MAINTENANCE POLICIES CONSIDERING DEGRADATION AND COST PROCESSES FOR A MULTICOMPONENT SYSTEM

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ABSTRACT. Condition-based maintenance (CbM) is a method for reducing the probability of system failures as well as the operating cost. Nowadays, a system is composed of multiple components. If the deteriorating process of each component can be monitored and then modelled by a stochastic process, the deteriorating process of the system is a stochastic process. The cost of repairing failures of the components in the system forms a stochastic process as well, and is known as a cost process.

This paper models the deterioration process of a multi-component system. Each deterioration process is modelled by the Wiener process. When a linear combination of the processes, which can be the deterioration processes and the cost processes, exceeds a pre-specified threshold, a replacement policy will be carried out to preventively maintain the system. Under this setting, this paper investigates maintenance policies based on the deterioration process and the cost process. Numerical examples are given to illustrate the optimisation process.

Keywords: Condition-based maintenance; age replacement policy; block replacement policy; cost process; Wiener process

1 Introduction

Condition-based maintenance (CbM) is a class of methods for scheduling maintenance policies that aims to reduce the probability of failures, help reduce the operation cost, and ensure the stable quality of the products. In the CbM related literature, stochastic processes such as the gamma process (Lawless and Crowder, 2004; Cholette et al., 2019), the inverse Gaussian process (Li et al., 2017; Hao et al., 2019), and the Wiener process (WP) (Wen et al., 2018; Xie et al., 2019; Wang and Kang, 2020) are widely used for modelling the deterioration processes under different applications.

Basically, CbM is performed on a piece of equipment once a parameter(s) related to the condition of the monitored system reaches a pre-specified value. Its purpose is to prevent the efficiency of the system from deteriorating to an unacceptable condition or the system stop working completely, due to the ageing or deterioration of the system. It is therefore important to assess the status or remaining useful life of a system, which can further be used in deciding the future operation in order to maintain the system at a certain level of availability.

In the real world, an engineering system is normally composed of multiple components. If the deterioration process of each component can be observed and modelled by a stochastic process, the deterioration process of the system forms a stochastic process. Sun et al.

(2017) optimised maintenance policies when the combination of multi-deterioration processes is assumed nonlinear using a Markov decision process for a k-out-of-n system, in which the deterioration process of each component follows the Wiener process. Other research considers multi-component systems under linear combinations. A real example is pavement defects, as discussed in Wu and Castro (2020), where the deterioration of a pavement was due to different defects such as cracking and potholes. Similar research could be seen in Coraddu et al. (2016) and Cheng et al. (2019). Zhang et al. (2018) discussed the application of both non-linear and linear Wiener processes degradation processes. Wu and Castro (2020) proposed the concept of the cost process for the scenario in which maintenance policies are considered for a linear combination of multiple gamma processes. Nevertheless, the concept of the cost process, has not been studied in other scenarios, including a linear combination of multiple Wiener processes. This knowledge gap is the main motivation for this research.

1.1 Related work

In the literature, publications relating to CbM are enormous. For example, Li and Nilkitsaranont (2009) proposed the combined regression techniques for CbM to assess the remaining useful life of gas turbine engines, which improves engine reliability and availability and reduces life cycle costs. Coraddu et al. (2016) used some machines approaches to effectively predicting potential failures of naval propulsion plants. Other research such as Zhu et al. (2015) presented a deterioration model that included two system deterioration processes: wear and shock and presents an optimal maintenance policy for the minimal cost criterion. These studies pointed a direction for future research: how to build a model which is more suitable for a complex system with multiple components or failure types.

As aforementioned, in existing literature, stochastic processes such as the gamma process citeplawless2004covariates,wu2020maintenance,wang2022condition, the inverse Gaussian process (Li et al., 2017; Chen et al., 2015), and the WP (Ebrahimi, 2005; Wen et al., 2018; Pedersen and Vatn, 2022) are widely used for different applications in CbM. Plenty of research is concentrated on the combination approach to dealing with the increasingly complex system (see Galar et al. (2013); Feng et al. (2017); Chang et al. (2019), for example). Caballé et al. (2015) proposed a condition-based maintenance policy by combining the non-homogeneous Poisson process(NHPP) and the gamma process(GP). They modelled a multiple deterioration processes with dependent deterioration-threshold-shock models. This was a typical example for multi-failure modes. It carried out two incremental processes in two different methodologies, so as to achieve the situation where the decline mode of a single system changes. They also pointed out that the dependence analysis between the causes of failures was a potential development and the variability of the threshold should be considered in future. Zhu et al. (2015) simulated the wear damage by a non-stationary gamma process and the random shock damage with a generalized Pareto distribution following Poisson arrivals. They derived the mathematical expression of the stationary behaviour of the system and calculated the long-term average total cost by using the semi-regenerative properties. It is worthwhile to notice that this study did not consider the impact of shocks or inspection costs which may influence the result of a long-term optimised maintenance policy. Liu et al. (2017a) proposed a new CbM model based on three-state deterioration and the influence of external environmental shocks. The deterioration process of the system was modelled by a two-state WP with a double stochastic Poisson process (DSPP). It considered two different thresholds, namely a normal threshold and a defective threshold, both of which depends on the system state. Other common methods such as the geometric process, regression analysis, artificial neural networks and support vector regression can be seen in these examples: Dong et al. (2014); Liu et al. (2017b); Lo et al. (2019). Zhang et al. (2018) reviewed some developments and applications of the WP. They also summarized some challenges and problems which mainly include: the WP with multiple time-scales, the WP integrating various types of data, the WP with state recoveries and the WP with non-Markovian feature. Change points on deterioration modelling and prognostics were largely occur randomly. Yang et al. (2019) proposed a two-phase preventive maintenance policy for a single-component system. The first stage was the imperfect maintenance phase which aims to keep the system working. The second stage was the postponed replacement phase which considers a preventive replacement. This meant that this maintenance policy would be sufficient and flexible for resource allocation due to its phase variability. Zhao et al. (2021) proposed a multi-criteria mission abort policy that considered the normal and defective stages based on the time threshold. They also indicated that performance of the optimal policy was compared in detail against several heuristic policies. Besides, the dynamic risk for controlling policy was also a possible extension for phased mission systems. Liu et al. (2021a) proposed a condition-based maintenance model in a finite-time horizon that consider a system with two heterogeneous dependent components with economic dependence. Moreover, this research pointed that the two-unit system in this paper could be extended to multi-unit systems by generalizing the deterioration process and Bellman equation, and the maintenance level could be extended to imperfect repair in future. For a multi-component system, in which each component had an observable deteriorating process, Wu and Castro (2020) developed a weighted linear combination of deterioration processes to optimise the time interval of maintenance for a pavement network.

