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Classical Solutions of the Degenerate Fifth Painlevé Equation

Peter A. Clarkson

School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury, CT2 7FS, UK

E-mail: P.A.Clarkson@kent.ac.uk
Orcid: 0000-0002-8777-5284

Abstract. In this paper classical solutions of the degenerate fifth Painlevé equation are classified, which include hierarchies of algebraic solutions and solutions expressible in terms of Bessel functions. Solutions of the degenerate fifth Painlevé equation are known to be expressible in terms of solutions of the third Painlevé equation. The classification and description of the classical solutions of the degenerate fifth Painlevé equation is done using the Hamiltonian associated with third Painlevé equation. Two applications of these classical solutions are discussed, deriving exact solutions of the complex sine-Gordon equation and of the coefficients in the three-term recurrence relation associated with generalised Charlier polynomials.

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1. Introduction

In this paper we are concerned with solutions of the equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \left(\frac{1}{2w}+\frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2(\alpha w^2+\beta)}{z^2w} + \frac{\gamma w}{z}, \tag{1}$$

with α, β and γ constants. Equation (1) is the special case of the fifth Painlevé equation ($\mathrm{P_V}$)

$$\begin{aligned}\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = &\left(\frac{1}{2w}+\frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2(\alpha w^2+\beta)}{z^2w}\\ &+ \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}.\end{aligned} \tag{2}$$

with α, β, γ and δ constants, when $\delta = 0$ and is known as the *degenerate fifth Painlevé equation* (deg-$\mathrm{P_V}$), cf. [39].

The six Painlevé equations ($\mathrm{P_I}$–$\mathrm{P_{VI}}$), were discovered by Painlevé, Gambier and their colleagues whilst studying second order ordinary differential equations of the form

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = F\left(z, w, \frac{\mathrm{d}w}{\mathrm{d}z}\right), \tag{3}$$

where F is rational in $\mathrm{d}w/\mathrm{d}z$ and w and analytic in z. The Painlevé functions can be thought of as nonlinear analogues of the classical special functions. The general solutions of the Painlevé equations are transcendental in the sense that they cannot be expressed in terms of known elementary functions and so require the introduction of a new transcendental function to describe their solution. However, it is well known that $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ possess rational solutions, algebraic solutions and solutions expressed in terms of the classical special functions — Airy, Bessel, parabolic cylinder, Kummer and hypergeometric functions, respectively — for special values of the parameters, see, e.g. [11, 21] and the references therein. These hierarchies are usually generated from "seed solutions" using the associated Bäcklund transformations and frequently can be expressed in the form of determinants. These solutions of the Painlevé equations are often called "classical solutions", cf. [50, 51].

It is well known that solutions of deg-$\mathrm{P_V}$ (1) are related to solutions of the third Painlevé equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}x^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}q}{\mathrm{d}x} + \frac{Aq^2+B}{x} + Cq^3 + \frac{D}{q}, \tag{4}$$

with A, B, C and D constants, a result originally due to Gromak [20]; see also [21, §34]. The relationship between solutions of deg-$\mathrm{P_V}$ and the third Painlevé equation is given in Lemma 2.1 below. The objective of this paper is to give a classification and description of the classical solutions of deg-$\mathrm{P_V}$ (1) using the associated Hamiltonian formalism, rather than through solutions of the third Painlevé equation (4).

In §2, the relationship between deg-$\mathrm{P_V}$ (1) and the third Painlevé equation (4) is discussed using the associated Hamiltonian. In §3, classical solutions of the third Painlevé equation (4) are reviewed, the rational solutions in §3.1 and the Bessel function solutions in §3.2. In §4, Bäcklund transformations of deg-$\mathrm{P_V}$ (1) are given, which can be used to derive a hierarchy of solutions from a "seed solution". In §5, classical solutions of deg-$\mathrm{P_V}$ (1) are classified, the algebraic solutions in §5.1 and the Bessel function solutions in §5.2. In §6, two applications of classical solutions of deg-$\mathrm{P_V}$

(1) are given to derive exact solutions of the complex sine-Gordon equation, which is equivalent to the Pohlmeyer-Lund-Regge model, and to derive explicit representations of the coefficients in the three-term recurrence relation satisfied by generalised Charlier polynomials, which are discrete orthogonal polynomials.

2. The relationship between deg-$\mathrm{P_V}$ and $\mathrm{P_{III}}$

In the generic case when $CD \neq 0$ in the third Painlevé equation (4), we set $C = 1$ and $D = -1$, without loss of generality (by rescaling the variables if necessary), and so consider the equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}q}{\mathrm{d}x} + \frac{Aq^2+B}{x} + q^3 - \frac{1}{q}. \tag{5}$$

In the sequel, we shall refer to this equation as $\mathrm{P_{III}}$ since it is the generic case.

Consider the Hamiltonian associated with $\mathrm{P_{III}}$ (5) given by

$$\mathcal{H}_{\mathrm{III}}(q,p,x;a,b,\varepsilon) = q^2p^2 - xq^2p - (2a+2b+1)qp + \varepsilon xp + 2bxq, \tag{6}$$

with a and b parameters and $\varepsilon = \pm 1$, see [27, 43]. Then $p(x)$ and $q(x)$ satisfy the Hamiltonian system

$$x\frac{\mathrm{d}q}{\mathrm{d}x} = \frac{\partial \mathcal{H}_{\mathrm{III}}}{\partial p} = 2q^2p - xq^2 - (2a+2b+1)q + \varepsilon x, \tag{7}$$

$$x\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{\partial \mathcal{H}_{\mathrm{III}}}{\partial q} = -2qp^2 + 2xqp + (2a+2b+1)p - 2bx. \tag{8}$$

Solving (7) for $p(x)$ gives

$$p(x) = \frac{1}{2q^2}\left\{x\frac{\mathrm{d}q}{\mathrm{d}x} + xq^2 + (2a+2b+1)q - \varepsilon x\right\},$$

and then substituting this in (8) gives

$$\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}q}{\mathrm{d}x} + \frac{2(a-b)q^2}{x} + \frac{2\varepsilon(a+b+1)}{x} + q^3 - \frac{1}{q}. \tag{9}$$

which is $\mathrm{P_{III}}$ (5), with parameters

$$A = 2(a-b), \qquad B = 2\varepsilon(a+b+1). \tag{10}$$

Solving (8) for $q(x)$ gives

$$q(x) = \frac{1}{2p(x-p)}\left\{x\frac{\mathrm{d}p}{\mathrm{d}x} - (2a+2b+1)p + 2bx\right\},$$

and then substituting this in (7) gives

$$\begin{aligned}\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} = {} & \frac{1}{2}\left(\frac{1}{p} + \frac{1}{p-x}\right)\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)^2 - \frac{p}{x(p-x)}\frac{\mathrm{d}p}{\mathrm{d}x} + 2\varepsilon p - \frac{2b^2}{p}\\ & - \frac{4a^2-1}{2(p-x)} + \frac{1-4(a^2-b^2)-4\varepsilon p^2}{2x}.\end{aligned} \tag{11}$$

Then making the transformation

$$p(x) = \frac{2\sqrt{z}\,w(z)}{w(z)-1}, \qquad x = 2\sqrt{z}, \tag{12}$$

in (11) gives

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \left(\frac{1}{2w}+\frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2(a^2w^2-b^2)}{2z^2w} + \frac{\varepsilon w}{z}, \tag{13}$$

which is deg-$\mathrm{P_V}$ (1) with parameters

$$\alpha = \tfrac{1}{2}a^2, \qquad \beta = -\tfrac{1}{2}b^2, \qquad \gamma = \varepsilon. \tag{14}$$

Hence we have the following result; see also [21, Theorem 34.2].

Lemma 2.1. *If $q(x)$ is a solution of (9) then*

$$w(z) = \frac{xq'(x)+xq^2(x)+(2a+2b+1)q(x)-\varepsilon x}{xq'(x)-xq^2(x)+(2a+2b+1)q(x)-\varepsilon x}, \qquad z=\tfrac{1}{2}x^2, \tag{15}$$

with $'\equiv \mathrm{d}/\mathrm{d}x$, is a solution of (13), provided that

$$x\frac{\mathrm{d}q}{\mathrm{d}x} - xq^2 + (2a+2b+1)q - \varepsilon x \neq 0.$$

Conversely, if $w(z)$ is a solution of (13), then

$$q(x) = \frac{1}{2\sqrt{z}\,w}\left\{z\frac{\mathrm{d}w}{\mathrm{d}z} + (w-1)(aw+b)\right\}, \qquad x=\sqrt{2z}, \tag{16}$$

is a solution of (9).

