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TOPICS IN THE THEORY OF INTERPOLATION

by

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PREFACE

The following work is concerned mainly with questions associated with the interpolation of a regular function in terms of its values at a set of discrete points $\{z_n\}$.

If the set $\{z_n\}$ contains only one limit point we may transform so that this becomes the point at infinity and the domain of regularity of the function is either the finite part of the plane or contains a sector. This is the situation dealt with in Chapters II and IV. Some new results on constructing an integral function of zero order from its values at an infinite sequence are obtained. These results are rather unwieldly for general use but are suitable for discussing problems connected with that of Iyer:

"What suitable conditions can we impose on a set $\{z_n\}$ in order to ensure that

$$\limsup_{r \to \infty} \frac{\log M(r,f)}{r^{\rho}} = \limsup_{|z_n| \to \infty} \frac{\log |f(z_n)|}{|z_n|^{\rho}}$$

for all integral functions $f(z)$ of growth less than type τ order ρ ?"

This question was raised as a result of work by Whittaker, Pólya, Pfluger, and Iyer himself, on the representation of integral functions in terms of their values at the lattice points $m + in$ $(m, n = 0, \pm 1, \pm 2, \ , \ )$. The results of Chapter IV are obtained by combining the approach of Pfluger, which depended on certain periodic properties of the Weierstrass σ-function, and that of Iyer, which only appealed to known facts about the size of $|\sigma(z)|$ and $|\sigma'(z)|$, to the lattice point situation. Indeed the interpolation techniques of Iyer are used throughout this thesis. We follow Pfluger, however, in seeking appropriate geometric distribution conditions on the set $\{z_n\}$, as opposed to assumptions concerning the behaviour of the canonical product of the z_n.

In Chapter III, following the work of Maitland and Noble, we invert the interpolation technique of Chapter II to give both new and improved results concerning the regions where an integral function is "large". As corollaries we deduce certain growth properties of $\log M(r, f)$.

Finally, in Chapter V, we discuss the situation where $\{z_n\}$ has more than one limit point at which the interpolating function is not necessarily regular. We may take the region of regularity to be the unit disc, in which case every point of the boundary may be a singularity, or a subset of the unit disc such that boundary singularities of the function only occur on that part of the boundary common with $|z| = 1$.

The results of Chapter IV have already appeared in print [58], the method of proof differing in detail in order to suit the more isolated situation.

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NOTATION

We shall assume many of the conventions of notation of the theory of functions of a complex variable and here clarify only some cases where confusion may arise.

By the letter K we shall denote a numerical constant independent of the main variables, $K(\alpha, \beta, \ldots)$ denotes a constant depending only on the variables listed in the parentheses. In both cases, the value of K is not necessarily the same at each appearance in an argument. Specific constants within a section are indicated by subscripts and absolute constants by other capital letters, usually A and B.

We shall often write

$$\limsup \varphi(x)$$

where x ranges over some countable set. This is to be interpreted as

$$\lim_{n \to \infty} \sup_{p \geq n} \varphi(x_p),$$

for some enumeration $\{x_p\}$ of the set. Obviously the value obtained is independent of the enumeration chosen.

Unless the context clearly indicates otherwise, the canonical product, without further qualification, of a set $\{z_n\}$ with exponent of convergence ρ means

$$\prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, [\rho]\right),$$

where $E(u, p)$ is the Weierstrass primary factor of genus p.

We denote non-increasing by $\downarrow$ and non-decreasing by $\uparrow$. The interior domain of a Jordan curve Γ is denoted by $D_i(\Gamma)$.

CONTENTS

CHAPTER I

INTRODUCTION

1.1. Generalities. The aspect of interpolation theory with which we are concerned here is that of interpolating a regular function in terms of its values at an infinite set of discrete points and, in particular, the growth properties which follow from the representation adopted.

One method of partially solving the problem, in the case where the set of points consists of the non-negative integers, is provide by the Gregory-Newton series (see, for example, Whittaker [56]).

"Any integral function $f(z)$ of exponential type less than $\log 2$ has a convergent representation

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{1}{n!}\left\{\prod_{p=0}^{n-1}(z-p)\right\} \Delta^n f(0), \tag{1.1.1}$$

where $\Delta f(z) = f(z+1) - f(z) \quad \Delta^n f(z) = \Delta(\Delta^{n-1} f(z))$."

This has as an almost immediate corollary the theorem of Pólya-Hardy,

"An integral function of exponential type less than $\log 2$ taking integral values at the non-negative integers is a polynomial."

Many other series of a similar nature can be formed, each depending on the set of points in question (see [2] and [56] for examples). We describe below a method of approach which deals succesfully with a wide range of sets of points.

<u>1.2. Cardinal series.</u> The general method depends on constructing an integral function $F(z)$ taking prescribed values $\{\omega_n\}$ at a set of points $\{z_n\}$, having no finite limit point. We can always form an integral function $\eta(z)$ which has simple zeros at the z_n and, given such a function and the sequence $\{\omega_n\}$, we can determine a sequence of integers $\{\nu_n\}$ such that

$$\sum_n \left| \frac{\omega_n}{\eta'(z_n)\, z_n^{1+\nu_n}} \right| < \infty . \tag{1.2.1}$$

The series

$$\eta(z) \sum_n \frac{\omega_n}{(z-z_n)\,\eta'(z_n)} \left(\frac{z}{z_n}\right)^{\nu_n} = F(z) = F(z;\{z_n\},\{\omega_n\},\{\nu_n\}) \tag{1.2.2}$$

will be called a cardinal series associated with the sequences $\{z_n\}$ and $\{\omega_n\}$. Since the series, with $\eta(z)$ taken inside the summation sign, is uniformly convergent in any bounded region of the complex plane, it follows that $F(z)$ is an integral function with the property that

$$F(z_n) = \omega_n . \tag{1.2.3}$$

The growth properties of such functions have been studied by Mursi and Winn [28] and Macintyre and Wilson [25].

If the ω_n are the functional values at z_n of some integral function $f(z)$, then

$$F(z) = \eta(z) \sum_n \frac{f(z_n)}{(z-z_n)\,\eta'(z_n)} \left(\frac{z}{z_n}\right)^{\nu_n} \qquad (1.2.4)$$

is an integral function which coincides with $f(z)$ at the sequence of points $\{z_n\}$ and we are immediately faced with the problem of under what, if any, conditions on the $\{z_n\}$ and the function $f(z)$ is it possible to identify the function with an associated cardinal series?

One method of answering this question is to restrict the classes of integral functions and point sets in such a manner that, for some fixed integer ν , there exists a sequence of curves $\{\Gamma_s\}$ with $\inf_{z \in \Gamma_s} |z| \to \infty$ as $s \to \infty$ and such that

$$\int_{\Gamma_s} \frac{f(\zeta)\, d\zeta}{\zeta^{\nu}\, \eta(\zeta)(\zeta - z)} \to 0 \quad \text{as} \quad s \to \infty. \qquad (1.2.5)$$

The calculus of residues then gives that

$$\frac{f(z)}{z^{\nu}\, \eta(z)} - \sum_{z_n \in D_i(\Gamma_s)} \frac{f(z_n)}{(z-z_n)\,\eta'(z_n)} \left(\frac{1}{z_n}\right)^{\nu} \longrightarrow 0, \qquad (1.2.6)$$

and thus

$$f(z) = \eta(z) \sum \frac{f(z_n)}{(z-z_n)\,\eta'(z_n)} \left(\frac{z}{z_n}\right)^{\nu}, \qquad (1.2.7)$$

with the series bracketed suitably. If equation (1.2.1) is satisfied with $\nu_n \equiv \nu$, it follows that the series in (1.2.7) converges absolutely and the remark about suitable bracketing becomes superfluous. In all applications of this method made in this thesis the pairs of class of admissible functions and the set of points $\{z_n\}$ are chosen so that we can satisfy conditions (1.2.1) and (1.2.5) with ν taken to be zero.

1.3. Early results. Certain sets of points have suitable associated functions $\eta(z)$ with well defined properties; with the set of all integers we can take $\eta(z)$ to be $\sin \pi z$ and with the set of all lattice points $m+in$ $(m, n = 0, \pm 1, \pm \ \ )$ we can use $\sigma(z)$, the Weierstrass σ-function. In the cases mentioned, many of the results take a very elegant form and we give here a few examples.

THEOREM 1. Suppose that $f(z)$ is an integral function of exponential type less than π and that, for all integer n, $f(n)$ is bounded; then

$$f(z) = \frac{\sin \pi z}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n f(n)}{z-n} . \tag{1.3.1}$$

Since the series involved is not always absolutely convergent this is not strictly a cardinal series as previously defined. In fact the representation of the theorem depends on the Abel summability of the series of (1.3.1) when the condition on $f(z)$ at the integers is relaxed to

$$\limsup |f(n)|^{1/n} \leq 1 \tag{1.3.2}$$

(Macintyre [24]).

A simple corollary to Theorem 1 is the following result, originally due to Carlson [3].

THEOREM 2. Suppose that $f(z)$ is an integral function satisfying

$$|f(z)| \leqslant Me^{k|z|}, \text{ with } k < \pi, \tag{1.3.3}$$

and $f(n) = 0$ for all integer n, then $f(z)$ is identically zero.

THEOREM 3. (Cartwright [6], Macintyre [24]). Suppose that $f(z)$ is an integral function satisfying (1.3.3) and

$$|f(n)| \leqslant A, \tag{1.3.4}$$

then for real x

$$|f(x)| \leqslant K(k)A. \tag{1.3.5}$$

In addition, if $f(z)$ tends to zero as z tends to infinity by positive integer values then $f(z)$ tends to zero as z tends to infinity by continuous positive values.

We also have the result of Polya ([41], p.606) that for functions satisfying (1.3.3) the growth along the real axis is determined by its growth at the integers, in the sense that

$$\limsup_{r \to \infty} \frac{1}{r} \log|f(r)| = \limsup_{n \to \infty} \frac{1}{n} \log|f(n)| \tag{1.3.6}$$

and, for a smaller class of functions, the following.

THEOREM 4. (Tschakalov [48]). Suppose that $f(z)$ is an integral function of growth at most minimal type of order 1 with $f(n)$ bounded, then $f(z)$ is constant.

The first results for the lattice points were those of J.M. Whittaker [54] who proved the following representation theorem.

THEOREM 5. (1) If $f(z)$ is an integral function for which

$$\liminf_{r \to \infty} \frac{1}{r^2} \log M(r,f) < \frac{1}{4}\pi \ , \tag{1.3.7}$$

then

$$f(z) = \sigma(z) \sum_{(m,n)} \frac{f(m+in)}{(z-m-in)\,\sigma'(m+in)} \ , \tag{1.3.8}$$

the series being bracketed suitably.

(2) If

$$\limsup_{r \to \infty} \frac{1}{r^2} \log M(r,f) < \frac{1}{2}\pi \tag{1.3.9}$$

then (1.3.8) holds, the series being absolutely convergent (in fact $\sigma'(m+in) = (-1)^{m+n+mn} \exp\{\frac{1}{2}\pi(m^2+n^2)\}$) Pfluger [35] and Iyer [13] both proved, independently and using different methods, Theorem 6.

THEOREM 6. If $f(z)$ is an integral function for which

$$\limsup_{r \to \infty} \frac{1}{r^2} \log M(r,f) < \frac{1}{2}\pi \tag{1.3.10}$$

and

$$|f(m+in)| \leq K \ , \tag{1.3.11}$$

then $f(z)$ is necessarily constant.

This extended a result obtained by both Whittaker [54,55] and Pólya [42] which, instead of (1.3.10), required

$$\limsup_{r\to\infty} \frac{1}{r^2} \log M(r,f) \leq 0 . \qquad (1.3.12)$$

<u>1.4. Generalisations.</u> A large amount of work has been done on generalising the interpolation results concerned with the behaviour of integral functions at the integers. This will not concern us here and we give only some references. Bernstein [1], Levinson [22], Macintyre [24] and, more recently, Rahman [43].

Theorem 6 is best possible in the sense that the value $\frac{1}{2}\pi$ cannot be allowed in (1.3.10), as the example of the σ-function itself shows. The result has however been generalised in several ways; the lattice points being replaced by more general sets and the boundedness condition by one of restricted rate of growth.

Iyer's method of proving Theorem 6, unlike that of Pfluger, made no appeal to the periodic properties of $\sigma(z)$ but merely to known facts concerning $|\sigma(z)|$ and $|\sigma'(z)|$. Developing this method [15, 16] he showed that the behaviour at the lattice points of a function satisfying (1.3.10) is typical of that of the function as a whole in the sense that

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^2} = \limsup \frac{\log|f(m+in)|}{\sqrt{m^2+n^2}} . \qquad (1.4.1)$$

He then generalised the problem to that of determining a set $\{z_n\}$ with the property that

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}} = \limsup_{|z_n|\to\infty} \frac{\log|f(z_n)|}{|z_n|^{\rho}} \qquad (1.4.2)$$

for all functions of growth less than type τ order ρ ; such a set he described as being "effective for ρ, τ" and gave some properties which it must posses in our next theorem.

THEOREM 7. If $\{z_n\}$ is effective for ρ, τ then:

(i) the exponent of convergence of $\{z_n\}$ cannot be less than ρ ;

(ii) if the exponent of convergence is ρ , then any integral function with zeros at the $\{z_n\}$ must be of growth at least type τ of order ρ unless it is identically zero;

(iii) the set of amplitudes of z_n is everywhere dense in $[0, 2\pi]$.

He also gave sufficient conditions that a set $\{z_n\}$ be effective but these depend on the canonical product of the z_n , not on their geometrical distribution directly.

THEOREM 8. The set $\{z_n\}$ will be effective for ρ, τ provided there exists an integral function $\eta(z)$ with simple zeros at $z = z_n$ and an $h > \rho$ such that both

$$\lim_{n\to\infty} \frac{\log|\eta'(z_n)|}{|z_n|^{\rho}} = \tau \qquad (1.4.3)$$

and

$$\frac{\log|\eta(z)|}{|z|^{\rho}} \longrightarrow \tau \qquad (1.4.4)$$

as $z \to \infty$ outside the set of discs

$$|z - z_n| \leq |z_n|^{-h} . \tag{1.4.5}$$

Pfluger approached the problem from a different direction. He developed a study of the Lindelöf $h(\theta)$ function and the extent to which it is typical of the growth of canonical products of sets of zeros measurable in a certain sense with respect to a proximate order $\rho(r)$ [37, 38]. He was concerned with sets $\{z_n\}$ which satisfy, for each pair θ_1, θ_2,

$$n(r, \theta_1, \theta_2) = \{N(\theta_2) - N(\theta_1)\} r^{\rho(r)} + o(r^{\rho(r)}), \tag{1.4.6}$$

where $n(r, \theta_1, \theta_2)$ denotes the number of z_n in the region

$$0 \leq |z| \leq r \quad , \quad \theta_1 \leq \arg z < \theta_2$$

and $N(\theta)$ is a non-decreasing function which satisfies

$$N(\theta + 2\pi) - N(\theta) = K > 0 \; ; \tag{1.4.7}$$

the proximate order function $\rho(r)$ being continuous for $0 < r < \infty$, having left- and right-hand derivatives which coincide in intervals and satisfying

$$\lim_{r \to \infty} \rho(r) = \rho > 0 \tag{1.4.8}$$

and

$$\lim_{r \to \infty} \rho'(r) r \log r = 0 . \tag{1.4.9}$$

(For any integral function $f(z)$ of finite positive order ρ such a $\rho(r)$ can be determined to ensure that

$$\limsup_{r \to \infty} r^{-\rho(r)} \log M(r, f)$$

is positive and finite.) Such a set is said to be measurable with respect to $r^{\rho(r)}$ with measure function $N(\theta)$. His principle result of this kind is

THEOREM 9. Suppose that $f(z)$ is an integral function of non-integral order ρ whose set of zeros $\{z_n\}$ is measurable with respect to $r^{\rho(r)}$ with measure function $N(\theta)$, then for any φ

$$h(\varphi) = \lim r^{-\rho(r)} \log|f(re^{i\varphi})| = \frac{\pi}{\sin \pi\rho}\int_{\theta=0}^{2\pi} \cos\rho(\theta-\pi)\, dN(\theta+\varphi) \quad (1.4.10)$$

as $r \to \infty$ through a set of unit density.

Using this, and similar results, and a modification of Iyer's method, Pfluger was able to establish very general representation theorems. He found it necessary to distinguish between cases of integral and non-integral order, for example

THEOREM 10. Suppose that the set $\{z_n\}$ is measurable with respect to r^{ρ} , with ρ not an integer, with measure function $N(\theta)$ and suppose, in addition, that the discs

$$|z - z_n| \leq \delta |z_n|^{1-\frac{1}{2}\rho} \quad (1.4.11)$$

are all disjoint, for some fixed positive δ . Let $\eta(z)$ be the canonical product of the z_n . If $f(z)$ is an integral function such that

$$h(\varphi) = \limsup_{r\to\infty} \frac{\log|f(re^{i\varphi})|}{r^{\rho}} < \frac{\pi}{\sin\pi\rho}\int_0^{2\pi} \cos\rho(\theta-\pi)\,dN(\theta+\varphi) \quad (1.4.12)$$

then

$$f(z) = \eta(z) \sum_n \frac{f(z_n)}{(z - z_n)\,\eta'(z_n)} \quad . \qquad (1.4.13)$$

He also gave [39, 40]

THEOREM 11. Suppose that $\{z_n\}$ is measurable with respect to r^ρ with measure function $(\rho\tau/2\pi)\theta$, that is

$$\lim_{r \to \infty} \frac{n(r, \theta_1, \theta_2)}{r^\rho} = \frac{\rho\tau}{2\pi}(\theta_2 - \theta_1) \; , \qquad (1.4.14)$$

and that the discs

$$|z - z_n| \leq \delta \, |z_n|^{1 - \frac{1}{2}\rho}$$

are disjoint. If ρ is an integer, suppose, in addition, that

$$\sum_{|z_n| \leq r} \frac{1}{z_n^\rho} \longrightarrow \alpha\rho \quad \text{as} \quad r \to \infty \quad . \qquad (1.4.15)$$

Then any integral function of growth less than type τ order ρ which is bounded at the z_n is a constant.

The proof of this is based on the representation result of the preceeding theorem and a similar result for the case of integer ρ, with $\eta(z)$ then defined to be the product of $\exp\{-\alpha z^\rho\}$ with the canonical product of the z_n. These results include earlier generalisations of Maitland [26] and Levin [21].

We shall describe a set satisfying the conditions of Theorem 11 as belonging to $P(\rho, \tau)$.

Noble [31] considered sets $\{z_n\}$ satisfying conditions which can be regarded as being complementary to those of Pfluger. He requires that they have a uniform angular distribution, in the sense that

$$\frac{n(r,\theta_1,\theta_2)}{n(r,\varphi_1,\varphi_2)} \longrightarrow \frac{\theta_2-\theta_1}{\varphi_2-\varphi_1} \quad \text{as} \quad r\to\infty, \qquad (1.4.16)$$

for all $\theta_1 < \theta_2$, $\varphi_1 < \varphi_2$, but considerably relaxes the condition that the set must have a density with respect to r^ρ in every sector. For integer ρ, the condition of (1.4.15) is retained. He obtained a very general representation lemma and using the basic argument of Iyer, which he was able to do under conditions far less severe than those of Theorem 8, obtained further results concerning the growth of integral functions. We now give a special case of one of these.

THEOREM 12. ([31], Theorem 2, Corollary 3). A set belonging to $P(\rho,\tau)$ is effective for ρ, τ, in the sense of Iyer.

The methods used by these authors have also been applied to the related problem of inferring properties of a function regular in a sector from its properties on certain sets of points in the sector. Noble [32] obtained several results which generalise earlier ones of Whittaker [54, 55] and Pfluger [36] for lattice points and of Maitland [26] and Cartwright [5] for more general sets. We give as an example a special case of one of his results.

THEOREM 13. ([32], Theorem A with $\alpha = D$, $\theta = 1$). Suppose that $f(z)$ is regular in the sector $|\arg z| \leq \alpha$ and satisfies:

(i) $\liminf\limits_{r\to\infty} \dfrac{\log M(r,\alpha,f)}{r^{\rho}} < \tau$; $M(r,\alpha,f) = \sup\limits_{|\theta|\leq\alpha} |f(re^{i\theta})|$ (1.4.17)

(ii) $\limsup\limits_{r\to\infty} \dfrac{\log|f(re^{\pm i\alpha})|}{r^{\rho}} \leq \gamma < \tau$; (1.4.18)

and (iii) $\limsup\limits_{|\arg z_n|\leq\alpha} \dfrac{\log|f(z_n)|}{|z_n|^{\rho}} \leq \varkappa < \tau$, (1.4.19)

where $\{z_n\}$ is a set of $P(\rho,\tau)$; then

$$\limsup_{r\to\infty} \frac{\log M(r,\alpha,f)}{r^{\rho}} \leq \sup\{\gamma,\varkappa\}. \quad (1.4.20)$$

Furthermore, a function $f(z)$ satisfying (1.4.17) which is bounded on the boundary of the sector and at those z_n lying in the sector is bounded throughout the sector.

This is similar to a result of Iyer [15] concerning sets satisfying the conditions of Theorem 8 and a further restriction on the manner in which the set of discs of (1.4.5) intersects the boundary of the sector.

Many of the results obtained, including that given above, require very heavy conditions on the function on the boundary of the sector; making, in some circumstances, a knowledge of the behaviour of the function at the set of points superfluous. An interesting result which does not have this drawback is the following theorem due, in a slightly weaker form, to Miss Cartwright.

THEOREM 14. (Cartwright [7], Iyer [15]). Suppose that $f(z)$ is regular in the region $S(\alpha, l)$, defined by $|\arg z| \leq \alpha$, $|z| \geq l$, and satisfies:

(i) $$\liminf_{r \to \infty} \frac{\log M(r, \alpha, f)}{r^2} < \tfrac{1}{2}\pi \text{ ;} \tag{1.4.21}$$

(ii) $$\limsup_{r \to \infty} \frac{\log |f(re^{\pm i\alpha})|}{r^2} \leq \gamma < \tfrac{1}{2}\pi \text{ ,} \tag{1.4.22}$$

and if $\alpha < \frac{1}{2}\pi$ suppose, in addition, that $\gamma < \frac{1}{2}\pi \sin^2 \alpha$;

and (iii) $$|f(m + in)| \leq A \tag{1.4.23}$$

for all lattice points $m + in$ in $S(\alpha, l)$; then as $r \to \infty$

$$|f(re^{i\theta})| = O(1) \text{ ,} \tag{1.4.24}$$

uniformly for $|\theta| \leq \beta < \alpha$, where $\sin^2(\alpha - \beta) > 2\gamma/\pi$.

CHAPTER II

INTEGRAL FUNCTIONS AND FUNCTIONS REGULAR IN A SECTOR

2.1. Introduction and statement of results. In this chapter we shall be concerned with interpolation sets $\{z_n\}$ having a high degree of uniformity. They are uniform in the sense that, for positive orders, they are measurable with respect to a proximate order $\rho(r)$ and have uniform angular distribution, as do those of Noble. They also satisfy a condition that the number of points in any region is 'just right'.

The conditions for such a set are rather unwieldly in any application but, fortunately, we can describe a set in terms of a natural covering property which has a subset satisfying all these restrictions to the necessary degree of accuracy, without any loss of sharpness in the results obtained. It is this simple description of the interpolation set and the fact that the theorems cover cases of zero order and give sharper growth theorems for the range of conditions considered that provide the motivation for this chapter.

Our main result is:

THEOREM 15. Suppose $\varphi(r)$ to be a twice differentiable function which is such that

$$\frac{d^2\varphi(r)}{d(\log r)^2} = r^{\nu(r)} \uparrow \to \infty \qquad (2.1.1)$$

as $r \to \infty$, with $\nu(r) \downarrow \to \rho \geq 0$. Suppose, also, that the union of the discs

$$|z - z_n| \leq \Delta |z_n|^{1 - \frac{1}{2}\nu(|z_n|)} \tag{2.1.2}$$

covers the complex plane. If $f(z)$ is an integral function satisfying

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{\varphi(r)} < \frac{4\pi}{\Delta^2 3\sqrt{3}} \tag{2.1.3}$$

and

$$\limsup \frac{\log |f(z_n)|}{\varphi(|z_n|)} \leq X < \frac{4\pi}{\Delta^2 3\sqrt{3}} , \tag{2.1.4}$$

then

$$\limsup_{r \to \infty} \frac{\log M(r, f)}{\varphi(r)} \leq X . \tag{2.1.5}$$

Furthermore, if $|f(z_n)| \leq A$ the conclusion becomes that $f(z)$ is identically constant.

The above results continue to hold if in the growth conditions, that is (2.1.4) or $f(z_n)$ bounded, $\{z_n\}$ is replaced by a set $\{z'_n\}$ satisfying

$$|z'_n - z_n| \leq K |z_n|^{1 - \frac{1}{2}\nu(|z_n|)} \tag{2.1.6}$$

and such that, for some fixed positive δ , the discs

$$|z - z'_n| \leq \delta |z'_n|^{1 - \frac{1}{2}\nu(|z'_n|)} \tag{2.1.7}$$

are all disjoint.

The separation condition of (2.1.7) is unnecessarily severe and could certainly be weakened; the form given, however, seems the most natural. The following theorem, a consequence of Theorem 16, gives a more accesible result.

THEOREM 16. Suppose that $\nu(r) \downarrow \to \rho$ and $\nu(r) \log r \uparrow \to \infty$. If $f(z)$ is an integral function satisfying

$$\liminf_{r\to\infty} \frac{\nu(r)^2 \log M(r,f)}{r^{\nu(r)}} < \frac{4\pi}{\Delta^2 3\sqrt{3}} \tag{2.1.8}$$

and

$$|f(z_n)| \leq A \ , \tag{2.1.9}$$

where the union of the discs

$$|z - z_n| \leq \Delta |z_n|^{1-\frac{1}{2}\nu(|z_n|)}$$

covers the complex plane, then $f(z)$ is identically a constant.

Further, if $\nu(r)$ also satisfies

$$\nu'(r)\, r \log r = o(\nu(r)) \quad \text{as} \quad r \to \infty \tag{2.1.10}$$

or the weaker condition

$$\liminf_{\delta\to 0} \frac{\nu(r+\delta) - \nu(r)}{\delta} = o(\nu(r)) \quad \text{as} \quad r \to \infty \ , \tag{2.1.11}$$

then

$$\limsup_{r\to\infty} \frac{\nu(r)^2 \log M(r,f)}{r^{\nu(r)}} = \limsup \frac{\nu(|z_n|)^2 \log|f(z_n)|}{|z_n|^{\nu(|z_n|)}} \ , \tag{2.1.12}$$

provided the right-hand side is less than $4\pi/\Delta^2 3\sqrt{3}$.

The proof of Theorem 15 requires only minor modifications to show that Theorem 16 is still valid if we only require that

$$\nu(r) \to \rho \qquad (0 \leq \rho < \infty) \quad \text{and} \quad \frac{r \log r\, \nu'(r)}{\nu(r)} \longrightarrow 0$$

as $r \to \infty$. We also note that Theorems 15 and 16 are sharp in the sense that the constant $4\pi/\Delta^2 3\sqrt{3}$ cannot be increased.

To establish Theorem 16 as a corollary to Theorem 15 it is only necessary to note that $\nu(r)$ must be continuous and that

$$\frac{(r+\delta)^{\nu(r+\delta)}}{\nu(r+\delta)^2} - \frac{r^{\nu(r)}}{\nu(r)^2} = \frac{(r+\delta)^{\nu(r)}}{\nu(r+\delta)^2} - \frac{r^{\nu(r)}}{\nu(r+\delta)^2} + \tag{2.1.13}$$
$$+ \frac{(r+\delta)^{\nu(r+\delta)} - (r+\delta)^{\nu(r)}}{\nu(r+\delta)^2} + \frac{r^{\nu(r)}}{\nu(r)^2\nu(r+\delta)^2}\left\{\nu(r)^2 - \nu(r+\delta)^2\right\},$$

and hence

$$\limsup_{\delta\to 0} \frac{r}{\delta}\left\{\frac{(r+\delta)^{\nu(r+\delta)}}{\nu(r+\delta)^2} - \frac{r^{\nu(r)}}{\nu(r)^2}\right\} = \frac{r^{\nu(r)}}{\nu(r)}\left\{1 + \left(1 - \frac{2}{\nu(r)\log r}\right) r\log r \limsup_{\delta\to 0}\frac{\nu(r+\delta)-\nu(r)}{\delta}\right\}$$
$$\leq \frac{r^{\nu(r)}}{\nu(r)}\left\{1 + \frac{2}{\nu(r)\log r}\right\} = \{1+o(1)\}\frac{r^{\nu(r)}}{\nu(r)}. \tag{2.1.14}$$

A similar process for $r^{\nu(r)}/\nu(r)$ shows that

$$\varphi(r) = \int_1^r \frac{1}{t}\int_1^t \frac{du}{u^{1-\nu(u)}}\,dt \geqslant \{1+o(1)\}\frac{r^{\nu(r)}}{\nu(r)^2}, \tag{2.1.15}$$

which is sufficient, with Theorem 15, for the first part of the theorem. If $\nu(r)$ satisfies (2.1.11) then we obtain

$$\liminf_{\delta\to 0} \frac{r}{\delta}\left\{\frac{(r+\delta)^{\nu(r+\delta)}}{\nu(r+\delta)^2} - \frac{r^{\nu(r)}}{\nu(r)^2}\right\} = \frac{r^{\nu(r)}}{\nu(r)}\left\{1 + \left(1 - \frac{2}{\nu(r)\log r}\right) r\log r \liminf_{\delta\to 0}\frac{\nu(r+\delta)-\nu(r)}{\delta}\right\}$$
$$\geqslant \{1+o(1)\}\frac{r^{\nu(r)}}{\nu(r)}. \tag{2.1.16}$$

Treating $r^{\nu(r)}/\nu(r)$ similarly leads to the reverse inequality to (2.1.16) and hence

$$\varphi(r) = \{1+o(1)\}\frac{r^{\nu(r)}}{\nu(r)^2}, \tag{2.1.17}$$

which is sufficient to complete the proof of the theorem.

There is also a result, corresponding to Theorem 13, for functions regular in a sector.

THEOREM 17. Suppose that $f(z)$ is regular in a sector $S(\alpha, \ell)$ and satisfies

$$\liminf_{r \to \infty} \frac{\log M(r,\alpha,f)}{\varphi(r)} < \frac{4\pi}{\Delta^2 3\sqrt{3}}, \qquad (2.1.18)$$

$$\limsup_{r \to \infty} \frac{\log |f(re^{\pm i\alpha})|}{\varphi(r)} \leq \gamma < \frac{4\pi}{\Delta^2 3\sqrt{3}}, \qquad (2.1.19)$$

and

$$\limsup \frac{\log |f(z_n)|}{\varphi(|z_n|)} \leq \varkappa < \frac{4\pi}{\Delta^2 3\sqrt{3}}, \qquad (2.1.20)$$

where the discs

$$|z - z_n| \leq \Delta |z_n|^{1-\frac{1}{2}\nu(|z_n|)} \qquad (2.1.21)$$

cover $S(\alpha, \ell)$ and $d^2\varphi(r)/d(\log r)^2 = r^{\nu(r)} \uparrow \to \infty$ with $\nu(r) \downarrow \to \rho$; then

$$\limsup_{r \to \infty} \frac{\log M(r,\alpha,f)}{\varphi(r)} \leq \sup\{\gamma, \varkappa\}. \qquad (2.1.22)$$

Furthermore, if both $f(re^{\pm i\alpha})$ and $\{f(z_n)\}$ are bounded then $f(z)$ is bounded in $S(\alpha, \ell)$.

2.2. Special products. We shall establish these results by using cardinal series interpolation techniques and we begin by constructing a suitable function $\eta(z)$. The construction of the function and the proof of its properties are based on a simpler product form which we describe in this section.

