UTILITARIAN SOCIAL WELFARE AND INSURANCE LOSS COVERAGE UNDER RESTRICTED RISK CLASSIFICATION

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Abstract

This thesis considers the effect of restrictions on insurance risk classification on utilitarian social welfare and insurance loss coverage. First, we consider two regimes: full risk classification, where insurers charge the actuarially fair premium for each risk, and pooling, where risk classification is banned and for institutional or regulatory reasons, insurers do not attempt to separate risk classes, but charge a common premium for all risks. For iso-elastic insurance demand, we derive sufficient conditions on higher and lower risks' demand elasticities which ensure that utilitarian social welfare is higher under pooling than under full risk classification. Using the concept of arc elasticity of demand, we extend the results to a form applicable to more general demand functions. Empirical evidence suggests that the required elasticity conditions for social welfare to be increased by a ban may be realistic for some insurance markets.

Next, we consider scenarios where the regulator does not ban risk classification, but instead imposes a price collar, i.e. a limit on the ratio of premiums for high risks relative to those for low risks. Pooling and full risk classification could be considered as limiting cases of a price collar. A regulator imposed price collar would force insurers to use partial risk classification - where some risk-groups might be merged to pay the same premium. We find that for iso-elastic demand, a price collar can give higher loss coverage than either pooling or full risk classification, but only if high and low risks have certain combinations of demand elasticities (both greater than one).

Contents

Chapter 1

Introduction

Restrictions on insurance risk classification are common in life insurance and other personal insurance markets. Examples include the ban on gender classification in the European Union, and restrictions in many countries on insurers' use of genetic test results. The restrictions are motivated by social objectives such as equality and access to insurance. But they also induce "adverse selection", which is usually seen as a bad thing.

Adverse selection arises from the different responses of individuals with high and low probabilities of loss ("high and low risks") to the changes in the insurance prices when restrictions on risk classification are imposed.

An individual's decision to buy insurance depends on her utility function. As attitude towards risk varies across individuals, the utility one draws from buying insurance would also vary. This results in an allocation of individual utilities in a population. In a population of size N, an allocation of utilities can be represented as an N-dimensional vector.

A utilitarian social welfare function essentially ranks all possible allocation of utilities within a population - or simply puts a value against any given N-tuple of utilities. A fair regulator would aim to achieve an allocation of utilities across the population, which would maximise the value of social welfare function.

Because high risks have higher probabilities of loss than low risks, it can be argued that insurance is worth more to the high risks than to the low risks. Stated differently, the utility of insurance is typically higher for high risks than for low risks, where utility is a measure of the subjective benefit or worth of insurance to the individual. So from the perspective of society as a whole, the shift in insurance purchasing from low to high risks under adverse selection may result in a better allocation of utilities, hence higher social welfare.

A full appraisal of the outcome depends on the form of individual utility functions; and also on any weighting scheme we use for the gains in utility by high risks under restricted risk classification versus losses in utility by low risks, i.e. it depends on the distributional preferences of low and high risks, captured in the social welfare function

Consider a market with two risk-groups, where risk classification is allowed. In this market, high-risk individuals would pay higher premiums than the low-risk individuals. Now, consider the effect of banning risk classification on this market. In this scenario, insurers must charge all risk-groups the same premium - i.e., a pooled premium. If the market is competitive, this pooled premium would lie somewhere between the risk-differentiated premiums of two risk-groups. Hence it would become cheaper for high risk individuals and dearer for low risk individuals to buy insurance. So, it would induce high risk-groups to buy more and low risk-groups to buy less insurance. As a result there would be a transfer of utilities from low to high risk-groups. But insurers will be subject to adverse selection.

Adverse selection, induced by restricting risk classification is usually perceived by economists as having negative effects on efficiency. In some extreme cases, participation of low risk-groups in the market may fall to such an extent, that the insurance market will cease to exist. However in real world such market breakdowns are extremely rare.

But because restrictions also make high-risk individuals better off and low-risk individuals worse off, they also have equity (distributional) effects. Therefore depending on distributional preferences expressed in the social welfare function, restrictions might either increase or decrease social welfare. Restricting risk classification - which induces a degree of adverse selection for the insurer as they have to take on more high risk individuals as customers - can on the other hand, be desirable from social welfare perspective.

The social welfare function used in the current thesis assumes cardinal and interpersonally comparable utilities. In other words, it assumes that utility (or expected utility) for society as a whole – utilitarian social welfare – can be measured as a weighted sum (or expectation) of individual utilities over the whole population. Our weighting scheme for high and low risks assigns equal weights to the utilities of all individuals. This equal-weights approach is based on the Harsanyi (1955) 'veil of ignorance' argument: that is, behind the (hypothetical) veil of ignorance, where one does not know what position in society (e.g. higher risk or lower risk) one occupies, the appropriate probability to assign to being any individual is $1/N$, where N is the number of individuals in society. Alternative risk classification regimes can then be compared by comparing expected utility in each regime for the (hypothetical) individual utility-maximiser behind the veil of ignorance.

We use this approach in defining our model and the measures of social welfare and insurance loss coverage. We assume that insurers compete only on price; for institutional or regulatory reasons, they do not offer partial cover, nor menus of contracts offering different levels of cover priced at different rates.

Under the pooling regime, it is intuitive that the equilibrium price – the pooled price at which insurers break even – will depend on demand elasticities of lower and higher risk-groups. Another intuition is that pooling implies a redistribution of utilities from lower risks towards higher risks. The welfare outcome will depend on how we evaluate the trade-off between the gains and losses of the two types. This research connects and builds on these intuitions, by establishing sufficient conditions on demand elasticities to ensure higher social welfare under pooling compared with full risk classification. The conditions encompass many plausible combinations of higher and lower risks' demand elasticities.

The typically higher utility of insurance to high risks broadly reflects their higher probability of loss. But utility also incorporates a subjective element of individual risk preference. Even individuals with the same probabilities of loss can have different subjective attitudes to risk (risk-averse or risk-loving), and so derive substantially different utilities from insurance. This element of individual risk preference means that the utilitarian concept of social welfare, whilst widely used in economics, is not directly observable by insurance regulators.

Thomas (2008) suggested "loss coverage" as a simpler policy metric for evaluating risk classification schemes. Loss coverage is defined as the expected losses compensated by insurance for the population as a whole. It depends only on probabilities of loss, not individual risk preferences, and so is more directly observable than social welfare.

A comparison between full risk classification and pooling regimes, in the context of loss coverage was done by Hao et al. (2018). Hao et al. (2019) also showed that if all risk groups have identical demand elasticities, and follows an iso-elastic demand function, then loss coverage can be used as a proxy measure for social welfare, because ranking of risk classification schemes by loss coverage and by social welfare would be identical. However this relationship between loss coverage and social welfare is not strictly true when demand elasticities vary across risk-groups or demand functions take more general forms, other than iso-elastic. As utilitarian social welfare is a more widely targeted objective in economic policy decisions, this motivates the investigations presented in this thesis.

The present thesis makes two main contributions.

First, we provide a direct evaluation of social welfare under the polar risk classification regimes of full risk classification and pooling. For iso-elastic insurance demand, we derive sufficient conditions on higher and lower risks' demand elasticities which ensure that utilitarian social welfare is higher under pooling than under full risk classification. Using the concept of arc elasticity of demand, we extend the results to a form applicable to more general demand functions.

Second, we investigate an intermediate regime in between the polar cases, where the regulator does not ban risk classification but instead imposes a price collar, i.e. a limit on the ratio of premiums for high risks relative to those for low risks. Pooling and full risk classification could be considered as limiting cases of a price collar. To obtain analytical results, it was necessary to use loss coverage, rather than social welfare, as our policy metric in this part of the thesis. For iso-elastic insurance demand, we derive sufficient conditions on higher and lower risks' demand elasticities which ensure that loss coverage is higher under a price collar regime than under either pooling or full risk classification.

The thesis is structured as follows:

Chapter 2 surveys previous literature and establishes the general rationale behind our model and highlights its similarities and deviations from the existing literature under three main headings: equilibrium and adverse selection, randomness of risk preference, and the measure of social welfare. A more detailed discussion of specific relevant literature is given in Appendix J

Chapter 3 gives the detailed set-up of our model, including all assumptions, and the formal definitions of social welfare and loss coverage.

Chapter 4 compares social welfare under the polar risk classification regimes: full risk classification and pooling. We first consider iso-elastic demand functions, with all risk-groups having similar demand elasticities. Next, we extend our approach to consider varying demand elasticities across risk-groups. Finally we derive results for general demand functions.

Chapter 5 compares loss coverage under a price collar, pooling and full risk classification. We show that for iso-elastic demands, a price collar can give higher loss coverage than either pooling or full risk classification, but only if high and low risks have certain combinations of demand elasticities (both greater than one).

Chapter 6 summarises our conclusions.

To facilitate the flow of discussion in this thesis, we have moved mathematical derivations and proofs of theorems in the appendices. However, mathematical derivations and proofs of theorems are integral parts of this thesis.

The following papers, Chatterjee et al. (2021) and Chatterjee et al. (2023), are based on the research presented in this thesis.

Chapter 2

Literature Review

The effect of risk classification on insurance markets is a widely researched topic in economics and actuarial science. Risk classification has long been preferred by the insurance industry as a tool for reducing information asymmetry, which is presumed to cause potential market failure via adverse selection.

In economics, the two canonical models for studying information asymmetry and its effect on insurance markets were developed by Akerlof (1970) and Rothschild and Stiglitz (1976). Both describe how information asymmetry can lead to a reduction in social welfare, or even complete market collapse. The assumed nature of insurance contracting and competition is different in each model, and hence the nature of the inefficiency arising from information asymmetry is also different (Hendren (2014)). In Akerlof (1970), all insurance contracts are for the same quantity of cover, and hence insurers compete only on price; inefficiency arises because lower risks are unwilling to pool with higher risks. In Rothschild and Stiglitz (1976), insurers compete on both price and quantity of cover, with different contracts having different price-quantity combinations designed to appeal only to lower or higher risks; inefficiency arises because lower risks can buy only partial cover at their actuarially fair price (rather than full cover, which they would prefer if it were offered).

However, much policy-oriented commentary suggests that in the real world, it is higher risks, not lower risks, that experience difficulty in obtaining insurance cover, or are priced out of the market (e.g. European Commission (2010), Joly et al. (2010), Prince (2019)). Thereby the purpose of insurance as a tool to protect against loss is impaired, as the individuals most likely to suffer a loss are not protected any more [1]. In the present thesis, we attempt to make a positive case for restricting risk classification, from the perspective of a policymaker or regulator who wishes to maximise utilitarian social welfare. Insurers in our model compete on price and only offer contracts with full coverage. Therefore our setup is closer to Akerlof than to the Rothschild Stiglitz model. We look at conditions where a full or partial ban on risk classification could make insurance work better and deliver higher social welfare.

The closest precedent to the present thesis is Hoy (2006), which shows that when potential losses are fixed and the fraction of high-risks in the population is sufficiently small, then a ban on risk classification will increase utilitarian welfare. Polborn et al. (2006) obtain a similar result in a dynamic model of life insurance, where the quantum of insurance which an individual can purchase is not fixed, but is subject to a cap.[2] Another strand of literature (e.g. Crocker and Snow (1986), Rothschild (2011)) argues that contract-specific taxes or partial social insurance are a Pareto-superior means to implement any welfare improvements achieved by a ban. Notwithstanding these arguments, bans remain of interest because for reasons of political feasibility or administrative convenience, they are invariably the preferred means in practice.

[1] This perspective has recently been quantified through the construct of "loss coverage" (Thomas (2008), Thomas (2017)). The general idea has earlier precedents: Gruber (2019) refers to a parable by Kenneth Arrow to show how more public information about individual risk could lead to less pooling of risk, and hence an overall welfare loss for society; the same point was also made by Hirshleifer (1971)

[2] 'Dynamic model' here denotes an initial period in which the individual is uninformed about her risk level and insurance needs, then a second period where she receives information about both, and finally a third period when she is exposed to risk; she may buy insurance in either the first or second periods.

To make our case in this thesis, we first establish that a stable market equilibrium could be reached in a regime with restricted risk classification, i.e. with a pooled premium. We consider a competitive insurance market where insurers would compete on premium for identical coverage. We also introduce randomness of choice across customers by introducing a probability distribution of individual risk aversion. This is a departure from the classical models of Akerlof (1970) and Rothschild and Stiglitz (1976), which do not consider variations in attitude towards risk. The Rothschild-Stiglitz model also concludes that a regime with pooled premium could not deliver a stable equilibrium, which differs from our findings.

Modelling the attitude towards risk is already a growing area of interest in economics. However, results from laboratory based methods of estimating risk preference are often inconsistent with real life decisions based on risk (see Charness et al. (2020)). Therefore estimating risk preference from empirical data obtained from market as opposed to laboratory based experiments, remains the preferred approach for economists. Barseghyan et al. (2018) reviewed an array of risk preference models using field data. Our model follows a similar approach as we assume an underlying risk preference model which would reveal itself via an observable characteristic of the market, viz. demand elasticity.

Finally, we consider an acceptable measure of social welfare. As a consequence of introducing a distribution of risk preference, in our model individuals would have a distribution of utility functions. A potential drawback of such a model is that the existence of a handful of individuals who derive unusually high utility for any given level of wealth could distort the measure for social welfare. To overcome this issue, we normalise the absolute value of derived utility to a scale of [0,1] and use the expected utility of an individual as the measure of social welfare. The approach is equivalent to relative utilitarianism, proposed by Dhillon and Mertens (1999) as a measure for a social welfare function.

In the next three sections we establish the rationale behind our model, its similarities and deviations from the existing literature on insurance risk classification in terms of:

- equilibrium and adverse selection;
- randomness of risk preference; and
- measure of social welfare.

A more detailed discussion on specific relevant literature is given in appendix J.

2.1 Equilibrium and Adverse Selection

In one of the most influential and cited works on insurance market, Rothschild and Stiglitz (1976) argued that an equilibrium would not exist under pooled premium. The authors developed a simple 2 risk-group model to show that a stable equilibrium cannot include a contract offering pooled premium for different risk-groups. The authors argue that it incentivises insurers to deviate from a pooled premium contract, hence a pooled premium contract can never be part of a stable equilibrium. A similar result would follow, even where the model is modified to include a continuous distribution of risk exposure.

The Rothschild-Stiglitz model assumes all individuals to be risk averse. It also assumes that every individual possesses full knowledge of her own risk exposure and behaves accordingly. Due to risk aversion, every individual would be willing to pay more than their fair premium. However, in reality attitude towards risk varies across individuals. In recent literature on estimating risk preferences, probability distortion models are often used to explain the consumer behaviour, (see Barseghyan et al. (2013)). This distortion could result from the individual's attitude towards risk or her perception about her own risk exposure. Whatever be the underlying reason, faced with the decision of buying insurance, in practice, an individual does not therefore always behave the way Rothschild and Stiglitz (1976) predicted. In our model, we assume individuals to be aware of their risk exposure. But we introduce heterogeneity in terms of attitude towards risk across individuals where everyone does not necessarily have to be risk averse. This leads to different, but realistic results.

In addition, the Rothschild-Stiglitz model examines the stability of the market equilibrium in a single period where each individual would buy insurance based on their private knowledge of risk exposure. It does not consider the scenario where the insurer can withdraw loss-making contracts from the market. For the details of the Rothschild-Stiglitz equilibrium concept please see Appendix 2.1. The argument, that a pooled premium contract cannot be part of a stable equilibrium, could therefore be re-evaluated under a different concept of equilibrium.

By using the concept of dynamic equilibrium over multiple periods Wilson (1976), argued that in all scenarios the market will iteratively reach a stable equilibrium. If in period n the equilibrium contract is a pooled contract, one insurer could potentially benefit by offering a new contract targeted only at low risks in period $n+1$; but it will not do so if it anticipates that the response of other insurers will be to withdraw the pooling contract, thus nullifying the advantage the one insurer gains by offering the new contract. Therefore we can also say that in long run the market would achieve a stable equilibrium where insurers would earn zero expected profit. This characterizes an insurance market with perfect competition with no entry or exit barrier for insurers.

Finally, the Rothschild and Stiglitz (1976) paper does not make any detailed observation on social welfare. Hoy (2006) used the Rothschild-Stiglitz model to conclude that if the high-risk population proportion is lower than a threshold where separating equilibrium is not achievable, it might be possible to deliver higher welfare by pooling the risks and let market achieve Wilson equilibrium. For low enough proportion of high-risk population this is certain to deliver higher social welfare (see Appendix J.4). In the present thesis we derive conditions where a regulator-induced pooled premium could unambiguously (i.e. irrespective of the proportion of high-risk individuals in the population) deliver a higher social welfare.

2.2 Randomness of Risk Preference

A principal departure of this thesis from all those just cited above is that rather than assuming all individuals have the same utility function, we assume a distribution of utility functions (not necessarily all risk-averse) across individuals who have the same probabilities of loss. The probability distribution of utility function captures the underlying distribution of risk preference across the individuals.

This assumption leads to qualitatively different results from simpler models, through two mechanisms. First, utility functions determine individuals' insurance purchasing decisions, which determine the insurance demand curve and hence the equilibrium price of insurance when all risks are pooled. Second, utility functions determine the expected utilities which individuals assign to their outcomes *given* an insurance price.

In the recent past, there has been a focus on estimating risk preferences of individuals based on field data. An extensive survey of literature on this topic is available in Barseghyan et al. (2018). To estimate risk preference from field data, researchers often looked at the demand curve for a product, such as insurance, whose outcome is random, against the price of the product. Each individual's demand for product would depend on her expected utility, which in turn depends on her risk preference, at a given price. Hence the demand function can be inverted to reveal the probability distribution of risk preference. We followed a similar approach in our study. The probability density function we have used for risk preference translates into an iso-elastic demand function, which has been generalised subsequently to any form of demand function.

Similar to the concept of "willingness to pay" in Einav and Finkelstein (2011), where the consumer pays a risk premium in addition to the fair premium, in our model an individual could pay higher than her fair premium. Einav and Finkelstein (2011) model is based on a framework similar to Akerlof, where insurers only offer full coverage and compete on price. Though the authors recognize variation in risk preference across individuals, they did not treat risk preference as a random variable independent of risk profile. In Appendix J.4 Figure J.5 (and J.6), reproduced from Einav and Finkelstein (2011), describes a scenario where risk aversion only decreases (or increases) with risk exposure. The present thesis goes beyond this scenario by allowing for random variation in risk preferences across individuals. We show that for some plausible parameterisation of risk preferences, a regulator can maximise social welfare by banning risk classification.

Risk preferences of individuals cannot be directly observed and therefore we need a "proxy measure" for risk preference. Collective risk preference for a given group of people with similar risk exposure, could be revealed by the group's aggregate demand against varying price. Therefore, in our model, risk preference is revealed by price sensitivity. An individual's price-sensitivity can be caused by external factors, e.g. her financial condition. But through her response to price changes, an individual would reveal her attitude towards risk. Price sensitivity is measured by demand elasticity and is often observable from the empirical studies of the insurance market. Therefore, it can be useful for regulators for prescribing policies.

2.3 Measure of Social Welfare

A social welfare function translates individual choices to a social choice representing the aggregate of individuals. In the words of Arrow (see Sen (2017), p. 271) it would map "the vector of utilities of individuals into a [collective] utility". Arrow (1963) defined a social welfare function as a functional relation specifying one social ordering R for any n-tuple of individual orderings R_i for each person, i.e. $R = f(R_i)$.

The individual orderings could be derived from a personal utility function. However with the utility function that assigns a cardinal value of utility to an individual, we face a problem of interpersonal comparability. Even if the individual utilities are normalised to a scale of [0,1], it can be shown to break the axiom for the Independence of Irrelevant

Alternative (IIA) (see Sen (2017), p. 144). IIA states that as long as individual preferences remain the same over a subset of social states, the social choice from that particular subset should also remain the same. IIA plays a central role in defining an Arrow social welfare function, as Arrow concentrated on understanding the voting paradox. But as his famous Impossibility Theorem shows, no rank-ordering social welfare function could satisfy all the axioms simultaneously.

However, by weakening IIA and defining an alternative set of axioms, it is possible to construct an alternative social welfare function. In "Relative Utilitarianism", Dhillon and Mertens (1999), presented such an alternative axiomatic framework, under which a sum of normalised individual von-Neumann-Morgenstern utility functions satisfy the conditions of a good social welfare function. It assumes that individuals with identical risk preferences and risk profile would derive same utility out of insurance at a given price. Hence inter-personal utilities are comparable. Normalising individual utilities to the range [0,1] also ensures that the social welfare measure cannot be influenced by a so-called "utility monster", i.e. an individual who derives more utility than all other individuals combined (see Bailey (1997), Nozick (1974)).

In the present thesis, we follow an approach similar to Dhillon and Mertens (1999) in assigning utility to individuals. Our measure of social welfare is expected utility *given* the distributions of loss probabilities and risk preferences in society, but evaluated behind a hypothetical veil of ignorance which screens off knowledge of the decision maker's own loss probability and preferences. In our model, individuals know their own risk exposure. Loss probabilities of different risk-groups, as well as the distribution of risk preferences across the population are also common knowledge.

This thesis is also related to Hao et al. (2018) which proposes 'loss coverage', defined as expected losses compensated by insurance for the whole population, as a criterion for comparing risk classification schemes. Loss coverage has the advantage that it depends on observable quantities, whereas utilitarian social welfare depends on unobservable utility functions.

Hao et al. (2019) showed that for iso-elastic insurance demand where the elasticity is same for all risk-groups, loss coverage can be used as a proxy measure for social welfare, as it always gives the same ranking of different risk classification schemes. But for other demand specifications, the 'common ranking' property of loss coverage and social welfare may not hold. The present thesis therefore focuses on direct evaluation of utilitarian social welfare, and derives sufficient conditions on demand elasticity for social welfare to be higher under pooling than under full risk classification.

Chapter 3

Model Framework

In this chapter, we develop a framework to evaluate utilitarian social welfare under different risk classification regimes. In Section 3.1, starting from individual insurance purchasing decisions, we develop insurance demand for a single risk-group as a function of premium. In Section 3.2, demand from different risk-groups constitutes an insurance market, where perfect competition yields different equilibria under different risk classification regimes. In Section 3.3, we formulate utilitarian social welfare for a given market equilibrium. Finally, in Section 3.4, we formally define loss coverage.

3.1 Insurance Demand for a Single Risk-Group

Typical theories of insurance demand assume that all individuals know their own probabilities of loss and have a common utility function. Given an offered premium, individuals with the same probabilities of loss then all make the same purchasing decision. This does not correspond well to the observable reality of many insurance markets, where individuals who appear to have similar probabilities of loss often make different decisions, and substantial fractions of the population do not purchase insurance at all.[1] This section gives a theory of insurance demand which accommodates the possibility that not all individuals with the same probabilities of loss make the same decision. Key assumptions which distinguish our model from other common models are highlighted at the points where the need for each assumption arises.

First, we consider demand from the perspective of a single individual. Suppose that an individual has wealth W and risks losing an amount L. The individual is offered insurance against the potential loss amount L at premium π (per unit of loss), i.e. for a payment of πL.

Assumption 1 (Non-satiation). *The individual's utility function $u(w)$, is increasing as a function of wealth, w, and differentiable, so that $u'(w) > 0$. The individual knows his own utility function.*

Note that in Assumption 1, no restriction is placed on the second derivative $u''(w)$, which may have either sign; we do not require that all individuals are risk-averse (i.e. $u''(w) < 0$). We will show later that this departure from typical models generates the partial take-up of insurance in our demand function.

Assumption 2 (Full-cover contracts). *Insurance is offered in a full-cover contract which is standardised across all insurers, who compete only on price. Insurers do not offer partial cover or other contract menus.*

We justify Assumption 2 by noting that separation via contract menus is not possible in some important markets, such as life insurance, which have non-exclusive contracting.

[1] For example, in life insurance, the Life Insurance Market Research Association (LIMRA) states that 57% of US households have some individual life insurance (LIMRA (2019)). The American Council of Life Insurers states that 138m individual policies were in force in 2018 (American Council of Life Insurers (2019, p66)); the US adult population (aged 18 years and over) at 1 July 2019 as estimated by the US Census Bureau was 255m.

It is also often not salient to practitioners in other markets where restrictions on risk classification apply.[2]

If the probability of loss is μ, the individual will choose to buy insurance if:

$$u(W - \pi L) > (1 - \mu)\, u(W) + \mu\, u(W - L). \tag{3.1.1}$$

Since certainty-equivalent decisions do not depend on the origin and scale of a utility function, it is convenient to define a normalised utility function as follows:

$$u_s(w) = \frac{u(w) - u(W - L)}{u(W) - u(W - L)}, \quad \text{for } (W - L) \leq w \leq W. \tag{3.1.2}$$

This normalisation ensures that $u_s(W - L) = 0$ and $u_s(W) = 1$, so that for all individuals, the normalised utilities at the 'end-points' are the same. It also preserves the curvatures of utility functions, and hence individual risk preferences and insurance purchasing behaviour remain unchanged. For now, the normalisation is just a matter of convenience, but we shall later state it as an assumption in Section 3.3, where it will be needed for our measure of social welfare.

Applying this normalisation (Equation 3.1.2) to Equation 3.1.1, the criterion becomes:

$$u_s(W - \pi L) > (1 - \mu). \tag{3.1.3}$$

From this point onwards, we use 'utility' to mean the normalised utility, $u_s(w)$, unless the context requires otherwise.

Next, we consider demand from the perspective of an insurer. The insurer observes a group of individuals comprising a *risk-group*, who all have the same probability of loss.

[2] Economists often postulate that insurers use menus of deductibles or other contract features as screening devices to separate high and low risks (e.g. Rothschild and Stiglitz (1976)). But most actuarial pricing textbooks make no reference to this concept (e.g. Gray and Pitts (2012), Friedland (2013), Parodi (2014)), and instead interpret deductibles as a device to limit moral hazard and the administrative costs of handling small claims.

The insurer knows the common probability of loss μ for all members of the risk-group. The individuals are, however, heterogeneous in terms of their utility functions, which the insurer cannot observe.

Assumption 3 (Heterogeneous utility functions). *Utility functions are heterogeneous across individuals, and unobservable by insurers.*

Hence for any risk-group, the insurer observes μ, π and possibly each individual's W and L, but not their utility functions. So from the insurer's perspective, given a premium π, the utility of insurance of an individual chosen at random from this risk-group, $u_s(W - \pi L)$, is unobservable and we denote it by the random variable: U_I (the subscript I indicates *insurance*), which depends on W, L and π.

So the insurer can at most observe the proportion of individuals who choose to buy insurance at a given premium π. We call this a (proportional) demand function and define it as:

$$d(\pi) = \mathrm{P}\left[\, U_I > (1-\mu)\,\right]. \tag{3.1.4}$$

Clearly, $0 \leq d(\pi) \leq 1$ and $d(\pi)$ is non-increasing in π (for a given value of μ) as increasing π decreases the utility of insurance for all individuals.

Assuming $d(\pi)$ to be differentiable, the (point price) elasticity of insurance demand, is defined as:

$$\epsilon(\pi) = -\frac{\partial \log d(\pi)}{\partial \log \pi} \tag{3.1.5}$$

which implies that demand can also be expressed as

$$d(\pi) = \tau \exp\left[-\int_{\mu}^{\pi} \epsilon(s)\, d\log s\right] \tag{3.1.6}$$

where $\tau = d(\mu)$ is the *fair-premium demand* for insurance.

3.2 Insurance Market Equilibrium with n Risk-Groups

Suppose a population consists of n distinct risk-groups with probabilities of loss given by $\mu_1, \mu_2, \ldots, \mu_n$. For convenience, we assume $0 < \mu_1 < \mu_2 < \ldots < \mu_n < 1$. Let the proportion of the population belonging to risk-group i be p_i, for $i = 1, 2, \ldots, n$.

Now let the occurrence of a loss event for an individual chosen at random from the whole population be represented by the indicator random variable, X, taking the value of 1 if a loss event occurs; and 0 otherwise. Then X, conditional on risk-group i, is a Bernoulli random variable with parameter μ_i.

Suppose insurers charge premiums (per unit of loss) $\pi_1, \pi_2, \ldots, \pi_n$ for the risk-groups $i = 1, 2, \ldots, n$, respectively. For brevity, we use the notation $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ to denote the premium regime under consideration. Define Π to be the premium which would be chargeable to an individual chosen at random from the population, if that individual purchased insurance. Then Π, conditional on risk-group i, takes the value π_i.

From insurers' perspective, the insurance purchasing decision of an individual chosen at random from the whole population can be represented by the indicator random variable Q, taking the value of 1 if insurance is purchased; and 0 otherwise. Then Q, conditional on risk-group i, is a Bernoulli random variable with parameter $d_i(\pi_i)$, where $d_i(\pi_i)$ is the demand for insurance within risk-group i at premium π_i (based on the model developed in Section 3.1). Then for an individual chosen at random from the population, the expected premium income is $\mathbb{E}[Q\Pi L]$ and the expected insurance claim is $\mathbb{E}[QXL]$.

We then need an assumption about the nature of insurance market competition and equilibrium, which we state as follows.

Assumption 4 (Competitive equilibrium). *Risk-neutral insurers have a common technology to classify diversifiable risks, with zero transaction costs. Competition between insurers leads to zero expected profits in equilibrium.*

Assumption 4 implies the following *equilibrium condition* under the premium regime $\underline{\pi}$, where $E(\underline{\pi})$ is the expected profit:

$$E(\underline{\pi}) = \mathbb{E}\,[\,Q\Pi L\,] - \mathbb{E}\,[\,QXL\,] = 0. \tag{3.2.1}$$

3.3 Definition of Social Welfare

We define social welfare, $S(\underline{\pi})$, for a particular premium regime $\underline{\pi}$, as the expected utility of an individual selected at random from the entire population, i.e.:

$$S(\underline{\pi}) = \mathbb{E}\,[\,Q\,U_I + (1-Q)\,[\,(1-X)\,U_W + X\,U_{W-L}\,]\,]\,, \tag{3.3.1}$$

where U_W and U_{W-L} are random variables denoting the utilities at individuals' initial wealth, W, and at their wealth after loss event, $(W-L)$, respectively. In Equation 3.3.1, the 'Q' term is the random utility if insurance is purchased, and the '$(1-Q)$' term is the random utility if insurance is not purchased.

In Section 3.1 we noted that certainty-equivalent decisions do not depend on the origins and scales of utility functions, and therefore the insurance decision for all individuals could be framed using normalised utility functions, irrespective of their different individual non-normalised utility functions. This was not a model requirement, but just a convenient normalisation.

However, this argument cannot be directly extended to Equation 3.3.1, because the utilitarian concept of social welfare does depend on the actual magnitudes of individuals' utilities at different levels of wealth. But without any normalisation, Equation 3.3.1 is susceptible to being dominated by a 'utility monster' who derives more utility from a given level of wealth than all other individuals combined (see Bailey (1997), Nozick (1974)). This makes it unsuitable for policy purposes. So in our measure of social welfare, we use the normalised utilities in Equation 3.1.2, as stated in the following assumption.

Assumption 5 (**Relative utilitarianism**). *Social welfare is expected normalised utility for an individual selected at random from the population. The normalisation uses $u_s(W) = 1$ and $u_s(W - L) = 0$, while preserving the shape of individual risk preferences at intermediate amounts of wealth.*

This "expectation of 0–1 normalised utilities" definition of social welfare can also be justified as the unique solution of (a slightly weakened version of) the Arrow (1963) axioms for a social welfare function (as shown in Dhillon and Mertens (1999), who call our approach "relative utilitarianism").[3]

Using Assumption 5, Equation 3.3.1 simplifies to:

$$S(\underline{\pi}) = \mathbb{E}\left[\, Q\, U_I + (1 - Q)\,(1 - X)\,\right]. \tag{3.3.2}$$

For many insurances, insurance premiums are typically relatively small compared to an individual's wealth.[4] We assume that the premium πL is 'small' in the following technical sense.

Assumption 6 (**Small premiums**). *All individuals' utility functions are such that for small premium amounts πL (compared to initial wealth W), the second and higher-order terms in the Taylor series of expansion of $u_s(W - \pi L)$ can be ignored as negligible.*

It is important to highlight here that we are not suggesting that the curvatures of individuals' utility functions are unimportant in general. Assumption 6 only requires

[3]There is nothing sacrosanct about this particular normalisation, but it has been used many times in the economics literature (for some recent examples see Segal (2000), Sobel (2001), Pivato (2008)), and seems well suited to the insurance context.

[4]There are some notable exceptions, such as health or life insurance at higher ages, or life insurance with a savings element, and our analysis will not apply in these cases.

that for small premium amounts πL, the utility function $u_s(w)$ over the short interval $(W - \pi L, W)$ can be approximated by a straight line.

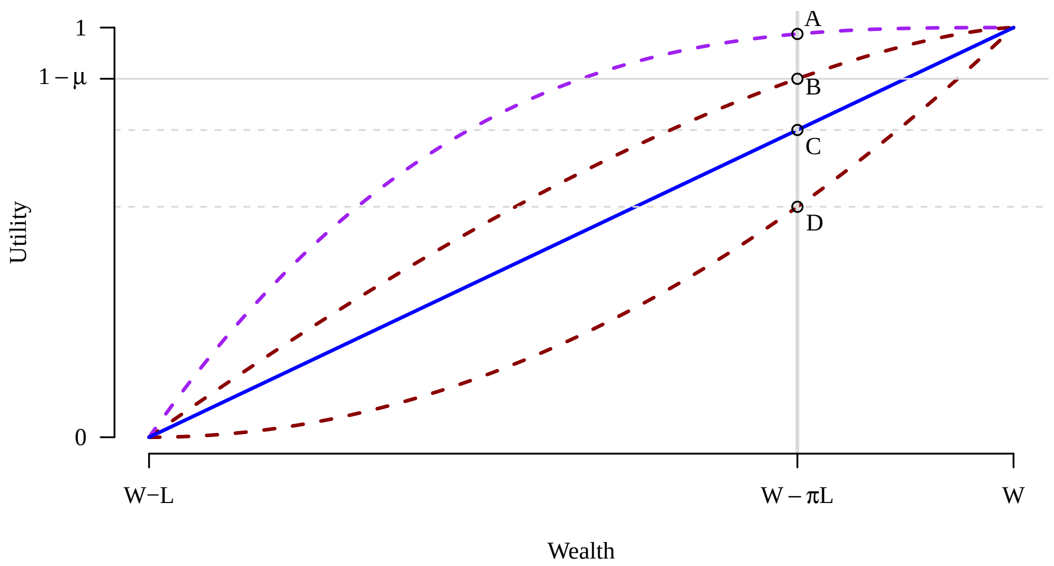


Figure 3.1: Intuition for $\gamma = L\,u_s'(W)$ as an index of risk preferences .

To illustrate the effect of Assumption 6, Figure 3.1 shows normalised utility functions over the range $(W-L, W)$ for four hypothetical individuals with different risk preferences. The straight diagonal line from $u_s(W-L)$ to $u_s(W)$ through point C represents a risk-neutral individual. The concave curves through points A and B each represent risk-averse individuals and the convex curve through point D represents a risk-loving individual.[5] The role of utility functions' slopes and curvatures, over the range $(W-L, W-\pi L)$ to portray individual risk preferences, is evident in the four distinctive curves and also in the relative differences in the values of $u_s(W-\pi L)$. Assumption 6 says that, for small πL, each

[5]Although 'risk-loving' or 'risk-seeking' are the usual stylised descriptions, it might be more appropriate to characterise this phenomenon as 'risk-neglecting'.

individual's utility curve over the short interval $(W - \pi L, W)$ can be approximated by a straight line.

