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Some Aspects of Kaluza-Klein Theory.

by

Andrew M. Samson.

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PREFACE.

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Except where indicated to the contrary the content of this thesis is my own original work.

Andrew M. Samson.

Andrew M. Samson.

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CONTENTS.

Abstract.

Five dimensional Kaluza-Klein theory is described from the historical viewpoint. This theory is then generalised firstly for a non-Abelian group and then for the specific group SU(2) x U(1). The calculation of the scalar curvature of the extended manifold is performed using Cartan's structure equations. The role of Killing vectors is heavily emphasized in this treatment.

Using certain results from this calculation an eight dimensional Dirac equation is derived and then simplified. Each term in the equation is then discussed.

The thesis ends with some concluding remarks.

Introduction.

Kaluza-Klein theory originated in the 1920's with Kaluza's attempt to unify electromagnetism with Einstein's new theory of gravitation. Later, Klein tried to reconcile Kaluza's theory with the new ideas on quantisation. Although this ended, by Klein's own admission, in failure, Kaluza-Klein theory had been born. It was comprehensively studied by Einstein and Bergmann in their search for the one true unified field theory, and, indeed the most complete account of Kaluza-Klein theory, at least in its five dimensional form, is given in Bergmann's book. (Bergmann 1942). After this it fell into the doldrums but was revived in 1968 with Kerner's proof that it could be extended to 4+n dimensions to describe the gauge fields associated with Yang-Mills theory. (Kerner 1968). The upsurge in interest in geometrical methods as applied to gravitational physics has carried Kaluza-Klein theory along too, and it can be safely said that it now stands at its strongest point ever.

The exact details of the theory will be found later on but, basically, the idea is to extend spacetime to 4+1 dimensions to describe electromagnetism, or to 4+n dimensions to describe n gauge fields. A special form of the metric tensor is hypothesized on the extended spacetime, the curvature scalar of which is used as the Lagrangian. This extended curvature scalar decomposes into the curvature scalar R^4 for gravity and the square of the field tensor for electromagnetism or the Yang-Mills field (or both).

There are a number of ways of formulating Kaluza-Klein theories. Historically, the starting point was to define the special Kaluza-Klein metric and to impose certain conditions on some components of this metric. The scalar curvature of the extended spacetime could then be calculated and could be taken, as in general relativity, to be the Lagrangian. Einstein and Bergmann showed however that the special form of the metric and the necessary simplification conditions occur naturally if a Killing vector field exists. In recent years in the treatment of generalised Kaluza-Klein theories the special form of the metric has become the centre of attention and the existence of Killing vectors is not overemphasized. The modern spirit is to try to describe theories in co-ordinate free language and some authors follow this ideal and use fibre bundles in their treatment of Kaluza-Klein theories. (McInnes 1982a, 1982b) Others avoid this approach and use classical tensor calculus. Some use both methods. (Cho 1975, Cho and Jang 1975). It is worthwhile noting at this point that the theory of fibre bundles is a mathematical theory and not a physical theory. The power of fibre bundle theory is conceptual but its use is neither necessary nor sufficient in the treatment of Kaluza-Klein theory. Most importantly there is no a priori way to define a metric on a fibre bundle. Certain authors (Cho 1975, Chang et al. 1978) have tried to circumvent this "problem" by defining a horizontal lift basis on the bundle manifold, and by demanding that the metric is diagonal with respect to this basis. It then follows that the metric expressed in a co-ordinate basis has the special Kaluza-Klein form. However, we take the view that it is the Killing vectors which determine the form of the metric and this point will be stressed throughout our work.

An increasingly popular developement in relation to Kaluza-Klein theory is the theories of spontaneous compactification (Cremmer et al., 1976a, 1976b, 1977a, 1977b, Scherk et al., 1979.). These theories use extra spacetime dimensions as does Kaluza-Klein theory, but the main emphasis is on the defenition of fields on the extended manifold. No attempt is made to hypothesize Killing vectors. We shall not consider spontaneous compactification here, but we do point out in our conclusions that such theories will go hand in hand with Kaluza-Klein theory in future developements.

The motivation of our work here is to try to unify Kaluza-Klein theory with the Weinberg-Salam model, that is to use the extra dimensions to describe the specific gauge group SU(2) x U(1). Accordingly we shall use four extra dimensions so that our extended spacetime is eight dimensional. We shall not use fibre bundle theory but instead use the more classical methods. Our main calculational technique will be to use Cartan's structure equations. This is less time consuming than tensor calculus and a further benefit is that when we go on to treat the Dirac equation we can use some of our previous results.

The structure of this work is as follows: Chapter 1 describes five dimensional Kaluza-Klein theory, most of which has been considered already by Bergmann. The second chapter describes the generalisation of this to eight dimensions in the particular case of the product group SU(2)xU(1). The geometrical calculations use Cartan's structure equations. The final chapter discusses the Dirac equation on the extended spacetime. The thesis ends with a discussion about the prospects for the future of Kaluza-Klein theory.

CHAPTER 1

FIVE DIMENSIONAL KALUZA-KLEIN THEORY

The Historical Treatment.

We begin with the hypothesis that spacetime is five dimensional and that the metric tensor can be expressed in some co-ordinate systems, known as K-systems, in the following way :

$$\gamma_{\alpha\beta} = \left(\begin{array}{c|c} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ \hline A_\nu & 1 \end{array} \right) \qquad (1.1)$$

The indices α, β refer to the entire manifold and take the values 1, 2, 3, 4, 5, whereas μ, ν are spacetime indices and take the values 1, 2, 3, 4. This means that the above matrix is in block form : the top left-hand block is 4 x 4, the top right-hand block is 4 x 1, the bottom left-hand block is 1 x 4 and the bottom right-hand block is 1 x 1.

The 1 co-ordinate is the time co-ordinate and so the Minkowski metric for this five dimensional spacetime is η_{AB} = diag (-1, 1, 1, 1, 1). A_μ is a four-vector and will, at a later stage, be identified with the vector potential of electromagnetism.

The metric (1.1) is a symmetric tensor of rank 2 in five dimensions, so it has fifteen independent components; ten of these describe gravity and four describe electromagnetism, which leaves one extra component. It is usual to put $\gamma_{55} = 1$. It can be shown that the sign of γ_{55} must be positive, that is x^5 is a spacelike dimension, to ensure that results from this theory are consistent with electromagnetism. (Thirring 1972). Some authors put γ_{55} equal to a scalar function $\sigma(x)$ in order to achieve unification with the Brans-Dicke theory of gravitation as opposed to Einstein's theory. Although this approach is interesting we shall not pursue it here.

As well as the condition $\gamma_{55} = 1$, we impose the following restrictions :

$$\partial_5 \, g_{\mu\nu} = 0 \tag{1.2}$$

$$\partial_5 \, A_\mu = 0 \tag{1.3}$$

We are now in a position to calculate the connection coefficients and thence the scalar curvature. The calculations are straightforward and easily verifiable.

The inverse of the metric tensor (1.1) is :

$$\gamma^{\alpha\beta} = \left(\begin{array}{c|c} g^{\mu\nu} & -g^{\mu\rho} A_\rho \\ \hline -g^{\rho\nu} A_\rho & 1 + g^{\sigma\rho} A_\sigma A_\rho \end{array} \right) \tag{1.4}$$

The index convention and the block convention are the same as before and $g^{\mu\nu}$ is the inverse of the four dimensional metric $g_{\mu\nu}$.

The connection coefficients are given by the formula :

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\gamma^{\alpha\delta}\left(\gamma_{\beta\delta,\gamma} + \gamma_{\gamma\delta,\beta} - \gamma_{\beta\gamma,\delta}\right) \qquad (1.5)$$

Explicit calculation gives the following results :

$$\Gamma^{\sigma}_{\mu\nu} = \Gamma^{④\sigma}_{\mu\nu} + \frac{1}{2}\left(f_{\mu}{}^{\sigma}A_{\nu} + f_{\nu}{}^{\sigma}A_{\mu}\right)$$

$$\Gamma^{\sigma}_{5\mu} = \frac{1}{2}f_{\mu}{}^{\sigma}$$

$$\Gamma^{5}_{\mu\nu} = \frac{1}{2}\left(A_{\mu|\nu} + A_{\nu|\mu}\right) - \frac{1}{2}A_{\rho}\left(f_{\mu}{}^{\rho}A_{\nu} + f_{\nu}{}^{\rho}A_{\mu}\right)$$

$$\Gamma^{5}_{5\mu} = -\frac{1}{2}f_{\mu}{}^{\rho}A_{\rho}$$

$$\Gamma^{\sigma}_{55} = 0 \quad ; \quad \Gamma^{5}_{55} = 0$$

$$(1.6)$$

where

$$f_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$$

$$f^{\mu}{}_{\nu} = g^{\mu\rho} f_{\rho\nu}$$

$$A_{\mu/\nu} = A_{\mu,\nu} - \Gamma^{(4)\rho}_{\mu\nu} A_{\rho}$$

(1.7)

and $\Gamma^{(4)\rho}_{\mu\nu}$ is the connection coefficient of the four dimensional metric $g_{\mu\nu}$. We are continuing the index convention that Greek letters near the beginning of the alphabet refer to the extended manifold and that those near the end refer to four dimensional spacetime.

The final step is to calculate the scalar curvature. This is done by contracting the Ricci tensor with the metric tensor :

$$R^{(5)} = \gamma^{\alpha\beta} R_{\alpha\beta}$$

(1.8)

where

$$R_{\alpha\beta} = \Gamma^{\delta}_{\alpha\beta,\delta} - \Gamma^{\delta}_{\beta\delta,\alpha} + \Gamma^{\epsilon}_{\alpha\beta}\Gamma^{\delta}_{\epsilon\delta} - \Gamma^{\epsilon}_{\alpha\delta}\Gamma^{\delta}_{\epsilon\beta}$$

(1.9)

Summing over α and β gives :

$$\gamma^{\mu\nu}\left\{ \Gamma^{\delta}_{\mu\nu,\delta} - \Gamma^{\delta}_{\nu\delta,\mu} + \Gamma^{\epsilon}_{\mu\nu}\Gamma^{\delta}_{\epsilon\delta} - \Gamma^{\epsilon}_{\mu\delta}\Gamma^{\delta}_{\epsilon\nu} \right\}$$

(A)

$$+\gamma^{\mu 5}\left\{ \Gamma^{\delta}_{\mu 5,\delta} - \Gamma^{\delta}_{5\delta,\mu} + \Gamma^{\epsilon}_{\mu 5}\Gamma^{\delta}_{\epsilon\delta} - \Gamma^{\epsilon}_{\mu\delta}\Gamma^{\delta}_{\epsilon 5} \right\}$$

(B)

$$+\gamma^{5\nu}\left\{ \Gamma^{\delta}_{5\nu,\delta} - \Gamma^{\delta}_{\nu\delta,5} + \Gamma^{\epsilon}_{5\nu}\Gamma^{\delta}_{\epsilon\delta} - \Gamma^{\epsilon}_{5\delta}\Gamma^{\delta}_{\epsilon\nu} \right\}$$

(C)

$$+\gamma^{55}\left\{ \Gamma^{\delta}_{55,\delta} - \Gamma^{\delta}_{5\delta,5} + \Gamma^{\epsilon}_{55}\Gamma^{\delta}_{\epsilon\delta} - \Gamma^{\epsilon}_{5\delta}\Gamma^{\delta}_{\epsilon 5} \right\}.$$

(D)

Now summation over δ and ϵ is performed and simplification is effected by using formulae (1.2), (1.3), and (1.6) and by noting that:

$$\Gamma^{\delta}_{5\delta} = 0 \qquad (1.10)$$

For clarity each part is calculated separately. Part (A) equals :

$$R^{(4)} + g^{\mu\nu} A_\nu \partial_\sigma f_\mu{}^\sigma - \Gamma^{(4)\rho}_{\sigma\mu} f_\rho{}^\sigma A_\nu g^{\mu\nu}$$

$$+ \Gamma^{(4)\rho}_{\sigma\rho} f_\mu{}^\sigma A_\nu g^{\mu\nu} - \frac{1}{4} g^{\mu\nu} A_\mu A_\nu f_\sigma{}^\rho f_\rho{}^\sigma$$

(A1)

and part (D) equals :

$$(1 + g^{\mu\nu} A_\mu A_\nu)\left(-\frac{1}{4} f_\sigma{}^\rho f_\rho{}^\sigma\right)$$

(D1)

Now because of the symmetry of both the metric tensor and the Ricci tensor, parts (B) and (C) are equal so either one can be calculated and then doubled. So part (B) is :

$$g^{\mu\lambda} A_\lambda \Big\{ -\frac{1}{2} \partial_\sigma f_\mu{}^\sigma + \frac{1}{2} \Gamma^{(4)\sigma}_{\rho\mu} f_\sigma{}^\rho$$

$$- \frac{1}{2} \Gamma^{(4)\rho}_{\sigma\rho} f_\mu{}^\sigma + \frac{1}{4} A_\mu f_\rho{}^\sigma f_\sigma{}^\rho \Big\} \; .$$

(B1)

Finally adding (A1) to (D1) to twice (B1) gives the final answer :

$$R^{(5)} = R^{(4)} - \frac{1}{4} f_{\sigma}{}^{\rho} f_{\rho}{}^{\sigma}$$

(1.11)

If we take as our action :

$$I^{(5)} = \int R^{(5)} \sqrt{-\gamma}\, d^5x$$

(1.12)

then since γ = det $\gamma_{\alpha\beta}$ = det $g_{\mu\nu}$ = g we can see that this will give us :

$$I^{(5)} = \int \left(R^{(4)} - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} \right) \sqrt{-g}\, d^5x$$

(1.13)

from which, by variation, we shall be able to derive Einstein's field equations and Maxwell's equations. Thus we are well justified in identifying A_μ with the vector potential of electromagnetism and in claiming that Kaluza-Klein theory unifies gravity and electromagnetism.

Two points are worth mentioning here in passing. Firstly, since we have put γ_{55} equal to the constant unity it cannot be varied. We shall see later that this is not a problem in the covariant formulation of Kaluza-Klein theory. However Ke and Ying have overcome this difficulty using the method of Lagrange multipliers to formally prove that Einstein's equations and Maxwell's equations are deriveable from the action (1.12). (Ke and Ying 1981).

Secondly, since the argument in (1.12) is independent of x^5, the x^5 co-ordinate must close on itself after a finite length, otherwise the integral is infinite. This is the origin of the idea of spontaneous compactification. Some authors state that the radius of curvature of the fifth dimension is the Planck length. While this may be a trifle rash we can defenitely say that the radius of curvature is very small due to its non-observability in daily life, even at the microscopic level of electromagnetic phenomena.

Covariant Formulation of Kaluza-Klein Theory

In the above paragraphs we began with a special form of the metric tensor and our reasoning continued under the assumption that we were still in a K-frame of reference. Although this procedure is not incorrect, it is a violation of the spirit of General Relativity where all reference frames are equally permissible. Since a co-ordinate transformation cannot affect the physics of the system, what must be shown is that Kaluza-Klein theory can be cast in a co-variant form and that when a co-ordinate transformation takes us out of our K-frame we observe no contradictions with our previous deductions. This procedure has been extensively carried out by Bergmann (Bergmann 1942) and we give here a brief resume of his ideas.

We begin with a five dimensional spacetime with co-ordinates ξ^α

and metric tensor $\gamma_{\alpha\beta}$ and introduce four functions of these co-ordinates x^a (ζ^α) where a = 1,2,3,4, with the proviso that the derivatives of these four functions are linearly independent. In this way the four functions define a set of curves x^a = constant in the five dimensional space, and this set of curves is itself a manifold, of dimesion four, on which can be defined vectors and tensors.

Writing the derivatives of these four functions with respect to the co-ordinates as :

$$x^a{}_{,\alpha} = \gamma^a{}_\alpha \qquad (1.14)$$

allows us to introduce a vector field A^α defined to satisfy:

$$\gamma^a{}_\alpha A^\alpha = 0 \qquad (1.15)$$

and

$$\gamma_{\alpha\beta} A^\alpha A^\beta = 1 \qquad (1.16)$$

We can define an inverse $\gamma^{\alpha}{}_{a}$ by the following conditions:

$$\gamma^{a}{}_{\alpha}\,\gamma^{\alpha}{}_{b} = \delta^{a}{}_{b} \qquad (1.17)$$

and

$$A_{\alpha}\,\gamma^{\alpha}{}_{a} = 0 \qquad (1.18)$$

Vectors, or tensors, in the five dimensional space will be referred to as ordinary while vectors, or tensors, in the four dimensional space will be referred to as parameter tensors, or p-tensors for short. With any ordinary vector V^{α} can be assosciated a p-vector and a scalar thus:

$$V^{a} = \gamma^{a}{}_{\alpha}\,V^{\alpha} \qquad (1.19)$$

$$V = A_{\alpha}\,V^{\alpha} \qquad (1.20)$$

and similarly for a covariant ordinary vector:

$$w_b = \gamma^{\beta}{}_{b}\, w_{\beta}$$

(1.21)

$$w = A^{\beta}\, w_{\beta}$$

(1.22)

It is natural to take the p-tensor assosciated with the ordinary metric tensor as the metric of the four dimensional space thus:

$$g_{ab} = \gamma^{\alpha}{}_{a}\, \gamma^{\beta}{}_{b}\, \gamma_{\alpha\beta}$$

(1.23)

To carry out analytic calculations in the four dimensional space it is necessary to define the operation of differentiation. There are two inequivalent types of differentiation that can be defined. Firstly there is A-differentiation, defined as being the derivative of the p-tensor contracted with the A-vector:

A-derivative of ($X^{ab}{}_{cd}$) = $X^{ab}{}_{cd,\alpha} A^{\alpha}$.

Secondly, there is p-differentiation, defined by:

$$\text{p-derivative of } (X^{ab}{}_{cd}) = X^{ab}{}_{cd/e} = X^{ab}{}_{cd,\alpha}\, \gamma^{\alpha}{}_{e}.$$

We can now perform analysis in the four dimensional space embedded in the five dimensional space. What has to be done now is to consider the connection between tensors in this covariant formulation and those in the K-frame.

We recover a K-frame by demanding that the first four co-ordinates $\xi^1 \ldots\ldots \xi^4$ are the same as $x^1 \ldots . x^4$, and ξ^5 is chosen so that $A^5 = 1$. In this case

$$\gamma^{a}{}_{\alpha} = \delta^{a}{}_{\alpha} \quad , \quad \alpha = 1, \ldots\ldots 4.$$

$$\gamma^{a}{}_{\alpha} = 0 \quad , \quad \alpha = 5.$$

$$A^{\alpha} = (\underline{0}, 1) \qquad (1.24a)$$

$$A_{\alpha} = (A_{\mu}, 1) \qquad (1.24b)$$

To go from one K-frame to another can be achieved by the following transformations:

$$\xi'^{\mu} = f^{\mu}(\xi^{\nu})$$

$$\xi'^{5} = \xi^{5} + f^{5}(\xi^{\nu})$$

(1.25)

To maintain invariance under these transformations means that A_μ has to transform thus:

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu f^5$$

(1.26)

which is the same as a gauge transformation in electromagnetism. Kaluza-Klein theory in a K-frame is therefore co-variant with respect to the transformations (1.25) which are co-ordinate transformations in the four dimensions and gauge transformations for the electromagnetic vector potential.

Finally, we must try to express the conditions (1.2) and (1.3) in co-variant language. A simple calculation shows that because of (1.24) if the A-derivative of the metric p-tensor is zero, that is if :

$$g_{ab,\alpha} A^{\alpha} = 0$$

(1.27)

then in a K-system this reduces to the conditions (1.2) and (1.3). Bergmann has shown that the condition for (1.27) to hold is that the

A-vector must be a Killing vector of the metric. This is an important fact and will be used to great effect in the next chapter.