Most existing maintenance policy optimisation approaches, such as Zhang et al. (2022a), Shi et al. (2020), and Liu et al. (2021b), aimed to minimise the relevant cost.

For a component in a system, it may have different failure modes. The deterioration process of a system with different failure modes can be modelled by multiple deterioration processes. Maintenance policies on such systems have been discussed in several papers. Zhu et al. (2016) studied the maintenance policies of a multi-component system with two independent failure modes. Qiu et al. (2017) considerd an optimal maintenance policy by both maximizing steady-state availability and minimizing long-term average cost for a system with multiple failure modes. They assumed that failure modes are independent. Zheng and Makis (2020) considered the failure state of a system changed from a soft failure to a hard failure and assumes that under different state, different maintenance activities can be taken (such as corrective replacement for soft failure and minimal repair for hard failure). Pedersen and Vatn (2022) considered a risk-averse decision maker of the CbM based on the Wiener process. They pointed out that a policy for reducing the cost of renewals or replacements may increase the risk of long downtime, and associated losses cannot be ignored. Zhang et al. (2022b) used the Wiener process to predict the remaining useful life of a system. The random effect of the operating environments and loading conditions were estimated by a continuous-time random walk.

In what follows, for convenience of expression, we regard the term *components* and *failure modes* interchangeable. That is, a system is composed of n components, or the deterioration process of a system is composed of n failure modes.

1.2 Novelty and contributions

From the above review, there is a need to explore the problem of the deterioration process of multi-component systems with. Consequently, this paper investigates the cost process relating to the linear combination, based on which maintenance policies are developed.

Hence, the contributions of this paper includes

- development of a cost process related to the linear combination of the deterioration processes.;
- development of maintenance policies for a system whose cost process can be modelled by a linear combination of Wiener processes.

1.3 Overview

The remainder of this paper is structured as follows. Section 2 describes notation and assumptions used in this paper. Section 3 develops deterioration processes and cost processes. Section 4 describes our maintenance policies under four situations. Section 5 shows some numerical examples. Section 6 concludes the paper.

2 Notation and Assumptions

2.1 Notation

Table 1 shows the notations used in this paper.

2.2 Assumption

- The system is new at time $t = 0$.
- Replacement is carried out every T_a units of time for age replacement policy or T_b for block replacement policy.
- Degradation processes of different failure modes are modelled by Wiener processes with different parameters.
- The deterioration process of each component develops from time $t = 0$. When a linear combination of the magnitudes of the deterioration exceeds a pre-specified value, the system needs replacement.
- The deterioration processes are independent from each other.

Although we assume that the deterioration processes are independent, other existing studies have discussed the different dependences between components of a multi-components system.

Tian and Liao (2011) proposed a proportional hazards model based CbM policy with the economic dependency among different components. In their work, the components are

k	Index of the k failure mode.
n	Number of components, or failure modes, in the system under consideration, $k = 1, 2, \ldots, n$
$X_k(t)$	Degradation state of kth failure modes at time t.
$Y(t)$	Overall deterioration of one system at time t.
μ_k	Drift of kth failure modes.
σ_k	Infinitesimal variance of kth failure modes.
a_k	Weight of failure mode k.
μ_Y	Drift of the overall deterioration of one system.
σ_Y	Infinitesimal variance of the overall deterioration of one system.
c_k	PM cost for every unit of the kth failure modes.
$U(t)$	Overall cost of a system at time t.
L	Threshold of the deterioration level for a system.
L_c	Threshold of the cost for a system.
$C_k(t)$	Total repair cost of the kth failure modes at t.
$C_{A,i}(T_a)$	Expected cost per time unit under the age replacement policy.
$C_{B,i}(T_b)$	Expected cost per time unit under the age replacement policy.
c_m	Expected repair cost incurred due to failures
c_r	Expected replacement cost
T_a	Interval time for the age replacement policy.
T_b	Interval time for the block replacement policy.

Table 1: Notation table

independent in their degradation and failure processes. They assumed that different components had different thresholds for determining which component should be preventively maintained. Song et al. (2014) studied the deterioration process of multi-components system under shocks. The number of shocks, which is caused by one component, has an effect on other components. The larger sum of the shocks leads to larger probabilities of failures. Li et al. (2016) considered both of stochastic dependence and economic dependences. The former is modelled by Levy copulas, and it will influenced by different dependence degrees. The latter will influence the performance of several maintenance policies, and the policy with the smallest long-term cost would be chosen by its decision rule. Liu et al. (2020) considered a life cycle cost model with multiple dependent degradation processes with random effect, which is due to environment. The dependence of the degradation process is evaluated by a copula in their work.

3 Model development

3.1 Deterioration process

We assume that the system has k deterioration processes, each of which follows a WP.

Let $X_k(t)$ be the deterioration level of the kth deterioration process at time t, where $k = 1, 2, \ldots, n$. Then, $X_k(t)$ have the following assumptions:

- $X_k(0) = 0$, which also means that $W_k(0) = 0$;
- $W_k(t)$ has independent increments that follows the normal distribution. That is,

for $0 < s < t$, $W_k(t-s) - W_k(s)$ follows $N(0, (t-s))$.

- $W_k(t)$ is continuous in t.

$X_k(t)$ is said having drift coefficient μ_k and variance parameter σ_k^2, the associated stochastic process is:

$$X_k(t) = \mu_k t + \sigma_k W_k(t), \tag{1}$$

where μ_k and σ_k are the parameters of failure mode k, respectively, $W_k(\cdot)$ is the standard WP, which also can be called as the Brownian motion. The estimation method of parameters can be seen in Shah et al. (2013).

3.1.1 Basic Properties

The unconditional probability density function, which follows the normal distribution with mean $= 0$ and variance $= t$, at a fixed time t:

$$f_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}.$$

We have $E[W_k(t)]/ = 0$ and $\mathrm{Var}[W_k(t)] = t$.

