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Research Paper

Delineating an ovoidal egg shape by length and breadth: A novel two-parametric mathematical model

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Mathematical characterisation of an egg shape implies, among other things, the possibility of its description with simple and accurate formulae. Of mathematical models delineating ovoidal egg shape, *Hügelschäffer's model* is deemed standard and theoretically established. However, for its application and in addition to measuring the egg length (L) and its maximum breadth (B), one more parameter (w) is needed, which is the shift of the B-axis from the middle point of egg length. Measurement of w is quite laborious and does not always meet the required accuracy criteria. Previously, we introduced *Narushin's model*, which uses only two parameters, L and B, but it does not exactly outline the egg profile. To grapple with this problem, here we have developed a combination of the two models. The new two-parametric model is based on Hügelschäffer's model, while the parameter w has a sliding character, changing its value at the egg's sharp end as calculated following Narushin's model, to zero at the blunt end. The new model was tested for accurate reproducing the egg profile with an average error of 4.17% when the value of the parameter w changes according to linear dependence. This error was lower than when using three parameters in Hügelschäffer's model. Thus, we propose a new egg shape model, which we named a *modified Hügelschäffer's model with two parameters*. With its use, new formulae were derived for calculating the egg volume and surface area, the empirical validation of which showed an average error of 1.72% and 0.83%, respectively.

1. Introduction

At present, the description of the bird's egg contours using mathematical indices, such as *shape index* (Romanoff & Romanoff, 1949) or *elongation* and *conicality* (Hays et al., 2020), cannot meet the increasing requirements for precise ovoidal egg shape characterization and its utility in engineering, construction, and research related to poultry production and evolution.

As we repeatedly noted (Narushin, Lu, et al., 2020; Narushin, Romanov, et al., 2020; Narushin et al., 2021a, 2021b; Narushin, Romanov, Lu, et al., 2021), it is a clear mathematical description of the geometrical profile of an egg that facilitates

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Nomenclature

a, b	Constant coefficients used to determine the function of changing the value of the parameter w
B	Egg maximum breadth
K_w	Coefficient used to recalculate the value of the parameter w depending on B and L
L	Egg length
n	A rational non-negative number and a function of the shape index (B/L)
S	Egg surface area
V	Egg volume
w	Parameter that corresponds to a distance between two vertical axes, one of which coincides with B and the other one is crossing the egg at the point of $L/2$ (Narushin, Romanov, & Griffin, 2021a)
w_{aln}	Average value of the parameter w when it changes along a linear function (see Eq. (20))
x_B	Value of the egg radius at the point on the horizontal axis corresponding to B
$x_{L/2}$	Value of the egg radius at the point on the horizontal axis corresponding to $L/2$
ε_H	Average error when comparing the Hügelschäffer's (Eq. (1)) model with actual egg profiles
ε_N	Average error when comparing the Narushin's model (Eq. (2)) with actual egg profiles
ε_{HJN}	Average error when comparing the Hügelschäffer's model (Eq. (1)) with actual egg profiles, when w values were calculated according to Eq. (10)
ε_{ln}	Average error when comparing the Hügelschäffer's model (Eq. (1)) with actual egg profiles, if w values were calculated according to Eq. (16)
ε_{el}	Average error when comparing the Hügelschäffer's model (Eq. (1)) with actual egg profiles, if w values were calculated according to Eq. (17)
ε_V	Average error when comparing actual values of the egg volume and those calculated by formula (25)
ε_{Vw}	Average error when comparing actual values of the egg volume and those calculated by formula (21) using measured values of the parameter w
ε_{Vln}	Average error when comparing actual values of the egg volume and those calculated by formula (21) using values of w_{aln}
ε_S	Average error when comparing actual values of the surface area and those calculated by formula (26)
ε_{Sw}	Average error when comparing actual values of the surface area and those calculated by formula (22) using measured values of the parameter w
ε_{Sln}	Average error when comparing actual values of the surface area and those calculated by formula (22) using values w_{aln}

solving the problems of commercial poultry farming. This is primarily due to developing the design of machines and mechanisms used for sorting, reloading and packaging of eggs. In many cases, the shape of the key mechanical elements should match the geometry of an actual egg. At the same time, the egg contour equation is indispensable in research work aimed at studying the effect of egg geometry on egg hatchability, shelf life, strength properties, etc. The need for a mathematical description of egg contours also arises for the whole industry of bio-inspired technologies (Narushin, Romanov, & Griffin, 2022). To talk more specifically, there are *egg-inspired engineering* approaches and applications that use designs based on, or inspired by, the shape of avian eggs and can be relevant to the design of thin-walled vessels and reservoirs (Lazarus et al., 2012; Zhang, Zhu, et al., 2017; Zhang, Wang, et al., 2017; Zhang et al., 2019, 2021; Guo et al., 2020; Narushin, Romanov, & Griffin, 2022), architectural structures (Freiberger, 2007; Petrović et al., 2011), many works of art (Gilbert, 1974; Herz-Fischler, 1990), and even computer games (e.g., Silverman, 2020). Nevertheless, the publication of Ursinus (1944) on applying *Hügelschäffer's model*, which quite properly replicates the egg profile, in the field of aircraft construction can be considered the beginning of such an egg-inspired engineering technologies era (Narushin, Romanov, & Griffin, 2022). Somewhat forgotten, this model was subsequently revived thanks to the publications of Petrović and Obradović (2010), Petrović et al. (2011), and Obradović et al. (2013) in relation to architectural structures. More recently, we successfully adapted it to the profile of bird eggs that have the so-called classic ovoid shape (Narushin, Romanov, et al., 2020; Narushin et al., 2021a,b):

