SYMMETRY STRUCTURE FOR DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract

Having infinitely many generalised symmetries is one of the definition of integrability for non-linear differential-difference equations. Therefore, it is important to develop tools by which we can produce these quantities and guarantee the integrability. Two different methods of producing generalised symmetries are studied throughout this thesis, namely recursion operators and master symmetries. These are objects that enable one to obtain the hierarchy of symmetries by recursive action on a known symmetry of a given equation.

Our first result contains new Hamiltonian, symplectic and recursion operators for several (1+1)-dimensional differential-difference equations both scalar and multi-component. In fact in chapter 5 we give the factorization of the new recursion operators into composition of compatible Hamiltonian and symplectic operators. For the list of integrable equations we shall also provide the inverse of recursion operators if it exists.

As the second result, we have obtained the master symmetry of differential-difference KP equation. Since for (2+1)-dimensional differential-difference equations recursion operators take more complicated form, master symmetries are alternative effective tools to produce infinitely many symmetries. The notion of master symmetry is thoroughly discussed in chapter 6 and as a result of this chapter we obtain the master symmetry for the differential-difference KP (DDKP) equation. Furthermore, we also produce time dependent symmetries through $\mathrm{sl}(2, \mathbb{C})$-representation of the DDKP equation.

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Contents

Chapter 1

Introduction

The interest on the study of nonlinear differential-difference equations (NDDEs) have been increased in recent years. For the works that have been devoted to these equations one can read [1, 2, 44, 45, 46, 95] and the references therein. The importance of NDDEs can be seen in their application as models for physical phenomena such as ladder type electric circuits, vibration of particles, collapse of Langmuir waves in plasma physics, population dynamics, etc.

In differential-difference equations at least one of the space variable is discretized while the time variable is still kept continuous. Usually the solution of NDDEs are obtained through numerical calculations. There are some equations that can be transformed into a linear equations or can be solved by the inverse scattering method [1]. These two types of integrable equations are respectively known as C-integrabe and S-integrable [96].

In this thesis, as its title indicates, we study the symmetry structure of nonlinear differential-difference equations. To avoid ambiguity, throughout the thesis wherever we use the word symmetry we mean generalised symmetries. Roughly speaking generalised symmetries are symmetries depending on higher order shifts of dependent variables. The notion of generalised symmetry plays an essential role in classification of integrable systems. We say an evolutionary equation is called integrable if it possesses infinitely many generalised symmetries.

Based on this definition of integrability, we are interested to find tools by which we can produce an infinite hierarchy of generalised symmetries and guarantee the integrability. One of the well-known tool is the so-called recursion operators. Throughout this thesis we discuss the structure of this object for NDDEs and show how it can construct the space of commuting symmetries.

The construction of recursion operators has a close relation with symplectic and Hamiltonian operators. In fact a recursion operator of an equation can be obtained by the composition of its corresponding symplectic and Hamiltonian operators. In this thesis we provide a number of new recursion operators where their factorization into composition of Hamiltonian and symplectic operators is proposed. The results are presented in sections 5.2.4, 5.2.14, 5.2.15, 5.2.16. It is one of the main parts of the paper [42] which is available online and is accepted to be published in the journal of Theoretical and Mathematical Physics.

Finding a recursion operator of a given equation is not always an easy task. Furthermore when we deal with NDDEs of higher dimension (i.e. with more than one continuous independent variables), the recursion operator has a very complicated structure. These two factors lead us to look for an alternative method of producing symmetries. This method is based on using the concept of master symmetry. Master symmetries are very handy tools when we are working with NDDEs in (2+1)-dimension (i.e. with two continuous and one discrete variable).

Since we are concerned with the definition of integrability based on generalised symmetries, we devote the thesis to the symmetry structure of differential-difference equations. In fact we shall discuss all the mentioned objects which are related with generalised symmetries and the concept of integrability.

The thesis is divided into five chapters as follows: In chapter 2 we construct the space of smooth difference function that is the foundation of all the discussions throughout this thesis. Furthermore in this chapter we introduce shift operators, difference operators and evolutionary vector fields which act on the space of smooth difference functions. The concepts of Fréchet derivative and weakly non-local difference operators are also introduced in this chapter.

Chapter 3 discusses the algebraic structures on the quotient space of difference functions and introduce three operators namely symplectic, Hamiltonian and Nijenhuis operators which play an important role in the theory of integrability. The interrelation of these operators is also given in this chapter.

In chapter 4 we first introduce (1+1)-dimensional differential-difference equations and then by taking equations of this type we shall apply the notions and theorems of the previous chapters on the given equation. Chapter 4 also contains the concept of generalised symmetry and definition of integrability.

The last two chapters contain the main results of this thesis. In particular, in chapter 5 we give a long list of differential-difference equations together with their symmetries, Hamiltonian, symplectic and recursion operators. Some of these quantities are already known for the given equations. We shall point out the new results (i.e. Hamiltonian, symplectic and recursion operators) at the end of Section 5.1.

In chapter 6 we discuss the extension of NDDEs to (2+1)-dimensional differential-difference equations in which a given equation depends on two continuous and one discrete variable. Then we introduce the notion of master symmetry and employ it to the differential-difference KP (DDKP) equation. The last section of this chapter presents the $\mathrm{sl}(2,\mathbb{C})$-representation of the DDKP equation which gives another interpretation of the master symmetries in constructing time dependent symmetries.

Chapter 2

Difference functions, functionals and pseudo-difference operators

2.1 Ring of smooth difference functions

Let $\mathbf{u} = (u^{(1)}(n,t), u^{(2)}(n,t), \cdots, u^{(m)}(n,t))^T$ be an m-tuple of unknown functions of two independent variables n and t where n takes its value in integers and t comes from the real numbers. For each component, the shift of length i in the discrete variable is denoted as

$$u^{(m)}(n+i,t) \equiv u_i^{(m)} . \tag{2.1.1}$$

This will be later obtained by the action of the so-called **shift** operator. In particular, when $i = 0$ we have

$$u^{(m)}(n,t) \equiv u_0^{(m)} , \tag{2.1.2}$$

which we often simply denote by $u^{(m)}$. In general for the vector valued $\mathbf{u}$ we have

$$\mathbf{u}_i = (u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(m)})^T .$$

When m is a small number (i.e. $m = 2, 3$), instead of components $u^{(1)}, u^{(2)}, u^{(3)}$ we use letters u, v, w as follows:

$$m = 2, \qquad \mathbf{u} = (u, v)^T,$$

$$m = 3, \qquad \mathbf{u} = (u, v, w)^T.$$

Now let $\mathfrak{F}$ denote the set of all smooth functions of the form

$$f[\mathbf{u}] := f(\cdots, \mathbf{u}_{-2}, \mathbf{u}_{-1}, \mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \cdots) .$$

Notice that f does not depend explicitly on the independent variable n. For example the following two functions are elements of $\mathfrak{F}$:

$$f = \frac{1}{u^2} + \sin(v_{-1}) + v_{-2}u_1, \qquad \mathbf{u} = (u, v) , \tag{2.1.3}$$

$$g = w_2u^2 + uvw_{-1}^2, \qquad \mathbf{u} = (u, v, w) . \tag{2.1.4}$$

As we can see in these examples f and g depend on the finite number of variables. Thus each element of $\mathfrak{F}$ can be presented in the following form:

$$f[\mathbf{u}] = f(\mathbf{u}_\ell, \mathbf{u}_{\ell-1}, \cdots, \mathbf{u}_{\ell'+1}, \mathbf{u}_{\ell'}), \qquad \ell \geq \ell' .$$

So in the above two examples we have

$$f(\mathbf{u}_1, \mathbf{u}, \mathbf{u}_{-1}, \mathbf{u}_{-2}), \qquad \mathbf{u} = (u, v) ,$$

$$g(\mathbf{u}_2, \mathbf{u}_1, \mathbf{u}, \mathbf{u}_{-1}), \qquad \mathbf{u} = (u, v, w) .$$

If $\mathbf{u}$ is a one component vector depending on just u, then for

$$f(u_\ell, u_{\ell-1}, \cdots, u_{\ell'+1}, u_{\ell'}) \in \mathfrak{F}, \qquad \ell \geq \ell',$$

we call ℓ, ℓ' respectively the left and right order of the smooth function. For example the scalar function $f = u_{-2}u + u_1 u^2 + u_2$ is a difference polynomial with the left and right order as

$$\ell = 2, \quad \ell' = -2.$$

Obviously $\mathfrak{F}$ has the structure of a ring and is called the ring of smooth difference functions. We now introduce the shift operator as the ring automorphism and denote it by $\mathcal{S} : \mathfrak{F} \longrightarrow \mathfrak{F}$ which satisfies

$$\begin{aligned} &\mathcal{S}^k(f) := f(\cdots, \mathbf{u}_{-2+k}, \mathbf{u}_{-1+k}, \mathbf{u}_k, \mathbf{u}_{1+k}, \mathbf{u}_{2+k}, \cdots) \equiv f_k, \qquad k \in \mathbb{Z}, \\ &\mathcal{S}(f + \tilde{f}) = \mathcal{S}(f) + \mathcal{S}(\tilde{f}) = f_1 + \tilde{f}_1, \\ &\mathcal{S}(f\tilde{f}) = \mathcal{S}(f)\mathcal{S}(\tilde{f}) = f_1\tilde{f}_1. \end{aligned} \tag{2.1.5}$$

For example the action of $\mathcal{S}^2$ on (2.1.4) gives

$$g_2 = \mathcal{S}^2(g) = w_4 u_2^2 + u_2 v_2 w_1^2.$$

Having the shift operator we shall introduce another operator called the **Difference** operator and is given by

$$\Delta = \mathcal{S} - 1. \tag{2.1.6}$$

According to the linearity of the shift operator one can easily obtain the same axiom for the difference operator, namely

$$\Delta(c_1 f + c_2 g) = c_1\Delta(f) + c_2\Delta(g), \qquad f, g \in \mathfrak{F},\ c_1, c_2 \in \mathbb{R}.$$

The difference operator Δ does not satisfy the Leibniz rule but the action of Δ on the product of two smooth functions obey the following relation:

$$\begin{aligned}
\Delta(fg) &= (\mathcal{S}-1)(fg) = f_1g_1 - fg \\
&= f_1g_1 - f_1g + f_1g - fg \\
&= f_1\Delta(g) + \Delta(f)g \\
&= g_1\Delta(f) + f\Delta(g)\,.
\end{aligned} \tag{2.1.7}$$

In the following proposition we see a formula for higher power of the difference operator [44].

Proposition 1. *For higher positive powers of the difference operator we have the general formula*

$$\Delta^m(fg) = \sum_{r=0}^{m} \binom{m}{r} \Delta^r(f_{m-r})\Delta^{m-r}(g)\,. \tag{2.1.8}$$

Proof. For $m = 1$ we get

$$\Delta(fg) = \sum_{r=0}^{1} \frac{\Delta^r(f_{1-r})\Delta^{1-r}(g)}{r!} = f_1\Delta(g) + \Delta(f)g\,,$$

which is valid due to (2.1.7). Now assume that it holds for $m \geq 0$, then we have

$$\Delta^{m+1}(fg) = \Delta\left(\sum_{r=0}^{m} \binom{m}{r} \Delta^r(f_{m-r})\Delta^{m-r}(g)\right)\,.$$

Using (2.1.7) it can be written as the sum of the following two terms:

$$\sum_{r=0}^{m}\binom{m}{r}\Delta^r(f_{m+1-r})\Delta^{m+1-r}(g)+\sum_{r=0}^{m}\binom{m}{r}\Delta^{r+1}(f_{m-r})\Delta^{m-r}(g)$$
$$=\sum_{r=0}^{m}\binom{m}{r}\Delta^r(f_{m+1-r})\Delta^{m+1-r}(g)+\sum_{r=1}^{m+1}\binom{m}{r-1}\Delta^r(f_{m+1-r})\Delta^{m+1-r}(g)$$
$$=f_{m+1}\Delta^{m+1}(g)+\left(\sum_{r=1}^{m}(\binom{m}{r}+\binom{m}{r-1})\Delta^r(f_{m+1-r})\Delta^{m+1-r}(g)\right)+\Delta^{m+1}(f)g\,.$$

Using the identity $\binom{m}{r}+\binom{m}{r-1}=\binom{m+1}{r}$, we get

$$\Delta^{m+1}(fg)=\sum_{r=0}^{m+1}\binom{m+1}{r}\Delta^r(f_{m+1-r})\Delta^{m+1-r}(g)\,.$$

□

Hence for two arbitrary elements $f,g\in\mathfrak{F}$ we have

$$\Delta^2(fg)=\mathcal{S}^2(f)\Delta^2(g)+2\Delta(f_1)\Delta g+\Delta^2(f)g\,.$$

From (2.1.7) we can easily derive the discrete version of integration by parts as the following relation:

$$\Delta^{-1}(\Delta(f)g)=fg-\Delta^{-1}(f_1\Delta(g))\,. \tag{2.1.9}$$

Remark 1. *Elements of* $\mathfrak{F}$ *are also known as* ***local*** *smooth difference functions.*

Having defined the space of difference functions we are now ready to define some operators acting on $\mathfrak{F}$.

2.2 Evolutionary vector fields

In this section we discuss the concept of evolutionary vector fields and how it acts on the space of smooth functions $\mathfrak{F}$. We also show that unlike the shift and difference operators, it satisfies the Leibniz rule. We begin the discussion by giving the definition of a vector field.

Definition 1. *A vector field on $\mathfrak{F}$ is defined as the following operator:*

$$X = \sum_{i \in \mathbb{Z}} \sum_{\alpha=1}^{m} X^{\alpha}_{(i)} \frac{\partial}{\partial u_i^{(\alpha)}} , \qquad X^{\alpha}_{(i)} \in \mathfrak{F} .$$

Notice that since elements of $\mathfrak{F}$ depend on the finite number of variables, thus the action of X on some smooth function in $\mathfrak{F}$ produces a finite sum. For example for $m = 3$ and the polynomial $f = u_1^2 v + w_{-2} v_2^2$ we have

$$\begin{aligned} X(f) &= \sum_i \left(X^1_{(i)} \frac{\partial f}{\partial u_i} + X^2_{(i)} \frac{\partial f}{\partial v_i} + X^3_{(i)} \frac{\partial f}{\partial w_i} \right) \\ &= X^1_{(1)}(2u_1 v) + X^2_{(0)} u_1^2 + X^2_{(2)} 2 v_2 w_{-2} + X^3_{(-2)} v_2^2 . \end{aligned}$$

One can easily check that for any two elements f and g in $\mathfrak{F}$ we have

$$\sum_{i \in \mathbb{Z}, \alpha} X^{\alpha}_{(i)} \frac{\partial (fg)}{\partial u_i^{(\alpha)}} = \sum_{i \in \mathbb{Z}, \alpha} X^{\alpha}_{(i)} \frac{\partial f}{\partial u_i^{(\alpha)}} g + f \sum_{i \in \mathbb{Z}, \alpha} X^{\alpha}_{(i)} \frac{\partial g}{\partial u_i^{(\alpha)}} ,$$

which can be rewritten as

$$X(fg) = (Xf)g + f(Xg) . \tag{2.2.1}$$

Relation (2.2.1) shows that for the product of two smooth difference functions, the action of a vector field obeys the Leibniz rule. We define the commutator of two vector fields X and Y naturally as

$$[X, Y] = X \circ Y - Y \circ X \,. \tag{2.2.2}$$

Basically we are interested in the so-called evolutionary vector fields which commute with the shift operator. Consider a vector field

$$X = \sum_{i \in \mathbb{Z}} X^{\alpha}_{(i)} \frac{\partial}{\partial u^{(\alpha)}_i} \,.$$

If the following equality holds:

$$0 = [\sum_{i \in \mathbb{Z}} X^{\alpha}_{(i)} \frac{\partial}{\partial u^{(\alpha)}_i}, \mathcal{S}] \,,$$

then through direct calculation we get

$$X^{\alpha}_{(i+1)} = \mathcal{S}(X^{\alpha}_{(i)}) \,.$$

Thus such a vector field can be identified by the following element:

$$\mathbf{p}\,[\mathbf{u}] := (p^1[\mathbf{u}], p^2[\mathbf{u}], \cdots, p^m[\mathbf{u}])^T, \qquad p^i[\mathbf{u}] \in \mathfrak{F} \,,$$

where the corresponding evolutionary vector field is denoted by $D_{\mathbf{p}}$ and has the form

$$D_{\mathbf{p}} = \sum_{i \in \mathbb{Z}} \sum_{\alpha=1}^{m} \mathcal{S}^i(p^{\alpha}) \frac{\partial}{\partial u^{(\alpha)}_i} \,. \tag{2.2.3}$$

From now on for convenience we denote $\mathfrak{F}^m$ by $\mathfrak{g}$. In relation (2.2.3), $\mathbf{p}$ is called the characteristic function of the evolutionary vector field. For example consider the following element of $\mathfrak{g}$:

$$\mathbf{p} = (u_1 v + v_{-1}^2, u_2 + v)^T .$$

Then according to the formula (2.2.3) the evolutionary vector field assigned to the characteristic $\mathbf{p}$ is of the form

$$D_{\mathbf{p}} = \sum_{i\in\mathbb{Z}} \left(\mathcal{S}^i(u_1 v + v_{-1}^2)\frac{\partial}{\partial u_i} + \mathcal{S}^i(u_2 + v)\frac{\partial}{\partial v_i} \right) .$$

Obviously when $\mathbf{u} = (u)$ is a one component variable, then for any scalar function $f[\mathbf{u}] \in \mathfrak{F}$ we have

$$D_f = \sum_{i\in\mathbb{Z}} \mathcal{S}^i(f)\frac{\partial}{\partial u_i} .$$

In fact as we expected each evolutionary vector field can be uniquely determined from some element of $\mathfrak{g}$. In the following theorem [16] we also show that the space of evolutionary vector fields forms a Lie algebra.

Theorem 1. *The space of evolutionary vector fields is a Lie algebra with the Lie bracket defined in (2.2.2).*

Proof. We just need to show that for two elements $\mathbf{p}, \mathbf{q} \in \mathfrak{g}$ the commutator of the corresponding evolutionary vector field is an evolutionary vector field

$$\begin{aligned} [D_{\mathbf{p}}, D_{\mathbf{q}}] &= D_{\mathbf{p}} D_{\mathbf{q}} - D_{\mathbf{q}} D_{\mathbf{p}} \\ &= D_{\mathbf{p}}(\sum_{i,\alpha} \mathcal{S}^i(q^\alpha)\frac{\partial}{\partial u_i^{(\alpha)}}) - D_{\mathbf{q}}(\sum_{i,\alpha} \mathcal{S}^i(p^\alpha)\frac{\partial}{\partial u_i^{(\alpha)}}) . \end{aligned}$$

By expanding the expression, $\sum_{i,\alpha} \mathcal{S}^i(q^\alpha)(D_{\mathbf{p}}\frac{\partial}{\partial u_i^{(\alpha)}})$ and $\sum_{i,\alpha} \mathcal{S}^i(p^\alpha)(D_{\mathbf{q}}\frac{\partial}{\partial u_i^{(\alpha)}})$ are canceled as

$$\sum_{i,\alpha} \mathcal{S}^i(q^\alpha)(D_{\mathbf{p}} \frac{\partial}{\partial u_i^{(\alpha)}}) - \sum_{i,\alpha} \mathcal{S}^i(p^\alpha)(D_{\mathbf{q}} \frac{\partial}{\partial u_i^{(\alpha)}}) = 0\,.$$

Therefore we get

$$[D_{\mathbf{p}}, D_{\mathbf{q}}] = \sum_{i,\alpha} D_{\mathbf{p}}(\mathcal{S}^i(q^\alpha)) \frac{\partial}{\partial u_i^{(\alpha)}} - \sum_{i,\alpha} D_{\mathbf{q}}(\mathcal{S}^i(p^\alpha)) \frac{\partial}{\partial u_i^{(\alpha)}}\,.$$

Now since $D_{\mathbf{p}}$ and $D_{\mathbf{q}}$ are evolutionary vector fields they commute with the shift operator and we get

$$\sum_{i,\alpha} \mathcal{S}^i \left(D_{\mathbf{p}}(q^\alpha) - D_{\mathbf{q}}(p^\alpha) \right) \frac{\partial}{\partial u_i^{(\alpha)}} = D_{\mathbf{h}}\,,$$

where

$$h^\alpha = D_{\mathbf{p}}(q^\alpha) - D_{\mathbf{q}}(p^\alpha)\,. \tag{2.2.4}$$

□

Example 1. *Consider two elements* $\boldsymbol{p}, \boldsymbol{q} \in \mathfrak{g}$ *as*

$$\boldsymbol{p} = (u_1 + v, v_{-1} + u)^T\,,$$
$$\boldsymbol{q} = (uv_2, uv)^T\,.$$

We have

$$\begin{aligned} [D_{\boldsymbol{p}}, D_{\boldsymbol{q}}] &= \sum_i \mathcal{S}^i(h^1)\frac{\partial}{\partial u_i} + \mathcal{S}^i(h^2)\frac{\partial}{\partial v_i} \\ &= \sum_i \mathcal{S}^i(D_{\boldsymbol{p}}(q^1) - D_{\boldsymbol{q}}(p^1))\frac{\partial}{\partial u_i} + \mathcal{S}^i(D_{\boldsymbol{p}}(q^2) - D_{\boldsymbol{q}}(p^2))\frac{\partial}{\partial v_i}\,, \end{aligned}$$

where

$$D_{\boldsymbol{p}}(q^1) = (u_1 + v)v_2 + (v_1 + u_2)u\,,$$
$$D_{\boldsymbol{q}}(p^1) = u_1v_3 + uv\,,$$
$$D_{\boldsymbol{p}}(q^2) = (u_1 + v)v + (v_{-1} + u)u\,,$$
$$D_{\boldsymbol{q}}(p^2) = uv_2 + u_{-1}v_{-1}\,.$$

Therefore the characteristic function is the following vector valued function:

$$\mathbf{h} = (u_1v_2 + vv_2 + uv_1 + uu_2 - u_1v_3 - uv, u_1v + v^2 + uv_{-1} + u^2 - uv_2 - u_{-1}v_{-1})^T\,.$$

The corresponding evolutionary vector field becomes

$$D_{\mathbf{h}} = \sum_i \mathcal{S}^i(u_1v_2 + vv_2 + uv_1 + uu_2 - u_1v_3 - uv)\frac{\partial}{\partial u_i}$$
$$+\mathcal{S}^i(u_1v + v^2 + uv_{-1} + u^2 - uv_2 - u_{-1}v_{-1})\frac{\partial}{\partial v_i}\,.$$

Notice that in the differential case (where all independent variables are continuous) the similar concept of evolutionary vector fields is defined with respect to the space derivation [68, 96]. In this case evolutionary vector fields are vector fields that commute with D_x. The difference between these two definitions lies in the fact that in discrete case $\mathcal{S}$ is treated as a ring automorphism not a derivation. In the next section we introduce the equivalence class among elements of the ring of smooth functions $\mathfrak{F}$.

2.3 Space of functionals

In this section we use the difference operator (2.1.6) to introduce the space of functionals. To do this we first explain what we mean by equivalent smooth functions.

Definition 2. *Two smooth functions f and g in $\mathfrak{F}$ are called equivalent (i.e. $f \sim g$) if there exists an element $h \in \mathfrak{F}$ such that*

$$f - g = \Delta h \,. \tag{2.3.1}$$

This is the discrete analogue of the total derivative in the differential case. An obvious example for the above definition is that any shift of an arbitrary element $f \in \mathfrak{F}$ is equivalent to the original expression

$$f_1 - f = \Delta f \,, \qquad f_1 \sim f \,,$$

where according to (2.1.5) by f_1 we mean $\mathcal{S}(f)$. Therefore according to the transitivity property of equivalence relations we have

$$f_j \sim f \,.$$

For a less trivial example one can easily check that $f = u_1 u_2^2 + 2uu_1^2 + u_{-1} + u$ is equivalent to $3uu_1^2 + 2u_{-1}$ since

$$(u_1 u_2^2 + 2uu_1^2 + u_{-1} + u) - (3uu_1^2 + 2u_{-1}) = \Delta(uu_1^2 + u_{-1}) \,.$$

When f is equivalent to zero we say f is a total difference, that is there exists $h \in \mathfrak{F}$ such that

$$f = \Delta h \,.$$

Based on Definition 2 we can define the quotient space on the ring of smooth difference functions as

$$\mathfrak{F}' = \mathfrak{F}/Im(\mathcal{S} - 1)\,.$$

For an arbitrary $f \in \mathfrak{F}$ we denote the equivalence class by the functional

$$\int f\,.$$

We define the action of evolutionary vector fields on the space of functionals as follows:

$$D_{\mathbf{p}}(\int f) = \int D_{\mathbf{p}}(f), \qquad \mathbf{p} \in \mathfrak{g},\, f \in \mathfrak{F}\,. \tag{2.3.2}$$

As an example, for the characteristic function $\mathbf{p} = (u_1 v, v_{-1}+u)$ and the functional $\int u^2 v_1$ we have

$$\begin{aligned} D_{\mathbf{p}}(\int f) &= \int \sum_i \mathcal{S}^i(u_1 v)\frac{\partial f}{\partial u_i} + \mathcal{S}^i(v_{-1}+u)\frac{\partial f}{\partial v_i} \\ &= \int (2uu_1vv_1 + vu^2 + u^2u_1)\,. \end{aligned}$$

The next theorem provides the necessary condition for some function to be a total difference [95].

Theorem 2. *Let $f(\boldsymbol{u}_\ell, \boldsymbol{u}_{\ell-1}, \cdots, \boldsymbol{u}_{\ell'+1}, \boldsymbol{u}_{\ell'})$ be an element of $\mathfrak{F}$ such that for some $1 \le \alpha, \beta \le m$ we have*

$$\frac{\partial f}{\partial u_\ell^{(\alpha)}} \neq 0, \qquad \frac{\partial f}{\partial u_{\ell'}^{(\beta)}} \neq 0\,.$$

Now if f is a total difference, then

$$\frac{\partial^2 f}{\partial u_\ell^{(\alpha)} \partial u_{\ell'}^{(\beta)}} = 0\,.$$

Proof. Suppose $f(\mathbf{u}_\ell, \mathbf{u}_{\ell-1} \cdots, \mathbf{u}_{\ell'})$ is a smooth function subject to the condition

$$\frac{\partial f}{\partial u_\ell^{(\alpha)}} \neq 0, \qquad \frac{\partial f}{\partial u_{\ell'}^{(\beta)}} \neq 0\,,$$

for some $1 \leq \alpha, \beta \leq m$. If f is a total difference then there exists an element $q(\mathbf{u}_{\ell-1}, \mathbf{u}_{\ell-2}, \cdots, \mathbf{u}_{\ell'}) \in \mathfrak{F}$ such that

$$f = q(\mathbf{u}_\ell, \cdots, \mathbf{u}_{\ell'+1}) - q(\mathbf{u}_{\ell-1}, \cdots, \mathbf{u}_{\ell'}) = q_1 - q\,. \tag{2.3.3}$$

Now according to the structure of q, by differentiating both sides of (2.3.3) with respect to $u_\ell^{(\alpha)}$ and $u_{\ell'}^{(\beta)}$ we get

$$\frac{\partial}{\partial u_{\ell'}^{(\beta)}}\left(\frac{\partial f}{\partial u_\ell^{(\alpha)}}\right) = \frac{\partial}{\partial u_{\ell'}^{(\beta)}}\left(\frac{\partial q_1}{\partial u_\ell^{(\alpha)}}\right) = 0\,.$$

□

According to this theorem it is obvious that $u_1 u + u_2 v$ is not a total difference. In our calculations we encounter so often the relation

$$\Delta(f) = 0\,,$$

where f is some smooth function. For instance this happens in obtaining the coefficients of formal shift operators or explicit expression of generalised symmetries that will be discussed in chapter 4. The following theorem helps one to specify the explicit form of f [95].

Theorem 3. *Let $f \in \mathfrak{F}$ and $\Delta f = 0$, then f is a constant function.*

Proof. Without loss of generality, let f be the non-constant smooth difference function of the form

$$f(u_\ell, u_{\ell-1}, \cdots, u_{\ell'+1}, u_{\ell'})\,,$$

where

$$\frac{\partial f}{\partial u_\ell} \neq 0, \qquad \frac{\partial f}{\partial u_{\ell'}} \neq 0 \tag{2.3.4}$$

and satisfies the relation

$$f(u_{\ell+1}, u_\ell, \cdots, u_{\ell'+2}, u_{\ell'+1}) - f(u_\ell, u_{\ell-1}, \cdots, u_{\ell'+1}, u_{\ell'}) = 0\,.$$

This leads to

$$\frac{\partial f}{\partial u_{\ell'}} = 0\,,$$

which contradicts with (2.3.4) and hence f is a constant function. □

For example let $g(u_2, u_1, u, u_{-1}, u_{-2})$ be a difference polynomial of its arguments and assume

$$(\mathcal{S} - 1)(\frac{1}{uu_1}\frac{\partial g}{\partial u_2}) = 0\,.$$

Then we get [95]

$$g = \alpha u u_1 u_2 + G(u_1, u, u_{-1}, u_{-2}),$$

where α is a constant.

Remark 2. *We can naturally extend Definition 2 to elements of* $\mathfrak{g}$*. In fact two elements* $\boldsymbol{p}, \boldsymbol{q} \in \mathfrak{g}$ *are equivalent if and only if each corresponding components are equivalent. In other words*

$$\boldsymbol{p} \sim \boldsymbol{q} \Longleftrightarrow \boldsymbol{p} - \boldsymbol{q} = (\mathcal{S} - 1)\boldsymbol{h}, \qquad \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{h} \in \mathfrak{g}\,, \tag{2.3.5}$$

where $(\mathcal{S}-1)$ *acts on each component of* $\boldsymbol{h}$.

In the next section we will encounter one of the most important notions which plays the fundamental role in the definition of integrability.

2.4 Fréchet derivative

Using the concept of Fréchet derivative we will later define other closely related notions of integrable systems such as generalised symmetries and recursion operators. We start this section with the definition of Fréchet derivative and then several examples are given to clarify the calculations in practice.

Definition 3. *Let* $\boldsymbol{p} \in \mathfrak{g}$. *Then the Fréchet derivative of* $\boldsymbol{p}$ *is a linear map acting as*

$$\boldsymbol{p}_*(\boldsymbol{q}) = \frac{d}{d\epsilon}\boldsymbol{p}[\boldsymbol{u}+\epsilon\boldsymbol{q}]\bigg|_{\epsilon=0}, \tag{2.4.1}$$

where $\boldsymbol{q}$ *is also an element of* $\mathfrak{g}$.

In fact to calculate the Fréchet derivative we replace each component $u_i^{(\alpha)}$ by $u_i^{(\alpha)} + \epsilon\mathcal{S}^i(q^\alpha)$ and then differentiating with respect to ϵ and set $\epsilon = 0$. From the definition it is clear that the Fréchet derivative is a linear operator and satisfies the Leibniz rule. In the first example we consider the case in which both elements are scalar functions depending on a single variable u.

Example 2. *Let* f *and* g *be the following scalar functions:*

$$f = u_1^2 + uu_{-1}\,,$$
$$g = u^2 + u_1\,.$$

Then we get

$$\begin{aligned} f_*(g) &= \frac{d}{d\epsilon}\left((u+\epsilon g)_1^2 + (u+\epsilon g)(u+\epsilon g)_{-1}\right)|_{\epsilon=0} \\ &= 2u_1\mathcal{S}(g) + u_{-1}g + u\mathcal{S}^{-1}(g) \\ &= 2u_1^3 + 2u_1u_2 + u^2u_{-1} + u_1u_{-1} + uu_{-1}^2 + u^2 \,. \end{aligned}$$

From this example we can easily see that if f and g are both scalar functions then we have the following formula for the Fréchet derivative:

$$f_*(g) = \sum_i \frac{\partial f}{\partial u_i} S^i(g)\,. \qquad (2.4.2)$$

We proceed with an example in which $\mathbf{u}$ and $\mathbf{p}$ are both vector valued functions.

Example 3. *Assume* $f = u_1v + v_{-1}u$ *and let* $\mathbf{p} = (p^1, p^2)^T$ *be an arbitrary element of* $\mathfrak{g}$ *then*

$$\begin{aligned} f_*(\mathbf{p}) &= \frac{d}{d\epsilon}\left((u_1 + \epsilon\mathcal{S}(p^1))(v+\epsilon p^2) + (v_{-1} + \epsilon\mathcal{S}^{-1}(p^2))(u+\epsilon p^1)\right)|_{\epsilon=0} \\ &= v\mathcal{S}(p^1) + u_1p^2 + v_{-1}p^1 + u\mathcal{S}^{-1}(p^2)\,. \end{aligned}$$

Notice that when f depends on two variable u and v we denote f_{*u} and f_{*v} respectively for the Fréchet derivative with respect to u and v. So in the above example we have

$$f_{*u} = v\mathcal{S} + v_{-1}\,,$$

$$f_{*v} = u_1 + u\mathcal{S}^{-1}\,.$$

This calculation can be written in matrix form as

$$f_*(\mathbf{p}) = \begin{pmatrix} f_{*u} & f_{*v} \end{pmatrix} \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}$$

and in general we have

$$f_*(\mathbf{p}) = \sum_{i=1}^{m} (\sum_{j} \frac{\partial f}{\partial u_j^{(i)}} \mathcal{S}^j(p^i)) .$$

In the following example we consider the general case where both functions are elements of $\mathfrak{g}$ and we shall find the Fréchet derivative of two components function.

