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COMPLETE NON-SELFADJOINTNESS FOR SCHRÖDINGER OPERATORS ON THE SEMI-AXIS

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ABSTRACT. In this note we investigate complete non-selfadjointness for all maximally dissipative extensions of a Schrödinger operator on a half-line with dissipative bounded potential and dissipative boundary condition. We show that all maximally dissipative extensions that preserve the differential expression are completely non-selfadjoint. However, it is possible for maximally dissipative extensions to have a one-dimensional reducing subspace on which the operator is selfadjoint. We give a characterisation of these extensions and the corresponding subspaces and present a specific example.

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1. INTRODUCTION

We investigate complete non-selfadjointness for all maximally dissipative extensions of a Schrödinger operator on a half-line with dissipative bounded potential and dissipative boundary condition. An operator is completely non-selfadjoint if it has no non-trivial reducing subspace on which it acts as a selfadjoint operator; see Section 2 for precise definitions. Complete non-selfadjointness is an important property of an operator on a Hilbert space which, in particular, plays a crucial role in the construction of a selfadjoint dilation of a maximally dissipative operator [3, 24]. It is often a surprisingly difficult property to prove.

For limit-point Schrödinger operators on a half-line with dissipative boundary condition, but real potential, constructions of the selfadjoint dilation and results on complete non-selfadjointness can be found in Pavlov's work [20, 21]. These results are extended to the limit circle case in [1]. Divergence-form operators on a bounded interval, again with real potentials, were treated in [12]. In 3-dimensional space, complete non-selfadjointness can be obtained for Schrödinger operators with dissipative potentials using Holmgren's theorem, see [22]. Only fairly recently the result has been obtained for the one-velocity transport operator [23]. Though the result will not be surprising to experts in the field, to the best of our knowledge there is no available proof of the property for a Schrödinger operator on a half-line with a dissipative potential and boundary condition, and we provide a proof here (see Theorem 3.8).

However, in this paper we go beyond maximally dissipative extensions that preserve the differential expression (so-called proper extensions for an appropriate choice of a dual pair, see, e.g. [2, 7, 9, 17, 18]). It has long been known that dissipative operators can have non-proper maximally dissipative extensions [5]. We refer to [8] for more on non-proper extensions; one result of that paper is a characterisation of maximally dissipative extensions for the half-line Schrödinger operator which we make use of here, see (3.11). We show that it is possible for half-line Schrödinger operators to have non-proper maximally dissipative extensions with a one-dimensional reducing subspace on which the operator is selfadjoint (see Theorems 3.14 and 3.15). We give a characterisation of these extensions and the corresponding subspaces and, in the final section, we present a specific example.

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2. BACKGROUND

The following notation will be used throughout this paper. We let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, which we assume to be linear in the second and anti-linear in the first component. A linear operator on $\mathcal{H}$ will typically be denoted by A , its domain by $\mathcal{D}(A)$ and its range by $\text{Ran}(A)$. We denote the (open) upper and lower half of the complex plane by $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively.

Definition 2.1. A densely defined linear operator A with domain $\mathcal{D}(A)$ in $\mathcal{H}$ is called *dissipative* if $\text{Im} \langle u, Au \rangle \geq 0$ for all $u \in \mathcal{D}(A)$. A is called *anti-dissipative* if $(-A)$ is dissipative. Dissipative operators which have no non-trivial dissipative extensions are called *maximally dissipative operators* (MDO).

An MDO consists of one part (corresponding to the set of eigenvectors of the real point spectrum) which looks like a selfadjoint operator and a remaining part. In many situations it makes sense to study the two parts separately. This idea leads to the introduction of the notion of completely non-selfadjoint operators (corresponding to the remaining part of the operator). We first need another definition.

Definition 2.2. Let A be an operator on a Hilbert space $\mathcal{H}$. A closed subspace $\mathcal{M} \subset \mathcal{H}$ is called a *reducing subspace* of A , or is said to *reduce* the operator A , if

$$(2.1) \quad \mathcal{D}(A) = \mathcal{D}(A) \cap \mathcal{M} + \mathcal{D}(A) \cap \mathcal{M}^\perp,$$

and if

$$(2.2) \quad A(\mathcal{D}(A) \cap \mathcal{M}) \subset \mathcal{M} \quad \text{and} \quad A(\mathcal{D}(A) \cap \mathcal{M}^\perp) \subset \mathcal{M}^\perp.$$

Remark 2.3. If A is an MDO on $\mathcal{H}$, we know that $\mathbb{C}^- \subset \rho(A)$. Instead of working with the possibly unbounded operator A , it sometimes proves useful to work with the bounded resolvent $(A - \lambda)^{-1}$ instead. It is well-known that a closed subspace $\mathcal{M}$ reduces A if and only if it reduces $(A - \lambda)^{-1}$ for any $\lambda \in \rho(A)$, and we will make use of this throughout the paper.

For later purposes, we need the following result on density.

Lemma 2.4. Let $\mathcal{D}$ be dense in $\mathcal{H}$ and assume that there exists a closed subspace $\mathcal{M}$ such that

$$(2.3) \quad \mathcal{D} = \mathcal{D} \cap \mathcal{M} \oplus \mathcal{D} \cap \mathcal{M}^\perp.$$

Then,

$$(2.4) \quad \overline{\mathcal{D} \cap \mathcal{M}} = \mathcal{M} \quad \text{and} \quad \overline{\mathcal{D} \cap \mathcal{M}^\perp} = \mathcal{M}^\perp.$$

Proof. We will only show the first identity in (2.4). Let $m \in \mathcal{M}$ be arbitrary. Since $\mathcal{D}$ is dense, there exists a sequence $\{d_n\}_{n=1}^\infty$ such that $d_n \rightarrow m$. For each $n \in \mathbb{N}$, decompose $d_n = \tilde{d}_n + d_n^\perp$, where $\tilde{d}_n \in \mathcal{D} \cap \mathcal{M}$ and $d_n^\perp \in \mathcal{D} \cap \mathcal{M}^\perp$. Since $\|m - d_n\|^2 = \|m - \tilde{d}_n\|^2 + \|d_n^\perp\|^2$, this implies that $\tilde{d}_n \rightarrow m$ and $d_n^\perp \rightarrow 0$. Consequently, $\{\tilde{d}_n\}_{n=1}^\infty$ is a sequence in $\mathcal{D} \cap \mathcal{M}$ that converges to m , which shows that $\overline{\mathcal{D} \cap \mathcal{M}} = \mathcal{M}$. $\square$

We next define the property of MDOs that particularly interests us in this paper.

Definition 2.5. Let A be an MDO. A is *completely non-selfadjoint (cns)* if there exists no non-trivial reducing subspace $\mathcal{H}_1 \subseteq \mathcal{H}$ such that $A|_{\mathcal{H}_1}$ is selfadjoint.

The following result gives an explicit formula for the completely non-selfadjoint part of the operator. In the case of a relatively bounded imaginary part the formula is simple. For more general situations the formula involves operators Δ and Δ_* which are regularisations of the (possibly non-existing) imaginary part of the operator.

Proposition 2.6. (*Langer decomposition, see [16, 19]*). Let A be an MDO. Then there exists a unique decomposition of $\mathcal{H} = \mathcal{H}_{sa}(A) \oplus \mathcal{H}_{cns}(A)$ into an orthogonal sum of two reducing subspaces for A such that $A|_{\mathcal{H}_{sa}(A)}$ is selfadjoint in $\mathcal{H}_{sa}(A)$ and $A|_{\mathcal{H}_{cns}(A)}$ is completely non-selfadjoint in $\mathcal{H}_{cns}(A)$.