From Equation 3.1.3, an individual's decision rule for purchasing insurance is:

$$u_s(W - \pi L) > (1 - \mu). \tag{3.3.3}$$

Using Assumption 6, the left-hand side of Equation 3.3.3 can be evaluated as:

$$u_s(W - \pi L) \approx u_s(W) - \pi L u_s'(W) = 1 - \pi L u_s'(W), \quad \text{as } u_s(W) = 1. \tag{3.3.4}$$

Now, if we define $\gamma = L u_s'(W)$, then the approximation in Equation 3.3.4 becomes:

$$u_s(W - \pi L) \approx 1 - \pi\gamma. \tag{3.3.5}$$

Then for a given individual, the decision rule in Equation 3.3.3 can be written as:

$$\gamma < \frac{\mu}{\pi}. \tag{3.3.6}$$

The quantity $\gamma = L u_s'(W)$ can be interpreted as a *risk preferences index*, in the sense illustrated in Figure 3.1. The straight diagonal line, representing a risk-neutral individual, has a slope of $1/L$, giving the index $\gamma = L u_s'(W) = 1$. The concave curves through points A and B representing risk-averse individuals have lower slopes $u_s'(W)$ than for the straight diagonal line, and hence the index $\gamma = Lu_s'(W) < 1$ for risk-averse individuals. For the convex curve through point D, representing a risk-loving individual, an analogous geometric intuition confirms $\gamma = L u_s'(W) > 1$. Provided that Assumption 6 holds, the index $\gamma = L u_s'(W)$ is then sufficient to characterise an individual's risk preferences at wealth $(W - \pi L)$.

As an example, consider the special case of power utility function $u_s(w) = w^\gamma$, with

$W = L = 1$. The parameter γ fully characterises an individual's risk preferences. For this particular example, Assumption 6 implies that for small premium π:

$$u_s(1-\pi) = (1-\pi)^\gamma \approx 1 - \pi\,\gamma, \quad \text{as } u_s(1) = 1 \text{ and } u_s'(1) = \gamma. \tag{3.3.7}$$

And for this specific power utility example, the decision rule then becomes:

$$u_s(1-\pi) > (1-\mu) \Leftrightarrow (1-\pi\,\gamma) > (1-\mu) \Leftrightarrow \gamma < \frac{\mu}{\pi}, \tag{3.3.8}$$

reproducing the same general decision rule as obtained in Equation 3.3.6.

Note that in accordance with the decision rule in Equation 3.3.3, insurance is purchased if $u_s(W - \pi\,L) > (1-\mu)$: so in this illustration, A purchases, B is indifferent, and C and D do not purchase. The variation across individuals in utility functions drives the partial take-up of insurance (i.e. $d(\pi) < 1$) in our model.

Since insurers cannot observe individuals' utility functions (Assumption 3), γ is not observable and appears to be sampled randomly from some underlying random variable Γ with distribution function $F_\Gamma(\gamma)$. Following on from Equation 3.3.6, the (proportional) insurance demand function in Equation 3.1.4 can be expressed as:

$$d(\pi) = \mathrm{P}\left[\,U_I > (1-\mu)\,\right] \approx \mathrm{P}\left[\,\Gamma < \frac{\mu}{\pi}\,\right]. \tag{3.3.9}$$

By applying Taylor series approximation as in Equation 3.3.5, the utility of a random individual, who is insured (identified by the indicator variable Q), is $(1 - \Pi\,\Gamma)$, where Π is the random variable representing the premium charged for this individual.

Therefore the Equation 3.3.2 can now be approximated by:

$$S(\underline{\pi}) \approx \mathbb{E}\left[\,Q\,(1-\Pi\,\Gamma) + (1-Q)\,(1-X)\,\right], \tag{3.3.10}$$

$$= \mathbb{E}\left[\,Q\,(X - \Pi\,\Gamma)\,\right] + K, \tag{3.3.11}$$

where $K = \mathbb{E}[1 - X]$ does not depend on the premium regime under consideration.

The development to this point accommodates the possibility that potential loss amounts L can vary across individuals. But to obviate the need to model this variation in this thesis, we make our next assumption:

Assumption 7 (**Fixed potential loss amount**). *For all individuals, the potential loss amount L is the same constant.*

Under this assumption, the equilibrium condition $E(\underline{\pi}) = 0$ from Equation 3.2.1 simplifies to:

$$\mathbb{E}\,[\,Q\Pi\,] - \mathbb{E}\,[\,QX\,] = 0. \tag{3.3.12}$$

To progress to a parameterised version of Equation 3.3.12, we need to assume that there is no moral hazard. Technically:

Assumption 8 (**No moral hazard**). *Conditional on a given risk-group, Q and X are independent.*

Given this assumption, conditioning over the different risk-groups and then taking conditional expectation, the equilibrium condition in Equation 3.3.12 yields:

$$\mathbb{E}\,[\,Q\,\Pi - Q\,X\,] = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} \mathrm{P}[\text{Risk-group } i]\ [\mathbb{E}\,[\,Q\,\Pi \mid \text{Risk-group } i] - \mathbb{E}\,[\,Q\,X \ \mid \text{Risk-group } i\,]] = 0 \tag{3.3.13}$$

$$\Leftrightarrow \sum_{i=1}^{n} p_i\ [\pi_i \mathbb{E}\,[\,Q \mid \text{Risk-group } i] - \mathbb{E}\,[\,Q \mid \text{Risk-group } i\,]\,\mathbb{E}\,[\,X \ \mid \text{Risk-group } i\,]] = 0 \tag{3.3.14}$$

(as $\Pi = \pi_i$ for risk-group i; and Q and X are independent given a risk-group),

$$\Leftrightarrow \sum_{i=1}^{n} p_i\, d_i(\pi_i)\ (\pi_i - \mu_i) = 0, \tag{3.3.15}$$

as given a risk-group i, Q and X are Bernoulli random variables with parameters $d_i(\pi_i)$ and μ_i respectively. Equation 3.3.15 is intuitively appealing as it can be interpreted as the demand-weighted average profits generated by different risk-groups.

By inspection, $\underline{\pi} = \underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ is a solution to Equation 3.3.15, and we will refer to this as the *full risk classification* regime.

At the other end of the spectrum is the *pooling* regime where risk classification is banned and all risk-groups are charged the same premium $\pi_i = \pi_0$ for $i = 1, 2, \ldots, n$. Since the insurance demand in our model is a continuous function of premium, there exists at least one premium π_0 where $\mu_1 \leq \pi_0 \leq \mu_n$ and $E(\pi_0) = 0$.[6]

Our final assumption is not a strict requirement, but is made for presentational convenience:

Assumption 9 (No full demand). *No risk-group is fully insured under any risk classification regimes.*

It is possible that an entire risk-group is insured, if the premium charged is sufficiently small; any further reduction in premium will then have no effect on demand from that risk-group. This special case can also be analysed using the same framework. However for ease of exposition, we present our findings based on Assumption 9 in the main text, and cover the case of full take-up for some risk-groups in Appendix F.

3.4 Definition of Loss Coverage

Although the first part of this thesis focuses on the impacts of premium regimes on social welfare, in some cases, it is computationally difficult to measure social welfare directly. In order to obtain analytical results, we have used loss coverage in our investigation of

[6] For notational convenience, we specify only one argument for multivariate functions if all arguments are equal, e.g. we write $E(\pi)$ for $E(\pi, \pi, \ldots, \pi)$.

partial risk classification in Chapter 5. The concept of loss coverage and its maximisation as a policy objective for insurance regulators has been discussed in Thomas (2008) and Thomas (2017). We use the same formulation of loss coverage as outlined in Hao et al. (2018).

Under an equilibrium premium, or risk classification, regime, $\underline{\pi}$, loss coverage is defined as the expected losses that are compensated by insurance, i.e.:

$$\text{Loss coverage: } LC\left(\underline{\pi}\right) = \mathbb{E}\left[\, QX \,\right] = \sum_{i=1}^{n} p_i \, d_i(\pi_i) \, \mu_i. \tag{3.4.1}$$

Using loss coverage under the full risk classification regime as a reference level, the *loss coverage ratio* is defined as follows:

$$\text{Loss coverage ratio: } C\left(\underline{\pi}\right) = \frac{LC\left(\underline{\pi}\right)}{LC\left(\underline{\mu}\right)} = \frac{\sum_{i=1}^{n} p_i \, d_i(\pi_i) \, \mu_i}{\sum_{i=1}^{n} p_i \, \tau_i \, \mu_i}. \tag{3.4.2}$$

Hao et al. (2019) showed that for iso-elastic insurance demand, where the elasticity is same for all risk groups, loss coverage is a valid proxy measure for social welfare. But this equivalence is not strictly true in other scenarios. Therefore, results for loss coverage cannot be generally extended to social welfare. However, as discussed below in Section 4.5, there is a large set of possible price elasticities, where the equivalence between loss coverage and social welfare holds true.

Chapter 4

Social Welfare under Pooling and Full Risk Classification

In this chapter, we use the model described in Chapter 3 to compare social welfare under the pooled premium regime against the full risk classification regime. We derive sufficient conditions on demand elasticities of different risk-groups to ensure higher social welfare under the pooled premium regime. In Section 4.1 we consider iso-elastic demand functions with same demand elasticity across all risk-groups. In Section 4.2 we consider iso-elastic demand function for all risk-groups, but demand elasticities are allowed to vary across risk-groups. In Section 4.3 we consider general demand functions, for which sufficiency conditions are derived using the concept of arc elasticity of demand. In Section 4.4 we have summarised our results and looked at some empirical evidence of observed demand elasticities in various insurance markets. In Section 4.5 we compared our results on social welfare with those obtained using the measure of loss coverage. In Section 4.6 we discuss our results in the context of existing literature, especially Hoy (2006).

4.1 Iso-elastic Insurance Demand

In this section, we apply the framework created in Chapter 3 to calculate Social Welfare under different premium regimes using the iso-elastic insurance demand.

Iso-elastic insurance demand is a tractable insurance demand function, which for risk-group i is given by:

$$d_i(\pi_i) = \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i}, \quad \text{(subject to a cap of 1)}, \tag{4.1.1}$$

producing a constant demand elasticity:

$$\epsilon(\pi_i) = -\frac{\partial \log(d_i(\pi_i))}{\partial \log \pi_i} = \lambda_i. \tag{4.1.2}$$

The parameter τ_i can be interpreted as the *fair-premium demand*, that is the demand when an actuarially fair premium is charged.

The above iso-elastic insurance demand can be constructed within our model set-up as follows. Consider an individual from risk-group i, with initial wealth W, who risks losing an amount L. Suppose her risk preferences are driven by a power utility function:

$$u_s(w) = \left[\frac{w-(W-L)}{L}\right]^{\gamma}, \tag{4.1.3}$$

so that $u_s(W) = 1$ and $u_s(W-L) = 0$. This particular form of utility function leads to:

$$u_s'(w) = \frac{\gamma}{L}\left[\frac{w-(W-L)}{L}\right]^{\gamma-1}, \text{ and so consequently:} \tag{4.1.4}$$

$$L\,u_s'(W) = \gamma. \tag{4.1.5}$$

So under the framework of power utility functions, the *risk preferences index*, $L\,u_s'(W)$, defined in Section 3.3, can be interpreted as the underlying parameter, γ, of the power utility function.

As outlined in Section 3.3, γ is sampled randomly from some underlying random variable Γ_i with distribution function $F_{\Gamma_i}(\gamma)$, and the demand for insurance for risk-group i at a given premium π_i is then:

$$d_i(\pi_i) = \mathrm{P}\left[\Gamma_i < \frac{\mu_i}{\pi_i}\right]. \tag{4.1.6}$$

The demand for insurance for risk-group i takes the form of iso-elastic demand given in Equation 4.1.1 if Γ_i has the following distribution:

$$F_{\Gamma_i}(\gamma) = \mathrm{P}\left[\Gamma_i \leq \gamma\right] = \begin{cases} 0 & \text{if } \gamma < 0 \\ \tau_i\, \gamma^{\lambda_i} & \text{if } 0 \leq \gamma \leq (1/\tau_i)^{1/\lambda_i} \\ 1 & \text{if } \gamma > (1/\tau_i)^{1/\lambda_i}, \end{cases} \tag{4.1.7}$$

where τ_i and λ_i are positive parameters. λ_i controls the shape of the distribution function and τ_i controls the range over which Γ_i takes its values.[1]

Using the specific form of iso-elastic demand, the analytical form of social welfare given in Equation 3.3.11 for a particular premium regime $\underline{\pi}$, is provided in Lemma 1 (proof in Appendix A).

[1]This is a generalised version of the Kumaraswamy distribution, which in its standard form takes values only over [0,1] (Kumaraswamy (1980)). Note that $\tau_i = \lambda_i = 1$ leads to a uniform distribution.

Lemma 1. *Suppose there are n risk-groups with risks $\mu_1 < \mu_2 < \cdots < \mu_n$ with iso-elastic demand elasticities $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, then for a given premium regime $\underline{\pi}$, the expression for social welfare is given by:*

$$S(\underline{\pi}) = \sum_{i=1}^{n} p_i \, \tau_i \, \frac{1}{(\lambda_i + 1)} \left(\frac{\mu_i}{\pi_i} \right)^{\lambda_i + 1} \pi_i + K, \tag{4.1.8}$$

where the premium regime $\underline{\pi}$ satisfies the equilibrium condition:

$$\sum_{i=1}^{n} p_i \tau_i \left(\frac{\mu_i}{\pi_i} \right)^{\lambda_i} (\pi_i - \mu_i) = 0, \tag{4.1.9}$$

and the constant K does not depend on the premium regime under consideration.

Lemma 1 provides the basis for comparing any two premium regimes. Specifically, we focus on comparing the pooling regime against the full risk classification regime.

Under pooling, it is sometimes notationally convenient to express the equilibrium condition and social welfare in terms of the *risk-premium ratios*: $v_i = \mu_i / \pi_0$. A risk-premium ratio of $v_i < 1$ indicates that the i-th risk-group pay more than their fair actuarial premium, and conversely for $v_i > 1$. Using this notation, the pooling equilibrium in Equation 4.1.9 becomes:

$$\sum_{i=1}^{n} \alpha_i v_i^{\lambda_i + 1} = \sum_{i=1}^{n} \alpha_i v_i^{\lambda_i}, \tag{4.1.10}$$

$$\text{or, equivalently:} \quad \sum_{i:\, v_i > 1} \alpha_i \left[v_i^{\lambda_i + 1} - v_i^{\lambda_i} \right] = \sum_{i:\, v_i \le 1} \alpha_i \left[v_i^{\lambda_i} - v_i^{\lambda_i + 1} \right], \tag{4.1.11}$$

where $\alpha_i = \frac{p_i \tau_i}{\sum_{j=1}^{n} p_j \tau_j}$ and the social welfare condition Equation 4.1.8 can be expressed as:

$$S(\pi_0) \gtreqless S(\underline{\mu}) \Leftrightarrow \sum_{i=1}^{n} \frac{\alpha_i \, v_i^{\lambda_i + 1}}{\lambda_i + 1} \gtreqless \sum_{i=1}^{n} \frac{\alpha_i \, v_i}{\lambda_i + 1},^{2} \tag{4.1.12}$$

$$\Leftrightarrow \sum_{i:\, v_i>1} \frac{\alpha_i}{\lambda_i+1}\left[v_i^{\lambda_i+1} - v_i\right] \gtreqless \sum_{i:\, v_i\leq 1} \frac{\alpha_i}{\lambda_i+1}\left[v_i - v_i^{\lambda_i+1}\right]. \qquad (4.1.13)$$

Equation 4.1.11 says that under the pooling equilibrium, losses from the high risk-groups are exactly offset by the profits from the low risk-groups. And Equation 4.1.13 can be interpreted as the comparison between the (aggregate) utility gains by the high risk-groups (from pooling as compared against full risk classification) against the (aggregate) utility losses of the low risk-groups.

We can now derive the conditions for which social welfare under pooling is higher than that under full risk classification. In the first instance, we make the simplest assumption that all risk-groups have the same positive constant demand elasticity λ. Under this assumption, we obtain the following result (proof in Appendix B) :

Theorem 1. *Suppose there are n risk-groups with risks $\mu_1 < \mu_2 < \cdots < \mu_n$ with the same positive constant demand elasticity λ for all risk-groups. Then:*

$$\lambda \lesseqgtr 1 \Rightarrow S(\pi_0) \gtreqless S(\underline{\mu}). \qquad (4.1.14)$$

Figure 4.1 provides a graphical representation of Theorem 1, showing the ratio of $(S(\pi_0) - K)$ to $\big(S(\underline{\mu}) - K\big)$ as a function of constant demand elasticity λ for two risk-groups with risks $(\mu_1, \mu_2) = (0.01, 0.04)$ and $(\alpha_1, \alpha_2) = (0.8, 0.2)$. Recall from Equation 4.1.8, in the expression for $S(\underline{\pi})$, K is a constant which does not depend on the premium regime $\underline{\pi}$. So the ratio of $(S(\pi_0) - K)$ to $\big(S(\underline{\mu}) - K\big)$ focuses solely on the effect of changes in premium regimes.

It can be clearly seen that $\lambda = 1 \Rightarrow S(\pi_0) = S(\underline{\mu})$, while $\lambda < 1 \Rightarrow S(\pi_0) > S(\underline{\mu})$ and vice versa, as postulated in Theorem 1.

[2] We use the notation $\gtreqless$ in the following sense: $A \gtreqless B \Rightarrow C \gtreqless D$ is shorthand for $A > B \Rightarrow C > D$ and $A = B \Rightarrow C = D$ and $A < B \Rightarrow C < D$. A similar interpretation applies for the notation $\lesseqgtr$.

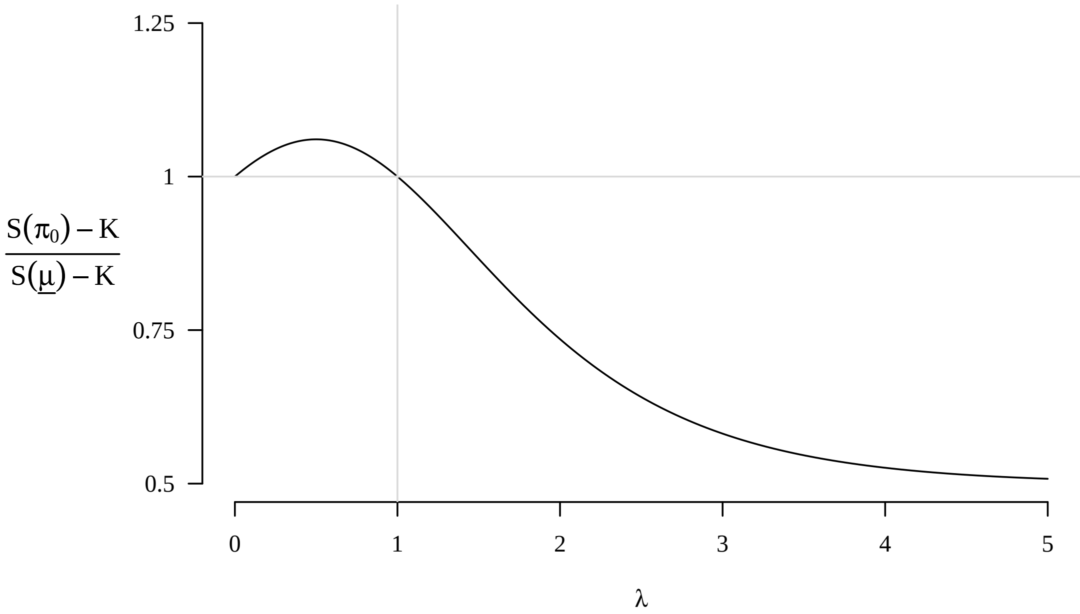


Figure 4.1: Illustration of Theorem 1: Social welfare under pooling is higher than under full risk classification for $\lambda < 1$.

Basis: $(\mu_1, \mu_2) = (0.01, 0.04)$ and $(\alpha_1, \alpha_2) = (0.8, 0.2)$. Similar pattern for any population structure and relative risk.

4.2 Varying Demand Elasticities across Risk-Groups

Theorem 1 assumes the same constant iso-elastic demand elasticity for all individuals. However, different risk-groups may have different sensitivities to price changes. In particular, for higher risk consumers, insurance premiums may represent a larger part of their total budget constraint, and so the effect of a small percentage change in price on their insurance demand might be larger. In this section, for ease of exposition, we first consider two risk-groups with iso-elastic demand, but with different demand elasticities. We then generalise our result to more than two risk-groups.

Typical insurance underwriting processes often classify a majority of insurance risks as *standard* (or low risks in the terminology of this thesis), with the remaining risks rated higher based on their individual characteristics. The empirical evidence (cited in Table 4.1 in Section 4.4) suggests that the more numerous low risk-group's demand elasticity may often be less than 1. But, as noted above, the high risk-group's demand elasticity is likely to be higher than that the low-risk-group, and may often exceed 1. This pattern motivates Theorem 2 (proof in Appendix C). Theorem 2.1 states a sufficient condition on λ_1 and λ_2 for social welfare to be higher under pooling than under full risk classification, for any population structures and underlying risks. Theorem 2.2 then extends it for some of the ranges of λ_2 not covered in Theorem 2.1, but this involves introduction of additional conditions.

Theorem 2. *Suppose there are two risk-groups with risks $\mu_1 < \mu_2$ with positive constant demand elasticities λ_1 and λ_2 respectively.*

2.1. *For any population structure:*

$$\lambda_1 \leq 1 \text{ and } \lambda_1 \leq \lambda_2 \leq \frac{1}{\lambda_1} \Rightarrow S(\pi_0) \geq S(\underline{\mu}). \tag{4.2.1}$$

2.2. *For any population structure there exists a threshold premium π^* such that:*

$$\lambda_1 \leq 1 \text{ and } \lambda_2 > \frac{1}{\lambda_1} \text{ and } \pi_0 \geq \pi^* \Rightarrow S(\pi_0) \geq S(\underline{\mu}). \tag{4.2.2}$$

Theorem 2 is illustrated in Figure 4.2, where $(\mu_1, \mu_2) = (0.01, 0.04)$ and the x and y axes represent the lower and higher demand elasticities λ_1 and λ_2. The two curves emanating from the origin show the boundary at which $S(\pi_0) = S(\underline{\mu})$ for two possible population structures. The bold red curve demarcates the boundary for a moderate population structure, $(\alpha_1, \alpha_2) = (0.8, 0.2)$; the dashed blue curve is the boundary for an extreme population structure with very few high risks, $(\alpha_1, \alpha_2) = (0.99, 0.01)$. Social welfare under pooling is higher than under full risk classification on the left of the boundary curves, and lower on the right. The sufficient conditions in Theorem 2.1 specify that in the green shaded region where $\lambda_1 \leq 1$ and $\lambda_1 \leq \lambda_2 \leq 1/\lambda_1$, social welfare under pooling is *always* higher than that under full risk classification, *irrespective* of the population structure (and also the risks μ_1 and μ_2).

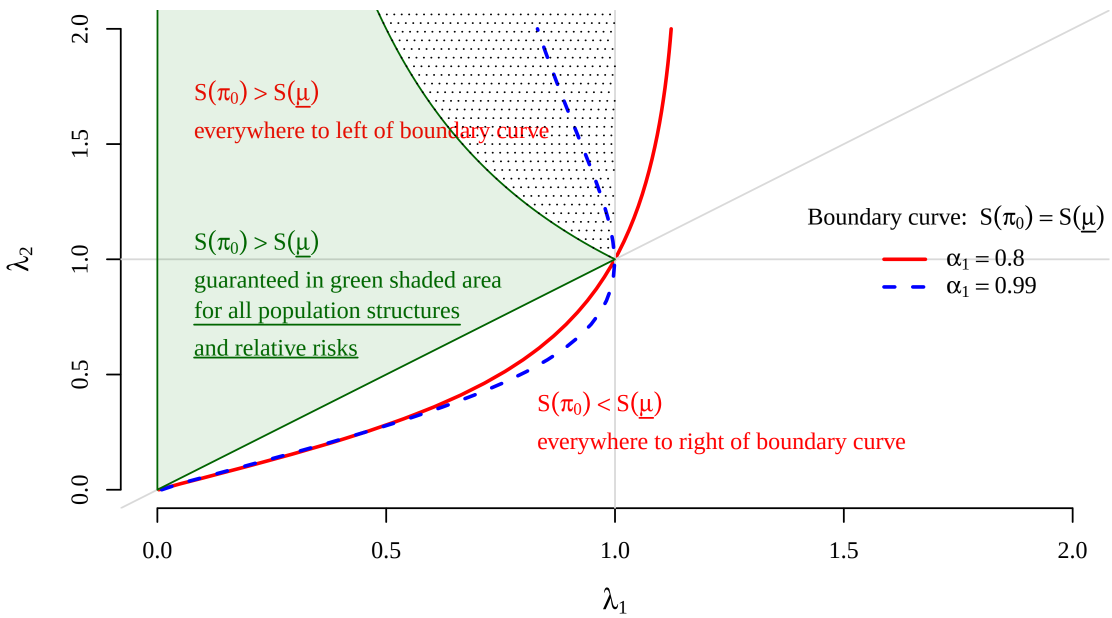


Figure 4.2: Illustration of Theorem 2: Social welfare under pooling is higher than under full risk classification in green area, for *all* population structures and relative risks. See text for interpretation of solid red and dashed blue boundary curves.

To understand the patterns in Figure 4.2, first note that moving from full risk classification to pooling always leads to (i) a beneficial *increase* in both the *number* of high-risks insured, and the *per capita* utility of insured high-risks and (ii) a detrimental *decrease* in both the *number* of low-risks insured, and the *per capita* utility of each insured low-risk.

An initial intuition is that pooling will tend to "work well" when lower risks' elasticity is low compared with higher risks' elasticity, i.e. towards the left of Figure 4.2.

As we move leftwards in the graph with λ_2 fixed, λ_1 eventually becomes *sufficiently low* compared with λ_2, so that pooling "works well" and effect (i) dominates. As we move upwards in the graph with λ_1 fixed (where $\lambda_1 \leq 1$), λ_2 eventually becomes *sufficiently high* compared with λ_1, so that pooling again "works well" and effect (i) dominates. This explains the position of the red curve.

However, if the high risk-group is small and has high demand elasticities, it may not have the required capacity to absorb all the aggregate utility losses of the low risk-group. This "capacity limit" on effect (ii) for a small high-risk-group is illustrated by the curvature of the dashed blue line for $\alpha_1 = 0.99$ (a very small fraction of high-risks) back towards the vertical axis for $\lambda_2 > 1$ (high elasticities of the high-risks). The green curve represents a limiting value of this "capacity limit" on effect (ii). To the left of this limit (i.e. inside the green shaded area specified by Theorem 2.1), effect (i) is guaranteed to dominate, for *any* population structure and risks.

Note that the conditions in Theorem 2.1 are sufficient, but *not* necessary. This non-necessity is illustrated by the white and dotted regions adjacent to the green shaded region, but to the left of the red boundary curve, where $S(\pi_0) > S(\underline{\mu})$ for the population structure $\alpha_1 = 0.8$ even though the conditions of Theorem 2.1 are not satisfied. Where the conditions of Theorem 2.1 are not satisfied, social welfare may still be higher under pooling than under full risk classification, but this might require additional conditions. For the region $\lambda_1 \leq 1$ and $\lambda_2 > 1/\lambda_1$ (dotted in Figure 4.2), Theorem 2.2 identifies the additional condition in the form of the equilibrium premium π_0 needing to exceed a threshold premium π^* for social welfare under pooling to be higher.

An implication of Theorem 2.2 is that the high risk-group needs to be of a large enough size to pull the equilibrium premium above the threshold. This can be interpreted as the need for the high risk-group to be of a reasonably large size to absorb the impact of aggregate utility losses for the low risk-group. The dashed blue boundary line for an extreme population structure with very few high-risks, $\alpha_1 = 0.99$, curves back into the dotted region, indicating that the condition $\pi_0 \geq \pi^*$ may not always be satisfied. In contrast, for a moderate population structure with $\alpha_1 = 0.8$, the bold red boundary curves back into the dotted region only at much higher values of λ_2 (not shown in the figure).

Theorem 2 can be generalised for more than two risk-groups with iso-elastic demand for all risk-groups. While generalising our results to more than two risk-groups, under pooling it will be convenient to classify the different risk-groups into two broad categories:

- 'lower' risk-groups, for whom pooled premium is higher than fair premium, i.e. $\mu_i \leq \pi_0$;
- 'higher' risk-groups, for whom pooled premium is lower than fair premium, i.e. $\mu_i > \pi_0$.

For these two broad categories, we define the following:

- $\lambda_{lo}^{min} = \min \{\lambda_i : \mu_i \leq \pi_0\}$, i.e. minimum demand elasticity for lower risk-groups;
- $\lambda_{lo}^{max} = \max \{\lambda_i : \mu_i \leq \pi_0\}$, i.e. maximum demand elasticity for lower risk-groups;
- $\lambda_{hi}^{min} = \min \{\lambda_i : \mu_i > \pi_0\}$, i.e. minimum demand elasticity for higher risk-groups;
- $\lambda_{hi}^{max} = \max \{\lambda_i : \mu_i > \pi_0\}$, i.e. maximum demand elasticity for higher risk-groups.

For the case of two risk-groups, we simply have: $\lambda_{lo}^{min} = \lambda_{lo}^{max} = \lambda_1$ and $\lambda_{hi}^{min} = \lambda_{hi}^{max} = \lambda_2$.

Using these notations, we present our general result (proof in Appendix C):

Theorem 3. *Suppose there are n risk-groups with risks $\mu_1 < \mu_2 < \cdots < \mu_n$ with iso-elastic demand elasticities $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively.*

3.1. *For any underlying population structures:*

$$\lambda_{lo}^{max} \leq \lambda_{hi}^{min} \leq \lambda_{hi}^{max} \leq 1 \Rightarrow S(\pi_0) \geq S(\underline{\mu}). \tag{4.2.3}$$

3.2. *For any underlying population structures:*

$$\lambda_{lo}^{max} \leq 1 \textit{ and } 1 \leq \lambda_{hi}^{min} \leq \lambda_{hi}^{max} \leq \frac{1}{\lambda_{lo}^{max}} \Rightarrow S(\pi_0) \geq S(\underline{\mu}). \tag{4.2.4}$$

3.3. *There exists a threshold premium π^* such that:*

$$\lambda_{lo}^{max} \leq 1 \textit{ and } \lambda_{hi}^{min} > \frac{1}{\lambda_{lo}^{min}} \textit{ and } \pi_0 \geq \pi^* \Rightarrow S(\pi_0) \geq S(\underline{\mu}). \tag{4.2.5}$$

It is easy to see that Theorem 2.1 can be obtained as a special case of Theorems 3.1 and 3.2; while Theorem 2.2 is a special case of Theorem 3.3.

4.3 Generalised Demand Function

So far, we have only considered constant demand elasticities, either for all individuals in the population, or for all individuals belonging to a particular risk-group. Iso-elastic demand functions are easy to understand and are also analytically convenient. However, they may also be criticised as being unrealistic. In this section, we use the concept of *arc elasticity of demand* to extend the results in Section 4 to a form applicable to more general demand functions.

The formulation of iso-elastic demand arose from the particular choice of distribution function in Equation 4.1.7 for the random variable Γ_i (denoting the risk preferences index) for risk-group i. However, the framework developed in Section 3 is general and can be applied to any distribution for the risk preferences index. In this section, we will just assume that Γ_i is a positive continuous random variable[3] with a distribution function:

$$F_{\Gamma_i}(\gamma) = \mathrm{P}\left[\, \Gamma_i \leq \gamma \,\right]. \tag{4.3.1}$$

Under this general framework, social welfare for a given premium regime $\underline{\pi}$ is given by Lemma 2 (for proof see Appendix D).

Lemma 2. *Suppose there are n risk-groups with risks $\mu_1 < \mu_2 < \cdots < \mu_n$ and any general demand functions. Then for a given premium regime $\underline{\pi}$, for which no risk-group is fully insured, the expression for social welfare is given by:*

$$S(\underline{\pi}) = \sum_{i=1}^{n} p_i \, G_i\left(\frac{\mu_i}{\pi_i}\right) \pi_i + K, \quad \textit{where} \quad G_i(g) = \int_0^g P\left[\, \Gamma_i < \gamma \,\right] d\gamma, \tag{4.3.2}$$

where the premium regime $\underline{\pi}$ satisfies the equilibrium condition:

$$\sum_{i=1}^{n} p_i \, d_i(\pi_i) \; (\pi_i - \mu_i) = 0, \tag{4.3.3}$$

and the constant K does not depend on the premium regime under consideration.

Comparing social welfare under pooling to that under full risk classification gives:

$$S(\pi_0) - S(\underline{\mu}) = \sum_{i=1}^{n} p_i \, G_i\left(\frac{\mu_i}{\pi_0}\right) \pi_0 - \sum_{i=1}^{n} p_i \, G_i\left(\frac{\mu_i}{\mu_i}\right) \mu_i, \tag{4.3.4}$$

[3]The derivations in this section can also be suitably adapted for any positive discrete random variable.

where the equilibrium premium π_0 satisfies:

$$\sum_{i=1}^{n} p_i \, d_i(\pi_0) \; (\pi_0 - \mu_i) = 0. \tag{4.3.5}$$

Using the notations involving risk-premium ratios, $v_i = \mu_i/\pi_0$, we get:

$$S(\pi_0) \gtreqless S(\underline{\mu}) \Leftrightarrow \sum_{i=1}^{n} p_i \; [\, G_i\,(v_i) - v_i \, G_i\,(1)\,] \gtreqless 0. \tag{4.3.6}$$

To make analytical progress with the general relationship in Equation 4.3.6, we need to establish a connection between general demand elasticity functions, $\epsilon_i(\cdot)$, and general distribution functions for the risk preferences index, $F_{\Gamma_i}(\cdot)$. The link arises from Equations 3.1.6 and 3.3.9, reproduced below with appropriate adaptation for risk-group i:

$$d_i(\pi) = \tau_i \exp\left[- \int_{\mu_i}^{\pi} \epsilon_i(s)\, d\log s \right], \tag{4.3.7}$$

$$d_i(\pi) = \mathrm{P}\left[\, \Gamma_i < \frac{\mu_i}{\pi} \,\right] = \mathrm{P}\,[\, \Gamma_i \leq v \,]\,, \quad \text{where } v = \frac{\mu_i}{\pi}. \tag{4.3.8}$$

Note the distinction between v_i (earlier in the thesis) and v for risk-group i: v_i is the risk-premium ratio at the equilibrium premium π_0, whereas v is the risk-premium ratio as a function of premium π.

We now need the concept of *arc elasticity of demand* (Vázquez (1995)), defined as:

$$\lambda_i(v) = \frac{\int_{\mu_i}^{\pi} \; \epsilon_i(s)\, d\log s}{\int_{\mu_i}^{\pi} \; d\log s}, \;\; \text{for } i = 1, 2, \ldots, n, \tag{4.3.9}$$

which can be interpreted as the weighted average of (point) elasticity for risk-group i, $\epsilon_i(s)$, over the arc of the demand curve from premium μ_i to premium π, where the weights are the log premiums.

Using the concept of arc elasticity of demand, Equation 4.3.8 can be written as:

$$d_i(\pi) = \mathrm{P}\left[\,\Gamma_i \leq v\,\right] = \tau_i\,\exp\left[-\lambda_i(v)\int_{\mu_i}^{\pi} d\log s\right] = \tau_i\,\left(\frac{\mu_i}{\pi}\right)^{\lambda_i(v)} = \tau_i\,v^{\lambda_i(v)}, \qquad (4.3.10)$$

and the equilibrium condition in Equation 4.3.5 as:

$$\sum_{i=1}^{n} p_i\,\tau_i\,v_i^{\lambda_i(v_i)+1} = \sum_{i=1}^{n} p_i\,\tau_i\,v_i^{\lambda_i(v_i)}, \text{ as } d_i(\pi_0) = \tau_i\,v_i^{\lambda_i(v_i)}. \qquad (4.3.11)$$

Now consider a *hypothetical* population with the same probabilities of loss, i.e. $\mu_1 < \mu_2 < \cdots < \mu_n$, as in the actual population. But suppose that in the hypothetical population, demand for insurance is iso-elastic with constant elasticity parameters set at values $\lambda_1(v_1), \lambda_2(v_2), \ldots, \lambda_n(v_n)$ respectively. Then all the results obtained in Section 4.2 are applicable for the hypothetical population with iso-elastic demand. This creates an avenue for extending the results for iso-elastic demand to general demand functions.