Example 4. *Consider two elements of* $\mathfrak{g}$ *given in Example 1 as*

$$\boldsymbol{p} = (u_1 + v, v_{-1} + u)^T ,$$
$$\boldsymbol{q} = (uv_2, uv)^T .$$

According to the definition of Fréchet derivative we have

$$\begin{aligned} \mathbf{p}_*(\boldsymbol{q}) &= \frac{d}{d\epsilon} \left((u + \epsilon q^1)_1 + (v + \epsilon q^2), (v + \epsilon q^2)_{-1} + (u + \epsilon q^1)\right) |_{\epsilon=0} \\ &= (\mathcal{S}(q^1) + q^2, \mathcal{S}^{-1}(q^2) + q^1) , \end{aligned}$$

which similar to the previous example it can be written in matrix form

$$\mathbf{p}_*(\mathbf{q}) = \begin{pmatrix} p^1_{*u} & p^1_{*v} \\ q^1_{*u} & q^1_{*v} \end{pmatrix} \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} .$$

This can be simply generalised in the following form to the case in which $\mathbf{p}, \mathbf{q}$ are

m component vector valued elements of $\mathfrak{g}$:

$$\mathbf{p}_*(\mathbf{q}) = \begin{pmatrix} p^1_{*u^{(1)}} & p^1_{*u^{(2)}} & \cdots & p^1_{*u^{(m)}} \\ p^2_{*u^{(1)}} & p^2_{*u^{(2)}} & \cdots & p^2_{*u^{(m)}} \\ \vdots & \vdots & \vdots & \vdots \\ p^m_{*u^{(1)}} & p^m_{*u^{(2)}} & \cdots & p^m_{*u^{(m)}} \end{pmatrix} \begin{pmatrix} q^1 \\ q^2 \\ \vdots \\ q^m \end{pmatrix}.$$

We recall that for two elements $\mathbf{p}, \mathbf{q} \in \mathfrak{g}$ the commutator of corresponding evolutionary vector fields is

$$[D_{\mathbf{p}}, D_{\mathbf{q}}] = D_{\mathbf{h}},$$

where components of $\mathbf{h}$ are defined as (2.2.4). Now the last example of this section indicates that the characteristic function $\mathbf{h}$ can be obtained by the use of Fréchet derivative.

Example 5. *Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be the two component functions given in Example 1 as*

$$\begin{aligned} \boldsymbol{p} &= (u_1 + v, v_{-1} + u)^T, \\ \boldsymbol{q} &= (uv_2, uv)^T. \end{aligned}$$

We calculate their Fréchet derivatives as follows:

$$\begin{aligned} \boldsymbol{p}_*(\boldsymbol{q}) &= \frac{d}{d\epsilon}\left((u+\epsilon q^1)_1 + (v+\epsilon q^2), (v+\epsilon q^2)_{-1} + (u+\epsilon q^1)\right)|_{\epsilon=0} \\ &= (\mathcal{S}(q^1) + q^2, \mathcal{S}^{-1}(q^2) + q^1), \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_*(\boldsymbol{p}) &= \frac{d}{d\epsilon}\left((u+\epsilon p^1)(v+\epsilon p^2)_2, (u+\epsilon p^1)(v+\epsilon p^2)\right)|_{\epsilon=0} \\ &= (v_2 p^1 + u\mathcal{S}^2(p^2), up^2 + vp^1). \end{aligned}$$

One can easily see that the function $\mathbf{h}$ in Example 1 can be obtained through the following formula:

$$\mathbf{h} = \mathbf{q}_*(\mathbf{p}) - \mathbf{p}_*(\mathbf{q}) \,.$$

Now looking back at the Lie algebra of evolutionary vector fields, clearly we can identify the space of evolutionary vector fields with $\mathfrak{g}$ through the following relation between their corresponding characteristic functions:

$$[\mathbf{p}, \mathbf{q}] := \mathbf{q}_*(\mathbf{p}) - \mathbf{p}_*(\mathbf{q}) \,. \tag{2.4.3}$$

The main concern in the next section is introducing the space of difference operators. Examples are also given to show the calculation of Fréchet derivatives on the elements of this space.

2.5 Pseudo-difference operator and weakly nonlocal difference operator

We shall study the structure of pseudo-difference operators and the notion of weakly nonlocal difference operator. We will encounter these operators when we talk about Hamiltonian structure and recursion operators.

Let $M_{\ell\times\ell}(\mathfrak{U})$ denote the space of $\ell \times \ell$ matrices over $\mathfrak{U}$ where $\mathfrak{U}$ is the space of pseudo-difference operators of the form

$$\mathfrak{U} = \{ \sum_{i=-\infty}^{N} f^{(i)}[\mathbf{u}]\mathcal{S}^i, \quad f^{(N)}[\mathbf{u}] \neq 0\} \,,$$

where $f^{(i)}[\mathbf{u}]$ is an element of $\mathfrak{F}$.

Example 6. *The following matrix is an element of* $M_{2\times 2}(\mathfrak{U})$*:*

$$\begin{pmatrix} u(\mathcal{S}^{-1}-\mathcal{S})u & u(1-\mathcal{S})v \\ v(\mathcal{S}^{-1}-1)u & 0 \end{pmatrix}.$$

If we define addition and multiplication on $\mathfrak{U}$ then naturally we can extend it to the space of $M_{\ell\times\ell}(\mathfrak{U})$. In $\mathfrak{U}$ addition of two elements is defined as the addition of coefficients of shift operators with the same power. Then $\mathfrak{U}$ forms an associative algebra with the following multiplication:

$$f^{(i)}\mathcal{S}^i \circ f^{(j)}\mathcal{S}^j = f^{(i)} f_i^{(j)} \mathcal{S}^{i+j}\,. \tag{2.5.1}$$

For example consider two elements $L_1 = v\mathcal{S} + u_{-2}v_1^2\mathcal{S}^{-3}$ and $L_2 = u\mathcal{S}^2 + u_1\mathcal{S}^{-1}$, then we have

$$L_1 \circ L_2 = vu_1\mathcal{S}^3 + vu_2 + u_{-3}u_{-2}v_1^2\mathcal{S}^{-1} + u_{-2}^2v_1^2\mathcal{S}^{-4}\,.$$

For an element $L \in \mathfrak{U}$ the inverse is also a pseudo-difference operator

$$L^{-1} = \sum_{i=-\infty}^{M} g^{(i)}[\mathbf{u}]\mathcal{S}^i, \qquad g^{(M)}[\mathbf{u}] \neq 0\,,$$

where the coefficients can be uniquely obtained through the relation

$$L^{-1}L = 1\,.$$

So we have

$$\begin{aligned}&(g^{(M)}\mathcal{S}^M + g^{(M-1)}\mathcal{S}^{M-1} + \cdots) \circ (f^{(N)}\mathcal{S}^N + f^{(N-1)}\mathcal{S}^{N-1} + \cdots)\\ &= g^{(M)}f_M^{(N)}\mathcal{S}^{N+M} + \left(g^{(M)}f_M^{(N-1)} + g^{(M-1)}f_{M-1}^{(N)}\right)\mathcal{S}^{M+N-1} + \cdots = 1\,.\end{aligned}$$

Since $g^{(M)}f_M^{(N)} \neq 0$ therefore we have $M = -N$ and the first two coefficients are expressed as

$$g^{(M)} = \frac{1}{f_{-N}^{(N)}}, \qquad g^{(M-1)} = -\frac{f_{-N}^{(N-1)}}{f_{-N}^{(N)} f_{-(N+1)}^{(N)}}.$$

So the inverse operator is a pseudo-difference operator of the following form:

$$L^{-1} = \frac{1}{f_{-N}^{(N)}}\mathcal{S}^{-N} - \frac{f_{-N}^{(N-1)}}{f_{-N}^{(N)} f_{-(N+1)}^{(N)}}\mathcal{S}^{-(N+1)} + \cdots$$

Example 7. *Consider the operator* $L = u_{-1}\mathcal{S}^2 + uv\mathcal{S}$, *then*

$$L^{-1} = \frac{1}{u_{-3}}\mathcal{S}^{-2} - \frac{u_{-2}v_{-2}}{u_{-3}v_{-4}}\mathcal{S}^{-3} + \cdots$$

Notice that the N-th root of pseudo-difference operators do not lie in the same space. In fact the coefficients will not depend on the finite number of variables. To clarify the issue consider the operator

$$L = \mathcal{S}^2 + u\,,$$

the task is to obtain the root of L. We start with the formal series:

$$L^{\frac{1}{2}} = \mathcal{S} + f^{(0)} + f^{(-1)}\mathcal{S}^{-1} + f^{(-2)}\mathcal{S}^{-2} + \cdots,$$

such that

$$L^{\frac{1}{2}} \circ L^{\frac{1}{2}} = \mathcal{S}^2 + u\,.$$

Expanding the left hand side and using (2.5.1) we have

$$\mathcal{S}^2 + u = \mathcal{S}^2 + (f_1^{(0)} + f^{(0)})\mathcal{S} + (f_1^{(-1)} + f^{(0)^2} + f^{(-1)}) + \cdots$$

Now if we compare the equal power of $\mathcal{S}$, the first two coefficients are expressed as

$$f^{(0)} = 0\,,$$
$$f^{(-1)} = u - u_{-1} + u_2 - u_{-3} + \cdots = \sum_{i=0}^{\infty} (-1)^i u_i\,.$$

So as we can see the coefficient is an infinite series and therefore $L^{\frac{1}{2}}$ is not an element of $\mathfrak{U}$. Considering relation (2.5.1) we can define the commutator of two pseudo-difference operators as

$$[L_1, L_2] = L_1 \circ L_2 - L_2 \circ L_1\,.$$

The concept of Fréchet derivative in Definition 3 can be extended to pseudo-difference operators. In the following example we calculate the Fréchet derivative of some pseudo-difference operator along a given difference function.

Example 8. *Let $L = u_1\mathcal{S}^2 + u\mathcal{S}^{-1}$ and consider the following element:*

$$g = u_{-1} + u_2 \in \mathfrak{g}\,,$$

then according to the definition of Fréchet derivative we have

$$\begin{aligned} L_*(g) &= \frac{d}{d\epsilon}\left((u+\epsilon g)_1\mathcal{S}^2 + (u+\epsilon g)\mathcal{S}^{-1}\right)|_{\epsilon=0} \\ &= g_1\mathcal{S}^2 + g\mathcal{S}^{-1} \\ &= (u+u_3)\mathcal{S}^2 + (u_{-1}+u_2)\mathcal{S}^{-1}\,. \end{aligned}$$

In general if L is represented by the following $m \times m$ matrix of difference operator:

$$(L)_{ij} = \sum_{k=-\infty}^{N} f_{ij}^{(k)} \mathcal{S}^k ,$$

then

$$(L_*(\mathbf{p}))_{ij} = \sum_{k=-\infty}^{N} (f_{ij}^{(k)})_*(\mathbf{p})\mathcal{S}^k .$$

Example 9. *Consider the element*

$$L = \begin{pmatrix} u_1 v\mathcal{S}^{-1} - u\mathcal{S} & u_{-2}\mathcal{S} + v_1 \\ v - u\mathcal{S} & 0 \end{pmatrix},$$

then for $\boldsymbol{p} = (p^1, p^2)^T \in \mathfrak{g}$ *we get*

$$L_*(\boldsymbol{p}) = \begin{pmatrix} (v\mathcal{S}(p^1) + u_1 p^2)\mathcal{S}^{-1} - p^1\mathcal{S} & \mathcal{S}^{-2}(p^1)\mathcal{S} + \mathcal{S}(p^2) \\ p^2 - p^1\mathcal{S} & 0 \end{pmatrix}.$$

Another important concept that we need to introduce is the notion of weakly nonlocal difference operators. The concept of weakly nonlocal pseudo-difference operators are the difference analogues of pseudo-differential operators [49].

Definition 4. *A weakly nonlocal difference operator is the following finite sum:*

$$\mathcal{W} = f^{(1)}(\mathcal{S}-1)^{-1} \circ g^{(1)} + f^{(2)}(\mathcal{S}-1)^{-1} \circ g^{(2)} + \cdots + f^{(\ell)}(\mathcal{S}-1)^{-1} \circ g^{(\ell)} , \quad (2.5.2)$$

where $f^{(i)}$ *and* $g^{(i)}$ *belong to the ring of smooth difference function* $\mathfrak{F}$.

Later on we see that the most of recursion operators with which we will deal contains (2.5.2). One can easily check that the weakly nonlocal pseudo-difference operators are not closed under the multiplication rule (2.5.1). In fact composition of two weakly nonlocal pseudo-difference operators is always a pseudo-difference operator but not necessarily weakly nonlocal [58].

Example 10. *Let*

$$\mathcal{W}_1 = (\mathcal{S}-1)^{-1}\frac{1}{u}, \qquad \mathcal{W}_2 = (\mathcal{S}-1)^{-1}u\,,$$

be two weakly nonlocal pseudo-difference operators. Then we have

$$(\mathcal{S}-1)^{-1}\frac{1}{u}\circ(\mathcal{S}-1)^{-1}u = 1 + (\frac{u_1}{u}+1)\mathcal{S} + (\frac{u_2}{u}+\frac{u_2}{u_1}+1)\mathcal{S}^2+\cdots$$

To proceed and introduce important features of integrable differential-difference equations we still need some background on the notion of complex over a Lie algebra and the Lie derivatives. The next section provides these purely algebraic concepts.

Chapter 3

Discrete variational calculus, Lie derivatives, Hamiltonian, symplectic and Nijenhuis operators

3.1 Complex over a Lie algebra

In this chapter we basically follow the general notations and algebraic structures given in [44, 58]. We shall give a quick review of the general theory regarding the construction of a complex over a Lie algebra. Let us start by recalling some basic definitions and notations used in the general setting of complex over Lie algebras. Notice that in what follows we use Ω and $\mathfrak{L}$ respectively to denote the vector space and the Lie algebra.

Definition 5. *A differential complex is a pair* (Ω, d) *where* Ω *admits the grading*

$$\Omega = \bigoplus_{q \geq 0} \Omega^q \tag{3.1.1}$$

and d *satisfies the axioms*

$$d : \Omega^q \longrightarrow \Omega^{q+1} , \tag{3.1.2}$$

$$d^2 = 0 . \tag{3.1.3}$$

Here d is called the exterior differential. Thus for any complex we have the following infinite sequence:

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots$$

Now we are ready to provide the concept of a complex over a Lie algebra.

Definition 6. *Let* $\mathfrak{L}$ *be a Lie algebra. For each element* $a \in \mathfrak{L}$ *assign the triple* (Ω, d, i_a) *where* i_a *is a linear operator*

$$i_a : \Omega^q \longrightarrow \Omega^{q-1} .$$

For any two arbitrary $a, b \in \mathfrak{L}$ *if*

$$i_a i_b + i_b i_a = 0 , \tag{3.1.4}$$

$$[i_a d + d i_a, i_b] = i_{[a,b]} , \tag{3.1.5}$$

then (Ω, d) *is called a complex over the Lie algebra* $\mathfrak{L}$.

Elements of the Lie algebra and Ω^q are respectively called vector fields and

q-forms. As a convention when ω is a 0-form then i_a gives the zero vector namely

$$i_a\omega = 0, \qquad \omega \in \Omega^0 .$$

Furthermore it satisfies the property

$$i_a i_a \omega = 0, \qquad \omega \in \Omega^q .$$

The pairing between 1-forms and the Lie algebra is defined as

$$< \, , \, > : \Omega^1 \times \mathfrak{L} \longrightarrow \Omega^0$$
$$(\omega \, , \, a) \longmapsto < \omega, a > = i_a \omega .$$

In a similar way we can define the notation for q-forms. In fact any q-form $\omega \in \Omega^q$ is an anti-symmetric operator $\omega : \mathfrak{L} \times \mathfrak{L} \times \cdots \times \mathfrak{L} \longrightarrow \Omega^0$ defined as

$$\omega(a_1, a_2, \cdots, a_q) = i_{a_q} i_{a_{q-1}} \cdots i_{a_1} \omega .$$

Since ω is an anti-symmetric operator we have

$$\omega(a_1, a_2, \cdots, a_i, \cdots, a_j, \cdots, a_q) = -\omega(a_1, a_2, \cdots, a_j, \cdots, a_i, \cdots, a_q) .$$

Example 11. *Let $\mathfrak{L}$ be a Lie algebra and Ω^0 a left $\mathfrak{L}$-module. So we have the following relations between Ω^0 and $\mathfrak{L}$:*

$$\mathfrak{L} \times \Omega^0 \longrightarrow \Omega^0$$
$$(a, \omega_0) \longmapsto a\omega_0 .$$

Such that for all $a, b \in \mathfrak{L}$ and $\omega \in \Omega^0$ we have

$$a(b\omega) - b(a\omega) = [a, b]\omega \,.$$

Now for $\omega \in \Omega^q$ we define the exterior differential d and the operator i_a as

$$\begin{aligned}(d\omega)(a_1, a_2, \cdots, a_{q+1}) &= \sum_i (-1)^{i+1} a_i \omega(a_1, \cdots, \hat{a}_i, \cdots, a_{q+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([a_i, a_j], \cdots, \hat{a}_i, \cdots, \hat{a}_j, \cdots, a_{q+1}), \quad (3.1.6)\end{aligned}$$

$$(i_a\omega)(a_1, a_2, \cdots, a_{q-1}) = \omega(a, a_1, \cdots, a_{q-1}) \,. \qquad (3.1.7)$$

By direct calculation one can see that d is nilpotent ($d^2 = 0$) and furthermore they meet relations (3.1.4) and (3.1.5). Throughout this thesis we will not deal with the general construction and restrict ourself to the q-forms where q takes the values $q = 0, 1, 2$.

3.2 Complex of discrete variational calculus and adjoint operators

In this section we construct a specific complex over the Lie algebra called the complex of discrete variational calculus. More details when all independent variables are continuous can be found in [15, 31, 30, 89]. The discrete version of the variational complex for difference equations is discussed in [44] and references therein. For differential-difference equations we follow the structured built up in [58]. Here we skip the complete proofs and provide the concepts that we use in our arguments.

Consider the Lie algebra $\mathfrak{g}$, the space of multi component smooth difference functions, together with the quotient space $\mathfrak{F}'$ which from now on we identify with the space of 0-forms Ω^0. According to Example 11 to obtain a complex it is sufficient to show that Ω^0 is a $\mathfrak{g}$-module. For this we first need to define the action of space $\mathfrak{g}$ on the space of functionals $\mathfrak{F}'$. Based on relation (2.3.2) we can naturally define the action of $\mathfrak{g}$ on $\mathfrak{F}'$ as follows:

$$\begin{aligned} \mathbf{p} \cdot \int f &:= \int D_{\mathbf{p}}(f) = \int \sum_{i \in \mathbb{Z}, \alpha} S^i(p^{(\alpha)}) \frac{\partial f}{\partial u_i^{(\alpha)}} \\ &= \int f_*(\mathbf{p}) \qquad \left(\int f \in \mathfrak{F}', \ \mathbf{p} \in \mathfrak{g} \right). \end{aligned} \tag{3.2.1}$$

Then we have the following proposition [16].

Proposition 2. *Let $\Omega^0 = \mathfrak{F}'$ be the space of 0-forms. Then Ω^0 is a $\mathfrak{g}$-module.*

Proof. To proof the statement we just need to show that

$$\mathbf{p}(\mathbf{q} \int f) - \mathbf{q}(\mathbf{p} \int f) = [\mathbf{p}, \mathbf{q}] \int f, \qquad \mathbf{p}, \mathbf{q} \in \mathfrak{g}, f \in \mathfrak{F}. \tag{3.2.2}$$

To do this we refer to the action of $\mathfrak{g}$ on the space of functionals. Therefore (3.2.2) can equivalently written in the form

$$\int (f_*\mathbf{q})_*(\mathbf{p}) - \int (f_*\mathbf{p})_*(\mathbf{q}) = \int f_*(\mathbf{q}_*\mathbf{p}) - \int f_*(\mathbf{p}_*\mathbf{q}).$$

To prove the above relation we start with the left hand side as

$$(f_*\mathbf{q})_*(\mathbf{p}) - (f_*\mathbf{p})_*(\mathbf{q}) = \left(\sum_i \sum_{j=1}^m \frac{\partial f}{\partial u_i^{(j)}} S^i(q^j) \right)_* (\mathbf{p}) - \left(\sum_i \sum_{j=1}^m \frac{\partial f}{\partial u_i^{(j)}} S^i(p^j) \right)_* (\mathbf{q}).$$

Expanding and rearranging the terms, we can write it in matrix form as

$$\begin{pmatrix} f_{*u^{(1)}} & \cdots & f_{*u^{(m)}} \end{pmatrix} \left(\begin{pmatrix} q^1_{*u^{(1)}} & q^1_{*u^{(2)}} & \cdots & q^1_{*u^{(m)}} \\ q^2_{*u^{(1)}} & q^2_{*u^{(2)}} & \cdots & q^2_{*u^{(m)}} \\ \vdots & \vdots & \vdots & \vdots \\ q^m_{*u^{(1)}} & q^m_{*u^{(2)}} & \cdots & q^m_{*u^{(m)}} \end{pmatrix} \begin{pmatrix} p^1 \\ p^2 \\ \vdots \\ p^m \end{pmatrix} - \begin{pmatrix} p^1_{*u^{(1)}} & p^1_{*u^{(2)}} & \cdots & p^1_{*u^{(m)}} \\ p^2_{*u^{(1)}} & p^2_{*u^{(2)}} & \cdots & p^2_{*u^{(m)}} \\ \vdots & \vdots & \vdots & \vdots \\ p^m_{*u^{(1)}} & p^m_{*u^{(2)}} & \cdots & p^m_{*u^{(m)}} \end{pmatrix} \begin{pmatrix} q^1 \\ q^2 \\ \vdots \\ q^m \end{pmatrix} \right).$$

Therefore we have

$$(f_*\mathbf{q})_*(\mathbf{p}) - (f_*\mathbf{p})_*(\mathbf{q}) = f_*(\mathbf{q}_*(\mathbf{p}) - \mathbf{p}_*(\mathbf{q})) = f_*(\mathbf{q}_*\mathbf{p}) - f_*(\mathbf{p}_*\mathbf{q}),$$

which is the desired conclusion. □

After establishing the space of 0-forms now we shall proceed by introducing the elements of 1-forms Ω^1. Every 1-form $\omega \in \Omega^1$ is defined as

$$\omega = \int \sum_{\alpha,i} f^{(\alpha,i)} du_i^{(\alpha)}, \qquad f^{(\alpha,i)} \in \mathfrak{F},$$

where $\sum$ is a finite sum and $du_i^{(\alpha)}$'s are generators of 1-forms and formal dual objects to $\frac{\partial}{\partial u_i^{(\alpha)}}$ in the following sense:

$$du_i^{(\alpha)}\left(\frac{\partial}{\partial u_j^{(\beta)}}\right) = \delta_{ij}\delta_{\alpha\beta}.$$

As an example the following expression denotes a 1-form:

$$\begin{aligned} \omega &= \int u_1^2 v_{-2} du_1 + u^3 v_1 dv + vudv_{-1} \\ &= \int f^{(1,1)} du_1 + f^{(2,0)} dv + f^{(2,-1)} dv_{-1}. \end{aligned}$$

As we know each element of the Lie algebra $\mathfrak{g}$ can be identified by an evolutionary vector field. Therefore the pairing between 1-forms and the Lie algebra is given as [44, 92]:

$$
\begin{aligned}
\omega(\mathbf{p}) :=< \omega, \mathbf{p} > &= < \int \sum_{i,\alpha} f^{(\alpha,i)} du_i^{(\alpha)}, \int \sum_{j,\beta} \mathcal{S}^j(p^{(\beta)}) \frac{\partial}{\partial u_j^{(\beta)}} > \\
&= \int \sum_{i,\alpha} f^{(\alpha,i)} \mathcal{S}^i(p^{(\alpha)}) \\
&= \int \sum_{i,\alpha} \mathcal{S}^{-i}(f^{(\alpha,i)}) p^\alpha . \qquad (3.2.3)
\end{aligned}
$$

In the last step we used the equivalent relation

$$
f^{(\alpha,i)} \mathcal{S}^i(p^{(\alpha)}) \sim \mathcal{S}^{-i}(f^{(\alpha,i)}) p^\alpha .
$$

Example 12. *Consider the following elements of 1-forms and the Lie algebra $\mathfrak{g}$:*

$$
\begin{aligned}
\omega &= \int u_1^2 v_{-2} du_1 + u^3 v_1 dv + vu dv_{-1} , \\
\boldsymbol{p} &= (uv, u_1 + v_1) ,
\end{aligned}
$$

then we have

$$
\begin{aligned}
< \omega, \boldsymbol{p} > &= \int p^1 \sum_i \mathcal{S}^{-i}(f^{(1,i)}) + p^2 \sum_i \mathcal{S}^{-i}(f^{(2,i)}) \\
&= \int p^1 \mathcal{S}^{-1}(f^{(1,1)}) + p^2 (f^{(2,0)} + \mathcal{S}(f^{(2,-1)})) \\
&= \int u^3(vv_{-3} + u_1 v_1) + u_1^2 v_1 + v_1^2(u^3 + u_1) .
\end{aligned}
$$

Relation (3.2.3) implies that any 1-form can be identified with an element of the Lie algebra $\mathfrak{g}$. In fact an arbitrary 1-form can be written in the form $\int \sum_i \mathcal{S}^{-i}(f^{(\alpha,i)}) du^{(\alpha)}$. So in the above example we have

$$
\begin{aligned}
< \omega, \mathbf{p} > &= < \left(\mathcal{S}^{-1}(u_1^2 v_{-2}), u^3 v_1 + \mathcal{S}(uv) \right) \ , \ (uv, u_1 + v_1) > \\
&= \int u^3(vv_{-3} + u_1 v_1) + u_1^2 v_1 + v_1^2(u^3 + u_1) .
\end{aligned}
$$

For the pairing (3.2.3) we have the following proposition [16].

Proposition 3. *The pairing*

$$< \ , \ >: \Omega^1 \times \mathfrak{g} \longrightarrow \Omega^0 ,$$

defined in (3.2.3) is non-degenerate.

Proof. To say the map $< \ , \ >$ is non-degenerate is equivalent to say that if

$$< \omega, \mathbf{p} >= \int \omega \cdot \mathbf{p} = 0 , \qquad \forall \omega \in \Omega^1 , \tag{3.2.4}$$

then $\mathbf{p} = 0$. The action of difference operator (2.1.6) is naturally extended to the Lie algebra $\mathfrak{g}$ in the following way:

$$(\mathcal{S} - 1)\mathbf{p} = ((\mathcal{S} - 1)p^1, (\mathcal{S} - 1)p^2, \cdots , (\mathcal{S} - 1)p^m) .$$

Now suppose (3.2.4) holds and assume $\mathbf{p}$ is a non-zero element of $\mathfrak{g}$. Let ω be an the identity element, namely $\omega \cdot \mathbf{p} = \mathbf{p}$ then $\mathbf{p} = (\mathcal{S} - 1)\mathbf{h}$ for some smooth difference function $\mathbf{h}$. Now pick the component $p^\alpha(\mathbf{u}_\ell, \cdots , \mathbf{u}_{\ell'})$ and without loss of generality we can assume for some $1 \leq i, j \leq m$ we have

$$\frac{\partial p^\alpha}{\partial u^{(i)}_\ell} \neq 0 , \qquad \frac{\partial p^\alpha}{\partial u^{(j)}_{\ell'}} \neq 0 .$$

According to Theorem 2, since p^α is a total difference $\frac{\partial^2 p^\alpha}{\partial u^{(i)}_\ell u^{(j)}_{\ell'}}$ vanishes. On the other hand if we set $\omega = \mathbf{u}_\ell$ then

$$\frac{\partial^2 p^\alpha}{\partial u^{(i)}_\ell u^{(j)}_{\ell'}} \neq 0 .$$

So this contradiction shows that fact that $\mathbf{p} \equiv 0$. □

In the following definition we present the action of exterior differential on an arbitrary element of 0-form.

Definition 7. *For any* 0*-form* $\omega = \int f$ *we define the action of exterior differential as*

$$
\begin{aligned}
< d\omega, \boldsymbol{p} > &= < d\int f, \boldsymbol{p} >= \int \sum \frac{\partial f}{\partial u_i^{(\alpha)}} \mathcal{S}^i(p^\alpha) \\
&= \int \sum \mathcal{S}^{-i}(\frac{\partial f}{\partial u_i^{(\alpha)}}) p^\alpha =< \frac{\delta f}{\delta \boldsymbol{u}}, \boldsymbol{p} > . \qquad (3.2.5)
\end{aligned}
$$

where $\frac{\delta f}{\delta \boldsymbol{u}} = (\frac{\delta f}{\delta u^{(1)}}, \frac{\delta f}{\delta u^{(2)}}, \cdots, \frac{\delta f}{\delta u^{(m)}})^T$ *and each component is defined*

$$
\frac{\delta f}{\delta u^{(\alpha)}} = \sum_{i=\ell'}^{\ell} \mathcal{S}^{-i}(\frac{\partial f}{\partial u_i^{(\alpha)}}) = \sum_{i=\ell'}^{\ell} (\frac{\partial f_{-i}}{\partial u^{(\alpha)}}) = \sum_{i=-\ell}^{-\ell'} \frac{\partial f_i}{\partial u^{(\alpha)}} . \qquad (3.2.6)
$$

Here the symbol $\frac{\delta}{\delta u^{(\alpha)}}$ *is called variational derivative.*

From the notion of variational derivative, given already in the above definition, we can deduce the Fréchet derivative of a 0-form $\omega = \int f$ along an element $\mathbf{p} \in \mathfrak{g}$

$$
\begin{aligned}
\omega_*(\mathbf{p}) &= \int f_*(\mathbf{p}) = \int \sum_{i=1}^{m} \left(\sum_j \frac{\partial f}{\partial {u^{(i)}}_j} \mathcal{S}^j(p^i) \right) \sim \int \sum_{i=1}^{m} \left(\sum_j \mathcal{S}^{-j}(\frac{\partial f}{\partial {u^{(i)}}_j}) \right) p^{(i)} \\
&= < \frac{\delta f}{\delta \mathbf{u}}, \mathbf{p} > .
\end{aligned}
$$

Example 13. *Consider the following* 0*-form and the element* $\boldsymbol{p} \in \mathfrak{g}$*:*

$$
\omega = \int u_1 v_{-1}, \qquad \boldsymbol{p} = (v_1, u_{-1}) .
$$

To obtain the action of the Fréchet derivative of ω *along* $\boldsymbol{p}$*, we first need to compute* *the variational derivative of f*

$$\frac{\delta f}{\delta \boldsymbol{u}} = \begin{pmatrix} \frac{\delta f}{\delta u} \\ \frac{\delta f}{\delta v} \end{pmatrix} = \begin{pmatrix} \mathcal{S}^{-1}(v_{-1}) \\ \mathcal{S}(u_1) \end{pmatrix}, \qquad \boldsymbol{u} = (u, v)\,.$$

Hence we have

$$\omega_*(\boldsymbol{p}) =< \frac{\delta f}{\delta(u,v)}, \boldsymbol{p} >=< (v_{-2}, u_2), (v_1, u_{-1}) >= v_1 v_{-2} + u_2 u_{-1}\,.$$

The following proposition shows the variational derivative does not depend on the choice of a representative f in the equivalence class $\int f$. We just give the sketch of proof, for further details one can read [95].

Proposition 4. *For two elements $f, g \in \mathfrak{g}$ if*

$$f \sim g\,,$$

then

$$\frac{\delta f}{\delta \boldsymbol{u}} = \frac{\delta g}{\delta \boldsymbol{u}}\,.$$

Proof. As f, g are two equivalent smooth difference function therefore for some $h \in \mathfrak{F}$ we have

$$f - g = (\mathcal{S} - 1)h\,.$$

Then by direct calculation and considering the dependency of h we obtain

$$\frac{\delta(\mathcal{S} - 1)h}{\delta \mathbf{u}} = 0\,.$$

□

What is still lacking is an explicit description for the action of exterior differential on the space of 1-forms. To do this we need the concept of adjoint operator which will be discussed in the next section.

3.3 Adjoint operator

Besides the elements of the Lie algebra and 1-forms we are interested in the operators acting between these two spaces. For any such operators we assign the notion of adjoint operator.

Definition 8. *Let* $A : \mathfrak{g} \longrightarrow \mathfrak{g}$ *be a linear operator. The corresponding adjoint operator is a map* $A^\dagger : \Omega^1 \longrightarrow \Omega^1$ *such that*

$$< A\boldsymbol{p}, \omega >=< \boldsymbol{p}, A^\dagger\omega >, \qquad \boldsymbol{p} \in \mathfrak{g},\ \omega \in \Omega^1 .$$

Through the similar way we can define the adjoint operators acting between 1-forms and $\mathfrak{g}$ and vise versa. Then we call an operator anti-symmetric if and only if

$$A = -A^\dagger .$$

In practice to obtain the explicit form of the adjoint operator one should notice since $< A\mathbf{p}, \omega >$ lies within the space of 0-forms we can use the equivalence classes to produce $A^\dagger$. Let us look how this works.

