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# Some extended geometric processes and their estimation methods

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***Abstract*—Modelling the failure process of a system is one of the most important problems in the reliability and maintenance research community. The geometric process (GP) is widely used for modelling the failure process because it can describe the phenomenon that the working times after repairs become shorter and shorter. This article reviews the geometric process and its extensions based on existing research. It also reviews relevant methods for estimating parameters, model performances, and widely used distributions for times to first failures. Future challenges for the GP-like processes will be discussed.**

***Index Terms*—Geometric process, Reliability, Stochastic process, Parameter estimations, Model performance**

## I. Introduction

Modelling the failure process of a system has been an important process in the reliability and maintenance research community. Its focus is on modelling the working times between failures (WTBF). A common phenomenon is that WTBF of a system after repairs may become shorter and shorter, and the repair times may become longer and longer. This phenomenon can be described by the geometric process (GP), which was first introduced by [1]. Since then, the GP has been widely applied in different fields such as software reliability analysis [2], [3], reliability analysis [4], maintenance policy optimisation [5], [6], warranty analysis [7] and electricity pricing, etc. [8]–[11]. Besides, the multiple extensions of the GP has been introduced: the arithmetic geometric process (AGP) [12], the $\alpha$-series process ($\alpha$-series) [13], the threshold geometric process (TGP) [14], the extended poisson process (EPP) [15], the exponent extended geometric process (EEGP) [16], the extended geometric process (EGP) [17], the doubly geometric process (DGP) [18], the semi-geometric process (SGP) [19], the alternating geometric process (alternating GP) [7], and the double ratio geometric process (DRGP) [20].

The flexibility of GP's extensions means that the GP-like models can be widely used in not only reliability and maintenance but also other fields. Therefore, the purposes of this paper are

- to conduct a brief review of the existing extensions of the GP;
- to review the estimation methods of the GP and its extensions; and
- to review the application of the GP and its extensions in other fields.

The remainder of the paper is structured as follows. Section II describes the GP and its extensions according to existing research. Section III describes the common estimation methods of the parameters and model performance. Section IV describes the application beyond the reliability and maintenance.

## II. The GP and its extensions

This section introduces the GP and its extensions in detail. Before that, relevant concepts will be introduced.

Assume that $X$ and $Y$ are two random variables. If for every real number $u$, the inequality

$$P(X \geq u) \geq P(Y \geq u)$$

holds, then $X$ is stochastically greater than or equal to $Y$, or $Y$ is stochastically less than or equal to $X$. Then, the monotonicity of a stochastic process can be defined as the following.

*Definition 1:* [21] Given a stochastic process $\{X_k = 1, 2...\}$, if $X_k \leq_{\text{st}} X_{k+1}$ or $X_k \geq_{\text{st}} X_{k+1}$ for $k = 1, 2...$ then $\{X_k, k = 1, 2, ...\}$ is stochastically increasing or decreasing.

[1] proposes the definition of the GP, as shown below.

*Definition 2:* [1] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, \dots\}$, if they are independent and the cdf of $X_k$ is given by $F(a^{k-1}x)$ for $k = 1, 2, \dots$, where $a$ is a positive constant, then $\{X_k, k = 1, 2, \cdots\}$ is called a geometric process (GP).

### A. The arithmetic geometric process (AGP)

The AGP is proposed by [12] with the following definition.

*Definition 3:* [12] Given a sequence of random variable $\{X_k, k = 1, 2, ...\}$, if for real positive number a and real number d, and the cdf of $X_k$ is given by $F(a^{k-1}x + a^{k-1}(k-1)d)$, where $d$ and $a$ are called the common difference and common ratio of the AGP, respectively, then $\{X_k, k = 1, 2, ...\}$ is called an AGP.

### B. The $\alpha$-series process ($\alpha$-series)

The $\alpha$-series process (short for: $\alpha$-series) is proposed by [13]. It considers that the expected number of counts at an arbitrary time does not exist for the decreasing geometric process. The $\alpha$-series process has a finite expected number of counts within certain conditions.

*Definition 4:* [13] alpha-seriesGiven a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of $X_k$ is given by $F(k^\alpha x)$ for $k = 1, 2...$, where $\alpha$ is a positive constant.