Consider a function $h(r) \downarrow \to 0$ as $r \to \infty$ which is such that $h(r) = \Lambda r^{-\frac{1}{2}\nu(r)}$ with $\nu(r) \downarrow \to \rho$. We first note that $r + 2\pi r h(r)$ is increasing for $r \geqslant \exp\{2\pi\Lambda/e\}$. For if $r_2 > r_1 \geqslant e^{2\pi\Lambda/e}$ then

$$r_2 + 2\pi r_2 h(r_2) = r_2 + 2\pi\Lambda r_2^{1-\frac{1}{2}\nu(r_2)} \geqslant r_2 + 2\pi\Lambda r_2^{1-\frac{1}{2}\nu(r_1)}$$

and this is greater than $r_1 + 2\pi\Lambda r_1^{1-\frac{1}{2}\nu(r_1)}$, since

$$\frac{d}{dr}(r + 2\pi\Lambda r^{1-\lambda}) = 1 + 2\pi\Lambda(1-\lambda)r^{-\lambda}$$

is positive for all r if $\lambda \leq 1$ and for all r greater than $\exp\{\frac{1}{\lambda}\log(2\pi\Lambda(\lambda-1))\}$ for $\lambda > 1$. The result follows from the inequality

$$\sup_{\lambda > 1} \frac{1}{\lambda} \log 2\pi\Lambda(\lambda-1) \leq \sup \frac{1}{\lambda} \log 2\pi\Lambda\lambda = \frac{2\pi\Lambda}{e}.$$

We now choose $r_1 > \exp\{2\pi\Lambda/e\}$ large enough to ensure that

$$\left[\frac{1}{h(r_1)}\right] > \left[\tfrac{1}{2}e\right] \tag{2.2.1}$$

and define a sequence $\{r_p\}$ by the recurrence relation

$$r_{p+1} - r_p = 2\pi r_p h(r_p) , \tag{2.2.2}$$

$r_p \uparrow \to \infty$. We set

$$\mu(z, \Lambda, \nu(r)) = \mu(z) = \prod_{p=1}^{\infty} \left\{ 1 - \left(\frac{z}{r_p}\right)^{1+[1/h(r_p)]} \right\}. \tag{2.2.3}$$

When $\nu(r)$ is defined clearly by the context we may abbreviate $\mu(z, \Lambda, \nu(r))$ to $\mu_\Lambda(z)$. Also, we denote by $n^*(r)$ the number of r_p not exceeding r and by $n(r)$ the number of zeros of $\mu(z)$ in $|z| \leqslant r$. We write

$$\mu(z) = \prod_{p=1}^{n^*(r)} \left(\frac{z}{r_p}\right)^{1+[1/h(r_p)]} (-1)^{n^*(r)} \prod_{p=1}^{n^*(r)} \left\{1 - \left(\frac{r_p}{z}\right)^{1+[1/h(r_p)]}\right\} \times$$

$$\times \prod_{p=n^*(r)+1}^{\infty} \left\{1 - \left(\frac{z}{r_p}\right)^{1+[1/h(r_p)]}\right\} , \qquad (2.2.4)$$

$$= \Pi_1 \, \Pi_2 \, \Pi_3 \, .$$

Since $\nu(r)$ is non-increasing, it follows that if $r'' \geq r'$ then

$$1 \leq \frac{h(r')}{h(r'')} = \frac{r''^{\frac{1}{2}\nu(r'')}}{r'^{\frac{1}{2}\nu(r')}} \leq \left(\frac{r''}{r'}\right)^{\frac{1}{2}\nu(r')} = \left(1 + \frac{r''-r'}{r'}\right)^{\frac{1}{2}\nu(r')} . \qquad (2.2.5)$$

Also

$$\frac{1}{h(r_p)}\left\{\frac{1}{1+2\pi h(r_p)}\right\} = \frac{r_p}{h(r_p)\, r_{p+1}} \leq \frac{1}{2\pi}\int_{r_p}^{r_{p+1}} \frac{dt}{h(t)^2 t} \leq \frac{h(r_p)}{h(r_{p+1})^2} = \frac{1}{h(r_p)} \left\{\frac{h(r_p)}{h(r_{p+1})}\right\}^2 . \qquad (2.2.6)$$

It follows from (2.2.5) that

$$1 \leq \frac{h(r_p)}{h(r_{p+1})} \leq \left(1 + h(r_p)\right)^{\frac{1}{2}\nu(r_p)} = 1 + O(\nu(r_p) h(r_p)) , \qquad (2.2.7)$$

and therefore we have

$$\frac{1}{h(r_p)} = \left\{1 + O(h(r_p))\right\} \frac{1}{2\pi} \int_{r_p}^{r_{p+1}} \frac{dt}{h(t)^2 t} \, . \qquad (2.2.8)$$

Since, in the manner of (2.1.15),

$$\int_1^r \frac{dt}{h(t)^2 t} \geq \frac{1+o(1)}{\nu(r)\, h(r)^2} , \qquad (2.2.9)$$

we have

$$\frac{1+O(h(r_{n^*(r)}))}{h(r_{n^*(r)})} = \frac{1}{2\pi}\int_{r_{n^*}}^{r_{n^*+1}} \frac{dt}{h(t)^2 t} = o\left(\int_1^r \frac{dt}{h(t)^2 t}\right) , \qquad (2.2.10)$$

and therefore

$$n(r) = \sum_{p=1}^{n^*(r)} \left\{1 + \left[\frac{1}{h(r_p)}\right]\right\} = \sum_{p=1}^{n^*(r)} \frac{\{1 + O(h(r_p))\}}{h(r_p)} , \quad (2.2.11)$$

$$= \{1 + o(1)\} \frac{1}{2\pi} \int_1^r \frac{dt}{h(t)^2\, t} , \quad (2.2.12)$$

since the right-hand side is unbounded. Now

$$\log |\Pi_1| = \sum_{p=1}^{n^*(r)} \left\{1 + \left[\frac{1}{h(r_p)}\right]\right\} \log \frac{r}{r_p} = \int_{t=0}^{t=r} \log \frac{r}{t}\, dn(t) ,$$

$$= \int_0^r \frac{n(t)}{t}\, dt , \quad (2.2.13)$$

and therefore

$$\log|\Pi_1| = \{1 + o(1)\} \frac{1}{2\pi} \int_1^r \frac{1}{t} \int_1^t \frac{du}{h(u)^2\, u}\, dt , \quad (2.2.14)$$

$$\geqslant \frac{\{1 + o(1)\}}{2\pi \nu(r)^2 h(r)^2} . \quad (2.2.15)$$

If $r_p \geqslant r$ then

$$-\frac{1}{h(r_p)} \log \frac{r}{r_p} \geqslant \frac{1}{h(r_p)} \left(1 - \frac{r}{r_p}\right) = \frac{r_p - r}{h(r_p)\, r_p} \quad (2.2.16)$$

and, since

$$r_{n^*+2} = r_{n^*+1} + 2\pi\, r_{n^*+1}\, h(r_{n^*+1}) \geqslant r + 2\pi\, h(r)\, r$$

and

$$r_{n^*+2} = r_{n^*+1} \{1 + 2\pi\, h(r_{n^*+1})\} = \{r_{n^*} + 2\pi h(r_{n^*})\, r_{n^*}\}\{1 + 2\pi\, h(r_{n^*+1})\}$$
$$\leqslant \{r + 2\pi r h(r)\}\{1 + 2\pi\, h(r)\} = r\{1 + 2\pi\, h(r)\}^2 ,$$

we have

$$\frac{r_{n^*+2} - r}{2\pi\, h(r_{n^*+2})\, r_{n^*+2}} \geqslant \frac{r\, h(r)}{h(r_{n^*+2})\, r_{n^*+2}} \geqslant \frac{r}{r_{n^*+2}} \geqslant \frac{1}{\{1 + 2\pi h(r)\}^2} ,$$

which, for $r \geqslant r_1$, is at least $(1+2\pi)^{-2} \geqslant 2^{-6}$. Now $\frac{1}{h(t)} \log \frac{r}{t}$ is an decreasing function of t for $t > r$ and

therefore

$$\frac{1}{h(r_p)} \log \frac{r}{r_p} \leq 2^{-6}$$

for $p \geq n^*(r)+2$, and hence

$$\left(\frac{r}{r_p}\right)^{1+[1/h(r_p)]} \leq \left(\frac{r}{r_p}\right)^{1/h(r_p)} \leq e^{-2^{-6}} < 1 . \qquad (2.2.17)$$

It follows that we can expand

$$\log\left\{1-\left(\frac{z}{r_p}\right)^{1+[1/h(r_p)]}\right\} = -\sum_{s=1}^{\infty} \frac{1}{s}\left(\frac{z}{r_p}\right)^{s(1+[1/h(r_p)])} \qquad (2.2.18)$$

to give

$$\left|\log\left\{1-\left(\frac{z}{r_p}\right)^{1+[1/h(r_p)]}\right\}\right| \leq \frac{1}{1-e^{-2^{-6}}}\left(\frac{r}{r_p}\right)^{1+[1/h(r_p)]}, \qquad (2.2.19)$$

uniformly for $p \geq n^*(r)+2$.

Now

$$\left(\frac{r}{r_p}\right)^{1+[1/h(r_p)]} \leq \left(\frac{r}{r_p}\right)^{1/h(r_p)} \leq \frac{\{1+h(r_{p-1})\}}{2\pi}\int_{r_{p-1}}^{r_p} \frac{(r/t)^{1/h(t)}}{h(t)\,t}\,dt, \qquad (2.2.20)$$

and therefore

$$\sum_{p=n^*+2}^{\infty} \left(\frac{r}{r_p}\right)^{1+[1/h(r_p)]} \leq \frac{K}{\Lambda}\int_r^{\infty} \frac{(r/t)^{1/h(t)}}{t^{1-\frac{1}{2}\nu(t)}}\,dt, \qquad (2.2.21)$$

and, since $(r/t)^{1/h(t)} \leq (r/t)^{1/h(r)}$ for $t \geq r$ and $\nu(t)$ is decreasing, this is at most

$$K r^{1/h(r)} \int_r^{\infty} t^{-1-1/h(r)+\frac{1}{2}\nu(r)}\,dt = \frac{K}{1-\frac{1}{2}\nu(r)\,h(r)} , \qquad (2.2.22)$$

provided that r is large enough to ensure that $1/h(r) - \frac{1}{2}\nu(r)$ is positive. We have therefore shown that

$$\left|\log \prod_{p=n^*+2}^{\infty}\left\{1-\left(\frac{z}{r_p}\right)^{1+[1/h(r_p)]}\right\}\right| \leq K , \qquad (2.2.23)$$

uniformly for all z .

The zeros of $\mu(z)$ are the points

$$z_{p,q}(\Lambda, \nu(r)) = z_{p,q} = r_p \exp\left\{\frac{2\pi i q}{1+[1/h(r_p)]}\right\} \tag{2.2.24}$$

($p = 1, 2, \ldots$; $q = 0, 1, \ldots, [1/h(r_p)]$). If z lies outside the set of discs

$$|z - z_{p,q}| \leqslant \delta\, r_p^{1-\frac{1}{2}\nu(r_p)} , \tag{2.2.25}$$

where δ ($<\frac{1}{4}\Lambda$) is a fixed positive number, then for $p \leqslant n^*(r)$,

$$\left|1 - \left(\frac{r_p}{z}\right)^{1+[1/h(r_p)]}\right| \geqslant \inf_{0\leqslant \chi \leqslant 2\pi} \left|1 - \left\{\frac{r_p}{r_p + \frac{\delta r_p h(r_p) e^{i\psi}}{\Lambda}}\right\}^{1+[1/h(r_p)]}\right| , \tag{2.2.26}$$

$$\sim \inf_{0\leqslant \psi \leqslant 2\pi} \left|1 - \exp\left\{-\frac{\delta e^{i\psi}}{\Lambda}\right\}\right| \quad \text{as } p \to \infty,$$

$$\geqslant K_1 > 0 ,$$

and therefore

$$\left|1 - \left(\frac{r_p}{z}\right)^{1+[1/h(r_p)]}\right| \geqslant K_2 > 0 . \tag{2.2.27}$$

We can make a similar estimate for the term

$$\left\{1 - \left(\frac{z}{r_{n^*+1}}\right)^{1+[1/h(r_{n^*+1})]}\right\} .$$

All of these terms have modulus at most 2 and there are $n^*(r)+1$ of them and therefore

$$\left|\log\left\{\prod_{p=1}^{n^*(r)}\left(1-\left(\frac{r_p}{z}\right)^{1+[1/h(r_p)]}\right)\left(1-\left(\frac{z}{r_{n^*+1}}\right)^{1+[1/h(r_{n^*+1})]}\right)\right\}\right| \leqslant (n^*+1)\log\frac{2}{K_2} . \tag{2.2.28}$$

Since

$$n^*(r) \leqslant K\int_1^r \frac{dt}{h(t)t} \leqslant \frac{K\log r}{h(r)} = \frac{2K\log\frac{1}{h(r)}}{\nu(r)^2 h(r)} = O\left(\frac{1}{\nu(r)h(r)^2}\right), \tag{2.2.29}$$

it follows from (2.2.13-15), (2.2.23) and the above that, for z outside

$$(B_\delta) \equiv \bigcup_{(p,q)} \left\{ |z - z_{p,q}| \le \delta |z_{p,q}|^{1-\frac{1}{2}\nu(|z_{p,q}|)} \right\}, \qquad (2.2.30)$$

$$\log|\mu(z)| = \frac{1+o(1)}{2\pi} \int_1^r \frac{1}{t} \int_1^t \frac{du}{h(u)^2 u}\, dt, \qquad (2.2.31)$$

$$= \frac{1+o(1)}{2\pi\Lambda^2} \int_1^r \frac{1}{t} \int_1^t \frac{du}{u^{1-\nu(u)}}\, dt. \qquad (2.2.32)$$

We now write

$$\tilde{\omega}_N(z, \Lambda, \nu(r)) = \tilde{\omega}_{N,\Lambda}(z) = \prod_{p=1}^{N} \left\{ 1 - \left(\frac{z}{r_p}\right)^{1+[1/h(r_p)]} \right\}, \qquad (2.2.33)$$

and note that we have also shown that

$$\log|\tilde{\omega}_{N,\Lambda}(z)| \le \frac{1+o(1)}{2\pi\Lambda^2} \int_1^r \frac{1}{t} \int_1^t \frac{du}{u^{1-\nu(u)}}\, dt \qquad (2.2.34)$$

as $z \to \infty$, uniformly in N.

<u>2.4. Definition and properties of $\eta(z)$.</u> We next consider a set $\{\zeta_{p,q}\} = \{\zeta_{p,q}(\Lambda, \nu(r))\}$ which satisfies, for some fixed positive constants D, δ;

$$|\zeta_{p,q} - z_{p,q}| \le D |z_{p,q}|^{1-\frac{1}{2}\nu(|z_{p,q}|)}, \qquad (2.3.1)$$

and is such that the discs

$$|z - \zeta_{p,q}| \le \delta |\zeta_{p,q}|^{1-\frac{1}{2}\nu(|\zeta_{p,q}|)} \qquad (2.3.2)$$

are disjoint. We denote the union of the discs of (2.3.2) by (A_δ) and assume that δ ($<\frac{1}{4}\Lambda$) is chosen small enough to ensure that the discs of (2.2.25) are also disjoint.

We define

$$\eta(z) = \eta(z, \Lambda, \nu(r)) = \prod_{p=1}^{\infty} \left\{ \prod_{q=0}^{[1/h(r_p)]} E\left(\frac{z}{\zeta_{pq}}, [\tfrac{1}{2}\rho]\right) \right\} . \quad (2.3.3)$$

Now, if ω_p is a primitive p-th root of unity and $p > s$ then

$$\sum_{q=1}^{p} \left(\frac{1}{\omega_p^q}\right)^s = \sum_{q=1}^{p} \left(\frac{1}{\omega_p^s}\right)^q = \frac{1}{\omega_p^s} \frac{1-\left(\frac{1}{\omega_p^s}\right)^p}{1-\frac{1}{\omega_p^s}} = 0 , \quad (2.3.4)$$

and hence, in view of (2.2.1), we have for z outside (B_δ),

$$\frac{\eta(z)}{\mu(z)} = \prod_{p=1}^{\infty} \left\{ \prod_{q=0}^{[1/h(r_p)]} \frac{1-\frac{z}{\zeta_{pq}}}{1-\frac{z}{z_{pq}}} \exp\left\{ \sum_{s=1}^{[\frac{1}{2}\rho]} \frac{1}{s}\left(\frac{z}{\zeta_{pq}}\right)^s - \sum_{s=1}^{[\frac{1}{2}\rho]} \frac{1}{s}\left(\frac{z}{z_{pq}}\right)^s \right\} \right\} , \quad (2.3.5)$$

with the convention that $\sum_{s=1}^{0} \equiv 0$. To estimate part of this product we shall use the following simple result.

LEMMA 2.3.1. Suppose that x and y are such that

$$|x| \leq \lambda < 1 \quad , \quad |y| \leq \lambda < 1 \quad ; \quad (2.3.6)$$

then

$$\left| \log \left\{ \frac{(1-x) \exp\left\{ \sum_{s=1}^{k} \frac{1}{s} x^s \right\}}{(1-y) \exp\left\{ \sum_{s=1}^{k} \frac{1}{s} y^s \right\}} \right\} \right| \leq K(\lambda) |y-x| \left(\sup\{|x|, |y|\}\right)^k . \quad (2.3.7)$$

Proof. In view of the inequalities of (2.3.6), we can expand the logarithm terms as power series in x and y to give

$$\left| \log \left\{ \frac{(1-x) \exp\left\{ \sum_{s=1}^{k} \frac{1}{s} x^s \right\}}{(1-y) \exp\left\{ \sum_{s=1}^{k} \frac{1}{s} y^s \right\}} \right\} \right| = \left| \sum_{s=k+1}^{\infty} \frac{1}{s} y^s - \sum_{s=k+1}^{\infty} \frac{1}{s} x^s \right| ,$$

and, since both the series are absolutely convergent, this is equal to

$$\left| \sum_{s=k+1}^{\infty} \frac{1}{s} (y^s - x^s) \right| = \left| (y-x) \sum_{s=k+1}^{\infty} \frac{1}{s} (y^{s-1} + y^{s-2}x + \cdots x^{s-1}) \right| ,$$

$$\leqslant |y-x| \sum_{s=k+1}^{\infty} \left(\sup\{|x|, |y|\}\right)^{s-1},$$

$$\leqslant \frac{|y-x| \left(\sup\{|x|, |y|\}\right)^{k}}{1-\lambda},$$

which is (2.3.8) with $K(\lambda) = \frac{1}{1-\lambda}$.

The contents of this lemma only represent in a convenient form a technique often used in the estimation of products.

For all sufficiently large $|z_{p,q}|$, we have

$$\left|\frac{z_{p,q}}{\zeta_{p,q}}\right| \leqslant \frac{1}{1 - D|z_{p,q}|^{-\frac{1}{2}\nu(|z_{p,q}|)}} \leqslant \frac{3}{2},$$

and if, in addition, $|z/z_{p,q}| \leq \frac{1}{2}$ then $|z/\zeta_{p,q}| \leq \frac{3}{4}$. Lemma 2.3.1, when applied to terms satisfying these conditions, gives

$$\left|\log\left\{\frac{\left(1-\frac{z}{\zeta_{p,q}}\right)\exp\left\{\sum_{s=1}^{[\frac{1}{2}\rho]}\frac{1}{s}\left(\frac{z}{\zeta_{p,q}}\right)^{s}\right\}}{\left(1-\frac{z}{z_{p,q}}\right)\exp\left\{\sum_{s=1}^{[\frac{1}{2}\rho]}\frac{1}{s}\left(\frac{z}{z_{p,q}}\right)^{s}\right\}}\right\}\right| \leqslant 4\left|\frac{z}{\zeta_{p,q}}-\frac{z}{z_{p,q}}\right|\left(\sup\left\{\left|\frac{z}{z_{p,q}}\right|, \left|\frac{z}{\zeta_{p,q}}\right|\right\}\right)^{[\frac{1}{2}\rho]},$$

$$\leqslant 4\left(\frac{3}{2}\right)^{[\frac{1}{2}\rho]}\left|\frac{\zeta_{p,q}-z_{p,q}}{\zeta_{p,q}}\right|\left|\frac{z}{z_{p,q}}\right|^{1+[\frac{1}{2}\rho]},$$

$$\leqslant K h(|z_{p,q}|)\left|\frac{z}{z_{p,q}}\right|^{1+[\frac{1}{2}\rho]}, \quad (2.3.9)$$

and therefore

$$\left|\sum_{q=0}^{[1/h(r_p)]} \log\left\{\frac{E(z/\zeta_{p,q}, [\frac{1}{2}\rho])}{E(z/z_{p,q}, [\frac{1}{2}\rho])}\right\}\right| \leqslant K\left|\frac{z}{z_{p,q}}\right|^{1+[\frac{1}{2}\rho]}, \quad (2.3.10)$$

which gives

$$\left|\sum_{|z_{p,q}|\geqslant 2r} \log\left\{\frac{E(z/\zeta_{p,q}, [\frac{1}{2}\rho])}{E(z/z_{p,q}, [\frac{1}{2}\rho])}\right\}\right| \leqslant K r^{[1+\frac{1}{2}\rho]} \sum_{r_p\geqslant 2r} \frac{1}{r_p^{[1+\frac{1}{2}\rho]}}. \quad (2.3.11)$$

Now

$$\frac{1}{r_p^{[1+\frac{1}{2}\rho]}} \leq \frac{1+O(h(r_p))}{2\pi} \int_{r_p}^{r_{p+1}} \frac{dt}{h(t)\, t^{[2+\frac{1}{2}\rho]}} \quad , \qquad (2.3.12)$$

and hence

$$\sum_{r_p \geq 2r} \frac{1}{r_p^{[1+\frac{1}{2}\rho]}} \leq K \int_{2r}^{\infty} \frac{dt}{h(t)\, t^{[2+\frac{1}{2}\rho]}} \quad ,$$

$$\leq K \int_{2r}^{\infty} \frac{dt}{t^{[2+\frac{1}{2}\rho] - \frac{1}{2}\nu(2r)}} \quad . \qquad (2.3.13)$$

Since, for some fixed positive ϵ , $[2+\frac{1}{2}\rho] - \frac{1}{2}\nu(2r) \geq 1+\epsilon$ for all sufficiently large r , this is equal to

$$\frac{K}{[\frac{1}{2}\rho+1] - \frac{1}{2}\nu(2r)} \left[\frac{-1}{t^{[1+\frac{1}{2}\rho] - \frac{1}{2}\nu(2r)}} \right]_{2r}^{\infty} \quad ,$$

which can be seen to be less than

$$\frac{K}{r^{[1+\frac{1}{2}\rho]}\, h(r)} \quad .$$

Hence, we have shown that

$$\left| \log \left| \prod_{|z_{p,q}| \geq 2r} \frac{E(z/\zeta_{p,q}, [\frac{1}{2}\rho])}{E(z/z_{p,q}, [\frac{1}{2}\rho])} \right| \right| \leq \frac{K}{h(r)} \quad . \qquad (2.3.14)$$

For $|z_{p,q}| < 2r$ with $\rho \geq 2$ we use

$$\left| \sum_{s=1}^{[\frac{1}{2}\rho]} \frac{1}{s} \left(\frac{z}{\zeta_{p,q}}\right)^s - \sum_{s=1}^{[\frac{1}{2}\rho]} \frac{1}{s} \left(\frac{z}{z_{p,q}}\right)^s \right| \leq \left| \frac{z}{\zeta_{p,q}} - \frac{z}{z_{p,q}} \right| \sum_{s=1}^{[\frac{1}{2}\rho]} \left(\sup\left\{ \left|\frac{z}{\zeta_{p,q}}\right|, \left|\frac{z}{z_{p,q}}\right| \right\} \right)^{s-1} \qquad (2.3.15)$$

$$\leq K \left| \frac{\zeta_{p,q} - z_{p,q}}{\zeta_{p,q}} \right| \left| \frac{z}{z_{p,q}} \right|^{[\frac{1}{2}\rho]} \leq K\, h(|z_{p,q}|) \left| \frac{z}{z_{p,q}} \right|^{[\frac{1}{2}\rho]} \quad , \qquad (2.3.16)$$

and therefore

$$\left| \sum_{q=0}^{[1/h(r_p)]} \left\{ \sum_{s=1}^{[\frac{1}{2}\rho]} \frac{1}{s} \left(\frac{z}{\zeta_{p,q}}\right)^s - \sum_{s=1}^{[\frac{1}{2}\rho]} \frac{1}{s} \left(\frac{z}{z_{p,q}}\right)^s \right\} \right| \leq K \left| \frac{z}{z_{p,q}} \right|^{[\frac{1}{2}\rho]} \quad . \qquad (2.3.17)$$

It follows that

$$\left| \sum_{|z_{p,q}|<2r} \left\{ \sum_{s=1}^{[\frac{1}{2}\rho]} \frac{1}{s}\left(\frac{z}{\zeta_{p,q}}\right)^s - \sum_{s=1}^{[\frac{1}{2}\rho]} \frac{1}{s}\left(\frac{z}{z_{p,q}}\right)^s \right\} \right| \leq K r^{[\frac{1}{2}\rho]} \int_1^{2r} \frac{dt}{h(t)\, t^{[\frac{1}{2}\rho+1]}} \quad , \tag{2.3.18}$$

and, since we can always find a number λ such that

$$0 > -\lambda > \tfrac{1}{2}\nu(t) - [\tfrac{1}{2}\rho+1] \geqslant -1 \tag{2.3.19}$$

for all sufficiently large t , we have

$$\int_1^{2r} \frac{dt}{h(t)\, t^{[\frac{1}{2}\rho+1]}} \leq K + \frac{1}{\Lambda}\int_0^{2r} t^{-\lambda}\, dt \leq K r^{1-\lambda} \ . \tag{2.3.20}$$

This gives

$$K r^{[\frac{1}{2}\rho]} \int_1^{2r} \frac{dt}{h(t)\, t^{[\frac{1}{2}\rho+1]}} \leq K r^{[\frac{1}{2}\rho]+1-\lambda} \leq \frac{K r^{-\lambda}}{h(r)^2} \quad , \tag{2.3.21}$$

since $\rho \geqslant 2$, and therefore we have, for all ρ ,

$$\left| \log \left| \prod_{|z_{p,q}|<2r} \exp\left\{ \sum_{s=1}^{[\frac{1}{2}\rho]} \frac{1}{s}\left(\frac{z}{\zeta_{p,q}}\right)^s - \sum_{s=1}^{[\frac{1}{2}\rho]} \frac{1}{s}\left(\frac{z}{z_{p,q}}\right)^s \right\} \right| \right| \leq \frac{K r^{-\lambda}}{h(r)^2} \ . \tag{2.3.22}$$

We are left to estimate the product of the factors

$$\frac{1 - z/\zeta_{p,q}}{1 - z/z_{p,q}} = 1 + \frac{z(\zeta_{p,q} - z_{p,q})}{\zeta_{p,q}(z_{p,q} - z)} \tag{2.3.23}$$

for which $|z_{p,q}| < 2r$. Now

$$\left| \frac{z(\zeta_{p,q} - z_{p,q})}{(z_{p,q} - z)\,\zeta_{p,q}} \right| \leq \frac{K\, r\, h(|z_{p,q}|)}{|z_{p,q} - z|} \ . \tag{2.3.24}$$

We consider first those terms for which $z_{p,q}$ satisfies

$$|z_{p,q} - z| \geqslant h(r)^{\frac{1}{2}}\, r \ . \tag{2.3.25}$$

It follows from the simple inequality

$$h(r^{2/3}) = \Lambda (r^{2/3})^{-\frac{1}{2}\nu(r^{2/3})} \leq \Lambda (r^{2/3})^{-\frac{1}{2}\nu(r)} = h(r)^{2/3} \tag{2.3.26}$$

that if $|z_{p,q}| \geq r^{2/3}$ then $h(|z_{p,q}|)/h(r)^{\frac{1}{2}} \leq h(r)^{1/6}$, and therefore

$$\left| \frac{1 - z/\zeta_{p,q}}{1 - z/z_{p,q}} \right| = 1 + O\left(h(r)^{1/6} \right) . \qquad (2.3.27)$$

Hence, for sufficiently large r,

$$\left| \log \left| \prod_{\substack{r^{2/3} < |z_{p,q}| < 2r \\ |z_{p,q} - z| \geq h(r)^2 r}} \left\{ \frac{1 - z/\zeta_{p,q}}{1 - z/z_{p,q}} \right\} \right| \right| \leq K \left\{ n(2r) - n(r^{2/3}) \right\} h(r)^{1/6}, \qquad (2.3.28)$$

$$\leq K\, h(r)^{1/6} \int_1^{2r} \frac{dt}{h(t)^2 t} \leq \frac{K\, h(r)^{1/6} \log r}{h(r)^2} . \qquad (2.3.29)$$

For $|z_{p,q}| \leq r^{2/3}$ we have

$$\left| \frac{z(\zeta_{p,q} - z_{p,q})}{(z_{p,q} - z)\, \zeta_{p,q}} \right| \leq \frac{K\, h(|z_{p,q}|)}{1 - r^{-1/3}} \leq K\, h(|z_{p,q}|), \qquad (2.3.30)$$

and therefore

$$\sum_{q=0}^{[1/h(r_p)]} \left| \frac{z(\zeta_{p,q} - z_{p,q})}{(z_{p,q} - z)\, \zeta_{p,q}} \right| \leq K . \qquad (2.3.31)$$

This gives

$$\sum_{|z_{p,q}| \leq r^{2/3}} \left| \frac{z(\zeta_{p,q} - z_{p,q})}{(z_{p,q} - z)\, \zeta_{p,q}} \right| \leq K\, n^*(r^{2/3}) , \qquad (2.3.32)$$

and this is easily seen to be less than

$$\frac{K \log r^{2/3}}{h(r)^{2/3}} \leq \frac{K \log r}{h(r)} = \frac{K\, h(r) \log \frac{1}{h(r)}}{\nu(r)\, h(r)^2} , \qquad (2.3.33)$$

and hence

$$\log \left| \prod_{|z_{p,q}| \leq r^{2/3}} \left\{ \frac{1 - z/\zeta_{p,q}}{1 - z/z_{p,q}} \right\} \right| = o\left(\frac{1}{\nu(r)\, h(r)^2} \right) . \qquad (2.3.34)$$

We have now only to consider those terms for which

$$|z_{p,q} - z| < h(r)^{\frac{1}{2}} r \, . \tag{2.3.35}$$

We now write

$$\frac{1 - z/\zeta_{p,q}}{1 - z/z_{p,q}} = \frac{(\zeta_{p,q} - z)\, z_{p,q}}{(z_{p,q} - z)\, \zeta_{p,q}} \, , \tag{2.3.36}$$

and, provided z lies outside the discs of (A_δ) and (B_δ), we have

$$\frac{\delta h(r) r}{K \Lambda h(r)^{\frac{1}{2}} r} \leqslant \left| \frac{(\zeta_{p,q} - z)\, z_{p,q}}{(z_{p,q} - z)\, \zeta_{p,q}} \right| \leqslant \frac{K \Lambda h(r)^{\frac{1}{2}} r}{\delta h(r) r} \, . \tag{2.3.37}$$

Therefore

$$\left| \log \left| \frac{(\zeta_{p,q} - z)\, z_{p,q}}{(z_{p,q} - z)\, \zeta_{p,q}} \right| \right| \leqslant K \log \frac{1}{h(r)} \, . \tag{2.3.38}$$

We now show that the number of $z_{p,q}$ satisfying (2.3.35) does not exceed

$$\frac{K}{h(r)} \, . \tag{2.3.39}$$

It is easy to see that if $z_{p,q}$ satisfies (2.3.35) then it also satisfies both

$$|\arg z_{p,q} - \arg z| \leqslant \sin^{-1}\left(h(r)^{\frac{1}{2}}\right) \leqslant \tfrac{1}{2}\pi\, h(r)^{\frac{1}{2}} \tag{2.3.40}$$

and

$$\big|\, |z_{p,q}| - r \,\big| \leqslant h(r)^{\frac{1}{2}} r \, . \tag{2.3.41}$$

It follows from (2.2.5) that if $r_p / r \to 1$ then $h(r_p)/h(r) \to 1$ and therefore the number of $z_{p,q}$ satisfying these conditions for fixed p does not exceed $K / h(r)^{\frac{1}{2}}$. Also, if $p_0 < p \leqslant p_1$, then

$$r_p - r_{p-1} = 2\pi\, h(r_{p-1})\, r_{p-1} > 2\pi\, h(r_{p_1})\, r_{p_0} \, , \tag{2.3.42}$$

and therefore

$$r_{p_1} - r_{p_0} \geqslant 2\pi(p_1 - p_0)\, h(r_{p_1})\, r_{p_0} \; . \qquad (2.3.43)$$

Hence the number of $z_{p,q}$ satisfying (2.3.40-41) is less than

$$\frac{K}{h(r)^{\frac{1}{2}}} \cdot \frac{2h(r)^{\frac{1}{2}} r}{h(r + rh(r)^{\frac{1}{2}})\; r\,(1 - h(r)^{\frac{1}{2}})} \sim \frac{2K}{h(r)} \; . \qquad (2.3.44)$$

This, together with (2.3.38), shows that

$$\left| \log \left| \prod_{|z_{p,q} - z| < h(r)^{\frac{1}{2}} r} \left\{ \frac{1 - z/\zeta_{p,q}}{1 - z/z_{p,q}} \right\} \right| \right| \leqslant \frac{K}{h(r)} \log \frac{1}{h(r)} \, . \qquad (2.3.45)$$

If we now combine the results contained in (2.3.14), (2.3.22), (2.3.29), (2.3.34) and (2.3.45), we have, for z outside (A_δ) and (B_δ),

$$\left| \log \left| \frac{\eta(z)}{\mu(z)} \right| \right| \leqslant \frac{K}{h(r)} + \frac{K r^{-\lambda}}{h(r)^2} + \frac{K h(r)^{1/4} \log \frac{1}{h(r)}}{\nu(r)\, h(r)^2} = O\left(\frac{1}{\nu(r)\, h(r)^{2}} \right) , \qquad (2.3.46)$$

and hence, in view of (2.1.14) and (2.2.32),

$$\log |\eta(z)| = \frac{\{1 + o(1)\}}{2\pi\Lambda^2} \int_1^r \frac{1}{t} \int_1^t \frac{du}{u^{1-\nu(u)}}\, dt \, , \qquad (2.3.47)$$

for the same set of z. The same result holds for z outside the discs of $(A_{\frac{1}{4}\delta})$ and $(B_{\frac{1}{4}\delta})$. Now if a disc of $(B_{\frac{1}{4}\delta})$ intersects any disc of $(A_{\frac{1}{4}\delta})$ then the corresponding disc of (A_δ) contains the disc of $(B_{\frac{1}{4}\delta})$ in its interior, for all sufficiently large $|z_{p,q}|$. This can easily be seen, for if $r' \leqslant r''$ and $r'(1 + \delta h(r')) \geqslant r''(1 - \delta h(r''))$ then

$$r'' \leqslant r' \left\{ \frac{1 + \delta h(r')}{1 - \delta h(r')} \right\} = \{1 + O(h(r'))\}\, r' \, ,$$

and therefore

$$\frac{r'' h(r'')}{r' h(r)} \longrightarrow 1$$

as r' or r'' tends to infinity. We can therefore apply the minimum and maximum modulus principles to those discs of $(B_{\frac{1}{4}\delta})$ which do not intersect $(A_{\frac{1}{4}\delta})$ and, since

$$\int_{r(1-h(r))}^{r(1+h(r))} \frac{1}{t}\int_1^t \frac{du}{u^{1-\nu(u)}}\,dt \leqslant 2rh(r)\left\{\frac{1}{r(1-h(r))}\int_1^{r(1+h(r))} \frac{du}{u^{1-\nu(u)}}\right\},$$

$$\leqslant K\,\frac{h(r)\log\{r+rh(r)\}}{\{1-h(r)\}\{h(r+h(r))\}^2} \leqslant \frac{K\log r}{h(r)} = o\left(\frac{1}{\nu(r)h(r)^2}\right), \quad (2.3.48)$$

this gives

$$\log|\eta(z)| = \frac{1+o(1)}{2\pi\Lambda^2}\int_1^{|z|}\frac{1}{t}\int_1^t\frac{du}{u^{1-\nu(u)}}\,dt\,, \quad (2.3.49)$$

for z outside (A_δ).

