline{\alpha}}} \longrightarrow 0, \ , \tag{5.3}$$

which leads to the long exact sequence segment

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{[\lambda]}(S^{\underline{\underline{\alpha}}}, S^{\underline{\underline{\beta}}}) \longrightarrow \mathrm{Hom}_{[\lambda]}(P(\underline{\underline{\alpha}}), S^{\underline{\underline{\beta}}}) \\ &\longrightarrow \mathrm{Hom}_{[\lambda]}(K(\underline{\underline{\alpha}}), S^{\underline{\underline{\beta}}}) \longrightarrow \mathrm{Ext}^1_{[\lambda]}(S^{\underline{\underline{\alpha}}}, S^{\underline{\underline{\beta}}}) \longrightarrow 0, \end{aligned} \tag{5.4}$$

of $kS[\lambda]$-modules.

Compare sequences (5.2) and (5.4). By Lemma 5.48, the first 3 terms of the first sequence coincide with the first 3 terms of the second sequence, and we see that we must have

$$\mathrm{Ext}^1_{RP_r}(V_\lambda \oslash S^{\underline{\underline{\alpha}}}, V_\lambda \oslash S^{\underline{\underline{\beta}}}) \cong \mathrm{Ext}^1_{[\lambda]}(S^{\underline{\underline{\alpha}}}, S^{\underline{\underline{\beta}}}),$$

which is 0 by Theorem 4.46. This concludes our proof. $\square$

In order to exhibit the stratifying system, we must choose a total order on the cell indices of the ramified partition algebra, and then use Lemma 4.43.

Firstly, complete the dominance order $\rhd$ on tuples of multipartitions to a total order $\rhd_{\text{tot}}$. Let $\rhd_{\text{rev}}$ be the reverse of this total order. Then $\underline{\underline{\alpha}} \rhd_{\text{rev}} \underline{\underline{\beta}} \implies \underline{\underline{\alpha}} \ntrianglerighteq \underline{\underline{\beta}}$, and $\underline{\underline{\alpha}} \trianglerighteq_{\text{rev}} \underline{\underline{\beta}} \implies \underline{\underline{\alpha}} \ntriangleright \underline{\underline{\beta}}$. In a similar way, complete the order on propagating indices $\succ$ to some total order $\succ_{\text{tot}}$ and reverse this to obtain $\succ_{\text{rev}}$, so that $\lambda \succ_{\text{rev}} \mu \implies \lambda \nsucceq \mu$ and so on.

Now, totally order the cell indices lexicographically according to these two reverse orders, so that $(\lambda, \underline{\underline{\alpha}}) >_{\text{lex}} (\mu, \underline{\underline{\beta}})$ if either $\lambda \succ_{\text{rev}} \mu$ or $\lambda = \mu$ and $\underline{\underline{\alpha}} \rhd_{\text{rev}} \underline{\underline{\beta}}$. In particular $(\lambda, \underline{\underline{\alpha}}) >_{\text{lex}} (\mu, \underline{\underline{\beta}})$ implies that $\lambda \nsucceq \mu$ or $\lambda = \mu$ and $\underline{\underline{\alpha}} \ntriangleright \underline{\underline{\beta}}$. Finally, we can collect the above results into the following overarching theorem.

Theorem 5.54. *Suppose $\delta^{\text{in}}\delta^{\text{out}} \neq 0$ and that the characteristic of k is not 2. Then*

$$\operatorname{Hom}_{RP_r}(V_\lambda \oslash S^{\underline{\underline{\alpha}}}, V_\mu \oslash S^{\underline{\underline{\beta}}}) = \begin{cases} 0 & \text{if } (\lambda, \underline{\underline{\alpha}}) >_{\text{lex}} (\mu, \underline{\underline{\beta}}) \\ k & \text{if } (\lambda, \underline{\underline{\alpha}}) = (\mu, \underline{\underline{\beta}}). \end{cases}$$

If we suppose further that the characteristic of k is also not 3, then also whenever $(\lambda, \underline{\underline{\alpha}}) \geq_{\text{lex}} (\mu, \underline{\underline{\beta}})$, we have

$$\operatorname{Ext}_{RP_r}(V_\lambda \oslash S^{\underline{\underline{\alpha}}}, V_\mu \oslash S^{\underline{\underline{\beta}}}) = 0.$$

Proof. This follows from Corollary 5.49 and Proposition 5.52, considering the definition of our new order. □

Therefore, we have exhibited a stratifying system for $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ consisting of the cell modules, provided that $\delta^{\text{in}}\delta^{\text{out}} \neq 0$ and the characteristic of the ground field is not 2 or 3. Knowing this, we can in particular apply Theorem 4.41 to realise the following Theorem.

Theorem 5.55 (Hemmer-Nakano for the Ramified Partition Algebra)**.** *Suppose that $\delta^{\text{in}}\delta^{\text{out}} \neq 0$ and the characteristic of k is not 2 or 3, and that M is a module for $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ which is filtered by cell modules. Then the multiplicities of* *cell modules appearing in a cell filtration of M are independent of the choice of filtration.*

Explicitly, suppose that M has a filtration $\mathcal{F}$ by cell modules. Then the multiplicities of cell modules occurring in any cell filtration $\mathcal{F}'$ of M are the same as those which occur in $\mathcal{F}$.

Proof. This follows from Theorem 5.54 $\square$

Chapter 6

Restriction of Ramified Cell Modules to the Partition Algebra

We work over the field $k = \mathbb{C}$ of complex numbers throughout this section. Assume $\delta^{\text{in}}\delta^{\text{out}} \neq 0$ and take $\delta = \delta^{\text{in}}\delta^{\text{out}}$ in this section. We freely abbreviate $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ to RP_r, and $P_r(\delta)$ to just P_r.

Recall that $P_r \subset RP_r$ via the diagonal map sending d to (d, d), so that S_r acts as usual, A^1 is sent to (A^1, A^1) and A^{12} is sent to (A^{12}, A^{12}).

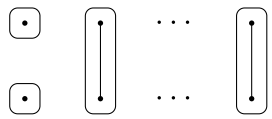


FIGURE 6.1: The generator A^1 is mapped to (A^1, A^1).

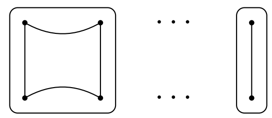
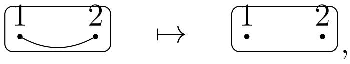


FIGURE 6.2: The generator A^{12} is mapped to (A^{12}, A^{12}).

We will be considering the restriction of cell modules for RP_r to P_r, looking only at *diagrammatic cell modules* V_λ. Morally, this restricted action is like that of the ramified partition algebra, but outer links cannot be made without also making inner links, and inner links cannot be broken without also breaking any outer links.

For example, there is no way to transform



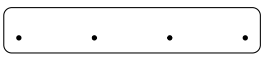
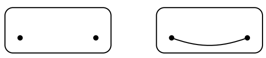
using partition algebra elements, since one cannot separate 1 from 2 in the inner part without also separating them in the outer part. We will freely write, for example, $A^1\mathbf{t}$ to represent the partition algebra action of A^1 via the ramified diagram (A^1, A^1) throughout.

6.1 The ℓ-filtration

In their paper [2], Bowman and Paget describe a filtration of the diagrammatic cell module $V_\emptyset \downarrow_{P_r}^{RP_r}$. They refer to $V_\emptyset \downarrow_{P_r}^{RP_r}$ as the "diagrammatic stable Foulkes module" $\mathbb{F}^r$. We summarise some of their results here in our own notation.

Definition 6.1. [2, §7.1] When t is a set partition, define $\ell(t)$ to be the number of parts of t. Given a ramified half-diagram $\mathbf{t} = (t^{\text{in}}, t^{\text{out}})$ in $V_\emptyset \downarrow_{P_r}^{RP_r}$, define $\ell(\mathbf{t}) = \ell(t^{\text{in}}) - \ell(t^{\text{out}})$.

Example 6.2. *Consider the half-diagrams* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *on* 4 *dots pictured below.*

$\mathbf{t}_1 =$  $\mathbf{t}_2 =$ .

The half-diagram $\mathbf{t}_1$ *has one outer part and four inner parts, and hence* $\ell(\mathbf{t}_1) = 4 - 1 = 3$. *The half-diagram* $\mathbf{t}_2$ *has two outer parts and three inner parts, so* $\ell(\mathbf{t}_2) = 3 - 2 = 1$.

Remark 6.3. Note that $\ell(\mathbf{t}) = 0$ means that there are the same number of inner and outer parts, and therefore every part is a "diagonal part" where a bubble is drawn around an inner part. We have already defined the map diag : $d \mapsto (d, d)$ for partition diagrams. We define a similar map on half-diagrams in the obvious way by setting $\text{diag}(t) = (t, t)$.

Theorem 6.4. *[2, Thm 7.2] Write M for the P_r-module $V_\emptyset \downarrow^{RP_r}_{P_r}$, and define*

$$M_i = \mathrm{span}\{\mathbf{t} \in M | \mathbf{t} \textit{ is a ramified half-diagram and } \ell(\mathbf{t}) \leq i\}.$$

Then M has a filtration

$$0 \subset M_0 \subset M_1 \subset \cdots \subset M_{r-1} = M$$

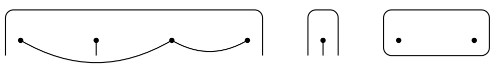
and the matrices representing the generators of P_r acting on M_i/M_{i-1} have only $0, 1$ and $\delta^{\mathrm{in}}\delta^{\mathrm{out}}$ as their entries for every i. Furthermore, the decomposition of M into simple modules depends only on $\delta^{\mathrm{in}}\delta^{\mathrm{out}}$, and not on the individual values of δ^{in} and δ^{out} separately.