Let

$$(2.5) \quad T = I - 2i(A + i)^{-1} = (A - iI)(A + iI)^{-1}$$

be the Cayley transform of A , define

$$(2.6) \quad \Delta = I - T^*T = 2i \left[(A + i)^{-1} - (A^* - i)^{-1} + 2i(A^* - i)^{-1}(A + i)^{-1} \right],$$

$$(2.7) \quad \Delta_* = I - TT^* = 2i \left[(A + i)^{-1} - (A^* - i)^{-1} + 2i(A + i)^{-1}(A^* - i)^{-1} \right]$$

and set

$$\mathcal{M} := \text{Ran}(\Delta) + \text{Ran}(\Delta_*) \subseteq \mathcal{H}.$$

Then the completely non-selfadjoint part $\mathcal{H}_{cns}(A)$ is given by the closure of the linear span of $\mathcal{M}$ developed by appropriate resolvents, namely

$$\mathcal{H}_{cns}(A) = \text{clos} \left(\text{Span}_{\text{Im } \lambda < 0} \{ (A - \lambda)^{-1} \mathcal{M} \} + \text{Span}_{\text{Im } \lambda > 0} \{ (A^* - \lambda)^{-1} \mathcal{M} \} \right).$$

If A has relatively bounded imaginary part, i.e. $A = L + iV$ with $L = L^*$, $V \geq 0$, V relatively L -bounded, then there is a simple explicit expression for the completely non-selfadjoint part $\mathcal{H}_{cns}(A)$:

$$\mathcal{H}_{cns}(A) = \text{clos} \left(\text{Span}_{\text{Im } \lambda \neq 0} \{ (L - \lambda)^{-1} \text{Ran } V \} \right) = \text{clos} \left(\text{Span}_{\lambda \notin (\sigma(A) \cup \mathbb{R})} \{ (A - \lambda)^{-1} \text{Ran } V \} \right),$$

i.e. $\mathcal{H}_{cns}(A)$ is generated by the range of the imaginary part V developed by the resolvent of the operator A or its real part L . Moreover, $A|_{\mathcal{H}_{sa}(A)} = L|_{\mathcal{H}_{sa}(A)}$.

For a more explicit description of the completely non-selfadjoint part in the general case which avoids determining the operators Δ and Δ_* by making use of the Lagrange identity, see [3, Theorem 7.6] and [4, Theorem 3.3].

We will also need the following characterisation of symmetric operators which are completely non-selfadjoint.

Proposition 2.7 (Kreĭn, [14]). *A closed symmetric operator S on a Hilbert space $\mathcal{H}$ is completely non-selfadjoint if and only if*

$$(2.8) \quad \text{clos} \left(\text{Span}_{\lambda \in \mathbb{C} \setminus \mathbb{R}} \ker(S^* - \lambda) \right) = \mathcal{H}.$$

Let us also introduce the symmetric subspace of an MDO:

Definition 2.8. Let A be an MDO. Then, the *symmetric subspace* $\mathcal{H}_{\text{sym}}(A)$ of A is given by

$$(2.9) \quad \mathcal{H}_{\text{sym}}(A) = \{f \in \mathcal{D}(A) : \forall g \in \mathcal{D}(A) : \langle f, Ag \rangle = \langle Af, g \rangle\}.$$

Remark 2.9. Clearly, we have $\mathcal{H}_{\text{sa}}(A) \cap \mathcal{D}(A) \subset \mathcal{H}_{\text{sym}}(A)$. Moreover, in [15, Prop. 1.1], it was shown that

$$(2.10) \quad \mathcal{H}_{\text{sym}}(A) = \ker(A - A^*) = \{f \in \mathcal{D}(A) \cap \mathcal{D}(A^*) : Af = A^*f\}.$$

Moreover, extending the definition of the symmetric subspace to maximally anti-dissipative operators in the obvious way, we have that $\mathcal{H}_{\text{sym}}(A) = \mathcal{H}_{\text{sym}}(A^*)$.

3. THE HALF-LINE SCHRÖDINGER OPERATOR

Let $\mathcal{H} = L^2(\mathbb{R}^+)$ and let $H^2(\mathbb{R}^+)$ denote the usual Sobolev space of order 2 over $\mathbb{R}^+$. We consider a Schrödinger operator on the half-line with dissipative potential. To this end, let the minimal operator A_{min} be given by

$$(3.1) \quad A_{\text{min}} : \quad \mathcal{D}(A_{\text{min}}) = \{f \in H^2(\mathbb{R}^+) : f(0) = f'(0) = 0\}, \quad (A_{\text{min}}f)(x) = -f''(x) + V(x)f(x),$$

where $V \in L^\infty(\mathbb{R}^+)$ such that $V_I(x) := \text{Im } V(x) \geq 0$ almost everywhere. For later purposes, it will also be convenient to write $A_{\text{min}} = S + iV_I$, where the symmetric operator S is given by

$$(3.2) \quad \mathcal{D}(S) = \{f \in H^2(\mathbb{R}^+) : f(0) = f'(0) = 0\}, \quad (Sf)(x) = -f''(x) + V_R(x)f(x),$$

where $V_R(x) := \text{Re } V(x)$ and V_I is the multiplication operator by $V_I(x)$. Moreover, note that S^* is given by

$$(3.3) \quad \mathcal{D}(S^*) = H^2(\mathbb{R}^+), \quad (S^*f)(x) = -f''(x) + V_R(x)f(x).$$

Let us now show a useful lemma about the elements of the defect spaces $\ker(S^* - \bar{\lambda})$, where $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 3.1. *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then, the initial value problem*

$$(3.4) \quad S^*f = -f'' + V_Rf = \bar{\lambda}f, \quad f(0) = 1$$

has a unique solution, which we denote by ϕ_λ . Moreover, we have

$$(3.5) \quad \lim_{\mu \rightarrow \lambda} \|\phi_\lambda - \phi_\mu\| = 0.$$

Proof. Firstly note that $\dim \ker(S^* - \bar{\lambda}) = 1$ since S is limit-circle at 0 and limit-point at infinity. Now, note that if $f \in \ker(S^* - \bar{\lambda})$, it is not possible that $f(0) = 0$. This follows from the fact that if $f(0) = 0$ and $f \in \ker(S^* - \bar{\lambda})$, then f would be in the domain of the selfadjoint operator

$$(3.6) \quad \hat{S} : \quad \mathcal{D}(\hat{S}) = \{f \in H^2(\mathbb{R}^+) : f(0) = 0\}, \quad f \mapsto -f'' + V_Rf.$$

Since in this case, we would have $\hat{S}f = S^*f = \bar{\lambda}f$, this would mean that f is an eigenvector of a selfadjoint operator corresponding to a non-real eigenvalue, which is impossible. Thus, there exists a unique element ϕ_λ of $\ker(S^* - \bar{\lambda})$ with $\phi_\lambda(0) = 1$.

Now, observe that for any $\mu \in \mathbb{C} \setminus \mathbb{R}$, we have

$$(3.7) \quad \phi_\mu = \left[\mathbb{1} - (\bar{\lambda} - \bar{\mu})(\hat{S} - \bar{\mu})^{-1} \right] \phi_\lambda.$$

To see this, note that it follows from a direct calculation that $\left[\mathbb{1} - (\bar{\lambda} - \bar{\mu})(\hat{S} - \bar{\mu})^{-1} \right] \phi_\lambda \in \ker(S^* - \bar{\mu})$. Moreover, since $\left[(\hat{S} - \bar{\mu})^{-1} \phi_\lambda \right](0) = 0$, the boundary condition at 0 is satisfied. We therefore get

$$(3.8) \quad \|\phi_\lambda - \phi_\mu\| = |\lambda - \mu| \|(\hat{S} - \bar{\mu})^{-1} \phi_\lambda\| \leq |\lambda - \mu| \frac{1}{|\operatorname{Im} \mu|} \xrightarrow{\mu \rightarrow \lambda} 0,$$

which finishes the proof. $\square$

Remark 3.2. Note that the proof of the lemma shows that $\mathbb{C} \setminus \mathbb{R}$ is contained in the residual spectrum of the operator S : Clearly, for such λ the operator $S - \lambda$ is injective, else λ would be a non-real eigenvalue of the symmetric operator S ; however, the range of $S - \lambda$ is not dense, as its orthogonal complement contains ϕ_λ .

