Some extended geometric processes and their estimation methods

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***Abstract*—Modelling the failure process of a system is one of the most important problems in the reliability and maintenance research community. The geometric process (GP) is widely used for modelling the failure process because it can describe the phenomenon that the working times after repairs become shorter and shorter. This article reviews the geometric process and its extensions based on existing research. It also reviews relevant methods for estimating parameters, model performances, and widely used distributions for times to first failures. Future challenges for the GP-like processes will be discussed.**

***Index Terms*—Geometric process, Reliability, Stochastic process, Parameter estimations, Model performance**

I. Introduction

Modelling the failure process of a system has been an important process in the reliability and maintenance research community. Its focus is on modelling the working times between failures (WTBF). A common phenomenon is that WTBF of a system after repairs may become shorter and shorter, and the repair times may become longer and longer. This phenomenon can be described by the geometric process (GP), which was first introduced by [1]. Since then, the GP has been widely applied in different fields such as software reliability analysis [2], [3], reliability analysis [4], maintenance policy optimisation [5], [6], warranty analysis [7] and electricity pricing, etc. [8]–[11]. Besides, the multiple extensions of the GP has been introduced: the arithmetic geometric process (AGP) [12], the α-series process (α-series) [13], the threshold geometric process (TGP) [14], the extended poisson process (EPP) [15], the exponent extended geometric process (EEGP) [16], the extended geometric process (EGP) [17], the doubly geometric process (DGP) [18], the semi-geometric process (SGP) [19], the alternating geometric process (alternating GP) [7], and the double ratio geometric process (DRGP) [20].

The flexibility of GP's extensions means that the GP-like models can be widely used in not only reliability and maintenance but also other fields. Therefore, the purposes of this paper are

- to conduct a brief review of the existing extensions of the GP;
- to review the estimation methods of the GP and its extensions; and
- to review the application of the GP and its extensions in other fields.

The remainder of the paper is structured as follows. Section II describes the GP and its extensions according to existing research. Section III describes the common estimation methods of the parameters and model performance. Section IV describes the application beyond the reliability and maintenance.

II. The GP and its extensions

This section introduces the GP and its extensions in detail. Before that, relevant concepts will be introduced.

Assume that X and Y are two random variables. If for every real number u, the inequality

$$P(X \geq u) \geq P(Y \geq u)$$

holds, then X is stochastically greater than or equal to Y, or Y is stochastically less than or equal to X. Then, the monotonicity of a stochastic process can be defined as the following.

Definition 1: [21] Given a stochastic process $\{X_k = 1, 2...\}$, if $X_k \leq_{\rm st} X_{k+1}$ or $X_k \geq_{\rm st} X_{k+1}$ for $k = 1, 2...$ then $\{X_k, k = 1, 2, ...\}$ is stochastically increasing or decreasing.

[1] proposes the definition of the GP, as shown below.

Definition 2: [1] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, \dots\}$, if they are independent and the cdf of X_k is given by $F(a^{k-1}x)$ for $k = 1, 2, \dots$, where a is a positive constant, then $\{X_k, k = 1, 2, \cdots\}$ is called a geometric process (GP).

A. The arithmetic geometric process (AGP)

The AGP is proposed by [12] with the following definition.

Definition 3: [12] Given a sequence of random variable $\{X_k, k = 1, 2, ...\}$, if for real positive number a and real number d, and the cdf of X_k is given by $F(a^{k-1}x + a^{k-1}(k-1)d)$, where d and a are called the common difference and common ratio of the AGP, respectively, then $\{X_k, k = 1, 2, ...\}$ is called an AGP.

B. The α-series process (α-series)

The α-series process (short for: α-series) is proposed by [13]. It considers that the expected number of counts at an arbitrary time does not exist for the decreasing geometric process. The α-series process has a finite expected number of counts within certain conditions.

Definition 4: [13] alpha-seriesGiven a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of X_k is given by $F(k^{\alpha}x)$ for $k = 1, 2...$, where α is a positive constant.

C. Threshold geometric process (TGP)

The threshold geometric process (TGP) is proposed by [14]. The main characterise of TGP is that it can describe non-monotonous trends in a failure process. According to [14], it has separate GPs and the kth GP is denoted

$$\text{GP}_k = \{X_i, T_n \leqslant i < T_{n+1}\}, k = 1, ..., K,$$

to each trend with turning point T_n. Then each GP has its a and the following definition can be introduced

Definition 5: [14] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of X_k is given by $F(a_k^{i-T_n}x)$ for $k = 1, 2...$, where a is a positive constant, then $\{X_k, k = 1, 2, ...\}$ is called a threshold geometric process.