Proof. Solving (7) for $p(x)$, substituting in (12) and solving for $w(z)$ gives (15). Also solving (8) for $q(x)$ and substituting (12) into the resulting expression gives (16). □

An alternative method of deriving solutions of (13) involves the second-order, second-degree equation satisfied associated with the Hamiltonian system (7,8), due to Jimbo and Miwa [27] and Okamoto [43], which is often called the "σ-equation".

Theorem 2.2. *If $\mathcal{H}_{\mathrm{III}}(q,p,x;a,b,\varepsilon)$ is given by (6), then*

$$\sigma(x;a,b,\varepsilon) = \mathcal{H}_{\mathrm{III}}(q,p,x;a,b,\varepsilon) + qp - \tfrac{1}{2}\varepsilon x^2 + (a+b)^2, \tag{17}$$

where $q(x)$ and $p(x)$ satisfy the system (7)-(8), satisfies the second-order, second-degree equation ($\mathrm{S_{III}}$)

$$\begin{aligned}\left(x\frac{\mathrm{d}^2\sigma}{\mathrm{d}x^2} - \frac{\mathrm{d}\sigma}{\mathrm{d}x}\right)^2 + 2\left\{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x}\right)^2 - x^2\right\}\left(x\frac{\mathrm{d}\sigma}{\mathrm{d}x} - 2\sigma\right)&\\ - 8\varepsilon(a^2-b^2)x\frac{\mathrm{d}\sigma}{\mathrm{d}x} &= 8(a^2+b^2)x^2.\end{aligned} \tag{18}$$

Conversely, if $\sigma(x;a,b,\varepsilon)$ satisfies (18) then the solution of the Hamiltonian system (7,8) is given by

$$\begin{aligned} q(x) &= \frac{\varepsilon x\sigma''(x) - \varepsilon(2a+2b+1)\sigma'(x) - 2(a-b)x}{x^2-[\sigma'(x)]^2},\\ p(x) &= \tfrac{1}{2}\varepsilon\sigma'(x) + \tfrac{1}{2}x.\end{aligned}$$

Proof. See Jimbo and Miwa [27] and Okamoto [43]. □

Consequently solutions of deg-$\mathrm{P_V}$ (13) can be expressed in terms of solutions of $\mathrm{S_{III}}$ (18).

Corollary 2.3. *If $\sigma(x;a,b,\varepsilon)$ is a solution of* $\mathrm{S_{III}}$ *(18), then*

$$w(z;a,b,\varepsilon)=\frac{\sigma'(x;a,b,\varepsilon)+\varepsilon x}{\sigma'(x;a,b,\varepsilon)-\varepsilon x},\qquad z=\tfrac{1}{2}x^2, \tag{19}$$

is a solution of (13).

Proof. This immediately follows from (12) and Theorem 2.2. □

Remark 2.4. From Lemma 2.1 and Corollary 2.3, it's clear that it's simpler to derive solutions of deg-$\mathrm{P_V}$ (13) from equation (19) rather than equation (15). Further as shown in §3, classical solutions of $\mathrm{S_{III}}$ involve one determinant, whereas classical solutions of $\mathrm{P_{III}}$ involve two determinants.

3. Classical solutions of $\mathrm{P_{III}}$ and $\mathrm{S_{III}}$

3.1. Rational solutions of $\mathrm{P_{III}}$ *and* $\mathrm{S_{III}}$

Rational solutions of $\mathrm{P_{III}}$ (5) are classified in the following theorem.

Theorem 3.1. *Equation (5) has a rational solution if and only if*

$$\varepsilon_1 A+\varepsilon_2 B=4n, \tag{20}$$

with $n\in\mathbb{Z}$ and $\varepsilon_1^2=1$, $\varepsilon_2^2=1$, independently.

Proof. For details see Lukashevich [30]; see also [36, 37]. □

Umemura [52]‡ derived special polynomials associated with rational solutions of $\mathrm{P_{III}}$ (5), which we now define; see also [9, 28, 29].

Definition 3.2. The *Umemura polynomial* $S_n(x;\mu)$ is given by the recursion relation

$$S_{n+1}S_{n-1}=-x\left\{S_n\frac{\mathrm{d}^2S_n}{\mathrm{d}x^2}-\left(\frac{\mathrm{d}S_n}{\mathrm{d}x}\right)^2\right\}-S_n\frac{\mathrm{d}S_n}{\mathrm{d}x}+(x+\mu)S_n^2, \tag{21}$$

where $S_{-1}(x;\mu)=S_0(x;\mu)=1$, with μ an arbitrary parameter.

Remarks 3.3.

(i) The Umemura polynomial $S_n(x;\mu)$ has the Wronskian representation

$$S_n(x;\mu)=c_n\,\mathrm{Wr}\left(\varphi_1,\varphi_2,\ldots,\varphi_{2n-1}\right),\qquad c_n=\prod_{k=0}^{n}(2k+1)^{n-k}, \tag{22}$$

where $\mathrm{Wr}(\varphi_1,\varphi_3,\ldots,\varphi_{2n-1})$ is the Wronskian defined by

$$\mathrm{Wr}(\varphi_1,\varphi_3,\ldots,\varphi_{2n-1})=\left|\begin{matrix}\varphi_1 & \varphi_3 & \cdots & \varphi_{2n-1}\\ \varphi_1^{(1)} & \varphi_3^{(1)} & \cdots & \varphi_{2n-1}^{(1)}\\ \vdots & \vdots & \ddots & \vdots\\ \varphi_1^{(n-1)} & \varphi_3^{(n-1)} & \cdots & \varphi_{2n-1}^{(n-1)}\end{matrix}\right|,\qquad \varphi_j^{(k)}=\frac{\mathrm{d}^k\varphi_j}{\mathrm{d}x^k},$$

and

$$\varphi_{2m-1}(x;\mu)=L_m^{(\mu-2m+1)}(-x),$$

with $L_k^{(\alpha)}(x)$ the Laguerre polynomial, for details see Kajiwara and Masuda [29]; see also [9, 28].

‡ The original manuscript was written by Umemura in 1996 for the proceedings of the conference "*Theory of nonlinear special functions: the Painlevé transcendents*" in Montreal, which were not published; for further details see [44].

(ii) Rational solutions of $\mathrm{P_{III}}$ (5) are expressed in terms of Umemura polynomials. For example,

$$w_n(z;\mu) = 1 + \frac{\mathrm{d}}{\mathrm{d}z} \ln \left\{ \frac{S_{n-1}(z;\mu-1)}{S_n(z;\mu)} \right\}, \tag{23}$$

satisfies (5) for the parameters

$$A = 2n + 2\mu - 1, \qquad B = 2n - 2\mu + 1.$$

To describe rational solutions of deg-$\mathrm{P_V}$ (1), it is more convenient to use rational solutions of $\mathrm{S_{III}}$ (18), which involve one Umemura polynomial and are discussed in the following theorem, whereas rational solutions of $\mathrm{P_{III}}$ (5) which involve two Umemura polynomials, as shown in (23).

Theorem 3.4. *The rational function solution of* $\mathrm{S_{III}}$ *(18) is given by*

$$\sigma_n(x;\mu,\varepsilon) = 2x \frac{\mathrm{d}}{\mathrm{d}x} \{\ln S_n(x;\mu)\} - \tfrac{1}{2}x^2 - 2\mu x - \tfrac{1}{4}, \quad n \geq 0, \tag{24}$$

with $S_n(x;\mu)$ *the Umemura polynomial, for the parameters*

$$a = n + \tfrac{1}{2}, \qquad b = \mu, \qquad \varepsilon = 1.$$

Proof. See Clarkson [9]. □

3.2. Special function solutions of $\mathrm{P_{III}}$ *and* $\mathrm{S_{III}}$

Special function solutions of $\mathrm{P_{III}}$ (5) are expressed in terms of Bessel functions. These are classified in the following Theorem.

