Quantification of Uncertainty of Warranty Claims

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1. Introduction

A warranty is a contractual obligation provided by a manufacturer to its product consumer in connection with the sale of the product. It guarantees that the product will meet certain requirements and perform its functions as specified in the warranty agreement. If the product does not meet these expectations, the manufacturer is obligated to either repair or replace the product at no charge to the buyer or at a partial cost to the buyer. Warranty plays an important role in protecting consumers' interests.

There are two types of warranty: base warranty and extended warranty. Base warranty is bundled with the product at no additional cost to the buyer and extended warranty is an agreement that covers repair costs after the base warranty has expired and is purchased separately and voluntarily by the buyer.

A warranty policy defines the duration of the warranty, how the item will be used, how the item will be repaired/replaced upon its failure, and how the cost of repair will be burdened. Warranty policies can be categorized into renewing or nonrenewing warranty policies. Under the renewing warranty policy, a failed product item can be replaced with a new identical item whose warranty will be renewed. Under the non-renewing warranty policy, a failed product item can either be repaired or replaced but the warranty of the repaired/replaced item cannot be renewed.

From the perspective of the length of warranty coverage, warranty can also be categorized into short term warranty and long-term warranty. Long term warranty can be

When a warranty covered failure occurs, typically, three remedy options are offered to the consumer: a free repair/replacement, a pro rata refund, or a combination of free replacement and pro rata refund. With free replacement, the seller is responsible for paying the entire cost of the remedy if the product fails before the end of the warranty period. Under a pro rata refund warranty, the seller is responsible for repair or replacement, and the cost extent of the seller's obligation is determined based upon the age or wear of the product at the time of failure [1]. For products sold with warranty, manufacturers bear additional cost incurred due to warranty servicing. Such warranty cost is generally substantial. For example, Volkswagen’s total warranty cost in 2021 was 9.27 billion euro with 4.5% claim rate, i.e. cost of claims as a percentage of sales revenue [2].

From the consumer's point of view, a warranty plays a protectional role in product purchase transactions, because it provides a means of redress if the item fails to perform as specified by the manufacturer under proper use. From the manufacturer's point of view, a warranty can be also a protection, as the warranty policy specify the proper use and conditions of use for the product. If the product is misused, the coverage of warranty will be limited or cancelled. The manufacturer is protected from the unexpected loss by the specification of requirements for care and maintenance of the product in the warranty policy [3]. Additionally, warranty also plays a promotional role in marketing a product. Many consumers believe a longer warranty indicates a more reliable product. Using warranty as a marketing tool is particularly important in promoting new products, which may be considered having a higher level of uncertainty than existing products [3]. In this sense, a warranty is an instrument which provides the consumer with a degree of assurance against uncertainty, which means the occurrence of failures of a product cannot be predicted with certainty.

A warranty does not eliminate the uncertainty of failures but transfers the burden of uncertainty from the consumer to the manufacturer during the specified period. From the manufacturer’s perspective, at the strategic level, uncertainty appears in all technical and commercial aspects, including design, manufacturing, marketing, etc., of the product. At the operational level, the manufacturer needs to deal with the uncertainty associated with warranty claims to achieve an efficient administration of warranty resulting in effective management of cost and reputation. Most of the warranty claims are triggered by the failures of product items; uncertainty appears, for example, in the time to failure, when a product is put into operation. Moreover, it should also be noted that some users’ behaviors (human factors) may contribute to warranty claims [4], the related uncertainty needs to be considered in some cases. For example, when a failure occurs, the user can subjectively decide whether they will make a claim or not [5].

The word “uncertainty” is explained as a situation in which something is not known, or something that is not known or certain in the Cambridge Dictionary. In daily conversation, risk is another term related to this situation, and sometimes these two words are interchangeable. However, in the context of business studies, risk refers to situations under which the outcome is unknown, but the likelihood of occurrences of all potential outcomes are known, whereas uncertainty refers to situations under which either the potential outcomes and/or their probabilities of occurrences are unknown [6]. Risk can be measured by appropriate measures like variance, value-at-risk, etc., and it can be used as a handle to discuss uncertainty.

