QUANTUM TESTS FOR NON-INERTIAL AND GENERAL RELATIVISTIC EFFECTS

BY

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A THESIS SUBMITTED TO

THE UNIVERSITY OF KENT AT CANTERBURY

IN THE SUBJECT OF THEORETICAL PHYSICS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY.

Canterbury, 21 September, 1999

To my parents who taught me how to think.

Abstract

A possible solution of combining General Relativity and Quantum Mechanics is given by means of writing the Dirac equation in curved space-time using Weyl's tetrad formalism.

First some background information is given about the necessity and problems of incorporating gravity into Quantum Mechanics. Then the fine details are described of how the Dirac Hamiltonian can be found in Riemannian spaces. The rest of the thesis is devoted to applying this method to describe the effects of stationary and rotating gravitational sources. This results a possible test of the Equivalence Principle in the quantum domain, as well as finding the limit of the applicability of various approximations of the Earth's field. Finally a general relativistic treatment of the COW experiment is given.

Acknowledgements

I would like to thank my supervisor, Dr. Lewis Ryder, for his persevering support over the four years I have spent in Canterbury, both scientific and personal. I am grateful to him for those thought-provoking conversations when he shared his penetrating insight into the nature of the physical world and also for his presence in my life as a good friend.

I gratefully acknowledge financial support from the Committee of Vice-Chancellors and Principals in form of an ORS Award, from the School of Physical Sciences in form of a grant and for Prof. John Strange for suitable arrangements of the latter one.

I could have not done this work without the love and encouragement from my family, Anya, Zsolt and Péter, who provided me with a sound background.

It was also important to have my friends at home and here to keep in touch and share, with good humour, the ups and downs of everyday life.

For all of these I am grateful.

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1 Introduction

1.1 Historical remarks

It is a truism that any experiment performed on the Earth is done under the effect of gravity. Gravitation, one of the four basic interactions governing the structure and behaviour of the material world, has by far the smallest effect. To detect a gravitational effect objects of large masses which are electrically and magnetically neutral have to be used. The gravitational coupling is so weak that the gravitational attraction between two protons is 10^{39} times less than the electric repulsion; an alternative comparison would show that the order of magnitude of the gravitational term in the Hamiltonian is approximately 10^{9} times less than the rest mass energy term when a particle in the Earth's field is considered. Even the Sun causes very little distortion of space-time; a ray passing by its disk is deflected only by 1.75 seconds of arc. It is very difficult to detect these effects, since the order of experimental error involved in these experiments, until recently, used to exceed gravitational effects by several orders of magnitudes.

For these reasons it is standard practice to ignore the effect of gravity in the case of laboratory experiments and to apply whatever physical theory is relevant; equivalently, to apply the physical theory in flat, rather than in curved space. It sounds even more plausible that this procedure is above all justifiable in the quantum regime: whoever thought that gravitational effects would manifest themselves at the quantum level?

It is now more than twenty years since Colella, Overhauser and Werner succeeded in performing an experiment which made it possible to detect gravitational effects in neutron interferometry. This experiment and the improved follow-up versions are commonly referred to as the COW experiments (for details see Ch. 2.4). When in 1975 Colella, *et al.* reported on their detection of gravitational effects in neutron interferometry, their paper meant a lot more than simply a report on an experiment no-one had done before. It proved that the standard practice of ignoring gravity when talking about quantum systems was wrong. To put it right was not a matter of putting an extra term in the calculations: a whole conceptual problem arose when one tried to combine general relativity and quantum mechanics. The phase shift in the experiment was explained by the authors using Newtonian mechanics, and this was a satisfactory approximation, because of the order of the experimental error involved. Since 1975, however, new experiments have been suggested, and the use of atomic interferometers is expected to increase the accuracy of the COW experiments by a factor of 10^{10}, which will take us to the regime where relativistic corrections become relevant. So apart from the matter of principle, that the proper description of gravitational effects is achieved by using Einstein's general relativity, there is a practical need too for a higher order description of gravitational effects on quantum systems.

1.2 Aim of thesis

As explained in the previous section, the direct evidence of gravitational effects manifesting themselves in neutron interferometry established a demand for describing general relativistic effects in quantum systems. In this work a synthesis of the distinct fields of General Relativity and Quantum Mechanics is attempted.

The aim of this thesis is to find the proper method of analysing the behaviour of quantum particles, especially spin-½ particles in an Earth-bound laboratory, i.e. to give a description of gravitational and non-inertial effects on them. The method used throughout this work is to find the Dirac Hamiltonian in whatever circumstance and approximation is appropriate.

First it is necessary to define the correct procedure for the determination of the frame, and solving various technical problems such as using the epsilon symbol in curved spaces, absorbing the determinental factor of the invariant volume element into the wavefunction, and taking the proper non-relativistic limit of the resulting Hamiltonian.

When these problems are clarified the effects of curved spaces and accelerated frames on spin-½ particles can be examined by giving an approximate description of the effect of the Earth's gravitational field. Comparing the effects of gravity and acceleration a test of the Equivalence Principle is gained in the quantum domain. Further studies of rotating frames in Minkowski, Schwarzschild and Kerr spaces would provide a higher order description of experiments performed in laboratories.

Using the results of these analyses I attempt to give a general relativistic description of the COW experiment.

1.3 Outline of thesis

This thesis is a report of my work concerning non-inertial and general relativistic effects on quantum systems, specifically on spin-½ particles.

Chapter 1 consists of general remarks, including a historical review of the topic, a statement of objectives, an outline of the thesis and general remarks on notation.

In Chapter 2 the theoretical and experimental backgrounds of this work is reviewed. This includes short discussions of the relevant theoretical concepts of Quantum Mechanics and General Relativity, as well as raising the problem of applying these two simultaneously. Then a brief overview of neutron and atomic interference experiments, testing gravitational and non-inertial effects on quantum systems, is given. In the end a summary of the preceding results in the field of finding the Dirac Hamiltonian in various spaces is presented.

Chapter 3 shows how the Dirac equation may be written in a general Riemannian space. It enters into details of the steps of the procedure such as choosing coordinates, determining the frame, various methods of finding the connection coefficients, using the epsilon symbol in curved spaces, absorbing the determinental factor of the invariant volume element into the wavefunction, and taking the proper non-relativistic limit of the resulting Hamiltonian. As examples of the use of the above, then, the form of the momentum operator is derived in isotropic and spherical polar coordinates, and the effects of rotation and position dependence of the frame are investigated.

The thesis proceeds in Chapter 4 to the application of the method described above, to give a description of the effect of stationary gravitational sources on spin-½ particles. The Dirac Hamiltonian is written in the Schwarzschild field and then being compared with the corresponding result in an accelerated Minkowski space. Then remarks are made and conclusions arising from this analysis are drawn concerning the Equivalence Principle.

Chapter 5 examines the effects of a rotating gravitational source on Dirac particles. To analyse the situation rotating frames in Schwarzschild and Kerr spaces are used and the resulting Hamiltonians are compared with each other, as well as, with the purely non-inertial effects of an accelerated rotating frame to determine the limits of the applicability of these three models when describing experimental results in Earth-based laboratories.

A reanalysis of the COW experiments takes place in Chapter 6 as a general relativistic derivation of the phase shift is presented.

A summary of the main results of the thesis and some directions for further study are given in Chapter 7.

Finally references of all the work cited in this thesis is included.

1.4 Conventions

Small Latin indices $(a, b, \ldots i, j, \ldots)$ run from 1 to 3 referring to spatial components, while small Greek indices $(\alpha, \beta, \ldots \kappa, \mu, \ldots)$ running from 0 to 3 note all space and time components. Unmarked indices, both Latin and Greek, $(a, b, \ldots \alpha, \beta, \ldots)$ refer to coordinate basis components, indices with hat $(\hat{a}, \hat{b}, \ldots \hat{\alpha}, \hat{\beta}, \ldots)$ refer to orthonormal basis components.

The Greek letter phi is used in several contexts, but different letter types are used: Φ denotes the "gravitational potential", ϕ the phase shift and φ is the polar angle. There is a similar "degeneracy" in the notation concerning theta: θ^{μ} denotes the basis 1-forms and ϑ is the other polar angle. Also, a g with a single index g_i means one component of the "gravitational acceleration" and with two indices $g_{\mu\nu}$ it refers to the metric tensor. Evidently $\underline{g}$ with a vector notation is the acceleration.

G denotes the universal gravitational constant, M the mass of the gravitating source. If the gravitational source is rotating, a is used to denote its angular momentum per unit mass, and ω or $\omega^i=(0,0,\omega)$ its angular velocity. ω is used also, as the angular velocity of a rotating frame of reference, which has the same value as the angular velocity of the rotating mass. An $\omega_{\mu\nu}$ with two indices refer to the connection 1-forms.

$\Gamma_{\mu\nu\kappa}$ and $C_{\mu\nu\kappa}$ are the connection coefficients or Christoffel symbols and the structure constants, respectively. The ordinary derivative is usually denoted with a comma: $A_{\mu,\nu} = \dfrac{\partial A_{\mu}}{\partial x^{\nu}}$. Square bracket [] is used for commutator, curly brackets {} are used for anticommutator relationships.

The signature of the space-time metric is $(+,-,-,-)$. $g_{\mu\nu}$ is used to denote the metric tensor in an arbitrary space in coordinate basis, and $g_{\hat{\mu}\hat{\nu}} = \eta_{\mu\nu}$ notes metric in orthonormal basis, i.e. the Minkowski metric tensor.

Basis 1-forms are denoted by θ^{μ}, and the dual vectors by e_{ν}. The duality is expressed as

$$\langle \theta^{\mu}, e_{\nu} \rangle = \langle e_{\nu}, \theta^{\mu} \rangle = \delta^{\mu}_{\nu}, \tag{1.1}$$

whereas the scalar product is denoted by ordinary brackets:

$$(\theta^{\mu}, \theta^{\nu}) = g^{\mu\nu} \text{ and } (e_{\mu}, e_{\nu}) = g_{\mu\nu}. \tag{1.2}$$

The wedge product of the 1-forms is antisymmetric

$$dx^{\mu} \wedge dx^{\nu} = -dx^{\nu} \wedge dx^{\mu}. \tag{1.3}$$

The line element in coordinate basis is

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}, \tag{1.4}$$

and in orthonormal basis

$$ds^2 = \eta_{\hat{\mu}\hat{\nu}}\, \theta^{\hat{\mu}}\, \theta^{\hat{\nu}}. \tag{1.5}$$

At some points in this thesis equations have been simplified by using the convention $\hbar = 1$ or $c = 1$ or both, though in other places they have been retained for clarity. The summation convention is used throughout the thesis, when the same index appears twice, once in covariant and once in a contravariant position.

2 Background

2.1 The theory of gravity - General Relativity

"Gravity is a habit that is hard to shake off" (T. Pratchett, *Small Gods*)

Physics, trying to explain the behaviour of the inanimate world, is a collection of mathematical models, consisting of differential equations, accompanied by rules correlating mathematical results and meaningful quantities of the physical world. In the case of the "gravitational interaction" it is Einstein's theory of general relativity which is the accepted model at present. Here the differential equations are geometric requirements on space and time together with the field equations describing the interaction of matter and space.

Studying gravitational effects is probably the oldest discipline in science: it can be considered as old as man who looked up at the stars in the early days of history. During this long time several concepts had to be demolished as more accurate observation techniques developed. Also, the theoretical study of gravitation has always relied on advancements in mathematics: inventing calculus provided a useful tool for Newton to formulate his theory, and general relativity could not exist without Riemannian geometry.

Shortly after the publication of Einstein's Special Theory of Relativity (SR) it became clear that it was inconsistent with Newton's Theory of Gravity, because of its space but not time dependence. The generalisation of SR (laws of physics are invariant under all, not only linear transformations) provided a new theory of gravity. "The extension of the principle of [special] relativity implies the necessity of the law of the equality of inertial and gravitational mass. The general theory of relativity must yield important results on the laws of gravitation." [Einstein, 1924]

Based on the idea of Galileo's falling body experiment Einstein generalised the theorem, that no experiment in mechanics can distinguish a gravitational field from an accelerated frame, to formulate the equivalence principle (EP): no experiment in physics can distinguish the local effects of gravity and acceleration. A consequence of this principle is that light travels on a curved path. Together with Fermat's least action principle it leads to the idea of curved spaces.

General Relativity (GR) is a theory of gravity describing it in terms of curved spaces. Picturesque examples for GR can be given: the Earth orbiting the Sun can be explained by saying that in a curved space curved orbits are natural or another example is the bending of light by massive stellar bodies. In this theory, terms such as gravitational field, force or gravitational interaction have no meaning any more. Many different mathematical entities are associated with gravitation: the metric, the Riemann curvature tensor, the covariant derivative, the connection coefficients, etc. Each of these plays an important role in gravitation theory, and they are all related to each other. Thus the terms "gravitational field" and "gravity" usually refer in a vague, collective sort of way to all of these entities.

Research in the field of GR may involve a purely mathematical analysis of the differential equations of the model (Einstein's field equations) finding as many exact solutions as possible. See for example Kerr [1963] or for a summary on the subject Kramer *et al.* [1980]. The other type of research in general relativity involves the mathematical and physical interpretation of the obtained solutions. In my research I contributed to this latter type of work using the known solutions for certain cases, such as Schwarzschild's, Kerr's or the accelerated frame metric, analysing their properties and effects on quantum systems.

Experimental tests of gravity can be done on two levels: with the technological advances of the last century the solar system, providing objects with large masses, became a good source of observational data, whereas experiments in Earth-bound laboratories allow controlling of various conditions. The dynamic progress of experimental techniques provoked the quotation "General relativity is no longer a theorist's paradise and an experimentalist's hell" [Misner *et al.*, 1973]. Overviews on experiments performed to test general relativity are by Vessot [1984] and Cook [1988].

I have no intention to give a complete description of the principles of GR here. Even a short summary would take up more space than this dissertation. I intend to give only a basic insight into its concepts, and refer the reader to various textbooks for details [see for example Misner *et al.*, 1973]. Another short and very picturesque, with hardly any equations involved, introduction is given in Feynman's book [Feynman *et al.*, 1975]. I will define each quantity when necessary as it turns up along the calculations.

In mathematics curved spaces are dealt by using Riemannian geometry. A space is characterised by a metric

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{2.1}$$

which carries all the information about the space. Characteristic quantities are usually calculated from the components of the metric tensor, carrying special information about the space, for example the connection coefficients are calculated in a coordinate basis, as

$$\Gamma^{\rho}{}_{\lambda\mu} = \frac{1}{2} g^{\rho\kappa}\left(g_{\kappa\lambda,\mu} + g_{\kappa\mu,\lambda} - g_{\lambda\mu,\kappa}\right). \tag{2.2}$$

Unlike in SR, where the metric has only a passive role, in GR the metric plays an active role, because the geometry of the space is not fixed in advanced, but determined by the mass distribution. To obtain a metric, Einstein's field equations should be solved. These are complicated nonlinear tensor equations, and no general solution is known. There are a few special cases, such as the field outside a spherically symmetric body at rest, in which the field equations can be solved. The metrics I use in my calculations are

- Schwarzschild space: outside the surface of a spherically symmetric, stationary gravitational source [Stephani, 1990]

$$ds^2 = \left(1 - 2\frac{GM}{rc^2}\right)c^2 dt^2 - \frac{1}{1 - 2\dfrac{GM}{rc^2}} dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta \ \ d\varphi^2\right) \tag{2.3}$$

- Kerr space: outside the surface of a spherically symmetric, rotating gravitational source [Hawking and Ellis, 1974]

$$ds^2 = \Sigma\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) + \left(r^2 + a^2\right)\sin^2\vartheta\, d\varphi^2 - dt^2 + \frac{2mr}{\Sigma}\left(a\sin^2\vartheta\, d\varphi - dt\right)^2 \tag{2.4}$$

  with

$$\Sigma = r^2 + a^2\cos^2\vartheta \quad \text{and} \quad \Delta = r^2 - 2mr + a^2 \ . \tag{2.5}$$

- flat space in an accelerating, rotating frame of reference [Hehl and Ni, 1990]

$$ds^2 = \left(1 + \frac{\underline{a}\cdot\underline{x}}{c^2}\right)^2 c^2 dt^2 - \left(dx^i + \left[\frac{\underline{\omega}}{c}\times\underline{x}\right]^i c\,dt\right)^2 \tag{2.6}$$

In GR contravariant, covariant and mixed tensors are defined by their transformation properties. The method of changing the position of indices is to apply the metric tensor. For example lowering the last index of the connection gives:

$$\Gamma_{\mu\nu\kappa} = g_{\kappa\rho}\,\Gamma_{\mu\nu}{}^{\rho} \tag{2.7}$$

where the usual summation convention is used.

A helping tool to deal with Riemannian geometry is the use of differential forms, which may make calculation easier. An example is given in Appendix 3.12.1.

Finally I should note here that in this thesis gravity is treated in the classical way, i. e. it is not quantised and torsion is not considered.

2.2 Quantum Mechanics

At the end of the 19th century, a series of experimental results (e.g. spectrum of blackbody radiation, photoelectric effect, electron diffraction) were presented, which were impossible to explain by the classical physics model. These observations led to the development of quantum theory. Quantum mechanics (QM) is the presently accepted way to describe the behaviour of matter and light in all its details on the atomic scale. As our everyday experience concerns large objects only, one may find that QM "represents an abrupt and revolutionary departure from classical ideas, calling forth a wholly new and radically counterintuitive way of thinking about the world" [Griffiths, 1995].

In QM particles and waves are characterised by the probability density, which is the square of the wavefunction. To determine the wavefunction, the wave-equation has to be solved. The first such wave-equation was written down by Schrödinger by substituting differential operators for T and $\mathbf{p}$ into the non-relativistic energy relation:

$$T = \frac{p^2}{2m} \tag{2.8}$$

to get the Schrödinger equation of a free particle

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi \tag{2.9}$$

A starting point for a relativistic equation could be Einstein's energy relation

$$E^2 = p^2c^2 + m^2c^4 \tag{2.10}$$

giving the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\Psi + m^2\Psi = 0 \tag{2.11}$$

with the usual $\hbar = c = 1$ convention. The Klein-Gordon equation expresses nothing more than the relation between energy, momentum and mass, so this equation has to hold for any particle. For particles with a spin, all the spin components have to satisfy the Klein-Gordon equation.

The wave-equations for spin ½ (Dirac and Weyl equations) and spin 1 (Maxwell and Proca equations) particles can be derived from the transformation properties of spinors under the Lorentz group [Ryder, 1996 Ch. 2. and references therein]. Therefore these equations simply express a relation between the components of the wavefunction; in Weinberg's words, they are a confession that we have too many spin components [Weinberg, 1964].

The focus of this thesis is the effects on massive spin ½ particles, therefore the Dirac equation will be used:

$$\left(\gamma^{\mu} p_{\mu} - m\right)\Psi = 0 \qquad (2.12)$$

which equation became famous by successfully predicting the existence of antiparticles and the correct value for the electron magnetic moment [for details see for example Shankar, 1988].

In QM observables are represented by operators, and measurement results correspond to the eigenvalues of the operators. Therefore physically meaningful results require real eigenvalues, i.e. the operators have to be Hermitian.

I would like to note here that the above mentioned wave-equations are relativistic in SR sense only, but they are not consistent with GR!

2.3 Combining QM and GR

Any theory of the fundamental nature of matter must of course be consistent with relativity as well as with quantum theory. GR usually concerns the behaviour of big objects, on the scale of the solar system or larger, whereas QM plays an important role in the micro world. Thus there seemed to be no need for these models to be applied simultaneously until the COW experiments proved the opposite.

To resolve the problem of wave-particle duality, William Bragg once suggested using the corpuscular theory on Monday, Wednesday and Friday, and the undulatory theory on Tuesday, Thursday and Saturday (Sunday is a day off). A similar phrase could easily be applied for the theories of QM and GR considering that the two models are really different. The language of GR is a language of scalars, four-vectors and tensors, while the Dirac equation describes the state of quantum systems by spinors. The possible combination of the two models was not even understood by Dirac, but later Weyl gave a solution to this by applying tetrad-fields. This method will be described in Chapter 3. In spite of all the differences in the essence of these theories, Anandan claims that "gravity appears to be deeply rooted in the wave-particle duality of matter" [Anandan, 1980 and references therein].

The general theory of relativity is compatible with all other classical theories, but a complete unification with quantum theory has not been achieved. "Quantum theory assumes a Minkowski space of infinite extent, whereas GR shows that the space is Riemannian." [Stephani, 1990] But it is possible to introduce a locally flat coordinate system at every point of space-time, and consequently get rid of the gravitational effect, which makes it possible for the two theories to work simultaneously at regions of small curvature.

There are various possibilities to combine GR and QM.

- The successful unification of the weak and electromagnetic interaction gave rise to the idea of including the strong and gravitational interactions, as well. Creating this "theory of everything" (superstring theory), has not been achieved, so far. In case of a source-free weak field the quantization of the gravitational interaction can be done and it results in massless, spin 2 quanta [Stephani, 1990 Ch. 13.2]. However, the general solution for quantizing the gravitational field has not been found.
- Another possible solution for the problem of the coexistence of GR and QM is the semiclassical gravity theory. In this case the gravitational field is treated classically, whereas the rest of the fields are quantized. Einstein himself was a supporter of this view. The main problem in this approach is the interpretation of states. Even the vacuum state is not universal: what one observer regards as vacuum, the other may regard as a mixture of particles.
- A third approach involves quantization in a given classical gravitational field. When one tries to carry out the quantization procedure in curved space-time difficulties arise because of the non-flat space-time. The most spectacular example of these difficulties is the prediction of the creation of particles by a gravitational field. Hawking [1975] found that in black holes particles are created, and they have a thermal spectrum equivalent to a black body of temperature $\dfrac{10^{26}K}{M}$, where M is the mass of the black body measured in grams. In this approach the problem of the back-reaction of the particle creation on the metric is still unsolved.

Let me comment here on the different roles of gravity in quantum and classical mechanics following Sakurai's [1994] argument. In the classical equation of motion of a falling body

$$m\ddot{x} = -m\nabla\Phi_{grav} \tag{2.13}$$

the mass term cancels, as a consequence of the equivalence of inertial and gravitational mass. As mass does not appear in the equation of a particle trajectory, gravity, in classical mechanics, is often said to be a geometric theory. On the other hand, in the wave-mechanical formulation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + m\,\Phi_{grav}\right]\Psi = i\hbar\frac{\partial}{\partial t}\Psi \tag{2.14}$$

mass does not cancel, and it always appears in the combination $\hbar/m$. To see a nontrivial quantum-mechanical effect of gravity, therefore, we must study effects in which $\hbar$ appears explicitly. In the analysis of the COW experiments (Chapter 2.1) it is found, that the phase shift depends on $\left(\hbar/m\right)^2$ proving that at the quantum level gravity is not a purely geometric concept.

2.4 Experimental results and interpretation

2.4.1 Neutron interference experiments

Gravity is known as a theory of the large scale and quantum mechanics is associated with the small scale. It was experimentally demonstrated that neutrons are subject to gravitational acceleration, and found that they fall on a parabolic trajectory [Dabbs *et al.*, 1965]. Though this incorporates small particles and gravity, it is a classical phenomenon, without any quantum mechanics involved. Some time later a neutron interference experiment was suggested by Overhauser and Colella [1974] in which gravity and quantum mechanics would play an essential role, **simultaneously**. The experiment was carried out and the report on it [Colella *et al.*, 1975] was the first to contain a formula with both the gravitational acceleration and Planck's constant in it. Therefore they provided, in principle, the first link between GR and QM. This experiment is usually referred to as the COW experiment.

The authors used a Bonse-Hart type [Bonse and Hart, 1965] interferometer, which is equivalent to a double slit arrangement (see Figure 2.1). The interferometer consisted of a silicon single crystal. Three slabs were cut from the crystal. The first two slabs served as beam splitter and mirror, whereas the last recombined the two beams. By means of this setup one does not observe any interference pattern directly. Instead it is designed for observing a phase shift induced by varying external parameters. If the apparatus is rotated around the incident beam to change the difference in height, and hence the gravitational potential, between the interfering beams, then a phase shift between the two beams can be observed. This phase shift was explained by the authors using Newtonian mechanics, assuming that neutrons travel in the gravitational potential of Earth. Considering the accuracy of the experiment this was a suitable approximation. They found

$$\Delta\phi_{grav} = q_{grav}\sin\alpha = 4\pi\lambda\frac{g}{h^2}m^2 d\left(d + a\cos\theta\right)\tan\theta\sin\alpha\,. \tag{2.15}$$

with α the angle of the rotation of the interferometer, λ and m the de Broglie wavelength and the mass of the neutrons, d and a the dimensions of the interferometer, and θ the Bragg angle. A 10% discrepancy was found between this formula and the experimental data. This was explained by taking into account the bending of the interferometer base during rotation out of the horizontal plane; after correcting for this, the discrepancy between the (Newtonian) theory and experiment was reduced to 1%.

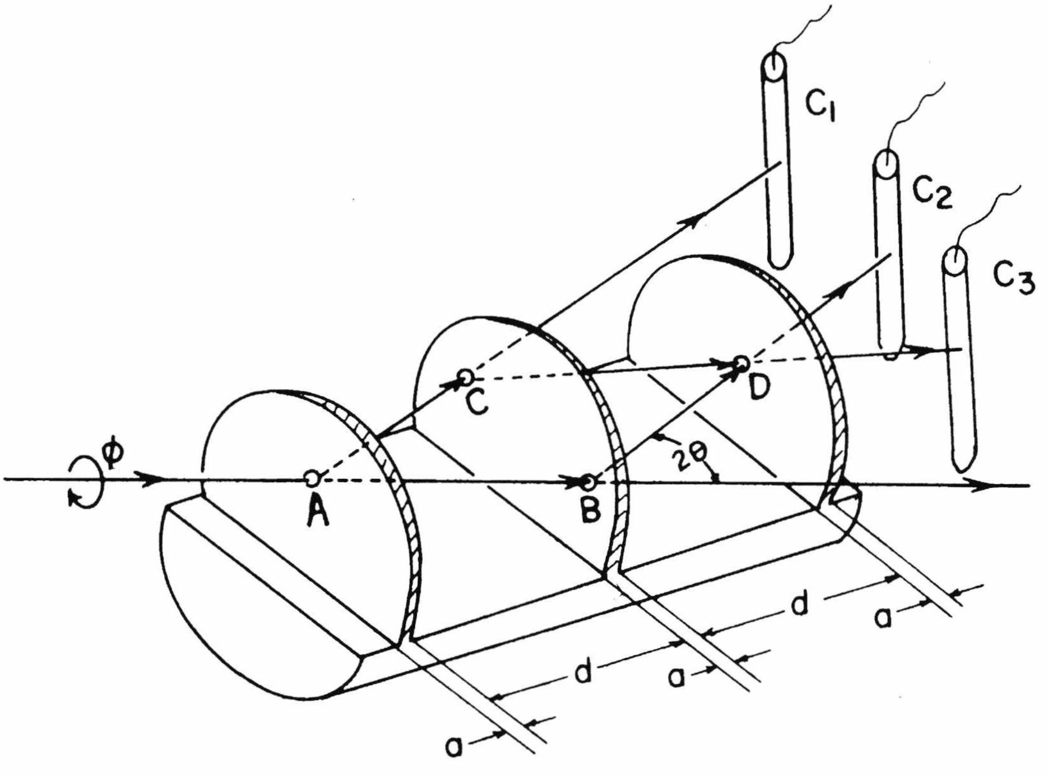

Figure 2.1: Schematic diagram of the neutron interferometer used in the COW experiment. Figure reproduced from paper by Colella *et al.*, 1975

Increasingly precise measurements were carried out [Staudenmann *et al.*, 1980 and Werner *et al.*, 1988] and the experiments were re-analysed by Horne [1986] taking into account the fact that the interferometer was an eight-path rather than a two-path device. Experimental and theoretical values were then found to agree within 0.8% [Werner, 1994].