$$y = \pm\frac{B}{2}\sqrt{\frac{L^2 - 4x^2}{L^2 + 8wx + 4w^2}} \tag{1}$$

where B is the egg maximum breadth, L is the egg length, and w is an additional parameter for the distance between two vertical axes, one coinciding with B and the other one with $L/2$ of the egg.

Parameters B and L are the most common egg characteristics, primarily due to the simplicity and accuracy by which they can be measured. This cannot be said about the parameter w; it was impossible to accurately determine w even with the help of 2-D imaging analysis (Narushin, Romanov, et al., 2020). In this respect, we proposed a method for recalculating w by measuring the egg diameter at points corresponding to $L/4$ at the sharp ($y_{L/4}$) and blunt ends ($y_{-L/4}$), respectively, as shown in Fig. 1. Thus, for an accurate construction of the ovoid profile of an actual egg, it is already necessary to perform not three, but four measurements. At the same time, in order to carry out the latter two measurements, it is necessary to obtain a computer image of an egg, and then process it using a basic programme such as MS Office Picture Manager application, or else specialised software programmes. In the event, however, this did not add too much value to the applicability and ease of practical use of the Hügelschäffer's model for creating mathematical analogues of bird eggs.

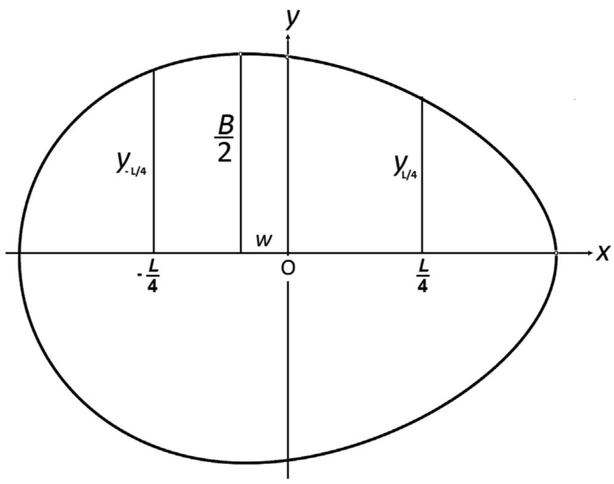

Fig. 1 – Schematic representation of the egg's geometric parameters required for calculating *w*.

In addition to Hügelschäffer's model, there are some other models that, from our point of view, are termini a quo for many subsequent similar studies. In this regard, they deserve more detailed consideration.

A three-parameter chicken egg profile model was also proposed by Carter (1968). In addition to *L* and *B*, the author suggested using the distance from the sharp end of the egg to the axis of maximum breadth. In essence, this is the same as the sum of $L/2 + w$.

Even earlier, Preston (1953) also relied on three parameters to arrive at a “simple” or “classical” ovoid as he characterised the figure. In addition to the classic measurements of *L* and *B*, he introduced a certain constant that changed from egg to egg. He assumed a number of values for this constant depending on the avian species of a particular egg. Perhaps, this method is acceptable for an approximate description of the egg shape but, unfortunately, it does not take into account all possible variants of individual variability, since the data presented were limited to only those egg samples that were involved in his research.

The only two-dimensional so-called “egg” model was proposed by Narushin (1997, 2001), which used only two main measurements, *L* and *B*. However, its egg profile gave some shape error at the sharp end.

Previously, Narushin (1997a) proposed a theoretical derivation of the egg profile using only two main parameters *B* and *L*, which was improved later (Narushin, 2001) to the following model that we will henceforth refer to as *Narushin's model*:

$$y = \pm\sqrt{L^{\frac{2}{n+1}} x^{\frac{2n}{n+1}} - x^2} \tag{2}$$

in which the coefficient *n* was defined as a rational non-negative number and a function of the shape index (*B*/*L*), and can be calculated approximately using the following formula (Narushin, 2001):

$$\frac{B}{L} = 2\sqrt{\frac{n^n}{(n+1)^{n+1}}} \tag{3}$$

This model was extremely convenient for creating engineering structures resembling a bird's egg in shape (e.g., Zhang, Zhu, et al., 2017; Zhang, Wang, et al., 2017; Zhang et al., 2021; Guo et al., 2020). However, judging from the graphic images presented in Narushin (1997a, 2001), this model had errors in comparison with the actual profiles of ovoid eggs.

The present study replaces *w* in Hügelschäffer's formula with a function of *B* and *L*. The choice of function is motivated by further consideration of Narushin's (1997a, 2001) model.

2. Theory

Narushin (2001) derived an equation for *n* (see Eqs. (2) and (3)) based on all possible variations of the *B*/*L* shape index that exist in nature, i.e., from 0.48 to 1.