These results follow immediately from the definition that increments have a normal distribution, centred at zero.

Thus, the expected value and the variance of $X_k(t)$ are given by: $E(X_k(t)) = \mu_k t$ and $V(X_k(t)) = \sigma_k^2 t$.

3.1.2 A linear combination of WPs

Now let us assume $Y(t)$ is a linear combination of n WPs. The overall deterioration $Y(t)$ of the system is represented by

$$Y(t) = \sum_{k=1}^{n} a_k X_k(t), t \geqslant 0, a_k \geqslant 0, \tag{2}$$

where a_k is the weight of failure mode k. Fig. 1 shows the realisation of a linear combination of two WPs.

Furthermore, the overall deterioration process $\{Y(t), t > 0\}$ is a stochastic process with the following properties (without the skew-normal random effects):

- $Y(0) = \sum_{k=1}^{n} a_k X_k(0) = 0$,
- $\Delta Y(t) = \sum_{k=1}^{n} a_k \Delta X_k(t)$ is an independent increment as well.

Thus, $Y(t)$ is given by

$$Y(t) = t \sum_{k=1}^{n} a_k \mu_k + \sum_{k=1}^{n} a_k \sigma_k W_k(t). \tag{3}$$

Let $\mu_Y = \sum_{k=1}^{n} a_k \mu_k$ and $\sigma_Y^2 = \sum_{k=1}^{n} a_k^2 \sigma_k^2$. Then $Y(t)$ follows the normal distribution $N(\mu_Y t, \sigma_Y^2 t)$.

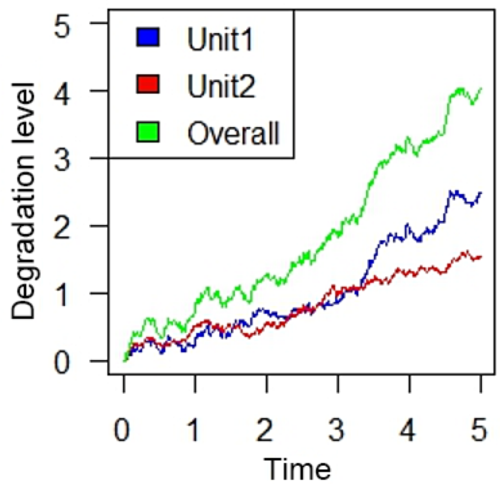

Figure 1: Example of two deterioration processes and a linear combination

3.1.3 First time t to exceed the pre-specified threshold L

The distribution of the first hitting time of the process $\{Y(t), t \geq 0\}$, which starts from $Y(0) = 0$ should be obtained. The first hitting time $\omega_{Y(t)}$ is defined when $Y(t)$ reaches the deterioration level L, according to the statistical characteristic of a WP, the first-passage-time, which is $\omega_{Y(t)}$, follows an inverse Gaussian distribution (Ross et al., 1996; Pan et al., 2017; Ye and Chen, 2014), then

$$\omega_L = \inf\{t > 0 \colon Y(t) \geq L\}, \tag{4}$$

then, the pdf of ω_L can be obtained by

$$\begin{aligned} f_{\omega_L}(t) &= \frac{L}{\sigma_Y\sqrt{2\pi t^3}}\exp(\frac{-(L-\mu_Y t)^2}{2{\sigma_Y}^2 t}) \\ &= \frac{L}{\sigma_Y\sqrt{\pi t^3}}\phi(\frac{-(L-\mu_Y t)}{\sigma_Y\sqrt{t}}), \end{aligned} \tag{5}$$

where $\phi(\cdot)$ denotes the standard normal pdf. Then, the cdf of ω_L is obtained by

$$\begin{aligned} F_{\omega_L}(t) &= P(Y(t) \geq L) \\ &= \Phi(\frac{-(L-\mu_Y t)}{\sigma_Y\sqrt{t}}) - \exp(\frac{2\mu_Y L}{{\sigma_Y}^2}), \end{aligned} \tag{6}$$

where $\Phi(\cdot)$ denotes the standard normal cdf.

3.2 Repair cost process

The repair costs of different failure modes are normally different. We consider that the actual cost is dependent on the deterioration level of the failure model. For example, the repair cost for a system with longer usage time is normally higher than a system

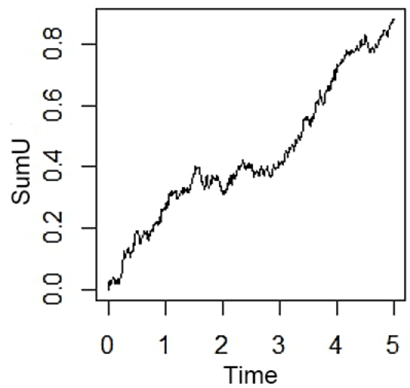

Figure 2: Example of the cost process of C(t)

with shorter usage time. Several references have considered this situation, see Liu et al. (2017a); Wu and Castro (2020), for example. It is worth noticing that, according to Wu and Castro (2020), the total cost $U(t)$, which is associated to $Y(t)$, is also a stochastic process and does not have a linear relationship with $Y(t)$. As $Y(t)$ is a WP, $U(t)$ is a WP which is a sum of $Y(t)$ with a drift.

Thus, the maintenance cost for the kth failure mode which is related to the deterioration level is given by,

$$C_k(t) = a_k c_k X_k(t), \tag{7}$$

where c_k is the maintenance cost for the kth failure mode. Then, the total cost of the whole system with multiple components or failure modes is given by

$$U(t) = \sum_{k=1}^{n} C_k(t) = \sum_{k=1}^{n} a_k c_k X_k(t), \tag{8}$$

where $U(t)$ is a WP with a linear drift related to its deterioration level.

3.2.1 Basic Properties

As $X_k(t)$ follows the normal distribution with mean $= \mu_k t$ and variance $= \sigma_k^2 t$, the expected value and the variance of $C_k(t)$ are given by: $E(C_k(t)) = a_k c_k \mu_k t$ and $V(C_k(t)) = a_k c_k^2 \sigma_k^2 t$.

Then $U(t)$ has expected value and variance,

$$E(U(t)) = \sum_{k=1}^{n} a_k c_k \mu_k t = \mu_U, \tag{9}$$

and

$$V(U(t)) = \sum_{k=1}^{n} a_k^2 c_k^2 \sigma_k^2 t = \sigma_U^2, \tag{10}$$

respectively.