Laboratories on the surface of the Earth rotate relative to the "fixed stars", therefore non-inertial effects, such as Coriolis force, are observable due to the rotation of the frame. In 1913 Sagnac demonstrated that optical interferometry is sensitive to rotations, and in 1925 Michelson, Gale and Pearson succeeded in constructing an interferometer in which the effect of the rotation of the earth was observable. A derivation using classical mechanical arguments shows, that on the rotating Earth, neutrons also experience a Sagnac-type shift [Werner, 1994]. The form of the phase-shift is

$$\Delta\phi_{Sagnac} = q_{Sagnac}\cos\alpha = \lambda\frac{4\pi m\omega A_0}{h}\cos\vartheta_L\cos\alpha \tag{2.16}$$

with ω the angular velocity of the Earth, A_0 the area of the interferometer and ϑ_L the colatitude at the place where the experiment is carried out. It was found that the effect of the rotation of the Earth adds only a small contribution to the gravitational effect; $\Delta\phi_{Sagnac} \approx 2\times10^{-4}\,\Delta\phi_{grav}$. Nevertheless, using an interferometer in a vertical plane (with consequently no gravitationally induced phase shift) this Sagnac shift was also verified [Werner *et al.*, 1979].

Following the logic of Einstein's equivalence principle in the quantum limit an experiment, corresponding to COW, searching for a phase shift in an accelerated frame of reference, rather than a gravitational field, was carried out by Bonse and Wroblewski [1983]. An interferometer oscillating in a horizontal plane was used, taking stroboscopic-type measurements at the inversion points of the movement. It was proved, to an accuracy of 4%, that the effects of acceleration and gravitation are the same.

I would like to note here that these experimental results provide a proof of the equivalence principle only within the limits of their accuracy. But the theoretical considerations for the phase shifts in gravitational field all relied on using Newtonian potential $V = mgh$ or a homogenous g-field, which is equivalent with the non-inertial effect of an accelerated frame, and not a genuine gravitational field!

As was made clear above, the theory with which the experimental data has been compared in these experiments is Newton's theory of gravity. From a fundamental point of view, however, this is unsatisfactory; the theoretical expression for the phase shift should be derived from General Relativity. General relativistic treatment of the COW experiment is presented in Chapter 6.

2.4.2 Atomic interferometry*

As compared with neutron-wave interferometers, atomic beam interferometers offer several advantages:

- atoms can be prepared with very low velocity by means of laser cooling;
- atoms have a larger mass and therefore a smaller de Broglie wavelength;
- sources of atomic beams are easier to handle;
- because of the internal degrees of freedom there are additional effects that can be tested;
- atoms may have larger spin and larger magnetic moments than single neutrons.

Atoms are of course more complex objects and should be described in an n-particle approach. In some approximation, this yields a Pauli-type equation with magnetic and electric dipole moments or its respective relativistic version. This represents a centre-of-mass motion with additional degrees of freedom. [Audretsch *et al.*, 1992b]

Apart from atomic interferometers based on a Young's double slit arrangement there are four other types in use: the most recent ones built by Kasevich and Chu [1991] and Shimizu *et al.* [1992].

In the COW experiment a sensitivity of $10^{-2}\,g$ was reported. At present the most accurate measurement of gravitational acceleration is done by using a superconducting gravimeter, which is able to measure up to $10^{-10}\,g$. Atomic interferometry promises further improvement, expecting to achieve a sensitivity of $10^{-12}\,g$. At these accuracies we have to ask the question whether we measure general relativistic or other types of corrections. Local fluctuations in the gravitational acceleration caused by tides ($10^{-7}\,g$) and changes of

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* Based on review papers of Adams *et al* [1994], and Audretsch *et al.*[1992b]

atmospheric pressure ($10^{-10}\, g/mbar$) can be subtracted, having a characteristic frequency. But other effects such as the vertical motion of the Earth's crust ($10^{-9}\, g/cm$) and changes in the local distribution of mass (a physicist at a distance of $1m$ produces $10^{-10}\, g$) produces anomalies at the order of the experimental accuracy. In the interference technique by the means of two nearby paths for particles the closer the two paths the less the effect of local fluctuations, but at the same time the relativistic effect is also reduced.

2.5 Examining the Dirac equation in non-inertial frames and gravitational fields

The above mentioned experiments, although involving atoms and neutrons, are not sensitive to spin effects. Therefore it was not necessary to use the Dirac equation in analysing them. In the studies of Wu [1988] and Xia and Wu, [1989] it was found that the spin polarisation of spin½ particles in the Earth's field is also affected, therefore in the analysis of experiments involving elementary particles in the Earth's field the use of the Dirac equation is necessary. The Dirac equation

$$i\hbar\gamma^{\mu} D_{\mu}\Psi = mc^{2}\Psi \tag{2.17}$$

is often rearranged in to the form

$$H\Psi = i\hbar\partial_{t}\Psi \tag{2.18}$$

and the Hamiltonian is used as characteristic quantity.

One such analysis was carried out by Fischbach [1980] who has determined the Hamiltonian for a Dirac particle in Schwarzschild space. In the calculation he has used isotropic coordinates which simplifies the form of expressions, therefore makes calculation easier. He has got

$$H = \beta mc^{2}(1-\Phi)+\frac{\beta}{2m}p^{2}+\frac{3}{2}\beta\left(-\frac{1}{m}\Phi\, p^{2}+\frac{i\hbar}{mc^{2}}\underline{g}\cdot\underline{p}+\frac{\hbar}{2\,mc^{2}}\underline{g}\cdot\underline{\sigma}\times\underline{p}\right). \tag{2.19}$$

In this expression the momentum is substituted for $-i\hbar\frac{\partial}{\partial x_{l}}$, differential with respect to isotropic coordinates, which should not have been done as will be explained in Chapter 3.9. A revised version of the above mentioned paper is published by Fischbach *et al.* [1981], but the same mistake was made. I shall present the derivation of the Dirac Hamiltonian in Cartesian coordinates in Chapter 4.1, and shall remark on how the Hamiltonian in isotropic coordinates should be interpreted.

Hehl and Ni [1990] have performed a similar calculation for particles in an accelerated and rotating frame. They found the Dirac Hamiltonian:

$$H = \beta mc^2\left(1+\frac{\underline{a}\cdot\underline{x}}{c^2}\right)+\frac{\beta}{2m}p^2+\frac{\beta}{2m}\underline{p}\cdot\frac{\underline{a}\cdot\underline{x}}{c^2}\underline{p}-\underline{\omega}\cdot\left(\underline{L}+\underline{S}\right)+\frac{\beta\hbar}{4mc^2}\underline{\sigma}\cdot\left(\underline{a}\times\underline{p}\right). \tag{2.20}$$

(with an error that the β was missing in the last term). Comparison of the resulting Hamiltonians in case of acceleration (setting $\omega = 0$ in (2.20)) and under the effect of gravity (modified (2.19)) furnishes a test for the equivalence principle, which will be carried out in Chapter 4.4. We shall see there signs of the equivalence principle not holding in the quantum domain.

Investigation of the Dirac equation in non-inertial frames was done by Chapman and Leiter [1976]. The analysis is in general terms, and the Hamiltonian is not calculated.

Further studies involving the use of the Dirac equation in the Earth's gravitational field were carried out using the Kerr metric by Lalak *et al.* [1995] and Wajima *et al.* [1997]. In these papers approximate forms of the Kerr metric are quoted, but the authors do not give a proper definition of coordinates. Moreover, to the same order of approximation, the expressions for the metric are found to be different (see Chapter 5). I shall therefore present a complete derivation of the Dirac Hamiltonian in Kerr space in Chapter 5.2. Then the Dirac Hamiltonian will be determined in Chapter 5.3.

3 On the Dirac equation in Riemannian spaces*

In this chapter it is explained how to write the Dirac equation in general Riemannian spaces using Weyl's tetrad formalism. This method is described in great detail, as are the problems of using different coordinate sets and moving reference frames. Some illustrative examples are provided here, some will be appended in Chapters 4 and 5.

3.1 Writing out the Dirac equation

The outcome of an experiment clearly depends on two things: on the space-time in which it is examined, and on the setting of the actual experimental setup which may be for example accelerating. Basically, given a metric, which carries all features of space-time, and choosing a frame, given by the basis vectors, we should be able to derive all characteristic

* A condensed version of the material in this chapter is to be submitted for publication [Varjú and Ryder, c]

quantities from this information. For example the Hamiltonian, i.e. the energy-function; to find the Hamiltonian in case of spin-½ particles, the Dirac equation has to be solved.

The way in which the general relativistic formalism of four-vectors and tensors on the one hand, and the spinor wavefunctions of quantum mechanics on the other, could be combined was not understood after the publication of general relativity. The relativistic wave-equation of Dirac (describing spin 1/2 particles) was only consistent with special relativity, but not with general relativity. The problem of compatibility was solved by Weyl who applied tetrad-fields. A tetrad defines a frame of reference at each point of space-time, a tangent space, which is locally inertial; in this frame space-time is Minkowski. Thus at each point of space-time a local flat frame is defined and Dirac's equation is reconstructed. It reads

$$i\hbar\gamma^{\mu} D_{\mu}\Psi = m\Psi \tag{3.1}$$

In writing the Dirac equation all effort is made to find the covariant derivative:

$$D_{\mu} = e_{\mu} + \frac{1}{8}\left[\gamma^{\nu}, \gamma^{\kappa}\right]\Gamma_{\nu\kappa\mu}\,. \tag{3.2}$$

From this we see immediately, what we are after: e_{μ}, γ^{ν}, $\Gamma_{\nu\kappa\mu}$; the basis vectors, spin matrices and connection coefficients.

This method is described in books and papers [see for example Sexl and Urbantke, 1983, Fischbach *et al.*, 1981 or Hehl and Ni, 1990] in certain special cases, but there are still unanswered questions when this simple-looking formula is used. In this chapter it will be illustrated what sort of problems turn up when different coordinates are used, and the equivalence of different-looking Hamiltonians is shown.

3.2 Coordinates

Coordinates are similar to some kind of "ruler"; we use them to determine the relative position of events (usually measured from the origin of the reference frame). The most frequently used coordinates are Cartesian, spherical polar and cylindrical polar coordinates. The choice of the coordinate system, however, influences the way in which the final result is written, in the same way that readings of a distance differ if rulers of centimetre or inch gratings are used. The distance is the same, only the expression describing it differs with the choice of coordinates. For example the momentum operator in spherical polars reads [Arthurs, 1970] $\mathbf{p} = -i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r},\ \frac{1}{r}\left(\frac{\partial}{\partial \vartheta}+\frac{\cot\vartheta}{2}\right),\ \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi}\right)$ while the Cartesian coordinate form is $\mathbf{p} = -i\hbar\left(\frac{\partial}{\partial x_C},\ \frac{\partial}{\partial y_C},\ \frac{\partial}{\partial z_C}\right)$.

In this chapter we are going to use three sets of space coordinates, while the time coordinate t is unchanged. These are

a, spherical polars r, ϑ, φ

b, Cartesian coordinates x_C, y_C, z_C

c, "isotropic" coordinates [Møller, 1972] x_I, y_I, z_I

The transformation relating Cartesian and spherical polar coordinates is:

$$\begin{aligned} x_C &= r\cdot\sin\vartheta\cdot\cos\varphi \\ y_C &= r\cdot\sin\vartheta\cdot\sin\varphi\ . \\ z_C &= r\cdot\cos\vartheta \end{aligned} \tag{3.3}$$

The isotropic radial coordinate is

$$r = r_I\cdot\left(1+\frac{m}{2r_I}\right)^2 \tag{3.4}$$

where m is the Schwarzschild radius. The advantage of using isotropic coordinates is that the Schwarzschild metric expressed in these coordinates takes a form when the spatial part has a common factor [see also Weinberg, 1972, p. 181]:

$$ds^2 = \frac{\left(1 - \frac{m}{2r_I}\right)^2}{\left(1 + \frac{m}{2r_I}\right)^2} dt^2 - \left(1 + \frac{m}{2r_I}\right)^4 \left(dr_I^2 + r_I^2 d\vartheta^2 + r_I^2 \sin^2 \vartheta \; d\varphi^2\right) \tag{3.5}$$

The definition of the isotropic Cartesian coordinates to first order in Φ is

$$\begin{aligned} x_I &= x_C \cdot (1 - \Phi) \\ y_I &= y_C \cdot (1 - \Phi) \qquad \text{with } \Phi = \frac{m}{r} \\ z_I &= z_C \cdot (1 - \Phi) \end{aligned} \tag{3.6}$$

It is known from the principle of general covariance that physics is independent of the choice of coordinates, so in a sense we can feel free to choose any sort of coordinates for our calculation. While this is true, we must be careful about interpreting the result in these arbitrarily chosen coordinates, and this is what we are going to illustrate below. In the words of Misner *et al.* [1973]: "The names given to coordinates have no intrinsic significance. A coordinate transformation is perfectly permissible, and has no influence on the physics or the mathematics of a relativistic problem. The only thing it affects is easy communication between the investigator who adopts it and his colleagues."

Choosing coordinates for any calculation always involves a trade-off; one set of coordinates will have advantages and disadvantages compared with other sets. Because of symmetry properties of the space the metric may look simple in one coordinate set, but the form of the momentum operator may be very complex. Also it often happens that one would like to compare Hamiltonians calculated in different spaces and frequently the relevant calculations are done using different coordinates: we end up with Hamiltonians expressed in different coordinates, and then the question arises how to compare them. For example, when the effects of gravitation and an accelerated frame are to be compared, we have to write the Dirac equation in Schwarzschild space and in an accelerated Minkowski space. The first calculation is undoubtedly of the simplest form using isotropic coordinates, while in the latter case it is advantageous to use Cartesians. This problem is worked out in detail in chapter 4.

3.3 Frames

Reference frames (or bases) are different type of objects from coordinate systems. They have a physical meaning and so cannot be chosen arbitrarily. They correspond to the room in which the experiment is done. A reference frame can also be rotating or accelerating and depending on this property the expression gained in the calculation will be different - although the space itself is the same; as the outcome of the corresponding experiment will differ when the setup is rotating or accelerating because of non-inertial effects. In Section 3.11 an example is given how the form of the Hamiltonian depends on the choice of the basis and not on coordinates.

When changing to a moving frame, often a coordinate transformation is performed. This coordinate transformation itself, however, does not correspond to a moving frame! But when the basis vectors are read off from the metric, the most natural one will be the one corresponding to the moving frame. It will be illustrated in Section 3.10 in case of a rotating frame.

We also make a distinction between a coordinate basis and an orthogonal basis of 1-forms. To make the difference clear, let us illustrate it with an example. The invariant line element in spherical polars in Minkowski metric reads:

$$ds^2 = dt^2 - dr^2 - r^2 d\vartheta^2 - r^2 \sin^2\vartheta \; d\varphi^2 . \tag{3.7}$$

The choice

$$\Theta^0 = dt\,,\ \Theta^1 = dr\,,\ \Theta^2 = d\vartheta\,,\ \Theta^3 = d\varphi \tag{3.8}$$

corresponds to a coordinate (holonomic) basis, while

$$\Theta^{\hat{0}} = dt\,,\ \Theta^{\hat{1}} = dr\,,\ \Theta^{\hat{2}} = r d\vartheta\,,\ \Theta^{\hat{3}} = r\sin\vartheta \, d\varphi \tag{3.9}$$

corresponds to an orthonormal basis, since

$$ds^2 = \left(\Theta^{\hat{0}}\right)^2 - \left(\Theta^{\hat{1}}\right)^2 - \left(\Theta^{\hat{2}}\right)^2 - \left(\Theta^{\hat{3}}\right)^2 \tag{3.10}$$

and the orthonormality condition

$$\left(\theta^{\hat{\alpha}}, \theta^{\hat{\beta}}\right) = \eta^{\hat{\alpha}\hat{\beta}} \tag{3.11}$$

or equivalently

$$\left(e_{\hat{\alpha}}, e_{\hat{\beta}}\right) = \eta_{\hat{\alpha}\hat{\beta}} \tag{3.12}$$

are satisfied. The relationship between coordinate and orthonormal bases is given by the tetrad (see Section 3.4).

For each calculation there is a choice of using coordinate or orthonormal bases. In a coordinate basis reading off the basis vectors and 1-forms from the metric is obvious, but finding the spin matrices is a non-trivial matter, whereas in an orthonormal basis finding one-forms is difficult but the form of spin matrices simply coincides with the special relativistic forms.

3.4 Tetrad formalism

To make clear the distinction between quantities expressed in an orthonormal basis on the one hand, and in a coordinate basis on the other, I shall use letters with hats to denote orthonormal indices and plain letters for coordinate indices. The tetrad components make the connection between orthonormal and coordinate 1-forms:

$$\Theta^{\hat{\kappa}} = h^{\hat{\kappa}}{}_{\alpha} dx^{\alpha} \quad . \tag{3.13}$$

From the duality condition (1.1) between the one-forms and the basis vectors, it follows that the basis vectors are related to the differentials by the inverse tetrad

$$e_{\hat{\kappa}} = h_{\hat{\kappa}}{}^{\alpha} \partial_{\alpha} \quad . \tag{3.14}$$

The tetrad components are used to transform tensors between coordinate and orthonormal form

$$K_{\hat{\kappa}} = h_{\hat{\kappa}}{}^{\alpha} K_{\alpha} \text{ and } K_{\alpha} = h^{\hat{\kappa}}{}_{\alpha} K_{\hat{\kappa}} . \tag{3.15}$$

A special case of the tensor transformation is the metric tensor. In coordinate basis the metric is denoted by $g^{\alpha\beta}$. From the definitions of orthonormal basis, (3.10)-(3.12), it follows that in orthonormal basis the metric tensor is Minkowski $g^{\hat{\kappa}\hat{\lambda}} = \eta^{\hat{\kappa}\hat{\lambda}}$. So changing from coordinate to orthonormal basis gives

$$h^{\hat{\kappa}}{}_{\alpha}\, h^{\hat{\lambda}}{}_{\beta}\, g^{\alpha\beta} = \eta^{\hat{\kappa}\hat{\lambda}} . \tag{3.16}$$

Tetrads can be also used to calculate the Dirac equation, as was done by Hehl and Ni [1990]. The object of anholonomicity is expressed in terms of the tetrad components as

$$C_{\hat{\kappa}\hat{\lambda}}{}^{\hat{\mu}} = h_{\hat{\kappa}}{}^{\alpha}\, h_{\hat{\lambda}}{}^{\beta} \left(\partial_{\beta}\, h^{\hat{\mu}}{}_{\alpha} - \partial_{\alpha}\, h^{\hat{\mu}}{}_{\beta} \right). \tag{3.17}$$

The Dirac spin matrices can also be given using orthonormal or coordinate bases. They have to fulfil

$$\left\{ \gamma^{\alpha}, \gamma^{\beta} \right\} = 2\, g^{\alpha\beta} \text{ or } \left\{ \gamma^{\hat{\kappa}}, \gamma^{\hat{\lambda}} \right\} = 2\, \eta^{\hat{\kappa}\hat{\lambda}} . \tag{3.18}$$

In the case of the orthonormal basis this relation is satisfied by the usual Dirac matrices:

$$\gamma^{\hat{0}} = \beta \, , \; \gamma^{\hat{i}} = \beta \alpha^{i} \, , \tag{3.19}$$

and in coordinate basis they can be expressed using the tetrad components,

$$\gamma^{\alpha} = h_{\hat{\kappa}}{}^{\alpha} \, \gamma^{\hat{\kappa}} \; . \tag{3.20}$$

3.5 The connection coefficients

There are alternative methods to calculate the connection coefficients. In the case of no torsion the connection coefficients are determined from the formula

$$\Gamma_{\kappa\lambda\mu} = \frac{1}{2}\left(g_{\kappa\lambda,\mu} + g_{\kappa\mu,\lambda} - g_{\lambda\mu,\kappa}\right) + \frac{1}{2}\left(C_{\kappa\lambda\mu} - C_{\lambda\mu\kappa} - C_{\mu\kappa\lambda}\right) \tag{3.21}$$

with

$$\left[e_\kappa, e_\lambda\right] = C_{\kappa\lambda}{}^{\mu}\, e_\mu . \tag{3.22}$$

When using a coordinate basis the basis vectors simply have the form of

$$e_\mu = \partial_\mu . \tag{3.23}$$

therefore they commute, and so the terms in the second bracket of (3.21) vanish. In this case then, the connection coefficients can be derived from differentials of the metric.

On the other hand in an orthonormal basis the metric is constant ($g_{\hat{\kappa}\hat{\lambda}} = diag(1,-1,-1,-1)$) so the first bracket in (3.21) vanishes and the connection coefficients are determined from the structure constants $C_{\hat{\kappa}\hat{\lambda}}{}^{\hat{\mu}}$ after lowering the third indices:

$$\Gamma_{\hat{\lambda}\hat{\mu}\hat{\nu}} = \frac{1}{2}\left(C_{\hat{\lambda}\hat{\mu}\hat{\nu}} - C_{\hat{\mu}\hat{\nu}\hat{\lambda}} - C_{\hat{\nu}\hat{\lambda}\hat{\mu}}\right) \tag{3.24}$$

The connection coefficients can also be determined using differential geometry. In this formalism a duality exists between space and functions [see for example: Israel, 1979, Flanders, 1989 or Ryder, 1998]. This may reduce the amount of calculations in certain circumstances, however in my calculations I have found it easier to use the other methods. In differential geometry one solves the Cartan-Maurer equations for the basis 1-forms θ^μ

$$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0 \tag{3.25}$$

and the metric compatibility condition

$$dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu} \tag{3.26}$$

to get the connection forms $\omega_{\mu\nu}$. Then the connection coefficients can be found from

$$\omega^{\mu}{}_{\nu} = \Gamma^{\mu}{}_{\nu\kappa}\, \theta^{\kappa}. \tag{3.27}$$

As mentioned above, the tetrad components can also be used to calculate the connection coefficients. I shall now show that expression (3.17) follows from the definition (3.22). To see this, use the duality condition

$$\left\langle e_{\hat{\mu}}, \Theta^{\hat{\nu}} \right\rangle = \delta^{\hat{\nu}}_{\hat{\mu}} \tag{3.28}$$

to give

$$C_{\hat{\kappa}\hat{\lambda}}{}^{\hat{\mu}} = \left\langle \left[e_{\hat{\kappa}}, e_{\hat{\lambda}} \right], \Theta^{\hat{\mu}} \right\rangle. \tag{3.29}$$

Changing to a coordinate basis (3.29) reads

$$\begin{aligned} C_{\hat{\kappa}\hat{\lambda}}{}^{\hat{\mu}} &= \left\langle \left[h^{\alpha}_{\hat{\kappa}}\, \partial_{\alpha}, h^{\beta}_{\hat{\lambda}}\, \partial_{\beta} \right],\ h^{\hat{\mu}}_{\eta}\, dx^{\eta} \right\rangle = \\ &= \left\langle h^{\alpha}_{\hat{\kappa}}\, \partial_{\alpha}\, h^{\beta}_{\hat{\lambda}}\, \partial_{\beta} - h^{\beta}_{\hat{\lambda}}\, \partial_{\beta}\, h^{\alpha}_{\hat{\kappa}}\, \partial_{\alpha},\ h^{\hat{\mu}}_{\eta}\, dx^{\eta} \right\rangle = \\ &= \Big\langle h^{\alpha}_{\hat{\kappa}} \left(\partial_{\alpha} h^{\beta}_{\hat{\lambda}} \right) \partial_{\beta} + h^{\alpha}_{\hat{\kappa}}\, h^{\beta}_{\hat{\lambda}}\, \partial_{\alpha}\, \partial_{\beta} - h^{\beta}_{\hat{\lambda}} \left(\partial_{\beta} h^{\alpha}_{\hat{\kappa}} \right) \partial_{\alpha} - h^{\beta}_{\hat{\lambda}}\, h^{\alpha}_{\hat{\kappa}}\, \partial_{\beta}\, \partial_{\alpha}, \\ &\qquad ,\ h^{\hat{\mu}}_{\eta}\, dx^{\eta} \Big\rangle = \\ &= \left\langle h^{\alpha}_{\hat{\kappa}} \left(\partial_{\alpha} h^{\beta}_{\hat{\lambda}} \right) \partial_{\beta} - h^{\beta}_{\hat{\lambda}} \left(\partial_{\beta} h^{\alpha}_{\hat{\kappa}} \right) \partial_{\alpha},\ h^{\hat{\mu}}_{\eta}\, dx^{\eta} \right\rangle \end{aligned} \tag{3.30}$$

The duality relation $\left\langle \partial_i\ ,\ dx^j \right\rangle = \delta_i^j$ then gives

$$C_{\hat{\kappa}\hat{\lambda}}{}^{\hat{\mu}} = h^{\alpha}_{\hat{\kappa}} \left(\partial_{\alpha} h^{\beta}_{\hat{\lambda}} \right) h^{\hat{\mu}}_{\beta} - h^{\beta}_{\hat{\lambda}} \left(\partial_{\beta} h^{\alpha}_{\hat{\kappa}} \right) h^{\hat{\mu}}_{\alpha}. \tag{3.31}$$

Using

$$0 = \partial_{\alpha} \left(\delta^{\hat{\mu}}_{\hat{\lambda}} \right) = \partial_{\alpha} \left(h^{\beta}_{\hat{\lambda}}\, h^{\hat{\mu}}_{\beta} \right) = \left(\partial_{\alpha} h^{\beta}_{\hat{\lambda}} \right) h^{\hat{\mu}}_{\beta} + h^{\beta}_{\hat{\lambda}} \left(\partial_{\alpha} h^{\hat{\mu}}_{\beta} \right) \tag{3.32}$$

gives for (3.31)

$$C_{\hat{\kappa}\hat{\lambda}}{}^{\hat{\mu}} = -\, h^{\alpha}_{\hat{\kappa}}\, h^{\beta}_{\hat{\lambda}} \left(\partial_{\alpha} h^{\hat{\mu}}_{\beta} \right) + h^{\beta}_{\hat{\lambda}}\, h^{\alpha}_{\hat{\kappa}} \left(\partial_{\beta} h^{\hat{\mu}}_{\alpha} \right) \tag{3.33}$$

which is (3.17). This has the opposite sign to the formula used by Hehl and Ni [1990].