For our purposes, it will be sufficient to confine ourselves to the *B*/*L* data, which are characteristic only for chicken eggs, since they can serve as model objects for most ovoid profiles. According to Romanoff and Romanoff (1949), *B*/*L* for chicken eggs varies in the range [0.65 … 0.82]. This is also confirmed by numerous data from our previous observations on egg quality parameters (Narushin, 1997a,b, 1998, 2001, 2005; Narushin, Lu, et al., 2020; Narushin, Romanov, et al., 2020).

Based on the *n* data given by Narushin (2001), values of this coefficient in the range [3.05 … 1.70] were used, which satisfied the required *B*/*L* interval (i.e., [0.65 … 0.82]). By substituting the value of *n* from the specified interval [3.05 … 1.70], with a step of 0.005, into formula (3), 270 values were generated and subsequently approximated by the following dependence:

$$n = 1.464\left(\frac{L}{B}\right)^2 - 0.47 \tag{4}$$

with $R^2 = 0.999999$.

Now let us determine what the parameter *w* in Narushin's model (Eq. (2)) is equal to. To do this, we find the difference between the values on the *x*-axis corresponding to (i) the egg maximum breadth (x_B), and (ii) half the egg length ($x_{L/2}$). The x_B value was derived in Narushin (2001) and corresponded to

$$x_B = L\left(\frac{n}{n+1}\right)^{\frac{n+1}{2}} \tag{5}$$

Then,

$$w = L\left(\frac{n}{n+1}\right)^{\frac{n+1}{2}} - \frac{L}{2} = L\left(\left(\frac{n}{n+1}\right)^{\frac{n+1}{2}} - \frac{1}{2}\right) \tag{6}$$

We rewrite (6) in the following form:

$$w = L \bullet K_w \tag{7}$$

where the coefficient K_w is equal to

$$K_w = \left(\frac{n}{n+1}\right)^{\frac{n+1}{2}} - \frac{1}{2} \tag{8}$$

Let us see how *w* will change depending on the *B*/*L* variations recalculated from the values of n using formula (3) (Fig. 2).

Approximation of the obtained dependence resulted in the following calculation formula:

$$K_w = 0.103 + 0.012\frac{B}{L} - 0.115\left(\frac{B}{L}\right)^2 \quad (9)$$

with $R^2 \approx 1$, and finally:

$$w = L\left(0.103 + 0.012\frac{B}{L} - 0.115\left(\frac{B}{L}\right)^2\right) \quad (10)$$

Thus, the parameter w in Hügelschäffer's model will match its counterpart in Narushin's model in cases where the value of w matches Eq. (10) data.

How each of the models will correspond to the profile of actual eggs, and also what variants of respective recalculations of the value of w according to Eq. (10), has become the goal of the experimental studies described below.

3. Materials and methods

For a comparative analysis, geometric profiles of 40 chicken eggs obtained as a result of 2-D image scanning (Narushin, Lu, et al., 2020) were used. The table chicken eggs were supplied by Woodlands Farm, Canterbury and Staveleys Eggs Ltd, Coppull, UK. The imaging procedure is described in detail in Narushin, Lu, et al. (2020). Average weight of the eggs involved in the experiment was 59.2 ± 4.7 g, with $L = 5.6 \pm 0.2$ cm and $B = 4.3 \pm 0.1$ cm. The shape index (B to L ratio) in this egg sample was about 0.767 ± 0.023.

The degree of correspondence of each of the theoretical profiles to actual eggs was assessed with approximating mean percentage error (ε; e.g., Makridakis et al., 1982):

$$\varepsilon = \frac{1}{k} \cdot \sum_{1}^{k} \left|\frac{v_1 - v_2}{v_1}\right| \cdot 100\% \quad (11)$$

where k is a number of x points on the x-axis, and v_1 and v_2 are their respective values on the y-axis, i.e., egg radii obtained by a direct measurement of the egg profile (v_1) and calculated from a corresponding theoretical model (v_2). The radii were measured along the horizontal axis of the egg at a shift of 1 pixel. Thus, depending on the value of L, the number of measurements k varied from 387 to 426.

Formula (1) was modified to better match Narushin's model:

$$y = \pm B\sqrt{\frac{x(L-x)}{L^2 + 4w(L-2x) + 4w^2}} \quad (12)$$

For each egg profile the following models were tested: (i) the Hügelschäffer's model with the measured parameter w as appeared in Narushin, Romanov, et al. (2020); (ii) the Narushin's model, for which only two measurements (B and L) are sufficient; (iii) the Hügelschäffer's model with the parameter w that conforms to the Narushin's model as recalculated with Eq. (10).

4. Results

4.1. Egg geometry

An example of three obtained geometric profiles is shown in Fig. 3. As an illustration, we used an egg #1 (Fig. 3a,b) in which the value of w for Hügelschäffer's model was very different from that obtained by recalculation with Eq. (10) for Narushin's model and also from the one (egg #2, Fig. 3c) for which a full compliance of this parameter was found.

Approximately the same nature of the profiles was obtained for the rest of the analysed eggs. The use of Hügelschäffer's model leads to a fairly accurate match with the actual profile under a correctly measured value of the parameter w. However, as noted in the Introduction section, this is by no means always possible. The average error of comparison of this model with actual eggs was: $\varepsilon_H = 4.62\%$.