### C. Threshold geometric process (TGP)

The threshold geometric process (TGP) is proposed by [14]. The main characterise of TGP is that it can describe non-monotonous trends in a failure process. According to [14], it has separate GPs and the $k$th GP is denoted

$$\text{GP}_k = \{X_i, T_n \leqslant i < T_{n+1}\}, k = 1, ..., K,$$

to each trend with turning point $T_n$. Then each GP has its $a$ and the following definition can be introduced

*Definition 5:* [14] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of $X_k$ is given by $F(a_k^{i-T_n} x)$ for $k = 1, 2...$, where $a$ is a positive constant, then $\{X_k, k = 1, 2, ...\}$ is called a threshold geometric process.

### D. The Extended Poisson Process (EPP)

The extended Poisson process(EPP) is proposed by [15]. Similar to the TGP, the EPP can model a failure process with non-monotonous trends.

*Definition 6:* [15] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of $X_k$ is given by $F((\alpha a^{k-1} + \beta b^{k-1})x)$ for $k = 1, 2...$, where $F(x)$ is an exponential cdf, $\alpha + \beta \neq 0$, $\alpha, \beta \geq 0$, $a \geq 0$ and $0 < b \leq 1$, then $\{X_k, k = 1, 2, ...\}$ is called as an extended poisson process (EPP).

### E. Binary geometric process (BGP)

The binary geometric process (BGP) is proposed by [22] for longitudinal binary data with rends. The definition is given by

*Definition 7:* [22] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, which is a GP, if $W_k$ is given by $W_k = I(X_k > 1) = I(Y_k > a^{k-1})$, and the cdf of $Y_k$ is $1 - F(a^{k-1}x)$, then $\{Y_k, k = 1, 2, ...\}$ is called as a binary GP model.

### F. Exponent extended geometric process (EEGP)

[16] proposed the EEGP with the following definition.

*Definition 8:* [16] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and non-negative, and the cdf of $X_k$ is given by $F(a^{b_k} Y_k)$ for $k = 1, 2...$, where $a$ is a positive constant, $(b_k)_{k\geq 1}$ forms a non-decreasing sequence such that $0 = b_1 \leq b_2, \ldots, \lim_{k\to\infty} b_k = \infty$ and $Y_k$ are the inter-arrival times of a RP, then $\{X_k, k = 1, 2, ...\}$ is called an exponent extended geometric process.

### G. Doubly geometric process (DGP)

The doubly geometric process (DGP) is proposed by [18]. It can model a situation that the shape parameters of the lifetime distributions of inter-arrival times $X_k$ changes $k$ with monotonously increasing or decreasing. The definition of a DPG is given in the following definition.

*Definition 9:* [18] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of $X_k$ is given by $F(a^{k-1}x^{h(k)})$ for $k = 1, 2...$, where $a$ is a positive constant, $h(k)$ is a function of k and the likelihood of the parameters in $h(k)$ has a known closed form, and $h(k) > 0$, then $\{X_k, k = 1, 2, ...\}$ is called as a doubly geometric process (DGP). The $a^{k-1}$ is refereed as the scale impact factor and $h(k)$ as the shape impact factor.

### H. Semi-geometric process (SGP)

[19] considered that the independence assumption of the GP is too restrictive due to a sequence of independent random variables $\{X_k, k = 1, 2, \dots\}$. Working times between occurrences of failures may be statistically dependent in the real world. In this paper, a relax assumption is that times between failures are independent.

The definition of the SGP is following

*Definition 10:* [19] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if $P\{X_k < x \mid X_{k-1} = x_{k-1}, \ldots, X_1 = x_1\} = P\{X_k < x \mid X_{k-1} = x_{k-1}\}$ and the marginal distribution of $X_k$ has cdf of $F(a^{k-1}x)$, where $a$ is a positive constant, then $\{X_k, k = 1, 2, ...\}$ is called an SGP.

### I. The extended geometric process (EGP)

[17] considered that some failures are slight and the influence of such failure after repair can be totally eliminated, which means that the system is not degenerative. The definition of the EGP is following.

*Definition 11:* [17] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of $X_k$ is $F_k = pF_{k-1}(x) + qF_{k-1}(ax)$, where $a$, $p$, $q$ are all positive constants, and $p + q = 1$ for $k = 1, 2, \ldots$, then $\{X_k, k = 1, 2, ...\}$ is called an EGP, and $p_k$ is called the extended factor.