We now use the separation condition of (2.3.2) to obtain a lower bound for the modulus of $\eta'(\zeta_{p,q})$. If we write, temporarily,

$$g(z) = \frac{\eta(z)}{z-\zeta_{p,q}}\,,\ z \neq \zeta_{p,q}\,,\quad g(\zeta_{p,q}) = \eta'(\zeta_{p,q});$$

then $g(z)$ is an integral function and, in particular, is regular and non-zero in

$$|z-\zeta_{p,q}| \leqslant \frac{\delta}{\Lambda}h(|\zeta_{p,q}|)|\zeta_{p,q}|\,.$$

An application of the minimum modulus principle gives

$$\log|\eta'(\zeta_{p,q})| \geqslant \inf_{0\leqslant\psi\leqslant2\pi}\log\left|f\left(\zeta_{p,q}+\frac{\delta}{\Lambda}h(|\zeta_{p,q}|)|\zeta_{p,q}|e^{i\psi}\right)\right| - \log\left\{\frac{\delta}{\Lambda}h(|\zeta_{p,q}|)|\zeta_{p,q}|\right\}$$

$$\geqslant \inf_{0\leqslant\psi\leqslant2\pi}\log\left|f\left(\zeta_{p,q}+\frac{\delta}{\Lambda}h(|\zeta_{p,q}|)|\zeta_{p,q}|e^{i\psi}\right)\right| - \log|\zeta_{p,q}|\,,$$

and this is easily seen to be at least

$$\frac{1+o(1)}{2\pi\Lambda^2}\int_1^{|\zeta_{p,q}|}\frac{1}{t}\int_1^t \frac{du}{u^{1-\nu(u)}}\,dt\,, \tag{2.3.50}$$

since

$$\log r = \frac{2}{\nu(r)}\log\frac{1}{h(r)} = o\left(\frac{1}{\nu(r)\,h(r)}\right).$$

We note here that the function $\varphi(r)$ of Theorem 15 and

$$\int_1^r \frac{1}{t}\int_1^t \frac{du}{u^{1-\nu(u)}}\,dt$$

differ by at most a constant and may, for our purposes, be regarded as being equal. We now collect the results of this section in the form of a theorem.

THEOREM 18. With $\{\zeta_{p,q}\}$ satisfying (2.3.1) and (2.3.2) and $\eta(z)$ as defined by (2.3.3) we can, for any positive ϵ, determine an $r_0(\epsilon)$ such that:

(i) $$\log|\eta(z)| \leqslant \frac{1+\epsilon}{2\pi\Lambda^2}\,\varphi(|z|) \tag{2.3.51}$$

for $|z| \geqslant r_0$;

(ii) $$\log|\eta(z)| \geqslant \frac{1-\epsilon}{2\pi\Lambda^2}\,\varphi(|z|) \tag{2.3.52}$$

for $|z| \geqslant r_0$ and z outside (A_δ);

(iii) $$\log|\eta'(\zeta_{p,q})| \geqslant \frac{1-\epsilon}{2\pi\Lambda^2}\,\varphi(|\zeta_{p,q}|) \tag{2.3.53}$$

for $|\zeta_{p,q}| \geqslant r_0$.

<u>2.4. Interpolation results for the sets $\{\zeta_{p,q}\}$.</u> We

are now in a position to prove our basic interpolation theorem, stated in terms of $\{\zeta_{p,q}\}$. With the notaition of the preceeding sections, we have

THEOREM 19. Suppose that $f(z)$ is an integral function which satisfies

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{\varphi(r)} < \frac{1}{2\pi\Lambda^2} = \tau \qquad (2.4.1)$$

and

$$\limsup \frac{\log|f(\zeta_{p,q})|}{\varphi(|\zeta_{p,q}|)} \leq \chi < \tau \ ; \qquad (2.4.2)$$

then

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{\varphi(r)} \leq \chi \ . \qquad (2.4.3)$$

Furthermore, if

$$|f(\zeta_{p,q})| \leq A \ , \qquad (2.4.4)$$

then $f(z)$ is a constant.

Proof. The principle of the proof is that of Iyer [14]. We first show that

$$f(z) = \eta(z) \sum_{(p,q)} \frac{f(\zeta_{p,q})}{(z-\zeta_{p,q})\,\eta'(\zeta_{p,q})} \ , \qquad (2.4.5)$$

the series being uniformly convergent in any compact set of z not containing a point $\zeta_{p,q}$. We can choose, for all sufficiently small ϵ , a sequence $\{r_n\}$ $r_n \uparrow\to \infty$ such that

$$\sup_n \frac{\log M(r_n,f)}{\varphi(r_n)} \leq \tau - 4\epsilon \ . \qquad (2.4.6)$$

We can then choose a closed contour Γ_n lying in $\{|z| \leq r_n\}$

and containing $\{|z| \leq r_n - h(r_n) r_n\}$ in its interior domain $D_i(\Gamma_n)$, such that Γ_n does not intersect any disc of (A_δ) and has length not exceeding $\pi^2 r_n$. To see this, we can take as Γ_n that part of the circle $|z| = r_n - \frac{1}{2}h(r_n) r_n$ which does not lie in any disc of (A_δ) and complete the curve by including the smaller section of the circumference of any disc of (A_δ) which this circle intersects.

Now consider

$$\frac{1}{2\pi i}\int_{\Gamma_n} \frac{f(\zeta)}{(\zeta - z)\,\eta(\zeta)}\, d\zeta = \frac{f(z)}{\eta(z)} - \sum_{\zeta_{p,q} \in D_i(\Gamma_n)} \frac{f(\zeta_{p,q})}{(z - \zeta_{p,q})\,\eta'(\zeta_{p,q})} \, . \qquad (2.4.7)$$

On Γ_n we have

$$|f(\zeta)| \leq \exp\{(\tau - 4\epsilon)\varphi(r_n)\}, \qquad (2.4.8)$$

and, for sufficiently large n, both

$$|\eta(\zeta)| \geq \exp\{(\tau - \epsilon)\varphi(r_n - h(r_n) r_n)\} \geq \exp\{(\tau - 2\epsilon)\varphi(r_n)\} \qquad (2.4.9)$$

and

$$|\zeta - z| \geq (1 - \epsilon) r_n \, . \qquad (2.4.10)$$

It follows that

$$\left| \int_{\Gamma_n} \frac{f(\zeta)}{\eta(\zeta)(\zeta - z)}\, d\zeta \right| \leq \frac{\pi^2}{1 - \epsilon} \exp\{-2\epsilon\varphi(r_n)\}, \qquad (2.4.11)$$

$$\longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

and therefore

$$\frac{f(z)}{\eta(z)} = \lim_{n \to \infty} \sum_{\zeta_{p,q} \in D_i(\Gamma_n)} \frac{f(\zeta_{p,q})}{(z - \zeta_{p,q})\,\eta'(\zeta_{p,q})} \, . \qquad (2.4.12)$$

Also, with z outside (A_δ),

$$\left|\frac{f(\zeta_{p,q})}{(z-\zeta_{p,q})\,\eta'(\zeta_{p,q})}\right| \leq K(\epsilon)\,\frac{\exp\{(\varkappa+\epsilon)\varphi(|\zeta_{p,q}|)\}}{\delta\, h(|\zeta_{p,q}|)\exp\{(\tau-\epsilon)\varphi(|\zeta_{p,q}|)\}}, \quad (2.4.13)$$

$$\leq K(\epsilon)\,\frac{\exp\{(\varkappa-\tau+2\epsilon)\varphi(|\zeta_{p,q}|)\}}{h(|\zeta_{p,q}|)},$$

and, provided $\epsilon < \frac{1}{4}(\tau-\varkappa)$, we have, in view of (2.1.14) and (2.3.48), that

$$\left|\frac{f(\zeta_{p,q})}{(z-\zeta_{p,q})\,\eta'(\zeta_{p,q})}\right| \leq K(\epsilon)\exp\{-\epsilon\,\varphi(r_p)\}. \quad (2.4.14)$$

It follows that

$$\sum_{(p,q)}\left|\frac{f(\zeta_{p,q})}{(z-\zeta_{p,q})\,\eta'(\zeta_{p,q})}\right| \leq K(\epsilon)\int_1^\infty \frac{e^{-\epsilon\varphi(t)}}{h(t)^2\,t}\,dt, \quad (2.4.15)$$

and, since

$$\varphi(r) = \int_1^r \frac{1}{t}\int_1^t \frac{du}{u^{1-\nu(u)}} \geq \int_1^r \frac{1}{t}\int_1^t \frac{du}{u}\,dt = (\log r)^2, \quad (2.4.16)$$

we have that the right-hand side of (2.4.15) is at most

$$K(\epsilon)\int_1^\infty t^{-\frac{1}{2}\epsilon\log t}\,dt \leq K_1(\epsilon). \quad (2.4.17)$$

We have therefore shown that

$$\sum_{(p,q)} \frac{f(\zeta_{p,q})}{(z-\zeta_{p,q})\,\eta'(\zeta_{p,q})}$$

is uniformly absolutely convergent for z outside (A_δ) and it follows from (2.4.12) that

$$\frac{f(z)}{\eta(z)} = \sum_{(p,q)} \frac{f(\zeta_{p,q})}{(z-\zeta_{p,q})\,\eta'(\zeta_{p,q})}. \quad (2.4.5)$$

It follows immediately from this that

$$\frac{z^m f(z)}{\eta(z)} = \sum_{(p,q)} \frac{\zeta_{p,q}{}^m f(\zeta_{p,q})}{(z-\zeta_{p,q})\,\eta'(\zeta_{p,q})}, \tag{2.4.18}$$

and similarly with z^m replaced by any polynomial, in particular

$$\frac{\tilde{\omega}_{N,\lambda}(z)\, f(z)}{\eta(z)} = \sum_{p,q} \frac{\tilde{\omega}_{N,\lambda}(\zeta_{p,q}) f(\zeta_{p,q})}{(z-\zeta_{p,q})\,\eta'(\zeta_{p,q})}. \tag{2.4.19}$$

If we choose λ so that

$$\frac{1}{2\pi\lambda^2} + \kappa < \frac{1}{2\pi\Lambda^2}, \tag{2.4.20}$$

it follows from (2.2.34) that the series in (2.4.19) is absolutely convergent, uniformly for all N and z outside (A_δ). Also, $\tilde{\omega}_{N,\lambda}(z)$ tends to $\mu_\lambda(z)$ as $N \to \infty$, uniformly on any comapct set of z, and therefore

$$\frac{\mu_\lambda(z)\, f(z)}{\eta(z)} = \sum_{(p,q)} \frac{\mu_\lambda(\zeta_{p,q}) f(\zeta_{p,q})}{(z-\zeta_{p,q})\,\eta'(\zeta_{p,q})}. \tag{2.4.21}$$

This is a departure from Iyer's method, which depends on the dominated absolute convergence of the power series for the function used instead of $\mu_\lambda(z)$ and an inversion in the order of summation. It follows that, if the regions $(B_{\delta(\lambda)})$, containing the zeros of $\mu_\lambda(z)$, and (A_δ) are excluded, then

$$|f(z)| \leq K \left|\frac{\eta(z)}{\mu_\lambda(z)}\right|, \tag{2.4.22}$$

$$\leq K \exp\left\{\left(\frac{1}{2\pi\Lambda^2} - \frac{1}{2\pi\lambda^2} + 2\epsilon\right)\varphi(|z|)\right\},$$

and using the maximum modulus principle that

$$|f(z)| \leq K \exp\left\{\left(\frac{1}{2\pi\Lambda^2} - \frac{1}{2\pi\lambda^2} + 3\epsilon\right)\varphi(|z|)\right\}, \tag{2.4.23}$$

for all z .Hence

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{\varphi(r)} \leq \frac{1}{2\pi\Lambda^2} - \frac{1}{2\pi\lambda^2} + 3\epsilon , \qquad (2.4.24)$$

for every $\epsilon>0$ and λ such that $\frac{1}{2\pi\Lambda^2} - \frac{1}{2\pi\lambda^2} > X$, and therefore

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{\varphi(r)} \leq X . \qquad (2.4.3)$$

To complete the proof of Theorem 19 we note that if $|f(\zeta_{p,q})| \leq A$ then

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{\varphi(r)} = \limsup \frac{\log|f(\zeta_{p,q})|}{\varphi(|\zeta_{p,q}|)} = 0, \qquad (2.4.25)$$

and it follows that

$$\limsup_{r\to\infty} \frac{\log M(r,f^N)}{\varphi(r)} = N \limsup_{r\to\infty} \frac{\log M(r,f)}{\varphi(r)} = 0, \qquad (2.4.26)$$

and hence we have, for all N ,

$$f(z)^N = \eta(z) \sum_{(p,q)} \frac{f(\zeta_{p,q})^N}{(z-\zeta_{p,q})\,\eta'(\zeta_{p,q})} . \qquad (2.4.27)$$

Therefore, for z outside (A_δ),

$$|f(z)|^N \leq |\eta(z)| \sum_{(p,q)} \frac{A^N}{|(z-\zeta_{p,q})\,\eta'(\zeta_{p,q})|} \leq K|\eta(z)|A^N, \qquad (2.4.28)$$

and hence

$$|f(z)| \leq A \limsup_{N\to\infty} \left\{K|\eta(z)|\right\}^{1/N} . \qquad (2.4.29)$$

For any bounded set of z , $\eta(z)$ is bounded and therefore

$$\limsup_{N\to\infty} \left\{K|\eta(z)|\right\}^{1/N} \leq 1 . \qquad (2.4.30)$$

It follows that

$$|f(z)| \leq A \ . \tag{2.4.31}$$

for z outside (A_δ); an application of the maximum modulus principle shows that the same result is true for all z, hence the conclusion of the theorem that $f(z)$ is bounded and therefore constant.

We also have a result for the case of functions regular in a sector.

THEOREM 20. Suppose that $f(z)$ is regular in a sector $S(\alpha, l)$ and satisfies:

(i) $$\liminf_{r \to \infty} \frac{\log M(r,\alpha,f)}{\varphi(r)} < \frac{1}{2\pi\Lambda^2} \ ; \tag{2.4.32}$$

(ii) $$\limsup_{r \to \infty} \frac{\log |f(re^{\pm i\alpha})|}{\varphi(r)} \leq \gamma < \frac{1}{2\pi\Lambda^2} \ ; \tag{2.4.33}$$

and (iii) $$\limsup_{\zeta_{p,q}(\Lambda,\nu(r)) \in S(\alpha,l)} \frac{\log |f(\zeta_{p,q})|}{\varphi(|\zeta_{p,q}|)} \leq \chi < \frac{1}{2\pi\Lambda^2} \ ; \tag{2.4.34}$$

then $$\limsup_{r \to \infty} \frac{\log M(r,\alpha,f)}{\varphi(r)} \leq \sup\{\chi, \gamma\} \ . \tag{2.4.35}$$

If (2.4.33) and (2.4.34) are replaced by

$$\sup_{z \in \partial(S(\alpha,l))} |f(z)| \leq A < \infty \ , \tag{2.4.36}$$

where $\partial(S(\alpha,l))$ is the boundary of $S(\alpha,l)$, and

$$|f(\zeta_{p,q})| \leq B < \infty \ , \tag{2.4.37}$$

then $$|f(z)| \leq A \ , \tag{2.4.38}$$

for all z in $S(\alpha, \ell)$.

Proof. We may suppose, without loss of generality, that no disc of (A_δ) intersects $\partial(S(\alpha,\ell))$. For we can certainly choose a set of $\{\zeta_{pq}(\Lambda, \nu(r))\}$ which agrees with any such given set in $S(\alpha,\ell) - S_\delta^*$ and has this property, where S_δ^* is defined by

$$S_\delta^* = \bigcup_{\zeta \in \partial(S(\alpha,\ell))} \left\{ |z-\zeta| < \delta |\zeta|^{1-\frac{1}{2}\nu(|\zeta|)} \right\}. \quad (2.4.39)$$

Also, we can, for some positive $\epsilon < \frac{1}{4}\left(\frac{1}{2\pi\Lambda^2} - \sup\{\chi, \gamma\}\right)$, choose a sequence $\{r_n\}$ $r_n \uparrow \to \infty$ such that

$$\sup_n \frac{\log M(r_n, \alpha, f)}{\varphi(r_n)} \leq \frac{1}{2\pi\Lambda^2} - 4\epsilon. \quad (2.4.40)$$

We can then define a sequence of curves $\{\Gamma_n\}$ as in the proof of Theorem 19 and define a system of curves $\{C_n\}$ by

$$C_n = \partial\{ D_i(\Gamma_n) \cap S(\alpha,\ell)\}. \quad (2.4.41)$$

Finally, we can choose a real number λ such that

$$\frac{1}{2\pi\Lambda^2} > \frac{1}{2\pi\lambda^2} + \sup\{\chi, \gamma\} + 4\epsilon, \quad (2.4.42)$$

and then define the function $\eta(z, \lambda, \nu(r)) = \eta_\lambda(z)$ as in (2.3.3) such that $\eta_\lambda(z)$ has no zeros in S_δ^*. We then consider, for z in $S(\alpha,\ell) - S_{\frac{1}{2}\delta}^*$,

$$\frac{1}{2\pi i}\int_{C_n} \frac{\eta_\lambda(\zeta) f(\zeta)}{\eta_\Lambda(\zeta)(\zeta - z)} d\zeta = \frac{\eta_\lambda(z) f(z)}{\eta_\Lambda(z)} - \sum_{\zeta_{pq} \in D_i(C_n)} \frac{\eta_\lambda(\zeta_{pq}) f(\zeta_{pq})}{(z-\zeta_{pq})\,\eta_\Lambda'(\zeta_{pq})}. \quad (2.4.43)$$

It is easy to show that the left-hand side of this is bounded and also that the series is uniformly absolutely convergent for z outside $(A_{\frac{1}{4}\delta}(\Lambda))$. We therefore have

$$|f(z)| \leq K \left| \frac{\eta_\Lambda(z)}{\eta_\lambda(z)} \right| , \tag{2.4.44}$$

for z outside $(A_{\frac{1}{4}\delta}(\Lambda))$ and, if we restrict z to be outside $(A_{\frac{1}{4}\delta}(\lambda))$ as well, we have

$$|f(z)| \leq K(\epsilon) \exp\left\{ \left(\frac{1}{2\pi\Lambda^2} - \frac{1}{2\pi\lambda^2} + 2\epsilon \right) \varphi(|z|) \right\} . \tag{2.4.45}$$

Applying the maximum modulus principle to the excluded discs, we obtain

$$\sup_{\substack{z \in S(\alpha,\ell) - S^*_{\frac{1}{2}\delta} \\ |z| \leq r}} |f(z)| \leq K(\epsilon) \exp\left\{ \left(\frac{1}{2\pi\Lambda^2} - \frac{1}{2\pi\lambda^2} + 3\epsilon \right) \varphi(|z|) \right\}, \tag{2.4.46}$$

for all $\epsilon > 0$ with λ satisfying (2.4.42), and hence

$$\limsup_{r \to \infty} \left\{ \frac{1}{\varphi(r)} \sup_{\substack{z \in S(\alpha,\ell) - S^*_{\frac{1}{2}\delta} \\ |z| \leq r}} |f(z)| \right\} \leq \sup\{X, \gamma\}. \tag{2.4.47}$$

To complete the proof of Theorem 20 we use a Phragmèn-Lindelöf argument to show that the behaviour of $f(z)$ in $S^*_{\frac{1}{2}\delta} \cap S(\alpha,\ell)$ is as indicated by the theorem. On the boundary of this region we have, from the above and (2.4.33), that

$$\limsup_{|z| \to \infty} \frac{\log|f(z)|}{\varphi(|z|)} \leq \sup\{X, \gamma\}, \tag{2.4.48}$$

and hence that

$$\sup_{z \in \partial(S^*_{\frac{1}{2}\delta} \cap S(\alpha,\ell))} \left| \frac{f(z)}{\eta_\Lambda(z)} \right| \leq K(\epsilon). \tag{2.4.49}$$

Also, on r_n,

$$\sup_{|z| = r_n,\, z \in S^*_{\frac{1}{2}\delta} \cap S(\alpha,\ell)} \left| \frac{f(z)}{\eta_\Lambda(z)} \right| \leq K(\epsilon) \exp\{-\epsilon\varphi(r_n)\} \leq K(\epsilon). \tag{2.4.50}$$

Now $f(z)/\eta_\Lambda(z)$ is regular in each of the regions

$$S^*_{\frac{1}{2}\delta} \cap S(\alpha, \ell) \cap \{ |z| \leq r_n \}$$

and uniformly bounded on their boundaries. It follows that $f(z)/\eta_\Lambda(z)$ is bounded in $S^*_{\frac{1}{2}\delta} \cap S(\alpha,\ell)$ and therefore we have

$$|f(z)| \leq K(\epsilon) \exp\left\{\left(\frac{1}{2\pi\Lambda^2} + \epsilon\right) \varphi(|z|)\right\} . \qquad (2.4.51)$$

In particular, it follows that $f(z)$ is of finite order ($\leq \rho$) in this region. Also, $f(z)\,\eta_\lambda(z)/\eta_\Lambda(z)$ is regular in $S^*_{\frac{1}{2}\delta} \cap S(\alpha,\ell)$ and on $\partial\left(S^*_{\frac{1}{2}\delta} \cap S(\alpha,\ell)\right)$ satisfies

$$\left| \frac{f(z)\,\eta_\lambda(z)}{\eta_\Lambda(z)} \right| \leq K(\epsilon) , \qquad (2.4.52)$$

and, since all sufficiently large parts of $S^*_{\frac{1}{2}\delta} \cap S(\alpha,\ell)$ can be contained in a sector of opening less than π/ρ, it follows by a standard argument (see e.g. [47], p. 177) that $f(z)\,\eta_\lambda(z)/\eta_\Lambda(z)$ is bounded in $S^*_{\frac{1}{2}\delta} \cap S(\alpha,\ell)$, and therefore

$$\limsup_{\substack{|z| \to \infty \\ z \in S^*_{\frac{1}{2}\delta} \cap S(\alpha,\ell)}} \frac{\log |f(z)|}{\varphi(|z|)} \leq \sup\{x, \gamma\}. \qquad (2.4.53)$$

For the case where $|f(\zeta_{p,q})| \leq B$ and $|f(z)| \leq A$ for $z \in \partial(S(\alpha,\ell))$, the above argument will yield

$$f(z)^N = \eta_\Lambda(z) \sum_{\zeta_{p,q} \in S(\alpha,\ell)} \frac{f(\zeta_{p,q})^N}{(z-\zeta_{p,q})\,\eta'_\Lambda(\zeta_{p,q})} + \frac{\eta_\Lambda(z)}{2\pi i} \int_{\partial(S(\alpha,\ell))} \frac{f(\zeta)^N d\zeta}{(\zeta - z)\,\eta_\Lambda(\zeta)} . \quad (2.4.54)$$

This will give, for z in $S(\alpha,\ell) - S^*_{\frac{1}{2}\delta}$,

$$|f(z)| \leq \limsup_{N \to \infty} \left\{ K_1 A^N + K_2 B^N \right\}^{1/N} = \sup\{A, B\}. \qquad (2.4.55)$$

The argument used above will then extend this to all z in $S(\alpha,\ell)$ and then, since $|f(z)|$ is bounded in $S(\alpha,\ell)$ and at most A on $\partial(S(\alpha,\ell))$, it follows that

$$|f(z)| \leq A \qquad (2.4.38)$$

for all z in $S(\alpha,\ell)$.

2.5. Connection between the sets $\{z_n\}$ and $\{\zeta_{p,q}\}$. In order to establish Theorems 15 and 17 it is now sufficient to prove

THEOREM 21. A set $\{z_n\}$ which is such that the union of the discs

$$|z - z_n| \leq \Delta |z_n|^{1-\frac{1}{2}\nu(|z_n|)}$$

covers the complex plane has a subset which can be represented as $\{\zeta_{p,q}(\Lambda, \nu(r))\}$, provided that

$$\frac{1}{2\pi\Lambda^2} < \frac{4\pi}{\Delta^2 3\sqrt{3}} . \qquad (2.5.1)$$

This follows since, given the result of this theorem, we can choose a Λ satisfying (2.5.1) such that $1/2\pi\Lambda^2$ is greater than each of

$\liminf\limits_{r\to\infty} \dfrac{\log M(r,f)}{\varphi(r)}$, $\limsup \dfrac{\log|f(z_n)|}{\varphi(|z_n|)}$ and, in the case of Theorem 17, $\liminf\limits_{r\to\infty} \dfrac{\log M(r,\alpha,f)}{\varphi(r)}$ and $\limsup\limits_{r\to\infty} \dfrac{\log|f(re^{\pm i\alpha})|}{\varphi(r)}$, and then use the results of Theorems 19 and 20.

Proof of Theorem 21. With Λ satisfying (2.5.1) we can form a sequence $\{r_p\}$, by using (2.2.2), and the corresponding set $\{z_{p,q}(\Lambda, \nu(r))\}$. We then divide the part of the complex plane lying outside the disc $\{|z| \leq r_{A^2}\}$ into quadrilaterals $Q_{k,l}$, defined by

$$r_{kA} \leq |z| < r_{(k+1)A},\ \frac{2\pi l}{1+[1/Ah(r_{kA})]} \leq \arg z < \frac{2\pi(l+1)}{1+[1/Ah(r_{kA})]} \quad (2.5.2)$$

$(k = A, A+1, \ldots \quad ; l = 0, 1, \ldots, [1/Ah(r_{kA})])$, for some fixed positive integer A. The number of $z_{p,q}$ in $Q_{k,l}$ for fixed p is at most

$$1 + \left\{\frac{1+[1/h(r_{(k+1)A})]}{1+[1/h(r_{kA})A]}\right\} \leq 1 + \frac{A(1+1/h(r_{(k+1)A}))}{1/h(r_{kA})}, \quad (2.5.3)$$

$$= A\{1+O(1/A)\}, \text{ as } k \to \infty .$$

The number of $z_{p,q}$ in $Q_{k,l}$ is therefore

$$A^2\left(1+O(1/A)\right), \text{ as } k \to \infty .$$

Now the union of the discs

$$|z-z_n| \leq \Delta |z_n|^{1-\frac{1}{2}\nu(|z_n|)} \quad (2.5.4)$$

covers the complex plane and therefore

$$\bigcup_{z_n \in Q_{k,l}} \left\{|z-z_n| \leq \Delta |z_n|^{1-\frac{1}{2}\nu(|z_n|)}\right\} \quad (2.5.5)$$

will cover $Q^*_{k,l}$, the region lying inside $Q_{k,l}$ with boundary at a constant distance $\Delta r_{(k+1)A} r_{kA}^{-\frac{1}{2}\nu(r_{kA})}$ from that of $Q_{k,l}$, provided that k is sufficiently large. For the distance $\Delta r_{(k+1)A} r_{kA}^{-\frac{1}{2}\nu(r_{kA})}$ is the most a disc of (2.5.4) with centre in a neighbouring quadrilateral can overlap $Q_{k,l}$. We now use the following easily verifiable result.

LEMMA 2.5.1. If we denote by Q the curvelinear quadrilateral

$$\mu \leq |z| \leq \mu + \nu , \quad \theta \leq \arg z \leq \theta + \alpha \qquad (\alpha < \tfrac{1}{2}\pi),$$

then there is a rectangle R lying inside Q with sides of length

$$2(\mu \tan \tfrac{1}{2}\alpha - \delta \sec \tfrac{1}{2}\alpha) \quad \text{and} \quad (\mu + \nu - \delta)\cos\tfrac{1}{2}\alpha - (\mu + \delta)$$

(these quantities being assumed positive), such that the distance from R to the complement of Q is at least δ.

This result shows that $Q^*_{k,l}$ contains a rectangle $R_{k,l}$ with sides of length

$$2 r_{kA} \left\{ \frac{\pi}{1 + [1/A h(r_{kA})]} - \frac{1}{2} \frac{\Delta}{\Lambda} r_{(k+1)A} h(r_{kA}) \right\} \left\{ 1 + O\left(\frac{\pi}{1 + [1/A h(r_{kA})]} \right)^2 \right\},$$

$$= 2\pi A r_{kA} h(r_{kA}) \left\{ 1 + O(1/A) \right\}, \text{ as } k \to \infty , \qquad (2.5.6)$$

and

$$\left\{ r_{kA} + 2\pi A r_{kA} h(r_{(k+1)A}) - \frac{\Delta}{\Lambda} r_{(k+1)A} h(r_{kA}) \right\} \left\{ 1 + O\left(\frac{\pi}{1 + [1/A h(r_{kA})]} \right)^2 \right\} -$$

$$- r_{kA} - \frac{\Delta}{\Lambda} r_{(k+1)A} h(r_{kA}) = 2\pi A r_{kA} h(r_{kA}) \left\{ 1 + O(1/A) \right\}. \qquad (2.5.7)$$

Kershner [18] obtained several results concerning the covering of a set by a system of discs. They effectively say that the most efficient covering of this sort is that obtained by a system of discs with centres at the centres of a hexagonal lattice with side equal to the radius of the discs. The following result is the one most suitable for our applications.

LEMMA 2.5.2. ([18], Lemma 5). The minimum number of discs of radius d necessary to cover a rectangle of area D is at least $2(D-2\pi d^2)/d^2 3\sqrt{3}$.

It follows from this that the number of discs of radius $\Delta r_{(k+1)A}\, r_{kA}^{-\frac{1}{2}\nu(r_{kA})}$ necessary to cover $Q^*_{k,\ell}$ is at least

$$\frac{8\pi^2 A^2 \Lambda^2}{\Delta^2 3\sqrt{3}}\left(1+O\left(\frac{1}{A}\right)\right), \tag{2.5.8}$$

and this exceeds $A^2(1+O(1/A))$, for sufficiently large A, provided that (2.5.1) is satisfied.

If we assume (2.5.1) we can choose $\Delta_1 > \Delta$ such that

$$\frac{1}{2\pi\Lambda^2} < \frac{4\pi}{\Delta_1^2 3\sqrt{3}},$$

and we can choose a subset $\{z_{n'}\}$ of the set $\{z_n\}$ which is such that the discs

$$|z - z_{n'}| \leq \tfrac{1}{2}(\Delta_1 - \Delta)|z_{n'}|^{1-\frac{1}{2}\nu(|z_{n'}|)} \tag{2.5.9}$$

are disjoint, for all sufficiently large $|z_{n'}|$, and also such that the union of the discs

$$|z - z_{n'}| \leq \Delta_1 |z_{n'}|^{1-\frac{1}{2}\nu(|z_{n'}|)} \tag{2.5.10}$$

covers the complex plane. Since the diameter of $Q_{k,\ell}$ is at most $K(A)\, r_{kA}\, r_{(k+1)A}^{-\frac{1}{2}\nu(r_{(k+1)A})}$, it follows that we can have a (1:1) correspondence between some subset $\{n''\}$ of $\{n'\}$ and the set $\{(p,q)\}$ such that

$$|z_{n''} - z_{p,q}| \leq K |z_{p,q}|^{1-\frac{1}{2}\nu(|z_{p,q}|)}. \tag{2.5.11}$$

This completes the proof of Theorem 21.

It is easy to show that the constant $4\pi/\Delta^2 3\sqrt{3}$ of Theorem 21 cannot be increased; a similar argument in reverse will show

THEOREM 22. Every set $\{\zeta_{p,q}(\Lambda, \nu(r))\}$ can be put in a (1:1) correspondence with a set $\{\xi_{p,q}\}$, such that

$$|\zeta_{p,q} - \xi_{p,q}| \leq K|\xi_{p,q}|^{1-\frac{1}{2}\nu(|\xi_{p,q}|)} \qquad (2.5.12)$$

and the union of the discs

$$|z - \xi_{p,q}| \leq \Delta|\xi_{p,q}|^{1-\frac{1}{2}\nu(|\xi_{p,q}|)} \qquad (2.5.13)$$

covers the complex plane, provided that

$$\frac{1}{2\pi\Lambda^2} > \frac{4\pi}{\Delta^2 3\sqrt{3}}. \qquad (2.5.14)$$

It follows, as a corollary to Theorem 22, that Theorem 19 is contained in Theorem 15.

CHAPTER III

THE LARGE REGIONS OF INTEGRAL FUNCTIONS

3.1. Introduction. We shall be mainly concerned with the following problem. Suppose that $f(z)$ is an integral function of mean type τ of order ρ ; what can be said about the distribution of regions where $|z|^{-\rho} \log|f(z)|$ is near to τ ? Similar problems in which $|f(z)|$ is compared with the maximum modulus or the Nevanlinna characteristic have been discussed by Whittaker [54,55] and Macintyre [57]. Whittaker's principle result is as follows.

THEOREM 23. There is an absolute constant H , not less than $16/729$, with the following property. If $f(z)$ is an integral function of order $\rho \leq 2$ satisfying the condition

$$n(r, f) = o(r^2) , \tag{3.1.1}$$

where $n(r,f)$ is the number of zeros of $f(z)$ in $|z| \leq r$, and h ($< H$), η ($< \rho$) and d are given positive constants, the values of ζ for which the inequalities

$$\log|f(z)| > h \log M(|\zeta|, f) > |\zeta|^{\eta} \tag{3.1.2}$$

are satisfied throughout the disc

$$|z - \zeta| \leq d \tag{3.1.3}$$

form a set of upper density greater than or equal to $(H-h)/(1-h)$.

For functions of any finite positive order he gave

THEOREM 24. Let $f(z)$ be an integral function of order ρ, and let σ be a given positive number less than $1-\frac{1}{2}\rho$. Then there is a positive constant h and a sequence $\{\zeta_n\}$ such that $|\zeta_n| \to \infty$ and

$$\log|f(z)| \geqslant h \log M(|\zeta_s|) \tag{3.1.4}$$

in the circle

$$|z - \zeta_s| \leqslant |\zeta_s|^{\sigma}. \tag{3.1.5}$$

Maitland [27], by inverting her earlier interpolation results of [26], was able to improve on this last result, and the improvement of it which results on specialising Macintyre's results to concern only integral functions. She proved the following.

