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CONTINUOUS UNIFORM FINITE TIME STABILIZATION OF PLANAR CONTROLLABLE SYSTEMS*

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Abstract. Continuous homogeneous controllers are utilized in a full state feedback setting for the uniform finite time stabilization of a perturbed double integrator in the presence of uniformly decaying piecewise continuous disturbances. Semiglobal strong $\mathcal{C}^1$ Lyapunov functions are identified to establish uniform asymptotic stability of the closed-loop planar system. Uniform finite time stability is then proved by extending the homogeneity principle of discontinuous systems to the continuous case with uniformly decaying piecewise continuous nonhomogeneous disturbances. A finite upper bound on the settling time is also computed. The results extend the existing literature on homogeneity and finite time stability by both presenting uniform finite time stabilization and dealing with a broader class of nonhomogeneous disturbances for planar controllable systems while also proposing a new class of homogeneous continuous controllers.

Key words. uniform finite time stability, Lyapunov functions, homogeneity

AMS subject classifications. 93D05, 93D15, 93D30, 93C10, 34A36, 34A60, 34H15

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1. Introduction.

Continuous finite time stabilization of linear and nonlinear control systems is an active area of research. The introduction of continuous finite time controllers [13] revealed the non-Lipschitzian nature of the closed-loop dynamics of planar finite time stable systems. Lyapunov and converse Lyapunov theorems were subsequently established, and the continuity properties of the settling time function were studied [8]. Homogeneous finite time controllers [6] and nonhomogeneous finite time controllers [13] find applications in robotics [11, 16, 35] as well as in aerospace engineering [7]. The finite time controllers, while being supported by a strict homogeneous Lyapunov function [7], prescribe better rejection of continuous disturbances than that achieved by Lipschitz controllers [8, Th. 5.2].

Earlier results on asymptotic stabilization [18, 31] of continuous homogeneous systems are based on the definition of a class of dilations where each state is dilated with a different weight [18]. The notion of geometric homogeneity and its application to stabilization were developed in [19, 20]. A detailed literature review on the topic of geometric homogeneity is presented in [9], where it is established that geometric homogeneity leads to finite time stability if the homogeneity degree of the asymptotically stable continuous homogeneous system is negative. A result on output feedback synthesis which combines a continuous finite time observer with a continuous finite time controller can be found in [15]. More recently, homogeneous approximations have been studied [3] that led to the development of tools to establish global asymptotic (and in some cases finite time) stability of nonlinear systems. This result used previous results on the so-called homogeneous domination approach (see [30], [3, section 5], and references therein for a detailed literature review).

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Finite time stability and uniform finite time stability of nonlinear time varying continuous systems was studied in [12]. Uniform finite time stability and the concept of quasi-homogeneity were established in [26] for discontinuous homogeneous systems with a negative homogeneity degree but with an additional requirement of uniform asymptotic stability. A settling time estimate and tuning for the planar discontinuous case with rectangular disturbances has been established recently via an alternative Lyapunov-based proof [29], which is a special case of the more general result [26]. Homogeneity-based finite time stability results also exist for the so-called higher order sliding mode controllers [24]. Several results also exist on continuous finite time stabilization of nonlinear systems of dimension higher than two (see [9] and references therein for linear controllable systems and [14, 17, 25, 35, 36] for nonlinear systems).

The main objective of this paper is to achieve continuous uniform finite time stabilization of planar controllable systems with piecewise continuous, nonhomogeneous disturbances. The proposed theoretical development considers a perturbed double integrator. An existing finite time stabilizing, continuous, homogeneous controller [6], [27] and a new homogeneous controller are utilized. The result on finite time stability of discontinuous systems [26] is utilized in place of the continuous counterpart [8] in order to extend the class of perturbations that can be successfully suppressed in finite time. Uniform asymptotic stability of the closed-loop system is achieved by identifying a class of semiglobal strong $\mathcal{C}^1$ Lyapunov functions for each of the two controllers. Uniform finite time stability then follows from the homogeneity principle which is extended for a continuous vector field. An explicit upper bound on the settling time is then computed using the homogeneity regions without the need to find a Lyapunov function satisfying a differential inequality. The main contribution is that the finite time stability attained in this paper is *uniform* in the initial data and in the piecewise continuous perturbation.

The theoretical motivation to propose a new Lyapunov and homogeneity framework for planar continuous homogeneous vector fields is to give *uniform* finite time stability, with respect to initial data and the disturbances. The motivation also lies in proving robustness to discontinuous disturbances, which is a stronger property than the existing methods for continuous disturbances which utilize the link between homogeneity [18, 31] and finite time stability [9]. The motivation also lies in computing the settling time for the class of homogeneous controllers [6, Example 2] in the presence of nonhomogeneous perturbations. Uniform finite time stability is a stronger feature than finite time stability and requires the Lyapunov stability to be uniform with respect to the initial time [12, Remark 3.1]. In this paper, uniformity with respect to disturbances, called *equiuniformity* according to [26], is emphasized. The result presented in the following sections of this paper achieves this for the class of controllers [6, Example 2] as well as the new result being proposed.

The method proposed in [7] relies on the homogeneity property of the strict Lyapunov function and that of its derivative. It is known that every controllable linear system admits a class of homogeneous finite time stabilizing controller (see [14, Corollary 3.1], [9, sect. 7,8], and references therein). However, construction of a strict Lyapunov function is required to find an explicit formula for the upper bound on the settling time for the given homogeneous controller. No Lyapunov function has been identified, for example, to prove that the controller [6, Example 2] finite time stabilizes a double integrator when it is perturbed by a nonhomogeneous disturbance. Furthermore, the proposed results can motivate similar developments for even arbitrarily higher order controllable systems in the presence of piecewise continuous perturbations, which is also an interesting problem from an engineering viewpoint. It should be noted that the proposed method proves uniform asymptotic stability of the origin, a result stronger than that appearing in the existing results [9].

The structure of the paper is outlined as follows. The notation, basic definitions, and problem statement are presented in section 2. Sections 3 and 4 present the main results. Section 5 outlines the conclusions and future scope.

2. Preliminaries. This section first presents definitions of the mathematical concepts that will be utilized throughout the paper.

2.1. Definitions. Consider the discontinuous dynamical system

$$\dot{x} = \phi(x,t), \tag{2.1}$$

where $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is the state vector, $t \in \mathbf{R}$ is the time variable, and function $\phi(x,t)$ is piecewise continuous. The function $\phi : \mathbf{R}^{n+1} \to \mathbf{R}^n$ is piecewise continuous iff $\mathbf{R}^{n+1}$ is partitioned into a finite number of domains $G_j^{\phi} \in \mathbf{R}^{n+1}, j = 1, \ldots, N^{\phi}$, with disjoint interiors and boundaries ∂G_j^{ϕ} of measure zero such that ϕ is continuous within each of these domains and for all $j = 1, \ldots, N^{\phi}$ it has a finite limit $\phi^j(x,t)$ as the argument $(x^*, t^*) \in G_j^{\phi}$ approaches a boundary point $(x,t) \in \partial G_j^{\phi}$. Throughout the paper, solutions to differential equations will be understood as defined in the following definition.

Definition 2.1 (solutions in the sense of Filippov [1]). *Given the differential equation* (2.1)*, let the smallest convex closed set* $\Phi(x,t)$ *be introduced for each point* $(x,t) \in \mathbf{R}^n \times \mathbf{R}$ *such that* $\Phi(x,t)$ *contains all the limit points of* $\phi(x^*,t)$ *as* $x^* \to x$, $t =$ *constant, and* $(x^*,t) \in \mathbf{R}^{n+1} \backslash (\cup_{j=1}^{N^{\phi}} \partial G_j^{\phi})$*. An absolutely continuous function* $x(\cdot)$ *defined on interval* $\mathbf{I}$ *is said to be a solution of* (2.1) *if it satisfies the differential inclusion*

$$\dot{x} \in \Phi(x,t) \tag{2.2}$$

almost everywhere on interval $\mathbf{I}$.

The emphasis of this paper is on studying robustness in the presence of perturbations. Let the perturbed version of (2.1) be given by

$$\dot{x} = \phi(x,t) + \psi(x,t), \tag{2.3}$$

where $\psi(x,t)$ is a piecewise continuous function, inducing the partition of $\mathbf{R}^{n+1}$ into a finite number of domains $G_j^{\psi} \in \mathbf{R}^{n+1}, j = 1, \ldots, N^{\psi}$, with disjoint interiors and boundaries ∂G_j^{ψ} of measure zero such that ψ is continuous within each of these domains and for all $j = 1, \ldots, N^{\psi}$ it has a finite limit $\psi^j(x,t)$ as the argument $(x^*,t^*) \in G_j^{\psi}$ approaches a boundary point $(x,t) \in \partial G_j^{\psi}$. The components $\psi_1, \psi_2, \ldots, \psi_n$ of the perturbation $\psi(x,t)$ are assumed throughout to be uniformly bounded according to

$$\operatorname*{ess\,sup}_{t \geq 0} \lim_{\substack{x^* \to x \\ t^* \to t}} |\psi_i(x^*,t^*)| \leq M_i \bar{\alpha}(\|x\|) \tag{2.4}$$

for all $x \in \mathbf{R}^n$, for some nonnegative constants $M_i,\ i = 1, 2, \ldots, n$, and some continuous positive definite function $\bar{\alpha}(\cdot)$ of a scalar argument such that $\lim_{\|x\| \to 0} \bar{\alpha}(\|x\|)) = 0$ for all $(x^*,t^*) \in G_j^{\psi}$. If confined to $(x^*,t^*) \in G_j^{\psi},\ j = 1, 2, \ldots, N^{\psi}$, the inequality (2.4) reduces to

$$\operatorname*{ess\,sup}_{t^* \geq 0} |\psi_i(x^*,t^*)| \leq M_i \bar{\alpha}(\|x^*\|). \tag{2.5}$$

Then, the following definition is in order for the perturbed system.

DEFINITION 2.2. *An absolutely continuous function $x(\cdot)$, defined on an interval* $\mathbf{I}$*, is said to be a solution of the uncertain differential equation* (2.3) *with the sectorial constraints* (2.4) *iff it is a solution of* (2.3) *on the interval* $\mathbf{I}$ *in the sense of Definition* 2.1 *for some piecewise continuous function ψ subject to* (2.4).

Analogously to [26, p. 1255], an uncertain system (2.3) can be represented as a differential inclusion of the form

$$\dot{x} \in \Phi(x,t) + \Psi(x), \tag{2.6}$$

where $\Phi(x,t)$ is the same as defined in Definition 2.1, whereas given $(x,t) \in G_j^{\psi},\ j = 1,2,\ldots,N^{\psi}$, the set $\Psi(x)$ is the Cartesian product of the closed intervals $\Psi(x) = [-M_i\bar{\alpha}(\|x\|),\ M_i\bar{\alpha}(\|x\|)]$ for the disturbance constraints (2.4).

The main focus of this paper is on uniform finite time stability with respect to initial time t_0 as well as uncertainty $\psi(x,t)$. It is important to highlight what is meant by uniformity. This is a well-studied area for systems with continuous dynamics, and many references are available [12, 21, 34] regarding uniformity with respect to initial time. It can be seen from the above references, however, that emphasis is seldom given to uniformity with respect to the disturbance. Indeed, finite time stability is the same as uniform finite time stability in the absence of perturbation. Definitions [26, Definitions 2.3–2.5] of (uniform) stability, (uniform) asymptotic stability, and (uniform) finite time stability of the inclusion (2.2) for the discontinuous vector field, which can be seen as the counterparts of the definitions available in the references [12, 21, 34] for similar stability concepts in the case of the continuous vector field, are not included here for brevity. The following definitions are inherited from [26, Definitions 2.6–2.8], which take into account the uniformity of stability with respect to the uncertainty ψ. It should be noted that the word "equiuniform" appearing in [26] is utilized in the following definitions to refer to uniformity of various stability concepts with respect to the initial conditions as well as the uncertainty ψ.

Suppose that $x = 0$ is an equilibrium point of the uncertain system (2.3), (2.4), i.e., that $x = 0$ is a solution of (2.3) for some function ψ_0, admissible in the sense of (2.4), and let $x(\cdot, t_0, x^0)$ denote a solution of (2.3) for some admissible function ψ under the initial conditions $x(t_0) = x^0$. The symbol B_δ in the following definitions represents a ball centered at the origin with radius δ.

DEFINITION 2.3 (equiuniform stability [26]). *The equilibrium point $x = 0$ of the uncertain system* (2.3), (2.4) *is equiuniformly stable iff for each $t_0 \in \mathbf{R}, \epsilon > 0$ there exists $\delta = \delta(\epsilon)$, dependent on ϵ and independent of t_0 and ψ, such that each solution $x(\cdot, t_0, x^0)$ of* (2.3), (2.4) *with the initial data $x^0 \in B_\delta$ exists on the semi-infinite time interval $[t_0, \infty)$ and satisfies the inequality*

$$\|(x(t,t_0,x^0))\| \le \epsilon \quad \text{for all } t \in [t_0,\infty). \tag{2.7}$$

DEFINITION 2.4 (equiuniform asymptotic stability [26]). *The equilibrium point $x = 0$ of the uncertain system* (2.3), (2.4) *is said to be equiuniformly asymptotically stable if it is equiuniformly stable and the convergence*

$$\lim_{t\to\infty} \|(x(t,t_0,x^0))\| \to 0 \tag{2.8}$$

holds for all the solutions $x(\cdot, t_0, x^0)$ of the uncertain system (2.3), (2.4) *initialized within some B_δ (uniformly in the initial data t_0 and x^0). If this convergence remains* *in force for each* $\delta > 0$*, the equilibrium point is said to be globally equiuniformly asymptotically stable.*

DEFINITION 2.5 (equiuniform finite time stability [26]). *The equilibrium point* $x = 0$ *of the uncertain system* (2.3), (2.4) *is said to be globally equiuniformly finite time stable if, in addition to the global equiuniform asymptotical stability, the limiting relation*

$$x(t, t_0, x^0) = 0 \tag{2.9}$$

holds for all the solutions $x(\cdot, t_0, x^0)$ *and for all* $t \geq t_0 + T(t_0, x^0)$*, where the settling time function*

$$T(t_0, x^0) = \sup_{x(\cdot,t_0,x^0)} \inf\{T \geq 0 : x(t, t_0, x^0) = 0 \quad \textit{for all } t \geq t_0 + T\} \tag{2.10}$$

is such that

$$T(B_\delta) = \sup_{x^0 \in B_\delta, t_0 \in \mathbf{R}} T(t_0, x^0) < \infty \quad \textit{for all } \delta > 0, \tag{2.11}$$

where $\delta = \delta(\epsilon)$ *is independent of* t_0 *and* ψ*.*

The infimum in (2.10) is to detect the first instant $t = T$ such that $x(t, t_0, x^0) = 0$ for all $t \geq t_0 + T$, and the supremum is for taking the worst case trajectory that takes the longest time to arrive at the origin.