Summarising then, Kaluza-Klein theory is built up by assuming that spacetime is five dimensional and that its metric tensor has a unit Killing vector. In a special frame of reference in which the first four co-ordinates of the extended spacetime are identified with the co-ordinates of ordinary four dimensional spacetime this Killing vector has the form specified in equations (1.24a) and (1.24b), which is to say that we identify the first four components of the Killing vector with the components of the electromagnetic four-vector. Treating the metric for the extended spacetime as the potential for five dimensional gravity in the same way that General Relativity does in four dimensions permits description of gravity and electromagnetism in the standard way. The five dimensional theory in a special frame of reference is covariant with respect to general co-ordinate transformations in the four dimensions and with respect to gauge transformations. The existence of the unit Killing vector maintains absolute covariance with respect to arbitrary transformations in the five dimensions.

Fibre Bundles

In the preceding discussion classical tensor calculus was used. Nowadays the preference is to formulate theories in a co-ordinate independent manner. As well as making many calculations simpler this carries the additional advantage of conceptual clarity. One branch of modern differential geometry which is of particular relevance to us is the theory of fibre bundles.

A fibre bundle is a manifold which can locally be expressed as a product space B x F where B and F are manifolds. B is known as the base space and F is called the typical fibre. If m is the dimension of the base space and n is the dimension of the fibre then the bundle has dimension m+n. A projection π is defined which maps the bundle onto the base space by mapping all the points on a fibre to the associated point on the base space. We can think of the action of the projection π as shrinking each fibre to a point. Conversely, if $x \in B$ then $\pi^{-1}(x)$ is the fibre over x and this can be thought of as a light shining through a pinprick, the pinprick being the point of the base manifold and the light ray being the fibre over that point.

Now consider a point x which is in two different overlapping regions U_i and U_j of the base space B, and consider the co-ordinate maps which maps points on a fibre onto a co-ordinate system. The mapping $\phi_i \circ \phi_j^{-1}$ maps F to F and it is a requirement that the transition functions, defined by

$$\phi_{ij} = \phi_i \circ \phi_j^{-1}$$

belong to a group G which is called the structure group of the bundle. When the typical fibre is a vector space and the structure group is a Lie group, so that each fibre is isomorphic to each other fibre, we have what is called a principal fibre bundle.

A cross section is a curve in the bundle which is nowhere parallel to a fibre, so when we have a principal fibre bundle a cross section defines a vector at each point on the base space, that is a vector field on the base space.

It is clear then that Kaluza-Klein theory can be described using fibre bundles, the base space being spacetime and the fibre space describing the additional dimensions. This approach can be particularly useful when Kaluza-Klein theory is generalised from the U(1) group to some other non-Abelian group. (Witten 1981). Where fibre bundle theory comes into its own is in conceptual understanding and the reader is referred to the paper by Benn et alia (Benn, Dereli and Tucker, 1980) as an example of the strength of this approach. However, it must be stressed that there is no standard way to define a metric on a fibre bundle and it is this procedure, or equivalently the postulated existence of a unit Killing vector field, which elevates Kaluza-Klein theory to the level of a unified field theory. We shall not use fibre bundles in our work because having chosen to generalise Kaluza-Klein theory for the case where the internal space is to describe the gauge group SU(2) x U(1) our conceptual faculties are temporarily suspended and we must use co-ordinate dependant geometry to obtain answers. Nevertheless it should be pointed out that fibre bundle theory is a valuable aid to the theoretical physicist particularly when it comes to general geometric considerations.

CHAPTER 2

GENERALISED KALUZA-KLEIN THEORY

Introduction.

The generalisation of Kaluza-Klein theory to 4+n dimensions has already been studied in the literature. (Kerner 1968, Cho 1975, Chang et al 1976, Mecklenburg 1980.) There are a number of ways of generalising the theory, all of which are roughly equivalent. Initially, it was considered sufficient merely to generalise the metric (1.1) and to impose conditions which guaranteed that the curvature scalar decomposed into the required additive combinations of the scalar curvature in four dimensions plus the Yang-Mills field tensor squared plus a term describing the scalar curvature in the internal space. Subsequently, the theory was described from a more geometric point of view which gave clearer insight into the theory and to the nature of the conditions which have to be imposed to afford simplification. It is with the benefit of hindsight that we can now pick and choose between these methods to suit our specific requirements. As mentioned previously we shall not use fibre bundles so our method will be more akin to the historically earlier treatments. It is worth pointing out that no ad hoc conditions have to be used : all the simplification techniques are rigourously deriveable from the hypothesis that certain Killing vectors exist.

The most powerful theory is the one which makes the most predictions from the fewest hypotheses. It is important then to be clear as to what are our initial assumptions and what the logical consequences of these are. The only assumption which has to made is that there exists a unit Killing vector of the metric of the extended spacetime, and that when we are in a special reference frame then the first four components of this Killing vector are identified with the Yang-Mills gauge fields. With this assumption, the special form of the metric occurs naturally. The authors who employ the more geometrically oriented arguments find that this special form of the metric occurs if a horizontal lift basis is defined on the bundle manifold and that the metric on this basis is diagonal.

The method we shall use to calculate the scalar curvature of the extended spacetime will be Cartan's structure equations. This has the advantage of being slightly more efficient than classical tensor calculus, since many terms drop out of before they actually have to be calculated. It has the further advantage that some parts of the calculation come in useful in the later chapter on the Dirac equation. This method has been used by Thirring in considering five dimensional Kaluza-Klein theory. (Thirring 1972)

We shall firstly discuss the generalisation of Kaluza-Klein theory from five dimensions to 4+n dimensions to describe a non-Abelian group, and then in greater detail we shall present the specific calculations for the group SU(2) x U(1). The reason for this choice is that this is the symmetry group of the Weinberg-Salam model, which has recieved much attention recently, and which, at going to press, has been well vindicated by the discovery of the intermediate vector bosons.

Kaluza-Klein Theory for a non-Abelian Group.

A group is well described by its Lie algebra (except for some global topological aspects). If we take a linearly independent set of left invariant vector fields $\vec{V}_s$ as a basis for the Lie algebra then the vectors satisfy the following relation :

$$[\vec{V}_s , \vec{V}_t] = c_{st}{}^{v} \vec{V}_v \qquad (2.1)$$

The c's are the components of a tensor, known as the structure tensor. They are also sometimes called the structure constants of the group.

The crucial point in what follows is that we can chose as a basis for the group manifold G either the left-invariant vectors in (2.1) which therefore form a non-co-ordinate basis, or alternatively another co-ordinate basis. We shall put co-ordinates x^i on the manifold G and so we shall use a co-ordinate basis. In this co-ordinate basis each left invariant vector field can be written as :

$$\vec{V}_s = K^{i}{}_{s} \vec{\partial}_i \qquad (2.2)$$

where $K^{i}{}_{s}$ are the components of the vector in the basis $\vec{\partial}_i$. Note that the subscript s here does not refer to components of the vector: it is the label appertaining to a particular vector. If the same vector $\vec{V}_s$ had been expressed in the non-co-ordinate basis it would have components $\delta^{t}{}_{s}$ thus:

$$\vec{V}_s = \delta^{t}{}_{s} \vec{V}_t \qquad (2.3) .$$

The K^{i}_{s} are written suggestively in this form because later we shall make the hypothesis that they are Killing vectors of the metric.

In Yang-Mills theory a set of gauge fields are introduced which act as a representation of the group G. In our language, having already described the group by its basis vectors and its Lie algebra, this amounts to introducing a vector $\vec{A}_{\mu}$ (the index μ refers to the spacetime component) which has components A^{s}_{μ} in the non-co-ordinate basis of the internal space, or A^{i}_{μ} in the co-ordinate basis.

We can now write down the generalisation of the metric (1.1) of the 4+n dimensional spacetime, again assuming that we are in a K frame of reference and that the basis vectors both for the internal space and for spacetime are co-ordinate based :

$$\gamma_{\alpha\beta} = \left(\begin{array}{c|c} g_{\mu\nu} + g_{ij} A^{i}_{\mu} A^{j}_{\nu} & g_{ij} A^{i}_{\mu} \\ \hline g_{ij} A^{j}_{\nu} & g_{ij} \end{array} \right) \qquad (2.4)$$

where α, β indices range from 1 to 4+n, μ, ν range over spacetime, that is from 1 to 4, and i, j range over the internal space that is from 4 to 4+n. $g_{\mu\nu}$ is the metric of spacetime and g_{ij} is the metric of the internal space. Notice that at this point we have not specified the exact form of the metric tensor of the internal space nor shall we do so at a later stage. Some authors specify that the internal metric has the Cartan-Killing form but all that we are saying is that there exists a metric in the internal space.

It is not difficult to show that the metric (2.4) has the same form in a non-co-ordinate basis, thus:

$\vec{\partial}_i$ = co-ordinate basis vectors

dx^j = co-ordinate based 1-forms, such that :

$$\langle \vec{\partial}_j , dx^i \rangle = \delta^i_{\cdot j} \qquad (2.5)$$

$\vec{V}_s$ = non-co-ordinate basis vectors :

$$\vec{V}_s = K^i_{\cdot s} \vec{\partial}_i \qquad (2.6)$$

ω^t = non-co-ordinate based 1-forms such that :

$$\langle \vec{V}_s , \omega^t \rangle = \delta^t_{\cdot s} \qquad (2.7)$$

Now let $\omega^t = K^t_{\cdot j} dx^j \qquad (2.8)$

then
$$\begin{aligned} \langle \vec{V}_s , \omega^t \rangle &= \langle K^i_{\cdot s} \vec{\partial}_i , K^t_{\cdot j} dx^j \rangle \\ &= K^i_{\cdot s} K^t_{\cdot j} \langle \vec{\partial}_i , dx^j \rangle \\ &= K^i_{\cdot s} K^t_{\cdot j} \delta^j_{\cdot i} \\ &= K^i_{\cdot s} K^t_{\cdot i} \end{aligned} \qquad (2.9)$$

and comparison with (2.7) shows formally that the $K^t_{\cdot j}$ are the inverse of $K^i_{\cdot s}$.

Since the metric tensor has components given by (Schutz 1980) :

$$g_{ij} = g\,(\vec{\partial}_i , \vec{\partial}_j)$$

then the components of the metric in the non-co-ordinate basis are:

$$g_{st} = g\,(\vec{V}_s , \vec{V}_t) = K^i_s K^j_t \, g_{ij} \qquad (2.10)$$

Now since the vector $\vec{A}_\mu$ has components A^i_μ in the co-ordinate basis and A^s_μ in the non-co-ordinate basis, that is :

$$\vec{A}_\mu = A^i_\mu \, \partial_i = A^s_\mu \, \vec{V}_s \qquad (2.11)$$

and since: $\vec{V}_s = K^i_s \, \vec{\partial}_i$

then $A^s_\mu K^i_s = A^i_\mu \qquad (2.12)$

So

$$g_{ij} A^i_\mu A^j_\nu = g_{st} A^s_\mu A^t_\nu \qquad (2.13)$$

So the metric is :

$$\gamma_{\alpha\beta} = \left(\begin{array}{c|c} g_{\mu\nu} + g_{st} A^s_\mu A^t_\nu & g_{st} A^s_\mu \\ \hline g_{st} A^t_\nu & g_{st} \end{array} \right) \qquad (2.14)$$

which has the same form as (2.4), but the basis vectors of the internal space do not form a co-ordinate basis, an important difference. Since we will be using the mathematical language of forms, which involves extensive use of the exterior derivative we shall use the co-ordinate based metric given by equation (2.4).

We now come to our main hypothesis, which is that the set of left invariant vectors fields $\vec{V}_s$ are Killing vector fields of the metric. A Killing vector field is a vector field such that the Lie derivative of the metric with respect to this vector field is zero :

$$\mathcal{L}_{\vec{V}} \gamma = 0 \qquad (2.15)$$

which in component notation is :

$$V^{\alpha} \partial_{\alpha} \gamma_{\beta\gamma} + \gamma_{\alpha\gamma} \partial_{\beta} V^{\alpha} + \gamma_{\alpha\beta} \partial_{\gamma} V^{\alpha} = 0 \qquad (2.16)$$

(For a fuller discussion of the concept of the Lie derivative and a proof of the last equation the reader is referred to Appendix Two.)

Now the basis vectors $\vec{V}_s$ of the group have no spacetime components, so that in equation (2.16) the 4+n components of V^{α} are :

$$V^{\alpha} = (0, 0, 0, 0, 0, K^{i}_{s}) \qquad (2.17)$$

and K^{i}_{s} is independent of spacetime.

In equation (2.16) the β and γ indices are free, so we shall in turn put them equal to :

a) $\beta = \mu \, , \; \gamma = \nu$ which gives :

$$K^i_s \partial_i \gamma_{\mu\nu} + \gamma_{i\nu} \partial_\mu K^i_s + \gamma_{i\mu} \partial_\nu K^i_s = 0 \qquad (2.18)$$

$$\therefore \quad K^i_s \partial_i \{ g_{jk} A^j_\mu A^k_\nu \} = 0 \qquad (2.19)$$

b) $\beta = \mu$, $\gamma = k$ which gives :

$$K^i_s \partial_i \gamma_{\mu k} + \gamma_{ik} \partial_\mu K^i_s + \gamma_{i\mu} \partial_k K^i_s = 0 \qquad (2.20)$$

$$\therefore \quad K^i_s \partial_i \{ g_{jk} A^j_\mu \} + g_{ij} A^j_\mu \partial_k K^i_s = 0 \qquad (2.21)$$

c) $\beta = j$, $\gamma = k$ which gives :

$$K^i_s \partial_i \gamma_{jk} + \gamma_{ik} \partial_j K^i_s + \gamma_{ij} \partial_k K^i_s = 0 \qquad (2.22)$$

$$\therefore \quad K^i_s \partial_i g_{jk} + g_{ik} \partial_j K^i_s + g_{ij} \partial_k K^i_s = 0 \qquad (2.23)$$

and these equations are satisfied if

$$\partial_i e^J_\mu = 0 \qquad \text{where} \qquad e^J_\mu = e^J_i A^i_\mu \qquad (2.24)$$

and

$$K^i_s \partial_i e^K_k + e^K_i \partial_k K^i_s = 0 \qquad (2.25)$$

where the e^{I}_{i} are the "internal vierbiens" given by:

$$g_{ij} = n_{IJ}\, e^{I}_{i}\, e^{J}_{j} \qquad (2.26)$$

with n_{IJ} = the n dimensional Euclidean metric.

Another set of internal vierbiens relative to the non-co-ordinate based metric tensor can be defined by :

$$g_{st} = e^{I}_{s}\, e^{J}_{t}\, n_{IJ}$$

and because of equation (2.10) :

$$e^{I}_{s} = e^{I}_{i}\, K^{i}_{s}$$

We shall return to the internal vierbiens in a later section when we perform our explicit calculation.

The case for the non-Abelian group being explicitly SU(2) x U(1) can now be considered.

Eight Dimensional Kaluza-Klein Theory for SU(2) x U(1).

The use of SU(2) x U(1) as the non-Abelian group is motivated by the success of the Weinberg-Salam model. It is also interesting from our point of view in that it is the explicit product of the U(1) group with a non-Abelian group. This means that we can use the U(1) part as a bookeeping device to keep checks on the results, since we already know the U(1) results from chapter 1.

The first step is to consider the form of the internal metric. This in turn requires consideration of the Lie algebra of SU(2) x U(1). There are four left invariant vector fields which we shall call $\vec{V_5}$ and $\vec{V_s}$ and they satisfy the following commutation relations :

$$[\vec{V_5}, \vec{V_5}] = 0$$

$$[\vec{V_5}, \vec{V_s}] = 0$$

$$[\vec{V_s}, \vec{V_t}] = C_{st}{}^{v} \vec{V_v}$$

Putting a basis on the group space with co-ordinates x^5 and x^i (i =1,2,3) means that the Lie algebra can be expressed thus :

$$\vec{V_5} = K^5_5 \vec{\partial_5} + K^i_5 \vec{\partial_i}$$

$$\vec{V_s} = K^5_s \vec{\partial_5} + K^i_s \vec{\partial_i}$$

so that the commutation relations imply that :

$$K^5_s = K^i_5 = 0$$

K^5_5 = a constant which we shall put equal equal to 1 for later convenience, and

$$K^i_s \partial_i K^j_t - K^i_t \partial_i K^j_s = C_{st}{}^v K^j_v \qquad (2.27)$$

The left invariant vector fields are then :

$$\vec{V}_5 = \vec{\partial}_5$$

$$\vec{V}_s = K^i_s \vec{\partial}_i$$

So if we assume that the metric for the internal space, in the co-ordinate basis is :

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g_{ij} \end{array}\right)$$

then it has the same form in the non-co-ordinate basis :

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g_{st} \end{array}\right)$$

This is why K^5_5 was chosen to equal 1.

Finally, in the expression (2.4) for the general non-Abelian metric, such terms as

$$g_{ij}\, A^i_\mu\, A^j_\nu$$

now become

$$g_{55}\, A^5_\mu\, A^5_\nu + g_{5j}\, A^5_\mu\, A^j_\nu + g_{i5}\, A^i_\mu\, A^5_\nu + g_{ij}\, A^i_\mu\, A^j_\nu$$

and writing A^5_μ simply as A_μ and A^i_μ as W^i_μ and remembering that g_{5j} = 0 gives us the form of the metric as :

$$\gamma_{\alpha\beta} = \left(\begin{array}{c|c|c} g_{\mu\nu} + A_\mu A_\nu + g_{ij}\, W^i_\mu W^j_\nu & A_\mu & g_{ij} W^i_\mu \\ \hline A_\nu & 1 & 0 \\ \hline g_{ij}\, W^j_\nu & 0 & g_{ij} \end{array}\right)$$

(2.28)

where the indices α, β range from 1 to 8, that is over the entire space; μ, ν range from 1 to 4 and refer to spacetime and i,j take values 6,7 or 8 and refer to the group space for SU(2).

The determinant of the above matrix is :

$$\det \gamma_{\alpha\beta} = \det (g_{\mu\nu}).\det (g_{ij})$$

and its inverse is :

$$
\gamma^{\alpha\beta} = \left(\begin{array}{c:c:c} g^{\mu\nu} & -g^{\mu\rho}A_\rho & -g^{\mu\rho}W^j_\rho \\ \hdashline -g^{\sigma\nu}A_\sigma & 1+g^{\sigma\rho}A_\sigma A_\rho & g^{\sigma\rho}A_\sigma W^j_\rho \\ \hdashline -g^{\sigma\nu}W^i_\sigma & g^{\sigma\rho}A_\rho W^i_\sigma & g^{ij}+g^{\sigma\rho}W^i_\sigma W^j_\rho \end{array} \right)
$$

(2.29)

where $g^{\mu\nu}$ and g^{ij} are the inverses, respectively, of $g_{\mu\nu}$ and g_{ij} .

In the five dimensional Kaluza-Klein theory the Killing vectors were:

$$A^\alpha = (0,0,0,0,1) \qquad\qquad A_\alpha = (A_\mu, 1)$$

and in extending the theory to eight dimensions we make the hypothesis that the Killing vectors are :

$$A^\alpha = (0,0,0,0,1,0,0,0)$$

$$W^\alpha = (0,0,0,0,0,K^i_s)$$

$$A_\alpha = (A_\mu, 1, 0, 0, 0)$$

$$W_\alpha = (g_{ij}K^i_s W^j_\nu, 0, g_{ij}K^i_s)$$

Substituting these into Killing's equation (2.16) gives the following conditions :

$$\partial_5 \gamma_{\alpha\beta} = 0 \qquad \text{and} \qquad \partial_5 K^i_5 = 0 \qquad (2.30a)$$

$$\partial_i g_{\mu\nu} = 0 \qquad \text{and} \qquad \partial_i A_\mu = 0 \qquad (2.30b)$$

$$\partial_\mu K^i_5 = 0 \qquad (2.30c)$$

$$\partial_i e^J_\mu = 0 \qquad (2.30d)$$

$$K^i_5 \partial_i e^K_k + e^K_i \partial_k K^i_5 = 0 \qquad (2.30e)$$

where

$$e^J_\mu = e^J_j W^j_\mu$$

and the e^J_j are as in equation (2.26) except that indices now take values 6,7 or 8 for the group space of SU(2).

There is one other condition which can be derived from the existence of the above Killing vectors. This condition will not be needed until the end of this chapter at which point it will be revealed.

We can now perform our main task which is to calculate the curvature of the extended space.

