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Joint Dynamic Optimization of Price and Two-Dimensional Warranty Policy

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***Abstract*—This paper studies the joint optimization of price and two-dimensional (2-D) warranty policy with limits of both warranty length and usage. For this purpose, in this paper, we first propose a new sales function to characterize the joint influence of price, warranty length, and usage limit on the sales rate while considering the influence of the heterogeneous customer usage rate. Then, we model the joint dynamic optimization of price and 2-D warranty policy as a nonlinear optimal control problem, and solve it via a numerical approach based on Pontryagin's maximum principle. We show that the proposed dynamic 2-D warranty policy can help achieve higher profits than the existing one-dimensional policies and static 2-D policy.**

***Index Terms*—Dynamic decision making, heterogeneous usage rate, joint optimization, reliability, two-dimensional (2-D) warranty.**

NOMENCLATURE

$P(t)$	Unit price of products sold at time t.
$W(t)$	Warranty length for products sold at time t.
$U(t)$	Usage limit for products sold at time t.
$q(t)$	Sales rate at time t.
$Q(t)$	Accumulated sales volume by time t.
Q_0	Accumulated sales volume at the initial time.
Q_M	Maximum sales potential.
δ	Discount rate.
k_1	Scaling parameter, $k_1 > 0$.
k_2	Parameter used to allow for nonzero sales when the warranty length equals zero, $k_2 > 0$.
a	Price elasticity, $a > 1$.
b	Elasticity of the displaced effective warranty length, $0 < b < 1$.
ψ	Parameter used to reflect the relative influence of innovators.
L	Product life cycle, which is the duration within which the product is sold in the market.
c_0	Initial production cost per unit.
γ	Learning parameter, $0 < \gamma < 1$.
$C(t)$	Unit production cost at time t.
c_{MR}	Unit cost of minimal repair.
$N(W(t), U(t) \| r)$	Number of warranty claims for a sold product as a function of $W(t)$ and $U(t)$, given that the usage rate equals r.
$M(W(t), U(t))$	Expected number of warranty claims for a sold product as a function of $W(t)$ and $U(t)$.
$\omega(W(t), U(t))$	Expected warranty cost for a sold product.
π	Expected discounted profit over the product's life cycle.
K	Parameter used to represent the effect of related factors such as inflation and production rate on production cost.
$g(x; \rho, \theta)$	pdf of the gamma distribution with scale parameter ρ and shape parameter θ.
R	Stochastic variable representing customer usage rate.
r	Realized value of the stochastic variable R.
r_0	Designed nominal usage rate.
r_M	Maximum usage rate.
$f_R(r; \alpha_R, \beta_R)$	pdf of the usage rate R following the censored gamma distribution over the interval $[0, r_M]$ with scale parameter α_R and shape parameter β_R.
η	Parameter of the proportional hazards model, $\eta \geq 0$.
$\lambda(t; \alpha, \beta \| r)$	Intensity function under the Weibull distributed failure time with scale parameter α and shape parameter β, given usage rate r.
$h(t; \alpha, \beta \| r)$	Observed hazard rate function under the Weibull distributed failure time with scale parameter α and shape parameter β, given usage rate r.
$h_0(t; \alpha, \beta)$	Baseline hazard rate function under the Weibull distributed failure time with scale parameter α and shape parameter β.

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$z(W(t), U(t)\|r)$	Effective warranty length as a function of $W(t)$ and $U(t)$, given the usage rate r.
$H_0(\ )$	Hamiltonian function.
$\mu_0(t)$	Adjoint variable at time t.
$H(\ )$	Current value Hamiltonian function.
$\mu(t)$	Current value of adjoint variable at time t.
α_A, β_A	Scale parameter and shape parameter of the gamma distribution for gamma series approximation model, respectively.
N_ϵ	Number of terms in the gamma series approximation model under tolerance ϵ.
S_n	Time of the *n*th replacement.
P_L, P_U	Lower bound and upper bound of the price, respectively.
W_L, W_U	Lower bound and upper bound of the warranty length, respectively.
U_L, U_U	Lower bound and upper bound of the usage limit, respectively.

Acronyms

pdf	Probability density function.
cdf	Cumulative distribution function.
PH	Proportional hazards.
NHPP	Nonhomogeneous Poisson process.

I. Introduction

A WARRANTY policy plays a significant role in attracting customers and promoting sales, as it can well-signal the reliability and quality of a product by specifying the manufacturer's obligation in dealing with product failures within the coverage [1]. Therefore, warranty has been widely used as a marketing tool in today's intensely competitive marketplace. For example, the warranty region for airplanes has been improved from 12 months or 1000 flight-hours in the early 1980s, to 60–120 months or 5000–10 000 flight-hours in 2013 [2].

Although an attractive warranty policy can help promote sales, it may also incur significant costs and, thus, hurt a manufacturer's profitability. According to Murthy [3], warranty servicing costs can account for as much as 2–10% of the sale price. It is reported that automakers worldwide spent US$42.2 billion and US$48.0 billion on warranty claims in 2015 and 2016, respectively.[1] Thus, a product's warranty policy needs to be carefully designed to balance its ability to attract customers against the incurred cost.

Besides warranty, price has also been widely recognized as a powerful marketing tool, and studied for decades [4]. Thus, it is of great interest to jointly optimize warranty policy and price to maximize profits, which was first studied under one-dimensional (1-D) warranty policy and static decision making [5]–[8]. That is, only one dimension of warranty limit (i.e., the length of time covered by warranty, or warranty length) was considered. Meanwhile, the price and warranty length, once determined, were held constant throughout the life cycle of a product (i.e., the duration within which the product is sold in the market).

Considering the importance of linking after-sales support strategies to product design, manufacturing, and marketing, it has been realized that warranty policies that are fixed throughout the product life cycle may not be suitable [9]. Therefore, the dynamic decision making on price and warranty length has been widely studied, by employing the Bass model [10]–[13]. That is, the price and warranty length of a product sold at different time can be different. It has been demonstrated in these studies that dynamically changing price and warranty length can help achieve much higher profits than holding them constant (i.e., static decision making).

The above studies [5]–[8], [10]–[13] are restricted to 1-D warranty policy (i.e., warranty length). However, each customer's usage rate (i.e., the amount of usage per unit of time) can vary dramatically [14]. As a result of usage degradation, if only a 1-D warranty policy is adopted, a customer with a very high usage rate will incur a much higher product hazard rate, which then leads to a much higher warranty cost, than a customer with a low usage rate [15]. To deal with this problem, the two-dimensional (2-D) warranty policy that has two dimensions of warranty limits (i.e., warranty length and usage limit) has been widely utilized. Under such a policy, the warranty expires as soon as either one of the two limits is exceeded.

Therefore, it is important to study the joint optimization of price and 2-D warranty policy. To deal with this problem, Huang *et al*. [16] modeled the total sales as a linear function of the warranty length and the usage limit, and Xie [17] considered the sales to be a function of the area of warranty region. In their models, the sales rate is invariant with the usage rate of customers. To consider the influence of customers' usage rate on the sales rate, Manna [18] incorporated mean coverage time, whereas He *et al*. [19] considered the extent of attractiveness and the demand multiplier. However, these studies focused on static rather than dynamic decision making [17]–[19].

As has been demonstrated by the studies on 1-D warranty policy [10]–[13], dynamic decision making on warranty policy and price can achieve higher profits than static decision making. However, to the best of our knowledge, this has not yet been explored under 2-D warranty policy. We bridge this gap by examining the joint dynamic optimization of price and 2-D warranty policy, while considering the influence of heterogeneous customer usage rate.

For this purpose, a new sales function is first proposed to characterize the joint influence of price and 2-D warranty policy on the sales rate. Compared to the existing sales functions [17]–[19], the proposed function explicitly considers the different attractiveness of a 2-D warranty policy for customers with different usage rates, and the diffusion effect. Moreover, this function can reduce to the traditional sales function under a 1-D warranty policy, when the usage limit becomes infinite, while the sales model proposed by He *et al*. [19] cannot. Based on the proposed sales function, the joint dynamic optimization of price and 2-D warranty policy is then described as a nonlinear optimal control problem. To deal with the analytically computational complexity incurred by its strong nonlinearity, the proposed problem is solved by a numerical approach while employing Pontryagin's maximum principle [20].

The main contribution of this manuscript is threefold. First, we propose a new function that can characterize the joint influence of price and 2-D warranty policy on the sales rate while

[1][Online]. Available: http://www.warrantyweek.com/archive/ww20170706.html

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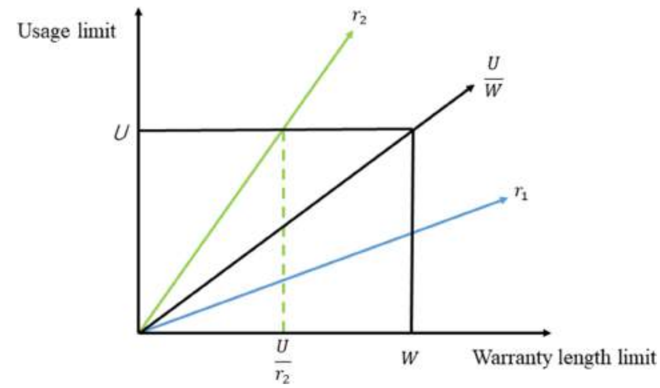

Fig. 1. 2-D warranty policy with a rectangular region.

considering the diffusion effect and the heterogeneous customer usage rate. Second, the influence of the heterogeneous customer usage rate on warranty cost under dynamic decision making is considered. Third, based on the proposed sales rate function, a nonlinear optimal control approach is developed to solve the problem of the joint dynamic optimization of price and 2-D warranty policy. Moreover, Pontryagin's maximum principle is employed to deal with the analytical complexity of the proposed problem.

The remainder of this paper is arranged as follows. Section II describes the formulation of the proposed problem, with the solving approach described in Section III. In Section IV, numerical experimentations are described to illustrate the proposed approach and explore insights. Finally, Section V concludes this paper.

II. Model Formulation

This paper focuses on the warranty policy with two dimensions (i.e., warranty length, denoted by W, and usage limit, denoted by U) and a rectangular warranty region, as shown in Fig. 1. Under the concerned 2-D warranty policy, warranty service terminates once either the warranty length or the usage limit is exceeded. We assume that the usage rate varies among customers, but is constant for each customer, as done in [15]. Let the stochastic variable R represent the usage rate of a customer, and r represent its realized value. Then, the effective warranty length actually enjoyed by a customer equals W, if $r \leq U/W$ (e.g., r_1 in Fig. 1), and U/r, if $r > U/W$ (e.g., r_2 in Fig. 1). It is important to note that the usage rate influences a product's deterioration and, thus, its warranty cost.

We seek to maximize the total expected discounted profit (π) of a product over its life cycle, the length of which is represented by L, by jointly optimizing the price and warranty policy. For this purpose, the sales rate of the product plays a significant role, and is mainly influenced by the price, the warranty length, the usage limit, the accumulated sales volume (through the diffusion effect), and the usage rate distribution of the customers (through the perception of the effective warranty length).

This paper also takes into account the learning effect, which reflects that the growing experience of producing a product can help reduce its unit production cost. As a building block to more realistic problems, it is assumed that the production rate is equal to the sales rate over the product's life cycle. The influencing relationships among the main elements involved in this paper are summarized in Fig. 2.

Let $P(t)$, $W(t)$, $U(t)$, and $C(t)$ represent the unit price, the warranty length, the usage limit, and the unit production cost of the concerned product sold at time t, respectively, and let $\delta \geq 0$ denote the discount rate. Then, the total profit can be obtained by subtracting the total warranty cost and the production cost from the total revenue, formulated by the following model:

$$\max_{\substack{P(t) \in [P_L, P_U] \\ W(t) \in [W_L, W_U] \\ U(t) \in [U_L, U_U]}} \pi = \int_0^L [P(t) - C(t) - \omega(W(t), U(t))]\, q(t)\, e^{-\delta t} dt \tag{1}$$

$$\text{s.t.} \quad \frac{dQ(t)}{dt} = q(t) \tag{2}$$

$$Q(0) = Q_0 \tag{3}$$

where $\omega(W(t), U(t))$ denotes the unit warranty cost of products sold with warranty limits $W(t)$ and $U(t)$, $q(t)$ represents the sales rate at time t, $Q(t)$ represents the accumulated sales volume at time t, and Q_0 denotes the accumulated sales volume at the initial time. The decision variables include $P(t)$, $W(t)$, and $U(t)$, whose lower bounds and upper bounds are assumed to be known *a priori*.

The proposed strategy, shortened as 2-D-MR-Dynamic, is to jointly and dynamically optimize price and 2-D warranty policy, conducting minimal repair for each failure within the warranty region. Minimal repair is to repair a failed product to the as bad as old condition (i.e., the failure rate right after repair is the same as that right before failure), while replacement is to replace a failed product with a new one [17]. As is often the case of reality, the cost of minimal repair is assumed lower than that of replacement. To demonstrate its advantages, the proposed strategy is compared with four alternative strategies, three of which are from existing studies with necessary adaptions to make them comparable with each other. All these strategies seek to jointly optimize price and warranty policy, and their main characteristics are summarized in Table I. Next, we will elaborate each of the compared strategies and its relation to the literature.