Lemma 3.3. *The operator S is completely non-selfadjoint.*

Proof. This follows from [10, Thm. 6.2], where it was shown that

$$(3.9) \quad \operatorname{clos}(\operatorname{span}\{\phi_\lambda : \lambda \in \mathbb{C} \setminus \mathbb{R}\}) = L^2(\mathbb{R}^+).$$

By Proposition 2.7, this implies that S is completely non-selfadjoint. $\square$

Remark 3.4. In the literature, a completely non-selfadjoint symmetric operator is commonly referred to as “simple”.

In [8], it was shown that all maximally dissipative extensions of $A_{\min}$ can be parametrised by $h \in \mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$ and $k \in \operatorname{Ran}(V_I^{1/2})$ satisfying

$$(3.10) \quad \frac{1}{4} \int_{\mathcal{E}} \frac{|k(x)|^2}{V_I(x)} dx \leq \operatorname{Im}(h) \quad \text{or equivalently} \quad \frac{1}{4} \|V_I^{-1/2} k\|^2 \leq \operatorname{Im}(h),$$

where $\mathcal{E} = \{x \in \mathbb{R}^+ : V_I(x) \neq 0\}$, which is determined up to a set of Lebesgue measure zero. The extensions are given by

$$(3.11) \quad \begin{aligned} A_{h,k} : \quad & \mathcal{D}(A_{h,k}) = \{f \in L^2(\mathbb{R}^+) : f'(0) = hf(0)\} \\ & (A_{h,k}f)(x) = -f''(x) + V(x)f(x) + f(0)k(x). \end{aligned}$$

Note that this means the differential expression “ $-\frac{d^2}{dx^2} + V(x)$ ” is preserved if and only if $k \equiv 0$. Moreover, from a direct calculation it can be seen that the adjoint operator $A_{h,k}^*$ is given by

$$(3.12) \quad \begin{aligned} A_{h,k}^* : \quad & \mathcal{D}(A_{h,k}^*) = \{f \in L^2(\mathbb{R}^+) : f'(0) = \bar{h}f(0) + \langle k, f \rangle\} \\ & (A_{h,k}^*f)(x) = -f''(x) + \overline{V(x)}f(x). \end{aligned}$$

The special case $h = \infty$ corresponds to a Dirichlet boundary condition and Condition (3.10) simplifies to $k = 0$ a.e. in this case. In other words, the only maximally dissipative extension of A_{min} which has a Dirichlet boundary condition at 0 is given by

$$(3.13) \quad A_{\infty,0} : \quad \mathcal{D}(A_{\infty,0}) = \{f \in H^2(\mathbb{R}^+) : f(0) = 0\}, \quad (A_{\infty,0}f)(x) = -f''(x) + V(x)f(x).$$

Next, let us introduce the subspace

$$\mathcal{D}_0 := \mathcal{D}(A_{min}) \cap \ker(V_I).$$

Note that for any $f \in \mathcal{D}_0$ we have $A_{h,k}f = Sf$. Thus, for any $f \in \mathcal{D}_0$ and $g \in \mathcal{D}(A_{h,k})$, we get

$$(3.14) \quad \begin{aligned} \langle f, A_{h,k}g \rangle &= \langle f, -g'' + V_Rg + iV_Ig + g(0)k \rangle = \langle f, -g'' + V_Rg \rangle = \langle -f'' + V_Rf, g \rangle \\ &= \langle Sf, g \rangle = \langle A_{h,k}f, g \rangle, \end{aligned}$$

where we used $f(0) = f'(0) = 0$ as well as $f \in \ker(V_I)$, which implies, since $k \in \text{Ran}(V_I^{1/2})$, that we have $f \perp k$. Hence, it is always true that $\mathcal{D}_0 \subset \mathcal{H}_{sym}(A_{h,k})$. If we assume in addition that $\mathcal{H}_{sym}(A_{h,k}) \subset \mathcal{D}_0$ – or equivalently that $\mathcal{H}_{sym}(A_{h,k}) = \mathcal{D}_0$ – let us show that $A_{h,k}$ is completely non-selfadjoint:

Lemma 3.5. *If $\mathcal{H}_{sym}(A_{h,k}) = \mathcal{D}_0$, then $A_{h,k}$ is completely non-selfadjoint.*

Proof. Let $\eta \in \mathcal{H}_{sa}(A_{h,k})$. For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, let ϕ_λ be such that $\ker(S^* - \bar{\lambda}) = \text{span}\{\phi_\lambda\}$. For any $\lambda \in \mathbb{C}^-$, we get

$$(3.15) \quad \langle \phi_\lambda, \eta \rangle = \langle \phi_\lambda, (A_{h,k} - \lambda)(A_{h,k} - \lambda)^{-1}\eta \rangle = \langle \phi_\lambda, (S - \lambda)(A_{h,k} - \lambda)^{-1}\eta \rangle = 0,$$

where we used that by assumption $(A_{h,k} - \lambda)^{-1}\eta \in \mathcal{D}(A_{h,k}) \cap \mathcal{H}_{sa}(A_{h,k}) \subset \mathcal{H}_{sym}(A_{h,k}) = \mathcal{D}_0$. By a similar argument, replacing $A_{h,k}$ by $A_{h,k}^*$, we find that $\langle \phi_\lambda, \eta \rangle = 0$ for all $\lambda \in \mathbb{C}^+$. By (3.9), this implies that $\eta = 0$ and thus $A_{h,k}$ is completely non-selfadjoint by Proposition 2.7. $\square$

3.1. The non-critical case. We refer to the case when we have strict inequality in (3.10) as the non-critical case. We are ready to prove our first main result:

Theorem 3.6. *If $h \in \mathbb{C}^+$ and $k \in \text{Ran}(V_I^{1/2})$ are such that we have strict inequality in (3.10), then $A_{h,k}$ is completely non-selfadjoint.*

Proof. Firstly, for any $f \in \mathcal{D}(A_{h,k})$, consider

$$\begin{aligned} \text{Im}\langle f, A_{h,k}f \rangle &= \text{Im}\langle f, -f'' \rangle + \|V_I^{1/2}f\|^2 + \text{Im}\langle f, f(0)k \rangle \\ &= \text{Im}(h)|f(0)|^2 + \|V_I^{1/2}f\|^2 + \text{Im}\langle V_I^{1/2}f, f(0)V_I^{-1/2}k \rangle \\ &= \left(\text{Im}(h) - \frac{1}{4}\|V_I^{-1/2}k\|^2 \right) |f(0)|^2 + \left\| V_I^{1/2}f - i\frac{f(0)}{2}V_I^{-1/2}k \right\|^2 \\ &\geq \left(\text{Im}(h) - \frac{1}{4}\|V_I^{-1/2}k\|^2 \right) |f(0)|^2. \end{aligned}$$

Now, if $f \in \mathcal{H}_{sym}(A_{h,k})$, this implies in particular $\text{Im}\langle f, A_{h,k}f \rangle = 0$. Consequently, if $f \in \mathcal{H}_{sym}(A_{h,k})$, we get

$$0 = \text{Im}\langle f, A_{h,k}f \rangle \geq \left(\text{Im}(h) - \frac{1}{4}\|V_I^{-1/2}k\|^2 \right) |f(0)|^2.$$

Since $\text{Im}(h) > \frac{1}{4}\|V_I^{-1/2}k\|^2$ by assumption, this implies $f(0) = 0$ and consequently, $f \in \mathcal{D}(A_{\min})$. We then get

$$(3.16) \quad 0 = \text{Im}\langle f, A_{h,k}f \rangle = \|V_I^{1/2}f\|^2,$$

and thus $f \in \ker(V_I^{1/2}) = \ker(V_I)$. This shows that $\mathcal{H}_{\text{sym}}(A_{h,k}) = \mathcal{D}_0$, which by Lemma 3.5 implies that $A_{h,k}$ is completely non-selfadjoint. $\square$

Remark 3.7. With this result, the case of a purely real potential V , corresponding to $V_I \equiv 0$ is completely covered. In this situation, all maximally dissipative extensions of $A_{\min}$ are given by $A_{h,0}$, where $h \in \mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$. Now, if $\text{Im}(h) > 0$, we can apply Theorem 3.6 and conclude that $A_{h,0}$ is completely non-selfadjoint. On the other hand, if $h = \infty$ or $\text{Im}(h) = 0$, the operator $A_{h,0}$ is obviously selfadjoint. Thus, in the following, we will always assume that $V_I \not\equiv 0$.