The Curvature of the Eight Dimensional Space.

The curvature of this eight dimensional spacetime is now about to become the focus of our attention. The calculation of the curvature can be carried out in the same way as before using Riemann calculus. This is the method employed firstly by Kerner and developed by Cho. This method is perfectly feasible but is complicated by the use of non-co-ordinate bases so that the connection coefficients are no longer given by Christoffels formula. This is the reason for the slight difference in the results of these two authors. An alternative method is to use Cartan's structure equations (O'Neill 1966) :

$$d\omega^A + \omega^A{}_B \wedge \omega^B = 0 \qquad (2.31)$$

and

$$R^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B \qquad (2.32)$$

where d denotes the exterior differentiation operator (Schutz 1980) and $\wedge$ denotes the wedge product. The ω^A are the one-forms related to the co-ordinate based one-forms dx^α by :

$$\omega^A = e^A{}_\alpha \, dx^\alpha \qquad (2.33)$$

where the e^A_α are the vierbien.

This method has been used by Thirring to calculate the scalar curvature for five dimensional Kaluza-Klein theory.

The first stage in the calculation is to calculate the vielbiens from the metric. They are defined by the relation :

$$\gamma_{\alpha\beta} = e^{A}_{\alpha}\, e^{B}_{\beta}\, \eta_{AB} \qquad (2.34)$$

where η_{AB} = diag (-1,1,1,1,1,1,1,1). In the above relation A and B are tangent space indices which take values from 1 to 8. They will be decomposed into tangent space indices for spacetime a,b, tangent space index for the fifth dimension, 5, and tangent space indices for the internal space I,J. This index convention is summarised in table 2.1. It can be seen from the table that we shall be using the same index, 5, to refer to the fifth world space dimension and to the fifth tangent space dimension. However, no ambiguity is incurred, as the reader will verify by reading on, since the type of index is indicated by position.

To find the vielbiens it is necessary to solve equations like :

$$\gamma_{\mu\nu} = e^{a}_{\mu}\, e^{b}_{\nu}\, \eta_{ab} + e^{5}_{\mu}\, e^{5}_{\nu}\, \eta_{55} + e^{I}_{\mu}\, e^{J}_{\nu}\, \eta_{IJ} \qquad (2.35)$$

and $$\gamma_{\mu 5} = e^{a}_{\mu}\, e^{b}_{5}\, \eta_{ab} + e^{5}_{\mu}\, e^{5}_{5}\, \eta_{55} + e^{I}_{\mu}\, e^{J}_{5}\, \eta_{IJ} \qquad (2.36)$$

and $$\gamma_{\mu i} = e^{a}_{\mu}\, e^{b}_{i}\, \eta_{ab} + e^{5}_{\mu}\, e^{5}_{i}\, \eta_{55} + e^{I}_{\mu}\, e^{J}_{i}\, \eta_{IJ} \qquad (2.37)$$

where η_{ab} = diag(-1,1,1,1), η_{55} = 1, η_{IJ} = diag(1,1,1).

It is not difficult to show that the vielbeins are given by :

$e_\mu^{\ a}$ = 4-d vierbien, that is $g_{\mu\nu} = e_\mu^{\ a} e_\nu^{\ b} \eta_{ab}$ (2.38)

$e_i^{\ I}$ = internal vierbein, that is $g_{ij} = e_i^{\ I} e_j^{\ J} \eta_{IJ}$ (2.39)

$$e_\mu^{\ 5} = A_\mu \qquad (2.40)$$

$$e_\mu^{\ I} = e_i^{\ I} W_\mu^{\ i} \qquad (2.41)$$

$$e_5^{\ 5} = 1 \qquad (2.42)$$

and all the others are zero :

$$e_5^{\ a} = e_5^{\ I} = e_i^{\ a} = e_i^{\ 5} = 0 . \qquad (2.43)$$

It is also possible to calculate the inverse vielbiens from the inverse metric tensor by :

$$\gamma^{\alpha\beta} = E^{\alpha}_{A} E^{\beta}_{B} \eta^{AB} \qquad (2.44)$$

and their values are :

$$E^{\mu}_{a} = \text{4-d vielbein, that is } g^{\mu\nu} = E^{\mu}_{a} E^{\nu}_{b} \eta^{ab} \qquad (2.45)$$

$$E^{i}_{I} = \text{internal vierbein, that is } g^{ij} = E^{i}_{I} E^{j}_{J} \eta^{IJ} \qquad (2.46)$$

$$E^{5}_{a} = - E^{\nu}_{a} A_{\nu} \qquad (2.47)$$

$$E^{i}_{a} = - E^{\mu}_{a} W^{i}_{\mu} \qquad (2.48)$$

$$E^{5}_{5} = 1 \qquad (2.49)$$

and all the others are, as before, zero :

$$E^{\mu}_{5} = E^{\mu}_{I} = E^{5}_{I} = E^{i}_{5} = 0 \qquad (2.50)$$

We have written these vielbiens with a capital E to emphasize that they are inverse vierbiens. The following relationships will be useful later on :

$$E^{\mu}_{A} e^{A}_{\nu} = E^{\mu}_{a} e^{a}_{\nu} + E^{\mu}_{5} e^{5}_{\nu} + E^{\mu}_{I} e^{I}_{\nu}$$

$$= E^{\mu}_{a} e^{a}_{\nu} = \delta^{\mu}_{\nu}$$

(2.51)

$$E^{i}_{A} e^{A}_{j} = E^{i}_{a} e^{a}_{j} + E^{i}_{5} e^{5}_{j} + E^{i}_{I} e^{I}_{j}$$

$$= E^{i}_{I} e^{I}_{j} = \delta^{i}_{j}$$

(2.52)

$$e^{a}_{\alpha} E^{\alpha}_{b} = e^{a}_{\mu} E^{\mu}_{b} + e^{a}_{5} E^{5}_{b} + e^{a}_{I} E^{I}_{b}$$

$$= e^{a}_{\mu} E^{\mu}_{b} = \delta^{a}_{b} .$$

(2.53)

$$e^{I}_{\alpha} E^{\alpha}_{J} = e^{I}_{\mu} E^{\mu}_{J} + e^{I}_{5} E^{5}_{J} + e^{I}_{i} E^{i}_{J}$$

$$= e^{I}_{i} E^{i}_{J} = \delta^{I}_{J} .$$

(2.54)

From these vielbiens we can write down the orthonormal basis one-forms. They are :

$$\omega^{a} = e^{a}_{\mu} dx^{\mu}$$

(2.55)

$$\omega^5 = A_\mu dx^\mu + dx^5 \tag{2.56}$$

$$\omega^I = e^I_i W^i_\mu dx^\mu + e^I_i dx^i \tag{2.57}$$

The first step in solving Cartan's first structure equation for the $\omega^A{}_B$ is to find the exterior derivative of the basis one-forms. This is quite straightforward :

$$\omega^a = e^a_\mu dx^\mu$$

so
$$d\omega^a = \partial_\nu e^a_\mu dx^\nu \wedge dx^\mu \tag{2.58}$$

since e^a_μ is a four dimensional quantity and so is independent of x^5 and x^i . In four dimensions the following relation is satisfied :

$$d\omega^a + \overline{\omega^a{}_b} \wedge \omega^b = 0 \tag{2.59}$$

where, following Thirring, the bar denotes a purely four dimensional one-form. The reason for this notation will become clear in due course.

Now $$\omega^5 = A_\mu dx^\mu + dx^5$$

so $$d\omega^5 = \partial_\nu A_\mu \, dx^\nu \wedge dx^\mu$$
$$= \tfrac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu)\, dx^\nu \wedge dx^\mu$$
$$\equiv \tfrac{1}{2} f_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

(2.60)

Notice in the above that we have antisymmetrised explicitly in the dummy suffixes. This practice will be used extensively in the calculations which follow. It is necessary to do this since in equating the coefficients of two wedge products, the wedge products, and hence the explicit antisymmetry, will disappear, and so the coefficients themselves must be made explicitly antisymmetric.

Finally
$$\omega^I = e^I_\mu\, dx^\mu + e^I_{\hat i}\, dx^i$$
$$= e^I_{\hat i} W^i_\mu\, dx^\mu + e^I_{\hat i}\, dx^i$$

so $$d\omega^I = \tfrac{1}{2} e^I_{\hat i}(\partial_\nu W^i_\mu - \partial_\mu W^i_\nu)\, dx^\nu \wedge dx^\mu + \tfrac{1}{2}(\partial_j e^I_{\hat i} - \partial_i e^I_{\hat j})\, dx^j \wedge dx^i$$

(2.61)

since $$\partial_\mu e^I_{\hat i} = \partial_j e^I_\mu = \partial_5 e^I_\mu = 0.$$

Having thus found the exterior derivatives of the one-forms we are now in a position to find the connection one-forms from the equation :

$$d\omega^{A} + \omega^{A}{}_{B} \wedge \omega^{B} = 0$$

The usual method is to guess a solution and see what happens, since once a solution is found it is known to be unique. However, in our case, because of the plethora of indices and the rampant antisymmetries we shall use the method outlined in the tome by Misner et al (Misner, Thorne and Wheeler, 1973) on page 358. Once the $d\omega^{A}$ are computed they are arranged in the format :

$$d\omega^{A} = -\tfrac{1}{2}\, C_{BC}{}^{A}\, \omega^{B} \wedge \omega^{C}$$

The ω_{AB} can then be worked out from :

$$\omega_{AB} = \tfrac{1}{2}\left(C_{ABC} + C_{ACB} - C_{BCA} \right) \omega^{C}$$

This is what we shall now do.

$$\begin{aligned} d\omega^{a} &= -\tfrac{1}{2}\, C_{BC}{}^{a}\, \omega^{B} \wedge \omega^{C} \\ &= -\tfrac{1}{2}\, C_{bc}{}^{a}\, \omega^{b} \wedge \omega^{c} - \tfrac{1}{2}\, C_{IJ}{}^{a}\, \omega^{I} \wedge \omega^{J} \\ &\quad - C_{b5}{}^{a}\, \omega^{b} \wedge \omega^{5} - C_{5J}{}^{a}\, \omega^{5} \wedge \omega^{J} \\ &\quad - C_{bJ}{}^{a}\, \omega^{b} \wedge \omega^{J} . \end{aligned}$$

but
$$d\omega^a = \partial_\nu e^a_\mu \, dx^\nu \wedge dx^\mu$$
$$= - \bar{\omega}^a{}_b \wedge \omega^b$$
$$= -\tfrac{1}{2} \overline{C_{bc}{}^a} \, \omega^b \wedge \omega^c$$

where the bar refers to a purely four-dimensional quantity. By inspection then :

$$C_{bc}{}^a = \overline{C_{bc}{}^a}$$

$$C_{IJ}{}^a = 0 \quad ; \quad C_{b5}{}^a = 0$$

$$C_{bJ}{}^a = 0 \quad ; \quad C_{5J}{}^a = 0 \qquad (2.62)$$

The same can be done for ω^5 :

$$d\omega^5 = -\tfrac{1}{2} C_{BC}{}^5 \omega^B \wedge \omega^C$$
$$= -\tfrac{1}{2} C_{ab}{}^5 \omega^a \wedge \omega^b - C_{a5}{}^5 \omega^a \wedge \omega^5$$
$$- C_{aJ}{}^5 \omega^a \wedge \omega^J - C_{5J}{}^5 \omega^5 \wedge \omega^J$$
$$- \tfrac{1}{2} C_{IJ}{}^5 \omega^I \wedge \omega^J$$

But $$d\omega^5 = \tfrac{1}{2} f_{\mu\nu} \, dx^\mu \wedge dx^\nu$$

so $-\frac{1}{2} C_{ab}{}^{5}\, \omega^{a} \wedge \omega^{b} = \frac{1}{2} f_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}$

$$\therefore \quad -\frac{1}{2} C_{ab}{}^{5}\, e_{\mu}^{a}\, e_{\nu}^{b}\, dx^{\mu} \wedge dx^{\nu} = \frac{1}{2} f_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}$$

Therefore

$$C_{ab}{}^{5} = -E^{\mu}_{a} E^{\nu}_{b} f_{\mu\nu} \equiv -f_{ab}. \qquad (2.63)$$

and all the others are zero :

$$C_{a5}{}^{5} = 0 \quad ; \quad C_{aJ}{}^{5} = 0$$

$$C_{5J}{}^{5} = 0 \quad ; \quad C_{IJ}{}^{5} = 0$$

$$(2.64)$$

Finally,

$$d\omega^{I} = \frac{1}{2} e_{i}^{I} f^{i}_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu} + \frac{1}{2}\left(\partial_{i} e_{j}^{I} - \partial_{j} e_{i}^{I}\right) dx^{i} \wedge dx^{j}$$

where $f^{i}_{\mu\nu} = \partial_{\mu} W^{i}_{\nu} - \partial_{\nu} W^{i}_{\mu}$. $\qquad (2.65)$

$$d\omega^{I} = -\frac{1}{2} C_{ab}{}^{I}\, \omega^{a} \wedge \omega^{b} - C_{a5}{}^{I}\, \omega^{a} \wedge \omega^{5}$$
$$- C_{aJ}{}^{I}\, \omega^{a} \wedge \omega^{J} - C_{5J}{}^{I}\, \omega^{5} \wedge \omega^{J}$$
$$-\frac{1}{2} C_{JK}{}^{I}\, \omega^{J} \wedge \omega^{K}.$$

In order to solve all these structure constants it is necessary to express the orthonormal basis one-forms in terms of the co-ordinate basis one-forms :

$$d\omega^I = -\frac{1}{2} C_{ab}{}^I e^a_\mu e^b_\nu \, dx^\mu \wedge dx^\nu$$
$$- C_{a5}{}^I e^a_\mu \, dx^\mu \wedge (A_\nu dx^\nu + dx^5)$$
$$- C_{aJ}{}^I e^a_\mu \, dx^\mu \wedge (e^J_\nu dx^\nu + e^J_j dx^j)$$
$$- C_{5J}{}^I (A_\nu dx^\nu + dx^5) \wedge (e^J_\mu dx^\mu + e^J_j dx^j)$$
$$-\frac{1}{2} C_{JK}{}^I (e^J_\mu dx^\mu + e^J_j dx^j) \wedge (e^K_\nu dx^\nu + e^K_k dx^k) .$$

By inspection

$$C_{a5}{}^I = C_{5J}{}^I = 0$$

and

$$\begin{aligned} &-\frac{1}{2} C_{ab}{}^I e^a_\mu e^b_\nu \, dx^\mu \wedge dx^\nu \\ &- C_{aJ}{}^I e^a_\mu e^J_\nu \, dx^\mu \wedge dx^\nu \\ &-\frac{1}{2} C_{JK}{}^I e^J_\mu e^K_\nu \, dx^\mu \wedge dx^\nu \end{aligned} = \frac{1}{2} e^I_i f^i_{\mu\nu} \, dx^\mu \wedge dx^\nu \qquad (2.66)$$

and

$$\begin{aligned} &- C_{aJ}{}^I e^a_\mu e^J_j \, dx^\mu \wedge dx^j \\ &- C_{JK}{}^I e^J_\mu e^K_j \, dx^\mu \wedge dx^j \end{aligned} = 0 \qquad (2.67)$$

and, finally,

$$-\tfrac{1}{2} C_{JK}{}^{I} e^{J}_{j} e^{K}_{k} \, dx^{j} \wedge dx^{k} = \tfrac{1}{2}\left(\partial_{i} e^{I}_{j} - \partial_{j} e^{I}_{i}\right) dx^{i} \wedge dx^{j} \qquad (2.68)$$

Using equations (2.27) and (2.30e) we obtain :

$$\partial_{i} e^{I}_{j} - \partial_{j} e^{I}_{i} = e^{I}_{k} K^{t}_{j} K^{s}_{i} K^{k}_{v} c_{st}{}^{v}$$

and from above

$$- C_{JK}{}^{I} e^{J}_{j} e^{K}_{k} = \partial_{j} e^{I}_{k} - \partial_{k} e^{I}_{j}$$

so

$$\begin{aligned} C_{JK}{}^{I} &= - E^{j}_{J} E^{k}_{K} \left(\partial_{j} e^{I}_{k} - \partial_{k} e^{I}_{j}\right) \\ &= - E^{j}_{J} E^{k}_{K} e^{I}_{i} K^{t}_{k} K^{s}_{j} K^{i}_{v} c_{st}{}^{v} \\ &\equiv - E^{s}_{J} E^{t}_{K} e^{I}_{v} c_{st}{}^{v} \end{aligned} \qquad (2.69)$$

Equation (2.67) is easy to solve :

$$\begin{aligned} C_{aJ}{}^{I} e^{a}_{\mu} e^{J}_{j} &= - C_{KJ}{}^{I} e^{K}_{\mu} e^{J}_{j} \\ \therefore \quad C_{aJ}{}^{I} &= - E^{\mu}_{a} e^{K}_{i} W^{i}_{\mu} C_{KJ}{}^{I} \\ &\equiv - W^{K}_{a} C_{KJ}{}^{I} \\ &= C_{JK}{}^{I} W^{K}_{a} \end{aligned} \qquad (2.70)$$

where W_a^K is shorthand for $E_a^\mu e_i^K w_\mu^i$.

Finally : $$\left.\begin{array}{l} -\frac{1}{2} C_{ab}{}^I e_\mu^a e_\nu^b \\ -\frac{1}{2} C_{aJ}{}^I e_\mu^a e_\nu^J - \frac{1}{2} c_{JK}{}^I e_\mu^J e_\nu^K \end{array}\right\} = \frac{1}{2} e_i^I f_{\mu\nu}^i .$$

so

$$\frac{1}{2} C_{ab}{}^I e_\mu^a e_\nu^b = -\frac{1}{2} e_i^I f_{\mu\nu}^i - C_{JK}^I W_a^K e_\mu^a e_\nu^J - \frac{1}{2} C_{JK}^I e_\mu^J e_\nu^K$$

$$= -\frac{1}{2} e_i^I f_{\mu\nu}^i - c_{JK}^I e_\mu^k e_\nu^J - \frac{1}{2} c_{JK}^I e_\mu^J e_\nu^K$$

$$= -\frac{1}{2} e_i^I f_{\mu\nu}^i - \frac{1}{2} c_{JK}{}^I e_\mu^K e_\nu^J$$

$$= -\frac{1}{2} e_i^I f_{\mu\nu}^i + \frac{1}{2} (E_J^S E_K^t e_v^I c_{st}^v) e_\mu^k e_\nu^J$$

$$= -\frac{1}{2} e_i^I f_{\mu\nu}^i + \frac{1}{2} C_{st}{}^v K_i^t K_j^s e_h^I K_v^k w_\mu^i w_\nu^j$$

$$= -\frac{1}{2} e_i^I f_{\mu\nu}^i - \frac{1}{2} C_{st}{}^v K_k^t K_j^s e_i^I K_v^i w_\mu^j w_\nu^k$$

$$= -\frac{1}{2} e_i^I \left\{ f_{\mu\nu}^i + C_{st}{}^v w_\mu^s w_\nu^t k_v^i \right\}$$

$$= -\frac{1}{2} e_v^I \left\{ f_{\mu\nu}^v + c_{st}^v w_\mu^s w_\nu^t \right\}$$

$$= -\frac{1}{2} e_V^I F_{\mu\nu}^V$$

where $F_{\mu\nu}^V$ is the Yang-Mills field tensor.

So

$$C_{ab}{}^{I} = -E^{\mu}_{a} E^{\nu}_{b} e^{I}_{\nu} F^{\nu}_{\mu\nu} \equiv -F_{ab}{}^{I} \qquad (2.71)$$

To summarise, then, all the structure constants are zero apart from the following :

$$C_{bc}{}^{a} = \overline{C_{bc}{}^{a}} \qquad (2.72)$$

$$C_{aJ}{}^{I} = C_{JK}{}^{I} W^{K}_{a} \qquad (2.73)$$

$$C_{ab}{}^{I} = -F_{ab}{}^{I} \qquad (2.74)$$

$$C_{ab}{}^{5} = -f_{ab} \qquad (2.75)$$

The reader should notice, in passing, that the structure constants are antisymmetric in the bottom two indices and that the topmost index occupies a unique position.