To illustrate the different methods for calculating the connection coefficients examples are included in Appendix 3.12.1.

3.6 The epsilon symbol

Equation (3.2) contains commutators of gamma matrices. Evaluating these commutators will lead to expressions involving the epsilon symbol. The totally antisymmetric Levi-Civita tensor was introduced in quantum mechanics, with all indices in covariant position. In GR, however, the (upper and lower) position of indices is also important because of the summation convention and the fixed position of the free indices, so the ε symbol needs to take up covariant and contravariant indices. The expression

$$i\left[\gamma^{i},\gamma^{j}\right]=2\varepsilon_{ijk}\ I_{2}\otimes\sigma^{k} \tag{3.34}$$

(Equations A15 & A22 of Itzykson and Zuber, 1980) looks improper having the free indices (i,j) upstairs on the left hand side, and downstairs on the right hand side. To eliminate this problem, I define ε symbols with mixed indices, e.g. the above expression would read

$$i\left[\gamma^{i},\gamma^{j}\right]=2\varepsilon^{ij}{}_{k}\ I_{2}\otimes\sigma^{k}. \tag{3.35}$$

For completeness I add that the other commutators have the form

$$\left[\gamma^{0},\gamma^{i}\right]=2\alpha^{i}. \tag{3.36}$$

In this thesis I am using this convention. As in most of my calculations orthonormal bases are used, lowering and raising indices is done by using the Minkowski metric, so

$$\varepsilon^{ij}{}_{k}=\eta^{il}\ \eta^{jm}\ \varepsilon_{lmk}, \tag{3.37}$$

therefore with a $(+,-,-,-)$ metric

$$\varepsilon_{123}=1,\ \varepsilon^{12}{}_{3}=1,\ \varepsilon^{123}=-1. \tag{3.38}$$

Please note that with my convention the sign of some identities are opposite to the special relativistic case

$$\varepsilon_{ijk}\ \varepsilon^{i}{}_{lm}=-\left(\eta_{jl}\ \eta_{km}-\eta_{jm}\ \eta_{kl}\right)\ \text{and}\ \varepsilon_{ijk}\ \varepsilon^{ijl}=-2\,\delta^{l}_{k}. \tag{3.39}$$

Here $\delta_{\mu}^{\nu} = diag(1, 1, 1, 1)$ is the Kronecker delta, which appears only with one covariant and one contravariant index. $\eta_{\mu\nu} = diag(1, -1, -1, -1) = \eta^{\mu\nu}$ are the components of the Minkowski metric with two covariant or two contravariant indices.

3.7 Invariant volume element

We recall from quantum mechanics that the quantity with a physical meaning is the expectation values of operators, which we expect to be invariant under change of coordinates. When comparisons are being made, it is expectation values that have to be compared.

From integral calculus it follows that under a general coordinate transformation $x' \to x$ the volume element d^4x transforms according to

$$d^4x' = \left\| \frac{\partial x'}{\partial x} \right\| d^4x, \tag{3.40}$$

where $\left\| \frac{\partial x'}{\partial x} \right\|$ is the inverse of the Jacobian of the transformation $x' \to x$.

Applying the transformation rule to the determinant of the metric tensor gives

$$\det g' = \left\| \frac{\partial x}{\partial x'} \right\|^2 \det g, \tag{3.41}$$

so in order to be able to form invariant integrals, we have to introduce a determinental factor

$$\sqrt{-\det g}\ d^4x \tag{3.42}$$

for the invariant volume element [see Weinberg, 1972, Dirac, 1975 or Adler *et al.*, 1965]. This implies that in a general curved space spatial integration has to be carried out using

$$\int d^3x\ \sqrt{-\det g_{ij}}\ \Psi^+ \Phi. \tag{3.43}$$

Therefore we get that the Hamiltonian is Hermitian when the spatial integration is carried out using the correct measure, i.e.,

$$\langle H \rangle = \int d^3x\ \sqrt{-\det g_{ij}}\ \Psi^+ H \Phi = \langle H^+ \rangle. \tag{3.44}$$

However, it is more convenient to absorb this factor into the wavefunction by performing a transformation. The required transformation, according to Audretsch and Schäfer [1978], is:

$$\Psi' = \left(\frac{-\det g_{\mu\nu}}{g_{00}} \right)^{1/4} \Psi = \chi \, \Psi . \tag{3.45}$$

Then the corresponding Hamiltonian

$$H' = \chi \, H \, \chi^{-1} . \tag{3.46}$$

is Hermitian when the integration is carried out in the usual (flat space) sense:

$$\left\langle H' \right\rangle = \int d^3x \; \Psi'^{+} \, H' \, \Phi' = \left\langle H'^{+} \right\rangle . \tag{3.47}$$

3.8 Nonrelativistic limit

We are considering experiments in the laboratory, where we are always dealing with non-relativistic events, therefore we must consider a proper non-relativistic limit for Dirac's theory. It is well known that in the non-relativistic limit, spin ½ particles are described by a two-component wavefunction in the Pauli theory. The usual method of demonstrating that the Dirac equation goes to the Pauli equation in the small momentum limit uses the fact that two of the four components of the Dirac spinor becomes small [see for example Ryder, Ch. 2.6].

One writes the four-spinor in the form of

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{3.48}$$

two two-spinors. Then with the Hamiltonian of the form

$$H = \begin{pmatrix} \mathcal{E} & \mathcal{O} \\ \mathcal{O} & -\mathcal{E} \end{pmatrix} \tag{3.49}$$

where $\mathcal{E}$ and $\mathcal{O}$ (referring to the "even" and "odd" parts) are each 2×2 matrices, the Dirac equation

$$E\,\Psi = H\,\Psi \tag{3.50}$$

can be written as two coupled equations:

$$\begin{aligned} E\,\varphi &= \mathcal{E}\,\varphi + \mathcal{O}\,\chi \\ E\,\chi &= \mathcal{O}\,\varphi - \mathcal{E}\,\chi \end{aligned} \tag{3.51}$$

Using the Dirac representation $\chi << \varphi$ in the non-relativistic limit we only keep terms of mc^2 as the coefficient of χ. Note here, that both E (the total energy) and $\mathcal{E}$ (the even part of the Hamiltonian) usually contains an mc^2 restmass term. Then from the second equation of (3.51) we get

$$\chi = \frac{\mathcal{O}}{2mc^2}\varphi . \tag{3.52}$$

Substituting into (3.52) the equation for φ gives

$$E\varphi = \mathcal{E}\varphi + \mathcal{O}\frac{\mathcal{O}}{2mc^2}\varphi . \tag{3.53}$$

Thus the non-relativistic Hamiltonian gets the form

$$H = \mathcal{E} + \frac{\mathcal{O}^2}{2mc^2} . \tag{3.54}$$

However if one goes beyond the lowest order approximation, the above method encounters several problems [Foldy and Wouthuysen, 1950]: in the presence of external fields the Hamiltonian associated with the large components is found to be not Hermitian and the components of the velocity operator do not commute. A systematic procedure developed by Foldy and Wouthuysen (FW transformation), which is a canonical transformation, decouples the Dirac equation into two two-component equations, and is free from the above mentioned problems. What's more, the transformation has very interesting consequences, for example the transformed position operator corresponds to a particle being spread out over a region of size of the Compton wavelength (rather than a point particle as in the Dirac representation).

The reason why four components are needed to describe the state of particles is, that the Dirac Hamiltonian contains odd operators. Essentially, the FW transformation brings the Hamiltonian into a form in which the odd terms vanish. Considering a particle in an external field, three successive FW transformations have to be applied for the odd terms to vanish in the nonrelativistic limit, i.e. keeping terms of order $\left({}^{kinetic\ energy}\!/_{m}\right)^3$.

Writing the Hamiltonian in the form $H = \beta mc^2 + \mathcal{O} + \mathcal{E}$ the result of applying these transformations [Bjorken and Drell, 1964], is:

$$H = \beta\left(mc^2 + \frac{\mathcal{O}^2}{2mc^2} - \frac{\mathcal{O}^4}{8m^3c^6} \right) + \mathcal{E} - \frac{1}{8m^2c^4}\left[\mathcal{O},\left[\mathcal{O},\mathcal{E}\right]\right] - \frac{i\hbar}{8m^2c^4}\left[\mathcal{O},\dot{\mathcal{O}}\right], \tag{3.55}$$

in which equation* we can recognise the terms of (3.54).

* Please note here, that the notation of $\mathcal{E}$ here does not refer to the full even part of the Hamiltonian, as was the case before in (3.54), but to the difference between the even part and mc^2. This notation is kept for historic reasons.

3.9 Momentum operator in different coordinates

The form of the momentum operator in Cartesian coordinates is the well known expression $\underline{p} = -i\hbar\left(\frac{\partial}{\partial x_C}, \frac{\partial}{\partial y_C}, \frac{\partial}{\partial z_C}\right)$. On the other hand, its form in other coordinates is not this straightforward. Fischbach *et al.* [1981] used the above mentioned isotropic coordinate set (3.6) and substituted $\underline{p} = -i\hbar\left(\frac{\partial}{\partial x_I}, \frac{\partial}{\partial y_I}, \frac{\partial}{\partial z_I}\right)$, which is clearly not identical to the differential operator with respect to the Cartesian coordinates. Therefore his Hamiltonian should be interpreted differently.

Below I shall show how one can determine the form of the momentum operator in an arbitrary set of coordinates. I use the sets x_I, y_I, z_I and r, ϑ, φ as examples. For the sake of simplicity I work only up to first order in Φ in the case of isotropic coordinates.

To achieve the aim of writing the momentum in an arbitrarily chosen coordinate set let us first have a look at the Dirac Hamiltonian in Minkowski space using Cartesian coordinates

$$H_C = \beta m + \underline{\alpha} \cdot \underline{p}_C \tag{3.56}$$

This formula suggests that writing the Dirac Hamiltonian in the chosen coordinates will help to determine the form of the momentum operator. So in the following I am going to write the Dirac equation in Minkowski space, using isotropic and spherical polar coordinates.

3.9.1 Isotropic coordinates

We have from (3.6)

$$x_I = x_C\,(1-\Phi) \Rightarrow x_C = x_I\,(1+\Phi) \Rightarrow dx_C = (1+\Phi)\,dx_I + d\,\Phi\,x_I \quad (3.57)$$

and similarly for the y and z coordinates. Thus the Minkowski line element will become

$$\begin{aligned} ds^2 &= dt^2 - \left(dx_C{}^2 + dy_C{}^2 + dz_C{}^2\right) = \\ &= dt^2 - (1+2\Phi)\left(dx_I{}^2 + dy_I{}^2 + dz_I{}^2\right) - \\ &\quad -2\,(1+\Phi)\left(dx_I\,d\Phi x_I + dy_I\,d\Phi y_I + dz_I\,d\Phi z_I\right) \end{aligned} \quad (3.58)$$

Please note here, that although Φ is the gravitational potential, (3.57) is only a coordinate transformation and this metric still refers to flat space (we shall see that all of the connection coefficients are zero).

Neglecting terms in Φ^2 as usual, and using

$$d\Phi = \frac{\partial\Phi}{\partial x_I}\,dx_I + \frac{\partial\Phi}{\partial y_I}\,dy_I + \frac{\partial\Phi}{\partial z_I}\,dz_I = -\underline{g}\cdot d\underline{x} \quad (3.59)$$

with

$$g_\mu = -\frac{\partial\Phi}{\partial x_I^\mu} \quad (3.60)$$

gives

$$ds^2 = dt^2 - (1+2\Phi)\left(dx_I{}^2 + dy_I{}^2 + dz_I{}^2\right) + 2\left(\underline{g}\cdot d\underline{x}\right)\left(x_I\,dx_I + y_I\,dy_I + z_I\,dz_I\right) \quad (3.61)$$

Note that from the definition of $\underline{g}$ it follows that as $\underline{x}_C$ and $\underline{x}_I$ are equal to zero order, $\underline{g}_C = \underline{g}_I$ to first order and also $\underline{g}_I\left(\underline{x}_I\cdot\dfrac{\partial}{\partial\underline{x}_I}\right) = \underline{g}_C\left(\underline{x}_C\cdot\dfrac{\partial}{\partial\underline{x}_C}\right)$ to first order. I.e. the metric tensor reads

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\left(1+2\Phi-2g_{x_I}x_I\right) & g_{x_I}y_I + g_{y_I}x_I & g_{x_I}z_I + g_{z_I}x_I \\ 0 & g_{x_I}y_I + g_{y_I}x_I & -\left(1+2\Phi-2g_{y_I}y_I\right) & g_{z_I}y_I + g_{y_I}z_I \\ 0 & g_{x_I}z_I + g_{z_I}x_I & g_{z_I}y_I + g_{y_I}z_I & -\left(1+2\Phi-2g_{z_I}z_I\right) \end{pmatrix} \quad (3.62)$$

We now write the Dirac equation, using an orthonormal basis. The orthogonal basis one forms will take the form

$$\Theta^{\hat{0}} = dt \text{ and } \Theta^{\hat{i}} = (1+\Phi)\, dx_I{}^i \;-\; g_I{}^i\,(\underline{x}\cdot d\underline{x}) \tag{3.63}$$

and the dual basis vectors are

$$e_{\hat{0}} = \frac{\partial}{\partial t} \text{ and } e_{\hat{i}} = (1-\Phi)\,\frac{\partial}{\partial x_I{}^i} + g_{Ii}\left(\underline{x}\cdot\frac{\partial}{\partial \underline{x}}\right). \tag{3.64}$$

The orthogonality of the time components is trivial, and for the space components we have

$$\begin{aligned}\left\langle \Theta^{\hat{i}},\, e_{\hat{j}}\right\rangle &= \left\langle (1+\Phi)dx_I^i,\, (1-\Phi)\frac{\partial}{\partial x_I^j}\right\rangle + \left\langle (1+\Phi)dx_I^i,\, g_j\left(x^k\frac{\partial}{\partial x^k}\right)\right\rangle - \\ &\quad -\left\langle g^i\left(x_k\, dx^k\right),\, (1-\Phi)\frac{\partial}{\partial x_I^j}\right\rangle\end{aligned} \tag{3.65}$$

which is to first order in Φ gives

$$\begin{aligned}\left\langle \Theta^{\hat{i}},\, e_{\hat{j}}\right\rangle &= \left\langle dx_I^i,\, \frac{\partial}{\partial x_I^j}\right\rangle + \left\langle dx_I^i,\, g_j\left(x^k\frac{\partial}{\partial x^k}\right)\right\rangle - \left\langle g^i\left(x_k\, dx^k\right),\, \frac{\partial}{\partial x_I^j}\right\rangle = \\ &= \delta_j^i + g_j\, x^k\, \delta_k^i - g^i\, x_k\, \delta_j^k = \\ &= \delta_j^i\end{aligned} \tag{3.66}$$

because g^i is proportional to x^i.

Calculating the commutators of the basis vectors gives

$$\left[e_{\hat{0}},\, e_{\hat{i}}\right] = 0 \tag{3.67}$$

$$\begin{aligned}\left[e_{\hat{i}},\, e_{\hat{j}}\right] &= \left[(1-\Phi)\,\frac{\partial}{\partial x_I^i},\, (1-\Phi)\,\frac{\partial}{\partial x_I^j}\right] + \left[(1-\Phi)\,\frac{\partial}{\partial x_I^i},\, g_j\, x^k\,\frac{\partial}{\partial x^k}\right] + \\ &\quad + \left[g_i\, x^k\,\frac{\partial}{\partial x^k},\, (1-\Phi)\,\frac{\partial}{\partial x_I^j}\right] =\end{aligned} \tag{3.68}$$

$$\begin{aligned}&= (1-\Phi)\left[\frac{\partial}{\partial x_I^i},\, (1-\Phi)\right]\frac{\partial}{\partial x_I^j} + (1-\Phi)\left[(1-\Phi),\, \frac{\partial}{\partial x_I^j}\right]\frac{\partial}{\partial x_I^i} + \\ &\quad + \left[\frac{\partial}{\partial x_I^i},\, g_j\, x^k\,\frac{\partial}{\partial x^k}\right] - \left[\frac{\partial}{\partial x_I^j},\, g_i\, x^k\,\frac{\partial}{\partial x^k}\right] =\end{aligned} \tag{3.69}$$

$$
\begin{aligned}
&= (1-\Phi)\, g_i \frac{\partial}{\partial x_I^j} + (1-\Phi)(-g_j)\frac{\partial}{\partial x_I^i} + x^k \frac{\partial}{\partial x^k}\frac{\partial g_j}{\partial x_I^i} + \\
&\qquad + g_j \frac{\partial}{\partial x_I^i} x^k \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^k}\frac{\partial g_i}{\partial x_I^j} - g_i \frac{\partial}{\partial x_I^j} x^k \frac{\partial}{\partial x^k} =
\end{aligned}
\tag{3.70}
$$

$$
\begin{aligned}
&= g_i \frac{\partial}{\partial x_I^j} + (-g_j)\frac{\partial}{\partial x_I^i} + g_j\, \delta_i^k \frac{\partial}{\partial x^k} - g_i\, \delta_j^k \frac{\partial}{\partial x^k} \,. \\
&= 0
\end{aligned}
\tag{3.71}
$$

We have from (3.21) and (3.22) that

$$\Gamma_{\hat{\kappa}\hat{\lambda}\hat{\mu}} = 0 \,. \tag{3.72}$$

As promised above, the space is flat Minkowski space. The covariant derivatives then reduce simply to the basis vectors,

$$D_{\hat{\mu}} = e_{\hat{\mu}} + \frac{1}{8}\left[\gamma^{\hat{\kappa}},\, \gamma^{\hat{\lambda}}\right]\Gamma_{\hat{\kappa}\hat{\lambda}\hat{\mu}} = e_{\hat{\mu}} \tag{3.73}$$

and so the Dirac equation reads

$$i\hbar \frac{\partial}{\partial t}\Psi = (\beta m - i\hbar\, \underline{\alpha} \cdot \underline{e})\Psi \tag{3.74}$$

$$H_I = \beta m - i\hbar\, \underline{\alpha} \cdot \underline{e} \tag{3.75}$$

We now compare our results in Cartesian and in isotropic coordinates. We use the equality of the expectation values: $\langle H_C \rangle = \langle H_I \rangle$, which gives:

$$\int dx_C\, dy_C\, dz_C\, \Phi^+ H_C \Psi = \int \sqrt{\det g_I}\, dx_I\, dy_I\, dz_I\, \Phi^+ H_I \Psi \tag{3.76}$$

As explained in Chapter 3.7., the next step is to absorb the determinental factor:

$$
\begin{aligned}
\det g_I &= (1 + 2\Phi - 2 g_{x_I} x_I)(1 + 2\Phi - 2 g_{y_I} y_I)(1 + 2\Phi - 2 g_{z_I} z_I) = \\
&= 1 + 6\Phi - 2(g_{x_I} x_I + g_{y_I} y_I + g_{z_I} z_I) = \\
&= 1 + 4\Phi
\end{aligned}
\quad . \tag{3.77}
$$

Then, according to (3.46)

$$
\begin{aligned}
H_I' &= (\det g_I)^{1/4}\, H_I\, (\det g_I)^{-1/4} = \\
&= H_I + (\det g_I)^{1/4}\left[H_I,\ (\det g_I)^{-1/4}\right] = \\
&= H_I - i\hbar\,(\det g_I)^{1/4}\left[(1-\Phi)\alpha^i\,\partial_{I_i} + \alpha^i\, g_i\,(\underline{x}\cdot\underline{\partial}),\ 1-\Phi\right]
\end{aligned}
\tag{3.78}
$$

which gives up to first order in Φ

$$
\begin{aligned}
H_I' &= H_I + i\hbar\left[\alpha^i\partial_{I_i},\Phi\right] = \\
&= H_I + i\hbar\alpha^i(-g_i) = \\
&= H_I - i\hbar\underline{\alpha}\cdot\underline{g} \qquad .
\end{aligned}
\tag{3.79}
$$

So the equality of the expectation values gives for the momentum

$$
\begin{aligned}
\underline{p} &= -i\hbar\,\underline{e} - i\hbar\,\underline{g} = \\
&= -i\hbar\left((1-\Phi)\frac{\partial}{\partial\underline{x}_I} + \underline{g}\left(\underline{x}_I\cdot\frac{\partial}{\partial\underline{x}_I}\right) + \underline{g}\right)
\end{aligned}
\tag{3.80}
$$

which to first order in Φ gives

$$
\underline{p} = -i\hbar(1-\Phi)\frac{\partial}{\partial\underline{x}_I} - i\hbar\,\underline{g}_C\,(\underline{x}_C\cdot\underline{p}) + i\hbar\underline{g}
\tag{3.81}
$$

or

$$
-i\hbar\frac{\partial}{\partial\underline{x}_I} = (1+\Phi)\,\underline{p} - \underline{g}_C\,(\underline{x}_C\cdot\underline{p}) - i\hbar\,\underline{g}
\tag{3.82}
$$

The main result of this section is the above expression for the momentum operator in isotropic coordinates, which is clearly not identical with $\underline{p} = -i\hbar\left(\frac{\partial}{\partial x_I},\ \frac{\partial}{\partial y_I},\ \frac{\partial}{\partial z_I}\right)$.

3.9.2 Spherical polar coordinates

As above we begin by writing the line element in terms of the chosen coordinates. Differentiating (3.3) and substituting into the Cartesian form of the line element gives:

$$\begin{aligned} ds^2 &= dt^2 - \left(dx_C{}^2 + dy_C{}^2 + dz_C{}^2\right) = \\ &= dt^2 - dr^2 - r^2\, d\vartheta^2 - r^2 \sin^2\vartheta\; d\varphi^2 \end{aligned} \quad (3.83)$$

Choosing

$$\Theta^{\hat{0}} = dt,\; \Theta^{\hat{1}} = dr,\; \Theta^{\hat{2}} = r\, d\vartheta,\; \Theta^{\hat{3}} = r \sin\vartheta\, d\varphi \quad (3.84)$$

and

$$e_{\hat{0}} = \frac{\partial}{\partial t},\; e_{\hat{1}} = \frac{\partial}{\partial r},\; e_{\hat{2}} = \frac{1}{r}\frac{\partial}{\partial \vartheta},\; e_{\hat{3}} = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi} \quad (3.85)$$

gives similarly to the previous case

$$\begin{aligned} \Gamma_{\hat{1}\hat{2}\hat{2}} &= -\Gamma_{\hat{2}\hat{1}\hat{2}} = -\Gamma_{\hat{2}\hat{2}\hat{1}} = r \\ \Gamma_{\hat{1}\hat{3}\hat{3}} &= -\Gamma_{\hat{3}\hat{1}\hat{3}} = -\Gamma_{\hat{3}\hat{3}\hat{1}} = r\sin^2\vartheta \\ \Gamma_{\hat{2}\hat{3}\hat{3}} &= -\Gamma_{\hat{3}\hat{2}\hat{3}} = -\Gamma_{\hat{3}\hat{3}\hat{2}} = r^2 \sin\vartheta\cos\vartheta \end{aligned} \quad (3.86)$$

Substituting into (3.1) gives after a rearrangement:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi &= H_{polar}\,\Psi = \\ &= \left\{\beta m - i\hbar\left(\alpha^1\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) + \alpha^2\left(\frac{1}{r}\frac{\partial}{\partial\vartheta}+\frac{\cot\vartheta}{2r}\right) + \alpha^3\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\right)\right\}\Psi \end{aligned} \quad (3.87)$$

This exhibits the Hamiltonian in spherical polar coordinates. It is seen, for example, that the radial component of the momentum operator is $-i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)$. This result can be gained using different methods [see Arthurs, 1970 or Dirac, 1974], too.

3.10 Effect of rotation of the reference frame; Fermi-Walker transport

The space-time metric defines the background geometry in which we are working and to describe a particular physical system we need also to specify a frame. Afterall, it is not the space that might be moving but the frame, relative to the gyroscope. Rotation can be intrinsically defined by using Fermi-Walker (FW) transport. For a non-rotating, non-accelerating vector, the FW derivative is zero. The Dirac equation depends on both the metric and the frame. The discussion of Hehl and Ni [1990] makes no mention of frames, but we shall show below how to cast the problem in such a way that the roles of frames and coordinate systems are kept distinct.

Let us consider a rotating reference frame in Minkowski space-time. Defining the coordinates

$$\begin{aligned} x_C &= x\cos\omega t - y\sin\omega t \\ y_C &= x\sin\omega t + y\cos\omega t \\ z_C &= z \end{aligned} \tag{3.88}$$

the line element becomes:

$$\begin{aligned} ds^2 &= dt^2 - \left(dx_C{}^2 + dy_C{}^2 + dz_C{}^2\right) = \\ &= dt^2 - \omega^2\left(x^2+y^2\right)dt^2 - \left(dx^2+dy^2+dz^2\right) + \\ &\qquad + 2\,\omega\, y\, dx\, dt - 2\,\omega\, x\, dy\, dt \end{aligned} \quad . \tag{3.89}$$

Please note that at this point we have the metric written in rotating coordinates, but it does not mean that anything would be rotating. One reads off the most natural frame with the orthonormal basis and dual vectors:

$$\Theta^{\hat{0}} = dt\,,\quad \Theta^{\hat{1}} = dx - \omega\, y\, dt\,,\quad \Theta^{\hat{2}} = dy + \omega\, x\, dt\,,\quad \Theta^{\hat{3}} = dz \tag{3.90}$$

$$e_{\hat{0}} = \frac{\partial}{\partial t} + \omega\, y\frac{\partial}{\partial x} - \omega\, x\frac{\partial}{\partial y}\,,\quad e_{\hat{1}} = \frac{\partial}{\partial x}\,,\quad e_{\hat{2}} = \frac{\partial}{\partial y}\,,\quad e_{\hat{3}} = \frac{\partial}{\partial z} \tag{3.91}$$

and hopefully it will correspond to a rotating frame, which will be examined below.