3.2. The critical case. Let us now focus on the critical case $\text{Im}(h) = \frac{1}{4}\|V_I^{-1/2}k\|^2$, which we will assume for the remainder of this paper. We will have to distinguish two cases: (i) the case of a selfadjoint boundary condition, corresponding to $\frac{1}{4}\|V_I^{-1/2}k\|^2 = \text{Im}(h) = 0$ and (ii) the case of a dissipative boundary condition, corresponding to $\frac{1}{4}\|V_I^{-1/2}k\|^2 = \text{Im}(h) > 0$.

3.2.1. Selfadjoint boundary condition. Since we are in the critical case $\frac{1}{4}\|V_I^{-1/2}k\|^2 = \text{Im}(h)$ but also assume a selfadjoint boundary condition $\text{Im}(h) = 0$, this implies that $k = 0$ a.e. Hence, we are considering only operators of the form

$$(3.17) \quad A_{h,0} : \quad \mathcal{D}(A_{h,0}) = \{f \in H^2(\mathbb{R}^+) : f'(0) = hf(0)\}, \quad (A_{h,0}f)(x) = -f''(x) + V(x)f(x),$$

where $h \in \mathbb{R} \cup \{\infty\}$. Let us now show that all these operators are completely non-selfadjoint. As usual, $h = \infty$ is interpreted as the Dirichlet condition at the endpoint.

Theorem 3.8. *Assume $V_I \not\equiv 0$. The operators $A_{h,0}$, where $h \in \mathbb{R} \cup \{\infty\}$ are completely non-selfadjoint.*

Proof. Let $|M|$ denote the Lebesgue measure of a set $M \subset \mathbb{R}^+$ and define

$$x_0 := \inf \{x > 0 : |(0, x + \varepsilon) \cap \{x \in \mathbb{R}^+ : V_I(x) > 0\}| > 0 \text{ for all } \varepsilon > 0\}.$$

Then $x_0 \geq 0$ and we have $V_I(x) = 0$ a.e. on $(0, x_0]$.

Choose a decreasing sequence (x_n) in (x_0, ∞) such that $x_n \rightarrow x_0$. Then

$$|(x_0, x_n + \varepsilon) \cap \{x \in \mathbb{R}^+ : V_I(x) > 0\}| > 0 \text{ for all } \varepsilon > 0.$$

Let $A_s \subset A_{h,0}$ such that A_s is selfadjoint on the subspace $\overline{\mathcal{D}(A_s)} \subseteq \mathcal{H}$. We need to show that $\mathcal{D}(A_s) = \{0\}$. Let $u \in \mathcal{D}(A_s)$, then

$$0 = \text{Im} \langle u, A_s u \rangle \geq \int_{x_0}^{x_n + \varepsilon} V_I |u|^2.$$

Thus, for any $\varepsilon > 0$, we can choose a sequence $(\tilde{x}_n)$ with $\tilde{x}_n \in (x_0, x_n + \varepsilon)$ such that $u(\tilde{x}_n) = 0$. As $u \in H^2(\mathbb{R}^+)$, it is continuous, so $u(x_0) = 0$.

Moreover, u' is continuous, so both its real and imaginary parts are. By the Mean Value Theorem there exist a sequence (x'_n) with $\tilde{x}_{n+1} \leq x'_n \leq \tilde{x}_n$ such that $\text{Re}(u')(x'_n) = 0$ and another sequence (x''_n) with $\tilde{x}_{n+1} \leq x''_n \leq \tilde{x}_n$ such that $\text{Im}(u')(x''_n) = 0$. By continuity, $\text{Re}(u')(x_0) = 0 = \text{Im}(u')(x_0)$, so $u'(x_0) = 0$.

First assume $x_0 > 0$ and consider the operator $A_{x_0} := A_{(0,x_0)} \oplus A_{(x_0,\infty)}$, where both $A_{(0,x_0)}$ and $A_{(x_0,\infty)}$ are given by the expression

$$f \mapsto -f'' + V_R f$$

with

$$\mathcal{D}(A_{(0,x_0)}) := \{f \in H^2(0, x_0) : f'(0) = hf(0), f(x_0) = 0\}$$

and

$$\mathcal{D}(A_{(x_0,\infty)}) := \{f \in H^2(x_0, \infty) : f(x_0) = 0\}.$$

Since $u(x_0) = 0$, we have that $u_0 := u|_{(0,x_0)} \in \mathcal{D}(A_{(0,x_0)})$ and $u_\infty = u|_{(x_0,\infty)} \in \mathcal{D}(A_{(x_0,\infty)})$.

Next, let $\lambda \in \mathbb{C}^-$ and consider $(A_s - \lambda)u = g$ for $u \in \mathcal{D}(A_s)$ and $g \in \overline{\mathcal{D}(A_s)}$. Then

$$(A_s - \lambda)u = g \iff -u'' + (V - \lambda)u = g \iff -u'' + (V_R - \lambda)u = g \iff (A_{x_0} - \lambda)u = g$$

and

$$u = (A_{x_0} - \lambda)^{-1}g = u_0 \oplus u_\infty = (A_{(0,x_0)} - \lambda)^{-1}g_0 \oplus (A_{(x_0,\infty)} - \lambda)^{-1}g_\infty,$$

where $g_0 = g|_{(0,x_0)}$ and $g_\infty = g|_{(x_0,\infty)}$.

Let $G_{(0,x_0)}$ denote the Green's function associated with $A_{(0,x_0)}$ and let φ_l, φ_r be the solutions to $-\varphi'' + V_R \varphi = \lambda \varphi$ in $(0, x_0)$ satisfying $\varphi'_l(0, \lambda) = h\varphi_l(0, \lambda)$ and $\varphi_r(x_0, \lambda) = 0$, respectively. Let $W(\lambda)$ be the corresponding Wronskian. Then for $x \in (0, x_0)$ we have

$$\begin{aligned} u_0(x) &= \int_0^{x_0} G_{(0,x_0)}(x, y) g_0(y) dy \\ &= \int_0^x \frac{\varphi_r(x, \lambda) \varphi_l(y, \lambda)}{W(\lambda)} g_0(y) dy + \int_x^{x_0} \frac{\varphi_l(x, \lambda) \varphi_r(y, \lambda)}{W(\lambda)} g_0(y) dy. \end{aligned}$$

Moreover,

$$(3.18) \quad u'_0(x) = \int_0^x \frac{\varphi'_r(x, \lambda) \varphi_l(y, \lambda)}{W(\lambda)} g_0(y) dy + \int_x^{x_0} \frac{\varphi'_l(x, \lambda) \varphi_r(y, \lambda)}{W(\lambda)} g_0(y) dy.$$

As shown above, since $u \in \mathcal{D}(A_s)$,

$$(3.19) \quad u'_0(x_0) = \frac{\varphi'_r(x_0, \lambda)}{W(\lambda)} \int_0^{x_0} \varphi_l(y, \lambda) g_0(y) dy = 0.$$

Since $\varphi'_r(x_0, \lambda) \neq 0$, Equation (3.19) implies that $\bar{g}_0 \perp \varphi_l(\cdot, \lambda)$ for all $\lambda \in \mathbb{C}^-$. By analyticity, $\bar{g}_0 \perp \varphi_l(\cdot, \lambda)$ for all $\lambda \in \mathbb{C}$. Choosing λ to run through all eigenvalues of $A_{(0,x_0)}$, we get that $\bar{g}_0$ is orthogonal to all eigenvectors and root vectors. These are complete, see e.g. [11, Chapter V.2] or [25], so therefore, $g_0(x) = 0$.