Specifically, if the relevant conditions of iso-elastic demand functions given in Theorem 3 of Section 4.2 apply for the hypothetical population, we know that pooling increases social welfare as compared to full risk classification. In that case, Equation 4.1.13 implies that for the hypothetical population:

$$\sum_{i=1}^{n} p_i\,\frac{\tau_i}{\lambda_i(v_i)+1}\left[v_i^{\lambda_i(v_i)+1} - v_i\right] \geq 0. \qquad (4.3.12)$$

However, insurance demand of the *actual* population is not necessarily iso-elastic. But, interestingly, by construction, the equilibrium condition in Equation 4.3.11 is the same for both the hypothetical population and the actual population, i.e. the pooled equilibrium premium, π_0, will be the same under both set-ups.

Now for the higher risk-groups, i.e. for those risk-groups for which $\mu_i > \pi_0$, it is shown in Lemma 4 in Appendix E that if the demand elasticity, $\epsilon_i(\pi)$, is either increasing or iso-elastic as a function of premium π, then:

$$G_i(v_i) - v_i G_i(1) \geq \frac{\tau_i}{\lambda_i(v_i)+1}\left[v_i^{\lambda_i(v_i)+1} - v_i\right]. \tag{4.3.13}$$

In other words: for the higher risk-groups, under the assumption of increasing or iso-elastic demand elasticities, the increase in social welfare in the actual population when we move to pooling is *higher* than that in the hypothetical population.

Conversely, for the lower risk-groups, i.e. for those risk-groups for which $\mu_i \leq \pi_0$, it is shown in Lemma 5 in Appendix E that if the demand elasticity, $\epsilon_i(\pi)$, is either decreasing or iso-elastic as a function of premium π, then:

$$v_i G_i(1) - G_i(v_i) \leq \frac{\tau_i}{\lambda_i(v_i)+1}\left[v_i - v_i^{\lambda_i(v_i)+1}\right]. \tag{4.3.14}$$

In other words: for the lower risk-groups, under the assumption of decreasing or iso-elastic demand elasticities, the fall in social welfare in the actual population when we move to pooling is *lower* than that in the hypothetical population.

Putting Equations 4.3.13 and 4.3.14 together, we get the following expression for the increase in social welfare in the actual population when we move to pooling:

$$\sum_{i=1}^{n} p_i \left[G_i(v_i) - v_i G_i(1)\right] \tag{4.3.15}$$

$$= \sum_{\mu_i > \pi_0} p_i \left[G_i(v_i) - v_i G_i(1)\right] - \sum_{\mu_i \leq \pi_0} p_i \left[v_i G_i(1) - G_i(v_i)\right], \tag{4.3.16}$$

$$\geq \sum_{\mu_i > \pi_0} p_i \frac{\tau_i}{\lambda_i(v_i)+1}\left[v_i^{\lambda_i(v_i)+1} - v_i\right] - \sum_{\mu_i \leq \pi_0} p_i \frac{\tau_i}{\lambda_i(v_i)+1}\left[v_i - v_i^{\lambda_i(v_i)+1}\right], \tag{4.3.17}$$

$$= \sum_{i=1}^{n} p_i \frac{\tau_i}{\lambda_i(v_i)+1}\left[v_i^{\lambda_i(v_i)+1} - v_i\right]. \tag{4.3.18}$$

This implies that if the actual population is such that the hypothetical population satisfies the relevant conditions of iso-elastic demand functions given in Theorem 3.1 of Section 4.2, then pooling gives higher social welfare than full risk classification in the actual population. The following theorem outlines the required conditions in the actual population.

Theorem 4. *Suppose there are n risk-groups with risks $\mu_1 < \mu_2 < \cdots \mu_n$. If the insurance demand elasticities have the following properties over their respective ranges from μ_i to the pooled premium π_0:*

(i) for each lower risk-group, demand elasticity is either decreasing or iso-elastic as a function of premium;

(ii) for each higher risk-group, demand elasticity is either increasing or iso-elastic as a function of premium;

(iii) risk-groups with higher risks have higher arc elasticities of demand; and

(iv) demand elasticities do not exceed 1

then pooling increases social welfare as compared against full risk classification.

Theorem 4 thus partly relaxes the iso-elasticity condition on higher risk-groups in Theorem 3.1. Specifically, condition (ii) allows higher risk-groups to have either iso-elastic *or increasing* demand elasticities (as a function of premium), *provided that* they also have higher *arc elasticities* than all lower risk-groups (condition (iii)) and their demand elasticities do not exceed 1 (condition (iv)).

Technically, Theorem 4 also partly relaxes the iso-elasticity condition on lower risk-groups. Specifically, condition (i) allows lower risk-groups to have either iso-elastic *or decreasing* demand elasticities (as a function of premium). However, as discussed previously, demand elasticities are more likely to be increasing as a function of premium.

So, for all practical purposes, condition (i) amounts to a restriction to iso-elastic demand functions.

We emphasise that the conditions presented in Theorem 4 are sufficient, but *not* necessary. In fact, experimentation using simple functions reveals that pooling can sometimes increase social welfare even where lower risk-groups have increasing demand elasticity (as a function of premium), as long as the marginal increase in their demand elasticities does not exceed a certain threshold which depends on the high risk-groups' demand elasticities. However, we do not include these results here as they are not generic and apply to specific analytic forms of demand elasticity functions.

4.4 Summary and Empirical Comparisons

The results obtained in this chapter give sufficient conditions for social welfare to be higher under pooling than under full risk classification. They can be summarised as follows.

(a) Theorem 1 for iso-elastic demand (common elasticity for all risk-groups) requires only that the common demand elasticity is less than 1.

(b) Theorem 2 (2 risk-groups) and Theorem 3 (n risk-groups) for iso-elastic demand (different elasticities for different risk-groups) require that all higher risk-groups' demand elasticities are higher than all lower risk-groups' demand elasticities, *and* all demand elasticities are less than 1. They also provide sufficient conditions when higher risk-groups' demand elasticities exceed 1, as long as all lower risk-groups' demand elasticities are less than 1.

(c) Theorem 4 then uses the concept of arc elasticity of demand to extend the results in a form applicable to more general demand functions.

The conditions above are stringent because they are sufficient for *any* population structures and relative risks. But the conditions are *not* necessary, and where they are not fully satisfied, social welfare under pooling may still be higher than under risk-differentiated premiums for some combinations of population structures and demand elasticities.

Given that the conditions all relate to demand elasticities, an obvious question is: what elasticities do we typically observe? Table 4.1 shows some relevant empirical estimates. It can be seen that most estimates are of magnitude significantly less than 1. This is at least suggestive of the possibility that social welfare in some insurance markets could be higher under pooling than under full risk classification.

Table 4.1: Estimates of demand elasticity for various insurance markets.

Market & country	Demand elasticities [a]	Authors
Term life insurance, USA	0.66	Viswanathan et al. (2006)
Yearly renewable term life, USA	0.4 to 0.5	Pauly et al. (2003)
Whole life insurance, USA	0.71 to 0.92	Babbel (1985)
Health insurance, USA	0 to 0.2	Chernew et al. (1997), Blumberg et al. (2001), Buchmueller and Ohri (2006)
Health insurance, Australia	0.35 to 0.50	Butler (1999)
Farm crop insurance, USA	0.32 to 0.73	Goodwin (1993)

[a]Estimates in empirical papers are generally given as negative values, but we have presented the absolute values here for consistency with the definition of demand elasticity used in this thesis.

The estimates in Table 4.1 are made in various contexts, some of which may not correspond closely to the set-up in this thesis. However, we wish to emphasise that they all appear to be *product* elasticities, not *brand* elasticities. Product elasticity is the response of *market* demand to a small change in *market* price. Brand elasticity is response of *one insurer's* demand to a (unilateral) small change in *one insurer's* price. Product elasticity is the relevant parameter for our analysis. Intuitively, in a competitive market, brand elasticity is likely to be many times higher than product elasticity.

Brand elasticities are of more immediate interest for competitive strategy, and so more likely to be estimated by insurers, but they are not informative for our analysis. More detailed empirical work on product elasticities, separately for different markets and risk-groups, is needed for policymakers to implement our results.

4.5 Relationship between Loss Coverage and Social Welfare Results

The results for social welfare can be compared with the analogous results for loss coverage in Hao et al. (2018). As a reminder, loss coverage is defined as expected losses compensated by insurance for the whole population. One of the advantages of using loss coverage, is that ex-post it is an observable quantity, whereas social welfare is based on an unobservable, notional utility functions.

The comparison is illustrated in Figure 4.3 with two risk-groups. The boundary curve is shown for a hypothetical population with 80% in low risk-group. The dotted area where pooling is sure to increase loss coverage (but increases social welfare only subject to further conditions) arises because the loss coverage criterion focuses on compensation of losses for the population as a whole, and places no weight on the premium cross-subsidies implied by pooling; on the other hand, social welfare takes account of the premium cross-subsidies. For moderate dispersion of elasticities (and hence utility functions), taking account of premium cross-subsidies typically does not change the ranking of pooling versus full risk classification. But with large dispersion of elasticities (and hence utility functions) – in particular, $\lambda_2 \gg \lambda_1$, that is where high-risks have much higher demand elasticities than low-risks – then pooling may be beneficial in terms of loss coverage, but not in terms of social welfare. However, $\lambda_2 \gg \lambda_1$ is probably an unrealistic parameterisation; for more realistic parameters (e.g. all elasticities not much more than 1), loss coverage and social

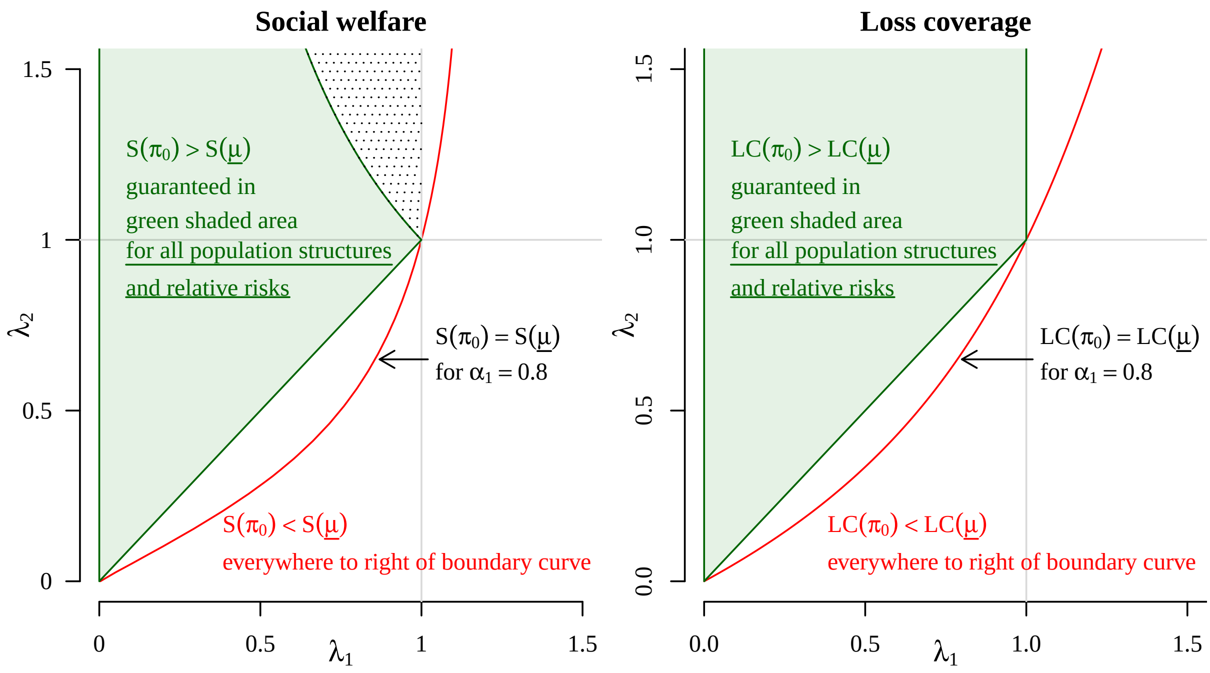


Figure 4.3: Elasticity conditions for pooling to beat full risk classification are more stringent for social welfare criterion (green area on left panel) than for loss coverage criterion (green area on right panel).

welfare usually give the same ranking of pooling versus full risk classification. This is shown by the similar positions of the red boundary curve, inside the unit square, in the left and right panels of Figure 4.3.

4.6 Discussion of Results

The results in this section can be compared with those of Hoy (2006), who finds that utilitarian welfare is increased by pooling, provided only that the fraction of high-risks is sufficiently small. Hoy (2006) assumes a utility function which is uniformly risk-averse for the whole population; this leads all individuals to buy insurance under either pooling or full risk classification, albeit the pooling contract provides only partial insurance.[4] When pooling is mandated, there is (i) a loss in efficiency because the pooling contract offers only partial insurance, and (ii) a redistribution from low risks (previously better off, because they paid lower premiums) to high-risks. Behind the veil of ignorance, effect (i) reduces welfare, but effect (ii) increases welfare. For a sufficiently small high-risk fraction, effect (ii) dominates (i.e. for a risk-averse utility function, expected utility behind the veil of ignorance is always increased by a sufficiently small redistribution towards the previously worse off).

In contrast, we allow for a distribution of utility functions in the population, such that not all individuals will purchase insurance at an actuarially fair price. In our model, if we pool a very small high-risk population with high elasticity with a large low-risk population with low elasticity, many of the high-risks who now choose to participate at the (cheap to them) pooled price have low-risk aversion, so their gain in utility from participating is relatively small. On the other hand, the low-elasticity lower risks' loss in utility from either leaving the market or paying the (expensive to them) pooled price, is relatively large. Therefore overall, pooling might not be advantageous, even with a very small high-risk fraction. Looking back at Figure 4.2, this is represented by the curvature of the dashed blue boundary for $\alpha_1 = 0.99$ (i.e. very few high-risks) back towards the vertical axis for $\lambda_2 \gg \lambda_1$.

But this feature in our model probably has little practical significance, because $\lambda_2 \gg \lambda_1$ is not a realistic parameterisation. For more typical parameter values (e.g. $\lambda_1 < \lambda_2 < 1$), the relative position of the dashed blue and solid red curves in Figure 4.2 suggests that reducing the size of the high risk-group makes pooling slightly more likely to be beneficial (in the sense that pooling gives higher social welfare for a slightly wider range of (λ_1, λ_2)

[4]The partial-cover pooling contract is that predicted by the anticipatory (E2) equilibrium concept in Wilson (1977).

parameter values). This is more in accordance with (albeit not the same as) Hoy's result.

Chapter 5

Loss Coverage under Partial Risk Classification

So far in this thesis, we have focused only on the two extreme premium regimes, pooling and full risk classification. A more common scenario in practice is partial risk classification, where risk classification is restricted but not completely banned. For example, in some cases, a regulator may allow risk classifications based on certain criteria (e.g. lifestyle choices) but not on others (e.g. gender). This leads to merging of some risk-groups and the insurer charging them same premium. For some markets, regulator may impose a limit on the ratio of premiums for high risk-groups relative to low risk-groups - which may also lead to merging of risk-groups. These scenarios can be compared with the polar cases by identifying and comparing against all possible intermediate classifications of the risk-groups permitted by the regulations. A full analysis of partial risk classification would require some extensions of our model.

Firstly, all possible solutions satisfying the equilibrium conditions may not be politically acceptable for the society or plausible in real life. For example, it would be unfair that the lower risk-groups are charged higher premiums than the higher risk-groups, and therefore politically unacceptable. Hence we introduce a political constraint to ensure fairness in premium charging. Secondly, it also makes sense to limit the investigation within the set of robust equilibria only, i.e. those equilibrium scenarios, where by unilateral deviation, a single insurer would not be able to gain in the long run. The robustness constraint essentially limits our investigation within the set of stable Nash equilibria.

As a deviation from earlier sections, in the current section we use loss coverage (Thomas (2008), Thomas (2017), Hao et al. (2018)) instead of social welfare, to compare premium regimes. To obtain analytical results, it was necessary to use loss coverage, rather than social welfare, as our policy metric in this part of the thesis.

5.1 Political, Regulatory and Economic Framework

Using the framework and notations developed in Section 3.2, consider a population consisting of n distinct risk-groups with probabilities of loss given by $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$, where $0 < \mu_1 < \mu_2 < \ldots < \mu_n < 1$. Let the proportion of the population belonging to risk-group i be p_i, for $i = 1, 2, \ldots, n$. Suppose for risk-group i, insurers charge premium (per unit of loss) π_i for $i = 1, 2, \ldots, n$, so that $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ is a premium, or risk classification, regime. The proportional insurance demand for risk-group i, at premium π_i, is denoted by $d_i(\pi_i)$.

Although any premium regime is theoretically possible, political, regulatory and economic constraints limit the possibilities of premium regimes which are feasible and stable. In this section, we discuss these constraints and their implications.

Political Constraints

Premiums need to be politically acceptable, rather than just being technically possible. For example, although an insurer can theoretically propose a premium regime, $\underline{\pi}$, where a high risk-group is charged a higher premium than their underlying risk and/or a low risk-group's premium is lower than their underlying risk. This might be politically unacceptable, as this can lead to a disagreeable perception that a disadvantaged high risk-group is over-charged to subsidise an already fortunate low risk-group.

On the other hand, it might also be considered inappropriate if a low risk-group is charged more than a high risk-group. Such a situation can arise, for example, in the case of three risk-groups, with say low, medium and high risks, where grouping the low and high risk-groups at one premium, and the medium risks at another premium, is theoretically possible; but if it leads to a premium regime where low risks are charged a higher premium than medium risks, this might be considered unacceptable.

Formally, these notions can be encapsulated as:

Constraint 1 (Political). *Given risks $\underline{\mu}$, a politically acceptable premium regime $\underline{\pi}$ needs to satisfy:*

$$\mu_1 \leq \pi_1 \leq \pi_2 \leq \cdots \leq \pi_n \leq \mu_n. \tag{5.1.1}$$

Other examples of politically unacceptable premium regimes include those which lack face validity (e.g. combine risk-groups having no apparent similarities), or which disadvantage socially protected classes (e.g. combine a low risk-group identified by disability with a high risk-group identified by participation in dangerous sports). We do not consider these situations here.

Regulatory Constraints: Price Collar

A regulator, or a policymaker, can implement policies which can directly influence the range of premium regimes available in an insurance market. For example, regulators can impose constraints on premiums to promote specific policy objectives, like banning of risk classifications based on certain protected characteristics, e.g. gender or genes.

In this thesis, we will analyse the impact of mandating a regulatory price collar, whereby insurers are allowed to charge any premiums to any risk-groups, subject to the condition that the highest premium charged cannot exceed the lowest premium by a certain prescribed multiple. The objective of a price collar is to ensure that no risk-group is charged an unacceptably large premium compared to the general population.

Although theoretically a regulator can impose different price collars for different categories of risk characteristics, it may not always be practical due to the complexity of implementation and cost of regulatory oversight. So, for the purposes of this thesis, we focus on the implications of operating a single regulatory price collar for all risk-groups.

Constraint 2 (Regulatory). *Given a prescribed price collar, κ, where $\kappa \geq 1$, any premium regime $\underline{\pi}$ needs to satisfy:*

$$\pi_{H} \leq \kappa \, \pi_{L}, \tag{5.1.2}$$

where $\pi_{L} = \min_i \pi_i$ and $\pi_{H} = \max_i \pi_i$.

Pooled premium regime, i.e. where all risk-groups are charged the same premium, can be achieved by setting $\kappa = 1$, which would imply $\pi_{L} = \pi_{H}$, and because of the political constraint, premiums of all risk-groups, sandwiched between the lowest and the highest risk-groups, would then all have to be the same.

The political constraint also requires that $\pi_{H} \leq \mu_n$ and $\pi_{L} \geq \mu_1$, implying:

$$\frac{\pi_{H}}{\pi_{L}} \leq \frac{\mu_n}{\mu_1}. \tag{5.1.3}$$

So for a price collar to have a tangible impact, it cannot exceed μ_n/μ_1. Hence, a price collar's effective range is:

$$1 \leq \kappa \leq \frac{\mu_n}{\mu_1}. \tag{5.1.4}$$

Economic constraints

A competitive insurance market would introduce further constraints on available premium regimes in the market to ensure that these regimes are economically viable. In particular, a perfectly competitive insurance market would require that any available premium regime in the market is also stable, in the sense of it not being susceptible to permitted unilateral decisions by competing insurers. Formally:

Constraint 3 (**Economic**)**.** *Perfect competition between insurers allows only stable premium regimes which are robust to permitted unilateral deviations, i.e. it is not possible for an insurer, operating in the same market, to profitably destabilise a stable premium regime by offering an alternative premium regime.*

A direct consequence of the economic constraint is that any stable premium regime leads to zero profits market equilibrium, because under perfect competition, no insurer can make sustained profits or endure sustained losses indefinitely. Henceforth, unless otherwise stated, we will only consider equilibrium premium regimes.

Using the notations developed in Chapter 3, in a perfectly competitive insurance market, we have:

$$\text{Premium income} = \sum_{i=1}^{n} p_i \, d_i(\pi_i) \, \pi_i. \tag{5.1.5}$$

$$\text{(Expected) insurance claim} = \sum_{i=1}^{n} p_i \, d_i(\pi_i) \, \mu_i. \tag{5.1.6}$$

$$\text{(Expected) profit} : E\left(\underline{\pi}\right) = \sum_{i=1}^{n} p_i \, d_i(\pi_i) \, \left(\pi_i - \mu_i\right). \tag{5.1.7}$$

$$\text{Market equilibrium} \Rightarrow E\left(\underline{\pi}\right) = 0. \tag{5.1.8}$$

Clearly, full risk classification regime, $\underline{\pi} = \underline{\mu}$ satisfies Equation 5.1.8. The full risk classification regime also satisfies the political constraint and also the regulatory constraint if the price collar is set at: $\kappa = \mu_n/\mu_1$.

At the other end of the spectrum, a regulator may ban all risk classification, by setting a price collar of $\kappa = 1$, resulting in a *pooled* regime, where all risk-groups are charged the same premium, $\pi_i = \pi_0$ for $i = 1, 2, \ldots, n$. Although multiple solutions are possible, only the cheapest of these solutions lead to a stable equilibrium, as any higher pooled equilibrium premium would be eliminated by competition.

In between the two extremes of full risk classification regime ($\kappa = \mu_n/\mu_1$) and pooled regime ($\kappa = 1$), a stable intermediate risk classification regime can also be achieved for a given price collar κ, where $1 < \kappa < \mu_n/\mu_1$, which satisfies the political, regulatory and economic constraints. This is presented in Theorem 5 (proof in Appendix G):

Theorem 5. *If there are n risk-groups, with risks $\mu_1 < \mu_2 < \cdots < \mu_n$, in presence of political, regulatory and economic constraints, with regulatory price collar of κ, where $1 \leq \kappa \leq \mu_n/\mu_1$, there exists a stable equilibrium premium regime $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$, such that:*

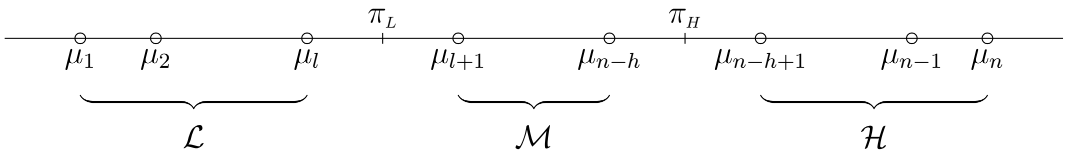
$$\pi_i = \begin{cases} \pi_L & \text{if } \mu_i < \pi_L; \\ \mu_i & \text{if } \pi_L \leq \mu_i \leq \pi_H; \\ \pi_H & \text{if } \mu_i > \pi_H. \end{cases} \tag{5.1.9}$$

where $\pi_L = \min_i \pi_i$, $\pi_H = \max_i \pi_i$ and $\pi_H = \kappa\,\pi_L$.

In other words, given a price collar κ, where $1 \leq \kappa \leq \mu_n/\mu_1$, a stable equilibrium premium regime will consist of three collection of risk-groups:

- l lowest risk-groups such that $\mu_i < \pi_L$, where $i \in \mathcal{L} = \{1, \ldots, l\}$;
- h highest risk-groups such that $\mu_i > \pi_H$, where $i \in \mathcal{H} = \{n - h + 1, \ldots, n\}$;
- remaining risk-groups such that $\pi_L \leq \mu_i \leq \pi_H$, where $i \in \mathcal{M} = \{l + 1, \ldots, n - h\}$;

where the premium regime $\underline{\pi}$ charges the premium π_L for all risk-groups in $\mathcal{L}$, the premium π_H for all risk-groups in $\mathcal{H}$ and the actuarially fair premiums for all remaining risk-groups in $\mathcal{M}$.



It is possible for $\mathcal{L}$, $\mathcal{M}$ or $\mathcal{H}$ to be empty. For example, for a full risk classification regime, all risk-groups, being charged the fair actuarial premium, belong to $\mathcal{M}$. For a pooled regime, $\mathcal{M}$ is empty, unless the pooled equilibrium premium happens to exactly match the risk of one of the risk-groups in the middle.

Given the nature of a stable market equilibrium under political, regulatory and economic constraints, a regulator might be interested in determining the optimal value of the price collar which would promote certain policy objectives. In this chapter, we will use iso-elastic insurance demand to determine the optimal price collar which would maximise loss coverage as a regulatory policy objective.

5.2 Loss Coverage under Iso-elastic Demand

In Equation 3.1.6 of Section 3.1.5 we have defined iso-elastic demand for insurance for risk-group i, as follows:

$$d_i(\pi_i) = \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i}, \text{ (subject to a cap of 1).} \tag{3.1.6}$$

The *risk-premium ratio*, μ_i/π_i, for risk-group i, which appears in Equation 3.1.6, plays a crucial role in our subsequent analysis, so we will denote it by v_i for ease of reference. Further, for the two extreme risk classification regimes, i.e. full risk classification and pooled, it would be convenient to extend the notation of v_i to v_{if} and v_{ip} respectively. Note that $v_{if} = 1$, and $v_{ip} \gtreqqless 1 \Leftrightarrow \pi_0 \lesseqqgtr \mu_i$, where π_0 is the pooled equilibrium premium.

For iso-elastic demand, the equilibrium condition in Equation 5.1.8 takes the form:

$$E\left(\underline{\pi}\right) = \sum_{i=1}^{n} p_i \, \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \left(\pi_i - \mu_i\right) = 0, \tag{5.2.1}$$

$$\Leftrightarrow \quad E\left(\underline{\pi}\right) = \sum_{i=1}^{n} p_i \, \tau_i \, \mu_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \left(\frac{\pi_i}{\mu_i} - 1\right) = 0, \tag{5.2.2}$$

$$\Leftrightarrow \quad E\left(\underline{v}\right) = \sum_{i=1}^{n} a_i \left(v_i^{\lambda_i - 1} - v_i^{\lambda_i}\right) = 0, \text{ where } a_i = \frac{p_i \, \tau_i \, \mu_i}{\sum_j p_j \, \tau_j \, \mu_j}. \tag{5.2.3}$$

In Equation 5.2.3, we have re-expressed the equilibrium condition $E\left(\underline{\pi}\right) = 0$, in terms of the risk-premium ratios, $\underline{v} = (v_1, v_2, \ldots, v_n)$, so that $E\left(\underline{v}\right) = 0$. We have also scaled the equilibrium condition in Equation 5.2.3, using the full risk classification regime's expected loss, so that the weights, a_i, add up to 1, i.e. $\sum_i a_i = 1$.

For iso-elastic demand, recalling the definition of loss coverage ratio from equation 3.4.2, the expression for loss coverage ratio, in terms of risk-premium ratios, takes the form:

$$C\left(\underline{v}\right) = \frac{\sum_{i=1}^{n} p_i \, \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i}{\sum_{i=1}^{n} p_i \, \tau_i \, \mu_i} = \sum_{i=1}^{n} a_i \, v_i^{\lambda_i}. \tag{5.2.4}$$

A regulator, or a policymaker, might be interested in maximising loss coverage ratio, so as to ensure that insurance purchased in the population covers the maximum possible losses in the population. To achieve this policy objective, a regulator would aim to set an optimal level of regulatory price collar, κ, so that loss coverage is maximised among all possible premium regimes, which satisfy the required political, regulatory and economic constraints.

Mathematically, the objective can be stated, in terms of premiums, as:

$$\max_{\kappa}\ C\left(\underline{\pi}\right),\ \text{subject to } E\left(\underline{\pi}\right)=0, \tag{5.2.5}$$

which, for the case of iso-elastic insurance demand, using risk-premium ratios, becomes:

$$\max_{\kappa}\ C\left(\underline{v}\right)=\sum_{i=1}^{n} a_i\, v_i^{\lambda_i},\ \text{subject to } E\left(\underline{v}\right)=\sum_{i=1}^{n} a_i\ \left(v_i^{\lambda_i-1}-v_i^{\lambda_i}\right)=0. \tag{5.2.6}$$

5.3 The Case of Two Risk-groups

For the simple case of iso-elastic demand with two risk-groups, the objective is to maximise:

$$C\left(\underline{v}\right)=a_1\, v_1^{\lambda_1}+a_2\, v_2^{\lambda_2}, \tag{5.3.1}$$

subject to the equilibrium condition:

$$E\left(\underline{v}\right)=a_1\ \left(v_1^{\lambda_1-1}-v_1^{\lambda_1}\right)+a_2\ \left(v_2^{\lambda_2-1}-v_2^{\lambda_2}\right)=0. \tag{5.3.2}$$

For two risk-groups, the political constraint, introduced in Section 5.1, requires:

$$\mu_1\leq\pi_1\leq\pi_2\leq\mu_2. \tag{5.3.3}$$

This translates into the following conditions for v_1 and v_2:

$$\mu_1\leq\pi_1\Rightarrow v_1=\frac{\mu_1}{\pi_1}\leq 1; \tag{5.3.4}$$

$$\pi_2\leq\mu_2\Rightarrow v_2=\frac{\mu_2}{\pi_2}\geq 1; \tag{5.3.5}$$

$$\pi_1\leq\pi_2\Rightarrow v_2=\frac{\mu_2}{\pi_2}=\frac{\mu_2}{\mu_1}\frac{\mu_1}{\pi_2}\leq\frac{\mu_2}{\mu_1}\frac{\mu_1}{\pi_1}=\frac{\mu_2}{\mu_1}\,v_1; \tag{5.3.6}$$

which, put together, lead to:

$$1 \leq \frac{v_2}{v_1} \leq \frac{\mu_2}{\mu_1}. \tag{5.3.7}$$

Note that the extremes, $v_1 = v_2 = 1$ signifies the full risk classification regime; and $v_2 = \frac{\mu_2}{\mu_1} v_1$, i.e. $\pi_1 = \pi_2 = \pi_0$, signifies the pooled regime.

Analysis of Premium Regimes under Equilibrium

Consider two premium regimes: $\underline{v} = (v_1, v_2)$ and $\underline{v} + \underline{dv} = (v_1 + dv_1, v_2 + dv_2)$, where both regimes satisfy the equilibrium condition in Equation 5.3.2, so that $E\left(\underline{v} + \underline{dv}\right) = E\left(\underline{v}\right) = 0$. If $\underline{dv}$ is "small", ignoring higher-order terms in the Taylor series expansion gives [1]:

$$dE = E\left(\underline{v} + \underline{dv}\right) - E\left(\underline{v}\right) = E_1\, dv_1 + E_2\, dv_2, \tag{5.3.8}$$

$$\text{where } E_i = \frac{\partial E}{\partial v_i} = -a_i\, \lambda_i\, v_i^{\lambda_i - 2} \left[v_i - \left(1 - \frac{1}{\lambda_i}\right) \right], \quad \text{for } i = 1, 2. \tag{5.3.9}$$

As $E\left(\underline{v} + \underline{dv}\right) = E\left(\underline{v}\right) = 0$, and thus $dE = 0$, the relationship between dv_1 and dv_2 can be expressed as:

$$\frac{dv_2}{dv_1} = -\frac{a_1\, \lambda_1\, v_1^{\lambda_1 - 2}}{a_2\, \lambda_2\, v_2^{\lambda_2 - 2}} \left[\frac{v_1 - \left(1 - \frac{1}{\lambda_1}\right)}{v_2 - \left(1 - \frac{1}{\lambda_2}\right)} \right]. \tag{5.3.10}$$

Analysis of Loss Coverage Ratios under Equilibrium

To compare loss coverage ratios under two equilibrium premium regimes: $\underline{v} = (v_1, v_2)$ and $\underline{v} + \underline{dv} = (v_1 + dv_1, v_2 + dv_2)$, Taylor series expansion ignoring higher-order terms gives:

$$dC = C_1\, dv_1 + C_2\, dv_2, \text{ where } C_i = \frac{\partial C}{\partial v_i} = a_i\, \lambda_i\, v_i^{\lambda_i - 1}, \text{ for } i = 1, 2. \tag{5.3.11}$$

[1] Readers, who are familiar with Lagrange multipliers and Kuhn-Tucker theorem (please see Dixit (1990) for an exposition from an economic perspective), would realise that the constrained maximisation problem, can be framed in terms of these optimisation approaches. However, instead of directly applying these methods mechanically, we provide a detailed analysis, so that the underlying economic interpretations are not overlooked.

Using the relationship between dv_1 and dv_2 in Equation 5.3.10, we get:

$$dC = a_1\,\lambda_1\,v_1^{\lambda_1-1}\,dv_1 + a_2\,\lambda_2\,v_2^{\lambda_2-1}\,dv_2, \tag{5.3.12}$$

$$\Rightarrow \frac{dC}{dv_1} = \underbrace{\left[\frac{a_1\,\lambda_1\,v_1^{\lambda_1-1}}{v_2 - \left(1-\frac{1}{\lambda_2}\right)}\right]}_{T_1}\;\underbrace{\left(1-\frac{1}{\lambda_1}\right)}_{T_2}\;\underbrace{\left[\frac{v_2}{v_1} - \frac{1-\frac{1}{\lambda_2}}{1-\frac{1}{\lambda_1}}\right]}_{T_3}. \tag{5.3.13}$$

The relative signs of the terms T_1, T_2 and T_3, in Equation 5.3.13, would vary according to the values of λ_1 and λ_2, which in turn would determine how C depends on v_1. Specifically, for given values of λ_1 and λ_2, analysis of dC/dv_1 would tell us if the loss coverage is maximised for a full risk classification regime or a pooled regime or a partial risk classification regime, which is intermediate between full risk classification and pooled regimes, or is indeterminate.

The result is presented in Theorem 6 (proof in Appendix H).

Theorem 6. *Suppose there are two risk-groups with risks $\mu_1 < \mu_2$ and iso-elastic demand elasticities λ_1 and λ_2 respectively. Consider the four segments, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, in the (λ_1, λ_2)–plane formed by the intersecting curves:*

$$\lambda_2 = \lambda_1; \tag{5.3.14}$$

$$\frac{1-\frac{1}{\lambda_2}}{1-\frac{1}{\lambda_1}} = \frac{\mu_2}{\mu_1}; \tag{5.3.15}$$

where

- $\mathcal{A} = \Big\{(\lambda_1, \lambda_2): \ \lambda_2 \geq \lambda_1\Big\} - \mathcal{D};$
- $\mathcal{B} = \Big\{(\lambda_1, \lambda_2): \ \lambda_2 \leq \lambda_1 \leq 1 \textit{ and } 1 \leq \big(1-\tfrac{1}{\lambda_2}\big)\big/\big(1-\tfrac{1}{\lambda_1}\big) \leq \frac{\mu_2}{\mu_1}\Big\};$
- $\mathcal{C} = \Big\{(\lambda_1, \lambda_2): \ \lambda_2 \leq \lambda_1\Big\} - \mathcal{B};$
- $\mathcal{D} = \Big\{(\lambda_1, \lambda_2): \ \lambda_2 \geq \lambda_1 \geq 1 \textit{ and } 1 \leq \big(1-\tfrac{1}{\lambda_2}\big)\big/\big(1-\tfrac{1}{\lambda_1}\big) \leq \frac{\mu_2}{\mu_1}\Big\}.$

For each of the segments, we have:

6.1. $\mathcal{A}$ *: Loss coverage is maximum for pooled and minimum for full risk classification regime, while partial risk classification is intermediate.*

6.2. $\mathcal{B}$ *: Loss coverage is minimum for a specific partial risk classification regime and maximum for either pooled or full risk classification.*

6.3. $\mathcal{C}$ *: Loss coverage is maximum for full risk classification regime and minimum for pooled, while partial risk classification is intermediate.*

6.4. $\mathcal{D}$ *: Loss coverage is maximum for a specific partial risk classification regime.*

Figure 5.1 provides a graphical representation of the partitions of (λ_1, λ_2)–plane into segments $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ for $\mu_2/\mu_1 = 2$, showing where the different risk classification regimes, i.e. pooled, full risk classification or partial, lead to maximum loss coverage. Figure 5.1 also shows subdivision of the segments $\mathcal{A}$ and $\mathcal{C}$ into three sub-segments each, by the vertical and horizontal axes $\lambda_1 = 1$ and $\lambda_2 = 1$ respectively. The sub-segments of $\mathcal{A}$ and $\mathcal{C}$ are denoted by $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ and $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ respectively.