It turns out that Dirac's equation is then

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi &= H_{rot}\Psi \\ &= \{\beta m + \underline{\alpha}\cdot\underline{p} - \omega(L_z + S_z)\}\Psi \end{aligned} \tag{3.92}$$

with

$$\underline{p} = -i\hbar\underline{\nabla}, \quad L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \quad S_z = \frac{\hbar}{2}\sigma^3. \tag{3.93}$$

Details of this calculation is given in Appendix 3.12.2.

Although this space is flat, we got additional terms, proportional to the angular velocity corresponding to non-inertial effects caused by the rotation of the frame. The spin-rotation coupling term was predicted by Mashhoon [1988], and a corresponding expression was found by Hehl and Ni [1990] as a special case of **a**=0.

To verify that the frame (3.91) is actually rotating one must calculate the Fermi-Walker derivatives of the basis vectors [Straumann, 1991]. The relevant expression is

$$\nabla^{FW}_{e_0} e_\beta = \nabla_{e_0} e_\beta - (e_0, e_\beta)A_{e_0} + (A_{e_0}, e_\beta)e_0 \tag{3.94}$$

with

$$\nabla_{e_\alpha} e_\beta = e_\mu \Gamma^\mu{}_{\alpha\beta} \tag{3.95}$$

being the covariant derivative of e_β in the direction of e_α; e_0 is tangent to the worldline and $A_{e_0} = \nabla_{e_0} e_0$ is the acceleration; $(\ ,\ )$ denotes scalar product of two covariant vectors. In the case of (3.91) using results (3.157) one gets

$$A_{e_0} = \nabla_0 e_0 = e_\mu \Gamma^\mu{}_{00} = 0 \tag{3.96}$$

$$\nabla^{FW}_0 e_1 = \nabla_0 e_1 = e_\mu \Gamma^\mu{}_{10} = e_2 \Gamma^2{}_{10} = \omega e_2 \tag{3.97}$$

$$\nabla^{FW}_0 e_2 = -\omega e_1 \tag{3.98}$$

which shows that the frame is not accelerating but is indeed rotating in the (1-2) plane.

As mentioned before, Hehl and Ni calculated the Dirac Hamiltonian in a rotating and accelerating frame. In their paper [Hehl and Ni, 1990] a reference to a coordinate transformation is made, which is not the proper way of handling non-inertial frames. In the following I am going to show that the basis they choose does really correspond to a rotating and accelerated frame, using the notion of the FW derivatives. They chose for the basis

$$e_{\hat{0}} = \frac{1}{1+\frac{\underline{a}\underline{x}}{c^2}}\left(e_0 - \left[\left(\frac{\underline{\omega}}{c}\right)\times \underline{x}\right]^k e_k\right) \tag{3.99}$$

$$e_{\hat{i}} = e_i \tag{3.100}$$

and they got for the connection

$$\begin{aligned} \Gamma_{\hat{i}\hat{j}\hat{0}} &= -\frac{\varepsilon_{ijk}\, \omega^k/c}{1+\frac{\underline{a}\cdot\underline{x}}{c^2}}\ ; \\ \Gamma_{\hat{0}\hat{i}\hat{0}} &= -\Gamma_{\hat{i}\hat{0}\hat{0}} = \frac{a_i/c^2}{1+\frac{\underline{a}\cdot\underline{x}}{c^2}}\ . \end{aligned} \tag{3.101}$$

Now we check the motion of this frame

$$\begin{aligned} A_{e_0} &= \nabla_{e_{\hat{0}}} e_{\hat{0}} = \\ &= \Gamma^{\hat{\mu}}{}_{\hat{0}\hat{0}}\, e_{\hat{\mu}} = \\ &= -\frac{a^i/c^2}{1+\frac{\underline{a}\cdot\underline{x}}{c^2}}\, e_i \approx \\ &\approx -a^i/c^2\, e_i\ , \end{aligned} \tag{3.102}$$

which corresponds to a frame being accelerated with acceleration $\boldsymbol{a}$, and

$$\begin{aligned} \nabla_0^{FW} e_{\hat{j}} &= \nabla_0 e_{\hat{j}} - (e_{\hat{o}},\ e_{\hat{j}})A_{e_0} + (A_{e_0},\ e_{\hat{j}})e_{\hat{o}} = \\ &= \Gamma^{\hat{\kappa}}{}_{\hat{j}0}\, e_{\hat{\kappa}} - \frac{a_j/c^2}{1+\frac{\underline{a}\cdot\underline{x}}{c^2}}\, e_{\hat{o}} = \end{aligned} \tag{3.103}$$

$$
\begin{aligned}
&= \frac{\varepsilon^{i}{}_{jk}\, \omega^{k}/c}{1+\dfrac{\underline{a}\cdot\underline{x}}{c^{2}}}\, e_{i} \approx \\
&\approx \varepsilon^{i}{}_{jk}\, \omega^{k}/c \; e_{i} \quad ,
\end{aligned}
\tag{3.104}
$$

which corresponds to a frame rotating with angular velocity ω, as claimed by the authors.

Please note that these relations hold only to first order in the acceleration and the angular velocity of the frame.

3.11 Dependence of the Hamiltonian on the choice of basis

Having the Minkowski metric in spherical polars, as in (3.83) the choice of basis one-forms of (3.84) seemed fairly obvious, and the result we gained was what one would expect from other references [Arthurs, 1970 or Dirac, 1974]. Below, an example is given of what result one obtains if one chooses a less trivial, position dependent basis:

$$\begin{aligned} \theta^{\hat{0}} &= dt \\ \theta^{\hat{1}} &= \sin\vartheta\cos\varphi\, dr + r\cos\vartheta\cos\varphi\, d\vartheta - r\sin\vartheta\sin\varphi\, d\varphi \\ \theta^{\hat{2}} &= \sin\vartheta\sin\varphi\, dr + r\cos\vartheta\sin\varphi\, d\vartheta + r\sin\vartheta\cos\varphi\, d\varphi \\ \theta^{\hat{3}} &= \cos\vartheta\, dr - r\sin\vartheta\, d\vartheta \end{aligned} \tag{3.105}$$

Then the basis vectors will be

$$\begin{aligned} e_{\hat{0}} &= \frac{\partial}{\partial t} \\ e_{\hat{1}} &= \sin\vartheta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\vartheta\cos\varphi\frac{\partial}{\partial\vartheta} - \frac{1}{r\sin\vartheta}\sin\varphi\frac{\partial}{\partial\varphi} \\ e_{\hat{2}} &= \sin\vartheta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\vartheta\sin\varphi\frac{\partial}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\cos\varphi\frac{\partial}{\partial\varphi} \\ e_{\hat{3}} &= \cos\vartheta\frac{\partial}{\partial r} - \frac{1}{r}\sin\vartheta\frac{\partial}{\partial\vartheta} \end{aligned} \tag{3.106}$$

As these expressions correspond to $\theta^{\hat{i}} = dx_C{}^{i}$ and $e_{\hat{i}} = \dfrac{\partial}{\partial x_C{}^{i}}$ it is trivial that they are orthonormal, and all the connection coefficient components are zero. Therefore the covariant differential operator will take the form of the basis vector, as in (3.73). So we get for the Dirac equation

$$\begin{aligned} 0 &= m\Psi + i\hbar\,\underline{\gamma}\,\underline{e}\Psi = \\ &= m\Psi + i\hbar\beta\left\{\left(\frac{\partial}{\partial t}\right) + \alpha^1\left(\sin\vartheta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\vartheta\cos\varphi\frac{\partial}{\partial\vartheta} - \frac{1}{r\sin\vartheta}\sin\varphi\frac{\partial}{\partial\varphi}\right) + \right. \\ &\qquad + \alpha^2\left(\sin\vartheta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\vartheta\sin\varphi\frac{\partial}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\cos\varphi\frac{\partial}{\partial\varphi}\right) + \\ &\qquad \left. + \alpha^3\left(\cos\vartheta\frac{\partial}{\partial r} - \frac{1}{r}\sin\vartheta\frac{\partial}{\partial\vartheta}\right)\right\}\Psi \end{aligned} \tag{3.107}$$

which is different from (3.87). The coefficients of the α matrices correspond to the components of the momentum in the directions of the basis vectors, and these are not the unit polar vectors, so this is the reason why these components are different from the ones given by (3.87).

The results of (3.87) can be derived from (3.107) using the method explained in the book of Sexl and Urbantke [1983]. Introducing

$$\begin{aligned} \tilde{\gamma}^0 &= \gamma^0 \\ \tilde{\gamma}^1 &= \sin\vartheta\cos\varphi\,\gamma^1 + \sin\vartheta\sin\varphi\,\gamma^2 + \cos\vartheta\,\gamma^3 \\ \tilde{\gamma}^2 &= \cos\vartheta\,\cos\varphi\,\gamma^1 + \cos\vartheta\sin\varphi\,\gamma^2 - \sin\vartheta\,\gamma^3 \\ \tilde{\gamma}^3 &= -\sin\varphi\,\gamma^1 + \cos\varphi\,\gamma^2 \end{aligned} \tag{3.108}$$

will give

$$0 = m\Psi + i\hbar\left\{\tilde{\gamma}^0 \frac{\partial\Psi}{\partial t} + \tilde{\gamma}^1 \frac{\partial\Psi}{\partial r} + \tilde{\gamma}^2 \frac{1}{r}\frac{\partial\Psi}{\partial\vartheta} + \tilde{\gamma}^3 \frac{1}{r\sin\vartheta}\frac{\partial\Psi}{\partial\varphi}\right\} \tag{3.109}$$

As $\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\,diag(1,-1,-1,-1)$ and $\{\gamma^\mu, \gamma^\nu\} = 2\,diag(1,-1,-1,-1)$ there must be a transformation such that $\tilde{\gamma}^\mu = S^{-1}\gamma^\mu S$ and this will imply $\tilde{\Psi} = S\Psi$. So (3.107) will give

$$\begin{aligned} 0 = m\,S^{-1}\tilde{\Psi} + i\hbar S^{-1}&\left\{\gamma^0\,S\,S^{-1}\frac{\partial\tilde{\Psi}}{\partial t} + \gamma^1\,S\,S^{-1}\frac{\partial\tilde{\Psi}}{\partial r} + \right. \\ &+\gamma^2\,S\frac{1}{r}\left(S^{-1}\frac{\partial\tilde{\Psi}}{\partial\vartheta} + \frac{\partial S^{-1}}{\partial\vartheta}\tilde{\Psi}\right) + \\ &\left.+\gamma^3\,S\frac{1}{r\sin\vartheta}\left(S^{-1}\frac{\partial\tilde{\Psi}}{\partial\varphi} + \frac{\partial S^{-1}}{\partial\varphi}\tilde{\Psi}\right)\right\} \end{aligned} \tag{3.110}$$

Multiplying by S gives

$$\begin{aligned} 0 = m\,\tilde{\Psi} + i\hbar\beta&\left\{\frac{\partial\tilde{\Psi}}{\partial t} + \alpha^1\frac{\partial\tilde{\Psi}}{\partial r} + \alpha^2\frac{1}{r}\frac{\partial\tilde{\Psi}}{\partial\vartheta} + \alpha^3\frac{1}{r\sin\vartheta}\frac{\partial\tilde{\Psi}}{\partial\varphi} + \right. \\ &\left.+\left(\alpha^2\frac{1}{r}\mathrm{S}\frac{\partial S^{-1}}{\partial\vartheta} + \alpha^3\frac{1}{r\sin\vartheta}S\frac{\partial S^{-1}}{\partial\varphi}\right)\tilde{\Psi}\right\} \end{aligned} \tag{3.111}$$

and now it is only to prove that

$$\alpha^1 = \alpha^2 \, S \, \frac{\partial S^{-1}}{\partial \vartheta} \tag{3.112}$$

and

$$\alpha^2 \frac{\cot \vartheta}{2} = \alpha^3 \frac{1}{\sin \vartheta} S \frac{\partial S^{-1}}{\partial \varphi} \tag{3.113}$$

are satisfied.

3.12 Appendices

3.12.1 Appendix: Examples of calculating the connection coefficients

Below examples are given of the calculation of the connection coefficients using different methods described in Chapter 3.5. The metric

$$ds^2 = dr^2 + r^2\, d\vartheta^2 + r^2 \sin^2\vartheta\; d\varphi^2 . \tag{3.114}$$

is used, describing E^3 in spherical polars.

The choice of a coordinate basis corresponds to

$$\Theta^1 = dr,\ \Theta^2 = d\vartheta,\ \Theta^3 = d\varphi \tag{3.115}$$

and

$$e_1 = \frac{\partial}{\partial r},\ e_2 = \frac{\partial}{\partial \vartheta},\ e_3 = \frac{\partial}{\partial \varphi} \tag{3.116}$$

with the metric tensor components

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\vartheta \end{pmatrix} \tag{3.117}$$

and

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\vartheta} \end{pmatrix}. \tag{3.118}$$

While an orthonormal basis would be

$$\Theta^{\hat{1}} = dr,\ \Theta^{\hat{2}} = r\, d\vartheta,\ \Theta^{\hat{3}} = r \sin\vartheta\; d\varphi \tag{3.119}$$

and

$$e_{\hat{1}} = \frac{\partial}{\partial r},\ e_{\hat{2}} = \frac{1}{r}\frac{\partial}{\partial \vartheta},\ e_{\hat{3}} = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi} \tag{3.120}$$

with the metric tensor components

$$g_{\hat{i}\hat{j}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.121}$$

and

$$g^{\hat{i}\hat{j}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.122}$$

A The traditional method using equation (3.21)

A.1 In coordinate basis

The non-zero derivatives of the metric components (3.117) are

$$\begin{aligned} g_{22,1} &= 2r \\ g_{33,1} &= 2r\sin^2\vartheta \\ g_{33,2} &= 2r\sin\vartheta\, r\cos\vartheta . \end{aligned} \tag{3.123}$$

Substituting these values into (3.21), and noting that in coordinate basis the structure constants C_{ijk} vanish, gives for the non-zero connection coefficients:

$$\begin{aligned} &\Gamma_{122} = -r \ , \ \Gamma_{212} = r \ , \ \Gamma_{221} = r \ , \\ &\Gamma_{133} = -r\sin^2\vartheta \ , \ \Gamma_{313} = r\sin^2\vartheta \ , \ \Gamma_{331} = r\sin^2\vartheta \ , \\ &\Gamma_{233} = -r\sin\vartheta\cos\vartheta \ , \ \Gamma_{323} = r\sin\vartheta\cos\vartheta \ , \ \Gamma_{332} = r\sin\vartheta\cos\vartheta . \end{aligned} \tag{3.124}$$

A.2 In orthonormal basis

In this case one has to find the non-vanishing commutators of the basis vectors. Using (3.120) these are

$$\left[e_{\hat{1}},\, e_{\hat{2}}\right] = -\frac{1}{r^2}\frac{\partial}{\partial\vartheta} \ , \tag{3.125}$$

$$\left[e_{\hat{1}},\, e_{\hat{3}}\right] = -\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\varphi} \tag{3.126}$$

and

$$\left[e_{\hat{2}},\, e_{\hat{3}}\right] = -\frac{\cos\vartheta}{r^2 \sin^2\vartheta}\frac{\partial}{\partial\varphi}. \tag{3.127}$$

The non-zero components of the structure constant C_{ijk} then turn out to be, using (3.127)

$$C_{\hat{1}\hat{2}}{}^{\hat{2}} = -C_{\hat{2}\hat{1}}{}^{\hat{2}} = -\frac{1}{r}, \tag{3.128}$$

$$C_{\hat{1}\hat{3}}{}^{\hat{3}} = -C_{\hat{3}\hat{1}}{}^{\hat{3}} = -\frac{1}{r}, \tag{3.129}$$

$$C_{\hat{2}\hat{3}}{}^{\hat{3}} = -C_{\hat{3}\hat{2}}{}^{\hat{3}} = -\frac{\cot\vartheta}{r}. \tag{3.130}$$

Lowering the third indices with the metric (3.121) makes no change to the values. Then with (3.21), on noting that in case of an orthonormal basis the derivatives of the metric tensor components vanish one gets

$$\begin{aligned}
\Gamma_{\hat{1}\hat{2}\hat{2}} &= -\Gamma_{\hat{2}\hat{1}\hat{2}} = -\frac{1}{r}, \\
\Gamma_{\hat{1}\hat{3}\hat{3}} &= -\Gamma_{\hat{3}\hat{1}\hat{3}} = -\frac{1}{r}, \\
\Gamma_{\hat{2}\hat{3}\hat{3}} &= -\Gamma_{\hat{3}\hat{2}\hat{3}} = -\frac{\cot\vartheta}{r}.
\end{aligned} \tag{3.131}$$

Please note here, that the connection coefficients found in (3.124) and (3.131) are different. This is because the connection is not a tensor, so it is not invariant, but depends on the choice of the basis, too. Also one can observe, that when working in an orthonormal basis, the connection coefficients are antisymmetric in the first two indices, and when working in a coordinate basis, the connection coefficients are symmetric in the last two indices.

B With differential geometry

B.1 In coordinate basis

Using the metric compatibility condition (3.26) gives

$$
\begin{aligned}
dg_{11} &= 2\omega_{11} = 0 \ , \\
dg_{22} &= 2\omega_{22} = 2r\,dr \ , \\
dg_{33} &= 2\omega_{33} = 2r\sin^2\vartheta\,dr + 2r^2\sin\vartheta\cos\vartheta\,d\vartheta .
\end{aligned}
\tag{3.132}
$$

and

$$
dg_{ij} = \omega_{ij} + \omega_{ji} = 0 \ \text{for} \ \ i \neq j . \tag{3.133}
$$

Raising indices is done using the metric tensor, so

$$
\begin{aligned}
\omega^1{}_1 &= 0 \\
\omega^2{}_2 &= \frac{1}{r^2}\,r\,dr = \frac{1}{r}\,dr \\
\omega^3{}_3 &= \frac{1}{r^2\sin^2\vartheta}\left(r\sin^2\vartheta\,dr + r^2\sin\vartheta\cos\vartheta\,d\vartheta\right) = \frac{1}{r}\,dr + \cot\vartheta\,d\vartheta .
\end{aligned}
\tag{3.134}
$$

Now the Cartan-Maurer equations (3.25) give, as $d\theta^\mu = 0$ from the Poincaré lemma,

$$
\begin{aligned}
&\omega^1{}_2 \wedge d\vartheta + \omega^1{}_3 \wedge d\varphi = 0 \ , \\
&\omega^2{}_1 \wedge dr + \frac{1}{r}\,dr \wedge d\vartheta + \omega^2{}_3 \wedge d\varphi = 0 \ , \\
&\omega^3{}_1 \wedge dr + \omega^3{}_2 \wedge d\vartheta + \frac{1}{r}\,dr \wedge d\varphi + \cot\vartheta\,d\vartheta \wedge d\varphi = 0 .
\end{aligned}
\tag{3.135}
$$

From the 2nd equation of (3.135) one can deduce that

$$
\omega^2{}_1 = \frac{1}{r}\,d\vartheta \ , \tag{3.136}
$$

and from the 3rd equation

$$
\omega^3{}_1 = \frac{1}{r}\,d\varphi \ \ \text{and} \ \ \omega^3{}_2 = \cot\vartheta\,d\varphi . \tag{3.137}
$$

The rest of the non-zero connection 1-forms can be obtained from these via lowering and raising indices and interchanging the indices using (3.133). Summarising the results for the connection 1-forms:

$$\begin{aligned} &\omega^1{}_1 = 0 \ , \ \omega^1{}_2 = -r\, d\vartheta \ , \ \omega^1{}_3 = -r \sin^2 \vartheta \, d\varphi \ , \\ &\omega^2{}_1 = \frac{1}{r} d\vartheta \ , \ \omega^2{}_2 = \frac{1}{r} dr \ , \ \omega^2{}_3 = -\sin\vartheta \cos\vartheta \, d\varphi \ , \\ &\omega^3{}_1 = \frac{1}{r} d\varphi \ , \ \omega^3{}_2 = \cot\vartheta \, d\varphi \ , \ \omega^3{}_3 = \frac{1}{r} dr + \cot\vartheta \, d\vartheta . \end{aligned} \tag{3.138}$$

Finally, using equation (3.27) gives for the non-zero connection coefficients

$$\begin{aligned} &\Gamma^1{}_{22} = -r \ , \ \Gamma^2{}_{12} = \frac{1}{r} \ , \ \Gamma^2{}_{21} = \frac{1}{r} \ , \\ &\Gamma^1{}_{33} = -r \sin^2 \vartheta \ , \ \Gamma^3{}_{13} = \frac{1}{r}, \ \Gamma^3{}_{31} = \frac{1}{r} \ , \\ &\Gamma^2{}_{33} = -\sin\vartheta \cos\vartheta \ , \ \Gamma^3{}_{23} = \cot\vartheta \ , \ \Gamma^3{}_{32} = \cot\vartheta \ , \end{aligned} \tag{3.139}$$

which is equivalent to (3.124) on lowering the first indices.

B.2 Using orthonormal basis

In this case all the derivatives of the metric tensor components vanish, so (3.26) gives

$$\omega_{\hat 1 \hat 1} = \omega_{\hat 2 \hat 2} = \omega_{\hat 3 \hat 3} = 0 \tag{3.140}$$

and

$$\omega_{\hat i \hat j} + \omega_{\hat j \hat i} = 0 \ \text{for} \ i \neq j . \tag{3.141}$$

Now the Cartan-Maurer equations (3.25) give,

$$\begin{aligned} &\omega^{\hat 1}{}_{\hat 2} \wedge r\, d\vartheta + \omega^{\hat 1}{}_{\hat 3} \wedge r \sin\vartheta \, d\varphi = 0 \ , \\ &dr \wedge d\vartheta + \omega^{\hat 2}{}_{\hat 1} \wedge dr + \omega^{\hat 2}{}_{\hat 3} \wedge r \sin\vartheta \, d\varphi = 0 \ , \\ &\sin\vartheta \, dr \wedge d\varphi + r \cos\vartheta \, d\vartheta \wedge d\varphi + \omega^{\hat 3}{}_{\hat 1} \wedge dr + \omega^{\hat 3}{}_{\hat 2} \wedge r\, d\vartheta = 0 . \end{aligned} \tag{3.142}$$

From the 2nd equation of (3.142) one can deduce that

$$\omega^{\hat 2}{}_{\hat 1} = d\vartheta \ , \tag{3.143}$$

and from the 3rd equation

$$\omega^{\hat{3}}{}_{\hat{1}} = \sin\vartheta\, d\varphi \quad \text{and} \quad \omega^{\hat{3}}{}_{\hat{2}} = \cos\vartheta\, d\varphi\,. \tag{3.144}$$

The rest of the non-zero connection 1-forms can be obtained from these via lowering and raising indices and interchanging the indices using (3.141). Summarising the results for the connection 1-forms:

$$\begin{aligned} &\omega^{\hat{1}}{}_{\hat{1}} = \omega^{\hat{2}}{}_{\hat{2}} = \omega^{\hat{3}}{}_{\hat{3}} = 0\ , \\ &\omega^{\hat{1}}{}_{\hat{2}} = -\omega^{\hat{2}}{}_{\hat{1}} = -d\vartheta\ , \\ &\omega^{\hat{1}}{}_{\hat{3}} = -\omega^{\hat{3}}{}_{\hat{1}} = -\sin\vartheta\, d\varphi\ , \\ &\omega^{\hat{2}}{}_{\hat{3}} = -\omega^{\hat{3}}{}_{\hat{2}} = -\cos\vartheta\, d\varphi\,. \end{aligned} \tag{3.145}$$

Finally, using equation (3.27) gives for the non-zero connection coefficients

$$\begin{aligned} &\Gamma^{\hat{1}}{}_{\hat{2}\hat{2}} = -\Gamma^{\hat{2}}{}_{\hat{1}\hat{2}} = -\frac{1}{r}\ , \\ &\Gamma^{\hat{1}}{}_{\hat{3}\hat{3}} = -\Gamma^{\hat{3}}{}_{\hat{1}\hat{3}} = -\frac{1}{r}\ , \\ &\Gamma^{\hat{2}}{}_{\hat{3}\hat{3}} = -\Gamma^{\hat{3}}{}_{\hat{2}\hat{3}} = -\frac{\cot\vartheta}{r}\ , \end{aligned} \tag{3.146}$$

which is equivalent to (3.131) as lowering indices in orthonormal basis makes no change to the value.

C Tetrad components

The tetrad transforming between bases (3.115) and (3.119) is

$$h^{\hat{1}}{}_{1} = 1,\ h^{\hat{2}}{}_{2} = r\ ,\ h^{\hat{3}}{}_{3} = r\sin\vartheta \tag{3.147}$$

and

$$h_{\hat{1}}{}^{1} = 1,\ h_{\hat{2}}{}^{2} = \frac{1}{r}\ ,\ h_{\hat{3}}{}^{3} = \frac{1}{r\sin\vartheta}\ . \tag{3.148}$$

The non-zero derivatives of the tetrad-components (3.147) are

$$\partial_1 h^{\hat{2}}{}_{2} = 1\ ,\ \partial_1 h^{\hat{3}}{}_{3} = \sin\vartheta \ \text{ and } \ \partial_2 h^{\hat{3}}{}_{3} = \mathrm{r}\cos\vartheta\ . \tag{3.149}$$

Using (3.17) gives for the structure constants

$$-C_{\hat{2}\hat{1}}{}^{\hat{2}} = C_{\hat{1}\hat{2}}{}^{\hat{2}} = h_{\hat{1}}{}^{1} h_{\hat{2}}{}^{2} \left(\partial_2 h^{\hat{2}}{}_1 - \partial_1 h^{\hat{2}}{}_2\right) = -1\frac{1}{r}1 = -\frac{1}{r}, \tag{3.150}$$

$$-C_{\hat{3}\hat{1}}{}^{\hat{3}} = C_{\hat{1}\hat{3}}{}^{\hat{3}} = h_{\hat{1}}{}^{1} h_{\hat{3}}{}^{3} \left(\partial_3 h^{\hat{3}}{}_1 - \partial_1 h^{\hat{3}}{}_3\right) = -1\frac{1}{r\sin\vartheta}\sin\vartheta = -\frac{1}{r}, \tag{3.151}$$

$$-C_{\hat{3}\hat{2}}{}^{\hat{3}} = C_{\hat{2}\hat{3}}{}^{\hat{3}} = h_{\hat{2}}{}^{2} h_{\hat{3}}{}^{3} \left(\partial_3 h^{\hat{3}}{}_2 - \partial_2 h^{\hat{3}}{}_3\right) = -\frac{1}{r}\frac{1}{r\sin\vartheta} r\cos\vartheta = -\frac{\cot\vartheta}{r}, \tag{3.152}$$

all the rest are zero. One can see that this is equivalent to (3.128)-(3.130).