DEFINITION 2.6 (homogeneity of differential inclusions and equations [26]). *The differential inclusion* (2.2) *(the differential equation* (2.1) *or the uncertain systems* (2.3), (2.4)*) is called homogeneous of degree* $q \in \mathbf{R}$ *with respect to dilation* $(r_1, r_2, \ldots, r_n)$*, where* $r_i > 0, i = 1, 2, \ldots, n$*, if there exists a constant* $c_0 > 0$*, called a lower estimate of the homogeneity parameter, such that any solution* $x(\cdot)$ *of* (2.2) *(respectively, that of the differential equation* (2.1)*, the uncertain systems* (2.3), (2.4)*) generates a parameterized set of solutions* $x^c(\cdot)$ *with components*

$$x_i^c(t) = c^{r_i} x_i(c^q t) \tag{2.12}$$

and any parameter $c \geq c_0$*.*

DEFINITION 2.7 (homogeneous piecewise continuous functions [26]). *A piecewise continuous function* $\phi : \mathbf{R}^{n+1} \to \mathbf{R}^n$ *is called homogeneous of degree* $q \in \mathbf{R}$ *with respect to dilation* $(r_1, r_2, \ldots, r_n)$*, where* $r_i > 0, i = 1, 2, \ldots, n$*, if there exists a constant* $c_0 > 0$ *such that*

$$\phi_i(c^{r_1} x_1, c^{r_2} x_2, \ldots, c^{r_n} x_n, c^{-qt}) = c^{q+r_i} \phi_i(x_1, x_2, \ldots, x_n, t) \tag{2.13}$$

for all $c \geq c_0$*.*

2.2. Problem statement. Let a controllable planar single input control system be given as follows:

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\, v(\bar{x}) + \bar{G} f(\bar{x}, t), \tag{2.14}$$

where $\bar{x} = (\bar{x}_1, \bar{x}_2)^T$ is the state vector, $v(\bar{x})$ is the control input, the matrix pair $(\bar{A}, \bar{B})$ is controllable, and $f(\bar{x}, t)$ is a scalar perturbation function. The following is assumed.

Assumption 1. $\mathcal{R}(\bar{G}) \subseteq \mathcal{R}(\bar{B})$, where $\mathcal{R}(\bar{G})$ is the range of $\bar{G}$ and $\mathcal{R}(\bar{B})$ is the range of $\bar{B}$.

The above assumption, known as the matching condition [10], means there exists a scalar $\bar{p} \in \mathbf{R}$ such that $\bar{G} = \bar{B}\bar{p}$ holds true. Furthermore, due to the controllability of the system (2.14), there always exists a nonsingular transformation matrix T such that the system (2.14) can be converted into one with coordinates (x_1, x_2) with a phase canonical structure [22, Th. 1.43]. Employing $v(x) = k_i x_i + u(x), i = 1, 2$, with k_i representing scalars appearing in the second row of the transformed system matrix $T\bar{A}T^{-1}$ with an opposite sign, and defining $T\bar{B}\bar{p}f(T\bar{x}, t) = \omega(x, t)$, the following perturbed double integrator results:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u(x_1, x_2) + \omega(x, t), \tag{2.15}$$

where $x = (x_1, x_2)^T \in \mathbf{R}^2$ is the state vector, u is the control input, and $\omega(x, t)$ is a piecewise continuous [1] disturbance. Consider the following two classes of homogeneous controllers:

$$u(x_1, x_2) = -\mu_1|x_2|^{\alpha}\mathrm{sign}(x_2) - \mu_2|x_1|^{\frac{\alpha}{2-\alpha}}\mathrm{sign}(x_1), \tag{2.16}$$

$$u(x_1, x_2) = -(\mu_1|x_2|^{\alpha} + \mu_3|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}})\mathrm{sign}(x_2) - \mu_2|x_1|^{\frac{\alpha}{2-\alpha}}\mathrm{sign}(x_1), \tag{2.17}$$

where $\alpha \in (0, 1)$ and μ_1, μ_2, μ_3 are positive constants.

Assumption 2. The piecewise continuous disturbance $\omega(x, t)$ is assumed to satisfy one of the following two inequalities:

$$\operatorname*{ess\,sup}_{t \geq 0} |\omega(x_1, x_2, t)| \leq M|x_2|^{\alpha}, \tag{2.18}$$

$$\operatorname*{ess\,sup}_{t \geq 0} |\omega(x_1, x_2, t)| \leq M|x_2|^{\frac{\alpha}{2}}(|x_1|^{\frac{\alpha}{2(2-\alpha)}} + |x_2|^{\frac{\alpha}{2}}), \tag{2.19}$$

where M is a positive constant.

Remark 1. The upper bound (2.19) can be shown to be conservatively larger than some norm of the vector x raised to some power multiplied by $|x_2|^{\frac{\alpha}{2}}$. Since the Lyapunov analysis in the following sections proves uniform asymptotic stability for the worst case given by the upper bound (2.19), the proposed synthesis is naturally robust to disturbances bounded by such functions of norms. For example, for the local case $\|x\|_\infty = \max\{|x_1|, |x_2|\} < 1$, which is relevant to the problem formulation, it is straightforward to derive the inequality $\|x\|_1^{\frac{\alpha}{2}}|x_2|^{\frac{\alpha}{2}} < (|x_1|^{\frac{\alpha}{2(2-\alpha)}} + |x_2|^{\frac{\alpha}{2}})|x_2|^{\frac{\alpha}{2}}$, where $\|x\|_1 = |x_1| + |x_2|$ (it suffices to utilize [4, Facts 1.12.30, 1.17.35] to compute the inequality).

The aim of this paper is to (i) prove uniform finite time stability and (ii) to establish a finite upper bound on the settling time $\mathcal{T}$ of the closed-loop systems (2.15), (2.16) and (2.15), (2.17) for $\alpha \in (\frac{2}{3}, 1)$ in the presence of disturbances that satisfy (2.18) and (2.19), respectively.

Global asymptotic stability of such a perturbed double integrator can be found in [27, Theorem 2]. Global finite time stability for the unperturbed case with the controller (2.16) was established via homogeneity in [6]. However, the class of nonhomogeneous disturbances (2.18) and (2.19) has not been studied explicitly in the literature while proving finite time stability. For example, the continuous terminal sliding mode control proposed in [35] makes no assumption on the continuity of lumped disturbances. However, the corresponding synthesis leads to finite time stability only if $\|\omega\| = 0$. In the presence of nonzero ω and with an upper bound of the form $\|\omega\| \leq b_0 + b_1\|x\| + b_2\|x\|^2$, where b_0, b_1, b_2 are positive scalars, the states are rendered only ultimately bounded. Furthermore, the existing result [3, Corollary 2.24] on homogeneous approximation cannot be applied to the problem under consideration since the right-hand side contains $\omega(x,t)$, which is discontinuous. The converse Lyapunov theorem presented in [32, section 1] for general discontinuous systems is applicable but results only in asymptotic stability. On the other hand, the existence of a homogeneous Lyapunov function that could potentially result in finite time stability was established in [32, section 2] for discontinuous dynamic systems but is inapplicable to the system in question because of the presence of nonhomogeneous time varying perturbations. The timeliness of the contribution of the results of this paper is reinforced by recent results on homogeneous inclusions where the principal result on the converse Lyapunov theorem [5, Th. 4.1] is presented without proof.

The definition of finite time stability presented here is superior to that presented in the literature to date as existing contributions do not incorporate robustness to discontinuous disturbances.

Remark 2. As opposed to the existing results, the results in the next section allow discontinuous disturbances. Consider, for example, the discontinuous disturbance $\omega(x_1,x_2,t) = |x_2|^\alpha \mathrm{sign}(x_1)\sin(t)$. As a matter of fact, in the presence of continuous disturbances with an upper bound $|x_2|^\alpha$, it is enough to apply [27, Th. 1] to establish global asymptotic stability and in turn [9, Th. 7.4] to establish global (but not uniform) finite time stability of the closed-loop system (2.15), (2.16).

Remark 3. The controller (2.16) does not belong to the class of controllers proposed in [13, Corollary 1]. The phase plane plot of the closed-loop system (2.15), (2.16) with $\alpha \in (0,1)$ can be found in [27], which shows that the trajectories spiral infinitely around the origin without approaching tangentially to the hyperplane $x_1 = 0$ as they move to the origin.

The following lemma extends the existing result [26, Lem. 2.12] to the present case with uniformly decaying piecewise continuous disturbances $\omega(x,t)$ and is utilized in the proof of the main result. It should be noted that the unperturbed closed-loop systems (2.15), (2.16) and (2.15), (2.17) with $M = 0$ are globally homogeneous of degree $q = -1$ with respect to dilations $(r_1, r_2) = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$ as per Definition 2.7 (see [6] and [27]).

LEMMA 2.8. *Let the function $\omega(x_1,x_2,t)$ be a piecewise continuous function which is uniformly bounded as defined in* (2.18)*. Then, the uncertain differential equation* (2.15)*,* (2.16) *with the uncertainty constraints* (2.18) *is homogeneous of degree $q = -1$ with respect to the dilation $(r_1,r_2) = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$.*

Proof. Let $x(\cdot) = (x_1(\cdot), x_2(\cdot))^T$ be a solution of (2.15), (2.16) under some piecewise continuous function $\omega(x,t)$, satisfying (2.18). Then it is straightforward to verify that for arbitrary $c \geq \max(1, c_0)$ the function $x^c(\cdot)$ with components $x_i^c(t) = c^{r_i} x_i(c^q t), i = 1, 2$, is a solution of (2.15), (2.16) with the piecewise continuous function $\omega(x_1,x_2,t) \triangleq \omega^c(x_1,x_2,t)$, which is as follows:

$$\omega^c(x_1,x_2,t) = c^{q+r_2}\omega(c^{-r_1}x_1, c^{-r_2}x_2, c^q t), \tag{2.20}$$

where the right-hand side represents a parameterized set of uncertainties. The following holds true due to the parameterization (2.20):

$$\begin{aligned} &|\omega^c(x_1,x_2,t)| = |c^{q+r_2}\omega(c^{-r_1}x_1, c^{-r_2}x_2, c^q t)| \\ &\Rightarrow |\omega^c(x_1,x_2,t)| \leq c^{q+r_2} M |c^{-r_2}x_2|^\alpha \leq c^{q+r_2-\alpha r_2} M |x_2|^\alpha. \end{aligned} \tag{2.21}$$

Hence, all parameterized disturbance functions represented by the right-hand side of (2.20) are admissible in the sense of (2.18) if the following holds true:

$$c^{q+r_2-\alpha r_2} \leq 1. \tag{2.22}$$

From the definitions $r_2 = \frac{1}{1-\alpha}, q = -1$, it is obtained that

$$q + r_2 - \alpha r_2 = 0 \Rightarrow c^{q+r_2-\alpha r_2} \leq 1 \tag{2.23}$$

and that the function $\omega^c(x_1, x_2, t)$ is admissible in the sense of (2.18). Recalling Definitions 2.6 and 2.7 and [26, Lemma 2.11], the solutions $x_1^c(t) = c^{r_1} x_1(c^q t)$, $x_2^c(t) = c^{r_2} x_2(c^q t)$ are solutions of the system (2.15), (2.16) with the piecewise continuous function $\omega(x_1, x_2, t) = \omega^c(x_1, x_2, t)$ given by (2.20). Thus, any solution of the differential equation (2.15), (2.16) generates a parameterized set of solutions $x_1^c(t), x_2^c(t)$ with the parameter c large enough. Hence, (2.15), (2.16) is homogeneous of degree $q = -1$ with the dilation $(r_1, r_2) = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$. This proves the statement of Lemma 2.8. ◻

LEMMA 2.9. *Let the function $\omega(x_1, x_2, t)$ be a piecewise continuous function which is uniformly bounded as defined in* (2.19)*. Then, the uncertain differential equation* (2.15)*,* (2.17) *with the uncertainty constraints* (2.19) *is homogeneous of degree $q = -1$ with respect to the dilation $(r_1, r_2) = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$.*

Proof. The proof, while being identical to that of Lemma 2.8, follows by noting that (i) the parameterization (2.20) can be utilized to analyze the upper bound

$$\begin{aligned} &|\omega^c(x_1, x_2, t)| = |c^{q+r_2} \omega(c^{-r_1} x_1, c^{-r_2} x_2, c^q t)| \\ \Rightarrow &|\omega^c(x_1, x_2, t)| \leq c^{q+r_2} M(|c^{-r_1} x_1|^{\frac{\alpha}{2(2-\alpha)}} |c^{-r_2} x_2|^{\frac{\alpha}{2}} + |c^{-r_2} x_2|^{\alpha}), \end{aligned}$$

(ii) the expression $\frac{-r_1 \alpha}{2(2-\alpha)} = \frac{-r_2 \alpha}{2}$ holds true, and, finally, (iii) all parameterized disturbance functions represented by the right-hand side of (2.20) are admissible in the sense of (2.19) since

$$|\omega^c(x_1, x_2, t)| \leq c^{q+r_2-r_2\alpha} M(|x_1|^{\frac{\alpha}{2(2-\alpha)}} |x_2|^{\frac{\alpha}{2}} + |x_2|^{\alpha}) \leq M(|x_1|^{\frac{\alpha}{2(2-\alpha)}} |x_2|^{\frac{\alpha}{2}} + |x_2|^{\alpha})$$

holds true when the expression (2.22) holds true. ◻

The importance of Lemmas 2.8 and 2.9 lies in the fact that proving uniform asymptotic stability of the perturbed system (2.15), (2.16) (or, respectively, (2.15), (2.17)) in the presence of disturbances $\omega(x_1, x_2, t)$ will render the existing result on finite time stability of discontinuous systems [26, Th. 3.1] applicable to the present case. Uniform asymptotic stability is proven next by identifying a class of semiglobal $\mathcal{C}^1$ Lyapunov functions for a limited range of $\alpha \in (\frac{2}{3}, 1)$.

3. Global equiuniform finite time stability. This section presents the main results of the paper by proving equiuniform finite time stability of the closed-loop system (2.15), (2.17) ((2.15), (2.16)) in Theorem 3.1 (respectively, in Theorem 3.2).

The closed-loop system (2.15), (2.17) exhibits rich and different qualitative behavior for different combinations of the values of the controller gains as depicted in Figures 1 and 2. For the unperturbed case, phase plane analysis may be possible, in a similar way to [7]. However, for the perturbed case (i.e., $\omega(x, t) \neq 0$), the analysis becomes tedious. One possible direction is the method of majorization as undertaken in [23, 29] for the discontinuous case. This remains an open problem since an explicit integration of trajectories of the system (2.15), (2.17) is not straightforward to compute. Hence, a systematic Lyapunov-based approach is established in the proof of Theorem 3.1 below.