It is a straightforward question of substitution to evaluate the connection one-forms. The results are :

$$\omega_{ab} = \bar{\omega}_{ab} + \frac{1}{2} C_{ab5}\, \omega^5 + \frac{1}{2} C_{abI}\, \omega^I \qquad (2.76)$$

$$\omega_{a5} = - \frac{1}{2} f_{ab}\, \omega^b \qquad (2.77)$$

$$\omega_{aI} = \frac{1}{2} C_{abI}\, \omega^b + \frac{1}{2} (C_{aIJ} + C_{aJI})\, \omega^J \qquad (2.78)$$

$$\omega_{IJ} = \frac{1}{2} (C_{IaJ} - C_{JaI})\, \omega^a + \frac{1}{2} (C_{IJK} + C_{IKJ} - C_{JKI})\, \omega^K \qquad (2.79)$$

and $\omega_{5I} = 0 \quad ; \quad \omega_{55} = 0 \quad .$ (2.80)

A further check could be made by subsituting these connection one-forms into Cartan's first structure equation. The details are not included here.

We can now calculate the scalar curvature of this eight dimensional space. The first step is to calculate the curvature two-form using Cartan's second structure equation :

$$R^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B$$

which is related to the Riemann tensor by :

$$R^A{}_B = \tfrac{1}{2} R^A{}_{BCD} \; \omega^C \wedge \omega^D \qquad (2.81)$$

This enables us to read off the values of the Riemann tensor and the scalar curvature is related to the Riemann tensor by contraction :

$$R = R^{AB}{}_{AB} \qquad (2.82)$$

which because of summation is :

$$\begin{aligned} R &= R^{AB}{}_{AB} \\ &= R^{ab}{}_{ab} + 2R^{a5}{}_{a5} \\ &\quad + 2R^{aI}{}_{aI} + 2R^{5I}{}_{5I} \\ &\quad + R^{IJ}{}_{IJ} \, . \end{aligned} \qquad (2.83)$$

The calculation is quite long but not too difficult :

$$\begin{aligned} R^a{}_b &= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \\ &\quad + \omega^a{}_5 \wedge \omega^5{}_b + \omega^a{}_I \wedge \omega^I{}_b \end{aligned}$$

Now

$$d(\omega^a{}_b) = d(\bar{\omega}^a{}_b + \tfrac{1}{2} C^a{}_{b5} \omega^5 + \tfrac{1}{2} C^a{}_{bI} \omega^I)$$

$$= d\bar{\omega}^a{}_b + \tfrac{1}{2} c^a{}_{b5,d}\, \omega^d \wedge \omega^5 + \tfrac{1}{2} c^a{}_{b5}\, d\omega^5$$

$$+ \tfrac{1}{2} c^a{}_{bI,d}\, \omega^d \wedge \omega^I + \tfrac{1}{2} c^a{}_{bI}\, d\omega^I$$

where we have used

$$d(f\omega^A) = \partial_\beta f\, dx^\beta \wedge \omega^A + f\, d\omega^A$$

and

$$\partial_5 (c^a{}_{bI}) = \partial_j (c^a{}_{bI}) = 0.$$

Now

$$\omega^a{}_c \wedge \omega^c{}_b = (\bar{\omega}^a{}_c + \tfrac{1}{2} c^a{}_{c5}\, \omega^5 + \tfrac{1}{2} c^a{}_{cI}\, \omega^I)$$

$$\wedge (\bar{\omega}^c{}_b + \tfrac{1}{2} c^c{}_{b5}\, \omega^5 + \tfrac{1}{2} c^c{}_{bJ}\, \omega^J)$$

and

$$\omega^a{}_5 \wedge \omega^5{}_b = (\tfrac{1}{2} c^a{}_{d5}\, \omega^d) \wedge (-\tfrac{1}{2} c_{bc5}\, \omega^c)$$

and

$$\omega^a{}_I \wedge \omega^I{}_b = (\tfrac{1}{2} c^a{}_{dI}\, \omega^d + \tfrac{1}{2}(c^a{}_{IJ} + c^a{}_{JI})\, \omega^J)$$

$$\wedge (-\tfrac{1}{2} c_{be}{}^I \omega^e - \tfrac{1}{2}(c_b{}^I{}_K + c_{bK}{}^I)\, \omega^K)$$

So adding all these terms gives :

$$R^a{}_b = d\bar{\omega}^a{}_b + \tfrac{1}{2} c^a{}_{b5,d}\, \omega^d \wedge \omega^5$$

$$+ \tfrac{1}{2} c^a{}_{bI,d}\, \omega^d \wedge \omega^I$$

$$+ \tfrac{1}{2} c^a{}_{b5} (\tfrac{1}{2} f_{de}\, \omega^d \wedge \omega^e)$$

$$
\begin{aligned}
+\tfrac{1}{2}\, C^a{}_{bI} \Big\{ & -\tfrac{1}{2}\, C_{de}{}^I\, \omega^d \wedge \omega^e \\
& - C_{JK}{}^I\, W^K_d\, \omega^d \wedge \omega^J \\
& -\tfrac{1}{2}\, C_{JK}{}^I\, \omega^J \wedge \omega^K \Big\}
\end{aligned}
$$

$$
+ \Big\{ \bar{\omega}^a{}_c + \tfrac{1}{2}\, C^a{}_{c5}\, \omega^5 + \tfrac{1}{2}\, C^a{}_{cI}\, \omega^I \Big\}
$$

$$
\wedge \Big\{ \bar{\omega}^c{}_b + \tfrac{1}{2}\, C^c{}_{b5}\, \omega^5 + \tfrac{1}{2}\, C^c{}_{bJ}\, \omega^J \Big\}
$$

$$
+ \left(\tfrac{1}{2}\, C^a{}_{d5}\, \omega^d \right) \wedge \left(-\tfrac{1}{2}\, C_{bc5}\, \omega^c \right)
$$

$$
+ \Big\{ \tfrac{1}{2}\, C^a{}_{dI}\, \omega^d + \tfrac{1}{2} \left(C^a{}_{IJ} + C^a{}_{JI} \right) \omega^J \Big\}
$$

$$
\wedge \Big\{ -\tfrac{1}{2}\, C_{be}{}^I\, \omega^e - \tfrac{1}{2} \left(C_b{}^I{}_K + C_{bK}{}^I \right) \omega^K \Big\}
$$

But

$$
\begin{aligned}
R^a{}_b = {} & \tfrac{1}{2}\, R^a{}_{bcd}\, \omega^c \wedge \omega^d \\
& + R^a{}_{bc5}\, \omega^c \wedge \omega^5 + R^a{}_{bcI}\, \omega^c \wedge \omega^I \\
& + R^a{}_{b5I}\, \omega^5 \wedge \omega^I + \tfrac{1}{2}\, R^a{}_{bIJ}\, \omega^I \wedge \omega^J
\end{aligned}
$$

So the only term in which we are interested is $R^a{}_{bcd}$ and by inspection we can say that

$$
\tfrac{1}{2}\, R^a{}_{bcd}\, \omega^c \wedge \omega^d = d\bar{\omega}^a{}_b + \tfrac{1}{2}\, C^a{}_{b5} \left(\tfrac{1}{2} f_{de}\, \omega^d \wedge \omega^e \right)
$$

$$+\tfrac{1}{2} C^a{}_{bI} \left\{ -\tfrac{1}{2} C_{de}{}^I \omega^d \wedge \omega^e \right\}$$

$$+ \bar{\omega}^a{}_c \wedge \bar{\omega}^c{}_b$$

$$+\left(\tfrac{1}{2} C^a{}_{d5}\, \omega^d\right) \wedge \left(-\tfrac{1}{2} C_{bc5}\, \omega^c\right)$$

$$+ \tfrac{1}{2} C^a{}_{dI}\, \omega^d \wedge \left(-\tfrac{1}{2} C_{be}{}^I \omega^e\right) .$$

Relabelling dummy suffixes :

$$\tfrac{1}{2} R^a{}_{bcd}\, \omega^c \wedge \omega^d = d\bar{\omega}^a{}_b + \bar{\omega}^a{}_c \wedge \bar{\omega}^c{}_b$$

$$+\tfrac{1}{4} C^a{}_{b5} \cdot f_{cd}\, \omega^c \wedge \omega^d - \tfrac{1}{4} C^a{}_{bI} C_{cd}{}^I \omega^c \wedge \omega^d$$

$$+\tfrac{1}{4} C^a{}_{d5} C_{bc5}\, \omega^c \wedge \omega^d + \tfrac{1}{4} C^a{}_{dI} C_{bc}{}^I \omega^c \wedge \omega^d$$

$$= d\bar{\omega}^a{}_b + \bar{\omega}^a{}_c \wedge \bar{\omega}^c{}_b + \tfrac{1}{4} C^a{}_{b5} f_{cd}\, \omega^c \wedge \omega^d$$

$$+ \tfrac{1}{8} \left(C^a{}_{d5} C_{bc5} - C^a{}_{c5} C_{bd5} \right) \omega^c \wedge \omega^d$$

$$-\tfrac{1}{4} \left(C^a{}_{bI} C_{cd}{}^I \right) \omega^c \wedge \omega^d$$

$$+\tfrac{1}{8} \left(C^a{}_{dI} C_{bc}{}^I - C^a{}_{cI} C_{bd}{}^I \right) \omega^c \wedge \omega^d$$

where we have made the coefficients explicitly antisymmetric.

Therefore :

$$\tfrac{1}{2} R^a{}_{bcd} = d\bar{\omega}^a{}_b + \bar{\omega}^a{}_c \wedge \bar{\omega}^c{}_b$$

$$-\frac{1}{4} f^a{}_b f_{cd} + \frac{1}{8}(f^a{}_d f_{bc} - f^a{}_c f_{bd})$$

$$-\frac{1}{4} F^a{}_{bI} F_{cd}{}^I$$

$$+\frac{1}{8}(F^a{}_{dI} F_{bc}{}^I - F^a{}_{cI} F_{bd}{}^I)$$

that is

$$R^a{}_{bcd} = \overline{R^a{}_{bcd}} - \frac{1}{2} f^a{}_b f_{cd} + \frac{1}{4}(f^a{}_d f_{bc} - f^a{}_c f_{bd})$$
$$-\frac{1}{2} F^a{}_{bI} F_{cd}{}^I + \frac{1}{4}(F^a{}_{dI} F_{bc}{}^I - F^a{}_{cI} F_{bd}{}^I)$$

So $R^{ab}{}_{ab} = R^{(4)} - \frac{1}{2} f^{ab} f_{ab} - \frac{1}{2} F^{ab}{}_I F_{ab}{}^I$

$$+\frac{1}{4}(f^a{}_b f^b{}_a - f^a{}_a f^b{}_b) + \frac{1}{4}(F^a{}_{bI} F^b{}_a{}^I - F^a{}_{aI} F^b{}_b{}^I)$$

and since

$$f^a{}_a = F^a{}_{aI} = 0$$

then

$$R^{ab}{}_{ab} = R^{(4)} - \frac{3}{4} f^{ab} f_{ab} - \frac{3}{4} F^{abI} F_{abI} \quad (2.84)$$

Also

$$R^a{}_5 = d\omega^a{}_5 + \omega^a{}_b \wedge \omega^b{}_5$$
$$+ \omega^a{}_5 \wedge \omega^5{}_5 + \omega^a{}_I \wedge \omega^I{}_5$$

$$= d\left(\tfrac{1}{2} C^{a}{}_{b5}\, \omega^{b}\right)$$

$$+ \left\{ \bar{\omega}^{a}{}_{b} + \tfrac{1}{2} C^{a}{}_{b5}\, \omega^{5} + \tfrac{1}{2} C^{a}{}_{bI}\, \omega^{I} \right\} \wedge \tfrac{1}{2} C^{b}{}_{c5}\, \omega^{c}$$

$$= \tfrac{1}{2} C^{a}{}_{b5,d}\, \omega^{d} \wedge \omega^{b}$$

$$+ \tfrac{1}{2} C^{a}{}_{b5} \left(-\tfrac{1}{2} \overline{C_{cd}{}^{b}}\, \omega^{c} \wedge \omega^{d} \right)$$

$$+ \left\{ \bar{\omega}^{a}{}_{b} + \tfrac{1}{2} C^{a}{}_{b5}\, \omega^{5} + \tfrac{1}{2} C^{a}{}_{bI}\, \omega^{I} \right\}$$

$$\wedge \left\{ \tfrac{1}{2} C^{b}{}_{c5}\, \omega^{c} \right\}$$

But

$$R^{a}{}_{5} = \tfrac{1}{2} R^{a}{}_{5cd}\, \omega^{c} \wedge \omega^{d}$$

$$+ R^{a}{}_{5c5}\, \omega^{c} \wedge \omega^{5} + R^{a}{}_{5cI}\, \omega^{c} \wedge \omega^{I}$$

$$+ R^{a}{}_{55I}\, \omega^{5} \wedge \omega^{I} + \tfrac{1}{2} R^{a}{}_{5IJ}\, \omega^{I} \wedge \omega^{J}$$

Therefore, by inspection :

$$R^{a}{}_{5c5} = -\tfrac{1}{4} C^{a}{}_{b5}\, C^{b}{}_{c5}$$

Therefore :

$$R^{a5}{}_{a5} = + \tfrac{1}{4} f^{a}{}_{b} f_{a}{}^{b} \qquad (2.85)$$

Now $R^a{}_I = d\omega^a{}_I + \omega^a{}_b \wedge \omega^b{}_I + \omega^a{}_J \wedge \omega^J{}_I$

$$= \tfrac{1}{2} c^a{}_{bI,d}\, \omega^d \wedge \omega^b + \tfrac{1}{2}(c^a{}_{IJ} + c^a{}_{JI})_{,d}\, \omega^d \wedge \omega^J$$

$$+ \tfrac{1}{2} c^a{}_{bI} \left(-\tfrac{1}{2} \bar{c}_{cd}{}^b\, \omega^c \wedge \omega^d\right)$$

$$+ \tfrac{1}{2}(c^a{}_{IJ} + c^a{}_{JI}) \Big\{ -\tfrac{1}{2} c_{db}{}^J\, \omega^d \wedge \omega^b$$

$$- c_{dK}{}^J\, \omega^d \wedge \omega^K - \tfrac{1}{2} c_{KL}{}^J\, \omega^K \wedge \omega^L \Big\}$$

$$+ \Big\{ \bar{\omega}^a{}_b + \tfrac{1}{2} c^a{}_{bs}\omega^s + \tfrac{1}{2} c^a{}_{bJ}\, \omega^J \Big\}$$

$$\wedge \Big\{ \tfrac{1}{2} c^b{}_{dI}\, \omega^d + \tfrac{1}{2}(c^b{}_{IJ} + c^b{}_{JI})\, \omega^J \Big\}$$

$$+ \Big\{ \tfrac{1}{2} c^a{}_{bJ}\, \omega^b + \tfrac{1}{2}(c^a{}_{JK} + c^a{}_{KJ})\, \omega^K \Big\}$$

$$\wedge \Big\{ \tfrac{1}{2}(c^J{}_{dI} - c_{Id}{}^J)\, \omega^d$$

$$+ \tfrac{1}{2}(c^J{}_{IM} + c^J{}_{MI} - c_{IM}{}^J)\, \omega^M \Big\}$$

But

$$R^a{}_I = \tfrac{1}{2} R^a{}_{Icd}\, \omega^c \wedge \omega^d$$

$$+ R^a{}_{Ics}\, \omega^c \wedge \omega^s + R^a{}_{IcJ}\, \omega^c \wedge \omega^J$$

$$+ R^a{}_{IsJ}\, \omega^s \wedge \omega^J + \tfrac{1}{2} R^a{}_{IJK}\, \omega^J \wedge \omega^K .$$

Therefore, by inspection and relabelling certain dummy suffixes :

$$R^a{}_{IcJ}\,\omega^c\wedge\omega^J = \tfrac{1}{2}\left(c^a{}_{IJ} + c^a{}_{JI}\right)_{,c}\,\omega^c\wedge\omega^J$$

$$+\ \bar{\omega}^a{}_b \wedge \left(\tfrac{1}{2}\left(c^b{}_{IJ} + c^b{}_{JI}\right)\omega^J\right)$$

$$-\tfrac{1}{4}\,c^a{}_{bJ}\,c^b{}_{cI}\,\omega^c\wedge\omega^J$$

$$-\tfrac{1}{2}\left(c^a{}_{IM} + c^a{}_{MI}\right)c_{cJ}{}^M\,\omega^c\wedge\omega^J$$

$$+\ \tfrac{1}{2}\,c^a{}_{cM}\,.\,\tfrac{1}{2}\left(c^M{}_{IJ} + c^M{}_{JI} - c_{IJ}{}^M\right)\omega^c\wedge\omega^J$$

$$+\left\{\tfrac{1}{2}\left(c^a{}_{MJ} + c^a{}_{JM}\right)\omega^J\right\}$$

$$\wedge\left\{\tfrac{1}{2}\left(c^M{}_{cI} - c_{Ic}{}^M\right)\omega^c\right\}$$

It is now necessary to use the explicit form of $\bar{\omega}^a{}_b$:

$$\bar{\omega}^a{}_b = \tfrac{1}{2}\left(\bar{c}^a{}_{bc} + \bar{c}^a{}_{cb} - \bar{c}_{bc}{}^a\right)\omega^c$$

Strictly speaking the bars above the c's are not absolutely necessary but they are kept for the sake of emphasis .

So

$$R^{a}{}_{IcJ} = \tfrac{1}{2}\left(c^{a}{}_{IJ} + c^{a}{}_{JI}\right)_{,c}$$

$$+\tfrac{1}{2}\left(\bar{c}^{a}{}_{bc} + \bar{c}^{a}{}_{cb} - \bar{c}_{bc}{}^{a}\right)\left(c^{b}{}_{IJ} + c^{b}{}_{JI}\right)$$

$$-\tfrac{1}{4}\, c^{a}{}_{bJ}\, c^{b}{}_{cI} \;-\tfrac{1}{2}\, c_{cJ}{}^{M}\left(c^{a}{}_{IM} + c^{a}{}_{MI}\right)$$

$$+\tfrac{1}{4}\, c^{a}{}_{cM} .\left(c^{M}{}_{IJ} + c^{M}{}_{JI} - c_{IJ}{}^{M}\right)$$

$$-\tfrac{1}{4}\left(c^{a}{}_{MJ} + c^{a}{}_{JM}\right)\left(c^{M}{}_{cI} - c_{Ic}{}^{M}\right)$$

Therefore :

$$R^{aI}{}_{aI} = \tfrac{1}{2}\left(c^{aI}{}_{I} + c^{a}{}_{I}{}^{I}\right)_{,a}$$

$$+\tfrac{1}{2}\left(\bar{c}^{a}{}_{ba} + \bar{c}^{a}{}_{ab} - \bar{c}_{ba}{}^{a}\right)\left(c^{bI}{}_{I} + c^{b}{}_{I}{}^{I}\right)$$

$$-\tfrac{1}{4}\, c^{a}{}_{bI}\, c^{b}{}_{aI} \;-\tfrac{1}{2}\, c_{aI}{}^{M}\left(c^{aI}{}_{M} + c^{a}{}_{M}{}^{I}\right)$$

$$+\tfrac{1}{4}\, c^{a}{}_{aM}\left(c^{MI}{}_{I} + c^{M}{}_{I}{}^{I} - c^{I}{}_{I}{}^{M}\right)$$

$$-\tfrac{1}{4}\left(c^{a}{}_{MI} + c^{a}{}_{IM}\right)\left(c^{M}{}_{a}{}^{I} - c^{I}{}_{a}{}^{M}\right)$$

Now the last term in this equation vanishes because the first part is symmetric under interchange of I and M, and the scalar product of an antisymmetric and a symmetric tensor is zero. The other terms vanish because

$$c^{a}{}_{aM} = c^{b}{}_{I}{}^{I} = c^{M}{}_{I}{}^{I} = 0$$

which are consequences of equations (2.69) and (2.74).