Using Theorem 6 and based on the actual values of λ_1 and λ_2 prevalent in the insurance market, a regulator would be able set appropriate levels of price collar to maximise loss coverage in the population. For example, if (λ_1, λ_2) falls in:

$\mathcal{A}$: set price collar $\kappa = 1$;

$\mathcal{B}$: set price collar $\kappa = 1$ or $\kappa = \mu_2/\mu_1$ depending on whether pooled or full risk classification regime maximises loss coverage;

$\mathcal{C}$: set price collar $\kappa = \mu_2/\mu_1$ (or not set a price collar at all);

$\mathcal{D}$: set price collar at a level κ, which produces the partial risk classification regime maximising loss coverage.

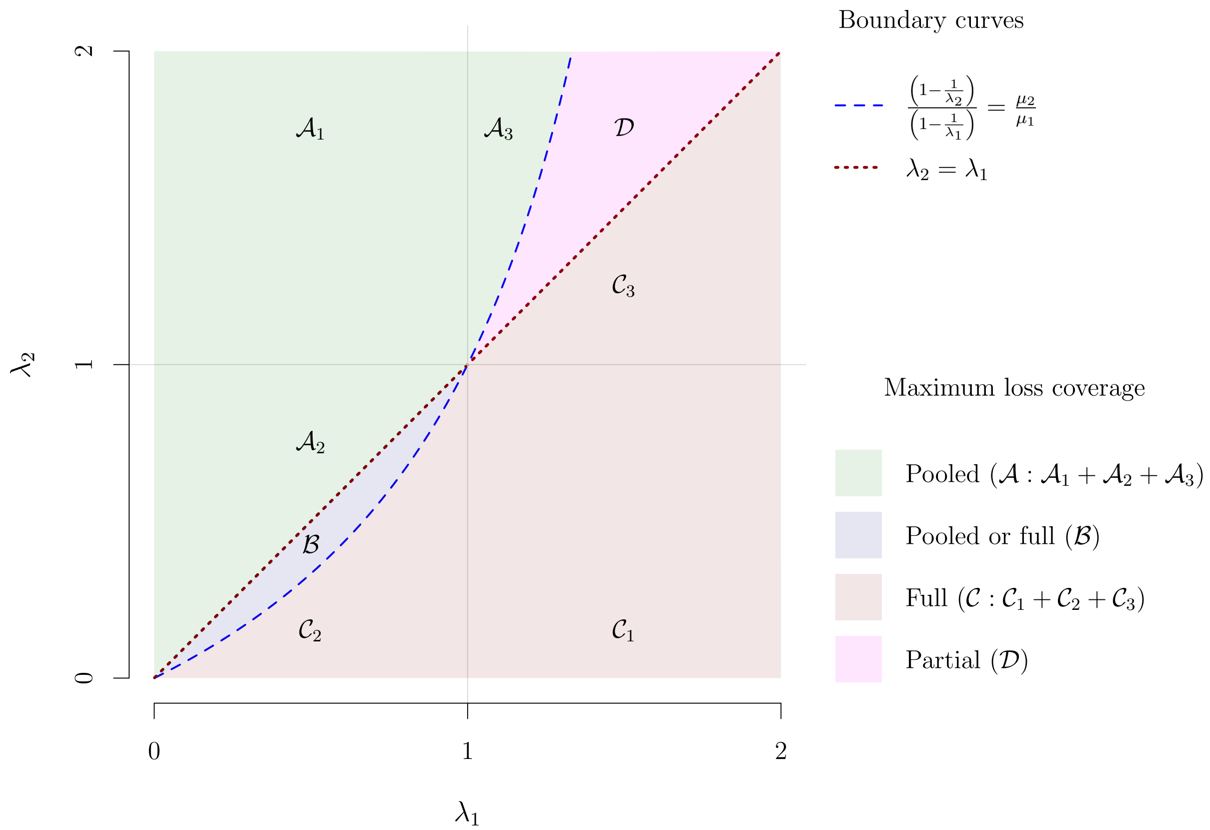


Figure 5.1: For $\mu_2/\mu_1 = 2$, the plot shows the partition of the (λ_1, λ_2)–plane into four segments, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, where different risk classification regimes lead to maximum loss coverage. The segments $\mathcal{A}$ and $\mathcal{C}$ are further subdivided into three sub-segments each, namely $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ and $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ respectively, by the vertical and horizontal axes $\lambda_1 = 1$ and $\lambda_2 = 1$ respectively.

5.4 Generalisation to n Risk-groups

In an insurance market with more than two risk-groups, we can generalise the result in Theorem 6, using the structure of the stable equilibrium premium regime in presence of the political, regulatory and economic constraints, as outlined in Theorem 5,.

Recall that, according to Theorem 5, the stable equilibrium premium regime $\underline{\pi}$ charges: premium π_L for all risk-groups in $\mathcal{L}$, i.e. with risks less than π_L; premium π_H for all risk-groups in $\mathcal{H}$, i.e. with risks more than π_H; and actuarially fair premium for all remaining risk-groups in $\mathcal{M}$.

While generalising our results to more than two risk-groups, we will assume that all risk-groups in $\mathcal{L}$ have the same iso-elastic demand elasticity λ_L and all risk-groups in $\mathcal{H}$ have the same iso-elastic demand elasticity λ_H. This is based on the premise that risk-groups with similar risks are likely to have similar sensitivities towards price changes.

Analysis of Premium Regimes Under Equilibrium

As the risk-groups in $\mathcal{M}$ do not contribute to profit or loss, the equilibrium condition can be expressed as:

$$E\left(\underline{\pi}\right) = \underbrace{\sum_{i \in \mathcal{L}} p_i\, \tau_i \left(\frac{\mu_i}{\pi_L}\right)^{\lambda_L} \left(\pi_L - \mu_i\right)}_{E_{\mathcal{L}}} + \underbrace{\sum_{j \in \mathcal{H}} p_j\, \tau_j \left(\frac{\mu_j}{\pi_H}\right)^{\lambda_H} \left(\pi_H - \mu_j\right)}_{E_{\mathcal{H}}} = 0. \tag{5.4.1}$$

The first term, $E_{\mathcal{L}}$, in Equation 5.4.1, can be split as follows:

$$E_{\mathcal{L}} = \sum_{i \in \mathcal{L}} p_i\, \tau_i \left(\frac{\mu_i}{\pi_L}\right)^{\lambda_L} \left(\pi_L - \mu_i\right); \tag{5.4.2}$$

$$= \sum_{i \in \mathcal{L}} p_i\, \tau_i \left(\frac{\mu_L}{\pi_L}\right)^{\lambda_L} \left(\frac{\mu_i}{\mu_L}\right)^{\lambda_L} \left[\left(\pi_L - \mu_L\right) + \left(\mu_L - \mu_i\right)\right]; \tag{5.4.3}$$

$$= \left(\frac{\mu_L}{\pi_{\scriptscriptstyle L}}\right)^{\lambda_L} \left[\sum_{i\in\mathcal{L}} p_i\,\tau_i \left(\frac{\mu_i}{\mu_L}\right)^{\lambda_L} (\pi_{\scriptscriptstyle L} - \mu_L) + \sum_{i\in\mathcal{L}} p_i\,\tau_i \left(\frac{\mu_i}{\mu_L}\right)^{\lambda_L} (\mu_L - \mu_i)\right]; \quad (5.4.4)$$

where μ_L is such that the second term in Equation 5.4.4, is zero, i.e.:

$$\sum_{i\in\mathcal{L}} p_i\,\tau_i \left(\frac{\mu_i}{\mu_L}\right)^{\lambda_L} (\mu_L - \mu_i) = 0, \quad (5.4.5)$$

so that μ_L can be interpreted as the pooled equilibrium premium, if the insurance market only consisted of the risk-groups in $\mathcal{L}$. Also, μ_L is unique and is given by:

$$\mu_L = \frac{\sum_{i\in\mathcal{L}} p_i\,\tau_i\,\mu_i^{\lambda_L+1}}{\sum_{i\in\mathcal{L}} p_i\,\tau_i\,\mu_i^{\lambda_L}}, \quad \text{so that:} \quad \mu_1 \le \mu_L \le \max_{i\in\mathcal{L}} \mu_i \le \pi_{\scriptscriptstyle L}. \quad (5.4.6)$$

Using such a μ_L, the expression for $E_{\mathcal{L}}$ in Equation 5.4.4 becomes:

$$E_{\mathcal{L}} = \left[\sum_{i\in\mathcal{L}} p_i\,\tau_i \left(\frac{\mu_i}{\mu_L}\right)^{\lambda_L}\right] \left(\frac{\mu_L}{\pi_{\scriptscriptstyle L}}\right)^{\lambda_L} (\pi_{\scriptscriptstyle L} - \mu_L) = p_L\tau_L \left(\frac{\mu_L}{\pi_{\scriptscriptstyle L}}\right)^{\lambda_L} (\pi_{\scriptscriptstyle L} - \mu_L); \quad (5.4.7)$$

$$\text{where } p_L = \sum_{i\in\mathcal{L}} p_i, \;\; \tau_L = \sum_{i\in\mathcal{L}} \left(\frac{p_i}{p_L}\right) \tau_i \left(\frac{\mu_i}{\mu_L}\right)^{\lambda_L}. \quad (5.4.8)$$

Note that p_L is the aggregate proportion of population belonging to the collection of risk-groups in $\mathcal{L}$ and τ_L can be interpreted as the 'fair-premium demand' when all risk-groups in $\mathcal{L}$ are pooled and charged the same pooled premium, μ_L.

A similar line of argument for the risk-groups in $\mathcal{H}$ leads to:

$$E_{\mathcal{H}} = p_{\scriptscriptstyle H}\tau_{\scriptscriptstyle H} \left(\frac{\mu_{\scriptscriptstyle H}}{\pi_{\scriptscriptstyle H}}\right)^{\lambda_H} (\pi_{\scriptscriptstyle H} - \mu_{\scriptscriptstyle H}); \text{ where } p_{\scriptscriptstyle H} = \sum_{j\in\mathcal{H}} p_j, \;\; \tau_{\scriptscriptstyle H} = \sum_{j\in\mathcal{H}} \left(\frac{p_j}{p_{\scriptscriptstyle H}}\right) \tau_j \left(\frac{\mu_j}{\mu_{\scriptscriptstyle H}}\right)^{\lambda_H}, \quad (5.4.9)$$

$$\text{where } \mu_{\scriptscriptstyle H} = \frac{\sum_{j\in\mathcal{H}} p_j\,\tau_j\,\mu_j^{\lambda_H+1}}{\sum_{j\in\mathcal{H}} p_j\,\tau_j\,\mu_j^{\lambda_H}}, \quad \text{so that:} \quad \pi_{\scriptscriptstyle H} \le \min_{j\in\mathcal{H}} \mu_j \le \mu_{\scriptscriptstyle H} \le \mu_n. \quad (5.4.10)$$

Using the expressions for $E_{\mathcal{L}}$ and $E_{\mathcal{H}}$ in Equations 5.4.7 and 5.4.9 respectively, Equation 5.4.1 becomes:

$$E\left(\underline{\pi}\right) = \underbrace{p_L\,\tau_L \left(\frac{\mu_L}{\pi_L}\right)^{\lambda_L} \left(\pi_L - \mu_L\right)}_{E_{\mathcal{L}}} \; + \; \underbrace{p_H\,\tau_H \left(\frac{\mu_H}{\pi_H}\right)^{\lambda_H} \left(\pi_H - \mu_H\right)}_{E_{\mathcal{H}}} = 0. \tag{5.4.11}$$

Equation 5.4.11 shows that it is possible to conceptualise $\mathcal{L}$ and $\mathcal{H}$ as *collective* risk-groups with demand elasticities λ_L and λ_H respectively, where the collective risks are the pooled equilibrium premiums, μ_L and μ_H, of the respective collections. Essentially, this reduces the problem involving more than two risk-groups to the simpler two risk-groups problem, so that the analysis of Section 5.3 can be extended directly to this situation.

Following the approach of Section 5.3, we express the equilibrium condition in Equations 5.4.1 in terms of the risk-premium ratios, as follows:

$$E\left(\underline{v}\right) = a_L\,\left(v_L^{\lambda L-1} - v_L^{\lambda L}\right) + a_H\,\left(v_H^{\lambda H-1} - v_H^{\lambda H}\right) = 0; \tag{5.4.12}$$

$$\text{where } v_L = \frac{\mu_L}{\pi_L}; \quad v_H = \frac{\mu_H}{\pi_H}; \tag{5.4.13}$$

$$a_L = \frac{p_L\,\tau_L\,\mu_L}{p_L\,\tau_L\,\mu_L + p_H\,\tau_H\,\mu_H + \sum_{m\in\mathcal{M}} p_m\,\tau_m\,\mu_m}; \tag{5.4.14}$$

$$a_H = \frac{p_H\,\tau_H\,\mu_H}{p_L\,\tau_L\,\mu_L + p_H\,\tau_H\,\mu_H + \sum_{m\in\mathcal{M}} p_m\,\tau_m\,\mu_m}. \tag{5.4.15}$$

Analysis of Loss Coverage Ratios Under Equilibrium

The loss coverage for the stable premium regime can then be expressed as:

$$C\left(\underline{\pi}\right) = \sum_{i\in\mathcal{L}} p_i\,\tau_i \left(\frac{\mu_i}{\pi_L}\right)^{\lambda_L} \mu_i \; + \; \sum_{j\in\mathcal{H}} p_j\,\tau_j \left(\frac{\mu_j}{\pi_H}\right)^{\lambda_H} \mu_j \; + \; \sum_{m\in\mathcal{M}} p_m\,\tau_m\,\mu_j; \tag{5.4.16}$$

$$= \sum_{i\in\mathcal{L}} p_i\,\tau_i \left(\frac{\mu_i}{\pi_L}\right)^{\lambda_L} \pi_L \; + \; \sum_{j\in\mathcal{H}} p_j\,\tau_j \left(\frac{\mu_j}{\pi_H}\right)^{\lambda_H} \pi_H \; + \; \sum_{m\in\mathcal{M}} p_m\,\tau_m\,\mu_j; \tag{5.4.17}$$

... by the equilibrium condition in Equation 5.4.1;

$$= p_L\, \tau_L \left(\frac{\mu_L}{\pi_L}\right)^{\lambda_L} \pi_L \;+\; p_H\, \tau_H \left(\frac{\mu_H}{\pi_H}\right)^{\lambda_H} \pi_H \;+\; \sum_{m\in\mathcal{M}} p_m\, \tau_m\, \mu_j; \tag{5.4.18}$$

... by the definitions of p_L, τ_L, μ_L, p_H, τ_H and μ_H;

$$= p_L\, \tau_L \left(\frac{\mu_L}{\pi_L}\right)^{\lambda_L} \mu_L \;+\; p_H\, \tau_H \left(\frac{\mu_H}{\pi_H}\right)^{\lambda_H} \mu_H \;+\; \sum_{m\in\mathcal{M}} p_m\, \tau_m\, \mu_j; \tag{5.4.19}$$

by the equilibrium condition in Equation 5.4.11.

Using the definition of loss coverage ratio in Equation 3.4.2 and the expression for loss coverage in Equation 5.4.19, we get:

$$C\left(\underline{\pi}\right) = \frac{LC\left(\underline{\pi}\right)}{LC\left(\underline{\mu}\right)} = \frac{p_L\, \tau_L \left(\frac{\mu_L}{\pi_L}\right)^{\lambda_L} \mu_L \;+\; p_H\, \tau_H \left(\frac{\mu_H}{\pi_H}\right)^{\lambda_H} \mu_H + \sum_{m\in\mathcal{M}} p_m\, \tau_m\, \mu_m}{\sum_{i=1}^{n} p_i\, \tau_i\, \mu_i}; \tag{5.4.20}$$

$$= \xi\, \frac{p_L\, \tau_L \left(\frac{\mu_L}{\pi_L}\right)^{\lambda_L} \mu_L \;+\; p_H\, \tau_H \left(\frac{\mu_H}{\pi_H}\right)^{\lambda_H} \mu_H + \sum_{m\in\mathcal{M}} p_m\, \tau_m\, \mu_m}{p_L\, \tau_L\, \mu_L + p_H\, \tau_H\, \mu_H + \sum_{m\in\mathcal{M}} p_m\, \tau_m\, \mu_m}; \tag{5.4.21}$$

$$\text{where } \xi = \frac{p_L\, \tau_L\, \mu_L + p_H\, \tau_H\, \mu_H + \sum_{m\in\mathcal{M}} p_m\, \tau_m\, \mu_m}{\sum_{i=1}^{n} p_i\, \tau_i\, \mu_i}. \tag{5.4.22}$$

The loss coverage ratio can then be expressed in terms of the risk-premium ratios as:

$$C\left(\underline{v}\right) = \xi\left[a_L\, v_L^{\lambda L} + a_H\, v_H^{\lambda_H} + a_M\right], \tag{5.4.23}$$

$$\text{where } a_M = \frac{\sum_{m\in\mathcal{M}} p_m\, \tau_m\, \mu_m}{p_L\, \tau_L\, \mu_L + p_H\, \tau_H\, \mu_H + \sum_{m\in\mathcal{M}} p_m\, \tau_m\, \mu_m}. \tag{5.4.24}$$

The role of the constant, ξ, is to provide the link of the loss coverage ratios between the insurance market with n distinct risk-groups and the conceptualised market involving the two collective risk-groups, $\mathcal{L}$ and $\mathcal{H}$ (along with the risk-groups in $\mathcal{M}$). ξ can also be interpreted as the loss coverage ratio for a premium regime charging the respective pooled premiums μ_L and μ_H for the collective risk-groups $\mathcal{L}$ and $\mathcal{H}$ (and fair actuarial premium for risk-groups in $\mathcal{M}$).

Assuming the compositions of $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{H}$ remain unaffected, changing π_L (or, equivalently, v_L) affects π_H (or, equivalently, v_H) without any implications for the risk-groups in $\mathcal{M}$. Also note that, as long as $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{H}$ remain unchanged, the constant ξ is unaffected. Then following the same steps as in Section 5.3, we get:

$$\frac{dC}{dv_L} = \xi \underbrace{\left[\frac{a_L\, \lambda_L\, v_L^{\lambda_L - 1}}{v_H - \left(1 - \frac{1}{\lambda_H}\right)}\right]}_{T_1} \underbrace{\left(1 - \frac{1}{\lambda_L}\right)}_{T_2} \underbrace{\left[\frac{v_H}{v_L} - \frac{\left(1 - \frac{1}{\lambda_H}\right)}{\left(1 - \frac{1}{\lambda_L}\right)}\right]}_{T_3}. \tag{5.4.25}$$

An alternative proof of Equation 5.4.25, from first principles, is given in Appendix I.

Noting that ξ is positive, the three multiplicative terms, T_1, T_2 and T_3, in Equation 5.4.25 can be analysed in the same way as in Theorem 6. Given that a_L, λ_L, λ_H and v_L are all positive:

$$\mu_H \geq \pi_H \text{ and } \lambda_H > 0 \Rightarrow T_1 \geq 0. \tag{5.4.26}$$

$$\text{Also: } \lambda_L \gtreqless 1 \Leftrightarrow T_2 \gtreqless 0. \tag{5.4.27}$$

T_3 can be analysed in the same way as for the case of two risk-groups. The general result is presented in Theorem 7.

Theorem 7. *Suppose there are n risk-groups, with risks $\mu_1 < \mu_2 < \cdots < \mu_n$ and iso-elastic demand elasticities $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively.*

Under political, regulatory and economic constraints, with regulatory price collar κ, where $1 \leq \kappa \leq \mu_n/\mu_1$, let $\underline{\pi}$ be the stable equilibrium premium regime, subdividing the risk-groups in three collections $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{H}$, where all risk-groups in $\mathcal{L}$ pay the same premium $\pi_{\scriptscriptstyle L}$, all risk-groups in $\mathcal{H}$ pay the same premium $\pi_{\scriptscriptstyle H}$, and all risk-groups in $\mathcal{M}$ pay their fair actuarial premiums. Further suppose:

$$\lambda_i = \begin{cases} \lambda_{\scriptscriptstyle L} & \text{if } i \in \mathcal{L}; \\ \lambda_{\scriptscriptstyle H} & \text{if } i \in \mathcal{H}. \end{cases} \tag{5.4.28}$$

Let μ_L and $\mu_{\scriptscriptstyle H}$ be the pooled equilibrium premiums of the risk-groups in $\mathcal{L}$ and $\mathcal{H}$ respectively. Consider the four segments, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, in the $(\lambda_{\scriptscriptstyle L}, \lambda_{\scriptscriptstyle H})$–plane formed by the intersecting curves:

$$\lambda_{\scriptscriptstyle H} = \lambda_{\scriptscriptstyle L}; \tag{5.4.29}$$

$$\frac{1 - \frac{1}{\lambda_H}}{1 - \frac{1}{\lambda_L}} = \frac{\mu_H}{\mu_L}; \tag{5.4.30}$$

where

- $\mathcal{A} = \Big\{(\lambda_{\scriptscriptstyle L}, \lambda_{\scriptscriptstyle H}) : \ \lambda_{\scriptscriptstyle H} \geq \lambda_{\scriptscriptstyle L}\Big\} - \mathcal{D};$
- $\mathcal{B} = \Big\{(\lambda_{\scriptscriptstyle L}, \lambda_{\scriptscriptstyle H}) : \ \lambda_{\scriptscriptstyle H} \leq \lambda_{\scriptscriptstyle L} \leq 1 \text{ and } 1 \leq \big(1-\tfrac{1}{\lambda_H}\big)/\big(1-\tfrac{1}{\lambda_L}\big) \leq \dfrac{\mu_H}{\mu_L}\Big\};$
- $\mathcal{C} = \Big\{(\lambda_{\scriptscriptstyle L}, \lambda_{\scriptscriptstyle H}) : \ \lambda_{\scriptscriptstyle H} \leq \lambda_{\scriptscriptstyle L}\Big\} - \mathcal{B};$
- $\mathcal{D} = \Big\{(\lambda_{\scriptscriptstyle L}, \lambda_{\scriptscriptstyle H}) : \ \lambda_{\scriptscriptstyle H} \geq \lambda_{\scriptscriptstyle L} \geq 1 \text{ and } 1 \leq \big(1-\tfrac{1}{\lambda_H}\big)/\big(1-\tfrac{1}{\lambda_L}\big) \leq \dfrac{\mu_H}{\mu_L}\Big\}.$

For each of the segments, we have:

7.1. $\mathcal{A}$ *: Loss coverage increases if price collar, κ, is decreased.*

7.2. $\mathcal{B}$ *: Loss coverage is minimum for a specific price collar, κ.*

7.3. $\mathcal{C}$ *: Loss coverage increases if price collar, κ, is increased.*

7.4. $\mathcal{D}$ *: Loss coverage is maximum for a specific price collar, κ.*

Theorem 7 follows directly from Theorem 6, with an added complication that if the price collar, κ, changes, then it is possible for the compositions of $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{H}$ to change. However, assuming insurance demands to be continuously differentiable functions of premiums, which is indeed the case for iso-elastic demand functions, the conclusions would remain unaffected if the constituents of $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{H}$ change when κ changes. Moreover, it is important to be mindful of the fact that (μ_H/μ_L) will change as the compositions of $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{H}$ change, and hence the actual boundaries of segments $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ would shift slightly as a result.

Note that, Theorem 7 requires that the risk-groups in $\mathcal{L}$ and $\mathcal{H}$ always have iso-elastic demand elasticities λ_L and λ_H respectively, even if the compositions of $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{H}$ change. This is not necessarily a constraint because, as κ changes, transfer of risk-groups would only happen between collections $\mathcal{L}$ and $\mathcal{M}$ or between collections $\mathcal{M}$ and $\mathcal{H}$. So, Theorem 7 only requires that if a risk-group in $\mathcal{M}$ joins $\mathcal{L}$, it has a demand elasticity of λ_L, and if a risk-group in $\mathcal{M}$ joins $\mathcal{H}$, it has a demand elasticity of λ_H.

5.5 Discussion of Results

In this chapter, we have dealt with the partial risk classification scenarios where regulator can introduce a price collar to restrict risk classifications. Note that the price collar, at its lower and upper extremes represent the pooled and full risk classification regimes respectively.

Our results for partial classification regimes are in agreement with our general finding that when demand elasticities are less than 1, and low risk-groups are less price sensitive than the high risk-groups, more pooling, characterised by lowering the price collar κ, increases social welfare and insurance loss coverage. We derived sufficient conditions on demand elasticities, which a regulator may find useful to devise a price collar to achieve higher loss coverage.

Chapter 6

Conclusions

This thesis has evaluated the welfare effects of restrictions on risk classification, in circumstances where institutional or regulatory factors lead insurers to restrict risk classification, and in particular the extreme case of pooling all risks at a common price. Such restrictions have both efficiency and equity effects. Depending on the distribution of utility functions in the population and the resulting insurance demand functions and elasticities, utilitarian social welfare and insurance loss coverage can increase or decrease.

In terms of social welfare, the distribution of utility functions in the population influences social welfare through two mechanisms. First, utility functions determine individuals' insurance purchasing decisions, which determine the insurance demand curve and hence the equilibrium prices when insurance risk classifications are restricted. Second, utility functions determine the utilities which individuals assign to their outcomes *given* the equilibrium insurance premium.

Because the distribution of utility functions implies the aggregate insurance demand function and vice versa, the distribution of utility functions across the population can be completely characterised by demand elasticities. Hence in this thesis, demand elasticity functions have been used to specify both demand and (implicitly) the distribution of utility functions in the population.

We derived sufficient conditions on demand elasticities of higher and lower risks which ensure that social welfare will be higher under pooling than under fully risk-differentiated premiums. The conditions were stated first for iso-elastic demand with a single elasticity parameter; then for iso-elastic demand with different elasticity parameters for different risk-groups; and then generalised in a form applicable to other demand functions using the concept of arc elasticity. The conditions for higher social welfare under pooling encompass many plausible combinations of higher and lower risks' demand elasticities, particularly in scenarios where all demand elasticities are less than 1.

In general, from the results we can see that as long as lower risk-groups have demand elasticities less than 1, and higher risk-groups have higher demand elasticities than the lower risk-groups, social welfare under pooling is higher. Having lower demand elasticity implies that the lower risk-groups are less price sensitive. Hence the demand from lower risk-groups would not fall significantly under pooling, whereas higher risk-groups being more price sensitive, would buy more insurance under pooling. Rise in expected utility of higher risk-groups under pooling would more than compensate the loss of utility of lower risk-groups caused by falling coverage and subsidising higher risk-groups. As a result social welfare at the aggregate level is expected to rise under pooling in these conditions.

Full risk classification or ban of risk classification are two extreme cases of equilibrium premium pricing regimes. Due to various reasons they may not be practicable in real life. For example, a full risk classification can be prohibitively expensive for insurers and therefore some pooling of risk-groups can be expected. Pooling of specific risk-groups can be mandated by regulators too. For example a regulator may ban genetic risk profiling but allow profiling based on lifestyle choice. On the other hand, complete ban on risk profiling can also be unimplementable due to political or ethical reasons. Hence, partial risk classification would be an area of interest for any real world policy maker.

In our analysis of partial risk classification, we have investigated the effect of a price collar on loss coverage. To obtain analytical results, it was necessary to use loss coverage, rather than social welfare, as our policy metric in this part of the thesis. Note that, full risk classification and pooling can be interpreted as special cases of a price collar. Under iso-elastic demand function, we provide a set of conditions on demand elasticities which determines the impact of changing price collar on loss coverage. When implementing a complete ban on classification is not plausible, a regulator can use price collars to achieve a relatively high level of loss coverage and social welfare.

In summary, the results show that there exists a large set of scenarios, especially where the lower risk-groups have demand elasticity less than 1, and are relatively less price sensitive than higher risk-groups, where imposition of some price collar to limit the price charged for high risk-groups would deliver higher benefit to the society as a whole, compared to an unrestricted risk classification regime. This rise in benefit can be quantified by expected utilitarian social welfare or by insurance loss coverage, both measures broadly producing similar conclusions.

Empirical data on insurance demand elasticities is available in many insurance markets. We hope that with the help of these results regulators and policymakers can set policies on risk classifications, with clear objectives of maximising social welfare and/or loss coverage, for these markets. This thesis shows that these societal benefits can be realised, regardless of the proportion of people in different risk-groups in the population.

6.1 Extensions

In the current thesis we have focused on a specific case of partial risk classification, viz. price collar, and analysed its effect on loss coverage. However, regulators could also pursue other forms of restricted risk classification. For example, a regulator may restrict risk classification for some risk categories (e.g. gender), but not for others (e.g. lifestyle choices). These scenarios can be compared with the polar cases (i.e. full classification or pooling) by identifying and comparing against all the possible intermediate groupings of the risk-groups permitted by any regulatory ban.

A detailed analysis would require us to systematically enumerate and analyse all possible partial risk classifications permitted under a given regulatory regime. For two risk-groups, only the polar risk classification regimes are possible. For three risk-groups, in addition to the two polar regimes, three partial risk classification regimes are possible, by grouping two of the risk-groups together while leaving out the third; this gives a total of five possible regimes. The number of possible regimes grows super-exponentially with the number of risk-groups. In combinatorial mathematics, this is equivalent to counting all possible partitions of a n-member set, known as the Bell number, B_n (for more details on Bell numbers see Sándor and Crstici (2004)). For six risk groups, the Bell number is $B_6 = 203$, and for ten risk groups, $B_{10} = 115{,}975$, which suggests that analysis of partial risk classification with a realistic number of risk-groups might require a more numerical approach.

Analysis of social welfare directly (i.e. without reference to the loss coverage) under partial risk classification and for generalised form of demand function could also be considered as a possible area of future research.

Appendices

Appendix A

Expressions for Social Welfare Under Iso-elastic Demand

Lemma 1. *Suppose there are n risk-groups with risks $\mu_1 < \mu_2 < \cdots < \mu_n$ with iso-elastic demand elasticities $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, then for a given premium regime $\underline{\pi}$, the expression for social welfare is given by:*

$$S(\underline{\pi}) = \sum_{i=1}^{n} p_i \, \tau_i \, \frac{1}{(\lambda_i + 1)} \left(\frac{\mu_i}{\pi_i} \right)^{\lambda_i + 1} \pi_i + K, \tag{4.1.8}$$

where the premium regime $\underline{\pi}$ satisfies the equilibrium condition:

$$\sum_{i=1}^{n} p_i \tau_i \left(\frac{\mu_i}{\pi_i} \right)^{\lambda_i} (\pi_i - \mu_i) = 0, \tag{4.1.9}$$

and the constant K does not depend on the premium regime under consideration.

Proof. The equilibrium condition follows directly by inserting the specific expression for iso-elastic insurance demand in Equation 3.3.15.

Now recall that, given a risk-group i, insurance is purchased when $\Gamma_i < \mu_i/\pi_i$ (a subscript i in Γ_i is used to denote the random variable specific to risk-group i). Hence:

$$[\,Q \mid \text{Risk-group } i\,] = I\left[\Gamma_i < \frac{\mu_i}{\pi_i}\right] \Rightarrow \mathbb{E}\,[\,Q \mid \text{Risk-group } i\,] = \mathrm{P}\left[\Gamma_i < \frac{\mu_i}{\pi_i}\right] = d_i(\pi_i), \tag{A.0.1}$$

where $I(\cdot)$ is the indicator function.

Using the expression for social welfare as given in Equation 3.3.11 we have:

$$S(\underline{\pi}) = \mathbb{E}\,[\,Q\,(X - \Pi\,\Gamma)\,] + K = \mathbb{E}\,[\,Q\,X\,] - \mathbb{E}\,[\,Q\,\Pi\,\Gamma\,] + K. \tag{A.0.2}$$

Evaluating each of these terms separately:

$$\mathbb{E}\,[\,Q\,X\,] = \sum_{i=1}^{n} \mathrm{P}[\text{Risk-group } i]\,\mathbb{E}\,[\,Q\,X \mid \text{Risk-group } i\,] \tag{A.0.3}$$

$$= \sum_{i=1}^{n} p_i\,\mathbb{E}\,[\,Q \mid \text{Risk-group } i\,]\;\mathbb{E}\,[\,X \mid \text{Risk-group } i\,], \quad \text{using Assumption 8,} \tag{A.0.4}$$

$$= \sum_{i=1}^{n} p_i\,d_i(\pi_i)\,\mu_i, \tag{A.0.5}$$

$$= \sum_{i=1}^{n} p_i\,\tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i, \tag{A.0.6}$$

$$= \sum_{i=1}^{n} p_i\,\tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i + 1} \pi_i, \tag{A.0.7}$$

and:

$$\mathbb{E}\,[\,Q\,\Pi\,\Gamma\,] = \sum_{i=1}^{n} \mathrm{P}[\text{Risk-group } i]\,\mathbb{E}\,[\,Q\,\Pi\,\Gamma \mid \text{Risk-group } i\,] \tag{A.0.8}$$

$$= \sum_{i=1}^{n} p_i\,\mathbb{E}\left[\,I\left[\Gamma_i < \frac{\mu_i}{\pi_i}\right]\,\Gamma_i\,\right]\pi_i, \tag{A.0.9}$$

$$= \sum_{i=1}^{n} p_i \left[\int_0^{\frac{\mu_i}{\pi_i}} \gamma \tau_i \lambda_i \gamma^{\lambda_i - 1} d\gamma \right] \pi_i, \quad \text{using the distribution of } \Gamma_i \text{ in Equation 4.1.7,} \tag{A.0.10}$$

$$= \sum_{i=1}^{n} p_i \, \tau_i \, \frac{\lambda_i}{(\lambda_i + 1)} \left(\frac{\mu_i}{\pi_i} \right)^{\lambda_i + 1} \pi_i. \tag{A.0.11}$$

Putting these together, we have:

$$S(\underline{\pi}) = \sum_{i=1}^{n} p_i \, \tau_i \, \frac{1}{(\lambda_i + 1)} \left(\frac{\mu_i}{\pi_i} \right)^{\lambda_i + 1} \pi_i + K, \tag{A.0.12}$$

where $K = E[1 - X]$ does not depend on the premium regime under consideration. $\square$

Appendix B

Proof of Theorem 1

Theorem 1. *Suppose there are n risk-groups with risks $\mu_1 < \mu_2 < \cdots < \mu_n$ with the same positive constant demand elasticity λ for all risk-groups. Then:*

$$\lambda \lesseqqgtr 1 \Rightarrow S(\pi_0) \gtreqqless S(\underline{\mu}). \tag{4.1.14}$$

Proof. Using the construction involving risk-premium ratios, $v_i = \mu_i/\pi_0$, we observe that, under the assumption of the same constant demand elasticity, λ, for all risk-groups, the equilibrium condition in Equation 4.1.10 simply becomes:

$$\sum_{i=1}^{n} \alpha_i v_i^{\lambda+1} = \sum_{i=1}^{n} \alpha_i v_i^{\lambda}. \tag{B.0.1}$$

And the condition comparing social welfare under pooling against that under the full risk classification regime in Equation 4.1.12 simplifies to:

$$S(\pi_0) \gtreqqless S(\underline{\mu}) \Leftrightarrow \sum_{i=1}^{n} \frac{\alpha_i\, v_i^{\lambda+1}}{\lambda+1} \gtreqqless \sum_{i=1}^{n} \frac{\alpha_i\, v_i}{\lambda+1} \Leftrightarrow \sum_{i=1}^{n} \alpha_i\, v_i^{\lambda+1} \gtreqqless \sum_{i=1}^{n} \alpha_i\, v_i. \tag{B.0.2}$$

We will consider the three cases $\lambda = 1$, $0 < \lambda < 1$ and $\lambda > 1$ separately:

Case: $\lambda = 1$**:** Due to the equilibrium condition in Equation B.0.1, for $\lambda = 1$:

$$\sum_{i=1}^{n} \alpha_i v_i^{\lambda+1} = \sum_{i=1}^{n} \alpha_i v_i^{\lambda} = \sum_{i=1}^{n} \alpha_i v_i \Rightarrow S(\pi_0) = S(\underline{\mu}). \tag{B.0.3}$$

Case: $0 < \lambda < 1$**:** (Weighted) Hölder's inequality (Hardy et al. (1988); Cvetkovski (2012)) states:

> **(Weighted) Hölder's inequality.** *Let $a_1, a_2, \ldots, a_n; b_1, b_2 \ldots, b_n; m_1, m_2, \ldots, m_n$ be three sequences of positive reals numbers and If $p, q > 1$ be such that $1/p + 1/q = 1$, Then:*
>
> $$\left(\sum_{i=1}^{n} m_i\, a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} m_i\, b_i^q\right)^{1/q} \geq \sum_{i=1}^{n} m_i\, a_i\, b_i. \tag{B.0.4}$$
>
> *Equality occurs if and only if* $\frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \cdots = \frac{a_n^p}{b_n^q}$.