THEOREM 25. Suppose that $f(z)$ is an integral function of mean type τ of order ρ and that d is a positive number less than $\frac{1}{\rho}\left(\frac{\pi}{2\tau}\right)^{1/2}$; then there exists a sequence $\{\zeta_n\}$ with $|\zeta_n| \to \infty$ such that

$$|z|^{-\rho} \log|f(z)| \to \tau \tag{3.1.6}$$

as $z \to \infty$ inside the discs

$$|z - \zeta_n| \leqslant d|\zeta_n|^{1-\frac{1}{2}\rho}. \tag{3.1.7}$$

Noble [34], using his more general interpolation results of [31], was able to give a lower bound to the upper density of the set of r for which there exists a $\zeta(r)$ with $|\zeta(r)| = r$ such that

$$\log|f(z)| \geq \mu r^{\rho} \qquad (\mu < \tau) \tag{3.1.8}$$

throughout the disc

$$|z - \zeta(r)| \leq d\, r^{1-\frac{1}{2}\rho}, \tag{3.1.9}$$

where d is again restricted to be less than $\frac{1}{e}\left(\frac{\pi}{2\tau}\right)^{\frac{1}{2}}$.

<u>3.2. Statement of results.</u> By developing the fairly exact interpolation theory of Chapter II to cover more general cases and by being more sophisticated in our choice of discs involved, we are able to improve the above results to give a best-possible value for the size of the discs and, for these larger discs, to give improved and much simpler estimates for the size of the set of moduli of the centres of the discs. Our first result is stated in terms which clearly demonstrate its connection with the results of the preceeding chapter.

THEOREM 26. Suppose that $\varphi(r)$ is such that

$$\frac{d^2\varphi(r)}{d(\log r)^2} = r^{\nu(r)} \uparrow\to \infty, \tag{3.2.1}$$

with $\nu(r) \downarrow\to \rho$. Then if $f(z)$ is an integral function which satisfies

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{\varphi(r)} = \frac{4\pi}{\Delta^2 3\sqrt{3}} \tag{3.2.2}$$

there exists a sequence $\{\zeta_n\}$ with $|\zeta_n| \to \infty$ such that

$$\log|f(z)| \geq \lambda \frac{4\pi}{\Delta^2 3\sqrt{3}} \varphi(|z|) \tag{3.2.3}$$

throughout the discs

$$|z - \zeta_n| \leq d\,|\zeta_n|^{1-\frac{1}{2}\nu(|\zeta_n|)}, \tag{3.2.4}$$

for every $d < \frac{1}{2}\Delta\sqrt{3}$ and $\lambda < 1$.

The case considered by Maitland and by Noble results on setting

$$\frac{4\pi}{\Delta^2 3\sqrt{3}} = \rho^2\tau \quad \text{and} \quad \nu(r) \equiv \rho > 0 , \tag{3.2.5}$$

for then $\varphi(r) = \frac{r^\rho}{\rho^2}$ and $\frac{1}{2}\Delta\sqrt{3}$ becomes $\frac{1}{\rho}\left(\frac{\pi}{\tau\sqrt{3}}\right)^{\frac{1}{2}}$. In fact we have

THEOREM 27. Let $f(z)$ be an integral function of mean type τ of order ρ . Then, as z tends to infinity in a set of discs

$$|z - \zeta| \leq \beta\,\frac{1}{\rho}\left(\frac{\pi}{\tau\sqrt{3}}\right)^{\frac{1}{2}} |\zeta|^{1-\frac{1}{2}\rho}, \tag{3.2.6}$$

$|z|^{-\rho}\log|f(z)|$ tends to the limit τ , $(0 \leq \beta < 1)$.

Furthermore, the set of the moduli of the admissible ζ has maximal density at least $1-\beta^2$.

The existence of such a set of ζ follows, as indicated above, from the preceeding theorem; the estimate of its size is an almost immediate corollary to our next result.

THEOREM 28. Let $f(z)$ be an integral function of mean type τ of order ρ . Denote by $\mathcal{E}(\mu,\beta)$ the set of r for which

$$|z|^{-\rho}\log|f(z)| \geq \mu\tau \tag{3.2.7}$$

throughout the disc

$$|z-\zeta| \leq \beta\,\frac{1}{\rho}\left(\frac{\pi}{\tau\sqrt{3}}\right)^{\frac{1}{2}}|\zeta|^{1-\frac{1}{2}\rho}, \tag{3.2.8}$$

for some ζ with $|\zeta| = r$, ($0 \leq \mu < 1$, $0 \leq \beta < 1$). Then $\mathcal{E}(\mu,\beta)$ has maximal density at least

$$1-\beta^2 \tag{3.2.9}$$

and upper density at least equal to

$$\inf\left\{1-\beta^2, \frac{1-\mu}{2-\mu}\right\}, \quad \text{if } \rho \leq 1, \tag{3.2.10}$$

and $$\inf\left\{1-\beta^{2/\rho}, 1-\left(\frac{1}{2-\mu}\right)^{1/\rho}\right\}, \quad \text{for } \rho \geq 1. \tag{3.2.11}$$

We note, as a corollary, that the upper density is, in particular, at least

$$\frac{1}{\rho}\inf\left\{1-\beta^2, \sup\{\tfrac{1}{2},\mu\}(1-\mu)\right\}, \tag{3.2.12}$$

for $\rho \geq 1$.

The size of the discs in the case considered by Noble and Maitland corresponds, in the notation used here, to the range

$$0 < \beta < \tfrac{1}{2}\sqrt{3}. \tag{3.2.13}$$

The estimates given by Noble for the lower bound to the upper density of $\mathcal{E}(\mu,\beta)$ are stated in terms of the function

$$\tau_\rho(\theta) = \rho^2 \sup_{\text{meas}\,E=\theta} \int_{E\cap(0,1)} x^{\rho-1}\log\frac{1}{x}\,dx\;; \tag{3.2.14}$$

if the upper density is denoted by $\bar{D}$ and λ is defined by

$$1 - \frac{2\beta^2}{\sqrt{3}}(1+\lambda)^2 = \frac{1}{2}\left(1 - \frac{2\beta^2}{\sqrt{3}}\right), \tag{3.2.15}$$

then

$$\tau_\rho\left(\frac{(1+\lambda)\bar{D}}{\lambda}\right) \geqslant \inf\left\{\frac{1}{2}\left(1 - \frac{2\beta^2}{\sqrt{3}}\right), 1-\mu\right\}. \tag{3.2.16}$$

If $\rho > 1$ there is the more accesible result

$$\bar{D} \geqslant \frac{e(\rho-1)}{\rho^2}\left(1 - \frac{2\beta^2}{\sqrt{3}}\right)\inf\left\{\frac{1}{2}\left(1 - \frac{2\beta^2}{\sqrt{3}}\right), 1-\mu\right\}. \tag{3.2.17}$$

If we let $\beta = 0$ in Theorem 28 we obtain results concerning the size of the set of r for which $r^{-\rho}\log M(r,f)$ is near to τ ; we state these as corollaries.

COROLLARY 1. If $f(z)$ is an integral function of finite type τ of order ρ , then

$$r^{-\rho}\log M(r,f) \to \tau \tag{3.2.18}$$

as $r \to \infty$ through a set of maximal density unity.

COROLLARY 2. With $f(z)$ as for Corollary 1, the set of r for which

$$r^{-\rho}\log M(r,f) \geqslant \mu\tau \qquad (0 \leqslant \mu \leqslant 1) \tag{3.2.19}$$

has upper density at least

$$\frac{1-\mu}{2-\mu}, \qquad \text{if} \quad \rho \leqslant 1, \tag{3.2.20}$$

and $$1 - \left(\frac{1}{2-\mu}\right)^{1/\rho}, \qquad \text{for} \quad \rho \geqslant 1. \tag{3.2.21}$$

Since to any function $\psi(r)$ convex with respect to $\log r$ can be made correspond an integral function $g(z)$ with

$$\log M(r, g) \sim \psi(r), \qquad (3.2.22)$$

for large r (Valiron [51], Clunie [8]), these corollaries are really concerned with the functions $\psi(r)$ of a real variable. It seems probable that they could be proved directly.

Results, of a similar nature to Theorems 27 and 28, corresponding to the more general growth allowed in Theorem 26 are also available but we restrict ourselves to the cases given in order to simplify the statements.

<u>3.3. Proof of Theorem 26.</u> The proof follows readily from the construction used in the proof of Theorem 21 of section 2.5. For any Δ_1, Δ_2 such that $0 < \Delta_1 < \Delta_2 < \Delta$, we can choose a Λ such that

$$\frac{4\pi}{\Delta^2 3\sqrt{3}} < \frac{1}{2\pi\Lambda^2} < \frac{4\pi}{\Delta_2^2\, 3\sqrt{3}}. \qquad (3.3.1)$$

We consider the intersection of a quadrilateral $Q_{k,\ell}(A, \Lambda, \nu(r))$ with a hexagonal lattice of side

$$\Delta_2\, r_{(k+1)A}\, r_{kA}^{-\frac{1}{2}\nu(r_{kA})}, \qquad (3.3.2)$$

and take as the members of a set $\{\xi_{p,q}\}$ associated (1:1) with the $z_{p,q}$ contained in $Q_{k,\ell}$ a subset of the centres of the hexagons lying completely within $Q_{k,\ell}$. This is possible provided that A is sufficiently large.

Now the discs

$$|z - \xi_{p,q}| \leq \tfrac{1}{2} \Delta_1 \sqrt{3}\, |\xi_{p,q}|^{1-\frac{1}{2}\nu(|\xi_{p,q}|)} \qquad (3.3.3)$$

are disjoint, since they lie completely within different hexagons of the lattice, and if $\zeta_{p,q}$ is any point of this disc then

$$|\zeta_{p,q} - \zeta_{r,s}| \geq \tfrac{1}{2}\sqrt{3}\,(\Delta_2 - \Delta_1) \sup\left\{|\zeta_{p,q}|^{1-\frac{1}{2}\nu(|\zeta_{p,q}|)},\ |\zeta_{r,s}|^{1-\frac{1}{2}\nu(|\zeta_{r,s}|)}\right\} \qquad (3.3.4)$$

for $(p,q) \neq (r,s)$.

If we assume the theorem is false for $d = \frac{1}{2}\Delta_1\sqrt{3}$, we can take, for all sufficiently large $|\xi_{p,q}|$, $\zeta_{p,q}$ to be a point such that

$$\log|f(\zeta_{p,q})| \leq \lambda \frac{4\pi}{\Delta^2 3\sqrt{3}}\, \varphi(|\zeta_{p,q}|), \qquad (3.3.5)$$

with $\lambda < 1$, and, since Λ is chosen so that $1/2\pi\Lambda^2$ is greater than $4\pi/\Delta^2 3\sqrt{3}$, it follows from Theorem 19 that

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{\varphi(r)} \leq \frac{\lambda 4\pi}{\Delta^2 3\sqrt{3}}, \qquad (3.3.6)$$

which contradicts (3.2.2). Since Δ_1 is any number less than Δ it follows that the result holds for any d less than $\frac{1}{2}\Delta\sqrt{3}$.

<u>3.4. Proof of Theorem 28.</u> The proof is based on an interpolation result similar in spirit to that of Theorem 19. It differs from Theorem 19 in that not all the $\zeta_{p,q}$ are available, only those associated with the $z_{p',q}$ having

modulus $r_{p'}$ belonging to some subset of the $\{r_p\}$. The principle is contained in our next theorem.

THEOREM 29. Suppose $\{r_{p'}\}$ to be a subset of the set $\{r_p\}$, defined by (2.2.2), which is such that

$$\liminf_{r\to\infty} \frac{\tilde{n}(r)}{n(r)} > 0, \tag{3.4.1}$$

where $\tilde{n}(r)$ is the number of $z_{p',q}$ in $|z| \leq r$ and $n(r)$ is the number of $z_{p,q}$ in $|z|\leq r$. Then, with $\{\zeta_{p,q}\}$ satisfying (2.3.1) and (2.3.2) and $\tilde{\eta}(z)$ defined by

$$\tilde{\eta}(z) = \prod_{k\in\{p'\}} \left\{ \prod_{q=0}^{[1/h(r_k)]} E\left(\frac{z}{\zeta_{p,q}}, [\tfrac{1}{2}\rho]\right)\right\}, \tag{3.4.2}$$

we have

$$\log|\tilde{\eta}(z)| = \{1+o(1)\}\int_1^{|z|} \frac{\tilde{n}(t)}{t}\,dt, \tag{3.4.3}$$

outside the discs

$$|z-\zeta_{p',q}| \leq \delta\,|\zeta_{p',q}|^{1-\frac{1}{2}\nu(|\zeta_{p',q}|)}, \tag{3.4.4}$$

and

$$\log|\tilde{\eta}'(\zeta_{p',q})| \geq \{1+o(1)\}\int_1^{|\zeta_{p',q}|} \frac{\tilde{n}(t)}{t}\,dt. \tag{3.4.5}$$

Proof of Theorem 29. This is an immediate corollary of the work contained in sections 2.2 and 2.3. Condition (3.4.1) ensures that

$$\int_0^r \frac{\tilde{n}(t)}{t}\,dt, \quad \text{which is equal to } \log\left|\prod_{|z_{p',q}|\leq r} \frac{z}{z_{p',q}}\right|,$$

is large enough to assymptotically dominate the terms which measure the deviation of $\log|\tilde{\eta}(z)|$ from $\log\left\{\prod_{|z_{p,q}|\le r}\left|\frac{z}{z_{p,q}}\right|\right\}$, since the estimates obtained for such deviation when all the r_p are included are not increased by going over to a subset.

Suppose now that

$$1 \geqslant \psi = \limsup_{r\to\infty} \frac{\tilde{n}(r)}{n(r)} \geqslant \liminf_{r\to\infty} \frac{\tilde{n}(r)}{n(r)} = \theta > 0 , \quad (3.4.6)$$

then Theorem 29 shows that:

$$\log|\tilde{\eta}(z)| \leq \{1+o(1)\} \frac{\psi}{2\pi\Lambda^2} \varphi(|z|) ; \quad (3.4.7)$$

$$\log|\tilde{\eta}(z)| \geqslant \{1+o(1)\} \frac{\theta}{2\pi\Lambda^2} \varphi(|z|), \quad (3.4.8)$$

for z outside the discs of (3.4.4); and

$$\log|\tilde{\eta}(\zeta_{p',q})| \geqslant \{1+o(1)\} \frac{\theta}{2\pi\Lambda^2} \varphi(|\zeta_{p',q}|). \quad (3.4.9)$$

Using these results we are able to prove

THEOREM 30. Suppose that $\{\zeta_{p',q}\}$ is as described above, with

$$1 \geqslant \psi = \limsup_{r\to\infty} \frac{\tilde{n}(r)}{n(r)} \geqslant \liminf_{r\to\infty} \frac{\tilde{n}(r)}{n(r)} = \theta > 0 . \quad (3.4.6)$$

If $f(z)$ is an integral function satisfying

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{\varphi(r)} < \frac{\theta}{2\pi\Lambda^2} \quad (3.4.10)$$

and $$\limsup \frac{\log|f(\zeta_{p',q})|}{\varphi(|\zeta_{p',q}|)} \leq \kappa < \frac{\theta}{2\pi\Lambda^2} , \quad (3.4.11)$$

then $$\limsup_{r\to\infty} \frac{\log M(r,f)}{\varphi(r)} \leqslant \varkappa + \frac{\psi-\Theta}{2\pi\Lambda^2}. \tag{3.4.12}$$

Proof. The inequalities of (3.4.8) and (3.4.9) are sufficient, with the argument of the proof of Theorem 19 as far as the consequence of equation (2.4.17), to show that

$$f(z) = \tilde{\eta}(z) \sum_{(p',q)} \frac{f(\zeta_{p',q})}{(z-\zeta_{p',q})\tilde{\eta}'(\zeta_{p',q})}. \tag{3.4.13}$$

Using a similar argument to that contained between equations (2.4.18) and (2.4.24), with λ chosen so that

$$\frac{1}{2\pi\lambda^2} + \varkappa < \frac{\Theta}{2\pi\Lambda^2}, \tag{3.4.14}$$

we obtain that

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{\varphi(r)} \leqslant \frac{\psi}{2\pi\Lambda^2} - \frac{1}{2\pi\lambda^2} + 3\epsilon, \tag{3.4.15}$$

which is sufficient to give the result of the theorem.

Having set up the necessary interpolation theory, we now turn to the problem of selecting suitable sets $\{\zeta_{p',q}\}$. The argument depends on the following proposition.

LEMMA 3.4.1. If $\mathcal{E}$ is a set of r such that

$$\frac{\operatorname{meas}\{\mathcal{E} \cap (r_{kA}, r_{(k+1)A})\}}{r_{(k+1)A} - r_{kA}} = \lambda_k \geqslant c > 0, \tag{3.4.16}$$

then we can describe a set of N_k discs, each of radius $\Delta\, r_{(k+1)A}\, r_{kA}^{-\frac{1}{2}\nu(r_{kA})}$, lying in the interior of $Q_{k,l}(A,\Lambda,\nu(r))$, at a distance at least $\frac{1}{4}(\Delta_1-\Delta)\, r_{(k+1)A}\, r_{kA}^{-\frac{1}{2}\nu(r_{kA})}$ from each other, with centres z_n having modulus belonging to $\mathcal{E}$ and such that

$$N_k \geqslant \lambda_k \frac{2\pi^2 \Lambda^2 A^2}{\Delta_1^2 \sqrt{3}} - K(c)\frac{\Lambda A}{\Delta_1}. \qquad (3.4.17)$$

Proof. We consider first all intervals of length more than $\frac{1}{2}(\Delta_1 - \Delta)\, r_{(k+1)A}\, r_{kA}^{-\frac{1}{2}\nu(r_{kA})}$ which lie in the complement of $\mathcal{E}$. There is no loss of generality in supposing, for notational convenience, that they are open; let them be

$$(a_1, b_1),\ (a_2, b_2),\ \ldots\ldots, (a_{n_k}, b_{n_k}), \qquad (3.4.18)$$

with $r_{kA} \le a_1 < b_1 < a_2 < \ldots < b_{n_k - 1} < a_{n_k} < b_{n_k} \le r_{(k+1)A}$.

We now translate the set of points of $Q_{k,\ell}$ which have modulus lying in $[b_{n_k}, r_{(k+1)A}]$ a distance $(b_{n_k} - a_{n_k})\sec\alpha_k$ in a direction negative to that of the radius vector of the centre of $Q_{k,\ell}$, where $2\alpha_k$ is the angle subtended by $Q_{k,\ell}$ at the origin. Next we translate the union of this translated set with the set of points of $Q_{k,\ell}$ which have modulus lying in the interval $[b_{n_k-1}, a_{n_k}]$ a distance $(b_{n_k-1}, a_{n_k-1})\sec\alpha_k$ in the same direction as before. We repeat this process for all the intervals of (3.4.18), taken in reverse order. Let the set which this process produces be denoted by $Q^{**}_{k,\ell}$; then $Q^{**}_{k,\ell}$ contains the curvelinear quadrilateral determined by

$$r_{kA} \le |z| \le r_{(k+1)A} - \sum_{p=1}^{n_k} (b_p - a_p)\sec\alpha_k, \qquad (3.4.19)$$

and

$$|\arg z - \varphi_{k,\ell}| \le \alpha_k, \qquad (3.4.20)$$

where $\varphi_{k,\ell}$ is the amplitude of the centre of $Q_{k,\ell}$. Now

$$\sum_{p=1}^{n_k} (b_p - a_p) \leq (1-\lambda_k)(r_{(k+1)A} - r_{kA}), \quad (3.4.21)$$

and it follows that

$$r_{(k+1)A} - \sum_{p=1}^{n_k} (b_p - a_p) \sec\alpha_k \geq r_{kA} + (r_{(k+1)A} - r_{kA})\{1 - (1-\lambda_k)\sec\alpha_k\},$$

$$\geq r_{kA} + \lambda_k(r_{(k+1)A} - r_{kA}) - (r_{(k+1)A} - r_{kA})(\sec\alpha_k - 1), \quad (3.4.22)$$

and therefore $Q^{**}_{k,l}$ contains the quadrilateral determined by (3.4.20) and

$$r_{kA} \leq |z| \leq r_{kA} + \lambda_k(r_{(k+1)A} - r_{kA}) - (r_{(k+1)A} - r_{kA})(\sec\alpha_k - 1). \quad (3.4.23)$$

That these equations do determine a real quadrilateral, for all sufficiently large k, is ensured by (3.4.16) and

$$\sec\alpha_k - 1 = \sec\left(\frac{\pi}{1 + [1/Ah(r_{kA})]}\right) - 1 \quad (3.4.24)$$

$$\leq K(Ah(r_{kA}))^2 \longrightarrow 0 \quad (3.4.25)$$

as $k \to \infty$.

It is now easy to show that this quadrilateral contains a rectangle with sides

$$2r_{kA}\tan\alpha_k = 2\pi A h(r_{kA})\left\{1 + O\left(\frac{1}{A}\right)\right\}, \quad (3.4.26)$$

as $k \to \infty$, and

$$\{r_{kA} + \lambda_k(r_{(k+1)A} - r_{kA}) - (r_{(k+1)A} - r_{kA})(\sec\alpha_k - 1)\}\cos\alpha_k - r_{kA}$$

$$= \{1 + O(A^2 h(r_{kA}))\}\lambda_k(r_{(k+1)A} - r_{kA}), \quad (3.4.27)$$

$$= \lambda_k 2\pi A r_{kA} h(r_{kA})\left\{1 + O\left(\frac{1}{A}\right)\right\}, \quad (3.4.28)$$

as $k \to \infty$. Next consider the intersection of this rectangle with a hexagonal lattice of side $\frac{2\Delta_1}{\sqrt{3}} r_{(k+1)A} r_{kA}^{-\frac{1}{2}\nu(r_{kA})}$ and denote by $\{z_p\}$ the set of centres of those hexagons which lie completely within the rectangle. The number N_k of z_p is easily shown to satisfy

$$N_k \geqslant \lambda_k \frac{4\pi^2 \Lambda^2 A^2}{\Delta_1^2 2\sqrt{3}} - \frac{K\Lambda A}{\Delta_1} . \qquad (3.4.29)$$

Now consider the component sets of $Q^{**}_{k,l}$ returned to their original positions with each z_p kept fixed relative to the component containing it. If a z_p is contained in more than one component it is kept fixed relative to only one of them. Let the new positions of the z_p be denoted by z'_p . Any z'_p must lie within a distance $\frac{1}{4}(\Delta_1 - \Delta) r_{(k+1)A} r_{kA}^{-\frac{1}{2}\nu(r_{kA})}$ of a point z''_p (possibly z'_p itself) which has modulus lying in $\mathcal{E}$. The distance apart of two discs having centres at z''_p and z''_q $(p \neq q)$ with radius $\Delta r_{(k+1)A} r_{kA}^{-\frac{1}{2}\nu(r_{kA})}$ is at least $\frac{1}{2}(\Delta_1 - \Delta) r_{(k+1)A} r_{kA}^{-\frac{1}{2}\nu(r_{kA})}$. This completes the proof of Lemma 3.4.1.

If we now choose Λ so that

$$\Lambda^2 > \frac{\Delta_1^2 \sqrt{3}}{2\pi^2} , \qquad (3.4.30)$$

we obtain, for all sufficiently large k , with λ_k satisfying (3.4.16),

$$N_k \geqslant \sigma \lambda_k A^2 \qquad (\sigma > 1) . \qquad (3.4.31)$$

Hence we can choose N^*_k points $z_{p',q}(\Lambda, \nu(r))$, that is all points of $\{z_{pq}(\Lambda, \nu(r))\}$ lying on some subset $\{|z| = r_{p'}\}$ of

$\{|z|=r_p\}$, in $Q_{k,\ell}$ with

$$N^*_k = \lambda_k A^2\left(1+\frac{h_k}{cA}\right), \qquad (3.4.32)$$

where $|h_k|\leq 1$, such that they are in a (1:1) correspondence with a subset of the centres of the discs constructed as in Lemma 3.4.1.

Now suppose that the set $\mathcal{E}$ of r has linear density θ. If $\lambda_k < c < \theta$ then we remove from $\mathcal{E}$ all those points which lie in the interval $(r_{kA}, r_{(k+1)A})$ to form a set $\mathcal{E}_1$ with upper density at most θ and lower density at least $\theta - c$. For $\mathcal{E}_1$ we have

$$\lambda^{(1)}_k \geq c \quad \text{or} \quad \lambda^{(1)}_k = 0. \qquad (3.4.33)$$

When $\lambda^{(1)}_k \geq c$ the conditions of the lemma apply and, for each $Q_{k,\ell}$ and each k, we can choose the N^*_k points $z_{p',q}$ as described above, the subsequence $\{r_{p'}\}$ being the same for all $Q_{k,\ell}$ with fixed k. When $\lambda^{(1)}_k = 0$ we choose no points.

We now determine the behaviour of $\tilde{n}(r)$ for the set of points so constructed in the case $\nu(r) \equiv \rho > 0$. Suppose that $\gamma_1(r)$ is the characteristic function of the set $\mathcal{E}_1$; that is $\gamma_1(r) = 1$ for $r \in \mathcal{E}_1$, $\gamma_1(r) = 0$ for $r \notin \mathcal{E}_1$. It is easy to show, using the arguments of Chapter II, that

$$\tilde{n}(r) \sim \left\{1+\frac{h(r)}{cA}\right\}\frac{1}{2\pi\Lambda^2}\int_1^r \frac{\gamma_1(t)}{t^{1-\rho}}\,dt, \qquad (3.4.34)$$

where $|h(r)|\leq 1$. Now

$$\int_1^r \frac{\gamma_1(t)}{t}\,dt = \frac{\int_1^r \gamma_1(t)\,dt}{r^{1-\rho}} - (\rho-1)\int_1^r \frac{\int_1^t \gamma_1(u)\,du}{t^{2-\rho}}\,dt, \quad (3.4.35)$$

and, since

$$\theta \geq \limsup_{r\to\infty} \frac{1}{r}\int_1^r \gamma_1(t)\,dt \geq \liminf_{r\to\infty} \frac{1}{r}\int_1^r \gamma_1(t)\,dt \geq \theta - c,$$

it follows that the left-hand side of (3.4.35) is assymptotically at least

$$(\theta-c)r^\rho - \frac{\theta(\rho-1)r^\rho}{\rho} = \frac{\theta r^\rho}{\rho}\left\{1 - \frac{c\rho}{\theta}\right\} \quad (3.4.36)$$

for $\rho \geq 1$, and

$$(\theta-c)r^\rho + \frac{(\theta-c)(1-\rho)r^\rho}{\rho} = \frac{\theta r^\rho}{\rho}\left\{1 - \frac{c}{\theta}\right\} \quad (3.4.37)$$

for $\rho \leq 1$, and assymptotically at most

$$\theta r^\rho - \frac{(\theta-c)(\rho-1)r^\rho}{\rho} = \frac{\theta r^\rho}{\rho}\left\{1 + \frac{c(\rho-1)}{\theta}\right\} \quad (3.4.38)$$

for $\rho \geq 1$, and

$$\theta r^\rho + \frac{\theta(1-\rho)r^\rho}{\rho} = \frac{\theta r^\rho}{\rho} \quad (3.4.39)$$

for $\rho \leq 1$. Therefore, in either case, it follows that

$$\theta\left(1+\frac{1}{cA}\right)\left(1+\frac{c\rho}{\theta}\right) \geq \limsup_{r\to\infty} \frac{\tilde{n}(r)}{n(r)} \geq \liminf_{r\to\infty} \frac{\tilde{n}(r)}{n(r)} \geq \theta\left(1-\frac{1}{cA}\right)\left(1-\frac{c(\rho+1)}{\theta}\right). \quad (3.4.40)$$

We are now in a position to prove the first proposition of Theorem 28. For if $\mathcal{E}$ lies in the complement of $\mathcal{E}(\mu,\beta)$, then in each of the discs of Lemma 3.4.1, with $\Delta = \beta\frac{1}{\rho}\left(\frac{\pi}{T\sqrt{3}}\right)^{\frac{1}{2}}$, there is a point where

$$|z|^{-\rho}\log|f(z)| \leq \mu T, \quad (3.4.41)$$

and we have shown that we can take a subset of these points as the set $\{\zeta_{p',q}\}$ associated with the $z_{p',q}$ considered above. If we suppose that

$$\tau = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}} < \frac{\theta}{2\pi\rho^2\Lambda^2}, \tag{3.4.42}$$

we can choose cA large enough and c small enough to ensure that

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}} < \frac{\theta}{2\pi\rho^2\Lambda^2}\left(1-\frac{1}{cA}\right)\left(1-\frac{c(\rho+1)}{\theta}\right), \tag{3.4.43}$$

and it then follows, from Theorem 30, that

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}} \le \mu\tau + \frac{\theta}{2\pi\rho^2\Lambda^2}\left\{\left(1+\frac{1}{cA}\right)\left(1+\frac{c\rho}{\theta}\right)-\left(1-\frac{1}{cA}\right)\left(1-\frac{c(\rho+1)}{\theta}\right)\right\}. \tag{3.4.44}$$

This holds for all small c and large cA and hence we have the contradiction

$$\tau \le \mu\tau . \tag{3.4.45}$$

It follows that we must have

$$\tau \geq \frac{\theta}{2\pi\rho^2\Lambda^2}, \tag{3.4.46}$$

and, since Λ is chosen to satisfy only (3.4.30) for any Δ_1 greater than Δ, it follows that

$$\Lambda^2 > \frac{\Delta^2\sqrt{3}}{2\pi^2} = \frac{\beta^2}{2\pi\rho^2\tau} \tag{3.4.47}$$

implies that

$$\theta = \text{density } \mathcal{E} \le 2\pi\rho^2\tau\Lambda^2. \tag{3.4.48}$$

Therefore

$$\text{density } \mathcal{E} \leq \beta^2. \tag{3.4.49}$$

Now $\mathcal{E}$ is any set having a density and lying in the complement of $\mathcal{E}(\mu, \beta)$; hence the conclusion that the maximal density of $\mathcal{E}(\mu, \beta)$ is at least $1 - \beta^2$.

If we suppose only that $\mathcal{E}$ has lower density θ , we can still form the set $\mathcal{E}_1$ having lower density at least $\theta - c$. The estimates for $\tilde{n}(r)$ in this case are provided as follows. For $\rho > 1$, $r^{\rho-1}$ is an increasing function of r and therefore

$$\int_1^r \frac{\gamma_1(t)}{t^{1-\rho}}\,dt \geqslant \{1 + o(1)\} \int_1^{(\theta-c)r} t^{\rho-1}\,dt, \tag{3.4.50}$$

$$= \{1 + o(1)\} (\theta - c)^{\rho} \frac{r^{\rho}}{\rho},$$

and hence

$$\liminf_{r \to \infty} \frac{\tilde{n}(r)}{n(r)} \geqslant (\theta - c)^{\rho}\left(1 - \frac{1}{cA}\right). \tag{3.4.51}$$

For $\rho \leq 1$ we use

$$\int_1^r \frac{\gamma_1(t)}{t^{1-\rho}}\,dt = \frac{\int_1^r \gamma_1(t)\,dt}{r^{1-\rho}} + (1-\rho)\int_1^r \frac{\int_1^t \gamma_1(u)\,du}{t^{2-\rho}}\,dt, \tag{3.4.52}$$

$$\geqslant \{1 + o(1)\}\left\{(\theta - c) r^{\rho} + \frac{1-\rho}{\rho}(\theta - c) r^{\rho}\right\}, \tag{3.4.53}$$

which gives

$$\liminf_{r \to \infty} \frac{\tilde{n}(r)}{n(r)} \geqslant (\theta - c)\left(1 - \frac{1}{cA}\right). \tag{3.4.54}$$

For both cases we have the trivial inequality

$$\limsup_{r \to \infty} \frac{\tilde{n}(r)}{n(r)} \leq 1. \tag{3.4.55}$$

With $\mathcal{E}$ lying in the complement of $\mathcal{E}(\mu,\beta)$, we can again choose as $\{\zeta_{p',q}\}$ a set for which

$$|\zeta_{p',q}|^{-\rho}\log|f(\zeta_{p',q})| \leq \mu\tau . \qquad (3.4.56)$$

We restrict ourselves, at first, to the case for $\rho \leq 1$. If the type τ of $f(z)$ satisfies

$$\tau < \frac{\theta}{2\pi\rho^2\Lambda^2}, \qquad (3.4.57)$$

then again we can choose c small enough and cA large enough to allow us to use Theorem 30 to give

$$\tau = \limsup_{r\to\infty}\frac{\log M(r,f)}{r^\rho} \leq \mu\tau + \frac{1-(\theta-c)(1-\frac{1}{cA})}{2\pi\rho^2\Lambda^2}. \qquad (3.4.58)$$

This holds for all small c and large cA and hence

$$\tau \leq \mu\tau + \frac{1-\theta}{2\pi\rho^2\Lambda^2}. \qquad (3.4.59)$$

We can restate this result in the following form. If $\theta > 2\pi\rho^2\Lambda^2\tau$, then $\theta \leq 1-(1-\mu)2\pi\rho^2\Lambda^2\tau$; that is

$$\theta \leq \sup\left\{2\pi\rho^2\Lambda^2\tau,\ 1-(1-\mu)2\pi\rho^2\Lambda^2\tau\right\}, \qquad (3.4.60)$$

and since Λ^2 is any number exceeding $\Delta^2\sqrt{3}/2\pi^2 = \beta^2/2\pi\rho^2\sqrt{3}$ this gives

$$\theta = \text{lower density }\mathcal{E} \leq \sup\{\beta^2,\ 1-(1-\mu)\beta^2\}. \qquad (3.4.61)$$

We can take $\mathcal{E}$ to be the complement of $\mathcal{E}(\mu,\beta)$ and hence

$$\text{upper density }\mathcal{E}(\mu,\beta) \geq \inf\{1-\beta^2,\ (1-\mu)\beta^2\}, \qquad (3.4.62)$$

for $\rho \leq 1$. A similar argument when $\rho \geq 1$ leads to

$$\theta^{\rho} \leq \sup\{\beta^2, 1-(1-\mu)\beta^2\} , \qquad (3.4.63)$$

and therefore, in this case,

$$\text{upper density } \mathcal{E}(\mu,\beta) \geq \inf\{1-\beta^{2/\rho}, 1-(1-(1-\mu)\beta^2)^{1/\rho}\}. \qquad (3.4.64)$$

Finally we note that the upper density of $\mathcal{E}(\mu,\beta)$ is a non-increasing function of μ and β , and therefore

$$\text{upper density } \mathcal{E}(\mu,\beta) \geq \sup_{\beta' \geq \beta, \mu' \geq \mu} \left(\inf\{1-\beta'^2, (1-\mu')\beta'^2\} \right), \qquad (3.4.65)$$

for $\rho \leq 1$, and since $1-\beta^2 \leq (1-\mu)\beta^2$ if and only if $\beta^2 \geq 1/(2-\mu)$ it follows that

$$\text{upper density } \mathcal{E}(\mu,\beta) \geq \inf\left\{1-\beta^2, \frac{1-\mu}{2-\mu}\right\}, \qquad (3.4.66)$$

for $\rho \leq 1$. Similarly for $\rho \geq 1$ we have

$$\text{upper density } \mathcal{E}(\mu,\beta) \geq \inf\left\{1-\beta^{2/\rho}, 1-\left(\frac{1}{2-\mu}\right)^{1/\rho}\right\}. \qquad (3.4.67)$$

This completes the proof of Theorem 28.