Quantifying the uncertainty of warranty claims has two tasks, i) modeling the uncertainty and ii) measuring the uncertainty. Quantifying the uncertainty of warranty claims is crucial for the manufacturer that offer warranty as they need to prepare resources for warranty claims. This motives our research in this chapter.

The remainder of this chapter is structured as follows. In the next section, we start with reviewing the warranty claim process, identifying the uncertainties implied in different warranty claim routes, and discussing the methods of modeling the uncertainties. Following it, the methods of measuring the uncertainty are reviewed in Section 3. Then, this chapter is concluded in Section 4.

2. Warranty Claim Process and Uncertainty

The uncertainty of warranty claims has multiple sources including, but not limited to, the uncertain quality of individual product, the random operating environment, the different using skills of users, and even the unknown misconception of failure of consumers. As shown in Fig 1, the causes of warranty claims can vary. They can be due to the following causes:

(a) poor performance and management of the customer care team, e.g., poor internal training programs, poor access to product information, etc,
(b) users-related, for example, abuse of product, abuse of the warranty claim process, wrong expectation of functions,
(c) product performance related, for example, hardware failure, software failure, etc., and
(d) product usability related, for example, missing accessories, poor design of product manual, etc.

The different causes of warranty claims result in different types of uncertainties, which are introduced into the warranty claim process at different stages. A today's product may be composed of a hardware subsystem and a software subsystem. Apparently, the main causes of warranty claims are related to the failures of hardware and software subsystems, on which a typical warranty claim process starts when the user believes the product is failed and results in five different routes [5], as shown in Fig 2.

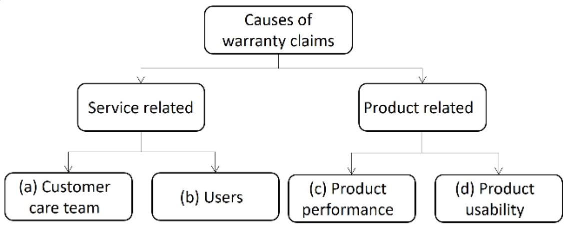

Figure 1: Causes of warranty claims

Route 1 shown in Fig. 2 describes the most typical and well-studied scenario, in which the warranty claims are triggered by the failures of the hardware subsystem and dealt with repair or replacement provided by the manufacturer. Routes 2 and 3 have been described by [4], the author indicates an absence of warranty claims is not a 'no failure' situation, it is possibly dur to failed-but-not-reported (FBNR) phenomena, i.e. some users do not make warranty claims when the product items fail during the warranty period; meanwhile, not all warranty claims result from product failures, because non-failed but reported (NFBR) claims can occur due to user's misconception of the product's designed functions. The FBNR and NFBR do not generate any conventional warranty claim cost but can lead to extra administration costs which should be taken into account. In these scenarios, the uncertainty is introduced by the random human behaviors. The methods of quantifying the uncertain human factors are provided by [4], which will be discussed later. Routes 4 and 5 represent the emerging scenarios in which the warranty claims result from the failures of the software subsystem. Nowadays, lots of products have software system embedded and been connected to the external network. The complex interplays within a product and with the external environment also influence the level of uncertainty. The difference in the reliabilities of the software subsystem and the hardware subsystem should be considered in modelling as well.

In the real world, new and used products are considered separately in developing warranty policies or conducting warranty studies. The studies of new product warranty are dominating. New products can be classified into three categories, consumer durables, industrial and commercial products for the provision of services (commercial products), and government acquisitions [8]. Most of the consumer durables are standard off-the-shelf products, while a considerable number of commercial products are custom-built or specialised. This taxonomy is used in the following discussions where it is necessary.