Setting $1/p = \lambda$, $1/q = 1 - \lambda$, $a_i = v_i^{\lambda^2}$, $b_i = v_i^{1-\lambda^2}$ and $m_i = \alpha_i$; and noting that the ratios, $a_i^p/b_i^q = 1/v_i$, are not constant (unless all $v_i = 1$), (weighted) Hölder's inequality gives:

$$\left[\sum_{i=1}^{n} \alpha_i \left(v_i^{\lambda^2}\right)^{\frac{1}{\lambda}}\right]^{\lambda} \left[\sum_{i=1}^{n} \alpha_i \left(v_i^{1-\lambda^2}\right)^{\frac{1}{1-\lambda}}\right]^{1-\lambda} > \sum_{i=1}^{n} \alpha_i\, v_i^{\lambda^2}\, v_i^{1-\lambda^2}, \tag{B.0.5}$$

$$\Rightarrow \left[\sum_{i=1}^{n} \alpha_i\, v_i^{\lambda}\right]^{\lambda} \left[\sum_{i=1}^{n} \alpha_i\, v_i^{1+\lambda}\right]^{1-\lambda} > \sum_{i=1}^{n} \alpha_i\, v_i, \tag{B.0.6}$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i\, v_i^{1+\lambda} > \sum_{i=1}^{n} \alpha_i\, v_i, \quad \text{by the equilibrium condition in Equation B.0.1,} \tag{B.0.7}$$

$$\Rightarrow S(\pi_0) > S(\underline{\mu}), \quad \text{by the social welfare condition in Equation B.0.2.} \tag{B.0.8}$$

Case: $\lambda > 1$**:** Young's inequality (Hardy et al. (1988); Cvetkovski (2012)) states that:

Young's inequality. *For $a, b > 0$ and $p, q > 1$ such that $1/p + 1/q = 1$:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{B.0.9}$$

Equality occurs if and only if $a^p = b^q$.

Setting $p = \lambda$, $q = \frac{\lambda}{\lambda-1}$, $a = v_i^{\frac{1}{\lambda}}$, $b = v_i^{\lambda - \frac{1}{\lambda}}$ and noting that $a^p \neq b^q$ unless $v_i = 1$, Young's inequality gives:

$$v_i^{\frac{1}{\lambda}} v_i^{\lambda - \frac{1}{\lambda}} < \frac{1}{\lambda}\, v_i^{\frac{1}{\lambda}\lambda} + \frac{\lambda - 1}{\lambda}\, v_i^{(\lambda - \frac{1}{\lambda})\frac{\lambda}{\lambda - 1}}, \tag{B.0.10}$$

$$\Rightarrow v_i^{\lambda} < \frac{1}{\lambda} v_i + \frac{\lambda - 1}{\lambda} v_i^{\lambda+1}, \tag{B.0.11}$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i\, v_i^{\lambda} < \frac{1}{\lambda} \sum_{i=1}^{n} \alpha_i\, v_i + \frac{\lambda - 1}{\lambda} \sum_{i=1}^{n} \alpha_i\, v_i^{\lambda+1}, \tag{B.0.12}$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i\, v_i^{\lambda+1} < \sum_{i=1}^{n} \alpha_i\, v_i, \quad \text{by the equilibrium condition in Equation B.0.1,} \tag{B.0.13}$$

$$\Rightarrow S(\pi_0) < S(\underline{\mu}), \quad \text{by the social welfare condition in Equation B.0.2.} \tag{B.0.14}$$

□

Appendix C

Proof of Theorem 3

In this section, we prove Theorem 3. As discussed in Section 4.2, Theorem 2 is a special case of Theorem 3.

Theorem 3. *Suppose there are n risk-groups with risks $\mu_1 < \mu_2 < \cdots < \mu_n$ with iso-elastic demand elasticities $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively.*

3.1. *For any underlying population structures:*

$$\lambda_{lo}^{max} \leq \lambda_{hi}^{min} \leq \lambda_{hi}^{max} \leq 1 \Rightarrow S(\pi_0) \geq S(\underline{\mu}). \tag{4.2.3}$$

3.2. *For any underlying population structures:*

$$\lambda_{lo}^{max} \leq 1 \text{ and } 1 \leq \lambda_{hi}^{min} \leq \lambda_{hi}^{max} \leq \frac{1}{\lambda_{lo}^{max}} \Rightarrow S(\pi_0) \geq S(\underline{\mu}). \tag{4.2.4}$$

3.3. *There exists a threshold premium π^* such that:*

$$\lambda_{lo}^{max} \leq 1 \text{ and } \lambda_{hi}^{min} > \frac{1}{\lambda_{lo}^{min}} \text{ and } \pi_0 \geq \pi^* \Rightarrow S(\pi_0) \geq S(\underline{\mu}). \tag{4.2.5}$$

Proof. (of Theorem 3.1) The proof is presented in the following steps:

Step 1: If $a > 0$ and $0 < b \leq 1$, then since Arithmetic Mean $\geq$ Geometric Mean:

$$(1-b)a^{b+1} + ba^{b} \geq a^{(b+1)(1-b)} \times a^{b^2} = a \Rightarrow \left(\frac{a^{b+1}-a}{b}\right) \geq \left(a^{b+1} - a^{b}\right). \quad \text{(C.0.1)}$$

Step 2: As $v_i > 0$ and $0 < \lambda_i \leq 1$ for all risk-groups, using Step 1, we get:

$$\sum_{i=1}^{n} \alpha_i \, \frac{v_i^{\lambda_i+1} - v_i}{\lambda_i} \geq \sum_{i=1}^{n} \alpha_i \left(v_i^{\lambda_i+1} - v_i^{\lambda_i}\right) = \sum_{i=1}^{n} \alpha_i v_i^{\lambda_i+1} - \sum_{i=1}^{n} \alpha_i v_i^{\lambda_i} = 0, \quad \text{(C.0.2)}$$

by equilibrium condition in Equation 4.1.10.

Step 3: Using Step 2, and separating out the terms involving $v_i > 1$ from $v_i \leq 1$ we get:

$$\sum_{i:\, v_i>1} \alpha_i \, \frac{v_i^{\lambda_i+1} - v_i}{\lambda_i} \geq \sum_{i:\, v_i \leq 1} \alpha_i \, \frac{v_i - v_i^{\lambda_i+1}}{\lambda_i} \geq 0. \quad \text{(C.0.3)}$$

Step 4: As $0 < x \leq y \Rightarrow \frac{x}{x+1} \leq \frac{y}{y+1}$, if $0 < v_j \leq 1 \leq v_k$, for some j and k, then

$$\lambda_j \leq \lambda_{lo}^{max} \leq \lambda_{hi}^{min} \leq \lambda_k \Rightarrow \frac{\lambda_j}{\lambda_j+1} \leq \frac{\lambda_{lo}^{max}}{\lambda_{lo}^{max}+1} \leq \frac{\lambda_{hi}^{min}}{\lambda_{hi}^{min}+1} \leq \frac{\lambda_k}{\lambda_k+1}. \quad \text{(C.0.4)}$$

Step 5: Using Steps 3 and 4, we get:

$$\sum_{i:\, v_i>1} \frac{\alpha_i}{\lambda_i+1} \left[v_i^{\lambda_i+1} - v_i\right] = \sum_{i:\, v_i>1} \alpha_i \, \frac{\lambda_i}{\lambda_i+1} \, \frac{v_i^{\lambda_i+1} - v_i}{\lambda_i}, \quad \text{(C.0.5)}$$

$$\geq \frac{\lambda_{hi}^{min}}{\lambda_{hi}^{min}+1} \sum_{i:\, v_i>1} \alpha_i \, \frac{v_i^{\lambda_i+1} - v_i}{\lambda_i}, \quad \text{(C.0.6)}$$

$$\geq \frac{\lambda_{lo}^{max}}{\lambda_{lo}^{miax}+1} \sum_{i:\, v_i \leq 1} \alpha_i \, \frac{v_i - v_i^{\lambda_i+1}}{\lambda_i}, \quad \text{(C.0.7)}$$

$$\geq \sum_{i:\, v_i \leq 1} \alpha_i \, \frac{\lambda_i}{\lambda_i+1} \, \frac{v_i - v_i^{\lambda_i+1}}{\lambda_i}, \quad \text{(C.0.8)}$$

$$= \sum_{i:\, v_i \le 1} \frac{\alpha_i}{\lambda_i + 1} \left[v_i - v_i^{\lambda_i+1} \right] \tag{C.0.9}$$

Hence by Equation 4.1.13, $S(\pi_0) \geq S(\underline{\mu})$. □

Proof. (of Theorem 3.2) The proof is presented in the following steps:

Step 1: Let $0 < a \leq 1$, $b \geq a$ such that $a\,b \leq 1$ and function $g(v)$ be defined as:

$$g(v) = (b-a)v^a + (a+1)v^{a-1} - (b+1), \text{ for } v > 0. \tag{C.0.10}$$

If $a = 1$, then $b = 1$ (as $b \geq a$ and $ab \leq 1$), in which case: $g(v) = 0$ for $v > 0$.

If $0 < a < 1$ i.e. $(a-1) < 0$, $\lim_{v \to 0+} g(v) = +\infty$, $g(1) = 0$ and:

$$g'(v) = (b-a)\,a\,v^{a-2} \left[v - \frac{1-a^2}{ab-a^2} \right] < 0, \text{ for } 0 < v < 1 \text{ as } ab \leq 1. \tag{C.0.11}$$

So $g(v)$ is a non-negative decreasing function over $0 < v \leq 1$. Hence $g(v) \geq 0$ for $0 < v \leq 1$.

Step 2: For $v_i \leq 1$, set $a = \lambda_i$ and $b = \lambda_{hi}^{max} \Rightarrow ab = \lambda_i\,\lambda_{hi}^{max} \leq \lambda_{lo}^{max}\,\lambda_{hi}^{max} \leq 1$. By Step 1:

$$(\lambda_{hi}^{max} - \lambda_i)v_i^{\lambda_i} + (\lambda_i + 1)v_i^{\lambda_i - 1} - (\lambda_{hi}^{max} + 1) \geq 0. \tag{C.0.12}$$

Rearranging and multiplying by $\alpha_i\, v_i$ on both sides, we get:

$$\frac{\alpha_i}{\lambda_{hi}^{max} + 1} \left[v_i^{\lambda_i} - v_i^{\lambda_i+1} \right] \geq \frac{\alpha_i}{\lambda_i + 1} \left[v_i - v_i^{\lambda_i+1} \right]. \tag{C.0.13}$$

As this holds for all $v_i \leq 1$, summing over all such risk-groups leads to:

$$\frac{1}{\lambda_{hi}^{max} + 1} \sum_{i:\, v_i \le 1} \alpha_i \left[v_i^{\lambda_i} - v_i^{\lambda_i+1} \right] \geq \sum_{i:\, v_i \le 1} \frac{\alpha_i}{\lambda_i + 1} \left[v_i - v_i^{\lambda_i+1} \right]. \tag{C.0.14}$$

Step 3: For all risk-groups with $v_i > 1$, $\lambda_i \geq 1$ (since $\lambda_{hi}^{min} \geq 1$). So:

$$\sum_{i:\, v_i>1} \frac{\alpha_i}{\lambda_i+1}\left[v_i^{\lambda_i+1} - v_i\right] \geq \frac{1}{\lambda_{hi}^{max}+1}\sum_{i:\, v_i>1}\alpha_i\left[v_i^{\lambda_i+1} - v_i\right], \text{ as } \lambda_{hi}^{max} \geq \lambda_i \quad \text{(C.0.15)}$$

$$\geq \frac{1}{\lambda_{hi}^{max}+1}\sum_{i:\, v_i>1}\alpha_i\left[v_i^{\lambda_i+1} - v_i^{\lambda_i}\right], \text{ as } v_i > 1 \text{ and } \lambda_i \geq 1 \quad \text{(C.0.16)}$$

$$= \frac{1}{\lambda_{hi}^{max}+1}\sum_{i:\, v_i\leq 1}\alpha_i\left[v_i^{\lambda_i} - v_i^{\lambda_i+1}\right], \text{ by Equation 4.1.11.} \quad \text{(C.0.17)}$$

Step 4: Combining Steps 2 and 3, we get:

$$\sum_{i:\, v_i>1} \frac{\alpha_i}{\lambda_i+1}\left[v_i^{\lambda_i+1} - v_i\right] \geq \frac{1}{\lambda_{hi}^{max}+1}\sum_{i:\, v_i\leq 1}\alpha_i\left[v_i^{\lambda_i} - v_i^{\lambda_i+1}\right] \geq \sum_{i:\, v_i\leq 1}\frac{\alpha_i}{\lambda_i+1}\left[v_i - v_i^{\lambda_i+1}\right], \quad \text{(C.0.18)}$$

Hence by Equation 4.1.13, $S(\pi_0) \geq S(\underline{\mu})$. □

Proof. (of Theorem 3.3) The proof is presented in the following steps:

Step 1: Let $0 < a \leq 1$, $b > a$ such that $a\,b > 1$ and function $h(v)$ be defined as:

$$h(v) = (b-a)v^b - (b+1)v^{b-1} + (a+1), \text{ for } v > 0. \quad \text{(C.0.19)}$$

$\lim_{v\to 0+} h(v) = a+1 > 1$, $\lim_{v\to+\infty} h(v) = +\infty$, $h(1) = 0$ and:

$$h'(v) = (b-a)\,b\,v^{b-2}\left[v - \frac{b^2-1}{b^2-ab}\right] \Rightarrow h'(v_m) = 0 \Rightarrow v_m = \frac{b^2-1}{b^2-ab} > 1. \quad \text{(C.0.20)}$$

$h''(v_m) > 0 \Rightarrow v_m$ is minimum. So there exists a $v^* > 1$ such that, $h(v) \leq 0$ for $1 < v \leq v^*$.

Step 2: For all $v_i > 1$, there exists a v_i^* such that for $1 < v_i \leq v_i^*$,

$$\sum_{i:\,v_i>1} \frac{\alpha_i}{\lambda_i+1}\left[v_i^{\lambda_i+1} - v_i\right] \geq \frac{1}{\lambda_{lo}^{min}+1} \sum_{i:\,v_i>1} \alpha_i \left[v_i^{\lambda_i+1} - v_i^{\lambda_i}\right]. \tag{C.0.21}$$

To prove this, set $a = \lambda_{lo}^{min}$ and $b = \lambda_i$, so $ab = \lambda_i\,\lambda_{lo}^{min} \geq \lambda_{hi}^{min}\,\lambda_{lo}^{min} > 1$. So, by Step 1:

$$(\lambda_i - \lambda_{lo}^{min})v_i^{\lambda_i} - (\lambda_i+1)v_i^{\lambda_i-1} + (\lambda_{lo}^{min}+1) \leq 0. \tag{C.0.22}$$

Rearranging and multiplying by $\alpha_i\,v_i$ on both sides, we get:

$$\frac{\alpha_i}{\lambda_i+1}\left[v_i^{\lambda_i+1} - v_i\right] \geq \frac{\alpha_i}{\lambda_{lo}^{min}+1}\left[v_i^{\lambda_i+1} - v_i^{\lambda_i}\right]. \tag{C.0.23}$$

As this holds for all $v_i > 1$, summing over all such risk-groups leads to Equation C.0.21.

Step 3: Based on all risk-groups for which $v_i \leq 1$:

$$\sum_{i:\,v_i\leq1} \frac{\alpha_i}{\lambda_i+1}\left[v_i - v_i^{\lambda_i+1}\right] \leq \frac{1}{\lambda_{lo}^{min}+1} \sum_{i:\,v_i\leq1} \alpha_i \left[v_i - v_i^{\lambda_i+1}\right], \text{ as } \lambda_{lo}^{min} \leq \lambda_i \tag{C.0.24}$$

$$\leq \frac{1}{\lambda_{lo}^{min}+1} \sum_{i:\,v_i\leq1} \alpha_i \left[v_i^{\lambda_i} - v_i^{\lambda_i+1}\right], \text{ as } v_i \leq 1 \text{ and } \lambda_i \leq 1 \tag{C.0.25}$$

$$= \frac{1}{\lambda_{lo}^{min}+1} \sum_{i:\,v_i>1} \alpha_i \left[v_i^{\lambda_i+1} - v_i^{\lambda_i}\right], \text{ by Equation 4.1.11.} \tag{C.0.26}$$

Step 4: Combining Steps 2 and 3, we get

$$\sum_{i:\,v_i>1} \frac{\alpha_i}{\lambda_i+1}\left[v_i^{\lambda_i+1} - v_i\right] \geq \frac{1}{\lambda_{lo}^{min}+1} \sum_{i:\,v_i>1} \alpha_i \left[v_i^{\lambda_i+1} - v_i^{\lambda_i}\right] \geq \sum_{i:\,v_i\leq1} \frac{\alpha_i}{\lambda_i+1}\left[v_i - v_i^{\lambda_i+1}\right], \tag{C.0.27}$$

for $1 < v_i \leq v_i^*$ for all $v_i > 1$.

As $v_i = \mu_i/\pi_0$, $v_i \leq v_i^* \Rightarrow \pi_0 \geq \mu_i/v_i^*$ for all risk-groups for which $v_i > 1$. So if we define $\pi^* = \max_{i:\, v_i > 1} (\mu_i/v_i^*)$, then $\pi_0 \geq \pi^* \Rightarrow S(\pi_0) \geq S(\underline{\mu})$ by Equation 4.1.13. $\square$

Appendix D

Expression for Social Welfare Under General Insurance Demand

Lemma 2. *Suppose there are n risk-groups with risks $\mu_1 < \mu_2 < \cdots < \mu_n$ and any general demand functions. Then for a given premium regime $\underline{\pi}$, for which no risk-group is fully insured, the expression for social welfare is given by:*

$$S(\underline{\pi}) = \sum_{i=1}^{n} p_i \, G_i \left(\frac{\mu_i}{\pi_i} \right) \pi_i + K, \quad \textit{where} \quad G_i(g) = \int_0^g P[\, \Gamma_i < \gamma \,] \, d\gamma, \tag{4.3.2}$$

where the premium regime $\underline{\pi}$ satisfies the equilibrium condition:

$$\sum_{i=1}^{n} p_i \, d_i(\pi_i) \; (\pi_i - \mu_i) = 0, \tag{4.3.3}$$

and the constant K does not depend on the premium regime under consideration.

Proof. Recall that, given a risk-group i, insurance is purchased when $\Gamma_i < \mu_i/\pi_i$. Hence:

$$[\,Q \mid \text{Risk-group } i\,] = I\left[\Gamma_i < \frac{\mu_i}{\pi_i}\right] \Rightarrow \mathbb{E}\,[\,Q \mid \text{Risk-group } i\,] = \mathrm{P}\left[\Gamma_i < \frac{\mu_i}{\pi_i}\right] = d_i(\pi_i). \tag{D.0.1}$$

Using the expression for social welfare as given in Equation 3.3.11 we have:

$$S(\underline{\pi}) = \mathbb{E}\,[\,Q\,(X - \Pi\,\Gamma)\,] + K, \tag{D.0.2}$$

$$= \mathbb{E}\,[\,Q\,X\,] - \mathbb{E}\,[\,Q\,\Pi\,\Gamma\,] + K, \tag{D.0.3}$$

$$= \mathbb{E}\,[\,Q\,\Pi\,] - \mathbb{E}\,[\,Q\,\Pi\,\Gamma\,] + K, \text{ as under equilibrium: } E\,[\,Q\,X\,] = E\,[\,Q\,\Pi\,] \tag{D.0.4}$$

$$= \mathbb{E}\,[\,(1-\Gamma)\,Q\,\Pi\,] + K, \tag{D.0.5}$$

$$= \sum_{i=1}^{n} p_i\,\mathbb{E}\left[(1-\Gamma_i)\,I\left[\Gamma_i \leq \frac{\mu_i}{\pi_i}\right]\right]\pi_i + K. \tag{D.0.6}$$

Now using Lemma 3:

$$S(\underline{\pi}) = \sum_{i=1}^{n} p_i\left[\left(1 - \frac{\mu_i}{\pi_i}\right)\mathrm{P}\left[\Gamma_i \leq \frac{\mu_i}{\pi_i}\right] + \int_0^{\frac{\mu_i}{\pi_i}} \mathrm{P}\,[\,\Gamma_i \leq \gamma\,]\;d\gamma\right]\pi_i + K, \tag{D.0.7}$$

$$= \sum_{i=1}^{n} p_i\left(1 - \frac{\mu_i}{\pi_i}\right)\mathrm{P}\left[\Gamma_i \leq \frac{\mu_i}{\pi_i}\right]\pi_i + \sum_{i=1}^{n} p_i\int_0^{\frac{\mu_i}{\pi_i}} \mathrm{P}\,[\,\Gamma_i \leq \gamma\,]\;d\gamma\,\pi_i + K, \tag{D.0.8}$$

$$= \sum_{i=1}^{n} p_i\,d_i(\pi_i)\;(\pi_i - \mu_i) + \sum_{i=1}^{n} p_i\,G_i\left(\frac{\mu_i}{\pi_i}\right)\pi_i + K, \quad \text{as } \mathrm{P}\left[\Gamma_i \leq \frac{\mu_i}{\pi_i}\right] = d_i(\pi_i), \tag{D.0.9}$$

$$= \sum_{i=1}^{n} p_i\,G_i\left(\frac{\mu_i}{\pi_i}\right)\pi_i + K, \quad \text{as in equilibrium: } \sum_{i=1}^{n} p_i\,d_i(\pi_i)\;(\pi_i - \mu_i) = 0. \tag{D.0.10}$$

as required. $\square$

Lemma 3. *For a positive continuous random variable, X:*

(i) $\mathbb{E}\,[\,X\,] = \int_0^\infty P\,[\,X > y\,]\;dy$*;*

(ii) $\mathbb{E}\,[\,X\,I\,[\,X \le c\,]\,] = c\,P\,[\,X \le c\,] - \int_0^c P\,[\,X \le y\,]\;dy$*;*

(iii) $\mathbb{E}\,[\,(1-X)\,I\,[\,X \le c\,]\,] = (1-c)\,P\,[\,X \le c\,] + \int_0^c P\,[\,X \le y\,]\;dy$*.*

Proof. Assuming the density function of X is given by $p(x)$

(i)

$$\begin{aligned}\mathbb{E}\,[\,X\,] &= \int_0^\infty x\,p(x)\,dx = \int_0^\infty \left[\int_0^x dy\right] p(x)\,dx = \int_0^\infty \left[\int_y^\infty p(x)\,dx\right] dy \\ &= \int_0^\infty \mathrm{P}\,[\,X > y\,]\;dy. \qquad \text{(D.0.11)}\end{aligned}$$

(ii)

$$\begin{aligned}\mathbb{E}\,[\,X\,I\,[\,X \le c\,]\,] &= \int_0^c x\,p(x)\,dx, && \text{(D.0.12)}\\ &= \int_0^c \left[\int_0^x dy\right] p(x)\,dx, && \text{(D.0.13)}\\ &= \int_0^c \left[\int_y^c p(x)\,dx\right] dy, \quad \text{by interchanging integrals,} && \text{(D.0.14)}\\ &= \int_0^c \mathrm{P}\,[\,y < X \le c\,]\;dy, && \text{(D.0.15)}\\ &= \int_0^c [\,\mathrm{P}\,[\,X \le c\,] - \mathrm{P}\,[\,X \le y\,]\,]\;dy, && \text{(D.0.16)}\\ &= c\,\mathrm{P}\,[\,X \le c\,] - \int_0^c \mathrm{P}\,[\,X \le y\,]\;dy. && \text{(D.0.17)}\end{aligned}$$

(iii)

$$
\begin{aligned}
&\mathbb{E}\left[\,(1-X)\, I\left[\, X \leq c\,\right]\right] \\
&= \mathbb{E}\left[\, I\left[\, X \leq c\,\right]\right] - \mathbb{E}\left[\, X\, I\left[\, X \leq c\,\right]\right], && \text{(D.0.18)} \\
&= \mathrm{P}\left[\, X \leq c\,\right] - \left[c\,\mathrm{P}\left[\, X \leq c\,\right] - \int_0^c \mathrm{P}\left[\, X \leq y\,\right]\, dy \right], && \text{(D.0.19)} \\
&= (1-c)\,\mathrm{P}\left[\, X \leq c\,\right] + \int_0^c \mathrm{P}\left[\, X \leq y\,\right]\, dy && \text{(D.0.20)}
\end{aligned}
$$

□

Appendix E

Derivations for General Demand Elasticities

First note that if demand elasticity is an increasing function of premium π, then it is a decreasing function of $v = \mu_i/\pi$; and hence a weighted average such as arc elasticity $\lambda_i(v)$ is also decreasing function of v. The inverse statements (i.e. with increasing replaced by decreasing and vice versa) also hold.

Lemma 4. *If for a risk-group i, $\mu_i > \pi_0$ (i.e. $v_i > 1$) and the demand elasticity, $\epsilon_i(\pi)$, is an increasing function of premium π, then:*

$$G_i(v_i) - v_i G_i(1) \geq \frac{\tau_i}{\lambda_i(v_i)+1}\left[v_i^{\lambda_i(v_i)+1} - v_i\right]. \tag{E.0.1}$$

Proof. Firstly:

$$\begin{aligned} G_i(v_i) - G_i(1) &= \int_1^{v_i} P\left[\Gamma_i \leq v\right]\, dv, && \text{(E.0.2)} \\ &= \int_1^{v_i} \tau_i\, v^{\lambda_i(v)} dv, \text{ by Equation 4.3.10}, && \text{(E.0.3)} \end{aligned}$$

$$\geq \int_1^{v_i} \tau_i\, v^{\lambda_i(v_i)} dv, \text{ as } \lambda_i(v) \text{ is a decreasing function,} \tag{E.0.4}$$

$$= \frac{\tau_i}{\lambda_i(v_i)+1}\left[v_i^{\lambda_i(v_i)+1} - 1\right]. \tag{E.0.5}$$

And,

$$(v_i - 1)G_i(1) = (v_i - 1)\int_0^1 P\left[\Gamma_i \leq v\right] dv, \tag{E.0.6}$$

$$= (v_i - 1)\int_0^1 \tau_i\, v^{\lambda_i(v)} dv, \text{ by Equation 4.3.10,} \tag{E.0.7}$$

$$\leq (v_i - 1)\int_0^1 \tau_i v^{\lambda_i(v_i)} dv, \text{ as } v < 1 \Rightarrow v^{\lambda_i(v)} \leq v^{\lambda_i(v_i)}, \tag{E.0.8}$$

$$= \frac{(v_i - 1)\tau_i}{\lambda_i(v_i)+1}. \tag{E.0.9}$$

Hence:

$$G_i(v_i) - v_i G_i(1) = \left[G_i(v_i) - G_i(1)\right] - \left[(v_i - 1)G_i(1)\right] \geq \frac{\tau_i}{\lambda_i(v_i)+1}\left[v_i^{\lambda_i(v_i)+1} - v_i\right], \tag{E.0.10}$$

as required. □

Lemma 5. *If for a risk-group i, $\mu_i \leq \pi_0$ (i.e. $v_i \leq 1$) and the demand elasticity, $\epsilon_i(\pi)$, is a decreasing function of premium π, then:*

$$v_i G_i(1) - G_i(v_i) \leq \frac{\tau_i}{\lambda_i(v_i)+1}\left[v_i - v_i^{\lambda_i(v_i)+1}\right]. \tag{E.0.11}$$

Proof. Firstly:

$$v_i\left[G_i(1) - G_i(v_i)\right] = v_i \int_{v_i}^1 P\left[\Gamma_i \leq v\right] dv, \tag{E.0.12}$$

$$= v_i \int_{v_i}^1 \tau_i\, v^{\lambda_i(v)} dv, \text{ by Equation 4.3.10,} \tag{E.0.13}$$

$$\leq v_i \int_{v_i}^{1} \tau_i\, v^{\lambda_i(v_i)} dv, \text{ as } v < 1 \Rightarrow v^{\lambda_i(v)} \leq v^{\lambda_i(v_i)}, \tag{E.0.14}$$

$$= \frac{v_i \tau_i}{\lambda_i(v_i) + 1} \left[1 - v_i^{\lambda_i(v_i)+1} \right]. \tag{E.0.15}$$

And

$$(1 - v_i) G_i(v_i) = (1 - v_i) \int_0^{v_i} P\left[\Gamma_i \leq v\right] dv, \tag{E.0.16}$$

$$= (1 - v_i) \int_0^{v_i} \tau_i\, v^{\lambda_i(v)} dv, \text{ by Equation 4.3.10,} \tag{E.0.17}$$

$$\geq (1 - v_i) \int_0^{v_i} \tau_i\, v^{\lambda_i(v_i)} dv, \text{ as } \lambda_i(v) \text{ is an increasing function,} \tag{E.0.18}$$

$$= \frac{(1 - v_i)\tau_i}{\lambda_i(v_i) + 1} \left[v_i^{\lambda_i(v_i)+1} \right]. \tag{E.0.19}$$

Hence, as required:

$$v_i G_i(1) - G_i(v_i) = v_i \left[G_i(1) - G_i(v_i) \right] - (1 - v_i) G_i(v_i) \leq \frac{\tau_i}{\lambda_i(v_i) + 1} \left[v_i - v_i^{\lambda_i(v_i)+1} \right]. \tag{E.0.20}$$

□

Appendix F

Social welfare when higher risks are fully insured under pooling

In the main text of the thesis, we have explicitly assumed that no risk-groups are fully insured under any premium regime. However, for sufficiently small pooled equilibrium premium, it is possible that all individuals purchase insurance, in some higher risk-groups.

If there are more than two risk-groups, the analysis of implications of full insurance would require consideration of many possible combinations. For ease of exposition, while analysing the case of full take-up of insurance, we will only consider two risk-groups, where the high risk-group is fully insured under pooling. We assume that fair-premium demand $\tau_i < 1$ for all risk-groups, which is consistent with most empirical evidence. (The special case of $\tau_i = 1$ can also be analysed using the same techniques.)

Assuming $\tau_i < 1$, social welfare under full risk classification follows from Lemma 1:

$$S(\underline{\mu}) = p_1 \, \tau_1 \frac{1}{(\lambda_1 + 1)} \mu_1 + p_2 \, \tau_2 \frac{1}{(\lambda_2 + 1)} \mu_2 + K. \tag{F.0.1}$$

For pooling we obtain the following lower bound for social welfare:

Lemma 6. *Suppose there are two risk-groups with risks $\mu_1 < \mu_2$ with positive constant demand elasticities λ_1 and λ_2 respectively. If the high risk-group is fully insured under pooling, then social welfare under pooled premium $S(\pi_0)$ satisfies:*

$$S(\pi_0) \geq p_1\,\tau_1\,\frac{1}{(\lambda_1+1)}\left(\frac{\mu_1}{\pi_0}\right)^{\lambda_1+1}\pi_0 + p_2\,\frac{1}{(\lambda_2+1)}\mu_2 + K, \tag{F.0.2}$$

where the pooled premium π_0 satisfies the equilibrium condition:

$$p_1\,\tau_1\left(\frac{\mu_1}{\pi_1}\right)^{\lambda_1}(\pi_0-\mu_1) + p_2\,(\pi_0-\mu_2) = 0, \tag{F.0.3}$$

and the constant K does not depend on the premium regime under consideration.

Proof. The equilibrium condition follows from Equation 3.3.15, by inserting the specific expression for iso-elastic insurance demand for low risk-group and noting that proportional demand for high risk-group is 1 under pooling.

Using the general expression for social welfare given in Equation 3.3.11, we have:

$$\begin{aligned} S(\pi_0) &= \mathbb{E}\,[\,Q\,X - \,Q\,\Pi\,\Gamma\,] + K, && \text{(F.0.4)}\\ &= \sum_{i=1}^{2}\mathbb{E}\,[\,Q\,X - \,Q\,\Pi\,\Gamma \mid \text{Risk-group } i\,]\,p_i + K. && \text{(F.0.5)} \end{aligned}$$

As not all low risks will purchase insurance, the same steps in Lemma 1 will give:

$$\mathbb{E}\,[\,Q\,X - \,Q\,\Pi\,\Gamma \mid \text{Risk-group } 1\,] = p_1\,\tau_1\,\frac{1}{(\lambda_1+1)}\left(\frac{\mu_1}{\pi_0}\right)^{\lambda_1+1}\pi_0. \tag{F.0.6}$$

But all high-risks buy insurance under pooling, i.e. $[Q \mid \text{Risk-group } 2] = 1$. So:

$$\mathbb{E}\,[\,Q\,X - \,Q\,\Pi\,\Gamma \mid \text{Risk-group } 2\,] = \mathbb{E}\,[\,X \mid \text{Risk-group } 2\,] - \mathbb{E}\,[\,\Pi\,\Gamma \mid \text{Risk-group } 2\,]\,, \tag{F.0.7}$$

$$= \mu_2 - \mathbb{E}\,[\,\Gamma \mid \text{Risk-group } 2\,]\ \pi_0, \tag{F.0.8}$$

$$= \mu_2 - \int_0^{\left(\frac{1}{\tau_2}\right)^{\frac{1}{\lambda_2}}} \gamma\tau_2\lambda_2\gamma^{\lambda_2-1} d\gamma\, \pi_0, \tag{F.0.9}$$

$$= \mu_2 - \frac{\lambda_2}{(\lambda_2+1)} \left(\frac{1}{\tau_2}\right)^{\frac{1}{\lambda_2}} \pi_0, \tag{F.0.10}$$

$$\geq \frac{1}{(\lambda_2+1)}\,\mu_2, \quad \text{since } \tau_2 \left(\frac{\mu_2}{\pi_0}\right)^{\lambda_2} \geq 1 \Rightarrow \left(\frac{1}{\tau_2}\right)^{\frac{1}{\lambda_2}} \pi_0 \leq \mu_2. \tag{F.0.11}$$

Using Equations F.0.6 and F.0.11 in Equation F.0.5 gives the required relationship in Equation F.0.2. □

Equation F.0.2 of Lemma 6 implies that, when high-risks are fully insured under pooling (but partially insured under full risk classification), social welfare under pooling exceeds that under full risk classification, i.e. $S(\pi_0) \geq S(\underline{\mu})$ if:

$$p_1\,\tau_1\,\frac{1}{(\lambda_1+1)}\left(\frac{\mu_1}{\pi_0}\right)^{\lambda_1+1} \pi_0 + p_2\,\frac{1}{(\lambda_2+1)}\,\mu_2 \geq p_1\,\tau_1\frac{1}{(\lambda_1+1)}\mu_1 + p_2\,\tau_2\frac{1}{(\lambda_2+1)}\mu_2, \tag{F.0.12}$$

$$\Leftrightarrow p_1\,\tau_1\,\frac{1}{(\lambda_1+1)}{v_1}^{\lambda_1+1} + p_2\,\frac{1}{(\lambda_2+1)}v_2 \geq p_1\,\tau_1\frac{1}{(\lambda_1+1)}\,v_1 + p_2\,\tau_2\frac{1}{(\lambda_2+1)}\,v_2, \tag{F.0.13}$$

using the notations involving risk-premium ratios: v_1 and v_2. And Equation F.0.3 becomes:

$$p_1\,\tau_1\,{v_1}^{\lambda_1}\ (1 - v_1) + p_2\ (1 - v_2) = 0 \tag{F.0.14}$$

We can then state the sufficient condition on λ_1 and λ_2, for social welfare to be higher under pooling than under full risk classification for any population structures and underlying risks, when high-risks are fully insured under pooling.