### J. Geometric Pólya-Aeppli process (GPAP)

The geometric Pólya-Aeppli process (GPAP) is proposed by [23], which is a combination of the GP and Pólya-Aeppli process. The definition of the GPAP is given by

*Definition 12:* [23] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of $X_k$ is given by $F_k(x) = 1 - (-\varrho)e^{\mu a^{k-1}}$ for $k = 1, 2...$, where $a_k$ is positive parameters and $\mu$ is the expectation of the exponential underlying distribution of $X_1$. Then $\{X_k, k = 1, 2, ...\}$ is a geometric Pólya-Aeppli process.

### K. Double ratio geometric process (DRGP)

The double ratio geometric process (DRGP) is proposed by [20]. Suppose that the hazard function of $X_k$ is denoted by $r_k(x)$ and that $\{X_k, k = 1, 2, \dots\}$ for follows the GP, then

$$h_k(x) = ah_{k-1}(ax), \tag{1}$$

where the two $a$'s play different roles in and have different implications in describing maintenance effectiveness: the first $a$ describes the effectiveness on how the hazard function is affected and the second $a$ (i.e., the one multiplying $x$ in the

parentheses) describes the effectiveness on how the age of the item under maintenance is affected. Therefore, the following definition of the DRGP can be given.

*Definition 13:* [20] Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of $X_k$ is given by $F_k(x) = 1 - (1 - F_1(a_k x))^{b_k/a_k}$ for $k = 1, 2...$, where $a_k$ and $b_k$ are positive parameters and $a_1 = b_1 = 1$. Then $\{X_k, k = 1, 2, ...\}$ is a double-ratio geometric process.

### *L. Alternating geometric process (Alternating GP)*

The alternating geometric process (alternating GP) is proposed by [24]. It combines the ideas of the GP and the alternating renewal process [21]. The following definition of the alternating GP is given by [24].

*Definition 14:* [24] $\{X_k\}_1^\infty$ and $\{Y_k\}_1^\infty$ are independent sequences, if $\{X_k\}_1^\infty$ is a stochastically decreasing GP with parameters $\{a, F_{X_1}(t)\}$, $a > 1$ and $\{Y_k\}_1^\infty$ is stochastically increasing GP with parameters $\{b, F_{Y_1}(t)\}, 0 < b < 1$. then the sequence of pairs of random variables $\{(X_1, Y_1), (X_2, Y_2), \ldots, \}$ is called an alternating GP with parameters $\{a, F_{X_1}(x); b, F_{Y_1}(x)\}$.

The alternating GP denotes $X_k$ as the $k$th operational time with cdf $F_{X_k}$ and $Y_k$ as the $k$th repair time with cdf $F_{Y_k}$. Therefore, comparing with other GP-like models, the main difference of the alternating GP is that it describes both working times and repair times between failures.

The following table summarizes the GP and its extensions.

TABLE I
SUMMARY OF THE GP AND ITS EXTENSIONS

| Survival distribution | Model | Reference |
|---|---|---|
| $F(a^{k-1}x)$ | GP | [1] |
| $F_k(x) = (g(k)x)$ | GRP | [25] |
| $F(a^{k-1}x + a^{k-1}(k-1)d)$ | AGP | [12] |
| $F(k^a x)$ | $\alpha$-series | [13] |
| $F(a^{k-M_i}x)$ | TGP | [14] |
| $F((\alpha a^{k-1} + \beta b^{k-1})x)$ | EPP | [15] |
| $1 - F(a^{k-1}x)$ | BGP | [22] |
| $F(a^{b_k}x)$ | EEGP | [16] |
| $F(a^{k-1}x)$ | SGP | [19] |
| $F_k = pF_{k-1}(x) + qF_{k-1}(ax)$ | EGP | [17] |
| $F(a^{k-1}x^{(1+\log(k))^b})$ | DGP | [18] |
| $F_k(x) = 1 - (-\varrho)e^{\mu a^{k-1}}$ | GPAP | [23] |
| $1 - (1 - F(a_k x))^{b_k/a_k}$ | DRGP | [20] |
| $\{(X_1, Y_1), (X_2, Y_2), \ldots, \}$ | Alternating GP | [24] |

## III. THE ESTIMATION METHODS OF THE GP AND ITS EXTENSIONS

The estimation of the parameters of the GP-like models, and the performance of the GP-like models are important. Besides, the distribution of the time to the first failure is another important issue. The choose of such a distribution influences the number of parameter in a GP-like model. Therefore, the statistical inference of the GP-like models would be influenced by the different distributions (for first occurrence time of failure) and the estimation methods of parameters. This section reviews several methods for parameter estimation, probability distributions for the time to first failure, methods for assessing model performance in the existing research.