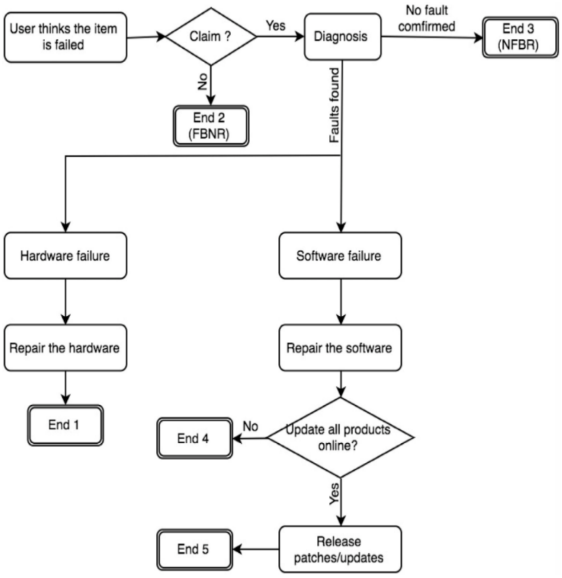

Fig. 2 Warranty claim process [2].

2.1 The Uncertainty of Warranty Claims on Hardware

Warranty claims due to the failures of the hardware subsystem are well-studied, most of the existing studies of warranty do not specify whether the research objective is due to the hardware subsystem or the software subsystem, but they are developed based on some assumptions or features of hardware subsystems, for example, the degradation of physic parts. Warranty claims are closely linked to the reliability of the hardware subsystem and influenced by the reliability of the software subsystem. In addition to the subjective decisions mentioned in Routes 2 and 3 above, several uncertain factors including usage (mode and intensity), operating environment, user skills, and maintenance also introduce uncertainty into warranty claims, as shown in Fig. 1. However, to initiate the discussion, we start with an assumption that all warranty claims are triggered by actual failures of product, and the user makes warranty claim immediately after a failure occurred.

Overall, there are two approaches to modelling the uncertainty of warranty claims, black-box and white-box approaches. The black-box approach literally means treating the product as a black box with unseen internal details, and the uncertainty of failures is modelled without considering the mechanisms that is responsible for failure. This approach is found very useful in modeling and analyzing the warranty claims of a large number of standard products [7]. Most of the models based on the black-box approach are data-based. In contrast, the white-box approach requires the knowledge of the product's internal design and the underlying mechanism of failure [7]. In most cases, a product comprises of many components, the failure of one or more components can lead to the failure of whole product. Therefore, the white-box models are largely physic-based and more complex than the black-box models, but they are very valuable in modeling the warranty claims of specialized products.

2.1.1. The Black-box Approach

In this approach, the products can be categorized into non-repairable and repairable types, and a product is considered being in working state or failed state at a moment. If a product is non-repairable, only the first failure, i.e. the first warranty claim, needs to be modeled. If the product is repairable, times between claims must be modeled differently, since the remedy action taken at the first claim can influence the occurrences of subsequent claims.

There are two variables associated with the warranty claims: the time to next claim and the cost of each claim. The uncertainty of the time to first claim, or the time to first failure, is modeled by a continuous probability distribution defined as $F(x;\theta) = P\{X \leq x|\theta\}$, where x and θ denote the time to first claim and the set of parameters of the distribution, respectively. The actual form of distribution used in a study is decided based on the availability of data and related information. There are two distributions extensively used in studies, the Weibull distribution and the exponential distribution. The cumulative distribution function (CDF) of Weibull distribution is defined as

$$F(x;\alpha,\beta) = 1 - e^{-\left(\frac{x}{\alpha}\right)^{\beta}}.$$

Here α is the scale parameter and β is the shape parameter. The exponential distribution is a special case of the Weibull distribution when $\beta = 1$. Normally, the CDF of the exponential distribution is defined as

$$F(x;\lambda) = 1 - e^{-\lambda x}.$$

where λ is often called the rate parameter.

Calculating the cost caused by a warranty claim could be a complex task in practice. For managerial and accounting purposes, the cost of supplies, the cost of labor, and all the cost of any related activities may be considered in the real cases. However, for the simplicity of modeling, in warranty studies, the simplest way is to assume a constant cost per claim or a constant expected cost per claim [9]. The former ignores the uncertainty of warranty claim cost and the latter recognizes it. Some studies express the structure of warranty claim cost as

$$C_i = A + Q_i,$$

where C_i is the cost of the ith claim, A is a constant, and Q_i is a random variable [10,11]. The individual warranty claim costs of a product over a time interval can be modeled by a sequence of random variables $\{X_i, i = 1, 2, ...\}$. When X_i are independent, the actual distribution of X_i is determined according to the information available in specific cases, and the lognormal distribution is commonly used [12].