Theorem 8. *Suppose there are two risk-groups with risks $\mu_1 < \mu_2$ with positive constant demand elasticities λ_1 and λ_2 respectively. If high-risks are fully insured under pooling while low-risks are not, and neither risk-group is fully insured under full risk classification, then:*

$$\lambda_1 \leq 1 \textit{ and } \lambda_2 \leq \left(1 + \frac{1}{\lambda_1}\right)(1 - \tau_2) - 1 \Rightarrow S(\pi_0) \geq S(\underline{\mu}). \tag{F.0.15}$$

Proof. The proof is presented in the following steps:

Step 1: The equilibrium condition in Equation F.0.14 leads to:

$$p_2\, v_2 = p_1\, \tau_1 \left({v_1}^{\lambda_1} - {v_1}^{\lambda_1+1}\right) + p_2. \tag{F.0.16}$$

Step 2: Using Equation F.0.16 in the social welfare condition in Equation F.0.13 gives:

$$S(\pi_0) \geq S(\underline{\mu}) \tag{F.0.17}$$

$$\text{if} \quad p_1\,\tau_1 \frac{1}{(\lambda_1+1)} {v_1}^{\lambda_1+1} + p_2 \frac{1}{(\lambda_2+1)} v_2 \geq p_1\,\tau_1 \frac{1}{(\lambda_1+1)} v_1 + p_2\,\tau_2 \frac{1}{(\lambda_2+1)} v_2, \tag{F.0.18}$$

$$\begin{aligned}\text{i.e. if} \quad & p_1\,\tau_1 \frac{1}{(\lambda_1+1)} {v_1}^{\lambda_1+1} + p_1\,\tau_1 \frac{1}{(\lambda_2+1)} \left({v_1}^{\lambda_1} - {v_1}^{\lambda_1+1}\right) + p_2 \frac{1}{(\lambda_2+1)} \\ & \geq p_1\,\tau_1 \frac{1}{(\lambda_1+1)} v_1 + p_1\,\tau_1 \frac{\tau_2}{(\lambda_2+1)} \left({v_1}^{\lambda_1} - {v_1}^{\lambda_1+1}\right) + p_2 \frac{\tau_2}{(\lambda_2+1)},\end{aligned} \tag{F.0.19}$$

$$\begin{aligned}\text{i.e. if} \quad & p_1\,\tau_1 \frac{1}{(\lambda_1+1)} {v_1}^{\lambda_1+1} + p_1\,\tau_1 \frac{1}{(\lambda_2+1)} \left({v_1}^{\lambda_1} - {v_1}^{\lambda_1+1}\right) \\ & \geq p_1\,\tau_1 \frac{1}{(\lambda_1+1)} v_1 + p_1\,\tau_1 \frac{\tau_2}{(\lambda_2+1)} \left({v_1}^{\lambda_1} - {v_1}^{\lambda_1+1}\right), \quad \text{as } \tau_2 < 1,\end{aligned} \tag{F.0.20}$$

$$\text{i.e. if} \quad \frac{(1-\tau_2)}{(\lambda_2+1)} \geq \frac{1}{(\lambda_1+1)} \frac{\left(v_1 - {v_1}^{\lambda_1+1}\right)}{\left({v_1}^{\lambda_1} - {v_1}^{\lambda_1+1}\right)}. \tag{F.0.21}$$

Step 3: As $0 < \lambda_1 \leq 1$ and $0 < v_1 < 1$, using Arithmetic Mean $\geq$ Geometric Mean:

$$(1-\lambda_1)\, v_1^{\lambda_1+1} + \lambda_1\, v_1^{\lambda_1} \geq v_1 \Rightarrow \frac{\lambda_1}{(\lambda_1+1)} \geq \frac{1}{(\lambda_1+1)} \frac{\left(v_1 - {v_1}^{\lambda_1+1}\right)}{\left({v_1}^{\lambda_1} - {v_1}^{\lambda_1+1}\right)}. \tag{F.0.22}$$

Step 4: Finally:

$$\lambda_2 \leq \left(1 + \frac{1}{\lambda_1}\right)(1-\tau_2) - 1 \Rightarrow \frac{(1-\tau_2)}{(\lambda_2+1)} \geq \frac{\lambda_1}{(\lambda_1+1)}, \tag{F.0.23}$$

$$\Rightarrow \frac{(1-\tau_2)}{(\lambda_2+1)} \geq \frac{1}{(\lambda_1+1)} \frac{\left(v_1 - {v_1}^{\lambda_1+1}\right)}{\left({v_1}^{\lambda_1} - {v_1}^{\lambda_1+1}\right)}, \quad \text{by Step 3,} \tag{F.0.24}$$

$$\Rightarrow S(\pi_0) \geq S(\underline{\mu}), \quad \text{by Step 2.} \tag{F.0.25}$$

□

Figure F.1 provides a graphical representation of Theorem 8, where the fair-premium demand is 50% for both low and high risk-groups. Social welfare is guaranteed to be higher under pooling for all population structures and risks in the shaded region to the left of the bold green curve.

For specific population structures and risk parameters, the region where social welfare is higher under pooling is a much larger area than the shaded region in Figure F.1. For example, social welfare is guaranteed to be higher under pooling in the region to the left of the blue dot-dashed line for $p_1 = 0.99$ and $(\mu_1, \mu_2) = (0.01, 0.04)$. Similarly, the region to the left of the red dashed line represents the region where social welfare is guaranteed to be higher under pooling for $p_1 = 0.9$ and $(\mu_1, \mu_2) = (0.01, 0.04)$. The region where social welfare is guaranteed to be higher under pooling increases with the size of the higher risk-

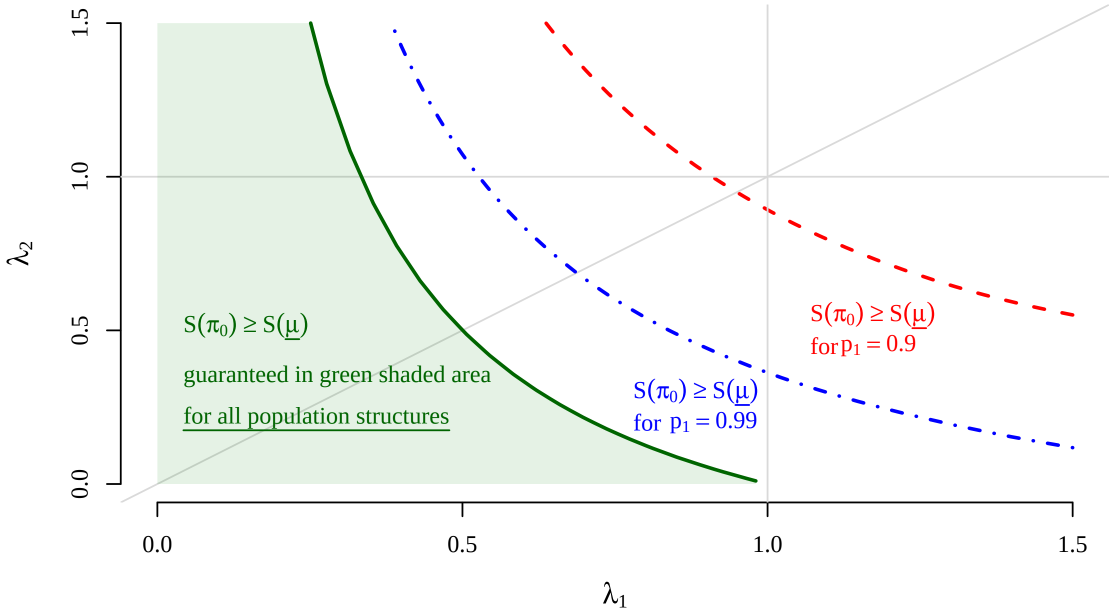


Figure F.1: Curve demarcating the regions where social welfare under pooling is greater than under full risk-differentiation where $(\mu_1, \mu_2) = (0.01, 0.04)$, fair-premium demand is 50% for both risk-groups and high-risks are fully insured under pooling.

group, because larger high risk-group's gain in welfare from pooling has greater capacity to offset the lower risk-group's loss in welfare from pooling.

Appendix G

Proof of Theorem 5

Theorem 5. *If there are n risk-groups, with risks* $\mu_1 < \mu_2 < \cdots < \mu_n$*, in presence of political, regulatory and economic constraints, with regulatory price collar of* κ*, where* $1 \leq \kappa \leq \mu_n/\mu_1$*, there exists a stable equilibrium premium regime* $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$*, such that:*

$$\pi_i = \begin{cases} \pi_L & \text{if } \mu_i < \pi_L; \\ \mu_i & \text{if } \pi_L \leq \mu_i \leq \pi_H; \\ \pi_H & \text{if } \mu_i > \pi_H. \end{cases} \tag{5.1.9}$$

where $\pi_L = \min_i \pi_i$*,* $\pi_H = \max_i \pi_i$ *and* $\pi_H = \kappa\, \pi_L$*.*

Proof. We will prove the theorem using the following steps:

1. An equilibrium premium regime with the structure proposed in Equation 5.1.9 exists.

2. If there are multiple equilibrium premium regimes with the same proposed structure, the regime with the smallest π_L is stable among all such regimes.

3. Given π_L and $\pi_H = \kappa\, \pi_L$, the premium regime with the proposed structure cannot be destabilised by any other equilibrium premium regime with the same π_L and π_H but having a different structure.

4. Given a regulatory price collar κ, the premium regime with the proposed structure cannot be destabilised by any other premium regime based on a smaller price collar.

Proof of step 1. Given a price collar κ, define the expected profit from setting the lowest premium π_L, where $\mu_1 \le \pi_L \le \mu_n/\kappa$, as follows:

$$e_\kappa(\pi_L) = E(\underline{\pi}) = \sum_{i=1}^{n} p_i\, d_i(\pi_i)\ (\pi_i - \mu_i)\,;\ \text{where } \pi_i = \begin{cases} \pi_L & \text{if } \mu_i < \pi_L; \\ \mu_i & \text{if } \pi_L \le \mu_i \le \kappa\,\pi_L; \\ \kappa\,\pi_L & \text{if } \mu_i > \kappa\,\pi_L. \end{cases} \tag{G.0.1}$$

If $\pi_L = \mu_1$, as $\kappa\,\mu_1 \le \mu_n$, expected profit cannot be positive, i.e.: $e_\kappa(\mu_1) \le 0$.

If $\pi_L = \mu_n/\kappa$, as $\mu_1 \le \mu_n/\kappa$, expected profit cannot be negative, i.e.: $e_\kappa(\mu_n/\kappa) \ge 0$.

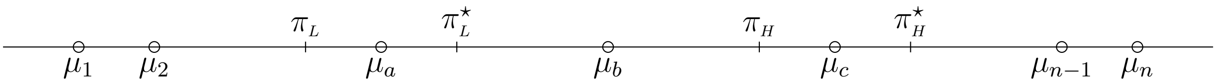
Assuming continuity of the demand functions $d_i(\pi_i)$ for all risk-groups, $e_\kappa(x)$ is also a continuous function. So, by the intermediate value theorem, there exists a value π_L, such that $\mu_1 \le \pi_L \le \mu_n/\kappa$, for which $e_\kappa(\pi_L) = 0$. This proves the existence of an equilibrium premium regime as outlined in the theorem. ■

Proof of step 2. If there are multiple solutions to the equation, $e_\kappa(\pi_L) = 0$, the premium regime, based on the smallest of these roots, cannot be destabilised by premium regimes based on any other solutions of $e_\kappa(\pi_L) = 0$. To show this, suppose if possible there are two premium regimes:

$$\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)\,,\ \text{with } \pi_H = \kappa\,\pi_L,\ \text{where } \pi_L = \min_i \pi_i \text{ and } \pi_H = \max_i \pi_i;$$

$$\underline{\pi}^\star = (\pi_1^\star, \pi_2^\star, \ldots, \pi_n^\star)\,,\ \text{with } \pi_H^\star = \kappa\,\pi_L^\star,\ \text{where } \pi_L^\star = \min_i \pi_i^\star \text{ and } \pi_H^\star = \max_i \pi_i^\star;$$

with $\pi_L < \pi_L^\star$ (and consequently $\pi_H < \pi_H^\star$), such as shown below:



Taking into account the choices available for all risk-groups, we have following possibilities:

- Risk-groups i, with $\mu_i \leq \pi_L$, will choose the cheaper premium regime $\underline{\pi}$ over $\underline{\pi}^\star$. These risk-groups are profitable, as they pay a premium higher than their risk.

- Risk-groups i, with $\pi_L < \mu_i \leq \pi_L^\star$, will choose the cheaper premium regime $\underline{\pi}$ over $\underline{\pi}^\star$, as under $\underline{\pi}$ they would pay actuarially fair premiums, whereas under $\underline{\pi}^\star$, they would pay $\pi_L^\star$. These risk-groups contribute towards neither profits or losses for $\underline{\pi}$, but losing these risk-groups is problematic for $\underline{\pi}^\star$ as they would have contributed towards profits.

- Risk-groups i, with $\pi_L^\star < \mu_i \leq \pi_H$, will be indifferent between the premium regimes as they pay fair actuarial premiums under both regimes.

- Risk-groups i, with $\pi_H < \mu_i \leq \pi_H^\star$, will choose the cheaper premium regime $\underline{\pi}$ over $\underline{\pi}^\star$. Retaining these risk-groups is not problematic for premium regime $\underline{\pi}$, as they have already captured all the required low risk-groups to recoup any losses from these high risk-groups. The premium regime $\underline{\pi}^\star$ would be indifferent about the non-retention of these risk-groups, as these risk-groups pay fair actuarial premiums under $\underline{\pi}^\star$.

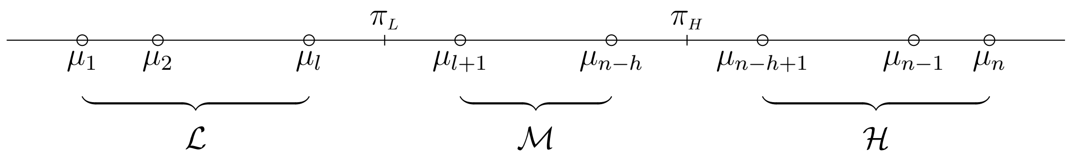
- Risk-groups i, with $\mu_i > \pi_H^\star$, will choose the cheaper premium regime $\underline{\pi}$ over $\underline{\pi}^\star$. Retaining these risk-groups is not problematic for premium regime $\underline{\pi}$, as they have already captured all the required low risk-groups to recoup any losses from these high risk-groups.

So premium regime $\underline{\pi}^{\star}$ is not attractive to any risk-groups, except those who are indifferent. Hence the smallest root of $e_{\kappa}(\pi_{L}) = 0$ provides a unique stable solution, among all possible roots. ■

Proof of step 3. The minimum and maximum premiums, π_{L} and π_{H}, create three collections of risk-groups:

- l lowest risk-groups for which $\mu_i < \pi_{L}$, where $i \in \mathcal{L} = \{1, \ldots, l\}$;
- h highest risk-groups for which $\mu_i > \pi_{H}$, where $i \in \mathcal{H} = \{n-h+1, \ldots, n\}$;
- remaining risk-groups for which $\pi_{L} \leq \mu_i \leq \pi_{H}$, where $i \in \mathcal{M} = \{l+1, \ldots, n-h\}$;

where the premium regime $\underline{\pi}$ charges π_{L} for all risk-groups in $\mathcal{L}$, π_{H} for all risk-groups in $\mathcal{H}$ and the actuarially fair premium for all remaining risk-groups in $\mathcal{M}$.



Consider an alternative equilibrium premium regime: $\underline{\pi}^{\star} = (\pi_1^{\star}, \pi_2^{\star}, \ldots, \pi_n^{\star})$, with the same minimum and maximum premiums, i.e. $\pi_{L} = \pi_{L}^{\star} = \min_i \pi_i^{\star}$ and $\pi_{H} = \pi_{H}^{\star} = \max_i \pi_i^{\star}$, but having a different structure. To check whether $\underline{\pi}^{\star}$ can destabilise $\underline{\pi}$, consider the following possibilities:

- For $i \in \mathcal{L}$, $\pi_i^{\star} \not< \pi_{L}$, as $\pi_{L} = \min_i \pi_i^{\star}$. Also, for $i \in \mathcal{L}$, $\pi_i^{\star} \not> \pi_{L}$, because then the profitable risk-group i would choose premium regime $\underline{\pi}$ over $\underline{\pi}^{\star}$, as $\underline{\pi}$ is cheaper. Then premium regime $\underline{\pi}^{\star}$ would lose out on the profits from risk-group i, which cannot be recouped from elsewhere, because increasing premiums would either be impossible (for risk-groups in $\mathcal{H}$) or make $\underline{\pi}^{\star}$ unattractive (for risk-groups in $\mathcal{M}$).

- For $i \in \mathcal{H}$, $\pi_i^{\star} \not> \pi_{H}$, as $\pi_{H} = \max_i \pi_i^{\star}$. Also, for $i \in \mathcal{H}$, $\pi_i^{\star} \not< \pi_{H}$, because then the loss-making risk-group i would choose premium regime $\underline{\pi}^{\star}$ over $\underline{\pi}$, as $\underline{\pi}^{\star}$ is cheaper. This means that $\underline{\pi}^{\star}$ would incur the losses from risk-group i, which cannot be recouped from elsewhere, because increasing premiums would make $\underline{\pi}^{\star}$ unattractive (for risk-groups in both $\mathcal{L}$ and $\mathcal{M}$). Note that, losing the loss-making risk-group i does not pose a problem for premium regime $\underline{\pi}$.

- For $i \in \mathcal{M}$, $\pi_i^{\star} \not> \mu_i$, as risk-group i would then choose premium regime $\underline{\pi}$ over $\underline{\pi}^{\star}$, as $\underline{\pi}$ is cheaper. Also, for $i \in \mathcal{M}$, $\pi_i^{\star} \not< \mu_i$, as risk-group i would then choose premium regime $\underline{\pi}^{\star}$ over $\underline{\pi}$, as $\underline{\pi}^{\star}$ is cheaper, and contribute losses for $\underline{\pi}^{\star}$, which cannot be recouped from elsewhere, because increasing premiums would either be impossible (for risk-groups in $\mathcal{H}$) or make $\underline{\pi}^{\star}$ unattractive (for risk-groups in $\mathcal{L}$). Note that, losing risk-group i does not pose a problem for premium regime $\underline{\pi}$, as these risk groups, paying the actuarially fair premiums, do not contribute to profits nor losses.

This proves that $\underline{\pi}$ cannot be destabilised by any other alternative equilibrium premium regime with the same minimum premium π_{L} and maximum premium π_{H}. ■

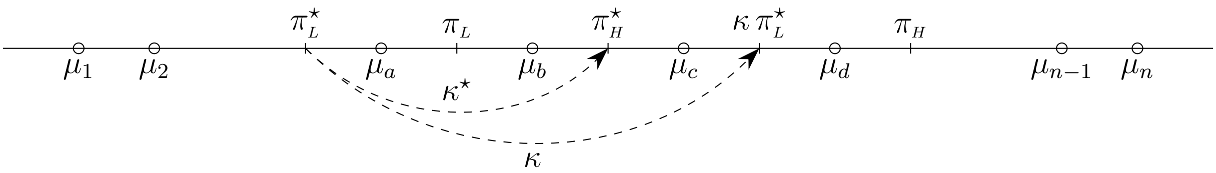
<u>*Proof of step 4.*</u> As a regulatory price collar would only require $\pi_{H} \leq \kappa\,\pi_{L}$, we need to show that $\underline{\pi}$ cannot be destabilised by an equilibrium premium regime using a smaller price collar.

Consider an alternative equilibrium premium regime: $\underline{\pi}^{\star} = (\pi_1^{\star}, \pi_2^{\star}, \ldots, \pi_n^{\star})$, with minimum and maximum premiums: $\pi_{L}^{\star} = \min_i \pi_i^{\star}$ and $\pi_{H}^{\star} = \max_i \pi_i^{\star}$ respectively, such that $\pi_{H}^{\star} = \kappa^{\star}\,\pi_{L}^{\star}$, for some $\kappa^{\star} < \kappa$. Also, suppose $\underline{\pi}^{\star}$ is stable and unique in the sense outlined in the proofs of steps 1, 2 and 3, but based on the smaller price collar $\kappa^{\star}$.

To check whether $\underline{\pi}^{\star}$ can destabilise $\underline{\pi}$, consider the following possibilities:

- If $\pi_{L} < \pi_{L}^{\star}$, then irrespective of the comparative values of π_{H} and $\pi_{H}^{\star}$, $\underline{\pi}^{\star}$ cannot destabilise $\underline{\pi}$, as all profitable low risk-groups i with risks $\mu_i < \pi_{L}$ will choose the cheaper premium regime $\underline{\pi}$.

- If $\pi_{L}^{\star} \leq \pi_{L}$, then $\kappa^{\star} < \kappa \Rightarrow \pi_{H}^{\star} = \kappa^{\star}\,\pi_{L}^{\star} < \kappa\,\pi_{L}^{\star} \leq \kappa\,\pi_{L} = \pi_{H}$, i.e.:



By construction of $\underline{\pi}^{\star}$, $e_{\kappa^{\star}}\left(\pi_{L}^{\star}\right) = 0$, where the profits generated from the low risk-groups by charging $\pi_{L}^{\star}$ is exactly offset by losses incurred from the high risk-groups by charging $\pi_{H}^{\star}$; while the risk-groups in between pay their actuarially fair premiums.

Now consider $e_{\kappa}\left(\pi_{L}^{\star}\right)$. In this case, the same amount of profit, as for $e_{\kappa^{\star}}\left(\pi_{L}^{\star}\right)$, is generated from the low risk-groups. But in $e_{\kappa}\left(\pi_{L}^{\star}\right)$, the premium charged for the high risk-groups, i.e. $\kappa\,\pi_{L}^{\star}$, is higher than that for $e_{\kappa^{\star}}\left(\pi_{L}^{\star}\right)$, which is $\pi_{H}^{\star} = \kappa^{\star}\,\pi_{L}^{\star}$. The higher premium, $\kappa\,\pi_{L}^{\star}$, reduces the high risk-groups' insurance demand, as well as lowers the losses; the overall impact being lower expected losses for $e_{\kappa}\left(\pi_{L}^{\star}\right)$.

Note that the risk-groups i, for which $\pi_{H}^{\star} < \mu_{i} < \kappa\,\pi_{L}^{\star}$, pay actuarially fair premium under $e_{\kappa}\left(\pi_{L}^{\star}\right)$ and thus do not contribute to profits or losses; whereas these risk-groups contribute losses for $e_{\kappa^{\star}}\left(\pi_{L}^{\star}\right)$. The risk-groups in between $\pi_{L}^{\star}$ and $\pi_{H}^{\star}$ pay their actuarially fair premiums under both $e_{\kappa}\left(\pi_{L}^{\star}\right)$ and $e_{\kappa^{\star}}\left(\pi_{L}^{\star}\right)$.

Hence $e_{\kappa}\left(\pi_{L}^{\star}\right) > 0$, implying that, there is a root of $e_{\kappa}\left(x\right) = 0$, which is smaller than $\pi_{L}^{\star} \leq \pi_{L}$, i.e. π_{L} cannot be the smallest root. This is a contradiction. So, $\pi_{L}^{\star} \nleq \pi_{L}$.

So, a smaller price collar leads to a higher minimum premium, i.e. $\pi_{L} < \pi_{L}^{\star}$, which cannot destabilise $\underline{\pi}$, as we have already proved. So, premium regime $\underline{\pi}$ cannot be destabilised by an equilibrium premium regime using a smaller price collar. $\blacksquare$

So, $\underline{\pi}$, as outlined in the theorem, exists, and is a unique stable equilibrium premium regime satisfying all political, regulatory and economic constraints. $\square$

Appendix H

Proof of Theorem 6

In order to prove Theorem 6, we first need a few preliminary results. First, we outline the relationship between v_1 and v_2:

Lemma 7. *Suppose there are two risk-groups with risks $\mu_1 < \mu_2$ and iso-elastic demand elasticities λ_1 and λ_2 respectively. For equilibrium premium regimes:*

$$\lambda_1 \leq 1 \Rightarrow \frac{dv_2}{dv_1} < 0. \tag{H.0.1}$$

Proof. Firstly note the following relationships:

$$v_2 \geq 1 \text{ and } \lambda_2 > 0 \Rightarrow v_2 - \left(1 - \frac{1}{\lambda_2}\right) = v_2 - 1 + \frac{1}{\lambda_2} > 0. \tag{H.0.2}$$

$$v_1 > 0 \text{ and } 0 < \lambda_1 \leq 1 \Rightarrow \left(1 - \frac{1}{\lambda_1}\right) \leq 0 \Rightarrow v_1 - \left(1 - \frac{1}{\lambda_1}\right) > 0. \tag{H.0.3}$$

Using Equations H.0.2 and H.0.3 in Equation 5.3.10:

$$\frac{dv_2}{dv_1} = -\frac{a_1\,\lambda_1\,v_1^{\lambda_1-2}}{a_2\,\lambda_2\,v_2^{\lambda_2-2}} \left[\frac{v_1 - \left(1 - \frac{1}{\lambda_1}\right)}{v_2 - \left(1 - \frac{1}{\lambda_2}\right)}\right] < 0. \tag{H.0.4}$$

□

Lemma 7 states that, if $\lambda_1 \leq 1$, v_2 is a decreasing function of v_1, or equivalently, π_2 is a decreasing function of π_1, irrespective of the value of λ_2. At first glance, the result in Lemma 7 seems obvious, as one would expect high risk-group's premium π_2 to fall, if low risk-group's premium π_1 increases, to maintain equilibrium. However, this is only partially true. Due to the compounding effect of insurance demand, as π_1 increases, although insurance profit increases as a result, this is tempered by the lowering of low risk-group's insurance demand. If the decrease in low risk-group's demand is not substantial, for example when $\lambda_1 \leq 1$, i.e. low risk-group's demand elasticity is low, a decrease in π_2 can be inferred with certainty. However, for higher demand elasticities for low risk-group, the impact of increasing π_1 on π_2 is indeterminate.

Next, we introduce the ratio of the risk-premium ratios v_2/v_1, which we will denote by ρ. Note that, for the case of two risk-groups, a price collar, $\kappa = \pi_2/\pi_1$, is related to ρ as follows:

$$\rho\,\kappa = \frac{\mu_2}{\mu_1}. \tag{H.0.5}$$

The relationship between the ratio of risk-premium ratios, ρ, and v_1 is as follows:

Lemma 8. *Suppose there are two risk-groups with risks $\mu_1 < \mu_2$ and iso-elastic demand elasticities λ_1 and λ_2 respectively. For equilibrium premium regimes:*

$$\lambda_1 \leq 1 \Rightarrow \frac{d\rho}{dv_1} < 0. \tag{H.0.6}$$

$$\lambda_2 \geq \lambda_1 \Rightarrow \frac{d\rho}{dv_1} < 0. \tag{H.0.7}$$

Proof. (of Equation H.0.6:) By Lemma 7.if $\lambda_1 \leq 1$, v_2 is a decreasing function of v_1. So $\rho = v_2/v_1$ is also a decreasing function of v_1 which proves Equation H.0.6. Mathematically:

$$v_2 = \rho\, v_1 \Rightarrow dv_2 = \rho\, dv_1 + v_1\, d\rho \Rightarrow \frac{d\rho}{dv_1} = \frac{1}{v_1}\left[\frac{dv_2}{dv_1} - \rho\right] < 0, \tag{H.0.8}$$

as $\frac{dv_2}{dv_1} < 0$ for $\lambda_1 \leq 1$ by Lemma 7. □

Proof. (of Equation H.0.7:) Using $v_2 = \rho\, v_1$ in the equilibrium condition in Equation 5.3.2 gives:

$$E\,(v_1, v_2) = a_1\,\left(v_1^{\lambda_1 - 1} - v_1^{\lambda_1}\right) + a_2\,\left(v_2^{\lambda_2 - 1} - v_2^{\lambda_2}\right) = 0, \tag{5.3.2}$$

$$\Leftrightarrow E\,(v_1, \rho) = a_1\,\left(v_1^{\lambda_1 - 1} - v_1^{\lambda_1}\right) + a_2\,\left(\rho^{\lambda_2 - 1}\,v_1^{\lambda_2 - 1} - \rho^{\lambda_2}\,v_1^{\lambda_2}\right) = 0, \tag{H.0.9}$$

$$\Leftrightarrow E^{\star}\,(v_1, \rho) = a_1\,(1 - v_1) + a_2\,\left(\rho^{\lambda_2 - 1}\,v_1^{\lambda_2 - \lambda_1} - \rho^{\lambda_2}\,v_1^{\lambda_2 - \lambda_1 + 1}\right) = 0, \tag{H.0.10}$$

after dividing by $v_1^{\lambda_1 - 1}$. Then ignoring higher-order terms in Taylor series expansion gives:

$$\begin{aligned} dE^{\star} = &- \left[a_1 + a_2\,\left\{(\lambda_2 - \lambda_1 + 1)\,\rho^{\lambda_2}\,v_1^{\lambda_2 - \lambda_1} - (\lambda_2 - \lambda_1)\,\rho^{\lambda_2 - 1}\,v_1^{\lambda_2 - \lambda_1 - 1}\right\}\right]\,dv_1 \\ &- a_2\,\left[\lambda_2\,\rho^{\lambda_2 - 1}\,v_1^{\lambda_2 - \lambda_1 + 1} - (\lambda_2 - 1)\,\rho^{\lambda_2 - 2}\,v_1^{\lambda_2 - \lambda_1}\right]\,d\rho = 0, \end{aligned} \tag{H.0.11}$$

$$\Rightarrow \frac{d\rho}{dv_1} = -\left[\frac{a_1 + a_2\,\left\{(\lambda_2 - \lambda_1 + 1)\,\rho^{\lambda_2}\,v_1^{\lambda_2 - \lambda_1} - (\lambda_2 - \lambda_1)\,\rho^{\lambda_2 - 1}\,v_1^{\lambda_2 - \lambda_1 - 1}\right\}}{a_2\,\left[\lambda_2\,\rho^{\lambda_2 - 1}\,v_1^{\lambda_2 - \lambda_1 + 1} - (\lambda_2 - 1)\,\rho^{\lambda_2 - 2}\,v_1^{\lambda_2 - \lambda_1}\right]}\right], \tag{H.0.12}$$

$$= -\left[\frac{a_1 + a_2\,\rho^{\lambda_2 - 1}\,v_1^{\lambda_2 - \lambda_1 - 1}\,(\lambda_2 - \lambda_1 + 1)\,\left[v_2 - \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1 + 1}\right]}{a_2\,\rho^{\lambda_2 - 2}\,v_1^{\lambda_2 - \lambda_1}\,\lambda_2\,\left[v_2 - \left(1 - \frac{1}{\lambda_2}\right)\right]}\right], \quad \text{as } v_2 = \rho\,v_1, \tag{H.0.13}$$

$$< 0, \quad \text{as } \lambda_2 \geq \lambda_1 \text{ and } v_2 \geq 1. \tag{H.0.14}$$

Equation H.0.7 asserts that even when $\lambda_1 \geq 1$, although the behaviour of v_2 as a function of v_1 is indeterminate, but if $\lambda_2 \geq \lambda_1$, ρ is still a decreasing function of v_1. □

Now recall the relationship between C and v_1 obtained in Equation 5.3.13, reproduced below:

$$\frac{dC}{dv_1} = \underbrace{\left[\frac{a_1\,\lambda_1\,v_1^{\lambda_1 - 1}}{v_2 - \left(1 - \frac{1}{\lambda_2}\right)}\right]}_{T_1}\,\underbrace{\left(1 - \frac{1}{\lambda_1}\right)}_{T_2}\,\underbrace{\left[\frac{v_2}{v_1} - \frac{1 - \frac{1}{\lambda_2}}{1 - \frac{1}{\lambda_1}}\right]}_{T_3}. \tag{5.3.13}$$

To analyse the three multiplicative terms, T_1, T_2 and T_3, in Equation 5.3.13, note that a_1, λ_1, λ_2 and v_1 are all positive and $v_2 \geq 1$. So:

$$T_1 > 0. \tag{H.0.15}$$

$$T_2 \gtreqless 0 \Leftrightarrow \lambda_1 \gtreqless 1. \tag{H.0.16}$$

The behaviour of T_3 depends on the values of λ_1 and λ_2. Using the segments and sub-segments of the (λ_1, λ_2)–plane given in Figure 5.1, the result outlining the behaviour of T_3 is presented in Lemma 9.

Lemma 9. *Suppose there are* 2 *risk-groups with risks* $\mu_1 < \mu_2$ *and iso-elastic demand elasticities* λ_1 *and* λ_2 *respectively. For* T_3 *defined as:*

$$T_3 = \left[\frac{v_2}{v_1} - \frac{1 - \frac{1}{\lambda_2}}{1 - \frac{1}{\lambda_1}}\right] = \left[\rho - \frac{1 - \frac{1}{\lambda_2}}{1 - \frac{1}{\lambda_1}}\right] : \tag{H.0.17}$$

9.1. $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *:* $T_3 \geq 0$.

9.2. $\mathcal{B}$ *and* $\mathcal{D}$ *: There exists a* $v_1^\star$ *where* $v_{1p} \leq v_1^\star \leq v_{1f}$ *such that* $T_3 \gtreqless 0 \Leftrightarrow v_1 \lesseqgtr v_1^\star$.

9.3. $\mathcal{C}_2$ *and* $\mathcal{A}_3$ *:* $T_3 \leq 0$.

9.4. $\mathcal{C}_1$ *and* $\mathcal{C}_3$ *:* $T_3 \geq 0$.

Proof. (of Lemma 9.1:)

$$\text{For } \mathcal{A}_1 : \lambda_1 \leq 1 \text{ and } \lambda_2 \geq 1 \Rightarrow \frac{1 - \frac{1}{\lambda_2}}{1 - \frac{1}{\lambda_1}} \leq 0. \tag{H.0.18}$$

$$\text{For } \mathcal{A}_2 : \lambda_1 \leq \lambda_2 \leq 1 \Rightarrow \frac{1 - \frac{1}{\lambda_2}}{1 - \frac{1}{\lambda_1}} \leq 1. \tag{H.0.19}$$

$$\text{As } \rho = \frac{v_2}{v_1} \geq 1, \text{ for } \mathcal{A}_1 \text{ and } \mathcal{A}_2 : T_3 = \left[\frac{v_2}{v_1} - \frac{1-\frac{1}{\lambda_2}}{1-\frac{1}{\lambda_1}}\right] \geq 0. \tag{H.0.20}$$

□

Proof. (of Lemma 9.2:) For $\mathcal{B}$, $\lambda_1 \leq 1$ and for $\mathcal{D}$, $\lambda_2 \geq \lambda_1$. So by Lemma 8, for both $\mathcal{B}$ and $\mathcal{D}$, $\rho = v_2/v_1$ is a decreasing function of v_1 and hence T_3 is a decreasing function of v_1.