### *A. Parameter estimation*

The two following tables show the abbreviation of several parameter estimations, the type of GP-like models, and the distribution of the time to first failure.

TABLE II
ABBREVIATION OF PARAMETER ESTIMATION METHODS

| Abbreviation | Parameter estimation methods | Times |
|---|---|---|
| ML | Maximum likelihood estimation | 15 |
| MML | Modified maximum likelihood estimation | 2 |
| MM | Modified moment estimation | 7 |
| MLS | Modified least square | 5 |
| MLM | Modified L-moments estimation | 3 |
| LSE | Least squared estimation | 3 |
| MMS | Maximum spacing estimation | 3 |
| BE | Bayesian estimation | 2 |

TABLE III
DISTRIBUTIONS AND PARAMETER ESTIMATION METHODS

| Model | First occurrence time | Parameter estimation method | Reference |
|---|---|---|---|
| GP | Weibull distribution | MM & MML | [26] |
| GP | Gamma distribution | ML & MML | [27] |
| GP | Weibull distribution | BE | [28] |
| GP | Rayleigh distribution | ML | [29] |
| GP | Lindley distribution | ML & BE | [30] |
| GP | Hjorth marginal distribution | ML & MM & LSE & MMS | [31] |
| GP | Scaled Muth | ML & MM & LSE & MMS | [32] |
| GP | Exponential distribution | ML & MM & MLS | [33] |
| GP | Lindley distribution | ML & MLS & MM & MLM | [34] |
| GP | Inverse gaussian distribution | ML & MM | [35] |
| GP | Power lindley distribution | ML & MLS & MM & MLM | [36] |
| GP | Exponential distribution | ML & MM & MLS | [33] |
| DGP | Weibull distribution | ML | [37] |
| DPG | Exponential distribution | ML | [38] |
| $\alpha$-series | Rayleigh Distribution | ML & MMS & MLS<br>MM & MLM | [39] |
| $\alpha$-series | Truncated normal distribution | ML & LSE | [40] |
| $\alpha$-series | Lognormal distribution | ML & MM | [41] |

The ML estimation is the most frequently used for estimating parameters of the GP-like models. Then, the MM estimation method is the second frequently used.

### *B. Model performance*

Table IV shows some common methods for assessing model performance of the GP-like models.

Bias and the MSE are most frequently used for discussing the statistical inference of the GP-like models.

## IV. THE APPLICATION OF THE GP

The GP-like models are normally used for modelling the working times after the repairs and the repair times for each maintenance activities. Commonly, the GP-like models can be applied into the reliability and maintenance and warranty analysis for estimating the repair or replacement interval time, planning the corrective and preventative maintenance

TABLE IV
METHODS FOR ASSESSING MODEL PERFORMANCE

| Abbreviation | Model performance | Reference |
|---|---|---|
| AIC | Akaike information criterion | [18] |
| AICc | Corrected AIC | [18] |
| BIC | Bayesian information criterion | [42], [43] |
| Bias | Bias | [28], [29], [31], [32], [34], [36]–[38], [41] |
| MSE | Mean squared error | [26]–[29], [31]–[34], [36]–[38], [40], [41], [43]–[45] |
| MPE | Mean percentage error | [44], [45] |
| ML | Maximum likelihood | [22], [34] |
| LSE | Least squared error | [22] |
| AMSE | Adjusted mean squared error | [14] |
| DIC | Deviance information criterion | [42], [45], [46] |
| APB | Absolute percentage bias | [42] |
| RMS | Root mean square error | [42] |
| CP | Coverage percentage | [42] |

and predicting the number of warranty claims etc. Several existing research publications have reviewed [7], [47]. In this article, we would like to focus on other applications of the GP-like models. The following table shows a summary of the application of the GP-like models beyond reliability and maintenance.