This leaves us with :

$$R^{aI}{}_{aI} = \frac{1}{4} F_{ab}{}^{I} F^{ab}{}_{I} - \frac{1}{2} C_{aI}{}^{M} (C^{aI}{}_{M} + C^{a}{}_{M}{}^{I}) \tag{2.86}$$

Also $R^{5}{}_{I} = d\omega^{5}{}_{I} + \omega^{5}{}_{a} \wedge \omega^{a}{}_{I} + \omega^{5}{}_{J} \wedge \omega^{J}{}_{I}$

$$= (-\tfrac{1}{2} C_{ab5}\, \omega^{b}) \wedge (\tfrac{1}{2} C^{a}{}_{dI}\, \omega^{d} + \tfrac{1}{2}(C^{a}{}_{IJ} + C^{a}{}_{JI})\, \omega^{J})$$

But $R^{5}{}_{I} = \frac{1}{2} R^{5}{}_{Icd}\, \omega^{c} \wedge \omega^{d} + R^{5}{}_{Ic5}\, \omega^{c} \wedge \omega^{5}$

$$+ R^{5}{}_{IcJ}\, \omega^{c} \wedge \omega^{J} + R^{5}{}_{I5J}\, \omega^{5} \wedge \omega^{J} + \tfrac{1}{2} R^{5}{}_{IJK}\, \omega^{J} \wedge \omega^{K}$$

So, by inspection :

$$R^{5I}{}_{5I} = 0 \tag{2.87}$$

Finally

$$R^{I}{}_{J} = d\omega^{I}{}_{J} + \omega^{I}{}_{a} \wedge \omega^{a}{}_{J} + \omega^{I}{}_{K} \wedge \omega^{K}{}_{J}$$

$$= d\left(\tfrac{1}{2}(C^{I}{}_{aJ} - C_{Ja}{}^{I})\,\omega^{a} + \tfrac{1}{2}(C^{I}{}_{JK} + C^{I}{}_{KJ} - C_{JK}{}^{I})\,\omega^{K}\right)$$

$$+ \left(-\tfrac{1}{2} C_{ab}{}^{I}\,\omega^{b} - \tfrac{1}{2}(C_{a}{}^{I}{}_{K} + C_{aK}{}^{I})\,\omega^{K}\right) \wedge \left(\tfrac{1}{2} C^{a}{}_{dJ}\,\omega^{d} + \tfrac{1}{2}(C^{a}{}_{JL} + C^{a}{}_{LJ})\,\omega^{L}\right)$$

$$+ \left(\tfrac{1}{2}(C^{I}{}_{aK} - C_{Ka}{}^{I})\,\omega^{a} + \tfrac{1}{2}(C^{I}{}_{MK} + C^{I}{}_{KM} - C_{MK}{}^{I})\,\omega^{K}\right)$$

$$\wedge \left(\tfrac{1}{2}(C^{M}{}_{dJ} - C_{Jd}{}^{M})\,\omega^{d} + \tfrac{1}{2}(C^{M}{}_{JL} + C^{M}{}_{LJ} - C_{JL}{}^{M})\,\omega^{L}\right)$$

$$d\left(\tfrac{1}{2}(c^I{}_{aJ} - c_{Ja}{}^I)\,\omega^a + \tfrac{1}{2}(c^I{}_{JK} + c^I{}_{KJ} - c_{JK}{}^I)\,\omega^K\right)$$

$$= \tfrac{1}{2}\,(c^I{}_{aJ} - c_{Ja}{}^I)_{,d}\;\omega^d \wedge \omega^a$$

$$+ \tfrac{1}{2}\,(c^I{}_{aJ} - c_{Ja}{}^I)(-\tfrac{1}{2}\;\overline{c}_{bc}{}^a\;\omega^b \wedge \omega^c)$$

$$+ \tfrac{1}{2}\,(c^I{}_{JK} + c^I{}_{KJ} - c_{JK}{}^I).$$

$$\left\{ -\tfrac{1}{2}\,c_{ab}{}^K\;\omega^a \wedge \omega^b + c_{aJ}{}^K\;\omega^J \wedge \omega^a \right.$$

$$\left. -\tfrac{1}{2}\,c_{LM}{}^K\;\omega^L \wedge \omega^M \right\}.$$

Now $R^I{}_J = \tfrac{1}{2}\,R^I{}_{Jab}\;\omega^a \wedge \omega^b$

$$+ R^I{}_{Jab}\;\omega^a \wedge \omega^b \; + R^I{}_{JaK}\;\omega^a \wedge \omega^K$$

$$+ R^I{}_{JbK}\;\omega^b \wedge \omega^K \; + \tfrac{1}{2}\,R^I{}_{JKL}\;\omega^K \wedge \omega^L$$

and so, by inspection : $\tfrac{1}{2}\,R^I{}_{JKL}\;\omega^K \wedge \omega^L =$

$$-\tfrac{1}{4}\,(c_a{}^I{}_K + c_{aK}{}^I)(c^a{}_{JL} + c^a{}_{LJ})\;\omega^K \wedge \omega^L$$

$$+\tfrac{1}{4}\,(c^I{}_{MK} + c^I{}_{KM} - c_{MK}{}^I)(c^M{}_{JL} + c^M{}_{LJ} - c_{JL}{}^M)\;\omega^K \wedge \omega^L$$

$$-\tfrac{1}{4}\,(c^I{}_{JM} + c^I{}_{MJ} - c_{JM}{}^I)\;c_{KL}{}^M\;\omega^K \wedge \omega^L$$

It is now necessary to make explicit the antisymmetry in K and L in the coefficients of ω^K and ω^L :

$$\tfrac{1}{2}\,R^I{}_{JKL}\;\omega^K \wedge \omega^L = -\tfrac{1}{4}\,c_{KL}{}^M(c^I{}_{JM} + c^I{}_{MJ} - c_{JM}{}^I)\;\omega^K \wedge \omega^L$$

$$-\frac{1}{8}\Big\{(c_a{}^I{}_K + c_{aK}{}^I)(c^a{}_{JL} + c^a{}_{LJ})$$
$$-(c_a{}^I{}_L + c_{aL}{}^I)(c^a{}_{JK} + c^a{}_{KJ})\Big\}\,\omega^K\wedge\omega^L$$

$$+\frac{1}{8}\Big\{(c^I{}_{MK} + c^I{}_{KM} - c_{MK}{}^I)(c^M{}_{JL} + c^M{}_{LJ} - c_{JL}{}^M)$$
$$-(c^I{}_{ML} + c^I{}_{LM} - c_{ML}{}^I)(c^M{}_{JK} + c^M{}_{KJ} - c_{JK}{}^M)\Big\}\,\omega^K\wedge\omega^L$$

Therefore : $R^I{}_{JKL} = -\frac{1}{2} c_{KL}{}^M(c^I{}_{JM} + c^I{}_{MJ} - c_{JM}{}^I)$

$$-\frac{1}{4}\Big\{(c_a{}^I{}_K + c_{aK}{}^I)(c^a{}_{JL} + c^a{}_{LJ})$$
$$-(c_a{}^I{}_L + c_{aL}{}^I)(c^a{}_{JK} + c^a{}_{KJ})\Big\}$$
$$+\frac{1}{4}\Big\{(c^I{}_{MK} + c^I{}_{KM} - c_{MK}{}^I)(c^M{}_{JL} + c^M{}_{LJ} - c_{JL}{}^M)$$
$$-(c^I{}_{ML} + c^I{}_{LM} - c_{ML}{}^I)(c^M{}_{JK} + c^M{}_{KJ} - c_{JK}{}^M)\Big\}$$

So $R^{IJ}{}_{IJ} = -\frac{1}{2} c_{IJ}{}^M(c^I{}_{JM} + c^I{}_{MJ} - c_{JM}{}^I)$

$$-\frac{1}{4}\Big\{(c_a{}^I{}_I + c_{aI}{}^I)(c^{aJ}{}_J + c^a{}_J{}^J)$$
$$-(c_a{}^I{}_J + c_{aJ}{}^I)(c^{aJ}{}_I + c^a{}_I{}^J)\Big\}$$
$$+\frac{1}{4}\Big\{(c^I{}_{MI} + c^I{}_{IM} - c_{MI}{}^I)(c^{MJ}{}_J + c^M{}_J{}^J - c^J{}_J{}^M)$$
$$-(c^I{}_{MJ} + c^I{}_{JM} - c_{MJ}{}^I)(c^{MJ}{}_I + c^M{}_I{}^J - c^J{}_I{}^M)\Big\}$$

So

$$R^{IJ}{}_{IJ} = \frac{1}{4}(c_a{}^I{}_J + c_{aJ}{}^I)(c^{aJ}{}_I + c^a{}_I{}^J)$$
$$-\frac{1}{2} c_{IJ}{}^M(c^I{}_{JM} + c^I{}_{MJ} - c_{JM}{}^I)$$

$$-\frac{1}{4}(C^I{}_{MJ}+C^I{}_{JM}-C_{MJ}{}^I)(C^{MJ}{}_I+C^M{}_I{}^J-C^J{}_I{}^M)$$

$$\therefore\ R^{IJ}{}_{IJ}=\frac{1}{4}(C_a{}^I{}_J+C_{aJ}{}^I)(C^{aJ}{}_I+C^a{}_I{}^J)$$
$$-\frac{1}{4}C_{IJ}{}^M(C^I{}_{JM}+C^I{}_{MJ}-C_{JM}{}^I)$$

(2.88)

Collecting all these terms together gives us our expression for the scalar curvature :

$$R^{(8)}=R^{ab}{}_{ab}+2R^{a5}{}_{a5}+2R^{aI}{}_{aI}+R^{IJ}{}_{IJ}$$

$$=R^{(4)}-\frac{3}{4}f^{ab}f_{ab}-\frac{3}{4}F^{abI}F_{abI}$$
$$+\frac{1}{2}f^{ab}f_{ab}+\frac{1}{2}F^{abI}F_{abI}-C_{aI}{}^M(C^{aI}{}_M+C^{aM}{}_I)$$
$$+\frac{1}{4}(C_a{}^I{}_J+C_{aJ}{}^I)(C^{aJ}{}_I+C^a{}_I{}^J)$$
$$-\frac{1}{4}C_{IJ}{}^M(C^I{}_{JM}+C^I{}_{MJ}-C_{JM}{}^I)$$

$$=R^{(4)}-\frac{1}{4}f^{ab}f_{ab}-\frac{1}{4}F^{abI}F_{abI}$$
$$-\frac{1}{2}C_a{}^{IM}(C^a{}_{IM}+C^a{}_{MI})$$
$$-\frac{1}{4}C_{IJ}{}^M(C^I{}_{JM}+C^I{}_{MJ}-C_{JM}{}^I).$$

(2.89)

The final term in this expression looks very much like a quantity which depends only on the internal space, and indeed a calculation similar to the one above but using only the metric of the internal space, that is the metric on page 30, would show that this term is exactly the scalar curvature of the internal space. The exact details of the calculation can be found in the appendix ; here we shall merely quote the result which is that :

$$R^{(int)} = -\frac{1}{4} C_{IJK} \left(C^{IJK} + C^{IKJ} - C^{JKI} \right) \tag{2.90}$$

Earlier in the chapter we mentioned that there remained one further condition which could be extracted from the Killing vector hypothesis and it is at this point that this condition will be revealed. The Lie derivative operator follows the Liebnitz rule (Schutz 1980) so that :

$$(\mathcal{L}_{\vec{X}}\, g)(\vec{Y},\vec{Z}) = \mathcal{L}_{\vec{X}}\, g(\vec{Y},\vec{Z}) + g(\mathcal{L}_{\vec{X}}\vec{Y},\vec{Z}) + g(\vec{Y},\mathcal{L}_{\vec{X}}\vec{Z})$$

where $\vec{X},\vec{Y}$ and $\vec{Z}$ are vectors and g is the metric tensor. Now consider the above equation with X,Y and Z replaced by the left invariant vector fields, that is by $\vec{V_s}$, $\vec{V_t}$ and $\vec{V_v}$. We have :

$$(\mathcal{L}_{\vec{V_s}}\, g)(\vec{V_t},\vec{V_v}) = \mathcal{L}_{\vec{V_s}}\, g(\vec{V_t},\vec{V_v}) + g(\mathcal{L}_{\vec{V_s}}\vec{V_t},\vec{V_v}) + g(\vec{V_t},\mathcal{L}_{\vec{V_s}}\vec{V_v})$$

But the Lie derivative of a vector field with respect to some other vector field is just the commutator of the two vector fields, so that in our case :

$$\mathcal{L}_{\vec{v}_s} \vec{v}_t = [\vec{v}_s , \vec{v}_t] = c_{st}{}^{x} \vec{v}_x$$

Therefore :

$$(\mathcal{L}_{\vec{v}_s} g|)(\vec{v}_t,\vec{v}_v) = \mathcal{L}_{\vec{v}_s} g|(\vec{v}_t,\vec{v}_v) + c_{st}{}^{x} g|(\vec{v}_x,\vec{v}_v)$$
$$+ c_{sv}{}^{x} g|(\vec{v}_t,\vec{v}_x)$$

But the hypothesis that the left invariant vector fields are Killing vectors means that the Lie derivative of the metric with respect to them is zero, so that :

$$c_{st}{}^{x} g_{xv} + c_{sv}{}^{x} g_{tx} = 0 \tag{2.91}$$

This is a very important result.
Equation (2.69) can be written as :

$$c_{st}{}^{x} = - e^{I}_{s} e^{J}_{t} E^{x}_{K} c_{IJ}{}^{K}$$

so

$$g_{xv} c_{st}{}^{x} = - g_{xv} e^{I}_{s} e^{J}_{t} E^{x}_{K} c_{IJ}{}^{K}$$

but $$g_{xv} = e_x^L e_v^M \eta_{LM}$$

so $$g_{xv} C_{st}{}^x = - e_x^L e_v^M e_s^I E_K^x e_t^J C_{IJ}{}^K \eta_{LM}$$

$$= -\delta_K^L \eta_{LM} e_v^M e_s^I e_t^J C_{IJ}{}^K$$

$$= - \eta_{KM} e_v^M e_s^I e_t^J C_{IJ}{}^K$$

$$\equiv - e_v^M e_s^I e_t^J C_{IJM}$$

and also

$$g_{xt} C_{sv}{}^x = - e_t^M e_s^I e_v^J C_{IJM}$$

$$= - e_v^M e_s^I e_t^J C_{IMJ}$$

therefore

$$C_{st}{}^x g_{xv} + C_{sv}{}^x g_{xt} = - e_v^M e_s^I e_t^J (C_{IJM} + C_{IMJ})$$

so the above equation (2.91) means that :

$$C_{IJM} + C_{IMJ} = 0 \qquad (2.92)$$

This condition will be used from now on to effect simplification whenever necessary.

Using the two equations (2.90) and (2.92) gives us our final answer :

$$R^{(8)} = R^{(4)} - \frac{1}{4} f^{ab} f_{ab} - \frac{1}{4} F^{abI} F_{abI} + R^{(int)} \tag{2.93}$$

This shows that if we use as our Lagrangian the curvature scalar in eight dimensions that it decomposes into the curvature scalar in four dimensions plus the U(1) field tensor squared plus the SU(2) field tensor squared plus $R^{(int)}$, the scalar curvature of the internal space. This last term is usually interpreted as a cosmological constant, but since the internal space is highly compactified its value is too large by far. In cosmology, simple arguments show that if the cosmological constant is not exactly zero it must be very small. (Misner et al., 1973). Note that since :

$$R^{(int)} = \frac{1}{4} C_{IJK} C^{IJK}$$

then it is always positive. It is interesting that this term, which gives an absurdly high value of the cosmological constant, is also responsible for predicting particles of very large mass in the Dirac equation. This will become apparent in the next chapter.

CHAPTER 3.

The Dirac Equation.

Introduction.

So far, we have found that gravity, electromagnetism, and Yang-Mills fields have separated out, as Pauli put it, "like oil and water". This is now about to change in considering the Dirac equation.

The position we are in now is that the electromagnetic field and the Yang-Mills fields have been absorbed into the space as properties of that space. The Dirac equation has, since its original inception, been seen in a more geometric light and Cartan has shown that Dirac's original equation is one particular example of a general equation which exists in an n-dimensional space which may be either curved or flat (Cartan 1966.). Utiyama (Utiyama 1956.) has shown that a curved space may be described as a patchwork of flat tangent spaces each related to the others by spacetime dependent Lorentz transformations, thus showing the exact way to generalise the Dirac equation. This means that in our case we can rigorously write down a Dirac equation and because the electromagnetic and the Yang-Mills fields are propeties of the space then they will enter into the equation naturally.

It was Thirring who first noted the existence of a Fierz-Pauli term in the Dirac equation in the five dimensional case (Thirring 1972.) and in extending his work to eight dimensions many more extra terms appear. Once the Dirac equation has been found there is a certain amount of freedom in interpreting these terms. As always, it is wise to proceed cautiously and so we shall make explicit the form of the Dirac equation before making any simplifying assumptions.

The Dirac Equation on a Curved Space.

The Dirac equation is generalised to a curved space by introducing the vierbein field and by replacing the ordinary derivative by the appropriate covariant derivative (Nieh and Yan 1982, Utiyama 1956.) thus :

$$m \psi = i \gamma^{a} E^{\mu}_{\ a} D_{\mu} \psi \tag{3.1}$$

where m is the mass of the particle concerned, ψ is a spinor on the flat tangent space, D_{μ} is the covariant derivative and γ^{a} is a set of Dirac matrices which satisfy :

$$\gamma^{a} \gamma^{b} + \gamma^{b} \gamma^{a} = 2 \eta^{ab} \tag{3.2}$$

where η^{ab} is the Minkowski tensor :

$$\eta^{ab} = \mathrm{diag}\,(\,-1,1,1,1\,) \qquad (\,3.3\,)$$

The covariant derivative D_μ is given by :

$$D_\mu = \partial_\mu - \frac{i}{4}\,\omega_{ab\mu}\,\sigma^{ab} \qquad (\,3.4\,)$$

where σ^{ab} = the generator of Lorentz transformations

$$\sigma^{ab} = \frac{i}{2}\left[\,\gamma^a\,,\,\gamma^b\,\right] \qquad (\,3.5\,)$$

and $\omega_{ab\mu}$ is the spin connection, which is related to the connection coefficients by the formula :

$$\Gamma^\sigma_{\mu\nu} = E^\sigma_a\left(\,e^a_{\mu,\nu} + \omega^a{}_{b\nu}\,e^b_\mu\,\right) \qquad (\,3.6\,)$$

If we make the assumption that there is no torsion then the connection coefficients are given by Christoffels formula :

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\,g^{\sigma\rho}\left(\,g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\,\right) \qquad (\,3.7\,)$$

and hence $\omega_{ab\mu}$ is given by :

$$\omega_{ab\mu} = \frac{1}{2} e^{c}_{\mu} \left(\gamma_{cab} - \gamma_{abc} - \gamma_{bca} \right)$$

(3.8)

where

$$\gamma^{c}_{\ ab} = \left(E^{\mu}_{\ a} E^{\nu}_{\ b} - E^{\nu}_{\ a} E^{\mu}_{\ b} \right) e^{c}_{\mu,\nu}$$

(3.9)

These $\gamma^{c}_{\ ab}$ are sometimes called the Ricci rotation coefficients, and we now show that they are in fact equal to the structure constants which were used in the previous chapter.