CHAPTER IV

INTEGRAL FUNCTIONS AND FUNCTIONS REGULAR IN A SECTOR, WITH INTERMEDIATE GROWTH CONDITIONS

4.1. Introduction. In this chapter we shall be concerned with results which are intermediate to, an in certain cases improvements on, Theorems 13 and 14 of Chapter I. As mentioned there, if in Theorem 13 we have $\alpha < \pi/\rho$ then the result follows simply from a Phragmèn-Lindelöf argument, without any appeal to the properties of $f(z)$ on the set of points $\{z_n\}$. Theorem 14, although not having this possible redundancy, is very specialised; it concerns only functions of growth less than type $\frac{1}{2}\pi$ of order 2 in the sector, the interpolation set is completely determinate and the growth condition is one of boundedness. In Theorem 31, we generalise Theorem 14 in a number of ways; we now include all cases of growth of finite type of positive order ρ and the interpolation sets are those of Chapter II specialised to the case $\nu(r) \equiv \rho$. Furthermore, in addition to giving to information when the condition on $f(z)$ at the interpolation set is one of finite type of order ρ as well as in the case where $f(z)$ is bounded on the set, as does Theorem 13, our theorem covers all cases of suitably smooth intermediate growth.

We are also, as a corollary to Theorem 31, able to establish similar results for integral functions having such intermediate growth on the interpolation set. The relaxation of the boundary conditions, compared to those of Theorems 13 and 17, achieved also allows us, in the final section of the chapter, to discuss some problems in the theory of integral functions with gap power series.

4.2. Principle results. The general theorem is as follows.

THEOREM 31. Suppose that $f(z)$ is regular in $S(\alpha, \ell)$ and satisfies:

(i) $$\liminf_{r\to\infty} \frac{\log M(r,\alpha,f)}{r^\rho} < \frac{4\pi}{\Delta^2 3\sqrt{3}} = \tau \; ; \qquad (4.2.1)$$

(ii) $$\limsup_{r\to\infty} \frac{\log |f(re^{\pm i\alpha})|}{r^\rho} \leq \gamma < \tau \; ; \qquad (4.2.2)$$

and (iii) $$\limsup_{z_n \in S(\alpha,\ell)} \frac{\log |f(z_n)|}{H(|z_n|)} \leq \kappa < \infty , \qquad (4.2.3)$$

where the union of the discs

$$|z - z_n| < \Delta |z_n|^{1-\frac{1}{2}\rho} \qquad (4.2.4)$$

covers $S(\alpha, \ell)$, the growth estimating function $H(r)$ is positive and non-decreasing and $r^{-\rho}H(r)$ is non-increasing and has limit c, possibly zero, with $c\kappa < \tau$. If $\alpha < \pi/\rho$ suppose, in addition, that

$$\sin^2 \tfrac{1}{2}\rho\alpha > \frac{\gamma}{\tau}\left\{1 - \frac{c\kappa(\tau-\gamma)}{\gamma(\tau - c\kappa)}\right\}. \qquad (4.2.5)$$

Then

$$\limsup_{r\to\infty} \frac{\log M(r,\beta,f)}{H(r)} \leq \varkappa\left\{1+\frac{c\varkappa}{\tau-c\varkappa}\right\}+K\lim_{r\to\infty}\frac{1}{H(r)}, \quad (4.2.6)$$

for all β ($0<\beta<\alpha$) which satisfy

$$\sin^2\left\{\tfrac{1}{2}\rho(\alpha-\beta)\right\} > \frac{\gamma}{\tau}\left\{1-\frac{c\varkappa(\tau-\gamma)}{\gamma(\tau-c\varkappa)}\right\}. \quad (4.2.7)$$

Furthermore, the conclusion remains unchanged if in (4.2.3) the set $\{z_n\}$ is replaced by a set $\{z'_n\}$, where for some finite positive numbers d, δ

$$|z'_n - z_n| \leq d\,|z_n|^{1-\frac{1}{2}\rho} \quad (4.2.8)$$

and for $m \neq n$

$$|z'_m - z'_n| \geq \delta \inf\left\{|z'_m|^{1-\frac{1}{2}\rho}, |z'_n|^{1-\frac{1}{2}\rho}\right\}. \quad (4.2.9)$$

If $c\varkappa = 0$ and $H(r)$ is unbounded the theorem gives

$$\limsup_{r\to\infty} \frac{\log M(r,\beta,f)}{H(r)} \leq \varkappa, \quad (4.2.10)$$

which is the best one could expect, and if $H(r)$ is bounded, that is $|f(z_n)| \leq A$ for $z_n \in S(\alpha, l)$, then $f(z)$ is bounded in $S(\beta, l)$. The angles α, β in these cases are such that

$$0<\beta<\alpha, \quad \sin^2\left\{\tfrac{1}{2}\rho(\alpha-\beta)\right\} > \frac{\gamma}{\tau}.$$

If $c\varkappa > 0$ we may assume that $H(r) = r^\rho$ and we then obtain

$$\limsup_{r\to\infty} \frac{\log M(r,\beta,f)}{r^\rho} \leq \varkappa\left\{\frac{\tau}{\tau-\varkappa}\right\}, \quad (4.2.11)$$

where α, β satisfy

$$0 < \beta < \alpha \;, \quad \sin^2\{\tfrac{1}{2}\rho(\alpha-\beta)\} > \frac{\gamma}{\tau}\left\{1 - \frac{\kappa(\tau-\gamma)}{\gamma(\tau-\kappa)}\right\}. \qquad (4.2.12)$$

This result, although not necessarily best possible, gives a smaller estimate for the type of $f(z)$ in $S(\beta,\ell)$ than that of Theorem 13 when

$$\kappa < \frac{\gamma\tau}{\tau+\gamma} \;; \qquad (4.2.13)$$

otherwise the result is inferior to that of Theorem 13.

For integral functions we have

THEOREM 32. Suppose that $f(z)$ is an integral function which satisfies both

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{r^\rho} < \tau \qquad (4.2.14)$$

and

$$\limsup \frac{\log|f(z_n)|}{H(|z_n|)} \leq \kappa < \infty, \qquad (4.2.15)$$

where the discs

$$|z - z_n| \leq \Delta |z_n|^{1-\frac{1}{2}\rho}$$

cover the complex plane, $\tau = 4\pi/\Delta^2 3\sqrt{3}$ and $H(r)\uparrow$, $r^{-\rho}H(r)\downarrow\to c$ as $r\to\infty$, with $c\kappa<\tau$. Then

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{H(r)} \leq \kappa . \qquad (4.2.16)$$

Theorem 32 only provides information additional to that of Theorem 15 in the case where $H(r)$ is unbounded and $r^{-\rho}H(r)$ tends to zero. There are a number of cases of special interest. For example, if

$$H(r) = (\log r)^2 \quad \text{then} \quad \log M(r,f) = O(\log r)^2,$$

and if we have, instead of (4.2.15),

$$|f(z_n)| \leq A|z_n|^k, \tag{4.2.17}$$

the conclusion is that $f(z)$ is a polynomial of degree at most k. Theorem 32 is an almost immediate corollary to the preceeding theorem; it follows from Theorem 15 that the type of $f(z)$ does not exceed $c\chi$ and therefore we can satisfy the conditions, suitably modified to allow for change of orientation, of Theorem 31 for an arbitary sector when $c = 0$. The part of Theorem 32 not contained in Theorem 15 follows, since we can take

$$|\arg z| \leq \tfrac{3}{4}\pi \quad \text{and} \quad |\arg z - \pi| \leq \tfrac{3}{4}\pi,$$

separately, as sectors in Theorem 31.

<u>4.3. Sequences of canonical products.</u> The proof of Theorem 31 is again based on cardinal series techniques. However, it is not now sufficient to have only one function $\eta(z)$ which, for integral functions of suitable growth, would ensure that

$$f(z) = \eta(z)\sum_n \frac{f(z_n)}{(z-z_n)\,\eta'(z_n)}, \tag{4.3.1}$$

where $\{z_n\}$ is the set of zeros of $\eta(z)$. We shall in fact construct an infinite set of functions, each member of which has this property.

We consider a set of points $\{z_{m,n}\}$ $(m, n = 0, \pm 1, \pm 2, \dots)$ for which

$$|z_{m,n} - (m+in)| \leq D < \infty \tag{4.3.2}$$

and, for all $(p,q) \neq (m,n)$,

$$|z_{p,q} - z_{m,n}| \geq \delta > 0. \tag{4.3.3}$$

For each pair (M, N) we form the integral function $\sigma_{M,N}(z)$ defined by

$$\sigma_{M,N}(z) = (z - z_{M,N}) \exp\left\{-\tfrac{1}{2}\alpha_{M,N}(z - z_{M,N})^2\right\} \prod_{(m,n)\neq(M,N)} E\left(\frac{z - z_{M,N}}{z_{m,n} - z_{M,N}}, 2\right), \tag{4.3.4}$$

where
$$\alpha_{M,N} = \lim_{r\to\infty} \sum_{\alpha < |z_{m,n} - z_{M,N}| \leq r} (z_{m,n} - z_{M,N})^{-2} \tag{4.3.5}$$

(we shall show later that this always exists).

In the special case $z_{m,n} = m + in$, $\sigma_{M,N}(z)$ becomes $\sigma(z-M-iN)$. Even in the general case the growth of these two functions is similar; if we denote by (A_η) the set of discs

$$|z - z_{m,n}| \leq \eta < \tfrac{1}{4}\delta, \tag{4.3.6}$$

the main properties of the set of functions $\{\sigma_{M,N}(z)\}$ are contained in the following theorem.

THEOREM 33. For the $\sigma_{M,N}(z)$ described above, we have:

(i) $\sigma_{M,N}(z_{m,n}) = 0$, for all (M,N), (m,n), and $\sigma_{M,N}(z)$ has no zeros outside the set $\{z_{m,n}\}$. Also, for every $\epsilon > 0$, we can determine $r_0 = r_0(\epsilon, \eta, \delta, D)$, independent of (M,N), such that:

(ii) $$\log|\sigma_{M,N}(z)| \leq (\tfrac{1}{2}\pi+\epsilon)|z-z_{M,N}|^2, \qquad (4.3.7)$$

for $|z-z_{M,N}| \geq r_0$;

(iii) $$\log|\sigma_{M,N}(z)| \geq (\tfrac{1}{2}\pi-\epsilon)|z-z_{M,N}|^2, \qquad (4.3.8)$$

for $|z-z_{M,N}| \geq r_0$ and z outside (A_η); and

(iv) $$\log|\sigma'_{M,N}(z_{m,n})| \geq (\tfrac{1}{2}\pi-\epsilon)|z_{m,n}-z_{M,N}|^2, \qquad (4.3.9)$$

for $|z_{m,n}-z_{M,N}| \geq r_0$. Furthermore,

(v) $$|\sigma_{M,N}(z)| \geq \omega > 0, \qquad (4.3.10)$$

for z outside (A_η), and

(vi) $$|\sigma'_{M,N}(z_{m,n})| \geq \omega' > 0, \qquad (4.3.11)$$

for all (m,n), (M,N).

Proof. Part (i) of the theorem follows directly from the definition of $\sigma_{M,N}(z)$.

The proof of parts (ii), (iii), (iv) is based on a comparison of $\sigma_{M,N}(z)$ with $\sigma(z-z_{M,N})$. We denote by (B_η) (different for each pair (M,N)) the set of discs

$$|z-(m+in+z_{M,N})| \leq \eta \quad (m,n=0,\pm 1,\dots). \qquad (4.3.12)$$

We then have

LEMMA 4.3.1. If z is outside (A_η) and (B_η), and $\epsilon>0$, then we can determine $r_1 = r_1(\epsilon,\eta,D)$, independent of (M,N), such that

$$\left| \log \left| \prod_{\mathcal{E}'} \left\{ \frac{1 - \dfrac{z - z_{M,N}}{z_{m+M,n+N} - z_{M,N}}}{1 - \dfrac{z - z_{M,N}}{m+in}} \right\} \right| \right| \leq \epsilon |z - z_{M,N}|^2, \qquad (4.3.13)$$

where $\mathcal{E}'$ is the set of (m,n) satisfying

$$0 < |m+in| \leq 2|z - z_{M,N}| .$$

Proof. Write $|z-z_{M,N}| = r$, and let $\mathcal{E}_1$ be the set of (m,n) satisfying either

$$0 < |m+in| \leq r^{\frac{1}{2}} \quad \text{or} \quad |m+M + i(n+N) - z| \leq r^{\frac{1}{2}} .$$

Now

$$\frac{1 - \dfrac{z - z_{M,N}}{m+in}}{1 - \dfrac{z - z_{M,N}}{z_{m+M,n+N} - z_{M,N}}} = 1 + \frac{(z - z_{M,N})(m+in+z_{M,N} - z_{m+M,n+N})}{(m+in)(z_{m+M,n+N} - z)}, \quad (4.3.14)$$

and if (m,n) belongs to $\mathcal{E}' - \mathcal{E}_1$, we have, writing $\delta_{p,q} = z_{p,q} - (p+iq)$,

$$\begin{aligned} |(m+in)(z_{m+M,n+N} - z)| &= |(m+in)\{ m+in - (z - M - iN) + \delta_{m+M,n+N}\}| , \\ &\geq \inf_{\zeta \notin \mathcal{E}^*} |\zeta\{\zeta - (z - M - iN) + \delta_{m+M,n+N}\}| , \end{aligned}$$

where $\mathcal{E}^*$ is the set of ζ satisfying either $|\zeta| \leq r^{\frac{1}{2}}$ or $|\zeta - (z - M - iN)| \leq r^{\frac{1}{2}}$. Now $\zeta\{\zeta - (z - M - iN) + \delta_{m+M,n+N}\}$ is a quadratic in ζ with no zeros outside $\mathcal{E}^*$ and hence its minimum modulus outside $\mathcal{E}^*$ occurs on the boundary of $\mathcal{E}^*$. This gives

$$|(m+in)(z_{m+M,n+N} - z)| \geq (r^{\frac{1}{2}} - D)(r - r^{\frac{1}{2}} - 2D), \qquad (4.3.15)$$

and therefore

$$\left| \frac{(z - z_{M,N})(m+in+z_{M,N} - z_{m+M,n+N})}{(m+in)(z_{m+M,n+N} - z)} \right| \leq \frac{2Dr}{(r^{\frac{1}{2}} - D)(r - r^{\frac{1}{2}} - 2D)}, \qquad (4.3.16)$$

$$\left|\log\left|\prod_{\mathcal{E}'}\left\{\frac{1-\dfrac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}}{1-\dfrac{z-z_{M,N}}{m+in}}\right\}\right|\right| \leq \epsilon|z-z_{M,N}|^2, \qquad (4.3.13)$$

where $\mathcal{E}'$ is the set of (m,n) satisfying

$$0<|m+in|\leq 2|z-z_{M,N}|.$$

Proof. Write $|z-z_{M,N}|=r$, and let $\mathcal{E}_1$ be the set of (m,n) satisfying either

$$0<|m+in|\leq r^{\frac{1}{2}} \quad \text{or} \quad |m+M+i(n+N)-z|\leq r^{\frac{1}{2}}.$$

Now

$$\frac{1-\dfrac{z-z_{M,N}}{m+in}}{1-\dfrac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}} = 1+\frac{(z-z_{M,N})(m+in+z_{M,N}-z_{m+M,n+N})}{(m+in)(z_{m+M,n+N}-z)}, \quad (4.3.14)$$

and if (m,n) belongs to $\mathcal{E}'-\mathcal{E}_1$, we have, writing $\delta_{p,q}=z_{p,q}-(p+iq)$,

$$|(m+in)(z_{m+M,n+N}-z)| = |(m+in)\{m+in-(z-M-iN)+\delta_{m+M,n+N}\}|,$$

$$\geq \inf_{\zeta\notin\mathcal{E}^*}|\zeta\{\zeta-(z-M-iN)+\delta_{m+M,n+N}\}|,$$

where $\mathcal{E}^*$ is the set of ζ satisfying either $|\zeta|\leq r^{\frac{1}{2}}$ or $|\zeta-(z-M-iN)|\leq r^{\frac{1}{2}}$. Now $\zeta\{\zeta-(z-M-iN)+\delta_{m+M,n+N}\}$ is a quadratic in ζ with no zeros outside $\mathcal{E}^*$ and hence its minimum modulus outside $\mathcal{E}^*$ occurs on the boundary of $\mathcal{E}^*$. This gives

$$|(m+in)(z_{m+M,n+N}-z)|\geq (r^{\frac{1}{2}}-D)(r-r^{\frac{1}{2}}-2D), \qquad (4.3.15)$$

and therefore

$$\left|\frac{(z-z_{M,N})(m+in+z_{M,N}-z_{m+M,n+N})}{(m+in)(z_{m+M,n+N}-z)}\right| \leq \frac{2Dr}{(r^{\frac{1}{2}}-D)(r-r^{\frac{1}{2}}-2D)}, \quad (4.3.16)$$

$$\leq \frac{4D}{r^{\frac{1}{2}}} ,$$

for $r \geqslant r_2(D)$, and if $r \geqslant r_3(\epsilon, D)$ we have

$$\left|\log\left|1 + \frac{(z-z_{M,N})(m+in+z_{M,N}-z_{m+M,n+N})}{(m+in)(z_{m+M,n+N}-z)}\right|\right| \leqslant \frac{\epsilon}{32} . \quad (4.3.17)$$

There are at most $\pi(2r+1)^2$ members of $\mathcal{E}'$ and hence for sufficiently large r ,

$$\left|\log\left|\prod_{\mathcal{E}'-\mathcal{E}_1}\left\{\frac{1-\frac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}}{1-\frac{z-z_{M,N}}{m+in}}\right\}\right|\right| \leqslant \tfrac{1}{2}\epsilon r^2 . \quad (4.3.18)$$

If $(m,n)\in\mathcal{E}_1$ and z is outside (A_η) and (B_η) we have the inequalities:

$$1 \leqslant |m+in| \leqslant 2r ;$$

$$\delta \leqslant |z_{m+M,n+N} - z_{M,N}| \leqslant |m+in| + 2D \leqslant 2r+2D ;$$

$$\eta \leqslant |z_{m+M,n+N} - z| \leqslant |z - z_{M,N}| + |m+in| + 2D \leqslant 3r+2D ;$$

$$\eta \leqslant |m+in+z_{M,N}-z| \leqslant 3r;$$

and therefore

$$\frac{\eta\delta}{2r(3r+2D)} \leqslant \left|\frac{(m+in+z_{M,N}-z)(z_{m+M,n+N}-z_{M,N})}{(m+in)(z_{m+M,n+N}-z)}\right| \leqslant \frac{3r(2r+2D)}{\eta} . \quad (4.3.19)$$

This gives, for $r \geqslant r_4(D)$,

$$\left|\log\left|\frac{(m+in+z_{M,N}-z)(z_{m+M,n+N}-z_{M,N})}{(m+in)(z_{m+M,n+N}-z)}\right|\right| \leqslant \log\frac{9r^2}{\eta\delta} . \quad (4.3.20)$$

There are at most $2\pi(r^{\frac{1}{2}}+1)^2$ members of $\mathcal{E}_1$ and therefore

$$\log\left|\left|\prod_{\mathcal{E}_1}\left\{\frac{1-\frac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}}{1-\frac{z-z_{M,N}}{m+in}}\right\}\right|\right| \leqslant 4\pi(r^{\frac{1}{2}}+1)^2\log\frac{3r}{\eta\delta} ,$$

$$\leqslant \tfrac{1}{2}\epsilon r^2 , \qquad (4.3.21)$$

for $r \geqslant r_5(\epsilon,\eta,\delta,D)$, which together with (4.3.18) establishes the lemma.

LEMMA 4.3.2. If $|z - z_{M,N}| \geqslant 4(D+1)$, there exists a constant $a_1 = a_1(D)$ such that

$$\left| \log \left| \prod_{\mathcal{G}} \frac{E\left(\frac{z - z_{M,N}}{z_{m+M,n+N} - z_{M,N}}, 1\right)}{E\left(\frac{z - z_{M,N}}{m+in}, 1\right)} \right| \right| \leqslant a_1 |z - z_{M,N}| ,$$

where $\mathcal{G}$ is the set of (m,n) such that $|m+in| > 2|z - z_{M,N}|$.

Proof. We have, under the given conditions, both

$$\left| \frac{z - z_{M,N}}{z_{m+M,n+N} - z_{M,N}} \right| \leqslant \frac{r}{2r - 2D} \leqslant \frac{2}{3} \quad \text{and} \quad \left| \frac{z - z_{M,N}}{m+in} \right| \leqslant \frac{1}{2} ,$$

and therefore, using Lemma 2.3.1,

$$\left| \log \frac{E\left(\frac{z - z_{M,N}}{z_{m+M,n+N} - z_{M,N}}, 1\right)}{E\left(\frac{z - z_{M,N}}{m+in}, 1\right)} \right| \leqslant \frac{6D\,|z - z_{M,N}|^2}{|m+in|\,(|m+in| - 2D)^2} , \qquad (4.3.22)$$

$$\leqslant \frac{24\,D\,r^2}{|m+in|^3} . \qquad (4.3.23)$$

Lemma 4.3.2 follows, since

$$\sum_{|m+in| \geqslant r} \frac{1}{|m+in|^2} = O\left(\frac{1}{r}\right) . \qquad (4.3.24)$$

LEMMA 4.3.3.

$$\alpha_{M,N} = \lim_{r \to \infty} \sum_{0 < |z_{m,n} - z_{M,N}| \leqslant r} \frac{1}{(z_{m,n} - z_{M,N})^2}$$

exists and is bounded.

Proof. Since

$$\sum_{r_1 < |m+in| \le r_2} \frac{1}{(m+in)^2} \equiv 0.$$

we have

$$\left| \left\{ \sum_{0<|m+in|\le r_1} - \sum_{0<|m+in|\le r_2} \right\} \frac{1}{(z_{m+N,n+N} - z_{M,N})^2} \right|$$

$$= \left| \sum_{r_1<|m+in|\le r_2} \left\{ \frac{1}{(z_{m+M,n+N} - z_{M,N})^2} - \frac{1}{(m+in)^2} \right\} \right| ,$$

$$\le \sum_{r_1<|m+in|\le r_2} \frac{2D \cdot 2(|m+in| + D)}{(|m+in| - 2D)^2 \, |m+in|^2} ,$$

$$\le \sum_{|m+in|\ge r_1} \frac{8D}{|m+in|^3} , \qquad \text{for } r_1 \ge K(D).$$

Consequently

$$\left| \sum_{r_1<|m+in|\le r_2} \frac{1}{(z_{m+M,n+N} - z_{M,N})^2} \right| \le \frac{K}{r_1} \longrightarrow 0 , \qquad (4.3.25)$$

as $r_1 \to \infty$. This shows that

$$\beta_{M,N} = \lim_{r\to\infty} \sum_{0<|m+in|\le r} \frac{1}{(z_{m+M,n+N} - z_{M,N})^2}$$

exists. Since the number of (m,n) satisfying $|m+in| \le r_1$ is at most $\pi(r_1+1)^2$, we have

$$\left| \sum_{0<|m+in|\le r_1} \frac{1}{(z_{m+M,n+N} - z_{M,N})^2} \right| \le \frac{\pi(r_1+1)^2}{\delta^2} . \qquad (4.3.26)$$

This together with (4.3.25) shows that $\beta_{M,N}$ is bounded. Now consider

$$\sum_{0<|z_{p,q} - z_{M,N}|\le r} \frac{1}{(z_{p,q} - z_{M,N})^2} - \sum_{0<|m+in|\le r+2D} \frac{1}{(z_{m+M,n+N} - z_{M,N})^2} .$$

Since every element of the first sum is a member of the second, and that the number of elements contained in the second sum but not the first is at most

$$\pi(r+2D+1)^2 - \pi(r-1)^2,$$

it follows that

$$\left|\sum_{0\le|z_{p,q}-z_{M,N}|\le r} \frac{1}{(z_{p,q}-z_{M,N})^2} - \sum_{0<|m+in|\le r+2D} \frac{1}{(z_{m+M,n+N}-z_{M,N})^2}\right| \le \frac{4\pi(D+1)(r+D)}{r^2},$$

$$\longrightarrow 0,$$

as $r\to\infty$. Hence, since $\beta_{M,N}$ exists so does $\alpha_{M,N}$ and $\alpha_{M,N}=\beta_{M,N}$. This completes the proof of the lemma.

Now

$$\left|(z-z_{M,N})\left\{\frac{1}{z_{m+M,n+N}-z_{M,N}} - \frac{1}{m+in}\right\}\right| \le \inf\left\{\frac{2Dr}{\big|\,|m+in|-2D\big|\,|m+in|}, \frac{2r}{\delta}\right\},$$

and, therefore, writing

$$\sum_{\varepsilon'} = \sum_{0<|m+in|<4D} + \sum_{4D<|m+in|\le 2r}$$

and using the estimate $2r/\delta$ in the first sum and $2Dr/(|m+in|-2D)|m+in|$ in the second, we have

$$\left|\sum_{\varepsilon'}(z-z_{M,N})\left\{\frac{1}{z_{m+M,n+N}-z_{M,N}} - \frac{1}{m+in}\right\}\right|$$

$$\le \frac{2r\pi(4D+1)^2}{\delta} + 4Dr\sum_{\varepsilon'}\frac{1}{|m+in|^2},$$

$$\le K_1(\delta,D)\,r + K_2(D)\,r\log r,$$

since $\displaystyle\sum_{0<|m+in|\le r}\frac{1}{|m+in|^2}=O(\log r)$ for large r . Hence we can find an a_2 such that, for $r \geq r(\delta, D)$,

$$\left| \log \left| \prod_{\mathcal{E}'} \frac{\exp\left\{ \dfrac{z - z_{M,N}}{z_{m+M,n+N} - z_{M,N}} \right\}}{\exp\left\{ \dfrac{z - z_{M,N}}{m+in} \right\}} \right| \right| \leq a_2 r \log r . \quad (4.3.27)$$

Also

$$\prod_{m+in \neq 0} \left\{ \frac{\exp\left\{ \frac{1}{2}\left(\dfrac{z - z_{M,N}}{z_{m+M,n+N} - z_{M,N}} \right)^2 \right\}}{\exp\left\{ \frac{1}{2}\left(\dfrac{z - z_{M,N}}{m+in} \right)^2 \right\}} \right\} = \exp\left\{ \tfrac{1}{2} d_{M,N} (z - z_{M,N})^2 \right\}. \quad (4.3.28)$$

Combining the results of Lemmas 4.3.1 and 4.3.2 with these last two expressions gives, for z outside (A_η) and (B_η),

$$\left| \log \left| \frac{\sigma_{M,N}(z)}{\sigma(z - z_{M,N})} \right| \right| \leq \epsilon |z - z_{M,N}|^2 + (a_1 + a_2)|z - z_{M,N}| \log |z - z_{M,N}|,$$

$$\leq 2\epsilon |z - z_{M,N}|^2 , \quad (4.3.29)$$

for $|z - z_{M,N}| \geq r_1(\epsilon, \eta, \delta, D)$. Hence, on rechoosing ϵ and using the properties of the σ-function, we have

$$\exp\left\{ (\tfrac{1}{2}\pi - \tfrac{1}{2}\epsilon)|z - z_{M,N}|^2 \right\} \leq |\sigma_{M,N}(z)| \leq \exp\left\{ (\tfrac{1}{2}\pi + \tfrac{1}{2}\epsilon)|z - z_{M,N}|^2 \right\} , \quad (4.3.30)$$

for $|z - z_{M,N}| \geq r_2(\epsilon, \eta, \delta, D)$, with z outside (A_η) and (B_η) .

The inequalities of (4.3.30) hold for any z outside $(A_{\frac{1}{4}\eta})$ and $(B_{\frac{1}{4}\eta})$, for $|z - z_{M,N}| \geq r_2(\epsilon, \frac{1}{4}\eta, \delta, D)$. Now if two discs of $(A_{\frac{1}{4}\eta})$ and $(B_{\frac{1}{4}\eta})$ intersect, then the corresponding disc of (A_η) contains the disc of $(B_{\frac{1}{4}\eta})$ in its interior. Hence we can apply the maximum and minimum modulus principles to those discs of $(B_{\frac{1}{4}\eta})$ which do not intersect a disc of $(A_{\frac{1}{4}\eta})$ to yield

$$\exp\left\{ (\tfrac{1}{2}\pi - \tfrac{1}{2}\epsilon)(|z - z_{M,N}| - \tfrac{1}{2}\eta)^2 \right\} \leq |\sigma_{M,N}(z)| \leq \exp\left\{ (\tfrac{1}{2}\pi + \tfrac{1}{2}\epsilon)(|z - z_{M,N}| + \tfrac{1}{2}\eta)^2 \right\}, \quad (4.3.31)$$

for all z outside (A_η), with $|z-z_{M,N}| \geq r_2(\epsilon, \frac{1}{4}\eta, \delta, D)$; that is

$$\exp\{(\tfrac{1}{2}\pi-\epsilon)|z-z_{M,N}|^2\} < |\sigma_{M,N}(z)| < \exp\{(\tfrac{1}{2}\pi+\epsilon)|z-z_{M,N}|^2\}, \quad (4.3.32)$$

for $|z-z_{M,N}| \geq r_3(\epsilon,\eta,\delta,D)$, with z outside (A_η). The left-hand inequality of (4.3.32) is conclusion (iii) of Theorem 33 while the right-hand inequality together with the maximum modulus principle applied to the discs of (A_η) gives conclusion (ii).

To obtain Theorem 33 (iv), we set, temporarily,

$$\zeta(z) = \sigma_{M,N}(z)/(z-z_{m,n}) \qquad (z \neq z_{m,n}),$$

and

$$\zeta(z_{m,n}) = \sigma'_{M,N}(z_{m,n}),$$

then $\zeta(z)$ is an integral function and

$$\left| \frac{\log|\zeta(z)|}{|z-z_{M,N}|^2} - \frac{1}{2}\pi \right| \leq \left| \frac{\log|\sigma_{M,N}(z)|}{|z-z_{M,N}|^2} - \frac{1}{2}\pi \right| + \left| \frac{\log|z-z_{m,n}|}{(z-z_{M,N})^2} \right|. \quad (4.3.33)$$

Now if z is any point of $|z-z_{m,n}| = \eta$, and if $|z_{m,n}-z_{M,N}| \geq r_3(\frac{1}{2}\epsilon,\eta,\delta,D)$, it follows that

$$\left| \frac{\log|\zeta(z)|}{|z-z_{M,N}|^2} - \frac{1}{2}\pi \right| < \frac{1}{2}\epsilon + \frac{|\log\eta|}{(|z_{m,n}-z_{M,N}|-\eta)^2}. \quad (4.3.34)$$

Since $\zeta(z)$ is regular and non-zero in $|z-z_{m,n}| \leq \eta$, we have

$$\left| \frac{\log|\zeta(z_{m,n})|}{|z_{m,n}-z_{M,N}|^2} - \frac{1}{2}\pi \right| < \epsilon, \quad (4.3.35)$$

if $|z_{m,n}-z_{M,N}| \geq r_4(\epsilon,\eta,\delta,D)$, as required.

For $|z-z_{M,N}|$, $|z_{m,n}-z_{M,N}| \geq r_0(\frac{1}{4}\pi,\eta,\delta,D) = r_0$ (say), results

(v) and (vi) of the theorem follow from parts (iii) and (iv). For $|z - z_{M,N}| \leq r_0$ with z outside (A_η) and $0 < |m+in| \leq 2r_0$,

$$\left| \left\{ \frac{z_{m+M,n+N} - z}{z_{m+M,n+N} - z_{M,N}} \right\} \exp\left\{ \frac{z - z_{M,N}}{z_{m+M,n+N} - z_{M,N}} + \frac{1}{2}\left(\frac{z - z_{M,N}}{z_{m+M,n+N} - z_{M,N}} \right)^2 \right\} \right|$$

$$\geq \frac{\eta \exp\left\{ -\frac{r_0^2}{2\delta^2} - \frac{r_0}{\delta} \right\}}{2(r_0 + D)} \quad . \qquad (4.3.36)$$

If $|m+in| > 2r_0 \geq 8D$,

$$\left| \log E\left(\frac{z - z_{M,N}}{z_{m+M,n+N} - z_{M,N}}, 2 \right) \right| = \left| \sum_{p=3}^{\infty} \frac{1}{p} \left(\frac{z - z_{M,N}}{z_{m+M,n+N} - z_{M,N}} \right)^p \right| ,$$

$$\leq \sum_{p=3}^{\infty} \frac{r_0^{\,p}}{p(|m+in| - 2D)^p} ,$$

$$\leq \frac{K_3 r_0^3}{|m+in|^3} ,$$

and, since $\sum_{m+in \neq 0} |m+in|^{-3}$ is convergent, we have

$$\left| \prod_{|m+in| \geq 2r_0} E\left(\frac{z - z_{M,N}}{z_{m+M,n+N} - z_{M,N}}, 2 \right) \right| \geq \exp\left\{ -K_4 r_0^3 \right\} . \qquad (4.3.37)$$

Combining equations (4.3.36) and (4.3.37) gives, since the $\alpha_{M,N}$ are bounded ($\leq K_5$, say),

$$\log|\sigma_{M,N}(z)| \geq -\pi(2r_0+1)^2 \left\{ \frac{r_0}{\delta} + \frac{1}{2}\frac{r_0^2}{\delta^2} + \log\frac{2(r_0+D)}{\eta} \right\} - K_4 r_0^3 - K_5 r_0^2 , \qquad (4.3.38)$$

for z outside (A_η). This establishes part (v) of Theorem 33.