TABLE V
SUMMARY OF THE APPLICATIONS OF THE GP-LIKE MODELS

| Application | GP-like model | Problems |
|---|---|---|
| Recruitment policy | GP | Loss of recruitment [48] |
| Crime analysis | GP | Cannabis offenses [49] |
| Crime analysis | GP | Number of arrests [46] |
| Electricity price | TGP | Forecasting [45] |
| Recruitment | GP | Forecasting & Recruitment policy [43] |
| Coal mining disasters | BGP | Simulation [22] |
| Coal mining disasters | GP | Simulation [44] |
| Market stock | TGP | High-low stock price [42] |
| SARS epidemic | TGP | Forecasting epidemic [14] |

[14] proposed the TGP, which can be considered a collection of several GPs with different ratio parameters $a_k$ based on the moving windows. The moving windows can separate a group of data into different subsets of fixed length starting from the first window with ratio $a_1$. In this research, the SARS data of Hong Kong, Singapore, Toronto and Taipei were separated into several subsets based on their moving windows. The model performance were be estimated by log-LSE and LSE method, respectively.

[45] used the TGP to modelling the changes of electricity market prices over time. One characteristic of the electricity market prices is that it is floating with high spikes at different time period for a whole day. The threshold of the TGP corresponds to the high spikes of the electricity price. This is the reason why the TGP had a good performance on this case.

[49] used the GP to analyse the cannabis offences in New South Wales of the Australia. According to their data analysis, they found that there exists underdispersed and overdispersed data, which shows a non-monotonicity trend. The Markov chain Monte Carlo was used introduced to overcome such problems. Besides, they discussed the model performance under both underdispersed and overdispersed data and used the result of an equidispersion situation as comparison.

Other research face to similar problems, such as [46]. It proposed the multivariate generalized poisson log-$t$ geometric process to analyse the number of arrests due to the use of two illicit drugs. [44] and [22] used the data of coal mining disasters to compare the model performance among several times series models and their extended GP-like models.

From the above-reviewed papers, the GP-like models can be applied into other fields such as the business analysis and health analysis. Besides, due to the properties of the GP-like models, we consider that they can be used in the fields where data can be described by the counting process such as the queuing problem and the transportation.

## V. CONCLUSIONS

This article briefly reviewed the GP and its extensions. Relevant statistical methods for estimating parameters and methods for assessing model performance were reviewed for the GP-like models. Besides, the application beyond the reliability and maintenance were reviewed with corresponding problems, such as the forecasting problem of the electricity price, the simulation of the coal mining disasters, and the predication of arrests etc.

We propose that the GP-like models can be used in more fields beyond the reliability and maintenance. As long as the data itself has a trend over time, which can be considered as a counting process, it may be modelled by the GP-like models. Furthermore, parameter estimation, assessment of model performance, and the distribution of $X_1$ of the GP-like models should be investigated and compared.

## REFERENCES


[1] L. Yeh, "A note on the optimal replacement problem," *Advances in Applied Probability*, vol. 20, no. 2, pp. 479–482, 1988.

[2] H. Pham and H. Wang, "A quasi-renewal process for software reliability and testing costs," *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, vol. 31, no. 6, pp. 623–631, 2001.

[3] S. Das, A. Dewanji, and S. Kundu, "Software reliability based on renewal process modeling for error occurrence due to each bug with periodic debugging schedule," *Probability in the Engineering and Informational Sciences*, pp. 1–18, 2020.

[4] M. Kamal, S. Khan, A. Rahman, R. A. Aldallal, M. M. A. El-Raouf, A. H. Muse, and A. Rabie, "Reliability analysis of hybrid system using geometric process in multiple level of constant stress accelerated life test through simulation study for type-ii progressive censored masked data," *Mathematical Problems in Engineering*, vol. 2022, 2022. [Online]. Available: www.scopus.com

[5] A. Yevkin and V. Krivtsov, "A generalized model for recurrent failures prediction," *Reliability Engineering & System Safety*, vol. 204, p. 107125, 2020.

[6] T. Yang and W. Gao, "Optimization of maintenance policy for cold standby system based on extended geometric process," *Jisuanji Jicheng Zhizao Xitong/Computer Integrated Manufacturing Systems, CIMS*, vol. 28, no. 1, pp. 84–92, 2022. [Online]. Available: www.scopus.com

[7] R. Arnold, S. Chukova, Y. Hayakawa, and S. Marshall, "Geometric and geometric-like processes and their applications in warranty analysis," *Mathematics Applied to Engineering and Management*, pp. 1–24, 2020.

[8] Y. L. Zhang, "An optimal geometric process model for a cold standby repairable system," *Reliability Engineering & System Safety*, vol. 63, no. 1, pp. 107–110, 1999.