Example 14. *Let* $\mathcal{H} : \Omega^1 \longrightarrow \mathfrak{g}$ *be the following operator:*

$$\mathcal{H} = \begin{pmatrix} 0 & u(\mathcal{S} - 1) \\ (1 - \mathcal{S}^{-1})u & 0 \end{pmatrix},$$

according to the definition we have

$$< \mathcal{H}\xi, \omega > = < \xi, \mathcal{H}^{\dagger}\omega > .$$

As we can identify any 1-form with an element of the Lie algebra $\mathfrak{g}$ we can write

$$< \mathcal{H}\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} > = < \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \mathcal{H}^{\dagger}\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} > .$$

So expanding the left hand side we have

$$\begin{aligned} < \mathcal{H}\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} > &= \int (\xi^2_1 u\omega^2 - \xi^2 u\omega^2 + u\xi^1\omega^1 - u_{-1}\xi^1_{-1}\omega^1) \\ &\sim \int (\xi^2 u_{-1}\omega^2_{-1} - \xi^2 u\omega^2 + u\xi^1\omega^1 - u\xi^1\omega^1_1) . \end{aligned}$$

Therefore

$$< \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \mathcal{H}^{\dagger}\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} > = \int (\xi^2 u_{-1}\omega^1_{-1} - \xi^2 u\omega^2 + u\xi^1\omega^1 - u\xi^1\omega^1_1) ,$$

that means

$$\mathcal{H}^{\dagger} = \begin{pmatrix} 0 & u(1-\mathcal{S}) \\ (\mathcal{S}^{-1} - 1)u & 0 \end{pmatrix} = -\mathcal{H} .$$

By the similar argument for the operator $A = f\mathcal{S}^n$ where $f \in \mathfrak{F}$ one can find that the adjoint operator is

$$A^{\dagger} = \mathcal{S}^{-n} \circ f .$$

Remark 3. *If L is a local anti-symmetric scalar difference operator, then it takes the form*

$$L = \sum_{i=1}^{N} (f^{(i)}\mathcal{S}^i - f^{(i)}_{-i}\mathcal{S}^{-i}) .$$

Now we are ready to obtain the explicit formula for $d\omega$ when ω is an arbitrary 1-form. Consider relations (3.1.6) and (3.2.1) then we deduce

$$
\begin{aligned}
(d\omega)(\mathbf{p},\mathbf{q}) &= \mathbf{p}\omega(\mathbf{q}) - \mathbf{q}\omega(\mathbf{p}) - \omega([\mathbf{p},\mathbf{q}]) \\
&= (\omega\cdot\mathbf{q})_*(\mathbf{p}) - (\omega\cdot\mathbf{p})_*(\mathbf{q}) - \omega(\mathbf{q}_*(\mathbf{p}) - \mathbf{p}_*(\mathbf{q})) \\
&= <\omega_*\mathbf{p},\mathbf{q}> - <\mathbf{p},\omega_*\mathbf{q}> \\
&= <(\omega_* - \omega_*^\dagger)\mathbf{p},\mathbf{q}> \,. \qquad (3.3.1)
\end{aligned}
$$

Notice that in the above computation we have used the following formula:

$$
(\omega(\mathbf{q}))_*(\mathbf{p}) = \omega_*[\mathbf{p}](\mathbf{q}) + \omega(\mathbf{p}_*(\mathbf{q}))\,, \qquad (3.3.2)
$$

where $\omega_*[\mathbf{p}]$ is an element of Ω^1 acting on $\mathbf{q}$. It is also worth to give the following remark.

Remark 4. *Suppose W is one of the spaces $\mathfrak{g}$, Ω^1 or the space of linear operators acting in $\mathfrak{g}$ and Ω^1 and between these two spaces. For $\Phi \in W$ the Fréchet derivative is the linear operator $\Phi_* : \mathfrak{g} \longrightarrow W$ obtained according to the formula (2.4.1).*

We say the 1-form ω is closed if $d\omega = 0$ and is called exact if there exists $\xi \in \Omega^0$ such that

$$
\omega = d\xi\,. \qquad (3.3.3)
$$

Since for any arbitrary 0-form ξ we have $d(d\xi) = 0$, then an exact 1-form ω satisfying (3.3.3) is always closed. The following proposition provides a necessary condition for a 1-form to be closed [16, 89].

Proposition 5. *Let $\omega \in \Omega^1$ be a 1-form. If ω is closed then*

$$
\omega_* = \omega_*^\dagger\,.
$$

Proof. As ω is a closed 1-form we have

$$d\omega = 0\,.$$

According to (3.3.1) for arbitrary $\mathbf{p}, \mathbf{q} \in \mathfrak{g}$ we have

$$d\omega(\mathbf{p}, \mathbf{q}) =< (\omega_* - \omega_*^\dagger)\mathbf{p}, \mathbf{q} >= 0\,.$$

Due to arbitrariness of $\mathbf{p}, \mathbf{q}$ we get the result $\omega_* = \omega_*^\dagger$. □

3.4 Lie derivatives

In this section we shall discuss another operator called the Lie derivative acting on the space of q-forms. This operator plays an essential role in the theory of integrability. In fact given an equation, we will define all the concepts of Hamiltonian structure, symmetries and conserved densities with respect to the Lie derivative.

Definition 9. *For any element $\boldsymbol{p} \in \mathfrak{g}$, the Lie derivative along that element is an element of the space $Aut(\Omega^q)$ defined as*

$$L_p = i_p d + d i_p\,. \tag{3.4.1}$$

According to the structure of the Lie derivative it sends q-forms to q-forms. We give two properties of Lie derivatives in the following proposition [16].

Proposition 6. *Let $\boldsymbol{p}, \boldsymbol{q} \in \mathfrak{g}$ be two arbitrary elements of the Lie algebra then we have*

1. $L_p d = d L_p$;

2. $[L_p, L_q] = L_{[p,q]}$.

Proof. To prove the first formula, we calculate the bracket of the Lie derivative $L_{\mathbf{p}}$ and d

$$[L_{\mathbf{p}}, d] = (i_{\mathbf{p}}d + di_{\mathbf{p}})d - d(i_{\mathbf{p}}d + di_{\mathbf{p}}) = 0 .$$

For the second relation we expand the left hand side as

$$\begin{aligned} [L_{\mathbf{p}}, L_{\mathbf{q}}] &= (i_{\mathbf{p}}di_{\mathbf{q}}d + di_{\mathbf{p}}i_{\mathbf{q}}d + di_{\mathbf{p}}di_{\mathbf{q}}) - (i_{\mathbf{q}}di_{\mathbf{p}}d + di_{\mathbf{q}}i_{\mathbf{p}}d + di_{\mathbf{q}}di_{\mathbf{p}}) \\ &= i_{[\mathbf{p},\mathbf{q}]}d + di_{[\mathbf{p},\mathbf{q}]} = L_{[\mathbf{p},\mathbf{q}]} . \end{aligned}$$

In the second line we used the property (3.1.5) to obtain the result. □

Based on relation (3.4.1), Lie derivatives act on the space of q-forms. We also define Lie derivatives on the elements of the Lie algebra as

$$L_{\mathbf{p}}\mathbf{q} = [\mathbf{p}, \mathbf{q}] = \mathbf{q}_*(\mathbf{p}) - \mathbf{p}_*(\mathbf{q}) , \qquad \mathbf{p}, \mathbf{q} \in \mathfrak{g} , \tag{3.4.2}$$

which is consistent with the second formula in Proposition 6. Now by the property (3.1.5) we have

$$L_{\mathbf{p}}i_{\mathbf{q}} = i_{[\mathbf{p},\mathbf{q}]} + i_{\mathbf{q}}L_{\mathbf{p}}$$

and thus the Lie derivative satisfies the Leibniz rule

$$L_{\mathbf{p}}(\omega(\mathbf{q})) = \omega(L_{\mathbf{p}}\mathbf{q}) + (L_{\mathbf{p}}\omega)(\mathbf{q}) \qquad \omega \in \Omega^1 .$$

Similarly applying the same argument recursively we obtain the following formula for the space of 2-forms:

$$L_{\mathbf{p}}(\omega(\mathbf{q}, \mathbf{h})) = \omega(L_{\mathbf{p}}\mathbf{q}, \mathbf{h}) + \omega(\mathbf{q}, L_{\mathbf{p}}\mathbf{h}) + (L_{\mathbf{p}}\omega)(\mathbf{q}, \mathbf{h}) \qquad \omega \in \Omega^2 .$$

In general if we extend the Lie derivative to linear operators acting on Ω^0, Ω^1 and $\mathfrak{g}$ or between these three spaces, then by Leibniz rule we have the relation [15]

$$(L_{\mathbf{p}}T)\sigma = L_{\mathbf{p}}(T\sigma) - T(L_{\mathbf{p}}\sigma)\,, \tag{3.4.3}$$

where T is a linear operator and σ can be either an element of 0, 1-forms or the Lie algebra $\mathfrak{g}$. At the end of this section we give the theorem [16, 89] which provides the action of Lie derivative expressed in the Fréchet derivatives. The theorem presents some handy formulae that we shall use in the next chapter where we deal with differential-difference equations.

Theorem 4. *Let $\boldsymbol{p} \in \mathfrak{g}$ be a vector field. The action of Lie derivative $L_{\boldsymbol{p}}$ is given by the formula*

1. $L_{\boldsymbol{p}} \int f = \int f_*(\boldsymbol{p})\,, \qquad \int f \in \mathfrak{F}'\,;$
2. $L_{\boldsymbol{p}}\omega = \omega_*[\boldsymbol{p}] + \boldsymbol{p}_*^{\dagger}\omega\,, \qquad \omega \in \Omega^1\,;$
3. $L_{\boldsymbol{p}}\mathcal{H} = \mathcal{H}_*[\boldsymbol{p}] - \boldsymbol{p}_*\mathcal{H} - \mathcal{H}\boldsymbol{p}_*^{\dagger}\,, \qquad \mathcal{H} : \Omega^1 \to \mathfrak{g}\,;$
4. $L_{\boldsymbol{p}}\mathcal{I} = \mathcal{I}_*[\boldsymbol{p}] + \mathcal{I}\boldsymbol{p}_* + \boldsymbol{p}_*^{\dagger}\mathcal{I}\,, \qquad \mathcal{I} : \mathfrak{g} \to \Omega^1\,;$
5. $L_{\boldsymbol{p}}\mathcal{R} = \mathcal{R}_*[\boldsymbol{p}] - \boldsymbol{p}_*\mathcal{R} + \mathcal{R}\boldsymbol{p}_*\,, \qquad \mathcal{R} : \mathfrak{g} \to \mathfrak{g}\,.$

Proof. The first formula can be obtained directly from the definition of Lie derivatives. For the second formula for an arbitrary $\mathbf{q} \in \mathfrak{g}$ we have

$$(L_{\mathbf{p}}\omega)\mathbf{q} \;=\; L_{\mathbf{p}}(\omega(\mathbf{q})) - \omega(L_{\mathbf{p}}\mathbf{q})\,.$$

Since $i_{\mathbf{p}}\omega(\mathbf{q}) = 0$ then the right hand side of the above relation is equal to

$$\omega_*[\mathbf{p}](\mathbf{q}) + \omega(\mathbf{p}_*(\mathbf{q})) = \omega_*[\mathbf{p}](\mathbf{q}) + \mathbf{p}_*^{\dagger}[\omega](\mathbf{q})\,.$$

Therefore the second property holds as $\mathbf{q}$ is an arbitrary element. Notice that in the above proof we used Relation (3.3.2). To prove other relations, we take into account the Leibniz rule and follow the same method. So to prove the third formula for an arbitrary $\omega \in \Omega^1$ we have

$$\begin{aligned}
(L_{\mathbf{p}}\mathcal{H})\omega &= L_{\mathbf{p}}(\mathcal{H}\omega) - \mathcal{H}(L_{\mathbf{p}}\omega) \\
&= [\mathbf{p}, \mathcal{H}\omega] - \mathcal{H}(\omega_*\mathbf{p} + \mathbf{p}_*^\dagger\omega) \\
&= (\mathcal{H}\omega)_*(\mathbf{p}) - \mathbf{p}_*(\mathcal{H}\omega) - \mathcal{H}(\omega_*\mathbf{p}) - \mathcal{H}(\mathbf{p}_*^\dagger\omega) \\
&= \mathcal{H}_*[\mathbf{p}](\omega) + \mathcal{H}(\omega_*\mathbf{p}) - \mathbf{p}_*(\mathcal{H}\omega) - \mathcal{H}(\omega_*\mathbf{p}) - \mathcal{H}(\mathbf{p}_*^\dagger\omega)\,.
\end{aligned}$$

Again due to the arbitrariness of ω we obtain the formula

$$L_{\mathbf{p}}\mathcal{H} = \mathcal{H}_*\mathbf{p} - \mathbf{p}_*\mathcal{H} - \mathcal{H}\mathbf{p}_*^\dagger\,.$$

For the next relation we calculate the action of the left hand side on some element $\mathbf{q}$ in the Lie algebra $\mathfrak{g}$

$$\begin{aligned}
(L_{\mathbf{p}}\mathcal{I})\mathbf{q} &= L_{\mathbf{p}}(\mathcal{I}\mathbf{q}) - \mathcal{I}(L_{\mathbf{p}}\mathbf{q}) \\
&= (\mathcal{I}\mathbf{q})_*\mathbf{p} + \mathbf{p}_*^\dagger\mathcal{I}\mathbf{q} - \mathcal{I}(\mathbf{q}_*(\mathbf{p}) - \mathbf{p}_*(\mathbf{q})) \\
&= (\mathcal{I}_*\mathbf{p} + \mathcal{I}\mathbf{p}_* + \mathbf{p}_*^\dagger\mathcal{I})(\mathbf{q})\,.
\end{aligned}$$

Finally for the last formula we pick an arbitrary element of the Lie algebra $\mathbf{q} \in \mathfrak{g}$, then we have

$$\begin{aligned}
(L_{\mathbf{p}}\mathcal{R})\mathbf{q} &= L_{\mathbf{p}}(\mathcal{R}\mathbf{q}) - \mathcal{R}(L_{\mathbf{p}}\mathbf{q}) \\
&= (\mathcal{R}\mathbf{q})_*(\mathbf{p}) - \mathbf{p}_*(\mathcal{R}\mathbf{q}) - \mathcal{R}\left(\mathbf{q}_*(\mathbf{p}) - \mathbf{p}_*(\mathbf{q})\right) \\
&= (\mathcal{R}_*[\mathbf{p}] - \mathbf{p}_*\mathcal{R} + \mathcal{R}\mathbf{p}_*)(\mathbf{p})\,.
\end{aligned}$$

Therefore all the identities hold for a given $\mathbf{p} \in \mathfrak{g}$. □

Definition 10. *Let* $\boldsymbol{q} \in \mathfrak{g}$, $\omega \in \Omega^q$ $(q = 0, 1, 2)$ *and* A *be an operator acting in the space of* $\mathfrak{g}$ *and* Ω^q *or acting between these spaces. Then they are called invariant along the vector field* $\boldsymbol{p} \in \mathfrak{g}$ *if*

$$L_{\boldsymbol{p}} \boldsymbol{q} = 0\,, \quad L_{\boldsymbol{p}} \omega = 0\,, \quad L_{\boldsymbol{p}} A = 0\,. \tag{3.4.4}$$

In the following section we discuss the operators acting between the space of 1-forms and the Lie algebra $\mathfrak{g}$.

3.5 Symplectic, Hamiltonian and Nijenhuis operators

This section is devoted to the concept of some crucial operators in the theory of integrability. Here we provide the algebraic structures of these operators based on the definitions and theorems given in previous sections. Later on we will show how these operators are related to some features of integrable systems such as symmetries and conserved densities.

For any anti-symmetric linear operator $\mathcal{I} : \mathfrak{g} \longrightarrow \Omega^1$, there is a 2-form $\omega_{\mathcal{I}} \in \Omega^2$ uniquely defined by

$$\omega_{\mathcal{I}}(\mathbf{p}, \mathbf{q}) = < \mathcal{I}\mathbf{p}, \mathbf{q} > = - < \mathcal{I}\mathbf{q}, \mathbf{p} > \,. \tag{3.5.1}$$

Definition 11. *An anti-symmetric operator* $\mathcal{I} : \mathfrak{g} \longrightarrow \Omega^1$ *is called symplectic iff the corresponding* 2*-form (3.5.1) is closed, that is,*

$$d\omega_{\mathcal{I}} = 0\,. \tag{3.5.2}$$

Here $\omega_{\mathcal{I}}$ has the following canonical form:

$$\omega_{\mathcal{I}} = \frac{1}{2}\int du \wedge \mathcal{I} du\,. \tag{3.5.3}$$

Remark 5. *Since there is a one-to-one correspondence between the symplectic operator $\mathcal{I} : \mathfrak{g} \longrightarrow \Omega^1$ and 2-form $\omega_{\mathcal{I}}$ we have*

$$L_p\mathcal{I} = (di_p + i_p d)\mathcal{I} = di_p\mathcal{I}\,. \tag{3.5.4}$$

Notice that as $\mathcal{I}$ can be treated as a 2-form, therefore $di_{\mathbf{p}}\mathcal{I}$ is well defined and is equal to $d\mathcal{I}(\mathbf{p})$. Also from relation (3.5.2), $d\mathcal{I} = 0$ which gives the above result. The following theorem [26] can be considered as an alternative equivalent definition of a symplectic operator.

Theorem 5. *An anti-symmetric operator $\mathcal{I} : \mathfrak{g} \longrightarrow \Omega^1$ is symplectic if and only if*

$$< \mathcal{I}_*[\boldsymbol{p}](\boldsymbol{q}), \boldsymbol{h} > + < \mathcal{I}_*[\boldsymbol{q}](\boldsymbol{h}), \boldsymbol{p} > + < \mathcal{I}_*[\boldsymbol{h}](\boldsymbol{p}), \boldsymbol{q} >= 0\,, \qquad \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{h} \in \mathfrak{g}. \tag{3.5.5}$$

Proof. The proof involves calculating the action of 3-form $d\omega_{\mathcal{I}}$ on arbitrary elements $\mathbf{p}, \mathbf{q}, \mathbf{h} \in \mathfrak{g}$

$$\begin{aligned} d\omega_{\mathcal{I}}(\mathbf{p}, \mathbf{q}, \mathbf{h}) &= \mathbf{p}\omega_{\mathcal{I}}(\mathbf{q}, \mathbf{h}) - \omega_{\mathcal{I}}([\mathbf{p}, \mathbf{q}], \mathbf{h}) + cycl. \\ &= \mathbf{p} < \mathcal{I}\mathbf{q}, \mathbf{h} > - < \mathcal{I}(\mathbf{q}_*(\mathbf{p}) - \mathbf{p}_*(\mathbf{q})), \mathbf{h} > + cycl. \\ &= (\mathcal{I}\mathbf{q})_*[\mathbf{p}]\mathbf{h} + \mathbf{h}_*(\mathbf{p})\mathcal{I}\mathbf{q} - < \mathcal{I}(\mathbf{q}_*(\mathbf{p}) - \mathbf{p}_*(\mathbf{q})), \mathbf{h} > + cycl. \\ &= < \mathcal{I}_*[\mathbf{p}](\mathbf{q}), \mathbf{h} > + < \mathcal{I}\mathbf{q}, \mathbf{h}_*(\mathbf{p}) > - < \mathcal{I}\mathbf{h}, \mathbf{p}_*(\mathbf{q}) > + cycl. \end{aligned}$$

If we write out all the terms we get

$$d\omega_{\mathcal{I}}(\mathbf{p}, \mathbf{q}, \mathbf{h}) =< \mathcal{I}_*[\mathbf{p}](\mathbf{q}), \mathbf{h} > + < \mathcal{I}_*[\mathbf{q}](\mathbf{h}), \mathbf{p} > + < \mathcal{I}_*[\mathbf{h}](\mathbf{p}), \mathbf{q} >,$$

which means $\mathcal{I}$ is a symplectic operator iff (3.5.5) holds. $\square$

Using the Lie derivative one can write an equivalent formula for (3.5.5) as

$$< L_{\mathbf{p}}(\mathcal{I}\mathbf{q}), \mathbf{h} > + < L_{\mathbf{q}}(\mathcal{I}\mathbf{h}), \mathbf{p} > + < L_{\mathbf{h}}(\mathcal{I}\mathbf{p}), \mathbf{q} >= 0 \,.$$

Example 15. *Consider the* 2×2 *matrix operator* $\mathcal{I} : \mathfrak{g} \longrightarrow \Omega^1$ *as*

$$\mathcal{I} = \begin{pmatrix} 0 & \frac{1}{1-uv_1}\mathcal{S} - \frac{1}{1+uv} \\ \frac{1}{1+uv} - \mathcal{S}^{-1}\frac{1}{1-uv_1} & 0 \end{pmatrix} .$$

According to Definition 8 one can see $\mathcal{I}$ is an anti-symmetric operator. We can check that $\mathcal{I}$ is a symplectic operator both based on Definition 11 and Theorem 5. To prove regarding Theorem 5, we should take into account the following relation:

$$\mathcal{I}_*[\mathbf{p}] = \frac{d}{d\epsilon}\mathcal{I}[(u + p^1, v + p^2)]\Big|_{\epsilon=0} .$$

Also we can show $\mathcal{I}$ is symplectic by using Definition 11. Then we have

$$\mathcal{I}\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{dv_1}{1-uv_1} - \frac{dv}{1+uv} \\ \frac{du}{1+uv} - \frac{du_{-1}}{1-u_{-1}v} \end{pmatrix} .$$

This leads to

$$\begin{aligned} \omega_I &= \frac{1}{2}\int \begin{pmatrix} du \\ dv \end{pmatrix} \wedge \begin{pmatrix} \frac{dv_1}{1-uv_1} - \frac{dv}{1+uv} \\ \frac{du}{1+uv} - \frac{du_{-1}}{1-u_{-1}v} \end{pmatrix} \\ &= \frac{1}{2}\int (\frac{1}{1-uv_1}du \wedge dv_1 - \frac{1}{1+uv}du \wedge dv + \frac{1}{1+uv}dv \wedge du - \frac{1}{1-u_{-1}v}dv \wedge du_{-1}) \\ &= \int (\frac{1}{1-uv_1}du \wedge dv_1 + \frac{1}{1+uv}dv \wedge du) . \end{aligned}$$

Since $dv_1 \wedge dv_1$, $dv \wedge dv$, $du \wedge du$ and $du_{-1} \wedge du_{-1}$ all vanishes, clearly $d\omega_{\mathcal{I}} = 0$

and $\mathcal{I}$ is a symplectic operator.

For an operator $\mathcal{H} : \Omega^1 \longrightarrow \mathfrak{g}$ we define the following bracket on the space of functionals:

$$\{\int f, \int g\}_{\mathcal{H}} =< d\int g, \mathcal{H}(d\int f) >=< \frac{\delta g}{\delta u}, \mathcal{H}(\frac{\delta f}{\delta u}) >= \int \frac{\delta g}{\delta u}\mathcal{H}(\frac{\delta f}{\delta u}). \quad (3.5.6)$$

Consider the formulae given in Theorem 4 and set $\xi_1 = \int f, \xi_2 = \int g$ then this bracket can be expressed in terms of the Lie derivative as

$$\{\int f, \int g\}_{\mathcal{H}} = L_{\mathcal{H}(d\xi_1)}\xi_2 \, .$$

Now we define a Hamiltonian operator with respect to the Poisson bracket as follows.

Definition 12. *An anti-symmetric operator $\mathcal{H} : \Omega^1 \longrightarrow \mathfrak{g}$ is called Hamiltonian if and only if the bracket (3.5.6) is a Poisson bracket, that is*

$$\{\int f, \{\int g, \int h\}_{\mathcal{H}}\}_{\mathcal{H}} + \{\int g, \{\int h, \int f\}_{\mathcal{H}}\}_{\mathcal{H}} + \{\int h, \{\int f, \int g\}_{\mathcal{H}}\}_{\mathcal{H}} = 0 \, .$$

The definition of Hamiltonian operators can be rewritten in terms of Fréchet derivative. In the following theorem [5] we give an alternative relation by which one can check if a given operator $\mathcal{H} : \Omega^1 \longrightarrow \mathfrak{g}$ is Hamiltonian.

Theorem 6. *An anti-symmetric operator $\mathcal{H} : \Omega^1 \longrightarrow \mathfrak{g}$ is Hamiltonian if and only if for any $\omega_1, \omega_2, \omega_3 \in \Omega^1$*

$$< \mathcal{H}_*[\mathcal{H}\omega_2](\omega_3), \omega_1 > + < \mathcal{H}_*[\mathcal{H}\omega_3](\omega_1), \omega_2 > + < \mathcal{H}_*[\mathcal{H}\omega_1](\omega_2), \omega_3 >= 0 \, . \quad (3.5.7)$$

The main idea of the proof is similar to the process given in the proof of Theorem 5. For the detailed proof we refer the reader to [5, 68, 91]. Relation

(3.5.7) can be expressed in terms of the Lie derivative as

$$< L_{\mathcal{H}\omega_1}\omega_2, \mathcal{H}\omega_3 > + < L_{\mathcal{H}\omega_2}\omega_3, \mathcal{H}\omega_1 > + < L_{\mathcal{H}\omega_3}\omega_1, \mathcal{H}\omega_2 >= 0\,. \qquad (3.5.8)$$

Example 16. *Consider the difference operator*

$$\mathcal{H} = uu_1\mathcal{S} - uu_{-1}\mathcal{S}^{-1}\,.$$

It is easy to check that $\mathcal{H} = -\mathcal{H}^{\dagger}$. Then relation (3.5.7) can be obtained through direct calculation, by substituting equivalent terms. Here we expand the last term $< \mathcal{H}_*[\mathcal{H}\omega_1](\omega_2), \omega_3 >$ as

$$\begin{aligned}
&< \mathcal{H}_*[\mathcal{H}\omega_1](\omega_2), \omega_3 >= u\mathcal{S}(\mathcal{H}(\omega_1))\mathcal{S}(\omega_2)\omega_3 + u_1\mathcal{H}(\omega_1)\mathcal{S}(\omega_2)\omega_3 \\
&-u\mathcal{S}^{-1}(\mathcal{H}(\omega_1))\mathcal{S}^{-1}(\omega_2)\omega_3 - u_{-1}\mathcal{H}(\omega_1)\mathcal{S}^{-1}(\omega_2)\omega_3 = uu_1u_2\mathcal{S}^2(\omega_1)\mathcal{S}(\omega_2)\omega_3 \\
&-u^2u_1\omega_1\mathcal{S}(\omega_2)\omega_3 + uu_1^2\mathcal{S}(\omega_1)\mathcal{S}(\omega_2)\omega_3 - uu_{-1}u_1\mathcal{S}^{-1}(\omega_1)\mathcal{S}(\omega_2)\omega_3 \\
&-u^2u_{-1}\omega_1\mathcal{S}^{-1}(\omega_2)\omega_3 + uu_{-1}u_{-2}\mathcal{S}^{-2}(\omega_1)\mathcal{S}^{-1}(\omega_2)\omega_3 \\
&-uu_{-1}u_1\mathcal{S}(\omega_1)\mathcal{S}^{-1}(\omega_2)\omega_3 + u_{-1}^2u\mathcal{S}^{-1}(\omega_1)\mathcal{S}^{-1}(\omega_2)\omega_3\,.
\end{aligned}$$

Now if we substitute the cyclic permutations of $\{\omega_1, \omega_2, \omega_3\}$, all terms cancel and we get the result.

Proposition 7. *Let $\mathcal{H} : \Omega^1 \longrightarrow \mathfrak{g}$ be a Hamiltonian operator then for some 1-form $\omega \in \Omega^1$ we have*

$$L_{\mathcal{H}(\omega)}\mathcal{H} = -\mathcal{H} \circ d\omega \circ \mathcal{H}\,. \qquad (3.5.9)$$

The idea of the proof is to consider $< \xi_1, (L_{\mathcal{H}(\omega)}\mathcal{H})\xi_2 > + < \xi_1, (\mathcal{H} \circ d\omega \circ \mathcal{H})\xi_2 >$ and use the property of adjoint operator and the fact that $\mathcal{H}$ is a Hamiltonian operator. Notice that we treat $d\omega$ as an operator from $\mathfrak{g}$ to Ω^1 and according to relation (3.3.1) we have $d\omega = \omega_* - \omega_*^\dagger$. The complete proof can be found in [5]. As we can see to check whether an anti-symmetric operator $\mathcal{H}$ satisfies condition (3.5.7) is sometimes a cumbersome task. In [68] relation (3.5.7) is formulated in terms of a tri-vector which makes the calculation much easier.

In general if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hamiltonian operators there is no guarantee that the linear combination is again a Hamiltonian operator.

Definition 13. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hamiltonian operators. They are called a compatible pair if $\mathcal{H}_1 + \lambda\mathcal{H}_2$ is a Hamiltonian operator for an arbitrary constant λ.*

For a symplectic and Hamiltonian operators we have the following definition as a compatible pair.

Definition 14. *Let $\mathcal{H}$ and $\mathcal{I}$ be respectively a Hamiltonian and a symplectic operator. We say that the pair $(\mathcal{H}, \mathcal{I})$ is compatible if $\mathcal{I}\mathcal{H}\mathcal{I}$ is also a symplectic operator.*

There is an equivalent definition for compatibility of Hamiltonian and symplectic operators which involves the notion of Nijenhuis operator which will be introduced in the next definition. In Theorem 8 we show how these two definition are related.

Remark 6. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ form a compatible Hamiltonian pair. If $\mathcal{H}_2$ is invertible then $(\mathcal{H}_1, \mathcal{H}_2^{-1})$ is also a compatible pair.*

In what follows we introduce and analyze the properties of another important operator in the theory of integrability called Nijenhuis or hereditary operators. The name was first given by Fuchssteiner in [23]. In [30] also the Nijenhuis operator is discussed under the name of regular operators. More details and properties of Nijenhuis operators and the interrelations between Hamiltonian pairs can be found in series of articles and books [16, 20, 26, 30].

Definition 15. *A linear map* $\mathcal{R} : \mathfrak{g} \longrightarrow \mathfrak{g}$ *is called Nijenhuis if it satisfies the condition*

$$[\mathcal{R}\boldsymbol{p}, \mathcal{R}\boldsymbol{q}] - \mathcal{R}[\mathcal{R}\boldsymbol{p}, \boldsymbol{q}] - \mathcal{R}[\boldsymbol{p}, \mathcal{R}\boldsymbol{q}] + \mathcal{R}^2[\boldsymbol{p}, \boldsymbol{q}] = 0\,, \qquad \boldsymbol{p}, \boldsymbol{q} \in \mathfrak{g}\,. \tag{3.5.10}$$

This relation can be expressed in terms of Lie derivatives as

$$L_{\mathcal{R}\mathbf{p}}\mathcal{R}\mathbf{q} - \mathcal{R}L_{\mathcal{R}\mathbf{p}}\mathbf{q} - \mathcal{R}L_{\mathbf{p}}\mathcal{R}\,.\mathbf{q} + \mathcal{R}^2 L_{\mathbf{p}}\mathbf{q} = 0\,.$$

Now if we consider the left hand side as an action of some operator on a vector field $\mathbf{q} \in \mathfrak{g}$ then it is equivalent to

$$L_{\mathcal{R}\mathbf{p}}\mathcal{R} = \mathcal{R}L_{\mathbf{p}}\mathcal{R}\,, \qquad \mathbf{p} \in \mathfrak{g}\,. \tag{3.5.11}$$

In the next chapter we will study one of the main feature of integrable systems namely generalised symmetries. Here the next theorem build up the basis of producing infinite number of such quantities (i.e. symmetries)[15].

Theorem 7. *Let* $\mathcal{R}$ *be a Nijenhuis operator and* $\boldsymbol{p} \in \mathfrak{g}$ *be an element such that* $L_p\mathcal{R} = 0$. *Moreover consider infinite elements of* $\mathfrak{g}$ *obtained by* $\mathcal{R}^i\boldsymbol{p}$ *for* $i \geq 0$. *Then the following conditions hold:*

1. $\mathcal{R}$ *is invariant along the vector field* $\mathcal{R}^i\boldsymbol{p}$ *which is equivalent to say*

$$L_{\mathcal{R}^i\boldsymbol{p}}\mathcal{R} = 0\,;$$

2. *For arbitrary* $i, j \geq 0$ *the bracket* $[\mathcal{R}^i(\boldsymbol{p}), \mathcal{R}^j(\boldsymbol{p})]$ *vanishes which in terms of*

the Lie derivative can be written as

$$L_{\mathcal{R}^i\mathbf{p}}\mathcal{R}^j\boldsymbol{p} = 0\,.$$

Proof. The first statement follows simply by induction on i and the definition of Nijenhuis operator. For the second part, using Leibniz rule of the Lie derivative (3.4.3) we have

$$[\mathcal{R}^i\mathbf{p}, \mathcal{R}^j\mathbf{p}] \;=\; L_{\mathcal{R}^i\mathbf{p}}(\mathcal{R}^j\mathbf{p}) = (L_{\mathcal{R}^i\mathbf{p}}\mathcal{R}^j)\mathbf{p} + \mathcal{R}^j(L_{\mathcal{R}^i\mathbf{p}}\mathbf{p})\,.$$

According to the first part of theorem $L_{\mathcal{R}^i\mathbf{p}}\mathcal{R}^j$ vanishes and we get

$$[\mathcal{R}^i\mathbf{p}, \mathcal{R}^j\mathbf{p}] = -\mathcal{R}^j(L_{\mathbf{p}}\mathcal{R}^i\mathbf{p})\,,$$

where clearly is equal to zero as $\mathcal{R}^j(L_{\mathbf{p}}\mathcal{R}^i\mathbf{p}) = \mathcal{R}^{j+i}(L_{\mathbf{p}}\mathbf{p}) = 0$. □

The relation between compatible pairs and Nijenhuis operator is presented in the following theorem [5, 26]. Later on this theorem provide the criteria for the construction of the recursion operators.