THEOREM 3.1. *Given $\alpha \in (\frac{2}{3}, 1)$, the closed-loop system* (2.15), (2.17) *is globally equiuniformly finite time stable, regardless of whichever disturbance $\omega(x, t)$, satisfying condition* (2.19) *with $0 < M < \min\{\mu_1, \mu_3\}, \mu_2 > \max\{\mu_1, \mu_3\}$, affects the system.*

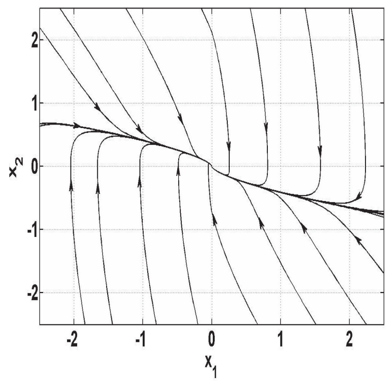

FIG. 1. *Phase plane plot of the closed-loop system* (2.15), (2.17) *with* $\omega(x,t) = 0, \mu_1 = 3, \mu_2 = 5, \mu_3 = 4$.

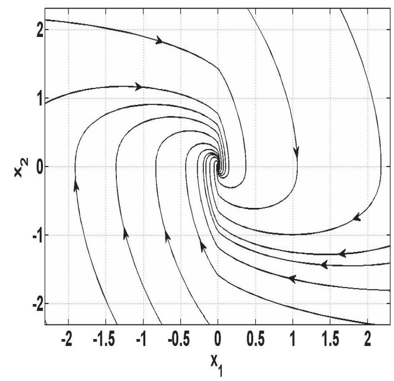

FIG. 2. *Phase plane plot of the closed-loop system* (2.15), (2.17) *with* $\omega(x,t) = 0, \mu_1 = 1, \mu_2 = 2, \mu_3 = 1$.

Proof. The proof is divided into several steps.

Step 1: *Global asymptotic stability.* Let the following candidate Lyapunov function V be considered [6, 27]:

$$V(x_1, x_2) = \mu_2 \frac{2-\alpha}{2} |x_1|^{\frac{2}{2-\alpha}} + \frac{1}{2}\, x_2^2. \tag{3.1}$$

Under the conditions of the theorem, the time derivative of the function $V(x_1, x_2)$, computed along the trajectories of (2.15), (2.17), is estimated as follows:

$$\begin{aligned}
\dot{V} &= \mu_2 |x_1|^{\frac{\alpha}{2-\alpha}} x_2 \mathrm{sign}(x_1) + x_2 \begin{pmatrix} -(\mu_1 |x_2|^\alpha + \mu_3 |x_1|^{\frac{\alpha}{2(2-\alpha)}} |x_2|^{\frac{\alpha}{2}}) \mathrm{sign}(x_2) \\ -\mu_2 |x_1|^{\frac{\alpha}{2-\alpha}} \mathrm{sign}(x_1) + \omega(x,t) \end{pmatrix} \\
&= -\mu_1 |x_2|^{\alpha+1} - \mu_3 |x_1|^{\frac{\alpha}{2(2-\alpha)}} |x_2|^{\frac{\alpha}{2}+1} + x_2 \omega(x,t) \\
&\le -\mu_1 |x_2|^{\alpha+1} - \mu_3 |x_1|^{\frac{\alpha}{2(2-\alpha)}} |x_2|^{\frac{\alpha}{2}+1} + |x_2| M (|x_1|^{\frac{\alpha}{2(2-\alpha)}} |x_2|^{\frac{\alpha}{2}} + |x_2|^\alpha) \\
&\le -(\mu_1 - M) |x_2|^{\alpha+1} - (\mu_3 - M) |x_1|^{\frac{\alpha}{2(2-\alpha)}} |x_2|^{\frac{\alpha}{2}+1}.
\end{aligned} \tag{3.2}$$

Noting that $M < \min\{\mu_1, \mu_3\}$ holds true by a condition of Theorem 3.1, expression $x_2 \dot{x}_2 \not< 0$ holds true, and the equilibrium point $x_1 = x_2 = 0$ is the only trajectory of (2.15), (2.17) on the invariance manifold $x_2 = 0$ where $\dot{V}(x_1, x_2) = 0$, the global asymptotic stability of (2.15), (2.17) is then established by applying the invariance principle [2, 33].

Step 2: *Semiglobal strong Lyapunov functions.* This step shows the existence of a parameterized family of semiglobal Lyapunov functions $V_{\tilde{R}}(x_1, x_2)$, with an a priori but arbitrarily given $\tilde{R} > 0$, such that each $V_{\tilde{R}}(x_1, x_2)$ is well-posed on the corresponding compact set

$$D_{\tilde{R}} = \{(x_1, x_2) \in \mathbf{R}^2 : V(x_1, x_2) \le \tilde{R}\}. \tag{3.3}$$

In other words, $V_{\tilde{R}}(x_1, x_2)$ is to be positive definite on $D_{\tilde{R}}$, and its derivative, computed along the trajectories of the uncertain system (2.15), (2.17) with initial conditions within $D_{\tilde{R}}$, is to be negative definite in the sense that

$$\dot{V}_{\tilde{R}}(x_1, x_2) \le -W_{\tilde{R}}(x_1, x_2) \tag{3.4}$$

for all $(x_1, x_2) \in D_{\tilde{R}}$ and for some $W_{\tilde{R}}(x_1, x_2)$ positive definite on $D_{\tilde{R}}$. A parameterized family of Lyapunov functions $V_{\tilde{R}}(x_1, x_2), \tilde{R} > 0$, with the properties defined above are constructed as follows by combining the Lyapunov function V of (3.1), whose time derivative (3.2) along the system motion is only negative semidefinite, with the indefinite functions $U_i, i = 1, 2, 3, 4$:

$$V_{\tilde{R}}(x_1, x_2) = V + \sum_{i=1}^{4} U_i, \tag{3.5}$$

where the indefinite functions $U_i, i = 1, 2, 3, 4$, are defined by the expressions

$$\begin{aligned} &U_1 = \kappa_1 x_1 x_2 |x_2|, \quad U_2 = \kappa_1 \kappa_2 |x_1|^{\frac{4-\alpha}{2(2-\alpha)}} \operatorname{sign}(x_1) x_2 |x_2|^{\alpha}, \\ &U_3 = 2\kappa_1\kappa_2\kappa_3 x_1^3 x_2 |x_2|^{\frac{\alpha}{2}}, \quad U_4 = \kappa_1\kappa_2\kappa_3\kappa_4 x_1^5 x_2, \end{aligned} \tag{3.6}$$

and the positive constants $\kappa_i, i = 1, 2, 3, 4$, are chosen a priori as follows:

$$\begin{aligned} &\kappa_4 < \frac{(2+\alpha)\mu_2}{\beta_5}, \quad \kappa_3 < \frac{(1+\alpha)\mu_2}{\beta_4}, \quad \kappa_2 < \frac{2\mu_2}{\beta_3}, \\ &\kappa_1 < \min\left\{ \frac{\mu_1 - M}{\beta_1}, \quad \frac{\mu_3 - M}{\beta_2}, \quad \frac{\mu_2(2-\alpha)}{\beta_6}, \quad \frac{1}{\beta_7} \right\}, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \rho = \frac{2\tilde{R}}{\mu_2(2-\alpha)}, \ &\beta_7 = ((2\tilde{R}) + \kappa_2(2\tilde{R})^{\alpha} + 2\kappa_2\kappa_3(2\tilde{R})^{\frac{\alpha}{2}} + \kappa_2\kappa_3\kappa_4), \\ &\beta_6 = (\rho^{1-\alpha} + \kappa_2\rho^{\frac{2-\alpha}{2}} + 2\kappa_2\kappa_3\rho^{5-3\alpha} + \kappa_2\kappa_3\kappa_4\rho^{9-5\alpha}), \\ &\beta_5 = (\mu_3 + M)\rho^{\frac{8-5\alpha}{4}}, \\ &\beta_4 = (2+\alpha)(\mu_3 + M)\rho^{\frac{4-3\alpha}{2}} + \kappa_4(\mu_1 + M)\rho^{\frac{16-11\alpha}{4}}, \\ &\beta_3 = (1+\alpha)(\mu_3 + M)(2\tilde{R})^{\frac{3\alpha-2}{4}} + \kappa_3(2+\alpha)(\mu_1 + M)\rho^{\frac{4-3\alpha}{2}}(2\tilde{R})^{\frac{3\alpha-2}{4}}, \\ &\beta_2 = 2(\mu_3 + M)\rho^{\frac{2-\alpha}{2}} + \kappa_2(1+\alpha)(\mu_1 + M)\rho^{\frac{2-\alpha}{2}}(2\tilde{R})^{\frac{3\alpha-2}{4}}, \\ &\beta_1 = (2\tilde{R})^{\frac{2-\alpha}{2}} + 2(\mu_1 + M)\rho^{\frac{2-\alpha}{2}} + \kappa_2\frac{4-\alpha}{2(2-\alpha)}\sqrt{2\tilde{R}}\rho^{\frac{\alpha}{4}} \\ &\qquad + 6\kappa_2\kappa_3\rho^{2-\alpha}(2\tilde{R})^{\frac{2-\alpha}{4}} + 5\kappa_2\kappa_3\kappa_4\rho^{2(2-\alpha)}(2\tilde{R})^{\frac{1-\alpha}{2}}. \end{aligned} \tag{3.8}$$

It should be noted that it is always possible to fix required parameters $\beta_j, j = 1, 2, \ldots, 7$, and $\kappa_i, i = 1, 2, 3, 4$, unambiguously satisfying (3.7) and (3.8) when the following tuning procedure is adhered to.

In the first step, once constants μ_1, μ_2, and μ_3 are fixed according to Theorem 3.1, relation (3.8) defines ρ corresponding to $\tilde{R}$ of (3.3). In the second step, the constant β_5, which does not depend on any κ_i, can be fixed using (3.8). Then κ_4 is fixed using (3.7). In the third step, β_4 is fixed using (3.8), and κ_4 is as defined in the previous step. Then, κ_3 is fixed using (3.7). In the fourth step, β_3 is fixed using (3.8), and κ_3 is as defined in the previous step. Then κ_2 is fixed using (3.7). In the fifth step, β_1, β_6, and β_7 are fixed using (3.8), and $\kappa_2, \kappa_3, \kappa_4$ are as defined in the previous steps. In the sixth step, fix β_2 using (3.8), and κ_2 is as defined in the fourth step. In the last step, fix κ_1 using (3.7), and $\beta_1, \beta_2, \beta_6, \beta_7$ are as defined in the previous steps.

Due to (3.2), all possible solutions of the uncertain system (2.15), (2.17), initialized at $t_0 \in \mathbf{R}$ within the compact set (3.3), are a priori estimated by

$$\sup_{t\in[t_0,\infty)} V(x_1, x_2) \leq \tilde{R}. \tag{3.9}$$

The following inequalities hold true:

$$|x_1|^{\frac{2}{2-\alpha}} \leq \rho = \frac{2\tilde{R}}{(2-\alpha)\mu_2}, \quad |x_2| \leq \sqrt{2\tilde{R}}. \tag{3.10}$$

Let the positive definiteness of the Lyapunov function (3.5) be verified. The positive definiteness of (3.5) is guaranteed as shown below:

$$\begin{aligned}
U_1 = \kappa_1 x_1 x_2 |x_2| &\geq -\frac{\kappa_1}{2}x_1^2 - \frac{\kappa_1}{2}x_2^2|x_2|^2 \geq -\frac{\kappa_1}{2}\rho^{1-\alpha}|x_1|^{\frac{2}{2-\alpha}} - \frac{\kappa_1}{2}(2\tilde{R})x_2^2,\\
U_2 = \kappa_1\kappa_2|x_1|^{\frac{4-\alpha}{2(2-\alpha)}}\operatorname{sign}(x_1)x_2|x_2|^{\alpha} &\geq -\frac{1}{2}\kappa_1\kappa_2|x_1|^{\frac{4-\alpha}{2-\alpha}} - \frac{1}{2}\kappa_1\kappa_2 x_2^2|x_2|^{2\alpha}\\
&\geq -\frac{1}{2}\kappa_1\kappa_2\left(\rho^{\frac{2-\alpha}{2}}|x_1|^{\frac{2}{2-\alpha}} + (2\tilde{R})^{\alpha}x_2^2\right),\\
U_3 = 2\kappa_1\kappa_2\kappa_3 x_1^3 x_2|x_2|^{\frac{\alpha}{2}} &\geq -\kappa_1\kappa_2\kappa_3\left(x_1^6 + x_2^2|x_2|^{\alpha}\right)\\
&\geq -\kappa_1\kappa_2\kappa_3\left(|x_1|^{\frac{2}{2-\alpha}}\rho^{5-3\alpha} + x_2^2(\sqrt{2\tilde{R}})^{\alpha}\right),\\
U_4 = \kappa_1\ \kappa_2\ \kappa_3\ \kappa_4\ x_1^5\ x_2 &\geq -\frac{\kappa_1\ \kappa_2\ \kappa_3\ \kappa_4}{2}x_1^{10} - \kappa_1\ \kappa_2\ \kappa_3\ \kappa_4 x_2^2\\
&\geq -\frac{\kappa_1\ \kappa_2\ \kappa_3\ \kappa_4}{2}|x_1|^{\frac{2}{2-\alpha}}\rho^{9-5\alpha} - \frac{\kappa_1\ \kappa_2\ \kappa_3\ \kappa_4}{2}x_2^2,
\end{aligned} \tag{3.11}$$

where (3.10) and the trivial inequality $2ab > -(a^2 + b^2)$ for all $a, b \in \mathbf{R}$ have been utilized. Hence, the Lyapunov function (3.5) is positive definite on compact set (3.3); for all $(x_1, x_2) \in D_{\tilde{R}}\backslash\{0, 0\}$ and $\kappa_i > 0, i = 1, 2, 3, 4$, satisfying (3.7), as shown below,

$$V_{\tilde{R}} \geq \frac{\mu_2(2-\alpha)}{2}|x_1|^{\frac{2}{2-\alpha}} + \frac{1}{2}x_2^2 - \frac{1}{2}\kappa_1\beta_6|x_1|^{\frac{2}{2-\alpha}} - \frac{1}{2}\kappa_1\beta_7 x_2^2 \geq L_{\tilde{R}}V, \tag{3.12}$$

where inequalities of (3.11) have been utilized, β_6, β_7 are defined in (3.8), and the positive scalar $L_{\tilde{R}}$ is defined as follows:

$$L_{\tilde{R}} < \min\left\{\mu_2\frac{2-\alpha}{2} - \frac{1}{2}\kappa_1\beta_6, \quad \frac{1}{2}(1-\kappa_1\beta_7)\right\}. \tag{3.13}$$