Our first objective is to extend this idea of an ℓ-filtration to general diagrammatic cell module V_λ where $\lambda \in \Lambda_r^{prop}$.

Definition 6.5. Write $M = V_\lambda \downarrow^{RP_r}_{P_r}$ for the module restricted to the partition algebra. For a half-diagram $\mathbf{t} = (t^{\mathrm{in}}, t^{\mathrm{out}}) \in M$, define $\ell(\mathbf{t}) = \ell(t^{\mathrm{in}}) - \ell(t^{\mathrm{out}})$. Furthermore, write $\ell(x\mathbf{t}) = \ell(\mathbf{t})$ when $x \in \mathbb{C} \setminus \{0\}$.

Remark 6.6. Note that no attention needs to be paid to which parts of $\mathbf{t}$ are propagating or not when calculating $\ell(\mathbf{t})$, this quantity depends entirely on the underlying set partitions.

Example 6.7. *The diagram $\mathbf{t}$ below has $\ell(t^{\mathrm{in}}) = 5$ and $\ell(t^{\mathrm{out}}) = 3$, so $\ell(\mathbf{t}) = 2$.*

$\mathbf{t} =$ 

Since by definition t^{in} is a refinement of t^{out}, it must have at least as many parts and thus $\ell(\mathbf{t}) \geq 0$ for all $\mathbf{t}$. Multiplication by diagrams in the partition algebra cannot increase the value of ℓ, as proven in the following lemma.

Lemma 6.8. *Suppose that d is a partition algebra diagram, and that $\mathbf{t}$ is some half-diagram in V_λ. Then either $d\mathbf{t} = 0$ or $\ell(d\mathbf{t}) \leq \ell(\mathbf{t})$*

Proof. The proof is similar to part of the proof of Theorem 7.2 in [2], however the presence of propagating parts requires some adaptation, and we give some more explanation of the different cases here. Write $M = V_\lambda$.

Elements $\tau \in S_r$ do not change the number of outer or inner parts, they just permute the dots between different parts. Therefore $\ell(\tau\mathbf{t}) = \ell(\mathbf{t})$ for all $\tau \in S_r$. The proof then reduces to observing the action of A^1 and A^{12} on the value of $\ell(\mathbf{t})$, which only depends on how the dots 1 and 2 are connected to each other and to other dots.

Consider the action of A^1 on a ramified half-diagram $\mathbf{t} = (t^{\text{in}}, t^{\text{out}})$. When t^{in} is a half-diagram, write $\tilde{t}^{\text{in}}$ for the half-diagram agreeing with t^{in} with the exception that 1 is made a non-propagating singleton, so that either $t^{\text{in}} = \tilde{t}^{\text{in}}$ or $\ell(\tilde{t}^{\text{in}}) = \ell(t^{\text{in}}) + 1$, with the same notation for t^{out}. Various situations are possible:

- We have $\{1\}$ as a propagating singleton in t^{in} or t^{out}. In this case, $A^1\mathbf{t} = 0$ in M since the propagating index will be decreased by multiplication.
- The singleton $\{1\}$ appears in both t^{in} and t^{out} but is not propagating. In this case, $A^1(t^{\text{in}}, t^{\text{out}}) = \delta^{\text{in}}\delta^{\text{out}}(t^{\text{in}}, t^{\text{out}})$ and so $\ell(A^1\mathbf{t}) = \ell(\mathbf{t})$.
- The dot 1 is a non-propagating singleton in t^{in}, but not singleton in t^{out}. In this case, $A^1(t^{\text{in}}, t^{\text{out}}) = \delta^{\text{in}}(t^{\text{in}}, \tilde{t}^{\text{out}})$ and $\ell(A^1\mathbf{t}) = \ell(\mathbf{t}) - 1$.
- The dot 1 is not singleton in t^{in} or t^{out}. In this case, $A^1(t^{\text{in}}, t^{\text{out}}) = (\tilde{t}^{\text{in}}, \tilde{t}^{\text{out}})$, and $\ell(A^1\mathbf{t}) = \ell(\mathbf{t})$ since $\ell(t^{\text{out}})$ and $\ell(t^{\text{in}})$ are both increased by 1.

In all cases, either $A^1\mathbf{t} = 0$ in M or $\ell(A^1\mathbf{t}) \in \{\ell(\mathbf{t}), \ell(\mathbf{t}) - 1\}$.

When considering the action of A^{12}, we also have various different situations. Given a half-diagram t^{in}, write $\bar{t}^{\text{in}}$ for the half-diagram obtained from t^{in} by merging the parts containing 1 and 2, so that either $t^{\text{in}} = \bar{t}^{\text{in}}$ or $\ell(\bar{t}^{\text{in}}) = \ell(t^{\text{in}}) - 1$, and do similarly for t^{out}. We have the following cases:

- The dots 1 and 2 are in separate inner or outer propagating parts. In this case $A^{12}\mathbf{t} = 0$ in M, since the propagating index will be decreased by the merging of these propagating parts.

- The dots 1 and 2 are in separate inner and outer parts, where at most one of the inner and at most one of the outer parts are propagating. In this case, $A^{12}\mathbf{t} = (\bar{t}^{\text{in}}, \bar{t}^{\text{out}})$. Therefore, $\ell(A^{12}\mathbf{t}) = \ell(\mathbf{t})$ as both $\ell(t^{\text{in}})$ and $\ell(t^{\text{out}})$ are reduced by 1.

- The dots 1 and 2 are in the same outer part but different inner parts, and at most one of the inner parts is propagating. In this case, $A^{12}\mathbf{t} = (\bar{t}^{\text{in}}, t^{\text{out}})$ and $\ell(A^{12}\mathbf{t}) = \ell(\mathbf{t}) - 1$.

- The dots 1 and 2 are in the same outer and inner part. In this case, $A^{12}\mathbf{t} = \mathbf{t}$, so $\ell(A^{12}\mathbf{t}) = \ell(\mathbf{t})$.

In all cases, either $A^{12}\mathbf{t} = 0$ in M or $\ell(A^{12}\mathbf{t}) \in \{\ell(\mathbf{t}), \ell(\mathbf{t}) - 1\}$.

We conclude that either $d\mathbf{t} = 0$ or $\ell(d\mathbf{t}) \leq \ell(\mathbf{t})$ when d is a generator of P_r, and hence the result follows for arbitrary diagrams $d \in P_r$. $\square$

Write $M_j := \langle \mathbf{t} | \mathbf{t}$ a diagram with $\ell(\mathbf{t}) \leq j \rangle$, so that

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_l = M$$

is a filtration of M as a P_r-module. Here, l is the maximal possible value of $\ell(\mathbf{t})$ for the propagating index λ. We will refer to this as the ℓ-filtration.

Example 6.9. *These submodules M_j of V_λ may be zero. For example, in $V_{(2)}$ for any number of dots, the minimal value of $\ell(\mathbf{t})$ is 1, since there must always be an outer part containing two distinct inner parts, and hence $M_0 = \{0\}$.*

The next lemma will show that the individual values of δ^{in} and δ^{out} do not have any bearing on the composition factors of V_λ as a partition algebra module, even though δ^{in} may appear in the coefficients for individual multiplications. The lemma follows from the proof of Lemma 6.8 and generalises [2, Thm 7.2], which is proved in the same fashion.

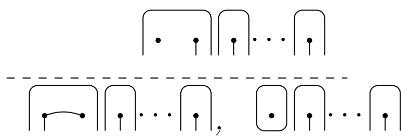
Lemma 6.10. *The composition factors of* $V_\lambda \downarrow^{RP_r}_{P_r}$*, being the restriction of the* $RP_r(\delta^{\mathrm{in}}, \delta^{\mathrm{out}})$*-module* V_λ *to* $P_r(\delta)$*, depend only on the value of* $\delta = \delta^{\mathrm{in}}\delta^{\mathrm{out}}$*, and not on the values of* δ^{in} *and* δ^{out} *individually.*

Proof. The proof of Lemma 6.8 implies that the matrices representing the action of $\tau \in S_r$, A^1 and A^{12} on subquotients of the ℓ-filtration contain only the scalars $0, 1$ and $\delta^{\mathrm{in}}\delta^{\mathrm{out}}$. Thus the composition factors of the module depend only on the value of $\delta^{\mathrm{in}}\delta^{\mathrm{out}}$. □

We will now consider some examples of this ℓ-filtration on diagrammatic cell modules, and note some irregularities which suggest that some better choice of filtration can be made.