3.12.2 Appendix: Details of the derivation of the Dirac Hamiltonian in Section 3.10

Again, in this calculation orthonormal basis will be used, with the traditional method described in Appendix 3.12.1.A.

The non-vanishing commutators of the basis vectors (3.91) are

$$\left[e_{\hat{0}},\, e_{\hat{1}}\right] = -\left[\omega x \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] = \omega \frac{\partial}{\partial y} \tag{3.153}$$

and

$$\left[e_{\hat{0}},\, e_{\hat{2}}\right] = -\omega \frac{\partial}{\partial x}. \tag{3.154}$$

The non-zero components of the structure constant are

$$C_{\hat{0}\hat{1}}{}^{\hat{2}} = -C_{\hat{1}\hat{0}}{}^{\hat{2}} = \omega \tag{3.155}$$

and

$$C_{\hat{0}\hat{2}}{}^{\hat{1}} = -C_{\hat{2}\hat{0}}{}^{\hat{1}} = -\omega. \tag{3.156}$$

Lowering the third indices with the metric $g_{\mu\nu} = diag(+,-,-,-)$ introduces a minus sign to the values of the above. Then with (3.21) one gets

$$\begin{aligned} \Gamma_{\hat{0}\hat{1}\hat{2}} &= -\Gamma_{\hat{1}\hat{0}\hat{2}} = 0\ , \\ \Gamma_{\hat{0}\hat{2}\hat{1}} &= -\Gamma_{\hat{2}\hat{1}\hat{0}} = 0\ , \\ \Gamma_{\hat{1}\hat{2}\hat{0}} &= -\Gamma_{\hat{2}\hat{1}\hat{0}} = \omega . \end{aligned} \tag{3.157}$$

The covariant derivatives then are

$$
\begin{aligned}
D_{\hat{0}} &= e_{\hat{0}} + \frac{1}{4}\left[\gamma^{\hat{1}}, \gamma^{\hat{2}}\right] \Gamma_{\hat{1}\hat{2}\hat{0}} = \\
&= \frac{\partial}{\partial t} + \omega\, y \frac{\partial}{\partial x} - \omega\, x \frac{\partial}{\partial y} - \frac{1}{2} i \sigma^{3} \omega
\end{aligned}
\tag{3.158}
$$

and

$$
D_{\hat{i}} = e_{\hat{i}} = \frac{\partial}{\partial x^{i}} .
\tag{3.159}
$$

So the Dirac equation reads

$$
m\Psi = \left\{ i\hbar\, \beta \frac{\partial}{\partial t} + \beta \frac{\hbar}{2} \sigma^{3} \omega - \beta\, \underline{\alpha}\, \underline{p} + i\hbar\, \beta\omega \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \right\} \Psi ,
\tag{3.160}
$$

which gives for the Hamiltonian

$$
H = \beta m - \frac{\hbar}{2} \sigma^{3} \omega + \underline{\alpha}\, \underline{p} - i\hbar\omega \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right),
\tag{3.161}
$$

as in (3.92).

4 Effect of a stationary gravitational source on Dirac particles*

In this chapter a study is performed of the effect of a stationary gravitational source on spin-½ particles. In Section 4.1 the proper, general relativistic treatment is followed, i.e. the particle is considered as being in a Schwarzschild field. In Section 4.2 the effect of an accelerated frame in Minkowski space is investigated. After a review of the Equivalence Principle in Section 4.3, the Hamiltonians found are being compared in Section 4.4 providing a test of the Equivalence Principle.

When the effect of gravity is mentioned, it is common to introduce the notation:

$$\Phi = \frac{m}{r} = \frac{GM_{\oplus}}{rc^{2}} \tag{4.1}$$

and

$$g_{i} = -\frac{\partial \Phi}{\partial x^{i}}, \tag{4.2}$$

* A condensed version of the material in this chapter has been published in Varjú and Ryder, 1998

where m is the Schwarzschild radius and r is the radius: $r = \left((x^1)^2 + (x^2)^2 + (x^3)^2\right)^{1/2}$ or $r^2 = -x_i\, x^i$.

Nonrelativistically Φ corresponds to the gravitational potential, and g to the gravitational acceleration. Then it follows that

$$g_i = -\frac{\partial}{\partial x^i}\frac{m}{r} = \frac{m}{r^2}\frac{\partial r}{\partial x^i} = -\frac{m}{r^3}x_i \tag{4.3}$$

and

$$g_i\, x^i = \Phi. \tag{4.4}$$

We can also see, that

$$\frac{\partial \Phi}{\partial x^i} = \frac{\partial}{\partial x^i}\left(g_j\, x^j\right) = -g_i \tag{4.5}$$

with

$$\frac{\partial g_i}{\partial x^j} = -\frac{m}{r^3}\eta_{ij} + 3\frac{x_i\, g_j}{r^2}. \tag{4.6}$$

Thus the derivative of the gravitational acceleration can not be neglected, the gravitational field in a Schwarzschild space is non-uniform. This accounts for the tidal effect.

4.1 Schwarzschild field

The Schwarzschild solution is most commonly referred to in spherical polar coordinates as in Equation (2.3)

$$ds^2 = \left(1 - 2\frac{GM_\oplus}{rc^2}\right)c^2 dt^2 - \frac{1}{1 - 2\dfrac{GM_\oplus}{rc^2}}dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta\ d\varphi^2\right), \tag{4.7}$$

i.e.

$$g_{\mu\nu} = diag\left(1-2\frac{GM_{\oplus}}{rc^2},\ \frac{1}{1-2\frac{GM_{\oplus}}{rc^2}},\ r^2,\ r^2\sin^2\vartheta \right). \tag{4.8}$$

However in certain situations it is more convenient to work in Cartesian coordinates. To obtain this form, a coordinate transformation

$$x^{\mu} = (ct,\ r,\ \vartheta,\ \varphi) \to x'^{\mu} = (ct,\ x^1,\ x^2,\ x^3) \tag{4.9}$$

has to be performed with

$$\begin{aligned} r &= \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \\ \vartheta &= \cos^{-1}\left(\frac{x^3}{r}\right) \\ \varphi &= \tan^{-1}\left(\frac{x^2}{x^1}\right). \end{aligned} \tag{4.10}$$

The metric tensor transforms according to $g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}$, hence to first order in Φ the metric becomes:

$$\begin{aligned} ds^2 = {} & (1-2\Phi)c^2dt^2 - \\ & -\left[\left(1+2\frac{g_1x^1}{c^2}\right)(dx^1)^2 + \left(1+2\frac{g_2x^2}{c^2}\right)(dx^2)^2 + \left(1+2\frac{g_3x^3}{c^2}\right)(dx^3)^2\right] - \\ & -\frac{2}{c^2}\left[(g_1x^2+g_2x^1)dx^1dx^2 + (g_2x^3+g_3x^2)dx^2dx^3 + (g_3x^1+g_1x^3)dx^3dx^1\right] \end{aligned} \tag{4.11}$$

with g_i defined in (4.3).

The space defined by the metric (4.11) will be used to calculate the Dirac Hamiltonian (2.18) following the same method as was used in Appendix 3.12.1.A.2. One finds the orthogonal basis one-forms $\Theta^{\hat{\mu}}$ to be

$$\Theta^{\hat{0}} = (1-\Phi)cdt \quad \text{and} \quad \Theta^{\hat{i}} = dx^i + \frac{g^i}{c^2}(\underline{x}\cdot d\underline{x}), \tag{4.12}$$

and the dual tetrad vectors are given by

$$e_{\hat{0}} = (1+\Phi)\frac{1}{c}\frac{\partial}{\partial t} \quad \text{and} \quad e_{\hat{i}} = \frac{\partial}{\partial x^i} - \frac{g_i}{c^2}\left(\underline{x}\cdot\frac{\partial}{\partial \underline{x}}\right). \tag{4.13}$$

For details see Appendix 4.5.1.

The non-zero structure-constants (3.22) turn out to be

$$\begin{aligned} C_{\hat{0}\hat{i}\hat{0}} &= -C_{\hat{i}\hat{0}\hat{0}} = \frac{g_i}{c^2}, \\ C_{\hat{i}\hat{j}\hat{k}} &= \frac{1}{c^2}\left(g_{\hat{j}\hat{k}}\, g_i - g_{\hat{i}\hat{k}}\, g_j\right), \end{aligned} \tag{4.14}$$

and the connection coefficients, defined by (3.24) are

$$\begin{aligned} \Gamma_{\hat{0}\hat{i}\hat{0}} &= -\Gamma_{\hat{i}\hat{0}\hat{0}} = \frac{g_i}{c^2}, \\ \Gamma_{\hat{i}\hat{j}\hat{k}} &= \frac{1}{c^2}\left(g_{\hat{j}\hat{k}}\, g_i - g_{\hat{i}\hat{k}}\, g_j\right). \end{aligned} \tag{4.15}$$

For details of the calculation, see Appendix 4.5.2.

It is then straightforward to write out the Dirac equation and find the Hamiltonian

$$H = (1-\Phi)\beta mc^2 + (1-\Phi)c(\underline{\alpha}\cdot\underline{p}) + \frac{i\hbar}{2c}(\underline{\alpha}\cdot\underline{g}) - \frac{1}{c}(\underline{\alpha}\cdot\underline{g})(\underline{x}\cdot\underline{p}). \tag{4.16}$$

Details of the calculation can be found in Appendix 4.5.3.

The expectation value of this Hamiltonian is

$$\langle H\rangle = \int dx^1\, dx^2\, dx^3 \sqrt{-\det g}\;\Psi^+ H \Psi, \tag{4.17}$$

where *(det g)* refers to the spatial part of the metric (4.11) and $dx^1 dx^2 dx^3$ is the Cartesian volume element. Absorbing, as explained in Chapter 3.7, the determinental factor into the wavefunction, we define $\Psi' = \left(1+\frac{\Phi}{2}\right)\Psi$ such that (4.17) simplifies to

$$\langle H\rangle = \int d^3r\, \Psi'^+ H'\Psi', \tag{4.18}$$

with:

$$
\begin{aligned}
H' &= \left(1+\frac{\Phi}{2}\right)H\left(1-\frac{\Phi}{2}\right) = \\
&= (1-\Phi)\beta mc^2 + (1-\Phi)c\left(\underline{\alpha}\cdot\underline{p}\right) - \frac{1}{c}\left(\underline{\alpha}\cdot\underline{g}\right)\left(\underline{x}\cdot\underline{p}\right).
\end{aligned}
\tag{4.19}
$$

This is the Hamiltonian in the usual sense. In the following I shall drop the prime, denoting this transformed Hamiltonian by H.

The proper non-relativistic limit will be obtained by applying three successive Foldy-Wouthuysen transformations as described in Section 3.8. Equation (4.19) then gives:

$$
\begin{aligned}
\mathcal{O} &= c\left(\underline{\alpha}\cdot\underline{p}\right) - \frac{1}{c}\left(\underline{g}\cdot\underline{x}\right)\left(\underline{\alpha}\cdot\underline{p}\right) - \frac{1}{c}\left(\underline{g}\cdot\underline{\alpha}\right)\left(\underline{x}\cdot\underline{p}\right) \\
\mathcal{E} &= -\beta m\left(\underline{g}\cdot\underline{x}\right).
\end{aligned}
\tag{4.20}
$$

Hence

$$
\begin{aligned}
\mathcal{O}^2 &= \alpha^i\alpha^j p_i p_j - \alpha^i p_i\left(g_j x^j \alpha^k p_k + \alpha^j g_j p_k x^k\right) - \\
&\qquad - \left(g_j x^j \alpha^k p_k + \alpha^j g_j p_k x^k\right)\alpha^i p_i = \\
&= p^2 - 2\underline{p}\cdot\left(\underline{g}\cdot\underline{x}\right)\underline{p} - 2\left(\underline{p}\cdot\underline{g}\right)\left(\underline{x}\cdot\underline{p}\right) + 2\hbar\underline{\sigma}\cdot\left(\underline{g}\times\underline{p}\right)
\end{aligned}
\tag{4.21}
$$

and

$$
\begin{aligned}
[\mathcal{O}, \mathcal{E}] &= -\left[\left(\underline{\alpha}\cdot\underline{p}\right), \beta m\left(\underline{g}\cdot\underline{x}\right)\right] = \\
&= 2\beta m\Phi\left(\underline{\alpha}\cdot\underline{p}\right) + i\hbar\beta m\left(\underline{\alpha}\cdot\underline{g}\right)
\end{aligned}
\tag{4.22}
$$

and

$$
\begin{aligned}
[\mathcal{O}, [\mathcal{O}, \mathcal{E}]] &= \left[\underline{\alpha}\cdot\underline{p},\ 2\beta m\Phi\left(\underline{\alpha}\cdot\underline{p}\right) + i\hbar\beta m\left(\underline{\alpha}\cdot\underline{g}\right)\right] = \\
&= -4\beta m\,\underline{p}\cdot\Phi\,\underline{p} + 4\hbar\beta m\underline{\sigma}\cdot\left(\underline{g}\times\underline{p}\right).
\end{aligned}
\tag{4.23}
$$

When the formulae

$$
[\Phi, p_i] = -i\hbar g_i\ ,
\tag{4.24}
$$

$$
\left[x^k, p_i\right] = i\hbar\delta_i^k\ ,
\tag{4.25}
$$

$$
\left[g_j, p_i\right] = -\frac{m}{r^3}g_{ji} + 3\frac{x_j g_i}{r^2}\ ,
\tag{4.26}
$$

$$
\alpha^i\alpha^j = -g^{ij} + i\varepsilon^{ij}{}_k\,\sigma^k\ ,
\tag{4.27}
$$

and

$$[\alpha^i, \beta] = 2\alpha^i\beta = -2\beta\alpha^i \tag{4.28}$$

are used.

The term containing $\mathcal{O}^4$ is of the order $\frac{1}{m^3}$ which is not of interest. Also, the term $[\mathcal{O}, \dot{\mathcal{O}}]$ vanishes, as the Hamiltonian is independent of time. Evaluating H to the desired accuracy gives:

$$H = \beta mc^2 - \beta m(\underline{g}\cdot\underline{x}) + \frac{\beta}{2m}p^2 - \frac{\beta}{2mc^2}\underline{p}\cdot(\underline{g}\cdot\underline{x})\underline{p} + \frac{\hbar\beta}{2mc^2}\underline{\sigma}\cdot(\underline{g}\times\underline{p}) - \frac{\beta}{mc^2}(\underline{p}\cdot\underline{g})(\underline{x}\cdot\underline{p}) \tag{4.29}$$

This is the main result of this Chapter, and expresses the Hamiltonian for a Dirac particle in a Schwarzschild field.* The table below shows the interpretation of these terms, and the approximate orders of magnitude in case of a thermal neutron (de Broglie wavelength of 2Å and kinetic energy of 20meV).

term	interpretation	order of magnitude
βmc^2	rest-mass energy	10^9 eV
$-\beta\ m(\underline{g}\cdot\underline{x})$	redshift of rest-mass energy (verified by COW)	1 eV
$\frac{\beta}{2m}p^2$	kinetic energy	10^{-2} eV
$-\frac{\beta}{2mc^2}\underline{p}\cdot(\underline{g}\cdot\underline{x})\underline{p}$	redshift of kinetic energy	10^{-11} eV
$+\frac{\hbar\beta}{2mc^2}\underline{\sigma}\cdot(\underline{g}\times\underline{p})$	spin-orbit coupling	10^{-29} eV
$-\frac{\beta}{mc^2}(\underline{p}\cdot\underline{g})(\underline{x}\cdot\underline{p})$	square of radial component of momentum	10^{-11} eV

Table 1: Meaning and approximate order of the terms of the Hamiltonian.

* The result is quoted in this form for easy comparison with the Hamiltonian in an accelerated frame [Hehl and Ni, 1990] in the next section. But for proper handling we note that here $\underline{p}\cdot(\underline{g}\cdot\underline{x})\underline{p} = p_1(\underline{g}\cdot\underline{x})p_1 + p_2(\underline{g}\cdot\underline{x})p_2 + p_3(\underline{g}\cdot\underline{x})p_3$. All the other scalar products are e.g. $(\underline{g}\cdot\underline{x}) = g_i x^i$, product of a covariant and a contravariant component, and summed over the spatial indices.

For the sake of comparison with other works I should like at this stage, however, to make the following observation. We may, instead of the coordinates (4.10), introduce isotropic coordinates (x_I, y_I, z_I) with the definitions

$$x_I = x^1\left(1-\Phi'/2\right)^2 \ , \quad y_I = x^2\left(1-\Phi'/2\right)^2 \ , \quad z_I = x^3\left(1-\Phi'/2\right)^2 \tag{4.30}$$

and $\Phi' = \dfrac{G M_{\oplus}}{c^2\sqrt{x_I^{\,2} + y_I^{\,2} + z_I^{\,2}}}$. Note that to first order $\Phi = \Phi'$. In terms of these coordinates it may be shown [Møller, 1972] that the Schwarzschild metric (4.7) becomes

$$ds^2 = \left(1+\Phi'/2\right)^4\left(dx_I^{\,2} + dy_I^{\,2} + dz_I^{\,2}\right) - \left(\frac{1-\Phi'/2}{1+\Phi'/2}\right)^2 c^2 dt^2 \tag{4.31}$$

which is exact to all orders in Φ'.

In previous work on the Dirac equation in a Schwarzschild field, Fischbach *et al.* [1981] assumed the form (4.31) for the metric with the momentum operator defined by $\underline{p} = -i\hbar\left(\dfrac{\partial}{\partial x_I}, \dfrac{\partial}{\partial y_I}, \dfrac{\partial}{\partial z_I}\right)$ in contrast with $\underline{p} = -i\hbar\left(\dfrac{\partial}{\partial x^1}, \dfrac{\partial}{\partial x^2}, \dfrac{\partial}{\partial x^3}\right)$. His calculations yielded the Hamiltonian* [Equation 2.37b of Fischbach *et al.*, 1981]

$$H_I = (1-\Phi)\beta mc^2 - i\hbar(1-2\Phi)c\left(\underline{\alpha}\cdot\underline{\partial}_I\right) - \frac{i\hbar}{c}\left(\underline{\alpha}\cdot\underline{g}\right). \tag{4.32}$$

In quantum mechanics momentum is defined by $\underline{p} = -i\hbar\dfrac{\partial}{\partial \underline{x}}$, differentiation with respect to the Cartesian coordinates. Using the results of Chapter 3.9.1 this means that in Fischbach's coordinates the momentum operator becomes

$$\underline{p} = -i\hbar(1-\Phi)\left(\frac{\partial}{\partial x_I}, \frac{\partial}{\partial y_I}, \frac{\partial}{\partial z_I}\right) - i\hbar\underline{g} - i\hbar\underline{g}\left(\underline{x_I}\cdot\frac{\partial}{\partial \underline{x_I}}\right). \tag{4.33}$$

The Hamiltonian (4.19) can be derived from (4.32) using the above form of the momentum operator. Substituting the expression for $-i\hbar\partial_{\underline{l}}$ gives

$$H_l = (1-\Phi)\beta\, mc^2 + (1-2\Phi)c\underline{\alpha}\cdot\left((1+\Phi)\underline{p} - \frac{1}{c^2}\underline{g}\left(\underline{x}\cdot\underline{p}\right) + \frac{i\hbar}{c^2}\underline{g}\right) - \frac{i\hbar}{c}\left(\underline{\alpha}\cdot\underline{g}\right) =$$

$$= (1-\Phi)\beta\, mc^2 + (1-\Phi)c\underline{\alpha}\cdot\underline{p} - \frac{1}{c}\left(\underline{\alpha}\cdot\underline{g}\right)\left(\underline{x}\cdot\underline{p}\right) + \frac{i\hbar}{c}\left(\underline{\alpha}\cdot\underline{g}\right) - \frac{i\hbar}{c}\left(\underline{\alpha}\cdot\underline{g}\right), \tag{4.34}$$

which is equivalent to (4.19).

Performing three successive FW transformations to (4.34) gives [Equation 2.44 of Fischbach *et al.*, 1981]

$$H = \beta\, mc^2\left(1-\Phi\right) + \frac{\beta}{2m}p^2 + \frac{3}{2}\beta\left(-\frac{\Phi}{m}p^2 + \frac{i\hbar}{mc^2}\underline{g}\cdot\underline{p} + \frac{\hbar}{2mc^2}\underline{g}\cdot\left(\underline{\sigma}\times\underline{p}\right)\right) \tag{4.35}$$

As far as the Dirac particle in a Schwarzschild field is concerned, our Hamiltonian (4.29) differs from Fischbach's (4.35), as it stands, in that (4.35) does not feature the last term of (4.29); and the coefficients of the gravitational correction terms are also different. But with redefining the momentum operator, one can see that the two expressions mean the same.

* Please note here, that Fischbach's g is defined with the opposite sign (cf. Equation 2.29 of Fischbach *et al.*, 1981 and (4.2)). Here the sign of the $\underline{\alpha g}$ is changed to opposite for correspondence with the convention applied in this thesis.

4.2 Accelerated frame in Minkowski space

The Equivalence Principle states the equality of the local effects of a gravitational field and a uniformly accelerated frame. When a gravitational field is compared to a uniformly accelerated frame, there is a trivial difference between the two effects, caused by the differences in structure of the two: one has a source, the other does not. For this reason the usual statement of the equivalence principle is restricted to small regions. Locality is a key point here, because the gravitational field being central, i.e. having a source, is never uniform, which results in tidal effects. A comparison of the Schwarzschild field and a uniformly accelerated frame is to be made in Section 4.4, where all tidal terms are to be neglected. Neglecting all tidal terms may be a case of throwing out the baby with the bath water, as we do not know whether the existence of these neglected terms arises from the fact that a curved space is considered, or because it is a central field.

Another possibility for the comparison of gravity and acceleration may be to consider an accelerated frame where the acceleration-field has a similar structure to the gravitational field, i.e. it is central, it has a "source", too. Such an accelerated frame could be produced if an electrically charged box was pulled by a fixed object with an opposite charge, but negligible mass (to avoid gravitational effects). To describe such a situation the Kerr-Newman space has to be considered. This situation can not be dealt with using the method of Hehl and Ni, by changing the constant a to $a_i = -\frac{k}{r^3}x_i$ in a rigid frame.

I would like to note here that there is no such thing as a homogeneous gravitational field which is supposed to be identical with an accelerated frame. According to my understanding of the Equivalence Principle, it is about the equivalence of a gravitational field and an accelerated frame in a **small region**, i.e. up to a **certain approximation**. This approximation is believed to be equivalent of neglecting tidal terms.

Also, tests for the Equivalence Principle are aimed to determine the order of the approximation up to what it is satisfied.

4.3 The Equivalence Principle

Of all the principles at work in gravitation, none is more central than the equivalence principle. It forms the foundation of General Relativity by stating that the effect of gravitational acceleration by a massive object is the same as that of an oppositely directed mechanical acceleration. "This assumption of exact physical equivalence makes it impossible for us to speak of the absolute acceleration of the system of reference, just as the usual theory of relativity forbids us to talk of the absolute velocity of a system; and it makes the equal falling of all bodies in a gravitational field seem a matter of course." [Einstein, 1911]

"Physics is simple only when viewed locally: that is Einstein's great lesson"
[Misner *et al.*, 1973, p. 19.]

The whole idea of the equivalence principle originates in the observation that all bodies, regardless of their composition, fall under gravity in the same way. We may recall Galileo's alleged experiments at the leaning tower of Pisa. Experimental tests looking for a discrepancy between the inertial and the gravitational masses, which would manifest itself in causing different gravitational acceleration for objects A and B, and characterised by the ratio:

$$\eta = \frac{a_A - a_B}{2\left(a_A + a_B\right)}, \tag{4.36}$$

have been sought for more than 300 years. Tests of this type was first recorded by Galileo using inclined planes to dilute gravity. Pendulums were used by Newton in 1687 (he had found that $\eta < 10^{-3}$) and by Bessel in 1832 ($\eta < 2\times10^{-5}$). Torsion balances were used by Eötvös in 1922 ($\eta < 5\times10^{-9}$) and by Dicke in 1960s ($\eta < 10^{-12}$). [see references of Vessot, 1984]

These tests confirmed the principle of equivalence to a very high accuracy, showing that gravitational acceleration is almost certainly independent of composition. However, with a nonzero experimental error involved, one cannot be sure that it is exactly true, and there is still a good reason for searching for an anomaly. This search has so far been done by means of theoretical reasoning, because if there is any discrepancy it is too small to be detected at the present level of experimental accuracy.

The above tests (of Galileo, Newton, *et al.*) all involved neutral matter, and a natural question would be whether or not the equivalence principle would hold for electrically charged objects. It was found by DeWitt and Brehme [1960] that a charged particle in a gravitational field experiences a self-interaction force, but it does not do so in an accelerated frame of reference. The authors claimed that the reason behind this is that a charged particle carries with it an electromagnetic field, which is by no means local, and therefore it "can not be considered as a local device".

Working out the electrostatic potential of a point charge in Schwarzschild space Léauté and Linet [1983] found that it is different from the potential resulting in an accelerated frame which fact violates the equivalence principle. Besides DeWitt's self-force they discovered an additional force arising from the electric field induced by the potential in Schwarzschild space. Piazzese and Rizzi [1991] examined the observability of this discrepancy, and found that for a gravitational source of very large angular momentum in a small neighbourhood of its "turning point" (where the reversal of the tidal force's direction takes place) this effect may be observable. This was the only case when they found the EP failing. Otherwise, including the case of the Schwarzschild space, the effect of the above mentioned discrepancy was found "quite unobservable" [Piazzese and Rizzi, 1991].