Alternative strategy 1, adapted from the strategy proposed by Huang *et al*. [13], considers just one dimension of warranty (i.e., warranty length) and makes dynamic decisions, using minimal repair. The remaining characteristics of this strategy (i.e., product failure distribution, the learning effect of production cost, usage degradation, etc.) are the same as the proposed strategy. The second alternative strategy also has only one dimension of warranty—the usage limit, with the remaining characteristics being the same as those of Alternative strategy 1. Alternative strategy 3 is the same as the proposed strategy, except that it employs the maintenance strategy described by Lin and Shue [21]—free replacement—to deal with all the failures within the warranty region. Alternative strategy 4 has only one difference

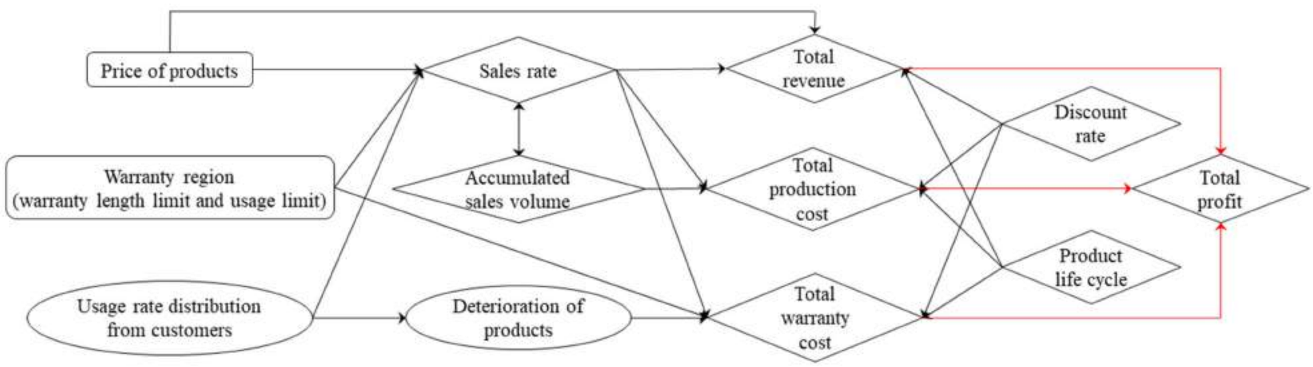

Fig. 2. Influencing relationship of the involved elements.

TABLE I
STRATEGIES TO COMPARE

Strategies	Dimensionalities of warranty	Maintenance policy	Dynamic or static
Proposed strategy	2-D warranty	Free minimal repair	Dynamic
Alternative strategy 1	1-D warranty with warranty length limit	Free minimal repair	Dynamic
Alternative strategy 2	1-D warranty with usage limit	Free minimal repair	Dynamic
Alternative strategy 3	2-D warranty	Free replacement	Dynamic
Alternative strategy 4	2-D warranty	Free minimal repair	Static

with the proposed strategy: it makes static decisions of price and 2-D warranty policy as done by Manna [18]. That is, the price (P), the warranty length (W), and the usage limit (U), once determined, stay constant throughout the whole life.

A. Sales Rate

In this paper, similar to Manna [18], it is assumed that the influence of a 2-D warranty on the sales rate can be transformed into the influence of the effective warranty length that each customer actually enjoys. However, instead of using mean coverage time as done by Manna [18], we directly employ the effective warranty length, which is represented by a function of usage rate r, $z(W(t), U(t)|r)$, as described by

$$z\left(W\left(t\right),U\left(t\right)|r\right)=\begin{cases}W\left(t\right), & r\leq U\left(t\right)/W\left(t\right)\\ U\left(t\right)/r, & r>U\left(t\right)/W\left(t\right)\end{cases}. \quad (4)$$

Gamma distribution has been widely used to describe the distribution of usage rate among customers for products sold with a 2-D warranty policy, such as automobiles [22], [23]. Thus, considering that customers with very high or very low usage rates are usually rare, the distribution of customer usage rate is described by the censored gamma distribution over interval $[0, r_M]$, with the pdf and cdf described, respectively, in

$$f_R\left(r;\alpha_R,\beta_R\right)=\frac{g\left(r;\alpha_R,\beta_R\right)}{G\left(r_M;\alpha_R,\beta_R\right)} \quad (5)$$

$$F_R\left(r;\alpha_R,\beta_R\right)=\int_0^r\frac{g\left(u;\alpha_R,\beta_R\right)}{G\left(r_M;\alpha_R,\beta_R\right)}du \quad (6)$$

where r_M represents the upper bound of usage rate, α_R and β_R represent the scale and shape parameter, and $g(\ )$ and $G(\ )$ represent the pdf and cdf of the gamma distribution, respectively.

Then, based on the Glickman–Berger [5] and Bass models [10], a new function is proposed to describe the joint influence of price and 2-D warranty on the sales rate while considering the diffusion effects and the influence of the heterogeneous customer usage rate. The general form of this function is described by

$$q\left(t\right)=\int_0^{r_M}k_1P(t)^{-a}(z\left(W\left(t\right),U\left(t\right)|r\right)+k_2)^b \left(1-\frac{Q\left(t\right)}{Q_M}\right)\left(\psi+\frac{Q\left(t\right)}{Q_M}\right)f_R\left(r;\alpha_R,\beta_R\right)dr \quad (7)$$

where k_1 ($k_1 > 0$) is a scaling parameter, k_2 ($k_2 > 0$) is a constant used to allow for nonzero sales when the effective warranty length equals zero, ψ is a constant (typically a few hundredths) employed to reflect the relative influence of innovators, and Q_M represents the maximum potential sales (i.e., market size) and is assumed to be known *a priori*. Parameter a $(a > 1)$ and b $(0 < b < 1)$ denote the elasticity of price and the displaced effective warranty length (i.e., $\mathrm{z}(W(t), U(t)|r) + k_2$), respectively.

Considering (4), (7) can be transformed to

$$q(t) = \begin{bmatrix} \int_0^{\frac{U(t)}{W(t)}} k_1 P(t)^{-a} (W(t) + k_2)^b \left(1 - \frac{Q(t)}{Q_M}\right) \\ \left(\psi + \frac{Q(t)}{Q_M}\right) f_R(r; \alpha_R, \beta_R)\, dr \\ + \int_{\frac{U(t)}{W(t)}}^{r_M} k_1 P(t)^{-a} \left(\frac{U(t)}{r} + k_2\right)^b \left(1 - \frac{Q(t)}{Q_M}\right) \\ \left(\psi + \frac{Q(t)}{Q_M}\right) f_R(r; \alpha_R, \beta_R)\, dr \end{bmatrix}. \quad (8)$$

It can be seen that the sales rate of Alternative strategy 3 can be determined using (8), and that of Alternative strategies 1, 2, and 4 can be determined by, respectively, by

$$q(t) = k_1 P(t)^{-a} (W(t) + k_2)^b \left(1 - \frac{Q(t)}{Q_M}\right)\left(\psi + \frac{Q(t)}{Q_M}\right) \quad (9)$$

$$q(t) = \int_0^{r_M} k_1 P(t)^{-a} \left(\frac{U(t)}{r} + k_2\right)^b \left(1 - \frac{Q(t)}{Q_M}\right) \left(\psi + \frac{Q(t)}{Q_M}\right) f_R(r; \alpha_R, \beta_R)\, dr \quad (10)$$

$$q(t) = \begin{bmatrix} \int_0^{\frac{U}{W}} k_1 P^{-a} (W + k_2)^b \left(1 - \frac{Q(t)}{Q_M}\right)\left(\psi + \frac{Q(t)}{Q_M}\right) \\ f_R(r; \alpha_R, \beta_R)\, dr \\ + \int_{\frac{U}{W}}^{r_M} k_1 P(t)^{-a} \left(\frac{U}{r} + k_2\right)^b \left(1 - \frac{Q(t)}{Q_M}\right)\left(\psi + \frac{Q(t)}{Q_M}\right) \\ f_R(r; \alpha_R, \beta_R)\, dr \end{bmatrix}. \quad (11)$$

B. Warranty Cost

This paper assumes that product failures within the determined warranty region have to be repaired by the manufacturer without charging the customers. In order to calculate the incurred warranty cost, the expected number of failures of a sold product within the warranty region needs to be determined first. For this purpose, the heterogeneous customer usage rate has to be considered, since it has been found to highly influence a product's failure rate [15]. Thus, the widely adopted proportional hazards (PH) model [15], [24], which describes the effects of the actual usage rate on the failure rate, is employed to describe the hazard rate as a function of the usage rate r, as shown in

$$h(t|r) = \left(\frac{r}{r_0}\right)^{\eta} h_0(t) \quad (12)$$

where r_0 denotes the designed nominal usage rate, $h_0(t)$ represents the baseline hazard rate under usage rate r_0, and parameter η $(\eta \geq 0)$ reflects the degree of the influence of the usage rate on the hazard rate.

This paper focuses on those products with failure times following the Weibull distribution, which has been widely utilized in the literature [25]. Then, the baseline hazard rate can be described by the following equation, where α and β, respectively, represent the scale and shape parameters:

$$h_0(t; \alpha, \beta) = \frac{\beta t^{\beta - 1}}{\alpha^{\beta}}. \quad (13)$$

For maintenance options, minimal repair is used in the proposed strategy and the three alternative strategies to be compared with, and replacement is employed in Alternative strategy 3. Assuming that the repair time is negligible, for minimal repair, the amount of failures can be calculated via the nonhomogeneous Poisson process (NHPP) [26]. Let $M(W(t), U(t))$ represent the expected number of warranty claims of a sold product, and $N(W(t), U(t)|r)$ represent the number of warranty claims of a sold product with usage rate r. Then, under minimal repair strategy, following can be obtained:

$$\begin{aligned} M(W(t), U(t)) &= E[N(W(t), U(t))] \\ &= \int_0^{r_M} E[N(W(t), U(t)|r)]\, f_R(r; \alpha_R, \beta_R)\, dr \\ &= \int_0^{r_M} \left[\int_0^{z(W(t),U(t)|r)} \lambda(u; \alpha, \beta|r)\, du\right] f_R(r; \alpha_R, \beta_R)\, dr. \end{aligned} \quad (14)$$

According to (4), under the proposed strategy (i.e., 2D-MR-Dynamic), which considers a 2-D warranty with a rectangular region, (14) can be transformed into

$$M(W(t), U(t)) = \begin{bmatrix} \int_0^{\frac{U(t)}{W(t)}} \left[\int_0^{W(t)} \lambda(u; \alpha, \beta|r)\, du\right] f_R(r; \alpha_R, \beta_R)\, dr \\ + \int_{\frac{U(t)}{W(t)}}^{r_M} \left[\int_0^{\frac{U(t)}{r}} \lambda(u; \alpha, \beta|r)\, du\right] f_R(r; \alpha_R, \beta_R)\, dr \end{bmatrix}. \quad (15)$$

Under Alternative strategy 1, since only the warranty length is applied, the effective warranty length can be thought of as $W(t)$. Thus, $M(W(t), U(t))$ can be calculated by

$$M(W(t), U(t)) = \int_0^{r_M} \left[\int_0^{W(t)} \lambda(u; \alpha, \beta|r)\, du\right] f_R(r; \alpha_R, \beta_R)\, dr. \quad (16)$$

Under Alternative strategy 2, the effective warranty length equals $U(t)/r$, and $M(W(t), U(t))$ can be determined by

$$M(W(t), U(t)) = \int_0^{r_M} \left[\int_0^{\frac{U(t)}{r}} \lambda(u; \alpha, \beta|r)\, du\right] f_R(r; \alpha_R, \beta_R)\, dr. \quad (17)$$

Under Alternative strategy 4, since the warranty length and the usage limit stay constant throughout a product's life, the calculation of $M(W(t), U(t))$ is similar to the proposed strategy except that W and U are now constants, as described in

$$M(W, U)$$

$$= \begin{bmatrix} \int_0^{\frac{U}{W}} \left[\int_0^{W} \lambda\left(u;\alpha,\beta|r\right)du\right] f_R\left(r;\alpha_R,\beta_R\right)dr \\ +\int_{\frac{U}{W}}^{r_M} \left[\int_0^{\frac{U}{r}} \lambda\left(u;\alpha,\beta|r\right)du\right] f_R\left(r;\alpha_R,\beta_R\right)dr \end{bmatrix}. \quad (18)$$

Once the expected number of warranty claims of a sold product under a free minimal repair policy is obtained, its expected warranty cost can be determined by

$$\omega\left(W\left(t\right),U\left(t\right)\right) = c_{\mathrm{MR}} M\left(W\left(t\right),U\left(t\right)\right) \quad (19)$$

where c_{MR} represents the cost incurred by each minimal repair.