Obviously, both of $Y(t)$ and $U(t)$ have the same values μ_k and σ_k, respectively, so the covariance between $Y(t)$ and $U(t)$ is given by

$$\begin{aligned}\mathrm{Cov}(Y(t),U(t)) &= \mathrm{Cov}(\sum_{k=1}^{n} a_k X_k(t) \sum_{j=1}^{n} c_k X_j(t)) \\ &= \sum_{k=1}^{n}\sum_{j=1}^{n} a_k c_k \mathrm{Cov}(X_k(t), X_j(t)) \\ &= \sum_{k=1}^{n} a_k c_k \mu_k^2 t. \end{aligned} \tag{11}$$

The characteristic function of the bivariate normal distribution is given by

$$\begin{aligned}&\phi_{(Y(t),U(t))}(t_1,t_2) = \mathbb{E}[\exp(it_1 Y(t) + it_2 U(t))] \\ &= \mathbb{E}[\exp(it_1 \sum_{k=1}^{n} a_k X_k(t) + it_2 \sum_{k=1}^{n} a_k c_k X_k(t)] \\ &= \mathbb{E}[\exp(i \sum_{k=1}^{n} (a_k t_1 + a_k c_k t_2) X_k(t))] \\ &= \mathbb{E}[\exp(i \sum_{k=1}^{n} (a_k t_1 + a_k c_k t_2) X_k(t))] \\ &= \prod_{k=1}^{n} \mathbb{E}[\exp(i(a_k t_1 + a_k c_k t_2) X_k(t))] \\ &= \prod_{k=1}^{n} \phi_{X_k(t)}(a_k t_1 + a_k c_k t_2), \end{aligned} \tag{12}$$

then we can obtain

$$f_{Y(t),U(t)}(y,u) \tag{13}$$

$$\begin{aligned}&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{(Y(t),U(t))}(t_1,t_2) e^{-it_1 y - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2 \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\prod_{k=1}^{n} \phi_{X_k(t)}(a_k t_1 + c_k t_2))^{-it_1 y - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2, \end{aligned} \tag{14}$$

the conditional probability $f_{U(t)|Y(t)(y,u)}$ is hence obtained by

$$\begin{aligned}f_{U(t)|Y(t)(y,u)} &= \frac{f_{U(t),Y(t)(y,u)}}{f_{Y(t)}(y)} \\ &= \frac{1}{4\pi^2 f_{Y(t)}(y)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\prod_{k=1}^{n} \phi_{X_k(t)}(a_k t_1 + c_k t_2))^{-it_1 y - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2. \end{aligned} \tag{15}$$

where

$$\phi_{X_k(t)}(a_k t_1 + c_k t_2) = \exp\{\frac{\sigma_k[1-(1-2i{\mu_k}^2(a_k t_1 + a_k c_k t_2)\sigma_k^{-1})^{1/2}]}{\mu_k}\}. \tag{16}$$

3.2.2 First time to exceed the pre-specified threshold L_U

However, if we consider a real situation: after a period of time, $U(t)$ becomes so high that using a new piece of equipment to replace the old one may be a better choice. Also, the owner of the equipment may have an expectation overall cost: when $U(t)$ is larger than this expectation, they will buy a new piece of equipment. For example, we assume this expectation cost is L_U, which will be described in the next section. Similarly, we define

$$\omega_U = \inf\{t > 0 : U(t) \geq L_U\}, \tag{17}$$

Then, the pdf of ω_U can be obtained as

$$\begin{aligned} f_{\omega_U}(t) &= \frac{L_U}{\sigma_U\sqrt{2\pi t^3}}\exp(\frac{-(L_U-\mu_U t)^2}{2\sigma_U^2 t}) \\ &= \frac{L_U}{\sigma_U\sqrt{\pi t^3}}\phi(\frac{-(L_U-\mu_U t)}{\sigma_U\sqrt{t}}). \end{aligned} \tag{18}$$

Then, the cdf of ω_U is obtained by

$$F_{\omega_{L_U}}(t) = P(U(t) \geq L_U) = \Phi(\frac{-(L_U-\mu_U t)}{\sigma_U\sqrt{t}}) - \exp(\frac{2\mu_U L_U}{\sigma_U^2}). \tag{19}$$

4 Maintenance policies

In this section, we will consider the maintenance policy under age replacement and block replacement policies.

We consider the following four maintenance policies:

- *Maintenance Policy A:* Under the deterioration process, when the deterioration level exceeds the pre-specified threshold L, then maintenance activities will be taken. We denote this event as A_1.
- *Maintenance Policy B:* Under the cost process, when the cost level exceeds the pre-specified threshold L_U, then maintenance activities will be taken. We denote this event as A_2.
- *Maintenance Policy C:* Only if both A_1 and A_2 have occurred, the age replacement will be conducted. Denote this event as $A_3 = A_1 \cap A_2$.
- *Maintenance Policy D:* If one of the two events, A_1 and A_2, occurs, the age replacement will be conducted. Denote this event as $A_4 = A_1 \cup A_2$.

Therefore, $G_1(t) \coloneqq P(A_1) = F_{\omega_L}(t)$ and $G_2(t) \coloneqq P(A_2) = F_{\omega_{L_U}}(t)$ and these functions can be obtained

$$
\begin{aligned}
G_3(t) &\coloneqq P(A_3) \\
&= P(A_1 \cap A_2) \\
&= P(A_1)P(A_2|A_1) \\
&= F_{\omega_L}(t)F_{\omega_{L_U}}(t|\omega_L), \qquad (20)
\end{aligned}
$$

and

$$
\begin{aligned}
G_4(t) &\coloneqq P(A_4) \\
&= P(A_1 \cup A_2) \\
&= P(A_1) + P(A_2) - P(A_1 \cap A_2) \\
&= P(A_1) + P(A_2) - P(A_3), \qquad (21)
\end{aligned}
$$

where symbol $\coloneqq$ is used to denote a definition.