Finally, $\zeta(z) = \sigma_{M,N}(z)/(z - z_{m,n})$ is such that $\zeta(z_{m,n}) = \sigma'_{M,N}(z_{m,n})$ and $|\zeta(z)| \geq \omega/\eta$ for z on the boundary of the disc $|z - z_{m,n}| \leq \eta$ and non-zero in its interior. An application of the minimum modulus principle gives part (vi).

4.4. Proof of Theorem 31. We shall first establish the result for the modified case where $\rho = 2$, $\tau = \frac{1}{2}\pi$ and, in (4.2.3), $\{z_n\}$ is a set $\{z_{m,n}\}$ satisfying (4.3.2) and (4.3.3). The starting point of our proof for this case is a generalisation of that used by Pfluger [35] and Cartwright [7] for lattice point interpolation.

In what follows we shall suppose that $0 < \alpha - \beta < \frac{1}{2}\pi$ and

$$\sin^2(\alpha-\beta) > \frac{2\gamma}{\pi}\left\{1 - \frac{cx(\frac{1}{2}\pi - \gamma)}{\gamma(\frac{1}{2}\pi - cx)}\right\} . \tag{4.4.1}$$

We can assume, without loss of generality, that no $z_{m,n}$ lies within a distance $\eta < \frac{1}{4}\delta$ of the boundary of $S(\alpha, l)$; for we can certainly construct such a sequence which agrees with the given one except within a distance η of the boundary.

With $\sigma_{M,N}(z)$ as defined in the preceding section, let

$$\phi_{M,N}(z) = \sigma_{M,N}(z) \sum_{z_{m,n} \in S(\alpha, l)} \frac{f(z_{m,n})}{(z - z_{m,n})\sigma'_{M,N}(z_{m,n})}, \tag{4.4.2}$$

where $z_{M,N}$ is any member of the sequence $\{z_{m,n}\}$ lying in $S(\beta, R_0)$, for some R_0 to be determined. Now $\phi(z_{m,n}) = f(z_{m,n})$ and so

$$\Phi_{M,N}(z) = \frac{f(z) - \phi_{M,N}(z)}{\sigma_{M,N}(z)} \tag{4.4.3}$$

is regular in $S(\alpha,\ell)$. Let $\{r_p\}$ $r_p\uparrow\to\infty$ be a sequence such that

$$\frac{\log M(r_p,\alpha,f)}{r_p^2} \leq h < \tfrac{1}{2}\pi , \tag{4.4.4}$$

and let C_p be a contour lying in the region $r_p - 1 \leq |z| \leq r_p$ and joining $\arg z = \pm\alpha$, such that the shortest distance from C_p to any point $z_{m,n}$ is at least η. If z is any point on C_p we have, from parts (iii) and (v) of Theorem 33,

$$\left|\frac{1}{\sigma_{M,N}(z)}\right| \leq K_1(\eta,\epsilon)\exp\left\{-\left(\tfrac{1}{2}\pi-\epsilon\right)|z-z_{M,N}|^2\right\}, \tag{4.4.5}$$

and therefore

$$\left|\frac{f(z)}{\sigma_{M,N}(z)}\right| \leq K_2(\eta,\epsilon)\exp\left\{-\left(\tfrac{1}{2}\pi-\epsilon\right)(r_p-|z_{M,N}|-1)^2+hr_p^2\right\}, \tag{4.4.6}$$

and, with $\epsilon < \frac{1}{4}\left(\frac{1}{2}\pi - h\right)$,

$$\left|\frac{f(z)}{\sigma_{M,N}(z)}\right| \leq K_3(M,N,\eta)\exp\left\{-\tfrac{1}{2}\left(\tfrac{1}{2}\pi-h\right)r_p^2\right\}. \tag{4.4.7}$$

For z on the boundary of $S(\alpha,\ell)$,

$$\left|\frac{f(z)}{\sigma_{M,N}(z)}\right| \leq K_4(\epsilon,\eta)\exp\left\{(\gamma+\epsilon)|z|^2-\left(\tfrac{1}{2}\pi-\epsilon\right)|z-z_{M,N}|^2\right\}, \tag{4.4.8}$$

and since, with $0 \leq A < B$,

$$\begin{aligned} A|re^{i\theta}|^2 - B|re^{i\theta}-Re^{i\varphi}|^2 &= (A-B)r^2 + 2BR\cos(\theta-\varphi) - BR^2, \\ &= -(B-A)\left\{r - \frac{BR\cos(\theta-\varphi)}{B-A}\right\}^2 + \frac{B^2R^2\cos^2(\theta-\varphi)}{B-A} - BR^2, \\ &\leq \frac{BR^2}{B-A}\left\{A - B\sin^2(\theta-\varphi)\right\}, \end{aligned}$$

it follows that

$$\left|\frac{f(z)}{\sigma_{M,N}(z)}\right| \leq K_4(\epsilon,\eta)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)|z_{M,N}|^2}{\frac{1}{2}\pi-\gamma-2\epsilon}\left\{\gamma+\epsilon-(\tfrac{1}{2}\pi-\epsilon)\sin^2(\alpha-\beta)\right\}\right\}. \quad (4.4.9)$$

Also, in view of (4.4.1), we can choose $\epsilon=\epsilon_0$ so small that

$$\frac{\frac{1}{2}\pi-\epsilon_0}{\frac{1}{2}\pi-\gamma-2\epsilon_0}\left\{\gamma+\epsilon_0-(\tfrac{1}{2}\pi-\epsilon_0)\sin^2(\alpha-\beta)\right\} < \frac{\frac{1}{2}\pi c\chi}{\frac{1}{2}\pi-c\chi},$$

which gives

$$\left|\frac{f(z)}{\sigma_{M,N}(z)}\right| \leq K_4(\epsilon_0,\eta)\exp\left\{\frac{\frac{1}{2}\pi c\chi}{\frac{1}{2}\pi-c\chi}|z_{M,N}|^2\right\}. \quad (4.4.10)$$

For z outside (A_η),

$$\left|\frac{\phi_{M,N}(z)}{\sigma_{M,N}(z)}\right| \leq \frac{1}{\eta}\sum_{z_{m,n}\in S(\alpha,\ell)}\left|\frac{f(z_{m,n})}{\sigma'_{M,N}(z_{m,n})}\right|. \quad (4.4.11)$$

From parts (iv) and (vi) of Theorem 33 we deduce that for any ϵ satisfying $0<\epsilon<\frac{1}{4}(\frac{1}{2}\pi-c\chi)$,

$$|\sigma'_{M,N}(z_{m,n})| \geq K_5(\epsilon)\exp\{(\tfrac{1}{2}\pi-\epsilon)|z_{m,n}-z_{M,N}|^2\}, \quad (4.4.12)$$

and, since for $z_{m,n}$ in $S(\alpha,\ell)$

$$|f(z_{m,n})| \leq K_6(\epsilon)\exp\{(\chi+\epsilon)H(|z_{m,n}|)\},$$

we have

$$\left|\frac{f(z_{m,n})}{\sigma'_{M,N}(z_{m,n})}\right| \leq K_7(\epsilon)\exp\{(\chi+\epsilon)H(|z_{m,n}|)-(\tfrac{1}{2}\pi-\epsilon)|z_{m,n}-z_{M,N}|^2\}. \quad (4.4.13)$$

If we now write $|z_{M,N}|=R(M,N)=R$, $|z_{m,n}-z_{M,N}|=r(M,N,m,n)=r$ and $r^{-2}H(r)=\mu(r)\downarrow\to c$, we have

$$(\chi+\epsilon)H(|z_{m,n}|)-(\tfrac{1}{2}\pi-\epsilon)|z_{m,n}-z_{M,N}|^2$$

$$\leq(\chi+\epsilon)H(R+r)-(\tfrac{1}{2}\pi-\epsilon)r^2,$$

$$= (\kappa+\epsilon)(R+r)^2\mu(R+r) - (\tfrac{1}{2}\pi-\epsilon)r^2,$$

$$= \frac{(\frac{1}{2}\pi-\epsilon)(\kappa+\epsilon)\mu(R+r)R^2}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R+r)} - \left\{\tfrac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R+r)\right\}\left\{r - \frac{(\kappa+\epsilon)\mu(R+r)R}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R+r)}\right\}^2,$$

$$\leq \frac{(\frac{1}{2}\pi-\epsilon)(\kappa+\epsilon)\mu(R)R^2}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R)},$$

$$= \frac{(\frac{1}{2}\pi-\epsilon)(\kappa+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R)}. \tag{4.4.14}$$

Also

$$(\kappa+\epsilon)H(R+r) - (\tfrac{1}{2}\pi-\epsilon)r^2 + \tfrac{1}{2}(\tfrac{1}{2}\pi-c\kappa)r^2$$

$$= (\kappa+\epsilon)\mu(R+r)(R+r)^2 - \tfrac{1}{2}(\tfrac{1}{2}\pi-2\epsilon+c\kappa)r^2,$$

$$= \frac{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)(\kappa+\epsilon)\mu(R+r)R^2}{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)-(\kappa+\epsilon)\mu(R+r)} -$$

$$- \left\{\tfrac{1}{2}(\tfrac{1}{2}\pi-2\epsilon+c\kappa)-(\kappa+\epsilon)\mu(R+r)\right\}\left\{r - \frac{(\kappa+\epsilon)\mu(R+r)R}{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)-(\kappa+\epsilon)\mu(R+r)}\right\}^2,$$

and this is negative or zero for all r satisfying

$$r \geq \frac{R}{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)-(\kappa+\epsilon)\mu(R+r)}\left\{(\kappa+\epsilon)\mu(R+r) + \left(\frac{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)(\kappa+\epsilon)\mu(R+r)}{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)-(\kappa+\epsilon)\mu(R+r)}\right)^{\frac{1}{2}}\right\},$$

or equivalently

$$\frac{r}{H(R)^{\frac{1}{2}}} \geq \frac{1}{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)-(\kappa+\epsilon)\mu(R+r)}$$

$$\times\left\{(\kappa+\epsilon)\frac{\mu(R+r)}{\mu(R)^{\frac{1}{2}}} + \left(\frac{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)(\kappa+\epsilon)}{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)-(\kappa+\epsilon)\mu(R+r)}\ \frac{\mu(R+r)}{\mu(R)}\right)^{\frac{1}{2}}\right\}. \tag{4.4.15}$$

Now the right-hand side of (4.4.15) is bounded (suppose by K_8) for all sufficiently large R and all $r \geq 0$, and

therefore

$$(\varkappa+\epsilon)\, H(R+r) - (\tfrac{1}{2}\pi-\epsilon)r^2 + \tfrac{1}{2}(\tfrac{1}{2}\pi - c\varkappa)r^2 \leq 0 \,, \qquad (4.4.16)$$

provided that $r \geqslant K_8 H(R)^{\frac{1}{2}}$, for all sufficiently large R.

We now write

$$\sum_{z_{m,n}\in S(\alpha,\ell)} \left| \frac{f(z_{m,n})}{\sigma'_{M,N}(z_{m,n})} \right| = \left\{ \sum\nolimits_1 + \sum\nolimits_2 \right\} \left| \frac{f(z_{m,n})}{\sigma'_{M,N}(z_{m,n})} \right| ,$$

where $\sum_1$ is the sum over the terms $z_{m,n}$ lying within the disc $|z - z_{M,N}| \leqslant K_8 H(R)^{\frac{1}{2}}$ (at most $K_9 H(R)$ in number) and $\sum_2$ is the sum over the remaining terms. We have, from (4.4.14) and the above, that

$$\sum\nolimits_1 \left| \frac{f(z_{m,n})}{\sigma'_{M,N}(z_{m,n})} \right| \leqslant K_9 H(R) \exp\left\{ \frac{(\frac{1}{2}\pi-\epsilon)(\varkappa+\epsilon)\, H(R)}{\frac{1}{2}\pi - \epsilon - (\varkappa+\epsilon)\,\mu(R)} \right\} , \qquad (4.4.17)$$

and, since for the relevant terms

$$(\varkappa+\epsilon)\, H(R+r) - (\tfrac{1}{2}\pi - \epsilon)r^2 \leqslant -\tfrac{1}{2}(\tfrac{1}{2}\pi - c\varkappa) ,$$

we also have

$$\sum\nolimits_2 \left| \frac{f(z_{m,n})}{\sigma'_{M,N}(z_{m,n})} \right| \leqslant \sum_{\substack{(m,n)\neq \\ (M,N)}} \exp\left\{ -\tfrac{1}{2}(\tfrac{1}{2}\pi - c\varkappa)|z_{m,n} - z_{M,N}|^2 \right\} . \qquad (4.4.18)$$

Now

$$\exp\left\{-\tfrac{1}{2}(\tfrac{1}{2}\pi - c\varkappa)|z_{m,n} - z_{M,N}|^2\right\} \leqslant \frac{1}{\pi(\frac{1}{2}\delta)^2} \iint\limits_{|re^{i\theta} - (z_{m,n} - z_{M,N})| \leqslant \frac{1}{2}\delta} \exp\left\{-\tfrac{1}{2}(\tfrac{1}{2}\pi - c\varkappa)(r - \tfrac{1}{2}\delta)^2\right\} r\,dr\,d\theta ,$$

and, since the regions of integration do not overlap, it follows that the right-hand side of (4.4.18) does not exceed

$$\frac{1}{\pi(\frac{1}{2}\delta)^2} \int_{\frac{1}{2}\delta}^{\infty} \int_0^{2\pi} \exp\left\{-\tfrac{1}{2}(\tfrac{1}{2}\pi - c\varkappa)(r - \tfrac{1}{2}\delta)^2\right\} r\,d\theta\,dr \leqslant \frac{8}{(\frac{1}{2}\pi - c\varkappa)\delta^2} . \qquad (4.4.19)$$

Therefore, from (4.4.11),

$$\left|\frac{\phi_{M,N}(z)}{\sigma_{M,N}(z)}\right| \leq K_{10}(\eta)\,H(R)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(X+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(X+\epsilon)\mu(R)}\right\}, \tag{4.4.20}$$

for z outside (A_η). Hence, for z on C_p we have

$$\left|\bar{\Phi}_{M,N}(z)\right| \leq K_{10}H(R)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(X+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(X+\epsilon)\mu(R)}\right\} + K_3(M,N,\eta)\exp\left\{-\tfrac{1}{2}(\tfrac{1}{2}\pi-h)r_p^2\right\}, \tag{4.4.21}$$

and on the boundary of $S(\alpha,l)$

$$\left|\bar{\Phi}_{M,N}(z)\right| \leq K_{10}H(R)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(X+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(X+\epsilon)\mu(R)}\right\} + K_4(\epsilon_0,\eta)\exp\left\{\frac{\frac{1}{2}\pi c X R^2}{\frac{1}{2}\pi-cX}\right\}. \tag{4.4.22}$$

The right-hand side of (4.4.21) tends to

$$K_{10}H(R)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(X+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(X+\epsilon)\mu(R)}\right\},$$

as $p\to\infty$, for every fixed pair (M,N) such that $z_{M,N}$ is in $S(\beta,R_0)$; since $\bar{\Phi}_{M,N}(z)$ is regular in the region enclosed by these curves, it follows that

$$\left|\bar{\Phi}_{M,N}(z)\right| \leq K_{10}H(R)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(X+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(X+\epsilon)\mu(R)}\right\} + K_4\exp\left\{\frac{\frac{1}{2}\pi c X R^2}{\frac{1}{2}\pi-cX}\right\}, \tag{4.4.23}$$

for all z in $S(\alpha,l)$ and in particular in the disc $|z-z_{M,N}|\leq 1+\delta+D$, provided that $R_0 \geq l+1+\delta+D$.

From Theorem 33 (ii) we see that

$$\left|\sigma_{M,N}(z)\right| \leq K_{11}(\delta,D) \tag{4.4.24}$$

in the disc $|z-z_{M,N}|\leq 1+\delta+D$. Combining this with (4.4.20) gives

$$\left|\phi_{M,N}(z)\right| \leq K_{12}H(R)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(X+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(X+\epsilon)\mu(R)}\right\} \tag{4.4.25}$$

on a contour lying inside $|z - z_{M,N}| \leq 1+\delta+D$, such that the shortest distance from it to any point of the set $\{z_{m,n}\}$ is at least η and containing the disc $|z - z_{M,N}| \leq 1+D$ in its interior. This, together with (4.4.23) and (4.4.24), gives

$$|f(z)| \leq |\sigma_{M,N}(z)||\Phi_{M,N}(z)| + |\phi_{M,N}(z)| ,$$

$$\leq K_{13} H(R) \exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(\varkappa+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(\varkappa+\epsilon)\mu(R)}\right\} + K_{14} \exp\left\{\frac{\frac{1}{2}\pi c\varkappa R^2}{\frac{1}{2}\pi - c\varkappa}\right\}, \quad (4.4.26)$$

on the contour. Since $f(z)$ is regular in the domain bounded by this contour the same inequality holds there; in particular it holds in the disc $|z-z_{M,N}| \leq 1+D$. Now $S(\beta, R_0)$ is covered by the union of these discs and it follows that

$$|f(z)| \leq K_{13} H(|z|+1+D) \exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(\varkappa+\epsilon)H(|z|+1+D)}{\frac{1}{2}\pi-\epsilon-(\varkappa+\epsilon)\mu(|z|-1-D)}\right\} +$$

$$+ K_{14} \exp\left\{\frac{\frac{1}{2}\pi c\varkappa(|z|+1+D)^2}{\frac{1}{2}\pi - c\varkappa}\right\}, \quad (4.4.27)$$

for all z in $S(\beta, R_0)$. If $c\varkappa > 0$ then $H(r) \sim cr^2$ and, since $(\frac{1}{2}\pi-\epsilon)/\{\frac{1}{2}\pi-\epsilon-c(\varkappa+\epsilon)\}$ exceeds $\frac{1}{2}\pi/(\frac{1}{2}\pi-c\varkappa)$, we have from the above that

$$\frac{\log M(r,\beta,f)}{H(r)} \leq \frac{\log\{2(K_{13}+K_{14})(1+H(r+1+D)\}}{H(r)} + \quad (4.4.28)$$

$$+ \frac{(\frac{1}{2}\pi-\epsilon)(\varkappa+\epsilon)H(r+1+D)}{\{\frac{1}{2}\pi-\epsilon-(\varkappa+\epsilon)\mu(r-1-D)\}H(r)} .$$

Now $$1 \leq \frac{H(r+1+D)}{H(r)} = \frac{\mu(r+1+D)}{\mu(r)} \cdot \frac{(r+1+D)^2}{r^2} ,$$

and, since $\mu(r+1+D) \leq \mu(r)$, it follows that

$$\lim_{r\to\infty} \frac{H(r+1+D)}{H(r)} = 1 .$$

We therefore have, if $H(r)$ is unbounded,

$$\limsup_{r\to\infty} \frac{\log M(r,\beta,f)}{H(r)} \leq \frac{(\frac{1}{2}\pi-\epsilon)(\chi+\epsilon)}{\frac{1}{2}\pi-\epsilon-(\chi+\epsilon)c} , \qquad (4.4.29)$$

and this is true for every sufficiently small positive ϵ . Therefore

$$\limsup_{r\to\infty} \frac{\log M(r,\beta,f)}{H(r)} \leq \frac{\frac{1}{2}\pi\chi}{\frac{1}{2}\pi-c\chi} = \chi\left\{1+\frac{c\chi}{\frac{1}{2}\pi-c\chi}\right\}, \qquad (4.4.30)$$

If $H(r)$ is bounded then (4.4.27) shows that for z in $S(\beta,R_0)$ we have

$$|f(z)| \leq K . \qquad (4.4.31)$$

These two inequalities establish the modified form of Theorem 31 under consideration. To obtain the general result as a consequence of this we shall need the following theorem, which is similar to Theorem 21.

THEOREM 34. Suppose that the set $\{z_n\}$ is such that the discs

$$|z-z_n| \leq \Delta_1 < \left(\frac{2}{3\sqrt{3}}\right)^{\frac{1}{2}} \qquad (4.4.32)$$

cover the complex plane, then $\{z_n\}$ has a subset which can be represented as a set $\{z_{m,n}\}$ satisfying (4.3.2) and (4.3.3).

The proof of Theorem 34 is based on a division of the plane into squares $S_{k,l}$ defined by

$$(k-1)A \leq \operatorname{Re} z \leq kA\,, \quad (l-1)A \leq \operatorname{Im} z \leq lA\,, \tag{4.4.33}$$

($k, l = 0, \pm 1, \pm$,) for some fixed integer A . The basic ideas of the proof of Theorem 21 carry over to this case, and it is unnecessary to give the detailed argument of this much simpler situation.

We now consider the effect of the transformation

$$\zeta = \lambda \left(\frac{2\tau}{\pi}\right)^{\frac{1}{2}} z^{\frac{1}{2}\rho}\,, \quad 0 < \lambda < 1\,, \tag{4.4.34}$$

on the situation of Theorem 31 when $\alpha < 2\pi/\rho$. The disc

$$|z - z_n| \leq \Delta\, |z_n|^{1-\frac{1}{2}\rho} \tag{4.4.35}$$

maps onto a region lying in the interior of

$$|\zeta - \zeta_n| \leq \lambda \left(\frac{2\tau}{\pi}\right)^{\frac{1}{2}} \tfrac{1}{2}\rho\Delta + \frac{K}{|\zeta_n|}\,, \tag{4.4.36}$$

where $\zeta_n = \lambda \left(\frac{2\tau}{\pi}\right)^{\frac{1}{2}} z_n^{\frac{1}{2}\rho}$. Now $\tau = 4\pi/\rho^2\Delta^2 3\sqrt{3}$ and hence this is the disc

$$|\zeta - \zeta_n| \leq \lambda \left(\frac{2}{3\sqrt{3}}\right)^{\frac{1}{2}} + \frac{K}{|\zeta_n|}\,, \tag{4.4.37}$$

and, since $S(\alpha, l)$ is covered by the discs of (4.4.35), it follows that the discs

$$(\zeta - \zeta_n) \leq \tfrac{1}{2}(1+\lambda) \left(\frac{2}{3\sqrt{3}}\right)^{\frac{1}{2}} \tag{4.4.38}$$

cover all parts of $S(\frac{1}{2}\rho\alpha, 0)$ having sufficiently large modulus. Also, $f(z)$ is transformed into a function $F(\zeta)$ satisfying

$$\liminf_{R \to \infty} \frac{\log M(R, \frac{1}{2}\rho\alpha, F)}{R^2} = \frac{\pi}{2\tau\lambda^2} \liminf_{r \to \infty} \frac{\log M(r, \alpha, f)}{r^\rho}, \tag{4.4.39}$$

$$\limsup_{R\to\infty} \frac{\log|F(Re^{\pm i\frac{1}{2}\rho\alpha})|}{R^2} = \frac{\pi}{2\tau\lambda^2} \limsup_{r\to\infty} \frac{\log|f(re^{\pm i\alpha})|}{r^\rho} \leq \frac{\pi\gamma}{2\tau\lambda^2} \quad (4.4.40)$$

and

$$\limsup \frac{\log|F(\zeta_n)|}{H^*(|\zeta_n|)} \leq \kappa < \infty, \quad (4.4.41)$$

where $H^*(R) = H\left(\left\{\left(\frac{\pi}{2\tau}\right)^{\frac{1}{2}} \frac{R}{\lambda}\right\}^{2/\rho}\right)$ is a non-decreasing function of R with $R^{-2}H^*(R)$ non-increasing and having limit $\pi c/2\tau\lambda^2$. We can choose λ sufficiently near to 1 to ensure that $\pi c\kappa/2\tau\lambda^2$ and the terms appearing in (4.4.39) and (4.4.40) are all less than $\frac{1}{2}\pi$ and therefore, using Theorem 34 and the modified form of Theorem 31 already established,

$$\limsup_{R\to\infty} \frac{\log M(R, \frac{1}{2}\rho\beta, F)}{H^*(R)} \leq \kappa\left\{1 + \frac{c\kappa}{\lambda^2\tau - c\kappa}\right\} + K \lim_{R\to\infty} \frac{1}{H^*(R)}. \quad (4.4.42)$$

Transforming back gives

$$\limsup_{r\to\infty} \frac{\log M(r, \beta, f)}{H(r)} \leq \kappa\left\{1 + \frac{c\kappa}{\lambda^2\tau - c\kappa}\right\} + K \lim_{r\to\infty} \frac{1}{H(r)}. \quad (4.4.43)$$

The conclusion of Theorem 31 follows for this case, since we can take λ arbitarily near to 1.

For the case where $\alpha \geq 2\pi/\rho$ we first note that the conditions of Theorem 31 imply those of Theorem 17, with $\nu(r) \equiv \rho$, and hence

$$\limsup_{r\to\infty} \frac{\log M(r, \alpha, f)}{r^\rho} \leq \sup\{c\kappa, \gamma\}. \quad (4.4.44)$$

If $0 < \gamma < c\kappa$ we have nothing further to prove and we may therefore assume that

$$\limsup_{r \to \infty} \frac{\log M(r, \alpha, f)}{r^{\rho}} \leq \gamma . \qquad (4.4.45)$$

We may also, without loss of generality, assume that $\alpha - \beta < \pi/\rho$. Now choose a real number Θ such that

$$4\pi/\rho > \Theta > 2(\alpha - \beta), \qquad (4.4.46)$$

and consider any sector of aperture Θ which is contained in $S(\alpha, l)$. The conditions, suitably modified to allow for the loss of symmetry, of the case for $\alpha < 2\pi/\rho$ proved above are satisfied by this sector and repeated application of that result will complete the proof of Theorem 31.

4.5. <u>Gap power series.</u> In this section we consider integral functions which have a power series representation

$$f(z) = \sum_{k=1}^{\infty} a_{n_k} z^{n_k} \qquad (4.5.1)$$

which satisfies the Fabry gap condition,

$$n_k / k \to \infty \quad \text{as} \quad k \to \infty . \qquad (4.5.2)$$

Several authors have obtained results which show that the behaviour of such functions is angularly uniform.

Turán [50], using a very general inequality on series, obtained

$$M(r, f)^{1+\epsilon} \leq \frac{16e\pi}{\beta - \alpha} M(2r)^{2\epsilon} M(r, \alpha, \beta, f), \qquad (4.5.3)$$

for $r \geq r_0(\epsilon)$, where $M(r, \alpha, \beta, f) = \sup_{\alpha \leq \theta \leq \beta} |f(re^{i\theta})|$. From this inequality one can easily deduce that if $f(z)$ is of finite type of order ρ , then

$$\lim \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r_n, f)}{r_n^\rho} = \lim \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r_n, \alpha, \beta, f)}{r_n^\rho}, \quad (4.5.4)$$

for any sequence $\{r_n\}$, $r_n \to \infty$. Also, if $f(z)$ is of finite order and is bounded in the sector $\alpha \leq \arg z \leq \beta$ then it is identically constant. One can also obtain the result of Pólya [41] that

$$\limsup_{r \to \infty} \frac{\log M(r, \alpha, \beta, f)}{\log M(r, f)} = 1 . \quad (4.5.5)$$

Kővari [19] improved this last result by showing that

$$\log M(r, \alpha, \beta, f) \geq (1-\epsilon) \log M(r, f) , \quad (4.5.6)$$

outside a set of r of zero logarithmic density; this result was completed by Fuchs [9] who showed that

$$\log m(r, f) \geq (1-\epsilon) \log M(r, f) , \quad (4.5.7)$$

outside a similar set of r , where $m(r, f) = \inf_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|$.
This also establishes the conjecture of Pólya that

$$\limsup_{r \to \infty} \frac{\log m(r, f)}{\log M(r, f)} = 1 . \quad (4.5.8)$$

The results of equations (4.5.4,5,6) do not depend on the aperture $\beta - \alpha$ of the sector. It might be asked how the behaviour of $f(z)$ on a set of points in some sector of arbitarily small opening determines the behaviour of the function throughout the plane. For example, it might be conjectured, for the kind of function under consideration, that boundedness at the members of a set of $P(\rho,\tau)$, that is a set satisfying the conditions of Theorem 11, lying in some sector, together with the assumption that the function is at most type $\varkappa < \tau$ of order ρ, would imply that the function is constant. Our next theorem is a result of this kind.

THEOREM 35. Suppose that $f(z)$ is an integral function at most of minimum type of order ρ which satisfies (4.5.1) and (4.5.2). Suppose, also, that

$$|f(z_n)| \leq K|z_n|^{\varkappa}, \tag{4.5.9}$$

where, for some finite Δ, the union of the discs

$$|z - z_n| \leq \Delta |z_n|^{1-\frac{1}{2}\rho}$$

covers a sector of positive aperture. Then $f(z)$ is a polynomial of degree at most $\varkappa$.

Proof of Theorem 35. We may suppose the sector covered by the discs to be $S(\alpha, l)$. It follows from (4.5.5) that

$$\log M(r_n, \alpha, f) \geq (1-\epsilon) \log M(r_n, f), \tag{4.5.10}$$

for some sequence $\{r_n\}$, $r_n \to \infty$. Also, from Theorem 31, we obtain that

$$\log M(r, \beta, f) \leq (\varkappa + \epsilon) \log r, \tag{4.5.11}$$

for any $\beta < \alpha$ with $r \geq r_0(\epsilon, \beta)$, and hence

$$\frac{\log M(r_n, f)}{\log r_n} \leq \frac{\varkappa + \epsilon}{1 - \epsilon} . \qquad (4.5.12)$$

This implies that

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{\log r} \leq \varkappa , \qquad (4.5.13)$$

which is sufficient for the result of the theorem.

We could combine Theorem 35 with the following result.

THEOREM 36. Suppose that $f(z)$ is an integral function at most of type $\gamma < \tau = 4\pi/\Delta^2(3\sqrt{3})\rho^2$ of order ρ satisfying (4.5.1,2). Suppose, also, that

$$\limsup \frac{\log |f(z_n)|}{|z_n|^\rho} = 0 , \qquad (4.5.14)$$

where the union of the discs

$$|z - z_n| \leq \Delta |z_n|^{1 - \frac{1}{2}\rho}$$

covers the sector $S(\alpha, l)$, and that $\sin \frac{1}{2}\rho\alpha > \gamma/\tau$. Then $f(z)$ is at most of minimum type of order ρ.

It should be noted that this result is stronger than that obtained by a direct appeal to Theorem 31, as this would require $\sin^2 \frac{1}{2}\rho\alpha > \gamma/\tau$, instead of the weaker $\sin \frac{1}{2}\rho\alpha > \gamma/\tau$.

Proof of Theorem 36. The proof is based on the result of the following.

LEMMA 4.5.1 . Suppose that $f(z)$ satisfies the conditions (4.2.1,2,3) of Theorem 31 with $c = 0$. If $\alpha < \pi/\rho$

suppose, in addition, that $\sin\frac{1}{2}\rho\alpha > \gamma/\tau$. Then, if β $(0<\beta<\alpha)$ satisfies $\sin\frac{1}{2}\rho(\alpha-\beta) > \gamma/\tau$,

$$\limsup_{r\to\infty} \frac{\log M(r,\beta,f)}{r^\rho} \leq \sup\left\{0, \gamma\left(1-\frac{2\eta\tau}{\tau-\gamma}\right)\right\}, \quad (4.5.15)$$

where $\eta = \sin\frac{1}{2}\rho(\alpha-\beta) - \gamma/\tau$.

The proof follows closely that of Theorem 31, the difference occuring in the estimate for $|f(z)/\sigma_{M,N}(z)|$ on the boundary of $S(\alpha, l)$ in (4.4.8-10).

It follows from the lemma that the type of $f(z)$ in $S(\beta, l)$ is at most $\theta\gamma$, where $\theta = 1-2\eta$ is a constant less than 1; combining this with (4.5.4) shows that $f(z)$ is at most of type $\theta\gamma$ of order ρ. Repeated application of this argument shows that $f(z)$ is at most of minimum type of order ρ.

CHAPTER V

FUNCTIONS REGULAR IN THE UNIT DISC AND SUBSETS OF THE UNIT DISC

5.1. Introduction. The problems dealt with up to this point have the common feature that there can be at most one singularity, taken as the point at infinity, of the function on the boundary of the regions under consideration. The problem considered in this chapter, that of functions regular in the unit disc, differs considerably from these previous cases; not only may we have two or more singularities on the boundary but also the singularities may be dense on arcs of the boundary.

For the classes H_p $(1 \leq p < \infty)$, consisting of functions for which

$$\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \qquad (5.1.1)$$

is bounded for $0 \leq r < 1$, and H_∞, the set of functions bounded in the unit disc, there is a fairly extensive interpolation theory in existence. For example, a necessary and sufficient condition that a function of H_p $(1 \leq p \leq \infty)$, or more generally a function of bounded characteristic, which has zeros at the points z_n of the unit disc be identically zero is that

$$\prod |z_n| = 0, \text{ or equivalently } \sum (1 - |z_n|) = \infty. \qquad (5.1.2)$$

Buck raised the following problem. "What conditions, if any, on a set $\{z_n\}$ of points lying in $|z|<1$ are sufficient to ensure that the interpolation problem

$$f(z_n) = \omega_n \qquad (5.1.3)$$

is soluble, for an arbitary bounded sequence $\{\omega_n\}$, by a function $f(z)$ regular and bounded for $|z|<1$?"

The classicial theory of Nevanlinna [29], and others, which deals with the possibility of such a solution for specific sequences $\{z_n\}$ and $\{\omega_n\}$ is very implicit and, in a concrete situation, gives little help in deciding whether such interpolation is possible. The first results were obtained, independently, by Hayman [11], Carleson [4] and Newman [30]. Hayman was able to show that such a set must necessarily satisfy

$$\prod_{\substack{m=1\\ m\neq n}}^{\infty} \left|\frac{z_m - z_n}{1 - z_m \bar{z}_n}\right| \geqslant \delta > 0 \qquad (5.1.4)$$

for all n, and that the slightly weaker condition

$$\prod_{\substack{m=1\\ m\neq n}}^{\infty} \left\{1 - \left(1 - \left|\frac{z_m - z_n}{1 - z_m \bar{z}_n}\right|\right)^{\lambda}\right\} \geqslant \delta > 0, \quad \lambda<1, \qquad (5.1.5)$$

is sufficient to ensure the existence of a bounded regular function satisfying (5.1.3). Carleson, using functional analytic techniques, showed that the condition (5.1.4) is both necessary and sufficient but his method, unlike that of Hayman when we have (5.1.5), does not give any means of constructing a specific interpolating function. A new proof

Carleson's result and a generalisation to different spaces of functions, for example H_p ($p \geq 1$), has been provided by Shapiro and Shields [44]. These were extended to the case of H_p ($p < 1$) by Kabaĭla [17].