In order to determine the warranty cost of Alternative strategy 3, in which replacement is used to deal with product failures, the renewal function (RF) of the Weibull distribution needs to be obtained first, which is known to be difficult, if not impossible, to achieve analytically [27]. Therefore, the gamma-series (GS) model proposed by Jiang [27] is employed in this paper to approximate the Weibull RF. As described in (20), the GS model approximates the Weibull distribution with the gamma distribution by considering their first two moments to be equal and replaces the first term of approximation series with the cdf of the Weibull distribution

$$M\left(t\right) = \sum_{n=1}^{\infty} F^{(n)}(t) \approx F\left(t\right) + \sum_{n=2}^{N_\epsilon} G^{(n)}(t;\alpha_A,\beta_A)$$
$$= F\left(t\right) + \sum_{n=2}^{N_\epsilon} G(t;\alpha_A,n\beta_A) \quad (20)$$

where $G(t;\alpha_A,n\beta_A)$ denotes the cdf of the gamma distribution used for the approximation with the scale parameter α_A and the shape parameter $n\beta_A$. N_ϵ is defined as

$$N_\epsilon = \inf\left\{n : G\left(t;\alpha_A,n\beta_A\right) < \epsilon\right\}. \quad (21)$$

Usually, $\epsilon = 10^{-6}$ or 10^{-7}. Note that the scale and shape parameter of the Weibull distribution employed in this research equal $(r_0/r)^{\eta/\beta}\alpha$ and β, respectively. Then, through equating the first two moments of the Weibull distribution and the gamma distribution, we can obtain α_A and β_A as

$$\alpha_A = \frac{\sigma_A^2}{\mu_A} = \left(\frac{r_0}{r}\right)^{\frac{\eta}{\beta}} \alpha \frac{\left\{\Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\}}{\Gamma\left(1+\frac{1}{\beta}\right)} \quad (22)$$

$$\beta_A = \frac{\mu_A^2}{\sigma_A^2} = \frac{\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2}{\left\{\Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\}}. \quad (23)$$

Then, the expected number of warranty claims of a sold product $M(W(t),U(t))$ under Alternative strategy 3, can be determined as follows:

$$M\left(W\left(t\right),U\left(t\right)\right) = E\left[N\left(W\left(t\right),U\left(t\right)\right)\right]$$
$$= \int_0^{r_M} E\left[N\left(W\left(t\right),U\left(t\right)|r\right)\right] f_R\left(r;\alpha_R,\beta_R\right)dr$$

$$= \begin{bmatrix} \int_0^{\frac{U(t)}{W(t)}} \sum_{n=1}^{\infty} Pr\left(S_n \le W\left(t\right)|r\right) f_R\left(r;\alpha_R,\beta_R\right)dr \\ +\int_{\frac{U(t)}{W(t)}}^{r_M} \sum_{n=1}^{\infty} Pr\left(S_n \le \frac{U(t)}{r}|r\right) f_R\left(r;\alpha_R,\beta_R\right)dr \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^{\frac{U(t)}{W(t)}} \sum_{n=1}^{\infty} F^{(n)}\left(W\left(t\right);\left(\frac{r_0}{r}\right)^{\frac{\eta}{\beta}}\alpha,\beta|r\right) \\ f_R\left(r;\alpha_R,\beta_R\right)dr \\ +\int_{\frac{U(t)}{W(t)}}^{r_M} \sum_{n=1}^{\infty} F^{(n)}\left(\frac{U(t)}{r};\left(\frac{r_0}{r}\right)^{\frac{\eta}{\beta}}\alpha,\beta|r\right) \\ f_R\left(r;\alpha_R,\beta_R\right)dr \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^{\frac{U(t)}{W(t)}} \left\{F\left(W\left(t\right);(r_0/r)^{\eta/\beta}\alpha,\beta|r\right)\right. \\ \left.+\sum_{n=2}^{N_\epsilon} G(W\left(t\right);\alpha_A,n\beta_A|r)\right\} f_R\left(r;\alpha_R,\beta_R\right)dr \\ +\int_{\frac{U(t)}{W(t)}}^{r_M} \left\{F\left(\frac{U(t)}{r};(r_0/r)^{\eta/\beta}\alpha,\beta|r\right)\right. \\ \left.+\sum_{n=2}^{N_\epsilon} G\left(\frac{U(t)}{r};\alpha_A,n\beta_A|r\right)\right\} f_R\left(r;\alpha_R,\beta_R\right)dr \end{bmatrix}. \quad (24)$$

The expected warranty cost of a sold product with free replacement policy can then be determined by the following equation, where $C(Q(t))$ represents the cost of each replacement

$$\omega\left(W\left(t\right),U\left(t\right)\right) = C\left(Q\left(t\right)\right) M\left(W\left(t\right),U\left(t\right)\right). \quad (25)$$

C. Production Cost

This paper takes into account the learning effect of the growing experience of producing products in reducing unit production costs, which could be significant, especially for expensive products with massive production. The learning effect can be described by a learning curve, which was first introduced by Wright [28]. As done in the literature [13], [29], it is assumed in this paper that the production rate strictly equals the sales rate and as such, there is no inventory or unmet demand. Then, the unit production cost at time t, $C(t)$, can be expressed as

$$C\left(t\right) = K c_0 \left(\frac{Q_0}{Q\left(t\right)}\right)^{\gamma} \quad (26)$$

where K represents the effect of the related factors (e.g., inflation and the production rate), c_0 denotes the initial production cost per unit, and γ $(0 < \gamma < 1)$ is the learning parameter.

III. Solution Approach

The proposed problem is a nonlinear optimal control problem. With the two decision variables, $W(t)$ and $U(t)$, appearing in the upper limit of the integral in the proposed sales function [i.e., (8)], our problem becomes strongly nonlinear and is impractical to solve via analytical methods, according to Stryk and Bulirsch [30]. Therefore, a numerical approach together with Pontryagin's maximum principle of optimal control theory [20] is employed to solve the proposed problem. To describe the approach, $\mu_0(t)$ is used as an adjoint variable. The Hamiltonian function can then be formulated as

$$H_0\left(t,Q\left(t\right),P\left(t\right),W\left(t\right),U\left(t\right),\mu_0\left(t\right)\right) = ((P\left(t\right) - C\left(t\right)$$

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$$-\omega\left(W\left(t\right),U\left(t\right)\right)e^{-\delta t}+\mu_{0}\left(t\right)\Big)\,q\left(t\right). \tag{27}$$

To simplify the conditions, the current value adjoint variable, represented by $\mu(t)$, and the current value Hamiltonian function, represented by $H(t, Q(t), P(t), W(t), U(t), \mu(t))$, are defined, respectively, by

$$\mu\left(t\right)=\mu_{0}\left(t\right)e^{\delta t} \tag{28}$$

$$\begin{aligned}&H\left(t,Q\left(t\right),P\left(t\right),W\left(t\right),U\left(t\right),\mu\left(t\right)\right)\\&=H_{0}\left(t,Q\left(t\right),P\left(t\right),W\left(t\right),U\left(t\right),\mu_{0}\left(t\right)\right)e^{\delta t}\end{aligned} \tag{29}$$

Then, the current value Hamiltonian function can be expressed as

$$\begin{aligned}H\left(t,Q\left(t\right),P\left(t\right),W\left(t\right),U\left(t\right),\mu\left(t\right)\right)=\big(P\left(t\right)-C\left(t\right)\\-\omega\left(W\left(t\right),U\left(t\right)\right)+\mu\left(t\right)\big)\,q\left(t\right).\end{aligned} \tag{30}$$

Based on optimal control theory and considering that $e^{\delta t}$ is a constant for any given t, certain necessary conditions on the optimal solution can be obtained in the current value form. First, $\mu(t)$ should satisfy the following equations:

$$\frac{d\mu\left(t\right)}{dt}=\delta\mu\left(t\right)-\frac{\partial H\left(t,Q\left(t\right),P\left(t\right),W\left(t\right),U\left(t\right),\mu\left(t\right)\right)}{\partial Q\left(t\right)} \tag{31}$$

$$\mu\left(L\right)=0. \tag{32}$$

Second, the state equation can be expressed as

$$\frac{dQ\left(t\right)}{dt}=q\left(t\right)=\frac{\partial H\left(t,Q\left(t\right),P\left(t\right),W\left(t\right),U\left(t\right),\mu\left(t\right)\right)}{\partial\mu\left(t\right)}. \tag{33}$$

Finally, Pontryagin's Maximum Principle indicates that

$$H\left(t,Q^{*}\left(t\right),P^{*}\left(t\right),W^{*}\left(t\right),U^{*}\left(t\right),\mu\left(t\right)\right)=\max_{\substack{P(t)\in[P_L,P_U]\\W(t)\in[W_L,W_U]\\U(t)\in[U_L,U_U]}} H\left(t,Q^{*}\left(t\right),P\left(t\right),W\left(t\right),U\left(t\right),\mu\left(t\right)\right). \tag{34}$$

The optimal solution can then be obtained numerically through the above necessary conditions. For this purpose, and as has been widely done in the literature [29], the product life is discretized into the number of L years. The *Global Search* algorithm and the *fmincon* function embedded in MATLAB are then utilized to identify the optimal solution. It is assumed that the upper bound of the customer usage rate is known to be r_M. Therefore, the value of $U(t)/W(t)$ is enforced to be less than or equal to r_M. The proposed approach can also be adapted to solve the alternative strategies with static decision making.

IV. Numerical Experimentation

In this section, to illustrate the proposed approach, an example of the joint optimization of a new car's 2-D warranty policy and price is studied. For the purpose of illustration only, the rational parameters required for implementing the proposed approach are shown in Table II. In practice, if real data become available, these parameters can be estimated by the approaches described by Xie [17] and He *et al.* [19].

We first obtained the optimal solution to the proposed strategy based on the aforementioned parameters, and, then, investigate the influence of warranty policy on sales and profits. Next, we analyze the influences of the usage rate, hazard rate, and maintenance cost at optimality. In order to demonstrate the advantages of the proposed strategy, the results are contrasted with those of the four alternative strategies described in Section III.

A. *Solution of the Proposed Strategy*

Implementing the approach described in Section III with the parameters listed in Table II, the proposed problem can be solved, and the solutions to the proposed strategy were obtained as shown in Fig. 3. It can be seen in Fig. 3(a) that the product price is relatively low at both the beginning and the end of the product's life. The reason for this is that at the beginning of the product's life, the product is new and relatively unknown, and thus, a low price is necessary to kick off the diffusion process. At the end of the product's life, a low price is essential in sustaining a moderate sales rate in the presence of market saturation. Moreover, at the end of the product's life, the production costs would be low owing to the learning effects and, thus, can offset the influence of low price on profitability. A comparison of Fig. 3(a) and (d) shows that as the price goes up, the sales rate goes down, and vice versa, which is consistent with the classical relationship between price and sales rate [5].

By comparing Fig. 3(b) and (c) with Fig. 3(d), it can be found that as warranty length or usage limits rise, the sales rate declines, and vice versa. This is contrary to what is expected (i.e., the higher warrant length or usage limits should be able to help promote sales). It is suspected that this is due to the stronger influence of price on the sales rate than that of warranty policy. To see this, observe Fig. 3(a)–(d). It can be found that when warrant length limit and usage limit go up, price goes up even faster. As a result, the increment of sales rate caused by higher warranty length and usage limit is lower than its decrement caused by a higher price, and thus, the sales rate declines. Similarly, the stronger influence of price than that of warranty policy can also be found when they go down. Therefore, it is a better choice to set low warranty length and usage limit to save warranty costs when a low price is used to promote sales, since the profitability is sensitive to cost under this situation. On the other hand, when price is high, the profitability is less sensitive to cost, and thus, high warranty length and usage limit can be used to promote sales to offset the influence of high prices.