Using Narushin's model eliminates the laborious and not always accurate measurement of w, although even when the value of this parameter corresponds to an actual egg, there is still an error in the exact reproduction of its shape. On average, it amounted to $\varepsilon_N = 6.95\%$ for all analysed eggs.

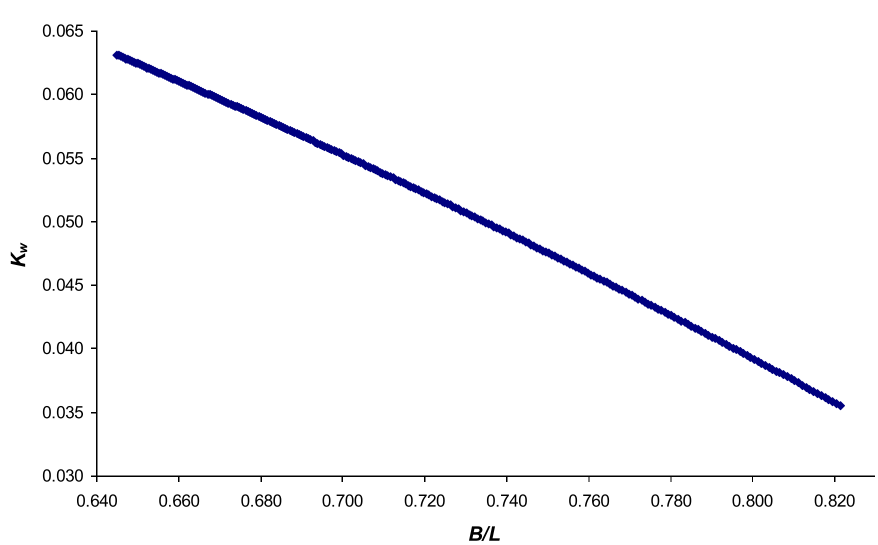

Fig. 2 – Graphic dependence of the coefficient K_w on the shape index of chicken eggs (B/L).

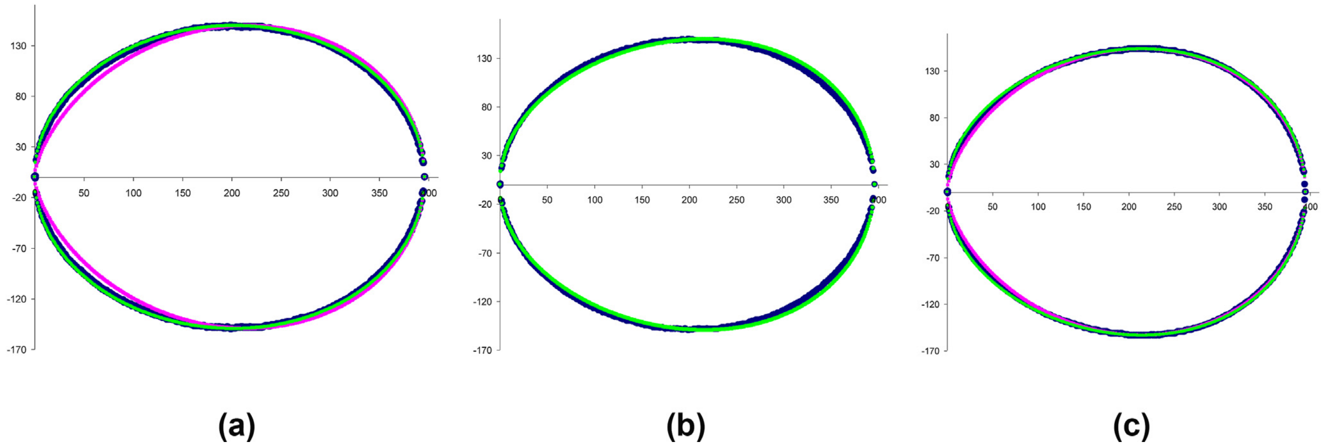

Fig. 3 – Comparative analysis of three geometric profiles of chicken eggs (dimensions are given in pixels). (a) The blue line matches the profile of an actual egg #1; the green line is plotted according to Hügelschäffer's model with the value of the parameter w conforming to an actual egg #1 and equal to 0.06 cm; and the purple line is consistent with Narushin's model, for which the w value was calculated from Eq. (10) to be 0.26 cm. (b) The blue line conforms to the profile of the same actual egg #1, and the green one to Hügelschäffer's model with the value of parameter w corresponding to that calculated from Eq. (10), i.e., 0.26 cm. (c) The blue line conforms to the profile of another actual egg #2; the green one to Hügelschäffer's model with the value of parameter w corresponding to the actual egg #2 equal to 0.23 cm; and the purple one to Narushin's model, for which the value of w was calculated from Eq. (10) and also amounted to 0.23 cm. (For interpretation of the references to color in this figure legend, the reader is referred to the Web version of this article.)

The use of w values calculated from equation (10) in Hügelschäffer's model somewhat improved the situation in comparison with Narushin's model, at $\varepsilon_{HJN} = 5.56\%$, although it did not allow us to speak of complete agreement, which is especially noticeable in Fig. 3b.