D. The Extended Poisson Process (EPP)

The extended Poisson process(EPP) is proposed by [15]. Similar to the TGP, the EPP can model a failure process with non-monotonous trends.

Definition 6: [15] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of X_k is given by $F((\alpha a^{k-1} + \beta b^{k-1})x)$ for $k = 1, 2...$, where $F(x)$ is an exponential cdf, $\alpha + \beta \neq 0$, $\alpha, \beta \geq 0$, $a \geq 0$ and $0 < b \leq 1$, then $\{X_k, k = 1, 2, ...\}$ is called as an extended poisson process (EPP).

E. Binary geometric process (BGP)

The binary geometric process (BGP) is proposed by [22] for longitudinal binary data with rends. The definition is given by

Definition 7: [22] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, which is a GP, if W_k is given by $W_k = I(X_k > 1) = I(Y_k > a^{k-1})$, and the cdf of Y_k is $1 - F(a^{k-1}x)$, then $\{Y_k, k = 1, 2, ...\}$ is called as a binary GP model.

F. Exponent extended geometric process (EEGP)

[16] proposed the EEGP with the following definition.

Definition 8: [16] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and non-negative, and the cdf of X_k is given by $F(a^{b_k}Y_k)$ for $k = 1, 2...$, where a is a positive constant, $(b_k)_{k\geq 1}$ forms a non-decreasing sequence such that $0 = b_1 \leq b_2, \ldots, \lim_{k\to\infty} b_k = \infty$ and Y_k are the inter-arrival times of a RP, then $\{X_k, k = 1, 2, ...\}$ is called an exponent extended geometric process.

G. Doubly geometric process (DGP)

The doubly geometric process (DGP) is proposed by [18]. It can model a situation that the shape parameters of the lifetime distributions of inter-arrival times X_k changes k with monotonously increasing or decreasing. The definition of a DPG is given in the following definition.

Definition 9: [18] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of X_k is given by $F(a^{k-1}x^{h(k)})$ for $k = 1, 2...$, where a is a positive constant, $h(k)$ is a function of k and the likelihood of the parameters in $h(k)$ has a known closed form, and $h(k) > 0$, then $\{X_k, k = 1, 2, ...\}$ is called as a doubly geometric process (DGP). The a^{k-1} is refereed as the scale impact factor and $h(k)$ as the shape impact factor.

H. Semi-geometric process (SGP)

[19] considered that the independence assumption of the GP is too restrictive due to a sequence of independent random variables $\{X_k, k = 1, 2, \ldots\}$. Working times between occurrences of failures may be statistically dependent in the real world. In this paper, a relax assumption is that times between failures are independent.

The definition of the SGP is following

Definition 10: [19] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if $P\{X_k < x \mid X_{k-1} = x_{k-1}, \ldots, X_1 = x_1\} = P\{X_k < x \mid X_{k-1} = x_{k-1}\}$ and the marginal distribution of X_k has cdf of $F(a^{k-1}x)$, where a is a positive constant, then $\{X_k, k = 1, 2, ...\}$ is called an SGP.

I. The extended geometric process (EGP)

[17] considered that some failures are slight and the influence of such failure after repair can be totally eliminated, which means that the system is not degenerative. The definition of the EGP is following.

Definition 11: [17] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of X_k is $F_k = pF_{k-1}(x) + qF_{k-1}(ax)$, where a, p, q are all positive constants, and $p + q = 1$ for $k = 1, 2, \ldots$, then $\{X_k, k = 1, 2, ...\}$ is called an EGP, and p_k is called the extended factor.

J. Geometric Pólya-Aeppli process (GPAP)

The geometric Pólya-Aeppli process (GPAP) is proposed by [23], which is a combination of the GP and Pólya-Aeppli process. The definition of the GPAP is given by

Definition 12: [23] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of X_k is given by $F_k(x) = 1 - (-\varrho)e^{\mu a^{k-1}}$ for $k = 1, 2...$, where a_k is positive parameters and μ is the expectation of the exponential underlying distribution of X_1. Then $\{X_k, k = 1, 2, ...\}$ is a geometric Pólya-Aeppli process.

K. Double ratio geometric process (DRGP)

The double ratio geometric process (DRGP) is proposed by [20]. Suppose that the hazard function of X_k is denoted by $r_k(x)$ and that $\{X_k, k = 1, 2, \ldots\}$ for follows the GP, then

$$h_k(x) = ah_{k-1}(ax), \tag{1}$$

where the two a's play different roles in and have different implications in describing maintenance effectiveness: the first a describes the effectiveness on how the hazard function is affected and the second a (i.e., the one multiplying x in the parentheses) describes the effectiveness on how the age of the item under maintenance is affected. Therefore, the following definition of the DRGP can be given.