For both $\mathcal{B}$ and $\mathcal{D}$, as v_1 increases from v_{1p} to v_{1f} (i.e. from pooled to full risk classification regime), ρ decreases from μ_2/μ_1 to 1. Now note that for both $\mathcal{B}$ and $\mathcal{D}$:

$$1 \leq \frac{1-\frac{1}{\lambda_2}}{1-\frac{1}{\lambda_1}} \leq \frac{\mu_2}{\mu_1}. \tag{H.0.21}$$

So there exists a $v_1^\star$ where $v_{1p} \leq v_1^\star \leq v_{1f}$ such that:

$$T_3 = \left[\frac{v_2}{v_1} - \frac{1-\frac{1}{\lambda_2}}{1-\frac{1}{\lambda_1}}\right] \gtreqqless 0 \Leftrightarrow v_1 \lesseqqgtr v_1^\star, \tag{H.0.22}$$

as required. □

Proof. (of Lemma 9.3:) For both $\mathcal{C}_2$ and $\mathcal{A}_3$:

$$\frac{1-\frac{1}{\lambda_2}}{1-\frac{1}{\lambda_1}} \geq \frac{\mu_2}{\mu_1} \Rightarrow T_3 = \left[\frac{v_2}{v_1} - \frac{1-\frac{1}{\lambda_2}}{1-\frac{1}{\lambda_1}}\right] \leq \left[\frac{v_2}{v_1} - \frac{\mu_2}{\mu_1}\right] \leq 0, \tag{H.0.23}$$

as $\rho = \frac{v_2}{v_1}$ never exceeds μ_2/μ_1. □

Proof. (of Lemma 9.4:)

$$\text{For } \mathcal{C}_1 : \lambda_1 \geq 1 \text{ and } \lambda_2 \leq 1 \Rightarrow \frac{1-\frac{1}{\lambda_2}}{1-\frac{1}{\lambda_1}} \leq 0. \tag{H.0.24}$$

$$\text{For } \mathcal{C}_3 : \lambda_1 \geq \lambda_2 \geq 1 \Rightarrow \frac{1-\frac{1}{\lambda_2}}{1-\frac{1}{\lambda_1}} \leq 1. \tag{H.0.25}$$

$$\text{As } \rho = \frac{v_2}{v_1} \geq 1, \text{ for } \mathcal{C}_1 \text{ and } \mathcal{C}_3 : T_3 = \left[\frac{v_2}{v_1} - \frac{1 - \frac{1}{\lambda_2}}{1 - \frac{1}{\lambda_1}} \right] \geq 0. \tag{H.0.26}$$

$\square$

Over the range of v_1, except for the two regions $\mathcal{B}$ and $\mathcal{D}$, where T_3 changes its sign, for all other regions, T_3 is either positive or negative.

Theorem 6. *Suppose there are two risk-groups with risks $\mu_1 < \mu_2$ and iso-elastic demand elasticities λ_1 and λ_2 respectively. Consider the four segments, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, in the (λ_1, λ_2)–plane formed by the intersecting curves:*

$$\lambda_2 = \lambda_1; \tag{5.3.14}$$

$$\frac{1-\frac{1}{\lambda_2}}{1-\frac{1}{\lambda_1}} = \frac{\mu_2}{\mu_1}; \tag{5.3.15}$$

where

- $\mathcal{A} = \left\{(\lambda_1, \lambda_2): \ \lambda_2 \geq \lambda_1\right\} - \mathcal{D};$
- $\mathcal{B} = \left\{(\lambda_1, \lambda_2): \ \lambda_2 \leq \lambda_1 \leq 1 \ \textit{and} \ 1 \leq \left(1-\frac{1}{\lambda_2}\right)/\left(1-\frac{1}{\lambda_1}\right) \leq \frac{\mu_2}{\mu_1}\right\};$
- $\mathcal{C} = \left\{(\lambda_1, \lambda_2): \ \lambda_2 \leq \lambda_1\right\} - \mathcal{B};$
- $\mathcal{D} = \left\{(\lambda_1, \lambda_2): \ \lambda_2 \geq \lambda_1 \geq 1 \ \textit{and} \ 1 \leq \left(1-\frac{1}{\lambda_2}\right)/\left(1-\frac{1}{\lambda_1}\right) \leq \frac{\mu_2}{\mu_1}\right\}.$

For each of the segments, we have:

6.1. $\mathcal{A}$ *: Loss coverage is maximum for pooled and minimum for full risk classification regime, while partial risk classification is intermediate.*

6.2. $\mathcal{B}$ *: Loss coverage is minimum for a specific partial risk classification regime and maximum for either pooled or full risk classification.*

6.3. $\mathcal{C}$ *: Loss coverage is maximum for full risk classification regime and minimum for pooled, while partial risk classification is intermediate.*

6.4. $\mathcal{D}$ *: Loss coverage is maximum for a specific partial risk classification regime.*

For the proof of Theorem 6, recall from Equations 5.3.13, H.0.15 and H.0.16:

$$\frac{dC}{dv_1} = T_1 \times T_2 \times T_3, \text{ where, } T_1 > 0;\ T_2 \gtreqless 0 \Leftrightarrow \lambda_1 \gtreqless 1; \tag{H.0.27}$$

and the behaviour of T_3 is outlined in Lemma 9.

Proof. (of Theorem 6.1:) For $\mathcal{A}$:

$$\text{For } \mathcal{A}_1 : T_2 \leq 0 \text{ and } T_3 \geq 0 \Rightarrow \frac{dC}{dv_1} \leq 0. \tag{H.0.28}$$

$$\text{For } \mathcal{A}_2 : T_2 \leq 0 \text{ and } T_3 \geq 0 \Rightarrow \frac{dC}{dv_1} \leq 0. \tag{H.0.29}$$

$$\text{For } \mathcal{A}_3 : T_2 \geq 0 \text{ and } T_3 \leq 0 \Rightarrow \frac{dC}{dv_1} \leq 0. \tag{H.0.30}$$

So, in all three cases, $\frac{dC}{dv_1} \leq 0$, implying that the loss coverage ratio, C, is a decreasing function of v_1, as v_1 increases from v_{1p} to v_{1f}. Hence, loss coverage is maximum for pooled equilibrium and minimum for full risk classification. Partial risk classification is intermediate. □

Proof. (of Theorem 6.2:) For $\mathcal{B}$:

$$T_2 \leq 0 \text{ and } \left[T_3 \gtreqless 0 \Leftrightarrow v_1 \lesseqgtr v_1^\star\right] \Rightarrow \left[\frac{dC}{dv_1} \lesseqgtr 0 \Leftrightarrow v_1 \lesseqgtr v_1^\star\right], \tag{H.0.31}$$

where $v_{1p} \leq v_1^\star \leq v_{1f}$. This implies that the loss coverage ratio, C, is a minimum at $v_1^\star$ as v_1 increases from v_{1p} to v_{1f}. Hence, loss coverage is maximum at either of the two extremes, pooled or full risk classification, depending on the other model parameters. □

Proof. (of Theorem 6.3:) For $\mathcal{C}$:

$$\text{For } \mathcal{C}_1 : T_2 \geq 0 \text{ and } T_3 \geq 0 \Rightarrow \frac{dC}{dv_1} \geq 0. \tag{H.0.32}$$

$$\text{For } \mathcal{C}_2 : T_2 \leq 0 \text{ and } T_3 \leq 0 \Rightarrow \frac{dC}{dv_1} \geq 0. \tag{H.0.33}$$

$$\text{For } \mathcal{C}_3 : T_2 \geq 0 \text{ and } T_3 \geq 0 \Rightarrow \frac{dC}{dv_1} \geq 0. \tag{H.0.34}$$

So, in all three cases, $\frac{dC}{dv_1} \geq 0$, implying that the loss coverage ratio, C, is an increasing function as v_1 increases from v_{1p} to v_{1f}. Hence, loss coverage is maximum for full risk classification and minimum for pooled equilibrium. Partial risk classification is intermediate. □

Proof. (of Theorem 6.4:) For $\mathcal{D}$:

$$T_2 \geq 0 \text{ and } \left[T_3 \gtreqless 0 \Leftrightarrow v_1 \lesseqgtr v_1^\star\right] \Rightarrow \left[\frac{dC}{dv_1} \gtreqless 0 \Leftrightarrow v_1 \lesseqgtr v_1^\star\right], \tag{H.0.35}$$

where $v_{1p} \leq v_1^\star \leq v_{1f}$. This implies that the loss coverage ratio, C, is a maximum at the partial risk classification regime $v_1^\star$, as v_1 increases from v_{1p} to v_{1f}. □

Appendix I

Proof of Equation 5.4.25

To show:

$$\frac{dC}{dv_L} = \xi \; \underbrace{\left[\frac{a_L \, \lambda_L \, v_L^{\lambda_L - 1}}{v_H - \left(1 - \frac{1}{\lambda_H}\right)}\right]}_{T_1} \; \underbrace{\left(1 - \frac{1}{\lambda_L}\right)}_{T_2} \; \underbrace{\left[\frac{v_H}{v_L} - \frac{\left(1 - \frac{1}{\lambda_H}\right)}{\left(1 - \frac{1}{\lambda_L}\right)}\right]}_{T_3}, \tag{5.4.25}$$

first note that by the definitions of a_L and ξ in Equations 5.4.14 and 5.4.22 respectively:

$$\xi \; a_L = \frac{p_L \, \tau_L \, \mu_L}{\sum_{i=1}^{n} p_i \, \tau_i \, \mu_i}; \tag{I.0.1}$$

$$= \frac{\sum_{i\in\mathcal{L}} p_i \, \tau_i \left(\frac{\mu_i}{\mu_L}\right)^{\lambda_L} \mu_L}{\sum_{i=1}^{n} p_i \, \tau_i \, \mu_i}; \quad \text{by definitions of } p_L \, \tau_L \text{ in Equation 5.4.8;} \tag{I.0.2}$$

$$= \frac{\sum_{i\in\mathcal{L}} p_i \, \tau_i \left(\frac{\mu_i}{\mu_L}\right)^{\lambda_L} \mu_i}{\sum_{i=1}^{n} p_i \, \tau_i \, \mu_i}; \quad \text{as } \mu_L \text{ is the pooled equilibrium premium for } \mathcal{L}; \tag{I.0.3}$$

$$= \sum_{i\in\mathcal{L}} a_i \left(\frac{\mu_i}{\mu_L}\right)^{\lambda_L}. \tag{I.0.4}$$

So, Equation 5.4.25 can be alternatively expressed as:

$$\frac{dC}{dv_L} = \left[\frac{\left(\sum_{i \in \mathcal{L}} a_i \left(\frac{\mu_i}{\mu_L} \right)^{\lambda_L} \right) \lambda_L \, v_L^{\lambda_L - 1}}{v_H - \left(1 - \frac{1}{\lambda_H} \right)} \right] \left(1 - \frac{1}{\lambda_L} \right) \left[\frac{v_H}{v_L} - \frac{\left(1 - \frac{1}{\lambda_H} \right)}{\left(1 - \frac{1}{\lambda_L} \right)} \right], \tag{I.0.5}$$

Proof. Equation I.0.5, and equivalently Equation 5.4.25, can be proved directly from first principles by following the method outlined in the proof of Theorem 6.

As the risk-groups in $\mathcal{M}$ do not contribute to profit or loss, the equilibrium condition can be expressed as:

$$E\left(\underline{\pi}\right) = \underbrace{\sum_{i \in \mathcal{L}} p_i \, \tau_i \left(\frac{\mu_i}{\pi_L} \right)^{\lambda_L} \left(\pi_L - \mu_i \right)}_{E_{\mathcal{L}}} + \underbrace{\sum_{j \in \mathcal{H}} p_j \, \tau_j \left(\frac{\mu_j}{\pi_H} \right)^{\lambda_H} \left(\pi_H - \mu_j \right)}_{E_{\mathcal{H}}} = 0. \tag{I.0.6}$$

The expression for the equilibrium condition in Equation I.0.6, can also be written in terms of the risk-premium ratios, as follows:

$$E\left(\underline{v}\right) = \sum_{i \in \mathcal{L}} a_i \left(v_i^{\lambda_L - 1} - v_i^{\lambda_L} \right) + \sum_{j \in \mathcal{H}} a_j \left(v_j^{\lambda_H - 1} - v_j^{\lambda_H} \right) = 0, \text{ where} \tag{I.0.7}$$

$$v_i = \frac{\mu_i}{\pi_L} \text{ for } i \in \mathcal{L} \text{ and } v_j = \frac{\mu_j}{\pi_H} \text{ for } j \in \mathcal{H}.$$

Based on the approach outlined in Section 5.3:

$$dE = E\left(\underline{v} + \underline{dv}\right) - E\left(\underline{v}\right) = \sum_{i \in \mathcal{L}} E_i \, dv_i + \sum_{j \in \mathcal{H}} E_j \, dv_j, \tag{I.0.8}$$

$$\text{where } E_i = \frac{\partial E}{\partial v_i} = -a_i \, \lambda_L \, v_i^{\lambda_L - 2} \left[v_i - \left(1 - \frac{1}{\lambda_L} \right) \right], \quad \text{for } i \in \mathcal{L}; \tag{I.0.9}$$

$$\text{and } E_j = \frac{\partial E}{\partial v_j} = -a_j \, \lambda_H \, v_j^{\lambda_H - 2} \left[v_j - \left(1 - \frac{1}{\lambda_H} \right) \right], \quad \text{for } j \in \mathcal{H}. \tag{I.0.10}$$

Now note that:

$$\text{For } i \in \mathcal{L}\text{: } v_i = \frac{\mu_i}{\mu_1}\, v_1 \text{ and } dv_i = \frac{\mu_i}{\mu_1}\, dv_1. \tag{I.0.11}$$

$$\text{For } j \in \mathcal{H}\text{: } v_j = \frac{\mu_j}{\mu_n}\, v_n \text{ and } dv_j = \frac{\mu_j}{\mu_1}\, dv_n. \tag{I.0.12}$$

Using these relationships, the component in Equation I.0.8 for risk-groups in $\mathcal{L}$ becomes:

$$\sum_{i\in\mathcal{L}} E_i\, dv_i$$

$$= -\sum_{i\in\mathcal{L}} a_i\, \lambda_L\, v_i^{\lambda_L - 2} \left[v_i - \left(1 - \frac{1}{\lambda_L}\right)\right] dv_i, \tag{I.0.13}$$

$$= -\sum_{i\in\mathcal{L}} a_i\, \lambda_L \left(\frac{\mu_i}{\mu_1}\right)^{\lambda_L - 2} v_1^{\lambda_L - 2} \left[\frac{\mu_i}{\mu_1}\, v_1 - \left(1 - \frac{1}{\lambda_L}\right)\right] \frac{\mu_i}{\mu_1}\, dv_1, \tag{I.0.14}$$

$$= -\lambda_L\, v_1^{\lambda_L - 2} \left[\sum_{i\in\mathcal{L}} a_i \left(\frac{\mu_i}{\mu_1}\right)^{\lambda_L}\right] \left[v_1 - \left(1 - \frac{1}{\lambda_L}\right) \frac{\sum_{i\in\mathcal{L}} a_i \left(\frac{\mu_i}{\mu_1}\right)^{\lambda_L - 1}}{\sum_{i\in\mathcal{L}} a_i \left(\frac{\mu_i}{\mu_1}\right)^{\lambda_L}} \right] dv_1, \tag{I.0.15}$$

$$= -\lambda_L\, v_1^{\lambda_L - 2} \left[\sum_{i\in\mathcal{L}} a_i \left(\frac{\mu_i}{\mu_1}\right)^{\lambda_L}\right] \left[v_1 - \left(1 - \frac{1}{\lambda_L}\right) \frac{\mu_1}{\mu_L}\right] dv_1, \tag{I.0.16}$$

where μ_L is the pooled equilibrium premium, if the insurance market only consisted of the risk-groups in $\mathcal{L}$, as shown in Equation 5.4.6 i.e.:

$$\mu_L = \frac{\sum_{i\in\mathcal{L}} p_i\, \tau_i\, \mu_i^{\lambda_L + 1}}{\sum_{i\in\mathcal{L}} p_i\, \tau_i\, \mu_i^{\lambda_L}} = \frac{\sum_{i\in\mathcal{L}} a_i\, \mu_i^{\lambda_L}}{\sum_{i\in\mathcal{L}} a_i\, \mu_i^{\lambda_{L-1}}} \quad \Rightarrow \quad \mu_1 \le \mu_L \le \max_{i\in\mathcal{L}} \mu_i \le \pi_L. \tag{I.0.17}$$

Similarly, the component in Equation I.0.8 for risk-groups in $\mathcal{H}$ becomes:

$$\sum_{i\in\mathcal{H}} E_j\, dv_j = -\lambda_H\, v_n^{\lambda_H - 2} \left[\sum_{j\in\mathcal{H}} a_j \left(\frac{\mu_j}{\mu_n}\right)^{\lambda_H}\right] \left[v_n - \left(1 - \frac{1}{\lambda_H}\right) \frac{\mu_n}{\mu_H}\right] dv_n, \tag{I.0.18}$$

$$\text{where } \mu_H = \frac{\sum_{j\in\mathcal{H}} a_j\, \mu_j^{\lambda_H}}{\sum_{j\in\mathcal{H}} a_j\, \mu_j^{\lambda_{H-1}}} \quad \Rightarrow \quad \pi_H \le \min_{j\in\mathcal{H}} \mu_j \le \mu_H \le \mu_n \tag{I.0.19}$$

As $dE = 0$, the relationship between dv_1 and dv_n can be expressed as:

$$\frac{dv_n}{dv_1} = -\left(\frac{\sum_{i\in\mathcal{L}} a_i \left(\frac{\mu_i}{\mu_1}\right)^{\lambda_L}}{\sum_{j\in\mathcal{H}} a_j \left(\frac{\mu_j}{\mu_n}\right)^{\lambda_H}}\right)\left(\frac{\lambda_L}{\lambda_H}\right)\left(\frac{v_1^{\lambda_L-2}}{v_n^{\lambda_H-2}}\right)\left(\frac{v_1-\left(1-\frac{1}{\lambda_L}\right)\frac{\mu_1}{\mu_L}}{v_n-\left(1-\frac{1}{\lambda_H}\right)\frac{\mu_n}{\mu_H}}\right). \qquad \text{(I.0.20)}$$

Note that, for two risk-groups, i.e. $n = 2$, Equation I.0.20 simplifies to Equation 5.3.10 as then $\mu_L = \mu_1$ and $\mu_H = \mu_n$.

The expression for loss coverage ratio for n risk-groups, in terms of risk-premium ratios, is given by:

$$C\left(\underline{v}\right) = \sum_{i\in\mathcal{L}} a_i\, v_i^{\lambda_L} + \sum_{j\in\mathcal{H}} a_j\, v_j^{\lambda_H}. \qquad \text{(I.0.21)}$$

Based on the approach outlined in Section 5.3:

$$dC = \sum_{i\in\mathcal{L}} C_i\, dv_i + \sum_{j\in\mathcal{H}} C_j\, dv_j, \qquad \text{(I.0.22)}$$

$$\text{where } C_i = \frac{\partial C}{\partial v_i} = a_i\, \lambda_L\, v_i^{\lambda_L-1}, \quad \text{for } i \in \mathcal{L}; \qquad \text{(I.0.23)}$$

$$\text{and } C_j = \frac{\partial C}{\partial v_j} = a_j\, \lambda_H\, v_j^{\lambda_H-1}, \quad \text{for } j \in \mathcal{H}. \qquad \text{(I.0.24)}$$

So:

$$dC = \sum_{i\in\mathcal{L}} a_i\, \lambda_L\, v_i^{\lambda_L-1}\, dv_i + \sum_{j\in\mathcal{H}} a_j\, \lambda_H\, v_j^{\lambda_H-1}\, dv_j. \qquad \text{(I.0.25)}$$

Using Equations I.0.11 and I.0.12, we get:

$$dC = \sum_{i\in\mathcal{L}} a_i\, \lambda_L \left(\frac{\mu_i}{\mu_1}\right)^{\lambda_L} v_1^{\lambda_L-1}\, dv_1 + \sum_{j\in\mathcal{H}} a_j\, \lambda_H \left(\frac{\mu_j}{\mu_n}\right)^{\lambda_H} v_n^{\lambda_H-1}\, dv_n. \qquad \text{(I.0.26)}$$

Now using the relationship between dv_1 and dv_n in Equation I.0.20, we get:

$$\frac{dC}{dv_1} = \sum_{i\in\mathcal{L}} a_i\,\lambda_L\,\left(\frac{\mu_i}{\mu_1}\right)^{\lambda_L}\,v_1^{\lambda_L-1}\left[1-\frac{v_n}{v_1}\left(\frac{v_1-\left(1-\frac{1}{\lambda_L}\right)\frac{\mu_1}{\mu_L}}{v_n-\left(1-\frac{1}{\lambda_H}\right)\frac{\mu_n}{\mu_H}}\right)\right]; \tag{I.0.27}$$

$$= \left[\frac{\left(\sum_{i\in\mathcal{L}} a_i\,\left(\frac{\mu_i}{\mu_1}\right)^{\lambda_L}\right)\,\lambda_L\,v_1^{\lambda_L-1}}{v_n-\left(1-\frac{1}{\lambda_H}\right)\frac{\mu_n}{\mu_H}}\right]\left(1-\frac{1}{\lambda_L}\right)\frac{\mu_1}{\mu_L}\left[\frac{v_n}{v_1}-\frac{\left(1-\frac{1}{\lambda_H}\right)\frac{\mu_n}{\mu_H}}{\left(1-\frac{1}{\lambda_L}\right)\frac{\mu_1}{\mu_L}}\right] \tag{I.0.28}$$

$$= \left[\frac{\left(\sum_{i\in\mathcal{L}} a_i\,\left(\frac{\mu_i}{\mu_1}\right)^{\lambda_L}\right)\,\lambda_L\,v_1^{\lambda_L-1}}{v_H-\left(1-\frac{1}{\lambda_H}\right)}\right]\left(1-\frac{1}{\lambda_L}\right)\left[\frac{v_H}{v_L}-\frac{\left(1-\frac{1}{\lambda_H}\right)}{\left(1-\frac{1}{\lambda_L}\right)}\right]; \tag{I.0.29}$$

where $v_L = \dfrac{\mu_L}{\pi_L}$ and $v_H = \dfrac{\mu_H}{\pi_H}$.

Further noting that:

$$v_1 = \frac{\mu_1}{\mu_L}\,v_L \text{ and } dv_1 = \frac{\mu_1}{\mu_L}\,dv_L, \tag{I.0.30}$$

we get:

$$\frac{dC}{dv_L} = \left[\frac{\left(\sum_{i\in\mathcal{L}} a_i\,\left(\frac{\mu_i}{\mu_1}\right)^{\lambda_L}\right)\,\lambda_L\,\left(\frac{\mu_1}{\mu_L}\right)^{\lambda_L}\,v_L^{\lambda_L-1}}{v_H-\left(1-\frac{1}{\lambda_H}\right)}\right]\left(1-\frac{1}{\lambda_L}\right)\left[\frac{v_H}{v_L}-\frac{\left(1-\frac{1}{\lambda_H}\right)}{\left(1-\frac{1}{\lambda_L}\right)}\right]; \tag{I.0.31}$$

$$= \left[\frac{\left(\sum_{i\in\mathcal{L}} a_i\,\left(\frac{\mu_i}{\mu_L}\right)^{\lambda_L}\right)\,\lambda_L\,v_L^{\lambda_L-1}}{v_H-\left(1-\frac{1}{\lambda_H}\right)}\right]\left(1-\frac{1}{\lambda_L}\right)\left[\frac{v_H}{v_L}-\frac{\left(1-\frac{1}{\lambda_H}\right)}{\left(1-\frac{1}{\lambda_L}\right)}\right]; \tag{I.0.32}$$

which proves Equation I.0.5 as required.

Again, note that, for two risk-groups, i.e. $n = 2$, Equation I.0.27 simplifies to Equation 5.3.13 as then $\mu_L = \mu_1$ and $\mu_H = \mu_n$. $\square$

Appendix J

Summary of relevant literature

J.1 Introduction

Most of the economic literature on insurance markets favour risk classification by insurers on the ground that it is perceived to be more efficient. The argument in favour of risk classification is that it reduces the information asymmetry between insurer and customer, and that in turn helps the market to operate more efficiently i.e. it reduces loss of utility for both insurer and customers. In practice however, regulators in many cases favour risk pooling, or at least bar insurers from using extensive risk differentiation. This is justified on the ground that risk classification and charging fair premium to customers may make insurance unaffordable to high-risk customers. A parable (said to have been posed by Kenneth Arrow, see Gruber (2019)) of two farmers facing a hurricane is relevant in this context. With the uncertainty of not knowing which farmer gets hit by the hurricane, it incentivizes both to pool their resources and provide cover for each other. But with the precise knowledge of which one will get hit, there is no incentive for the farmer, who will be unaffected by the storm to help the other, who faces a certain disaster . Therefore more information about individual risks may result in a society with a sub-optimal allocation of resources.

In the following sub-sections we present a detailed summary of some of the relevant literature we have covered in chapter 2 above.

J.2 Equilibrium and Adverse Selection

In this section we present a summary of results from Rothschild and Stiglitz (1976). In a simple two risk-group model, the authors showed that with information asymmetry, a pure-strategy Nash equilibrium can never exist in the insurance market with pooled premium. The authors defined an equilibrium in a single period, when customers would choose a contract from an offered set to maximize their utilities. Each insurer would act independently of its competitors, and :

1. No contract would generate negative expected profit for insurer.

2. No contract outside the set will make the insurer better off.

However this characterization of equilibrium is not universally accepted, as other authors, e.g. Wilson (1976) have defined equilibrium differently.

Summary of Rothschild and Stiglitz (1976)

Rothschild and Stiglitz (1976) describes a model where individuals are offered insurance contracts $\underline{\alpha} = (\alpha_1, \alpha_2)$ consisting of a premium α_1 and a payout α_2. The individual can belong to either of two risk-groups with probabilities of loss given by p^H or p^L, where $p^H > p^L$. The proportion of high-risk customers is given by λ. W_1 and W_2 denote their wealth before and after the loss inducing event respectively. Insurers' expected profit from a contract, which would be zero under perfect competition, would be given by "fair-odds line" :

$$\pi(\overline{p}, \underline{\alpha}) = (1 - \overline{p})\alpha_1 - \overline{p}\alpha_2 = 0 \tag{J.2.1}$$

where

$$\overline{p} = \lambda p^H + (1 - \lambda)p^L. \tag{J.2.2}$$

The authors argued that a pooled contract can never achieve market equilibrium. As shown in the figure J.1 below, for any point α on the fair odds line, it would be possible for an insurer to offer an alternative pooled contract β which would only be attractive to lower risk customers and also generate positive profit for the insurer. u^H and u^L denote the indifference curves of the high-risk and low-risk customers respectively. Therefore the equilibrium would be disrupted and only high-risk individuals will be left with the original contract.

Even when insurers are able to offer separating contracts, i.e. a pair of contracts which would reveal the risk of loss as perceived by the individual customer, a market equilibrium is not guaranteed. It would depend on the relative population of high and low risk individuals in the market.

In a separating scenario, high-risk customers will be offered a contract that would lie on the line with slope $-\left(\frac{1-p^H}{p^H}\right)$, and low-risk customers would be offered a contract on the line with slope $-\left(\frac{1-p^L}{p^L}\right)$.They are shown in the figure J.2 by lines EG and EF respectively E represents the initial wealth of any individual.

Note that α^H must always be part of separating equilibrium. Also note that the contract offered to low-risk individuals should be such that it would not incentivize high-risk customers to switch. Therefore low-risk customers would be offered the contract α^L. The authors argue that the contracts represented by α^H and α^L are the only possible

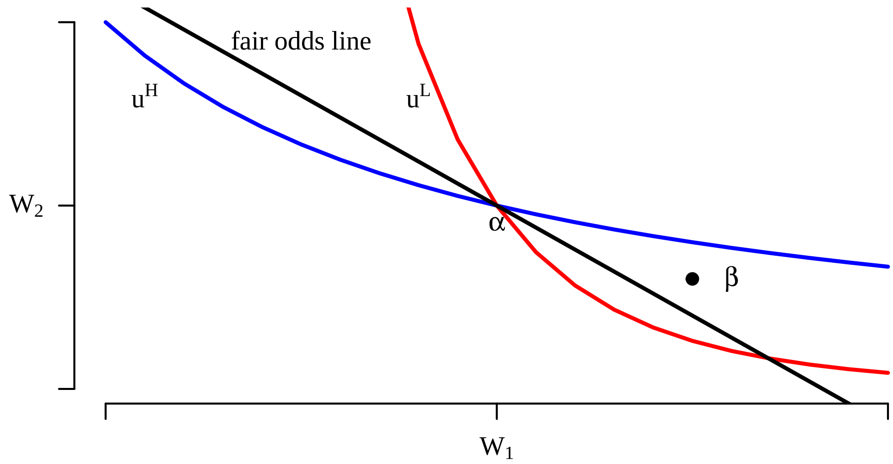


Figure J.1: Pooled premium regime

separating equilibrium in this scenario.

However in the scenario in figure J.3, it would be possible to disrupt this equilibrium too. Suppose the market fair odds line is given here by EH. A pooled contract offered in the region below EH but above u^L curve (e.g. γ) will attract both high and low risk customers and will produce positive profit for insurer. However it was argued earlier that a pooled equilibrium would never be achievable. Therefore in this case there would be no possible market equilibrium. Some of the observations from Figure J.3 are

- If there are very few high-risk individuals, then cost of pooling would be low. EH and EF would be closer. In this case, insurer may offer a pooled contract.

- Similarly, if cost of separating is too high for low-risk customers (i.e. Utility loss for low-risk customer for the separating contract in figure J.2 is high compared to full information scenario), a pooled premium may be offered by the insurer. Graphically

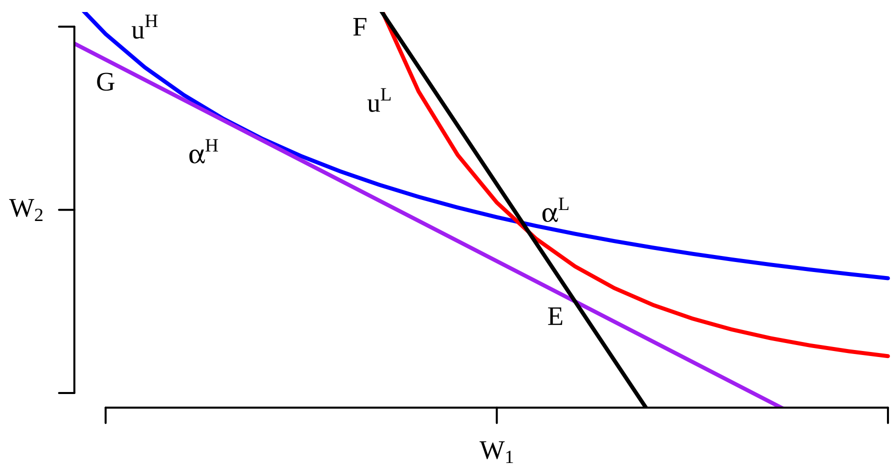


Figure J.2: Separating equilibrium

this means that in figure J.3 the gap between EF and u^L is bigger.

- If p^L is close to zero, then it would not pay the low-risk individuals to pool. Therefore low-risk individuals will not have complete coverage.

- Low cost of pooling implies EH and EF are close. Cost of pooling arises from low-risk individuals subsidizing high-risk individuals.

The authors argue that an equilibrium would not exist even if we assume that the individual risk probabilities follow a continuous distribution. However if there are other types of heterogeneity introduced in the model (e.g. risk aversion), then an equilibrium under pooled premium may exist, and the authors have not ruled it out. Also, if a separating equilibrium doesn't exist then some alternative concept of equilibria could be used to explain the insurance market. Using a different concept of equilibrium, e.g. Wilson

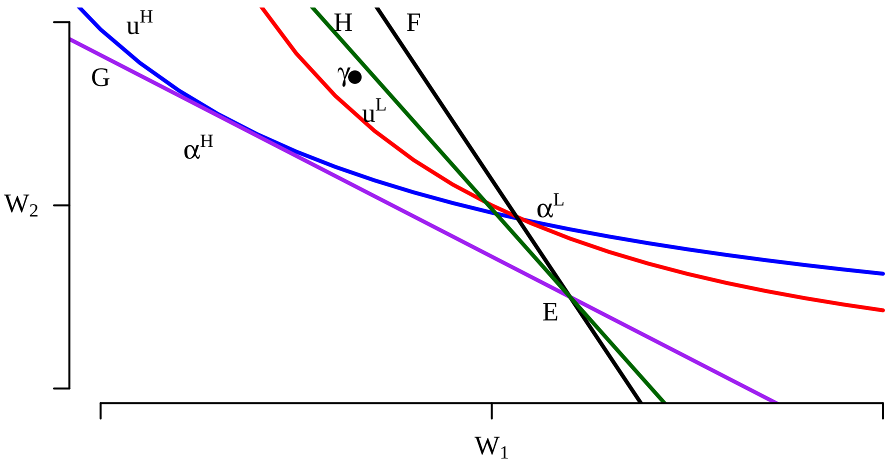


Figure J.3: Disrupting equilibrium

(1976), which assumes that loss-making contracts can be withdrawn from the market by insurers, it could be argued that the market will iteratively reach an equilibrium in all scenarios.

J.3 Randomness of risk preference

Economists generally prefer estimating risk preference from field data, i.e. from observable economic behaviour of people in the real world, as laboratory experiment based results are often found to be true for a specific experiment setup, but not applicable in a more general scenario. Barseghyan et al. (2018) reviewed different models of risk preference from the available literature in this area.

Summary of Barseghyan et al. (2018)

Demand for insurance providing full cover, say $Q^F(\pi)$ at premium π is observable from data. Assuming the slope of demand function for a given risk-group (i.e. a group where all individuals are having same probability of loss), arises from heterogeneity of risk aversion coefficient, for a given functional form of utility function, e.g. constant absolute risk aversion (CARA) or constant relative risk aversion (CRRA), one can derive a distribution function of the risk aversion coefficient itself.

Specifically for the CARA utility function where $u(y) = -e^{(-ry)}/r$, where r is the coefficient of absolute risk aversion, an individual would be willing to pay a maximum premium π to insure against loss L where:

$$e^{(r\pi)} = \mu e^{(rL)} + (1-\mu) \tag{J.3.1}$$

From equation J.3.1 the coefficient of absolute risk aversion $r^F(\pi)$ for an individual, who is willing to pay at most a premium π for full insurance, can be estimated. Any individual with a risk aversion coefficient $r > r^F(\pi)$ would then buy insurance. From the observable demand for full insurance , a distribution function $F(r^F(\pi))$ for $r^F(\pi)$ can be derived, as

$$Q^F(\pi) = 1 - F(r^F(\pi)) \tag{J.3.2}$$

Now, if a deductible d is introduced , then J.3.1 would be modified to:

$$\mu e^{(r(\pi+d))} + (1-\mu)e^{(r\pi)} = \mu e^{(rL)} + (1-\mu) \tag{J.3.3}$$

From the distribution of risk aversion coefficient , which is known from the equation J.3.2, it would now be possible to construct the demand function for deductible insurance $Q^D(\pi)$. As the deductible insurance provides less coverage, $Q^D(\pi) < Q^F(\pi)$.

A different underlying model of risk aversion however can lead to a different conclusion about the level of demand. Let's assume that the individual uses a distorted probability of loss, say $\Omega(\mu) = \overline{\Omega} > \mu$, i.e. the individual's perceived risk is higher than her actual risk exposure. Hence the decision for buying the deductible insurance driven by the equation J.3.3 can be written as

$$\overline{\Omega}e^{(r(\pi+d))} + (1-\overline{\Omega})e^{(r\pi)} = \overline{\Omega}e^{(rL)} + (1-\overline{\Omega}) \tag{J.3.4}$$

which would produce a demand function for deductible insurance with distorted probability $Q^D_\Omega(\pi)$. It can be shown that, despite each individual having a higher perceived risk than actual, for certain values of deductible, d , $Q^D_\Omega(\pi) < Q^D(\pi)$, i.e. the demand for insurance will actually fall.