A more careful analysis of the obtained contours allowed us to suggest that the use of Hügelschäffer's model with the value of the parameter w calculated from Eq. (10) in the range of $x = [0 \ldots L/2]$ (which corresponds to the sharp end of the egg) does not affect the distortion as compared to the actual egg contour. The error already occurs in the interval of $x = [L/2 \ldots L]$ (which corresponds to the blunt end of the egg). At the same time, the blunt end of the egg is almost identical to the ellipse, i.e., when $w = 0$.

In this respect, we hypothesised that the value of the parameter w may not be constant, but with each value of x from 0 to L, it smoothly changes from its maximum value (i.e., calculated according to Eq. (10)) at the point $x = 0$ and up to the value $w = 0$ at the point $x = L$.

To evaluate this hypothesis, we tested a few possible functional dependencies of the change in the values of w as presented below.

4.1.1. *Linear function*

For this case, the change in the values of w will occur according to the following function (also shown conventionally in Fig. 4):

$$w = ax + b \tag{13}$$

where a and b are constants to be determined.

According to our hypothesis, at $x = 0$ the value of w will be maximum, i.e., calculated by formula (10). Then, starting from Eq. (13),

$$b = L\left(0.103 + 0.012\frac{B}{L} - 0.115\left(\frac{B}{L}\right)^2\right) \tag{14}$$

At $x = L$, the value $w = 0$ (Fig. 4). Taking Eq. (14) into account, formula (13) can be rewritten as

$$a = -\left(0.103 + 0.012\frac{B}{L} - 0.115\left(\frac{B}{L}\right)^2\right) \tag{15}$$

Then, substituting the values of the obtained coefficients a (Eq. (15)) and b (Eq. (14)) into Eq. (12), we have the following final function of changing w:

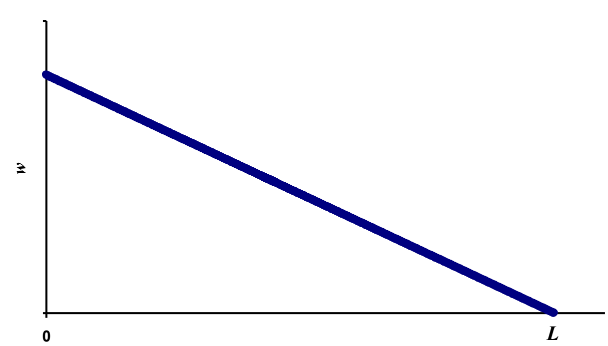

Fig. 4 – Linear change in the values of w over the interval from 0 to L.

$$w = \left(0.103 + 0.012\frac{B}{L} - 0.115\left(\frac{B}{L}\right)^2\right)(L - x) \tag{16}$$

Using Eq. (16) to determine w in Eq. (12) for Hügelschäffer's model, variants of chicken egg profiles were constructed. For ease of comparison, Fig. 5 shows graphic images corresponding to the same eggs #1 and #2 as in Fig. 3.

The average error in comparing the obtained contours with actual eggs was: $\varepsilon_{\mathrm{ln}} = 4.17\%$.

4.1.2. *Ellipse equation*

What will happen if the values of w change curvilinearly? The easiest way to test this hypothesis is using the ellipse equation due to the smoothness and uniformity of the change in the values of y depending on x. That is, the value of w will gradually decrease along its 1st quadrant (Fig. 6).

Using the ellipse formula and an approach similar to the one we applied for the linear function, the following equation for changing the parameter w was obtained:

$$w = \left(0.103 + 0.012\frac{B}{L} - 0.115\left(\frac{B}{L}\right)^2\right)\sqrt{L^2 - x^2} \tag{17}$$

Using Eq. (17) to determine w in Eq. (12) for Hügelschäffer's model, we also obtained variants of chicken egg profiles. Two of them are presented in Fig. 7 and conform to actual eggs that have already been used by us in Figs. 3 and 5.

The average error $\varepsilon_{\mathrm{el}}$ in comparing the obtained contours with actual eggs was 4.38%.

The use of other curvilinear functions was not feasible for two probable reasons: (i) due to the presence of more than two unknown constants, which made it impossible to determine them in our case; or (ii) due to obtaining incorrect values for the case $w = 0$. The latter could occur, for example, when using power, exponential or logarithmic functions.

Comparison of the values of the average errors ε showed that the linear dependence of the change in the parameter w leads to a more accurate match with the actual egg contour ($\varepsilon_{\mathrm{ln}} = 4.17\%$). In this case, the calculation accuracy was even higher than when using a correctly measured constant value of w ($\varepsilon_H = 4.62\%$).

Given the linear change in w according to Eq. (16), Eq. (12) of Hügelschäffer's model can be rewritten as follows:

$$y = \pm B\sqrt{\frac{x(L-x)}{L^2 + 4\left(0.103 + 0.012\frac{B}{L} - 0.115\left(\frac{B}{L}\right)^2\right)(L-x)(L-2x) + 4\left(0.103 + 0.012\frac{B}{L} - 0.115\left(\frac{B}{L}\right)^2\right)^2(L-x)^2}} \tag{18}$$

Thus, Eq. (18) was taken as optimal for the geometric description of the contours of chicken eggs. We conditionally coined this model as *modified Hügelschäffer's model with two parameters*.