We have already obtained the conditional probability $f_{U(t)|Y(t)(y,u)}$, using $f_{\omega_L}(t)$ and $f_{\omega_{L_U}}(t)$ to replace $f_{Y(t)}$ and $f_{U(t)}$, respectively, then

$$
\begin{aligned}
f_{\omega_{L_U}|\omega_L(y,u)} &= \frac{f_{\omega_{L_U},\omega_L(y,u)}}{f_{\omega_L}(y)} \\
&= \frac{1}{4\pi^2 f_{\omega_L}(y)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\prod_{k=1}^{n} \phi_{X_k(t)}(a_k t_1 + c_k t_2))^{-it_1 y - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2, \qquad (22)
\end{aligned}
$$

where

$$
\phi_{X_k(t)}(a_k t_1 + c_k t_2) = \exp\{\frac{\sigma_k[1-(1-2i\mu_k^2(a_k t_1 + a_k c_k t_2)\sigma_k^{-1})^{1/2}]}{\mu_k}\}, \qquad (23)
$$

and

$$
\begin{aligned}
F_{\omega_{L_U}}(t|\omega_L) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\omega_L}^{-1}(t)(\prod_{k=1}^{n}\phi_{X_k(t)}(a_k t_1 + c_k t_2))^{-it_1 t - it_2 u}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t \\
&= \frac{\ln f_{\omega_L}(t)}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\prod_{k=1}^{n}\phi_{X_k(t)}(a_k t_1 + c_k t_2))^{-it_1 t - it_2 u}\mathrm{d}t_1\mathrm{d}t_2. \qquad (24)
\end{aligned}
$$

Therefore, the distribution of both $G_3(t)$ and $G_4(t)$ can be obtained.

The distribution of $G_3(t)$ is given by

$$
G_3(t) \coloneqq \frac{F_{\omega_L}(t)\ln f_{\omega_L}(t)}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\prod_{k=1}^{n}\phi_{X_k(t)}(a_k t_1 + c_k t_2))^{-it_1 t - it_2 u}\mathrm{d}t_1\mathrm{d}t_2, \qquad (25)
$$

and $G_4(t)$ now can be presented by

$$
G_4(t) \coloneqq F_{\omega_L}(t) + F_{\omega_{L_U}}(t) - \frac{F_{\omega_L}(t)\ln f_{\omega_L}(t)}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\prod_{k=1}^{n}\phi_{X_k(t)}(a_k t_1 + c_k t_2))^{-it_1 t - it_2 u}\mathrm{d}t_1\mathrm{d}t_2. \qquad (26)
$$

4.1 Age replacement policy

For the age replacement policy, a preventive replacement is conducted after a continuous working time T_a when there is no failure occurs (Barlow and Hunter, 1960).

- The replacement time interval is T_a.
- Immediately after a preventive or corrective maintenance, the system rests its age to 0.
- Both c_r and c_m are constants.

Then the mean time between replacements $M(T_a)$ will be

$$\begin{aligned} M(T_a) &= \int_0^{T_a} tf(t)dt + t_0 P(X > T_a) \\ &= \int_0^{T_a} tf(t)dt + t_0(t - F(T_a)) \\ &= \int_0^{T_a} (1 - F(t))dt. \end{aligned} \tag{27}$$

Then, the expected cost per time unit is given by

$$C_{A,i}(T_a) = \frac{c_r + c_m G_i(T_a)}{\int_0^{T_a}(1 - G_i(t))dt}, \tag{28}$$

where $i = 1, 2, 3, 4$, corresponding to maintenance policies A, B, C, and D, respectively and T_a is the decision variable, c_r is the expected replacements cost and c_m is the expected repair cost incurred due to failures.

Property 1. *For given t, if $G_1(t) \geq G_3(t)$, $G_2(t) \geq G_3(t)$, $G_1(t) \leq G_4(t)$ and $G_2(t) \leq G_4(t)$, then $C_{A,1}(T_a) \geq C_{A,3}(T_a)$, $C_{A,2}(T_a) \geq C_{A,3}(T_a)$, $C_{A,1}(T_a) \leq C_{A,4}(T_a)$, and $C_{A,2}(T_a) \leq C_{A,4}(T_a)$.*

Proof. Since $G_1(t) \geq G_3(t)$, $c_r + c_m G_1(T_a) \geq c_r + c_m G_3(T_a)$ and $\int_0^{T_a}(1 - G_1(t))dt \leq \int_0^{T_a}(1 - G_3(t))dt$. Hence, $C_{A,1}(T_a) = \dfrac{c_r + c_m G_1(T_a)}{\int_0^{T_a}(1 - G_i(t))dt} \geq \dfrac{c_r + c_m G_3(T_a)}{\int_0^{T_a}(1 - G_3(t))dt} = C_{A,3}(T_a)$.

Similar proofs can be established on the other inequality. ■

By minimising $C_{A,i}(T_a)$, we can obtain the optimum T_a^* for the age replacement policy based on maintenance policies A, B, C, and D, respectively.

4.2 Block replacement policy

For the block replacement policy, which is introduced by Barlow and Hunter (1960), a unit is replaced at a scheduled time regardless of time since its last repair. Any failure between replacements will be repaired with the minimal repair, which restores the failed system to the status just before the failure occurred.

We have following assumptions.

- The inspection will be taken every T_b.
- Immediately after a preventive or corrective maintenance, the system rests its age to 0.
- Both c_r and c_m are constants.

Then, the expected cost per time unit for the block replacement policy is given by

$$C(T) = \frac{c_r + c_m M(T)}{T}, \tag{29}$$

where $M(t)$ is a renewal functions. To approximate this renewal function, given a

$$C_{B,i}(T_b) = \frac{c_r + c_m M_{\omega_L}(T_b)}{T_b}, \tag{30}$$

where $M_{\omega_L}(T_b)$ is the expected number of failed units with the CDF (cumulative distribution function) $F_{\omega_L}(t)$, during the interval $(0, T_b]$, c_r is the replacement cost and c_m is the maintenance cost. Assume that the replacement interval is so short that the probability of two or more failures occurring within $(0, T_b)$ is zero. Denote that $N(T_b)$ is the number of failures within an interval of length T_b, then

$$B(T_b) = E[M_{\omega_L}(T_b)], \tag{31}$$

then the expected cost per time unit is given by

$$C_{B,i}(T_b) = \frac{c_r + c_m B(T_b)}{T_b}, \tag{32}$$

According to our four maintenance policies, then

$$C_{B,i}(T_b) = \frac{c_r + c_m B_i(T_b)}{T_b}, \tag{33}$$

where $i = 1, 2, 3, 4$, corresponding to maintenance policies A, B, C, and D, the optimal scheduled replacement time T_b could be obtained by minimizing the $C_{B,i}(T_b)$. Similarly, we can obtain this property.