We should also note that spinning neutral particles deviate from geodetic motion by terms involving the Riemann tensor explicitly, which is an expression of the fact that spin is a nonlocal phenomenon [Papapetrou, 1951]. It may therefore be expected that terms involving spin may violate the Equivalence Principle.

According to the general theory of relativity, we must include as part of the mass of an object, the binding energy holding it together. This includes the nuclear binding energy, the energy from the electromagnetic forces holding the atom together, the intermolecular forces holding solids together, and the gravitational energy that holds such massive bodies as the Earth together. These very different forms of energy might contribute to mass or with Einstein's words [1906]: "the mass of a body is a measure of its energy-content". This is the basis of these very precise experimental tests with material bodies of widely different composition. Tests of the equivalence principle involve the question of how various forms of energy contribute to mass.

It is common to make a distinction between various forms of the EP. The EP is called **strong** if it says that locally all laws of nature are the same in a gravitational field and in an accelerated frame, i.e. locally the acceleration caused by gravity can be transformed to zero for point particles provided there are no fields present other than gravity. We call the EP **weak** if it concerns not all the laws of nature but only laws of motion of freely falling particles (the experiments of Eötvös *et al.* and Dicke *et al.* provided direct evidence for the weak and indirect for the strong EP). In other words it leads to the universality of free fall. For a classical point-like particle it means that in the absence of any interaction other than gravity, particles with the same prescribed velocity in some point of space-time move along the same path irrespective of their mass. We may also find that some books divide the strong EP into two: the **very strong** EP applies to all phenomena, whereas the **medium strong** EP to all but gravitational phenomena [Weinberg, 1972, Ch. 3.1 and Ciufolini and Wheeler, 1995].

4.4 Conclusions on the Equivalence Principle

When the Equivalence Principle is tested, what is involved is essentially a comparison of results in an accelerated Minkowski frame and a frame in Schwarzschild space. The relevant metrics in spherical polars are, as in Equations (2.6) and (2.3)

$$ds^2 = \left(1 + 2\frac{ax}{c^2}\right)c^2 dt^2 - dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta\ d\varphi^2\right) \tag{4.37}$$

and

$$ds^2 = \left(1 - 2\frac{GM_\oplus}{rc^2}\right)c^2 dt^2 - \frac{1}{1 - 2\frac{GM_\oplus}{rc^2}}dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta\ d\varphi^2\right), \tag{4.38}$$

respectively.

Considering these equations, one can see that there are already differences between the two cases at the level of the metrics. For example, in Equation (4.37) only the temporal part of the metric has a coefficient different form unity, whereas in Equation (4.38) both g_{00} and g_{11} depend on position. We can notice this difference, which seems to be a fundamental one, but can not deduce any physical differences between the two cases. One has to keep in mind that only invariant quantities carry physical information and a metric is not such a thing. Even if we derive quantities from the metric, and they are found to be different, one has to be careful about which observer measures the given quantity.

Below a comparison is made involving the Hamiltonian of a Dirac particle in the two cases. In an accelerated frame the Hamiltonian has the form of

$$H = \beta mc^2 + \beta m\left(\underline{a}\cdot\underline{x}\right) + \frac{\beta}{2m}p^2 + \frac{\beta}{2mc^2}\underline{p}\cdot\left(\underline{a}\cdot\underline{x}\right)\underline{p} + \frac{\hbar\beta}{4mc^2}\underline{\sigma}\cdot\left(\underline{a}\times\underline{p}\right) \tag{4.39}$$

[results of Hehl and Ni, 1990 substituting $\omega = 0$].

Comparing (4.29) and (4.39), which equations describe the effect of gravitational field and acceleration on spin ½ particles, yields a test of the medium strong equivalence principle.

Here we mean the usual view about an accelerated frame and a frame in gravitational field being locally indistinguishable. Putting $\underline{a} = -\underline{g}$, into equation (4.39), gives us that the flat-space energy-mass terms and their redshifted forms are the same in the two cases. On the other hand in case of the higher order correction terms we do not get agreement. Although both Hamiltonians contain a spin-orbit coupling term, which first turned up as a result of Hehl and Ni's calculation [1990], the coefficients of these are different by a factor of 2. Also, an additional term appears in our calculation in the gravitational case, which has not been mentioned before, and is the same order of magnitude as the redshift to the kinetic energy term. This term is proportional to $(\underline{x} \cdot \underline{p})^2$, i.e. the radial component of momentum squared, as $\underline{x}$ and $\underline{g}$ are both in the direction of the normal to the surface of the Earth.

On neglecting all quantum corrections, the Hamiltonians (4.29) and (4.39) can be rearranged. The fourth term in (4.29) can be written as

$$-\frac{\beta}{2mc^2}\underline{p} \cdot \underline{p}(\underline{g} \cdot \underline{x}), \tag{4.40}$$

which is of the form of a Darwin term [Bjorken and Drell, 1964]. Rewriting it in the form

$$\begin{aligned} -\frac{\beta}{2mc^2}\nabla^2\Phi &= -\frac{\beta}{2mc^2} div\ grad\ \Phi = \\ &= -\frac{\beta}{2mc^2} div\, \underline{g} \end{aligned} \tag{4.41}$$

makes it clear that this term vanishes in vacuum. Similarly, this can be applied to the fourth term in (4.39), which Hehl and Ni [1990] called a redshift to the kinetic energy. Still neglecting quantum corrections, the last term in (4.29) can be written as

$$-\frac{\beta}{mc^2} p_i\, p_j\, g^i\, x^j \tag{4.42}$$

which is of the form of a second derivative of the potential, and therefore it represents a tidal term, and hence curvature. A test of the equivalence principle, applying as it does neglecting tidal terms, will ignore this term.

We conclude that the difference between the Hamiltonians in the cases of a uniformly accelerated frame and a frame in a Schwarzschild space consists only of quantum terms. This is the order up to which we find the EP holding. We have also found, that the discrepancy contains a spin term, as was suggested earlier.

For completeness we must add, that although the comparison was made on the level of Hamiltonians, the difference between the two cases will manifest itself at the level of expectation values as well; this makes the statement physically meaningful. This follows from the fact that both Hamiltonians are formulated for the same scalar product (4.18), i.e. when integrating the different Hamiltonians over the same volume element the expectation values of the Hamiltonians are going to be different, as well. This is now a statement about observables and therefore offers a possibility to distinguish an accelerated frame from one in a gravitational field by a measurement. Experimental verification of this might not be too remote, as the use of atomic interferometers is capable of increasing the accuracy of the COW experiments by a factor of 10^{10} [Adams *et al.*, 1994].

The above reasoning holds in case of spin ½ particles only, as the use of the Dirac equation has been crucial in obtaining our results. The Dirac equation is a first order wave-equation, and such equations exist for particles of all spins except spin 0 [Weinberg, 1964]. It may be the case, then, that a similar problem with the equivalence principle holds for particles of all non-zero spins; but that this problem disappears for spin 0 particles. It should be remembered that spin 0 particles obey the Klein-Gordon equation, but that equation expresses nothing other than the Einstein relation between energy, momentum and rest-mass (see Section 2.2). Such a relation holds for every component of a spin non-zero particle.

4.5 Appendices

4.5.1 Appendix: Basis 1-forms and vectors in the calculation

To see that the basis 1-forms satisfy the criteria of an orthonormal basis, one has to check if (3.10) is satisfied.

$$\left(\Theta^{\hat{0}}\right)^2 - \left(\Theta^{\hat{1}}\right)^2 - \left(\Theta^{\hat{2}}\right)^2 - \left(\Theta^{\hat{3}}\right)^2 = ((1-\Phi)dt)^2 - \sum_{i=1,2,3}\left(dx^i + g^i\left(x_j dx^j\right)\right)^2 \tag{4.43}$$

gives after dropping terms of second and higher order in Φ

$$\begin{aligned}(1-2\Phi)dt^2 - \sum_{i=1,2,3}\left(\left(dx^i\right)^2 + 2g^i dx^i\left(x_j\, dx^j\right)\right) &= \\ = (1-2\Phi)dt^2 - \sum_{i=1,2,3}\left(\left(dx^i\right)^2 + 2x^i dx^i\left(g_j\, dx^j\right)\right),\end{aligned} \tag{4.44}$$

which after rearranging is equivalent to the metric (4.11).

The duality of the basis 1-forms and vectors can be checked using equation (3.28). There is no mixing of the temporal and spatial 1-forms in the basis 1-forms, so only the space-space duality has to be checked:

$$\begin{aligned}\left\langle e_{\hat{i}}, \Theta^{\hat{j}}\right\rangle &= \left\langle \partial_i - g_i x^k \partial_k, dx^j + g^j x_m dx^m\right\rangle = \\ &= \delta_i^j + \left\langle \partial_i, g^j x_m dx^m\right\rangle - \left\langle g_i x^k \partial_k, dx^j\right\rangle = \\ &= \delta_i^j + g^j x_m \delta_i^m - g_i x^k \delta_k^j = \delta_i^j .\end{aligned} \tag{4.45}$$

4.5.2 Appendix: Finding the connection coefficients

The method described in Appendix 3.12.1.A.2 is followed. The commutators of the basis vectors give, up to first order in Φ

$$\begin{aligned}\left[e_{\hat{0}}, e_{\hat{i}}\right] &= \left[(1+\Phi)\partial_t, \partial_i - g_i x^k \partial_k\right] = \\ &= \left[\Phi, \partial_i\right]\partial_t = g_i \partial_t\end{aligned} \tag{4.46}$$

and

$$\begin{aligned} \left[e_{\hat{i}},\, e_{\hat{j}}\right] &= \left[\partial_i - g_i x^k \partial_k, \partial_j - g_j x^l \partial_l\right] = \\ &= \left[\partial_i, - g_j x^l \partial_l\right] - \left[g_i x^k \partial_k, \partial_j\right] = \end{aligned} \tag{4.47}$$

$$\begin{aligned} &= -\left(\partial_i g_j\right) x^l \partial_l - g_j \left(\partial_i x^l\right)\partial_l + \left(\partial_j g_i\right) x^k \partial_k + g_i \left(\partial_j x^k\right)\partial_k = \\ &= - g_j \partial_i + g_i \partial_j \,. \end{aligned} \tag{4.48}$$

This gives for the structure constant:

$$C_{\hat{0}\hat{i}}{}^{\hat{0}} = -C_{\hat{i}\hat{0}}{}^{\hat{0}} = g_i \tag{4.49}$$

$$C_{\hat{i}\hat{j}}{}^{\hat{k}} = \delta_i^k g_j + \delta_j^k g_i \,. \tag{4.50}$$

Lowering the third indices gives (4.14).

4.5.3 Appendix: Writing the Dirac equation

The covariant derivatives are

$$\begin{aligned} D_{\hat{0}} &= e_{\hat{0}} + \frac{1}{4}\left[\gamma^{\hat{0}},\, \gamma^{\hat{i}}\right]\Gamma_{\hat{0}\hat{i}\hat{0}} = \\ &= \left(1+\Phi\right)\partial_t + \frac{1}{2}\underline{\alpha}\cdot\underline{g} \end{aligned} \tag{4.51}$$

$$\begin{aligned} D_{\hat{k}} &= \partial_k - g_k x^l \partial_l + \frac{1}{8}\left[\gamma^{\hat{i}}, \gamma^{\hat{j}}\right]\Gamma_{\hat{i}\hat{j}\hat{k}} = \\ &= \partial_k - g_k x^l \partial_l - \frac{1}{4} i\varepsilon^{ij}{}_l \sigma^l \left(- g_{\hat{i}\hat{k}} g_j + g_{\hat{j}\hat{k}} g_i\right) = \\ &= \partial_k - g_k x^l \partial_l - \frac{1}{2} i\varepsilon^{ij}{}_l \sigma^l g_{\hat{j}\hat{k}} g_i \,. \end{aligned} \tag{4.52}$$

So the Dirac equation reads

$$m\Psi = i\hbar\beta \left\{ \left(1+\Phi\right)\partial_t + \frac{1}{2}\underline{\alpha}\cdot\underline{g} + \alpha^k \left(\partial_k - g_k x^l \partial_l - \frac{1}{2} i\varepsilon^{ij}{}_l \sigma^l g_{\hat{j}\hat{k}} g_i \right)\right\}\Psi \,, \tag{4.53}$$

which gives, using (3.38),

$$m\Psi = i\hbar\beta \left\{ \left(1+\Phi\right)\partial_t + \frac{1}{2}\underline{\alpha}\cdot\underline{g} + \alpha^k \left(\partial_k - g_k x^l \partial_l\right) - \frac{1}{2} 2\delta^i_m \alpha^m g_i \right\}\Psi \,. \tag{4.54}$$

Substituting $\underline{p} = -i\hbar\underline{\partial}$ and rearranging (4.54) gives for the Hamiltonian (4.16).

5 Effect of a rotating gravitational source on Dirac particles*

"According to present plans the next gravitational project in space will be a measurement of the frame-dragging effect predicted to result from the Earth's rotation." (Vessot, 1984)

Earth is a rotating massive body, therefore all terrestrial experiments are performed in the field of a rotating gravitational source. Such a field is described by the Kerr metric. The Kerr metric is quoted in a wide range of forms in various textbooks and papers. It should be a simple matter to find the relation between these using coordinate transformations, but in practice this is less straightforward. For example in d'Inverno's book [1992] the coordinate transformation from Boyer-Lindquist coordinates to Kerr coordinates is wrongly quoted; the correct transformation is given in [Hawking and Ellis, 1974]. (See Appendix 5.5.1.)

In papers [Wajima *et al.*, 1997 and Lalak *et al.*, 1995] approximate forms of the Kerr metric are quoted, but without a proper definition of coordinates. Up to the same order the expressions for the metric are found to be different. Wajima *et al.* [1997] have

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* A condensed version of the material in this chapter is to be submitted for publication [Varjú and Ryder, b]

$$ds^2 = \left(c^2 + 2\Phi + \frac{2\Phi^2}{c^2} + \frac{2GMa}{c^2 r^3}\left(x^2 + y^2\right)\right)dt^2 + \\ + \frac{4GMa}{c^2 r^3}(x\,dy - y\,dx)dt - \left(1 - \frac{2\Phi}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right), \tag{5.1}$$

where the substitution $\Phi = -\dfrac{GM}{r}$ was made and a is the angular momentum per unit mass of the source; while Lalak *et al.* [1995] use

$$ds^2 = \left(c^2 + 2\Phi + \frac{\Phi^2}{c^2}\right)dt^2 - \frac{2\,\Phi\,a}{c^2 r^2}(x\,dy - y\,dx)dt - \\ -\left(1 - \frac{2\Phi}{c^2} + \frac{2\Phi^2}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right), \tag{5.2}$$

substituting $\dfrac{\Phi}{c^2} = -\dfrac{1}{2}\dfrac{r_g}{r}$. (The quantity Φ is introduced here as the new parameter for easy comparison of the two metrics, because the authors expressed the metric using different parameters: G and r_g.) These metrics are clearly different, although the authors claim to work up to the same order, using asymptotically static coordinates in both cases. The last term in the first parenthesis of (5.1), $\dfrac{2\,GMa}{c^2 r^3}\left(x^2 + y^2\right)dt^2$, is not even correct dimensionally. As the authors do not refer to the source where they have derived their metrics, it is difficult to tell what the cause of disagreement is. Therefore I find it necessary to present a complete derivation of the Dirac Hamiltonian in Kerr space working up to the order of $\left(\dfrac{m}{r}\right)^2$ and $\left(\dfrac{ma}{r^2}\right)$. This approximation should be used in the case of the Earth, as we have $\dfrac{m}{r} \approx 6\cdot 10^{-10}$ and $\dfrac{a}{r} \approx 10^{-13}$. I also present the corresponding calculation in Schwarzschild space, and refer to the result of an accelerated frame, for comparison with the Kerr case. Then I investigate the differences between these cases at different levels of experimental accuracy. The effects should in fact be studied in a rotating frame, as the experiments are done on the Earth, so laboratories fixed to the Earth rotate relative to the fixed stars. Even in the case of the Kerr metric a rotation of the frame should be performed as the Kerr metric in its form (5.13) describes the gravitational field of a rotating massive body, as viewed from a fixed point outside it.

To obtain the Dirac Hamiltonian we use the method described in Chapter 3. Throughout this chapter $c=1$ convention is used.

5.1 Rotating frame in Schwarzschild space

The exact Schwarzschild metric in isotropic coordinates $(\rho, \vartheta, \varphi)$ with the relation

$$r = \rho\left(1+\frac{m}{2r}\right)^2 \tag{5.3}$$

reads [Møller, 1972]

$$ds^2 = \left(1+\frac{m}{2\rho}\right)^4\left(d\rho^2+\rho^2 d\vartheta^2+\rho^2\sin^2\vartheta\, d\varphi^2\right)-\frac{\left(1-\frac{m}{2\rho}\right)^2}{\left(1+\frac{m}{2\rho}\right)^2}dt^2 . \tag{5.4}$$

Changing to static isotropic Cartesian coordinates,

$$x_S = \rho\sin\theta\cos\varphi \ , \quad y_S = \rho\sin\theta\sin\varphi \ , \quad z_S = \rho\cos\theta \ , \tag{5.5}$$

we get

$$ds^2 = \left(1+\frac{m}{2\rho}\right)^4\left(dx_S{}^2+dy_S{}^2+dz_S{}^2\right)-\frac{\left(1-\frac{m}{2\rho}\right)^2}{\left(1+\frac{m}{2\rho}\right)^2}dt^2 . \tag{5.6}$$

Because an observer on the Earth is rotating with the Earth, we must consider a frame rotating relative to the fixed stars:

$$x_S = x\cos\omega t - y\sin\omega t \ , \quad y_S = x\sin\omega t + y\cos\omega t \ , \quad z_S = z \ , \tag{5.7}$$

which gives for the relevant order of approximation

$$\begin{aligned} ds^2 &= \left(1+\frac{2m}{\rho}+\frac{3m^2}{2\rho^2}\right)\left(dx^2+dy^2+dz^2\right)-\left(1-\frac{2m}{\rho}+\frac{2m^2}{\rho^2}\right)dt^2 + \\ &\quad + 2\omega\left(1+\frac{2m}{\rho}\right)\left(x\,dy - y\,dx\right)dt . \end{aligned} \tag{5.8}$$

We identify the metric components:

$$\begin{aligned} g_{00} &= -\left(1-\frac{2m}{\rho}+\frac{2m^2}{\rho^2}\right), \quad g_{01} = g_{10} = -\omega\left(1+\frac{2m}{\rho}\right)y \ , \\ g_{02} &= g_{20} = \omega\left(1+\frac{2m}{\rho}\right)x \ , \quad g_{ij} = \left(1+\frac{2m}{\rho}+\frac{3m^2}{2\rho^2}\right)\eta_{ij} \end{aligned} \tag{5.9}$$

The tetrad components satisfying $g_{\alpha\beta} = \eta_{\hat{\mu}\hat{\nu}}\, h^{\hat{\mu}}{}_{\alpha}\, h^{\hat{\nu}}{}_{\beta}$ are

$$\begin{aligned} h^{\hat{0}}{}_{0} &= \left(1-\frac{m}{\rho}+\frac{m^2}{2\rho^2}\right), \\ h^{\hat{i}}{}_{0} &= \left(1+\frac{m}{\rho}\right)f^{i}, \\ h^{\hat{i}}{}_{j} &= \left(1+\frac{m}{\rho}+\frac{m^2}{4\rho^2}\right)\delta^{i}{}_{j}, \end{aligned} \tag{5.10}$$

with the inverse components:

$$\begin{aligned} h_{\hat{0}}{}^{0} &= \left(1+\frac{m}{\rho}+\frac{m^2}{2\rho^2}\right), \\ h_{\hat{0}}{}^{i} &= -\left(1+\frac{m}{\rho}\right)f^{i} \ , \\ h_{\hat{i}}{}^{j} &= \left(1-\frac{m}{\rho}+\frac{3m^2}{4\rho^2}\right)\delta_{i}{}^{j} \ . \end{aligned} \tag{5.11}$$

We shall have recourse to the following definitions below:

$$\begin{aligned} &\Phi = \frac{m}{\rho}, \quad g_i = -\frac{\partial\Phi}{\partial x^i} = -\frac{x_i m}{\rho^3}, \\ &\underline{\omega} = (0,0,\omega), \quad \underline{f} = (\underline{\omega}\times\underline{x}) = (\omega\, y, -\omega\, x, 0). \end{aligned} \tag{5.12}$$

5.2 Rotating frame in Kerr space

The exact Kerr metric is [Stephani, 1990]

$$ds^2 = \Sigma\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) + \left(r^2 + a^2\right)\sin^2\vartheta\, d\varphi^2 - dt^2 + \\ + \frac{2mr}{\Sigma}\left(a\sin^2\vartheta\, d\varphi - dt\right)^2 , \tag{5.13}$$

with

$$\Sigma = r^2 + a^2\cos^2\vartheta \quad \text{and} \quad \Delta = r^2 - 2mr + a^2 \ . \tag{5.14}$$

Up to the order of $\left(\frac{m}{r}\right)^2$ and $\left(\frac{ma}{r^2}\right)$ one finds

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 + \frac{2m}{r} + \frac{4m^2}{r^2}\right)dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) - \\ - \frac{4ma}{r^2}\, r\sin^2\vartheta\, d\varphi\, dt . \tag{5.15}$$

The transformation to isotropic coordinates, with (5.3) leads to

$$ds^2 = -\left(1 - \frac{2m}{\rho} + \frac{2m^2}{\rho^2}\right)dt^2 + \\ + \left(1 + \frac{2m}{\rho} + \frac{3m^2}{2\rho^2}\right)\left(d\rho^2 + \rho^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right) - \\ - \frac{4ma}{\rho^2}\,\rho\sin^2\vartheta\, d\varphi\, dt . \tag{5.16}$$

Changing to static isotropic Cartesian coordinates using (5.5) gives

$$ds^2 = -\left(1 - \frac{2m}{\rho} + \frac{2m^2}{\rho^2}\right)dt^2 + \\ + \left(1 + \frac{2m}{\rho} + \frac{3m^2}{2\rho^2}\right)\left(dx_S^2 + dy_S^2 + dz_S^2\right) - \\ - \frac{4ma}{\rho^2}\,\frac{x_S\, dy_S - y_S\, dx_S}{\rho}\, dt . \tag{5.17}$$

Transforming to an Earth-bound, rotating frame, as in (5.7) above, gives for the relevant order of approximation

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2m}{\rho}+\frac{2m^2}{\rho^2}\right)dt^2+\left(1+\frac{2m}{\rho}+\frac{3m^2}{2\rho^2}\right)\left(dx^2+dy^2+dz^2\right)+ \\
&\quad +2\omega\left(1+\frac{6m}{5\rho}\right)(x\,dy-y\,dx)\,dt\,.
\end{aligned}
\tag{5.18}
$$

In the above we have used the relationship

$$
a=\frac{2}{5}\omega r^2\approx\frac{2}{5}\omega\rho^2, \tag{5.19}
$$

which holds for a spherical, rotating gravitational source.

The presence of the last term in (5.18) shows that the rotation of the gravitational source and the effect of the rotation of the reference frame are different. The metric components are then

$$
\begin{aligned}
g_{00} &= -\left(1-\frac{2m}{\rho}+\frac{2m^2}{\rho^2}\right), \quad g_{01}=g_{10}=-\omega\left(1+\frac{6m}{5\rho}\right)y\,, \\
g_{02} &= g_{20}=\omega\left(1+\frac{6m}{5\rho}\right)x\,, \quad g_{ij}=\left(1+\frac{2m}{\rho}+\frac{3m^2}{2\rho^2}\right)\eta_{ij}
\end{aligned}
\tag{5.20}
$$

and hence the tetrad components are (calculated as before):

$$
\begin{aligned}
h^{\hat{0}}{}_0 &= \left(1-\frac{m}{\rho}+\frac{m^2}{2\rho^2}\right), \quad h^{\hat{i}}{}_0=\left(1+\frac{1}{5}\frac{m}{\rho}\right)f^{\,i}\,, \\
h^{\hat{i}}{}_j &= \left(1+\frac{m}{\rho}+\frac{m^2}{4\rho^2}\right)\delta^i{}_j\,,
\end{aligned}
\tag{5.21}
$$

and the inverse tetrad components:

$$
\begin{aligned}
h_{\hat{0}}{}^0 &= \left(1+\frac{m}{\rho}+\frac{m^2}{2\rho^2}\right), \quad h_{\hat{0}}{}^i=-\left(1+\frac{1}{5}\frac{m}{\rho}\right)f^{\,i}\,, \\
h_{\hat{i}}{}^j &= \left(1-\frac{m}{\rho}+\frac{3m^2}{4\rho^2}\right)\delta_i{}^j\,.
\end{aligned}
\tag{5.22}
$$

5.3 Dirac equation in the Earth's field

We notice that the Kerr case (5.21) (5.22) and the Schwarzschild case (5.10) (5.11) differ only in a factor of one of the tetrad components. Introducing a constant b for this factor we can treat the two cases together:

$$\begin{aligned} h^{\hat{0}}{}_{0} &= \left(1-\frac{m}{\rho}+\frac{m^2}{2\rho^2}\right), \\ h^{\hat{i}}{}_{0} &= \left(1+b\frac{m}{\rho}\right)f^{i}, \\ h^{\hat{i}}{}_{j} &= \left(1+\frac{m}{\rho}+\frac{m^2}{4\rho^2}\right)\delta^{i}{}_{j}, \end{aligned} \tag{5.23}$$

$$\begin{aligned} h_{\hat{0}}{}^{0} &= \left(1+\frac{m}{\rho}+\frac{m^2}{2\rho^2}\right), \\ h_{\hat{0}}{}^{i} &= -\left(1+b\frac{m}{\rho}\right)f^{i}, \\ h_{\hat{i}}{}^{j} &= \left(1-\frac{m}{\rho}+\frac{3m^2}{4\rho^2}\right)\delta_{i}{}^{j}, \end{aligned} \tag{5.24}$$

with $b=1$ in case of a Schwarzschild space, and $b=\frac{1}{5}$ in a Kerr space.