Next we consider g_∞ . Let $\tilde{\varphi}_l$ be the solution to $-\varphi'' + V_R \varphi = \lambda \varphi$ in (x_0, ∞) satisfying $\tilde{\varphi}_l(x_0, \lambda) = 0$. Let f be the L^2 -solution of $-f'' + V_R f = \lambda f$ and $\tilde{W}(\lambda)$ be the corresponding Wronskian. Then for $x \in (x_0, \infty)$ we have

$$(3.20) \quad u_\infty(x) = \int_{x_0}^x \frac{f(x, \lambda) \tilde{\varphi}_l(y, \lambda)}{\tilde{W}(\lambda)} g_\infty(y) dy + \int_x^\infty \frac{\tilde{\varphi}_l(x, \lambda) f(y, \lambda)}{\tilde{W}(\lambda)} g_\infty(y) dy.$$

Moreover,

$$(3.21) \quad u'_\infty(x) = \int_{x_0}^x \frac{f'(x, \lambda) \tilde{\varphi}_l(y, \lambda)}{\tilde{W}(\lambda)} g_\infty(y) dy + \int_x^\infty \frac{\tilde{\varphi}'_l(x, \lambda) f(y, \lambda)}{\tilde{W}(\lambda)} g_\infty(y) dy.$$

Since $u \in \mathcal{D}(A_s)$,

$$(3.22) \quad u'_\infty(x_0) = \frac{\tilde{\varphi}'_I(x_0, \lambda)}{\tilde{W}(\lambda)} \int_{x_0}^{\infty} f(y, \lambda) g_\infty(y) dy = 0.$$

This implies that $\overline{g_\infty} \perp f(\cdot, \lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By [10, Theorem 6.2] we get $g_\infty(x) = 0$.

Therefore, $g(x) = g_0(x) + g_\infty(x) = 0$ a.e. and this implies that $u = 0$ and so $\mathcal{D}(A_s)$ is trivial.

Finally, we note that if $x_0 = 0$, then $u = u_\infty$ which can be shown to be zero also in this case by the same argument as above. $\square$

3.2.2. Dissipative boundary condition. In this section, we will only consider the case where $\frac{1}{4}\|V_I^{-1/2}k\|^2 = \text{Im}(h) > 0$. We investigate when the operators $A_{h,k}$ are completely non-selfadjoint and when they possess a non-trivial reducing selfadjoint subspace. Since we are only considering the selfadjoint/completely non-selfadjoint/symmetric subspaces of the operator $A_{h,k}$, we will drop the dependence on $A_{h,k}$ and only write $\mathcal{H}_{sa}$, $\mathcal{H}_{cns}$ and $\mathcal{H}_{sym}$, respectively.

The next two lemmas give a more explicit description of $\mathcal{H}_{sym}$.

Lemma 3.9. *We have $\mathcal{H}_{sym} = \ker(A_{h,k} - A_{h,k}^*) = \{f \in \mathcal{D}(A_{h,k}) : V_I f = \frac{i}{2}f(0)k\}$.*

Proof. The first equality was established in Remark 2.9, so we only need to show the second. It is obvious that $\ker(A_{h,k} - A_{h,k}^*) \subset \{f \in \mathcal{D}(A_{h,k}) : V_I f = \frac{i}{2}f(0)k\}$. Hence, we need to show that if $f \in \mathcal{D}(A_{h,k})$ such that $V_I f = \frac{i}{2}f(0)k$, then this implies that $f \in \mathcal{D}(A_{h,k}^*) = \{f \in H^2(\mathbb{R}^+) : f'(0) = \bar{h}f(0) + \langle k, f \rangle\}$, see also (3.12). But this follows from

$$(3.23) \quad f'(0) = hf(0) = \bar{h}f(0) + 2i \text{Im}(h)f(0) = \bar{h}f(0) + \left\langle V_I^{-1/2}k, V_I^{-1/2} \left(\frac{i}{2}f(0)k \right) \right\rangle$$

$$(3.24) \quad = \bar{h}f(0) + \langle k, f \rangle,$$

where we have used that $\text{Im}(h) = \frac{1}{4}\|V_I^{-1/2}k\|^2$ and $V_I f = \frac{i}{2}f(0)k$. $\square$

Lemma 3.10. *We have the following two cases:*

- (i) *If there exists a function $K_V \in \mathcal{D}(A_{h,k})$ with $K_V(0) \neq 0$ such that $V_I K_V = \frac{i}{2}K_V(0)k$, then $\mathcal{H}_{sym} = \mathcal{D}_0 + \text{span}\{K_V\}$.*
- (ii) *If there does not exist such a function, then $\mathcal{H}_{sym} = \mathcal{D}_0$.*

Proof. By Lemma 3.9, we have $\mathcal{H}_{sym} = \ker(A_{h,k} - A_{h,k}^*) = \{f \in \mathcal{D}(A_{h,k}) : V_I f = \frac{i}{2}f(0)k\}$. Case (i): clearly, we have $\text{span}\{K_V\} + \mathcal{D}_0 \subset \mathcal{H}_{sym}$. Let us now show the other inclusion. Let $f \in \mathcal{H}_{sym} = \ker(A_{h,k} - A_{h,k}^*)$. If $f(0) = 0$, then $f \in \mathcal{H}_{sym}$ implies $V_I f = \frac{i}{2}f(0)k = 0$ and thus $f \in \mathcal{D}_0$. Hence, assume $f(0) \neq 0$ from now on. By decomposing

$$(3.25) \quad f = f - \frac{f(0)}{K_V(0)}K_V + \frac{f(0)}{K_V(0)}K_V,$$

it is sufficient to show that $f - \frac{f(0)}{K_V(0)}K_V \in \mathcal{D}_0$. This follows by observing that

$$(3.26) \quad V_I \left(f - \frac{f(0)}{K_V(0)}K_V \right) = \frac{i}{2}f(0)k - \frac{i}{2} \frac{f(0)}{K_V(0)}K_V(0)k = 0,$$

and using that $f'(0) = hf(0)$ and $K_V'(0) = hK_V(0)$ since $f, K_V \in \mathcal{D}(A_{h,k})$. Thus, we have shown that $f \in \mathcal{D}_0 + \text{span}\{K_V\}$.

Case (ii): Again, it is clear that $\mathcal{D}_0 \subset \mathcal{H}_{sym}$. Now, let $f \in \mathcal{H}_{sym}$. If $f(0) \neq 0$, this would contradict the assumptions of Case (ii) since one could choose $K_V := f$ in this case. Thus,

since $f(0) = 0$ and since $f \in \mathcal{H}_{sym}$, we get $V_I f = \frac{i}{2}f(0)k = 0$ and thus $f \in \mathcal{D}_0$. This shows the lemma. $\square$

Before we proceed, we need to prove the following technical result.