In our eight dimensional index notation (3.9) reads

$$\gamma^{c}_{\ AB} = \left(E^{\alpha}_{A} E^{\beta}_{B} - E^{\alpha}_{B} E^{\beta}_{A} \right) e^{c}_{\alpha,\beta}$$

(3.10)

which by relabelling dummy suffixes is equal to :

$$\gamma^{c}_{\ AB} = E^{\alpha}_{A} E^{\beta}_{B} \left(e^{c}_{\alpha,\beta} - e^{c}_{\beta,\alpha} \right)$$

(3.11)

The structure constants were defined as follows :

$$d\omega^C = -\tfrac{1}{2} C_{AB}{}^C \omega^A \wedge \omega^B$$

But

$$d\omega^C = d(e^C_\alpha dx^\alpha)$$

$$= \partial_\beta e^C_\alpha dx^\beta \wedge dx^\alpha$$

$$= \tfrac{1}{2}(e^C_{\alpha,\beta} - e^C_{\beta,\alpha}) dx^\beta \wedge dx^\alpha$$

$$= \tfrac{1}{2}(e^C_{\beta,\alpha} - e^C_{\alpha,\beta}) dx^\alpha \wedge dx^\beta$$

and

$$-\tfrac{1}{2} C_{AB}{}^C \omega^A \wedge \omega^B = -\tfrac{1}{2} C_{AB}{}^C e^A_\alpha e^B_\beta dx^\alpha \wedge dx^\beta$$

so

$$-\tfrac{1}{2} C_{AB}{}^C e^A_\alpha e^B_\beta = \tfrac{1}{2}(e^C_{\beta,\alpha} - e^C_{\alpha,\beta})$$

and comparing this with (3.11) shows that:

$$\gamma^C{}_{AB} = C_{AB}{}^C$$

Notice that the γ's as they have been written are antisymmetric in their last two indices whereas the c's are antisymmetric in their first two. Some authors write it one way, some the other. Notice in my case, having chosen to use the notation of Misner et al., that the lowered top index takes the third place, whereas for the γ's it will take the first place, that is :

$$\gamma_{CAB} \equiv c_{ABC}$$

The Spin Connections.

As a last step before proceeding to the Dirac equation we need to calculate the spin connections for our eight dimensional space. Formula (3.8) can, by the arguments of the previous section be written in terms of the structure constants as :

$$\omega_{AB\alpha} = \frac{1}{2} e^{C}_{\alpha} (c_{ABC} + c_{ACB} - c_{BCA}) \tag{3.12}$$

The indices A,B can take the value a, 5, or I, and the α index can take the value μ , 5, or i so that we shall have connections like $\omega_{ab\mu}$, ω_{a55} etcetera. They are presented here in a fairly logical fashion :

$$\begin{aligned} \omega_{ab\mu} = & \frac{1}{2} e^{c}_{\mu} (c_{abc} + c_{acb} - c_{bca}) \\ & + \frac{1}{2} e^{5}_{\mu} (c_{ab5} + c_{a5b} - c_{b5a}) \\ & + \frac{1}{2} e^{I}_{\mu} (c_{abI} + c_{aIb} - c_{bIa}) \end{aligned}$$

$$= \frac{1}{2} e^c_\mu (C_{abc} + C_{acb} - C_{bca})$$

$$+ \frac{1}{2} A_\mu f_{ab} + \frac{1}{2} e^I_\mu F_{abI}$$

$$= \overline{\omega}_{ab\mu} + \frac{1}{2} A_\mu f_{ab} + \frac{1}{2} e^I_\mu F_{abI}$$

(3.13)

where $\overline{\omega_{ab\mu}}$ is the four dimensional spin connection thus :

$$\omega_{ab\mu} = \frac{1}{2} e^c_\mu (C_{abc} + C_{acb} - C_{bca})$$

$$\omega_{ab5} = \frac{1}{2} e^5_5 (C_{ab5} + C_{a5b} - C_{b5a})$$

$$= \frac{1}{2} f_{ab} .$$

(3.14)

$$\omega_{abi} = \frac{1}{2} e^I_i (C_{abI} + C_{aIb} - C_{bIa})$$

$$= \frac{1}{2} e^I_i F_{abI} .$$

(3.15)

$$\omega_{a5\mu} = \frac{1}{2} e^{c}_{\mu} (C_{a5c} + C_{ac5} - C_{5ca})$$
$$+ \frac{1}{2} e^{5}_{\mu} (C_{a55} + C_{a55} - C_{55a})$$
$$+ \frac{1}{2} e^{I}_{\mu} (C_{a5I} + C_{aI5} - C_{5Ia})$$

$$= \frac{1}{2} e^{c}_{\mu} f_{ac}$$

(3.16)

$$\omega_{a55} = \frac{1}{2} e^{c}_{5} (C_{a5c} + C_{ac5} - C_{5ca})$$
$$+ \frac{1}{2} e^{5}_{5} (C_{a55} + C_{a55} - C_{55a})$$
$$+ \frac{1}{2} e^{I}_{5} (C_{a5I} + C_{aI5} - C_{5Ia})$$

$$= 0$$

(3.17)

$$\omega_{a5\lambda} = \frac{1}{2} e^{c}_{\lambda} (C_{a5c} + C_{ac5} - C_{5ca})$$
$$+ \frac{1}{2} e^{5}_{\lambda} (C_{a55} + C_{a55} - C_{55a})$$
$$+ \frac{1}{2} e^{I}_{\lambda} (C_{a5I} + C_{aI5} - C_{5Ia})$$

$$\therefore \quad \omega_{a5i} = 0$$

$$\omega_{aI\mu} = \frac{1}{2} e^{c}_{\mu} (C_{aIc} + C_{acI} - C_{Ica}) + \frac{1}{2} e^{5}_{\mu} (C_{aI5} + C_{a5I} - C_{I5a}) + \frac{1}{2} e^{J}_{\mu} (C_{aIJ} + C_{aJI} - C_{IJa}) \tag{3.18}$$

$$= \frac{1}{2} e^{c}_{\mu} F_{acI} + \frac{1}{2} e^{J}_{\mu} (C_{aIJ} + C_{aJI})$$

$$= \frac{1}{2} e^{c}_{\mu} F_{acI}$$

$$\omega_{aI5} = \frac{1}{2} e^{c}_{5} (C_{aIc} + C_{acI} - C_{Ica}) + \frac{1}{2} e^{5}_{5} (C_{aI5} + C_{a5I} - C_{I5a}) + \frac{1}{2} e^{J}_{5} (C_{aIJ} + C_{aJI} - C_{IJa}) \tag{3.19}$$

$$= 0$$

$$\omega_{aIj} = \frac{1}{2} e^{c}_{j} (C_{aIc} + C_{acI} - C_{Ica}) + \frac{1}{2} e^{5}_{j} (C_{aI5} + C_{a5I} - C_{I5a}) + \frac{1}{2} e^{J}_{j} (C_{aIJ} + C_{aJI} - C_{IJa}) \tag{3.20}$$

$$\therefore \quad \omega_{aIj} = \frac{1}{2} e^{J}_{j} \left(c_{aIJ} + c_{aJI} \right)$$

$$= 0$$

(3.21)

$$\omega_{5I\mu} = \frac{1}{2} e^{c}_{\mu} \left(c_{5Ic} + c_{5cI} - c_{Ic5} \right)$$
$$+ \frac{1}{2} e^{5}_{\mu} \left(c_{5I5} + c_{55I} - c_{I55} \right)$$
$$+ \frac{1}{2} e^{J}_{\mu} \left(c_{5IJ} + c_{5JI} - c_{IJ5} \right)$$

$$= 0$$

(3.22)

All the c's in the above equation are identically zero so :

$$\omega_{5I5} = 0 \qquad \text{and} \qquad \omega_{5Ij} = 0 \qquad (3.23)$$

$$\omega_{IJ\mu} = \frac{1}{2} e^{c}_{\mu} \left(c_{IJc} + c_{IcJ} - c_{JcI} \right)$$
$$+ \frac{1}{2} e^{5}_{\mu} \left(c_{IJ5} + c_{I5J} - c_{J5I} \right)$$
$$+ \frac{1}{2} e^{K}_{\mu} \left(c_{IJK} + c_{IKJ} - c_{JKI} \right)$$

$$= \frac{1}{2} e^{c}_{\mu} \left(c_{IcJ} - c_{JcI} \right)$$
$$+ \frac{1}{2} e^{K}_{\mu} \left(c_{IJK} + c_{IKJ} - c_{JKI} \right)$$

Now, since $$C_{IcJ} = - C_{IKJ} W^{K}_{c}$$

and since $$e^{K}_{\mu} = W^{k}_{c} e^{c}_{\mu}$$

then $$\tfrac{1}{2} e^{c}_{\mu} (C_{IcJ} - C_{JcI}) = \tfrac{1}{2} e^{k}_{\mu} (- C_{IKJ} + C_{JKI})$$
$$= -\tfrac{1}{2} e^{K}_{\mu} (C_{IKJ} - C_{JKI})$$

hence
$$\omega_{IJ\mu} = \tfrac{1}{2} e^{k}_{\mu} C_{IJK} \qquad (3.24)$$

$$\begin{aligned}\omega_{IJ5} &= \tfrac{1}{2} e^{c}_{5} (C_{IJc} + C_{IcJ} - C_{JcI}) \\ &\quad + \tfrac{1}{2} e^{5}_{5} (C_{IJ5} + C_{I5J} - C_{J5I}) \\ &\quad + \tfrac{1}{2} e^{K}_{5} (C_{IJK} + C_{IKJ} - C_{JKI}) \\ &= 0\end{aligned}$$

(3.25)

$$\begin{aligned}\omega_{IJi} &= \tfrac{1}{2} e^{c}_{i} (C_{IJc} + C_{IcJ} - C_{JcI}) \\ &\quad + \tfrac{1}{2} e^{5}_{i} (C_{IJ5} + C_{I5J} - C_{J5I}) \\ &\quad + \tfrac{1}{2} e^{K}_{i} (C_{IJK} + C_{IKJ} - C_{JKI})\end{aligned}$$

$$= \frac{1}{2} e^{K}_{\lambda} (- c_{JKI})$$

$$= - \frac{1}{2} e^{K}_{\lambda} c_{IJK} \qquad (3.26)$$

This completes the calculations of the spin connections. Notice that they are antisymmetric in their first two indices so that, for example, in the above terms like ω_{555} and $\omega_{55\mu}$ and $\omega_{55\lambda}$ are automatically zero.

The Eight Dimensional Dirac Equation.

We can now explicitly work out the Dirac equation using the results of the previous section.

The equation is a generalisation of (3.1) with μ replaced by α and a replaced by A :

$$m\psi = i\gamma^{A} E^{\alpha}_{A} D_{\alpha}\psi$$

The gamma matrices are a set of matrices which satisfy a relation like (3.2) but since the Minkowski tensor is now eight dimensional then there are eight matrices in all. Cartan has shown (Cartan 1966) that the minimum rank of a Dirac matrix in an n dimensional space is $2^{[\frac{n+1}{2}]}$, where the square bracket means the biggest integer less than the argument. Hence in four dimensions the Dirac matrices have rank four ($2^{[2\frac{1}{2}]} = 2^{2} = 4$) and in eight dimensions they have rank sixteen ($2^{[4\frac{1}{2}]} = 2^{4} = 16$). We shall make explicit these 16 x 16 matrices at a later stage.

After performing the summation :

$$m\psi = i\gamma^a E^\mu_a D_\mu\psi + i\gamma^a E^5_a D_5\psi + i\gamma^a E^i_a D_i\psi + i\gamma^5 E^5_5 D_5\psi + i\gamma^I E^i_I D_i\psi$$

(3.27)

since $E^\mu_5 = E^i_5 = E^\mu_I = E^5_I = 0$.

Now $$D_\mu\psi = (\partial_\mu - \frac{i}{4} \omega_{AB\mu}\sigma^{AB})\psi$$

$$= (\partial_\mu - \frac{i}{4} \omega_{ab\mu}\sigma^{ab} - \frac{i}{2} \omega_{a5\mu}\sigma^{a5} - \frac{i}{2} \omega_{aI\mu}\sigma^{aI} - \frac{i}{4} \omega_{IJ\mu}\sigma^{IJ})$$

(3.28)

$$D_5\psi = (\partial_5 - \frac{i}{4} \omega_{AB5}\sigma^{AB})\psi$$

$$= (\partial_5 - \frac{i}{4} \omega_{ab5}\sigma^{ab})\psi$$

(3.29)

$$D_i\psi = (\partial_i\psi - \frac{i}{4} \omega_{ABi}\sigma^{AB})$$

$$\therefore \quad D_i\psi = \left(\partial_i - \frac{i}{4}\,\omega_{ab\,i}\,\sigma^{ab} - \frac{i}{2}\,\omega_{aI}\,\sigma^{aI} - \frac{i}{4}\,\omega_{IJ\,i}\,\sigma^{IJ}\right)\psi$$

(3.30)

So the Dirac equation in full glorious detail is :

$$\begin{aligned}
m\psi = {} & i\gamma^a E^{\mu}_{a}\,\partial_\mu\psi + i\gamma^a E^{5}_{a}\,\partial_5\psi \\
& + i\gamma^5\,\partial_5\psi + i\gamma^a E^{i}_{a}\,\partial_i\psi + i\gamma^I E^{i}_{I}\,\partial_i\psi \\
& + i\gamma^d E^{\mu}_{d}\left\{ -\frac{i}{4}\,\omega_{ab\mu}\,\sigma^{ab} - \frac{i}{2}\,\omega_{a5\mu}\,\sigma^{a5} \right. \\
& \qquad \left. - \frac{i}{2}\,\omega_{aI\mu}\,\sigma^{aI} - \frac{i}{4}\,\omega_{IJ\mu}\,\sigma^{IJ} \right\}\psi \\
& + i\gamma^d\left(-E^{\mu}_{d} A_\mu\right)\left(-\frac{i}{4}\,\omega_{ab5}\,\sigma^{ab}\right)\psi \\
& + i\gamma^d\left(-E^{\mu}_{d} W^{i}_{\mu}\right)\left\{ -\frac{i}{4}\,\omega_{ab\,i}\,\sigma^{ab} - \frac{i}{2}\,\omega_{aI\,i}\,\sigma^{aI} \right. \\
& \qquad \left. - \frac{i}{4}\,\omega_{IJ\,i}\,\sigma^{IJ} \right\}\psi \\
& + i\gamma^5\left(-\frac{i}{4}\,\omega_{ab5}\,\sigma^{ab}\right)\psi \\
& + i\gamma^I E^{i}_{I}\left\{ -\frac{i}{4}\,\omega_{ab\,i}\,\sigma^{ab} - \frac{i}{2}\,\omega_{aJ\,i}\,\sigma^{aJ} \right. \\
& \qquad \left. - \frac{i}{4}\,\omega_{JK\,i}\,\sigma^{JK} \right\}\psi
\end{aligned}$$

Let us look at the part of this equation containing the spin connections that is the part which doesn't contain any derivatives :

$$i\gamma^{\alpha} E^{\mu}_{\alpha} \Big\{ -\frac{i}{4}\big(\overline{w}_{ab\mu} + \underbrace{\tfrac{1}{2} A_\mu f_{ab}}_{(1)} + \underbrace{\tfrac{1}{2} e^{I}_{i} w^{i}_{\mu} F_{abI}}_{(2)} \big)\sigma^{ab}$$

$$-\frac{i}{2}\big(\tfrac{1}{2} e^{c}_{\mu} f_{ac}\big)\sigma^{a5}$$

$$-\frac{i}{2}\big(\tfrac{1}{2} e^{c}_{\mu} F_{acI}\big)\sigma^{aI}$$

$$-\frac{i}{4}\big(\tfrac{1}{2} c_{IJK}\, e^{K}_{i} w^{i}_{\mu}\big)\sigma^{IJ}\Big\}\psi$$

$$+ i\gamma^{\alpha}\underbrace{(-E^{\mu}_{\alpha} A_\mu)\big(-\tfrac{i}{4}(\tfrac{1}{2} f_{ab})\sigma^{ab}}_{(1)}\big)\psi$$

$$+ i\gamma^{\alpha}(-E^{\mu}_{\alpha} w^{i}_{\mu})\Big\{-\frac{i}{4}\big(\underbrace{\tfrac{1}{2} e^{I}_{i} F_{abI}}_{(2)}\big)\sigma^{ab}$$

$$-\frac{i}{4}\big(-\tfrac{1}{2} e^{K}_{i} c_{IJK}\big)\sigma^{IJ}\Big\}\psi$$

$$+ i\gamma^{5}\big(-\tfrac{i}{4}(\tfrac{1}{2} f_{ab})\sigma^{ab}\big)\psi$$

$$+ i\gamma^{M} E^{i}_{M}\big(-\tfrac{i}{4}(\tfrac{1}{2} e^{I}_{i} F_{abI})\sigma^{ab}$$

$$-\frac{i}{4}\big(-\tfrac{1}{2} e^{I}_{i} c_{JKI}\big)\sigma^{JK}\big)\psi$$

and the terms underlined in wiggly red lines cancel out as indicated.

This leaves us with the following spin connection parts :

$$i\gamma^d E^\mu_d \Big\{ -\frac{i}{4} (\bar{w}_{ab\mu}\, \sigma^{ab} + e^c_\mu f_{ac}\, \sigma^{a5} + e^c_\mu F_{acI}\, \sigma^{aI})$$

$$-\frac{i}{4} (c_{IJK}\, e^K_i\, w^i_\mu)\, \sigma^{IJ} \Big\} +$$

$$+ i\gamma^5 \Big\{ -\frac{i}{8} f_{ab}\, \sigma^{ab} \Big\} +$$

$$+ i\gamma^M E^i_M \Big\{ -\frac{i}{8} e^I_i F_{abI}\, \sigma^{ab} + \frac{i}{8} e^I_i\, c_{JKI}\, \sigma^{JK} \Big\} +$$

Further simplification can be made by noting that :

$$i\gamma^d E^\mu_d \left(-\frac{i}{4} e^c_\mu f_{ac}\, \sigma^{a5} \right) = i\gamma^d f_{ad} \left(-\frac{i}{4} \sigma^{a5} \right)$$

$$= \frac{i}{8} f_{ad} (\gamma^d \gamma^a \gamma^5 - \gamma^d \gamma^5 \gamma^a)$$

and that :

$$i\gamma^5 \left(-\frac{i}{8} f_{ab}\, \sigma^{ab} \right) = \frac{i}{16} f_{ab} (\gamma^5 \gamma^a \gamma^b - \gamma^5 \gamma^b \gamma^a)$$

Now since
$$\gamma^A \gamma^B + \gamma^B \gamma^A = 2 \eta^{AB}$$

then

$$\gamma^a \gamma^5 + \gamma^5 \gamma^a = 2 \eta^{a5} = 0$$

so that in effect γ^a and γ^5 anticommute :

$$\gamma^a \gamma^5 = - \gamma^5 \gamma^a$$

so

$$i\gamma^5 \left(-\frac{i}{8} f_{ab} \sigma^{ab}\right) = \frac{i}{16} f_{ab} \left(\gamma^a \gamma^b \gamma^5 + \gamma^b \gamma^5 \gamma^a\right) \qquad \text{(A)}$$

$$= \frac{i}{16} f_{ab} \left(\gamma^b \gamma^5 \gamma^a - \gamma^b \gamma^a \gamma^5\right) \qquad \text{(B)}$$

$$= i\gamma^b \left(\frac{-i}{8} f_{ab} \sigma^{a5}\right)$$

so that :

$$i\gamma^d f_{ad} \left(\frac{-i}{4} \sigma^{a5}\right) + i\gamma^5 \left(\frac{-i}{8} f_{ab} \sigma^{ab}\right)$$

$$= i\gamma^d f_{ad} \left(\frac{-i}{8} \sigma^{a5}\right)$$

(Note that in going from line (A) to line (B) above we have used

$$f_{ab} \gamma^a \gamma^b \gamma^5 = - f_{ab} \gamma^b \gamma^a \gamma^5$$

which is true because of dummy suffixes. Alternatively we could have said that :

$$f_{ab}\gamma^a\gamma^b\gamma^5 = f_{ab}\left(-\gamma^b\gamma^a\gamma^5 + 2\eta^{ab}\gamma^5\right)$$

and the last term vanishes since it is the scalar product of a symmetric and an antisymmetric tensor.)