Theorem 8. *Suppose a symplectic operator $\mathcal{I}$ and a Hamiltonian operator $\mathcal{H}$ form the compatible pair. Then the operator $\mathcal{R} = \mathcal{H}\mathcal{I}$ is a Nijenhuis operator.*

Proof. To prove the theorem it is sufficient to show that

$$L_{\mathcal{R}\mathbf{p}}\mathcal{R} = \mathcal{R}L_{\mathbf{p}}\mathcal{R}\,, \qquad \mathcal{R} = \mathcal{H}\mathcal{I},\; \mathbf{p} \in \mathfrak{g}\,.$$

From (3.4.3) we have

$$L_{\mathcal{R}\mathbf{p}}(\mathcal{H}\mathcal{I}) - \mathcal{R}L_{\mathbf{p}}\mathcal{R} = \mathcal{H}(L_{\mathcal{H}\mathcal{I}(\mathbf{p})}\mathcal{I}) + (L_{\mathcal{H}\mathcal{I}(\mathbf{p})}\mathcal{H})\mathcal{I} - \mathcal{R}L_{\mathbf{p}}\mathcal{R}\,.$$

The second term on the right hand side can be written in an equivalent way by considering relations (3.5.4) and (3.5.9)

$$\begin{aligned} L_{\mathcal{R}\mathbf{p}}(\mathcal{H}\mathcal{I}) - \mathcal{R}L_{\mathbf{p}}\mathcal{R} &= \mathcal{H}(L_{\mathcal{H}\mathcal{I}(\mathbf{p})}\mathcal{I}) - (\mathcal{H} \circ d\mathcal{I}\mathbf{p} \circ \mathcal{H})\mathcal{I} - \mathcal{R}L_{\mathbf{p}}\mathcal{R} \\ &= \mathcal{H}(L_{\mathcal{H}\mathcal{I}(\mathbf{p})}\mathcal{I}) - \mathcal{H}(L_{\mathbf{p}}\mathcal{I})\mathcal{H}\mathcal{I} - \mathcal{R}L_{\mathbf{p}}\mathcal{R} \\ &= \mathcal{H}\left(L_{\mathcal{R}(\mathbf{p})}\mathcal{I} - (L_{\mathbf{p}}\mathcal{I})\mathcal{R} - \mathcal{I}L_{\mathbf{p}}\mathcal{R}\right) \\ &= \mathcal{H}\left(L_{\mathcal{R}(\mathbf{p})}\mathcal{I} - L_{\mathbf{p}}(\mathcal{I}\mathcal{R})\right) . \end{aligned}$$

From (3.5.4) and the fact that $\mathcal{I}\mathcal{R} = \mathcal{I}\mathcal{H}\mathcal{I}$ is a symplectic operator the left hand side is equal to

$$\mathcal{H}\left(d\mathcal{I}\mathcal{R}(\mathbf{p}) - L_{\mathbf{p}}(\mathcal{I}\mathcal{R})\right) = 0 \,,$$

which proves the theorem. □

To summarize these two chapters: we have constructed the ring of differential-difference functions and the corresponding quotient space as the space of 0-forms. The concepts of Hamiltonian, symplectic and Nijenhuis operators were also discussed in terms of Lie derivatives and Fréchet derivative. In the next chapter we will see the application of these algebraic structures to differential-difference equations.

Chapter 4

Evolutionary differential-difference equations

This chapter is basically devoted to the integrable differential-difference equations in (1+1)-dimension. As it is known, there is not a unique definition for the integrability of nonlinear evolutionary equations. For instance equations which are solvable through the inverse scattering method are usually called S-integrable [10, 94]. Also equations that are linearizable or in other words can be transformed to a linear equations are called C-integrable. In this chapter we will define integrability based on the concept of generalised symmetries. The common feature of C-integrable and S-integrable equations is that these equations possess infinitely many generalised symmetries.

4.1 (1+1)-dimensional differential-difference equations

A (1+1)-dimensional evolutionary differential-difference equation has the following general form:

$$\mathbf{u}_t = (K^1[\mathbf{u}], K^2[\mathbf{u}], \cdots, K^m[\mathbf{u}]) = \mathbf{K}\,[\mathbf{u}]\,, \tag{4.1.1}$$

where $\mathbf{u} = (u^{(1)}, u^{(2)}, \cdots, u^{(m)})^T$, $K^i \in \mathfrak{F}$ and the time derivative is naturally denoted as

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u}_t = (u_t^{(1)}, u_t^{(2)}, \cdots, u_t^{(m)})^T\,.$$

In fact to each element $\mathbf{K} \in \mathfrak{g}$ one can associate an equation (4.1.1) and the derivation

$$D_t = \sum_{i\in\mathbb{Z}} \sum_{\alpha=1}^{m} \mathcal{S}^i(K^\alpha) \frac{\partial}{\partial u_i^{(\alpha)}}\,. \tag{4.1.2}$$

If $m = 1$ then (4.1.1) is called the scalar equation. A well-known example of scalar differential-difference equation is the Volterra equation [88]

$$u_t = u(u_1 - u_{-1})\,. \tag{4.1.3}$$

In the case that $m \geq 2$ we have system of equations called m-component differential-difference equations. Toda is an example of two component differential-difference equation [85]

$$\begin{cases} u_t = u(v_1 - v)\,, \\ v_t = u - u_{-1}\,. \end{cases} \tag{4.1.4}$$

It is known that the above equations possess rich mathematical structures associated with integrability such as infinitely many generalised symmetries and conserved densities, recursion operators and master symmetries. In the following sections we explain these properties and provide more examples.

4.2 Generalised symmetries and integrability

Traditionally symmetries of equations are defined as transformations which map solutions of the equation into solutions [54]. Symmetries that are function of both dependent and independent variables are called point symmetries. The following two examples are examples of point symmetries (i.e. scaling symmetry) respectively for the Volterra (4.1.3) and the Toda (5.2.6) equations:

$$(u,t) \longmapsto (\lambda\ u, \lambda^{-1}t) = (\hat{u}, \hat{t})\,, \tag{4.2.1}$$

$$(u,v,t) \longmapsto (\lambda\ u, \lambda^{2}v, \lambda^{-1}t) = (\hat{u}, \hat{v}, \hat{t})\,. \tag{4.2.2}$$

In fact one can easily check that $(\hat{u}, \hat{t})$ and $(\hat{u}, \hat{v}, \hat{t})$ satisfy the same relations as (4.1.3) and (4.1.4). We are interested in other type of symmetries which depend only on the dependent variable and its shifts. These kind of symmetries are known as generalised symmetries. Before we give more precise definition of the generalised symmetry we discussed the steps that clarify our definition. Suppose

$$\mathbf{u}_\tau = (G^1[\mathbf{u}], G^2[\mathbf{u}], \cdots, G^m[\mathbf{u}]) = \mathbf{G}\ [\mathbf{u}]\,,$$

is an equation which has a common solution with the differential-difference equation (4.1.1). Therefore we have

$$u_{\tau t}^{(i)} = G_t^i[\mathbf{u}] = \sum_{\beta=1}^{m}\sum_{j_\beta} u_{j_\beta t}^{(\beta)} \frac{\partial G^i}{\partial u_{j_\beta}^{(\beta)}} = \sum_{\beta=1}^{m}\sum_{j_\beta} \mathcal{S}^{j_\beta}(K^\beta)\frac{\partial G^i}{\partial u_{j_\beta}^{(\beta)}} = \sum_{\beta=1}^{m} G^i_{*u^{(\beta)}}(K^\beta)\,.$$

On the other hand in a similar way we can obtain

$$u_{t\tau}^{(i)} = \sum_{\beta=1}^{m} K^i_{*u^{(\beta)}}(G^\beta)\,.$$

If we calculate all components as above we get

$$\mathbf{u}_{t\tau} - \mathbf{u}_{\tau t} = \begin{pmatrix} K^1_{*u^{(1)}} & \cdots & K^1_{*u^{(m)}} \\ K^2_{*u^{(1)}} & \cdots & K^2_{*u^{(m)}} \\ \vdots & \vdots & \vdots \\ K^m_{*u^{(1)}} & \cdots & K^m_{*u^{(m)}} \end{pmatrix} \begin{pmatrix} G^1 \\ G^2 \\ \vdots \\ G^m \end{pmatrix} - \begin{pmatrix} G^1_{*u^{(1)}} & \cdots & G^1_{*u^{(m)}} \\ G^2_{*u^{(1)}} & \cdots & G^2_{*u^{(m)}} \\ \vdots & \vdots & \vdots \\ G^m_{*u^{(1)}} & \cdots & G^m_{*u^{(m)}} \end{pmatrix} \begin{pmatrix} K^1 \\ K^2 \\ \vdots \\ K^m \end{pmatrix} = 0.$$

This procedure can be formulated more compact in the following definition.

Definition 16. *Let* $\boldsymbol{G}\,[\boldsymbol{u}] = (G^1[\boldsymbol{u}], G^2[\boldsymbol{u}], \cdots, G^m[\boldsymbol{u}])$ *be an element of* $\mathfrak{g}$. *It is called a generalised symmetry of equation (4.1.1) if and only if*

$$[\boldsymbol{K}, \boldsymbol{G}] = \boldsymbol{G}_*(\boldsymbol{K}) - \boldsymbol{K}_*(\boldsymbol{G}\,) = 0\,. \tag{4.2.3}$$

Thus according to above definition one can easily check that the following expressions $\mathbf{G}$ and $\mathbf{G}'$ are respectively generalised symmetry of the scalar Volterra and the two component Toda equations:

$$\mathbf{G} = uu_1(u_2 + u_1 + u) - uu_{-1}(u + u_{-1} + u_{-2})\,,$$

$$\mathbf{G}' = \begin{cases} u(v_1^2 - v^2 + u_1 - u_{-1})\,, \\ u(v_1 + v) - u_{-1}(v_{-1} + v)\,. \end{cases}$$

Definition 16 can be written in an equivalent way with respect to Lie derivatives. In fact $\mathbf{G} \in \mathfrak{g}$ is a generalised symmetry for differential-difference equation (4.1.1) if and only if

$$L_{\mathbf{K}}\mathbf{G} = 0\,. \tag{4.2.4}$$

Now we are ready to give the concept of integrability based on generalised symmetries.

Definition 17. *Differential-difference equation (4.1.1) is called integrable iff it possesses infinitely many commuting generalised symmetries.*

According to this definition we are motivated to find some methods or tools by which one can construct infinite series of generalised symmetries and consequently guarantee the integrability. In section (4.4) we introduce one of these tools called recursion operators. The next section discusses another feature of integrable systems namely conserved densities.

4.3 Conserved densities and co-symmetries

Here we define the notion of conservation laws regarding some differential-difference equation (4.1.1). As given in [95], by introducing the canonical conservation laws one can obtain integrability conditions for a differential-difference equation (4.1.1). In fact they are necessary conditions for the existence of infinitely many generalised symmetries which guarantees integrability.

Definition 18. *Let $\int \rho$ be an element of $\mathfrak{F}'$. We call $\int \rho$ a conserved density of equation (4.1.1) if it is invariant along the vector field $\boldsymbol{K}$, i.e.*

$$L_{K}\int \rho = 0\,. \tag{4.3.1}$$

According the formulae in Theorem 4 if we expand Relation (4.3.1) we get

$$L_{\mathbf{K}} \int \rho = \int \rho_*(\mathbf{K}) =< d \int \rho, \mathbf{K} >=< \frac{\delta \rho}{\delta u}, \mathbf{K} >= 0\,. \tag{4.3.2}$$

So based on the above relation in order to check if a given $\int \rho$ is a conserved density we first calculate the variational derivative of ρ and then multiplying by the equation $\mathbf{K}$ should produce some expression which is a total difference.

Example 17. *Functionals $\int \ln u$ and $\int (u + \frac{1}{2}v^2)$ are respectively the conserved densities of the Volterra and the Toda equations with the following conservation laws:*

$$(\frac{\delta \ln u}{\delta u}) u(u_1 - u_{-1}) = (\mathcal{S} - 1)(u + u_{-1})\,,$$
$$\frac{\delta(\int u + \frac{1}{2}v^2)}{\delta u} \cdot (u(v_1 - v), u - u_{-1}) = (\mathcal{S} - 1)u_{-1}v\,.$$

In fact relation (4.3.1) can be expressed in terms of the difference operator as follows:

$$L_{\mathbf{K}} \int \rho = (\mathcal{S} - 1)J, \qquad J \in \mathfrak{F}\,.$$

In this relation J is known as the corresponding flux for the conserved density $\int \rho$. In [32, 35] one can find the algorithmic process of constructing the conserved densities and their corresponding fluxes which are polynomial on their variables. We proceed by introducing the so-called **co-symmetries** that are closely related to conserved densities.

Definition 19. *A 1-form $\omega \in \Omega^1$ is called a co-symmetry of differential-difference equation (4.1.1) if*

$$L_{\boldsymbol{K}} \omega = 0\,. \tag{4.3.3}$$

Example 18. *Consider the 1-form $\omega = \int (u_{-1} + u + u_1) du$. From (3.2.3) we can identify ω with the vector field $(u_{-1} + u + u_1)$. Using the formula in Theorem 4 we can show* *that ω is invariant along the Volterra equation*

$$L_{u(u_1-u_{-1})}\omega = (\mathcal{S}^{-1}+1+\mathcal{S})(u(u_1-u_{-1})) + (u_1-u_{-1})\omega + u_{-1}\mathcal{S}^{-1}(\omega) - u_1\mathcal{S}(\omega) = 0\,.$$

If we look more closely at the structure of ω we can see that it is the variational derivative of the 0-form

$$\int (uu_1 + \frac{1}{2}u^2)\,,$$

where $\int(uu_1+\frac{1}{2}u^2)$ itself is a conserved density of the Volterra equation. Actually it is not an exception and we have the following proposition [89] for the relation between conserved densities and co-symmetries.

Proposition 8. *Consider the differential-difference equation (4.1.1). If $\int\rho \in \Omega^0$ is a conserved density then $\frac{\delta\rho}{\delta u}$ is its co-symmetry.*

Proof. Consider an element of 0-forms $\int\rho \in \Omega^0$. Then we have

$$L_{\mathbf{K}} d\int \rho = L_{\mathbf{K}}\frac{\delta\rho}{\delta u}\,.$$

On the other hand according to Proposition 6 that $[L_{\mathbf{K}}, d] = 0$ we have

$$L_{\mathbf{K}} d\int \rho = dL_{\mathbf{K}}\int \rho = 0\,.$$

Since ρ is a conserved density, we have

$$L_{\mathbf{K}}\frac{\delta\rho}{\delta u} = 0\,.$$

□

In the next section we discuss operators that are closely related to the integrability of differential-difference equations. In fact they provide a tool to produce the series of generalised symmetries.

4.4 Hamiltonian, symplectic and recursion operators

In this section we employ the concepts of Hamiltonian, symplectic and recursion operators for a differential-difference equation (4.1.1). We will also provide the definition of Hamiltonian systems which guarantees the integrability of the system.

Definition 20. *Hamiltonian operator* $\mathcal{H} : \Omega^1 \longrightarrow \mathfrak{g}$ *is a Hamiltonian operator for equation (4.1.1) if and only if*

$$L_{\mathbf{K}}\mathcal{H} = 0\,. \tag{4.4.1}$$

As we saw earlier in Example 16, the difference operator

$$\mathcal{H} = uu_1\mathcal{S} - uu_{-1}\mathcal{S}^{-1}\,, \tag{4.4.2}$$

is a Hamiltonian operator. Now one can check that it is the Hamiltonian operator for the Volterra equation (4.1.3) indeed we have

$$\begin{aligned}
\mathcal{H}_*[\mathbf{K}] &= (u_1\mathbf{K} + u\mathcal{S}(\mathbf{K}))\,\mathcal{S} - (u_{-1}\mathbf{K} + u\mathcal{S}^{-1}(\mathbf{K}))\mathcal{S}^{-1}\,,\\
\mathbf{K}_* &= (u_1 - u_{-1}) + u\mathcal{S} - u\mathcal{S}^{-1}\,,\\
\mathbf{K}_*^\dagger &= (u_1 - u_{-1}) + u_{-1}\mathcal{S}^{-1} - u_1\mathcal{S}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&L_{\mathbf{K}}\mathcal{H} = \mathcal{H}_*[\mathbf{K}] - \mathbf{K}_*\mathcal{H} - \mathcal{H}\mathbf{K}_*^\dagger = (uu_1(u_1 - u_{-1}) + uu_1(u_2 - u))\,\mathcal{S}\\
&- (uu_{-1}(u_1 - u_{-1}) + uu_{-1}(u - u_{-2}))\,\mathcal{S}^{-1} - uu_1(u_1 - u_{-1})\mathcal{S} + uu_{-1}(u_1 - u_{-1})\mathcal{S}^{-1}\\
&-uu_1u_2\mathcal{S}^2 + u^2u_1 + u^2u_{-1} - uu_{-1}u_{-2}\mathcal{S}^{-2} - uu_1(u_2 - u)\mathcal{S} - u^2u_1 + uu_1u_2\mathcal{S}^2\\
&+uu_{-1}(u - u_{-2})\mathcal{S}^{-1} + uu_{-1}u_{-2}\mathcal{S}^{-2} - u^2u_{-1} = 0\,.
\end{aligned}$$

We can define Hamiltonian equation as follows.

Definition 21. *A differential-difference equation (4.1.1) is called Hamiltonian if there exists a Hamiltonian operator $\mathcal{H}$ and a functional $\int f$ called Hamiltonian density such that*

$$\boldsymbol{u}_t = \mathcal{H}(\frac{\delta f}{\delta u}) = \boldsymbol{K}.$$

The reason that we refer to $\int f$ as a Hamiltonian density is simply due to the fact that it is a conserved density of equation (4.1.1)

$$L_{\mathbf{K}} \int f = < d \int f, \mathbf{K} > = < d \int f, \mathcal{H}(\frac{\delta f}{\delta u}) > = \{\int f, \int f\}_{\mathcal{H}} = 0.$$

Therefore in practice to claim that equation (4.1.1) is a Hamiltonian vector field we shall look for a Hamiltonian operator $\mathcal{H}$ satisfying (4.4.1) and an appropriate conserved density. For instance the Volterra equation is a Hamiltonian equation with the Hamitonian operator (4.4.2) and Hamiltonian density $\int u$ since

$$u_t = \mathcal{H}(\frac{\delta u}{\delta u}) = u(u_1 - u_{-1}).$$

Moreover if $\mathcal{H}$ is a Hamiltonian operator of equation (4.1.1), we can see that $\mathcal{H}$ maps a co-symmetry to a symmetry. Let $\omega \in \Omega^1$ be a co-symmetry then

$$L_{\mathbf{K}}(\mathcal{H}\omega) = (L_{\mathbf{K}}\mathcal{H})\omega + \mathcal{H}(L_{\mathbf{K}}\omega) = 0.$$

Naturally we call (4.1.1) a bi-Hamiltonian equation if there exist two Hamiltonian operators $\mathcal{H}_1$ and $\mathcal{H}_2$ and two Hamiltonians $\int f_1$ and $\int f_2$ such that

$$\mathbf{u}_t = \mathcal{H}_1(\frac{\delta f_1}{\delta u}) = \mathcal{H}_2(\frac{\delta f_2}{\delta u}).$$

Example 19. *The Volterra equation (4.1.3) is a bi-Hamiltonian equation with*

$$\mathcal{H}_1 = u(\mathcal{S}-\mathcal{S}^{-1})u, \qquad f_1 = u\,,$$
$$\mathcal{H}_2 = u(\mathcal{S}u\mathcal{S}+u\mathcal{S}+\mathcal{S}u-u\mathcal{S}^{-1}-\mathcal{S}^{-1}u-\mathcal{S}^{-1}u\mathcal{S}^{-1})u, \qquad f_2 = \frac{1}{2}\ln u\,.$$

Definition 22. *Symplectic operator $\mathcal{I}:\mathfrak{g}\longrightarrow\Omega^1$ is a symplectic operator for equation (4.1.1) if and only if*

$$L_{\boldsymbol{K}}\mathcal{I} = 0\,. \tag{4.4.3}$$

Example 20. *The anti-symmetric operator $\mathcal{I}=\frac{1}{u}(\mathcal{S}-\mathcal{S}^{-1})^{-1}\frac{1}{u}$ is a symplectic operator for (4.1.3) since*

$$\begin{aligned}\mathcal{I}_*[\boldsymbol{K}] &= -\frac{1}{u}(\mathcal{S}-\mathcal{S}^{-1})^{-1}\frac{\boldsymbol{K}}{u^2}-\frac{\boldsymbol{K}}{u^2}(\mathcal{S}-\mathcal{S}^{-1})^{-1}\frac{1}{u}\\ &= -\frac{1}{u}(\mathcal{S}-\mathcal{S}^{-1})^{-1}\frac{(u_1-u_{-1})}{u}-\frac{(u_1-u_{-1})}{u}(\mathcal{S}-\mathcal{S}^{-1})^{-1}\frac{1}{u};\end{aligned}$$

$$\begin{aligned}\mathcal{I}\boldsymbol{K}_* &= \frac{1}{u}(\mathcal{S}-\mathcal{S}^{-1})^{-1}\frac{(u_1-u_{-1})}{u}+\frac{1}{u}(\mathcal{S}-\mathcal{S}^{-1})^{-1}\mathcal{S}-\frac{1}{u}(\mathcal{S}-\mathcal{S}^{-1})^{-1}\mathcal{S}^{-1}\\ &= \frac{1}{u}(\mathcal{S}-\mathcal{S}^{-1})^{-1}\frac{(u_1-u_{-1})}{u}+\frac{1}{u};\end{aligned}$$

$$\boldsymbol{K}_*\mathcal{I} = \frac{(u_1-u_{-1})}{u}(\mathcal{S}-\mathcal{S}^{-1})^{-1}\frac{1}{u}-\frac{1}{u}.$$

The next definition is the main concept of this chapter. We will define the notion of recursion operator and shows the importance of having this tool.

Definition 23. *An operator $\mathcal{R}:\mathfrak{g}\longrightarrow\mathfrak{g}$ is called recursion operator for equation (4.1.1) if and only if*

$$L_{\boldsymbol{K}}\mathcal{R} = 0\,. \tag{4.4.4}$$

From the definitions of Hamiltonian and symplectic operators one can prove that $\mathcal{H}\mathcal{I}$ is a recursion operator when $\mathcal{H}$ and $\mathcal{I}$ are respectively the Hamiltonian and the symplectic operators for a given differential-difference equation.

$$\begin{aligned}
(L_{\mathbf{K}}\mathcal{H}\mathcal{I})\mathbf{p} &= L_{\mathbf{K}}(\mathcal{H}(\mathcal{I}\mathbf{p})) - \mathcal{H}\mathcal{I}(L_{\mathbf{K}}\mathbf{p}) \qquad \forall \mathbf{p} \in \mathfrak{g} \\
&= (L_{\mathbf{K}}\mathcal{H})(\mathcal{I}\mathbf{p}) + \mathcal{H}(L_{\mathbf{K}}\mathcal{I}\mathbf{p}) - \mathcal{H}\mathcal{I}(L_{\mathbf{K}}\mathbf{p}) \\
&= (\mathcal{H}_*[\mathbf{K}] - \mathbf{K}_*\mathcal{H} - \mathcal{H}\mathbf{K}_*^\dagger)(\mathcal{I}\mathbf{p}) + \mathcal{H}((L_{\mathbf{K}})\mathbf{p} + \mathcal{I}(L_{\mathbf{K}\mathbf{p}})) - \mathcal{H}\mathcal{I}(\mathbf{p}_*\mathbf{K} - \mathbf{K}_*\mathbf{p}) \\
&= (\mathcal{H}\mathcal{I})_*[\mathbf{K}](\mathbf{p}) - (\mathbf{K}_*\mathcal{H}\mathcal{I})\mathbf{p} + (\mathcal{H}\mathcal{I}\mathbf{K}_*)\mathbf{p}\,.
\end{aligned}$$

For more details regarding factorization of recursion operators into composition of Hamiltonian and symplectic operator one can read [5, 15, 48, 77, 91]. For the recursion operator of a given differential -difference equation we have the following significant proposition [5, 95].

Proposition 9. *Suppose $\mathcal{R} : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a recursion operator for equation (4.1.1), then $\mathcal{R}$ maps a symmetry of (4.1.1) to a symmetry.*

Proof. Let $\mathbf{G} \in \mathfrak{g}$ be a generalised symmetry of equation (4.1.1). We have

$$L_{\mathbf{K}}(\mathcal{R}\mathbf{G}) = (L_{\mathbf{K}}\mathcal{R})\mathbf{G} + \mathcal{R}(L_{\mathbf{K}}\mathbf{G}) = 0\,.$$

□

As we mentioned earlier in the previous chapter the above proposition shows the significant role of recursion operators in the theory of integrability. In fact if we can find the recursion operator of some differential-difference equation and starting from one symmetry $\mathbf{G} \in \mathfrak{g}$ then we can generate the hierarchy of generalised symmetries according to the formula

$$\mathbf{G}_n = \mathcal{R}^n(\mathbf{G})\,.$$

It is not always an easy task to check if a given recursion operator satisfies the Nijenhuis property. According to Theorem 8 in the case that the recursion operator is the composition of a compatible Hamiltonian and symplectic operator it satisfies the Nijenhuis property. In fact being a Nijenhuis operator is the key point which shows the power of recursion operators. Suppose $\mathcal{R}$ is a recursion operator which is also a Nijenhuis operator, then from Theorem 7 for a given symmetry $\mathbf{G}$ as a seed point we can produce the following commuting generalised symmetries:

$$[\mathcal{R}^i\mathbf{G}, \mathcal{R}^j\mathbf{G}], \qquad i, j \geq 0 .$$

If we look at the recursion operator of the Volterra equation we see that it contains a weakly nonlocal term $u(u_1 - u_{-1})(\mathcal{S}-1)^{-1}\frac{1}{u}$. In fact this is the case that contains most of the known recursion operators. The weakly nonlocal term has the form

$$\mathbf{p} \otimes (\mathcal{S}-1)^{-1}\omega \,, \tag{4.4.5}$$

where $\mathbf{p}$ and ω are respectively the symmetry and the co-symmetry of the equation and $\otimes$ denotes the outer product of two $m \times 1$ and $1 \times m$ matrices defined as

$$\begin{pmatrix} p^1 \\ p^2 \\ \vdots \\ p^m \end{pmatrix} \otimes (\mathcal{S}-1)^{-1} \begin{pmatrix} \omega^1 & \omega^2 & \dots & \omega^m \end{pmatrix} = \begin{pmatrix} p^1(\mathcal{S}-1)^{-1}\omega^1 & \dots & p^1(\mathcal{S}-1)^{-1}\omega^m \\ p^2(\mathcal{S}-1)^{-1}\omega^1 & \dots & p^2(\mathcal{S}-1)^{-1}\omega^m \\ \vdots & & \\ p^m(\mathcal{S}-1)^{-1}\omega^1 & \dots & p^m(\mathcal{S}-1)^{-1}\omega^m \end{pmatrix} .$$

More details about the structure of weakly nonlocal terms can be found in [73, 76, 89, 91].

Example 21. *Consider the Two component Volterra lattice [66]*

$$\begin{cases} u_t = u(v_1 - v) \,, \\ v_t = v(u - u_{-1}) \,. \end{cases} \tag{4.4.6}$$

Its recursion operator has the form

$$\mathcal{R} = \begin{pmatrix} u + v_1 & u\mathcal{S} + \frac{uv_1}{v} \\ v + v\mathcal{S}^{-1} & u + v \end{pmatrix} + \begin{pmatrix} u(v_1 - v) \\ v(u - u_{-1}) \end{pmatrix} \otimes (\mathcal{S} - 1)^{-1} \begin{pmatrix} \frac{1}{u} & \frac{1}{v} \end{pmatrix},$$

where $\mathbf{p}$ *is the equation itself and the co-symmetry is obtained by*

$$\begin{pmatrix} \frac{1}{u} & \frac{1}{v} \end{pmatrix} = \frac{\delta}{\delta(u,v)}(\ln u + \ln v).$$

An important concern regarding the recursion operator is whether the action of recursion operators with weakly nonlocal terms of the from (4.4.5) on some local symmetry produces a new local symmetry. In the next theorem [76] we discuss the case where a recursion operator contains one weakly nonlocal terms. For more general cases one can read [58, 76, 89, 91].

Theorem 9. *Let for a differential-difference equation (4.1.1),* $\mathcal{R}$ *be a recursion operator of the form*

$$\mathcal{R} = \mathfrak{M} + \boldsymbol{p} \otimes (\mathcal{S} - 1)^{-1}\omega,$$

where $\mathfrak{M}$ *is an element of* $M_{\ell\times\ell}(\mathfrak{U})$. *Furthermore assume* $\mathcal{R}$ *is a Nijenhuis and both* ω *and* $\mathcal{R}^\dagger(\omega)$ *are closed. Then all* $\boldsymbol{K}^{(i)} = \mathcal{R}^i(\boldsymbol{K})$, $i = 1, 2, \cdots$ *are local; i.e. elements of the Lie algebra* $\mathfrak{g}$.

Proof. Notice that the variational derivative satisfies

$$\frac{\delta(\omega(\mathbf{K}))}{\delta u} = \mathbf{K}_*^\dagger(\omega) + \omega_*^\dagger[\mathbf{K}].$$

On the other hand since $\omega_*^\dagger = \omega_*$ we get

$$L_{\mathbf{K}}\omega = \frac{\delta(\omega(\mathbf{K}))}{\delta u}.$$

Now we prove the theorem by induction on i. For $i = 1$ since ω is a co-symmetry we have

$$L_{\mathbf{K}}\omega = \frac{\delta(\mathbf{K}\cdot\omega)}{\delta u} = 0\,.$$

Therefore from Proposition 4 the element $\mathbf{K}\cdot\omega$ is a total difference and according to the structure of $\mathcal{R}$ it yields that

$$\mathcal{R}(\mathbf{K}) \in \mathfrak{g}\,.$$

Now assume that $\mathbf{K}^{(i)} = \mathcal{R}(\mathbf{K}^{(i-1)})$ is a local expression then due to the structure of $\mathcal{R}$ it implies that

$$\mathbf{K}^{(i-1)}\cdot\omega \in Im(\mathcal{S}-1)\,.$$

Now to prove that $\mathbf{K}^{(i+1)}$ is a local expression we have to show that $\omega(\mathbf{K}^{(i)})$ is also a total difference. We calculate $L_{\mathbf{K}^{(i)}}\omega$ as

$$\begin{aligned}
L_{\mathbf{K}^{(i)}}\omega &= L_{\mathcal{R}(\mathbf{K}^{(i-1)})}\omega \\
&= \frac{\delta\left(\mathcal{R}(\mathbf{K}^{(i-1)})\cdot\omega\right)}{\delta u} = \frac{\delta\left(\mathbf{K}^{(i-1)}\cdot\mathcal{R}^{\dagger}(\omega)\right)}{\delta u} \\
&= L_{\mathbf{K}^{(i-1)}}\mathcal{R}^{\dagger}(\omega)\,.
\end{aligned}$$

In the second line we used the property of adjoint operators. Now as the Lie derivative satisfies Leibniz rule we get

$$\begin{aligned}
L_{\mathbf{K}^{(i-1)}}\mathcal{R}^{\dagger}(\omega) &= (L_{\mathbf{K}^{(i-1)}}\mathcal{R}^{\dagger})\omega + \mathcal{R}^{\dagger}(L_{\mathbf{K}}\omega) \\
&= (L_{\mathbf{K}^{(i-1)}}\mathcal{R}^{\dagger})\omega \\
&= (L_{\mathbf{K}^{(i-1)}}\mathcal{R})^{\dagger}\omega
\end{aligned}$$

and since $\mathcal{R}$ is a Nijenhuis operator we end up with the desired result that is

$$L_{\mathbf{K}^{(i)}}\omega_{\alpha} = 0\,.$$

We use this relation to show that $\mathcal{R}^i(\mathbf{K})$ is an element of the Lie algebra $\mathfrak{g}$. We have

$$\frac{\delta(\mathbf{K}^{(i)} \cdot \omega)}{\delta u} = L_{\mathbf{K}^{(i)}}\omega = 0\,,$$

which again according to the property of variational derivatives, the pairing of $\mathbf{K}^{(i)}$ and 1-forms ω lies in the image of the difference operator and therefore

$$\mathbf{K}^{(i+1)} = \mathcal{R}(\mathbf{K}^{(i)}) \in \mathfrak{g}\,,$$

which proves the claim. □

4.5 Construction of recursion operators

There exists no general algorithmic process by which one can always obtain the corresponding recursion operator. In [34] one can find the construction of recursion operators based on a given Lax representation. In [91] the construction of recursion operators based on the composition of Hamiltonian and symplectic operator for the two dimensional periodic Volterra chain is discussed. Another method is starting from a proper candidate for the recursion operator and using relation (4.4.4) to derive the explicit form. This method is proposed in [33, 35] for number of differential-difference equations. In particular the authors discuss in details the algorithm to obtain the recursion operator of the Toda system 4.1.4. This method involves the concept of homogeneous equations and the candidate for the recursion operator is based on the scaling point symmetry. Here we basically employ the methods given in [33, 35, 91]. Using relation (4.4.4) it is important to find the order of the ansazt with which we shall start. To do this we need to look at the structure of the equation itself and its non-trivial generalised symmetry.

In this section we take the Merola-Ragnisco-Tu lattice and explain the process through which we can obtain its recursion operator. In fact we obtain the recursion operator through several steps. In the first step we try to find a proper candidate for the co-symmetry. Then we will find the maximum and minimum order of the shift operator that can appear in the recursion operator. After that we determine the coefficients with the help of relation (4.4.4) and the structure of the non-trivial symmetry. Notice that for different differential-difference equations one might need to consider different methods or more information and analysis of the form of known recursion operators.