Property 2. *For given t, $G_1(t) \geq G_3(t)$, $G_2(t) \geq G_3(t)$, $G_1(t) \leq G_4(t)$, and $G_2(t) \leq G_4(t)$, then $C_{B,1}(T_b) \geq C_{B,3}(T_b)$, $C_{B,2}(T_b) \geq C_{B,3}(T_b)$, $C_{B,1}(T_b) \leq C_{B,4}(T_b)$, and $C_{B,2}(T_b) \leq C_{B,4}(T_b)$.*

5 Numerical examples

We consider a system with two different failure modes. The deterioration process of the two failure modes is modelled with two WPs, respectively, each of which has different parameters α, β and σ. We assume that two modes have weights as following $a_1 = 0.3$ and $a_2 = 0.7$. α_1 , β_1 and σ_1 are 0.8, 0.5 and 0.2 for the first failure mode, respectively. α_2, β_2 and σ_2 are 0.7,1 and 0.5, respectively. We also assume that $c_r = 100$ and $c_m = 50$, then we can obtain the following result.

Thus, the linear combination of the two processes is given by

$$Y(t) = 0.3X_1 + 0.7X_2.$$

We assume that the system needs to be repaired when the deterioration levels exceed the threshold L_{w_L} and the threshold $L_{w_{L_c}}$, respectively. Replacement activities will be taken and the deterioration level will be restored to zero when the component is completely replaced. We obtain the result under $L_{w_L} = \{3, 3.5, 2\}$ and $L_{w_{L_c}} = \{1.5, 1, 2.5\}$ under policies A, B, C and D, respectively. It is worth noticing that all parameters can be estimated based on historical data or expert elicitation (Shah et al., 2013).

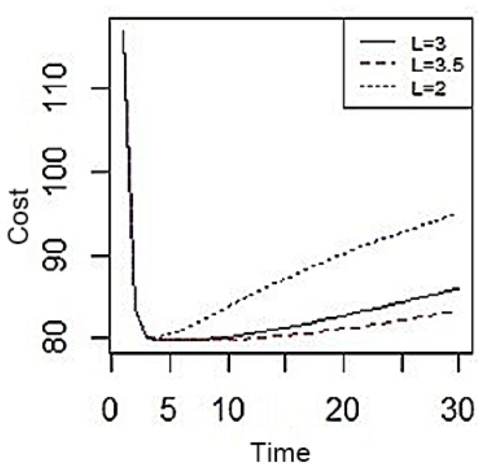

Figure 3: Maintenance Policy A

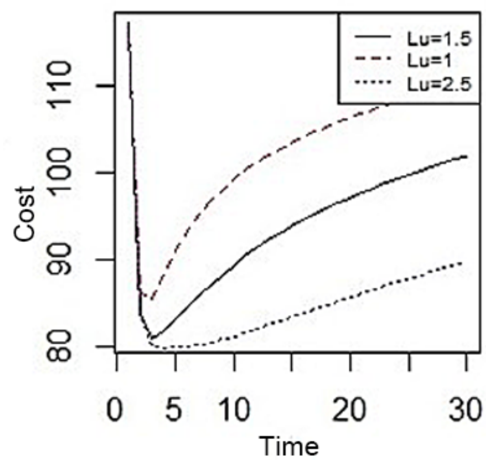

Figure 4: Maintenance Policy B

Figure 3 shows the expected cost per unit time under the maintenance policy A.

- When the threshold L_{w_L} is 3, the optimised time interval is ($T_{opt} = 4.318$) and the expected unit cost per time is 79.793.
- When the threshold L_{w_L} is 3.5, the optimised time interval is ($T_{opt} = 4.745$) and the expected unit cost per time is 79.789.
- When the threshold L_{w_L} is 3, the optimized time interval is ($T_{opt} = 3.410$) and the expected unit cost per time is 79.966

Figure 4 shows the expected cost per unit time under the maintenance policy B.

- When the threshold $L_{w_{L_c}}$ is 1.5, the optimized time interval is ($T_{opt} = 2.934$) and the expected unit cost per time is 80.787.
- When the threshold $L_{w_{L_c}}$ is 1, the optimized time interval is ($T_{opt} = 2.483$) and the expected unit cost per time is 84.731.
- When the threshold $L_{w_{L_c}}$ is 2.5, the optimized time interval is ($T_{opt} = 3.874$) and the expected unit cost per time is 79.817.

Figure 5 shows the expected cost per unit time under the maintenance policy C.

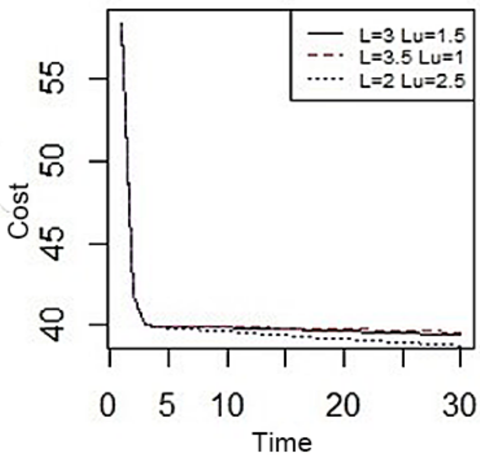

Figure 5: Maintenance Policy C

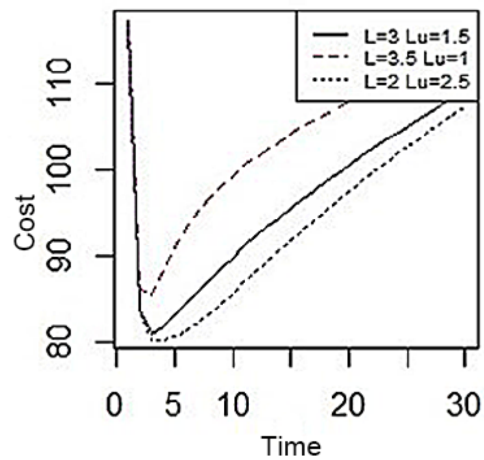

Figure 6: Maintenance Policy D

- When the thresholds for L_{w_L} and $L_{w_{L_c}}$ are 3 and 1.5, respectively, the expected unit cost per time is 39.85863.
- When the thresholds for L_{w_L} and $L_{w_{L_c}}$ are 3.5 and 1, respectively, the expected unit cost per time is 39.88409.
- When the thresholds for L_{w_L} and $L_{w_{L_c}}$ are 2 and 2.5, respectively, the expected unit cost per time is 39.62653.