For the more general class of functions of finite order, that is functions $f(z)$ for which

$$\rho = \limsup_{r \to 1-0} \frac{\log \log M(r,f)}{\log \frac{1}{1-r}} \tag{5.1.6}$$

is finite, the theory is far from complete. Lammel [20] and Walsh [53] studied approximations to $f(z)$ by sequences of rational functions. For the case of interpolating given values ω_n at a sequence $\{z_n\}$, however, they required the z_n to lie in a closed subset of $|z|<1$. V.L. and M.K. Gončarov [10] obtained some results on the representation of $f(z)$ as series of the form

$$c_0 + \sum_{n=1}^{\infty} c_n \left\{ \prod_{k=1}^{n} \frac{z - z_k}{1 - z\bar{z}_k} \right\} \tag{5.1.7}$$

for the case where the z_k are real and tend to 1 with $\prod |z_k| = 0$. Obviously the coefficients c_n can be completely determined in terms of the sequence of values $\{f(z_k)\}$ but the expressions are very complicated. Slobodeckiĭ [45] stated

THEOREM 37. Denote by $L(r)$ the number of z_n in $|z| \leq r$. If

$$(1-r)^{\rho} L(r) \geq A > 0, \tag{5.1.8}$$

where $\rho > 1$, and

$$(1-r)^q \log M(r,f) = O(1) \qquad (5.1.9)$$

for some q less than $\frac{1}{2}(\rho-1)$, then the expression (5.1.7) is convergent to $f(z)$ for $|z|<1$, uniformly on closed subsets.

He also gave some results, similar to those of [10], when $\{z_n\}$ consists of a finite number of subsequences each tending radially to a point on $|z|=1$, [46].

In his thesis [33], Noble raised the question of whether a representation theory in terms of cardinal series is possible in the unit disc. The difficulty, as he pointed out, lies in finding a suitable analogue for the functions $\eta(z)$ of 1.2 . The Blaschke product is not suitable since it is bounded and direct generalisations, such as

$$\prod E\left\{1 - \frac{|z_n|(z_n - z)}{z_n(1-z\bar{z}_n)}, p\right\},$$

seem to be difficult to handle. We shall show, under conditions similar to those of 2.3 on $\{\zeta_{p,q}\}$, that such a representation theory is possible. Furthermore, we obtain theorems concerning the growth properties of functions regular in the unit disc which correspond to those of Chapters II and IV; these results are given in the following section. The proofs again consist mainly of the construction of a suitable function $\eta(z)$ and a demonstration of its properties, the arguments being completed in a manner similar to that used previously.

5.2. Growth theorems. Our first theorem covers interpolation sets for functions having very general majorising functions and corresponds to Theorem 15 of Chapter II.

THEOREM 38. Let $\varphi(r)$ be a twice differentiable function of r, for $0 \leq r < 1$, which satisfies

$$(1-r)^2 \frac{d^2\varphi(r)}{dr^2} = (1-r)^{-\nu(r)} \uparrow \to \infty \tag{5.2.1}$$

with $\nu(r) \downarrow \to \rho$, as $r \uparrow \to 1-0$. Further, let $\{z_n\}$ be a set of points lying in $|z|<1$ such that the union of the discs

$$|z - z_n| \leq \frac{\Delta\,(1-|z_n|)^{1+\frac{1}{2}\nu(|z_n|)}}{1 + \Delta\,(1-|z_n|)^{\frac{1}{2}\nu(|z_n|)}} \tag{5.2.2}$$

covers $|z|<1$. Then if $f(z)$ is a function regular for $|z|<1$ which satisfies

$$\liminf_{r\to 1-0} \frac{\log M(r,f)}{\varphi(r)} < \frac{4\pi}{\Delta^2 3\sqrt{3}} \tag{5.2.3}$$

and

$$\limsup \frac{\log|f(z_n)|}{\varphi(|z_n|)} \leq \kappa < \frac{4\pi}{\Delta^2 3\sqrt{3}}, \tag{5.2.4}$$

we must also have

$$\limsup_{r\to 1-0} \frac{\log M(r,f)}{\varphi(r)} \leq \kappa. \tag{5.2.5}$$

If, also, $|f(z_n)| \leq A$ then $|f(z)| \leq A$ for $|z|<1$.

We note that if $\nu(r)$ is differentiable then

$$0 \le -\nu'(r)(1-r)\log\frac{1}{1-r} \le \nu(r),$$

and if $\nu'(r)(1-r)\log\frac{1}{1-r} = o(\nu(r))$ then we can take

$$\varphi(r) \sim \frac{r^{\nu(r)}}{\nu(r)(1+\nu(r))} .$$

When $\nu(r) \equiv \rho > 0$ (5.2.3) and (5.2.4) become

$$\liminf_{r\to 1-0} (1-r)^{\rho} \log M(r,f) < \frac{4\pi}{\rho(\rho+1)\Delta^2 3\sqrt{3}} \qquad (5.2.6)$$

and $$\limsup (1-|z_n|)^{\rho}\log|f(z_n)| \le \lambda < \frac{4\pi}{\rho(\rho+1)\Delta^2 3\sqrt{3}}, \qquad (5.2.7)$$

respectively, and our conclusion is that

$$\limsup (1-r)^{\rho}\log M(r,f) \le \lambda, \qquad (5.2.8)$$

or when $|f(z_n)| \le A$ that $|f(z)| \le A$. We can, however, say more about the behaviour of $\log M(r,f)$, for this case, when the behaviour of $\log|f(z_n)|$ is more exactly specified; corresponding to Theorem 32 of Chapter IV, we have

THEOREM 39. Suppose that $f(z)$ is a function regular for $|z|<1$ which satisfies

$$\liminf_{r\to 1-0}(1-r)^{\rho}\log M(r,f) < \frac{4\pi}{\rho(\rho+1)\Delta^2 3\sqrt{3}} \qquad (5.2.9)$$

and $$\limsup \frac{\log|f(z_n)|}{H(|z_n|)} \le K < \infty, \qquad (5.2.10)$$

where the union of the discs

$$|z-z_n| \le \frac{\Delta(1-|z_n|)^{1+\frac{1}{2}\rho}}{1+\Delta(1-|z_n|)^{\frac{1}{2}\rho}} \qquad (5.2.11)$$

covers $|z|<1$, and the non-decreasing function $H(r)$ satisfies $(1-r)^\rho H(r) \downarrow \to c$ with $c\kappa < 4\pi/\rho(\rho+1)\Delta^2 3\sqrt{3}$. Then

$$\limsup_{r\to 1-0} \frac{\log M(r,f)}{H(r)} \le K. \qquad (5.2.12)$$

The extreme cases, $H(r)$ bounded and $c>0$, follow from the theorem above and, once that result has been established it will only be necessary to prove the remaining cases.

For subsets of the unit disc we have the following result.

THEOREM 40. Let Ω be a simply connected domain contained in $|z|<1$ with rectifiable boundary $\partial(\Omega)$. We suppose that $\partial(\Omega)\cap\{|z|=1\}$ is non-null. Suppose that $f(z)$ is regular in Ω and on $\partial(\Omega)-\{|z|=1\}$ and satisfies

(i) $$\liminf_{r\to 1-0} \frac{\log M(r,\Omega,f)}{\varphi(r)} < \frac{4\pi}{\Delta^2 3\sqrt{3}}, \qquad (5.2.13)$$

where $$M(r,\Omega,f) = \sup_{\theta \ni re^{i\theta}\in\Omega} |f(re^{i\theta})|;$$

(ii) $$\limsup_{|z|\to 1-0,\, z\in\partial(\Omega)} \frac{\log|f(z)|}{\varphi(|z|)} \le \gamma < \frac{4\pi}{\Delta^2 3\sqrt{3}}; \qquad (5.2.14)$$

and (iii) $$\limsup_{z_n\in\Omega} \frac{\log|f(z_n)|}{\varphi(|z_n|)} \le \kappa < \frac{4\pi}{\Delta^2 3\sqrt{3}}, \qquad (5.2.15)$$

where the union for $z_n\in\Omega$ of the discs of (5.2.2) covers Ω . Then

$$\limsup_{r\to 1-0} \frac{\log M(r,\Omega,f)}{\varphi(r)} \le \sup\{\kappa,\gamma\}. \qquad (5.2.16)$$

Furthermore, if conditions (5.2.14) and (5.2.15) are ammended to

$$|f(z)| \leq B \quad \text{for} \quad z \in \partial(\Omega) - \{|z| = 1\} \tag{5.2.17}$$

and
$$|f(z_n)| \leq B \quad \text{for} \quad z_n \in \Omega \,, \tag{5.2.18}$$

then the conclusion becomes

$$|f(z)| \leq B \quad \text{for all } z \text{ in } \Omega \,.$$

Theorem 38 can be regarded as the special case of Theorem 40 for which $\partial(\Omega) - \{|z| = 1\}$ is null, that is $\Omega = \{|z| \leq 1\}$. The results of Theorems 38, 39 and 40 will remain valid if in the growth conditions $\{z_n\}$ is replaced by a set $\{z'_n\}$ lying in $|z| < 1$ which satisfies

$$|z'_n - z_n| \leq K(1 - |z_n|)^{1 + \frac{1}{2}\nu(|z_n|)} \tag{5.2.19}$$

and is such that the discs

$$|z - z'_n| \leq \delta(1 - |z'_n|)^{1 + \frac{1}{2}\nu(|z'_n|)} \tag{5.2.20}$$

are disjoint. As with our earlier theorems, the separation condition is unnecessarily strong but seems to be the most natural.

5.3. Special canonical products. This section is similar in spirit to 2.2 but requires the solution of different technical problems. Section 5.4 will correspond in the same manner with 2.3 .

For any positive constant Λ , with the $\nu(r)$ of

(5.2.1), choose r_0 to be a positive number less than 1 such that

$$\Lambda(1-r_0)^{\frac{1}{2}\nu(r_0)} < 1 \tag{5.3.1}$$

and define $h(r)$ by

$$h(r) = \Lambda(1-r)^{\frac{1}{2}\nu(r)}. \tag{5.3.2}$$

It follows from (5.2.1) that $h(r)$ is a non-increasing function which tends to zero as r tends to 1 from below. It is easy to show that

$$r + h(r)(1-r) \tag{5.3.3}$$

is an increasing function of r for $r_0 \leq r < 1$, and hence we can define a sequence $\{r_p\}$, $r_p \uparrow \rightarrow 1$, by

$$r_{p+1} - r_p = (1-r_p)h(r_p) \tag{5.3.4}$$

and a function $\mu(z)$ by

$$\mu(z) = \prod_{p=1}^{\infty}\left\{1 - \left(\frac{z}{r_p}\right)^{1+[1/(1-r_p)h(r_p)]}\right\}. \tag{5.3.5}$$

Let $n^*(r)$ denote the number of r_p not greater than r and $n(r)$ the number of zeros of $\mu(z)$ lying in $|z| \leq r$. We write, with $|z| = r$,

$$\mu(z) = \prod_{p=1}^{n^*(r)}\left(\frac{z}{r_p}\right)^{1+[1/(1-r_p)h(r_p)]} (-1)^{n^*(r)}\prod_{p=1}^{n^*(r)}\left\{1 - \left(\frac{r_p}{z}\right)^{1+[1/(1-r_p)h(r_p)]}\right\} \times$$

$$\times \prod_{p=n^*(r)+1}^{\infty}\left\{1 - \left(\frac{z}{r_p}\right)^{1+[1/(1-r_p)h(r_p)]}\right\}. \tag{5.3.6}$$

Since $\nu(r)$ is non-increasing, it follows that if

$r' \le r'' < 1$ then

$$\frac{h(r')}{h(r'')} = \frac{(1-r')^{\frac{1}{2}\nu(r')}}{(1-r'')^{\frac{1}{2}\nu(r'')}} \le \left(\frac{1-r'}{1-r''}\right)^{\frac{1}{2}\nu(r')} = \left(1 - \frac{r''-r'}{1-r'}\right)^{-\frac{1}{2}\nu(r')} . \quad (5.3.7)$$

Also

$$\frac{1}{h(r_p)(1-r_p)} \le \int_{r_p}^{r_{p+1}} \frac{dt}{h(t)^2(1-t)^2} \le \frac{h(r_p)(1-r_p)}{h(r_{p+1})^2(1-r_{p+1})^2} = \frac{1}{h(r_p)(1-r_p)}\left\{\frac{h(r_p)}{h(r_{p+1})(1-h(r_p))}\right\}^2 \quad (5.3.8)$$

It follows from (5.3.7) that

$$\frac{h(r_p)}{h(r_{p+1})} \le \left(1-h(r_p)\right)^{-\frac{1}{2}\nu(r_p)} = 1 + O\left(\nu(r_p)\,h(r_p)\right), \quad (5.3.9)$$

and therefore we have

$$\frac{1}{h(r_p)(1-r_p)} = \left\{1 + O\left(h(r_p)\right)\right\}\int_{r_p}^{r_{p+1}} \frac{dt}{h(t)^2(1-t)^2} . \quad (5.3.10)$$

Since

$$\int_{r_1}^{r} \frac{dt}{h(t)^2(1-t)^2} \ge \frac{1}{\Omega^2}\int_{r_1}^{r} \frac{dt}{(1-t)^{2+\nu(r)}} ,$$

$$= \frac{1}{(1+\nu(r))\,h(r)^2(1-r)} - K(r_1) \quad (5.3.11)$$

which shows that

$$\frac{1+O\left(h(r_{n^*(r)})\right)}{h(r_{n^*(r)})(1-r_{n^*(r)})} = \int_{r_{n^*}}^{r_{n^*+1}} \frac{dt}{h(t)^2(1-t)^2} = o\left\{\int_{r_1}^{r} \frac{dt}{h(t)^2(1-t)^2}\right\}, \quad (5.3.12)$$

we have

$$n(r) = \{1+o(1)\}\int_0^r \frac{dt}{h(t)^2(1-t)^2} . \quad (5.3.13)$$

Therefore

$$\log\left|\prod_{p=1}^{n^*(r)}\left(\frac{z}{r_p}\right)^{1+[1/(1-r_p)h(r_p)]}\right| = \int_0^r \frac{n(t)}{t}\,dt , \quad (5.3.14)$$

$$= \{1+o(1)\}\int_0^r\int_0^t \frac{du}{h(u)^2(1-u)^2}\,dt\,, \qquad (5.3.15)$$

$$\geqslant \frac{1+o(1)}{\nu(r)(1+\nu(r))h(r)^2}\,. \qquad (5.3.16)$$

If $(1-r)\geqslant\{1+h(r_p)\}(1-r_p)$, which is so if $p\geqslant n^*+2$, then

$$\frac{(1-r)-(1-r_p)}{h(r_p)(1-r_p)} = \frac{r_p-r}{h(r_p)(1-r_p)} \geqslant 1\,. \qquad (5.3.17)$$

This implies that

$$\frac{-\log r/r_p}{h(r_p)(1-r_p)} > 1\,, \qquad (5.3.18)$$

since

$$\frac{-\log r/r_p}{h(r_p)(1-r_p)} = \frac{-1}{h(r_p)(1-r_p)}\log\left\{1-\left(1-\frac{r}{r_p}\right)\right\} = \frac{1}{h(r_p)(1-r_p)}\sum_{q=1}^{\infty}\frac{1}{q}\left(1-\frac{r}{r_p}\right)^q,$$

$$\geqslant \frac{1}{r_p}\,\frac{r_p-r}{h(r_p)(1-r_p)} \geqslant \frac{1}{r_p} > 1\,.$$

Therefore

$$\left(\frac{r}{r_p}\right)^{1+[1/(1-r_p)h(r_p)]} \leqslant \left(\frac{r}{r_p}\right)^{1/(1-r_p)h(r_p)} \leqslant \frac{1}{e}\,. \qquad (5.3.19)$$

Now for $r_p \geqslant r$

$$\left(\frac{r}{r_p}\right)^{1+[1/(1-r_p)h(r_p)]} \leqslant \left(\frac{r}{r_p}\right)^{1/(1-r_p)h(r_p)} = \exp\left\{\frac{-1}{h(r_p)(1-r_p)}\left(\sum_{q=1}^{\infty}\frac{1}{q}\left((1-r)^q-(1-r_p)^q\right)\right)\right\} \qquad (5.3.20)$$

$$\leqslant \exp\left\{\frac{(1-r)-(1-r_p)}{h(r_p)(1-r_p)}\right\},$$

$$\leqslant \int_{r_{p+1}}^{r_p}\frac{\exp\left\{-\frac{1}{h(r)}\left(\frac{1-r}{1-t}-1\right)\right\}}{h(t)(1-t)}\,. \qquad (5.3.21)$$

Therefore

$$\sum_{p=n^*+2}^{\infty}\left(\frac{r}{r_p}\right)^{1+[1/h(r_p)(1-r_p)]} \le \int_{r_{n^*+1}}^{1}\frac{\exp\left\{-\frac{1}{h(t)}\left(\frac{1-r}{1-t}-1\right)\right\}}{h(t)(1-t)}\,dt, \quad (5.3.22)$$

$$\le \int_{r}^{1}\frac{\exp\left\{-\left(\frac{1-r}{1-t}-1\right)\right\}}{(1-t)^{1+\frac{1}{2}\nu(r)}}\,dt,$$

$$\le \int_{0}^{\infty}\frac{\exp\{-y\}}{(1-r)^{\frac{1}{2}\nu(r)}(1+y)^{1-\frac{1}{2}\nu(r)}}\,dy,$$

$$\le \frac{K}{(1-r)^{\frac{1}{2}\nu(r)}}\int_{0}^{\infty}(1+y)^{K}\exp\{-y\}\,dy,$$

$$\le \frac{K}{h(r)}, \quad (5.3.23)$$

and therefore, in view of (5.3.19),

$$\left|\log\left|\prod_{n^*+2}^{\infty}\left\{1-\left(\frac{z}{r_p}\right)^{1+[1/h(r_p)(1-r_p)]}\right\}\right|\right| \le \frac{K}{h(r)}. \quad (5.3.24)$$

Now if

$$z_{p,q} = r_p\exp\left\{\frac{2\pi i q}{1+[1/h(r_p)(1-r_p)]}\right\} \quad (p=1,2,\ldots;\ q=0,1,\ldots,[1/h(r_p)(1-r_p)])$$

and z does not lie in any of the discs

$$|z-z_{p,q}| \le \delta h(r_p)(1-r_p), \quad (5.3.25)$$

where δ is a fixed positive number, then for $p \le n^*(r)$

$$\left|1-\left(\frac{r_p}{z}\right)^{1+[1/h(r_p)(1-r_p)]}\right| \ge \inf_{0\le\psi\le2\pi}\left|1-\left(\frac{r_p}{r_p+\delta h(r_p)(1-r_p)e^{i\psi}}\right)^{1+[1/h(r_p)(1-r_p)]}\right| \quad (5.3.26)$$

$$\sim \inf_{0\le\psi\le2\pi}\left|1-\exp\{-\delta e^{i\psi}\}\right|,$$

$$\ge K_1 > 0, \quad (5.3.27)$$

and therefore

$$\left| 1 - \left(\frac{r_p}{z}\right)^{1+[1/h(r_p)(1-r_p)]} \right| \geq K_2 > 0 . \qquad (5.3.28)$$

We can make a similar estimate for

$$\left| 1 - \left(\frac{z}{r_{n^*+1}}\right)^{1+[1/h(r_{n^*+1})(1-r_{n^*+1})]} \right| .$$

All these terms have modulus at most 2 and there are only n^*+1 of them. Since

$$n^*(r) \sim \int_0^r \frac{dt}{h(t)(1-t)} = O\left\{\int_0^r \frac{dt}{h(t)^2(1-t)}\right\}, \qquad (5.3.29)$$

it follows that

$$\log|\mu(z)| = \{1+o(1)\}\int_0^r \int_0^t \frac{du}{h(u)^2(1-u)^2}\, dt \qquad (5.3.30)$$

outside the union of the discs of (5.3.25).

We note that we have also shown that

$$\log|\tilde{\omega}_N(z)| \leq \{1+o(1)\}\int_0^r \int_0^t \frac{du}{h(u)^2(1-u)^2}\, dt, \qquad (5.3.31)$$

uniformly in N, where $\tilde{\omega}_N(z)$ is the partial product up to $p=N$ of the equation (5.3.5) defining $\mu(z)$.

It follows from (5.3.30) that $\mu(z)$ is of order ρ, defined in terms of the maximum modulus or the Nevanlinna characteristic, and hence we can use a more general form of canonical product, due to Tsuji [49], having the same zeros as $\mu(z)$. We define

$$\mathcal{E}(z,\zeta,p) = \frac{\bar{\zeta}(\zeta - z)}{1 - z\bar{\zeta}} \exp\left\{\sum_{s=1}^{p} \frac{1}{s}\left(\frac{1-|\zeta|^2}{1-z\bar{\zeta}}\right)^s\right\} = E\left(\frac{1-|\zeta|^2}{1-z\bar{\zeta}}, p\right), \qquad (5.3.32)$$

and then define $\eta_0(z)$ by

$$\eta_0(z) = \prod_{(p,q)} \mathcal{E}(z, z_{p,q}, [\rho+1]). \qquad (5.3.33)$$

That this does represent a function regular in $|z|<1$ which is of order at most $\sup\{\rho, 1\}$ follows from a result of Linden ([23], Theorem III) but we shall not need this; it is sufficient for our purposes that the sequence of partial products

$$\prod_{p=1}^{n} \left\{ \prod_{q=0}^{[1/h(r_p)(1-r_p)]} \mathcal{E}(z, z_{p,q}, [\rho+1]) \right\}$$

converges uniformly on compact subsets of $|z|<1$. We shall show that $\mu(z)$ can be replaced by $\eta_0(z)$ in (5.3.30). To this end we first prove the following lemma.

LEMMA 3.5.1 . Let ω_p be a primitive p-th root of unity. Then for $|z|<1$ and positive integer s

$$\left| \sum_{q=1}^{p} \frac{1}{(1-\omega_p^q z)^s} - p \right| \leq \frac{K(s)\, p^s |z|^p}{(1-|z|^p)^s}. \qquad (5.3.34)$$

Proof. Write

$$g_{s,p}(z) = \sum_{q=1}^{p} \frac{1}{(1-\omega_p^q z)^s}. \qquad (5.3.35)$$

For $s \geq 2$ we have

$$g_{s,p}(z) = \sum_{q=1}^{p} \frac{1}{(1-\omega_p^q z)^{s-1}} + z \sum_{q=1}^{p} \frac{\omega_p^q}{(1-\omega_p^q z)^s},$$

$$= g_{s-1,p}(z) + z \sum_{q=1}^{p} \frac{1}{s-1} \frac{d}{dz}\left\{ \frac{1}{(1-\omega_p^q z)^{s-1}} \right\},$$

$$= g_{s-1,p}(z) + \frac{z}{s-1} g'_{s-1,p}(z) . \qquad (5.3.36)$$

Also,

$$g_{1,p}(z) = \frac{p}{1-z^p} = p + \frac{pz^p}{1-z^p} . \qquad (5.3.37)$$

The result can now be established inductively on the hypothesis

$$g_{s,p}(z) = p + \sum_{u=1}^{s} \sum_{v=1}^{s} \sum_{w=1}^{s} a_{u,v,w}(s) \frac{p^u z^{vp}}{(1-z^p)^w} . \qquad (5.3.38)$$

We have

$$\eta_0(z) = \prod_{p=1}^{\infty} \prod_{q=0}^{[1/h(r_p)(1-r_p)]} \left\{ \frac{\bar{z}_{p,q}(z_{p,q}-z)}{1-z\bar{z}_{p,q}} \exp\left\{ \sum_{s=1}^{[\rho+1]} \frac{1}{s}\left(\frac{1-|z_{p,q}|^2}{1-z\bar{z}_{p,q}} \right)^s \right\} \right\},$$

$$= \prod_{p=1}^{\infty} \frac{r_p^{2+2[1/h(r_p)(1-r_p)]} \left(1-\left(\frac{z}{r_p}\right)^{1+[1/h(r_p)(1-r_p)]}\right)}{1-(zr_p)^{1+[1/h(r_p)(1-r_p)]}} \exp\left\{ \sum_{s=1}^{[\rho+1]} \sum_{q=0}^{[1/h(r_p)(1-r_p)]} \frac{1}{s}\left(\frac{1-r_p^2}{1-z\bar{z}_{p,q}}\right)^s \right\},$$

$$= \mu(z) \prod_{p=1}^{\infty} \frac{(1-(1-r_p^2))^{1+[1/h(r_p)(1-r_p)]}}{1-(zr_p)^{1+[1/h(r_p)(1-r_p)]}} \exp\left\{ (1+[1/h(r_p)(1-r_p)]) \sum_{s=1}^{[\rho+1]} \frac{(1-r_p^2)^s}{s} + \right.$$

$$\left. + O\left(\frac{(rr_p)^{1+[1/h(r_p)(1-r_p)]}}{h(r_p)^{[\rho+1]}} \right) \right\}, \qquad (5.3.39)$$

by Lemma 5.3.1 . Hence

$$\frac{\eta_0(z)}{\mu(z)} = \prod_{p=1}^{\infty} \exp\left\{ \sum_{s=[\rho+2]}^{\infty} \frac{(1+[1/h(r_p)(1-r_p)])(1-r_p^2)^s}{s} + O\left(\frac{(rr_p)^{1+[1/h(r_p)(1-r_p)]}}{h(r_p)^{[\rho+1]}} \right) \right\} \qquad (5.3.40)$$

Now

$$(1+[1/h(r_p)(1-r_p)]) \sum_{s=[\rho+2]}^{\infty} \frac{(1-r_p^2)^s}{s} \leq \frac{K(1-r_p)^{[\rho+1]}}{h(r_p)},$$

and, since $(1-t)^{[\rho+1]-\frac{1}{2}\nu(t)}$ is a decreasing function of t , for t sufficiently near to 1 ,

$$\frac{(1-r_p)^{[\rho+1]}}{h(r_p)} \leq \int_{r_{p-1}}^{r_p} \frac{(1-t)^{[\rho+1]}}{h(t)^2(1-t)} dt = \int_{r_{p-1}}^{r_p} (1-t)^{[\rho]-\nu(t)} dt. \quad (5.3.41)$$

Also, $\nu(t)-[\rho]$ is less than 1 for t sufficiently near to 1 . Hence $\int_0^1 (1-t)^{[\rho]-\nu(t)}$ converges and therefore

$$\sum_{p=1}^{\infty} \frac{(1-r_p)^{[\rho+1]}}{h(r_p)} < \infty . \quad (5.3.42)$$

We are left to consider

$$\sum_{p=1}^{\infty} \frac{(r r_p)^{1/h(r_p)(1-r_p)}}{h(r_p)^{[\rho+1]}} .$$

By expanding $\log r = \log\{1-(1-r)\}$ and $\log r_p = \log\{1-(1-r_p)\}$ as series, we have

$$\frac{(r r_p)^{1/h(r_p)(1-r_p)}}{h(r_p)^{[\rho+1]}} \leq \frac{1}{h(r_p)^{[\rho+1]}} \exp\left\{-\frac{1}{h(r_p)}\left(\frac{1-r}{1-r_p}+1\right)\right\}, \quad (5.3.43)$$

$$\leq K \int_{r_{p-1}}^{r_p} \frac{\exp\left\{-\frac{1}{h(t)}\left(\frac{1-r}{1-t}+1\right)\right\}}{h(t)^{[\rho+1]}(1-t)} dt. \quad (5.3.44)$$

Now

$$\int_0^1 \frac{\exp\left\{-\frac{1}{h(t)}\left(\frac{1-r}{1-t}+1\right)\right\}}{h(t)^{[\rho+2]}(1-t)} dt = \int_0^1 \frac{(1-t)^{[\rho+1]}}{(1-r+1-t)^{[\rho+2]}} \left\{\frac{1}{h(t)}\left(\frac{1-r}{1-t}+1\right)\right\}^{[\rho+2]} \exp\left\{-\frac{1}{h(t)}\left(\frac{1-r}{1-t}+1\right)\right\} dt,$$

$$\leq [\rho+2]^{[\rho+2]} \exp\{-[\rho+2]\} \int_0^1 \frac{(1-t)^{[\rho+1]}}{(1-r+1-t)^{[\rho+2]}} ,$$

$$\leq K \int_0^1 \frac{dt}{(1-r+1-t)} \leq K \log\frac{1}{1-r} . \quad (5.3.45)$$

Therefore

$$\sum_{p=1}^{\infty} \frac{(r r_p)^{1/h(r_p)(1-r_p)}}{h(r_p)^{[\rho+1]}} \leq K \log\frac{1}{1-r} . \quad (5.3.46)$$

It follows from (5.3.40) that

$$\left|\log\left|\frac{\eta_0(z)}{\mu(z)}\right|\right| \le K\left(1+\log\frac{1}{1-r}\right) = o\left(\frac{1}{v(r)\,h(r)^2}\right), \quad (5.3.47)$$

which, together with (5.3.16) and (5.3.30), shows that

$$\log|\eta_0(z)| = \{1+o(1)\}\int_0^r\int_0^t \frac{du}{h(u)^2(1-u)^2}\,dt \quad (5.3.48)$$

outside the union of the discs of (5.3.25).

5.4. Canonical products on more general sets. We now consider a set of points $\{\zeta_{p,q}\}$ in $|z|<1$ which satisfies

$$|\zeta_{p,q} - z_{p,q}| \le D\,h(|z_{p,q}|)(1-|z_{p,q}|), \quad (5.4.1)$$

and define

$$\eta(z) = \prod_{(p,q)} \mathcal{E}(z, \zeta_{p,q}, [\rho+1]). \quad (5.4.2)$$

We investigate the behaviour of $\eta(z)$ by comparing it with $\eta_0(z)$. For z outside the union (B_δ) of the discs of (5.3.25) we have

$$\frac{\eta(z)}{\eta_0(z)} = \prod_{(p,q)} \frac{\bar{\zeta}_{p,q}(\zeta_{p,q}-z)(1-z\bar{z}_{p,q})}{\bar{z}_{p,q}(z_{p,q}-z)(1-z\bar{\zeta}_{p,q})}\exp\left\{\sum_{s=1}^{[\rho+1]}\frac{1}{s}\left(\left(\frac{1-|\zeta_{p,q}|^2}{1-z\bar{\zeta}_{p,q}}\right)^s-\left(\frac{1-|z_{p,q}|^2}{1-z\bar{z}_{p,q}}\right)^s\right)\right\}. \quad (5.4.3)$$

Now

$$\begin{aligned}|1-z\bar{\zeta}_{p,q}| &= |1-z\bar{z}_{p,q} - z(\bar{\zeta}_{p,q}-\bar{z}_{p,q})|,\\ &\ge |1-z\bar{z}_{p,q}| - D|z|\,h(|z_{p,q}|)(1-|z_{p,q}|),\\ &\ge (1-D\,h(|z_{p,q}|))\,|1-z\bar{z}_{p,q}|. \quad (5.4.4)\end{aligned}$$

In particular, it follows that

$$\left|\frac{1 - z\bar{z}_{p,q}}{1 - z\bar{\zeta}_{p,q}}\right| \leq K_1 . \tag{5.4.5}$$

for all (p,q) and z in $|z| \leq 1$.

We shall need the following, almost immediately obvious, result.

LEMMA 5.4.1 . For some constant K_2,

$$\left| \frac{1 - |z_{p,q}|^2}{1 - z\bar{z}_{p,q}} - \frac{1 - |\zeta_{p,q}|^2}{1 - z\bar{\zeta}_{p,q}} \right| \leq \frac{K_2\, h(|z_{p,q}|)(1 - |z_{p,q}|)}{|1 - z\bar{z}_{p,q}|} . \tag{5.4.6}$$

Proof. We have

$$\frac{1-|z_{p,q}|^2}{1 - z\bar{z}_{p,q}} - \frac{1-|\zeta_{p,q}|^2}{1 - z\bar{\zeta}_{p,q}} = \frac{|\zeta_{p,q}|^2 - |z_{p,q}|^2 - z(\bar{\zeta}_{p,q} - \bar{z}_{p,q}) - z\bar{z}_{p,q}\bar{\zeta}_{p,q}(\zeta_{p,q} - z_{p,q})}{(1 - z\bar{z}_{p,q})(1 - z\bar{\zeta}_{p,q})} \tag{5.4.7}$$

and, since

$$|\zeta_{p,q}|^2 - |z_{p,q}|^2 - z(\bar{\zeta}_{p,q} - \bar{z}_{p,q}) - z\bar{z}_{p,q}\bar{\zeta}_{p,q}(\zeta_{p,q} - z_{p,q})$$

$$= |\zeta_{p,q} - z_{p,q}|\left\{2(1 - z\bar{z}_{p,q}) - 2(1 - |z_{p,q}|) - \frac{z\bar{z}_{p,q}(1 - |z_{p,q}|^2)}{|z_{p,q}|}\right\}\cos\left\{\arg\left(\frac{\zeta_{p,q}}{z_{p,q}} - 1\right)\right\} + \tag{5.4.8}$$

$$+ \frac{i\,|\zeta_{p,q} - z_{p,q}|\, z\bar{z}_{p,q}(1 - |z_{p,q}|^2)}{|z_{p,q}|}\sin\left\{\arg\left(\frac{\zeta_{p,q}}{z_{p,q}} - 1\right)\right\} + |\zeta_{p,q} - z_{p,q}|^2(1 - z\bar{z}_{p,q}) ,$$

this gives

$$\left| \frac{1 - |z_{p,q}|^2}{1 - z\bar{z}_{p,q}} - \frac{1 - |\zeta_{p,q}|^2}{1 - z\bar{\zeta}_{p,q}} \right| \leq \frac{8K_1 |\zeta_{p,q} - z_{p,q}|}{|1 - z\bar{z}_{p,q}|} , \tag{5.4.9}$$

$$\leq \frac{K_2\, h(|z_{p,q}|)(1 - |z_{p,q}|)}{|1 - z\bar{z}_{p,q}|} . \tag{5.4.10}$$

In order to estimate the product of (5.4.3), we divide it into products over subsets of the (p,q) ; the first set $\mathcal{G}_1 = \mathcal{G}_1(z)$ is the set of (p,q) such that

$$\left| \frac{1-|z_{p,q}|^2}{1-z\bar{z}_{p,q}} \right| \leq \frac{1}{2} . \tag{5.4.11}$$

It follows from (5.4.6), for $|z_{p,q}|$ sufficiently near to 1, that then

$$\left| \frac{1-|\zeta_{p,q}|^2}{1-z\bar{\zeta}_{p,q}} \right| \leq \frac{3}{4} . \tag{5.4.12}$$

Therefore, for $(p,q) \in \mathcal{G}_1$, we can expand the terms involved in the product as series to obtain, by the argument of Lemma 2.3.1 and the inequality of (5.4.6),

$$\left| \log \frac{\mathcal{E}(z, \zeta_{p,q}, [\rho+1])}{\mathcal{E}(z, z_{p,q}, [\rho+1])} \right| \leq K_3\, h(|z_{p,q}|) \left(\frac{1-|z_{p,q}|}{1-z\bar{z}_{p,q}} \right)^{[\rho+2]} . \tag{5.4.13}$$

We now need the following result.