If the warranty duration does not expire at the first claim, the consumer may make more than one claim over the warranty period, no matter the product is repairable or not. If the product is non-repairable, the failed product will be replaced by a new one at the first warranty claim. The uncertainty of the time to next claim can be modeled by the same distribution of the time to first claim. The time spent on replacement is uncertain in some cases, but it could be negligible if it is relatively small compared to the time to failure. If the product is repairable, the times between subsequent claims are influenced by the type of repair. One commonly cited type of repair is minimal repair, which restore the failed product to the state as just before it failed, in other words, minimal repair brings the product back to a working state but does not improve the reliability of it [13].

When the repair time is negligible, the number of warranty claims on a product over the warranty period and its uncertainty can be modeled by a continuous time discrete valued stochastic process [3], which is also called a point stochastic process. Such a process is denoted by $\{N(t), t \geq 0\}$ with a value space $\{0, 1, 2, ...\}$, where $N(t)$ is the total number of points/claims in time interval $(0, t]$. If the numbers of points in disjoint time intervals, i.e. $N(t_3) - N(t_2)$ and $N(t_2) - N(t_1)$ for $t_1 < t_2 < t_3$, are independent from each other, the process has independent increment. If the numbers of point in any two time intervals of equal length, i.e. $N(t_2) - N(t_1)$ and $N(t_2 + \gamma) - N(t_1 + \gamma)$ for $t_2 > t_1$ and $\forall \gamma > 0$, have the same distribution the process has stationary increment [14].

For a non-repairable product covered by a free replacement warranty, the distributions of the time to first claim and the time between subsequent claims are independent and identical. Then, the number of claims over the warranty period can be modeled by a stochastic point process with independent and stationary increment. If the time between claims follows an exponential distribution with CDF $F(x; \lambda) = 1 - e^{-\lambda x}$, this process is a homogeneous Poisson process (HPP), in which

$$P\{N(t) = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, n = 0, 1, 2, ...$$

The expected time between claims is $\frac{1}{\lambda}$, and the expected number of claims in $(0, t]$ is λt.

For a repairable product covered by a free repair warranty, normally, minimal repairs are conducted at warranty claims. The reliability of the hardware subsystem is changing over time, largely declining, when imperfect repairs including minimal repairs are applied [13]. The claim arrival process is rarely regarded to be stationary. If the process has independent increment, and the time between claims follows the exponential distribution with time-varying rate parameter, the process is a non-homogeneous Poisson process (NHPP), in which

$$P\{N(t)=n\}=\frac{\left(\Lambda(t)\right)^n e^{-\Lambda(t)}}{n!};\ n=0,1,2,\dots,$$
$$\Lambda(t)=\int_{\gamma}^{\gamma+t}\lambda(t)\,dt;\ \forall\gamma\geq 0.$$

Here $\Lambda(t)$ is called the cumulative rate, and it is a positive-valued, continuous, non-decreasing function [14, 15]. $\Lambda(t)$ is the expected number of claims in $(0,t]$. In practice, the extensively used cumulative rate is defined by the power-law function,

$$\lambda(t)=\alpha\beta t^{\beta-1};\ \alpha>0,\beta>0,$$
$$\Lambda(t)=\int_{\gamma}^{\gamma+t}\lambda(t)\,dt=\alpha t^{\beta}.$$

This process is also called a Weibull process and the time to the first claim follows the Weibull distribution. It should be noted that HPP is a special case of NHPP when $\lambda(t)$ is a constant λ. Different from the HPP, the inter arrival times, i.e. the time between claims, of NHPP are neither independent nor identically distributed [14].

In above discussion, the time to claim are modeled by the exponential distribution and the Weibull distribution, but in practice the distribution of time is not limited to these two. If we relax this assumption and allow the times following an arbitrary non-negative distribution, the stochastic process becomes a renewal process [16, 17]. For instance, the ordinary renewal process (ORP) can be quoted to model the claim occurrence when a product receives perfect repairs at warranty claims and is restored to the as-good-as-new condition because of repair, and the times between claims are independent and identically distributed [17]. The expected number of claims in $(0,t]$ is defined by a renewal equation:

$$\Lambda(t)=F(t)+\int_{0}^{t}F(t-s)\,d\Lambda(s),$$

where $F(t)$ is the CDF of the time to first claim and the time between subsequent claims. Obviously, the HPP is a special case of ORP, and it is one of a few special ORPs who have closed form solution of the above renewal equation [17, 18, 19].