Theorem 3.5. *Equation (5) has solutions expressible in terms of the Riccati equation*

$$x \frac{\mathrm{d}q}{\mathrm{d}x} = \varepsilon_1 x q^2 + (A\varepsilon_1 - 1)q + \varepsilon_2 x, \tag{25}$$

if and only if

$$\varepsilon_1 A + \varepsilon_2 B = 4n + 2, \tag{26}$$

with $n \in \mathbb{Z}$ *and* $\varepsilon_1^2 = 1$, $\varepsilon_2^2 = 1$, *independently. Further, the Riccati equation (25) has the solution*

$$q(x) = -\varepsilon_1 \frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_\nu(x), \tag{27}$$

where $\psi_\nu(x)$ *satisfies*

$$x \frac{\mathrm{d}^2\psi_\nu}{\mathrm{d}x^2} + (1 - 2\varepsilon_1\nu) \frac{\mathrm{d}\psi_\nu}{\mathrm{d}x} + \varepsilon_1\varepsilon_2 x \psi_\nu = 0, \tag{28}$$

which has solution

$$\psi_\nu(x) = \begin{cases} x^{\nu} \{C_1 J_\nu(x) + C_2 Y_\nu(x)\}, & \text{if} \quad \varepsilon_1 = 1, \quad \varepsilon_2 = 1, \\ x^{-\nu} \{C_1 J_\nu(x) + C_2 Y_\nu(x)\}, & \text{if} \quad \varepsilon_1 = -1, \quad \varepsilon_2 = -1, \\ x^{\nu} \{C_1 I_\nu(x) + C_2 K_\nu(x)\}, & \text{if} \quad \varepsilon_1 = 1, \quad \varepsilon_2 = -1, \\ x^{-\nu} \{C_1 I_\nu(x) + C_2 K_\nu(x)\}, & \text{if} \quad \varepsilon_1 = -1, \quad \varepsilon_2 = 1, \end{cases} \tag{29}$$

with C_1, C_2 *arbitrary constants, and* $J_\nu(x)$, $Y_\nu(x)$, $I_\nu(x)$, $K_\nu(x)$ *Bessel functions.*

Proof. For details see Okamoto [43]; see also [11, 21, 34, 36, 37]. □

Determinantal representations of special function solutions of $\mathrm{P_{III}}$ (5) were given by Okamoto [43]; see also [18, 35]. As for rational solutions, to describe special function solutions of deg-$\mathrm{P_V}$ (1), it is more convenient to use special function solutions of $\mathrm{S_{III}}$ (18), which are discussed in the following theorem.

Theorem 3.6. *Suppose $\tau_n(x;\mu,\varepsilon)$ is the determinant given by*

$$\tau_n(x;\mu,\varepsilon) = \det\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j+k}\varphi_\mu(x;\varepsilon)\right]_{j,k=0}^{n-1}, \tag{30}$$

where

$$\varphi_\mu(x;\varepsilon) = \begin{cases} c_1 J_\mu(x) + c_2 Y_\mu(x), & \text{if} \quad \varepsilon = 1,\\ c_1 I_\mu(x) + c_2 K_\mu(x), & \text{if} \quad \varepsilon = -1,\end{cases}$$

with c_1, c_2 arbitrary constants, and $J_\mu(z)$, $Y_\mu(z)$, $I_\mu(z)$, $K_\mu(z)$ Bessel functions. The Bessel function solution of $\mathrm{S_{III}}$ *(18) is given by*

$$\sigma_n(x;\mu,\varepsilon) = 2x\frac{\mathrm{d}}{\mathrm{d}x}\left\{\ln\tau_n(x;\mu,\varepsilon)\right\} + \tfrac{1}{2}\varepsilon x^2 + \mu^2 - n^2 + 2n, \tag{31}$$

for the parameters

$$a = n, \qquad b = \mu. \tag{32}$$

Lemma 3.7. *The determinant $\tau_n(x;\mu,\varepsilon)$ given by (30) satisfies the equation*

$$x^2\left\{\tau_n\frac{\mathrm{d}^2\tau_n}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}\tau_n}{\mathrm{d}x}\right)^2\right\} + x\tau_n\frac{\mathrm{d}\tau_n}{\mathrm{d}x} = \tau_{n+1}\tau_{n-1}, \tag{33}$$

or equivalently the Toda equation

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 \ln\tau_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}. \tag{34}$$

Proof. See Okamoto [43, Theorem 2]. □

4. Bäcklund transformations

A *Bäcklund transformation* relates the solution of a Painlevé equation either to another solution of the same equation with different values of the parameters, or to another Painlevé equation. All Painlevé equations, except the first Painlevé equation, have Bäcklund transformations. Hierarchies of classical solutions of the Painlevé equations can be obtained by applying Bäcklund transformations to a "seed solution".

Let $w_j(z_j;\alpha_j,\beta_j,\gamma_j)$, $j = 0,1,2$, be solutions of deg-$\mathrm{P_V}$ (1) with

$$\begin{aligned} z_1 &= -z_0, \qquad & (\alpha_1,\beta_1,\gamma_1) &= (\alpha_0,\beta_0,-\gamma_0),\\ z_2 &= z_0, & (\alpha_2,\beta_2,\gamma_2) &= (-\beta_0,-\alpha_0,-\gamma_0).\end{aligned}$$

Then deg-$\mathrm{P_V}$ (1) has the symmetries

$$\mathcal{S}_1: \qquad w_1(z_1) = w_0(-z_0), \tag{35}$$

$$\mathcal{S}_2: \qquad w_2(z_2) = 1/w_0(z_0). \tag{36}$$

Theorem 4.1. *Suppose that $W_0 = w(z;\alpha,\beta,\gamma)$ satisfies deg-$\mathrm{P_V}$ (1) with parameters*

$$\alpha = \tfrac{1}{2}a^2, \qquad \beta = -\tfrac{1}{2}b^2, \qquad \gamma = c.$$

Let $W_j = w(z;\alpha_j,\beta_j,\gamma_j)$, $j = 1,2,3,4$, be solutions of (1) with parameters

$$\begin{array}{lll} \alpha_1 = \tfrac{1}{2}(a+1)^2, & \beta_1 = -\tfrac{1}{2}b^2, & \gamma_1 = c,\\ \alpha_2 = \tfrac{1}{2}(a-1)^2, & \beta_2 = -\tfrac{1}{2}b^2, & \gamma_2 = c,\\ \alpha_3 = \tfrac{1}{2}a^2, & \beta_3 = -\tfrac{1}{2}(b+1)^2, & \gamma_3 = c,\\ \alpha_4 = \tfrac{1}{2}a^2, & \beta_4 = -\tfrac{1}{2}(b-1)^2, & \gamma_4 = c, \end{array}$$

respectively. Then these solutions can be obtained from W_0 as follows

$$W_1 = \frac{\{zW_0' + (W_0-1)(aW_0-b)\}\{zW_0' + (W_0-1)(aW_0+b)\}}{z^2(W_0')^2 + 2azW_0(W_0-1)W_0' + 2cz^2W_0(W_0-1) + (W_0-1)^2(a^2W_0^2-b^2)},$$

$$W_2 = \frac{\{zW_0' - (W_0-1)(aW_0-b)\}\{zW_0' - (W_0-1)(aW_0+b)\}}{z^2(W_0')^2 - 2azW_0(W_0-1)W_0' + 2cz^2W_0(W_0-1) + (W_0-1)^2(a^2W_0^2-b^2)},$$

$$W_3 = \frac{z^2(W_0')^2 + 2bz(W_0-1)W_0' + 2cz^2W_0^2(W_0-1) - (W_0-1)^2(a^2W_0^2-b^2)}{\{zW_0' - (W_0-1)(aW_0-b)\}\{zW_0' + (W_0-1)(aW_0+b)\}},$$

$$W_4 = \frac{z^2(W_0')^2 - 2bz(W_0-1)W_0' + 2cz^2W_0^2(W_0-1) - (W_0-1)^2(a^2W_0^2-b^2)}{\{zW_0' - (W_0-1)(aW_0-b)\}\{zW_0' + (W_0-1)(aW_0+b)\}}.$$

Proof. See Adler [2]; also Filipuk and Van Assche [17]. □

5. Classical solutions of deg-$\mathrm{P_V}$

To discuss classical solutions of deg-$\mathrm{P_V}$ (1), it is convenient to make the transformation

$$w(z) = u(x), \qquad z = \tfrac{1}{2}x^2, \tag{37}$$

in (1), which gives

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \left(\frac{1}{2u} + \frac{1}{u-1}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{4(u-1)^2(\alpha u^2+\beta)}{x^2u} + 2\gamma u. \tag{38}$$

We could fix the parameter γ in (38), by rescaling x if necessary, but it is more convenient not to do so. Instead classical solutions will be classified for $\gamma = \pm 1$. From Corollary 2.3 and (37), we have that if $\sigma(x;a,b,\varepsilon)$ is a solution of $\mathrm{S_{III}}$ (18), then

$$u(x;a,b,\varepsilon) = \frac{\sigma'(x;a,b,\varepsilon) + \varepsilon x}{\sigma'(x;a,b,\varepsilon) - \varepsilon x}, \tag{39}$$

is a solution of (38) with $\gamma = \varepsilon$. As remarked above, it is easier to derive classical solutions of deg-$\mathrm{P_V}$ (1) from $\mathrm{S_{III}}$ rather than $\mathrm{P_{III}}$, compare equations (19) and (15).