Definition 13: [20] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of X_k is given by $F_k(x) = 1 - (1 - F_1(a_k x))^{b_k/a_k}$ for $k = 1, 2...$, where a_k and b_k are positive parameters and $a_1 = b_1 = 1$. Then $\{X_k, k = 1, 2, ...\}$ is a double-ratio geometric process.

L. Alternating geometric process (Alternating GP)

The alternating geometric process (alternating GP) is proposed by [24]. It combines the ideas of the GP and the alternating renewal process [21]. The following definition of the alternating GP is given by [24].

Definition 14: [24] $\{X_k\}_1^\infty$ and $\{Y_k\}_1^\infty$ are independent sequences, if $\{X_k\}_1^\infty$ is a stochastically decreasing GP with parameters $\{a, F_{X_1}(t)\}$, $a > 1$ and $\{Y_k\}_1^\infty$ is stochastically increasing GP with parameters $\{b, F_{Y_1}(t)\}, 0 < b < 1$. then the sequence of pairs of random variables $\{(X_1, Y_1), (X_2, Y_2), \ldots, \}$ is called an alternating GP with parameters $\{a, F_{X_1}(x); b, F_{Y_1}(x)\}$.

The alternating GP denotes X_k as the kth operational time with cdf F_{X_k} and Y_k as the kth repair time with cdf F_{Y_k}. Therefore, comparing with other GP-like models, the main difference of the alternating GP is that it describes both working times and repair times between failures.

The following table summarizes the GP and its extensions.

TABLE I
SUMMARY OF THE GP AND ITS EXTENSIONS

Survival distribution	Model	Reference
$F(a^{k-1}x)$	GP	[1]
$F_k(x) = (g(k)x)$	GRP	[25]
$F(a^{k-1}x + a^{k-1}(k-1)d)$	AGP	[12]
$F(k^a x)$	α-series	[13]
$F(a^{k-M_i}x)$	TGP	[14]
$F((\alpha a^{k-1} + \beta b^{k-1})x)$	EPP	[15]
$1 - F(a^{k-1}x)$	BGP	[22]
$F(a^{b_k}x)$	EEGP	[16]
$F(a^{k-1}x)$	SGP	[19]
$F_k = pF_{k-1}(x) + qF_{k-1}(ax)$	EGP	[17]
$F(a^{k-1}x^{(1+\log(k))^b})$	DGP	[18]
$F_k(x) = 1 - (-\varrho)e^{\mu a^{k-1}}$	GPAP	[23]
$1 - (1 - F(a_k x))^{b_k/a_k}$	DRGP	[20]
$\{(X_1, Y_1), (X_2, Y_2), \ldots, \}$	Alternating GP	[24]

III. THE ESTIMATION METHODS OF THE GP AND ITS EXTENSIONS

The estimation of the parameters of the GP-like models, and the performance of the GP-like models are important. Besides, the distribution of the time to the first failure is another important issue. The choose of such a distribution influences the number of parameter in a GP-like model. Therefore, the statistical inference of the GP-like models would be influenced by the different distributions (for first occurrence time of failure) and the estimation methods of parameters. This section reviews several methods for parameter estimation, probability distributions for the time to first failure, methods for assessing model performance in the existing research.

A. Parameter estimation

The two following tables show the abbreviation of several parameter estimations, the type of GP-like models, and the distribution of the time to first failure.

TABLE II
ABBREVIATION OF PARAMETER ESTIMATION METHODS

Abbreviation	Parameter estimation methods	Times
ML	Maximum likelihood estimation	15
MML	Modified maximum likelihood estimation	2
MM	Modified moment estimation	7
MLS	Modified least square	5
MLM	Modified L-moments estimation	3
LSE	Least squared estimation	3
MMS	Maximum spacing estimation	3
BE	Bayesian estimation	2

TABLE III
DISTRIBUTIONS AND PARAMETER ESTIMATION METHODS

Model	First occurrence time	Parameter estimation method	Reference
GP	Weibull distribution	MM & MML	[26]
GP	Gamma distribution	ML & MML	[27]
GP	Weibull distribution	BE	[28]
GP	Rayleigh distribution	ML	[29]
GP	Lindley distribution	ML & BE	[30]
GP	Hjorth marginal distribution	ML & MM & LSE & MMS	[31]
GP	Scaled Muth	ML & MM & LSE & MMS	[32]
GP	Exponential distribution	ML & MM & MLS	[33]
GP	Lindley distribution	ML & MLS & MM & MLM	[34]
GP	Inverse gaussian distribution	ML & MM	[35]
GP	Power lindley distribution	ML & MLS & MM & MLM	[36]
GP	Exponential distribution	ML & MM & MLS	[33]
DGP	Weibull distribution	ML	[37]
DPG	Exponential distribution	ML	[38]
α-series	Rayleigh Distribution	ML & MMS & MLS MM & MLM	[39]
α-series	Truncated normal distribution	ML & LSE	[40]
α-series	Lognormal distribution	ML & MM	[41]

The ML estimation is the most frequently used for estimating parameters of the GP-like models. Then, the MM estimation method is the second frequently used.