The scalar $L_{\tilde{R}}$ is always greater than zero due to the condition (3.7) on κ_1. Similarly, the upper bound on $V_{\tilde{R}}$ in terms of a $\mathcal{C}^1$ function can be obtained as follows:

$$V_{\tilde{R}} \leq \frac{\mu_2(2-\alpha)}{2}|x_1|^{\frac{2}{2-\alpha}} + \frac{1}{2}x_2^2 + \frac{1}{2}\kappa_1\beta_6|x_1|^{\frac{2}{2-\alpha}} + \frac{1}{2}\kappa_1\beta_7 x_2^2 \leq M_{\tilde{R}}V, \tag{3.14}$$

where the trivial inequality $2ab < (a^2 + b^2)$ for all $a, b \in \mathbf{R}$ is used and

$$M_{\tilde{R}} > \max\left\{\mu_2\frac{2-\alpha}{2} + \frac{1}{2}\kappa_1\beta_6, \quad \frac{1}{2}(1+\kappa_1\beta_7)\right\}. \tag{3.15}$$

Having established the positive definiteness of $V_{\tilde{R}}$, its derivative computed along the trajectories of the closed-loop system (2.15), (2.17) is to be negative definite in the sense of (3.4). The derivative of U_1 can be obtained as follows:

$$\begin{aligned}
\dot{U}_1 &= \kappa_1|x_2|^3 + \kappa_1 x_1|x_2|\dot{x_2} + \kappa_1 x_1 x_2 \operatorname{sign}(x_2)\dot{x_2} \\
&= \kappa_1|x_2|^3 + 2\kappa_1 x_1|x_2|\dot{x_2} \\
&= \kappa_1|x_2|^3 + 2\kappa_1 x_1|x_2| \begin{pmatrix} -(\mu_1|x_2|^\alpha + \mu_3|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}})\operatorname{sign}(x_2) \\ -\mu_2|x_1|^{\frac{\alpha}{2-\alpha}}\operatorname{sign}(x_1) + \omega(x,t) \end{pmatrix} \\
&\le \kappa_1|x_2|^3 + 2\kappa_1\mu_1|x_1||x_2|^{\alpha+1} + 2\kappa_1\mu_3|x_1||x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}+1} \\
&\qquad + 2\kappa_1 M|x_1||x_2|(|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}} + |x_2|^\alpha) - 2\kappa_1\mu_2|x_1|^{\frac{2}{2-\alpha}}|x_2| \\
&\le \kappa_1|x_2|^3 + 2\kappa_1(\mu_1 + M)|x_1||x_2|^{\alpha+1} + 2\kappa_1(\mu_3 + M)|x_1|^{\frac{4-\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}+1} \\
&\qquad - 2\kappa_1\mu_2|x_1|^{\frac{2}{2-\alpha}}|x_2|.
\end{aligned} \tag{3.16}$$

Similarly, the derivative of U_2 can be obtained as follows:

$$\begin{aligned}
\dot{U}_2 &= \kappa_1\kappa_2\frac{4-\alpha}{2(2-\alpha)}|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\alpha+2} + \kappa_1\kappa_2|x_1|^{\frac{4-\alpha}{2(2-\alpha)}}\operatorname{sign}(x_1)|x_2|^\alpha\dot{x}_2 \\
&\quad + \kappa_1\kappa_2|x_1|^{\frac{4-\alpha}{2(2-\alpha)}}\operatorname{sign}(x_1)x_2\alpha|x_2|^{\alpha-1}\operatorname{sign}(x_2)\dot{x}_2 \\
&= \kappa_1\kappa_2\frac{4-\alpha}{2(2-\alpha)}|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\alpha+2} + (1+\alpha)\kappa_1\kappa_2|x_1|^{\frac{4-\alpha}{2(2-\alpha)}}\operatorname{sign}(x_1)|x_2|^\alpha\dot{x}_2 \\
&= \kappa_1\kappa_2\frac{4-\alpha}{2(2-\alpha)}|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\alpha+2} \\
&\quad + (1+\alpha)\kappa_1\kappa_2|x_1|^{\frac{4-\alpha}{2(2-\alpha)}}\operatorname{sign}(x_1)|x_2|^\alpha \begin{pmatrix} -\mu_1|x_2|^\alpha\operatorname{sign}(x_2) - \mu_2|x_1|^{\frac{\alpha}{2-\alpha}}\operatorname{sign}(x_1) \\ -\mu_3|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}}\operatorname{sign}(x_2) + \omega(x,t) \end{pmatrix} \\
&\le \kappa_1\kappa_2\frac{4-\alpha}{2(2-\alpha)}|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\alpha+2} + \kappa_1\kappa_2(1+\alpha)\mu_1|x_1|^{\frac{4-\alpha}{2(2-\alpha)}}|x_2|^{2\alpha} \\
&\quad + \kappa_1\kappa_2(1+\alpha)\mu_3|x_1|^{\frac{2}{2-\alpha}}|x_2|^{\frac{3\alpha}{2}} \\
&\quad + \kappa_1\kappa_2(1+\alpha)M|x_1|^{\frac{4-\alpha}{2(2-\alpha)}}|x_2|^\alpha(|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}} + |x_2|^\alpha) \\
&\quad - \kappa_1\kappa_2(1+\alpha)\mu_2|x_1|^{\frac{4+\alpha}{2(2-\alpha)}}|x_2|^\alpha \\
&\le \kappa_1\kappa_2\frac{4-\alpha}{2(2-\alpha)}|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\alpha+2} + \kappa_1\kappa_2(1+\alpha)(\mu_1 + M)|x_1|^{\frac{4-\alpha}{2(2-\alpha)}}|x_2|^{2\alpha} \\
&\quad + \kappa_1\kappa_2(1+\alpha)(\mu_3 + M)|x_1|^{\frac{2}{2-\alpha}}|x_2|^{\frac{3\alpha}{2}} - \kappa_1\kappa_2(1+\alpha)\mu_2|x_1|^{\frac{4+\alpha}{2(2-\alpha)}}|x_2|^\alpha.
\end{aligned} \tag{3.17}$$

Similarly, the derivative of U_3 can be obtained as follows:

$$\begin{aligned}
\dot{U}_3 &= 6\kappa_1\kappa_2\kappa_3 x_1^2|x_2|^{\frac{\alpha}{2}+2} + 2\kappa_1\kappa_2\kappa_3 x_1^3|x_2|^{\frac{\alpha}{2}}\dot{x}_2 + \alpha\kappa_1\kappa_2\kappa_3 x_1^3 x_2|x_2|^{\frac{\alpha}{2}-1}\operatorname{sign}(x_2)\dot{x}_2 \\
&= 6\kappa_1\kappa_2\kappa_3 x_1^2|x_2|^{\frac{\alpha}{2}+2} + (2+\alpha)\kappa_1\kappa_2\kappa_3 x_1^3|x_2|^{\frac{\alpha}{2}}\dot{x}_2 \\
&= 6\kappa_1\kappa_2\kappa_3 x_1^2|x_2|^{\frac{\alpha}{2}+2} \\
&\quad + (2+\alpha)\kappa_1\kappa_2\kappa_3 x_1^3|x_2|^{\frac{\alpha}{2}} \begin{pmatrix} -\mu_1|x_2|^\alpha\operatorname{sign}(x_2) - \mu_2|x_1|^{\frac{\alpha}{2-\alpha}}\operatorname{sign}(x_1) \\ -\mu_3|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}}\operatorname{sign}(x_2) + \omega(x,t) \end{pmatrix} \\
&= 6\kappa_1\kappa_2\kappa_3 x_1^2|x_2|^{\frac{\alpha}{2}+2} - (2+\alpha)\kappa_1\kappa_2\kappa_3\mu_1 x_1^3|x_2|^{\frac{3\alpha}{2}}\operatorname{sign}(x_2) \\
&\quad - (2+\alpha)\kappa_1\kappa_2\kappa_3\mu_2 x_1^2|x_1|^{\frac{2}{2-\alpha}}|x_2|^{\frac{\alpha}{2}} \\
&\quad - (2+\alpha)\kappa_1\kappa_2\kappa_3\mu_3 x_1^3|x_2|^{\frac{\alpha}{2}}|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}}\operatorname{sign}(x_2) + (2+\alpha)\kappa_1\kappa_2\kappa_3 x_1^3|x_2|^{\frac{\alpha}{2}}\omega \\
&\le 6\kappa_1\kappa_2\kappa_3 x_1^2|x_2|^{\frac{\alpha}{2}+2} + (2+\alpha)\kappa_1\kappa_2\kappa_3\mu_1|x_1|^3|x_2|^{\frac{3\alpha}{2}} - (2+\alpha)\kappa_1\kappa_2\kappa_3\mu_2 x_1^2|x_1|^{\frac{2}{2-\alpha}}|x_2|^{\frac{\alpha}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+(2+\alpha)\kappa_1\kappa_2\kappa_3\mu_3|x_1|^3|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\alpha}\\
&+(2+\alpha)\kappa_1\kappa_2\kappa_3|x_1|^3|x_2|^{\frac{\alpha}{2}}(|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}}+|x_2|^{\alpha})\\
&\le 6\kappa_1\kappa_2\kappa_3x_1^2|x_2|^{\frac{\alpha}{2}+2}+(2+\alpha)\kappa_1\kappa_2\kappa_3(\mu_1+M)|x_1|^3|x_2|^{\frac{3\alpha}{2}}\\
&\quad -(2+\alpha)\kappa_1\kappa_2\kappa_3\mu_2x_1^2|x_1|^{\frac{2}{2-\alpha}}|x_2|^{\frac{\alpha}{2}}\\
&\quad +(2+\alpha)\kappa_1\kappa_2\kappa_3(\mu_3+M)|x_1|^3|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\alpha}.
\end{aligned}
\tag{3.18}
$$

Finally, the derivative of U_4 can be obtained as follows:

$$
\begin{aligned}
\dot U_4 &= 5\kappa_1\kappa_2\kappa_3\kappa_4x_1^4x_2^2+\kappa_1\kappa_2\kappa_3\kappa_4x_1^5\begin{pmatrix}-\mu_1|x_2|^{\alpha}\operatorname{sign}(x_2)-\mu_2|x_1|^{\frac{\alpha}{2-\alpha}}\operatorname{sign}(x_1)\\ -\mu_3|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}}\operatorname{sign}(x_2)+\omega(x,t)\end{pmatrix}\\
&= 5\kappa_1\kappa_2\kappa_3\kappa_4x_1^4x_2^2-\kappa_1\kappa_2\kappa_3\kappa_4\mu_1x_1^5|x_2|^{\alpha}\operatorname{sign}(x_2)-\kappa_1\kappa_2\kappa_3\kappa_4\mu_2x_1^4|x_1|^{\frac{2}{2-\alpha}}\\
&\quad -\kappa_1\kappa_2\kappa_3\kappa_4\mu_3x_1^5|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}}\operatorname{sign}(x_2)+\kappa_1\kappa_2\kappa_3\kappa_4x_1^5\omega\\
&\le 5\kappa_1\kappa_2\kappa_3\kappa_4x_1^4x_2^2+\kappa_1\kappa_2\kappa_3\kappa_4\mu_1|x_1|^5|x_2|^{\alpha}-\kappa_1\kappa_2\kappa_3\kappa_4\mu_2x_1^4|x_1|^{\frac{2}{2-\alpha}}\\
&\quad +\kappa_1\kappa_2\kappa_3\kappa_4\mu_3|x_1|^5|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}}+\kappa_1\kappa_2\kappa_3\kappa_4Mx_1^5(|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}}+|x_2|^{\alpha})\\
&\le 5\kappa_1\kappa_2\kappa_3\kappa_4x_1^4x_2^2+\kappa_1\kappa_2\kappa_3\kappa_4(\mu_1+M)|x_1|^5|x_2|^{\alpha}-\kappa_1\kappa_2\kappa_3\kappa_4\mu_2x_1^4|x_1|^{\frac{2}{2-\alpha}}\\
&\quad +\kappa_1\kappa_2\kappa_3\kappa_4(\mu_3+M)|x_1|^5|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}}.
\end{aligned}
\tag{3.19}
$$

The following inequality shows the negative definiteness of the derivative of $V+U=V+U_1+U_2+U_3+U_4$ which can be formulated by collecting all the derivatives from (3.2), (3.16), (3.17), (3.18), and (3.19) as follows:

$$
\begin{aligned}
\dot V+\sum_{i=1}^{4}\dot U_i \le{}& -\overbrace{(\mu_1-M)|x_2|^{\alpha+1}}-\overbrace{(\mu_3-M)|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}+1}}\\
&+\kappa_1|x_2|^3+2\kappa_1(\mu_1+M)|x_1||x_2|^{\alpha+1}+2\kappa_1(\mu_3+M)|x_1|^{\frac{4-\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}+1}\\
&\overbrace{-2\kappa_1\mu_2|x_1|^{\frac{2}{2-\alpha}}|x_2|}\\
&+\kappa_1\kappa_2\frac{4-\alpha}{2(2-\alpha)}|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\alpha+2}+\kappa_1\kappa_2(1+\alpha)(\mu_1+M)|x_1|^{\frac{4-\alpha}{2(2-\alpha)}}|x_2|^{2\alpha}\\
&+\kappa_1\kappa_2(1+\alpha)(\mu_3+M)|x_1|^{\frac{2}{2-\alpha}}|x_2|^{\frac{3\alpha}{2}}-\overbrace{\kappa_1\kappa_2(1+\alpha)\mu_2|x_1|^{\frac{4+\alpha}{2(2-\alpha)}}|x_2|^{\alpha}}\\
&+6\kappa_1\kappa_2\kappa_3x_1^2|x_2|^{\frac{\alpha}{2}+2}+(2+\alpha)\kappa_1\kappa_2\kappa_3(\mu_1+M)|x_1|^3|x_2|^{\frac{3\alpha}{2}}\\
&\overbrace{-(2+\alpha)\kappa_1\kappa_2\kappa_3\mu_2x_1^2|x_1|^{\frac{2}{2-\alpha}}|x_2|^{\frac{\alpha}{2}}}\\
&+(2+\alpha)\kappa_1\kappa_2\kappa_3(\mu_3+M)|x_1|^3|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\alpha}\\
&+5\kappa_1\kappa_2\kappa_3\kappa_4x_1^4x_2^2+\kappa_1\kappa_2\kappa_3\kappa_4(\mu_1+M)|x_1|^5|x_2|^{\alpha}-\overbrace{\kappa_1\kappa_2\kappa_3\kappa_4\mu_2x_1^4|x_1|^{\frac{2}{2-\alpha}}}\\
&+\kappa_1\kappa_2\kappa_3\kappa_4(\mu_3+M)|x_1|^5|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}}.
\end{aligned}
\tag{3.20}
$$