Example 6.11. *We will first investigate the family of modules* $M = V_{(1^{r-1})} \downarrow^{RP_r}_{P_r}$ *for* P_r*. Suppose that* $\mathbf{t} \in M$ *is a ramified half-diagram. One can see that* $\ell(\mathbf{t})$ *is either* 0 *or* 1*, since there can either be* r *or* $r-1$ *outer parts, and either* r *or* $r-1$ *inner parts of* $\mathbf{t}$*. We will see that the* $\ell = 1$ *layer is an inflated symmetric group module, while the* $\ell = 0$ *layer is a diagrammatic cell module for the partition algebra.*

The ℓ*-filtration of* M *is readily understood from the following schematic, showing only one diagram from each* S_r*-orbit.*



The top layer $\ell = 1$ *contains* $r(r-1)$ *diagrams, all of which are in the* S_r*-orbit of the diagram at the top of the schematic, which we will refer to as* $\mathbf{e}$*. In the quotient* $Q_1 = M_1/M_0$*, each diagram is annihilated by* A^1 *and* A^{12} *and thus* Q_1 *is an inflated* S_r*-module. On closer inspection, this module can be seen to be* $\mathrm{Inf}^{P_r}_{S_r}(M^{(r-2,1^2)})$ *by* *the following easy isomorphism. Suppose* $\mathbf{t}$ *is the half-diagram in the orbit of* $\mathbf{e}$ *with* i *and* j *in the same outer propagating part, such that* i *is inner propagating but* j *is not. Then map* $\mathbf{t}$ *to the* $(r-2,1^2)$*-tabloid*

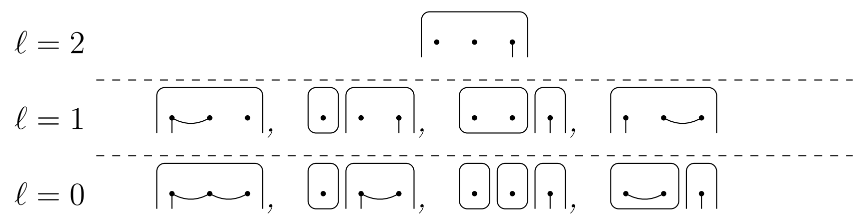
$$\begin{array}{l} \hline 1 \ \cdots \ \hat{i} \ \ \hat{j} \ \cdots \ r \\ \hline i \\ \cline{1-1} j \\ \cline{1-1} \end{array},$$

where the symbol $\hat{i}$ *shows that* i *has been omitted from that row. By comparing the permutation action of* S_r *on tabloids and half-diagrams, one sees that this map yields a isomorphism of* kS_r*-modules.*

Now, turn to the bottom layer $\ell = 0$*. Note that the ramified half-diagrams in the bottom layer are diagonalisations of partition half-diagrams of propagating number* $r-1$*, and in fact* $M_0 = \mathrm{diag}(W_{r-1})$.

Thus the cell modules appearing in the decomposition of M *are* $W_{r-1} \cong W_{r-1} \oslash S^{(r-1)}$ *with multiplicity* 1 *and* $W_r \oslash S^\lambda$ *for* $\lambda \vdash r$ *with multiplicity* $|SSYT(\lambda,(r-2,1^2))|$ *by Young's Rule.*

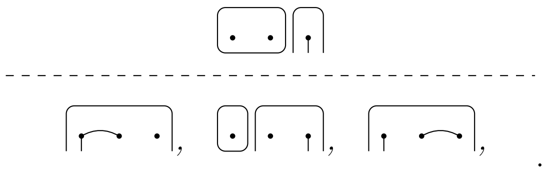
Example 6.12. *We now consider the module* $M = V_{(1)}$ *for* 3 *dots, where we have* $r - \mathrm{env}((1)) = 2$ *"spare dots", and therefore we expect layers for* $\ell = 0,1,2$*. We shall quickly see that the top and bottom layers are not challenging to deal with, however the middle layer* $\ell = 1$ *requires some additional thought. As before, we show the structure in a diagram, using only one representative from each* S_3*-orbit.*



Using a similar method to the previous example, one may see that $Q_1 = M_2/M_1 \cong W_3 \oslash M^{(2,1)} = \mathrm{Inf}_{S_r}^{P_r}(M^{(2,1)})$*. Similarly, one may quickly observe that* $M_0 = \mathrm{diag}(W_1)$ *where* W_1 *is the diagrammatic cell module of propagating number* 1 *for*

P_3. *Thus we only need to consider the middle layer* M_1/M_0. *Note that this cannot be an inflated symmetric group module - for example, several diagrams are not in the kernel of the* A^{12}*-action.*

One can find a further P_3*-submodule structure within the module* M_1/M_0 *as in the following schematic, showing one diagram from each* S_3*-orbit:*



The top quotient in this structure is easily identified as $\mathrm{Inf}_{S_3}^{P_3}(M^{(21)})$ *by similar methods as we have already used. Write* N *for the lower layer spanned by the* S_3*-orbits of the diagrams below the dotted line - one can check that these orbits are closed under the* P_3*-action.*

We will now express N *as a direct sum of* $P_3(\delta^{\mathrm{in}}\delta^{\mathrm{out}})$*-module. We will define submodules* N_1 *and* N_2*, observe that they are direct summands and identify them as cell modules for* P_3.

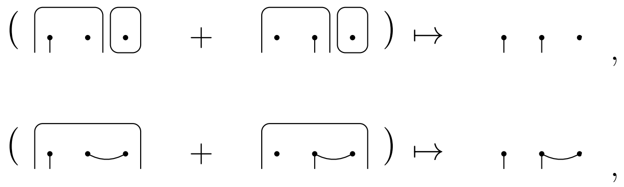
The module N_1 *is generated under the* P_r*-action by*

The rest of the six element basis of N_1 *also consists of sums of two diagrams. In particular each diagram is in a pair with another diagram which is the same apart from the inner propagating part is no longer propagating, and the other inner part in its outer part is now inner propagating. For example, another element is*

The module N_2 has a very similar basis, one just has to change addition in the expression for subtraction, so that the module is generated by

One may observe that each half-diagram is a sum or difference of a generator of N_1 and a generator of N_2, and thus $N = N_1 + N_2$. A dimension argument then shows that $N = N_1 \oplus N_2$.

We can show $N_1 \cong W_2$ by declaring



and observing that this extends to a P_3-homomorphism by checking the action of generators. The injectivity of this function is clear from its diagrammatic nature, and since both source and target are 6-dimensional, it must be an isomorphism. From here it becomes very clear that $N_2 \cong W_2 \oslash S^{(1^2)}$.

Therefore, our module N is isomorphic to

$$(W_2 \oslash S^{(2)}) \oplus (W_2 \oslash S^{(1^2)}).$$

Indeed, $S^{(2)} \oplus S^{(1^2)}$ is the decomposition of the Young permutation module $M^{(1^2)}$, so our module is isomorphic to $W_2 \oslash M^{(1^2)}$.

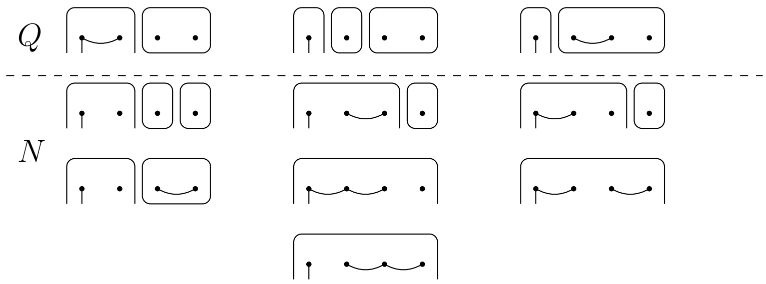
In summary, we have found that the quotients in the ℓ-filtration of M include two copies of $W_3 \oslash M^{(2,1)}$, one copy of $W_2 \oslash M^{(1^2)}$, and one copy of W_1.

This example suggests that the ℓ-filtration may not be the correct one to use, since the layer M_1/M_0 in the previous example had cell modules of two different propagating numbers, 2 and 3, inside.

Example 6.13. *We partially investigate the module $V_{(1)}$ for 4 dots. The module $V_{(1)}$ for P_4 is of large dimension, with over 20 types of diagrams, and therefore we will not examine it in full detail.*

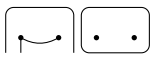
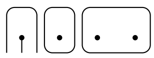
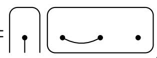
We can use our methods from the previous two examples to analyse the layers $\ell = 3, 2, 0$, but we will not describe how to do so here.

We focus on the $\ell = 1$ layer in this example, where one can find a P_4-submodule N of M_1/M_0 with 7 types of diagrams. We call the quotient of this layer Q, and note that it consists of 3 types of diagrams. A schematic of the $\ell = 1$ layer is directly below, with one diagram displayed from each S_4-orbit.



Upon use of similar methods to the previous example it is easy to see that $N \cong W_2 \oslash M^{(1^2)}$. The upper layer Q has something more interesting about it, and thus we should use more explicit detail in this case.

The representatives of the S_4 orbits of diagrams appearing in Q are:

$\mathbf{e} =$  $\mathbf{f} =$  $\mathbf{g} =$ ,

with 6 diagrams in the orbit of e and 12 each in the orbits of f and g. This is a 30-dimensional module for P_4, so let us try our previous ideas in identifying it as a cell module or otherwise.