Now let us start the process of finding the recursion operator for a concrete example. As we mentioned in the previous section, most of recursion operators contain finite number of terms of the form (4.4.5) where $\mathbf{p}$ and ω are respectively the symmetry and the co-symmetry of the given equation. In many cases $\mathbf{p}$ is the equation itself although it is not a general rule. Consider the Merola-Ragnisco-Tu lattice given in Section 5.2.16

$$\mathbf{K} = \begin{cases} u_t = u_1 - u^2 v \,, \\ v_t = -v_{-1} + v^2 u \,. \end{cases}$$

The following element of $\mathfrak{g}$ is a non-trivial generalised symmetry:

$$\mathbf{G} = \begin{cases} u_\tau = u_2 - u_1^2 v_1 - u^2 v_{-1} - 2uvu_1 + u^3 v^2 \,, \\ v_\tau = -v_{-2} + v_{-1}^2 u_{-1} + v^2 u_1 + 2uvv_{-1} - u^2 v^3 \,. \end{cases}$$

As we know for the weakly nonlocal terms ω is a co-symmetry. To find a proper co-symmetry, we should note that the action of the recursion operator produces local symmetries. Therefore we should consider the pairing between ω and the equation $\mathbf{K}$. In fact this pairing should vanish or in other words it must lie in the image of the difference operator. Therefore looking more closely at the equation, an appropriate candidate for the co-symmetry ω can be $\omega = \begin{pmatrix} v & u \end{pmatrix}$ since

$$< \omega, \mathbf{K} >= vu_1 - uv_{-1} = (\mathcal{S} - 1)uv_{-1} \,.$$

In fact $\omega = \begin{pmatrix} v & u \end{pmatrix}$ is the variational derivative of the conserved density

$$\rho = uv \,.$$

From Theorem 9 we know that recursion operators can be split into local and weakly nonlocal parts. Let us start with the following generic ansatz in which just ω is specified:

$$\mathcal{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} + \mathbf{p} \otimes (\mathcal{S}-1)^{-1} \begin{pmatrix} v & u \end{pmatrix}.$$

For the next step we find the order of shift operators for the local part. To do this we compare the terms in the equation to the terms with highest and lowest length of the shift in the symmetry $\mathbf{G}$. Since the action of $\mathcal{R}$ on $\mathbf{K}$ should give the symmetry $\mathbf{G}$, $R_{11}u_t + R_{12}v_t$ and $R_{21}u_t + R_{22}v_t$ respectively produce the term with the highest shift u_2 and lowest shift v_{-2}. Thus we can split the above expression into the sum of shift operators of order 1,0 and -1 as

$$\mathcal{R} = \begin{pmatrix} R_{11}^{(1)} & R_{12}^{(1)} \\ R_{21}^{(1)} & R_{22}^{(1)} \end{pmatrix} \mathcal{S} + \begin{pmatrix} R_{11}^{(0)} & R_{12}^{(0)} \\ R_{21}^{(0)} & R_{22}^{(0)} \end{pmatrix} + \begin{pmatrix} R_{11}^{(-1)} & R_{12}^{(-1)} \\ R_{21}^{(-1)} & R_{22}^{(-1)} \end{pmatrix} \mathcal{S}^{-1}$$
$$+\mathbf{p} \otimes (\mathcal{S}-1)^{-1} \begin{pmatrix} v & u \end{pmatrix}.$$

To obtain the coefficients we consider relation (4.4.4) and formulae in Theorem 4 from which we can obtain the following relation:

$$\mathcal{R}_*(\mathbf{K}) = [\mathbf{K}_*, \mathcal{R}], \tag{4.5.1}$$

where $\mathbf{K}_*$ is of the form

$$\mathbf{K}_* = \begin{pmatrix} \mathcal{S} - 2uv & -u^2 \\ v^2 & 2uv - \mathcal{S}^{-1} \end{pmatrix}.$$

Notice that in $\mathcal{R}_*(\mathbf{K})$ the coefficient of $\mathcal{S}^2$ is zero and on the right hand side we have

$$\begin{pmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{11}^{(1)} & R_{12}^{(1)} \\ R_{21}^{(1)} & R_{22}^{(1)} \end{pmatrix} \mathcal{S} - \begin{pmatrix} R_{11}^{(1)}\mathcal{S} & R_{12}^{(1)}\mathcal{S} \\ R_{21}^{(1)}\mathcal{S} & R_{22}^{(1)}\mathcal{S} \end{pmatrix} \begin{pmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{pmatrix},$$

which gives the relations

$$\mathcal{S}(R_{11}^{(1)}) - R_{11}^{(1)} = 0, \qquad \mathcal{S}(R_{12}^{(1)}) = R_{21}^{(1)} = 0\,.$$

Thus $R_{12}^{(1)} = R_{21}^{(1)} = 0$ and $\mathcal{R}_{11}^{(1)} = \alpha$ where α is a constant. Furthermore since $v_{\mathcal{T}}$ does not contain v_1 we have $R_{22}^{(1)} = 0$. In a similar way as u_{-1} does not appear in $u_{\mathcal{T}}$, if we solve relation (4.5.1) for $\mathcal{S}^{-2}$ we get

$$R_{21}^{(-1)} = R_{12}^{(-1)} = R_{11}^{(-1)} = 0, \qquad R_{22}^{(-1)} = \beta\,,$$

where β is an arbitrary constant. The constants α and β can be determined by looking at the coefficients of u_2 and v_{-2} respectively in $u_{\mathcal{T}}$ and $v_{\mathcal{T}}$. Hence we find $\alpha = \beta = 1$. To obtain the matrix $\begin{pmatrix} R_{11}^{(0)} & R_{12}^{(0)} \\ R_{21}^{(0)} & R_{22}^{(0)} \end{pmatrix}$ we collect the coefficients of $\mathcal{S}$ and set to zero. Expanding the right hand side of relation (4.5.1) for $\mathcal{S}$ gives

$$\begin{pmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} R_{11}^{(0)} & R_{12}^{(0)} \\ R_{21}^{(0)} & R_{22}^{(0)} \end{pmatrix} + \begin{pmatrix} -2uv & -u^2 \\ v^2 & 2uv \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & R_{22}^{(1)} \end{pmatrix}\mathcal{S}$$
$$-\begin{pmatrix} \alpha\mathcal{S} & 0 \\ 0 & R_{22}^{(1)}\mathcal{S} \end{pmatrix}\begin{pmatrix} -2uv & -u^2 \\ v^2 & 2uv \end{pmatrix} - \begin{pmatrix} R_{11}^{(0)} & R_{12}^{(0)} \\ R_{21}^{(0)} & R_{22}^{(0)} \end{pmatrix}\begin{pmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{pmatrix},$$

which provides the following relations:

$$(\mathcal{S} - 1)(R_{11}^{(0)} + 2uv) = 0\,,$$
$$\mathcal{S}(R_{11}^{(0)}) + u_1^2 = 0\,,$$
$$v^2 - R_{21}^{(0)} = 0\,.$$

Therefore we have

$$R_{11}^{(0)} = -2uv, \; R_{11}^{(0)} = -u^2, \; R_{21}^{(0)} = v^2\,.$$

Now we can rewrite $\mathcal{R}$ in a more precise form as

$$\mathcal{R} = \begin{pmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -2uv & -u^2 \\ v^2 & R_{22}^{(0)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{S}^{-1} \end{pmatrix}$$
$$+\mathbf{p} \otimes (\mathcal{S}-1)^{-1} \begin{pmatrix} v & u \end{pmatrix},$$

To determine the rest of coefficients we need to calculate the action of $\mathcal{R}$ on $\mathbf{K}$ and compare the result to the terms appearing in the symmetry $\mathbf{G}$. Notice that since

$$< \omega, \mathbf{K} >= (\mathcal{S}-1)uv_{-1},$$

therefore the action of quasi-local term $\mathbf{K}$ produces the following local term:

$$\begin{aligned} p^1(\mathcal{S}-1)^{-1}vu_t + p^1(\mathcal{S}-1)^{-1}uv_t &= p^1(\mathcal{S}-1)^{-1}(vu_t + uv_t) \\ &= p^1(\mathcal{S}-1)^{-1} < \omega, \mathbf{K} > \\ &= p^1 uv_{-1}. \end{aligned}$$

By the similar calculation we get the term p^2uv_{-1}. Now if we compute $\mathcal{R}(\mathbf{K})$ and equate to $\mathbf{G}$, the following equalities hold:

$$u^2v_{-1} + p^1uv_{-1} = -u^2v_{-1}, \tag{4.5.2}$$

$$-R_{22}^{(0)}v_{-1} + R_{22}^{(0)}v^2u + p^2uv_{-1} = 2uvv_{-1}. \tag{4.5.3}$$

Equation (4.5.2) simply gives $p^1 = -2u$ and from relation (4.5.3) we get the system

$$\begin{cases} -R_{22}^{(0)}v_{-1} + p^2uv_{-1} = 2uvv_{-1}, \\ R_{22}^{(0)}v_2u = 0. \end{cases}$$

Hence coefficients $R_{22}^{(0)}$ and p^2 are determined as

$$R_{22}^{(0)} = 0, \qquad p^2 = 2v.$$

Now that all coefficients of $\mathcal{R}$ are obtained, we can write it as the following explicit form:

$$\mathcal{R} = \begin{pmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -2uv & -u^2 \\ v^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{S}^{-1} \end{pmatrix} + \begin{pmatrix} -2u \\ 2v \end{pmatrix} \otimes (\mathcal{S}-1)^{-1} \begin{pmatrix} v & u \end{pmatrix}.$$

But still the last step remains. In fact since some coefficients are obtained just by the action of $\mathcal{R}$ on $\mathbf{K}$, we have to prove $\mathcal{R}$ is indeed a recursion operator. This means we have to show that it satisfies relation (4.5.1). As we have the explicit form of $\mathcal{R}$ and $\mathbf{K}$, we can compute $\mathbf{K}_*\mathcal{R} - \mathcal{R}\mathbf{K}_*$ and equate to $R_*(\mathbf{K})$. Notice that in computing $\mathcal{R}\mathbf{K}_*$ we come up with expressions such as

$$2u(\mathcal{S}-1)^{-1}v\mathcal{S}, \qquad 2u(\mathcal{S}-1)^{-1}u\mathcal{S}^{-1}.$$

To simplify these terms we use the following identities:

$$\begin{aligned} f(\mathcal{S}-1)^{-1}g_1\mathcal{S} &= f\mathcal{S}(\mathcal{S}-1)^{-1}g = fg + f(\mathcal{S}-1)^{-1}g\,, \\ f(\mathcal{S}-1)^{-1}g_{-1}\mathcal{S}^{-1} &= f\mathcal{S}^{-1}(\mathcal{S}-1)^{-1}g = -fg_{-1}\mathcal{S}^{-1} + f(\mathcal{S}-1)^{-1}g\,. \end{aligned}$$

To summarize this section, we tried to obtain the recursion operator through the direct method using relation (4.5.1). We also were looking closely at the structure of the given equation and the terms appearing in the symmetry. This helped us to specify the order of local terms and the coefficients of constant terms.

As we mentioned, recursion operators are not obtained through straightforward processes and there exists no general algorithm to apply on a given equation and produce its recursion operator. For recursion operators which are the product of Hamiltonian and symplectic operators we use some similar logic to construct appropriate Hamiltonian and symplectic or two Hamiltonian operators. Notice that the structure of weakly nonlocal terms in Hamiltonian and symplectic operators are respectively as follows:

$$\mathbf{p}_1(\mathcal{S}-1)^{-1}\mathbf{p}_2\,,$$
$$\omega_1(\mathcal{S}-1)^{-1}\omega_2\,,$$

where $\mathbf{p}_1, \mathbf{p}_2$ are symmetries and ω_1 and ω_2 are co-symmetries of the given equation. A thorough treatment of this structure is discussed in [91]. The recursion operator of Merola-Ragnisco-Tu system has also an weakly nonlocal inverse as follows:

$$\mathcal{R}^{-1} = \begin{pmatrix} \frac{1}{(u_{-1}v+1)^2}\mathcal{S}^{-1} & \frac{u_{-1}^2}{(u_{-1}v+1)^2} \\ -\frac{v_1^2}{(uv_1+1)^2} & \frac{1}{(uv_1+1)^2}\mathcal{S} - \frac{2u_{-1}v_1}{(u_{-1}v+1)(uv_1+1)} \end{pmatrix}$$
$$+2\begin{pmatrix} \frac{u_{-1}}{u_{-1}v+1} \\ -\frac{v_1}{uv_1+1} \end{pmatrix} \otimes (\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{v_1}{uv_1+1} & \frac{u_{-1}}{u_{-1}v+1} \end{pmatrix}.$$

This does not happen for all recursion operators. In fact it is still an open problem if for a given recursion operator its inverse exists.

The next chapter provides the list of integrable differential-difference equations along with the quantities and operators defined in this chapter. There are several new results on the Hamiltonian, symplectic and recursion operators. We also give the inverse of the recursion operator in the case if it exists.

Chapter 5

List of integrable differential-difference equations

5.1 Introduction

In this chapter we apply definitions and theorems provided in the last three chapters. To do so, a long list of integrable differential-difference equations is given. The list contains equations themselves, their Hamiltonian structures, recursion operators and non-trivial generalised symmetries. We not only put in an effort to compute all the objects in the list, but also include some new results. In fact this chapter forms the main part of the result of the paper [42]. We mainly refer to the source where we learned about each system although some attempt has been made to track the historical origins. In fact scalar equations are coming mainly from the review paper [95] and articles [12, 13, 56, 58, 66, 92]. For multi component equations we first tried to find equations from different sources such as [62, 66, 81] and then looking for other known properties form references therein. The new results of the list are also obtained from [86] in which equations and in some cases the simple form of Hamiltonian operators are given.

We also include partial results on master symmetries of equations. Roughly speaking, a master symmetry for an integrable differential-difference equation is a vector field which maps a symmetry to a new symmetry. Results on master symmetries are closely related to scaling symmetries of the corresponding equations. A scaling symmetry for a differential-difference equation $\mathbf{K}$ is a vector field $\mathfrak{Q} \in \mathfrak{g}$ which satisfies the following relation:

$$L_{\mathfrak{Q}}\mathbf{K} = \lambda\mathbf{K},$$

where λ is a scalar. Then the action of recursion operators, which also satisfy the Nijenhuis property, on a scaling symmetry produces a master symmetry. In fact we have to show that the action of $\mathcal{R}(\mathfrak{Q})$ on $\mathbf{K}$ produces a symmetry. This can be seen through the following calculation:

$$[\mathcal{R}(\mathfrak{Q}), \mathbf{K}] = L_{\mathcal{R}(\mathfrak{Q})}\mathbf{K} = \mathcal{R}(L_{\mathfrak{Q}}\mathbf{K}) = \lambda\mathcal{R}(\mathbf{K}),$$

where as $\mathcal{R}$ is a recursion operator $\mathcal{R}(\mathbf{K})$ gives a symmetry. The thorough study of master symmetries is done in the next chapter. For some equations we add further notes on their links with other known equations and the weakly nonlocal inverses of recursion operators if existing. This list is far from being complete.

As we mentioned in the previous chapter the majority of pseudo-difference operators we studied are weakly nonlocal, that is, according to Definition 4 they are pseudo-difference operators with only a finite number of nonlocal terms of the form

$$\mathbf{p} \otimes (\mathcal{S} - 1)^{-1}\omega.$$

We should note that while dealing with the above weakly nonlocal terms the action of the recursion operator is not uniquely determined. In fact since the field of scalars are the kernel of the difference operator $(\mathcal{S} - 1)$, for the action of $(\mathcal{S} - 1)^{-1}$ we should also consider a constant term. For example consider the recursion operator of the Volterra equation

$$\mathcal{R} = u\mathcal{S} + u + u_1 + u\mathcal{S}^{-1} + u(u_1 - u_{-1})(\mathcal{S} - 1)^{-1}\frac{1}{u}\,.$$

The action of $\mathcal{R}$ on $u(u_1 - u_{-1})$ gives

$$u(u_1u_2 + uu_1 + u_1^2 - uu_{-1} - u_{-1}^2 - u_{-1}u_{-2}) + \alpha u(u_1 - u_{-1})\,,$$

where α is an arbitrary constant. For some weakly nonlocal recursion operators of multi component integrable differential-difference equations, there exists a weakly nonlocal inverse recursion operator. This also enables us to generate infinitely many inverse symmetry flows. The Ablowitz-Ladik lattice (cf. Section 5.2.12 for more algebraic properties of this equation)

$$\begin{cases} u_t = (1 - uv)(\alpha u_1 - \beta u_{-1})\,, \\ v_t = (1 - uv)(\beta v_1 - \alpha v_{-1})\,. \end{cases}$$

possesses a recursion operator

$$\begin{aligned} \mathcal{R} = &\begin{pmatrix} (1-uv)\,\mathcal{S} - u_1 v - uv_{-1} & -uu_1 \\ vv_{-1} & (1-uv)\,\mathcal{S}^{-1} \end{pmatrix} + \begin{pmatrix} -u \\ v \end{pmatrix} \otimes (\mathcal{S}-1)^{-1} \begin{pmatrix} v_{-1} & u_1 \end{pmatrix} \\ &- \begin{pmatrix} (1-uv)u_1 \\ -(1-uv)v_{-1} \end{pmatrix} \otimes (\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{v}{1-uv} & \frac{u}{1-uv} \end{pmatrix}. \end{aligned}$$

We usually do not explicitly write out $\otimes$ if this causes no confusion.

The operator $\mathcal{R}$ is weakly nonlocal and it has a weakly nonlocal inverse

$$\begin{aligned} \mathcal{R}^{-1} = &\begin{pmatrix} (1-uv)\,\mathcal{S}^{-1} & uu_{-1} \\ -vv_1 & (1-uv)\,\mathcal{S} - uv_1 - u_{-1}v \end{pmatrix} + \begin{pmatrix} u \\ -v \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} v_1 & u_{-1} \end{pmatrix} \\ &+ \begin{pmatrix} (1-uv)u_{-1} \\ -(1-uv)v_1 \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{v}{1-uv} & \frac{u}{1-uv} \end{pmatrix}. \end{aligned}$$

The phenomenon has been explored for the Bruschi-Ragnisco lattice in [62]. In this chapter, we compute the weakly nonlocal inverse recursion operators for all multi component integrable differential-difference equations if existing. For a given weakly nonlocal difference operator, to answer whether there exists a weakly nonlocal inverse operator is still an open problem.

To the best of our knowledge, the following results are new:

- The Hamiltonian operators, symplectic operators and recursion operators given in section 5.2.4 for equation (5.2.3).
- The Hamiltonian operators, symplectic operators and recursion operators for the Kaup-Newell lattice (section 5.2.14), the Chen-Lee-Liu lattice (section 5.2.15) and the Ablowitz-Ramani-Segur (Gerdjikov-Ivanov) lattice (section 5.2.16);
- The weakly nonlocal inverse recursion operators, if they exist, except for the Ablowitz-Ladik lattice and the Bruschi-Ragnisco lattice.

5.2 A list of integrable differential-difference equations

5.2.1 The Volterra chain

- Equation [88]:

$$u_t = u(u_1 - u_{-1}) \tag{5.2.1}$$

- Hamiltonian structure [12, 66]:

$$\mathcal{H}_1 = u(\mathcal{S} - \mathcal{S}^{-1})u, \qquad f_1 = u$$
$$\mathcal{H}_2 = u(\mathcal{S}u\mathcal{S} + u\mathcal{S} + \mathcal{S}u - u\mathcal{S}^{-1} - \mathcal{S}^{-1}u - \mathcal{S}^{-1}u\mathcal{S}^{-1})u, \qquad f_2 = \frac{1}{2}\ln u$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1} = u\mathcal{S} + u + u_1 + u\mathcal{S}^{-1} + u(u_1 - u_{-1})(\mathcal{S} - 1)^{-1}\frac{1}{u}$$

- Non-trivial symmetry [12, 95]:

$$\mathcal{R}(u_t) = u\left(u_1u_2 + u_1^2 + uu_1 - uu_{-1} - u_{-1}^2 - u_{-1}u_{-2}\right)$$

- Master symmetry [12, 13, 46]:

$$\mathcal{R}(u) = nu_t + u(2u_1 + u + u_{-1})$$

This equation is also known as the Lotka-Volterra model, the Kac-van Moerbeke lattice or the Langmuir lattice. The so-called Kac-van Moerbeke-Langmuir equation [66]

$$w_\tau = w(w_1^\epsilon - w_{-1}^\epsilon), \qquad \epsilon \neq 0 \text{ is a constant.}$$

is related to (5.2.1) by the point transformation $u = w^\epsilon$ and $t = \epsilon\tau$. This equation is also written as

$$w_t = \exp(w + w_1) - \exp(w + w_{-1}),$$

which can be transformed into (5.2.1) by the transformation $u = \exp(w + w_1)$.

5.2.2 Modified Volterra equation

- Equation [37, 95]:

$$u_t = u^2(u_1 - u_{-1})$$

- Hamiltonian structure [87, 95]:

$$\mathcal{H}_1 = u(\mathcal{S}-1)(\mathcal{S}+1)^{-1}u, \qquad f_1 = uu_1$$
$$\mathcal{H}_2 = u^2(\mathcal{S}-\mathcal{S}^{-1})u^2, \qquad f_2 = \ln u$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1} = u^2\mathcal{S} + 2uu_1 + u^2\mathcal{S}^{-1} + 2u^2(u_1 - u_{-1})(\mathcal{S}-1)^{-1}\frac{1}{u}$$

- Non-trivial symmetry [95]:

$$\mathcal{R}(u_t) = u^2u_1^2(u_2+u) - u^2u_{-1}^2(u+u_{-2})$$

- Master symmetry [95]:

$$\mathcal{R}(\frac{u}{2}) = nu_t + \frac{u^2}{2}(3u_1 + u_{-1})$$

The Modified Volterra equation is also known as the discrete modified Korteweg-de Vries equation. Under the Miura transformation $w = uu_1$ it can be transformed into the Volterra Chain $w_t = w(w_1 - w_{-1})$ as in section 5.2.1.

5.2.3 Yamilov's discretisation of the Krichever-Novikov equation

- Equation [94]:

$$u_t = \frac{R(u_1, u, u_{-1})}{u_1 - u_{-1}} := K^{(1)},$$

where R is the polynomial with constant coefficients $\alpha, \beta, \gamma, \delta, \epsilon$ defined by

$$R(u,v,w) = (\alpha v^2 + 2\beta v + \gamma)uw + (\beta v^2 + \lambda v + \delta)(u+w) + \gamma v^2 + 2\delta v + \epsilon \qquad (5.2.2)$$

- Two non-trivial symmetries [58, 56, 78, 93]:

$$K^{(2)} = \frac{R(u,u_{-1},u)R(u_1,u,u_1)}{(u_1-u_{-1})^2}\left(\frac{1}{u_2-u}+\frac{1}{u-u_{-2}}\right);$$

$$K^{(3)} = \frac{R(u_1,u,u_1)R(u,u_{-1},u)}{(u_1-u_{-1})^2}\left(\frac{\mathcal{S}^2 K^{(1)}}{(u_2-u)^2}+\frac{\mathcal{S}^{-2}K^{(1)}}{(u-u_{-2})^2}\right)$$

$$+K^{(1)}K^{(2)}\left(\frac{1}{u_2-u}+\frac{1}{u-u_{-2}}\right)$$

- Hamiltonian structure [55, 58]:

$$\mathcal{H} = A\,\mathcal{S} - \mathcal{S}^{-1}\,A + 2\,K^{(1)}\,(\mathcal{S}-1)^{-1}\mathcal{S}K^{(2)} + 2\,K^{(2)}\,(\mathcal{S}-1)^{-1}K^{(1)};$$

$$\hat{\mathcal{H}} = \hat{A}\,\mathcal{S}^2 - \mathcal{S}^{-2}\hat{A} + \hat{B}\mathcal{S} - \mathcal{S}^{-1}\hat{B} + K^{(2)}\,(\mathcal{S}-1)^{-1}(\mathcal{S}+1)K^{(2)}$$

$$+2\,K^{(1)}\,(\mathcal{S}-1)^{-1}\mathcal{S}K^{(3)} + 2\,K^{(3)}\,(\mathcal{S}-1)^{-1}K^{(1)},$$

where

$$A = \frac{R(u_2,u_1,u_2)R(u_1,u,u_1)R(u,u_{-1},u)}{(u_1-u_{-1})^2(u_2-u)^2};$$

$$\hat{A} = \frac{R(u_3,u_2,u_3)R(u_2,u_1,u_2)R(u_1,u,u_1)R(u,u_{-1},u)}{(u_1-u_{-1})^2(u_2-u)^2(u_3-u_1)^2};$$

$$\hat{B} = 2A\left(\frac{K^{(1)}}{u-u_{-2}} - \frac{\partial_u R(u_1,u,u_1)}{2(u_1-u_{-1})} + \frac{\partial^2 R(u_1,u,u_1)}{4\partial u \partial u_1}\right)$$

$$+\frac{2R(u,u_{-1},u)}{(u_1-u_{-1})^2}\mathcal{S}\left(K^{(1)}K^{(2)}\right)$$

- Symplectic operator [58]:

$$\mathcal{I} = \frac{1}{R(u_1,u,u_1)}\,\mathcal{S} - \mathcal{S}^{-1}\,\frac{1}{R(u_1,u,u_1)}$$

- Recursion operator [55, 58]:

$$\mathcal{R} = \mathcal{H}\mathcal{I} \quad \text{and} \quad \hat{\mathcal{R}} = \hat{\mathcal{H}}\mathcal{I}$$

5.2.4 Integrable Volterra type equations

The classification of integrable Volterra type equations of the form

$$u_t = f(u_{-1}, u, u_1),$$

where f is a smooth function of all its variables was obtained by Yamilov using the symmetry approach. In his remarkable review paper [95], he presented the following complete list of integrable Volterra type equations (with higher order conservation laws) up to point transformations:

$$\text{V1} \quad u_t = P(u)(u_1 - u_{-1}) \tag{5.2.3}$$

$$\text{V2} \quad u_t = P(u^2)\left(\frac{1}{u_1 + u} - \frac{1}{u + u_{-1}}\right) \tag{5.2.4}$$

$$\text{V3} \quad u_t = Q(u)\left(\frac{1}{u_1 - u} + \frac{1}{u - u_{-1}}\right) \tag{5.2.5}$$

$$\text{V4} \quad u_t = \frac{R(u_1, u, u_{-1}) + \nu R(u_1, u, u_1)^{1/2} R(u_{-1}, u, u_{-1})^{1/2}}{u_1 - u_{-1}} \tag{5.2.6}$$

$$\text{V5} \quad u_t = y(u_1 - u) + y(u - u_{-1}), \qquad y' = P(y) \tag{5.2.7}$$

$$\text{V6} \quad u_t = y(u_1 - u)\, y(u - u_{-1}) + \mu, \qquad y' = \frac{P(y)}{y} \tag{5.2.8}$$

$$\text{V7} \quad u_t = \frac{1}{y(u_1 - u) + y(u - u_{-1})} + \mu, \qquad y' = P(y^2) \tag{5.2.9}$$

$$\text{V8} \quad u_t = \frac{1}{y(u_1 + u) - y(u + u_{-1})}, \qquad y' = Q(y) \tag{5.2.10}$$

$$\text{V9} \quad u_t = \frac{y(u_1 + u) - y(u + u_{-1})}{y(u_1 + u) + y(u + u_{-1})}, \qquad y' = \frac{P(y^2)}{y} \tag{5.2.11}$$

$$\text{V10} \quad u_t = \frac{y(u_1 + u) + y(u + u_{-1})}{y(u_1 + u) - y(u + u_{-1})}, \qquad y' = \frac{Q(y)}{y} \tag{5.2.12}$$

$$\text{V11} \quad u_t = \frac{(1 - y(u_1 - u))(1 - y(u - u_{-1}))}{y(u_1 - u) + y(u - u_{-1})} + \mu, \quad y' = \frac{P(y^2)}{1 - y^2} \tag{5.2.13}$$

where $\mu \in \mathbb{C}$, $\nu \in \{0, \pm 1\}$ and P and Q are polynomials with constant coefficients $\alpha, \beta, \gamma, \delta, \epsilon$ defined by

$$P(u) = \alpha u^2 + \beta u + \gamma, \tag{5.2.14}$$

$$Q(u) = \alpha u^4 + \beta u^3 + \gamma u^2 + \delta u + \epsilon, \tag{5.2.15}$$

and the polynomial R is defined by (5.2.2). As stated in the paper [95], the problem of constructing the generalised symmetries for all equations (V1)–(V11) remains open although the master symmetries for some forms of equations in the list are known [12, 13]. We know that the Miura transformation $\tilde{u} = y(u_1 - u)$ transforms equations (V5) and (V6) to (V1) and equations (V7) and (V11) into (V2), and the Miura transformation $\tilde{u} = y(u_1 + u)$ transforms equation (V9) to (V2) and equations (V8) and (V10) into (V3) [95]. In what follows we just provide the new Hamiltonian, symplectic and recursion operators for (V1). For new results on (V2), (V3) and (V4) one can read [42]. For other equations one can obtain the corresponding Hamiltonian, symplectic and recursion operators via Miura transformations.

V1 equation (5.2.3)

- Hamiltonian structure [95]:

$$\mathcal{H} = P(u)(\mathcal{S} - \mathcal{S}^{-1})P(u), \qquad f = \int \frac{u}{P(u)} du$$

- Symplectic operator:

$$\mathcal{I} = \alpha(\mathcal{S}-\mathcal{S}^{-1})+(\alpha u_1+\beta+\alpha u_{-1})\mathcal{S}(\mathcal{S}-1)^{-1}\frac{P'(u)}{P(u)}+\frac{P'(u)}{P(u)}(\mathcal{S}-1)^{-1}(\alpha u_1+\beta+\alpha u_{-1})$$

- Recursion operator:

$$\mathcal{R} = P(u)\mathcal{S} + 2\alpha u u_1 + \beta(u + u_1) + P(u)\mathcal{S}^{-1} + u_t(\mathcal{S} - 1)^{-1}\frac{P'(u)}{P(u)}$$

- Non-trivial symmetry :

$$\mathcal{R}(u_t) = P(u)\left(P(u_1)u_2 + \alpha u u_1^2 + \beta(u + u_1)u_1 - P(u_{-1})u_{-2} - \alpha u u_{-1}^2 - \beta(u + u_{-1})u_{-1}\right)$$

- Master symmetry:

$$nu_t + P(u)(cu_1 + \frac{\beta}{\alpha} + (2-c)u_{-1}), \quad c \in \mathbb{C}, \quad \text{when } \alpha \neq 0$$
$$nu_t + P(u)(cu_1 + u + (3-c)u_{-1}), \quad c \in \mathbb{C}, \quad \text{when } \alpha = 0$$

This equation includes both the Volterra chain in section 5.2.1 and the modified Volterra equation in section 5.2.2.

5.2.5 The Narita-Itoh-Bogoyavlensky lattice

- Equation [8, 39, 60]:

$$u_t = u(\sum_{k=1}^{p} u_k - \sum_{k=1}^{p} u_{-k}), \qquad p \in \mathbb{N}$$

- Hamiltonian structure [62]:

$$\mathcal{H} = u(\sum_{i=1}^{p} \mathcal{S}^i - \sum_{i=1}^{p} \mathcal{S}^{-i})u, \qquad f = u$$

- Recursion operator [92]:

$$\mathcal{R} = u(\mathcal{S} - \mathcal{S}^{-p})(\mathcal{S} - 1)^{-1} \prod_{i=1}^{\rightarrow p} (\mathcal{S}^{p+1-i}u - u\mathcal{S}^{-i})(\mathcal{S}^{p-i}u - u\mathcal{S}^{-i})^{-1},$$

  where the notation $\prod_{i=1}^{\rightarrow p}$ is denoted the order of the value i, from 1 to p, that is, $\prod_{i=1}^{\rightarrow p} a_i = a_1 a_2 \cdots a_p$.

- Non-trivial symmetry:

$$\mathcal{R}(u_t) = u(1 - \mathcal{S}^{-(p+1)})\mathcal{S}^{1-p} \sum_{0 \leq i \leq j \leq 2p-1} u_j u_{i+p}$$

- Master symmetry [92]: $\mathcal{R}(u)$

For $p = 1, 2$ or 3, a few higher order symmetries are explicitly given in [62], where the authors also studied their Hamiltonian operator, recursion operator and master symmetry for $p = 1, 2$. The Narita-Itoh-Bogoyavlensky lattice is known as an integrable discretisation for the Korteweg-de Vries equation. It can also be presented as

$$v_t = v(\prod_{k=1}^{p} v_k - \prod_{k=1}^{p} v_{-k}),$$

which is related to the Narita-Itoh-Bogoyavlensky lattice via the transformation $u = \prod_{k=0}^{p-1} v_k$ for fixed p. Taking $p = 1$, we get the well-known Volterra chain in section 5.2.1. Thus they can be regarded as the generalisation of the Volterra chain.

Let $u = \prod_{k=0}^{p} w_k$. Then w satisfies the so-called the modified Bogoyavlensky chain

$$w_t = w^2(\prod_{k=1}^{p} w_k - \prod_{k=1}^{p} w_{-k}).$$

The recursion operator given above for the Narita-Itoh-Bogoyavlensky lattice is highly nonlocal (so is the master symmetry). Recently, Svinin [83] derived the explicit formulas for its generalised symmetries in terms of a family of homogeneous difference polynomials.

5.2.6 The Toda lattice

- Equation [85]:

$$q_{tt} = \exp(q_1 - q) - \exp(q - q_{-1})$$

In the Manakov-Flaschka coordinates [18, 50] defined by $u = \exp(q_1 - q)$, $v = q_t$, it can be rewritten as two component evolution system:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u(v_1 - v) \\ u - u_{-1} \end{pmatrix} \tag{5.2.16}$$

- Hamiltonian structure [13, 44, 66]:

$$\mathcal{H}_1 = \begin{pmatrix} 0 & u(\mathcal{S}-1) \\ (1-\mathcal{S}^{-1})u & 0 \end{pmatrix}, \qquad f_1 = u + \frac{v^2}{2}$$

$$\mathcal{H}_2 = \begin{pmatrix} u(\mathcal{S}-\mathcal{S}^{-1})u & u(\mathcal{S}-1)v \\ v(1-\mathcal{S}^{-1})u & u\mathcal{S}-\mathcal{S}^{-1}u \end{pmatrix}, \qquad f_2 = v$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1} = \begin{pmatrix} v_1 + u(v_1-v)(\mathcal{S}-1)^{-1}\frac{1}{u} & u\mathcal{S}+u \\ 1+\mathcal{S}^{-1} + (u-u_{-1})(\mathcal{S}-1)^{-1}\frac{1}{u} & v \end{pmatrix}$$

$$= \begin{pmatrix} v_1 & u\mathcal{S}+u \\ 1+\mathcal{S}^{-1} & v \end{pmatrix} + \begin{pmatrix} u(v_1-v) \\ u-u_{-1} \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{1}{u} & 0 \end{pmatrix}$$

- Non-trivial symmetry:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u(v_1^2 - v^2 + u_1 - u_{-1}) \\ u(v_1+v) - u_{-1}(v_{-1}+v) \end{pmatrix}$$

- Master symmetry [13]:

$$\mathcal{R}\begin{pmatrix} u \\ \frac{v}{2} \end{pmatrix} = \begin{pmatrix} nu_t + \frac{3}{2}uv_1 + \frac{1}{2}uv \\ nv_t + u + u_{-1} + \frac{v^2}{2} \end{pmatrix}$$

The Hirota nonlinear Lumped Network equation [36]

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} v_1 - v \\ v(u-u_{-1}) \end{pmatrix}$$

is related to the Toda lattice (5.2.16) by a simple invertible transformation. Namely, let $u = q$ and $v = p_{-1}$. Then the variables p and q satisfy the Toda equation. All its properties can be obtained via those for the Toda lattice.