Lemma 3.11. *Suppose that $g \in H^1(\mathbb{R}^+)$ and V is measurable. If $Vg = 0$ a.e., then this implies that $Vg' = 0$ a.e.*

Proof. Let $B := \{x \in \mathbb{R}^+ : V(x) = 0\}$; this is defined up to a null set $\mathcal{K}$. Since $Vg = 0$ a.e., there is a null set $\mathcal{N}$ such that $g(x) = 0$ if $x \in \mathbb{R}^+ \setminus (B \cup \mathcal{N} \cup \mathcal{K})$. Then, up to a countable set I , no points in $\mathbb{R}^+ \setminus (B \cup \mathcal{N} \cup \mathcal{K})$ are isolated. So, for every $x \in \mathbb{R}^+ \setminus (B \cup \mathcal{N} \cup \mathcal{K} \cup I)$, we can find a sequence $\{x_n\} \subset \mathbb{R}^+ \setminus (B \cup \mathcal{N} \cup \mathcal{K} \cup I)$, where $x_n \neq x$, such that $x_n \rightarrow x$. Since $g \in H^1(\mathbb{R}^+)$, we have that g is differentiable – up to a null set F . Consequently, we have

$$(3.27) \quad g'(x) = \lim_{n \rightarrow \infty} \frac{g(x) - g(x_n)}{x - x_n} = 0$$

for every $x \in \mathbb{R}^+ \setminus (B \cup \mathcal{N} \cup \mathcal{K} \cup I \cup F)$ and hence, we conclude that $V(x)g'(x) = 0$ for every $x \in \mathbb{R}^+ \setminus (\mathcal{N} \cup \mathcal{K} \cup I \cup F)$. But since $\mathcal{N}, \mathcal{K}, I$, and F are all null sets, this implies that $Vg' = 0$ a.e. and thus the lemma. $\square$

We are now ready to prove a uniqueness result in $\mathcal{H}_{sa}$.

Lemma 3.12. *Let $f \in \mathcal{H}_{sa}$ and assume there exists a $\lambda \in \mathbb{C}^-$ such that $(A_{h,k} - \lambda)^{-1}f \in \mathcal{D}_0$. Then $f = 0$.*

Proof. Take any $\mu \in \mathbb{C}^-$. By the resolvent identity, we get

$$(3.28) \quad (A_{h,k} - \mu)^{-1}f = (A_{h,k} - \lambda)^{-1}f + (\mu - \lambda)(A_{h,k} - \mu)^{-1}(A_{h,k} - \lambda)^{-1}f.$$

By assumption, we have $g := (A_{h,k} - \lambda)^{-1}f \in \mathcal{D}_0$. Let us thus focus on the second term $(A_{h,k} - \mu)^{-1}g$. Using that $(A_{h,k} - \mu)^{-1}g \in \mathcal{H}_{sa} \cap \mathcal{D}(A_{h,k}) \subset \mathcal{H}_{sym}$, we know from Lemma 3.10 that there exist $g_0 \in \mathcal{D}_0$ and $\tau \in \mathbb{C}$ such that

$$(3.29) \quad (A_{h,k} - \mu)^{-1}g = g_0 + \tau K_V$$

or – equivalently –

$$(3.30) \quad \begin{aligned} g &= (A_{h,k} - \mu)(A_{h,k} - \mu)^{-1}g \\ &= -g_0'' + V_R g_0 - \mu g_0 + \tau(-K_V'' + V_R K_V + iV_I K_V + K_V(0)k - \mu K_V) \\ &= -g_0'' + V_R g_0 - \mu g_0 + \tau(-K_V'' + V_R K_V - iV_I K_V - \mu K_V). \end{aligned}$$

Now, observe that since $g, g_0 \in \mathcal{D}_0$ we have $g \in \ker(V_I)$ as well as $-g_0'' + V_R g_0 - \mu g_0 \in \ker(V_I)$, which follows from a twofold application of Lemma 3.11 to g_0 . Next, let us consider $-K_V'' + V_R K_V - iV_I K_V - \mu K_V$. We want to show that there exists at most one $\tilde{\mu} \in \mathbb{C}^-$ such that $-K_V'' + V_R K_V - iV_I K_V - \tilde{\mu} K_V \in \ker(V_I)$. To this end, assume the following equality holds

$$(3.31) \quad \langle V_I K_V, -K_V'' + V_R K_V - iV_I K_V - \mu K_V \rangle = 0.$$

Then,

$$(3.32) \quad \begin{aligned} 0 &= \langle V_I K_V, -K_V'' + V_R K_V - iV_I K_V - \mu K_V \rangle \\ &= -\langle V_I K_V, K_V'' + V_R K_V \rangle - i\|V_I K_V\|^2 - \mu\|V_I^{1/2} K_V\|^2. \end{aligned}$$

Since $K_V \notin \ker(V_I)$, we have $\|V_I^{1/2}K_V\|^2 > 0$ and thus – depending on whether

$$(3.33) \quad \tilde{\mu} := \frac{-\langle V_I K_V, K_V'' + V_R K_V \rangle - i\|V_I K_V\|^2}{\|V_I^{1/2}K_V\|^2}$$

is an element of $\mathbb{C}^-$ or not – there exists at most one solution $\tilde{\mu} \in \mathbb{C}^-$ such that (3.31) is satisfied. Hence, if $\mu \in \mathbb{C}^- \setminus \{\tilde{\mu}\}$, we have $\langle V_I K_V, -K_V'' + V_R K_V - iV_I K_V - \mu K_V \rangle \neq 0$ and thus $-K_V'' + V_R K_V - iV_I K_V - \mu K_V \notin \ker(V_I)$. So, if $\mu \neq \tilde{\mu}$, then Equation (3.30) implies that $\tau = 0$. Consequently, $(A_{h,k} - \mu)^{-1}g = (A_{h,k} - \mu)^{-1}(A_{h,k} - \lambda)^{-1}f = g_0 \in \mathcal{D}_0$ and (3.28) therefore implies that $(A_{h,k} - \mu)^{-1}f \in \mathcal{D}_0$ for all $\mu \in \mathbb{C}^- \setminus \{\tilde{\mu}\}$. Letting ϕ_μ be as in the proof of Lemma 3.1, we get by the same argument as in (3.15) that $\langle \phi_\mu, f \rangle = 0$ for all $\mu \in \mathbb{C}^- \setminus \{\tilde{\mu}\}$.

Next, since $f \in \mathcal{H}_{sa}$, we have $g \in \mathcal{H}_{sa}$ and therefore, $(A_{h,k}^* - \bar{\mu})^{-1}g \in \mathcal{H}_{sa} \cap \mathcal{D}(A_{h,k}^*)$. Now, $\mathcal{H}_{sa} \cap \mathcal{D}(A_{h,k}^*) \subset \mathcal{H}_{sym}(A_{h,k}^*) = \mathcal{H}_{sym}$. Thus, there exist $\hat{g}_0 \in \mathcal{D}_0$ and $\hat{\tau} \in \mathbb{C}$ such that $(A_{h,k}^* - \bar{\mu})^{-1}g = \hat{g}_0 + \hat{\tau}K_V$. By repeating the same argument as was presented after Equation (3.29), we may conclude that there exists at most one $\hat{\mu} \in \mathbb{C}^+$ such that $(A_{h,k}^* - \bar{\mu})^{-1}f \in \mathcal{D}_0$ for all $\bar{\mu} \in \mathbb{C}^+ \setminus \{\hat{\mu}\}$. Consequently, we have shown

$$(3.34) \quad \langle \phi_\mu, f \rangle = 0$$

for all $\mu \in \mathbb{C} \setminus (\mathbb{R} \cup \{\tilde{\mu}, \hat{\mu}\})$. By Lemma 3.1, we have

$$(3.35) \quad 0 \leq |\langle \phi_{\tilde{\mu}}, f \rangle| = |\langle \phi_{\tilde{\mu}}, f \rangle - \langle \phi_\mu, f \rangle| \leq \|\phi_{\tilde{\mu}} - \phi_\mu\| \|f\| \xrightarrow{\mu \rightarrow \tilde{\mu}} 0.$$

Analogously, it also follows that $\langle \phi_{\hat{\mu}}, f \rangle = 0$. Arguing similarly as in Lemma 3.5, we get $f = 0$, which finishes the proof. $\square$

Corollary 3.13. *If $\mathcal{H}_{sym} = \mathcal{D}_0 + \text{span}\{K_V\}$, then $\dim \mathcal{H}_{sa} \leq 1$.*