Focusing more on our first example diagram, let us examine the dots 3 and 4. Applying A^3 will cut this part into two, and therefore annihilate the diagram due to the ℓ value being decreased. In the quotient by lower ℓ layers, $A^4 e$ is also equal *to* 0 *by the same logic. Similarly, applying* A^{34} *will combine the two inner parts and reduce the* ℓ *value, meaning* $A^{34}e = 0$ *in the quotient.*

We can identify this module as $W_3 \oslash M^{(2,1)}$ *with the map given by*

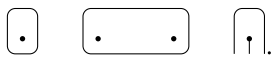
$e \mapsto$ 

$f \mapsto$ 

$g \mapsto$ 

One checks that this is indeed a P_4*-homomorphism by checking the action of generators, similar to the discussion in the previous paragraph. The general formula is to make inner nodes propagating unless they are in a non-propagating outer singleton, and then delete the outer parts, attaching a tableaux to record which inner propagating parts were contained in the same outer part, and which was by itself in an outer part. For example,*

is the image of the diagram



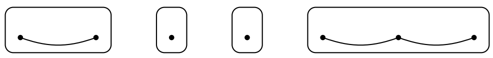
We will not summarise the module structure in order to save space, as these examples serve to show some limitations of the ℓ*-filtration and give some motivation for the* p^**-filtration, which we will now define. The interesting thing to take away from this last example is that effectively the partition algebra 'sees' two non-propagating inner parts which are in the same non-propagating outer parts as propagating in this context.*

6.2 The p^*-filtration

In our study of $V_{(1)}$ for 3 and 4 dots, we saw that modules of different propagating numbers are mixed together in the same ℓ-layer. This suggests that the ℓ-filtration may not be the correct one to decompose the module. We saw in our example that within layers, the partition algebra 'sees' any inner part within an outer propagating part as propagating itself. Similarly, in an outer part containing more than one inner part, the partition algebra 'sees' all inner parts as propagating. Therefore, the only inner parts that are 'non-propagating' according to the partition algebra are those inner parts which are the only part in their non-propagating outer part.

We will define a new filtration of V_λ based on this logic, the *p^*-filtration.*

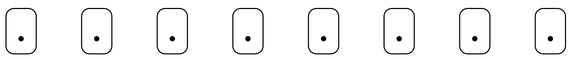
Notation. We will refer to inner parts which are also outer parts as *diagonal parts*, regardless of whether or not they are propagating. For example,

$\mathbf{t} =$ 

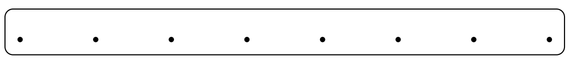
consists entirely of diagonal parts.

Definition 6.14. Let $\mathbf{t} = (t^{\text{in}}, t^{\text{out}})$ be a ramified half-diagram. Define $p^*(\mathbf{t})$ to be the number of parts of t^{in}, minus the number of non-propagating diagonal parts of $\mathbf{t}$. This is equal to the number of parts of t^{in} which are non-diagonal or within an outer propagating part. If x is a nonzero scalar, we also write $p^*(x\mathbf{t}) = p^*(\mathbf{t})$.

Example 6.15. *The diagram*

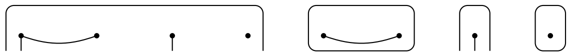
$\mathbf{t}_1 =$ 

has all parts non-propagating diagonal, and therefore has $p^*(\mathbf{t}_1) = 0$. On the other extreme, the diagram

$\mathbf{t}_2 =$ 

has $p^*(\mathbf{t}_2) = 8$*, even though none of the dots are inner or outer propagating in the sense of the ramified partition algebra.*

The diagram

$\mathbf{t}_3 =$ 

has $p^*(t_3) = 4$*.*

Remark 6.16. The p^*- and ℓ-filtrations are not the same. For example the two diagrams in $V_\emptyset$ for 4 dots pictured below have different $\ell(\mathbf{t})$ but the same $p^*(\mathbf{t})$.

Lemma 6.17. *Suppose that* d *is a partition diagram, and* $\mathbf{t}$ *is a ramified half-diagram in* V_λ*. Then either* $d\mathbf{t} = 0$ *in* V_λ *or* $p^*(d\mathbf{t}) \leq p^*(\mathbf{t})$*.*

Proof. To prove this, we need only look at the cases where d is a symmetric group element, or one of the generators A^1 and A^{12}.

It is easy to see that $p^*(\tau\mathbf{t}) = p^*(\mathbf{t})$ when $\tau \in S_r$, since parts are not meaningfully changed apart from having their elements relabelled.

When $\mathbf{d} = A^1$, several cases occur:

- The dot 1 is an inner or outer propagating singleton, in which case $A^1\mathbf{t} = 0$.
- The dot 1 is a non-propagating inner and outer singleton, in which case $A^1\mathbf{t} = \delta^{\text{in}}\delta^{\text{out}}\mathbf{t}$ and $p^*(A^1\mathbf{t}) = p^*(\mathbf{t})$.
- The dot 1 is in a non-singleton inner part. In this case, the inner part containing 1 splits into two, and so $A^1\mathbf{t}$ will have one more inner part than $\mathbf{t}$. One of these two parts will be the diagonal singleton $\{1\}$, so $A^1\mathbf{t}$ will have one more diagonal part than $\mathbf{t}$. Putting this all together we see that in this case $p^*(A^1\mathbf{t}) = p^*(\mathbf{t})$.

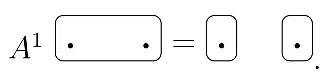
- The dot 1 is in a non-propagating singleton inner part, which is in a non-singleton outer part. In this case, $A^1\mathbf{t}$ has the same number of inner parts as $\mathbf{t}$. In the resulting diagram, $\{1\}$ will become a non-propagating diagonal part, therefore the number of non-propagating diagonal parts increases by at least one. Therefore $p^*(A^1\mathbf{t}) \leq p^*(\mathbf{t})$.

Suppose that $d = A^{12}$.

- If dots 1 and 2 are in different inner propagating parts, or different outer propagating parts, then $A^{12}\mathbf{t} = 0$ since the propagating index is reduced.
- If dots 1 and 2 are in the same inner part, then $A^{12}\mathbf{t} = \mathbf{t}$ and $p^*(A^{12}\mathbf{t}) = p^*(\mathbf{t})$.
- Suppose dots 1 and 2 are in different inner parts, and not in either of the cases above. Then certainly $A^{12}\mathbf{t}$ has exactly one less inner part than $\mathbf{t}$. There are two subcases:
  - If neither of the inner parts containing 1 or 2 are non-propagating diagonal, then merging them also cannot be non-propagating diagonal, and the number of non-propagating diagonal parts is the same in $\mathbf{t}$ and $A^{12}\mathbf{t}$. In this case, $p^*(A^{12}\mathbf{t}) = p^*(\mathbf{t}) - 1$.
  - If one or both of the parts containing 1 or 2 are non-propagating diagonal, then either two such parts are merged into one, or one non-propagating diagonal is connected with something else, and the result is no longer a non-propagating diagonal part. Thus the number of non-propagating diagonal parts is reduced by one. In this case $p^*(A^{12}\mathbf{t}) = p^*(\mathbf{t})$.

In all cases, either $d\mathbf{t} = 0$ or $p^*(d\mathbf{t}) \leq p^*(\mathbf{t})$ when d is a generator, and thus the same is true when d is any diagram. □

Example 6.18. *Note that p^* may be decreased by more more than 1 at a time. For example, consider the multiplication:*



This multiplication takes a diagram with $p^* = 2$ and results directly in a diagram where $p^* = 0$.

6.3 The Depth Radical

Throughout this section we set $\delta = \delta^{\text{in}}\delta^{\text{out}} \neq 0$. We abbreviate $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$ to RP_r and $P_r(\delta)$ to P_r.

Notation. In this section we must take great care to acknowledge the number r of dots in our partition algebra $P_r(\delta)$. We will write $V_\lambda^{[r]}$ for the module labelled V_λ for $RP_r(\delta)$. For example, $V_\emptyset^{[2]}$ and $V_\emptyset^{[4]}$ are different modules for different algebras, though both are naturally labelled by the tail partition $\emptyset$.

We label the simple module for $P_r(\delta)$ corresponding to the partition ν by $L^{[r]}(\nu)$. In the semisimple case this is just $W_i \oslash S^\nu$.

In [2, §7.2], Bowman and Paget use the concept of a *depth radical* to decompose the P_r-module $V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}$. They present the following definitions and results.

Definition 6.19. [2, Def 7.5] Define the depth radical of $V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}$ to be the subspace of $V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}$ spanned by those ramified diagrams $(t^{\text{in}}, t^{\text{out}})$ such that either

- The diagram t^{in} contains a non-singleton part;
- The diagram t^{out} contains a singleton part.

Write $DR(V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r})$ for the depth radical of $V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}$.

Lemma 6.20. *[2, Prop 7.7] The depth radical $DR(V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r})$ is a P_r-submodule of $V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}$.*

Definition 6.21. [2, Def 7.8] Define

$$DQ(V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}) := V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r} / DR(V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}),$$

the *depth quotient* of $V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}$.

Example 6.22. *[2, Ex 7.6] The module $V_\emptyset^{[4]} \downarrow_{P_4}^{RP_4}$ is a 60-dimensional module. The depth radical is 56-dimensional including all diagrams apart from the S_4-orbits of the following two:*

.

The depth quotient is therefore 4-dimensional, and is just the inflated S_4-permutation module consisting of the orbits above. One can verify that this module decomposes as $S^{(4)} \oplus H^{(2^2)}$, where $H^{(2^2)}$ is the Foulkes module as described in §2.3. One may look at the plethysm $s_{(2)} \circ s_{(2)} = s_{(4)} + s_{(2,2)}$ to see that $H^{(2^2)}$ decomposes as $S^{(4)} \oplus S^{(2,2)}$, and thus we may see that our depth quotient decomposes as $(W_4 \oslash S^{(4)}) \oplus (W_4 \oslash S^{(4)}) \oplus (W_4 \oslash S^{(2,2)})$.