5.2.7 The Relativistic Toda system

- Equation [69]:

$$q_{tt} = q_t q_{-1t} \frac{\exp(q_{-1} - q)}{1 + \exp(q_{-1} - q)} - q_t q_{1t} \frac{\exp(q - q_1)}{1 + \exp(q - q_1)}$$

Let us introduce the following dependent variables [80, 28]:

$$u = \frac{q_t \exp(q - q_1)}{1 + \exp(q - q_1)}, \qquad v = \frac{q_t}{1 + \exp(q - q_1)}.$$

Then the equation can be written as

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u(u_{-1} - u_1 + v - v_1) \\ v(u_{-1} - u) \end{pmatrix} \tag{5.2.17}$$

- Hamiltonian structure [28]:

$$\mathcal{H}_1 = \begin{pmatrix} 0 & u(1 - \mathcal{S}) \\ (\mathcal{S}^{-1} - 1)u & u\mathcal{S} - \mathcal{S}^{-1}u \end{pmatrix}, \qquad f_1 = \frac{1}{2}(u^2 + v^2) + uv + u_1 u + uv_1$$

$$\mathcal{H}_2 = \begin{pmatrix} u(\mathcal{S}^{-1} - \mathcal{S})u & u(1 - \mathcal{S})v \\ v(\mathcal{S}^{-1} - 1)u & 0 \end{pmatrix}, \qquad f_2 = u + v$$

- Recursion operator [28]:

$$\mathcal{R} = \begin{pmatrix} u\mathcal{S} + u + v_1 + u_1 + u\mathcal{S}^{-1} & u\mathcal{S} + u \\ v + v\mathcal{S}^{-1} & v \end{pmatrix} - \begin{pmatrix} u_t \\ v_t \end{pmatrix} (\mathcal{S} - 1)^{-1} \begin{pmatrix} \frac{1}{u} & 0 \end{pmatrix}$$

- Non-trivial symmetry :

$$\mathcal{R} \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

where

$$G_1 = uu_{-1}(u + u_{-1} + u_{-2} + 2v + v_{-1}) - uu_1(u_2 + u_1 + u + 2v_1 + v_2)$$
$$+u^2(v - v_1) + u(v^2 - v_1^2)$$
$$G_2 = vu_{-1}(u_{-2} + u_{-1} + v + v_{-1}) - uv(u_1 + u + v_1 + v)$$

- Master symmetry [28]:

$$\mathcal{R}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -nu_t + u(v + 2v_1 + u + 2u_1 + u_{-1}) \\ -nv_t + v(u + v + u_{-1}) \end{pmatrix}, \qquad ((\mathcal{S}-1)^{-1}1 = n)$$

As noticed in [28], the inverse of this recursion operator $\mathcal{R}$ is also weakly nonlocal:

$$\mathcal{R}^{-1} = \mathcal{H}_1\mathcal{H}_2^{-1} = \begin{pmatrix} \frac{1}{v_1} & -\frac{u}{v_1^2}\mathcal{S} + \frac{u}{v^2} - \frac{2u}{vv_1} \\ -\mathcal{S}^{-1}\frac{1}{v} - \frac{1}{v_1} & \frac{u}{v_1^2}\mathcal{S} + \mathcal{S}^{-1}\frac{u}{v^2} + \frac{2u}{vv_1} + \frac{1}{v} \end{pmatrix}$$
$$+\begin{pmatrix} \frac{u}{v_1} - \frac{u}{v} \\ \frac{u_{-1}}{v_{-1}} - \frac{u}{v_1} \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{1}{u} & -\frac{2}{v} \end{pmatrix}.$$

However, recursion operators $\mathcal{R}$ and $\mathcal{R}^{-1}$ have different starting point, i.e., seeds. For $\mathcal{R}$, it starts with acting on the right hand of the equation while the seed for $\mathcal{R}^{-1}$ is $\sigma = \begin{pmatrix} \frac{u}{v_1} - \frac{u}{v} \\ \frac{u_{-1}}{v_{-1}} - \frac{u}{v_1} \end{pmatrix}$. Moreover, $\mathcal{R}$ acting on σ and $\mathcal{R}^{-1}$ acting on the right hand of the equation do not give rise to new symmetries.

5.2.8 Two component Volterra lattice

- Equation [66]:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u(v_1 - v) \\ v(u - u_{-1}) \end{pmatrix} \tag{5.2.18}$$

- Hamiltonian structure [66]:

$$\mathcal{H}_1 = \begin{pmatrix} 0 & u(\mathcal{S}-1)v \\ v(1-\mathcal{S}^{-1})u & 0 \end{pmatrix}, \qquad f_1 = u+v$$

$$\mathcal{H}_2 = \begin{pmatrix} u(\mathcal{S}v - v\mathcal{S}^{-1})u & u(u\mathcal{S} - u + \mathcal{S}v - v)v \\ v(u - \mathcal{S}^{-1}u + v - v\mathcal{S}^{-1})u & v(u\mathcal{S} - \mathcal{S}^{-1}u)v \end{pmatrix}, \qquad f_2 = \ln u$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1} = \begin{pmatrix} u+v_1 & u\mathcal{S} + \frac{uv_1}{v} \\ v + v\mathcal{S}^{-1} & u+v \end{pmatrix} + \begin{pmatrix} u(v_1 - v) \\ v(u - u_{-1}) \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{1}{u} & \frac{1}{v} \end{pmatrix}$$

- Non-trivial symmetry:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u^2(v_1 - v) + u\left({v_1}^2 - v^2 + v_1u_1 - vu_{-1}\right) \\ v^2(u - u_{-1}) + v(u^2 - u_{-1}^2 + uv_1 - u_{-1}v_{-1}) \end{pmatrix}$$

- Master symmetry [13]:

$$\mathcal{R}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2nu_t + u^2 + 3uv_1 \\ 2nv_t + vu_{-1} + 2uv + v^2 \end{pmatrix}$$

This system comes from the Volterra chain in section 5.2.1 written in the variable w, that is,

$$w_t = w(w_1 - w_{-1}), \tag{5.2.19}$$

by renaming $u(n,t) = w(2n,t)$ and $v(n,t) = w(2n-1,t)$. It is related to the Toda equation (5.2.16), written in variables $\bar{u}$ and $\bar{v}$, by the Miura transformation [82]

$$\bar{u} = uv \quad \text{and} \quad \bar{v} = u_{-1} + v. \tag{5.2.20}$$

5.2.9 The Relativistic Volterra lattice

- Equation [43, 81, 82]:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u(v - v_{-1} + uv - u_{-1}v_{-1}) \\ v(u_1 - u + u_1 v_1 - uv) \end{pmatrix}$$

- Hamiltonian structure [81]:

$$\mathcal{H}_1 = \begin{pmatrix} 0 & u(1-\mathcal{S}^{-1})v \\ v(\mathcal{S}-1)u & 0 \end{pmatrix}, \qquad f_1 = u + v + uv$$

$$\mathcal{H}_2 = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \qquad f_2 = \ln u \text{ or } f_2 = \ln v$$

where

$$h_{11} = uv(1+u)\mathcal{S}u - u\mathcal{S}^{-1}uv(1+u)$$

$$h_{12} = uv(u+v+uv) - u(\mathcal{S}^{-1}uv\mathcal{S}^{-1} + u\mathcal{S}^{-1} + \mathcal{S}^{-1}v)v$$

$$h_{21} = v(\mathcal{S}uv\mathcal{S} + v\mathcal{S} + \mathcal{S}u)u - uv(u+v+uv)$$

$$h_{22} = v\mathcal{S}uv(1+v) - uv(1+v)\mathcal{S}^{-1}v$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1} = \begin{pmatrix} uv_{-1}\mathcal{S}^{-1} + u + v + uv & u(1+u_{-1})\mathcal{S}^{-1} + u(1+u) \\ v(1+v_1)\mathcal{S} + \frac{u_1 v(1+v_1)}{u} & u_1 v\mathcal{S} + u_1 + v + u_1 v_1 \end{pmatrix}$$

$$+ \begin{pmatrix} u(v - v_{-1} + uv - u_{-1}v_{-1}) \\ v(u_1 - u + u_1 v_1 - uv) \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{1}{u} & \frac{1}{v} \end{pmatrix}$$

- Non-trivial symmetry:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} uv(1+u)(u+u_1+u_1v_1) + uv^2(1+u)^2 - uv_{-1}^2(1+u_{-1})^2 - u^2v_{-1} \\ -u_{-1}uv_{-1}(1+u+v_{-2}+u_{-1}+u_{-2}v_{-2}); \\ u_1vv_1(1+2u_1+u_2v_2+u_2) + u_1^2v + u_1vv_1^2(1+u_1) + v^2u_1(1+v_1) \\ -uv(1+v)(v+v_{-1}+u_{-1}v_{-1}) - u^2v(1+v)^2 \end{pmatrix}$$

It is related to the Relativistic Toda equation (5.2.17), written in variables $\bar{u}$ and $\bar{v}$, by the Miura transformation $\bar{u} = -uv$ and $\bar{v} = -(u+v_{-1}+1)$ [82]. This transformation is similar to (5.2.20), which explains the name of this equation.

5.2.10 The Merola-Ragnisco-Tu lattice

- Equation [51, 62]:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u_1 - u^2v \\ -v_{-1} + v^2u \end{pmatrix}$$

- Hamiltonian structure [62]:

$$\mathcal{H} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad f = u_1v - \frac{u^2v^2}{2}$$

- Recursion operator [62]:

$$\mathcal{R} = \begin{pmatrix} \mathcal{S} - 2uv & -u^2 \\ v^2 & \mathcal{S}^{-1} \end{pmatrix} + 2\begin{pmatrix} -u \\ v \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} v & u \end{pmatrix}$$

- Non-trivial symmetry [62]:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u_2 - u_1^2v_1 - u^2v_{-1} - 2uvu_1 + u^3v^2 \\ -v_{-2} + v_{-1}^2u_{-1} + v^2u_1 + 2uvv_{-1} - u^2v^3 \end{pmatrix}$$

- Master symmetry [62]:

$$\mathcal{R}\begin{pmatrix} (n+1)u \\ -nv \end{pmatrix} = \begin{pmatrix} nu_t + 2u_1 - 2u^2v - 2u(\mathcal{S}-1)^{-1}uv \\ nv_t + v_{-1} + uv^2 + 2v(\mathcal{S}-1)^{-1}uv \end{pmatrix}$$

The recursion operator $\mathcal{R}$ has a weakly nonlocal inverse:

$$\mathcal{R}^{-1} = \begin{pmatrix} \frac{1}{(u_{-1}v+1)^2}\mathcal{S}^{-1} & \frac{u_{-1}^2}{(u_{-1}v+1)^2} \\ -\frac{v_1^2}{(uv_1+1)^2} & \frac{1}{(uv_1+1)^2}\mathcal{S} - \frac{2u_{-1}v_1}{(u_{-1}v+1)(uv_1+1)} \end{pmatrix}$$
$$+2\begin{pmatrix} \frac{u_{-1}}{u_{-1}v+1} \\ -\frac{v_1}{uv_1+1} \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{v_1}{uv_1+1} & \frac{u_{-1}}{u_{-1}v+1} \end{pmatrix}.$$

The symmetry $(u,\ -v)^{\mathrm{tr}}$ is a seed for both $\mathcal{R}$ and $\mathcal{R}^{-1}$.

5.2.11 The Kaup lattice

- Equation [3]:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} (u+v)(u_1-u) \\ (u+v)(v-v_{-1}) \end{pmatrix}$$

- Hamiltonian structure [3]:

$$\mathcal{H} = \begin{pmatrix} 0 & u+v \\ -(u+v) & 0 \end{pmatrix}, \qquad f = u_1v - uv$$

- Recursion operator:

$$\mathcal{R} = \begin{pmatrix} (u+v)\mathcal{S}+u_1-u & 0 \\ 0 & (u+v)\mathcal{S}^{-1}+u_1-u \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{1}{u+v} & \frac{1}{u+v} \end{pmatrix}$$
$$+\begin{pmatrix} 1 \\ -1 \end{pmatrix}\mathcal{S}(\mathcal{S}-1)^{-1}\begin{pmatrix} v_{-1}-v & u_1-u \end{pmatrix}$$

- Non-trivial symmetry:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} (u+v)(uu_1 + uv_{-1} + u_1v_1 - u_1u_2 - u_2v_1 - u_1v_{-1}) \\ (u+v)(u_{-1}v_{-2} + v_{-2}v_{-1} - u_1v - u_{-1}v_{-1} + u_1v_{-1} - v_{-1}v) \end{pmatrix}$$

There exists another weakly nonlocal recursion operator

$$\mathcal{R}' = \begin{pmatrix} \frac{u+v}{(u_{-1}+v)^2}\mathcal{S}^{-1} & -\frac{(u-u_{-1})}{(u_{-1}+v)^2} \\ -\frac{(v_1-v)}{(u+v_1)^2} & \frac{u+v}{(u+v_1)^2}\mathcal{S} + \frac{u-u_{-1}-v_1+v}{(u_{-1}+v)(u+v_1)} \end{pmatrix}$$

$$- \begin{pmatrix} 1 \\ -1 \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{1}{u+v_1} - \frac{1}{u+v} & \frac{1}{u_{-1}+v} - \frac{1}{u+v} \end{pmatrix}$$

$$- \begin{pmatrix} \frac{u-u_{-1}}{u_{-1}+v} \\ \frac{v_1-v}{u+v_1} \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{2}{u+v_1} - \frac{1}{u+v} & \frac{2}{u_{-1}+v} - \frac{1}{u+v} \end{pmatrix}.$$

The symmetry $(1,\ -1)^{\mathrm{tr}}$ is a seed for both $\mathcal{R}$ and $\mathcal{R}'$. In fact, operator $\mathcal{R}'$ is the inverse operator of $\mathcal{R}$.

5.2.12 The Ablowitz-Ladik lattice

- Equation [2]:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} (1-uv)(\alpha u_1 - \beta u_{-1}) \\ (1-uv)(\beta v_1 - \alpha v_{-1}) \end{pmatrix} =: \alpha K_1 + \beta K_{-1}$$

- Hamiltonian structure [62]:

$$\mathcal{H} = \begin{pmatrix} 0 & 1-uv \\ -(1-uv) & 0 \end{pmatrix}, \qquad f = (\alpha u_1 - \beta u_{-1})v$$

- Recursion operator [33, 35, 62]

$$\mathcal{R} = \begin{pmatrix} (1-uv)\mathcal{S} - u_1 v - u v_{-1} & -u u_1 \\ v v_{-1} & (1-uv)\mathcal{S}^{-1} \end{pmatrix} + \begin{pmatrix} -u \\ v \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} v_{-1} & u_1 \end{pmatrix}$$
$$- \begin{pmatrix} (1-uv)u_1 \\ -(1-uv)v_{-1} \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{v}{1-uv} & \frac{u}{1-uv} \end{pmatrix}$$

- Non-trivial symmetry [62]:

$$\mathcal{R} \begin{pmatrix} (1-uv)u_1 \\ -(1-uv)v_{-1} \end{pmatrix} = \begin{pmatrix} (1-uv)\left((1-u_1v_1)u_2 - v u_1^2 - u u_1 v_{-1}\right) \\ (1-uv)\left(-(1-u_{-1}v_{-1})v_{-2} + u v_{-1}^2 + u_1 v_{-1} v\right) \end{pmatrix}$$

- Master symmetry [62]:

$$\mathcal{R} \begin{pmatrix} nu \\ -nv \end{pmatrix} = \begin{pmatrix} (n+1)(1-uv)u_1 - u^2 v_{-1} - u(\mathcal{S}-1)^{-1} u v_{-1} \\ (1-n)(1-uv)v_{-1} + u v v_{-1} + v(\mathcal{S}-1)^{-1} u v_{-1} \end{pmatrix}$$

The coefficients for α and β, namely, K_1 and K_{-1}, are commuting symmetries for the equation. The inverse of this recursion operator $\mathcal{R}$ is of weakly nonlocal form:

$$\mathcal{R}^{-1} = \begin{pmatrix} (1-uv)\,\mathcal{S}^{-1} & u u_{-1} \\ -v v_1 & (1-uv)\,\mathcal{S} - u v_1 - u_{-1} v \end{pmatrix} + \begin{pmatrix} u \\ -v \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} v_1 & u_{-1} \end{pmatrix}$$
$$+ \begin{pmatrix} (1-uv)u_{-1} \\ -(1-uv)v_1 \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{v}{1-uv} & \frac{u}{1-uv} \end{pmatrix}.$$

Both $\mathcal{R}$ and $\mathcal{R}^{-1}$ share a common the seed $\sigma = \begin{pmatrix} u \\ -v \end{pmatrix}$. Starting form it, we can generate the commuting symmetries $\mathcal{R}^{-i}(\sigma)$ and $\mathcal{R}^{i}(\sigma)$ for $i \in \mathbb{N}$.

5.2.13 The Bruschi-Ragnisco lattice

- Equation [9, 66, 81]:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u_1 v - u v_{-1} \\ v(v - v_{-1}) \end{pmatrix}$$

- Hamiltonian structure [9, 62]:

$$\mathcal{H}_1 = \begin{pmatrix} 0 & (1-\mathcal{S}^{-1})v \\ v(\mathcal{S}-1) & 0 \end{pmatrix}, \qquad f_1 = u_1 v$$

$$\mathcal{H}_2 = \begin{pmatrix} v\mathcal{S}u - u\mathcal{S}^{-1}v & v(\mathcal{S}-1)v \\ v(1-\mathcal{S}^{-1})v & 0 \end{pmatrix}, \qquad f_2 = u$$

- Recursion operator [62, 66]:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1} = \begin{pmatrix} v\mathcal{S} & u_1 + u\mathcal{S}^{-1} + (u_1 v - u v_{-1})(\mathcal{S}-1)^{-1}\frac{1}{v} \\ 0 & v\mathcal{S}^{-1} + v(v - v_{-1})(\mathcal{S}-1)^{-1}\frac{1}{v} \end{pmatrix}$$

$$= \begin{pmatrix} v\mathcal{S} & u_1 + u\mathcal{S}^{-1} \\ 0 & v\mathcal{S}^{-1} \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} 0 & \frac{1}{v} \end{pmatrix}$$

- Non-trivial symmetry [62]:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} v v_1 u_2 - v_{-2} v_{-1} u \\ v(v v_{-1} - v_{-1} v_{-2}) \end{pmatrix}$$

- Master symmetry [62]:

$$\mathcal{R}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} n u_t + 2u_1 v + u_{-1} u \\ n v_t + v v_{-1} \end{pmatrix}$$

The recursion operator $\mathcal{R}$ has a weakly nonlocal inverse:

$$\mathcal{R}^{-1} = \mathcal{H}_1\mathcal{H}_2^{-1} = \begin{pmatrix} \mathcal{S}^{-1}\frac{1}{v} & -\mathcal{S}^{-1}\frac{u}{v^2} - \frac{u}{v^2} + (\frac{u_{-1}}{v_{-1}} - \frac{u}{v})(\mathcal{S}-1)^{-1}\frac{1}{v} \\ 0 & v\mathcal{S}\frac{1}{v^2} - \frac{1}{v} + \frac{1}{v_1} + (\frac{v}{v_1} - 1)(\mathcal{S}-1)^{-1}\frac{1}{v} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{S}^{-1}\frac{1}{v} & -\mathcal{S}^{-1}\frac{u}{v^2} - \frac{u}{v^2} \\ 0 & v\mathcal{S}\frac{1}{v^2} - \frac{1}{v} + \frac{1}{v_1} \end{pmatrix} + \begin{pmatrix} \frac{u_{-1}}{v_{-1}} - \frac{u}{v} \\ \frac{v}{v_1} - 1 \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} 0 & \frac{1}{v} \end{pmatrix}.$$

However, recursion operators $\mathcal{R}$ and $\mathcal{R}^{-1}$ have different seeds similar to the Relativistic Toda system in section 5.2.7. The starting point for operator $\mathcal{R}$ is the equation itself while the seed for $\mathcal{R}^{-1}$ is $\sigma = \begin{pmatrix} \frac{u_{-1}}{v_{-1}} - \frac{u}{v} \\ \frac{v}{v_1} - 1 \end{pmatrix}$. Operator $\mathcal{R}^{-1}$ acting on the equation itself and $\mathcal{R}$ on σ do not generate new symmetries. Notice that the scalar lattice $v_t = v(v - v_{-1})$ can be linearised into $w_t = w_{-1}$ by the transformation $v = -\frac{w_{-1}}{w}$.

5.2.14 The Kaup-Newell lattice

- Equation [86]:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} a\left(\frac{u_1}{1-u_1v_1} - \frac{u}{1-uv}\right) + b\left(\frac{u}{1+uv_1} - \frac{u_{-1}}{1+u_{-1}v}\right) \\ a\left(\frac{v}{1-uv} - \frac{v_{-1}}{1-u_{-1}v_{-1}}\right) + b\left(\frac{v_1}{1+uv_1} - \frac{v}{1+u_{-1}v}\right) \end{pmatrix} =: aK_1 + bK_{-1}$$

- Symplectic operator:

$$\mathcal{I} = \begin{pmatrix} 0 & \frac{1}{1-uv} \\ -\frac{1}{1-uv} & 0 \end{pmatrix} - \begin{pmatrix} \frac{v}{1-uv} \\ \frac{u}{1-uv} \end{pmatrix} (\mathcal{S}+1)(\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{v}{1-uv} & \frac{u}{1-uv} \end{pmatrix}$$

- Hamiltonian structure [86]:

$$\mathcal{H} = \begin{pmatrix} 0 & \mathcal{S}-1 \\ 1-\mathcal{S}^{-1} & 0 \end{pmatrix}, \qquad f = -a\ln(1-uv) + b\ln(1+uv_1)$$

- Recursion operator:

$$\mathcal{R}=\mathcal{H}\mathcal{I}=\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} - 2K_1(\mathcal{S}-1)^{-1}\left(\begin{array}{cc} \frac{v}{1-uv} & \frac{u}{1-uv} \end{array}\right),$$

where

$$\begin{aligned}
R_{11} &= -\frac{1}{(1-u_1v_1)^2}\mathcal{S}+\frac{1}{(1-uv)^2}-\frac{2u_1v}{(1-u_1v_1)(1-uv)}\\
R_{12} &= -\frac{u_1^2}{(1-u_1v_1)^2}\mathcal{S}+\frac{u^2}{(1-uv)^2}-\frac{2uu_1}{(1-uv)(1-u_1v_1)}\\
R_{21} &= -\frac{v_{-1}^2}{(1-u_{-1}v_{-1})^2}\mathcal{S}^{-1}-\frac{v^2}{(1-uv)^2}\\
R_{22} &= -\frac{1}{(1-u_{-1}v_{-1})^2}\mathcal{S}^{-1}+\frac{1-2uv}{(1-uv)^2}
\end{aligned}$$

- Non-trivial symmetry :

$$\mathcal{R}(K_1)=\begin{pmatrix} \frac{1}{(1-u_1v_1)^2}(u_1-\frac{u_2}{1-u_2v_2}-\frac{u_1^2v}{1-uv})-\frac{1}{(1-uv)^2}(u-\frac{u_1}{1-u_1v_1}-\frac{u^2v_{-1}}{1-u_{-1}v_{-1}}) \\ \frac{1}{(1-uv)^2}(v-\frac{u_1v^2}{1-u_1v_1}-\frac{v_{-1}}{1-u_{-1}v_{-1}})-\frac{1}{(1-u_{-1}v_{-1})^2}(v_{-1}-\frac{uv_{-1}^2}{1-uv}-\frac{v_{-2}}{1-u_{-2}v_{-2}}) \end{pmatrix}$$

The recursion operator $\mathcal{R}$ has a seed $\sigma=\begin{pmatrix} -u \\ v \end{pmatrix}$ and $\mathcal{R}(\sigma)=K_1$. Similar to the Ablowitz-Ladik Lattice in section 5.2.12, the coefficients for a and b, namely, K_1 and K_{-1}, are commuting symmetries for the equation. Indeed, there exists another weakly nonlocal recursion operator

$$\mathcal{R}'=\mathcal{H}\mathcal{I}'=\begin{pmatrix} \frac{1}{(1+u_{-1}v)^2}\mathcal{S}^{-1}-\frac{1+2uv_1}{(1+uv_1)^2} & -\frac{u^2}{(1+uv_1)^2}\mathcal{S}+\frac{u_{-1}^2}{(1+u_{-1}v)^2}-\frac{2uu_{-1}}{(1+u_{-1}v)(1+uv_1)} \\ -\frac{v^2}{(1+u_{-1}v)^2}\mathcal{S}^{-1}-\frac{v_1^2}{(1+uv_1)^2} & \frac{1}{(1+uv_1)^2}\mathcal{S}-\frac{1}{(1+u_{-1}v)^2}-\frac{2u_{-1}v_1}{(1+u_{-1}v)(1+uv_1)} \end{pmatrix}$$
$$-2K_{-1}(\mathcal{S}-1)^{-1}\left(\begin{array}{cc} \frac{v_1}{1+uv_1} & \frac{u_{-1}}{1+u_{-1}v} \end{array}\right),$$

where the symplectic operator $\mathcal{I}'$ is

$$\mathcal{I}' = \begin{pmatrix} 0 & \frac{1}{1+uv_1}(\mathcal{S}-u_{-1}v_1)\frac{1}{1+u_{-1}v} \\ \frac{1}{1+u_{-1}v}(u_{-1}v_1-\mathcal{S}^{-1})\frac{1}{1+uv_1} & 0 \end{pmatrix}$$
$$-\begin{pmatrix} \frac{v_1}{1+uv_1} \\ \frac{u_{-1}}{1+u_{-1}v} \end{pmatrix}(\mathcal{S}+1)(\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{v_1}{1+uv_1} & \frac{u_{-1}}{1+u_{-1}v} \end{pmatrix}$$

5.2.15 The Chen-Lee-Liu lattice

- Equation [86]:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} a(1+uv)(u_1-u)+b(1+u_{-1}v)^{-1}(u-u_{-1}) \\ a(1+uv)(v-v_{-1})+b(1+uv_1)^{-1}(v_1-v) \end{pmatrix} =: aK_1+bK_{-1}$$

- Hamiltonian structure:

$$\mathcal{H} = \begin{pmatrix} 0 & 1+uv \\ -(1+uv) & 0 \end{pmatrix}, \qquad f = a(uv_{-1}-uv)+b\ln\frac{1+uv}{1+uv_1}$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}^{-1} = \begin{pmatrix} (1+uv)\mathcal{S}-2uv+u_1v+uv_{-1} & uu_1-u^2 \\ v^2-vv_{-1} & (1+uv)\mathcal{S}^{-1} \end{pmatrix}$$
$$+K_1(\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{v}{1+uv} & \frac{u}{1+uv} \end{pmatrix} - \begin{pmatrix} u \\ -v \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} v-v_{-1} & u-u_1 \end{pmatrix},$$

where Hamiltonian operator $\mathcal{H}_2$ is given by

$$\mathcal{H}_2 = \begin{pmatrix} 0 & (1+uv)\left(\mathcal{S}(1+uv)-uv+u_1v\right) \\ \left(uv-u_1v-(1+uv)\mathcal{S}^{-1}\right)(1+uv) & 0 \end{pmatrix}$$
$$-K_1(\mathcal{S}-1)^{-1}\begin{pmatrix} u & -v \end{pmatrix} - \begin{pmatrix} u \\ -v \end{pmatrix}\mathcal{S}(\mathcal{S}-1)^{-1}K_1^{\mathrm{tr}}.$$

- Non-trivial symmetry :

$$\mathcal{R}(K_1) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$$

where

$$G_1 = (1+uv)(u_1u_2v_1 + u_2 + u_1^2 v + uu_1v_{-1} + u^2v - u_1 - u^2v_{-1} - 2uu_1v - u_1^2v_1)\,,$$
$$G_2 = (1+uv)(u_1v^2 + 2uv_{-1}v + u_{-1}v_{-1}^2 + v_{-1} - v_{-2} - uv^2 - uv_{-1}^2$$
$$- u_1v_{-1}v - u_{-1}v_{-2}v_{-1})\,.$$

The coefficients for a and b, namely, K_1 and K_{-1}, are commuting symmetries for the equation. The above recursion operator $\mathcal{R}$ has a seed $\sigma = \begin{pmatrix} -u \\ v \end{pmatrix}$ and $\mathcal{R}(\sigma) = K_1$.

There exists another weakly nonlocal recursion operator

$$\mathcal{R}' = \mathcal{H}_2'\mathcal{H}^{-1} = \begin{pmatrix} \frac{1+uv}{(1+u_{-1}v)^2}\mathcal{S}^{-1} & -\frac{u_{-1}(u-u_{-1})}{(1+u_{-1}v)^2} \\ -\frac{v_1(v_1-v)}{(1+uv_1)^2} & \frac{1+uv}{(1+uv_1)^2}\mathcal{S} + \frac{v_1u-2u_{-1}v_1+u_{-1}v}{(1+u_{-1}v)(1+uv_1)} \end{pmatrix}$$
$$-K_{-1}(\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{2v_1}{1+uv_1} - \frac{v}{1+uv} & \frac{2u_{-1}}{1+u_{-1}v} - \frac{u}{1+uv} \end{pmatrix}$$
$$-\begin{pmatrix} -u \\ v \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{v_1}{1+uv_1} - \frac{v}{1+uv} & \frac{u_{-1}}{1+u_{-1}v} - \frac{u}{1+uv} \end{pmatrix},$$

where the Hamiltonian operator $\mathcal{H}_2'$ is

$$\mathcal{H}_2' = \begin{pmatrix} 0 & \frac{1+uv}{1+u_{-1}v}\left(\mathcal{S}^{-1}\frac{1+uv}{1+uv_1} + \frac{v_1(u-u_{-1})}{1+uv_1}\right) \\ -\left(\frac{1+uv}{1+uv_1}\mathcal{S} + \frac{v_1(u-u_{-1})}{1+uv_1}\right)\frac{1+uv}{1+u_{-1}v} & 0 \end{pmatrix}$$
$$-K_{-1}(\mathcal{S}+1)(\mathcal{S}-1)^{-1}K_{-1}^{\mathrm{tr}} - K_{-1}\mathcal{S}(\mathcal{S}-1)^{-1}\begin{pmatrix} -u & v \end{pmatrix} - \begin{pmatrix} -u \\ v \end{pmatrix}(\mathcal{S}-1)^{-1}K_{-1}^{\mathrm{tr}}\,.$$

Again σ is a seed for $\mathcal{R}'$. In fact, operator $\mathcal{R}'$ is the inverse operator of $\mathcal{R}$.

5.2.16 The Ablowitz-Ramani-Segur (Gerdjikov-Ivanov) lattice

- Equation [86]:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} (au_1 - bu_{-1})(1+uv)(1-uv_1) \\ (bv_1 - av_{-1})(1+uv)(1-u_{-1}v) \end{pmatrix} =: aK_1 + bK_{-1}$$

The equation given in [86] is

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} (b-a)u \\ (a-b)v \end{pmatrix} + aK_1 + bK_{-1}.$$

Since the vector $\sigma = \begin{pmatrix} -u \\ v \end{pmatrix}$ commutes with both K_1 and K_{-1}, we removed this term in our consideration.