Theorem 5.1. *Supppose that $u_0 = u(x;\alpha,\beta,\gamma)$ satisfies (38) with parameters*

$$\alpha = \tfrac{1}{2}a^2, \qquad \beta = -\tfrac{1}{2}b^2, \qquad \gamma = c.$$

Let $u_j = u(x;\alpha_j,\beta_j,\gamma_j)$, $j = 1.2.3, 4$ be solutions of (38) with parameters

$$\begin{array}{lll} \alpha_1 = \tfrac{1}{2}(a+1)^2, & \beta_1 = -\tfrac{1}{2}b^2, & \gamma_1 = c,\\ \alpha_2 = \tfrac{1}{2}(a-1)^2, & \beta_2 = -\tfrac{1}{2}b^2, & \gamma_2 = c,\\ \alpha_3 = \tfrac{1}{2}a^2, & \beta_3 = -\tfrac{1}{2}(b+1)^2, & \gamma_3 = c,\\ \alpha_4 = \tfrac{1}{2}a^2, & \beta_4 = -\tfrac{1}{2}(b-1)^2, & \gamma_4 = c, \end{array}$$

respectively. Then these solutions can be obtained from u_0 *as follows*

$$u_1 = \frac{\{xu' + 2(u-1)(au-b)\}\{xu' + 2(u-1)(au+b)\}}{x^2(u')^2 + 4axu(u-1)u' + 4cu(u-1)x^2 + 4(u-1)^2(a^2u^2-b^2)},$$
$$u_2 = \frac{\{xu' - 2(u-1)(au-b)\}\{xu' - 2(u-1)(au+b)\}}{x^2(u')^2 - 4axu(u-1)u' + 4cu(u-1)x^2 + 4(u-1)^2(a^2u^2-b^2)},$$
$$u_3 = \frac{x^2(u')^2 + 4bx(u-1)u' + 4cx^2u^2(u-1) - 4(u-1)^2(a^2u^2-b^2)}{\{xu' - 2(u-1)(au-b)\}\{xu' + 2(u-1)(au+b)\}},$$
$$u_4 = \frac{x^2(u')^2 - 4bx(u-1)u' + 4cx^2u^2(u-1) - 4(u-1)^2(a^2u^2-b^2)}{\{xu' - 2(u-1)(au-b)\}\{xu' + 2(u-1)(au+b)\}}.$$

Proof. This is easily proved by applying (37) to the Bäcklund transformations in Theorem 4.1. □

5.1. *Algebraic solutions*

Since deg-$\mathrm{P_V}$ (1) and equation (38) are related by the transformation (37) then algebraic solutions of deg-$\mathrm{P_V}$ (1), which are rational functions of $\sqrt{z}$, are equivalent to rational solutions of (38), which are rational functions of x. Therefore we discuss rational solutions of (38), which are classified in the following Theorem.

Theorem 5.2. *Necessary and sufficient conditions for the existence of rational solutions of (38) are either*

$$(\alpha,\beta,\gamma) = \left(\tfrac12(n+\tfrac12), -\tfrac12\mu^2, 1\right), \tag{40}$$

or

$$(\alpha,\beta,\gamma) = \left(\tfrac12\mu^2, -\tfrac12(n+\tfrac12), -1\right), \tag{41}$$

where $n \in \mathbb{Z}$ *and* μ *is an arbitrary constant.*

Proof. For details see Gromak, Laine and Shimomura [21, §38]; see also [36, 37]. □

We remark that the solutions of (38) satisfying (40) are related to those satisfying (41) through the analog of the symmetry (36). Consequently we shall be concerned only with rational solutions of (38) for the parameters given by (40).

Theorem 5.3. *The rational solution of (38) for the parameters (40) is given by*

$$u_n(x;\mu) = 1 - \frac{xS_n^2(x;\mu)}{S_{n+1}(x;\mu)S_{n-1}(x;\mu)}, \qquad n \ge 0, \tag{42}$$

where $S_n(x;\mu)$ *is the Umemura polynomial (22).*

Proof. Substituting the rational solution of $\mathrm{S_{III}}$ (18) given by (24) into (39) and then using the reccurence relation (21) gives the result. □

Remark 5.4. The Umemura polynomial $S_n(x;\mu)$ satisfies the difference equation

$$S_{n+1}(x;\mu)S_{n-1}(x;\mu) = xS_n^2(x;\mu) + \mu S_n(x;\mu+1)\,S_n(x;\mu-1). \tag{43}$$

Hence from (42) there are two alternative representations of the rational solution

$$u_n(x;\mu) = \frac{\mu S_n(x;\mu+1)\,S_n(x;\mu-1)}{\mu S_n(x;\mu+1)\,S_n(x;\mu-1) + xS_n^2(x;\mu)},$$
$$u_n(x;\mu) = \frac{\mu S_n(x;\mu+1)\,S_n(x;\mu-1)}{S_{n+1}(x;\mu)S_{n-1}(x;\mu)}.$$

5.2. Bessel function solutions

Theorem 5.5. *Necessary and sufficient conditions for the existence of Bessel function solutions of (38) are either*

$$(\alpha,\beta,\gamma)=\left(\tfrac12 n^2,-\tfrac12\mu^2,\varepsilon\right), \tag{44}$$

or

$$(\alpha,\beta,\gamma)=\left(\tfrac12 \mu^2,-\tfrac12 n^2,-\varepsilon\right), \tag{45}$$

with $\varepsilon=\pm1$, and where $n\in\mathbb{Z}^+$ and μ is an arbitrary constant.

Proof. From (10) and (14), the parameters in $\mathrm{P_{III}}$ (5) and deg-$\mathrm{P_V}$ (38) are given by

$$(A,B)=\big(2(a-b),2\varepsilon(a+b+1)\big),\qquad (\alpha,\beta,\gamma)=(\tfrac12 a^2,-\tfrac12 b^2,\varepsilon),$$

respectively, for parameters a, b and ε. The result then follows from Theorem 3.5. □

Theorem 5.6. *The Bessel function solution of (38) for the parameters*

$$(\alpha,\beta,\gamma)=\left(\tfrac12 n^2,-\tfrac12\mu^2,\varepsilon\right),$$

is given by

$$u_n(x;\mu,\varepsilon)=1+\frac{\varepsilon x^2\tau_n^2(x;\mu,\varepsilon)}{\tau_{n+1}(x;\mu,\varepsilon)\,\tau_{n-1}(x;\mu,\varepsilon)},\qquad n\geq1, \tag{46}$$

where

$$\tau_n(x;\mu,\varepsilon)=\det\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j+k}\varphi_\mu(x;\varepsilon)\right]_{j,k=0}^{n-1}, \tag{47}$$

and $\tau_0(x;\mu,\varepsilon)=1$, with

$$\varphi_\mu(x;\varepsilon)=\begin{cases} c_1J_\mu(x)+c_2Y_\mu(x), & \text{if}\quad \varepsilon=1,\\ c_1I_\mu(x)+c_2K_\mu(x), & \text{if}\quad \varepsilon=-1,\end{cases} \tag{48}$$

c_1 and c_2 arbitrary constants, and $J_\mu(x)$, $Y_\mu(x)$, $I_\mu(x)$ and $K_\mu(x)$ Bessel functions.

Proof. Substituting the Bessel function solution of $\mathrm{S_{III}}$ (18) given by (31) into (39) and then using (33) gives the result. □

Corollary 5.7. *The Bessel function solution of (38) for the parameters*

$$(\alpha,\beta,\gamma)=\left(\tfrac12 n^2,-\tfrac12\mu^2,2\varepsilon\right),$$

is given by

$$w_n(z;\mu,\varepsilon)=1+\frac{\varepsilon z\mathcal{T}_n^2(z;\mu,\varepsilon)}{\mathcal{T}_{n+1}(z;\mu,\varepsilon)\,\mathcal{T}_{n-1}(z;\mu,\varepsilon)},\qquad n\geq1, \tag{49}$$

where

$$\mathcal{T}_n(z;\mu,\varepsilon)=\det\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j+k}\psi_\mu(z;\varepsilon)\right]_{j,k=0}^{n-1}, \tag{50}$$

and $\mathcal{T}_0(z;\mu,\varepsilon)=1$, with

$$\varphi_\mu(z;\varepsilon)=\begin{cases} c_1J_\mu(2\sqrt{z})+c_2Y_\mu(2\sqrt{z}), & \text{if}\quad \varepsilon=1,\\ c_1I_\mu(2\sqrt{z})+c_2K_\mu(2\sqrt{z}), & \text{if}\quad \varepsilon=-1,\end{cases} \tag{51}$$

c_1 and c_2 arbitrary constants, and $J_\mu(x)$, $Y_\mu(x)$, $I_\mu(x)$ and $K_\mu(x)$ Bessel functions.

In the next Lemma, it is shown that the first solution $u_1(x;\mu,\varepsilon)$, the "seed solution", satisfies a first-order, second-degree equation.