The module $DQ(V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r})$ is always an inflated symmetric group module, since it is annihilated by each A^i and $A^{i,j}$. One can decompose $V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}$ by considering the depth quotient only, and then reducing the number of dots to look at $V_\emptyset^{[r-1]} \downarrow_{P_{r-1}}^{RP_{r-1}}$, proceeding inductively. This is put precisely in the following main theorem of Bowman and Paget.

Theorem 6.23. *[2, Cor 7.10] Write $V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}$. In the case where $P_r(\delta)$ is semisimple,*

$$[V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r} : L^{[r]}(\nu)]_{P_r} = \begin{cases} [DQ(V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}) : L^{[r]}(\nu)]_{P_r} & \textit{if } |\nu| = r, \\ [V_\emptyset^{[r-1]} \downarrow_{P_{r-1}}^{RP_{r-1}} : L^{[r-1]}(\nu)]_{P_{r-1}} & \textit{if } |\nu| < r. \end{cases}$$

The decomposition of $V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}$ can therefore be determined by decomposing $DQ(V_\emptyset^{[r]} \downarrow_{P_r}^{RP_r}), DQ(V_\emptyset^{[r-1]} \downarrow_{P_{r-1}}^{RP_{r-1}}), \cdots, DQ(V_\emptyset^{[1]} \downarrow_{P_1}^{RP_1})$ as modules for $P_r(\delta)$, $P_{r-1}(\delta), \ldots, P_1(\delta)$ respectively. In fact, since each $DQ(V_\emptyset^{[s]} \downarrow_{P_s}^{RP_s})$ is an inflated kS_s module, we need only decompose modules for the symmetric group (though as mentioned before, this is often not easy).

This concludes our exposition on the depth radical as defined and studied in [2], which was defined only in the case where the propagating index was $\emptyset$. We now define the depth radical of $V_\lambda^{[r]} \downarrow_{P_r}^{RP_r}$ for any propagating index $\lambda \in \Lambda_r^{\text{prop}}$. For the rest of this section, label $M^{[r]} := V_\lambda^{[r]} \downarrow_{P_r}^{RP_r}$.

Definition 6.24. Define $DR(M^{[r]})$ to be the subspace of $M^{[r]}$ spanned by those diagrams $\mathbf{t} = (t^{\text{in}}, t^{\text{out}})$ such that either

- The diagram t^{in} has a non-singleton part.
- The diagram $\mathbf{t}$ has a non-propagating diagonal part.

Upon closer inspection, one sees that diagrams $\mathbf{t}$ in $DR(M^{[r]})$ are exactly those with $p^*(\mathbf{t})$ less than maximal. From this and Lemma 6.17, we may immediately observe that the depth radical is always a submodule.

Lemma 6.25. *The subspace $DR(M^{[r]})$ is a P_r-submodule of $M^{[r]}$.*

Let us return to Definition 6.19 and show consistency when $\lambda = \emptyset$. Write $M^{[r]} = V_\lambda^{[r]}$. In this case, there are no propagating parts, so diagrams in $DR(M^{[r]})$ have either a non-singleton inner part or a diagonal part. If all inner parts are singletons, then the existence of a diagonal part is equivalent to there being a singleton outer part. Therefore, diagrams in the depth radical $DR(M^{[r]})$ either have an inner non-singleton or an outer singleton. This matches exactly with our previous notion of depth radical in the $\lambda = \emptyset$ case.

Definition 6.26. Write $DQ(M^{[r]})$ for the quotient P_r-module $M^{[r]}/DR(M^{[r]})$, the *depth quotient* of $M^{[r]}$.

Remark 6.27. The module $DQ(M^{[r]})$ has basis those half diagrams such that all inner parts are singletons and there are no non-propagating outer singletons.

We now use this generalised concept of a depth radical to decompose the module $M^{[r]} = V_\lambda^{[r]} \downarrow_{P_r}^{RP_r}$ by considering the modules indexed by λ for smaller numbers of dots, as Bowman and Paget did for $\lambda = \emptyset$ in [2]. We begin with the following lemma, which is simply [2, Prop 7.9] in a more general setting.

Lemma 6.28. *Fix the idempotent* $e = \frac{1}{\delta}A^1$ *in* P_r *and write* $M^{[s]} := 0$ *if* $s < \mathrm{env}(\lambda)$. *Then the following three facts hold:*

- $P_r eDR(M^{[r]}) = DR(M^{[r]})$,
- $eDQ(M^{[r]}) = 0$,
- $eDR(M^{[r]}) \cong M^{[r-1]}$ *as a* $P_{r-1} \cong eP_re$*-module.*

Proof. We now follow essentially the same proof as [2, Prop 7.9].

It is clear that $P_r eDR(M^{[r]}) \subseteq DR(M^{[r]})$. For the reverse inclusion, suppose that $\mathbf{t} \in DR(M^{[r]})$. There are two cases.

- Suppose that $\mathbf{t}$ has a non-singleton inner part. This must contain at least two dots, call them i and j. We can then see that

  $$\mathbf{t} = (1,i)(2,j)A^{12}e(1,i)(2,j)\mathbf{t},$$

  with $(1,i)(2,j)A^{12} \in P_r$ and $(1,i)(2,j)\mathbf{t} \in DR(M^{[r]})$.

- Suppose that $\mathbf{t}$ has a non-propagating diagonal part. If this is non-singleton, we can reduce to the first case, so assume that $\mathbf{t}$ has a non-propagating diagonal singleton at dot i. In this case

  $$\mathbf{t} = (1,i)e(1,i)\mathbf{t},$$

  with $(1,i) \in P_r$ and $(1,i)\mathbf{t} \in DR(M^{[r]})$.

In either case, we see that $\mathbf{t} \in P_r eD_r(M^{[r]})$ and thus $P_r eDR(M^{[r]}) = M^{[r]}$

For the second statement, suppose $\mathbf{t} \in M^{[r]}$ is a diagram representing a nonzero element of $DQ(M^{[r]})$. Then $e\mathbf{t}$ must either be zero in $M^{[r]}$ due to removal of a propagating part, or have non-propagating singleton $\{1\}$ and hence in either case $e\mathbf{t} \in DR(M^{[r]})$. Therefore $eDQ(M^{[r]}) = 0$.

If $r = \text{env}(\lambda)$, then multiplying a diagram by e will always destroy a propagating part, annihilating the diagram. Therefore in this case $eDR(M^{[r]}) = 0$. If $r > \text{env}(\lambda)$, the module $eDR(M^{[r]})$ has basis exactly those diagrams in $M^{[r]}$ with a non-propagating diagonal singleton $\{1\}$. The isomorphism $eDQ \to M^{[r-1]}$ is given by the map taking a diagram $\mathbf{t}$ on r dots to the diagram on $r-1$ dots given by restricting $\mathbf{t}$ to the dots $2, \cdots, r$. $\square$

Recall from Lemma 5.31 [19, 6.2b] that if e is a nonzero idempotent in A and M is a simple A-module, then eM is either zero or a simple eAe-module. We arrive at a generalisation of [2, Cor 7.10] which is proved in the same way, but we give a slightly more explicit proof than is written in [2].

Theorem 6.29. *Write $M^{[r]} = V_\lambda^{[r]} \downarrow_{P_r}^{RP_r}$. In the case where P_r is semisimple, the following equality of partition algebra composition multiplicities holds:*

$$[M^{[r]} : L^{[r]}(\nu)]_{P_r} = \begin{cases} [DQ(M^{[r]}) : L^{[r]}(\nu)]_{P_r} & \textit{if } |\nu| = r, \\ [M^{[r-1]} : L^{[r-1]}(\nu)]_{P_{r-1}} & \textit{if } |\nu| < r. \end{cases}$$

Proof. We know that $eDQ(M^{[r]}) = 0$, and thus every composition factor of $DQ(M^{[r]})$ must be annihilated by $e = \frac{1}{\delta^{\text{in}}\delta^{\text{out}}} A^1$. The simple modules, in this case the cell modules, for P_r which are annihilated by e are precisely those with corresponding partitions ν such that $|\nu| = r$. Furthermore, $P_r eDR(M^{[r]}) = DR(M^{[r]})$, so no composition factor of $DR(M^{[r]})$ is annihilate by e. Thus

$$[M^{[r]} : L^{[r]}(\nu)] = \begin{cases} [DQ(M^{[r]}) : L^{[r]}(\nu)] & \text{if } |\nu| = r, \\ [DR(M^{[r]}) : L^{[r]}(\nu)] & \text{if } |\nu| < r, \end{cases}$$

where all composition multiplicites are over P_r. Now, when $|\nu| < r$, $eL^{[r]}(\nu)$ is nonzero and thus simple, so

$$[DR(M^{[r]}) : L^{[r]}(\nu)]_{P_r}$$

is equal to

$$[eDR(M^{[r]}) : eL^{[r]}(\nu)]_{P_r}.$$

However, $eDR(M^{[r]}) \cong M^{[r-1]}$ and $eL^{[r]}(\nu) \cong L^{[r-1]}(\nu)$ as $eP_re \cong P_{r-1}$-modules, so

$$[eDR(M^{[r]}) : eL^{[r]}(\nu)]_{P_r} = [V_\lambda^{[r-1]} : L^{[r-1]}(\nu)]_{P_{r-1}}$$

as required. □

Now, it is clear to us that in order to understand the composition factors of the module $M^{[r]} = V_\lambda^{[r]} \downarrow_{P_s}^{RP_s}$, we may just look at the depth quotients

$$DQ(M^{[r]}), DQ(M^{[r-1]}), \ldots, DQ(M^{[\mathrm{env}(\lambda)]}).$$

The following lemma shows that $DQ(M^{[s]})$ is a permutation module for kS_s.