The expression (3.20) can be simplified by using the conservative upper bounds for the positive definite terms within the compact set $D_{\tilde R}$ defined in (3.3). Utilizing the expressions (3.9) and (3.10) produces the following:

$$
\begin{gathered}
|x_2|^3\le(2\tilde R)^{\frac{2-\alpha}{2}}|x_2|^{\alpha+1}\ ,\quad |x_1||x_2|^{\alpha+1}\le\rho^{\frac{2-\alpha}{2}}|x_2|^{\alpha+1},\\
|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\alpha+2}\le\rho^{\frac{\alpha}{4}}\sqrt{2\tilde R}|x_2|^{\alpha+1}\ ,\quad x_1^2|x_2|^{\frac{\alpha}{2}+2}\le\rho^{2-\alpha}(2\tilde R)^{\frac{2-\alpha}{4}}|x_2|^{\alpha+1},\\
x_1^4x_2^2\le\rho^{2(2-\alpha)}(2\tilde R)^{\frac{1-\alpha}{2}}|x_2|^{\alpha+1}\ ,\quad |x_1|^{\frac{4-\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}+1}\le\rho^{\frac{2-\alpha}{2}}|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}+1},
\end{gathered}
$$

$$\begin{aligned}(3.21)\qquad & |x_1|^{\frac{4-\alpha}{2(2-\alpha)}}|x_2|^{2\alpha} \le \rho^{\frac{2-\alpha}{2}}(2\tilde{R})^{\frac{3\alpha-2}{4}}|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}+1},\\ & |x_1|^{\frac{2}{2-\alpha}}|x_2|^{\frac{3\alpha}{2}} \le (2\tilde{R})^{\frac{3\alpha-2}{4}}|x_1|^{\frac{2}{2-\alpha}}|x_2| \ , \ \ |x_1|^3|x_2|^{\frac{3\alpha}{2}} \le \rho^{\frac{4-3\alpha}{2}}(2\tilde{R})^{\frac{3\alpha-2}{4}}|x_1|^{\frac{2}{2-\alpha}}|x_2|,\\ & |x_1|^3|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\alpha} \le \rho^{\frac{4-3\alpha}{2}}|x_1|^{\frac{4+\alpha}{2(2-\alpha)}}|x_2|^{\alpha} \ , \ \ |x_1|^5|x_2|^{\alpha} \le \rho^{\frac{16-11\alpha}{4}}|x_1|^{\frac{4+\alpha}{2(2-\alpha)}}|x_2|^{\alpha},\\ & |x_1|^5|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}} \le \rho^{\frac{8-5\alpha}{4}}x_1^2|x_1|^{\frac{2}{2-\alpha}}|x_2|^{\frac{\alpha}{2}}.\end{aligned}$$

Combining the overbraced terms in (3.20), which are the most dominant negative terms, with the corresponding weaker positive definite terms as given by the upper bounds (3.21) produces the following compact form of (3.20):

$$\begin{aligned}& \dot{V} + \sum_{i=1}^{4}\dot{U}_i \le -((\mu_1 - M) - \kappa_1\beta_1)|x_2|^{\alpha+1} - ((\mu_3 - M) - \kappa_1\beta_2)|x_1|^{\frac{\alpha}{2(2-\alpha)}}|x_2|^{\frac{\alpha}{2}+1}\\ (3.22)\qquad & -\kappa_1(\mu_2 - \kappa_2\beta_3)|x_1|^{\frac{2}{2-\alpha}}|x_2| - \kappa_1\kappa_2((1+\alpha)\mu_2 - \kappa_3\beta_4)|x_1|^{\frac{4+\alpha}{2(2-\alpha)}}|x_2|^{\alpha}\\ & -\kappa_1\kappa_2\kappa_3((2+\alpha)\mu_2 - \kappa_4\beta_5)x_1^2|x_1|^{\frac{2}{2-\alpha}}|x_2|^{\frac{\alpha}{2}} - \kappa_1\kappa_2\kappa_3\kappa_4x_1^4|x_1|^{\frac{2}{2-\alpha}},\end{aligned}$$

where $\beta_i, i = 1, 2, 3, 4, 5$, are defined as in (3.8). It can be seen from the definitions (3.7) that the derivative $V_{\tilde{R}}$ is negative definite. Ignoring the semidefinite terms in (3.22) containing $|x_1||x_2|$, the temporal derivative $\dot{V}_{\tilde{R}}$ can be obtained as

$$(3.23)\quad \dot{V}_{\tilde{R}} = \dot{V} + \sum_{i=1}^{4}\dot{U}_i \le -((\mu_1 - M) - \kappa_1\beta_1)|x_2|^{\alpha+1} - \kappa_1\kappa_2\kappa_3\kappa_4x_1^4|x_1|^{\frac{2}{2-\alpha}}.$$

Furthermore, the following inequalities hold true within the compacta (3.3):

$$(3.24)\qquad \begin{aligned}& x_2^2 = |x_2|^2 = |x_2|^{\alpha+1}|x_2|^{1-\alpha} \le |x_2|^{\alpha+1}\left(\sqrt{2\tilde{R}}\right)^{1-\alpha}\\ & \Rightarrow -|x_2|^{\alpha+1} \le -\frac{x_2^2}{\left(\sqrt{2\tilde{R}}\right)^{1-\alpha}}.\end{aligned}$$

Hence, (3.23) can be simplified as follows:

$$(3.25)\qquad \dot{V}_{\tilde{R}} \le -c_{\tilde{R}}\left(|x_1|^{\frac{10-4\alpha}{2-\alpha}} + x_2^2\right),$$

where

$$(3.26)\qquad c_{\tilde{R}} = \min\left\{\frac{(\mu_1 - M) - \kappa_1\beta_1}{\left(\sqrt{2\tilde{R}}\right)^{1-\alpha}}, \quad \kappa_1\ \kappa_2\ \kappa_3\ \kappa_4\ \mu_2\right\} > 0.$$

Case 1: $|x_1| \ge 1$. The following inequality holds true for $|x_1| \ge 1$:

$$(3.27)\qquad \frac{10-4\alpha}{2-\alpha} \ge \frac{2}{2-\alpha} \Leftrightarrow |x_1|^{\frac{10-4\alpha}{2-\alpha}} \ge |x_1|^{\frac{2}{2-\alpha}}.$$

Also, the following can be obtained from (3.14):

$$(3.28)\qquad \frac{M_{\tilde{R}}}{2}\max\{1, \mu_2(2-\alpha)\}(|x_1|^{\frac{2}{2-\alpha}} + x_2^2) \ge V_{\tilde{R}}(x_1, x_2).$$

Hence, the following inequality is then obtained for $|x_1| \geq 1$ by combining (3.25), (3.27), and (3.28):

$$\dot{V}_{\tilde{R}} \leq -\bar{\kappa}_1 V_{\tilde{R}}, \tag{3.29}$$

where

$$\bar{\kappa}_1 = \frac{2c_{\tilde{R}}}{M_{\tilde{R}} \max\{1, \mu_2(2-\alpha)\}} > 0. \tag{3.30}$$

Case 2: $|x_1| < 1$. Note that the following inequalities hold true for $|x_1| < 1$ and for some $\gamma > 5 - 2\alpha$:

$$|x_1|^{\frac{10-4\alpha}{2-\alpha}} > |x_1|^{\frac{2\gamma}{2-\alpha}} \Leftrightarrow \frac{10-4\alpha}{2-\alpha} < \frac{2\gamma}{2-\alpha} \Leftrightarrow \gamma > 5 - 2\alpha. \tag{3.31}$$

Noting that $5 - 2\alpha < \frac{11}{3}$ always holds true due to $\alpha \in (\frac{2}{3}, 1)$, $\gamma \geq \frac{11}{3}$ is a valid choice. In the following, $\gamma = 4$ is chosen. It can be seen that the following equality holds true:

$$\begin{aligned}\left(|x_1|^{\frac{2}{2-\alpha}} + x_2^2\right)^4 &= |x_1|^{\frac{8}{2-\alpha}} + 4\,|x_1|^{\frac{6}{2-\alpha}} x_2^2 + 6\,|x_1|^{\frac{4}{2-\alpha}} x_2^4 + 4\,|x_1|^{\frac{2}{2-\alpha}} x_2^6 + x_2^8 \\ &\leq \max\{\rho^{2\alpha-1}, K_2\} \left(|x_1|^{\frac{10-4\alpha}{2-\alpha}} + x_2^2\right),\end{aligned} \tag{3.32}$$

where the bounds (3.10) have been utilized, resulting in the following definition of K_2:

$$K_2 = \max\left\{4\rho^3, 6\rho^2(2\tilde{R}), 4\rho(2\tilde{R})^2, (2\tilde{R})^3\right\} > 0. \tag{3.33}$$

Note that the following can be obtained from (3.14):

$$\left(\frac{M_{\tilde{R}}}{2} \max\{1, \mu_2(2-\alpha)\}(|x_1|^{\frac{2}{2-\alpha}} + x_2^2)\right)^4 \geq \left(V_{\tilde{R}}(x_1, x_2)\right)^4. \tag{3.34}$$

Then, the following can be obtained by combining (3.25), (3.32), and (3.34):

$$\dot{V}_{\tilde{R}}(x_1, x_2) \leq -c_{\tilde{R}}\left(|x_1|^{\frac{10-4\alpha}{2-\alpha}} + x_2^2\right) \leq -\bar{\kappa}_2 \left(V_{\tilde{R}}\right)^4, \tag{3.35}$$

where

$$\bar{\kappa}_2 = \frac{c_{\tilde{R}}}{\left(\frac{M_{\tilde{R}}}{2} \max\{1, \mu_2(2-\alpha)\}\right)^4 \max\{\rho^{2\alpha-1}, K_2\}} > 0. \tag{3.36}$$

Hence, the desired uniform negative definiteness (3.4) is obtained by combining (3.29) and (3.35) as follows:

$$W_{\tilde{R}}(x_1, x_2) = \min\left\{\bar{\kappa}_1 V_{\tilde{R}}, \bar{\kappa}_2 \left(V_{\tilde{R}}\right)^4\right\}. \tag{3.37}$$

Step 3: *Global equiuniform asymptotic stability.* Since the inequality (3.4) holds on the solutions of the uncertain system (2.15), (2.17), initialized within the compact set (3.3), the decay of the function $V_{\tilde{R}}(x_1, x_2)$ can be found by considering the majorant solution $\nu(t)$ of $V_{\tilde{R}}$ as follows:

$$\dot{\nu}(t) = \begin{cases} -\bar{\kappa}_1 \nu(t) & \text{if } |x_1| \geq 1, \\ -\bar{\kappa}_2 \nu^{\gamma} & \text{if } |x_1| < 1, \end{cases} \tag{3.38}$$

where $\gamma > 5 - 2\alpha$ is introduced for generality. A more conservative decay than that in (3.38) can be computed. There are two possible subcases, namely, $\nu(t) \geq 1$ and $\nu(t) < 1$, for each of the cases $|x_1| \geq 1$ and $|x_2| < 1$. The following expressions hold true for a positive definite function $\nu(t)$ and a scalar $\gamma > 1$:

$$\begin{aligned} &\nu(t)^\gamma \geq \nu(t) \Rightarrow -\nu(t)^\gamma \leq -\nu(t) \quad \text{if } \nu(t) \geq 1, \\ &\nu(t)^\gamma \leq \nu(t) \Rightarrow -\nu(t) \leq -\nu(t)^\gamma \quad \text{if } \nu(t) < 1. \end{aligned} \tag{3.39}$$

Hence, the decay (3.38) is modified by utilizing (3.39) independently of the magnitude of $|x_1|$ and dependent on $\nu(t)$ as follows:

$$\dot{\nu}(t) = \begin{cases} -\bar{\kappa}\nu & \text{if } \nu(t) \geq 1, \\ -\bar{\kappa}\nu^\gamma & \text{if } \nu(t) < 1, \end{cases} \tag{3.40}$$

where

$$\bar{\kappa} = \min\{\bar{\kappa}_1, \bar{\kappa}_2\} > 0. \tag{3.41}$$

The solution for the case $\nu(t) < 1$ can be obtained as follows:

$$\int_{\nu_0}^{\nu(t)} \frac{\mathrm{d}\zeta(t)}{\zeta^\gamma} = -\bar{\kappa} \int_{t_1}^{t} \mathrm{d}\tau, \tag{3.42}$$

where $\nu_0 = \nu(t_1)$, where t_1 is the time instant when the solution $\nu(t)$ satisfies the condition $\nu(t) = 1$. The general solution of $\nu(t)$ of (3.40) can then be obtained as follows:

$$\nu(t) = \begin{cases} \nu(t_0)\, e^{-\bar{\kappa}(t-t_0)} & \text{if } \nu(t) \geq 1, \\ \nu(t_1) \left(\frac{1}{\bar{\kappa}(t-t_1)(\gamma-1)\nu_0^{\gamma-1}+1} \right)^{\frac{1}{\gamma-1}} & \text{if } \nu(t) < 1. \end{cases} \tag{3.43}$$

It is noted that $t_1 = t_0$ if $\nu(t_0) \leq 1$. It can be easily seen that the solution $\nu(t) \to 0$ as $t \to \infty$ and that the decay rate depends on the gain parameters μ_1, μ_2, μ_3 and bound M on the disturbance $\omega(x,t)$. On the compact set (3.3), the following inequality holds (see (3.12) and (3.14)):

$$L_{\tilde{R}} V(x_1, x_2) \leq V_{\tilde{R}}(x_1, x_2) \leq M_{\tilde{R}} V(x_1, x_2) \tag{3.44}$$

for all $(x_1, x_2) \in D_{\tilde{R}}$ and positive constants $L_{\tilde{R}}, M_{\tilde{R}}$. The above inequalities (3.43) and (3.44) ensure that the globally radially unbounded function $V(x_1, x_2)$ decays exponentially:

$$V(x_1(t), x_2(t)) \leq \begin{cases} L_{\tilde{R}}^{-1} M_{\tilde{R}} \tilde{R} e^{-\bar{\kappa}(t-t_0)} & \text{if } V_{\tilde{R}} \geq 1, \\ L_{\tilde{R}}^{-1} M_{\tilde{R}} \tilde{R} \left(\frac{1}{\bar{\kappa}(t-t_1)(\gamma-1)\nu_0^{\gamma-1}+1} \right)^{\frac{1}{\gamma-1}} & \text{if } V_{\tilde{R}} < 1 \end{cases} \tag{3.45}$$

on the solutions of (2.15), (2.17) uniformly in $\omega(x,t)$ and the initial data, located within an arbitrarily large set (3.3). This proves that the uncertain system (2.15), (2.17) is globally equiuniformly asymptotically stable around the origin $(x_1, x_2) = (0, 0)$.