B. *Experimentation on the Influences of Warranty*

This section teases out the effects of warranty and price on sales rates and profits by conducting separate optimization over one variable at a time. We first solve the problem of dynamic pricing to maximize the total profits, under different values of fixed warranty length and usage limit. The solutions are shown

TABLE II
PARAMETERS OF THE EXAMPLE

Parameter	Value	Parameter	Value
α_R	1×10^4	k_1	1×10^{16}
β_R	2.5	k_2	0.1
r_M	20×10^4 km per year	a	2
K	1	b	0.3
c_0	5×10^4 CNY	ψ	0.04
γ	0.03	δ	0.08
α	1.5	L	13 years
β	2.5	P_L	0 CNY
c_{MR}	5000 CNY	P_U	1×10^6 CNY
η	1.5	W_L	0 year
r_0	2×10^4 km per year	W_U	100 year
Q_0	5×10^5	U_L	0×10^4 km
Q_M	3×10^6	U_U	300×10^4 km

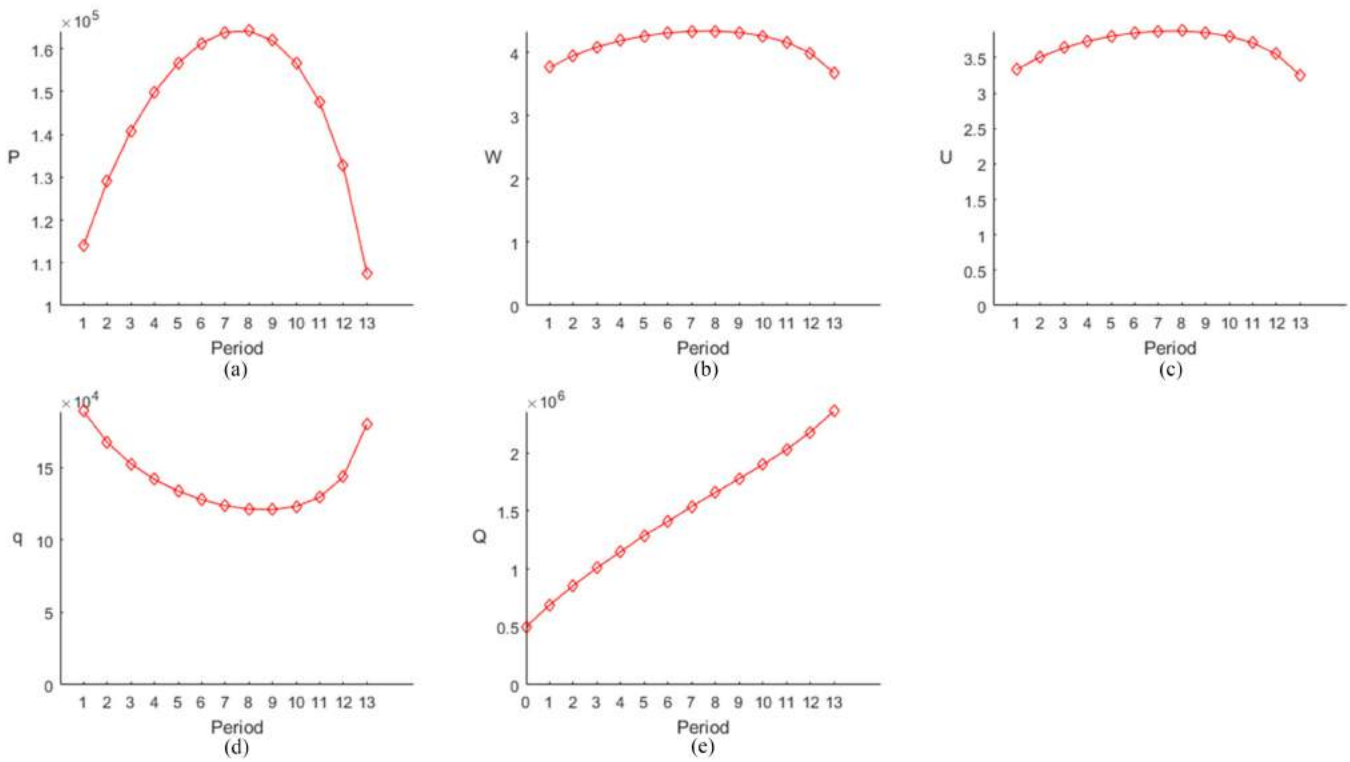

Fig. 3. Solution of the proposed strategy.

in Fig. 4. It can be found in Fig. 4(a)–(j) that, similar to the results shown in Fig. 3, under fixed warranty length and usage limit, the price also initially rises and then declines. Moreover, under higher values of warranty length and/or usage limit, the price is also higher. This is reasonable considering that under a higher warranty length and/or usage limit, the unit warranty cost would be also high. and thus, a higher price is necessary to maintain the profitability. In addition, higher values of warranty length and/or usage limit can help promote sales and, thus, can tolerate a higher price and, to a certain extent, offset its adverse influence on the sales rate.

From Fig. 4(s), it can be found that the total warranty cost increases as the warranty region becomes larger (i.e., either only the warranty length or the usage limit increases, or they increase

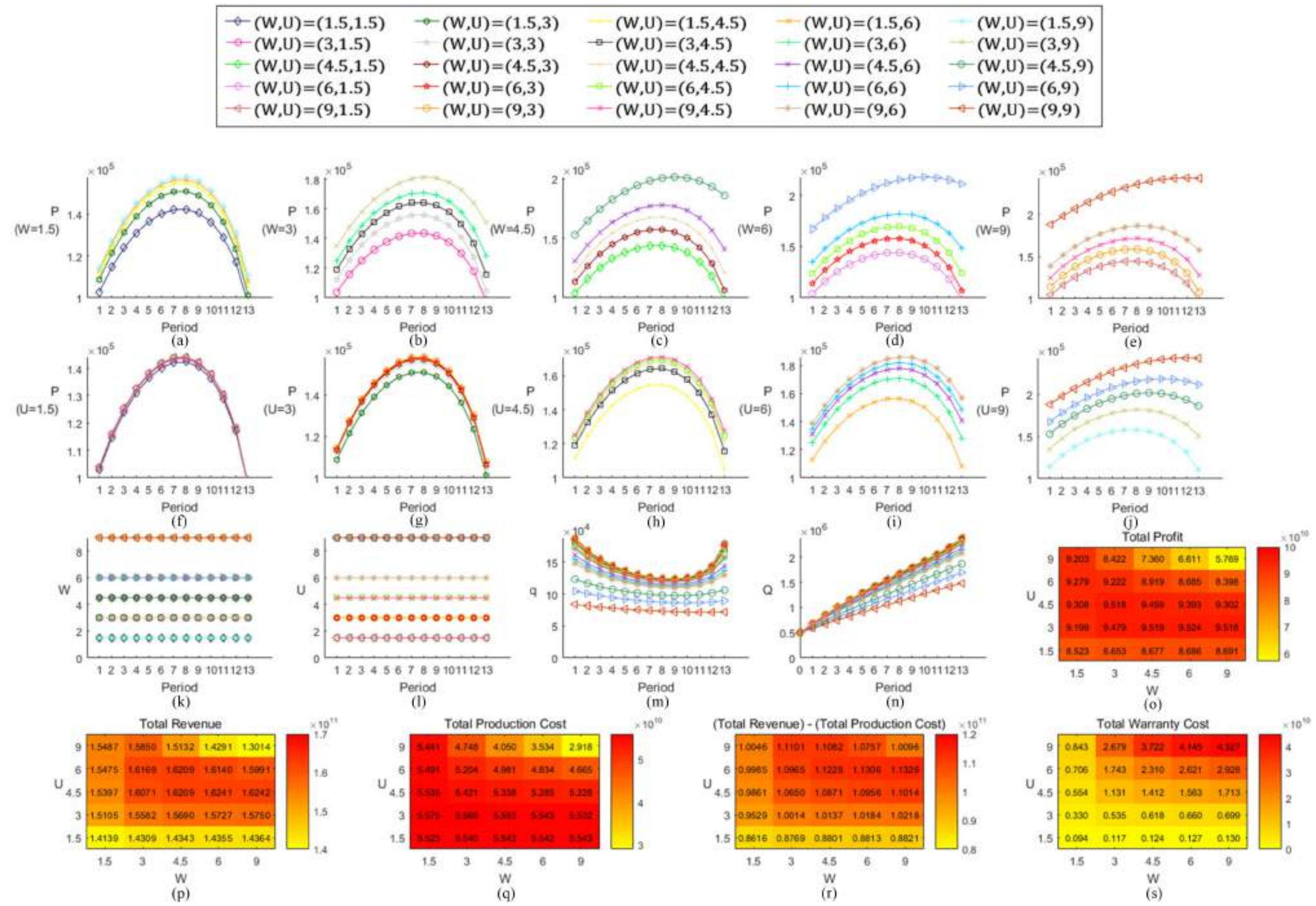

Fig. 4. Solution of the proposed strategy given the fixed warranty length and usage limit.

simultaneously). This is because a larger warranty region results in more product failures within the region. From Fig. 4(p) and (r), it can be seen that the total revenue and the difference between the total revenue and production costs first increase and then decrease, as the warranty region becomes larger. As a result, there exist optimal values of warranty length and usage limit leading to the highest profit (i.e., around 9.524×10^{10} units), as shown in Fig. 4(o).

To better investigate the influences of warranty on sales and profits, the results under different values of fixed price and also fixed warranty region are compared. The results determined by the proposed approach under different fixed prices (130 000 CNY, 145 000 CNY, and 160 000 CNY) are shown in Figs. 5–7, respectively. For each of the values of price, different combinations of the values of warranty length and usage limit are also considered and compared. It is important to note that the values of price, warranty length, and usage limit considered may not be the optimal solutions of the proposed problem.

From Figs. 5(a)–(h), 6(a)–(h), and 7(a)–(h), we can see that under each combination of the fixed price, warranty length, and usage limit, the sales rate rises initially and then declines. This is consistent with the Bass model [10]: The sales rate is low when the product is less known and few are sold at the beginning, and it is high when the product obtains adequate market recognition. Due to the limitation of the overall market potential, the sales rate decreases when approaching the end of a product's life.

Another interesting finding is that under higher warranty length and/or usage limit, the sales rate increases faster at the beginning, but also decreases faster when approaching the end of a product's life. This reflects the ability of a warranty policy to promote sales and accelerate penetration of the market at the beginning of a product's life. However, due to the limitation of the overall market potential, faster market penetration leads to a faster decrease of sales rate when getting close to the end of a product's life. A similar phenomenon can also be found by comparing these figures: Under a lower price, the sales rate increases and decreases faster. The reason is also similar: A lower price can accelerate market penetration at the beginning, but, a higher peak sales rate also means a faster decrease when approaching the end of a product's life.

From Figs. 5(i)–(p), 6(i)–(p), and 7(i)–(p), it can be realized that, for the same price, higher values of the warranty length and/or usage limit lead to higher accumulated sales volumes during the whole product life. This then results in higher total revenues, production costs, and warranty costs, as shown in

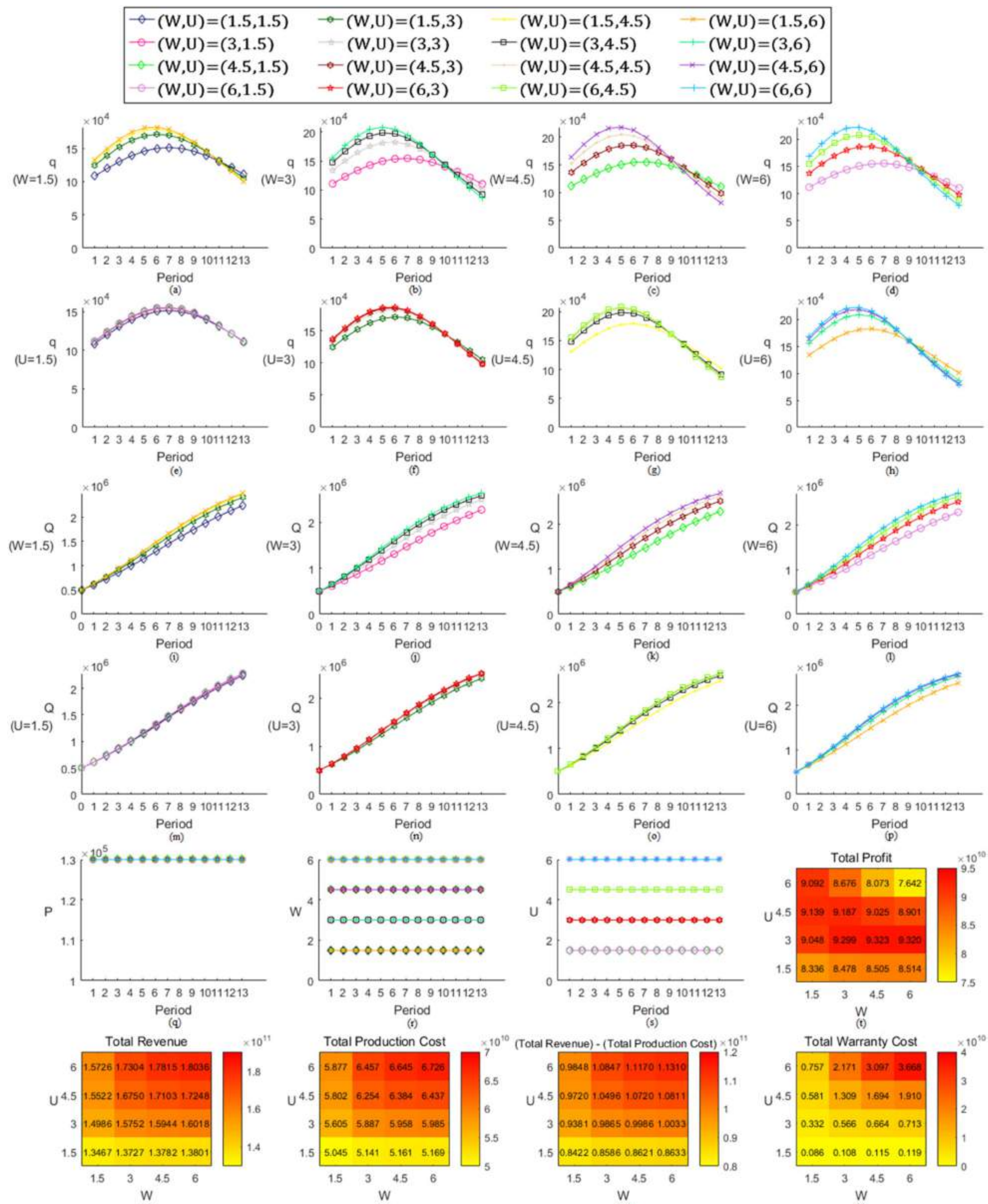

Fig. 5. Results under the fixed price ($P = 130\,000$ CNY), warranty length, and usage limit.