- Symplectic operator:

$$\mathcal{I} = \begin{pmatrix} 0 & \frac{1}{1-uv_1}\mathcal{S} - \frac{1}{1+uv} \\ \frac{1}{1+uv} - \mathcal{S}^{-1}\frac{1}{1-uv_1} & 0 \end{pmatrix}$$

and we have

$$\mathcal{I}(aK_1 + bK_{-1}) = \delta_{(u,v)}\left(a(uv_{-1} - uv - uvu_1v_1) + b(u_{-1}v_1 - uv_1 + u_{-1}uvv_1)\right).$$

- Hamiltonian structure:

$$\mathcal{H} = \begin{pmatrix} 0 & h_{12} \\ h_{21} & 0 \end{pmatrix} - K_1\mathcal{S}(\mathcal{S}-1)^{-1}\begin{pmatrix} u & -v \end{pmatrix} - \begin{pmatrix} u \\ -v \end{pmatrix}(\mathcal{S}-1)^{-1}K_1^{\mathrm{tr}},$$

where

$$h_{12} = (1+uv)(\mathcal{S}(1+uv)+uv_{-1})(1-u_{-1}v)$$
$$h_{21} = -(1-u_{-1}v)((1+uv)\mathcal{S}^{-1}+uv_{-1})(1+uv)$$

- Recursion operator:

$$\mathcal{R} = \begin{pmatrix} (1+uv)(1-uv_1)\mathcal{S}+u_1v-u_1v_1 & -uu_1(1+uv)\mathcal{S}-u^2(1+u_{-1}v_{-1}) \\ +uv_{-1}-uv(1+u_{-1}v_{-1}+2u_1v_1) & +\frac{1-uv_1}{1-u_{-1}v}u_1(u-u_{-1}-2uu_{-1}v) \\ & \\ -(1-u_{-1}v)v_{-1}v-(1+uv)v_{-1}v\mathcal{S}^{-1} & (1+uv)(1-u_{-1}v)\mathcal{S}^{-1}+uvu_{-1}v_{-1} \end{pmatrix}$$
$$+\begin{pmatrix} u_1(1+uv)(1-uv_1) \\ -v_{-1}(1+uv)(1-u_{-1}v) \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{v}{1+uv}-\frac{v_1}{1-uv_1} & \frac{u}{1+uv}-\frac{u_{-1}}{1-u_{-1}v} \end{pmatrix}$$
$$-\begin{pmatrix} u \\ -v \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} v-v_{-1}+u_{-1}v_{-1}v+u_1vv_1 & u-u_1+uu_{-1}v_{-1}+uu_1v_1 \end{pmatrix}$$
$$= \mathcal{HI}+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Non-trivial symmetry $\mathcal{R}(K_1)$:

$$\mathcal{R}(K_1) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

where

$$G_1 = (1+uv)(1-uv_1)[(1+u_1v_1)(1-u_1v_2)u_2-u_1^2v_1(1+uv)+uu_1v_{-1}(1-u_{-1}v)$$
$$-u_1v(u-u_1)],$$
$$G_2 = (1+uv)(u_{-1}v-1)[(1+u_{-1}v_{-1})(1-u_{-2}v_{-1})v_{-2}-u_{-1}v_{-1}^2(1+uv)$$
$$+u_1v_{-1}v(1-uv_1)+uv_{-1}(v_{-1}-v)].$$

There exists another weakly nonlocal recursion operator

$$\mathcal{R}' = \begin{pmatrix} (1+uv)(1-uv_1)\mathcal{S}^{-1} + uvu_{-1}v_1 & \begin{matrix} uu_{-1}(1+uv)\mathcal{S} + uu_{-2}(1+u_{-1}v_{-1}) \\ -\frac{1-uv_1}{1-u_{-1}v}u_{-1}(u - u_{-1} - 2uu_{-1}v) \end{matrix} \\ & \\ (1-u_{-1}v)v_1v + (1+uv)v_1v\mathcal{S}^{-1} & \begin{matrix} (1+uv)(1-u_{-1}v)\mathcal{S}+uv_1-2u_{-1}uvv_1 \\ -u_{-1}v_1+u_{-1}v-u_{-2}v(1+u_{-1}v_{-1}) \end{matrix} \end{pmatrix}$$

$$+ \begin{pmatrix} -u_{-1}(1+uv)(1-uv_1) \\ v_1(1+uv)(1-u_{-1}v) \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{v}{1+uv} - \frac{v_1}{1-uv_1} & \frac{u}{1+uv} - \frac{u_{-1}}{1-u_{-1}v} \end{pmatrix}$$

$$+ \begin{pmatrix} u \\ -v \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} v_2(1+u_1v_1)+v_1(u_{-1}v-1) & u_{-2}(1+u_{-1}v_{-1})+u_{-1}(uv_1-1) \end{pmatrix}$$

$$= \mathcal{H}'\mathcal{I} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the Hamiltonian operator $\mathcal{H}'$ is

$$\mathcal{H}' = \begin{pmatrix} 0 & -(1+uv)(1-u_{-1}v) \\ (1+uv)(1-u_{-1}v) & 0 \end{pmatrix}$$

$$-K_{-1}\mathcal{S}(\mathcal{S}-1)^{-1}\begin{pmatrix} u & -v \end{pmatrix} - \begin{pmatrix} u \\ -v \end{pmatrix} (\mathcal{S}-1)^{-1}K_{-1}^{\mathrm{tr}}.$$

Operator $\mathcal{R}'$ is the inverse operator of $\mathcal{R}$. The vector σ is them seed for both of them and $\mathcal{R}'(K_{-1})$ is

$$\mathcal{R}'(K_{-1}) = \begin{pmatrix} G_1' \\ G_2' \end{pmatrix},$$

where

$$G'_1 = (1-uv_1)(1+uv)[(1+u_{-1}v_{-1})(1-u_{-1}v)u_{-2} - uu_{-1}v_2(1+u_1v_1) - u_{-1}^2v_1(1+uv)$$
$$+u_{-1}(u_{-1}v + uv_1)]\,,$$
$$G'_2 = (1-u_{-1}v)(1+uv)[-(1+u_1v_1)(1-uv_1)v_2 + u_{-2}vv_1(1+u_{-1}v_{-1})$$
$$+u_{-1}v_1(v_1vu + v_1 - v) - uv_1^2]\,.$$

5.2.17 The Heisenberg ferromagnet lattice

- Equation [79]:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} (u-v)(u-u_1)(u_1-v)^{-1} \\ (u-v)(v_{-1}-v)(u-v_{-1})^{-1} \end{pmatrix}$$

- Hamiltonian structure:

$$\mathcal{H} = \begin{pmatrix} 0 & (u-v)^2 \\ -(u-v)^2 & 0 \end{pmatrix}, \qquad f = \ln(u-v) - \ln(u-v_{-1})$$

- Symplectic operator:

$$\mathcal{I} = \begin{pmatrix} 0 & -(u-v_{-1})^{-2}\mathcal{S}^{-1} + \frac{(u-u_1)(v-v_{-1})}{(u-v)^2(u-v_{-1})(u_1-v)} \\ (u_1-v)^{-2}\mathcal{S} - \frac{(u-u_1)(v-v_{-1})}{(u-v)^2(u-v_{-1})(u_1-v)} & 0 \end{pmatrix}$$
$$-\begin{pmatrix} \frac{v-v_{-1}}{(u-v)(u-v_{-1})} \\ \frac{u-u_1}{(u-v)(u_1-v)} \end{pmatrix} (\mathcal{S}+1)(\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{v-v_{-1}}{(u-v)(u-v_{-1})} & \frac{u-u_1}{(u-v)(u_1-v)} \end{pmatrix}$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}\mathcal{I} = \begin{pmatrix} \frac{(u-v)^2}{(u_1-v)^2}\mathcal{S} - \frac{2(u-u_1)(v-v_{-1})}{(u-v_{-1})(u_1-v)} & -\frac{(u-u_1)^2}{(u_1-v)^2} \\ \frac{(v-v_{-1})^2}{(u-v_{-1})^2} & \frac{(u-v)^2}{(u-v_{-1})^2}\mathcal{S}^{-1} \end{pmatrix}$$

$$-2\begin{pmatrix} u_t \\ v_t \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{v-v_{-1}}{(u-v)(u-v_{-1})} & \frac{u-u_1}{(u-v)(u_1-v)} \end{pmatrix}$$

- Non-trivial symmetry:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \frac{(u-v)}{(u_1-v)^2}\left(\frac{(u-v)(u_1-v_1)(u_1-u_2)}{(u_2-v_1)} + \frac{(u-u_1)^2(v_{-1}-v)}{(u-v_{-1})}\right) \\ \frac{(u-v)}{(u-v_{-1})^2}\left(\frac{(u-v)(u_{-1}-v_{-1})(v_{-2}-v_{-1})}{(u_{-1}-v_{-2})} + \frac{(v-v_{-1})^2(u-u_1)}{(u_1-v)}\right) \end{pmatrix}$$

The recursion operator $\mathcal{R}$ has a weakly nonlocal inverse:

$$\mathcal{R}^{-1} = \mathcal{H}\mathcal{I}' = \begin{pmatrix} \frac{(u-v)^2}{(u_{-1}-v)^2}\mathcal{S}^{-1} & \frac{(u-u_{-1})^2}{(u_{-1}-v)^2} \\ -\frac{(v-v_1)^2}{(u-v_1)^2} & \frac{(u-v)^2}{(u-v_1)^2}\mathcal{S} - \frac{2(u-u_{-1})(v-v_1)}{(u-v_1)(u_{-1}-v)} \end{pmatrix}$$

$$-2\begin{pmatrix} \frac{(u-v)(u_{-1}-u)}{u_{-1}-v} \\ \frac{(u-v)(v-v_1)}{u-v_1} \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{v-v_1}{(u-v)(u-v_1)} & \frac{u-u_{-1}}{(u-v)(u_{-1}-v)} \end{pmatrix},$$

where the symplectic operator $\mathcal{I}'$ is given by

$$\mathcal{I}' = \begin{pmatrix} 0 & -(u-v_1)^{-2}\mathcal{S} + \frac{(u-u_{-1})(v-v_1)}{(u-v)^2(u-v_1)(u_{-1}-v)} \\ (u_{-1}-v)^{-2}\mathcal{S}^{-1} - \frac{(u-u_{-1})(v-v_1)}{(u-v)^2(u-v_1)(u_{-1}-v)} & 0 \end{pmatrix}$$

$$+\begin{pmatrix} \frac{v-v_1}{(u-v)(u-v_1)} \\ \frac{u-u_{-1}}{(u-v)(u_{-1}-v)} \end{pmatrix}(\mathcal{S}+1)(\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{v-v_1}{(u-v)(u-v_1)} & \frac{u-u_{-1}}{(u-v)(u_{-1}-v)} \end{pmatrix}.$$

5.2.18 The Belov-Chaltikian lattice

- Equation [4]:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u(v_2 - v_{-1}) \\ u_{-1} - u + v(v_1 - v_{-1}) \end{pmatrix}$$

- Hamiltonian structure [4]:

$$\mathcal{H}_1 = \begin{pmatrix} u(\mathcal{S}-\mathcal{S}^{-1})(\mathcal{S}+1+\mathcal{S}^{-1})u & u(\mathcal{S}-1)(\mathcal{S}+1+\mathcal{S}^{-1})v \\ v(1-\mathcal{S}^{-1})(\mathcal{S}+1+\mathcal{S}^{-1})u & v(\mathcal{S}-\mathcal{S}^{-1})v+\mathcal{S}^{-1}u-u\mathcal{S} \end{pmatrix}, \quad f_1 = v$$

$$\mathcal{H}_2 = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \qquad f_2 = -\frac{1}{3}\ln u$$

$$h_{11} = u(1+\mathcal{S}+\mathcal{S}^2)(\mathcal{S}^{-1}v-v\mathcal{S})(1+\mathcal{S}^{-1}+\mathcal{S}^{-2})u$$

$$h_{12} = u(1+\mathcal{S}+\mathcal{S}^2)\left(u\mathcal{S}-\mathcal{S}^{-2}u\right)+u(1+\mathcal{S}+\mathcal{S}^2)(\mathcal{S}^{-1}v-v\mathcal{S})(1+\mathcal{S}^{-1})v$$

$$h_{21} = \left(u\mathcal{S}^2-\mathcal{S}^{-1}u\right)(1+\mathcal{S}^{-1}+\mathcal{S}^{-2})u+v(1+\mathcal{S})(\mathcal{S}^{-1}v-v\mathcal{S})(1+\mathcal{S}^{-1}+\mathcal{S}^{-2})u$$

$$h_{22} = v(1+\mathcal{S})(u\mathcal{S}-\mathcal{S}^{-2}u)+(u\mathcal{S}^2-\mathcal{S}^{-1}u)(1+\mathcal{S}^{-1})v$$

$$+v(1+\mathcal{S})(\mathcal{S}^{-1}v-v\mathcal{S})(1+\mathcal{S}^{-1})v$$

- Non-trivial symmetry :

$$\mathcal{H}_2\delta f_1 = \begin{pmatrix} uv_{-1}\left(v+v_{-1}+v_{-2}\right)-uv_2(v_1+v_2+v_3)+u\left(u_1+u_2-u_{-1}-u_{-2}\right) \\ \left(u-vv_1\right)\left(v+v_1+v_2\right)+\left(vv_{-1}-u_{-1}\right)\left(v+v_{-1}+v_{-2}\right)-v\left(u_{-2}-u_1\right) \end{pmatrix}$$

- Master symmetry [70]:

$$\begin{pmatrix} nu_t + uv_1 + 4uv_{-1} + uv \\ nv_t + u - vv_1 - 4u_{-1} + 4vv_{-1} + v^2 \end{pmatrix}$$

5.2.19 The Blaszak-Marciniak lattice

- Equation [6]:

$$\begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix} = \begin{pmatrix} w_1 - w_{-1} \\ u_{-1}w_{-1} - uw \\ w(v - v_1) \end{pmatrix}$$

- Hamiltonian structure [6]:

$$\mathcal{H}_1 = \begin{pmatrix} \mathcal{S}-\mathcal{S}^{-1} & 0 & 0 \\ 0 & 0 & (\mathcal{S}^{-1}-1)w \\ 0 & -w(\mathcal{S}-1) & 0 \end{pmatrix}, \qquad f_1 = uw + \frac{1}{2}v^2$$

$$\mathcal{H}_2 = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}, \qquad f_2 = v$$

where

$$\begin{aligned}
h_{11} &= \mathcal{S}v - v\mathcal{S}^{-1} - u(\mathcal{S}+1)^{-1}(1-\mathcal{S})u \\
h_{12} &= \mathcal{S}w\mathcal{S} - \mathcal{S}^{-1}w \\
h_{13} &= u(\mathcal{S}+1)^{-1}(1-\mathcal{S})w \\
h_{21} &= w\mathcal{S} - \mathcal{S}^{-1}w\mathcal{S}^{-1} \\
h_{22} &= \mathcal{S}^{-1}uw - uw\mathcal{S} \\
h_{23} &= v(\mathcal{S}^{-1}-1)w \\
h_{31} &= w(\mathcal{S}+1)^{-1}(1-\mathcal{S})u \\
h_{32} &= -w(\mathcal{S}-1)v \\
h_{33} &= w(\mathcal{S}^{-1}-\mathcal{S})w - w(\mathcal{S}+1)^{-1}(1-\mathcal{S})w
\end{aligned}$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1},$$

where

$$\mathcal{H}_1^{-1} = \begin{pmatrix} \frac{1}{2}(\mathcal{S}-1)^{-1} + \frac{1}{2}(\mathcal{S}+1)^{-1} & 0 & 0 \\ 0 & 0 & -(\mathcal{S}-1)^{-1}\frac{1}{w} \\ 0 & -\frac{1}{w}\mathcal{S}(\mathcal{S}-1)^{-1} & 0 \end{pmatrix}$$

- Non-trivial symmetry :

$$\mathcal{H}_2\delta f_1 = \mathcal{H}_2 \begin{pmatrix} w \\ v \\ u \end{pmatrix} = \begin{pmatrix} w_1(v_1+v_2) - w_{-1}(v+v_{-1}) \\ u_{-1}w_{-1}(v+v_{-1}) - uw(v+v_1) - w_{-1}w_{-2} + ww_1 \\ w(v^2-v_1^2) + w(w_{-1}u_{-1} - w_1u_1) \end{pmatrix}$$

- Master symmetry:

$$\mathcal{R} \begin{pmatrix} u/2 \\ v \\ 3w/2 \end{pmatrix}$$

We do not explicitly write out its recursion operator, which is no longer weakly nonlocal although both operators $\mathcal{H}_2$ and $\mathcal{H}_1^{-1}$ are weakly nonlocal. The statement that such recursion operator generates local symmetries can be proved in the same way as in [91], for weakly nonlocal differential recursion operators. Its master symmetry, which is highly nonlocal, is explicitly given in [70].

5.3 Conclusion

In this chapter we presented a list of integrable differential-difference equations containing equations themselves, Hamiltonian structures, recursion operators and a nontrivial generalised symmetry. First of all we should note that the list is not complete. We tried to collect as many integrable equations as we could, along with their properties, from known references. Also for most equations, we provided the corresponding weakly nonlocal inverse recursion operator. To say whether there exists an inverse recursion operator for a given equation is still an open problem.

Chapter 6

Master symmetries and representation theory of the differential-difference KP equation

6.1 Introduction

As we mentioned earlier in the introduction of the thesis, both integrable linearizable evolution PDEs and the evolution partial differential equations solvable by inverse scattering method (S-integrable) enjoy having the infinite series of higher generalised symmetries. This property can be used as an integrability criteria for both S-integrable and C-integrable equations. In recent years there have been works exploring the similar feature for the differential-difference and lattice equations [47, 95, 94].

It is quite desirable to find methods in order to generate the explicit form of generalised symmetries. As we saw in previous chapters, recursion operators which map a symmetry to a new symmetry can be used to produce an infinite hierarchy of generalised symmetries [41, 67]. Finding the recursion operator for a given equation is not a trivial task. Master symmetries are an effective tool by which one can generate the hierarchy of generalised symmetries. The concept of master symmetry was first introduced in [21].

Master symmetries for (1+1)-dimensional differential equations and differential - difference equations are not always local. For example for the KdV

$$u_t = u_{xxx} + 6uu_x \,,$$

the master symmetry has the form [15, 19, 63]

$$\tau = x(6uu_x + u_{xxx}) + 4u_{xx} + 8u^2 + 2u_x D_x^{-1} u \,, \tag{6.1.1}$$

where D_x^{-1} is the right inverse of total x-derivative. The efficiency of master symmetries can be clearly seen in the (2+1)-dimensional integrable systems for which the recursion operators have more complicated structure than in (1+1)-dimension [22, 75]. For the Kadomtsev-Petviashvili equation, which can be viewed as the two-dimensional KdV, that is

$$u_t = 6uu_x - u_{xxx} - 3\alpha^2 D^{-1} u_{yy} = K(u) \,,$$

the master symmetry is given in [65] as follows:

$$\tau(u) = yK(u) - 2\alpha^2 x u_y - 4\alpha^2 D_x^{-1} u_y \,. \tag{6.1.2}$$

The nonlinear partial integro-differential Benjamin-Ono equation

$$u_t = Hu_{xx} + 2uu_x = K(u) \,, \tag{6.1.3}$$

where H stands for the Hilbert transform

$$(Hf)(x) = \frac{1}{\pi} \int \frac{f(\xi)}{\xi - x} dx \,,$$

has the master symmetry given in [21] as

$$\tau(u) = xK(u) + u^2 + \frac{3}{2}Hu_x \,. \tag{6.1.4}$$

Master symmetries are also used to obtain the t-dependent symmetries. In [24, 74] this relation is discussed and some examples are given. It is shown that if one has a master symmetry of evolution equation, then one can produce an infinite number of t-dependent symmetries which are polynomials in time.

Unlike partial differential equations many differential-difference equations possess local master symmetries. For more details regarding local master symmetries of discrete equations see [95] and the references therein. As we saw in the previous chapter the well-known Volterra chain (5.2.1) has the following master symmetry :

$$u_t = nu(u_1 - u_{-1}) + u(2u_1 + u + u_{-1}) \,.$$

Comparing the works done on (1+1)-dimensional case with (2+1)-dimensional differential-difference equations, less research has been devoted to the latter. In Section 6.4 we will construct the master symmetry of the differential-difference KP (DDKP) equation

$$u_t = u_{xx} + 2uu_x + 2\Delta^{-1}D_x(u_x) \,.$$

This chapter is structured as follows. In section 6.2 we shall give the definition of quasi-local terms and define the ring extension with respect to these terms. Section 6.3 first introduces the structure of (2+1)-dimensional differential-difference evolution equations and later discusses the notion of K-generators and K-master symmetries and theorems that we employ through the arguments. In the two last sections 6.4 and 6.5 respectively we obtain the master symmetry of the DDKP equation and employ the method in [74] to make the irreducible representations of the equation. In section 6.5 we also construct the generators of t-dependent symmetries of this equation.

6.2 Quasi-local difference polynomials

The concept of quasi-locality was first introduced in [57]. In [57] the main feature of integrable equations in $(2+1)$-dimension is pointed out. In fact evolutionary integrable (2+1)-dimensional equations along with their higher symmetries and conservation laws are not local anymore which means they are not only polynomials in u_m,$u_{m,x}$,$u_{m,xx}$, etc. They provide a new concept of quasi-local functions. In [90] one can find the proof for the symmetry structure of integrable (2+1)-dimensional equations. It is proved through the method called the symbolic representation. In this section we discuss an analogous concept of quasi-local terms for integrable (2+1)-dimensional differential-difference equations.

Let $u(n,t,x)$ be a function of discrete variable $n \in \mathbb{Z}$ and be analytic in two variables (t,x) where t and x are varying over the complex numbers. Analogue to the first chapter by writing $u_{m,j}$ we mean m steps shift in the discrete variable and the second index stands for the derivative of u with respect to x. For example $u_{1,2}$ shows one step shift in the discrete variable and second derivative with respect to x. When j is not very big (i.e. less that 4) we may also show it by the x index. So in this case we can write as u_{1xx}. A monomial of degree ℓ has the following form:

$$u^{\alpha_1}_{m_1,i_1} u^{\alpha_2}_{m_2,i_2} \cdots u^{\alpha_k}_{m_k,i_k}, \qquad \sum_{j=1}^{k} \alpha_j = \ell\,.$$

Naturally the difference operator (2.1.6) acts on the discrete variable as

$$\Delta(u_{mj}) = u_{m+1,j} - u_{mj}\,.$$

To proceed we need to define the so-called quasi-local terms and consequently the concept of quasi-local polynomials. Consider $\hat{\mathfrak{R}}^\ell$ be the ring of polynomials generated by all monomials of degree ℓ as

$$\hat{\mathfrak{R}}^\ell =< u^{\alpha_1}_{m_1,i_1} u^{\alpha_2}_{m_2,i_2} \cdots u^{\alpha_k}_{m_k,i_k} | \sum_{j=1}^{k} \alpha_j = \ell > \,.$$

Let $\mathfrak{R}^\ell$ also contains the combination of monomials of degree ℓ where the coefficients are polynomials in n and x. Then suppose we define $\hat{\mathfrak{R}}$ and $\mathfrak{R}$ such that their elements are linear combination of elements in $\hat{\mathfrak{R}}^\ell$ and $\mathfrak{R}^\ell$ for different values of ℓ. Therefore $\hat{\mathfrak{R}}$ and $\mathfrak{R}$ can be expressed as the following direct sums:

$$\hat{\mathfrak{R}} = \bigoplus_{\ell \geq 1} \hat{\mathfrak{R}}^\ell ,$$
$$\mathfrak{R} = \bigoplus_{\ell \geq 1} \mathfrak{R}^\ell .$$

Now let us consider two new operators Θ and Θ^{-1} defined as

$$\Theta = (\mathcal{S} - 1)^{-1} D_x = \Delta^{-1} D_x , \tag{6.2.1}$$

$$\Theta^{-1} = (\mathcal{S} - 1) D_x^{-1} = \Delta D_x^{-1} , \tag{6.2.2}$$

where D_x denotes derivative with respect to x. To introduce the concept of quasi-local polynomials we need to extend the ring $\hat{\mathfrak{R}}$ with respect to the new operators $\Theta^{\pm 1}$. Let

$$\hat{\mathfrak{R}}_{k+1}(\Theta) = \hat{\mathfrak{R}}_k(\Theta) \cup \Theta \hat{\mathfrak{R}}_k(\Theta) \cup \Theta^{-1} \hat{\mathfrak{R}}_k(\Theta), \qquad k \geq 0 ,$$

where $\hat{\mathfrak{R}}_0(\Theta) = \hat{\mathfrak{R}}$ and we have

$$\Theta \hat{\mathfrak{R}}_k(\Theta) = \{\Theta(P) \mid P \in \hat{\mathfrak{R}}_k(\Theta)\} ,$$
$$\Theta^{-1} \hat{\mathfrak{R}}_k(\Theta) = \{\Theta^{-1}(P) \mid P \in \hat{\mathfrak{R}}_k(\Theta)\} .$$

According to the structure of $\hat{\mathfrak{R}}_k(\Theta)$, index k shows the maximal number of the nesting for $\Theta^{\pm 1}$ in an expression. For example the following expressions lie respectively in $\hat{\mathfrak{R}}_2(\Theta)$ and $\hat{\mathfrak{R}}_3(\Theta)$:

$$u_{1xx} + \Theta^2(u_{-1}) + \Theta(u_x \Theta^{-1}(u^2 u_{1x})) ,$$
$$\Theta^2(u_{-1x}\Theta(u_{3xx})) + \Theta(u_{1x}^2 u_{-2} \Theta^{-1}(u\Theta(u^3 u_{1xx}))) .$$

Now we define two ring extensions when k reaches infinity. First consider $\hat{\mathfrak{G}}$ defined as the following limit:

$$\hat{\mathfrak{G}} = \lim_{k\to\infty} \hat{\mathfrak{R}}_k(\Theta)\,.$$

In a similar way we can have the ring extension

$$\mathfrak{G} = \lim_{k\to\infty} \mathfrak{R}_k(\Theta)\,.$$

In fact $\mathfrak{G}$ is a ring containing all elements of $\hat{\mathfrak{G}}$ along with the ones that also have dependency on x and n. For example we have

$$u_{1x}\Theta(u_{xx}) + \Theta(u\Theta(u_x)) \in \hat{\mathfrak{G}}\,,$$
$$nu_{xx} + x^2\Theta(u_1\Theta(u_{2xxx})) \in \mathfrak{G}\,.$$

According to the concept of Fréchet derivative defined in Section 2.4, for two elements F and G in $\mathfrak{G}$ we define the Lie bracket of two elements as

$$[F, G] = F_*(G) - G_*(F)\,.$$

The anti-symmetric axiom is clear and bilinearity also is deduced from the linearity of derivations and the shift operator. For the Jacobi identity it can be obtained through direct calculation and taking into account the following symmetry property:

$$(H_*)_*[G](F) = (H_*)_*[F](G), \qquad F, G, H \in \mathfrak{G}\,.$$

If we use the chain rule the explicit formula for the Fréchet derivative of elements in $\mathfrak{G}$ is given as

$$F_* = \sum_{i=-\infty}^{N} \sum_{j\geq 0} \frac{\partial F}{\partial u_{i,j}} \mathcal{S}^i D_x^j\,. \tag{6.2.3}$$

Example 22. *Let $F \in \mathfrak{G}$ be the following expression:*

$$nu_{1xx} + \Theta(xuu_x)\,,$$

then for the element $G = \Theta(u)$ we get

$$\begin{aligned} F_*(G) &= nSD_x^2(G) + \Theta(xu_xG) + \Theta(xuD_x(G)) \\ &= n\Theta(u_{1xx}) + \Theta(xu_x\Theta(u)) + \Theta(xu\Theta(u_x))\,. \end{aligned}$$

6.3 (2+1)-dimensional differential-difference equations and master symmetries

A (2+1)-dimensional differential-difference equation is defined as an evolutionary equation of the following form:

$$u_t = K, \qquad K \in \hat{\mathfrak{G}}\,. \tag{6.3.1}$$

Equation (6.3.1) is also known as a lattice field equation. Through this chapter we shall discuss the master symmetry of the differential-difference KP (DDKP) equation [11, 14, 61, 84]

$$u_t = u_{xx} + 2uu_x + 2\Theta(u_x) = K\,. \tag{6.3.2}$$

This equation is also derived in [7, 40]. In [7] it is constructed through the procedure of central extension and the author refers to the equation as the lattice field Benjamin-Ono equation. Naturally we call an element $G \in \hat{\mathfrak{G}}$ a generalised symmetry for equation (6.3.1) if and only if the following bracket vanishes:

$$[K, G] = 0\,. \tag{6.3.3}$$

One can simply check that u_x is a symmetry of the DDKP equation (6.3.2).

For the concept of master symmetry we basically follow the notions given in [21, 24, 25, 27, 64, 65]. Consider equation (6.3.1), a K-generator of degree m is an element $T \in \mathfrak{G}$ which satisfies the following relation:

$$ad_K^{m+1} T := \underbrace{[K \cdots, [K, [K}_{(m+1)-times}, T]]] = 0\,.$$

Obviously if T is a K-generator of degree zero and an element of $\hat{\mathfrak{G}}$, then it is a generalised symmetry of equation (6.3.1), i.e.

$$m = 0, \qquad ad_K^1 T = [K, T] = 0\,.$$

Example 23. *For the DDKP equation (6.3.2), the constant* 1 *is a K-generator of degree one since*

$$[u_{xx} + 2uu_x + 2\Theta(u_x), 1] = 2u_x$$

and

$$[u_{xx} + 2uu_x + 2\Theta(u_x), 2u_x] = 0\,.$$

Notice that when we say T is a K-generator of degree one the only information we get is that $[T, K]$ commutes with K. If it also is an element of $\hat{\mathfrak{G}}$ then it produces a symmetry for the given equation. In fact we are interested in such generators where $[T, K] \in \hat{\mathfrak{G}}$. This property is the main part of the notion of master symmetry.

Definition 24. *An element $W \in \mathfrak{G}$ is called master symmetry if it is a K-generator of degree one and $[W, K] \in \hat{\mathfrak{G}}$. i.e*

$$[K, [K, W]] = 0, \qquad [W, K] \in \hat{\mathfrak{G}}\,. \tag{6.3.4}$$

In practice we are interested to find a W such that its recursive action on K is well-defined and produces a new element in $\hat{\mathfrak{G}}$.

To do this let us see what we really need to show. Let $[W, K] = G_1 \in \hat{\mathfrak{G}}$ and $[K, G_1] = 0$. If we continue this procedure, in each step we get $[W, G_m] = G_{m+1}$. One can use the Jacobi identity to show that G_2 is a symmetry for K which means $[K, G_2] = 0$. Now for G_2 we have

$$[K, G_3] = [K, [W, G_2]] = [W, [K, G_2]] + [G_2, [W, K]] = [G_1, G_2] .$$

This vanishes if G_1 and G_2 are commuting elements. In general to show that an element W is a master symmetry we refer to the following theorem by I. Dorfman [16].

Theorem 10. *Let L be a Lie algebra and fix an element $a_0 \in L$. Suppose $\tau \in L$ is an element by which one can obtain an infinite series of elements $a_n \in L$ according to the following recurrence relation:*

$$[\tau, a_m] = a_{m+1}, \qquad m \geq 0 .$$

Furthermore suppose for some $N \geq 1$ and $M \geq 2$ there exist elements $\tau_{-N}, \tau_{-N+1}, \cdots, \tau_0, \tau_1 \in L$ such that the following conditions hold:

1. $\tau_{s+1} = [\tau, \tau_s], \qquad s = -N, \cdots, 0,$

   $\tau_1 = [\tau, \tau_0] = \lambda\tau, \qquad \lambda \in \mathbb{R};$

2. $[\tau_{-N}, a_0] = [\tau_{-N+1}, a_0] = \cdots = [\tau_{-1}, a_0] = 0,$

   $[\tau_0, a_0] = \mu a_0, \qquad \mu \in \mathbb{R}; \qquad \mu\lambda < 0;$

3. $[a_{M-1}, a_M] = [a_{M-2}, a_{M-1}] = \cdots = [a_0, a_1] = 0;$

4. *From* $[p, a_{M-1}] = [p, \tau_{-N}] = 0$ *we get* $p = \sum_{k=0}^{N-1} r_k a_k, \qquad r_k \in \mathbb{R}.$

Then we deduce that all a_n mutually commute

$$[a_i, a_j] = 0, \qquad i, j = 0, 1, \cdots$$

Remark 7. *If a_i and p belong to some Lie subalgebra of L while τ is an element of L, the theorem remains valid. More details can be found in [16].*

In the next section we will use the above theorem along with this remark to obtain and prove the master symmetry for DDKP equation.

6.4 Master symmetry for the DDKP equation

Consider the differential-difference KP equation (6.3.2). The first non-trivial generalised symmetry of (6.3.2) is given in [7]. We rewrite it in our notation as follows:

$$\begin{aligned} G &= u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x + 3\Theta(uu_x) + 3u_x\Theta(u) \\ &+ 3u\Theta(u_x) + 3\Theta^2(u_x) + 3\Theta(u_{xx})\,. \end{aligned} \tag{6.4.1}$$

From (6.2.3) one can check that the bracket $[K, G]$ vanishes. Now according to Definition 24 and the generalised symmetry (6.4.1) we shall think of W such that $[W, K] = \gamma G$ where γ is a scalar. This will be discussed in more details in the next proposition but to proceed we first need to define the homogeneous elements of $\mathfrak{G}$.Let λ be the weight of u denoted by $\omega(u) = \lambda$. Then the weight of monomial

$$u^\alpha = u_{m_1,i_1}^{\alpha_1} u_{m_2,i_2}^{\alpha_2} \cdots u_{m_k,i_k}^{\alpha_k}\,,$$

is defined as

$$\lambda(\sum_{j=1}^{k} \alpha_j) + \sum_{j=1}^{k} \alpha_j i_j \omega(D_x)\,.$$

Notice that the shift has no effect on the weight of monomials and therefore Θ has the same weight as D_x. In other words we have

$$\omega(u_{m,j}) = \omega(u_{m+i,j}), \qquad j \in \mathbb{Z}\,.$$

If an element $K \in \mathfrak{G}$ consists of monomials of weight ℓ then we call the expression ℓ-homogenous. Consider the DDKP equation, in order to have a homogeneous expression on the right hand side of the equation we have

$$\omega(u) + 2\omega(D_x) = 2\omega(u) + \omega(D_x) = \omega(\Theta) + \omega(u) + \omega(D_x),$$

which means u an x-derivative have the same weight. For example if we set $\lambda = 1$ then the right hand side of the equation is a 3-homogeneous polynomial. According to the concept of homogenous elements one can easily see that the symmetry (6.4.1) is a 4-homogenous expression. This provides the first hint to start with the proper form of the master symmetry W. On the other hand notice that the Lie bracket of two expressions with weights ω_1 and ω_2 produces an element with weight $\omega_1+\omega_2-1$. Therefore W should be a 2-homogenous expression. To find the explicit form of W we start with an ansatz where the structure is obtained based on the known examples of master symmetries for continuous (2+1)-dimensional evolutionary equations. Expressions (6.1.1), (6.1.2) and (6.1.4) are some examples which guide one to start with a suitable ansatz.

Consider the following 2-homogeneous expression:

$$W = xK + cnu_x + \alpha_1 u^2 + \alpha_2 \Theta(u), \tag{6.4.2}$$

where c is an arbitrary constant and $\omega(x) = -\omega(D_x)$. The terms $f = \alpha_1 u^2 + \alpha_2 \Theta(u)$ are obtained from the remaining choices for monomials of weight two that have no dependency on x and n. Now consider the generalised symmetry G and assume $[W, K] = \gamma G$. Then we have the following proposition.