Step 4: *Global equiuniform finite time stability.* The piecewise continuous uncertainty $\omega(x_1, x_2, t)$ in the right-hand side of the system (2.15), (2.17) is uniformly bounded by $M|x_2|^{\frac{\alpha}{2}}(|x_1|^{\frac{\alpha}{2(2-\alpha)}} + |x_2|^{\frac{\alpha}{2}})$. The feedback is globally homogeneous with homogeneity degree $q = -1$ with respect to dilation $(r_1, r_2) = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$. In the presence of piecewise continuous disturbances $\omega(x_1, x_2, t)$, Lemma 2.9 proves that the closed-loop system (2.15), (2.17) is homogeneous of degree $q = -1$ with respect to dilations $(r_1, r_2) = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$. Thus, coupling the homogeneity of the perturbed system (2.15), (2.17) within the arbitrarily large compact set (3.3), with the global equiuniform asymptotic stability of the system (2.15), (2.17), it is obtained that the closed-loop system (2.15), (2.17) is globally equiuniformly finite time stable according to [26, Theorem 3.1]. ∎

Remark 4. In the case of purely continuous disturbances $\omega(x_1, x_2, t)$, the existing result ([3, Corollary 2.24] applied to [3, Example 2.19]), which utilizes some positive constants p, q such that $0 < q < p < 2$, may provide a finite time stability with superior robustness properties when compared to the controller (2.16). Under the hypothesis of [3, Corollary 2.24], for example, the result [3, Corollary 2.24] is superior if parameter α is chosen such that $0 < q < p < \alpha < 1$ holds true. However, the class of controllers presented in Theorem 3.1 is able to maintain robustness for discontinuous disturbances, which admit an upper bound that is a function of both the state variables, without requiring any condition such as that on homogeneous approximation appearing in [3, Corollary 2.24].

The controller (2.17) coincides with the controller (2.16) when $\mu_3 = 0$ is chosen. The controller (2.16) is robust to disturbances bounded by the vanishing bounds given by (2.18). This result is captured in the following theorem, the proof of which is very similar to that of Theorem 3.1 in that equiuniform asymptotic stability is to be proven by identifying semiglobal strong Lyapunov functions followed by the application of Lemma 2.8 to establish equiuniform finite time stability. A sketch of a proof is included in the following.

THEOREM 3.2. *Given $\alpha \in (\frac{2}{3}, 1)$, the closed-loop system* (2.15), (2.16) *is globally equiuniformly finite time stable, regardless of whichever disturbance $\omega(x, t)$, satisfying condition* (2.18) *with $0 < M < \mu_1 < \mu_2 - M$, affects the system.*

Proof. The sketch of the proof is given briefly.

Step 1: *Global asymptotic stability.* The same Lyapunov function V considered in Theorem 3.1 is a valid candidate. Under the conditions of the theorem, the time derivative of the function $V(x_1, x_2)$, computed along the trajectories of (2.15), (2.16), is estimated as follows [27, Th. 1]:

$$\dot{V} \le -(\mu_1 - M)|x_2|^{\alpha+1}. \tag{3.46}$$

Noting that $M < \mu_1$ by a condition of the theorem and that the equilibrium point $x_1 = x_2 = 0$ is the only trajectory of (2.15), (2.16) on the invariance manifold $x_2 = 0$ where $\dot{V}(x_1, x_2) = 0$, the global asymptotic stability of (2.15), (2.16) is then established by applying the invariance principle [2, 33].

Step 2: *Semiglobal strong Lyapunov functions.* The indefinite functions similar to Theorem 3.1 are identified as follows:

$$\begin{aligned} U(x_1, x_2) &= U_1(x_1, x_2) + U_2(x_1, x_2) + U_3(x_1, x_2), \\ U_1(x_1, x_2) &= \kappa_1 |x_1|^{\frac{2\alpha}{2-\alpha}} \operatorname{sign}(x_1)\ |x_2|^{2\alpha}\ x_2, \\ U_2(x_1, x_2) &= \kappa_1\ \kappa_2\ x_1^3\ x_2\ |x_2|^{\alpha}, \quad U_3(x_1, x_2) = \kappa_1\ \kappa_2\ \kappa_3\ x_1^5\ x_2. \end{aligned} \tag{3.47}$$

This produces a class of semiglobal Lyapunov functions as follows:

$$V_{\tilde{R}}(x_1, x_2) = V(x_1, x_2) + \sum_{i=1}^{3} U_i(x_1, x_2), \tag{3.48}$$

where the positive weight scalars $\kappa_i, i = 1, 2, 3$, are chosen small enough so that

$$\begin{aligned} &\kappa_2 < \frac{(1+2\alpha)\mu_2}{(1+\alpha)(\mu_1+M)\ \rho^{3(1-\alpha)}}, \quad \kappa_3 < \frac{(1+\alpha)\mu_2}{(\mu_1+M)\ \rho^{\frac{4-3\alpha}{2}}}, \\ &\kappa_1 < \min\left\{ \frac{\mu_1 - M}{K_1}, \quad \frac{\mu_2(2-\alpha)}{\kappa_2\rho^{5-3\alpha}\left(1+\kappa_3\rho^{2(2-\alpha)}\right)}, \quad \frac{1}{\rho^{2\alpha}+\left(2\tilde{R}\right)^{2\alpha-1}+\kappa_2\left(\left(2\tilde{R}\right)^{\alpha}+\kappa_3\right)} \right\}, \end{aligned} \tag{3.49}$$

$$\begin{aligned} K_1 = {} & \frac{2\alpha}{2-\alpha}\rho^{\frac{3\alpha-2}{2}}\left(2\tilde{R}\right)^{\frac{1+\alpha}{2}} + (\mu_1+M)(1+2\alpha)\rho^{\alpha}\left(2\tilde{R}\right)^{\frac{2\alpha-1}{2}} \\ & + 3\kappa_2\rho^{2-\alpha}\left(2\tilde{R}\right)^{\frac{1}{2}} + 5\kappa_2\kappa_3\rho^{2(2-\alpha)}\left(2\tilde{R}\right)^{\frac{1-\alpha}{2}}, \end{aligned} \tag{3.50}$$

and ρ is defied in (3.10). Following a similar tuning procedure to that outlined in Theorem 3.1, scalars κ_2, κ_3, K_1, and κ_1 can be fixed unambiguously, in that order, for given values of μ_1, μ_2, $\tilde{R}$, and M. A similar semiglobal analysis to that of Theorem 3.1 leads to the temporal derivative

$$\dot{V}_{\tilde{R}} \leq -\left(\mu_1 - M - \kappa_1 K_1\right)|x_2|^{\alpha+1} - \kappa_1\ \kappa_2\ \kappa_3\ \mu_2\ x_1^4\ |x_1|^{\frac{2}{2-\alpha}}, \tag{3.51}$$

which includes exactly the same powers of the terms $|x_1|$ and $|x_2|$ as there are in (3.23) with a slight difference in the multipliers of these terms. Equiuniform asymptotic stability then follows as the temporal derivative (3.51) is negative definite due to (3.49).

Step 3: *Global equiuniform finite time stability.* Coupling the homogeneity of the perturbed system (2.15), (2.16) (see Lemma 2.8) within the arbitrarily large compact set (3.3) with the global equiuniform asymptotic stability of the system (2.15), (2.16), it is obtained that the closed-loop system (2.15), (2.16) is globally equiuniformly finite time stable according to [26, Theorem 3.1]. ∎

4. Settling time estimate. A finite upper bound on the settling time of the closed-loop system (2.15), (2.16) is computed in this section, which presents the second main result of the paper. Since Theorems 3.1 and 3.2 arrive at similar expressions while proving equiuniform asymptotic stability (see (3.23) and (3.51)), the method of deriving the settling time presented in this section applies to both the closed-loop systems (2.15), (2.17) and (2.15), (2.16). The identification of the ellipsoids $E_\delta, E_{\frac{1}{2}\delta}$ (see [26, Theorem 3.1]) can lead to an explicit formula for the finite settling time. A method similar to that developed for a discontinuous controller [28] is employed to identify the required parameters for the computation of the settling time.

4.1. Parameters for settling time computation. The process of identifying the parameters for the computation of the settling time can be listed as follows:

1. Identify the radius $\bar{r}$ of the ball

$$\mathcal{B}_{\bar{r}} = \left\{(x_1, x_2) : \frac{x_1^2}{\bar{r}^2} + \frac{x_2^2}{\bar{r}^2} \leq 1\right\}. \tag{4.1}$$

2. Identify the scalar $\delta > 0$ such that the following definition of the ellipsoid E_δ holds true [26]:

$$E_\delta = \left\{ (x_1, x_2) : \sqrt{\left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2} \le 1 \right\} \subseteq \mathcal{B}_{\bar{r}}, \tag{4.2}$$

where r_1, r_2 are dilation weights.

3. Identify the scalars $R' > 0, \bar{R} > 0$ such that the following expressions of the level sets of the Lyapunov function $V_{\tilde{R}}$ hold true in addition to (4.2):

$$\begin{aligned} &\Omega_2 = \left\{(x_1, x_2) : V_{\tilde{R}}(x_1, x_2) \le \bar{R}\right\} \subseteq E_\delta, \\ &E_\delta \subseteq \Omega_1 = \left\{(x_1, x_2) : V_{\tilde{R}}(x_1, x_2) \le R'\right\}. \end{aligned} \tag{4.3}$$

4. Identify the scalar $\hat{R} > 0$ of the level set Ω_3 corresponding to the ellipsoid $E_{\frac{1}{2}\delta}$ [26] in a similar way such that the following expressions are satisfied:

$$\begin{aligned} E_{\frac{1}{2}\delta} \;&=\; \left\{ (x_1, x_2) : \sqrt{\left(\frac{x_1}{\left(\frac{1}{2}\delta\right)^{r_1}}\right)^2 + \left(\frac{x_2}{\left(\frac{1}{2}\delta\right)^{r_2}}\right)^2} \le 1 \right\}, \\ \Omega_3 = \;&\left\{(x_1, x_2) : V_{\tilde{R}}(x_1, x_2) \le \hat{R}\right\} \subseteq E_{\frac{1}{2}\delta}. \end{aligned} \tag{4.4}$$

The motivation to achieve the above results is the fact that the estimate of the finite settling time can be obtained by utilizing the exponential decay (3.43) once the definitions of the parameters $\bar{r}, \delta, \bar{R}, R', \hat{R}$ are obtained (recall that the finite time stability results [26, Theorem 3.2] and Theorem 3.2 apply in the vicinity of the origin defined by the ball $\mathcal{B}_r$ and ellipsoids E_δ, $E_{\frac{1}{2}\delta}$). The stated steps can be established as follows.

Step 1: *Definition of radius* $\bar{r}$ *of the ball* $\mathcal{B}_{\bar{r}}$.

Lemma 4.1. *Given a positive scalar* $M_0 \in (M(\sqrt{2\tilde{R}})^{\frac{\alpha}{2}}, 1 + M(\sqrt{2\tilde{R}})^{\frac{\alpha}{2}})$ *and conditions* $\mu_1 > \max\{1, M\}, \mu_2 > \mu_1$, $\mu_2 > \mu_3$, *the following upper bound on* 1*-norm* $\|x\|_1 = |x_1| + |x_2|$ *holds true in finite time:*

$$|x_1| + |x_2| \le \frac{M_0 - M\left(2\tilde{R}\right)^{\frac{\alpha}{2}}}{\mu_2} \tag{4.5}$$

for some arbitrary scalar $\alpha \in (\frac{2}{3}, 1)$.

Proof. It should be noted that this lemma was not needed in the proof of Theorem 3.1. Theorem 3.1 asserts the fact that the trajectories of the closed-loop system (2.15), (2.17) decay exponentially within the vicinity $D_{\tilde{R}}$ of the origin if the condition $q+r_2-\alpha r_2 \le 0$ is met. Hence, in finite time, the system trajectories enter a region close to origin where the homogeneous part $-\mu_1|x_2|^\alpha \text{sign}(x_2) - \mu_2|x_1|^{\frac{\alpha}{2-\alpha}}\text{sign}(x_1) - \mu_3|x_2|^{\frac{\alpha}{2}}|x_1|^{\frac{\alpha}{2(2-\alpha)}}\text{sign}(x_2)$ of the second differential equation of the closed-loop system (2.15), (2.17) dominates the nonhomogeneous part $\omega(x_1, x_2, t)$. Due to equiuniform asymptotic stability of the closed-loop system (2.15), (2.17), this region can be chosen such that the expression $|x_1|^{\frac{\alpha}{2(2-\alpha)}} + |x_2|^{\frac{\alpha}{2}} < 1$ holds true. This means $M|x_2|^{\frac{\alpha}{2}}(|x_1|^{\frac{\alpha}{2(2-\alpha)}} + |x_2|^{\frac{\alpha}{2}}) \le M|x_2|^{\frac{\alpha}{2}} \le M(2\tilde{R})^{\frac{\alpha}{2}}$. Hence, a conservative region can be chosen such that the following holds true:

$$\mu_2(|x_1|^{\frac{\alpha}{2-\alpha}} + |x_2|^\alpha + |x_2|^{\frac{\alpha}{2}}|x_1|^{\frac{\alpha}{2(2-\alpha)}}) + M(2\tilde{R})^{\frac{\alpha}{2}} < M_0 \tag{4.6}$$

with the positive scalar M_0. Since the Lyapunov function $V_{\tilde{R}}$ decays exponentially, the above constant M_0 can be arbitrarily chosen and the trajectories are guaranteed to enter the region (4.6) in finite time. Let the choice be $M_0 \in (M(2\tilde{R})^{\frac{\alpha}{2}}, 1 + M(2\tilde{R})^{\frac{\alpha}{2}})$. Noting that $|\omega(x_1, x_2, t)| \leq M|x_2|^{\frac{\alpha}{2}}(|x_1|^{\frac{\alpha}{2(2-\alpha)}} + |x_2|^{\frac{\alpha}{2}}) \leq M|x_2|^{\frac{\alpha}{2}} \leq M(2\tilde{R})^{\frac{\alpha}{2}}$ for the case when $|x_1|^{\frac{\alpha}{2(2-\alpha)}} + |x_2|^{\frac{\alpha}{2}} < 1$ holds true and noting that $|x_1|^{\frac{\alpha}{2-\alpha}} + |x_2|^{\alpha} \leq |x_1|^{\frac{\alpha}{2-\alpha}} + |x_2|^{\alpha} + |x_2|^{\frac{\alpha}{2}}|x_1|^{\frac{\alpha}{2(2-\alpha)}}$ for all x_1, x_2, the following is obtained from (4.6):

$$\mu_2|x_1|^{\frac{\alpha}{2-\alpha}} + \mu_1|x_2|^{\alpha} + |\omega(x_1, x_2, t)| \leq \mu_2(|x_1|^{\frac{\alpha}{2-\alpha}} + |x_2|^{\alpha}) + M\left(2\tilde{R}\right)^{\frac{\alpha}{2}} \leq M_0. \tag{4.7}$$

Hence, (4.6) is a conservatively large upper bound for the chosen scalar M_0 on the nonhomogeneous right-hand side. The following is obtained from (4.7):

$$\mu_2(|x_1|^{\frac{\alpha}{2-\alpha}} + |x_2|^{\alpha}) + M(2\tilde{R})^{\frac{\alpha}{2}} < M_0 \Rightarrow |x_1|^{\frac{\alpha}{2-\alpha}} + |x_2|^{\alpha} \leq \frac{M_0 - M(2\tilde{R})^{\frac{\alpha}{2}}}{\mu_2}. \tag{4.8}$$

Noting that the bound appearing in the right-hand side of (4.8) is always less than unity due to the conditions $\mu_2 > 1, 0 < M_0 - M(2\tilde{R})^{\frac{\alpha}{2}} < 1$ of Lemma 4.1, the inequality $|x_1| + |x_2| \leq |x_1|^{\frac{\alpha}{2-\alpha}} + |x_2|^{\alpha}$ also holds true. Hence a conservative estimate of the region within the compact set (3.3) in terms of the 1-norm can be obtained from (4.8) as follows:

$$|x_1| + |x_2| \leq \frac{M_0 - M\left(2\tilde{R}\right)^{\frac{\alpha}{2}}}{\mu_2}, \tag{4.9}$$

where $\tilde{R} = V(x_1(t_0), x_2(t_0))$. $\square$

It is recalled here that the uncertainty $\omega(x_1, x_2, t)$ is treated as a nonhomogeneous perturbation. The finite time stability of discontinuous homogeneous systems in the presence of nonhomogeneous perturbations was established in the previous section (see Theorem 3.1). The following is a well-known relationship between the Euclidean norm $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$ and 1-norm $\|x\|_1 = |x_1| + |x_2|$ of vector $x = (x_1, x_2)^T$ (see [4, Fact 1.12.34]):

$$\|x\|_1 \leq \sqrt{2}\|x\|_2. \tag{4.10}$$

From (4.9) and (4.10), a conservative bound on the radius $\bar{r}$ of the ball $\mathcal{B}_{\bar{r}}$ can be obtained as follows:

$$\bar{r} = \sqrt{x_1^2 + x_2^2} \leq \frac{M_0 - M\left(2\tilde{R}\right)^{\frac{\alpha}{2}}}{\sqrt{2}\mu_2}. \tag{4.11}$$

The inequalities (4.10) and (4.11), when combined, will always ensure that the inequality (4.9) holds true.