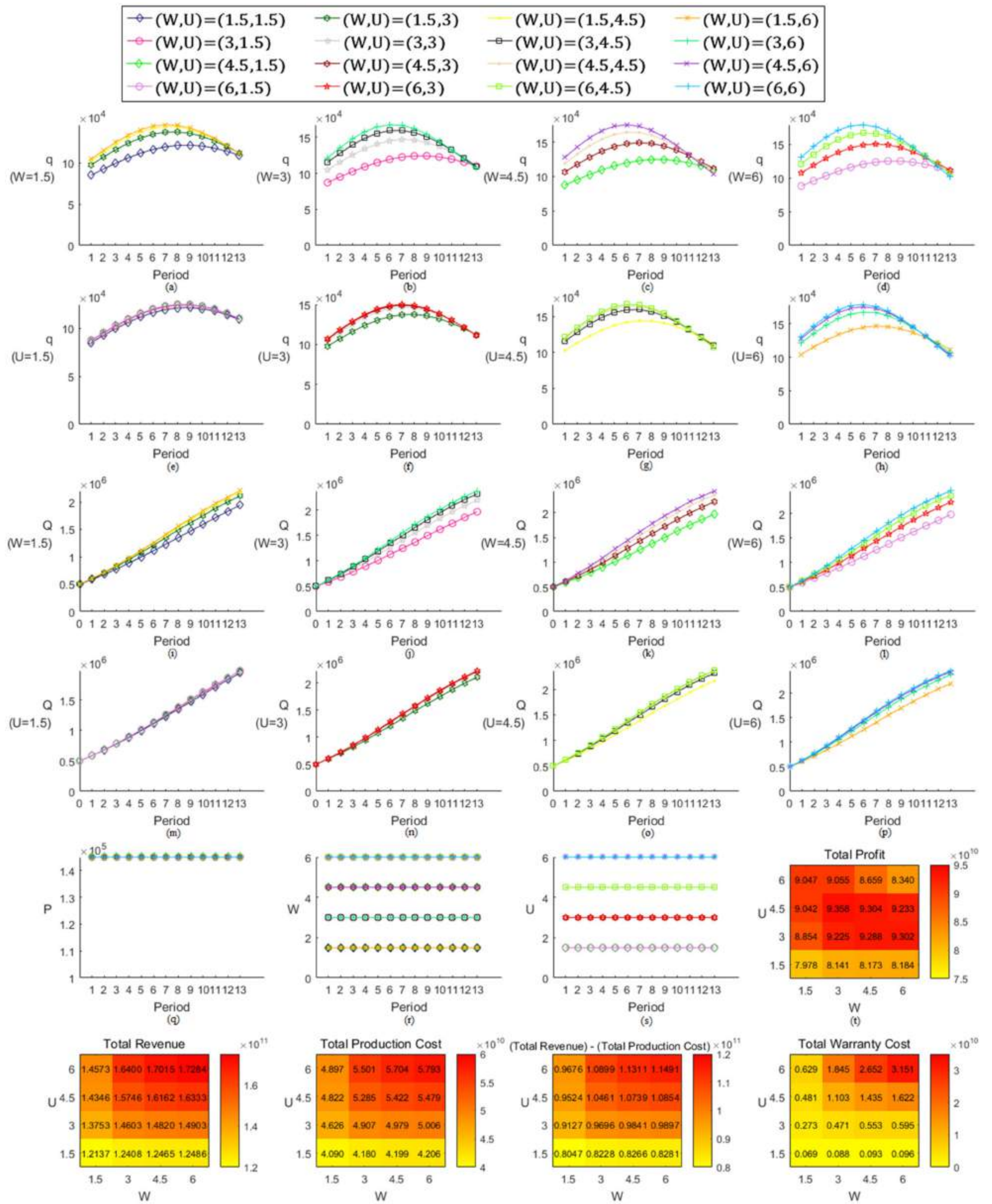

Fig. 6. Results under the fixed price ($P = 145\ 000$ CNY), warranty length, and usage limit.

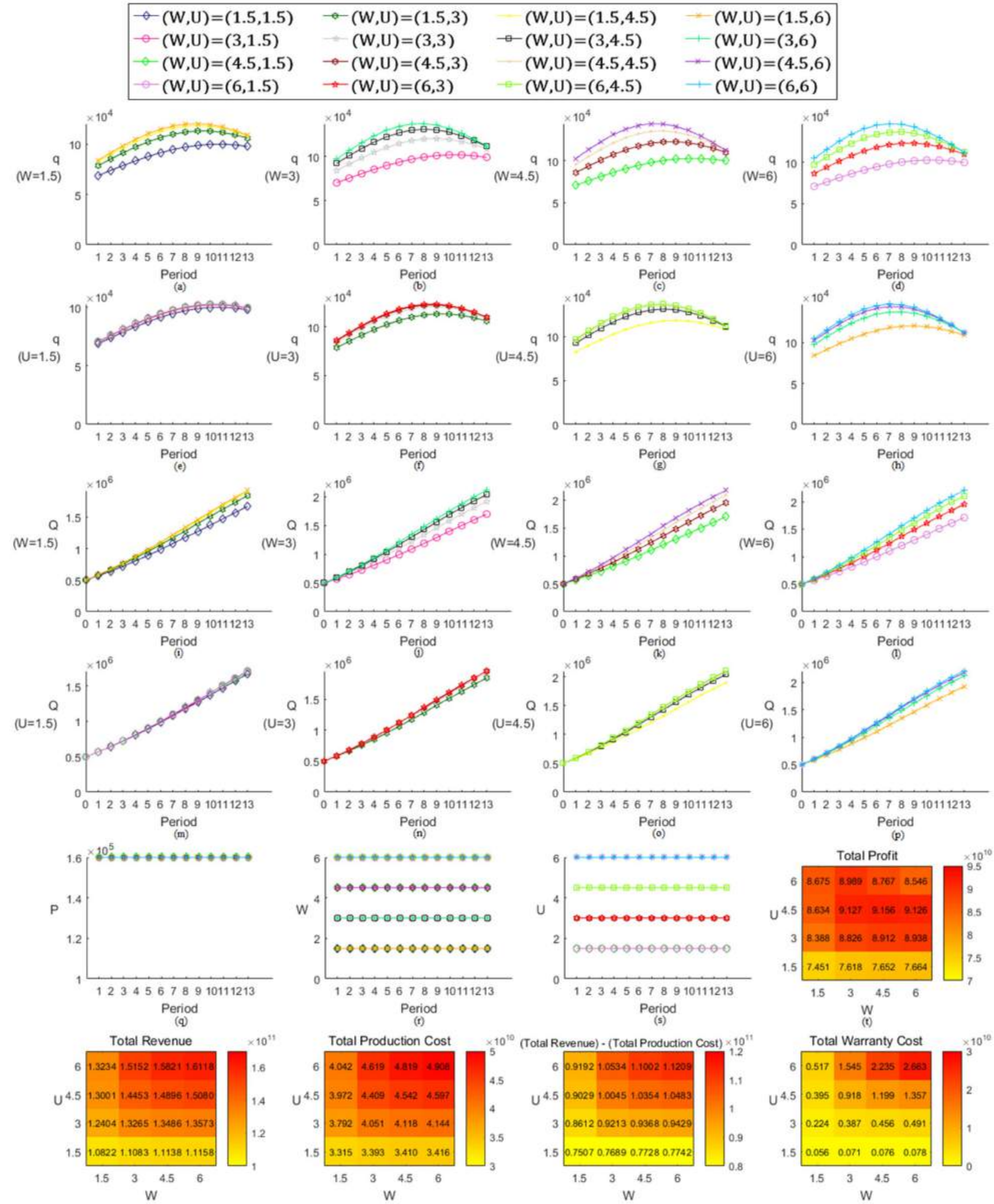

Fig. 7. Results under the fixed price ($P = 160\,000$ CNY), warranty length, and usage limit.

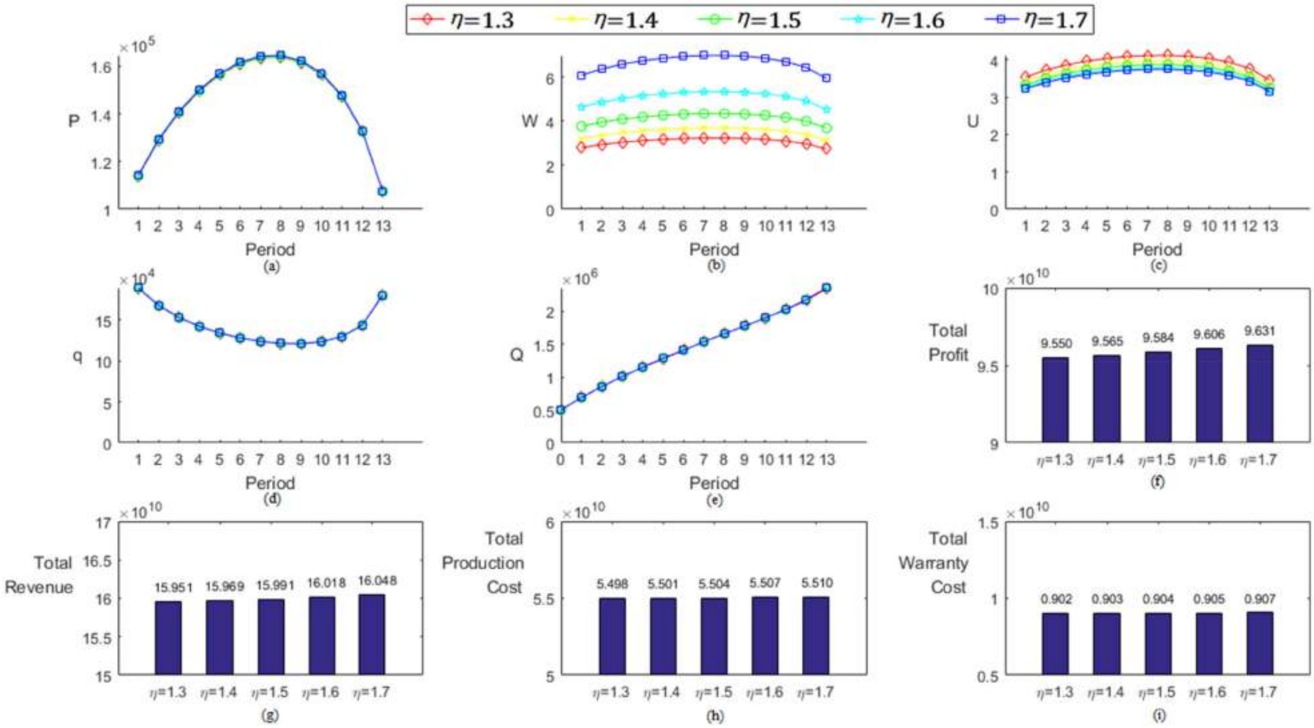

Fig. 8. Sensitivity analysis on η.

Fig. 5(u), (v), and (x); Fig. 6(u), (v), and (x); and Fig. 7(u), (v), and (x), respectively.

To separate the influence of warranty on the profit from that of production cost, the difference between the total revenue and the production cost is calculated as a function of the warranty region, as shown in Figs. 5(w), 6(w), and 7(w). By comparing these with the total profits shown in Figs. 5(t), 6(t), and 7(t), respectively, we can find that, as the warranty length and/or usage limit increase, while the difference between the total revenue and the production cost always increases, the total profit first increases and then decreases. This means that there exist optimal values of the warranty length and the usage limit leading to the highest profits. This demonstrates the significant influence of warranty on the sales and profits.

C. Sensitivity Analysis

1) Influences of Usage Rate: In this section, experiments on the influence of parameter η of the PH model described in (12) are first conducted. The solutions of the proposed problem under different values of η are shown in Fig. 8. It can be seen in Fig. 8(b) and (c) that much higher values of warranty length and slightly lower values of usage limit are obtained under higher values of η. This is reasonable since a higher value of η reflects stronger influence of the usage rate r on the hazard rate when it is higher than the nominal usage rate r_0, according to (12). Therefore, under a higher value of η, a lower usage limit is necessary to control the warranty cost of customers with high usage rates, while higher values of warranty length can be employed to attract customers with low usage rates, without incurring additional warranty costs. As a result, the total warranty costs under different values of η are quite similar, whereas the total profits under higher values of η are slightly higher, as can be seen in Fig. 8(i) and (f), respectively.