A similar calculation shows that :

$$-\frac{i}{4}\gamma^d F_{adI}\,\sigma^{aI} - \frac{i}{8}\gamma^I F_{adI}\,\sigma^{ad} = -\frac{i}{8}\gamma^d F_{adI}\,\sigma^{aI}$$

Therefore the Dirac equation in this curved eight dimensional space is :

$$\begin{aligned} m\psi = {} & i\gamma^a E^{\mu}_{a}\partial_\mu\psi - i\gamma^a E^{\nu}_{a} A_\nu \partial_5\psi \\ & + i\gamma^5\partial_5\psi - i\gamma^a E^{\nu}_{a} w^i_\nu \partial_i\psi \\ & + i\gamma^I E^i_I \partial_i\psi \\ & + \Big\{ \frac{1}{4}\gamma^d E^{\mu}_{d}\, \overline{w}_{ab\mu}\,\sigma^{ab} + \frac{1}{8}\gamma^d f_{ad}\,\sigma^{a5} \\ & + \frac{1}{8}\gamma^d F_{adI}\,\sigma^{aI} \\ & + \frac{1}{4}\gamma^d E^{\mu}_{d}\, e^K_i w^i_\mu\, C_{IJK}\,\sigma^{IJ} \\ & - \frac{1}{8}\gamma^I C_{JKI}\,\sigma^{JK} \Big\}\psi \end{aligned}$$

(3.31)

At this juncture it is worth pointing out that this equation has been derived using completely rigorous methods and that no assumptions have been made. This then will be the foundation stone of further work.

It is commonly stated in the literature that Kaluza-Klein theory predicts particles of very high bare mass because in any matter field equation the radius of the curvature of the internal space appears as an eigenvalue of the derivative operator, and since this radius is very small this means that the particle has very high mass (Witten 1981). To see how this comes about remember that the scalar curvature is given by :

$$R^{(int)} = \frac{1}{4} c_{IJK} c^{IJK}.$$

Now, the scalar curvature has dimensions of inverse length squared so that the structure constants in equation (3.31) will have dimension of inverse length and will, if the internal space is highly compactified, have magnitude of the order of the Planck length. Accordingly, any mass entering into the equation via the last term in equation (3.31) will have magnitude of the order of the Planck mass. This point will become clearer when each term in equation (3.31) is considered in turn. It is easier to do this if we choose a specific representation of the Dirac matrices, which is what the next section is all about.

The Dirac Matrices.

According to Cartan (Cartan 1966) in an n dimensional space the Dirac matrices are $2^{[\frac{n}{2}]} \times 2^{[\frac{n}{2}]}$ matrices, so that in eight dimensions the Dirac matrices are 16 x 16 matrices. Extending the procedures of Mecklenburg and Domokos and Kovesi-Domokos (Mecklenburg 1980, Domokos and Kovesi-Domokos 1977) we shall use the following 16 x 16 Dirac matrices :

$$\gamma^a_{16\times16} = \gamma^a_{4\times4} \otimes I_{4\times4}$$

$$\gamma^I_{16\times16} = \gamma^a_{4\times4} \otimes \gamma^5_{4\times4}$$

(3.32)

(In the above equation the index I refers to the fifth dimension too.)

where $\gamma^a_{4\times4}$ = the normal 4 x 4 Dirac matrices which satisfy (3.2). Explicitly they are :

$$\gamma^0 = i \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\gamma^1 = i\begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$$

$$\gamma^3 = i\begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

so that $(\gamma^5)^2 = 1$ and $\gamma^a\gamma^5 + \gamma^5\gamma^a = 0.$

The 16 x 16 Dirac matrices are therefore :

$$\gamma^a_{16\times16} = \begin{pmatrix} \gamma^a & & & \\ & \gamma^a & & \\ & & \gamma^a & \\ & & & \gamma^a \end{pmatrix}$$

$$\gamma^5_{16\times 16} = \begin{pmatrix} -\gamma^5 & & & \\ & -\gamma^5 & & \\ & & \gamma^5 & \\ & & & \gamma^5 \end{pmatrix}$$

$$\gamma^6_{16\times 16} = \begin{pmatrix} & & & i\gamma^5 \\ & & i\gamma^5 & \\ & -i\gamma^5 & & \\ -i\gamma^5 & & & \end{pmatrix}$$

$$\gamma^7_{16\times 16} = \begin{pmatrix} & & & \gamma^5 \\ & & -\gamma^5 & \\ & -\gamma^5 & & \\ \gamma^5 & & & \end{pmatrix}$$

$$\gamma^8_{16\times 16} = \begin{pmatrix} & & i\gamma^5 & \\ & & & -i\gamma^5 \\ -i\gamma^5 & & & \\ & i\gamma^5 & & \end{pmatrix}$$

It is not difficult to work out the generators of the Lorentz transformations :

$$\sigma^{AB} = \frac{i}{2}\left[\gamma^{A}, \gamma^{B}\right]$$

The results are :

$$\sigma^{ab}_{16\times16} = \begin{pmatrix} \sigma^{ab}_{4\times4} & & & \\ & \sigma^{ab}_{4\times4} & & \\ & & \sigma^{ab}_{4\times4} & \\ & & & \sigma^{ab}_{4\times4} \end{pmatrix}$$

$$\sigma^{a5}_{16\times16} = \begin{pmatrix} -\sigma^{a5}_{4\times4} & & & \\ & -\sigma^{a5}_{4\times4} & & \\ & & \sigma^{a5}_{4\times4} & \\ & & & \sigma^{a5}_{4\times4} \end{pmatrix}$$

$$\sigma^{a6}_{16\times16} = \begin{pmatrix} & & & i\sigma^{a5}_{4\times4} \\ & & i\sigma^{a5}_{4\times4} & \\ & -i\sigma^{a5}_{4\times4} & & \\ -i\sigma^{a5}_{4\times4} & & & \end{pmatrix}$$

$$\sigma^{a7}_{16\times16} = \begin{pmatrix} & & & \sigma^{a5}_{4\times4} \\ & & -\sigma^{a5}_{4\times4} & \\ & -\sigma^{a5}_{4\times4} & & \\ \sigma^{a5}_{4\times4} & & & \end{pmatrix}$$

$$\Gamma^{a8}_{16\times 16} = \begin{pmatrix} & & i\sigma^{a5}_{4\times 4} & \\ & & & -i\sigma^{a5}_{4\times 4} \\ -i\sigma^{a5}_{4\times 4} & & & \\ & i\sigma^{a5}_{4\times 4} & & \end{pmatrix}$$

(Henceforth the subscripts 16 x 16 and 4 x 4 will be dropped unless special emphasis is necessary.)

$$\Gamma^{56} = \begin{pmatrix} & & & I \\ & & I & \\ & I & & \\ I & & & \end{pmatrix}$$

$$\Gamma^{57} = \begin{pmatrix} & & & -iI \\ & & iI & \\ & -iI & & \\ iI & & & \end{pmatrix}$$

$$\Gamma^{58} = \begin{pmatrix} & & I & \\ & & & -I \\ I & & & \\ & -I & & \end{pmatrix}$$

$$\Gamma^{67} = \begin{pmatrix} -I & & & \\ & I & & \\ & & -I & \\ & & & I \end{pmatrix}$$

$$\sigma^{68} = \begin{pmatrix} & -iI & & \\ iI & & & \\ & & & -iI \\ & & iI & \end{pmatrix}$$

$$\sigma^{78} = \begin{pmatrix} & -I & & \\ -I & & & \\ & & & -I \\ & & -I & \end{pmatrix}$$

This concludes all the necessary algebra of the Dirac matrices, so that we can now consider each of the terms in the equation (3.31).

1) The mass term $\frac{1}{8}\gamma^{I} C_{JKI}\, \sigma^{JK} \psi$.

This term is referred to as the mass term since in the (unrealistic) limit of no U(1) and no SU(2) and no gravitational interaction, and a spinor ψ which is independent of the x^5 and the x^i co-ordinates, and m = 0, then this is the only term which remains and the Dirac equation (3.31) becomes :

$$i\gamma^{a} E^{m}_{a} \partial_{m} \psi = \frac{1}{8} \gamma^{I} C_{JKI}\, \sigma^{JK} \psi$$

Performing the summation and using the following relationships :

$$\gamma^I \sigma^{IJ} = i\gamma^J$$

$$\gamma^I \sigma^{JI} = -i\gamma^J$$

$$\gamma^I \sigma^{JK} = \gamma^J \sigma^{KI} = \gamma^K \sigma^{IJ}$$

gives

$$i\gamma^a \bar{E}^{\mu}_{a} \partial_\mu \psi = 3/4\, \gamma^6 C_{678} \sigma^{78} \psi$$

and since

$$\sigma^{78} = \begin{pmatrix} & -I & & \\ -I & & & \\ & & & -I \\ & & -I & \end{pmatrix}$$

and

$$\gamma^6 = \begin{pmatrix} & & & i\gamma^5 \\ & & i\gamma^5 & \\ & -i\gamma^5 & & \\ -i\gamma^5 & & & \end{pmatrix}$$

and writing

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

gives us :

$$i\gamma^a E^\mu_a \partial_\mu \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \frac{3}{4} C_{678} \begin{pmatrix} -i\gamma^5\psi_3 \\ -i\gamma^5\psi_4 \\ i\gamma^5\psi_1 \\ i\gamma^5\psi_2 \end{pmatrix}$$

For this to be an eigenvalue equation means that ψ has to have the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ i\psi_1 \\ i\psi_2 \end{pmatrix}$$

and our equation becomes two separate equations :

$$i\gamma^a E^\mu_a \partial_\mu \psi_1 = \frac{3}{4} C_{678} \gamma^5 \psi_1$$

$$i\gamma^a E^\mu_a \partial_\mu \psi_2 = \frac{3}{4} C_{678} \gamma^5 \psi_2$$

Now since :

$$R^{(int)} = \frac{1}{4} C_{IJK} C^{IJK} = \frac{6}{4} C_{678} C^{678} \sim 1/r^2$$

then :

$$C_{678} \sim 1/r$$

where r is the radius of curvature of the internal space. If r is of the order of the Planck length then any mass arising from this term will be of the order of the Planck mass. Note also that if ψ does not have the above form then we do not have an eigenvalue equation and we cannot interpret this as a mass term.

2) The terms $i\gamma^a E_a^{\ \mu} A_\mu \partial_5 \psi$ and $i\gamma^a E_a^{\ \mu} W_\mu^i \partial_i \psi$.

In the literature most authors assume that the spinor has some kind of exponential dependence on the internal co-ordinates (see, for example, Souriau 1963, Thirring 1972, Mecklenburg 1980). Carrying on this tradition if we assume our spinor has the following dependence on the internal co-ordinates :

$$\psi(x^\mu, x^5, x^i) = e^{-iex^5} e^{-i g/2\, x^i} \psi(x^\mu)$$

then these two terms give us the standard covariant derivative terms in the equation thus :

$$i\gamma^a E_a^{\ \mu}(\partial_\mu - ieA_\mu) \quad \text{and} \quad i\gamma^a E_a^{\ \mu}(\partial_\mu - i\tfrac{g}{2} W_\mu^i)$$

3) The terms $i\gamma^5 \partial_5 \psi$ and $i\gamma^I E_I^{\ i} \partial_i \psi$.

These terms will decompose under the above scheme into :

$$i\gamma^5 \partial_5 \psi = i\gamma^5(-ie\psi)$$

$$i\gamma^I E_I^{\ i} \partial_i \psi = i\gamma^I(-i g/2\, \psi)$$

Notice that, since in our representation of the Dirac matrices the elements of γ^{I} do not lie on the leading diagonal, we have a further case of component mixing, the phenomenon we first noticed when we considered the mass term.

4) The term $\frac{1}{4}\gamma^{d}\bar{E}_{d}^{\mu}\,\bar{\omega}_{ab\mu}\,\sigma^{ab}$.

This term represents a gravitational interaction term and is usually put equal to zero in the literature by demanding that four dimensional spacetime is flat.

5) The terms $\frac{1}{8}\gamma^{d}f_{ad}\,\sigma^{a5}$ and $\frac{1}{8}\gamma^{d}F_{adI}\,\sigma^{aI}$.

These terms represent contributions from the U(1) and the SU(2) field tensors. The first of these terms also appears in the five dimensional case considered by Thirring and that author argues that it is so small as to be unobservable.

6) The term $\frac{1}{4}\gamma^{d}\bar{E}_{d}^{\mu}\,W_{\mu}^{i}\,e_{i}^{I}\,C_{IKJ}\,\sigma^{JK}$.

This term represents an interaction from the W-vector. Notice again that there will be component mixing in the same way as in the mass term.

The reader will observe that we have avoided coming to any direct conclusions concerning the above Dirac equation : this is because we wish to emphasize the degree of freedom there is in the equation. For example we have shown that because the radius of curvature of the internal space appears in the equation then we expect particles of very high mass to appear. However we can "tune" our mass by changing the original m which appears in the equation. The phenomenon of component mixing awaits interpretation, but may offer us another degree of freedom in dealing with the equation.

Conclusion.

Kaluza's initial hypothesis to extend spacetime from four to five dimensions probably seemed highly speculative at the time. Furthermore the special form of the metric tensor and the necessity to impose certain conditions was more open to criticism than admiration. It was the work of Einstein and Bergmann, however, which honed the theory to perfection, showing that it could now be formulated with only one hypothesis namely that there exists a unit Killing vector field on the extended manifold. The question as to why there should be such a Killing vector field remained unanswered.

In the extension of the theory to $4 + n$ dimensions to describe gauge fields, the concept of a group manifold is introduced. From this we take the hypothesis that the basis vectors of the Lie algebra are Killing vectors of the group manifold, which in turn leads to the same hypothesis for the covariant form of these vectors (see page 32), the first four components of which are identified with the Yang-Mills gauge fields. The question as to the origin of the Killing vectors has now been answered : they come from the geometric unification of spacetime with a group manifold.

The calculation of the scalar curvature of the extended space is the cornerstone of Kaluza-Klein theory and chapter two shows the result for the particular case where the gauge group is $SU(2) \times U(1)$. In this calculation the Killing vector hypothesis is used to great effect and the fundamental result is obtained using completely rigorous methods. No additional hypotheses are necessary. As expected, a cosmological term appears in the Lagrangian in the form of the scalar curvature of the internal space. Since the internal space is assumed to be highly compactified this means that the cosmological term is very large. It may be possible to compensate for this by adding in by hand an additional multidimensional cosmological constant.

In chapter three the exact form of invariant equation of the Dirac type was derived from the multidimensional metric tensor. Having obtained this equation we are faced with the problem of its interpretation. We have shown explicitly how Witten's idea of very massive particles comes about, but other than that all we can say is that there exists a great deal of freedom in treating the finished equation : the fact that the spinor is a sixteen component spinor may make it possible for us to describe four particles (that is four four-component spinors) in one fell swoop but then this freedom is reduced by the phenomenon of component mixing. On an optimistic note, the standard covariant derivatives of Yang-Mills theory are easily obtained by hypothesising that the spinor has an exponential dependence on the internal co-ordinates.

In the text it was stated that the reason for choosing SU(2) x U(1) as the gauge group was to investigate the possible link between this theory and the Weinberg-Salam model. We have been successful, as anticipated, in obtaining the Lagrangian for the U(1) and the SU(2) fields and, in chapter three, in displaying the usual covariant derivative terms but beyond this no further unification has been demonstrated. The measure of the lack of our success lies in the fact that the Weinberg-Salam theory goes much further than this : the left-hand electron and the neutrino are doublets under SU(2) and the right-handed electron is a singlet with U(1) symmetry, this scheme being repeated for muons and muon neutrinos and for any other generations of leptons, for example the tau leptons ; covariant derivatives are defined in the usual way thus introducing the W-vector field and the A-vector field, and then the crucially important phenomenon of spontaneous symmetry breaking is used to give the fields mass, as a result of which some mixing occurs so that the observed photon field is :

$$A_\mu^{(obs)} = \sin\theta_W \, W_\mu^3 + \cos\theta_W \, A_\mu$$

and the neutral intermediate vector boson is :

$$Z_\mu = \cos\theta_W \, W_\mu^3 - \sin\theta_W \, A_\mu$$

where θ_W is the Glashow-Weinberg angle (Aitchison and Hey, 1982).In fairness though, we could not expect much more than that which we have achieved since a complete geometric reproduction of the Weinberg-Salam theory from Kaluza-Klein theory would be a major breakthrough.

In considering the future of Kaluza-Klein theory we must ask how it could be improved. Initial advances will come from the nature of the theory itself : the work here has been done to permit generalisation to larger groups by extending the dimensions of the internal space which, if we are searching for a complete unified field theory, will be necessary to describe the strong interactions too. As far as the internal space itself is concerned we have treated it as a group manifold. Witten, however, has suggested a more economical method which is to consider the internal space as a manifold acted on by the relevant symmetry group. For example the group manifold of

SU(2) is the three-sphere which has as a symmetry group SU(2) x SU(2), that is it invariant with respect to both left and right translation. In his paper " Search for a realistic Kaluza-Klein theory " (Witten 1981) he argues that to effect a complete description of particle interactions the internal space must at least have SU(3) x SU(2) x U(1) as its symmetry group, and that the manifold of minimum dimension with this symmetry group is $CP^2 \times S^2 \times S^1$ which is a seven dimensional manifold. A Kaluza-Klein theory with this internal space would have 4 + 7 = 11 dimensions. This freedom to specify the internal space exactly will be one area whence improvements to Kaluza-Klein theory will come.

In the above paper Witten goes on to point out that supergravity is widely believed not to exist in a space of dimension greater than eleven. This tentative suggestion of a link between the two theories could be a fruitful line of future research and is in fact being actively pursued by those authors who study spontaneous compactification (Cremmer et al 1976a, 1976b, 1977a, 1977b).

One aspect of the theory which has lain dormant is that of the gravitational unification. If ultimately the theory meets all its demands in the particle sector then interest will be focussed on how it is modified by gravity. It should be pointed out though that there is a large degree of separation of the interactions in the theory : in the calculation of the scalar curvature each term stands isolated from its neighbour, and in the Dirac equation there is only one term, $\bar{\omega}_{ab\mu}$, which represents a gravitational interaction. Chodos and Detweiler have looked at the five dimensional theory from the cosmological point of view and shown how it could have happened that the fifth dimension has become so highly compactified. It would be interesting to examine their ideas in the case where spacetime has more dimensions.

The theory of fibre bundles will be another tool open to the future researcher. The geometrical methods now available to the physicist represent a formidable array of techniques and these will be particularly useful when the nature of the four dimensional gravitation is being taken into account (see, for example, Eguchi, Gilkey and Hanson 1981).

Further research is needed and will no doubt quickly come. The steady trickle of papers in Kaluza-Klein theory is growing and, indeed the two scientists who recently won the Nobel prize for their work on unified weak and electromagnetic theory have now turned their attention to Kaluza-Klein theory (Salam and Strathdee 1982, Weinberg 1983).

In summing up our opinions of this theory we point out that it has survived for sixty years and, like a good tree, has bent with the winds of progress and, although not yet able to "explain everything" is still flourishing and looks like continuing to grow for some time yet.

APPENDIX ONE.

The Curvature of the Internal Space.

The purpose of this appendix is to show that the expression for the curvature of the internal space is as quoted in the text. Again Cartan's structure equations will be used so that this derivation mimics part of the main calculation.

The group is described by the left invariant vector fields $\vec{V}_\sigma$ and $\vec{V}_s$ which satisfy :

$$[\vec{V}_\sigma , \vec{V}_s] = 0$$

$$[\vec{V}_\sigma , \vec{V}_\sigma] = 0$$

$$[\vec{V}_s , \vec{V}_t] = C_{st}{}^{v} \vec{V}_v$$

(A.1.)

whose components on a co-ordinate basis are, as before :

$$\vec{V}_\sigma = \vec{\partial}_\sigma$$

$$\vec{V}_s = K^{i}_{s} \vec{\partial}_i$$

(A.2.)

so that :

$$K_s^i \, \partial_i K_t^j - K_t^i \, \partial_i K_s^j = c_{st}{}^v K_v^j$$

(A.3.)

Using the metric on page 30 :

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g_{ij} \end{array} \right)$$

(A.4.)

and making the Killing vector hypothesis gives us :

$$K_s^i \, \partial_i g_{jk} + g_{ik} \, \partial_j K_s^i + g_{ij} \, \partial_k K_s^i = 0$$

(A.5.)

Decomposing the above metric gives the vierbien as :

$$e_5^5 = 1 \quad ; \quad e_i^5 = 0$$

$$e_5^I = 0 \quad ; \quad e_i^I e_j^J \eta_{IJ} = g_{ij}$$

(A.6.)