Step 2: *Definition of the parameter* δ. The aim is to find $\delta > 0$ such that every point (x_1, x_2) contained within the ellipsoid E_δ is also contained within the ball $\mathcal{B}_{\bar{r}}$. Having computed the radius $\bar{r}$ in Step 1, if $\delta > 0$ is chosen such that the equalities

$$\min\left\{\frac{1}{\delta^{2r_1}}, \frac{1}{\delta^{2r_2}}\right\} = \frac{1}{\bar{r}^2} \Rightarrow \max\left\{\delta^{2r_1}, \delta^{2r_2}\right\} = \bar{r}^2 \Rightarrow \max\left\{\delta^{r_1}, \delta^{r_2}\right\} = \bar{r} \tag{4.12}$$

are satisfied, then, due to the fact that the equality

$$\min\left\{\frac{1}{\delta^{2r_1}}, \frac{1}{\delta^{2r_2}}\right\}\left(x_1^2 + x_2^2\right) = \frac{1}{\bar{r}^2}\left(x_1^2 + x_2^2\right) \tag{4.13}$$

always holds true, the inequality

$$\min\left\{\frac{1}{\delta^{2r_1}}, \frac{1}{\delta^{2r_2}}\right\}\left(x_1^2+x_2^2\right) = \frac{1}{\bar{r}^2}\left(x_1^2+x_2^2\right) \le \left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2 \tag{4.14}$$

also holds true. If the given point $(x_1, x_2) \in E_\delta$, then the inequality

$$\sqrt{\left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2} \le 1 \tag{4.15}$$

holds true, which, using (4.14), leads to the inequality

$$\frac{1}{\bar{r}^2}\left(x_1^2+x_2^2\right) \le 1. \tag{4.16}$$

Hence $(x_1, x_2) \in \mathcal{B}_{\bar{r}}$, and the choice (4.12) of δ is indeed valid, which, upon further simplification, satisfies

$$\delta = \min\left\{\bar{r}^{\frac{1}{r_1}}, \bar{r}^{\frac{1}{r_2}}\right\}. \tag{4.17}$$

The aim of computing $\delta > 0$ such that $E_\delta \subseteq \mathcal{B}_{\bar{r}}$ is thus achieved.

Step 3: *Definition of scalars* $\bar{R}, R'$ *of the level sets* Ω_1, Ω_2. The first aim is to compute $\bar{R} > 0$ such that the level set Ω_2 satisfies $\Omega_2 \subseteq E_\delta$. Combining the definition of the level set Ω_2 with the inequality (3.14), it suffices that the inequality $V \le \frac{\bar{R}}{M_{\tilde{R}}}$ holds true in order that $\Omega_2 \subseteq E_\delta$ is satisfied for any given (x_1, x_2) in a small vicinity of the origin. Hence the following must be satisfied:

$$\frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2\bar{R}}\,|x_1|^{\frac{2}{2-\alpha}} + \frac{M_{\tilde{R}}}{2\bar{R}}x_2^2 \le 1 \Rightarrow \left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2 \le 1. \tag{4.18}$$

Having computed the ellipsoid parameter δ in Step 2, if $\bar{R} > 0$ is chosen such that the inequalities

$$\left(\frac{x_1}{\delta^{r_1}}\right)^2 \le \frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2\bar{R}}\,|x_1|^{\frac{2}{2-\alpha}}, \quad \left(\frac{x_2}{\delta^{r_2}}\right)^2 \le \frac{M_{\tilde{R}}}{2\bar{R}}x_2^2 \tag{4.19}$$

are satisfied, then the inequality

$$\left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2 \le \frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2\bar{R}}\,|x_1|^{\frac{2}{2-\alpha}} + \frac{M_{\tilde{R}}}{2\bar{R}}x_2^2 \tag{4.20}$$

always holds true. For a given point $(x_1, x_2) \in \Omega_2$, the inequality

$$\frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2\bar{R}}\,|x_1|^{\frac{2}{2-\alpha}} + \frac{M_{\tilde{R}}}{2\bar{R}}x_2^2 \le 1 \tag{4.21}$$

holds true, which, using (4.20), leads to the inequality

$$\left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2 \le 1. \tag{4.22}$$

Hence $(x_1, x_2) \in E_\delta$, and the choice (4.19) of $\bar{R}$ is indeed valid. Noting from (3.10) that $x_1^2 < |x_1|^{\frac{2}{2-\alpha}}\rho^{1-\alpha}$, requirement (4.19) can be reformulated as follows:

$$\begin{aligned}&\left(\frac{x_1}{\delta^{r_1}}\right)^2 \le \frac{|x_1|^{\frac{2}{2-\alpha}}}{\delta^{2r_1}}\left(\frac{2\tilde{R}}{(2-\alpha)\mu_2}\right)^{1-\alpha} \le \frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2\bar{R}}\,|x_1|^{\frac{2}{2-\alpha}},\\ &\left(\frac{x_2}{\delta^{r_2}}\right)^2 \le \frac{M_{\tilde{R}}}{2\bar{R}}x_2^2.\end{aligned} \tag{4.23}$$

The above inequalities (4.23) result in the following definition of $\bar{R}$:

$$\bar{R} = \frac{M_{\tilde{R}}}{2} \min \left\{ \delta^{2r_1} \mu_2^{2-\alpha} \frac{(2-\alpha)^{2-\alpha}}{\left(2\tilde{R}\right)^{1-\alpha}}, \quad \delta^{2r_2} \right\}. \tag{4.24}$$

The second aim is to compute $R' > 0$ such that the the expression $E_\delta \subseteq \Omega_1$ is satisfied. Combining the definition of the level set Ω_1 with the inequality (3.14), it suffices that the inequality $V \leq \frac{R'}{M_{\tilde{R}}}$ holds true in order that $E_\delta \subseteq \Omega_1$ is satisfied for any given (x_1, x_2) in a small vicinity of the origin. Hence the following must be satisfied:

$$\left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2 \leq 1 \Rightarrow \frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2R'}|x_1|^{\frac{2}{2-\alpha}} + \frac{M_{\tilde{R}}}{2R'}x_2^2 \leq 1. \tag{4.25}$$

If the inequality

$$\frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2R'}|x_1|^{\frac{2}{2-\alpha}} + \frac{M_{\tilde{R}}}{2R'}x_2^2 \leq \left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2 \tag{4.26}$$

holds true, then (4.25) always holds true for all $(x_1, x_2) \in E_\delta$. Inequality (4.26) always holds true if the following is ensured:

$$\frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2R'}|x_1|^{\frac{2}{2-\alpha}} \leq (1-\epsilon_1), \quad \frac{M_{\tilde{R}}}{2R'}x_2^2 \leq \epsilon_1 \left(\frac{x_2}{\delta^{r_2}}\right)^2, \tag{4.27}$$

where $0 < \epsilon_1 < 1$ is an arbitrary constant. The fact that $(x_1, x_2) \in E_\delta$ leads to $|x_1| \leq \delta^{r_1}$ by definition. Hence (4.27) can be further simplified to derive a formula for R' by enforcing the following subconditions:

$$\begin{gathered} \frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2R'}|x_1|^{\frac{2}{2-\alpha}} \leq \frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2R'}\delta^{\frac{2r_1}{2-\alpha}} \leq (1-\epsilon_1), \\ \frac{M_{\tilde{R}}}{2R'}x_2^2 \leq \epsilon_1 \left(\frac{x_2}{\delta^{r_2}}\right)^2. \end{gathered} \tag{4.28}$$

Hence the formula

$$R' = \frac{M_{\tilde{R}}}{2} \max \left\{ \delta^{\frac{2r_1}{2-\alpha}} \frac{\mu_2(2-\alpha)}{1-\epsilon_1}, \quad \frac{\delta^{2r_2}}{\epsilon_1} \right\} \tag{4.29}$$

can be deduced from (4.28). The aims of computing $R' > 0, \bar{R} > 0$ such that $\Omega_2 \subseteq E_\delta \subseteq \Omega_1$ are thus achieved.

Step 4: *Definition of the parameter* $\hat{R}$ *of the level set* Ω_3. Arguments similar to those outlined in Step 3 produce the following formula:[1]

$$\hat{R} = \frac{M_{\tilde{R}}}{8} \min \left\{ \frac{\delta^{2r_1}\left(\mu_2(2-\alpha)\right)^{2-\alpha}}{\left(2\tilde{R}\right)^{1-\alpha}}, \delta^{2r_2} \right\}. \tag{4.30}$$

[1] All constants $M_{\tilde{R}}, M_{\bar{R}}, M_{\hat{R}}, M_{\acute{R}}$ corresponding to the semiglobal regions $D_{\tilde{R}}, \Omega_i, i = 1, 2, 3$, can be chosen as $M_{\tilde{R}} = M_{\bar{R}} = M_{\hat{R}} = M_{R'}$ since only the lower bound (3.14) is to be satisfied.

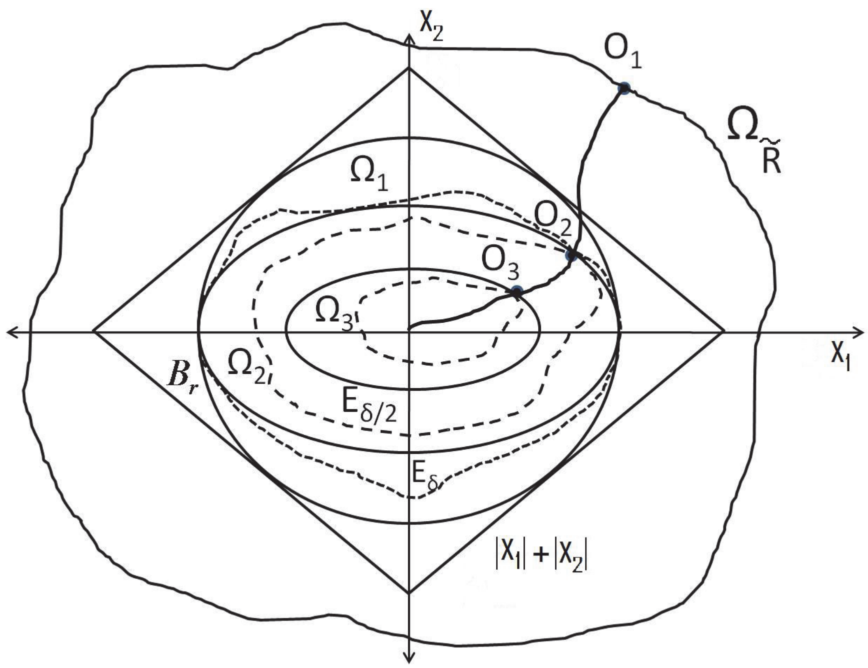

FIG. 3. *Finite time behavior: Regions* $\|(x_1, x_2)\|_1$*, ball* $\mathcal{B}_{\bar{r}}$*, ellipsoids* (E_δ*,* $E_{\frac{1}{2}\delta}$)*, and level sets* $\Omega_R, \Omega_1, \Omega_2, \Omega_3$*.*

4.2. Computation of settling time. The finite time behavior[2] is geometrically depicted in Figure 3. Trajectories of the system (2.15), (2.17) in the phase plane (x_1, x_2) are also schematically shown. The existence of a uniformly decaying global Lyapunov function $V_{\tilde{R}}$ is utilized (see (3.43)). The point O_1 is the system initial condition which corresponds to the boundary of the level set $\Omega_{\tilde{R}} = \{(x_1, x_2) : V_{\tilde{R}}(x_1, x_2) \leq M_{\tilde{R}}\tilde{R}\}$, where $\tilde{R} = V(x_1(t_0), x_2(t_0))$. Then, due to the fact that the system decays exponentially toward the origin, it can be deduced that the trajectory enters the ball $\mathcal{B}_{\bar{r}}$ in finite time, where $\bar{r}$ is defined by (4.11), and subsequently enters the ellipsoid E_δ. This in turn causes the trajectories of the closed-loop system to satisfy the definition of the level set $\Omega_2 = \left\{(x_1, x_2) : V_{\tilde{R}}(x_1, x_2) \leq \bar{R}\right\} \subseteq E_\delta$ of the Lyapunov function $V_{\tilde{R}}$ in finite time. This corresponds to the point O_2. Finally, finite time stability follows from the homogeneity principle once the system trajectories are inside the ellipsoid E_δ (see Theorem 3.1 and [26, Th. 3.1]). As a consequence, the settling time of the system is the summation of the following:

$$\mathcal{T} = \mathcal{T}_{O_1 - O_2} + \mathcal{T}_h, \tag{4.31}$$

where $\mathcal{T}_{O_1 - O_2}$ is the time taken by the state trajectories of the closed-loop system to attain the level set Ω_2 (point O_2) from the initial condition level set $\Omega_{\tilde{R}}$ (point O_1) and $\mathcal{T}_h$ is finite settling time of the system to attain equilibrium point $(0, 0)$ from the boundary of $E_\delta \subseteq \Omega_1$, which can be readily computed using the expression (3.12) of

[2] This figure is inspired from [28], where a discontinuous counterpart $\alpha = 0$ of the control law (2.16) was studied.