Experiments are then conducted concerning the influence of the scale parameter α_R of the censored gamma distribution, which is employed to describe the distribution of a customer's usage rate. From the results under different values of α_R, as shown in Fig. 9(b) and (c), it can be seen that larger values of α_R result in lower warranty length and higher usage limit. The reason for that can be attributed to the increased number of customers with higher usage rates under higher values of α_R. Consequently, under higher α_R, higher usage limit can attract more customers (i.e., those with higher usage rates), while warranty length can be lowered down since there are fewer customers with low usage rate.

The influence of the shape parameter β_R of the usage rate distribution is also studied, with the results shown in Fig. 10. Within the considered range [2.3, 2.7], a larger β_R implies more customers with higher usage rates, and vice versa. Thus, from Fig. 10, findings and explanations similar to those in the experiment on α_R, can be achieved, which are omitted here for the sake of brevity.

2) Influences of the Elasticity of the Displaced Effective Warranty Length: Fig. 11 shows the results of the sensitivity analysis on the elasticity b of the displaced effective warranty length in the sales rate function described in (7). The elasticity b reflects the degree of the influence of the displaced effective warranty length on the sales rate. Customers with a high value of b will

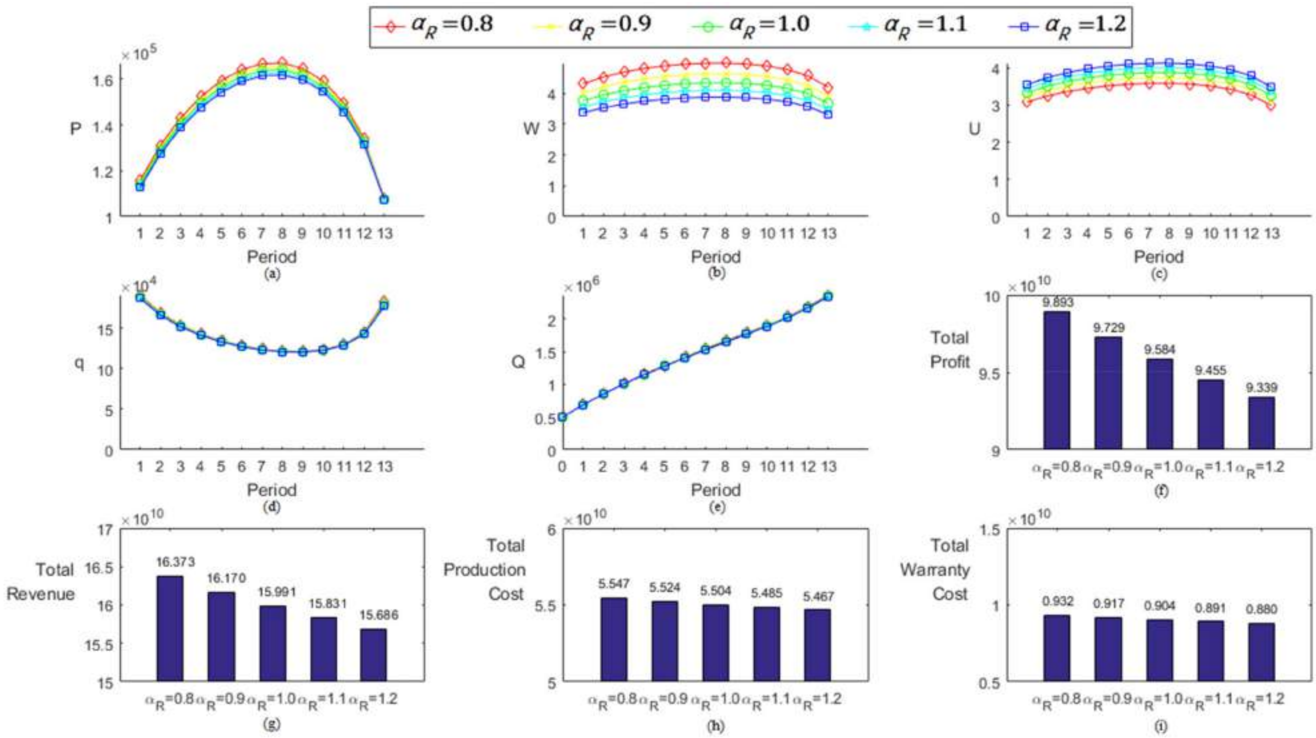

Fig. 9. Sensitivity analysis on α_R.

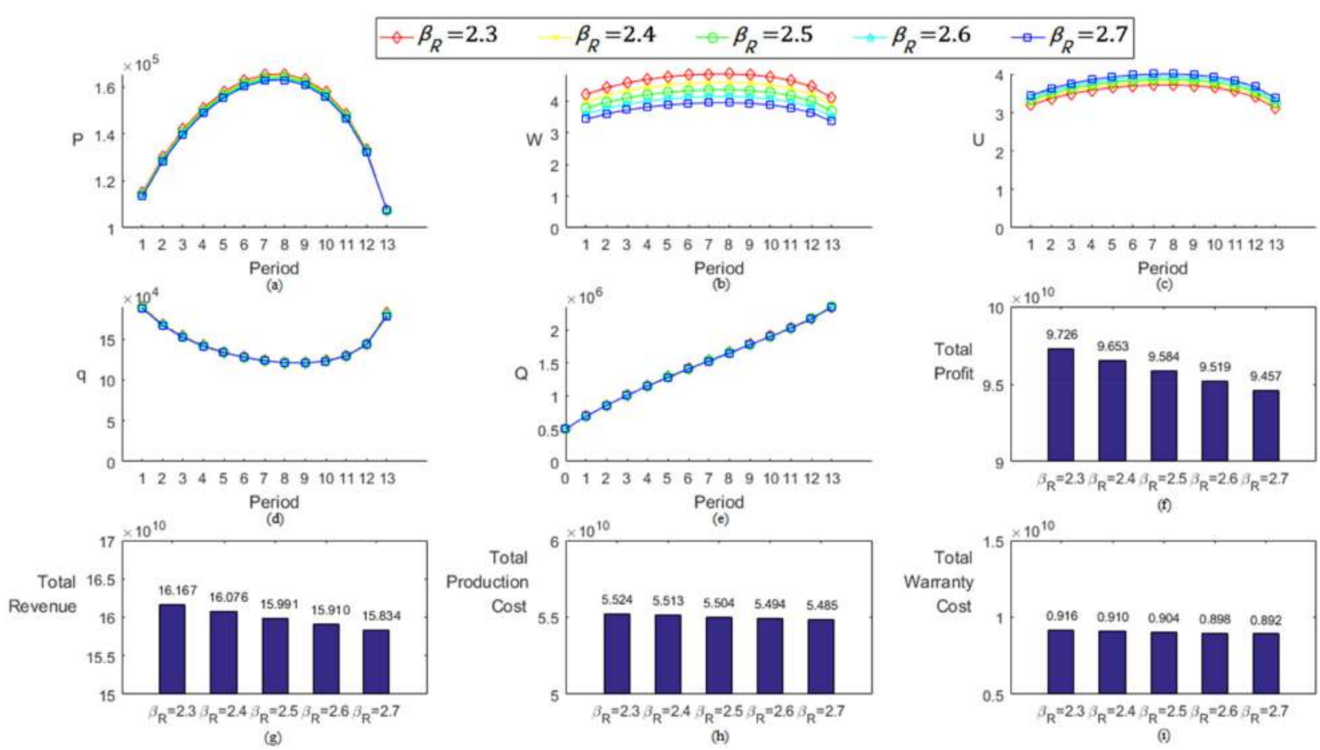

Fig. 10. Sensitivity analysis on β_R.

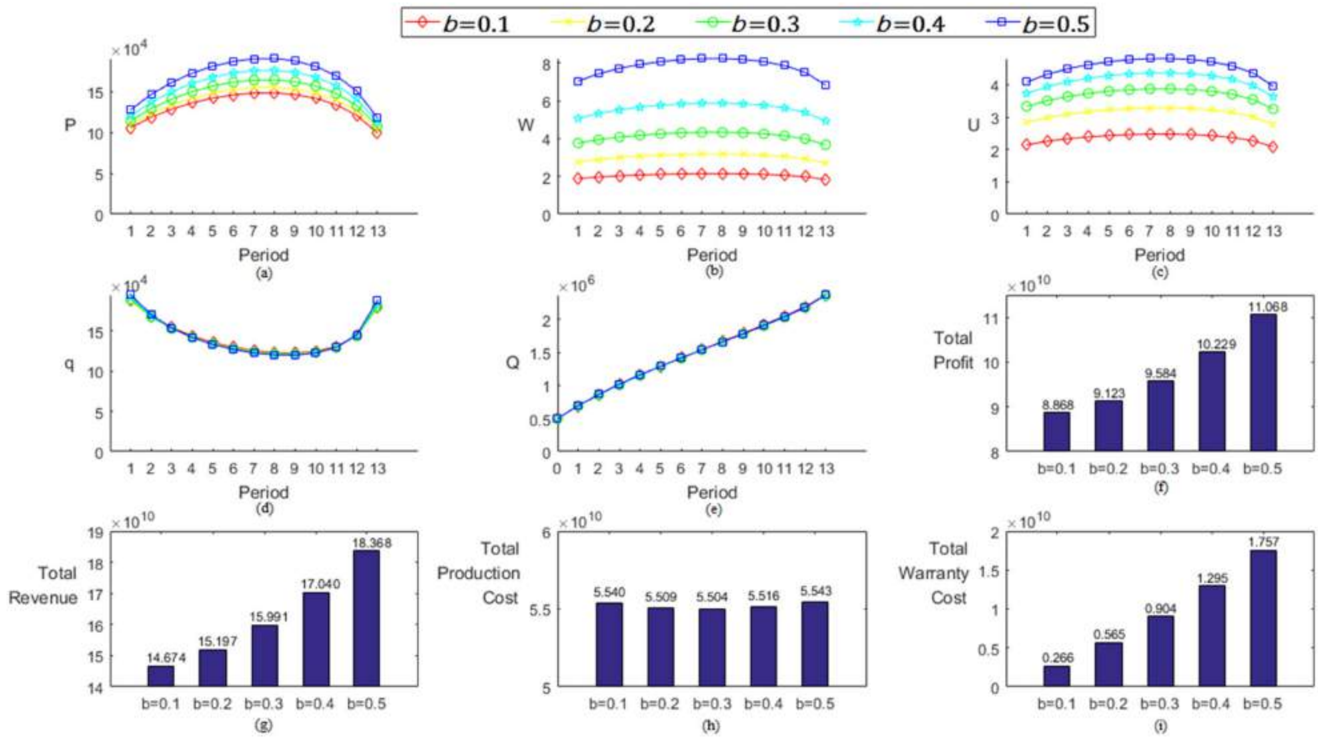

Fig. 11. Sensitivity analysis on b.

be more sensitive to the displaced effective warranty length than those with a low value of b. Therefore, higher values of b result in higher warranty length and usage limit, as shown in Fig. 11(b) and (c), respectively.

Although higher warranty length and usage limit can promote sales, they also lead to higher warranty costs, as shown in Fig. 11(i). Therefore, a higher price is necessary to achieve high profitability under high values of b, as shown in Fig. 11(a). Moreover, under high values of b, the adverse influence of price on the sales rate can be well mitigated by the positive influence of a larger warranty region, as indicated by the very close sales rate under different scenarios in Fig. 11(d). As a result, the total revenue and profit under high values of b can be much greater than those under low values of b, as shown in Fig. 11(g) and (f).

3) Influences of Hazard Rate: This section studies the influences of the hazard rate. The first focus is on the scale parameter α of the failure time distribution employed in this paper (i.e., the Weibull distribution). The results of the proposed problem under different values of α are shown in Fig. 12. From Fig. 12(a)–(c), it can be found that the higher the value of α, the higher the price, the warranty length, and the usage limit. This is reasonable since a higher value of α means a lower hazard rate, which makes it possible to increase the warranty length and usage limit without increasing the warranty cost much, compared to those under lower values of α, as illustrated in Fig. 12(i). With higher warranty length and usage limit to mitigate its adverse influence on sales, a higher price can then be used to generate greater revenue. Consequently, the total revenues and profits under higher values of α are much higher than those under lower values of α, as shown in Fig. 12(g) and (f), respectively.

This section then studies the influences of the shape parameter β of the failure time distribution. Fig. 13 shows the results under different values of β, ranging from 2.3 to 2.7. It is well known that for β within the range [2.3, 2.7], the hazard rate under the Weibull distribution increases exponentially with time *t*, and the higher the value of β, the faster it increases. Thus, for higher values of β, the warranty length needs to be low to ensure that the warranty expires before the hazard rate becomes too high to control warranty cost, as shown in Fig. 13(b). According to (12), a higher value of β also leads to a higher hazard rate at the same usage and time, through its influence on the baseline hazard rate $h_0(t)$. Thus, to control warranty cost under higher values of β, the usage limit also needs to be lowered, albeit only slightly, since the hazard rate is mainly influenced by using time instead of usage (i.e., the usage rate distribution is unchanged).