B. Model performance

Table IV shows some common methods for assessing model performance of the GP-like models.

Bias and the MSE are most frequently used for discussing the statistical inference of the GP-like models.

IV. THE APPLICATION OF THE GP

The GP-like models are normally used for modelling the working times after the repairs and the repair times for each maintenance activities. Commonly, the GP-like models can be applied into the reliability and maintenance and warranty analysis for estimating the repair or replacement interval time, planning the corrective and preventative maintenance

TABLE IV
METHODS FOR ASSESSING MODEL PERFORMANCE

Abbreviation	Model performance	Reference
AIC	Akaike information criterion	[18]
AICc	Corrected AIC	[18]
BIC	Bayesian information criterion	[42], [43]
Bias	Bias	[28], [29], [31], [32], [34], [36]–[38], [41]
MSE	Mean squared error	[26]–[29], [31]–[34], [36]–[38], [40], [41], [43]–[45]
MPE	Mean percentage error	[44], [45]
ML	Maximum likelihood	[22], [34]
LSE	Least squared error	[22]
AMSE	Adjusted mean squared error	[14]
DIC	Deviance information criterion	[42], [45], [46]
APB	Absolute percentage bias	[42]
RMS	Root mean square error	[42]
CP	Coverage percentage	[42]

and predicting the number of warranty claims etc. Several existing research publications have reviewed [7], [47]. In this article, we would like to focus on other applications of the GP-like models. The following table shows a summary of the application of the GP-like models beyond reliability and maintenance.

TABLE V
SUMMARY OF THE APPLICATIONS OF THE GP-LIKE MODELS

Application	GP-like model	Problems
Recruitment policy	GP	Loss of recruitment [48]
Crime analysis	GP	Cannabis offenses [49]
Crime analysis	GP	Number of arrests [46]
Electricity price	TGP	Forecasting [45]
Recruitment	GP	Forecasting & Recruitment policy [43]
Coal mining disasters	BGP	Simulation [22]
Coal mining disasters	GP	Simulation [44]
Market stock	TGP	High-low stock price [42]
SARS epidemic	TGP	Forecasting epidemic [14]

[14] proposed the TGP, which can be considered a collection of several GPs with different ratio parameters a_k based on the moving windows. The moving windows can separate a group of data into different subsets of fixed length starting from the first window with ratio a_1. In this research, the SARS data of Hong Kong, Singapore, Toronto and Taipei were separated into several subsets based on their moving windows. The model performance were be estimated by log-LSE and LSE method, respectively.

[45] used the TGP to modelling the changes of electricity market prices over time. One characteristic of the electricity market prices is that it is floating with high spikes at different time period for a whole day. The threshold of the TGP corresponds to the high spikes of the electricity price. This is the reason why the TGP had a good performance on this case.

[49] used the GP to analyse the cannabis offences in New South Wales of the Australia. According to their data analysis, they found that there exists underdispersed and overdispersed data, which shows a non-monotonicity trend. The Markov chain Monte Carlo was used introduced to overcome such problems. Besides, they discussed the model performance under both underdispersed and overdispersed data and used the result of an equidispersion situation as comparison.

Other research face to similar problems, such as [46]. It proposed the multivariate generalized poisson log-t geometric process to analyse the number of arrests due to the use of two illicit drugs. [44] and [22] used the data of coal mining disasters to compare the model performance among several times series models and their extended GP-like models.

From the above-reviewed papers, the GP-like models can be applied into other fields such as the business analysis and health analysis. Besides, due to the properties of the GP-like models, we consider that they can be used in the fields where data can be described by the counting process such as the queuing problem and the transportation.

V. CONCLUSIONS

This article briefly reviewed the GP and its extensions. Relevant statistical methods for estimating parameters and methods for assessing model performance were reviewed for the GP-like models. Besides, the application beyond the reliability and maintenance were reviewed with corresponding problems, such as the forecasting problem of the electricity price, the simulation of the coal mining disasters, and the predication of arrests etc.

We propose that the GP-like models can be used in more fields beyond the reliability and maintenance. As long as the data itself has a trend over time, which can be considered as a counting process, it may be modelled by the GP-like models. Furthermore, parameter estimation, assessment of model performance, and the distribution of X_1 of the GP-like models should be investigated and compared.

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