[26] as follows:

$$\mathcal{T}_h = \frac{{c_0}^q}{1-2^q}\left\{\sup_{(x_1,x_2)\in E_\delta} \mathcal{T}_{O_2-O_3}\right\}, \tag{4.32}$$

where q is the homogeneity degree, c_0 is a lower estimate of the homogeneity parameter, and $\mathcal{T}_{O_2-O_3}$ is the time taken by the state trajectories of the closed-loop system to travel from the boundary $E_\delta \subseteq \Omega_1$ to the boundary $\Omega_3 \subseteq E_{\frac{1}{2}\delta}$ (point O_3). While computing $\mathcal{T}_{O_2-O_3}$, the necessity to use the boundary of the level set Ω_1 in place of Ω_2 stems from the fact that the supremum of $\mathcal{T}_{O_2-O_3}$ has to be taken into consideration while computing the worst possible decay of the Lyapunov function. Hence, the boundary given by Ω_2 has to be utilized to compute $\mathcal{T}_{O_1-O_2}$ and that given by Ω_1 to compute $\mathcal{T}_{O_2-O_3}$ in order to encompass the worst case scenario. Although an overlap of time contributions may occur in the summation (4.31) leading to a conservative result since $\Omega_2 \subseteq \Omega_1$ holds true, the estimate of the settling time thus obtained is a true upper bound nevertheless. The terms $\mathcal{T}_{O_1-O_2}$ and $\mathcal{T}_{O_2-O_3}$ can be estimated for $\gamma = 4$ from the decay (3.43) as follows:

$$\begin{aligned}
&\mathcal{T}_{O_1-O_2}(\mu_1,\mu_2,M,\tilde{R},\bar{R}) = \begin{cases} t_{O_1} + \bar{t} & \text{if } M_{\tilde{R}}\tilde{R} \geq 1, \bar{R} > 1,\\ t_{O_1} + t_1 + \frac{1-\bar{R}^3}{3\bar{\kappa}\bar{R}^3} & \text{if } M_{\tilde{R}}\tilde{R} \geq 1, \bar{R} < 1,\\ t_{O_1} + \frac{((\frac{M_{\tilde{R}}\tilde{R}}{\bar{R}})^3-1)}{3\bar{\kappa}(\tilde{R}M_{\tilde{R}})^3} & \text{if } M_{\tilde{R}}\tilde{R} < 1,\end{cases}\\
&\mathcal{T}_{O_2-O_3}(\mu_1,\mu_2,M,\tilde{R},\hat{R}) = \begin{cases} \frac{\ln(\frac{R'}{\hat{R}})}{\bar{\kappa}} & \text{if } R' > 1, \hat{R} > 1,\\ \frac{\ln R'}{\bar{\kappa}} + \frac{1-\hat{R}^3}{3\bar{\kappa}\hat{R}^3} & \text{if } R' > 1, \hat{R} < 1,\\ \frac{((\frac{R'}{\hat{R}})^3-1)}{3\bar{\kappa}(\hat{R})^3} & \text{if } R' < 1,\end{cases}
\end{aligned} \tag{4.33}$$

where $\bar{t}$ and t_1 can be obtained from the first equality of the exponential decay (3.43) as

$$\bar{t} = \frac{\ln\frac{M_{\tilde{R}}\tilde{R}}{\bar{R}}}{\bar{\kappa}}, \quad t_1 = \frac{\ln\left(M_{\tilde{R}}\tilde{R}\right)}{\bar{\kappa}}, \tag{4.34}$$

and the substitutions $V_{\tilde{R}}(t_{O_1}) = M_{\tilde{R}}\tilde{R}, V_{\tilde{R}}(t_{O_2}) = \bar{R}$ have been utilized corresponding to the level sets $\Omega_{\tilde{R}}$ and Ω_2 at time instants t_{O_1} and t_{O_2}, respectively, in the first equality and substitutions $V_{\tilde{R}}(t_{O_2}) = R', V_{\tilde{R}}(t_{O_3}) = \hat{R}$ have been utilized corresponding to the level sets Ω_1 and Ω_3 at time instants t_{O_2} and t_{O_3}, respectively, in the second equality while utilizing (3.43).

Under the stated assumptions, the parameters $\bar{r}, \delta, \bar{R}, \hat{R}, R'$ outlined in section 4.1 and in turn the settling time estimate (4.31) can be computed a priori for a given $\tilde{R}$. It remains to give conditions under which the estimates $\mathcal{T}_{O_1-O_2}, \mathcal{T}_h$ are guaranteed to be positive or, in other words, expressions $M_{\tilde{R}}\tilde{R} > \bar{R}$ and $R' > \hat{R}$ always hold true.

LEMMA 4.2. *Given a positive scalar* $M_0 \in (M(\sqrt{2\tilde{R}})^{\frac{\alpha}{2}}, 1 + M(\sqrt{2\tilde{R}})^{\frac{\alpha}{2}})$, *conditions* $\mu_1 > \max\{1, M\}, \mu_2 > \max\{\mu_1, \frac{1-\epsilon_1}{(2-\alpha)\epsilon_1}, \frac{2-\alpha}{4}\}$, $\tilde{R} > 1$, *and the condition* $1 > \epsilon_1 > 1 - \bar{\epsilon}^2 > 0$, *with some positive scalar* $\bar{\epsilon} \in (0,1)$, *the expressions* $M_{\tilde{R}}\tilde{R} > \bar{R}$ *and* $R' > \hat{R}$ *always hold true.*

Proof. Due to the choice $M_0 \in (M(\sqrt{2\tilde{R}})^{\frac{\alpha}{2}}, 1 + M(\sqrt{2\tilde{R}})^{\frac{\alpha}{2}})$, the inequality

$$(M_0 - M(\sqrt{2\tilde{R}})^{\frac{\alpha}{2}})^2 < 1 \tag{4.35}$$

holds true. Using the condition $\mu_2 > \frac{2-\alpha}{4}$, (4.35) can be modified as follows:

$$\frac{(M_0 - M(\sqrt{2\tilde{R}})^{\frac{\alpha}{2}})^2}{4\mu_2^2}\mu_2(2-\alpha) < 1. \tag{4.36}$$

The inequality $\tilde{R} > 1$ holds true by assumption, and hence (4.36) can be rewritten as

$$\frac{(M_0 - M(\sqrt{2\tilde{R}})^{\frac{\alpha}{2}})^2}{4\mu_2^2}\mu_2(2-\alpha) < \tilde{R}. \tag{4.37}$$

Rearranging (4.37) and raising the power by $2-\alpha$ results in

$$\left(\frac{M_0 - M(\sqrt{2\tilde{R}})^{\frac{\alpha}{2}}}{\sqrt{2}\mu_2}\right)^{2(2-\alpha)} (\mu_2(2-\alpha))^{2-\alpha} < 2\tilde{R}^{2-\alpha}. \tag{4.38}$$

Recalling the definition of δ from (4.17), multiplying by $M_{\tilde{R}}$ on both sides, and noting that $\frac{2r_1}{r_2} = 2(2-\alpha)$ produces

$$M_{\tilde{R}}\delta^{2r_1}\frac{(\mu_2(2-\alpha))^{2-\alpha}}{2\tilde{R}^{1-\alpha}} < 2M_{\tilde{R}}\tilde{R}, \tag{4.39}$$

which, in turn, recalling the definition of $\bar{R}$ from (4.24), gives

$$M_{\tilde{R}}\delta^{2r_1}\frac{(\mu_2(2-\alpha))^{2-\alpha}}{2\tilde{R}^{1-\alpha}} < M_{\tilde{R}}\tilde{R}. \tag{4.40}$$

Using a similar analysis, (4.35) produces

$$\frac{M_{\tilde{R}}}{2}\delta^{2r_2} < M_{\tilde{R}}\tilde{R}. \tag{4.41}$$

Hence, $M_{\tilde{R}}\tilde{R} > \bar{R}$ follows from (4.40), (4.41). The second claim to be proved is $R' > \hat{R}$. First, the following, which stems from the condition $\mu_2 > \frac{(1-\epsilon_1)}{\epsilon_1(2-\alpha)}$, is in order to simplify (4.29):

$$\begin{aligned} \mu_2 > \frac{(1-\epsilon_1)}{\epsilon_1(2-\alpha)} &\Rightarrow \frac{\mu_2(2-\alpha)}{1-\epsilon_1} > \frac{1}{\epsilon_1} \\ &\Rightarrow \delta^{\frac{2r_1}{2-\alpha}-2r_2}\frac{\mu_2(2-\alpha)}{1-\epsilon_1} > \frac{1}{\epsilon_1} \quad \text{because } \frac{2r_1}{2-\alpha} - 2r_2 = 0. \\ &\Rightarrow \delta^{\frac{2r_1}{2-\alpha}}\frac{\mu_2(2-\alpha)}{1-\epsilon_1} > \frac{\delta^{2r_2}}{\epsilon_1} \end{aligned} \tag{4.42}$$

Recalling the definition (4.29) of R', (4.42) produces

$$R' = \frac{M_{\tilde{R}}}{2}\delta^{\frac{2r_1}{2-\alpha}}\frac{\mu_2(2-\alpha)}{1-\epsilon_1}. \tag{4.43}$$

It can be seen from the definition of $\hat{R}$ in (4.30) that the terms inside the $\min\{\cdot\}$ function are less than unity since $\delta < 1$ due to the condition $M_0 \in (M(\sqrt{2\tilde{R}})^{\frac{\alpha}{2}}, 1 + M(\sqrt{2\tilde{R}})^{\frac{\alpha}{2}})$. Hence, in order to prove $R' > \hat{R}$, it suffices to prove that the right-hand side of (4.43) is greater than $\frac{M_{\tilde{R}}}{8}$. Let the scalar $\bar{\epsilon} \in (0,1)$ be selected small enough such that

$$1 > \frac{(M_0 - M(2\tilde{R})^{\frac{\alpha}{2}})}{\sqrt{2}} > \bar{\epsilon}\mu_2 > 0, \quad \epsilon_1 > 1 - \bar{\epsilon}^2 \Leftrightarrow \frac{\bar{\epsilon}^2}{1-\epsilon_1} > 1 \tag{4.44}$$

holds true. This is always possible for fixed values of μ_2. Then, the following holds from (4.44):

$$\begin{aligned}
\frac{(M_0 - M(2\tilde{R})^{\frac{\alpha}{2}})}{\sqrt{2}} > \bar{\epsilon}\mu_2 &\Rightarrow \frac{(M_0 - M(2\tilde{R})^{\frac{\alpha}{2}})}{\sqrt{2}\mu_2} > \bar{\epsilon} \\
&\Rightarrow \delta > \bar{\epsilon}^{\frac{1}{r_2}} \quad (\text{since (4.17) and } \bar{r} < 1, r_1 > r_2 \text{ produces } \delta = \bar{r}^{\frac{1}{r_2}}) \\
&\Rightarrow \delta^{\frac{2r_1}{2-\alpha}} > \bar{\epsilon}^{\frac{2r_1}{(2-\alpha)r_2}} = \bar{\epsilon}^2 \\
&\Rightarrow \delta^{\frac{2r_1}{2-\alpha}} \frac{\mu_2(2-\alpha)}{1-\epsilon_1} > \frac{\mu_2(2-\alpha)}{1-\epsilon_1}\bar{\epsilon}^2 > 1 \\
&\qquad \text{since } \mu_2 > 1, 2-\alpha > 1, \tfrac{\bar{\epsilon}^2}{1-\epsilon_1} > 1.
\end{aligned} \tag{4.45}$$

Noting that $\min\{\frac{\delta^{2r_1}(\mu_2(2-\alpha))^{2-\alpha}}{(2\tilde{R})^{1-\alpha}}, \delta^{2r_2}\} < 1$ and in turn $\hat{R} < \frac{M_{\tilde{R}}}{8}$, combining the last inequality of (4.45) with (4.43) produces

$$\begin{aligned}
\frac{M_{\tilde{R}}}{2}\delta^{\frac{2r_1}{2-\alpha}} \frac{\mu_2(2-\alpha)}{1-\epsilon_1} > \frac{M_{\tilde{R}}}{2}\frac{\mu_2(2-\alpha)}{1-\epsilon_1}\bar{\epsilon}^2 > \frac{M_{\tilde{R}}}{8} > \frac{M_{\tilde{R}}}{8}\min\left\{\frac{\delta^{2r_1}(\mu_2(2-\alpha))^{2-\alpha}}{(2\tilde{R})^{1-\alpha}}, \delta^{2r_2}\right\} \\
\Rightarrow R' > \hat{R}.
\end{aligned} \tag{4.46}$$

Thus, the estimate (4.31) proves to be a positive real constant. ∎

Remark 5. The estimate of the homogeneity parameter c should satisfy $c \geq c_0$ for the chosen c_0, where c_0 is the lower estimate of the homogeneity parameter. It can be seen from the above development that the closed-loop system is homogeneous inside the ellipsoid E_δ. The identity $\delta\, R_0^{-1} = c$ then leads to $c = 1$ because $R_0 = \delta$ is chosen to facilitate the application of (4.32), where the scalar $R_0 > 0$ represents the largest ellipsoid E_{R_0} (see (3.12) of [26] for more details). Hence $c_0 = 1$ is a valid choice.

5. Conclusions. Uniform asymptotic stability of planar controllable systems is established for two classes of continuous homogeneous controllers by identifying corresponding $\mathcal{C}^1$ smooth, strong Lyapunov functions. The homogeneity principle of discontinuous systems is extended to the case of continuous systems with uniformly decaying, but piecewise continuous, nonhomogeneous disturbances to establish uniform finite time stability of planar controllable systems. In turn, it gives finite time stability results superior to the existing results which only cover either homogeneous or continuous disturbances. Interestingly, the Lyapunov function V given in (3.1) is homogeneous in the sense of the definition given in the statement of [31, Lemma 2] since for all $c = \max\{1, c_0\}, c_0 > 0$, and $r_1 = \frac{2-\alpha}{1-\alpha}, r_2 = \frac{1}{2-\alpha}, r_3 = \frac{2}{1-\alpha}$ the equality

$$V(c^{r_1}x_1, c^{r_2}x_2) = c^{r_3}V(x_1, x_2) \tag{5.1}$$

holds true, where $r_3 > r_1, r_3 > r_2$. Thus, the Lyapunov analysis of previous sections combines a $\mathcal{C}^1$ smooth homogeneous Lyapunov function that satisfies (5.1) with indefinite functions to form a semiglobal Lyapunov function $V_{\tilde{R}}$ such that the time derivative $\dot{V}_{\tilde{R}}$ is negative definite along the closed-loop system trajectories. The future scope of this line of research is to investigate if the semiglobal Lyapunov analysis can be successfully extended to the case of dimension n, i.e., $\frac{\mathrm{d}^n}{\mathrm{d}t^n}x(t) = u(x, \dot{x}, \ldots, \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}x,) + \omega(x, \dot{x}, \ldots, \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}x, t)$, in the presence of nonhomogeneous time varying discontinuous disturbances $\omega(x, \dot{x}, \ldots, \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}x, t)$, where the existing results for homogeneous systems with discontinuous right-hand sides [32] may prove to be instrumental.

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