Figure 6 shows the expected cost per unit time under the maintenance policy D.

- When the thresholds for L_{w_L} and $L_{w_{L_c}}$ are 3 and 1.5, respectively, the optimized time interval is ($T_{opt} = 2.934$) and the expected unit cost per time is 80.787.
- When the thresholds for L_{w_L} and $L_{w_{L_c}}$ are 3.5 and 1, respectively, the optimized time interval is ($T_{opt} = 2.483$) and the expected unit cost per time is 84.731.
- When the thresholds for L_{w_L} and $L_{w_{L_c}}$ are 2 and 2.5, respectively, the optimized time interval is ($T_{opt} = 3.360$) and the expected unit cost per time is 79.994.

Figure 7 shows the comparison among policy A, B, C and D. Table 2 is the optimized result which is related to Figure 7.

Optimized result	**A1**	**A2**	**A3**	**A4**
Optimized expected unit cost per time	79.803	81.518	39.819	81.518
Time interval	4.024	2.777	-	2.777

Table 2: Comparison result among policy A, B, C and D

Then, we set 10 scenarios. The following table shows parameters we used for these 10 scenarios. Table 3 shows parameters we used for 10 scenarios.

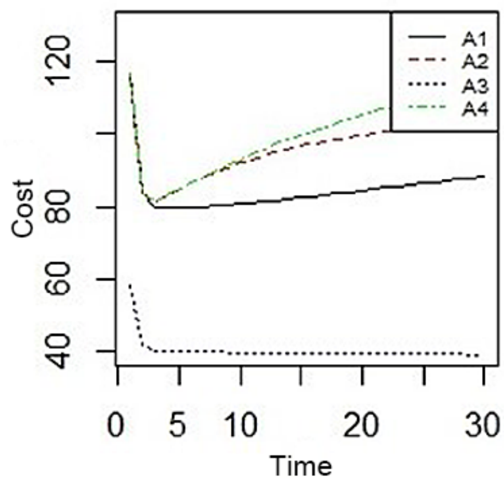

Figure 7: Comparison among policy A, B, C and D

Parameters	**S1**	**S2**	**S3**	**S4**	**S5**	**S6**	**S7**	**S8**	**S9**	**S10**
c_r	100	100	100	100	80	85	90	100	120	100
c_m	50	50	50	50	40	80	70	80	90	120
L	3.00	2.00	2.50	3.50	3.00	3.00	2.00	2.50	3.50	3.00
L_u	1.50	1.80	2.00	2.50	1.50	1.50	1.80	2.00	2.50	1.50
a_1	0.30	0.30	0.40	0.50	0.30	0.30	0.30	0.40	0.50	0.30
a_2	0.70	0.70	0.60	0.50	0.70	0.70	0.70	0.60	0.50	0.70
α_1	0.80	0.60	0.70	0.65	0.80	0.80	0.60	0.70	0.65	0.80
α_2	0.70	0.50	0.80	0.55	0.70	0.70	0.50	0.80	0.55	0.70
β_1	0.50	0.60	0.80	1.20	0.50	0.50	0.60	0.80	1.20	0.50
β_2	1.00	0.90	0.80	0.70	1.00	1.00	0.90	0.80	0.70	1.00
σ_1	0.20	0.40	0.60	0.80	0.20	0.20	0.40	0.60	0.80	0.20
σ_2	0.50	0.55	0.65	0.45	0.50	0.50	0.55	0.65	0.45	0.50

Table 3: Parameters for 10 scenario

- S1, S5, S6 and S10 have same parameters exclude the replacement cost and repair cost.
- S2 and S7 have same parameters exclude the replacement cost and repair cost.
- S3 and S8 have same parameters exclude the replacement cost and repair cost.
- S4 and S9 have same parameters exclude the replacement cost and repair cost.
- S1, S2, S3, S4 have same replacement cost and repair cost. However, other parameters are different.

The following table shows the expected cost per time unit with its time interval based on our 10 scenarios. The value outside the brackets is the optimized expected cost per time unit and the value inside the brackets is the time interval.

Scenario	A1	A2	A3	A4
S1	80.787(2.934)	80.787(2.934)	39.859	80.787(2.934)
S2	80.197(3.186)	80.534(3.018)	39.498	80.776(2.894)
S3	80.006(3.358)	80.621(2.988)	39.600	80.715(2.929)
S4	79.801(4.067)	80.019(3.354)	39.827	80.012(3.348)
S5	63.835(4.280)	64.769(2.870)	31.880	64.770(2.870)
S6	67.826(4.210)	69.154(2.752)	33.853	69.154(2.752)
S7	72.312(3.074)	72.731(2.897)	35.351	73.009(2.775)
S8	80.091(3.247)	80.956(2.856)	39.424	81.073(2.801)
S9	95.766(3.993)	96.099(3.258)	47.752	96.102(3.253)
S10	79.796(4.169)	81.644(2.682)	39.809	81.644(2.682)

Table 4: Numerical examples for 10 scenario

According to Table 4, we can find that the result is satisfied with property 1 in section 4, $C_{A,1}(T_a) \geq C_{A,3}(T_a), C_{A,2}(T_a) \geq C_{A,3}(T_a), C_{A,1}(T_a) \leq C_{A,4}(T_a)$, and $C_{A,2}(T_a) \leq C_{A,4}(T_a)$.

We compare these results from two aspects: the influence of cost and the influence of other parameter exclude cost. According to Table 4, we use results of S1, S5, S6, S10 for the first aspect and S1, S2, S3, S4 for the second aspect.

5.1 Comparison among S1, S5, S6, S10

We focus on the influence of cost in this part.