Lemma 6.30. *Let M be the module $V_\lambda^{[s]} \downarrow_{P_s}^{RP_s}$ for P_s, where $\lambda \in \Lambda_s^{\mathrm{prop}}$. Then $DQ(M)$ is an inflation of a permutation module for the group algebra kS_s.*

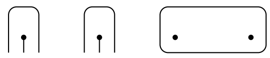
Proof. Let $\mathbf{t}$ be a half-diagram which is not zero in $DQ(M)$, so $\mathbf{t}$ has s singleton inner parts and no non-propagating diagonal parts. The diagram $A^i\mathbf{t}$ has a non-propagating diagonal singleton at i, so $A^i\mathbf{t} \in DR(M)$ and is equal to 0 in the quotient. The diagram $A^{i,j}\mathbf{t}$ has a nonsingleton inner part containing i and j, and hence $A^{i,j}\mathbf{t} \in DR(M)$ is 0 in the quotient.

The fact that A^i and $A^{i,j}$ both annihilate $DQ(M)$ tells us that it is an inflated module for the symmetric group S_s. The group S_s permutes the ramified half-diagrams in $DQ(M)$ and thus we do indeed have a permutation module. □

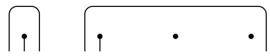
We pause to consider some explicit examples, starting with small examples for a fixed number of dots.

Example 6.31. *We can partially decompose the P_4-module $V_{(1^2)}^{[4]} \downarrow_{P_4}^{RP_4}$. Consider the module $DQ(V_{(1^2)}^{[4]} \downarrow_{P_4}^{RP_4})$ for P_4. There are three orbits, giving rise to modules isomorphic to $M^{(2,2)}, M^{(2,1^2)}$ and H for kS_4, where we describe H in a moment.*

Representatives for the first two, together with a tabloid giving showing the isomorphism, are:

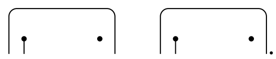


$$\mapsto \begin{array}{cc} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$



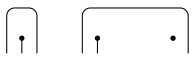
$$\mapsto \begin{array}{cc} \hline 3 & 4 \\ \hline 1 & \\ \hline 2 & \\ \hline \end{array},$$

where tabloids are obtained by putting the number of a dot on the same line as others from which it is indistinguishable in terms of inner and outer connections to other dots. The final orbit representative is



One can see that this clearly has stabiliser $(S_1 \times S_1) \wr S_2$*, and thus the module* H *is the trivial module induced from this group to* S_4*.*

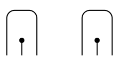
The basis for $DQ(V^{[3]}_{(1^2)} \downarrow^{RP_3}_{P_3})$ *consists of the* S_3 *orbit of*



$$\mapsto \begin{array}{c} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array},$$

which gives rise to the kS_3*-module* $M^{(1^3)}$*.*

Finally, $DQ(V^{[2]}_{(1^2)} \downarrow^{RP_2}_{P_2})$ *has basis the single diagram*

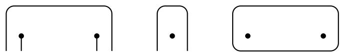


$$\mapsto \begin{array}{cc} \hline 1 & 2 \\ \hline \end{array},$$

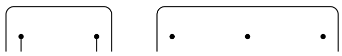
giving rise to a copy of the kS_2*-module* $M^{(2)}$*.*

Thus one can deduce that the simple constituents of $V^{[4]}_{(1^2)} \downarrow^{RP_4}_{P_4}$ *are the simple constituents of* $W_4 \oslash M^{(2,2)}$*,* $W_4 \oslash M^{(2,1^2)}$*,* $W_4 \oslash H$*,* $W_3 \oslash M^{(1^3)}$ *and* $W_2 \oslash M^{(2)}$*.*

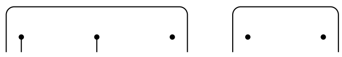
Example 6.32. *We decompose the P_5-module $V^{[5]}_{(2,0)} \downarrow^{RP_5}_{P_5}$. Consider the module $DQ(V^{[5]}_{(2,0)} \downarrow^{RP_5}_{P_5})$ for P_5. The S_5-orbits forming a basis are those of the following four diagrams, where we give a tabloid for each to be mapped to where dot numbers are grouped when their dots are identical in terms of inner and outer connection to others:*



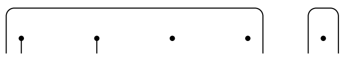
$$\mapsto \begin{array}{cc}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline 3 & \\ \cline{1-1}\end{array}$$



$$\mapsto \begin{array}{ccc}\hline 3 & 4 & 5 \\ \hline 1 & 2 & \\ \cline{1-2}\end{array}$$



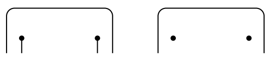
$$\mapsto \begin{array}{cc}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline 3 & \\ \cline{1-1}\end{array}$$



$$\mapsto \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \cline{1-1}\end{array}$$

These orbits give rise to kS_5-modules $M^{(2^2,1)}$, $M^{(3,2)}$, $M^{(2^2,1)}$ and $M^{(2^2,1)}$ respectively.

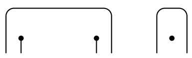
The module $DQ(V^{[4]}_{(2,0)} \downarrow^{RP_4}_{P_4})$ has basis the S_4-orbits of the following two diagrams:



$$\mapsto \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$$

$$\mapsto \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} 4 & \\ \cline{1-1}\end{array}$$

which give rise to the kS_4-modules $M^{(2^2)}$ and $M^{(2,1^2)}$ respectively. Finally, the module $DQ(V^{[3]}_{(2,0)} \downarrow^{RP_3}_{P_3})$ has basis the S_3-orbit of



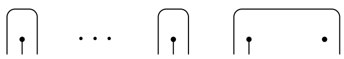
$$\mapsto \begin{array}{ll} 1 & 2 \\ 3 & \end{array},$$

giving rise to the kS_3-module $M^{(2,1)}$.

Thus we see that the simple constituents of $V^{[5]}_{(2,0)} \downarrow^{RP_5}_{P_5}$ are the same as those of three copies of $W_5 \oslash M^{(2^2,1)}$, as well as a copy each of $W_5 \oslash M^{(3,2)}$, $W_4 \oslash M^{(2^2)}$, $W_4 \oslash M^{(2,1^2)}$ and $W_3 \oslash M^{(2,1)}$.

We give some examples of easier infinite families of modules.

Example 6.33. *We look at the P_r-module $V^{[r]}_{(1^{r-1})} \downarrow^{RP_r}_{P_r}$. Consider the module $DQ(V^{[r]}_{(1^{r-1})} \downarrow^{RP_r}_{P_r})$ on r dots. This has basis consisting of the orbit with representative*



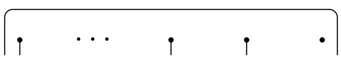
$$\mapsto \begin{array}{l} 1 \cdot\cdot (r-2) \\ (r-1) \\ r \end{array}$$

giving rise to the kS_r-modules $M^{(r-2,1^2)}$. The module $DQ(V^{[r-1]}_{(1^{(r-1)})} \downarrow^{RP_{r-1}}_{P_{r-1}})$ has basis the single diagram

$$\mapsto \begin{array}{l} 1 \cdot\cdot (r-1) \end{array},$$

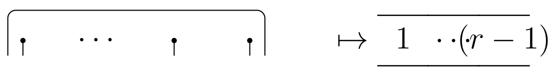
giving rise to a copy of $M^{(r-1)}$. Thus the composition factors of $V^{[r]}_{(1^{r-1})} \downarrow^{RP_r}_{P_r}$ are the same as those of $W_r \oslash M^{(r-2,1^2)}$ and $W_{r-1} \oslash M^{(r-1)}$.