Proof. Fix an arbitrary $\lambda \in \mathbb{C}^-$. Now, assume there exist two linearly independent $f, g \in \mathcal{H}_{sa}$. Since $\mathcal{H}_{sa} \cap \mathcal{D}(A_{h,k}) \subset \mathcal{H}_{sym}$, we know that there exist $f_0, g_0 \in \mathcal{D}_0$ and $\tau_f, \tau_g \in \mathbb{C} \setminus \{0\}$ such that

$$(3.36) \quad (A_{h,k} - \lambda)^{-1}f = f_0 + \tau_f K_V \text{ and } (A_{h,k} - \lambda)^{-1}g = g_0 + \tau_g K_V.$$

This implies

$$(3.37) \quad (A_{h,k} - \lambda)^{-1}(\tau_g f - \tau_f g) = \tau_g f_0 - \tau_f g_0 \in \mathcal{D}_0,$$

which by Lemma 3.12 implies $\tau_g f - \tau_f g = 0$ and hence that f and g are linearly dependent. $\square$

We are now prepared to show the main result for the critical case with dissipative boundary conditions. It turns out that for the operator $A_{h,k}$ to have a non-trivial reducing selfadjoint subspace, a few rather restrictive conditions have to be met:

Theorem 3.14. *Suppose that equality holds in (3.10). The maximally dissipative extension $A_{h,k}$ has a non-trivial reducing selfadjoint subspace if and only if there exists a function $K_V \in \mathcal{D}(A_{h,k})$ with $K_V(0) \neq 0$ such that*

- (i) $V_I K_V = \frac{i}{2} K_V(0)k$ and
- (ii) $A_{h,k} K_V \in \text{span}\{K_V\}$.

Moreover, if both Conditions (i) and (ii) are satisfied, then $\mathcal{H}_{sa} = \text{span}\{K_V\}$.

Proof. Let us first assume that $A_{h,k}$ has a non-trivial reducing selfadjoint subspace. Combining Lemma 3.5 with Lemma 3.10, this implies that there exists a function $\tilde{K}_V$ with $\tilde{K}_V(0) \neq 0$ such that $V_I \tilde{K}_V = \frac{i}{2} \tilde{K}_V(0)k$ and that $\mathcal{H}_{sym} = \mathcal{D}_0 + \text{span}\{\tilde{K}_V\}$. By Corollary 3.13, we know that $\dim \mathcal{H}_{sa} = 1$. As this is finite-dimensional, by Lemma 2.4, we have $\mathcal{H}_{sa} \cap \mathcal{D}(A_{h,k}) = \mathcal{H}_{sa}$. Consequently, we have $\mathcal{H}_{sa} \subset \mathcal{H}_{sym} = \mathcal{D}_0 + \text{span}\{\tilde{K}_V\}$. Now, it is not possible that $\mathcal{H}_{sa} = \text{span}\{f_0\}$ for some $f_0 \in \mathcal{D}_0$, since we would have $(A_{h,k} + i)^{-1}f_0 \in \mathcal{D}_0$ and thus by Lemma 3.12, we would have $f_0 = 0$. Hence, $\mathcal{H}_{sa} = \text{span}\{f_0 + \tilde{K}_V\}$ for some $f_0 \in \mathcal{D}_0$. Set $K_V = f_0 + \tilde{K}_V$. Since $f_0 \in \mathcal{D}_0$, the function K_V satisfies $K_V(0) \neq 0$ as well as Condition (i). Since $\mathcal{H}_{sa}$ is a reducing subspace, $A_{h,k}K_V \in \text{span}\{K_V\}$, as required.

Let us now assume that there exists a $K_V \in \mathcal{D}(A_{h,k})$ with $K_V(0) \neq 0$ such that Conditions (i) and (ii) are both satisfied. From Condition (ii), we get $A_{h,k}K_V = zK_V$ for some $z \in \mathbb{C}$. Moreover, since $K_V \in \mathcal{H}_{sym}$, we have $A_{h,k}K_V = A_{h,k}^*K_V = zK_V$ and therefore $\text{span}\{K_V\}$ reduces $A_{h,k}$. This also implies that $z = \bar{z} \in \mathbb{R}$ and thus $\mathcal{H}_{sa} = \text{span}\{K_V\}$. $\square$

We summarize the results shown so far in this chapter:

- (1) $\text{Im } h > 0$:
 - (a) $\text{Im}(h) > \frac{1}{4}\|V_I^{-1/2}k\|^2$: $A_{h,k}$ is completely non-selfadjoint;
 - (b) $\text{Im}(h) = \frac{1}{4}\|V_I^{-1/2}k\|^2$:
 - (i) the conditions (i) and (ii) in Theorem 3.14 are not satisfied: $A_{h,k}$ is completely non-selfadjoint;
 - (ii) the conditions (i) and (ii) in Theorem 3.14 are satisfied: $A_{h,k}$ has a non-trivial reducing selfadjoint subspace $\mathcal{H}_{sa} = \text{span}\{K_V\}$;
- (2) $h \in \mathbb{R} \cup \{\infty\}$:
 - (a) $V_I \equiv 0$: $A_{h,k}$ is selfadjoint;
 - (b) $V_I \not\equiv 0$: $A_{h,k}$ is completely non-selfadjoint.

3.3. Construction of maximally dissipative extensions with a real eigenvalue.

Theorem 3.15. *Let the operator A_{min} be given by (3.1) and let $\lambda \in \mathbb{R}$. If there exists a non-zero solution $g \in H^2(\mathbb{R}^+)$ to the differential equation*

$$(3.38) \quad A_{min}^*g = -g'' + V_Rg - iV_Ig = \lambda g,$$

then there exists a unique maximally dissipative extension A_λ of A_{min} for which $\ker(A_\lambda - \lambda) = \text{span}\{g\}$ is a non-trivial reducing selfadjoint subspace.

Proof. Let $g \in H^2(\mathbb{R}^+)$ be a non-trivial solution of (3.38). Multiplying both sides of (3.38) by $\bar{g}$ and integrating from 0 to ∞ yields – using integration by parts –

$$(3.39) \quad \overline{g(0)}g'(0) = \int_0^\infty (-V_R(x) + iV_I(x) + \lambda)|g(x)|^2 dx - \int_0^\infty |g'(x)|^2 dx.$$

Let us now argue that $\text{Im } \overline{g(0)}g'(0) \neq 0$. By way of contradiction, assume that $\text{Im } \overline{g(0)}g'(0) = 0$. Then by comparing the imaginary parts in (3.39), we get

$$(3.40) \quad \int_0^\infty V_I(x)|g(x)|^2 dx = 0,$$

which is equivalent to $g \in \ker(V_I)$. Now, since $\text{Im } \overline{g(0)}g'(0) = 0$, we have $g'(0) = hg(0)$ for some $h \in \mathbb{R} \cup \{\infty\}$. Then $g \in \mathcal{D}(A_{h,0}) = \mathcal{D}(A_{h,0}^*)$ – the maximally dissipative extension of

A_{min} with the selfadjoint boundary condition $g'(0) = hg(0)$. Since $g \in \ker(V_I)$, we get

$$(3.41) \quad A_{h,0}g = -g'' + V_Rg + iV_Ig = -g'' + V_Rg - iV_Ig = A_{h,0}^*g = A_{min}^*g = \lambda g.$$

But this would mean that $g \in \mathcal{H}_{sa}(A_{h,0})$ contradicting the fact that $A_{h,0}$ is completely non-selfadjoint, which was shown in Theorem 3.8. Hence, $\text{Im } \overline{g(0)}g'(0) \neq 0$, which implies in particular that $g(0) \neq 0$. Now, let $\eta := \frac{-2i}{g(0)}g$ be the unique element in $\text{span}\{g\}$ which satisfies $\eta(0) = -2i$. Plugged into (3.39), this yields

$$(3.42) \quad \overline{\eta(0)}\eta'(0) = 2i\eta'(0) = \int_0^\infty (-V_R(x) + iV_I(x) + \lambda)|\eta(x)|^2 dx - \int_0^\infty |\eta'(x)|^2 dx$$

and by taking imaginary parts, we get

$$(3.43) \quad \text{Re } \eta'(0) = \frac{1}{2} \int_0^\infty V_I(x)|\eta(x)|^2 dx > 0.$$

We now claim that the operator $A_\lambda := A_{\frac{i}{2}\eta'(0), V_I\eta}$ (defined in (3.11)) is a maximally dissipative extension of A_{min} , which has a non-trivial reducing selfadjoint subspace, which is given by $\ker(A_\lambda - \lambda) = \text{span}\{\eta\}$. Firstly, note that it follows from (3.43) that we are in the critical case. It can now be directly verified that $\eta \in \mathcal{D}(A_{\frac{i}{2}\eta'(0), V_I\eta})$ satisfies Conditions (i) and (ii) in Theorem 3.14. Thus, it has a reducing selfadjoint subspace spanned by η corresponding to the eigenvalue λ . Uniqueness follows from the fact that there exists at most one solution to (3.38) in $H^2(\mathbb{R}^+)$ (the problem is limit-circle at zero and limit-point at infinity). $\square$

In what follows, we will work with the following definition of the essential spectrum of A from [6].