[9] Y. Lam, "Geometric processes and replacement problem," *Acta Mathematicae Applicatae Sinica*, vol. 4, pp. 366–377, 1988.

[10] G. J. Wang and R. C. Yam, "Generalized geometric process and its application in maintenance problems," *Applied Mathematical Modelling*, vol. 49, pp. 554–567, 2017.

[11] C. Niu, J. Jiang, B. Ge, and Y. Chen, "Preventive maintenance model based on the renewal-geometric process," *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, vol. 236, no. 2, pp. 348–356, 2022.

[12] L. K.-N. Francis, "Optimal replacement policies determined using arithmetico-geometric processes," *Engineering Optimization*, vol. 33, no. 4, pp. 473–484, 2001.

[13] W. J. Braun, W. Li, and Y. Q. Zhao, "Properties of the geometric and related processes," *Naval Research Logistics (NRL)*, vol. 52, no. 7, pp. 607–616, 2005.

[14] J. S. Chan, P. L. Yu, Y. Lam, and A. P. Ho, "Modelling sars data using threshold geometric process," *Statistics in medicine*, vol. 25, no. 11, pp. 1826–1839, 2006.

[15] S. Wu and D. Clements-Croome, "A novel repair model for imperfect maintenance," *IMA Journal of Management Mathematics*, vol. 17, no. 3, pp. 235–243, 2006.

[16] L. Bordes and S. Mercier, "Extended geometric processes: semiparametric estimation and application to reliability," *Journal of The Iranian Statistical Society*, vol. 12, no. 1, pp. 1–34, 2013.

[17] Y. L. Zhang and G. J. Wang, "An extended geometric process repair model for a cold standby repairable system with imperfect delayed repair," *International Journal of Systems Science: Operations & Logistics*, vol. 3, no. 3, pp. 163–175, 2016.

[18] S. Wu, "Doubly geometric processes and applications," *Journal of the Operational Research Society*, pp. 1–13, 2018.

[19] S. Wu and G. Wang, "The semi-geometric process and some properties," *IMA Journal of Management Mathematics*, vol. 29, no. 2, pp. 229–245, 2018.

[20] S. Wu, "The double ratio geometric process for the analysis of recurrent events," *Naval Research Logistics (NRL)*, pp. 484–495, 2022.

[21] S. M. Ross, "Stochastic processes 2nd edition john wiley & sons," 1996.

[22] J. S. Chan and D. Y. Leung, "Binary geometric process model for the modeling of longitudinal binary data with trend," *Computational Statistics*, vol. 25, no. 3, pp. 505–536, 2010.

[23] S. Chukova and L. Minkova, "Geometric pólya-aeppli process," *Stochastics*, vol. 92, no. 8, pp. 1261–1275, 2020.

[24] R. Arnold, S. Chukova, Y. Hayakawa, and S. Marshall, "Mean and variance of an alternating geometric process: An application in warranty cost analysis," *Quality and Reliability Engineering International*, vol. 38, no. 6, pp. 2968–2985, 2022.

[25] M. S. Finkelstein, "A scale model of general repair," *Microelectronics Reliability*, vol. 33, no. 1, pp. 41–44, 1993.

[26] H. Aydoğdu, B. Şenoğlu, and M. Kara, "Parameter estimation in geometric process with weibull distribution," *Applied Mathematics and Computation*, vol. 217, no. 6, pp. 2657–2665, 2010.

[27] M. Kara, G. Güven, B. Şenoğlu, and H. Aydoğdu, "Estimation of the parameters of the gamma geometric process," *Journal of Statistical Computation and Simulation*, pp. 1–11, 2022.

[28] I. Usta, "Bayesian estimation for geometric process with the weibull distribution," *Communications in Statistics-Simulation and Computation*, pp. 1–27, 2022.

[29] C. Biçer, H. D. Biçer, K. Mahmut, and H. Aydoğdu, "Statistical inference for geometric process with the rayleigh distribution," *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, vol. 68, no. 1, pp. 149–160, 2019.

[30] A. Yilmaz, M. Kara, and H. Kara, "Bayesian inference for geometric process with lindley distribution and its applications," *Fluctuation and Noise Letters*, 2022.

[31] H. Demirci Bicer, C. Bicer, and H. S. H. Bakouch, "A geometric process with hjorth marginal: Estimation, discrimination, and reliability data modeling," *Quality and Reliability Engineering International*, 2022.