Lemma 5.8. *The solution of (38) for the parameters*

$$(\alpha,\beta,\gamma)=\left(\tfrac{1}{2},-\tfrac{1}{2}\mu^2,\varepsilon\right),$$

is

$$u_1(x;\mu,\varepsilon)=\frac{\varphi_{\mu+1}(x;\varepsilon)\left[x\varphi_{\mu+1}(x;\varepsilon)-2\varepsilon\mu\varphi_\mu(x;\varepsilon)\right]}{x\varphi_{\mu+1}^2(x;\varepsilon)-2\varepsilon\mu\varphi_{\mu+1}(x;\varepsilon)\varphi_\mu(x;\varepsilon)+\varepsilon x\varphi_\mu^2(x;\varepsilon)}, \quad (52)$$

where

$$\varphi_\mu(x;\varepsilon)=\begin{cases} c_1J_\mu(x)+c_2Y_\mu(x), & \text{if} \quad \varepsilon=1,\\ c_1I_\mu(x)+c_2K_\mu(x), & \text{if} \quad \varepsilon=-1,\end{cases}$$

with c_1 and c_2 constants, satisfies the first-order, second-degree equation

$$x^2\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2-4xu(u-1)\frac{\mathrm{d}u}{\mathrm{d}x}+4\varepsilon x^2u(u-1)+4(u-1)^2(u^2-\mu^2)=0. \quad (53)$$

Proof. Define

$$\Phi_\mu(x;\varepsilon)=\frac{\varphi_{\mu+1}(x;\varepsilon)}{\varphi_\mu(x;\varepsilon)},$$

then from (52)

$$u_1(x;\mu,\varepsilon)=1-\frac{x}{\varepsilon x\Phi_\mu^2-2\mu\Phi_\mu+x}, \quad (54)$$

and $\Phi_\mu(x;\varepsilon)$ satisfies the Riccati equation

$$x\frac{\mathrm{d}\Phi_\mu}{\mathrm{d}x}=\varepsilon x\Phi_\mu^2-(2\mu+1)\Phi_\mu+x. \quad (55)$$

Next we assume that $u_1(x;\mu,\varepsilon)$ satisfies a first-order, second-degree equation of the form

$$x^2\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+x\left[f_2(x,\mu,\varepsilon)u^2+f_1(x,\mu,\varepsilon)u+f_0(x,\mu,\varepsilon)\right]\frac{\mathrm{d}u}{\mathrm{d}x} + \sum_{j=0}^{4}g_j(x,\mu,\varepsilon)u^j=0, \quad (56)$$

where $\{f_j(x,\mu,\varepsilon)\}_{j=0}^2$ and $\{g_j(x,\mu,\varepsilon)\}_{j=0}^4$ are to be determined. Then substituting (54) into (56), using the fact that $\Phi_\mu(x;\varepsilon)$ satisfies (55) and equating coefficients of powers of Φ_μ yields

$$\begin{aligned}&f_2=-4,\quad f_1=4,\quad f_0=0,\\ &g_4=4,\quad g_3=-8,\quad g_2=4\varepsilon x^2-4\mu^2+4,\quad g_1=-4\varepsilon x^2+8\mu^2,\quad g_0=-4\mu^2.\end{aligned}$$

Hence we obtain equation (53), as required. □

This demonstrates that special function solutions of (38), and hence also deg-$\mathrm{P_V}$ (1), are different from special function solutions of $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ where the "seed solution" satisfies a Riccati equation, a first-order, first-degree equation.

Remark 5.9. Gromak, Laine and Shimomura [21, equation (38.7)] give, without proof, a first-order, second-degree equation associated with Bessel function solutions of deg-$\mathrm{P_V}$ (1); see also Filipuk and Van Assche [17, §2.3].

6. Applications

6.1. Complex sine-Gordon equation

Consider the two-dimensional complex sine-Gordon equation

$$\nabla^2\psi + \frac{(\nabla\psi)^2\overline{\psi}}{1-|\psi|^2} + \psi\left(1-|\psi|^2\right) = 0, \tag{57}$$

where $\nabla\psi = (\psi_x, \psi_y)$. Making the transformation

$$\psi(x,y) = \cos(\varphi(x,y))\exp\{\mathrm{i}\eta(x,y)\}, \qquad \overline{\psi}(x,y) = \cos(\varphi(x,y))\exp\{-\mathrm{i}\eta(x,y)\},$$

in the complex sine-Gordon equation (57) yields

$$\begin{aligned}\nabla^2\varphi + \frac{\cos\varphi}{\sin^3\varphi}(\nabla\eta)^2 - \tfrac{1}{2}\sin(2\varphi) &= 0,\\ \sin(2\varphi)\,\nabla^2\eta = 4(\varphi_x\eta_x + \varphi_y\eta_y),&\end{aligned}$$

which is the *Pohlmeyer-Lund-Regge model* [31, 32, 47].

The complex sine-Gordon equation (57) has a separable solution in polar coordinates given by $\psi(r,\theta) = R_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}$, where $R_n(r)$ satisfies

$$\frac{\mathrm{d}^2R_n}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R_n}{\mathrm{d}r} + \frac{R_n}{1-R_n^2}\left\{\left(\frac{\mathrm{d}R_n}{\mathrm{d}r}\right)^2 - \frac{n^2}{r^2}\right\} + R_n\left(1-R_n^2\right) = 0, \tag{58}$$

We remark that this equation also arises in extended quantum systems [4, 5, 6], in relativity [19] and in coefficients in the three-term recurrence relation for orthogonal polynomials with respect to the weight $w(\theta) = \mathrm{e}^{t\cos\theta}$ on the unit circle, see [53, equation (3.13)]. The orthogonal polynomials for this weight on the unit circle are related to unitary random matrices [46].

Equation (58) can be shown to possess the Painlevé property, though it is not in the list of 50 equations given in [24, Chapter 14]. Equation (58) can be transformed to the fifth Painlevé equation (2) in two different ways.

(i) If $R_n(r)$ satisfies (58) then making the transformation

$$R_n(r) = \frac{1+u_n(z)}{1-u_n(z)}, \qquad r = \tfrac{1}{2}z, \tag{59}$$

yields

$$\begin{aligned}\frac{\mathrm{d}^2u_n}{\mathrm{d}z^2} = \left(\frac{1}{2u_n} + \frac{1}{u_n-1}\right)\left(\frac{\mathrm{d}u_n}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}u_n}{\mathrm{d}z}&\\ + \frac{n^2(u_n-1)^2(u_n^2-1)}{8z^2u_n} - \frac{u_n(u_n+1)}{2(u_n-1)},&\end{aligned} \tag{60}$$

which is $\mathrm{P_V}$ (2) with $\alpha = \frac{1}{8}n^2$, $\beta = -\frac{1}{8}n^2$, $\gamma = 0$ and $\delta = -\frac{1}{2}$.

(ii) If $R_n(r)$ satisfies (58) then making the transformation

$$R_n(r) = \frac{1}{\sqrt{1-v_n(x)}}, \qquad r = \sqrt{x}, \tag{61}$$

yields

$$\frac{\mathrm{d}^2v_n}{\mathrm{d}x^2} = \left(\frac{1}{2v_n} + \frac{1}{v_n-1}\right)\left(\frac{\mathrm{d}v_n}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}v_n}{\mathrm{d}x} - \frac{n^2(v_n-1)^2}{2x^2v_n} + \frac{v_n}{2x}, \tag{62}$$

which is deg-$\mathrm{P_V}$ (1) with $\alpha = 0$, $\beta = -\frac{1}{2}n^2$ and $\gamma = \frac{1}{2}$ so is equivalent to $\mathrm{P_{III}}$ (5), as mentioned above.

This shows that solutions of equations (60) and (62) are related by

$$v_n(x) = \frac{4u_n(z)}{1+u_n^2(z)}, \qquad x = \tfrac{1}{4}z^2.$$

The function $R_n(r)$ satisfies the ordinary differential equation (58), the differential-difference equations

$$\frac{\mathrm{d}R_n}{\mathrm{d}r} + \frac{n}{r}R_n - \left(1-R_n^2\right)R_{n-1} = 0, \tag{63}$$

$$\frac{\mathrm{d}R_{n-1}}{\mathrm{d}r} - \frac{n-1}{r}R_{n-1} + \left(1-R_{n-1}^2\right)R_n = 0, \tag{64}$$

since solving (63) for $R_{n-1}(r)$ and substituting in (64) yields equation (58). Also eliminating the derivatives in (63)-(64), after letting $n \to n+1$ in (64), yields the difference equation

$$R_{n+1} + R_{n-1} = \frac{2n}{r}\,\frac{R_n}{1-R_n^2}, \tag{65}$$

which is known as the discrete Painlevé II equation [38, 46].