LEMMA 5.4.2 . For $s \geqslant 2$ and p-th primitive root of unity ω_p,

$$\sum_{q=1}^{p} \frac{1}{|1-z\omega_p^q|^s} \leq \frac{K_4\, p}{(1-|z|)^{s-1}} \cdot - \frac{1}{(1-|z|)^s} \tag{5.4.14}$$

Proof. The result follows from comparing the sum with

$$\frac{p}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{(1-2r\cos\theta + r^2)^{\frac{1}{2}s}} .$$

It follows from (5.4.13) and the lemma that

$$\sum_{q=0}^{[1/h(r_p)(1-r_p)]} \left| \log \frac{\mathcal{E}(z, \zeta_{p,q}, [\rho+1])}{\mathcal{E}(z, z_{p,q}, [\rho+1])} \right| \leq K_5 \left(\frac{1-r_p}{1-rr_p} \right)^{[\rho+1]} . \tag{5.4.15}$$

Now

$$\left(\frac{1-r_p}{1-rr_p}\right)^{[\rho+1]} \leq \frac{1-r_p}{1-rr_p} \leq \int_{r_p}^{r_{p+1}} \frac{dt}{h(t)(1-rt)}, \qquad (5.4.16)$$

and therefore

$$\sum_{p=1}^{\infty} \left(\frac{1-r_p}{1-rr_p}\right)^{[\rho+1]} \leq \int_0^1 \frac{dt}{h(t)(1-rt)}. \qquad (5.4.17)$$

We now show that

$$\int_0^1 \frac{dt}{h(t)(1-rt)} = o\left(\frac{1}{\nu(r)\,h(r)^2}\right), \qquad (5.4.18)$$

as $r \to 1-0$. Firstly

$$h(r)^2 \int_r^1 \frac{dt}{h(t)(1-rt)} \leq \Lambda^2 \int_r^1 \frac{dt}{h(t)(1-rt)^{1+\nu(r)}}, \qquad (5.4.19)$$

$$\leq \int_r^1 \frac{\Lambda\, dt}{(1-t)^{1-\frac{1}{2}\nu(r)}} = \frac{2h(r)}{\nu(r)}. \qquad (5.4.20)$$

Also,

$$\int_0^r \frac{dt}{h(t)(1-rt)} \leq \frac{1}{h(r)} \int_0^r \frac{dt}{1-t} = \frac{1}{h(r)} \log \frac{1}{1-r}, \qquad (5.4.21)$$

and hence

$$\nu(r)\, h(r)^2 \int_0^r \frac{dt}{h(t)(1-rt)} \leq 2\,h(r) \log \frac{1}{h(r)} \to 0, \qquad (5.4.22)$$

as $r \to 1-0$. This and (5.4.20) are sufficient to give (5.4.18).

For $(p,q) \in \mathcal{G}_2$, the complement of $\mathcal{G}_1$, we have

$$\left| \frac{1-|z_{p,q}|^2}{1-z\bar{z}_{p,q}} \right| > \frac{1}{2}. \qquad (5.4.23)$$

Or, writing $z = re^{i\theta}$ and $z_{p,q} = r_p e^{i\varphi}$,

$$(1 - r_p^2)^2 > \frac{1}{4}\left(1 - 2rr_p\cos(\varphi - \theta) + r^2 r_p^2\right). \tag{5.4.24}$$

This implies that

$$1 - 2rr_p\cos(\theta - \varphi) + r^2 r_p^2 < 16(1 - r_p)^2, \tag{5.4.25}$$

or, equivalently,

$$(1 - rr_p)^2 + 4rr_p \sin^2\left\{\tfrac{1}{2}(\varphi - \theta)\right\} < 16(1 - r_p)^2. \tag{5.4.26}$$

If we consider φ to be such that $|\varphi - \theta| \leq \pi$, this gives

$$(1 - rr_p)^2 + \frac{4rr_p}{\pi^2}(\varphi - \theta)^2 < 16(1 - r_p)^2, \tag{5.4.27}$$

and hence for $(p,q) \in \mathcal{G}_2$

$$\sup_q |\arg z_{p,q} - \arg z| \leq K_6(1 - r_p). \tag{5.4.28}$$

The number of (p,q) in $\mathcal{G}_2$ for fixed p is at most

$$\frac{1}{\pi}\left\{1 + \frac{1}{h(r_p)(1 - r_p)}\right\} \sup_q \left\{|\arg z_{p,q} - \arg z|\right\} + 1, \tag{5.4.29}$$

and it follows from the above that this does not exceed

$$K_7 / h(r_p).$$

Therefore, on summing over these elements, in view of (5.4.6) and that

$$\left|\frac{1 - |\zeta|^2}{1 - z\bar{\zeta}}\right| \leq 2 \quad \text{for all } |\zeta| < 1,\ |z| < 1, \tag{5.4.30}$$

we obtain

$$\left|\sum_{q \ni (p,q) \in \mathcal{G}_2} \sum_{s=1}^{[\rho+1]} \frac{1}{s}\left\{\left(\frac{1 - |\zeta_{p,q}|^2}{1 - z\bar{\zeta}_{p,q}}\right)^s - \left(\frac{1 - |z_{p,q}|^2}{1 - z\bar{z}_{p,q}}\right)^s\right\}\right| \leq \frac{K_7 K_2 h(r_p)(1 - r_p)}{h(r_p)(1 - rr_p)}[\rho+1]2^{[\rho]} \leq K_8 \tag{5.4.31}$$

If $1-r_{p-1} > \frac{1}{4}(1-r)$ then

$$p-1 \leq \int_0^{1-\frac{1}{4}(1-r)} \frac{dt}{h(t)(1-t)} \leq \frac{\log \frac{4}{1-r}}{h\left(1-\frac{1}{4}(1-r)\right)} . \qquad (5.4.32)$$

It follows from (5.3.9) that this is less than

$$\frac{K \log \frac{1}{1-r}}{h(r)} = o\left(\frac{1}{\nu(r)h(r)^2}\right). \qquad (5.4.33)$$

Hence, if r_{p^*} is the smallest r_p satisfying $1-r_p \leq \frac{1}{4}(1-r)$ then

$$p^* = o\left(\frac{1}{\nu(r)\,h(r)^2}\right) \qquad (5.4.34)$$

as $r \to 1-0$, and therefore

$$\left|\sum_{(p,q)\in \mathcal{G}_2} \sum_{s=1}^{[e+1]} \frac{1}{s}\left\{\left(\frac{1-|\zeta_{p,q}|^2}{1-z\bar{\zeta}_{p,q}}\right)^s - \left(\frac{1-|z_{p,q}|^2}{1-z\bar{z}_{p,q}}\right)^s\right\}\right| \leq K_8\, p^* = o\left(\frac{1}{\nu(r)h(r)^2}\right). \qquad (5.4.35)$$

Let $\mathcal{G}_3$ consist of those (p,q) for which

$$|z_{p,q} - z| \leq (1-r)\,h(r)^{\frac{1}{2}}. \qquad (5.4.36)$$

Now

$$\frac{\bar{\zeta}_{p,q}(\zeta_{p,q}-z)(1-z\bar{z}_{p,q})}{\bar{z}_{p,q}(z_{p,q}-z)(1-z\bar{\zeta}_{p,q})} = 1 + \frac{|\zeta_{p,q}|^2-|z_{p,q}|^2 - z(\bar{\zeta}_{p,q}-\bar{z}_{p,q}) - z\bar{z}_{p,q}\bar{\zeta}_{p,q}(\zeta_{p,q}-z_{p,q})}{\bar{z}_{p,q}(1-z\bar{\zeta}_{p,q})(z_{p,q}-z)},$$

$$= 1 + \lambda(z,p,q), \qquad (5.4.37)$$

and, as with Lemma 5.4.1 , it is easily shown that

$$|\lambda(z,p,q)| \leq \frac{K_9\, h(|z_{p,q}|)(1-|z_{p,q}|)}{|z_{p,q}-z|}. \qquad (5.4.38)$$

If

$$1-r_p \geq 2(1-r) \qquad (5.4.39)$$

then $|z_{p,q} - z| \geqslant \frac{1}{2}(1-r_p)$ and therefore

$$|\lambda(z,p,q)| \leqslant K\, h(|z_{p,q}|). \tag{5.4.40}$$

Hence, if we denote the set of (p,q) in $\mathcal{G}_2$ which also satisfy (5.4.39) by $\mathcal{G}_4$, we have

$$\left|\log\left|\prod_{(p,q)\in\mathcal{G}_4}\{1+\lambda(z,p,q)\}\right|\right| \leqslant K \sum_{(p,q)\in\mathcal{G}_4} h(|z_{p,q}|), \tag{5.4.41}$$

$$\leqslant K \sum_{p \ni (p,q)\in\mathcal{G}_4 \text{ for some } q} 1 \quad , \tag{5.4.42}$$

$$\leqslant K \int_0^{1-2(1-r)} \frac{dt}{h(t)(1-t)} = o\left(\frac{1}{\nu(r)h(r)^2}\right). \tag{5.4.43}$$

If p is such that $2(1-r) > (1-r_p) > \frac{1}{4}(1-r)$ and $(p,q) \in (\mathcal{G}_2 - \mathcal{G}_3) - \mathcal{G}_4$ we have

$$|\lambda(z,p,q)| \leqslant K\, h(r)^{\frac{1}{2}}. \tag{5.4.44}$$

Also, the number of (p,q) in $\mathcal{G}_2$ does not exceed

$$K \int_0^{1-\frac{1}{4}(1-r)} \frac{dt}{h(t)^2(1-t)} \leqslant \frac{K \log \frac{1}{1-r}}{h(r)^2}. \tag{5.4.45}$$

Therefore

$$\left|\log\left|\prod_{(p,q)\in(\mathcal{G}_2-\mathcal{G}_3)-\mathcal{G}_4}\{1+\lambda(z,p,q)\}\right|\right| \leqslant \frac{K\log\frac{1}{1-r}}{h(r)^{3/2}} = o\left(\frac{1}{\nu(r)h(r)^2}\right). \tag{5.4.46}$$

We denote by (A_δ) the union of the discs

$$|z - \zeta_{pq}| \leqslant \delta\, h(|\zeta_{pq}|)(1-|\zeta_{pq}|) \quad . \tag{5.4.47}$$

If $(p,q) \in \mathcal{G}_3$ and z is outside (A_δ) and (B_δ), then

$$\frac{\delta\, h(r)}{K} \leqslant \left|\frac{\bar{\zeta}_{p,q}(\zeta_{pq}-z)(1-z\bar{z}_{pq})}{\bar{z}_{pq}(z_{pq}-z)(1-z\bar{\zeta}_{pq})}\right| \leqslant \frac{K}{\delta\, h(r)} \quad . \tag{5.4.48}$$

The disc

$$|\zeta - z| \leq h(r)^{\frac{1}{2}}(1-r)$$

is contained in the curvelinear quadrilateral defined by

$$(1-r)\{1-h(r)^{\frac{1}{2}}\} \leq 1-|\zeta| \leq (1-r)\{1+h(r)^{\frac{1}{2}}\},$$

$$|\arg\zeta - \arg z| \leq \frac{(1-r)h(r)^{\frac{1}{2}}}{\{r^2-(1-r)^2h(r)\}^{\frac{1}{2}}},$$

and therefore the number of (p,q) in $\mathcal{G}_3$ does not exceed

$$\sum_{\substack{(1-r)\{1+h(r)^{\frac{1}{2}}\} \geq \\ 1-r_p \geq (1-r)\{1-h(r)^{\frac{1}{2}}\}}} \frac{K}{h(r_p)^{\frac{1}{2}}} \leq K\int_{1-(1-r)\{1+h(r)^{\frac{1}{2}}\}}^{1-(1-r)\{1-h(r)^{\frac{1}{2}}\}} \frac{dt}{h(t)^{3/2}(1-t)} \leq \frac{K}{h(r)}. \qquad (5.4.49)$$

Therefore

$$\left|\log\left|\prod_{(p,q)\in\mathcal{G}_3} \frac{\bar{\zeta}_{p,q}(\zeta_{p,q}-z)(1-z\bar{z}_{p,q})}{\bar{z}_{p,q}(z_{p,q}-z)(1-z\bar{\zeta}_{p,q})}\right|\right| \leq \frac{K}{h(r)}\log\frac{K}{\delta h(r)} = o\left(\frac{1}{\nu(r)h(r)^2}\right). \qquad (5.4.50)$$

Combining the results of equations (5.4.15-18), (5.4.35), (5.4.43), (5.4.46) and (5.4.50) with (5.4.3) gives

$$\left|\log\left|\frac{\eta(z)}{\eta_0(z)}\right|\right| = o\left(\frac{1}{\nu(r)h(r)^2}\right) \qquad (5.4.51)$$

for z outside (A_δ) and (B_δ). Equation (5.3.48) then shows that

$$\log|\eta(z)| = \{1+o(1)\}\int_0^{|z|}\int_0^t \frac{du}{h(u)^2(1-t)^2}\,dt, \qquad (5.4.52)$$

for the same set of z. An argument similar to that contained between equations (2.3.47) and (2.3.39) shows that the same result holds for all z outside (A_δ). If we also suppose that the discs of (5.4.47) are disjoint, we can show, using a minimum modulus argument, that

$$\log|\eta'(\zeta_{p,q})| = \{1+o(1)\}\int_0^{|\zeta_{p,q}|}\int_0^t \frac{du}{h(u)^2(1-u)^2}\,dt. \qquad (5.4.53)$$

We collect these results in the form of a theorem.

THEOREM 41. Suppose that $\{\zeta_{p,q}\}$ satisfies (5.4.1), that the discs of (5.4.47) are disjoint and that $\eta(z)$ is as defined by (5.4.2), then with the $\varphi(r)$ of (5.2.1):

$$\log|\eta(z)| \leq \frac{1+\epsilon}{\Lambda^2}\,\varphi(|z|), \qquad (5.4.54)$$

for $|z| \geq r_0(\epsilon)$;

$$\log|\eta(z)| \geq \frac{1-\epsilon}{\Lambda^2}\,\varphi(|z|), \qquad (5.4.55)$$

for $|z| \geq r_0(\epsilon)$, with z outside the discs of (5.4.47);

$$\log|\eta'(\zeta_{p,q})| \geq \frac{1-\epsilon}{\Lambda^2}\,\varphi(|\zeta_{p,q}|), \qquad (5.4.56)$$

for $|\zeta_{p,q}| \geq r_0(\epsilon)$.

<u>5.5. Proof of Theorems 38 and 40.</u> We shall give only an outline of the procedure followed, since it is very similar to that used to establish Theorems 15 and 17 of Chapter II. We first establish the following result.

THEOREM 42. A set $\{z_n\}$ $|z_n|<1$, which is such that the union of the discs

$$|z-z_n| \leq \frac{\Delta(1-|z_n|)^{1+\frac{1}{2}\nu(|z_n|)}}{1+\Delta(1-|z_n|)^{\frac{1}{2}\nu(|z_n|)}} \qquad (5.5.1)$$

covers $|z|<1$, contains a subset which is representable as a set $\{\zeta_{pq}\}$, satisfying the conditions of Theorem 41, provided that

$$\frac{4\pi\Lambda^2}{\Delta^2 3\sqrt{3}} > 1. \tag{5.5.2}$$

The method for proving Theorem 42 is analoguous to the proof of Theorem 21; the theorem follows from considering the points of each set which lie in each of the quadrilaterals $Q_{k\ell}(A,\nu(r))$, defined by

$$\begin{gathered} r_{kA} \leq |z| < r_{(k+1)A}, \\ \frac{2\pi\ell}{1+[1/Ah(r_{kA})(1-r_{kA})]} \leq \arg z \leq \frac{2\pi(\ell+1)}{1+[1/Ah(r_{kA})(1-r_{kA})]}, \end{gathered} \tag{5.5.3}$$

where A is a fixed, sufficiently large, positive integer.

In view of Theorem 42, it is now only necessary to prove Theorems 38 and 40, modified so that the set $\{z_n\}$ is replaced by a set ζ_{pq}, where Λ is such that $4\pi\Lambda^2/\Delta^2 3\sqrt{3}$ exceeds but is sufficiently near to 1. This can be done by an almost verbatim repetition of the argument used to establish Theorems 19 and 20. We note, however, that the use of partial products of the form $\tilde{\omega}_N(z)$ of (5.3.31) is essential to the argument, since we do not have any suitable estimate for $\log\{\sum_{n=1}^{\infty}|a_n|r^n\}$ in terms of $\log\{\sup_{|z|=r}|\sum_{n=1}^{\infty}a_n z^n|\}$ when $\limsup |a_n|^{1/n} = 1$.

<u>5.6. Proof of Theorem 39.</u> As mentioned previously, the extreme cases $H(r)$ bounded and $c>0$ follow from

Theorem 38. For $c = 0$ we have, again by Theorem 38, that

$$\limsup_{r \to 1-0} (1-r)^{\rho} \log M(r, f) = 0. \tag{5.6.1}$$

Now consider the effect of the transformation

$$\zeta = \zeta_{\theta}(z) = \frac{1}{e^{i\theta}(e^{i\theta} - z)} \tag{5.6.2}$$

in this situation. Write

$$F_{\theta}(\zeta) = f\left(\frac{\zeta e^{i\theta} - e^{i\theta}}{\zeta}\right), \tag{5.6.3}$$

and, for each θ, consider the sector $S(\alpha, \sec\alpha)$ $(0 < \alpha < \frac{1}{2}\pi)$ of the ζ-plane; we denote this sector by S_{α}. $F_{\theta}(\zeta)$ is regular in S_{α} and satisfies

$$\frac{\log M(R, S_{\alpha}, F_{\theta})}{R^{\rho}} \leq \frac{\log M\left(\left\{1 - \frac{2\cos\alpha}{R} + \frac{1}{R^2}\right\}^{\frac{1}{2}}, f\right)}{R^{\rho}}, \tag{5.6.4}$$

$$= \frac{\left(1 - \left\{1 - \frac{2\cos\alpha}{R} + \frac{1}{R^2}\right\}^{\frac{1}{2}}\right)^{\rho} \log M\left(\left\{1 - \frac{2\cos\alpha}{R} + \frac{1}{R^2}\right\}^{\frac{1}{2}}, f\right)}{\left\{R\left(1 - \left\{1 - \frac{2\cos\alpha}{R} + \frac{1}{R^2}\right\}^{\frac{1}{2}}\right)\right\}^{\rho}}.$$

Now $R - (R^2 - 2R\cos\alpha + 1)^{\frac{1}{2}} \uparrow \to \cos\alpha$ as $R \uparrow \to \infty$, and therefore

$$\frac{\log M(R, S_{\alpha}, F_{\theta})}{R^{\rho}} \leq \frac{1}{\cos^{\rho}\alpha}\left(1 - \left\{1 - \frac{2\cos\alpha}{R} + \frac{1}{R^2}\right\}^{\frac{1}{2}}\right)^{\rho} \log M\left(\left\{1 - \frac{2\cos\alpha}{R} + \frac{1}{R^2}\right\}^{\frac{1}{2}}, f\right), \tag{5.6.5}$$

$$\longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty,$$

by (5.6.1), uniformly for $0 \leq \theta \leq 2\pi$. Also,

$$\frac{\Delta |e^{i\theta} - z_n|^{1+\frac{1}{2}\rho}}{1 + \Delta |e^{i\theta} - z_n|^{\frac{1}{2}\rho}} \geq \frac{\Delta(1 - |z_n|)^{1+\frac{1}{2}\rho}}{1 + \Delta(1 - |z_n|)^{\frac{1}{2}\rho}}, \tag{5.6.6}$$

for all real θ, and therefore $|z| < 1$ is covered by the union of the discs

$$|z - z_n| \leqslant \frac{\Delta |e^{i\theta} - z_n|^{1+\frac{1}{2}\rho}}{1 + \Delta |e^{i\theta} - z_n|^{\frac{1}{2}\rho}} . \qquad (5.6.7)$$

The transform of this disc is contained in

$$|\zeta - \zeta_\theta(z_n)| \leqslant \Delta |\zeta_\theta(z_n)|^{1-\frac{1}{2}\rho} , \qquad (5.6.8)$$

and therefore S_α is covered by the union of the discs of (5.6.8).

It follows from (5.2.10) that

$$\limsup_{\zeta_\theta(z_n) \in S_\alpha} \frac{\log |F_\theta(\zeta_\theta(z_n))|}{H\left(\left\{1 - \frac{2\cos\alpha}{|\zeta_\theta(z_n)|} + \frac{1}{|\zeta_\theta(z_n)|^2}\right\}^{\frac{1}{2}}\right)} \leqslant \varkappa , \qquad (5.6.9)$$

uniformly in θ , and that

$$\frac{H\left(\left\{1 - \frac{2\cos\alpha}{R} + \frac{1}{R^2}\right\}^{\frac{1}{2}}\right)}{R^\rho} = \frac{\left(1 - \left\{1 - \frac{2\cos\alpha}{R} + \frac{1}{R^2}\right\}^{\frac{1}{2}}\right)^\rho H\left(\left\{1 - \frac{2\cos\alpha}{R} + \frac{1}{R^2}\right\}^{\frac{1}{2}}\right)}{\left(R - \{R^2 - 2R\cos\alpha + 1\}^{\frac{1}{2}}\right)^\rho} , \qquad (5.6.10)$$

$$\downarrow \longrightarrow \frac{1}{\cos^\rho \alpha} \qquad \lim_{r \to 1-0} (1-r)^\rho H(r) = 0 ,$$

as $R \uparrow \to \infty$.

The conditions of Theorem 31 of Chapter IV are satisfied (with $\gamma = 0$, $c = 0$ and $H(R)$ replaced by $H\left(\left\{1 - \frac{2\cos\alpha}{R} + \frac{1}{R^2}\right\}^{\frac{1}{2}}\right)$, which for the case under consideration is unbounded) and hence for any β with $0 < \beta < \alpha$,

$$\frac{\log M(R, S_\beta, F_\theta)}{H\left(\left\{1 - \frac{2\cos\alpha}{R} + \frac{1}{R^2}\right\}^{\frac{1}{2}}\right)} \leqslant \varkappa + \epsilon , \qquad (5.6.11)$$

for $R \geqslant R_0(\epsilon)$; that this holds uniformly for $0 \leqslant \theta \leqslant 2\pi$, while not a direct consequence of the theorem, can easily be seen to be implied by the proof of Theorem 31.

Transforming the result of (5.6.11) back to the unit disc gives

$$\limsup_{r\to 1-0} \frac{\log M(r,f)}{H(r)} \leq \varkappa \limsup_{R\to\infty} \frac{H\left(\left\{1-\frac{2\cos\alpha}{R}+\frac{1}{R^2}\right\}^{\frac{1}{2}}\right)}{H\left(1-\frac{1}{R}\right)}, \quad (5.6.12)$$

and, since

$$1 \leq \frac{H\left(\left\{1-\frac{2\cos\alpha}{R}+\frac{1}{R^2}\right\}^{\frac{1}{2}}\right)}{H\left(1-\frac{1}{R}\right)} \leq \frac{\left(1-\left\{1-\frac{1}{R}\right\}\right)^{\rho}}{\left(1-\left\{1-\frac{2\cos\alpha}{R}+\frac{1}{R^2}\right\}^{\frac{1}{2}}\right)^{\rho}},$$

$$= \frac{1}{\left(R-\left\{R^2-2R\cos\alpha+1\right\}^{\frac{1}{2}}\right)^{\rho}} \longrightarrow \frac{1}{\cos^{\rho}\alpha},$$

as $R\to\infty$, we have

$$\limsup_{r\to 1-0} \frac{\log M(r,f)}{H(r)} \leq \frac{\varkappa}{\cos^{\rho}\alpha}. \quad (5.6.13)$$

This is true for every positive α $(\alpha < \frac{1}{2}\pi)$ and hence the proof of Theorem 39 is complete.

CHAPTER VI

CONCLUSION

In this part of the thesis we discuss further the position of some of our results within the general theory and mention some simple extensions and some open questions.

<u>6.1. The classes of growth estimating functions $\varphi(r)$ of Chapters II and V.</u> These classes contain all suitably smooth functions corresponding to growth of finite order except that we cannot have $\varphi(r) = O(\log r)^2$ in Chapter II or $\varphi(r) = O\left(\log \frac{1}{1-r}\right)$ in Chapter V; otherwise we would contradict (2.1.25) or (5.3.16) respectively. Since an integral function $f(z)$ satisfying

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{\log r} < \infty \tag{6.1.1}$$

is a polynomial, its assymptotic growth is determined by its behaviour at any unbounded sequence $\{z_n\}$.

This gap, between $\log M(r,f) = O(\log r)$ and $\log M(r,f) = O(\log r)^2$, in our results for integral functions can be filled by considering a function $\varphi(r)$ which is such that

$$\frac{d\varphi(r)}{d\log r} \;\exists = (\log r)^{\lambda(r)}, \tag{6.1.2}$$

where $0 \leq \lambda(r) \leq 1$ and $\lambda(r)\downarrow$. Suppose $\{z_p\}$ to be a set of points satisfying

$$n(r) \sim A(\log r)^{\lambda(r)}, \tag{6.1.3}$$

and such that the discs

$$|z - z_p| \leq \delta |z_p| \tag{6.1.4}$$

are disjoint for some fixed positive δ . We then define

$$\eta(z) = \prod_{p=1}^{\infty} \left\{ 1 - \frac{z}{z_p} \right\}. \tag{6.1.5}$$

Direct estimations show that, for z outside the discs of (6.1.4),

$$\log |\eta(z)| \sim A\,\varphi(|z|), \tag{6.1.6}$$

and it follows from this that

$$\log |\eta'(z_p)| \sim A\,\varphi(|z_p|) \tag{6.1.7}$$

and that

$$\sum \left| \frac{1}{\eta'(z_p)} \right|^h < \infty, \tag{6.1.8}$$

for any fixed positive h . Obvious analogues to the theorems of Chapter II can now be established with only minor changes in proof. For example,

THEOREM 43. Suppose that $\{z_p\}$ satisfies (6.1.3) and (6.1.4), and that $\varphi(r)$ satisifes (6.1.2). If $f(z)$ is an integral function for which

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{\varphi(r)} < A, \tag{6.1.9}$$

then

$$\limsup_{r \to \infty} \frac{\log M(r,f)}{\varphi(r)} = \limsup \frac{\log |f(z_n)|}{\varphi(|z_n|)} , \qquad (6.1.10)$$

provided that the right-hand side is less than A . Furthermore, if $f(z_n)$ is bounded then $f(z)$ is identically constant.

In fact, the results would appear more powerful in that there is nothing in the conditions which ensures that each sector of positve aperture contains an infinite set of points of $\{z_n\}$ and it could be asked if the same is true for a wider class of functions.

The growth results for the case considered above are also suggested by more general theorems of Hayman [12] and Valiron [52]. For example, Hayman has proved

THEOREM 44. If $f(z)$ satisfies

$$\limsup_{r \to \infty} \frac{\log M(r,f)}{(\log r)^2} < \infty , \qquad (6.1.11)$$

then

$$\log |f(re^{i\theta})| \sim \log M(r,f) \qquad (6.1.12)$$

outside a set of discs subtending angles at the origin which have finite sum.

Since this result does not extend to larger classes of functions, it seems unlikely that it is possible to remove conditions involving the angular distribution of the set $\{z_n\}$ from our interpolation theorems for a larger class than that attained above. We note, also, that if $\rho = 0$, then $\varphi(\lambda r)/\varphi(r) \to 1$, as $r \to \infty$, for any fixed positive λ, that is $\varphi(r)$ belongs to the class of slowly oscillating functions.

Whether the gap, $\log M(r,f) = O\left(\log \frac{1}{1-r}\right)$, in our results for the unit disc can be filled remains an open question.

6.2. Regions where $|f(z)|$ is small. In Chapter III we obtained results on regions where $|z|^{-\rho} \log |f(z)|$ is near to its upper limit, as $|z| \to \infty$. A very direct and simple argument using the same ideas will also give information about regions where $|z|^{-\rho} \log |f(z)|$ may be small. Using the separation condition of (2.1.7), we have

THEOREM 45. Suppose $f(z)$ to be an integral function of mean type τ of positive order ρ. Let E be the set of ζ for which

$$|z|^{-\rho} \log |f(z)| \leq \lambda < \tau \tag{6.2.1}$$

throughout the disc

$$|z - \zeta| \leq \delta |\zeta|^{1-\frac{1}{2}\rho}, \tag{6.2.2}$$

for some fixed positive δ. Then

$$\bigcup_{\zeta \in E} \left\{ |z-\zeta| \leq \Delta |\zeta|^{1-\frac{1}{2}\rho} \right\} \tag{6.2.3}$$

does not cover the complex plane for any finite Δ.

There are also corresponding results for the wider range of estimator functions $\varphi(r)$. Also, if we weaken the separation condition (2.1.7), as is certainly possible, we can increase the size of the set $\mathcal{E}$. It seems likely that an appropriate determination of $\mathcal{E}$ would be the set of ζ for which (6.2.1) holds throughout the disc $|z-\zeta| \leq |\zeta|^{-k}$ for some fixed positive k.

6.3. Intermediate growth conditions. Our results, Theorems 31, 32 and 40, of Chapters IV and V concerning functions whose growth an the interpolation set is dominated by $\exp\{(1+o(1))H(r)\}$ are restricted to cases of growth at most mean type of finite order. It remains an open question as to whether or not they can be extended to wider classes of functions. For example, could we prove the following proposition?

Suppose

(i) $\dfrac{d^2\varphi(r)}{d(\log r)^2} \ \exists = r^{\nu(r)} \uparrow \to \infty$ with $\nu(r) \downarrow \to \rho$;

(ii) $H(r)\uparrow$, $H(r)/\varphi(r) \downarrow \to c$;

and (iii) the discs $|z-z_n| \leq \Delta |z_n|^{1-\frac{1}{2}\nu(|z_n|)}$ cover the complex plane. Suppose, in addition, that $f(z)$ is an integral function satisfying

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{\varphi(r)} < \frac{4\pi}{\Delta^2 3\sqrt{3}},$$

then

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{H(r)} = \limsup \frac{\log|f(z_n)|}{H(|z_n|)}$$

provided that the right-hand side is less than $4\pi/c\Delta^2 3\sqrt{3}$.

There appear to be two methods of approach to this kind of problem; one could try to establish a sector theorem, corresponding to Theorem 31, by determining a (1:1) regular map $\zeta(z)$ of $S(\alpha, \ell)$ onto a region which is sufficiently approximate to a sector $S(\alpha', \ell')$ and such that $F(\zeta) = f(z)$ satisfies conditions which would allow appeal to the results already established. Alternatively one could select a suitable subset $\{z_{n'}\}$ ($=\{\zeta_{p,q}\}$) of the z_n and form the set of canonical products

$$\eta_N(z) = (z - z_N) \prod_{n' \neq N} E\left(\frac{z - z_N}{z_{n'} - z_N}, [\rho]\right) \tag{6.3.1}$$

with a suitable modifying factor $\exp\{-K(N)(z-z_N)^\rho\}$ when ρ is an integer, and consider representations

$$f(z) = \eta_N(z) \sum_{p \in \{n'\}} \frac{f(z_p)}{(z-z_p)\,\eta_N'(z_p)} \tag{6.3.2}$$

for integral functions, or

$$\Phi_N(z) = f(z) - \sum_{z_{n'} \in S(\alpha,\ell)} \frac{\eta_N(z)\, f(z_{n'})}{(z-z_{n'})\,\eta_N'(z_{n'})} \tag{6.3.3}$$

for functions regular in a sector.

It would seem to be necessary to establish results which are uniform in N on the assymptotic behaviour of

$\log|\eta_N(z)|$ in terms of $|z - z_N|$, outside regions which may depend on N containing its zeros. For example:

If $|z-z_N| \leq K_1 |z_N|^{1-\frac{1}{2}\nu(|z_N|)}$ then $\log|\eta_N(z)| \leq K_2$, where K_1 and K_2 are independent of N .

The difficulties of such estimations on $\{\eta_N(z)\}$ are obvious.

<u>6.4. Gap power series.</u> In section 4.5 , we studied integral functions having power series satisfying the Fabry gap condition. We note here that if the gap condition is strengthened to

$$a_{n_{k+1}} - a_{n_k} \to \infty \tag{6.4.1}$$

then, since we would then have

$$\log M(r, f) \sim \log M(r, \alpha, \beta, f) \tag{6.4.2}$$

for any sector $\alpha \leq \arg z \leq \beta$, with no exceptional set of r , we can prove the following.

THEOREM 46. Suppose that $f(z)$ is an integral function at most of minimum type of order ρ such that its Taylor series

$$\sum_{k=1}^{\infty} a_{n_k} z^{n_k} \tag{6.4.3}$$

satisfies (6.4.1). Suppose, also, that

$$\limsup \frac{\log|f(z_n)|}{H(|z_n|)} \leq X \tag{6.4.4}$$

where $H(r)\uparrow$, $r^{-\rho}H(r)\downarrow\to 0$, and for some finite $\underline{\Delta}$ the union of the discs

$$|z - z_n| \leq \Delta |z_n|^{1-\frac{1}{2}\rho}$$

covers a sector of positive aperture. Then

$$\limsup_{r \to \infty} \frac{\log M(r, f)}{H(r)} \leq \chi . \qquad (6.4.6)$$

One could also obtain a similar conclusion under conditions involving the aperture of the sector, the separation of the non-zero coefficients, the size of Δ and the maximum growth of the class of admissible functions.

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