As described in Fig. 13(a), a lower price is required to offset the adverse influence of a smaller warranty region and maintain the sales rate under higher values of β. As a result, compared to a smaller value of β, larger β values result in very similar sales rates and accumulated sales, as shown in Fig. 13(d) and (e), respectively, which, together with the lower price, lead to lower total revenues and profits, as shown in Fig. 13(g) and (f), respectively.

4) Influences of Minimal Repair Cost: In this section, the influences of the minimal repair cost (c_{MR}) are studied. The results are shown in Fig. 14. It can be seen, from Fig. 14(b) and (c), that the higher the value of c_{MR}, the lower the warranty length and usage limit. This is reasonable considering that it is necessary to lower the number of product failures by shortening the warranty region to control warranty costs, when the unitary

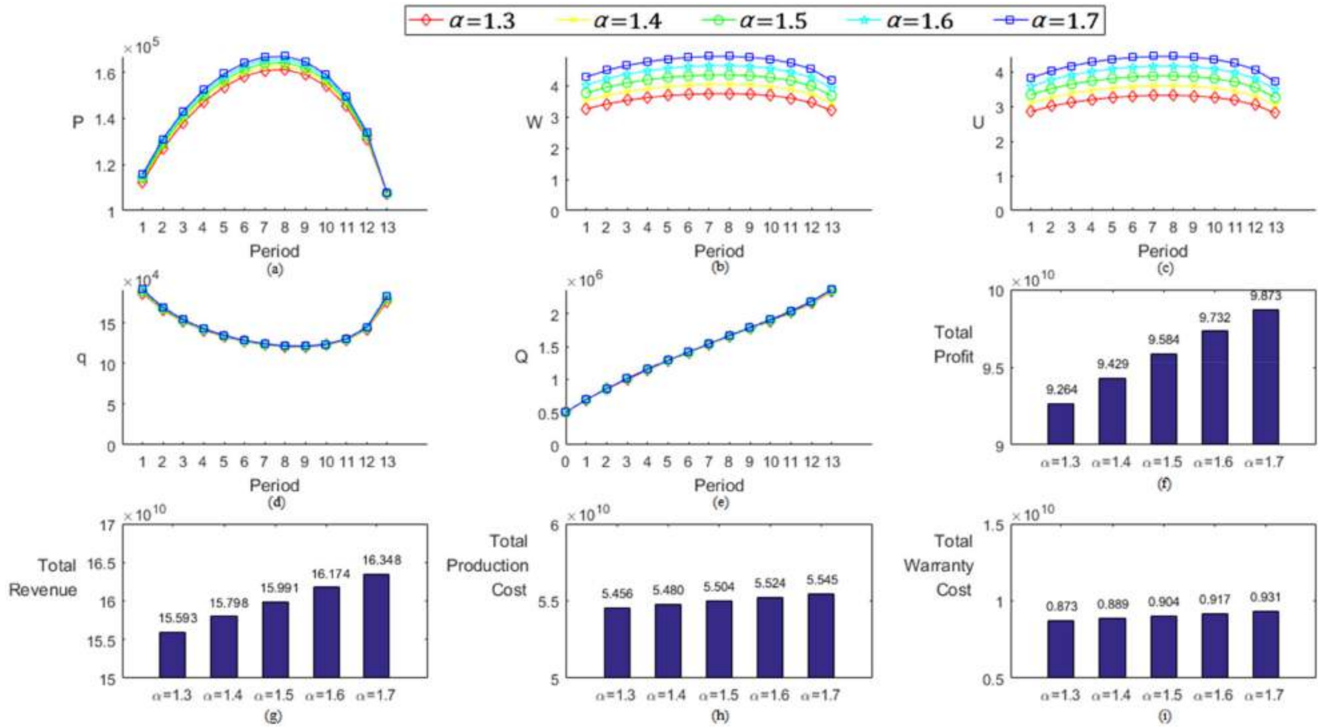

Fig. 12. Sensitivity analysis on α.

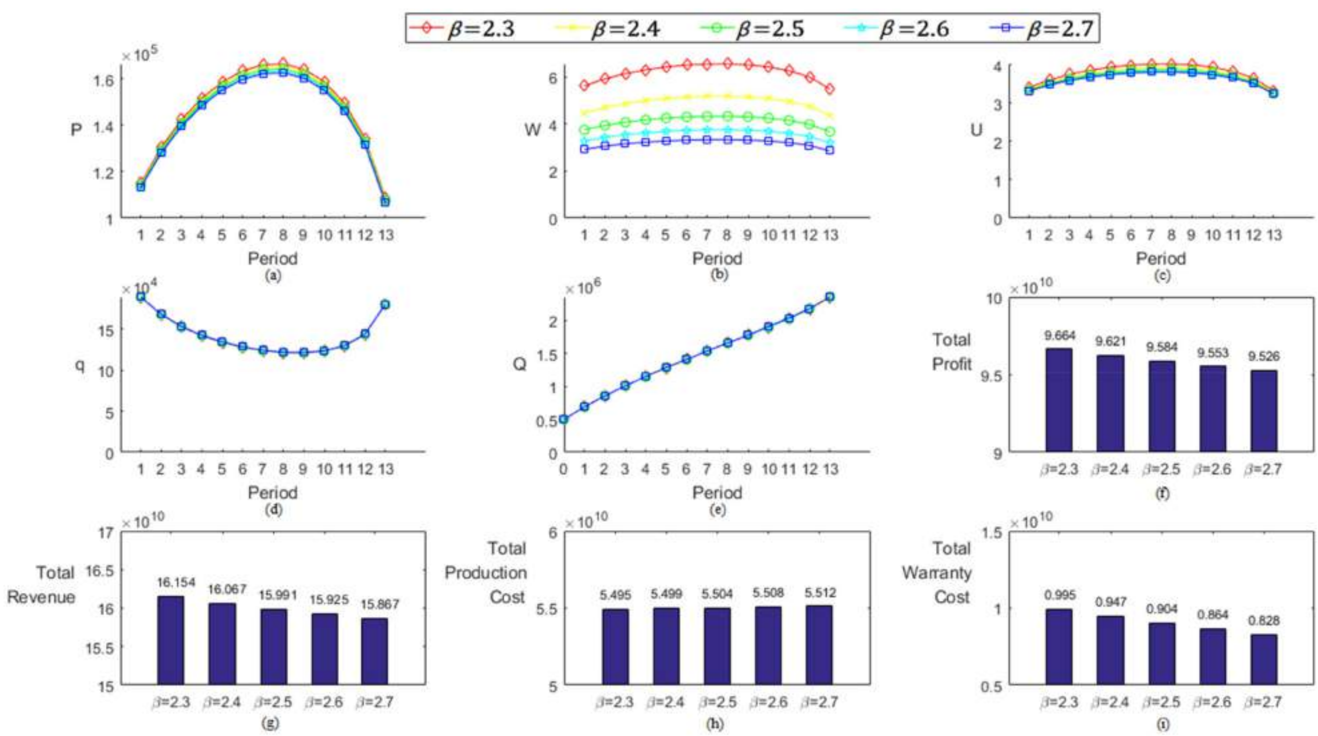

Fig. 13. Sensitivity analysis on β.

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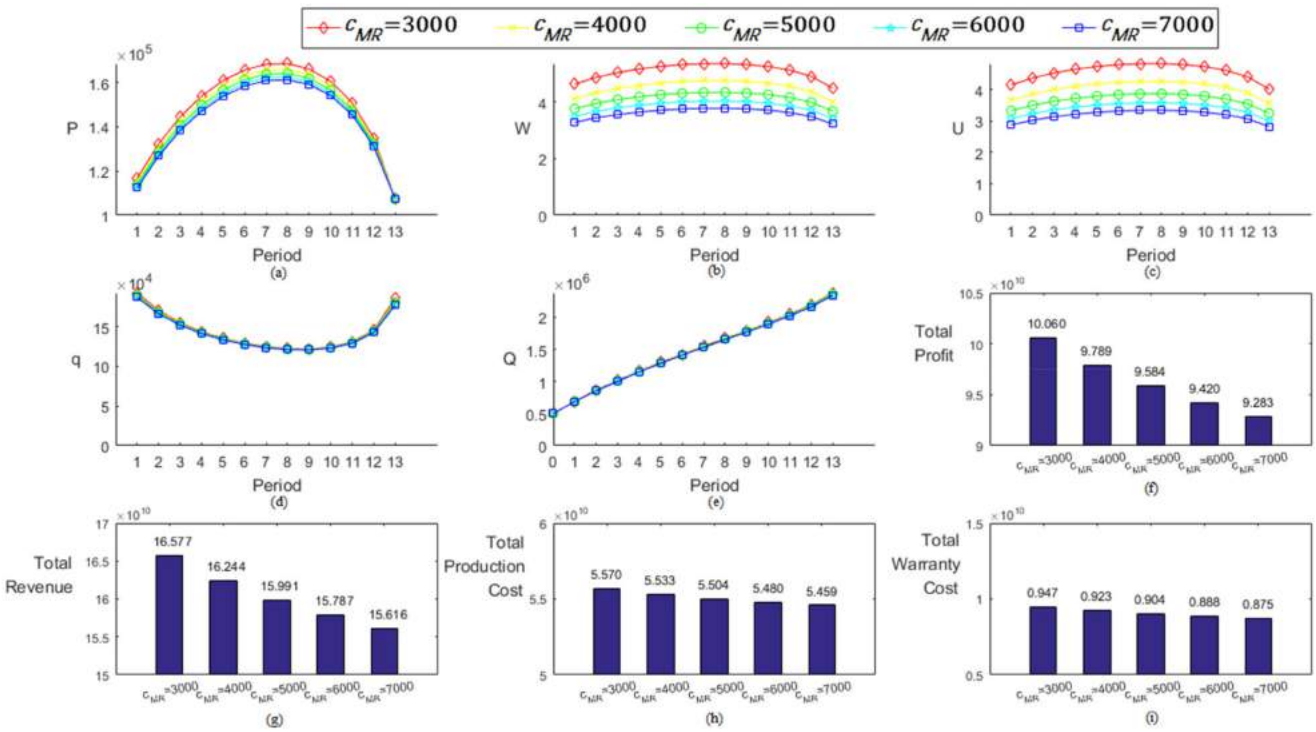

Fig. 14. Sensitivity analysis on c_{MR}.

cost c_{MR} increases. Consequently, the sales rate would be adversely influenced. This influence needs to be mitigated by a lower price, as shown in Fig. 14(a), which eventually leads to lower total revenue and profit under higher values of c_{MR}, as shown in Fig. 14(g) and (f), respectively.

D. Comparisons Between Warranty Policies

This section compares our proposed strategy (i.e., dynamic pricing and 2-D warranty policy) with the four alternative strategies described in Section III, in order to illustrate its merits.

1) Comparison Between the 2-D and 1-D Warranty Policies: To illustrate the merits of the proposed 2-D warranty policy, this section compares it with the two 1-D warranty policies (i.e., Alternative strategy 1 with only the warranty length limit and Alternative strategy 2 with only the usage limit). By implementing the approaches described in Sections II and III, their solutions are obtained and compared, as shown in Fig. 15.

Note that, according to (12) and (13), the hazard rate can be increased by two factors: high usage rate and aging. Moreover, a higher hazard rate leads to more product failures and, thus, higher warranty costs. Compared to the 2-D warranty policy, Alternative strategy 1 fails to control the warranty cost incurred by a high usage rate, while Alternative strategy 2 fails to control the warranty cost incurred by aging under a low usage rate. Consequently, these two strategies need lower warranty length and usage limit, respectively, than those under the proposed policy for the purpose of cost control, as described in Fig. 15(b) and (c), respectively. As a result, by simultaneously limiting the two dimensions of the warranty region, the 2-D warranty policy can employ a higher limit for each dimension and achieve a very close warranty cost, as illustrated in Fig. 15(i), when the sales are similar, as shown in Fig. 15(e), compared with the 1-D warranty policies.

When its adverse influence on sales can be mitigated by the positive influence of a higher warranty region, a higher price should be used to strengthen the product's profitability. Therefore, it can be found in Fig. 15(a) that the price under the 2-D warranty is higher than that under the 1-D warranty, while their sales rates and accumulated sales are very similar, as shown in Fig. 15(d) and (e), respectively. Consequently, the total revenue and profit can be highly improved by implementing the 2-D warranty rather than the 1-D warranty, as described in Fig. 15(g) and (f), respectively. The total profit under the 2-D warranty is about 4.1 billion CNY and 4.5% higher than that under the one-dimensional warranty with warranty length limit, and about 770 million CNY and 0.8% higher than that under the one-dimensional warranty with usage limit.

2) Comparison Between Free Minimal Repair and Free Replacement: This section compares the influences of maintenance policy within a warranty region: free minimal repair employed in the proposed strategy, and free replacement used in Alternative strategy 3. To solve the latter, let $N_\epsilon = 45$ to keep ϵ less than 10^{-7}, W_U equals 10 years and U_U equals 100×10^4 km. The solutions of the two strategies are shown and compared in Fig. 16.