If $n = 1$ then equations (63)-(64) have the solution

$$R_0(r) = 1, \qquad R_1(r) = \frac{C_1 I_1(r) - C_2 K_1(r)}{C_1 I_0(r) + C_2 K_0(r)},$$

where $I_0(r)$, $K_0(r)$, $I_1(r)$ and $K_1(r)$ are the imaginary Bessel functions and C_1 and C_2 are arbitrary constants. For solutions which are bounded at $r = 0$ then necesssarily $C_2 = 0$ and so

$$R_0(r) = 1, \qquad R_1(r) = \frac{I_1(r)}{I_0(r)}. \tag{66}$$

Hence one can use the difference equation (65) to determine $R_n(r)$, for $n \geq 2$, which yields

$$R_2(r) = -\frac{rR_1^2(r) + 2R_1(r) - r}{r\left[R_1^2(r) - 1\right]},$$

$$R_3(r) = \frac{R_1^3(r) - rR_1^2(r) - 2R_1(r) + r}{R_1(r)\left[rR_1^2(r) + R_1(r) - r\right]},$$

$$R_4(r) = \frac{r(r^2+5)R_1^4(r) + 4R_1^3(r) - 2r(r^2+3)R_1^2(r) + r^3}{r\left[(r^2-1)R_1^4(r) + 4rR_1^3(r) - 2(r^2+2)R_1^2(r) - 4rR_1(r) + r^2\right]}.$$

These results suggest that (58) should be solvable in terms of $\mathrm{P_{III}}$ (5), which is illustrated in the following theorem.

Theorem 6.1. *If $R_n(r)$ satisfies (58) then $w_n(r) = R_{n+1}(r)/R_n(r)$ satisfies*

$$\frac{\mathrm{d}^2 w_n}{\mathrm{d}r^2} = \frac{1}{w_n}\left(\frac{\mathrm{d}w_n}{\mathrm{d}r}\right)^2 - \frac{1}{r}\frac{\mathrm{d}w_n}{\mathrm{d}r} - \frac{2n}{r}w_n^2 + \frac{2(n+1)}{r} + w_n^3 - \frac{1}{w_n}, \tag{67}$$

which is $\mathrm{P_{III}}$ (5) with parameters $\alpha = -2n$ and $\beta = 2(n+1)$.

Proof. See Hisakado [22] and Tracy & Widom [49]; see also [53, §3.1]. □

We note that since the parameters in (67) satisfy $-\alpha + \beta = 4n + 2$, with $n \in \mathbb{Z}^+$, then the equation has solutions expressible in terms of the modified Bessel functions $I_0(r)$ and $I_1(r)$ (as well as $K_0(r)$ and $K_1(r)$, but these are not needed here).

Theorem 6.2. *Let $\tau_n(r;\nu)$ be the $n\times n$ determinant*

$$\tau_n(r;\nu)=\det\left[\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)^{j+k} I_\nu(r)\right]_{j,k=0}^{n-1}, \tag{68}$$

with $I_\nu(r)$ the modified Bessel function, then

$$w_n(r;\nu)=\frac{\tau_{n+1}(r;\nu+1)\,\tau_n(r;\nu)}{\tau_{n+1}(r;\nu)\,\tau_n(r;\nu+1)}\equiv\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\frac{\tau_{n+1}(z;\nu)}{\tau_n(z;\nu+1)}\right\}-\frac{n+\nu}{z}, \tag{69}$$

for $n\geq 0$, satisfies $\mathrm{P_{III}}$ (5) with $\alpha=2(\nu-n)$ and $\beta=2(\nu+n+1)$.

Proof. See, for example, [18, 35]. □

Theorem 6.3. *Equation (58) has the solution*

$$R_n(r)=\frac{\tau_n(r;1)}{\tau_n(r;0)}, \tag{70}$$

where $\tau_n(r;\nu)$ is the determinant given by (68).

Proof. The proof is straightforward using induction. From (66) we have

$$R_1(r)=\frac{I_1(r)}{I_0(r)}=\frac{\tau_1(r;1)}{\tau_1(r;0)},$$

so (70) is true if $n=1$. Assuming (70) holds then from Theorems 6.1 and 6.2

$$R_{n+1}(r)=w_n(r;0)R_n(r)=\frac{\tau_{n+1}(r;1)\,\tau_n(r;0)}{\tau_{n+1}(r;0)\,\tau_n(r;1)}\times\frac{\tau_n(r;1)}{\tau_n(r;0)}=\frac{\tau_{n+1}(r;1)}{\tau_{n+1}(r;0)},$$

as required, and so the result follows by induction. □

Corollary 6.4. Equations (60) and (62) have the Bessel function solutions

$$u_n(z)=\frac{\tau_n(\tfrac{1}{2}z;1)+\tau_n(\tfrac{1}{2}z;0)}{\tau_n(\tfrac{1}{2}z;1)-\tau_n(\tfrac{1}{2}z;0)},\qquad v_n(x)=1-\frac{\tau_n^2(\sqrt{x};0)}{\tau_n^2(\sqrt{x};1)},$$

respectively, with $\tau_n(r;\nu)$ the determinant given by (68).

Lemma 6.5. *The formal asymptotic behaviour of the vortex solution $R_n(r)$ is given by*

$$R_n(r)=\frac{r^n}{2^n\,n!}\left\{1-\frac{r^2}{4(n+1)}+\mathcal{O}\left(r^4\right)\right\},\qquad \textit{as}\quad r\to 0, \tag{71}$$

$$R_n(r)=1-\frac{n}{2r}-\frac{n^2}{8r^2}-\frac{n(n^2+1)}{16r^3}+\mathcal{O}(r^{-4}),\qquad \textit{as}\quad r\to\infty. \tag{72}$$

Proof. These are determined from (65) and (66). □

6.2. Generalised Charlier polynomials

The Charlier polynomials $C_n(k;z)$ are a family of orthogonal polynomials introduced in 1905 by Charlier [7] given by

$$C_n(k;z) = {}_2F_0\left(-n,-k;;-1/z\right) = (-1)^n n! L_n^{(-1-k)}\left(-1/z\right), \quad z>0, \tag{73}$$

where ${}_2F_0(a,b;;z)$ is the hypergeometric function and $L_n^{(\alpha)}(z)$ is the associated Laguerre polynomial, see, for example, [45, §18.19]. The Charlier polynomials are orthogonal on the lattice $\mathbb{N}$ with respect to the Poisson distribution

$$\omega(k) = \frac{z^k}{k!}, \qquad z>0, \tag{74}$$

and satisfy the orthogonality condition

$$\sum_{k=0}^{\infty} C_m(k;z)C_n(k;z)\frac{z^k}{k!} = \frac{n!\,\mathrm{e}^z}{z^n}\delta_{m,n}.$$

Smet and Van Assche [48] generalized the Charlier weight (74) with one additional parameter through the weight function

$$\omega(k;\nu) = \frac{\Gamma(\nu+1)\,z^k}{\Gamma(\nu+k+1)\,\Gamma(k+1)}, \qquad z>0,$$

with ν a parameter such that $\nu>-1$. This gives the discrete weight

$$\omega(k;\nu) = \frac{z^k}{(\nu+1)_k\,k!}, \qquad z>0, \tag{75}$$

where $(\nu+1)_k = \Gamma(\nu+1+k)/\Gamma(\nu+1)$ is the Pochhammer symbol, on the lattice $\mathbb{N}$. Discrete orthogonal polynomials are characterized by the discrete Pearson equation

$$\Delta\big[\sigma(k)\omega(k)\big] = \tau(k)\omega(k), \tag{76}$$

where Δ is the forward difference operator

$$\Delta f(k) = f(k+1) - f(k).$$

The weight (75) satisfies the discrete Pearson equation (76) with

$$\sigma(k) = k(k+\nu), \qquad \tau(k) = -k^2 - \nu k + z,$$

and so the generalised Charlier polynomials are semi-classical orthogonal polynomials since $\tau(k)$ is a polynomial with $\deg(\tau)>1$. The special case $\nu=0$ was first considered by Hounkonnou, Hounga and Ronveaux [23] and later studied by Van Assche and Foupouagnigni [54].

For the generalised Charlier weight (75), the orthonormal polynomials $p_n(k;z)$ satisfy the orthogonality condition

$$\sum_{k=0}^{\infty} p_m(k;z)p_n(k;z)\frac{z^k}{(\nu+1)_k\,k!} = \delta_{m,n},$$

and the three-term recurrence relation

$$kp_n(k;z) = a_{n+1}(z)p_{n+1}(k;z) + b_n(z)p_n(k;z) + a_n(z)p_{n-1}(k;z), \tag{77}$$

with $p_{-1}(k;z)=0$ and $p_0(k;z)=1$. Our interest is in the coefficients $a_n(z)$ and $b_n(z)$ in the recurrence relation (77).