[32] C. Biçer, H. S. Bakouch, and H. D. Biçer, "Inference on parameters of a geometric process with scaled muth distribution," *Fluctuation and Noise Letters*, vol. 20, no. 01, p. 2150006, 2021.

[33] N. H. Idrees and M. S. Sulaiman, "Estimating the rate of occurrence of geometric process with exponential distribution," in *2021 7th International Conference on Contemporary Information Technology and Mathematics (ICCITM)*. IEEE, 2021, pp. 291–297.

[34] H. Demirci Biçer, "Statistical inference for geometric process with the two-parameter lindley distribution," *Communications in Statistics-Simulation and Computation*, vol. 49, no. 11, pp. 2979–3000, 2020.

[35] M. Kara, H. Aydoğdu, and Ö. Türkşen, "Statistical inference for geometric process with the inverse gaussian distribution," *Journal of Statistical Computation and Simulation*, vol. 85, no. 16, pp. 3206–3215, 2015.

[36] C. Bicer, "Statistical inference for geometric process with the power lindley distribution," *Entropy*, vol. 20, no. 10, p. 723, 2018.

[37] M. H. Pekalp, G. Eroğlu İnan, and H. Aydoğdu, "Statistical inference for doubly geometric process with weibull interarrival times," *Communications in Statistics-Simulation and Computation*, vol. 51, no. 6, pp. 3428–3440, 2022.

[38] M. H. Pekalp, G. E. İnan, and H. Aydoğdu, "Statistical inference for doubly geometric process with exponential distribution," *Hacettepe Journal of Mathematics and Statistics*, vol. 50, no. 5, pp. 1560–1571, 2021.

[39] H. Demirci Biçer, "Statistical inference for alpha-series process with the generalized rayleigh distribution," *Entropy*, vol. 21, no. 5, p. 451, 2019.

[40] C.-x. Yan, D.-q. Yue, Z. Hui-juan, and C. Bian-juan, "Statistical inference for alpha-series process with truncated normal distribution," in *International Conference on Automatic Control and Artificial Intelligence (ACAI 2012)*. IET, 2012, pp. 1926–1929.

[41] M. Kara, Ö. Altındağ, M. H. Pekalp, and H. Aydoğdu, "Parameter estimation in alpha-series process with lognormal distribution," 2019.

[42] J. S.-K. Chan, C. P. Lam, P. Yu, S. B. Choy, and C. W. Chen, "A bayesian conditional autoregressive geometric process model for range data," *Computational Statistics & Data Analysis*, vol. 56, no. 11, pp. 3006–3019, 2012.

[43] J. Chan, W. Wan, and P. Yu, "A poisson geometric process approach for predicting drop-out and committed first-time blood donors," *Journal of Applied Statistics*, vol. 41, no. 7, pp. 1486–1503, 2014.

[44] Y. Lam, L.-x. Zhu, J. S. Chan, and Q. Liu, "Analysis of data from a series of events by a geometric process model," *Acta Mathematicae Applicatae Sinica, English Series*, vol. 20, no. 2, pp. 263–282, 2004.

[45] J. S. Chan, S. B. Choy, and C. P. Lam, "Modeling electricity price using a threshold conditional autoregressive geometric process jump model," *Communications in Statistics-Theory and Methods*, vol. 43, no. 10-12, pp. 2505–2515, 2014.

[46] J. S. K. Chan and W. Y. Wan, "Multivariate generalized poisson geometric process model with scale mixtures of normal distributions," *Journal of Multivariate Analysis*, vol. 127, pp. 72–87, 2014.

[47] D. Wu, R. Peng, and S. Wu, "A review of the extensions of the geometric process, applications, and challenges," *Quality and Reliability Engineering International*, vol. 36, no. 2, pp. 436–446, 2020.

[48] D. Samundeeswari, A. Kanchana, and V. N. Varuvel, "A man power model forthree grade system with univariate policy of recruitment using geometric process for inter decision times," in *Recent Advances in Internet of Things and Machine Learning*. Springer, 2022, pp. 235–241.

[49] J. S. K. Chan and W. Y. Wan, "Bayesian analysis of cannabis offences using generalized poisson geometric process model with flexible dispersion," *Journal of Statistical Computation and Simulation*, vol. 86, no. 16, pp. 3315–3336, 2016.