Proposition 10. *Consider the DDKP equation (6.3.2) together with its non-trivial symmetry G (6.4.1) and the ansatz (6.4.2) satisfying*

$$[W, K] = \gamma G,$$

where γ is a constant. The explicit form of W has the form

$$W = xu_{xx} + 2xuu_x + 2x\Theta(u_x) + nu_x + u^2 + 3\Theta(u)\,. \tag{6.4.3}$$

Proof. According to the definition of generalised symmetries we can split the bracket as

$$[f, K] = \gamma G - [xu_t, u_t] - c[nu_x, u_t]\,. \tag{6.4.4}$$

Calculating the right hand side of (6.4.4) we get:

$$\begin{aligned}
[xu_t, u_t] &= -4u^2u_x - 4u\Theta(u_x) + 2x\Theta(u_{xxx}) - 6uu_{xx} + 4x\Theta(u_x^2)\\
&\quad + 4x\Theta(uu_{xx}) - 2u_{xxx} - 6\Theta(u_{xx}) - 4u_x^2 - 2\Theta(xu_{xxx})\\
&\quad - 4\Theta(uu_x) - 4\Theta(xuu_{xx}) - 8\Theta^2(u_x)\,,\\
[nu_x, u_t] &= 2c(n\Theta(u_{xx}) - \Theta(nu_{xx}))\,.
\end{aligned}$$

Collecting the equal terms we have

$$\begin{aligned}
[f, K] &= (\gamma+2)u_{xxx} + (3\gamma+6)uu_{xx} + (3\gamma+4)u_x^2 + (3\gamma+4)u^2u_x\\
&+ (3\gamma+8)\Theta(uu_x) + 3\gamma u_x\Theta(u) + (3\gamma+4)u\Theta(u_x)\\
&+ (3\gamma+8-2c)\Theta^2(u_x) + (3\gamma+8-2c)\Theta(u_{xx})\,.
\end{aligned} \tag{6.4.5}$$

Now for the expression $f = \alpha_1 u^2 + \alpha_2\Theta(u)$ the Fréchet derivative takes the form

$$\begin{aligned}
f_* &= 2\alpha_1 u + \alpha_2\Theta\,,\\
K_* &= D_x^2 + 2uD_x + 2u_x + 2\Theta D_x\,,
\end{aligned}$$

and consequently we get

$$\begin{aligned}
[f, K] &= -2\alpha_1u^2u_x + 4\alpha_1u\Theta(u_x) + 2\alpha_2\Theta(uu_x) - 2\alpha_1u_x^2\\
&- 4\alpha_1\Theta(uu_x) - 2\alpha_2u\Theta(u_x) - 2\alpha_2u_x\Theta(u)\,.
\end{aligned} \tag{6.4.6}$$

Comparing (6.4.6) and (6.4.5) we obtain

$$\gamma = -2, \qquad c = 1\,,$$

$$\alpha_1 = 1, \qquad \alpha_2 = 3\,.$$

Therefore W takes the following form:

$$W = x(u_{xx} + 2uu_x + 2\Theta(u_x)) + nu_x + u^2 + 3\Theta(u)\,.$$

□

So W is a master symmetry of the DDKP equation. Having obtained W in order to produce the commutative algebra of symmetries, we need to show that it meets the conditions given in Theorem 10. In the next theorem the proof is described step by step but to proceed we need the following proposition.

Proposition 11. *Assume for $P \in \hat{\mathfrak{G}}$ we have $[P, K] = [P, 1] = 0$ where K is (6.3.2), then*

$$P = \alpha u_x\,.$$

Proof. First we show that P is an expression with no shift. To do this suppose m is the maximum length of shift appears in P. Let $P_m^{(n)}$ be the terms with maximum degree n and highest shift m. Since $P_m^{(n)}$ contains the terms with maximum degree and $[P, K] = 0$, therefore it should commute with maximum degree term in the DDKP equation that is

$$[P_m^{(n)}, uu_x] = 0\,.$$

Furthermore let $Q_m^{(d)}$ be terms in $P_m^{(n)}$ where d is the maximum number of u with shift m. In fact d is the total number of u_{mj} $(j \geq 0)$ appears in each summand of $P_m^{(n)}$.Therefore

$[Q_m^{(d)}, uu_x]$ does not vanish as $(Q_m^{(d)})_*(uu_x)$ produces an expression

$$Q_m^{(d+1)} .$$

This implies that $m = 0$. In Proposition 13 in the appendix we will show that P does not possess any quasi-local terms. Hence P is the linear combination of monomials of the form

$$u_{j1}^{\alpha_1} u_{j2}^{\alpha_2} \cdots u_{jk}^{\alpha_k}, \qquad u_{j_\ell} = \frac{\partial^{j_\ell} u}{\partial x^{j_\ell}} .$$

Comparing the number of terms in $[u_{j1}^{\alpha_1} u_{j2}^{\alpha_2} \cdots u_{jk}^{\alpha_k}, uu_x]$ we find P is generated by the monomials $u^{n-1}u_x$. Now from $[P, 1] = 0$ we get

$$P = \alpha u_x ,$$

which is our assertion. □

We use this proposition in the proof of next theorem.

Theorem 11. *Let W be the expression defines as (6.4.3). Define $a_{m+1} = [W, a_m]$ where $a_0 = u_x$. Then*

$$[a_i, a_j] = 0, \qquad i, j \geq 0 .$$

Proof. Consider the elements

$$a_0 = u_x, \quad \tau_{-1} = 1, \quad \tau = W .$$

We show that the requirements in Theorem 10 are satisfied. For this we find

1. $\tau_0 = [\tau, \tau_{-1}] = 2xu_x + 2u,$

   $\tau_1 = [\tau, \tau_0] = 2\tau;$

2. $[\tau_{-1}, a_0] = [1, u_x] = 0$,

   $[\tau_0, a_0] = [2xu_x + 2u, u_x, u_x] = -2a_0$;

   We also can obtain the following elements:

$$a_1 = [W, a_0] = -K,$$

$$a_2 = [W, a_1] = -[W, K] = 2G,$$

   which obviously satisfy

3. $[a_1, a_2] = [a_0, a_1] = 0$.

Therefore if we compare the above results with Theorem 10 we can see that the above three conditions hold for $N = 1$ and $M = 2$. The last condition is deduced from Remark 7 and Proposition 11. For $a_{m+1} = [W, a_m]$ the following bracket vanishes:

$$[a_i, a_j] = 0, \qquad i, j \geq 0\,,$$

which completes the proof. □

We have almost reached the main result of this section. To proceed we just need the proposition below which guarantees elements obtained from action of W belongs to $\hat{\mathfrak{G}}$.

Proposition 12. *Let a_i be the elements produced recursively from*

$$a_i = [W, a_{i-1}]\,.$$

Then each a_i lies in $\hat{\mathfrak{G}}$.

Proof. Notice that the bracket $[W, a_i]$ can be decomposed as

$$[W, a_i] = [xK, a_i] + [nu_x, a_i] + [u^2 + 3\Theta(u), a_i]\,,$$

where the troublesome terms can be $[nu_x, a_i]$ and $[xK, a_i]$as $[u^2 + 3\Theta(u), a_i]$ lies in $\hat{\mathfrak{G}}$. If we calculate the first two brackets we find

$$[xK, a_i] = x[K, a_i] + h, \qquad h \in \hat{\mathfrak{G}}\,,$$

$$[nu_x, a_i] = n[u_x, a_i] + \hat{h}, \qquad \hat{h} \in \hat{\mathfrak{G}}\,,$$

which the result can be derived since a_i commutes with K and u_x. □

Thus we found the master symmetry of the DDKP equation and furthermore have constructed the commutative space of symmetries. In the next section we shall see the relation of master symmetries with t-dependent symmetries.

6.5 sl$(2,\mathbb{C})$-representation and t-dependent symmetries of evolution equations

As we mentioned in the previous section, the concept of master symmetry can be used to build up symmetries which depend explicitly on the independent variable t. In [74] it is shown that the representation of Burgers, Ibragimov-Shabat and KP equation can help one to construct the t-dependent symmetries. In the following we also use the Lie algebra $sl(2,\mathbb{C})$ to derive t-dependent symmetries of the DDKP equation. Consider a lattice field equation (6.3.1). Let $G(u,t)$ be an element of $\mathfrak{G}[t]$ where $\mathfrak{G}[t]$ is an extension of $\mathfrak{G}$ to the space that the coefficients are polynomials in t. We call $G(u,t)$ a t-dependent symmetry of (6.3.1) if

$$\frac{\partial G}{\partial t} = [G, K]\,. \tag{6.5.1}$$

To construct t-dependent symmetries we use the following theorem.

Theorem 12. *Consider equation (6.3.1) and let G_0 be an element of $\mathfrak{G}$. If for some ℓ*

the following relation holds:

$$ad_K^{\ell}(G_0) = 0\,, \tag{6.5.2}$$

then $G = \exp(-t\,ad_K)(G_0)$ *is a* t*-dependence symmetry.*

Proof. According to (6.5.2) if we expand G we get the following finite set:

$$G = G_0 - t\,ad_K G_0 + \frac{t^2}{2!} ad_K^2 G_0 - \frac{t^3}{3!} ad_K^3 G_0 + \cdots + (-1)^{\ell-1} \frac{t^{\ell-1}}{(\ell-1)!} ad_K^{\ell-1} G_0\,. \tag{6.5.3}$$

Now we have

$$[G, K] = -ad_K G_0 + t\,ad_K^2 G_0 + \cdots = \sum_{i=1}^{\ell} (-1)^i \frac{t^{i-1}}{(i-1)!} ad_K^i G_0$$

and one can simply check that this is equal to $\frac{\partial G}{\partial t}$. ◻

This theorem shows another advantage of having K-generators. If we find a K-generator then through the above construction we can produce t-dependent symmetries which are polynomial in t. In next theorem we will see that in fact all t-dependent symmetries that are polynomial in t can be written as (6.5.3).

Theorem 13. *Let* $G \in \mathfrak{G}[t]$ *be a* t*-dependent generalised symmetry of (6.3.1) which is polynomial in* t*. Then* G *is expressed as*

$$G = \sum_{\ell} \frac{t^{\ell}}{k!} ad_K^{\ell} G_0, \qquad G_0 \in \mathfrak{G}\,. \tag{6.5.4}$$

Proof. Since G is a polynomial in t we can write as

$$G = G_0 + tG_1 + t^2 G_2 + \cdots + t^m G_m, \qquad G_i \in \mathfrak{G}\,.$$

Furthermore G is a symmetry and from (6.5.1) we have

$$G_1 + 2tG_2 + 3t^2 G_3 + \cdots + m t^{m-1} G_m = [G, K]\,.$$

Comparing both sides of relation with equal powers of t we have

$$
\begin{aligned}
&[G_0, K] = G_1 \longrightarrow -ad_K G_0 = G_1\,,\\
&[G_1, K] = 2G_2 = [[G_0, K], K] = 2G_2 \longrightarrow \frac{1}{2} ad_K^2 G_0 = G_2\,,\\
&[G_2, K] = \frac{1}{2}[[[G_0, K], K], K] = 3G_3 \longrightarrow \frac{-1}{3!} ad_K^3 G_0 = G_3\,,\\
&\vdots\\
&[G_{m-1}, K] = mG_n \longrightarrow \frac{(-1)^m}{m!} ad_K^m G_0 = G_m\,,\\
&[G_m, K] = 0 \longrightarrow ad_K^{m+1} G_0 = 0\,.
\end{aligned}
$$

Therefore G can be expressed as (6.5.4) where $G_0 \in \mathfrak{G}$ is a K-generator of degree m. □

For example for $m = 1$, any t-dependent symmetry of the form $G = G_0 + tG_1$ provides G_0 as a master symmetry. From Theorem 12 and 13 we see that K-generators (K-master symmetries) and t-dependent symmetries are in one-to-one correspondence. In what follows we use $sl(2, \mathbb{C})$-representation to produce the hierarchy of t-dependent symmetries. Consider elements

$$
M = K, \quad N = -\frac{1}{2}x, \quad H = -xu_x - u\,, \tag{6.5.5}
$$

where K is the right hand side of equation (6.3.2). Calculating the Lie bracket of these elements we get

$$
[M, N] = H, \quad [H, N] = -2N, \quad [H, M] = 2M\,. \tag{6.5.6}
$$

Therefore we have the following Lie algebra isomorphisms:

$$
sl(2, \mathbb{C}) \cong span\{M, N, H\}\,.
$$

We know from the representation theory of $sl(2, \mathbb{C})$ (more details can be found in [17, 38]) that if V is a finite dimensional irreducible representation of $sl(2, \mathbb{C})$ then there is a vector $v_0 \in V$ where $Hv_0 = \lambda v_0$ and $Mv_0 = 0$ and furthermore V is spanned by the

$\lambda + 1$ linear independent vectors $\{v_0, ad_N v_0, ad_N^2 v_0, \cdots, ad_N^\lambda v_0\}$. The element v_0 is called the highest weight vector.

Consider the element $v_0 = K$, as we have $[M, K] = 0$ and $[H, K] = 2K$ we get a three dimensional representation of $sl(2, \mathbb{C})$ spanned by $V_3 = span\{K, ad_N K, ad_N^2 K\}$. Now let N_2 be the master symmetry of the DDKP equation. Hence

$$[M, ad_{N_2}^m K] = 0\,.$$

Then by help of the following theorem we can produce $sl(2, \mathbb{C})$ representation of the DDKP equation.

Theorem 14. *For elements defined in (6.5.5) the following relation holds:*

$$[H, ad_{N_2}^m K] = (m+2) ad_{N_2}^m K\,.$$

Proof. We prove the theorem by induction on m. For $m = 0$ according to (6.5.6) we have

$$[H, K] = 2K\,.$$

Now assume that it holds for $[H, ad_{N_2}^m K] = (m+2) ad_{N_2}^m K$ then we have

$$\begin{aligned}
[H, ad_{N_2}^{m+1} K] &= [H, [N_2, ad_{N_2}^m K]] \\
&= [N_2, [H, ad_{N_2}^m K]] + [[H, N_2], ad_{N_2}^m K] \\
&= [N_2, (m+2) ad_{N_2}^m K] + [[H, N_2], ad_{N_2}^m K] \\
&= (m+2) ad_{N_2}^{m+1} K + [[H, N_2], ad_{N_2}^m K]\,.
\end{aligned}$$

One can check that $[H, N_2] = N_2$ and therefore we have the relation

$$[H, ad_{N_2}^{m+1} K] = (m+3) ad_{N_2}^{m+1} K\,,$$

which proves our assertion. □

This theorem implies that for different value of m, $ad_{N_2}^m K$ is a highest weight vector of the $(m+3)$-dimensional representation

$$V_{m+3} = span\{ad_{N_2}^{m}K, ad_N ad_{N_2}^{m}K, ad_N^2 ad_{N_2}^{m}K, \cdots, ad_N^{m+2} ad_{N_2}^{m}K\},$$

where

$$ad_N^{m_2} ad_{N_2}^{m_1}K = \underbrace{[N,[N,\cdots,[N,}_{m_2-times}\overbrace{[N_2,[N_2,\cdots[N_2,}^{m_1-times}K]]]]]] .$$

So starting from K we can draw the following diagram for the DDKP equation in which the first element of each row is the symmetry of equation (6.3.2) and the horizontal lines contain the representations of $sl(2,\mathbb{C})$:

$$
\begin{array}{ccccccccc}
K & \xrightarrow{N} & ad_N K & \xrightarrow{N} & ad_N^2 K & & & & \\
\downarrow{\scriptstyle N_2} & & & & & & & & \\
ad_{N_2}K & \xrightarrow{N} & ad_N ad_{N_2}K & \xrightarrow{N} & ad_N^2 ad_{N_2}K & \xrightarrow{N} & ad_N^3 ad_{N_2}K & & \\
\downarrow{\scriptstyle N_2} & & & & & & & & \\
ad_{N_2}^2 K & \xrightarrow{N} & ad_N ad_{N_2}^2 K & \xrightarrow{N} & ad_N^2 ad_{N_2}^2 K & \xrightarrow{N} & ad_N^3 ad_{N_2}^2 K & \xrightarrow{N} & ad_N^4 ad_{N_2}^2 K \\
\downarrow{\scriptstyle N_2} & & & & & & & & \\
\vdots & & & & & & & &
\end{array}
$$

In the following theorem we see that horizontal lines are not just elements of the representation but K-generators of different degrees.

Theorem 15. *Consider $ad_{N_2}^{m}K \in V_{m+3}$, then the basis elements*

$$ad_N^{\ell} ad_{N_2}^{m}K ,$$

commute with $K^{\ell+1}, (0 \leq \ell \leq m+2)$. In other words we get the K-generator of degree

ℓ

$$ad_K^{\ell+1}(ad_N^{\ell}ad_{N_2}^m K) = 0\,.$$

Proof. The proof is by induction on ℓ. For $\ell = 0$ as all $ad_{N_2}^m K$ are symmetries of (6.3.2) we have

$$[K, ad_{N_2}^m K] = 0\,.$$

Assume the statement holds for

$$ad_K^{\ell+1}(ad_N^{\ell}ad_{N_2}^m K) = 0\,,$$

then we have

$$\begin{aligned} ad_K^{\ell+2}ad_N^{\ell+1}ad_N^m K &= [K, ad_K^{\ell+1}[N, ad_N^{\ell}ad_{N_2}^m K]] \\ &= [K, [ad_K^{\ell+1}N, ad_K^{\ell+1}ad_N^{\ell}ad_{N_2}^m K]] = 0\,. \end{aligned}$$

□

Hence from Theorem 12 and 13 each basis element lying on the horizontal line provide a generator for the t-dependence symmetries of the DDKP equation. In fact $\exp(-tad_K)(ad_N^{\ell}N_2^m K)$, $0 \leq \ell \leq m+2$ are t-dependent symmetries of the DDKP equation.

For instance

$$\exp(-tad_K)(ad_N K) = [K, N] - t[K, [K, N]] = (-xu_x - u) + 2t(u_{xx} + 2uu_x + 2\Theta(u_x))\,,$$

$$\exp(-tad_K)(ad_N^2 K) = ad_N^2 K - t(ad_K ad_N^2 K) + \frac{t^2}{2!}(ad_K^2 ad_N^2 K) = x - 2t(xu_x - u)$$

$$+4t^2(u_{xx} + 2uu_x + 2\Theta(u_x))\,.$$

In the appendix we discuss the symbolic representation method. It is shown how we can simplify some calculations, given in this chapter, by means of this method.

6.6 Conclusion

Master symmetries of differential-difference equations can produce an infinite hierarchy of commuting symmetries. We also showed how systematically, using the Lie algebra $sl(2, \mathbb{C})$, one can obtain t-dependent symmetries that are polynomials in t. In fact the Lie algebra $sl(2, \mathbb{C})$ is constructed around a scaling symmetry of a given equation and a corresponding master symmetry. We can apply the similar process to lattice field equations that possess quasi-local master symmetries. This structure can be also used for (1+1)-dimensional equations in which the equation, symmetries and the basis elements of $sl(2, \mathbb{C})$ are all local.

Chapter 7

Conclusion

This thesis mainly concerns integrable (1+1)-dimensional differential-difference and (2+1)-lattice field equations. In particular, in chapter 2 we first build up the space of smooth difference functions denoted by $\mathfrak{F}$ with elements of the form

$$f[\mathbf{u}] := f(\cdots, \mathbf{u}_{-2}, \mathbf{u}_{-1}, \mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \cdots), \qquad \mathbf{u} = (u^{(1)}, u^{(2)}, \cdots, u^{(m)})^T,$$

where $u^{(m)}(n+i,t) \equiv u_i^{(m)}$.Therefore the general form of NDDEs is given as

$$\mathbf{u}_t = (K^1[\mathbf{u}], K^2[\mathbf{u}], \cdots, K^m[\mathbf{u}])^T = \mathbf{K}\,[\mathbf{u}]\,,$$

where $\mathbf{u} = (u^{(1)}, u^{(2)}, \cdots, u^{(m)})^T$ and K^i's are elements of $\mathfrak{F}$. Then the first result was given in chapter 5 in which we provide a long list containing both scalar and multi component differential-difference equations. The equations appear with their non-trivial generalized symmetry, symplectic operators, Hamiltonian operators and recursion operators. For some equations their master symmetry is also presented. Section 5.2.4 provides new results on Hamiltonian, symplectic and recursion operators for scalar equations. In sections 5.2.14, 5.2.15 and 5.2.16 one can find results for multi component integrable systems. For multi component systems we have also provided the inverse of recursion operators. In the case that the inverse exists, it is a weakly nonlocal operator.

We have also seen that master symmetries are an alternative powerful tool in (2+1)-dimensional case to produce higher symmetries. The comprehensive treatment of master symmetries is given in chapter 6 in which we obtain the master symmetry of the differential-difference KP (DDKP) equation

$$u_t = u_{xx} + 2uu_x + 2\Theta(u_x) = K, \qquad \Theta = (\mathcal{S} - 1)^{-1} D_x \,,$$

given by

$$W = xu_{xx} + 2xuu_x + 2x\Theta(u_x) + nu_x + u^2 + 3\Theta(u) \,.$$

We proved that W can produce the infinite hierarchy of commuting symmetries of the DDKP equation and guarantee the integrability. We finish this chapter by indicating the application of master symmetry in constructing t-dependent symmetries. This is done by the $sl(2,\mathbb{C})$-representation of the equation which provides a neat picture of symmetries, master symmetries and generators of t-dependent symmetries.

Symbolic representation was a method we briefly described in the Appendix. We showed how symbolic representation provides a very handy tool in simplifying some tedious calculations. The advantage of using this method can also be seen while working with (2+1)-dimensional differential-difference equations. In fact since in this case we face quasi-local terms, by the symbolic representation we can rewrite the expression in a way that can be calculated algebraically.

Moreover there exists some ongoing research in which the symbolic representation can be a helpful method tackling certain problems. For instance the structure of symmetries of (2+1)-dimensional integrable differential equations was proved by the symbolic representation in [90]. For the discrete case the symmetry structures still remains as a conjecture that may progress with the help of symbolic representation.

Appendix A

Symbolic representation

In recent years there have been number of researches using the symbolic representation for classification and testing the integrability of partial differential equations. This was first introduced by Gelfand and Dikii in [29] and later it was developed to both (1+1)-dimension [52, 59, 53, 71, 72] and (2+1)-dimension [90]. It is shown that the symbolic representation can provide a powerful device for testing the integrability and constructing symmetries of evolutionary equations. It also eases some tedious calculations such as obtaining the coefficients of formal recursion operators and thus leading to the result much quicker. Here we provide a brief review of the symbolic representation for (2+1)-dimension differential-difference equations and discuss some applications. As a first step we shall introduce the linear map on monomials to obtain the one-to-one correspondence between difference polynomials and elements in the symbolic space. For comprehensive description in differential case one can find [72] and papers [59, 90].

A.1 Basic calculations in symbolic representation

Consider an arbitrary monomial of degree n as

$$u_{m_1,i_1}u_{m_2,i_2}\cdots u_{m_n,i_n}\,.$$

Then we define its corresponding symbolic representation as follows:

$$u_{m_1,i_1}u_{m_2,i_2}\cdots u_{m_n,i_n}\longmapsto \tilde{u}^n\langle \xi_1^{m_1}\eta_1^{i_1}\xi_2^{m_2}\eta_2^{i_2}\cdots\xi_n^{m_n}\eta_n^{i_n}\rangle_{S_n}\,.$$

We assign to each $u_{i,j}$ a couple (ξ,η) in which the power of ξ and η respectively match with indices i and j (i.e. $\xi^i\eta^j$). The symbol $\tilde{u}^n$ shows the degree of the monomial or in other words the number of variable in the symbolic expression. Finally we have $\langle\rangle_{S_n}$ which denotes the average over the permutation group of n elements, i.e.

$$\langle a(\xi_1,\eta_1,\xi_2,\eta_2\cdots,\xi_n,\eta_n)\rangle_{S_n}=\frac{1}{n!}\sum_{\sigma\in S_n}a(\xi_{\sigma(1)},\eta_{\sigma(1)},\cdots,\xi_{\sigma(n)},\eta_{\sigma(n)})\,.$$

Since we assume that $u(n,x,t)$ lies in some commutative algebra so we need the symmetrization in order to get the unique expression. To make the notation clear let us consider the following example:

$$\begin{aligned}
u_{-3y}^2u_{1yy}\longmapsto \tilde{u}^3<\xi_1^{-3}\eta_1\xi_2^{-3}\eta_2\xi_3\eta_3^2>_{S_3} &= \frac{1}{3}\tilde{u}^3\Big(\frac{\eta_1\eta_2\xi_3\eta_3^2}{\xi_1^3\xi_2^3}+\frac{\eta_2\eta_3\xi_1\eta_1^2}{\xi_2^3\xi_3^3}+\frac{\eta_3\eta_1\xi_2\eta_2^2}{\xi_3^3\xi_1^3}\Big)\\
&= \frac{1}{3}\tilde{u}^3\Big(\frac{\eta_1\eta_2\xi_3^4\eta_3^2+\eta_2\eta_3\xi_1^4\eta_1^2+\eta_3\eta_1\xi_2^4\eta_2^2}{\xi_1^3\xi_2^3\xi_3^3}\Big).
\end{aligned}$$

In the symbolic space we have the symmetric polynomial in tuple $(\frac{1}{\xi},\xi,\eta)$. In compact form the symbolic representation is a rational function which the denominator is non-negative integer power of multiplication of ξ_i's and the numerator is the symmetric polynomial with respect to (ξ,η). To obtain the difference monomial from the symbolic representation we can split the rational function into different summands and obtain the corresponding expression

$$
\begin{aligned}
\frac{\tilde{u}^2}{2}\left(\frac{\eta_1\xi_1^2\xi_2^5+\eta_2\xi_2^2\xi_1^5+4\xi_1^4\eta_1^2\xi_2^4\eta_2^2}{\xi_1^2\xi_2^2}\right) &= \frac{\tilde{u}^2}{2}\left(\frac{\eta_1\xi_2^3}{\xi_1^2}+\frac{\eta_2\xi_1^3}{\xi_2^2}\right)+2\tilde{u}^2\xi_1^2\eta_1^2\xi_2^2\eta_2^2 \\
&= = \tilde{u}^2 < \frac{\eta_1\xi_2^3}{\xi_1^2} >_{S_2} +2\tilde{u}^2 < \xi_1^2\eta_1^2\xi_2^2\eta_2^2 >_{S_2} \\
&\mapsto 2u_{-2y}u_3 + 2u_{2yy}^2 .
\end{aligned}
$$

If we set $\bar{\xi} = (\xi, \eta)$, the multiplication rule in this ring takes the form:

$$
\begin{aligned}
&\tilde{u}^n a(\bar{\xi}_1, \bar{\xi}_2, \cdots, \bar{\xi}_n) \circ \tilde{u}^m b(\bar{\xi}_1, \bar{\xi}_2, \cdots, \bar{\xi}_m) = \\
&\tilde{u}^{n+m} < a(\bar{\xi}_1, \bar{\xi}_2, \cdots, \bar{\xi}_n) b(\bar{\xi}_{n+1}, \bar{\xi}_{n+2}, \cdots, \bar{\xi}_{n+m}) >_{S_{n+m}} .
\end{aligned} \tag{A.1.1}
$$

Example 24. *For two expressions*

$$
\begin{aligned}
P_1 &= \frac{\tilde{u}^3}{3}\left(\frac{\xi_1^3\eta_1^2}{\xi_2\xi_3}+\frac{\xi_2^3\eta_2^2}{\xi_3\xi_1}+\frac{\xi_3^3\eta_3^2}{\xi_1\xi_2}\right) \mapsto u_{3yy}u_{-1}^2, \\
P_2 &= \tilde{u}\frac{\eta_1}{\xi_1^2} \mapsto u_{-2y},
\end{aligned}
$$

we have

$$
P_1 \circ P_2 = \tilde{u}^4 < \frac{\xi_1^3\eta_1^2\eta_4}{\xi_2\xi_3\xi_4^2} >_{S_4} .
$$

For a monomial of degree n the action of the shift operator and x-derivative takes the form

$$
\mathcal{S}\tilde{u}^n a(\bar{\xi}_1, \cdots, \bar{\xi}_n) = \tilde{u}^n a(\bar{\xi}_1, \cdots, \bar{\xi}_n)(\xi_1\xi_2\cdots\xi_n), \tag{A.1.2}
$$

$$
D_x(\tilde{u}^n a(\bar{\xi}_1, \cdots, \bar{\xi}_n)) = \tilde{u}^n a(\bar{\xi}_1, \cdots, \bar{\xi}_n)(\eta_1+\eta_2+\cdots+\eta_n) . \tag{A.1.3}
$$

One of the advantages of symbolic representation is in calculation of Fréchet derivatives. For an arbitrary monomial the Fréchet derivative is a symmetric function over its arguments (including $\mathcal{S}, D_x$)

$$(\tilde{u}^n a(\bar{\xi_1}, \cdots, \bar{\xi_n}))_* = n\tilde{u}^{n-1} a(\bar{\xi_1}, \cdots, \bar{\xi}_{n-1}, S, D_y)\,. \tag{A.1.4}$$

Example 25. *Consider the expression*

$$P = \frac{\tilde{u}^2}{2}(\xi_1^3\xi_2\eta_2 + \xi_2^3\xi_1\eta_1)\,.$$

Then its Fréchet derivative has the form

$$P_* = \tilde{u}(\xi_1^3 \mathcal{S} D_y + \xi_1\eta_1\mathcal{S}^3)\,.$$

To give the space of symbolic representation the structure of Lie algebra we require to define the corresponding Lie bracket. From A.1.1, A.1.2, A.1.3 and A.1.4 the Lie bracket of two elements is expressed as

$$[\tilde{u}^n a(\bar{\xi_1}, \bar{\xi_2}, \cdots, \bar{\xi_n}), \tilde{u}^m b(\bar{\xi_1}, \bar{\xi_2}, \cdots, \bar{\xi_m})] =$$
$$\tilde{u}^{n+m-1}\langle na(\bar{\xi_1}, \cdots, \bar{\xi}_{n-1}, \xi_n\xi_{n+1}\cdots\xi_{n+m-1}, \eta_n + \cdots + \eta_{n+m-1}) b(\bar{\xi_n}, \cdots, \bar{\xi}_{n+m-1}) -$$
$$mb(\bar{\xi_1}, \cdots, \bar{\xi}_{m-1}, \xi_m\xi_{m+1}\cdots\xi_{n+m-1}, \eta_m + \eta_{m+1} + \cdots + \eta_{n+m-1}) a(\bar{\xi_m}, \cdots, \bar{\xi}_{n+m-1})\rangle_{S_{n+m-1}}\,.$$

The last two relations are the symbolic representation of non-local terms

$$\Theta P(\bar{\xi_1}, \bar{\xi_2}, \cdots, \bar{\xi_n}) = P(\bar{\xi_1}, \bar{\xi_2}, \cdots, \bar{\xi_n})\frac{\eta_1 + \cdots + \eta_n}{\xi_1\xi_2\cdots\xi_n - 1},$$
$$\Theta^{-1} P(\bar{\xi_1}, \bar{\xi_2}, \cdots, \bar{\xi_n}) = P(\bar{\xi_1}, \bar{\xi_2}, \cdots, \bar{\xi_n})\frac{\xi_1\xi_2\cdots\xi_n - 1}{\eta_1 + \cdots + \eta_n}\,.$$

In the next section we shall see some applications of symbolic representation which simplify calculation.

A.2 Application of symbolic representation

As we mentioned earlier by symbolic representation we can simplify some of cumbersome calculations. Consider the generalised symmetry of the DDKP equation given in (6.4.1) as

$$\begin{aligned} G &= u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x + 3\Theta(uu_x) + 3u_x\Theta(u) \\ &+ 3u\Theta(u_x) + 3\Theta^2(u_x) + 3\Theta(u_{xx}) . \end{aligned}$$

To show G is a symmetry we should show the bracket $[K, G]$ vanishes. This becomes a bit complicated when we are dealing with the non-local terms. For example let us consider terms of degree two that appear in the bracket $[K, G]$. Hence we get

$$\begin{aligned} &[u_{xx}, 3\Theta(uu_x) + 3u_x\Theta(u) + 3u\Theta(u_x)] + [2uu_x, 3\Theta^2(u_x) + 3\Theta(u_{xx})] \\ &+[2\Theta(u_x), 3uu_{xx} + 3u_x^2 + 3\Theta(uu_x) + 3u_x\Theta(u) + 3u\Theta(u_x)] . \end{aligned}$$

Expanding the brackets and simplifying the calculations we have

$$\Theta(u_xu_{xx}) + \Theta(u_{xx}\Theta(u)) + \Theta(u_x\Theta(u_x)) - \Theta(u)\Theta(u_{xx}) - (\Theta(u_x))^2 .$$

As we can see, this is not an easy task to show the expression is zero. Expressing the terms in symbolic representation we get

$$\begin{aligned} &\frac{(\eta_1+\eta_2)}{(\xi_1\xi_2-1)}\left(< \eta_1\eta_2^2 >_{s_2} + < \frac{\eta_1^2\eta_2}{(\xi_2-1)} >_{s_2} + < \frac{\eta_1\eta_2^2}{(\xi_2-1)} >_{s_2} \right) \\ &- < \frac{\eta_1\eta_2^3}{(\xi_2-1)(\xi_1-1)} >_{s_2} - < \frac{\eta_1^2\eta_2^2}{(\xi_1-1)(\xi_2-1)} >_{s_2} . \end{aligned}$$

Since we end up with an algebraic expression, although it looks longer we can simply expand the terms and see that all summands are canceled.

We proceed this section to show how the symbolic representation can help us in proving a missing part of Proposition 11 for which we have to show that P contains no non-local terms.

Proposition 13. *Let $P \in \hat{\mathfrak{G}}$ and $[P, uu_x] = 0$ then P does not posses any quasi-local terms*

Proof. Let $P^{(n)}$ denotes quasi-local terms with the maximum degree n. Hence for $P^{(n)}$ and uu_x respectively we have the following symbolic representation:

$$P^{(n)} \mapsto \tilde{u}^n a(\xi_1, \eta_1, \xi_2, \eta_2, \ldots, \xi_n, \eta_n),$$
$$uu_x \mapsto \tilde{u}^2 b(\eta_1, \eta_2) = \frac{\tilde{u}^2}{2}(\eta_1 + \eta_2)\,.$$

Therefore the Fréchet derivative of $P_*^{(n)}(uu_x)$ contains the expression

$$a(\xi_1, \eta_1, \xi_2, \eta_2, \ldots, \xi_n \xi_{n+1}, \eta_n + \eta_{n+1}),$$

which increase the number of variables in the denominator by one as the symbolic representation of Θ gives

$$\frac{\eta_i + \eta_{i+1} + \cdots + \eta_n}{\xi_i \xi_{i+1} \cdots \xi_n - 1}, \qquad 1 \le i \le n\,.$$

On the other hand $(uu_x)_* P^{(n)}$ has the same number of variables in the denominator as $P^{(n)}$. Therefore the bracket does not vanish and this contradiction proves that no quasi-local terms appear in P. □

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