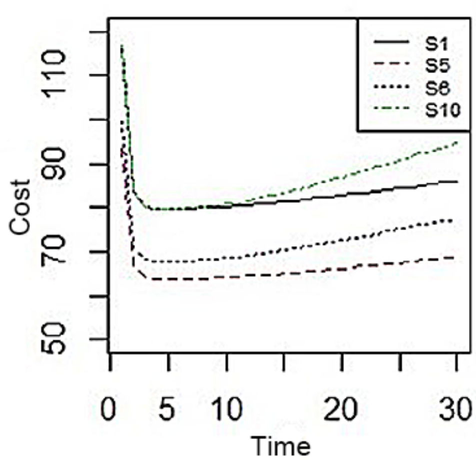

Figure 8: Policy A for S1, S5, S6 and S10

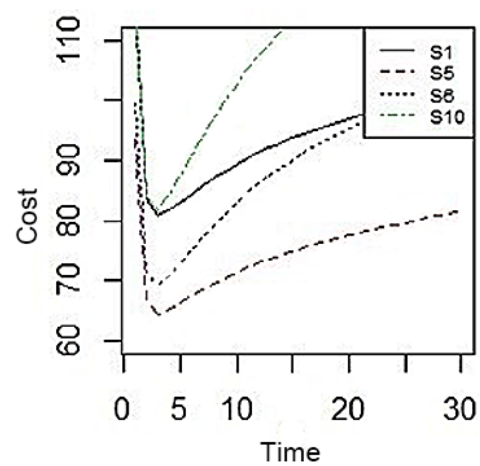

Figure 9: Policy B for S1, S5, S6 and S10

- According to Figures 8, 9 and 10, with the increase of cost, all of policy A, B and D have increasing expected cost.
- Among them, maintenance policy D is the most sensitive to price changes. The expected cost of S6 is gradually higher than that of S5.

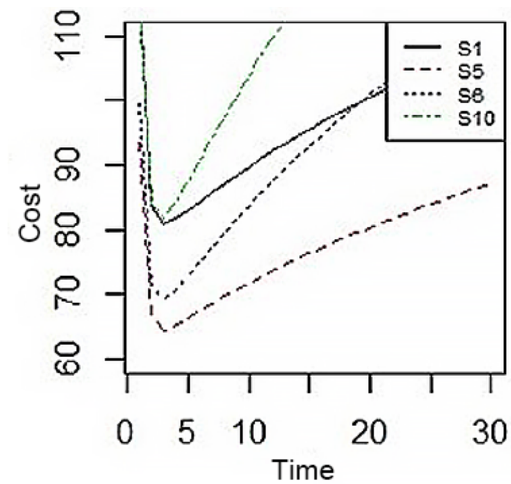

Figure 10: Policy D for S1, S5, S6 and S10

5.2 Comparison among S1, S2, S3, S4

We focus on the influence of other parameters exclude cost in this part.

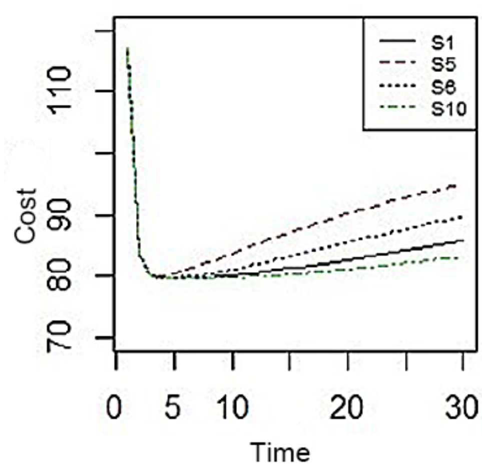

Figure 11: Policy A for S1, S2, S3 and S4

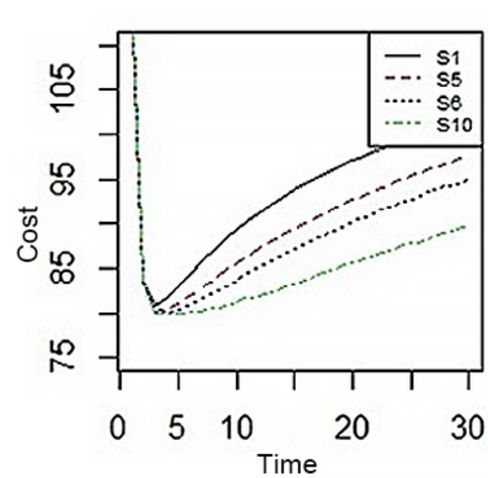

Figure 12: Policy B for S1, S2, S3 and S4

- According to Figure 11, the ratios of cost changing from the highest to the lowest are: $S5 > S6 > S1 > S10$.
- According to Figure 12, the ratios of cost changing from the highest to the lowest are: $S1 > S5 > S6 > S10$.
- According to Figure 13, the ratios of cost changing from the highest to the lowest are: $S1 > S5 > S6 > S10$ before the turning point $t = 10$ and $S5 > S1 > S6 > S10$ after the turning point.

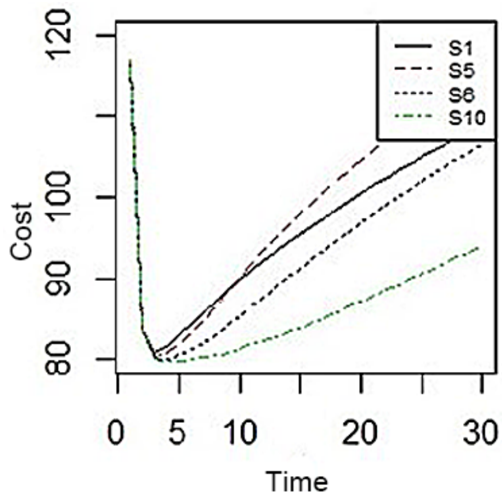

Figure 13: Policy D for S1, S2, S3 and S4

6 Conclusions

This paper investigated maintenance policies for a system whose deterioration process is a linear combination of Wiener processes. It proposed four maintenance policies with both degradation and cost thresholds for a multi-component system and then compared them. This paper also discussed two properties based on these four maintenance policies. Numerical examples were given to illustrate the optimisation process.

However, there are several limitations in our research.

1. The deterioration process of a system may be a non-linear combination of deterioration processes. A non-linear combination of deterioration processes based on other models, such as the gamma process and the geometric process, can be considered in future.

2. The dependence among failure modes or failure components has not been considered in this paper. Besides, the economic dependence is another possible problem for designing the maintenance policy. Such problems will be investigated in our future work.

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