In this chapter the connection coefficients are calculated from the tetrad components, using formulae (3.17) and (3.24). These give for the tetrad (5.23) and (5.24)

$$\begin{aligned} \Gamma_{\hat{0}\hat{i}\hat{0}} &= -\Gamma_{\hat{i}\hat{0}\hat{0}} = (1-\Phi)g_i, \\ \Gamma_{\hat{i}\hat{j}\hat{0}} &= \frac{1}{2}(1-b)(f_i g_j - f_j g_i) + (1+b\Phi)\varepsilon_{ij}{}^{l}\omega_l, \\ \Gamma_{\hat{0}\hat{i}\hat{j}} &= -\Gamma_{\hat{i}\hat{0}\hat{j}} = \frac{1}{2}(1-b)(f_i g_j + f_j g_i), \\ \Gamma_{\hat{i}\hat{j}\hat{k}} &= (\eta_{jk} g_i - \eta_{ik} g_j)\left(1-\frac{3}{2}\Phi\right) \end{aligned} \tag{5.25}$$

(for details see Appendix 5.5.2).

Then one writes the Dirac equation similarly to the previous cases to get the Dirac Hamiltonian,

$$
\begin{aligned}
H = & \left(1-\Phi+\frac{1}{2}\Phi^2\right)\beta m - i\hbar\left(1-2\Phi+\frac{9}{4}\Phi^2\right)(\underline{\alpha}\cdot\underline{\partial})+ \\
& +i\hbar\left(1-(1-b)\Phi\right)(\underline{f}\cdot\underline{\partial})+\frac{i\hbar}{2}(1-3\Phi)(\underline{\alpha}\cdot\underline{g})- \\
& -\frac{\hbar}{4}(1-b)\underline{\sigma}\cdot(\underline{f}\times\underline{g})+\frac{\hbar}{2}\left(1-(1-b)\Phi\right)(\underline{\sigma}\cdot\underline{\omega}).
\end{aligned} \tag{5.26}
$$

The expressions of the basis vectors and covariant derivatives are given in Appendix 5.5.3.

The determinant of the spatial part of the metric tensor is

$$
\det g_{ij} = \left(1+2\Phi+\frac{3}{2}\Phi^2\right)^3, \tag{5.27}
$$

because the terms containing off-diagonal components are of second order in ω.

Absorbing the determinental factor into the wavefunction, and transforming the Hamiltonian as described in Chapter 3.7 gives, after relabelling $H' \to H$,

$$
\begin{aligned}
H = & \left(1-\Phi+\frac{1}{2}\Phi^2\right)\beta m - i\hbar\left(1-2\Phi+\frac{9}{4}\Phi^2\right)(\underline{\alpha}\cdot\underline{\partial})+ \\
& +i\hbar\left(1-(1-b)\Phi\right)(\underline{f}\cdot\underline{\partial})-i\hbar\left(1-\frac{9}{4}\Phi\right)(\underline{\alpha}\cdot\underline{g})- \\
& -\frac{\hbar}{4}(1-b)\underline{\sigma}\cdot(\underline{f}\times\underline{g})+\frac{\hbar}{2}\left(1-(1-b)\Phi\right)(\underline{\sigma}\cdot\underline{\omega})+\frac{3}{2}i\hbar(\underline{f}\cdot\underline{g}).
\end{aligned} \tag{5.28}
$$

As the above Hamiltonian still depends on the value of b we can see that the effect of Kerr space is different from the effect of Schwarzschild space. However, to see the differences caused in laboratory experiments, we have to take the non-relativistic limit. The proper non-relativistic limit can be obtained by applying three successive Foldy-Wouthuysen transformations as explained in Chapter 3.8. To calculate all these terms would be difficult, but as we are only interested in the order at which the difference between the two cases becomes manifest, it is sufficient to consider the leading terms. The odd and even parts of the Hamiltonian are

$$\begin{aligned}\mathcal{O} &= -i\hbar\left(1-2\Phi+\frac{9}{4}\Phi^2\right)\underline{\alpha}\cdot\underline{\partial}-i\hbar\left(1-\frac{9}{4}\Phi\right)\underline{\alpha}\cdot\underline{g}\\ \mathcal{E} &= -\left(1-\frac{1}{2}\Phi\right)\Phi\beta m+i\hbar\left(1-(1-b)\Phi\right)\underline{f}\cdot\underline{\partial}+\frac{3}{2}i\hbar\underline{f}\cdot\underline{g}-\\ &\qquad -\frac{\hbar}{4}(1-b)\underline{\sigma}\cdot\left(\underline{f}\times\underline{g}\right)+\frac{\hbar}{2}\left(1-(1-b)\Phi\right)\left(\underline{\sigma}\cdot\underline{\omega}\right).\end{aligned} \tag{5.29}$$

The odd terms contain no b, so they are the same in the two cases. The difference will come from the terms $\mathcal{E}-\frac{1}{8m^2c^4}[\mathcal{O},[\mathcal{O},\mathcal{E}]]$. To the leading order we have for the difference:

$$i\hbar(1-b)\Phi\,\underline{f}\cdot\underline{\partial}. \tag{5.30}$$

As only the leading order correction is of interest, one may use the approximate expression

$$\underline{p}=-i\hbar\,\underline{\partial}. \tag{5.31}$$

Higher order corrections to the momentum can be obtained using the method described in Chapter 3.9.

In case of a thermal neutron (kinetic energy of 20 meV), the momentum

$$p = \sqrt{2mE_{kin}} = \sqrt{2\times 940\,{MeV}/{c^2}\times 2meV} = 1.9\times 10^3\,{eV}/{c} \tag{5.32}$$

is of order $p\approx 2\times 10^3\,\mathrm{eV}/\mathrm{c}$. Therefore the order of the difference term is

$$i\hbar(1-b)\Phi\underline{f}\cdot\underline{\partial} \approx \Phi\frac{a}{r}p \approx 6\times 10^{-10}\times 10^{-13}\times 2\times 10^3\,\mathrm{eV} \approx 10^{-19}\,\mathrm{eV}. \tag{5.33}$$

5.4 Conclusion

When analysing terrestrial experiments it has to be taken into account that the Earth's gravitational effect is properly described by a rotating frame in Kerr space, which is a rather difficult calculation. The effect can be approximated by using a rotating frame in Schwarzschild space, or even a rotating accelerated frame. The question is, at different levels of experimental accuracy, which approximation is sufficient. In this chapter the comparison between a Kerr and a Schwarzschild field was carried out, and it was found that the difference between the cases becomes apparent at energies of 10^{-19} eV. A comparison between an accelerated frame and the Schwarzschild field was carried out in Chapter 4 and it was found that the difference between the two is of the same order as the redshift of the kinetic energy, that is 10^{-11} eV. This is the level of accuracy where the differences between the gravitational effect and the effect of acceleration become distinct.

For comparison we note here that the gravitational term detected in the COW experiment (redshift of the restmass term) is of the order of 1 eV. Atomic interferometers are expected to increase this accuracy by a factor of 10^{10} so it is becoming clear that further experimental developments will make it necessary to use general relativity in analysing the behaviour of quantum systems.

5.5 Appendices

5.5.1 Appendix: Comparing the Kerr and Boyer-Lindquist forms

Equation (19.27) of d'Inverno [1992] gives the Kerr metric in Boyer-Lindquist coordinates:

$$ds_{BL}^2 = \frac{\Delta}{\rho^2}\left(a\sin^2\vartheta\, d\varphi - dt\right)^2 - \frac{\sin^2\vartheta}{\rho^2}\left(\left(r^2+a^2\right)d\varphi - a\,dt\right)^2 - \\ -\frac{\rho^2}{\Delta}dr^2 - \rho^2 d\vartheta^2 \ , \tag{5.34}$$

with $(\rho, \vartheta, \varphi)$ the standard polar coordinates, and

$$r^2 = \rho^2 - a^2\cos^2\vartheta \quad \text{and} \quad \Delta = r^2 - 2mr + a^2 \ . \tag{5.35}$$

For the same metric in Kerr coordinates Equation (19.28) of the same reference gives

$$ds_K^2 = d\bar{t}^2 - dx^2 - dy^2 - dz^2 - \\ -\frac{2mr^3}{r^4+a^2z^2}\left(d\bar{t} + \frac{r}{a^2+r^2}(xdx+ydy) + \frac{a}{a^2+r^2}(ydx-xdy) + \frac{z}{r}dz\right)^2 \tag{5.36}$$

and from equations (19.29) and (19.66) the transformation connecting the two cases,

$$\begin{aligned} x &= r\sin\vartheta\cos\varphi + a\sin\vartheta\sin\varphi \\ y &= r\sin\vartheta\sin\varphi - a\sin\vartheta\cos\varphi \\ z &= r\cos\vartheta \\ d\bar{t} &= \mathrm{dt} + \frac{2mr}{\Delta}dr . \end{aligned} \tag{5.37}$$

On substituting (5.37) and (5.35) into (5.36) one can see, that

$$ds_K^2 - ds_{BL}^2 = -2\sin^2\vartheta\, a\, d\varphi\, dr - \frac{a^2}{r^2}\sin^2\vartheta\, dr^2 . \tag{5.38}$$

On the other hand, Hawking and Ellis [1974] has the same form of the metrics as d'Inverno (equations (5.29) and (5.30)), but the transformation connecting them are different. They have

$$
\begin{aligned}
&x + iy = (r + ia)\sin\vartheta \cdot \exp\left(i\int d\varphi + \frac{a}{\Delta}dr \right) \\
&z = r\cos\vartheta \qquad\qquad (5.39)\\
&\bar{t} = \int \mathrm{dt} + \frac{r^2 + a^2}{\Delta}dr - r \ ,
\end{aligned}
$$

which agrees with the last two equations of (5.37) but, instead of the first two there, this transformation provides

$$
x + iy = (r + ia)\sin\vartheta(\cos\alpha + i\sin\alpha) \ , \qquad (5.40)
$$

with

$$
\alpha = \varphi + \frac{a}{\sqrt{a^2 - m^2}}\arctan\frac{r - m}{\sqrt{a^2 - m^2}} \qquad (5.41)
$$

instead of $\alpha = \varphi$. The good news is, that substituting (5.39) and (5.35) into (5.36) gives (5.34).

5.5.2 Appendix: Calculation of the connection coefficients

First the structure constants have to be calculated from the tetrad components, using the formula (3.17). These give:

$$
\begin{aligned}
C_{\hat{0}\hat{i}}{}^{\hat{0}} &= h_{\hat{0}}{}^{\alpha} h_{\hat{i}}{}^{\beta}\left(\partial_{\beta} h^{\hat{0}}{}_{\alpha} - \partial_{\alpha} h^{\hat{0}}{}_{\beta}\right) = \\
&= h_{\hat{0}}{}^{0} h_{\hat{i}}{}^{k}\left(\partial_{k} h^{\hat{0}}{}_{0} - \partial_{0} h^{\hat{0}}{}_{k}\right) + h_{\hat{0}}{}^{j} h_{\hat{i}}{}^{k}\left(\partial_{k} h^{\hat{0}}{}_{j} - \partial_{j} h^{\hat{0}}{}_{k}\right) = \qquad (5.42)
\end{aligned}
$$

$$
\begin{aligned}
&= h_{\hat{0}}{}^{0} h_{\hat{i}}{}^{k}\partial_{k} h^{\hat{0}}{}_{0} = \\
&= \left(1 + \Phi + \frac{1}{2}\Phi^2\right)\left(1 - \Phi + \frac{3}{4}\Phi^2\right)\delta_{i}^{k}\partial_{k}\left(1 - \Phi + \frac{1}{2}\Phi^2\right) = \qquad (5.43)\\
&= g_{i}(1 - \Phi) \ ,
\end{aligned}
$$

$$
\begin{aligned}
C_{\hat{0}\hat{i}}{}^{\hat{j}} &= h_{\hat{0}}{}^{\alpha} h_{\hat{i}}{}^{\beta}\left(\partial_{\beta} h^{\hat{j}}{}_{\alpha} - \partial_{\alpha} h^{\hat{j}}{}_{\beta}\right) = \\
&= h_{\hat{0}}{}^{0} h_{\hat{i}}{}^{k}\left(\partial_{k} h^{\hat{j}}{}_{0} - \partial_{0} h^{\hat{j}}{}_{k}\right) + h_{\hat{0}}{}^{l} h_{\hat{i}}{}^{k}\left(\partial_{k} h^{\hat{j}}{}_{l} - \partial_{l} h^{\hat{j}}{}_{k}\right) = \qquad (5.44)\\
&= h_{\hat{0}}{}^{0} h_{\hat{i}}{}^{k}\left(\partial_{k} h^{\hat{j}}{}_{0}\right) + h_{\hat{0}}{}^{l} h_{\hat{i}}{}^{k}\left(\partial_{k} h^{\hat{j}}{}_{l} - \partial_{l} h^{\hat{j}}{}_{k}\right) =
\end{aligned}
$$

$$= \left(1+\Phi+\frac{1}{2}\Phi^2\right)\left(1-\Phi+\frac{3}{4}\Phi^2\right)\delta_i^k\left(\partial_k\left(1+b\Phi\right)a^j\right)-$$
$$-\left(1+b\Phi\right)a^l\left(1-\Phi+\frac{3}{4}\Phi^2\right)\delta_i^k\left(\delta_l^j\partial_k-\delta_k^j\partial_l\right)\left(1+\Phi+\frac{1}{4}\Phi^2\right)= \tag{5.45}$$

$$= \left(\partial_i\left(1+b\Phi\right)a^j\right)+a^l\delta_i^k\left(\delta_l^j g_k-\delta_k^j g_l\right)=$$
$$= -bg_i a^j-\left(1+b\Phi\right)\varepsilon_i^{\ jm}\omega_m+a^j g_i-\delta_i^j a^l g_l \tag{5.46}$$
$$= \left(1-b\right)g_i a^j-\left(1+b\Phi\right)\varepsilon_i^{\ jm}\omega_m .$$

Similarly

$$C_{\hat{i}\hat{j}}^{\ \ \hat{0}} = 0 \tag{5.47}$$

and

$$C_{\hat{i}\hat{j}}^{\ \ \hat{k}} = \left(1-\frac{3}{2}\Phi\right)\left(\delta_j^k g_i-\delta_i^k g_j\right). \tag{5.48}$$

Lowering the last indices and using (3.24) gives (5.25).

5.5.3 Appendix: Basis vectors and covariant derivatives

Using (5.24) gives

$$e_{\hat{0}} = \left(1+\frac{m}{\rho}+\frac{m^2}{2\rho^2}\right)\partial_t-\left(1+b\Phi\right)f^i\partial_i ,$$
$$e_{\hat{i}} = \left(1-\frac{m}{\rho}+\frac{3m^2}{4\rho^2}\right)\partial_i \tag{5.49}$$

and the covariant derivatives turn out to be using (5.25)

$$D_{\hat{0}} = \left(1+\Phi+\frac{1}{2}\Phi^2\right)\partial_0-\left(1+b\Phi\right)f^i\partial_i+\frac{1}{2}\left(1-\Phi\right)\left(\underline{\alpha}\cdot\underline{g}\right)-$$
$$-\frac{1}{4}i\left(1-b\right)\underline{\sigma}\cdot\left(\underline{f}\times\underline{g}\right)+\frac{i}{2}\left(1+b\Phi\right)\left(\underline{\omega}\cdot\underline{\sigma}\right)$$

$$D_{\hat{k}} = \left(1-\Phi+\frac{3}{4}\Phi^2\right)\partial_k+\frac{1}{4}\left(1-b\right)\left(\left(\underline{\alpha}\cdot\underline{f}\right)g_k+\left(\underline{\alpha}\cdot\underline{g}\right)f_k\right)+\frac{1}{2}i\left(1-\frac{3}{2}\Phi\right)\left(\varepsilon_{ilk}\sigma^l g^i\right). \tag{5.50}$$

6 General relativistic treatment of the COW experiment*

We may recall, from Chapter 2.1.1, the gravitational phase shift derivable from Newtonian mechanics

$$\Delta\phi_{grav} = q_{grav} \sin\alpha = 2\pi\lambda \frac{g}{h^2} m^2 A_0 \sin\alpha \,, \tag{6.1}$$

and the Sagnac-shift:

$$\Delta\phi_{Sagnac} = q_{Sagnac} \cos\alpha = \frac{4\pi m \omega A_0}{h} \cos\vartheta_L \cos\alpha \,. \tag{6.2}$$

As was made clear above, the theory with which the experimental data has been compared in the experiments is Newton's theory of gravity. From a fundamental point of view, however, this is somewhat unsatisfactory; the theoretical expression for the phase shift should be derived from General Relativity. A step in this direction has been taken by Anandan [1977], who gave a special relativistic discussion of the behaviour of neutrons in a gravitational field. Anandan used the Klein-Gordon equation, simulating the gravitational and rotational effects of the Earth by passing to an accelerating and rotating frame of reference. The discussion is *special* relativistic in the sense that the Klein-Gordon equation is used; and it is consistent with *general* relativity since it makes use of the Equivalence Principle. Crucially, however, the Klein-Gordon equation is not capable of exhibiting spin effects, and the neutron is a spin ½ particle. To find any gravitational spin effects, the correct procedure is to write down the Dirac equation in a curved space.

Anandan found the following expression for the phase shift:

$$\begin{aligned}\Delta\phi &= \Delta\phi_{grav} + \Delta\phi_{rot} = \\ &= \frac{gAm^2}{\hbar^2\kappa} + \frac{gA\kappa}{c^2} - \frac{2\Omega_n Am}{\hbar} - \frac{\hbar\Omega_n A\kappa^2}{mc^2}\end{aligned} \tag{6.3}$$

($\kappa = \dfrac{2\pi}{\lambda}$ is the wavenumber of the neutron, $\Omega_n = \omega\cos\vartheta_L$ is the component of the angular velocity of the Earth normal to the interferometer surface). The first two phase terms are caused by gravity and the second two by the rotation of the reference frame. The first term in (6.3) is equivalent to (6.1) (Anandan assigns normal vector to the area, so $gA = gA_0 \sin\alpha$ and $\Omega_n A = \omega\cos\vartheta_L\, A_0 \cos\alpha$ where α is the tilt angle of the interferometer). The third term corresponds to (6.2), the Sagnac term . The other two terms are too small to have been detected (yet).

It may be of interest to note that a completely classical derivation of the phase shift has been given by Mannheim [1998]. His calculation is based on the fact that particles moving in a gravitational potential at higher paths have greater gravitational potential energy and therefore a smaller kinetic energy, than particles on lower paths, and it therefore takes them longer to arrive at the place of interference. Mannheim finds:

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* A condensed version of the material in this chapter is accepted for publication at the American Journal of Physics [Varjú and Ryder, a].

$$\Delta\phi = \frac{gAm}{\hbar \mathrm{v}}, \tag{6.4}$$

which is equivalent to (6.1), on noting that $\mathrm{v} = \frac{p}{m}$, $p = \frac{h\kappa}{2\pi}$ and again $gA = gA_0 \sin\alpha$.

In this chapter I present my calculation to obtain the formula for the phase shift using general relativistic arguments. When this is done the calculated phase is almost, but not exactly, the same as the one found by Anandan.

6.1 The Dirac Hamiltonian

As it has already been mentioned in Chapter 2.5, following the studies of Xia *et al.* [1989] it became known that there are spin polarisation effects of spin ½ particles in the Earth's gravitational field. Since here I also want to draw attention to spin effects, in particular the Mashhoon spin-rotation coupling, the correct procedure is clearly to start with the Dirac equation in the Schwarzschild field of the Earth, and then to take, in an appropriate manner, its non-relativistic limit. It is my aim to show that in this limit we finish up with terms like (6.1) and (6.2) above, as well as correction terms; and, in addition to these, terms involving spin. It is clear, of course that in obtaining this result we shall work to certain orders of approximation in the various "small" quantities in the theory, such as v/c and Φ, the gravitational potential. Before proceeding, however, I should like to make an explanatory remark about the procedure. Some of this have been explained above, but for completeness I feel it helpful to repeat them here.

The gravitational field of the Earth is, strictly speaking, described by the Kerr solution, which is the generalisation of the Schwarzschild solution to a rotating source. The Kerr solution, as usually quoted, is given in a frame of reference which is *not* rotating; this can be envisaged as an asymptotically inertial frame, from which one "looks down" on the rotating source. In the present problem, however, the interference apparatus is on the surface of the Earth, which *is* rotating! The correct procedure is then to write the Kerr solution, but in a *rotating frame*. The exact application of the Kerr solution to the Dirac equation is, however, very complicated, and appropriate approximations have to be made; and even then the resulting Hamiltonian is not in a very tractable form.

So much for the Kerr solution. The next best procedure is to consider the Schwarzschild solution, again in a rotating frame of reference. The philosophy of this step is that the contribution of the rotation of the Earth to its *gravitational field* may be ignored; we need only retain the fact that, in whatever form we choose to represent the gravitational field of the Earth, our observations are made in a rotating frame. The Dirac Hamiltonian may be calculated in this case, to a suitable order of approximation, but this turns out also not to be tractable enough to deal with. The essence of the intractability, here and above, is that the form of the momentum operator as well as the integration measure, *in curved space*, are not trivial (see Fischbach [1980] and Varjú and Ryder [1998] or Chapter 3). To enter into the details of these would cause unnecessary trouble, particularly in view of the fact that the final result will be, by virtue of our approximations, unchanged. Finally, the Equivalence Principle* may be appealed to, and the Dirac equation written down in

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* "It may be worth remarking that the *usual* Equivalence Principle is considered to be that which describes as equivalent the gravitational field of a non-rotating body, and an accelerating frame of reference. Strictly speaking, in our view, it should be borne in mind that there are two types of non-inertial forces – accelerations and rotations – and therefore there should be two Equivalence Principles, so that, taken together, they would have the consequence that gravitational mass is equal to inertial mass both as measured by acceleration and as measured by rotation. As usually presented in General Relativity, the Equivalence

Minkowski space, but in an accelerating and rotating frame. This calculation was first performed by Hehl and Ni [1990] and provides the most suitable form of the non-relativistic Hamiltonian for our present purposes. It is important to remark once more that to the order of approximation which concerns us [see the conclusions of Chapter 5], the three calculations described above are *equivalent,* so we are perfectly justified in choosing the approximation which gives the Hamiltonian which is easiest to work with.

The Dirac Hamiltonian found by Hehl and Ni [1990] is

$$H = \beta mc^2\left(1 + \frac{\underline{a}\cdot\underline{x}}{c^2}\right) + \frac{\beta}{2m}p^2 + \frac{\beta}{2m}\underline{p}\cdot\frac{\underline{a}\cdot\underline{x}}{c^2}\underline{p} - \underline{\omega}\cdot\left(\underline{L}+\underline{S}\right) + \frac{\beta\hbar}{4mc^2}\underline{\sigma}\cdot\left(\underline{a}\times\underline{p}\right). \qquad (6.5)$$

This Hamiltonian enables us to find the phase shift, as will be explained in the next section.

Principle equates gravitational mass to inertial mass measured by acceleration, but it should be noted that the Eötvös experiment is actually concerned with rotations." [L. H. Ryder]

6.2 The phase shift

We write the total Dirac Hamiltonian in the form

$$H = H_{free} + H_{non-in}\,, \tag{6.6}$$

where the indices refer to the free particle and non-inertial terms, respectively. The phase shift is defined relative to the free particle situation, and is therefore caused by the second term. Subtracting the free particle terms from (6.5) gives

$$H_{non-in} = m(\underline{a}\cdot\underline{x}) + \frac{1}{2mc^2}\underline{p}\cdot(\underline{a}\cdot\underline{x})\underline{p} - \underline{\omega}\cdot(\underline{L}+\underline{S}) + \frac{\hbar}{4mc^2}\underline{\sigma}\cdot(\underline{a}\times\underline{p}). \tag{6.7}$$

The phase difference, to be measured in the experiment, is

$$\Delta\phi = -\frac{1}{\hbar}\oint H_{non-in}\,\mathrm{d}t\,. \tag{6.8}$$

We now consider the interferometer, consisting notionally of two paths. Because the size of the wavepacket can be assumed to be much smaller than the macroscopic dimension of the loop formed by the two alternate paths, we can apply the concept of a classical trajectory. For simplicity consider a rectangular interferometer OABC, with the beam split up at O, travelling along OAB and OCB, and finally interfering at B, as shown in the diagram. Here R is the radius of the Earth, a is the acceleration due to gravity, and x_0 and y_0 are the dimensions of the interferometer.

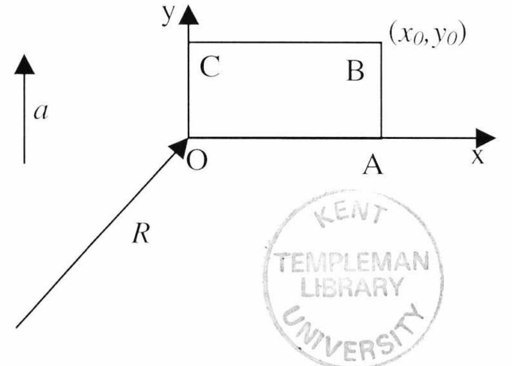

Figure 6.1: The interferometer loop.

We use the simple nonrelativistic relation

$$\oint p \mathrm{d}t = \oint m \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t = m \oint \mathrm{d}x . \tag{6.9}$$

In addition we denote by T_1 and T_2 the times taken for neutrons to travel along the lower and higher horizontal sides x_0, respectively; and by p_1 and p_2 the momentum along these paths.

The first term in (6.7), which corresponds to a "redshift of the rest mass", gives, in its contribution to (6.8)

$$\oint (\underline{a} \cdot \underline{x}) \mathrm{d}t = \underline{a} \cdot \underline{R} (T_1 - T_2) + \underline{a} \cdot \underline{y_0} (-T_2) . \tag{6.10}$$

The next term in (6.7), which corresponds to a redshift of the kinetic energy, gives:

$$\oint \underline{p} \cdot (\underline{a} \cdot \underline{x}) \underline{p} \, \mathrm{d}t = m\underline{a} \cdot \underline{R} x_0 (p_1 - p_2) - m\underline{a} \cdot \underline{y_0} p_2 x_0 . \tag{6.11}$$

The Sagnac term in (6.7) gives, in its contribution to (6.8)

$$\oint \underline{\omega} \cdot \underline{\mathrm{L}} \, \mathrm{d}t = 2 m\underline{\omega} \cdot \underline{A_0} . \tag{6.12}$$

The spin-rotation term can only be detected if the spin is flipped along one of the paths [Mashhoon, 1988]; we then have

$$\oint \underline{\omega} \cdot \underline{\mathrm{S}} \, \mathrm{d}t = 2\underline{\omega} \cdot \underline{\mathrm{S}} T_{tot} . \tag{6.13}$$

The spin-orbit coupling term gives

$$\oint \underline{\sigma} \cdot (\underline{a} \times \underline{p}) \mathrm{d}t = 2 m \, a x_0 \sigma . \tag{6.14}$$

Details of the integrations are given in Appendix 6.4.