Smet and Van Assche [48, Theorem 2.1] proved the following theorem for recurrence relation coefficients associated with the generalised Charlier weight (75).

Theorem 6.6. *The recurrence relation coefficients $a_n(z)$ and $b_n(z)$ for orthonormal polynomials associated with the generalised Charlier weight (75) on the lattice $\mathbb{N}$ satisfy the discrete system*

$$\begin{aligned}&(a_{n+1}^2-z)(a_n^2-z)=z(b_n-n)(b_n-n+\nu),\\&b_n+b_{n-1}-n+\nu+1=nz/a_n^2,\end{aligned}\tag{78}$$

with initial conditions

$$a_0^2=0,\qquad b_0=\frac{\sqrt{z}\,I_{\nu+1}(2\sqrt{z})}{I_\nu(2\sqrt{z})}=z\frac{\mathrm{d}}{\mathrm{d}z}\big\{\ln I_\nu(2\sqrt{z})\big\}-\frac{\nu}{2},\tag{79}$$

with $I_\nu(k)$ the modified Bessel function.

Remark 6.7. The discrete system such as (78) for the recurrence relation coefficients is sometimes known as the *Laguerre-Freud equations*, cf. [3, 23, 33].

The recurrence relation coefficients $a_n(z)$ and $b_n(z)$ also satisfy the Toda lattice, cf. [53, Theorem 3.8]

$$z\frac{\mathrm{d}}{\mathrm{d}z}a_n^2=a_n^2(b_n-b_{n-1}),\qquad z\frac{\mathrm{d}}{\mathrm{d}z}b_n=a_{n+1}^2-a_n^2.\tag{80}$$

Letting $a_n^2(z)=x_n(z)$ and $b_n(z)=y_n(z)$ in (78) and (80) yields

$$\begin{aligned}&(x_{n+1}-z)(x_n-z)=t(y_n-n)(y_n-n+\nu), && z\frac{\mathrm{d}x_n}{\mathrm{d}t}=x_n(y_n-y_{n-1}),\\&y_n+y_{n-1}-n+\nu+1=\frac{nz}{x_n}, && z\frac{\mathrm{d}y_n}{\mathrm{d}z}=x_{n+1}-x_n.\end{aligned}$$

Eliminating x_{n+1} and y_{n-1} in these equations yields the differential system

$$z\frac{\mathrm{d}x_n}{\mathrm{d}z}=x_n(2y_n+\nu-n+1)-nz,\tag{81}$$

$$z\frac{\mathrm{d}y_n}{\mathrm{d}z}=-x_n+z+\frac{(y_n-n)(y_n-n+\nu)z}{x_n-z}.\tag{82}$$

Solving (81) for y_n gives

$$y_n=\frac{z}{2x_n}\frac{\mathrm{d}x_n}{\mathrm{d}z}+\frac{nz}{2x_n}+\frac{n-\nu-1}{2},$$

and substituting this into (82) yields

$$\begin{aligned}\frac{\mathrm{d}^2x_n}{\mathrm{d}z^2}=\frac{1}{2}\left(\frac{1}{x_n}+\frac{1}{x_n-z}\right)-\frac{x_n}{z(x_n-z)}\frac{\mathrm{d}x_n}{\mathrm{d}z}-\frac{2x_n^2}{z^2}\\+\frac{4x_n+n^2-\nu^2+1}{2z}-\frac{n^2}{2x_n}+\frac{1-\nu^2}{2(x_n-z)}.\end{aligned}\tag{83}$$

Making the transformation

$$x_n(z)=\frac{z}{1-w_n(z)}.\tag{84}$$

in (83) yields

$$\frac{\mathrm{d}^2w_n}{\mathrm{d}z^2}=\left(\frac{1}{2w_n}+\frac{1}{w_n-1}\right)\left(\frac{\mathrm{d}w_n}{\mathrm{d}z}\right)^2-\frac{1}{z}\frac{\mathrm{d}w_n}{\mathrm{d}z}+\frac{(w_n-1)^2(n^2w_n^2-\nu^2)}{2w_nz^2}-\frac{2w_n}{z},\tag{85}$$

which is deg-$\mathrm{P_V}$ (1) with parameters $\alpha=\tfrac12 n^2$, $\beta=-\tfrac12\nu^2$ and $\gamma=-2$.

Solving (82) for x_n gives

$$x_n = -\tfrac{1}{2} z \frac{\mathrm{d}y_n}{\mathrm{d}z} + z + \tfrac{1}{2} X_n, \tag{86}$$

where

$$X_n^2 = z^2 \left(\frac{\mathrm{d}y_n}{\mathrm{d}z} \right)^2 + 4z(y_n - n)(y_n - n + \nu). \tag{87}$$

From (87) we get

$$\begin{aligned} \frac{\mathrm{d}X_n}{\mathrm{d}z} = \frac{z^2}{X_n} \frac{\mathrm{d}^2 y_n}{\mathrm{d}z^2} \frac{\mathrm{d}y_n}{\mathrm{d}z} + \frac{z}{X_n} \left(\frac{\mathrm{d}y_n}{\mathrm{d}z} \right)^2 + \frac{2z(2y_n - 2n + \nu)}{X_n} \frac{\mathrm{d}y_n}{\mathrm{d}z} \\ + \frac{2(y_n - n)(y_n - n + \nu)}{X_n} \end{aligned} \tag{88}$$

Substituting (86) into (81), then using (88), solving for X_n, and substituting into (87) yields the second-order, second-degree equation

$$\begin{aligned} &\left(2z \frac{\mathrm{d}^2 y_n}{\mathrm{d}z^2} + \frac{\mathrm{d}y_n}{\mathrm{d}z} + 8y_n - 8n + 4\nu \right)^2 \\ &\qquad = \frac{(4y_n - 2n + 2\nu + 1)^2}{z} \left\{ z \left(\frac{\mathrm{d}y_n}{\mathrm{d}z} \right)^2 + 4(y_n - n)(y_n - n + \nu) \right\}. \end{aligned} \tag{89}$$

Making the transformation

$$y_n(z) = \tfrac{1}{2} v_n(x) + \tfrac{1}{2} n - \tfrac{1}{2}\nu - \tfrac{1}{4}, \qquad x = 2\sqrt{z},$$

in (89) yields

$$\begin{aligned} &\left(\frac{\mathrm{d}^2 v_n}{\mathrm{d}x^2} + 4v_n - 4n - 2 \right)^2 \\ &\qquad = \frac{4v_n^2}{x^2} \left\{ \left(\frac{\mathrm{d}v_n}{\mathrm{d}x} \right)^2 + 4v_n^2 - 4(2n+1)v_n + (2n+1)^2 - 4\nu^2 \right\}. \end{aligned} \tag{90}$$

Equation (A.5) in [14] is

$$\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - av - b \right)^2 = \frac{4v^2}{x^2} \left\{ \left(\frac{\mathrm{d}v}{\mathrm{d}x} \right)^2 - av^2 - 2bv - c \right\}, \tag{91}$$

with a, b and c parameters, an equation derived by Chazy [8], and is the primed version of equation SD-III in [15]. Hence equation (90) is the special case of equation (91) with

$$a = -4, \qquad b = 4n + 2, \qquad c = 4\nu^2 - (2n+1)^2.$$

Cosgrove [14] showed that equation (91) is solvable in terms of solutions of $\mathrm{P_{III}}$ (5). Consequently, the solution of (90) is given by

$$v_n(x) = \frac{x}{2q} \left(\frac{\mathrm{d}q}{\mathrm{d}x} + q^2 + 1 \right),$$

where $q(x)$ satisfies $\mathrm{P_{III}}$ (5) for the parameters $A = 2\nu - 2n - 2$ and $B = 2\nu + 2n$.

Theorem 6.8. *The recurrence relation coefficients $a_n(z)$ and $b_n(z)$ are given by*

$$a_n^2(z) = x_n(z) = \frac{\mathcal{T}_{n+1}(z;\nu)\mathcal{T}_{n-1}(z;\nu)}{\mathcal{T}_n^2(z;\nu)}, \tag{92}$$

$$b_n(z) = y_n(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\frac{\mathcal{T}_{n+1}(z;\nu)}{\mathcal{T}_n(z;\nu)}\right\} - \frac{\nu}{2}, \tag{93}$$

where

$$\mathcal{T}_n(z;\nu) = \det\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j+k} I_\nu\big(2\sqrt{z}\big)\right]_{j,k=0}^{n-1},$$

with $\mathcal{T}_0(z;\nu)=1$, and $I_\nu(x)$ is the modified Bessel function.