Example 6.34. *We look at the P_r-module $V^{[r]}_{(r-1)} \downarrow^{RP_r}_{P_r}$. Consider the module $DQ(V^{[r]}_{(r-1)} \downarrow^{RP_r}_{P_r})$ on r dots. This has basis a single orbit with representative*



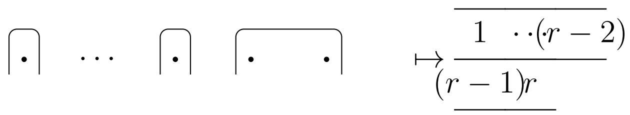
$$\mapsto \begin{array}{l} 1 \cdot\cdot (r-1) \\ r \end{array},$$

giving rise to the kS_r*-module* $M^{(r-1,1)}$*. The module* $DQ(V^{[r-1]}_{(r-1))} \downarrow^{RP_{r-1}}_{P_{r-1}})$ *has basis the single diagram*

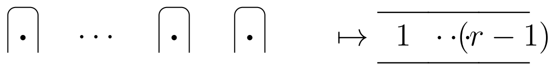


giving rise to a copy of $M^{(r-1)}$*. Thus the composition factors of* $V^{[r]}_{(r-1)} \downarrow^{RP_r}_{P_r}$ *are the same as those of* $W_r \oslash M^{(r-1,1)}$ *and* $W_{r-1} \oslash M^{(r-1)}$*.*

Example 6.35. *We look at the* P_r*-module* $V^{[r]}_{(0^{r-1})} \downarrow^{RP_r}_{P_r}$*. Consider the module* $DQ(V^{[r]}_{(0^{r-1})} \downarrow^{RP_r}_{P_r})$ *on* r *dots. This has a single orbit with representative*



giving rise to the kS_r*-module* $M^{(r-2,2)}$*. The module* $DQ(V^{[r-1]}_{(0^{r-1})} \downarrow^{RP_{r-1}}_{P_{r-1}})$ *has basis the single diagram*

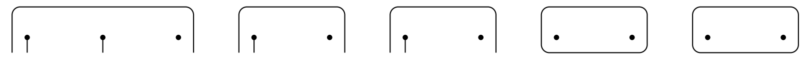


giving rise to a copy of $M^{(r-1)}$*. Thus the composition factors of* $V^{[r]}_{(0^{r-1})} \downarrow^{RP_r}_{P_r}$ *are the same as those of* $W_r \oslash M^{(r-2,2)}$ *and* $W_{r-1} \oslash M^{(r-1)}$*.*

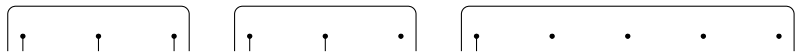
After this collection of examples, we consider the general decomposition of $DQ(V^{[r]}_{\lambda} \downarrow^{RP_r}_{P_r})$ into its orbits, and try to say something about its constituent modules. We commence by describing more carefully the pattern of dots in a ramified half-diagram in the depth quotient. Recall that a ramified half-diagram $\mathbf{t} \in DQ(V^{[r]}_{\lambda} \downarrow^{RP_r}_{P_r})$ has all inner parts singleton, with no outer non-propagating singleton parts.

Definition 6.36. Let $\mathbf{t}$ be a ramified half-diagram in $DQ(V^{[r]}_{\lambda} \downarrow^{RP_r}_{P_r})$. Define a propagating outer part of $\mathbf{t}$ to be of pattern (a, b) if a of the inner dots are propagating, and b are not. Say that an outer propagating part is of standard pattern (a, b) if it has a propagating dots, each of which are to the left of its b non-propagating dots. Define a non-propagating outer part of $\mathbf{t}$ to be of (standard) pattern (e) if it contains e dots.

Example 6.37. *Consider the diagrams*

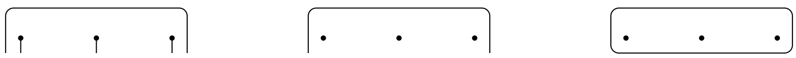
$\mathbf{t}_1 =$ 

and

$\mathbf{t}_2 =$ .

The half-diagram $\mathbf{t}_1$ *is of propagating index* $(2, 1^2)$ *and has, from left to right, outer parts of patterns* $(2, 1), (1, 1), (1, 1), (2)$ *and* (2). *The half-diagram* $\mathbf{t}_2$ *is of propagating index* $(3, 2, 1)$ *and has, from left to right, outer parts of patterns* $(3, 0), (2, 1)$ *and* $(1, 4)$. *All outer parts in both diagrams are standard.*

Remark 6.38. Note that $(x, 0), (0, x)$ and (x) are all valid patterns of outer parts when $x \geq 1$, and that none of them are the same. The first two are patterns of outer propagating parts, and the second of an outer non-propagating part. Below are outer parts of patterns $(3, 0)$, $(0, 3)$ and (3) respectively.



Now that we have described the possible patterns of outer parts, we define the pattern of a half-diagram $\mathbf{t} \in DQ(V_\lambda^{[r]} \downarrow_{P_r}^{RP_r})$ as follows.

Definition 6.39. The pattern Ω of a half-diagram $\mathbf{t} \in DQ(V_\lambda^{[r]} \downarrow_{P_r}^{RP_r})$ is the list of the patterns of its outer parts, ordered so that propagating parts (a, b) come first, with (a, b) occuring before (c, d) whenever $a > c$ or $a = c$ and $b > d$, followed by non-propagating parts (e) where (e) occurs before (f) if $e > f$. We abbreviate using index notation, for example writing $(2, 1)^2$ instead of $(2, 1), (2, 1)$.

Example 6.40. *Consider the diagrams* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *from Example 6.37. The half-diagram* $\mathbf{t}_1$ *has pattern* $((2, 1), (1, 1)^2, (2)^2)$, *whereas the half-diagram* $\mathbf{t}_2$ *has pattern* $((3, 0), (2, 1), (1, 4)))$.

Now, the module $DQ(V_\lambda^{[r]} \downarrow_{P_r}^{RP_r})$, when considered as an inflated S_r-module, can be partitioned into S_r-orbits, each of which consists of all half-diagrams of a given pattern Ω.

It remains to describe which patterns appear in a depth quotient, and then to identify the kS_r-module that each orbit gives rise to.

Lemma 6.41. *The S_r-orbits appearing in $DQ(V_\lambda^{[r]} \downarrow_{P_r}^{RP_r})$ are indexed by those patterns*

$$((a_1, b_1)^{s_1}, \ldots, (a_m, b_m)^{s_m}, (c_1)^{t_1}, \ldots, (c_n)^{t_n})$$

such that

- $\sum_{i=1}^m s_i(a_i + b_i) + \sum_{j=1}^n t_j c_j = r,$
- *the list* $(a_1^{s_1}, \ldots, a_m^{s_m})$ *is exactly the propagating index* λ*,*
- $a_i + b_i \geq 1$ *for each* i *and*
- $c_j \geq 2$ *for each* j*.*

Proof. The first constraint simply says that there are exactly r dots in a half-diagram. The second says that diagrams of this pattern are indeed of propagating index λ. The third says that no propagating part is empty. The final constraint guarantees that we do not have any outer non-propagating singletons, and thus are genuinely looking at a pattern of a nonzero element of $DQ(V_\lambda^{[r]} \downarrow_{P_r}^{RP_r})$. $\square$

Given a pattern Ω, we choose a representative $\mathbf{t}_\Omega$ of the associated orbit as follows.

Definition 6.42. Let Ω be the pattern indexing an S_r-orbit in the basis of $DQ(V_\lambda^{[r]} \downarrow_{P_r}^{RP_r})$. Let $\mathbf{t}_\Omega$ be the unique ramified half-diagram of pattern Ω such that:

- Each outer part of Ω contains only consecutive dots, so that there are no overlaps in the diagram of $\mathbf{t}_\Omega$.

- All outer propagating parts occur to the left of all outer non-propagating parts, and are of standard pattern.

- Outer propagating parts of pattern (a, b) occur to the left of outer propagating parts of pattern (c, d) whenever $a > c$ or $a = c$ and $b > d$.

- Outer non-propagating parts of pattern (e) occur to the left of outer non-propagating parts of pattern (f) whenever $e > f$.

In words, given an S_r-orbit associated to the pattern Ω in the basis of $DQ(V_\lambda^{[r]} \downarrow_{P_r}^{RP_r})$, the half-diagram $\mathbf{t}_\Omega$ is the element of Ω where propagating outer parts are written from left to right in order of how many propagating dots, and then by how many non-propagating dots they have, followed by non-propagating outer parts in order of how large they are. Within each outer part, all propagating dots occur to the left of all non-propagating dots.

Example 6.43. *The half-diagrams* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *in Example 6.37 are the representative diagrams* $\mathbf{t}_\Omega$ *for their respective patterns.*

The half-diagram basis of $DQ(V_\lambda^{[r]} \downarrow_{P_r}^{RP_r})$ can be partitioned into orbits associated to patterns Ω_i for $i = 1, \ldots, s$. If Ω is a given pattern, the kS_r-module with basis the orbit associated to Ω is the permutation module

$$\mathbb{1} \uparrow_{\mathrm{stab}(\mathbf{t}_\Omega)}^{S_r},$$

which we can concretely write down, even if we cannot decompose it. It is clear that if

$$\Omega = ((a_1, b_1)^{s_1}, \ldots, (a_m, b_m)^{s_m}, (c_1)^{t_1}, \ldots, (c_n)^{t_n}),$$

then, since we are working purely diagrammatically,

$$\mathrm{stab}(\mathbf{t}_\Omega) = \prod_{i=1}^{m} (S_{a_i} \times S_{b_i}) \wr S_{s_i} \times \prod_{j=1}^{n} (S_{c_j} \wr S_{t_j}).$$

Example 6.44. *Referring back to Example 6.37, we have*

$$\mathrm{stab}(\mathbf{t}_1) = (S_2 \times S_1) \times (S_1 \times S_1) \wr S_2 \times S_2 \wr S_2,$$

whereas

$$\mathrm{stab}(\mathbf{t}_2) = S_3 \times (S_2 \times S_1) \times (S_1 \times S_4).$$

It is at this point that we run into extreme difficulty, as we now want to understand the decomposition of the module $\mathbb{1} \uparrow_{\mathrm{stab}(\mathbf{t}_\Omega)}^{S_r}$. In some cases, such as when all s_i and t_j are 1, this is as simple as a Young permutation module. For example, $\mathbb{1} \uparrow_{\mathrm{stab}(\mathbf{t}_2)}^{S_r} \cong M^{(4,3,2,1^2)}$ which we know about fully. However, if $\mathbf{t}_\Omega$ is of pattern, for example, $((c)^t)$, then $\mathbb{1} \uparrow_{\mathrm{stab}(\mathbf{t}_\Omega)}^{S_r} \cong H^{(c^t)}$, the Foulkes module which we are far from being able to decompose. Indeed, the decomposition of an orbit of completely general pattern would include a mass of plethysm coefficients for which we do not have general formulae. This would be an interesting scope for future research. Indeed, this is the subject of current research by Bowman, Paget and Wildon.