4.2. *Egg volume and surface area*

The next step in the evaluation of this two-parameter approach was the derivation of equations for calculating the main geometrical egg parameters, volume (V) and surface area (S).

Previously, we have proposed a number of approaches for calculating V and S based on Hügelschäffer's model. For this, it was assumed both the use of measured values of w (Narushin, Romanov, et al., 2020; Narushin et al., 2021; Narushin, Romanov, Mishra, et al., 2022) and calculations without it (Narushin et al., 2021a).

Nonetheless, the development of a new model can be considered completed when, on its basis, derived formulae for the main characteristics of an object are also obtained. Thus, we considered it necessary to expand research on Eq. (18), i.e., modified Hügelschäffer's model with two parameters, by inferring new dependencies:

$V = f(L,B)$ and $S = f(L,B)$.

In doing so, we took two approaches.

4.2.1. *Using averages*

Here, we tested possibility to use the average value of w calculated using Eq. (16).

To calculate the mean value w that we will denote as w_a, we use the classical formula of a mean value theorem (e.g., Besenyei, 2012):

$$f_a = \frac{1}{b-a}\int_a^b f(x)dx, \tag{19}$$

which for our cases will be rewritten in the following form:

$$w_{a\ \mathrm{ln}} = 0.5L\left(0.103 + 0.012\frac{B}{L} - 0.115\left(\frac{B}{L}\right)^2\right) \tag{20}$$

To test this approach and the average value of the parameter w of the above function, we used the formula for calculating egg volume from Narushin, Romanov, et al. (2020) and its surface area formula from Narushin, Romanov, Lu, et al. (2021), respectively:

$$V = \frac{\pi B^2}{256 w_a^3}\left(4w_a L\left(L^2 + 4w_a^2\right) - \left(L^2 - 4w_a^2\right)^2 \cdot \ln\left|\frac{L + 2w_a}{L - 2w_a}\right|\right) \tag{21}$$

$$S = \frac{2.48B}{L} \cdot \left(L^2 - 0.34Lw_a - 4.27w_a^2\right) \tag{22}$$

Empirical data on egg volumes measured by Archimedes' principle and surface areas examined with the 2-D digital imaging and subsequent image processing techniques were used from the same papers.

Comparison of the calculated values for V (Eq. (21)) and S (Eq. (22)) with the experimental data values showed the following error values: for a linear function, $\varepsilon_{V\mathrm{ln}} = 1.72\%$ and $\varepsilon_{S\mathrm{ln}} = 0.83\%$.

Similarly, we checked the possibility of calculating V and S by recalculating the value of w that varies following an elliptic function (formula (17)); however, the resulting errors