By comparing the solutions under these two strategies, our first finding is that both the warranty length and usage limit under free replacement are much lower than those under free minimal repair, as shown in Fig. 16(b) and (c), respectively. This can be

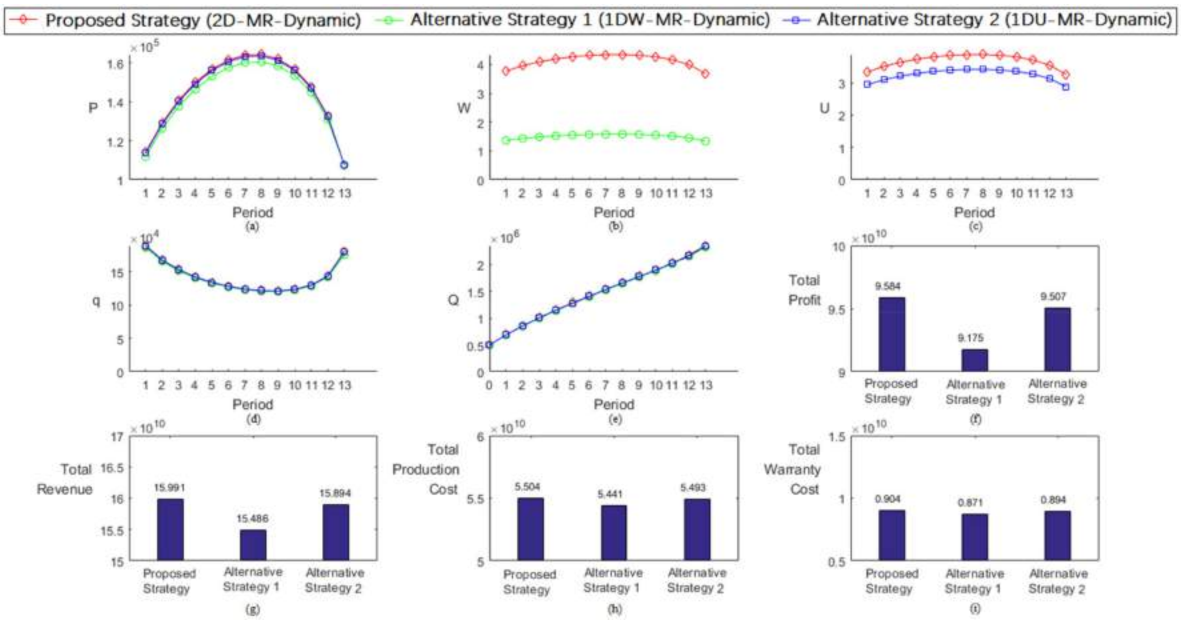

Fig. 15. Comparison between 2-D and 1-D warranty policies.

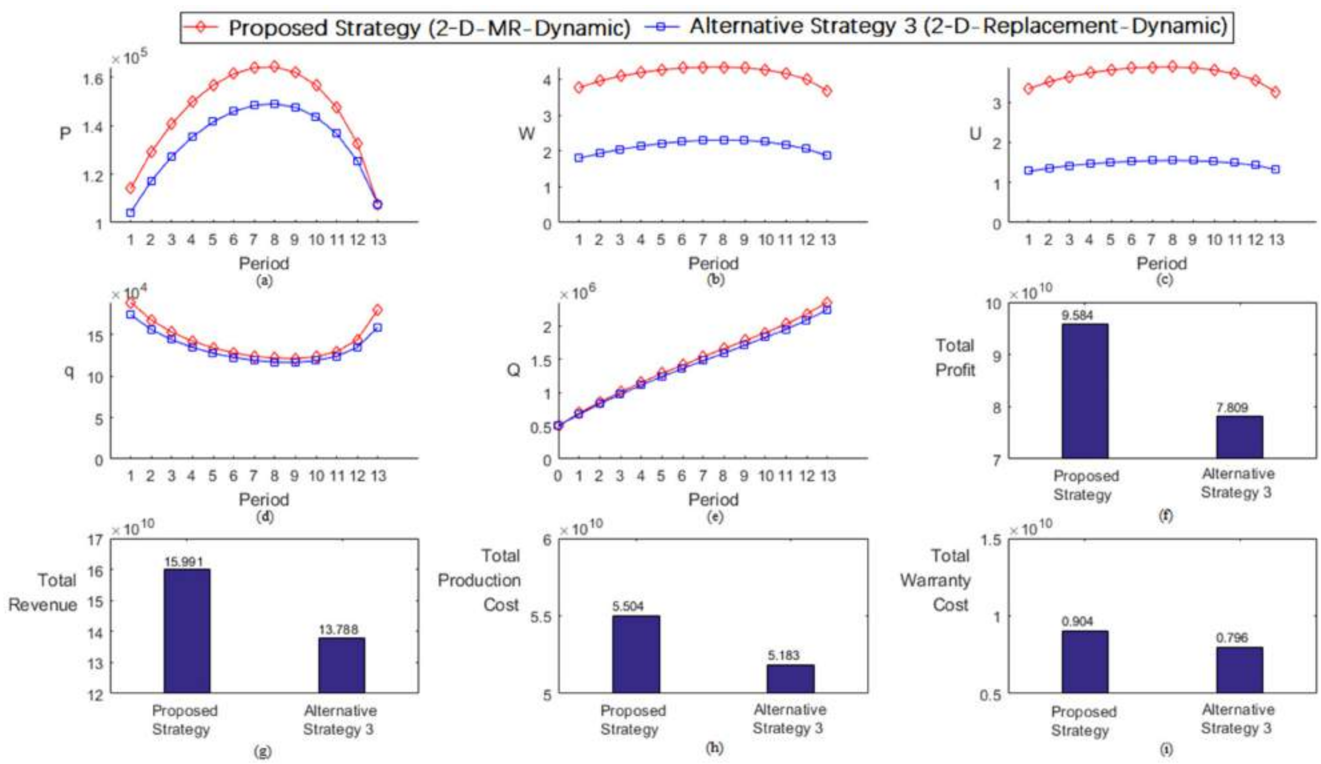

Fig. 16. Comparison between free minimal repair and free replacement.

attributed to the much higher cost of dealing with each warranty claim by replacement than by minimal repair. Consequently, the warranty length and usage limit under replacement have to be much lower than those under minimal repair, in order to control warranty cost by controlling the amount of failures within the warranty region.

To mitigate the adverse influences of a small warranty region on the sales rate, the price under replacement needs to be lower than that under minimal repair, as can be seen in Fig. 16(a). Nevertheless, as seen in Fig. 16(d) and (e), respectively, the sales rate and the accumulated sales volume under replacement are still lower than those under minimal repair. As a result, the total

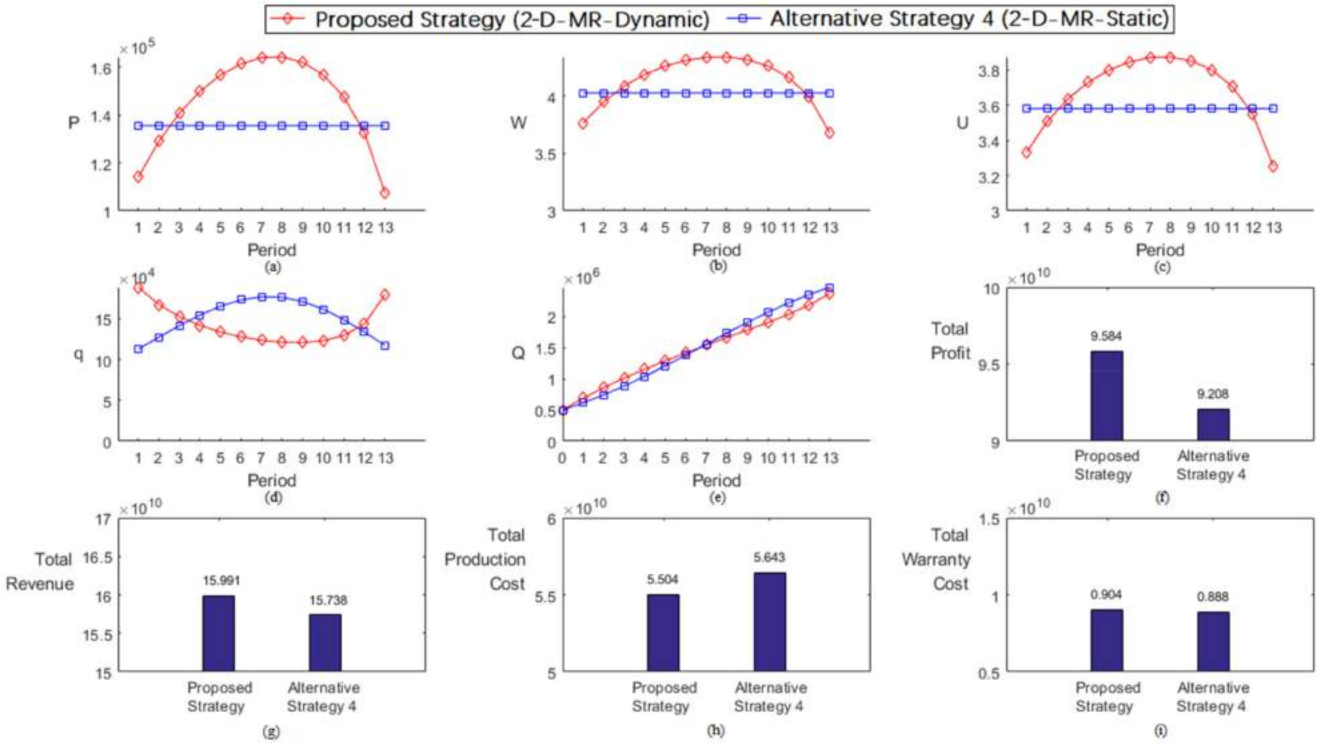

Fig. 17. Comparison between dynamic and static decision making.

revenue, total profit, and total production cost under free replacement are all much lower than those under free minimal repair, as shown in Fig. 16(g), (f), and (h), respectively. More specifically, the total profit under replacement is about 17.8 billion CNY and 22.7% lower than that under minimal repair.

3) Comparison Between Dynamic and Static Decision Making: To see the advantages of dynamic decision making over static decision making, a comparison is made between the proposed strategy and Alternative strategy 4. Their only difference is that the product price, the warranty length, and the usage limit can be dynamically changed under the proposed strategy, while, under Alternative strategy 4, the variables are set at the beginning of the horizon and held constant throughout the product's life cycle. The solutions of the two strategies are contrasted in Fig. 17.

As previously explained in Section IV-A, the price, the warranty length, and the usage limit initially increase, and then decrease, while the sales rate first decreases and then increases, under dynamic decision making, as shown in Fig. 17(a)–(d). Comparatively, as shown in Fig. 17(d), under static decision making, the sales rate increases initially as accumulated sales increase, and then decreases due to the limitation of the total market potential, which is consistent with the Bass model [10], as previously explained in Section IV-B.

Although the accumulated sales throughout a product's life under static decision making is higher than that under dynamic decision making, as can be seen in Fig. 17(e), the total revenue under the former is much lower than that under the latter, as shown in Fig. 17(g). Together with its higher production cost and slightly lower warranty cost, as shown in Fig. 17(h) and (i), respectively, the static strategy results in 3.76 billion CNY and 4.1% lower total profit than the dynamic strategy, as shown in Fig. 17(f). This finding illustrates the advantage of dynamic decision making in achieving higher profits through the improved balancing of sales promotion, revenue earning, and cost saving, compared to static decision making.

V. Conclusion

In this paper, we studied the joint dynamic optimization of price and 2-D warranty policy with free minimal repair or replacement, and modeled it into a nonlinear optimal control problem. For this purpose, in this paper, we proposed a new function to describe the joint influence of price and 2-D warranty on the sales rate, while considering heterogeneous customer usage rate. The proposed sales function could also characterize the diffusion process of a product. The heterogeneous usage rate and usage degradation had been taken into consideration, when modeling product failures to calculate the warranty cost.

The maximum principle of optimal control theory and the global search algorithm were employed to solve the proposed optimization problem. Numerical experimentations were then conducted to illustrate the proposed strategy and compare it with four alternative strategies. Based on this comparison, it was found that 2-D warranty policy was superior to 1-D warranty policies, and dynamic decision making was better than static decision making in achieving higher profit.

As a building block to the more realistic problem, only corrective maintenance within a warranty region had been considered. Employing preventive maintenance together with corrective maintenance to further decrease warranty cost is of great interest and will be studied in future. In this paper, minimal repair and replacement were considered as separate strategies. In future, the combined minimal repair and replacement [31] will be studied to further lowering the maintenance cost within the warranty coverage. Our future research effort will also be devoted to study the strategic interaction among competitors and warranty region with shapes other than rectangle, such as L-shaped region.

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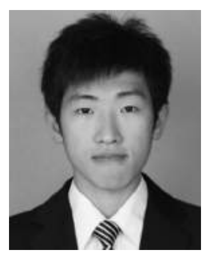

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