Then, the G-renewal process, aka, generalized renewal process, relaxes the assumptions of perfect repair and minimal repair. This process provides high flexibility in modeling the failure/claim arrival process by covering almost all repair assumptions from worse-than-old to better-than-new [17, 18, 19]. In [17], To adopt the different assumptions of repair, the repair effectiveness parameter, q, is created; and the concept virtual age is introduced

$$A_n=A_{n-1}+qX_n, n=1,2,\dots$$

where A_n is the virtual age of the product immediately after the nth repair, $A_0 = 0$, and X_n is the time between $(n-1)$th and nth claims. The distribution of the time to $(n+1)$th claims after the nth repair is defined by the following conditional CDF

$$F(x|A_n = y) = \frac{F(x+y) - F(y)}{1 - F(y)},$$

where $F(x)$ is the CDF of X_1, the time to first claim of a new product. The expected number of claims in $(0, t]$ is defined by the renewal equation is defined by a generalized renewal equation:

$$\Lambda(t) = \int_0^t \left(g(\tau|0) + \int_0^\tau h(x) g(\tau - x|x)\, dx \right) d\tau,$$

where

$$g(t|x) = \frac{f(t+qx)}{1 - F(qx)}$$

is the conditional probability density function (PDF) and $g(t|0) = f(t)$. When $q = 0$, the above GRP models the process under the perfect repair assumption. When $q = 1$, it models the process under the minimal repair assumption. However, due to the complexity, the closed form solution of the above generalize renewal equation has not been obtained so far. There are many different formalizations of GRP developed by researchers, [17] has been cited circa 300 times after published in the 1980s.

One of the popular generalizations of the renewal process is the geometric process (GP), which is used to model the monotonous stochastic point process with increasing or decreasing times between events/claims [20, 21]. For example, GP can be applied to model times-between-failures of the hardware subsystem whose successive operating times after repair are decreasing due to deterioration [20]. If the times between claims of a product is modelled by a sequence of independent nonnegative random variable $\{X_n, n = 1, 2, \dots\}$, and the distribution of X_n is defined by a CDF $F(a^{n-1}x)$ for $a > 0$; this claim arrival process is said to be a geometric process, and a is the ratio of this GP. When $0 < a \leq 1$, this GP is stochastically increasing; and when $a \geq 1$, it is stochastically decreasing. Denote the CDF and PDF of X_1 as F and f, respectively, and set the expected value of X_1, $E[X_1] = \frac{1}{\lambda}$. Then we have $E[X_n] = \frac{1}{\lambda a^{n-1}}$.

GP is extended to the doubly geometric process (DGP) by [21]. That is, GP becomes a special case of DGP. DGP can model a nonmonotone point process in which the distribution of times to claim can have varying shape parameters. Based on a dataset of warranty claims on network cards, which were collected from an network card manufacturer in the UK, DGP outperforms GP in terms of the Akaike information criterion.

So far, repair is assumed to be instantaneous in above models. However, in the real world, it is not rarely to see that the mean time to failure (MTTF) is not far longer than the mean repair time, and the time spent on repair is not negligible. The repair time can be random, and the uncertainty of this variable should be modeled in these cases as well [22].

2.1.2. The White-box Approach

The white-box approach requires the knowledge of the internal design of a product including the details of subsystems, components, and interactions between them. In this approach, the claims or failures are modeled at the component level based on the understanding of relevant degradation mechanisms, which can be classified into two categories, over-stress and wear-out failures [3].

Over-stress failures may occur with the following physical phenomena, brittle fracture, ductile fracture, yield, buckling, large and elastic deformation, due to the degradation of material strength or/and changing stress over time. The time to failure is the first time instant the strength falls below the stress. Wear-out failures are due to the damage of wear accumulates over time. The accumulated damage can be modeled by a stochastically increasing variable, and the failure occurs when the value of this variable reaches a threshold [3].