Putting all these together the expression for the phase shift is

$$\begin{aligned} \hbar \Delta\phi &= -\oint H_{non-in} \, \mathrm{d}t = \\ &= -m\underline{a} \cdot \underline{R} (T_1 - T_2) + m\underline{a} \cdot \underline{y_0} T_2 - \frac{1}{2mc^2} \left(m\underline{a} \cdot \underline{R} x_0 (p_1 - p_2) - m\underline{a} \cdot \underline{y_0} p_2 x_0 \right) - \\ &\qquad - 2m\underline{\omega} \cdot \underline{A_0} - 2\underline{\omega} \cdot \underline{S} T_{tot} + \frac{\hbar}{4mc^2} 2m \, a x_0 \sigma \end{aligned} \tag{6.15}$$

$$\begin{aligned}\hbar\,\Delta\phi = & -m\underline{a}\cdot\underline{R}\left(T_1-T_2\right)+m\underline{a}\cdot\underline{y}_0T_2-\frac{1}{2c^2}\underline{a}\cdot\underline{R}x_0\left(p_1-p_2\right)+\\ & +\frac{1}{2c^2}\underline{a}\cdot\underline{y}_0p_2x_0-2m\underline{\omega}\cdot\underline{A}_0-2\underline{\omega}\cdot\underline{S}T_{tot}+\frac{\hbar}{2c^2}ax_0\sigma\,.\end{aligned} \tag{6.16}$$

It is useful to rewrite this after introducing the "gravitational potential"*

$$\Phi = \frac{\underline{a}\cdot\underline{R}}{c^2}. \tag{6.17}$$

The expression for time T is

$$T = \frac{x_0}{\mathrm{v}} = \frac{mx_0}{p} = \frac{m\lambda x_0}{h} \tag{6.18}$$

so that the time difference between the journeys along the upper and lower paths is

$$T_1-T_2 = mx_0\left(\frac{1}{p_1}-\frac{1}{p_2}\right) = mx_0\frac{p_1-p_2}{p_1p_2} = mx_0\frac{\Delta p}{p_2^2}. \tag{6.19}$$

Neglecting terms in $\left(\frac{\Delta p}{p_2}\right)^2$, the phase difference then becomes

$$\begin{aligned}\hbar\,\Delta\phi = & -mc^2\Phi\left(T_1-T_2\right)-\frac{1}{2}\left(p_1-p_2\right)\Phi x_0+\underline{a}\cdot\underline{y}_0\left(mT_2+\frac{1}{2c^2}p_2x_0\right)-\\ & -2m\underline{\omega}\cdot\underline{A}_0-2\underline{\omega}\cdot\underline{S}T_{tot}+\frac{\hbar}{2c^2}ax_0\sigma\end{aligned} \tag{6.20}$$

$$\begin{aligned}\hbar\,\Delta\phi = & -\frac{1}{2}\Delta p\,x_0\Phi\left(1+2\frac{m^2c^2}{p_2^2}\right)+\underline{a}\cdot\underline{y}_0\left(\frac{m^2x_0}{p_2}+\frac{1}{2c^2}p_2x_0\right)-\\ & -2m\underline{\omega}\cdot\underline{A}_0-2\underline{\omega}\cdot\underline{S}T_{tot}+\frac{\hbar}{2c^2}ax_0\sigma\,.\end{aligned} \tag{6.21}$$

The next step is to find an expression for Δp using general relativistic arguments. In the literature [see for example Werner, 1994] an expression for Δp is found using a Newtonian argument based on energy conservation, whereas below a general relativistic derivation is presented, based on Dirac's argument [Dirac, 1975]. Consider the metric

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* Strictly speaking there is no gravitational potential in GR, what's more here we are dealing with non-inertial effects purely. This phrase is used for convenience.

$$ds^2 = \left(1 + 2\frac{a \cdot r}{c^2}\right)dt^2 - dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) \tag{6.22}$$

which corresponds to an accelerating frame. Under radial free fall $d\vartheta = d\varphi = 0$, and we have

$$1 = \left(1 + 2\frac{a \cdot r}{c^2}\right)\dot{t}^2 - \dot{r}^2 , \tag{6.23}$$

where dots denote differentiation with respect to proper time. Rearranging (6.23) gives

$$1 = \left(1 + 2\frac{a \cdot r}{c^2} - \left(\frac{dr}{dt}\right)^2\right)\dot{t}^2 . \tag{6.24}$$

As a boundary condition we require that if a particle falls from $r = \rho$ the starting velocity be zero $\left.\frac{dr}{dt}\right|_{r=\rho} = 0$, therefore

$$\left.\frac{dt}{ds}\right|_{r=\rho} = \left(1 + 2\frac{a \cdot \rho}{c^2}\right)^{-1/2} . \tag{6.25}$$

In the standard way we can express the following quantity as a constant:

$$\left(1 + 2\frac{a \cdot r}{c^2}\right)\frac{dt}{ds} = b = const. = \left(1 + 2\frac{a \cdot \rho}{c^2}\right)^{1/2} . \tag{6.26}$$

Hence

$$\frac{dt}{ds} = \left(1 + 2\frac{a \cdot \rho}{c^2}\right)^{1/2}\left(1 + 2\frac{a \cdot r}{c^2}\right)^{-1} . \tag{6.27}$$

Substituting this into (6.24) gives, after rearrangement

$$\left(\frac{dr}{dt}\right)^2 = \left(1+2\frac{\underline{a}\cdot\underline{r}}{c^2}\right) - \left(1+2\frac{\underline{a}\cdot\underline{r}}{c^2}\right)^2\left(1+2\frac{\underline{a}\cdot\underline{\rho}}{c^2}\right)^{-1} =$$

$$= \left(1+2\frac{\underline{a}\cdot\underline{r}}{c^2}\right)\left(1-\frac{1+2\frac{\underline{a}\cdot\underline{r}}{c^2}}{1+2\frac{\underline{a}\cdot\underline{\rho}}{c^2}}\right) = \tag{6.28}$$

$$= \frac{1+2\frac{\underline{a}\cdot\underline{r}}{c^2}}{1+2\frac{\underline{a}\cdot\underline{\rho}}{c^2}}\left(2\frac{\underline{a}\cdot\underline{\rho}}{c^2}-2\frac{\underline{a}\cdot\underline{r}}{c^2}\right) \approx$$

$$\approx \left(1-2\frac{\underline{a}}{c^2}\cdot\left(\underline{\rho}-\underline{r}\right)\right)2\frac{\underline{a}\cdot\left(\underline{\rho}-\underline{r}\right)}{c^2}.$$

Up to 1st order in $\underline{a}$ and $\underline{\Delta r}$ this gives

$$\mathrm{v}^2 = \left(\frac{dr}{dt}\right)^2 \approx 2\frac{\underline{a}\cdot\left(\underline{\rho}-\underline{r}\right)}{c^2}. \tag{6.29}$$

This expression for v^2 was found by assuming the boundary condition that $\mathrm{v}=0$ at $r=\rho$. In our case, however, the particles travelling along the vertical arms of the interferometer never have zero velocity. To find an expression for the velocity of a particle in this situation, consider an object falling from an imaginary (higher) point, where its velocity was zero. Let us use the notation of ρ, r_1 and r_2 for distances measured from the centre of the Earth as indicated in Figure 6.2.

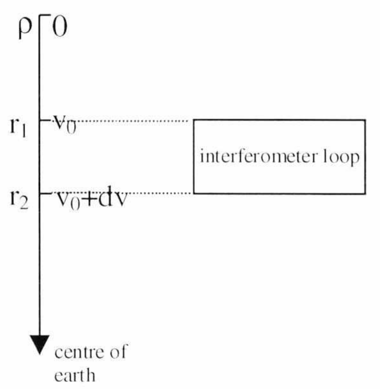

Figure 6.2: Notations used in the calculation for the derivation of the momentum.

The interferometer loop is positioned between coordinates r_1 and r_2, and ρ is the distance of a fictitious point from which a particle dropped with zero velocity gains a velocity of v_0 by the time it reaches coordinate r_1.

Using the notations in the above diagram and equation (6.29), and assuming that $\rho - r_1, r_1 - r_2 << R$ we get

$$\begin{aligned}(v_0 + dv)^2 - v_0^2 &= 2\frac{a}{c^2}(\rho - r_2 - \rho + r_1) = \\ &= 2\frac{a}{c^2}\Delta r \approx 2 v_0 \Delta v ,\end{aligned} \tag{6.30}$$

which gives

$$\Delta v \approx \frac{a}{c^2}\frac{\Delta r}{v_0} \tag{6.31}$$

and hence

$$\Delta p = \frac{ma\,\Delta r}{v_0} = \frac{m^2 a\,\Delta r}{p_0} = \frac{m^2 a\,\Delta r \lambda}{h} . \tag{6.32}$$

This expression is the same as the one obtained using a Newtonian potential, which is not surprising, considering a first order approximation was used; this is, nevertheless, a gratifying result. In addition, of course, this method enables one to find a higher order expression for Δp, if needed.

If the OABC interference loop is tilted about a horizontal angle by an angle α, then

$$\Delta r = y_0 \sin\alpha . \tag{6.33}$$

It is clear that the first term in (6.21) is second order in g. The second term, using the relation

$$\frac{p_2^2}{2m} = E_k , \tag{6.34}$$

gives

$$\underline{a} \cdot \underline{y}_0 \left(\frac{m^2 x_0}{p_2} + \frac{1}{2c^2} p_2 x_0 \right) = \frac{1}{2c^2} p_2 x_0 \underline{a} \cdot \underline{y}_0 \left(\frac{mc^2}{E_k} + 1 \right), \tag{6.35}$$

which may be written as

$$\frac{h}{2\lambda c^2} a\, A_0 \sin\alpha \left(\frac{mc^2}{E_k} + 1 \right), \tag{6.36}$$

where A_0 is the area of the interferometer. This can be re-expressed as follows

$$\begin{aligned} \frac{\hbar\pi}{\lambda c^2} aA_0 \sin\alpha \left(\frac{mc^2}{E_k} + 1 \right) &= \frac{\hbar\pi m}{\lambda} aA_0 \sin\alpha \frac{2m}{p_2^2} + \frac{\hbar\pi}{\lambda c^2} aA_0 \sin\alpha = \\ &= \frac{\hbar\pi m}{\lambda} aA_0 \sin\alpha \frac{2m\lambda^2}{h^2} + \frac{\hbar\pi}{\lambda c^2} aA_0 \sin\alpha = \\ &= \frac{2\hbar\pi m^2}{h^2} aA_0\lambda \sin\alpha + \frac{\hbar\pi}{\lambda c^2} aA_0 \sin\alpha \ . \end{aligned} \tag{6.37}$$

Finally, putting equations (6.21) and (6.37) together, the phase shift (6.8) will take the form

$$\Delta\phi = \frac{2\pi m^2}{h^2} aA_0\lambda \sin\alpha + \frac{\pi}{\lambda c^2} aA_0 \sin\alpha - \frac{2}{\hbar} m\underline{\omega} \cdot \underline{A_0} - \frac{2}{\hbar} \underline{\omega} \cdot \underline{S}\, T_{tot} + \frac{1}{2c^2} ax_0\sigma \ . \tag{6.38}$$

6.3 Conclusion

Equation (6.38) gives the phase shift expected in neutron interference experiments in a rotating frame in a gravitational field. The first and third terms in (6.38), representing the acceleration effect and the Sagnac effect, have already been detected. The second term, which is ${v^2}/{2c^2}$ times the acceleration term, is beyond the accuracy of present experiments. The fourth, Mashhoon, term, should be detectable using atomic interferometers in the near future [Audretsch and Lämmerzahl, 1992a]. The final term, originating in spin-orbit coupling, is, for thermal neutrons, approximately 10^{-10} times the Mashhoon term, so is surely a "next generation" effect. It is interesting to note that the first three terms are proportional to the area of the interferometer, whereas the last two terms are proportional to its linear dimension.

	Audretsch *et al.* [1992b]	Anandan [1977]	Werner [1994]	Mannheim [1998]	Varjú and Ryder [a]
acceleration term	$\frac{mA}{\hbar v}a$	$\frac{gAm^2}{\hbar^2\kappa}$	$2\pi m^2\frac{g}{h^2}\lambda A_0$	$\frac{gAm}{\hbar v}$	$\frac{2\pi m^2}{h^2}aA_0\lambda$
correction to acceleration	$\frac{v^2}{2c^2}\frac{mA}{\hbar v}a$	$\frac{gA\kappa}{c^2}$	–	–	$\frac{\pi}{\lambda c^2}aA_0$
Sagnac term	$\frac{2m}{\hbar}\underline{\omega}\cdot\underline{A}$	$-\frac{2\underline{\omega}\cdot\underline{A}m}{\hbar}$	$-\frac{2m}{\hbar}\omega A_0\cos\theta_L$	–	$-\frac{2}{\hbar}\underline{\omega}\cdot\underline{A_0}$
Mashhoon term	$\frac{2l_{tot}}{v}\underline{\omega}\cdot\underline{J}$	–	–	–	$-\frac{2}{\hbar}\underline{\omega}\cdot\underline{ST}_{tot}$
spin-orbit coupling	$\frac{al}{c^2}J$	–	–	–	$\frac{1}{2c^2}ax_0\sigma$
other terms	$\frac{lh^\alpha}{2\hbar v}R_{0\alpha\alpha}{}^{\beta}J^{\alpha}{}_{\beta}$ $\frac{lh^\alpha}{2\hbar v}R_{0\alpha\alpha}{}^{\beta}S^{\alpha}{}_{\beta}$	$-\frac{\hbar\underline{\omega}\cdot\underline{A}\kappa^2}{mc^2}$ $\frac{GMS_n\omega A}{mc^2R^3}$	–	–	–

Table 6.1: COW phase shifts in the literature.

Table 6.1 shows a summary of the different contributions to the phase shift for the COW experiment, as calculated by various authors. Using the formulae

$$A = A_0 \sin\alpha \ ; \ \underline{\omega}\cdot\underline{A} = \omega A_0 \cos\theta_L \cos\alpha \ ;$$
$$2\pi\frac{\lambda}{h^2} = \frac{1}{\hbar^2\kappa} \ ; \ \omega = \kappa c \ ; \tag{6.39}$$

$$\mathrm{v} = \frac{\hbar\kappa}{m} \ ; \ \frac{mA}{\hbar\mathrm{v}}a = \frac{aAm^2}{\hbar^2\kappa} \ ;$$
$$\frac{\mathrm{v}^2}{2c^2}\frac{mA}{\hbar\mathrm{v}}a = \frac{aA\kappa}{2c^2} \ ; \tag{6.40}$$

$$T_{tot} = \frac{l_{tot}}{\mathrm{v}} \ ; \ \underline{S} = \frac{1}{2}\hbar\underline{\sigma} \tag{6.41}$$

relating the different notations used in various references, one can see that:

- the leading order, acceleration, term is identical in all accounts.
- the correction to the acceleration term is not shown in Werner's and Mannheim's accounts, since they only worked to a lower order.
- in the Sagnac term there is agreement where applicable, apart from a minus sign in Audretsch's case.
- the Mashhoon term only appears in two accounts and they agree apart from the sign. In Audretsch's formula the angular momentum J includes orbital angular momentum and is in units of Planck's constant. This explains the missing factor $\hbar$.
- the two spin orbit coupling terms agree, since we may put $J = \frac{\sigma}{2}$. Audretsch's J, however, also includes an orbital contribution.
- the two other terms in Audretsch' depend on curvature, so we do not expect to get this type of contribution, as we are working in Minkowski space. Anandan's two extra terms, which are not equivalent to Audretsch's as they depend on the rotation of the source, also do not appear anywhere else.

6.4 Appendix: Evaluating the integrals

$$\oint(\underline{a}\cdot\underline{x})\mathrm{d}t = \int_O^A \underline{a}\cdot(\underline{R}+\underline{x})\mathrm{d}t + \int_A^B \underline{a}\cdot(\underline{R}+\underline{x_0}+\underline{y})\mathrm{d}t + \\ + \int_B^C \underline{a}\cdot(\underline{R}+\underline{x}+\underline{y_0})\mathrm{d}t + \int_C^O \underline{a}\cdot(\underline{R}+\underline{y})\mathrm{d}t = \\ = \underline{a}\cdot\underline{R}(T_1-T_2)+\underline{a}\cdot\underline{y_0}(-T_2)\ , \tag{6.42}$$

where we have used the fact that in the case of a vertical acceleration $\underline{a}\cdot\underline{x}=0$.

$$\oint \underline{p}\cdot(\underline{a}\cdot\underline{x})\underline{p}\,\mathrm{d}t = \oint(\underline{a}\cdot\underline{x})p^2\,\mathrm{d}t - i\hbar\oint(\underline{a}\cdot\underline{p})\mathrm{d}t = \\ = \oint(\underline{a}\cdot\underline{x})p^2\,\mathrm{d}t - i\hbar m\oint\underline{a}\cdot\mathrm{d}\underline{s} = \tag{6.43}$$

$$= \oint(\underline{a}\cdot\underline{x})p^2\,\mathrm{d}t = \\ = m\oint(\underline{a}\cdot\underline{x})\,p\mathrm{d}s = \\ = m\int_O^A p_1\underline{a}\cdot(\underline{R}+\underline{x})\mathrm{d}x + m\int_A^B p\,\underline{a}\cdot(\underline{R}+\underline{x_0}+\underline{y})\mathrm{d}y + \\ + m\int_B^C p_2\underline{a}\cdot(\underline{R}+\underline{x}+\underline{y_0})\mathrm{d}x + m\int_C^O p\,\underline{a}\cdot(\underline{R}+\underline{y})\mathrm{d}y = \tag{6.44}$$

$$= m\int_O^A p_1\underline{a}\cdot\underline{R}\,dx + m\int_B^C p_2\underline{a}\cdot(\underline{R}+\underline{y_0})dx = \\ = m\underline{a}\cdot\underline{R}\,x_0\,(p_1-p_2) - m\underline{a}\cdot\underline{y_0}p_2x_0 \tag{6.45}$$

$$\oint\underline{\omega}\cdot\underline{\mathrm{L}}\,\mathrm{d}t = \oint\underline{\omega}\cdot(\underline{\mathrm{r}}\times\underline{p})\mathrm{d}t = \\ = \oint(\underline{\omega}\times\underline{\mathrm{r}})\cdot\underline{p}\,\mathrm{d}t = \\ = m\oint(\underline{\omega}\times\underline{\mathrm{r}})\cdot\mathrm{d}\underline{s} = \tag{6.46}$$

$$= m\int_O^A\underline{\omega}\times(\underline{R}+\underline{x})\mathrm{d}x + m\int_A^B\underline{\omega}\times(\underline{R}+\underline{x_0}+\underline{y})\mathrm{d}y + \\ + m\int_B^C\underline{\omega}\times(\underline{R}+\underline{x}+\underline{y_0})\mathrm{d}x + m\int_C^O\underline{\omega}\times(\underline{R}+\underline{y})\mathrm{d}y = \tag{6.47}$$

$$
\begin{aligned}
&= m\underline{\omega}\times\underline{R}\oint ds + m\int_A^B \underline{\omega}\times\underline{x_0}\,\mathrm{d}y + m\int_B^C \underline{\omega}\times\underline{y_0}\,\mathrm{d}x = \\
&= m\underline{\omega}\times\underline{x_0}\cdot\underline{y_0} - m\underline{\omega}\times\underline{y_0}\cdot\underline{x_0} = \\
&= 2\,m\underline{\omega}\cdot\underline{x_0}\times\underline{y_0} = \\
&= 2\,m\underline{\omega}\cdot\underline{A_0}
\end{aligned}
\tag{6.48}
$$

$$
\oint\underline{\omega}\cdot\underline{\mathrm{S}}\,\mathrm{d}t = \int_O^A\underline{\omega}\cdot\underline{\mathrm{S}}\,\mathrm{d}t + \int_A^B\underline{\omega}\cdot\underline{\mathrm{S}}\,\mathrm{d}t - \int_O^C -\underline{\omega}\cdot\underline{\mathrm{S}}\,\mathrm{d}t - \int_C^B -\underline{\omega}\cdot\underline{\mathrm{S}}\,\mathrm{d}t = 2\underline{\omega}\cdot\underline{\mathrm{S}}\,T_{tot} \tag{6.49}
$$

$$
\begin{aligned}
\oint\underline{\sigma}\cdot\left(\underline{\mathrm{a}}\times\underline{p}\right)\mathrm{d}t &= \oint\left(\underline{\sigma}\times\underline{a}\right)\cdot\underline{p}\,\mathrm{d}t = \\
&= m\oint\left(\underline{\sigma}\times\underline{a}\right)\cdot\mathrm{d}\underline{s} = \\
&= m\left(\underline{\sigma}\times\underline{a}\right)\cdot\left(\int_O^A d\underline{s} + \int_A^B d\underline{s} - \int_O^C - d\underline{s} - \int_C^B - d\underline{s}\right) = \\
&= 2\,m\left(\underline{\sigma}\times\underline{a}\right)\cdot\underline{s} = 2\,m\left(\underline{a}\times\underline{s}\right)\cdot\underline{\sigma} = \\
&= 2\,m\,ax_0\sigma
\end{aligned}
\tag{6.50}
$$

7 Conclusions, final remarks

For the most part, the thesis has been concerned with the possibility of describing gravitational effects on spin-½ particles.

It is beyond any doubt that experiments carried out in our laboratories are affected by the Earth's gravity, still it is common practice to ignore this circumstance. It had been believed that, because of the order of magnitude of the effect, gravity would not manifest on the level of quantum experiments, until Colella, Overhauser and Werner proved the opposite with their remarkable experiment. This experiment created a need for a theory combining Quantum Mechanics and General Relativity. In this thesis a work has been summarised concerning the consequences of using Weyl's tetrad formalism to describe gravitational effects on quantum systems.

Writing the Dirac equation in Riemannian spaces has been the topic of textbooks and papers since 1980. Still I have not found anything in the literature of sufficiently detailed coverage of this topic. I made an attempt in Chapter 3 to give a thorough description of the problem, providing solutions to the questions I have not found being answered in the literature. I have touched subjects such as what is determined by the choice of coordinates, and how the Hamiltonian depends on the choice of the frame. I have summarised the various methods of finding the connection coefficients with examples in the Appendix. I have also given a possible solution of dealing with the epsilon symbol and a recipe of finding the form of the momentum operator in curved spaces.

The thesis proceeded in Chapter 4 to the application of the method described previously, to give a description of the effect of stationary gravitational sources on spin-½ particles. The Dirac Hamiltonian has been written in a Schwarzschild field and then compared with the corresponding result in an accelerated Minkowski space.

Comparing the Hamiltonians describing the effects of gravitational field and acceleration on spin ½ particles yields a test of the medium strong equivalence principle; that is, the statement that physical effects in an accelerated frame and a gravitational field are locally indistinguishable. The comparison gives us that the flat-space energy-mass terms and their redshifted forms are the same in the two cases. On the other hand in the case of the higher order correction terms we do not get agreement. Although both Hamiltonians contain a spin-orbit coupling term the coefficients of these are different by a factor of 2. Also, an additional term appears in our calculation in the gravitational case, which has not been mentioned before, and is the same order of magnitude as the redshift to the kinetic energy term. This term is proportional to $(\underline{x} \cdot \underline{p})^2$, i.e. the radial component of momentum, as $\underline{x}$ and $\underline{g}$ are both in the direction of the normal vector to the surface of the Earth. On neglecting all quantum corrections, we see that the differences between the two cases vanish; one term being in the form of a Darwin term that vanishes in vacuum, and the other of the form of a second derivative of the potential, which therefore represents a tidal term, and hence curvature. A test of the equivalence principle, applying as it does only to a uniform gravitational field, will take no account of this term. We conclude that the difference between the Hamiltonians in the cases of a uniformly accelerated frame and a frame in a Schwarzschild space consists only of quantum terms. This reasoning holds only in the case of spin ½ particles, as the use of the Dirac equation has been crucial in obtaining the results. It may be the case, then, that a similar problem with the equivalence principle holds for particles of all non-zero spins; but that this problem disappears for spin 0 particles.

Chapter 5 examined the effect of the Earth's field on Dirac particles. When analysing of terrestrial experiments it has to be taken into account that the Earth's gravitational effect is properly described by a rotating frame in Kerr space. The calculation of this is rather difficult, but the effect can be approximated by using a rotating frame in Schwarzschild space, or even a rotating accelerated frame. The question is, at different levels of experimental accuracy, which approximation is appropriate. To decide about the applicability of these three models when describing experimental results in Earth-based laboratories, the Hamiltonians have been calculated and compared with each other. The analysis showed that the difference between a Kerr and a Schwarzschild field becomes apparent at energies of 10^{-19} eV. From the results of Chapter 4 we concluded that the difference between an accelerated frame and the Schwarzschild field is of the same order as of the redshift of the kinetic energy, that is 10^{-11} eV. For comparison we note here that the gravitational term detected in the COW experiment (redshift of the restmass term) is of the order of 1 eV. Atomic interferometers are expected to increase this accuracy by a factor of 10^{10} so it is becoming clear that further experimental developments will make it necessary to use general relativity in analysing the behaviour of quantum systems.

A reanalysis of the COW experiments was made in Chapter 6 and a General Relativistic derivation of the phase shift was presented. The acceleration and the Sagnac terms have already been detected. The Mashhoon term is expected to be detectable using atomic interferometers in the near future, but there are terms which are far beyond the accuracy of present experiments. It is interesting to note that three of the terms in the expression for the phase shift are proportional to the area of the interferometer, whereas the other two terms are proportional to its linear dimension.

A further step in this study could be a study of torsion, and the effect it may have on quantum systems. Due to Einstein it is said that mass curves space-time and in this way gravitation takes on the aspect of a geometrical entity. In special relativity, however, mass and spin have in common that they are two conserved quantities connected to space-time. It would therefore be nice if spin also had a dynamical manifestation; this would be a generalisation of GR and the idea of torsion. Theories of torsion have a long history, but the attempts to verify it experimentally on the cosmological scale have not yet been successful. The extension of the above exercise using the theory of torsion might suggest a possible test for it in the quantum domain.

Another possibility of extending this study is to carry out the above calculations up to higher order that would enable one to describe situations where the mass or the angular velocity of the gravitating source is more substantial than in case of the Earth, such as in rotating black holes, or at the Big Bang.

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