Appendix A

SAGE Calculations

In this section, we include SAGE [48] code and outputs to verify those plethysm coefficient calculations which are not unreasonably large to be calculated by computer.

One may replicate the following computations by inputting the following code, which calculates $s_\nu \circ s_\mu$.

```
s = SymmetricFunctions(QQ).schur();
s[nu].plethysm(s[mu]);
```

The partitions nu and mu must be listed without brackets, without using index notation and with a comma separating each entry.

For example, the code to calculate $s_{(3,2,1)} \circ s_{(1^3)}$ would be as follows.

```
s = SymmetricFunctions(QQ).schur();
s[3,2,1].plethysm(s[1,1,1]);
```

We will truncate the output to only those plethysm coefficients we are interested in, so for example we will never include coefficients with first entry $|\nu| - 3$ or less, and only sometimes include coefficients with first entry $|\nu| - 2$.

SAGE Computation A.1. *The SAGE output for* $s_{(2)} \circ s_{(2)}$ *is:*

```
s[2, 2] + s[4]
```

SAGE Computation A.2. *The SAGE output for* $s_{(1^2)} \circ s_{(2)}$ *is:*

```
s[3, 1]
```

SAGE Computation A.3. *Part of the SAGE output for* $s_{(3,2,2,1)} \circ s_{(2)}$ *is:*

```
... + s[10, 2, 2, 1, 1] + s[10, 2, 2, 2] + s[10, 3, 1, 1, 1]
    + 2*s[10, 3, 2, 1] +  s[10, 4, 1, 1] + s[10, 4, 2]
    + s[11, 2, 2, 1].
```

SAGE Computation A.4. *Part of the SAGE output for* $s_{(3,3,1)} \circ s_{(2)}$ *is:*

```
... + s[9, 2, 2, 1] + s[9, 3, 1, 1] + 2*s[9, 3, 2]
    + s[9, 4, 1] + s[9, 5] + s[10, 3, 1].
```

SAGE Computation A.5. *Part of the SAGE output for* $s_{(3,2,1)} \circ s_{(3)}$ *is:*

```
... + s[14, 2, 1, 1] + s[14, 2, 2] + 2*s[14, 3, 1]
    + s[14, 4] + s[15, 2, 1].
```

SAGE Computation A.6. *Part of the SAGE output for* $s_{(3,3,2)} \circ s_{(3)}$ *is:*

```
... + s[18, 2, 2, 2] + 2*s[18, 3, 2, 1] + s[18, 3, 3]
    + s[18, 4, 1, 1] + s[18, 4, 2] + s[18, 5, 1]
    + s[19, 3, 2].
```

SAGE Computation A.7. *Part of the SAGE output for* $s_{(5,2)} \circ s_{(1^2)}$ *is:*

```
... + s[6, 4, 2, 1, 1] + s[6, 4, 3, 1] + s[6, 5, 1, 1, 1]
    + s[6, 5, 2, 1] + s[6, 6, 1, 1] + s[7, 5, 2].
```

SAGE Computation A.8. *Part of the SAGE output for* $s_{(3^3)} \circ s_{(1^2)}$ *is:*

```
... + s[8, 3, 3, 2, 1, 1] + s[8, 4, 3, 2, 1]
    + s[8, 4, 4, 2] + s[9, 3, 3, 3].
```

SAGE Computation A.9. *The SAGE output for* $s_{(1^7)} \circ s_{(1^2)}$ *is:*

```
s[4, 3, 3, 3, 1] + s[4, 4, 2, 2, 2] + s[5, 3, 2, 2, 1, 1]
    + s[6, 2, 2, 1, 1, 1, 1] + s[7, 1, 1, 1, 1, 1, 1, 1].
```

SAGE Computation A.10. *Part of the SAGE output for* $s_{(4,1^5)} \circ s_{(1^2)}$ *is:*

```
... + s[8, 3, 1, 1, 1, 1, 1, 1, 1]
    + s[8, 3, 2, 1, 1, 1, 1, 1]
    + s[8, 3, 2, 2, 1, 1, 1] + s[8, 4, 1, 1, 1, 1, 1, 1]
    + s[8, 4, 2, 1, 1, 1, 1] + s[8, 4, 2, 2, 1, 1]
    + s[8, 5, 2, 1, 1, 1] + s[9, 4, 1, 1, 1, 1, 1].
```

SAGE Computation A.11. *Part of the SAGE output for* $s_{(2,1^3)} \circ s_{(1^2)}$ *is:*

```
... + s[4, 1, 1, 1, 1, 1, 1] + s[4, 2, 1, 1, 1, 1]
    + s[4,2,2,1,1] + s[4, 2, 2, 2]
    + s[4, 3, 2, 1] + s[5, 2, 1, 1, 1].
```

SAGE Computation A.12. *Part of the SAGE output for* $s_{(7,3)} \circ s_{(1^2)}$ *is:*

```
... + s[9, 6, 3, 1, 1] + s[9, 6, 4, 1] + s[9, 7, 2, 1, 1]
    + s[9, 7, 3, 1] + s[9, 8, 2, 1] + s[10, 7, 3].
```

SAGE Computation A.13. *Part of the SAGE output for* $s_{(5,4)} \circ s_{(1^2)}$ *is:*

```
... + s[8, 4, 4, 1, 1] + s[8, 5, 3, 1, 1] + s[8, 5, 4, 1]
    + s[8, 6, 3, 1] + s[9, 5, 4].
```

SAGE Computation A.14. *Part of the SAGE output for* $s_{(4,4)} \circ s_{(1^2)}$ *is:*

```
... + s[7, 4, 3, 1, 1] + s[7, 5, 3, 1] + s[8, 4, 4].
```

SAGE Computation A.15. *Part of the SAGE output for* $s_{(3^3)} \circ s_{(2,1)}$ *is:*

```
... + s[17, 3, 3, 2, 1, 1] + s[17, 3, 3, 2, 2]
    + s[17, 3, 3, 3, 1] + 2*s[17, 4, 3, 2, 1]
    + s[17, 4, 3, 3] + s[17, 4, 4, 2]
    + s[17, 5, 3, 2] + s[18, 3, 3, 3].
```

SAGE Computation A.16. *Part of the SAGE output for* $s_{(5,3)} \circ s_{(2,1)}$ *is:*

```
... + s[15, 4, 3, 1, 1] + s[15, 4, 3, 2]
    + 2*s[15, 4, 4, 1] + s[15, 5, 2, 1, 1]
    + s[15, 5, 2, 2] + 3*s[15, 5, 3, 1]
    + 2*s[15, 5, 4] + 2*s[15, 6, 2, 1]
    + 2*s[15, 6, 3] + s[15, 7, 2] + s[16, 5, 3].
```

SAGE Computation A.17. *Part of the SAGE output for* $s_{(3,1^2)} \circ s_{(2,1)}$ *is:*

```
... + s[9, 2, 1, 1, 1, 1] + 2*s[9, 2, 2, 1, 1] + s[9, 2, 2, 2]
    +2*s[9, 3, 1, 1, 1] + 3*s[9, 3, 2, 1]
    + s[9, 3, 3] + 2*s[9, 4, 1, 1]
    + 2*s[9, 4, 2] + s[9, 5, 1] + s[10, 3, 1, 1].
```

SAGE Computation A.18. *Part of the SAGE output for* $s_{(4,1^2)} \circ s_{(1^2)}$ *is:*

```
... + s[4, 2, 1, 1, 1, 1, 1, 1] + s[4, 2, 2, 1, 1, 1, 1]
    + 2*s[4, 2, 2, 2, 1, 1] + s[4, 3, 1, 1, 1, 1, 1]
    + 2*s[4, 3, 2, 1, 1, 1] + 2*s[4, 3, 2, 2, 1]
    + s[4, 3, 3, 1, 1] + s[4, 3, 3, 2] + 2*s[4, 4, 2, 1, 1]
    + s[4, 4, 3, 1] + ...
```

where we truncate all output apart from partitions beginning with 4.

Appendix B

Further Directions for Research

We comment on some further directions for research.

In Chapter 3, we calculated plethysm coefficients using a formula of Law and Okitani in the case where $m = 2$ and $\ell(\lambda)$ is close to $|\nu|$. One could consider lower values of $\ell(\lambda)$ or, perhaps more interestingly, the case $m = 3$.

An obvious direction for further study is the semisimplicity of $RP_r(\delta^{\text{in}}, \delta^{\text{out}})$, for which we stated a conjecture. One might try to prove that $RP_r(2r-1, 2r-1)$ is semisimple, or indeed the conjecture in general. More ambitiously, one might seek a result characterising homomorphisms between cell modules, as Doran and Wales did for the partition algebra in [11].

It would also be interesting to continue the line of thinking around the depth radical in our final chapter, and to try and link this to plethysm coefficients. One might try to express plethysm coefficients in terms of the composition multiplicities of restricted ramified cell modules. One might also follow through our description of the decomposition of the restriction of ramified cell modules to the partition algebra to obtain an expression involving plethysm coefficients. These avenues of research are currently being pursued by Bowman, Paget and Wildon.

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