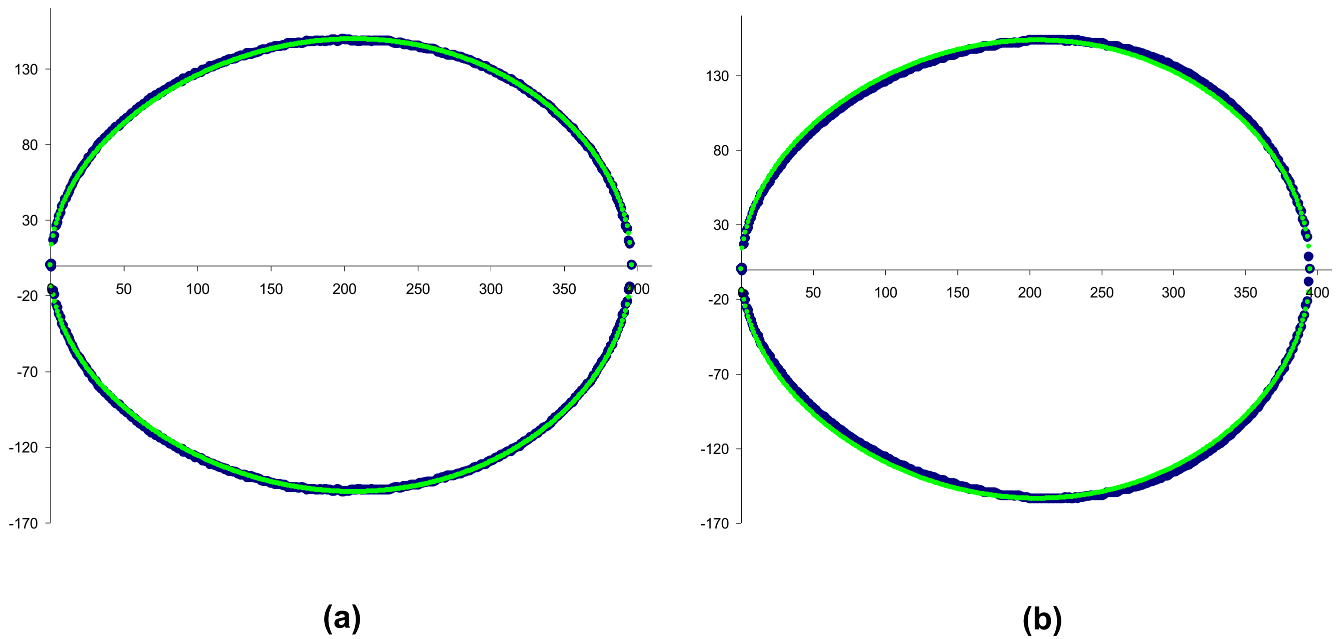

Fig. 5 – Construction of geometric profiles of two different chicken eggs according to Hügelschäffer's model with a linear change in the parameter w (dimensions are given in pixels). (a) The blue line conforms to the profile of an actual egg #1 ($w = 0.06$ cm), and the green one to Hügelschäffer's model with a linear change in the parameter w from 0.26 to 0 cm. (b) The blue line corresponds to the profile of an actual egg #2 ($w = 0.23$ cm), and the green one to Hügelschäffer's model with a linear change in the parameter w from 0.23 to 0 cm. (For interpretation of the references to color in this figure legend, the reader is referred to the Web version of this article.)

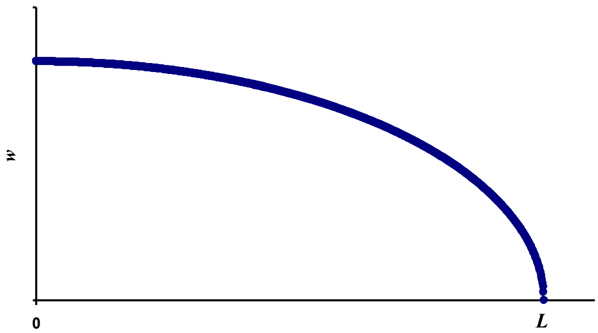

Fig. 6 – Ellipsoid change in w values from 0 to L.

exceeded those for a linear function. As a result, it was concluded that it would be inappropriate to use the curvilinear w change function.

Checking the error in calculating V and S using the same formulas and considering the measured (i.e., actual) value of the parameter w, gave the following results: $\varepsilon_{Vw} = 1.73\%$, and $\varepsilon_{Sw} = 0.95\%$.

4.2.2. *Using integral geometry formulae*

This approach is considered standard when deriving such formulae, although sometimes the complexity of calculating the integral, associated with the cumbersomeness of the integrand, does not always lead to an exact accurate result. However, we decided to follow this testing path in order to choose the most optimal solution for sure.

In this instance, the volume of the rotation figure described using the following integral geometry formulae:

$$V = \pi \int_0^L y^2 dx \tag{23}$$

and

$$S = 2\pi \int_0^L y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{24}$$

where function y is defined by formula (18)

Integrals (23) and (24) cannot be converted to a tabular form, therefore, their solution is possible only with the help of approximate methods. As such, the method for determining the area of a flat figure formed by the respective curves from the integrands in Eqs. (23) and (24) was chosen. For that purpose, we exploited a numerical method in MS Excel as was proposed elsewhere (Piessens et al., 1983).

As a consequence of the implementation of this approach, the solution of the above integrals (Eqs. (23) and (24)) resulted in the following mathematical expressions:

$$V = 0.034LB^2\left(\frac{B}{L} + 14.389\right) \tag{25}$$

$$S = 1.038LB\left(\frac{B}{L} + 2.005\right) \tag{26}$$

A detailed transformation of Eqs. (23) and (24) and the derivation of the final formulas are presented in Supplementary Data A.

The L and B values from our experimental data were substituted into Eqs. (25) and (26). After that, the obtained values were compared with the measured egg volumes and surfaces (similar to the procedure in Section 4.2.1. Using averages).

A comparison of the calculated values for V (Eq. (25)) and S (Eq. (26)) with the experimental ones showed the following errors: $\varepsilon_V = 2.55\%$, and $\varepsilon_S = 17.70\%$, respectively.

For the convenience of subsequent analysis, we transformed Eqs. (21) and (22) by substituting Eq. (20) into them and