Proof. The expression (92) for $a_n^2(z)$ follows immediately by substituting (49) in (84). To prove the result (93) for $b_n(z)$ we use induction and the fact that from equation (80), $a_n^2(z)=x_n(z)$ and $b_n(z)=y_n(z)$ are related by

$$z\frac{\mathrm{d}x_n}{\mathrm{d}t} = x_n(y_n - y_{n-1}),$$

and initially

$$y_0(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\big\{\ln\mathcal{T}_1(z;\nu))\big\} - \frac{\nu}{2}.$$

Hence

$$\begin{aligned} y_1(z) &= z\frac{\mathrm{d}}{\mathrm{d}z}\big\{\ln x_1(z)\big\} + y_0(z) \\ &= z\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\frac{\mathcal{T}_2(z;\nu)\mathcal{T}_0(z;\nu)}{\mathcal{T}_1^2(z;\nu)}\right\} + z\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\mathcal{T}_1(z;\nu)\right\} - \frac{\nu}{2} \\ &= z\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\frac{\mathcal{T}_2(z;\nu)}{\mathcal{T}_1(z;\nu)}\right\} - \frac{\nu}{2}, \end{aligned}$$

since $\mathcal{T}_0(z;\nu)=1$, so (93) is true for $n=1$. Now suppose that (93) is true, then

$$\begin{aligned} y_{n+1}(z) &= z\frac{\mathrm{d}}{\mathrm{d}z}\big\{\ln x_n(z)\big\} + y_n(z) \\ &= z\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\frac{\mathcal{T}_{n+2}(z;\nu)\mathcal{T}_n(z;\nu)}{\mathcal{T}_{n+1}^2(z;\nu)}\right\} + z\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\frac{\mathcal{T}_{n+1}(z;\nu)}{\mathcal{T}_n(z;\nu)}\right\} - \frac{\nu}{2} \\ &= z\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\frac{\mathcal{T}_{n+2}(z;\nu)}{\mathcal{T}_{n+1}(z;\nu)}\right\} - \frac{\nu}{2}, \end{aligned}$$

as required, and so the result follows by induction. We remark that equation (80) is identically satisfied by $a_n^2(z)$ and $b_n(z)$ given by (92) and (93), respectively. □

In a recent paper, Fernández-Irisarri and Mañas [16, §2] discuss the generalised Charlier weight (75), in particular properties of the coefficients in the recurrence relation. The relationship between the notations in [16] and those here are $x_n(z) = \gamma_n(\eta)$ and $y_n(z) = \beta_n(\eta)$, with $z=\eta$. Fernández-Irisarri and Mañas [16] relate $x_n(z)$ and $y_n(z)$ to Okamoto's Hamiltonian for $\mathrm{P}_{\mathrm{III}'}$ [43] and derive two ordinary differential equations for $x_n(z)$.

(i) Equation (45) in [16, Theorem 4] is the third-order equation

$$\delta_z\left(\frac{x_n}{z}\left\{\delta_z^2(\ln x_n)+2x_n\right\}+\frac{n^2z}{x_n}\right)=2x_n,\qquad \delta_z(f)=z\frac{\mathrm{d}f}{\mathrm{d}z},$$

i.e.

$$\begin{aligned}\frac{\mathrm{d}^3x_n}{\mathrm{d}z^3}=\frac{1}{zx_n^2}\left(z\frac{\mathrm{d}x_n}{\mathrm{d}z}-x_n\right)&\left\{2x_n\frac{\mathrm{d}^2x_n}{\mathrm{d}z^2}-\left(\frac{\mathrm{d}x_n}{\mathrm{d}z}\right)^2+n^2\right\}\\ &-\frac{4x_n}{z^2}\frac{\mathrm{d}x_n}{\mathrm{d}z}+\frac{2x_n(x_n+z)}{z^3},\end{aligned}\tag{94}$$

and the authors state that this equation "should have the Painlevé property". Equation (94) can be integrated to give equation (83), with ν^2 as the constant of integration. Since equation (83) is equivalent to deg-$\mathrm{P_V}$ (38) then equation (94) does have the Painlevé property.

(ii) Equation (60) in [16, Theorem 5] is the second-order equation

$$\begin{aligned}&\left(1-\frac{x_n}{z}\right)\left\{\delta_z\left(\frac{\delta_z(x_n)+nz}{x_n}\right)+2x_n\right\}+2\{x_n-z+(n-b)n\}\\ &\quad=-\tfrac{1}{2}\left(\frac{\delta_z(x_n)+nz}{x_n}\right)^2+(n+1)\left(\frac{\delta_z(x_n)+nz}{x_n}\right)+(n-b-1)(3n-b+1),\end{aligned}$$

which is equation (83) with

$$\nu^2=2(b-n)^2+n^2-2n-1.$$

7. Discussion

In this paper the classical solutions of deg-$\mathrm{P_V}$ (38) have been classified. Ohyama and Okumura [40, Theorem 2.1] give a list of classical solutions of $\mathrm{P_I}$ to $\mathrm{P_V}$ and state that "deg-P5 with $\alpha=\frac{1}{2}a^2$, $\beta=-\frac{1}{8}$, $\gamma=-2$ has the algebraic solution $w(z)=1+2\sqrt{z}/a$"§ and "deg-P5 with $\beta=0$ has the Riccati type solutions". The results in this paper show that there are more classical solutions of deg-$\mathrm{P_V}$ (1). The algebraic solution is equivalent to the "seed solution" obtained by setting $n=0$ in (42), i.e.

$$u_0(x;\mu)=\frac{\mu}{x+\mu},$$

and there is a more general hierarchy of "Riccati type solutions" which are described in Theorem 5.6.

All solutions of $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ that are expressible in terms of special functions satisfy a first-order equation of the form

$$\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^n=\sum_{j=0}^{n-1}F_j(u,x)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^j,\tag{95}$$

where $F_j(u,x)$ is polynomial in u with coefficients that are rational functions of x. It can be shown that the Bessel function solutions of $\mathrm{P_{III}}$ (5) satisfy a first-order equation of the form (95) for n odd, whereas the Bessel function solutions of deg-$\mathrm{P_V}$ (38) satisfy a first-order equation of the form (95) for n even.

§ As noted in [1], there is typo in [40] who say $\beta=-8$ rather than $\beta=-\frac{1}{8}$.

The relationship between P_{III} (5) and deg-P_V (1) is similar to that between the second Painlevé equation (P_{II})

$$\frac{d^2q}{dx^2} = 2q^3 + xq, \tag{96}$$

with α a parameter, and Painlevé XXXIV equation (P_{34})

$$\frac{d^2p}{dx^2} = \frac{1}{2p}\left(\frac{dp}{dx}\right)^2 + 2p^2 - xp - \frac{(\alpha+\frac{1}{2})^2}{2p}, \tag{97}$$

which is equivalent to equation XXXIV of Chapter 14 in [24], in that both pairs of equations arise from a Hamiltonian. The Hamiltonian associated with P_{II} (96) and P_{34} (97) is

$$\mathcal{H}_{II}(q,p,z;\alpha) = \tfrac{1}{2}p^2 - (q^2+\tfrac{1}{2}z)p - (\alpha+\tfrac{1}{2})q \tag{98}$$

and so

$$\frac{dq}{dz} = p - q^2 - \tfrac{1}{2}z, \qquad \frac{dp}{dz} = 2qp + \alpha + \tfrac{1}{2}, \tag{99}$$

see [27, 41]. It is known that P_{II} (96) and P_{34} (97) have special function solutions in terms of Airy functions, cf. [13]. It can be shown that the Airy function solutions of P_{II} (96) satisfy first-order equation of the form (95) for n odd, whereas the Airy function solutions of P_{34} (97) satisfy a first-order equation of the form (95) for n even. Further the function $\sigma(z;\alpha) = \mathcal{H}_{II}(q,p,z;\alpha)$ given by (98), with q and p satisfying (99), satisfies the second-order, second degree equation (S_{II})

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2\frac{d\sigma}{dz}\left(z\frac{d\sigma}{dz} - \sigma\right) = \tfrac{1}{4}(\alpha+\tfrac{1}{2})^2, \tag{100}$$

see [27, 41]. Conversely, if $\sigma(z;\alpha)$ is a solution of (100), then

$$q(z;\alpha) = \frac{4\sigma''(z;\alpha) + 2\alpha + 1}{8\sigma'(z;\alpha)}, \qquad p(z;\alpha) = -2\sigma'(z;\alpha), \tag{101}$$

with $' \equiv d/dz$, are solutions of (96) and (97), respectively. Consequently it is easier to express classical solutions of P_{34} (97) in terms of classical solutions of S_{II} (100), which involve one determinant, rather than solutions of P_{II} (96), which involve two determinants.

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