In recent years, warranty for complex systems or products with multiple components attract more attentions from researchers. In the daily life, lots of smart products are introduced into the market, these products, at least, have two subsystems, hardware and software. If these products are treated as single-component products or black-box systems, the important internal structure information is ignored, then, the estimation of warranty cost may be inaccurate [23], because the failure of individual components and the interaction between them can influence the reliability of product.

The configuration of internal system structure can determine the influence of component failure on the system reliability. Basically, if the system has a parallel configuration, the product probably does not fail until all components fail. If the configuration is series, one component fails the product fails [23]. In reality, the system configuration can be a combination of parallel and series structures and even have a more complex structure. The interaction between components can influence the system reliability. [24] describes three types of failure interactions: Type I interaction is called induced failure interaction. It assumes that the failure of a component can induce a simultaneous failure of the other components. Type II interaction is called a failure rate interaction, it assumes the failure of a component can change the failure rate of another component. Type III interaction is a combination of Types I and II.

However, a component may consist of multiple components at a lower level. It is not possible to model the details of all structures. To balance the efficiency and accuracy, the individual component at a certain level can be treated as a black-box. Then, the failures of individual component will be modeled in the black-box approach, and the failures of the product will be modeled with considering the system configuration and the interaction between components. For instance, if the system configuration of a product is series, whenever a component fails the product fails. If the failed component will be replaced by a brand new one, the failure process of the component can be modeled by a renewal process. Therefore, the failure process of this product can be modeled by a superimposed renewal process [25].

2.2. The Uncertainty of Warranty Claims on Software

Generally, if the warranty claims are made immediately after the software failed, subsystem the warranty claim arrival process of a software product can be modeled by a stochastic point process like those applied on hardware products. However, the main difference between hardware failure and software failure by nature is the potential faults or causes of failures are introduced into the software during its development process. Once a fault of the software is diagnosed and removed in operation, the relevant errors in codes may be debugged. These activities can result in a growing reliability of the software system. In recent years, to deal with customer's concern of software reliability, manufacturers may provide warranty on the embedded software. Similar to hardware warranty, during the warranty period, the manufacturer provides assurance to the customers that the software will work properly and if any defect is found, the manufacturer may either repair or replace the software without charging the customer. Different from the remedy actions conducted on hardware, the software faults can be repaired by releasing patches or updates online, and these updates can improve the reliability of all software products in the same batch [5].

2.3. Summary

The common interest of warranty management to be analysed and evaluated is the expected total warranty cost over the warranty period as well as the lifecycle of the product. This measure and the uncertainty associated summarize the financial risk or burden carried by manufacturer and even consumer [n].

3. Measuring the Uncertainty

If we come back to the discussion in the introduction section, uncertainty refers to situations under which either the potential outcomes and/or their probabilities of occurrences are unknown. Once the uncertainty is modeled by appropriate mathematic tools, the likelihood of occurrences of all potential outcomes are known, then we can talk about risk and give numeric measures of uncertainty [6].

Risk measures are initially introduced in financial area to meet the requirement of quantifying the losses that may be incurred. The variance of a random variable is a dominating risk measure in financial studies. However, since the variance is a symmetric risk measure, researchers turn to using downside risk measures, such as Value-at-Risk (VaR) and Conditional VaR (Expected Shortfall) to highlight the possible worst loss [27]. Furthermore, if multiple components or products are considered, using variance as a risk measure is normally applied under the assumption that the correlations between the variables of interest are linear. However, this an assumption is not imposed in the VaR and CVaR theories, potentially, VaR and CVaR can be adopted in more scenarios.

4. Conclusion

This chapter provided an introductory review on modelling the uncertainty of warranty claims. It first introduced different types of warranty claims, causes of warranty claims, and then discussed black-box and white-box approaches to modelling times between warranty claims and time to the first claims.

Our future work aims to develop more realistic models to reflect the real-world applications. For example, many articles in www.warrantyweek.com show that warranty claim rates are normally quite low, which implied that performing preventive maintenance policies, which are the main focus of many research papers, is not cost-effective. A challenge is then on the development of cost-effective approaches to reducing the cost on warranty claims.

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