Definition 3.16. The *essential spectrum* of an operator A is defined as

$$\sigma_{ess}(A) = \{\lambda \in \mathbb{C} : \text{Ran}(A - \lambda) \text{ is not closed or } \dim \ker(A - \lambda) = \infty \text{ or } \dim \ker(A^* - \bar{\lambda}) = \infty\}.$$

Remark 3.17. Note that we will only consider operators A such that $\dim \ker(A - \lambda)$ and $\dim \ker(A^* - \bar{\lambda})$ are both at most one-dimensional. In this case, we get the following simpler description of the essential spectrum

$$(3.44) \quad \sigma_{ess}(A) = \{\lambda \in \mathbb{C} : \text{Ran}(A - \lambda) \text{ is not closed}\}$$

and in particular, since $\text{Ran}(A - \lambda)$ is closed if and only if $\text{Ran}(A^* - \bar{\lambda})$ is closed [13, Chapter I, Thm. 5.13], we get $\sigma_{ess}(A) \cap \mathbb{R} = \sigma_{ess}(A^*) \cap \mathbb{R}$.

Proposition 3.18. *Let $\lambda \in \mathbb{R}$. If $\lambda \notin \sigma_{ess}(A_{min})$, then, the equation $A_{min}^*g = -g'' + V_Rg - iV_Ig = \lambda g$ has a non-zero solution in $H^2(\mathbb{R}^+)$.*

Proof. Assume that no such solution exists, i.e. $\ker(A_{min}^* - \lambda) = \{0\}$. This implies that $\overline{\text{Ran}(A_{min} - \lambda)} = \mathcal{H}$. Since $\lambda \notin \sigma_{ess}(A_{min})$, this implies $\text{Ran}(A_{min} - \lambda) = \mathcal{H}$. Since from the argument in the proof of Theorem 3.15 using (3.39), we have $\ker(A_{min} - \lambda) = \{0\}$, this would imply that $(A_{min} - \lambda)$ is boundedly invertible and thus $\lambda \in \rho(A_{min})$. Since resolvent sets are open, there exists an $\varepsilon > 0$ such that $(\lambda - i\varepsilon) \in \rho(A_{min})$, thus implying that A_{min} is maximally dissipative, which is a contradiction. Hence, $\text{Ran}(A_{min} - \lambda)$ is not closed and thus $\lambda \in \sigma_{ess}(A_{min})$. $\square$

Altogether, we have shown the following result:

Theorem 3.19. *For any $\lambda \in \mathbb{R} \setminus \sigma_{ess}(A_{min})$, there exists a unique maximally dissipative extension A_λ which has a one-dimensional reducing selfadjoint subspace corresponding to the eigenvalue λ .*

4. EXAMPLE

As an application, consider a finitely supported potential well of the form $V(x) = i\chi_{(0,1)}(x)$, i.e.

$$(4.1) \quad A_{min} : \quad \mathcal{D}(A_{min}) = \{f \in H^2(\mathbb{R}^+) : f(0) = f'(0) = 0\}, \quad (A_{min}f)(x) = -f''(x) + i\chi_{(0,1)}(x)f(x).$$

Here, $\chi_{(0,1)}$ is the indicator function over the interval $(0,1)$. Firstly, let us argue that $\sigma_{ess}(A_{min}) = [0, \infty)$. To see this, consider the extension $A_{\infty,0}$ of A_{min} , which is of the form $A_{\infty,0} = S_{\infty,0} + V$, where $S_{\infty,0}$ is the selfadjoint Dirichlet Laplacian on the half-line. It is well-known that $\sigma_{ess}(S_{\infty,0}) = [0, \infty)$. From a direct calculation, it can be seen that $V(S_{\infty,0} - i)^{-1}$ is Hilbert-Schmidt. This implies that V is a relatively compact perturbation of $S_{\infty,0}$ and therefore $\sigma_{ess}(A_{\infty,0}) = \sigma_{ess}(S_{\infty,0}) = [0, \infty)$. Now, since $\dim(\mathcal{D}(A_{\infty,0})/\mathcal{D}(A_{min})) = 1$, this implies that $\text{Ran}(A_{min} - \lambda)$ is closed if and only if $\text{Ran}(A_{\infty,0} - \lambda)$ is closed. Consequently, we have $\sigma_{ess}(A_{min}) = [0, \infty)$ as well. According to Theorem 3.19, for every $\lambda < 0$, there is a unique maximally dissipative extension A_λ of A_{min} which has a non-trivial selfadjoint subspace corresponding to the eigenvalue λ . In what follows, we set $\lambda = -\xi^2$, where $\xi > 0$.

In order to determine A_λ , we need to find the function $\eta \in H^2(\mathbb{R}^+)$ with $\eta(0) = -2i$ such that

$$(4.2) \quad -\eta''(x) - i\chi_{(0,1)}(x)\eta(x) = -\xi^2\eta(x).$$

We introduce the numbers $\sigma_\pm$, which are given by

$$(4.3) \quad \sigma_\pm = (\xi \pm \sqrt{\xi^2 - i}) \exp(\pm \sqrt{\xi^2 - i}).$$

Then, the solution to this initial value problem is given by

$$(4.4) \quad \eta(x) = \begin{cases} \frac{2i\sigma_-}{\sigma_+ - \sigma_-} \exp(\sqrt{\xi^2 - i} \cdot x) + \frac{-2i\sigma_+}{\sigma_+ - \sigma_-} \exp(-\sqrt{\xi^2 - i} \cdot x) & \text{if } x \in (0, 1) \\ \left[\frac{2i\sigma_-}{\sigma_+ - \sigma_-} \exp(\sqrt{\xi^2 - i} + \xi) + \frac{-2i\sigma_+}{\sigma_+ - \sigma_-} \exp(-\sqrt{\xi^2 - i} + \xi) \right] \exp(-\xi \cdot x) & \text{if } x \in (1, \infty) \end{cases}.$$

Let us now determine $h := \frac{i}{2}\eta'(0)$ and $k(x) := \chi_{(0,1)}(x)\eta(x)$ in order to construct the maximally dissipative extension $A_\lambda = A_{h,k}$. We find

$$(4.5) \quad \begin{aligned} h &= \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} \sqrt{\xi^2 - i} \quad \text{and} \\ k(x) &= \begin{cases} \frac{2i\sigma_-}{\sigma_+ - \sigma_-} \exp(\sqrt{\xi^2 - i} \cdot x) + \frac{-2i\sigma_+}{\sigma_+ - \sigma_-} \exp(-\sqrt{\xi^2 - i} \cdot x) & \text{if } x \in (0, 1) \\ 0 & \text{if } x \in (1, \infty) \end{cases}, \end{aligned}$$

which parametrise the special maximally dissipative extension $A_{h,k}$ of A_{min} such that it has a non-trivial reducing selfadjoint subspace spanned by η corresponding to the eigenvalue $\lambda = -\xi^2$.

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