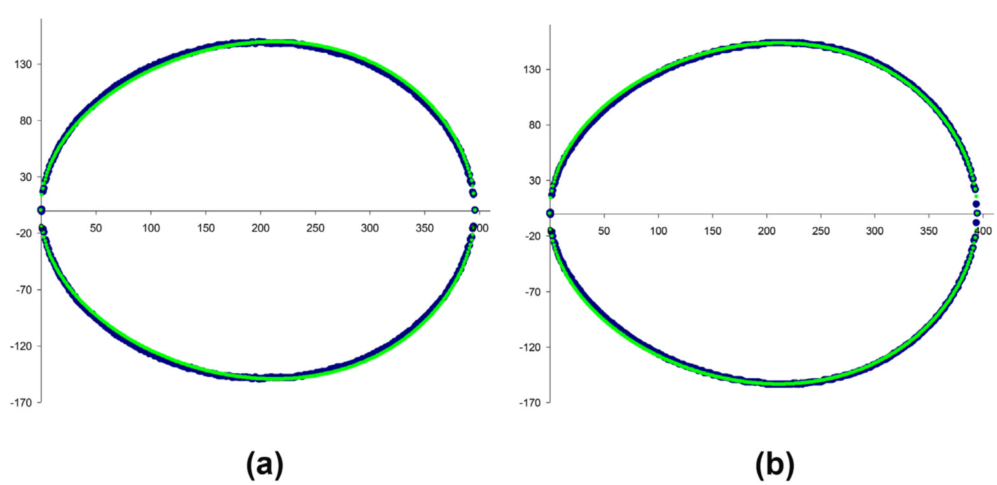

Fig. 7 – Construction of geometric profiles of two different chicken eggs according to Hügelschäffer's model with an ellipsoidal change in the parameter w (dimensions are given in pixels). (a) The blue line conforms to the profile of an actual egg #1 ($w = 0.06$ cm), and the green one to Hügelschäffer's model with ellipsoidal change in parameter w from 0.26 to 0 cm. (b) Blue line corresponds to the profile of an actual egg #2 ($w = 0.23$ cm), and the green one to Hügelschäffer's model with an ellipsoidal change in the parameter w from 0.23 to 0 cm. (For interpretation of the references to color in this figure legend, the reader is referred to the Web version of this article.)

some transformation to a simpler form. As a result, we obtained the following final calculation formulae:

$$V = 0.5115LB^2 + 0.0035B^3\left(1 - 0.343\frac{B}{L}\right) \quad (27)$$

$$S = 2.48LB\left(1 - 0.02\left(1 - \frac{B}{L}\right)\left(\frac{B}{L} + 0.896\right) - 0.014\left(1 - \frac{B}{L}\right)^2\left(\frac{B}{L} + 0.896\right)^2\right) \quad (28)$$

Detailed output of Eqs. (27) and (28) is provided in Supplementary Data B.

5. Discussion

All previous successful attempts to describe the profile of chicken eggs mathematically required, at a minimum, the measurement of three parameters. Two of them, i.e., the egg length (L) and its maximum breadth (B), have always been considered the most popular and easiest to implement. Nevertheless, another one was normally required, for example, w, which most often turned out to be quite laborious in terms of its measurement.

In the present study, this problem was solved by our suggesting to change the variant of the constant value of the parameter w for a particular egg to a "sliding" one, i.e., replacing w in Hügelschäffer's formula with a function of B/L along the egg profile. Such a functional change in w enabled to combine two fundamental principles – the accuracy of matching with the actual egg profile and the use of the minimum possible number of measurements. When tested, this hypothesis resulted in an unexpectedly high accuracy estimate, exceeding not only Narushin's model, which is more distant from the actual profile of a chicken egg, but also the classical Hügelschäffer's model, for which the value of w was carefully selected by sorting and comparing with the digitised shape of each egg involved in the experiment. In this case, the two considered variants (linear and elliptical) of the functional change in w led to a smaller error than its constant value ($\varepsilon_{ln} = 4.17\%$ and $\varepsilon_{el} = 4.38\%$, respectively, vs $\varepsilon_H = 4.62\%$). It was the linear function that was taken as the basic one and the most accurate and easy to use, resulting in a new mathematical model of a chicken egg as described using Eq. (18).

Since the derivation of mathematical dependence often pursues purely applied problems, we also performed the respective calculations. And since the most relevant parameters, being most often in demand both in research work and in poultry industry (Baydevlyatova et al., 2009; Tagirov et al., 2009; Shomina et al., 2009), are the egg volume and surface area, we have made a number of steps to generate the appropriate calculation formulae for these two main egg characteristics also. Since field zootechnical studies often do not require a specialized analytical equipment for measuring accurately a whole number of geometric parameters of eggs, the availability of simpler calculation formulae based on just two measurements makes it easy to carry out these investigations.

The results of theoretical calculations and experimental verification enabled us to assert that our approach using the average value of the "sliding" parameter w turned out to be more accurate than the expressions obtained using the integral calculus formula. However, based on a comparison of the obtained errors of these calculations, we chose the option that gives the minimum ε.

We produced the final formulae for calculating V (Eq. (27)) and S (Eq. (28)) exactly in this form in order to demonstrate the adequacy of the controversy we raised earlier about the coefficients for LB^2 (for V) and LB (for S) (Narushin, 2005; Narushin et al., 2021a). The coefficient of 0.5115 in Eq. (27) fits exactly into the interval of 0.5163 ± 0.0065 demonstrated by Narushin et al. (2021a). At the same time, the remaining part of Eq. (27) is designed to level out possible calculated errors depending on the values of L and B for a particular egg.

Similarly, the coefficient of 2.48 in Eq. (28) is fairly close to that found by Narushin et al. (2021a), i.e., 2.186. The differences between them are smoothed out by an additional functional factor, which in this case has the form of a quadratic dependence, in contrast to the linear dependence that we obtained earlier (Narushin et al., 2021a).

6. Conclusions

On the basis of our findings, we suggest that the designed ovoidal egg shape mathematical model (Eq. (18)) can be considered useful in terms of accuracy, simplicity and compactness.

Thus, in this investigation, we have synthesised a new modified, two-parameter mathematical model of a chicken egg. As a result of the study with its help, we also obtained theoretical expressions for calculating the egg volume and surface area, the adequacy and relevance of which are confirmed not only by the current performed observations, but also by other previous studies in this field (e.g., Narushin et al., 2021a).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.biosystemseng.2022.11.003.

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