The authors have also discussed other models of risk aversion e.g. constant relative risk aversion (CRRA), hyperbolic absolute risk aversion (HARA) and negligible third derivative (NTD). However in our case we assume that individual's attitude towards risk is independent of her initial wealth, hence CARA would be the appropriate model in this case.

J.4 Social Welfare

Two models developed respectively by Akerlof (1970) and Rothschild and Stiglitz (1976), are the basis of most of the economic studies around asymmetric information in insurance market. However neither of these deals with the aspect of social welfare in great detail. We can mention Einav and Finkelstein (2011) as an example of a model based on Akerlof framework to examine social welfare implications. Similarly, Hoy (2006) has used the Rothschild-Stiglitz model to study the social welfare implications of insurance. In the following sections we summarise both the models.

Summary of Einav and Finkelstein (2011)

Einav and Finkelstein (2011) used the concepts of traditional commodity market equilibrium in a graphical framework to explain social welfare within the insurance market. A risk averse individual would be ready to pay more than his fair premium due to the certainty offered by insurance. The maximum quantity that the risk averse customer can be charged above his fair premium is termed by the authors as "risk premium". For a risk loving customer on the hand, the price of insurance has to be subsidized before he will buy, as he would not be prepared to buy insurance at fair premium. In the case of risk loving customer risk premium would be negative. An individual's "willingness to pay" is defined as the sum of his fair premium and risk premium.

In the graphical framework of Einav and Finkelstein (2011) willingness to pay determines the demand for insurance (given by BE in figure J.4). The social welfare has been calculated as the sum of consumers' surplus and producers' surplus. The model predicts that the absence of perfect information would cause a deadweight loss in social welfare. In case of a population where all individuals are risk averse, the equilibrium is depicted in figure J.4.

Assuming that the risk premium of the population does not vary with their risk profile, as the insurer lowers the price, the marginal customer gained would have lower risk than the average customer, who has already bought the insurance. This implies the marginal cost curve of the supplier would be downward sloping, and will be below the average cost curve. In figure J.4, AF and AG are the marginal and average cost curves respectively.

Note that in a commodity market, a producer would be willing to produce as long as marginal cost (i.e. cost of producing one additional unit) is lower than price he gets for it. In other words, marginal cost curve is below demand curve. Marginal cost curve is therefore considered as supply curve of the commodity.

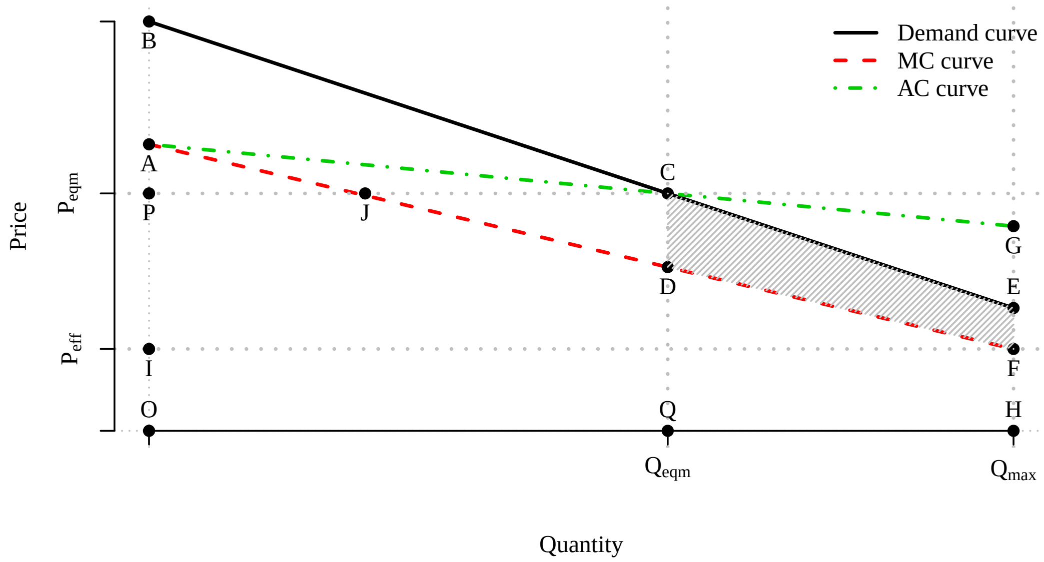


Figure J.4: Insurance market equilibrium: risk averse customers

However, in the insurance market, the true marginal cost of the customer remains unknown to insurer. Hence the supply of insurance in this case would be determined by the average cost. The equilibrium price and coverage in the figure is given by P_{eqm} and Q_{eqm} respectively.

Customer surplus in this case would be represented by the area of △BPC (this area represents the difference between the willingness to pay and the actual price paid by the insured population).

Producer's revenue is represented by area of □POQC.

Producer's total cost is given by □AOQD.

Therefore producer's surplus is represented by △JCD - △APJ

Hence total surplus = △BPC + △JCD - △APJ = Area of □BADC

Note that in absence of information asymmetry, the insurer could charge a differentiated price to each customer and due to perfect competition each customer would have paid fair premium. A full insurance coverage would have been possible in that case, generating a surplus represented by area of □BAFE.

Therefore a deadweight loss equal to the area of □CDFE is the result of information asymmetry.

Note that in figure J.4, the demand curve is always above the MC curve. This implies that all the individuals are willing to pay more than their fair premium. However if there are customers who are risk loving, i.e. who have a negative risk premium, then these customers will have a lower willingness to pay than their marginal cost. They are represented by the segment QH in figure J.5. In this scenario there would still be a deadweight loss represented by area of △CJD.

If insurance is mandated then the uninsured will be forced to buy insurance at equilibrium price, and in this case total deadweight loss (compared to perfect information scenario) would be represented by △DEF in figure J.5. Therefore mandated buying of insurance may in some case increase social welfare if area of △DEF is less than that of

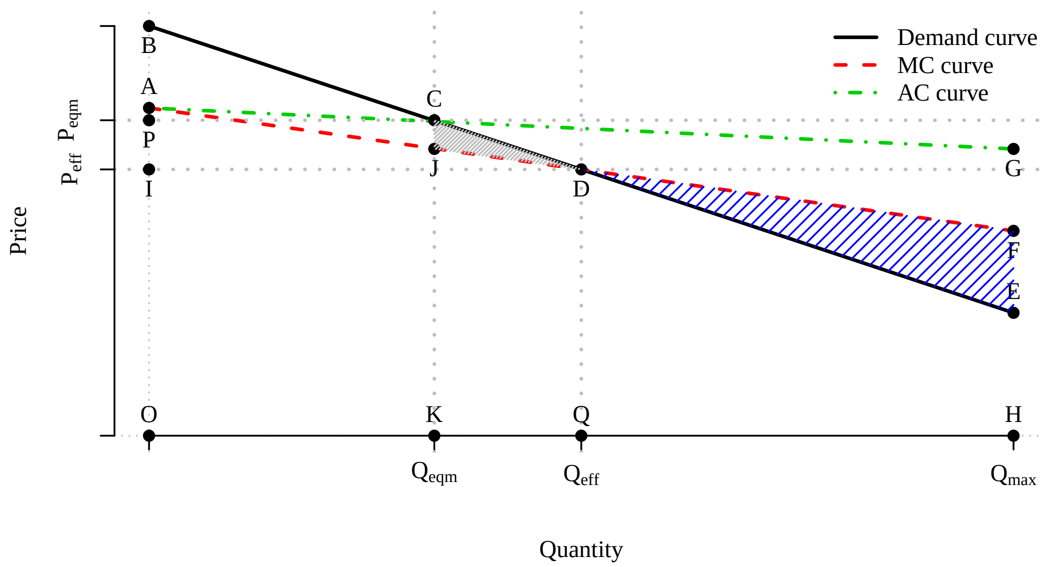


Figure J.5: Insurance market equilibrium: with risk loving customers

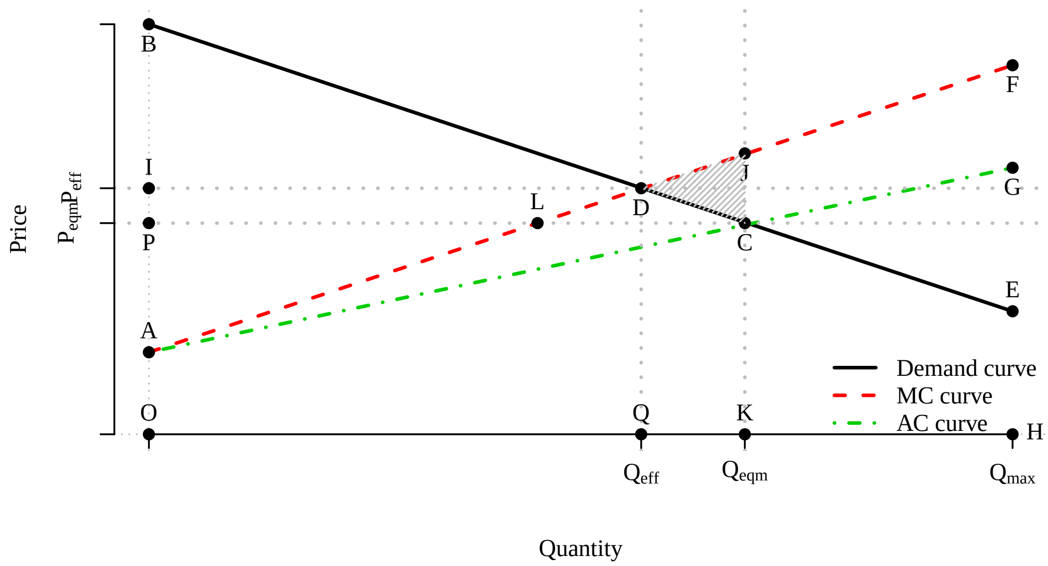


Figure J.6: Insurance market equilibrium: propitious selection

△CJD.

The authors also considered a scenario where low-risk customers are highly risk averse and high-risk individuals are risk lovers. In this case, depicted in J.6, the marginal cost curve would be increasing with quantity of sold insurance and the average cost curve will be below marginal cost.

The authors argued that in this case too there will be deadweight loss and the deadweight loss will result from subsidizing customers who would not have been insured in perfect information regime. These customers would have a lower willingness to pay than their fair premium. As opposed to "adverse Selection" this is called "propitious selection".

In Equilibrium, customer surplus = the area of △BPC

Producer's surplus = Area of □POKC - Area of AOKJ

= Area of △PAL - Area of △LCJ

Hence total social welfare = $\triangle$BAD - $\triangle$DCJ

Under symmetric information the social welfare would be represented by area of $\triangle$BAD.

Therefore the deadweight loss is represented by Area of $\triangle$DCJ

Observations from Einav and Finkelstein (2011) include the following:

- The "willingness to pay" is a sum of the customer's actual risk and his risk aversion. The scenarios explored in the paper assumes simple upward or downward sloping marginal cost curves. It would be interesting to translate the willingness to pay into its component parts i.e. risk and risk aversion (or affection) and look at their individual effects on the demand and social welfare in detail, using the same graphical framework.

- The model assumes that firms compete on price and offer full coverage. Unlike Rothschild and Stiglitz (1976) it does not allow contracts that can specify both premium and coverage. It could be investigated further to allow coverage as part of contract in the model.

- It is possible to show from figures J.4 and J.5 under adverse selection, a regulator can set price at P_{eff} externally to maximize social welfare. However in this case insurers will have negative surplus. But regulator can tax the consumers to transfer some customer surplus to insurers to ensure non negative surplus for insurers.

- If consumers are subsidized to buy insurance, it would move demand curve upward. That would increase coverage and hence reduce deadweight loss. Therefore subsidies to consumers can be used by a regulator to achieve the socially optimal consumption level, i.e. Q_{eff}. On the other hand if administrative cost of insurance increases, then marginal cost and average cost curves would move up. As a result coverage will fall.

- If by segregating the markets it is possible to remove the information asymmetry then segregating (e.g. on the basis of gender) could increase social welfare. However in most real life scenarios it would not eliminate information asymmetry and therefore no inference could be drawn on the impact of segregation on social welfare.

- From policymaker's point of view it may be important to find out the cost of adverse selection. However a "more adversely selective market" (i.e. where average cost of insured and uninsured population differ by higher margin) does not necessarily imply bigger deadweight loss. It can be shown that the size of deadweight loss would actually depend on the slope of the insurance demand curve, not just on the difference in average costs.

The model in Einav and Finkelstein (2011) only considers full coverage contracts, and therefore closer to Akerlof framework. In the next section we consider the welfare concept of Hoy (2006), which examines the utilitarian impacts arising out of the effect of insurance on the income distribution, in a Rothschild-Stiglitz framework.

Summary of Hoy (2006)

The author in this paper used the basic simple model of Rothschild and Stiglitz, with a utilitarian social welfare function. The utility functions are assumed to be interpersonally comparable cardinal functions. Behind a veil of ignorance, maximizing social welfare is essentially the problem of maximizing expected utility of an individual.

The paper uses a theorem from Atkinson (1970) which considers $F(x)$ and $G(x)$, two income distributions with equal means and with density functions $f(x)$ and $g(x)$ respectively (i.e. $f = F'$ and $g = G'$). It says that if $L_F(k)$ and $L_G(k)$ represent their respective Lorenz curves, then for a function $U(x)$ where $U'(x) > 0$ and $U''(x) < 0$,

$$L_F(k) \geq L_G(k), \forall k \in [0,1] \Leftrightarrow \int U(x)f(x)dx \geq \int U(x)g(x)dx \qquad \text{(J.4.1)}$$

As $U(x)$ can be an utility function, the above theorem means that if F and G represent two possible income distributions with equal mean, then Lorenz dominance of $F(k)$ over $G(k)$ implies higher social welfare under distribution $F(x)$ compared to $G(x)$. In this case one needs to consider the distribution of expected income (*ex ante*, i.e. before individuals are exposed to the loss inducing event) under different premium pricing regimes to consider their welfare implications.

As argued by Rothschild and Stiglitz, in the case of perfect information, as per figure J.7 α^A and α^B would be separating contracts sold to high and low risk customers. With information asymmetry, α^A and α^C would be separating contracts sold to high and low risk customers respectively. Lines *EH* and *EL* denote high and low risk customers' fair odds lines. *EG* denotes pooled fair odds line. As the pooled market odds line cuts through the low risk indifference curve u_C^L for separating contract, in this case the Rothschild-Stiglitz equilibrium can be disrupted. However, considering the non-myopic equilibrium concept of Wilson (1976), a contract that would lie on fair odds line and is most preferred contract by low-risk individuals, i.e. α^D ,would be a Wilson equilibrium contract in this case.

Hoy argues that there would be two scenarios which we need to consider while considering social welfare:

1. When the high-risk population proportion λ is high then pooled fair odds line would not pass through u_C^L and in that case separating equilibrium ,as shown in figure J.2 by α^A and α^C would be stable.

2. When high-risk population proportion λ is low enough then pooled fair odds line would pass through u_C^L and in that case separating equilibrium would not be stable. But Wilson equilibrium as shown in figure J.7 by α^D would be stable.

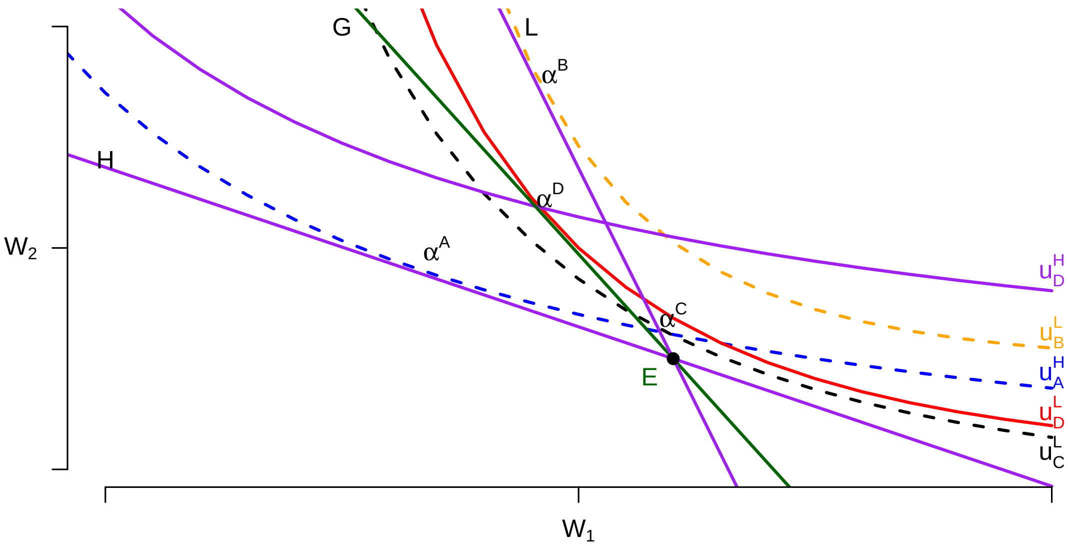


Figure J.7: Wilson equilibrium

Based on the above, the following scenarios are of interest:

1. Scenario 1 - Contract A and B: Contracts offered in the perfect information scenario (represented by α^A and α^B in figure J.7): In this case loss probabilities are common knowledge. So each customer would pay fair premium and buy full insurance. If W_0 indicates initial wealth and d denotes loss, then utility for high-risk customers (say, buying contract A) would be $U_{H1} = u(W_0 - p^H d)$ in both accident and no-accident state.

   For low-risk customers (say, buying contract B), it would be $U_{L1} = u(W_0 - p^L d)$ in both accident and no-accident state.

2. Scenario 2 - Contract A and C: Separating contracts offered in the asymmetric information scenario with Rothschild-Stiglitz equilibrium (represented by α^A and α^C in figure J.7): As low-risk customers would not buy full coverage, we denote the fraction of coverage bought by low-risk customers as r^S. The expected utility in this case for low-risk customers (contract C) would be:

$$U_{L2} = (1 - p^L)u(W_0 - p^L r^S d) + p^L u(W_0 - p^L r^S d - d + r^S d) \tag{J.4.2}$$

   High-risk customers would still be buying contract A. Utility of high-risk customers would be the same as in scenario 1. High-risk customers would remain indifferent between the two contracts on offer. Therefore high-risk individuals' utility is given by:

$$U_{H2} = u(W_0 - p^H d) \tag{J.4.3}$$

   Note that in the accident state, low-risk individuals' wealth would be lower than high-risk individuals' wealth (as they will not be buying full coverage). Hence

$$W_0 - p^L r^S d - d + r^S d < W_0 - p^H d \tag{J.4.4}$$

$$\Rightarrow r^S(1 - p^L) < 1 - p^H \Rightarrow r^S < (1 - p^H)/(1 - p^L) \tag{J.4.5}$$

3. Scenario 3 - Contract D: Pooled contract offered in Wilson equilibrium (represented by α^D in figure J.7): This is the contract (say, contract D) that would lie on pooled fair odds line and maximize low-risk individuals' utility. The pooled premium price for unit loss would be $\overline{p} = \lambda p^H + (1-\lambda)p^L$, as explained in equation J.2.2. Therefore, if r^* denotes the proportion of loss covered by insurance, then

$$r^* = argmax_r (1 - p^L)u(W_0 - \overline{p}rd) + p^L u(W_0 - \overline{p}rd + rd - d) \tag{J.4.6}$$

The utilities would be given by:

$$U_{L3} = (1 - p^L)u(W_0 - \overline{p}r^*d) + p^L u(W_0 - \overline{p}r^*d - d + r^*d) \tag{J.4.7}$$

$$U_{H3} = (1 - p^H)u(W_0 - \overline{p}r^*d) + p^H u(W_0 - \overline{p}r^*d - d + r^*d) \tag{J.4.8}$$

For each of above scenarios we can derive the Lorenz curves (figure J.8) based on expected wealth distribution. Then using the result J.4.1, social welfare implications can be drawn from Lorenz curve dominance. Note that slope of Lorenz curve at any point is equal to the ratio of the wealth of the marginal individual to the average wealth of the population. As the average expected wealth of the population is same in all scenarios ($W_0 - \overline{p}d$), the slope of the Lorenz curves in each scenario would actually indicate the expected wealth.

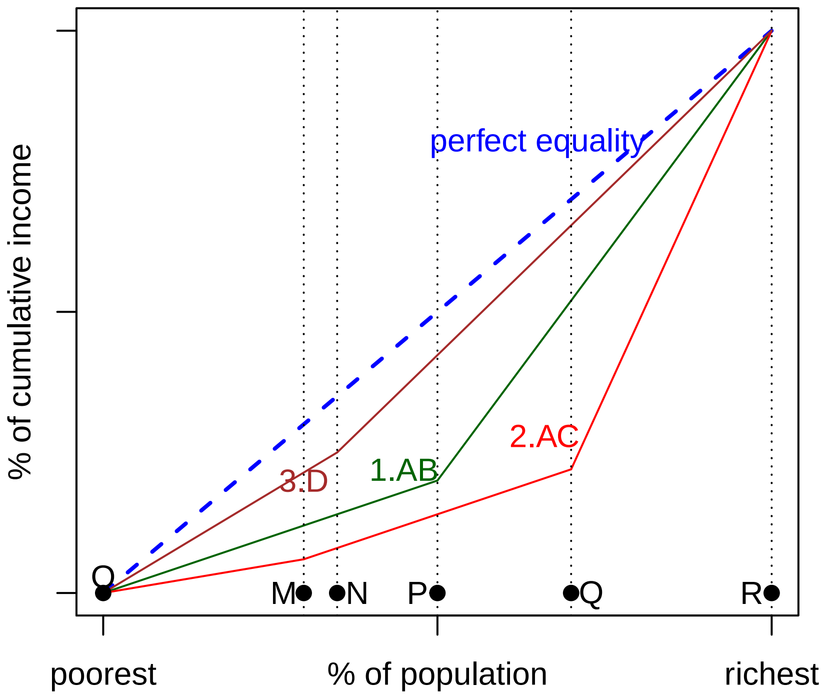


Figure J.8: Lorenz curve analysis

The following conclusions are based on the Lorenz curve analysis of figure J.8.

1. Scenario 1 - Green line AB: OP denotes the proportion of high-risk individuals i.e. λ. The slope of the curve between O and P represents $W_0 - p^H d$ and the slops to the right of P represents $W_0 - p^L d$.

2. Scenario 2 - Red Line AC: OM is the proportion of low-risk individuals expected to face loss : $(1-\lambda)p^L$. MQ is the proportion of high-risk individuals : λ. The slope of the curve between O and M represents $W_0 - r^S p^L d + r^S d - d$. As all high-risk individuals would buy full coverage, the slope of the curve between M and Q represents $W_0 - p^H d$, i.e. same as the slope of first segment of scenario 1 curve. The slope of the curve to the right of Q represents $W_0 - r^S p^L d$. This is the expected wealth of low-risk individuals who do not face loss, so length of QR is $(1-\lambda)(1-p^L)$.

3. Scenario 3 - Brown line D: Here the contract on offer is a pooled contract. So the wealth distribution will only depend on whether a person faces a loss or not. Hence ON represents the proportion of people expected to face a loss , i.e. $\overline{p} = \lambda p^H + (1-\lambda)p^L$, and NR=$1-\overline{p}$. The slope of the curve to the left of N represents the expected wealth of individuals who suffer a loss, i.e. $W_0 - \overline{p}r^* d + r^* d - d$. The slope of the curve to the right of N represents $W_0 - \overline{p}r^* d$.

We can note that if $W_0 - \overline{p}r^* d + r^* d - d > W_0 - p^H d$ and $W_0 - \overline{p}r^* d < W_0 - p^L d$ then brown scenario 3 Lorenz curve will dominate the scenario 1 curve.

$$W_0 - \overline{p}r^* d + r^* d - d > W_0 - p^H d \tag{J.4.9}$$

$$\Rightarrow r^* > (1-p^H)/(1-\overline{p}) \tag{J.4.10}$$

and

$$W_0 - \overline{p}r^*d < W_0 - p^L d \tag{J.4.11}$$

$$\Rightarrow r^* > p^L/\overline{p} \tag{J.4.12}$$

Essentially if r^* is close to 1 then Lorenz curve for Wilson equilibrium or Scenario 3 will completely dominate the Scenario 1 (perfect information) Lorenz curve. On the other hand if r^* is small then the perfect information scenario will dominate.
Note that this analysis does not allow *ex post* redistribution of wealth via tax/subsidy. The author also mentions that if information about risk profiles can be collected without cost to the insurers and customers are fully aware of their own risk probabilities, then more information for insurers about their customers would always lead to increase in Pareto efficiency. Now expected *ex ante* utility under full information is given by:

$$EU^{AB} = (1-\lambda)u(W_0 - p^L d) + \lambda u(W_0 - p^H d) \tag{J.4.13}$$

Expected *ex ante* utility under Wilson equilibrium contract D is given by:

$$EU^D = (1-\overline{p})u(W_0 - \overline{p}r^*d) + \overline{p}u(W_0 - \overline{p}r^*d + r^*d - d) \tag{J.4.14}$$

The author proves the following :

1. There exists some $\lambda^c > 0$, such that for any $\lambda < \lambda^c$, $EU^D > EU^{AB}$.

2. If loss probability for high-risk individuals, p^H, increases while proportion of high-risk population λ falls, so that λp^H is constant, then EU^{AB} would fall, but EU^D will remain constant.

Together the above results mean that if the proportion of high-risk individuals is high enough so that separating equilibrium of Rothschild-Stiglitz is possible, then full information delivers higher welfare and therefore regulator or government should not impose ban on classification.

However if high-risk population proportion is lower than a threshold where separating equilibrium is not achievable, it might be possible to deliver higher welfare by pooling the risks and let market achieve Wilson equilibrium . For low enough proportion of high-risk population this is certain to deliver higher social welfare.

Additionally, if the loss probability of the high-risk population is higher, (i.e. p^H is higher) then the threshold proportion of high-risk population λ, below which Wilson equilibrium under pooling unambiguously delivers higher welfare, would be smaller.

J.5 Measures of Social Welfare

A good measure of social welfare has always been a fundamental question among economists. A social welfare function (SWF) is essentially a function on the set of alternative states which would allow us to rank the alternatives in terms of their goodness from a social point of view. In the words of Arrow (Sen (2017), p. 271) it would map "the vector of utilities of individuals into a [collective] utility".

Arrow (1963) defined a SWF as a functional relation specifying one social ordering R for any n-tuple of individual orderings R_i for each person, i.e. $R = f(R_i)$.

Bergson (1938) and Samuelson (1947) proposed a SWF as an ordering of choices. Note that Arrow SWF produces a Bergson-Samuelson SWF as an output of individual preferences. Arrow imposed a number of conditions that a reasonable SWF would be expected to satisfy. One of the central conditions that Arrow proposed is the Independence of Irrelevant Alternatives (IIA). IIA states that as long as individual preferences remain the same over a subset of social states, the social choice from that particular subset should also remain the same. IIA plays a central role in defining an Arrow SWF as Arrow concentrated on understanding the voting paradox. The mutual inconsistency of the axioms is proven in Arrow's impossibility theorem.

However on the question of finding a "good SWF" Dhillon and Mertens (1999) departed from Arrow's axioms. Replacing IIA by an alternative set of weaker axioms, Dhillon and Mertens (1999) presented "relative utilitarianism" as an alternative SWF. It assumes that utility, as opposed to preferences, exists objectively. Also, persons with similar preferences and expressive reactions are assumed to derive same utilities from similar situations. In this framework, relative utilitarianism - which consists of first normalising von-Neumann-Morgenstern utilities to a scale of 0-1, and then aggregating them – is shown to be a good SWF. Thus our measure of social welfare is expected utility *given* the distributions of loss probabilities and preferences in society, but evaluated *behind* a hypothetical veil of ignorance which screens off knowledge of the decision maker's own loss probability and preferences.

We also explored "loss coverage" in the context of social welfare. The concept of "loss coverage" is introduced in Thomas (2008) as the total expected loss covered by insurance. Hao et al. (2019) also links the concept of loss coverage to the utilitarian social welfare concept. They show that under iso elastic demand the ranking of two policies in the order of their loss coverage would be same as their ranking in order of social welfare achieved under them. Note that, due to the unobservable nature of utility functions, utilitarian social welfare is difficult to measure. But loss coverage under a given premium pricing regime is an observable quantity ex-post. Therefore from the policymakers' point of view, loss coverage could be used to compare social welfare achieved under different pricing regimes (e.g. pooled premium, partial risk classification or full risk classification).

In the model of Hao et al. (2019), the decision to buy insurance for an individual would depend on the individual's utility function $U(.)$, which is private knowledge. If an individual's initial wealth is given by W and he faces risk of loss L with probability μ then his expected utility is given by:

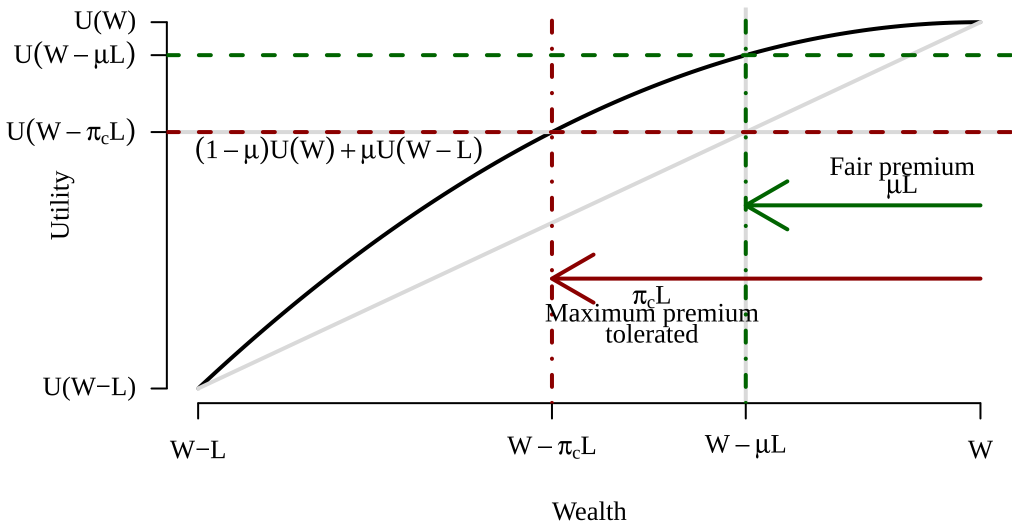


Figure J.9: Insurance utility function

$$\mu U(W-L)+(1-\mu)U(W)$$

If π is the premium for insuring each unit of loss, then by buying insurance, an individual derives an utility of $U(W-\pi L)$. Therefore the individual would buy insurance if

$$U(W-\pi L)>\mu U(W-L)+(1-\mu)U(W) \tag{J.5.1}$$

This implies that an individual with concave utility function would always buy insurance at fair premium i.e. when $\pi=\mu$. Actually the above condition also implies that he would be willing to pay above his fair premium. This can be related to risk premium mentioned in Einav and Finkelstein (2011) and discussed above. Figure J.9 shows why an individual would be willing to pay up to π_c where (π_c - μ) actually represents the risk premium explained in Einav and Finkelstein (2011).

Applying the normalization principle of Dhillon and Mertens (1999), and assuming all individuals have same utility at the end points , i.e. $U(W) = 1$ and $U(W - L) = 0$, decision to buy insurance will be determined by the condition

$$U(W - \pi L) > 1 - \mu \tag{J.5.2}$$

An individual's preference of risk would be represented by a random variable Γ, so that an individual's utility function would be represented by $U_\Gamma(.)$. The distribution function of the individual utilities (which really is the relative utility as defined by Dhillon and Mertens (1999)) can be translated into a demand function for insurance as follows:

$$d(\pi) = P[U_\Gamma(W - \pi L) > 1 - \mu] \tag{J.5.3}$$

$d(\pi)$ can be considered as a demand function of insurance.
If we consider a class of power utility functions of the form $U(w) = w^\gamma$, then:

$$P[U_\Gamma(W - \pi L) > 1 - \mu] \approx P[\gamma < \frac{\mu}{\pi}] \tag{J.5.4}$$

Note that elasticity of demand by definition is given by:

$$\epsilon(\pi) = -\frac{\partial \log d(\pi)}{\partial \log \pi} \tag{J.5.5}$$

An iso-elastic demand function implies that the elasticity is constant, say λ. Then the demand would be given by:

$$d(\pi) = \tau \exp[-\int_\mu^\pi \epsilon(s) d \log s] = \tau \left(\frac{\mu}{\pi}\right)^\lambda \tag{J.5.6}$$

where $d(\mu) = \tau$ is the demand at fair premium.

It is possible to find the distribution function of γ such that the resulting demand function is iso-elastic.

Now assume a population of n risk-groups where the proportion of population in each group is given by $p_i, i = 1, 2 \ldots n$ and risk of loss for each group is μ_i ($\mu_1 < \mu_2 < \ldots < \mu_n$). Assume each group's insurance demand function is given by $d_i(.)$. If each group is charged a premium of π_i, then under perfect competition, the market equilibrium condition , under perfect competition implies that the insurer's profit :

$$\rho(\underline{\pi}) = L \sum_i d_i(\pi_i) p_i(\pi_i - \mu_i) = 0 \tag{J.5.7}$$

It can be shown that there exists a pooled premium π^*, where $\mu_i < \pi^* < \mu_n$, for which, the equilibrium condition would be met.

Note that in above expression, premium earned by insurers is $L \sum_i d_i(\pi_i) p_i \pi_i$ and expected loss coverage is $L \sum_i d_i(\pi_i) p_i \mu_i$. Maximizing loss coverage under equilibrium is equivalent to finding a solution vector $(\underline{\pi})$, such that:

$$LC(\underline{\pi}) = \textstyle\sum_i d_i(\pi_i) p_i \mu_i \text{ is maximum, where } \rho(\underline{\pi}) = 0$$

In Hao et al. (2019), the authors had shown that if n risk-groups have iso elastic demand functions where elasticity of i-th group given by λ_i, and π_0 denotes equilibrium premium under pooled regime; defining $\lambda_{lo} = \max\{\lambda_i : \mu_i < \pi_0\}$ and $\lambda_{hi} = \min\{\lambda_i : \mu_i > \pi_0\}$

$$[\lambda_{hi} \geq \lambda_{lo}] \cap [\lambda_{lo} < 1] \Rightarrow LC(\pi_0) \geq LC(\underline{\mu}) \tag{J.5.8}$$

The result proves that under certain condition on demand elasticities is possible to achieve higher loss coverage with adverse selection than under full risk classification. The condition on demand elasticity implies that the high risk-groups would have to be more price sensitive than low-risk individuals.

J.6 Discussion

The body of literature discussed above in this section have been used in current thesis, as a springboard, to develop our model. We looked at them to identify areas where expansion and further generalisation could lead to qualitatively different results.

Heterogeneity in risk aversion was clearly one area where the existing literature could be expanded. To estimate the distribution of risk preference in the population we used demand elasticity as a proxy measure. This approach is similar to Hao et al. (2019). However unlike Hao et al. (2019), which uses loss coverage to infer about social welfare in a limited set of scenarios, our focus has been primarily on direct estimation of social welfare in a more generalized setup. We also use the same approach in partial risk classification scenarios and examining their implications on social welfare and loss coverage.

For measuring social welfare we used the approach of relative utilitarianism, as defined by Dhillon and Mertens (1999). This addresses the issue of interpersonal comparability of individual utilities. In our model, a regulator's goal is to maximize the expected utility of a randomly selected individual in the population, behind a hypothetical veil of ignorance which screens off the regulator's knowledge about the individual's risk preference and loss probability . In our model, insurers offer full coverage with competitive price. The question of social welfare in a similar model has previously been discussed in Einav and Finkelstein (2011). But Einav and Finkelstein (2011) did not use a probability distribution for risk preference in the model. In the current thesis, we show, that introducing randomness of risk preference across individuals in the model could lead to useful results related to social welfare. It may help a regulator to look at empirical data from the market (e.g. demand elasticities) and prescribe policies to increase welfare. Qualitatively our findings are similar to Hoy (2006). But unlike Hoy (2006), we derive conditions on demand elasticities which could deliver higher social welfare irrespective of the proportion of different risk-groups in the population.

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