The above metric can also be expressed in a non-co-ordinate basis :

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g_{st} \end{array}\right)$$

so that the vierbien in this basis are :

$$e^{5}_{5} = 1 \qquad ; \qquad e^{5}_{s} = 0$$

$$e^{I}_{5} = 0 \qquad ; \qquad e^{I}_{s}\, e^{J}_{t}\, \eta_{IJ} = g_{st}$$

and since

$$g_{st} = K^{i}_{s}\, K^{j}_{t}\, g_{ij}$$

then

$$e^{I}_{s} = e^{I}_{i}\, K^{i}_{s}$$

The one-forms are :

$$\omega^{5} = dx^{5}$$

$$\omega^{I} = e^{I}_{i}\, dx^{i}$$

(A.7.)

Cartan's first structure equation is :

$$d\omega^{A} + \omega^{A}{}_{B} \wedge \omega^{B} = 0$$

(A.8.)

and defining the structure constants by :

$$d\omega^{A} = -\frac{1}{2} C_{BC}{}^{A} \omega^{B} \wedge \omega^{C}$$

(A.9.)

gives for the connection one-forms :

$$\omega_{I5} = 0$$

$$\omega_{IJ} = \frac{1}{2} (C_{IJK} + C_{IKJ} - C_{JKI}) \omega^{K}$$

(A.10.)

Cartan's second structure equation is :

$$R^{A}{}_{B} = d\omega^{A}{}_{B} + \omega^{A}{}_{C} \wedge \omega^{C}{}_{B}$$

(A.11.)

so $$R^I{}_J = d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J$$

$$= \tfrac{1}{2}\left(c^I{}_{JM} + c^I{}_{MJ} - c_{JM}{}^I\right)_{,i} E^i_L \; \omega^L \wedge \omega^M$$

$$+\tfrac{1}{2}\left(c^I{}_{JK} + c^I{}_{KJ} - c_{JK}{}^I\right) d\omega^K$$

$$+\tfrac{1}{2}\left(c^I{}_{KL} + c^I{}_{LK} - c_{KL}{}^I\right) \omega^L$$

$$\wedge \tfrac{1}{2}\left(c^K{}_{JM} + c^K{}_{MJ} - c_{JM}{}^K\right) \omega^M$$

Antisymmetrising :

$$R^I{}_J = \tfrac{1}{2}\Big\{ \tfrac{1}{2}\left(c^I{}_{JM} + c^I{}_{MJ} - c_{JM}{}^I\right)_{,i} E^i_L$$

$$-\tfrac{1}{2}\left(c^I{}_{JL} + c^I{}_{LJ} - c_{JL}{}^I\right)_{,i} E^i_M \Big\}\, \omega^L \wedge \omega^M$$

$$+\tfrac{1}{2}\left(c^I{}_{JK} + c^I{}_{KJ} - c_{JK}{}^I\right)\left(-\tfrac{1}{2} c_{LM}{}^K\right) \omega^L \wedge \omega^M$$

$$+\tfrac{1}{8}\Big\{ \left(c^I{}_{KL} + c^I{}_{LK} - c_{KL}{}^I\right)\left(c^K{}_{JM} + c^K{}_{MJ} - c_{JM}{}^K\right)$$

$$- \left(c^I{}_{KM} + c^I{}_{MK} - c_{KM}{}^I\right)\left(c^K{}_{JL} + c^K{}_{LJ} - c_{JL}{}^K\right)\Big\}$$

$$\cdot\, \omega^L \wedge \omega^M$$

But $$R^I{}_J = \frac{1}{2} R^I{}_{JLM}\, \omega^L \wedge \omega^M$$

so

$$R^{IJ}{}_{IJ} = \frac{1}{2}\left(C^I{}_{JJ} + C^I{}_{JJ} - C_{JJ}{}^I\right)_{,i} E^i_I$$
$$-\frac{1}{2}\left(C^I{}_{JI} + C^I{}_{IJ} - C_{JI}{}^I\right)_{,i} E^i_J$$

$$-\frac{1}{2} C_{IJ}{}^K \left(C^{IJ}{}_K + C^I{}_K{}^J - C^J{}_K{}^I\right)$$

$$+\frac{1}{4}\Big\{\left(C^I{}_{KI} + C^I{}_{IK} - C_{KI}{}^I\right)\left(C^{KJ}{}_J + C^K{}_J{}^J - C^J{}_J{}^K\right)$$
$$-\left(C^I{}_{KJ} + C^I{}_{JK} - C_{KJ}{}^I\right)\left(C^{KJ}{}_I + C^K{}_I{}^J - C^J{}_I{}^K\right)\Big\}$$

But

$$C^I{}_{KI} = C^I{}_{JJ} = 0$$

so

$$R = -\frac{1}{2} C_{IJ}{}^K \left(C^{IJ}{}_K + C^I{}_K{}^J - C^J{}_K{}^I\right)$$
$$+\frac{1}{4} C_{KJ}{}^I \left(C^{KJ}{}_I + C^K{}_I{}^J - C^J{}_I{}^K\right)$$

where we have used

$$(c^{I}{}_{KJ} + c^{I}{}_{JK})(c^{KJ}{}_{I} + c^{K}{}_{I}{}^{J} - c^{J}{}_{I}{}^{K}) = 0$$

which follows by reasons of symmetry-antisymmetry.

Therefore :

$$R = -\frac{1}{4} c_{IJ}{}^{K}(c^{IJ}{}_{K} + c^{I}{}_{K}{}^{J} - c^{J}{}_{K}{}^{I})$$

(A.12.)

which is as claimed in equation (2.90).

Note that because of equation (2.92) this expression can also be written as :

$$\begin{aligned} R &= -\frac{1}{4} c_{IJ}{}^{K}(- c^{J}{}_{K}{}^{I}) \\ &= \frac{1}{4} c_{IJ}{}^{K} c^{J}{}_{K}{}^{I} \\ &= \frac{1}{4} c_{IJ}{}^{K} c^{IJ}{}_{K}. \end{aligned}$$

This calculation can also be performed by finding the Riemann tensor from the antisymmetrised covariant derivative of a Killing vector. The starting point is equation (2.27) :

$$K_s^i \partial_i K_t^j - K_t^i \partial_i K_s^j = c_{st}{}^v K_v^j$$

and because the connection coefficients are symmetric we can replace ordinary derivatives by covariant ones :

$$K_s^i K_{t\,;i}^j - K_t^i K_{s\,;i}^j = c_{st}{}^v K_v^j$$

Multiply by K_k^t :

$$K_k^t K_s^i K_{t\,;i}^j - \delta_k^i K_{s\,;i}^j = c_{st}{}^v K_v^j K_k^t$$

$$K_k^t K_s^i K_{t\,;i}^j - K_{s\,;k}^j = c_{st}{}^v K_v^j K_k^t$$

(A.13.)

Now the covariant derivative of the metric tensor is zero :

$$g_{ij\,;k} = 0 \;.$$

and using

$$g_{ij} = K_i^s K_j^t g_{st}$$

gives

$$K_i^t K_{tj\,;k} + K_j^t K_{ti\,;k} = 0$$

where we have written $g_{st} K_j^s$ as K_{tj} .

Raising the j index :

$$K_i^t K_{t\,;k}^j + K^{tj} K_{ti\,;k} = 0$$

(A.14.)

so that (A.13.) becomes :

$$-K_s^i K^{tj} K_{tk\,;i} - K_{s\,;k}^j = c_{st}{}^v K_v^j K_k^t$$

Now using Killing's equation in the form :

$$K_{k\,;i}^t + K_{i\,;k}^t = 0$$

gives

$$K_s^i K^{tj} K_{ti\,;k} - K_{s\,;k}^j = c_{st}{}^v K_v^j K_k^t$$

and using (A.14) once again :

$$-K^i_s K^t_i K^j_{t;k} - K^j_{s;k} = c_{st}{}^{v} K^j_v K^t_k$$

$$-\delta_s^t K^j_{t;k} - K^j_{s;k} = c_{st}{}^{v} K^j_v K^t_k$$

$$-K^j_{s;k} - K^j_{s;k} = c_{st}{}^{v} K^j_v K^t_k$$

Therefore :

$$K^j_{s;k} = -\frac{1}{2} c_{st}{}^{v} K^j_v K^t_k \tag{A.15.}$$

Now the Riemann tensor can be found from :

$$K^i_{s;j;k} - K^i_{s;k;j} = -K^l_s R^i{}_{ljk}$$

$$= \left\{ K^i_{s;j} \right\}_{;k} - j \leftrightarrow k.$$

$$= \left\{ -\frac{1}{2} c_{st}{}^{v} K^i_v K^t_j \right\}_{;k} - j \leftrightarrow k$$

$$= -\frac{1}{2} c_{st}{}^{v} K^i_v \left\{ K^t_{j;k} \right\} - \frac{1}{2} c_{st}{}^{v} K^t_j \left\{ K^i_{v;k} \right\} - j \leftrightarrow k.$$

$$= -\tfrac{1}{2} c_{st}{}^{v} K^{i}_{v} \left\{ -\tfrac{1}{2} c^{t}{}_{y}{}^{x} K_{jx} K^{y}_{k} \right\}$$

$$-\tfrac{1}{2} c_{st}{}^{v} K_{j}{}^{t} \left\{ -\tfrac{1}{2} c_{vy}{}^{x} K^{y}_{k} K^{i}_{x} \right\} \quad - \quad j \leftrightarrow k$$

$$= \tfrac{1}{4} c_{st}{}^{v} c^{t}{}_{y}{}^{x} K^{i}_{v} K_{jx} K^{y}_{k}$$

$$+\tfrac{1}{4} c_{st}{}^{v} c_{vy}{}^{x} K^{t}_{j} K^{y}_{k} K^{i}_{x}$$

$$- \quad j \leftrightarrow k$$

So, explicitly :

$$R^{i}{}_{ljk} = -K^{s}_{l} \left\{ \tfrac{1}{4} c_{st}{}^{v} c^{t}{}_{y}{}^{x} K^{i}_{v} K_{jx} K^{y}_{k} \right.$$

$$\left. +\tfrac{1}{4} c_{st}{}^{v} c^{t}{}_{y}{}^{x} K^{t}_{j} K^{y}_{k} K^{i}_{x} \right\} \quad - \quad j \leftrightarrow k$$

So $\quad R = R^{jk}{}_{jk}$

$$= -\tfrac{1}{4} c_{st}{}^{v} c^{t}{}_{y}{}^{x} K^{sk} K^{j}_{v} K_{jx} K^{y}_{k}$$

$$-\tfrac{1}{4} c_{st}{}^{v} c_{vy}{}^{x} K^{sk} K^{t}_{j} K^{y}_{k} K^{j}_{x}$$

$$+\tfrac{1}{4} c_{st}{}^{v} c^{t}{}_{y}{}^{x} K^{j}_{v} K_{kx} K^{sk} K^{y}_{j}$$

$$+\tfrac{1}{4} c_{st}{}^{v} c_{vy}{}^{x} K^{sk} K^{t}_{k} K^{y}_{j} K^{j}_{x}$$

$$= -\frac{1}{4} C_{st}{}^{v} C^{t}{}_{y}{}^{x} \; g^{sy} g_{vx}$$

$$-\frac{1}{4} C_{st}{}^{v} C_{vy}{}^{x} \; g^{sy} \delta^{t}_{x}$$

$$+\frac{1}{4} C_{st}{}^{v} C^{t}{}_{y}{}^{x} \; \delta^{y}_{v} \delta^{s}_{x}$$

$$+\frac{1}{4} C_{st}{}^{v} C_{vy}{}^{x} \; g^{st} \delta^{y}_{x}$$

and the last term vanishes because of the antisymmetry in the first two indices in the structure constants.Therefore :

$$R = -\frac{1}{4} C_{st}{}^{v} C^{ts}{}_{v} - \frac{1}{4} C_{st}{}^{v} C_{v}{}^{st}$$

$$+\frac{1}{4} C_{st}{}^{v} C^{t}{}_{v}{}^{s}$$

$$= -\frac{1}{4} C_{st}{}^{v} \left\{ C^{ts}{}_{v} + C_{v}{}^{st} - C^{t}{}_{v}{}^{s} \right\}$$

$$= -\frac{1}{4} C_{st}{}^{v} \left\{ -C^{st}{}_{v} - C^{s}{}_{v}{}^{t} - C^{t}{}_{v}{}^{s} \right\}$$

because of equation (2.91),

$$R = \frac{1}{4} C_{st}{}^{v} C^{st}{}_{v}$$

and because of equation (2.69) this is the same as our previous answer.

APPENDIX TWO.

Lie Groups and Lie Derivatives.

The point of this brief appendix is to outline some of the ideas used in the text concerning group theory, Lie derivatives and Killing vectors. It is not intended to be an exhaustive account of these topics and for further elucidation the reader is referred to the literature (see, for example, Schutz 1980).

A group is a collection of elements under a binary operation if it exhibits the usual group axioms, namely closure, associativity, the existence of an identity element and the existence of inverses for every element. If the elements are continuous then the group can be considered as a manifold. Now any element g of the group maps any other element h under left translation to gh and under right translation to hg. These mappings will carry tangent vectors at one point to tangent vectors at the image point. A vector field is a rule which defines a vector at each point so if a mapping carries a vector at one point to the corresponding vector at the image point then that vector field is said to be invariant under the mapping. In particular, if a vector field is invariant under left translation then it is known as a left-invariant vector field. Similarly, each vector at a particular point defines a left invariant vector field by the operation of left translation. If $\bar{v}$ and $\bar{w}$ are two left invariant vector fields then so is $[\bar{v}, \bar{w}]$ so that the left invariant vector fields form a Lie algebra, known as the Lie algebra of the group. If we take some set of linearly independent left-invariant vector fields then we can write :

$$[\bar{V}_s , \bar{V}_t] = C_{st}{}^{v} \bar{V}_v$$

which defines the structure constants of the group. The significance of this lies in the fact that every Lie group has its own uniqe set of structure constants.

The Lie derivative is a co-ordinate independent way of taking derivatives of functions or vectors or tensors on a manifold. Loosely, the Lie derivative of a function is the difference in value of that function at two points on a curve separated by a parameter value $\Delta\lambda$, divided by that parameter value $\Delta\lambda$. Similar definitions can be made for vectors, one-forms or tensors. Now a vector field defines a set of curves to which it is always tangent, called the integral curves, so that the Lie derivative with respect to a given vector field involves taking the difference along these integral curves.

If the Lie derivative of the metric tensor is zero with respect to a given vector field then that vector field is known as a Killing vector field, and an analytic expression for its components can be derived as follows :

The metric tensor is a rank 2 covariant tensor as is its Lie derivative, and, furthermore, the Lie derivative operation follows the Liebnitz rule, so that :

$$\mathcal{L}_{\bar{V}}\, g\!\!\mid(\bar{X},\bar{Y}) = (\mathcal{L}_{\bar{V}}\, g\!\!\mid)(\bar{X},\bar{Y}) + g\!\!\mid(\mathcal{L}_{\bar{V}}\bar{X},\bar{Y}) + g\!\!\mid(\bar{X},\mathcal{L}_{\bar{V}}\bar{Y}) \qquad (\text{A2.1})$$

where $\bar{X}$ and $\bar{Y}$ are two arbitrary vectors. But

$$\mathcal{L}_{\bar{V}}\bar{X} = [\bar{V},\bar{X}]$$

which in components is :

$$(\mathcal{L}_{\bar{V}}\bar{X})^i = V^j \partial_j X^i - X^j \partial_j V^i$$

so that (A2.1) can be written in components as :

$$V^i \partial_i (g_{jk} X^j Y^k) = (\mathcal{L}_{\bar{V}}\, g\!\!\mid)_{jk} X^j Y^k + g_{jk}(V^l \partial_l X^j - X^l \partial_l V^j) Y^k + g_{jk} X^j (V^l \partial_l Y^k - Y^l \partial_l V^k)$$

Therefore :

$$(\mathcal{L}_{\bar{V}}\, g\!\!\mid)_{jk} X^j Y^k = V^i \partial_i (g_{jk} X^j Y^k) - g_{jk}(V^l \partial_l X^j - X^l \partial_l V^j) Y^k - g_{jk} X^j (V^l \partial_l Y^k - Y^l \partial_l V^k)$$

$$
\begin{aligned}
&= v^i(\partial_i g_{jk}) X^j Y^k + \underbrace{g_{jk} v^i (\partial_i X^j) Y^k}_{(1)} \\
&\quad + \underbrace{g_{jk} v^i(\partial_i Y^k) X^j}_{(2)} - \underbrace{g_{jk} v^i (\partial_i X^j) Y^k}_{(1)} \\
&\quad + g_{jk} X^i (\partial_i v^j) Y^k - \underbrace{g_{jk} X^j v^i (\partial_i Y^k)}_{(2)} \\
&\quad + g_{jk} X^j Y^i (\partial_i v^k)
\end{aligned}
$$

and the underlined terms cancel as indicated. Changing dummy suffixes gives us :

$$
\begin{aligned}
(\mathcal{L}_{\bar{V}}\, g)_{jk} X^j Y^k &= v^i \partial_i g_{jk} X^j Y^k \\
&\quad + g_{ik} X^j Y^k \partial_j v^i \\
&\quad + g_{ij} X^j Y^k \partial_k v^i
\end{aligned}
$$

so that finally because of the arbitrariness of $\bar{X}$ and $\bar{Y}$:

$$
(\mathcal{L}_{\bar{V}}\, g)_{jk} = v^i \partial_i g_{jk} + g_{ik} \partial_j v^i + g_{ij} \partial_k v^i
$$

and by demanding that this is zero gives equation (2.16).

TABLE 2.1 - The Index Convention.

	WORLD INDICES	TANGENT SPACE INDICES
The extended manifold :	α, β	A, B
Four dimensional spacetime :	μ, ν	a, b
The fifth dimension :	5	$\bar{5}$
The internal space :	i, j, k	I, J, K

TABLE 2.2 - Reference Guide to Various Quantities.

The Vierbien.

The four dimensional vierbien and the internal vierbien are defined normally:

$$e_\mu^{\ a} e_\nu^{\ b} \eta_{ab} = g_{\mu\nu}$$

$$e_i^{\ I} e_j^{\ J} \eta_{IJ} = g_{ij}$$

Most of the others are zero :

$$e_5^{\ a} = e_5^{\ J} = e_i^{\ a} = e_i^{\ 5} = 0$$

but

$$e_\mu^{\ 5} = A_\mu$$

$$e_\mu^{\ I} = e_i^{\ I} W_\mu^{\ i}$$

$$e_5^{\ 5} = 1$$

The Inverse Vierbien.

As expected :

$$E^{\mu}_{a} E^{\nu}_{b} \eta^{ab} = g^{\mu\nu}$$

$$E^{i}_{I} E^{j}_{J} \eta^{IJ} = g^{ij}$$

and

$$E^{\mu}_{5} = E^{\mu}_{I} = E^{5}_{I} = E^{i}_{5} = 0$$

and

$$E^{5}_{a} = -E^{\nu}_{a} A_{\nu}$$

$$E^{i}_{a} = -E^{\mu}_{a} W^{i}_{\mu}$$

$$E^{5}_{5} = 1$$

The Structure Constants.

$$C_{bc}{}^{a} = \overline{c_{bc}{}^{a}}$$

$C_{b5}{}^{a} = 0$

$C_{bJ}{}^{a} = 0$

$C_{5J}{}^{a} = 0$

$C_{IJ}{}^{a} = 0$

$C_{a5}{}^{5} = 0$

$C_{aJ}{}^{5} = 0$

$C_{5J}{}^{5} = 0$

$C_{IJ}{}^{5} = 0$

$C_{a5}{}^{I} = 0$

$C_{5J}{}^{I} = 0$

$$C_{ab}{}^{5} = -f_{ab}$$

$$C_{ab}{}^{I} = -F_{ab}{}^{I}$$

$$C_{aJ}{}^{I} = C_{JK}{}^{I} W_{a}{}^{K}$$

The Ricci Rotation Coefficients.

$$\omega_{ab\mu} = \bar{\omega}_{ab\mu} + \frac{1}{2} A_\mu f_{ab} + \frac{1}{2} e^I_\mu F_{abI}$$

$$\omega_{ab5} = \frac{1}{2} f_{ab}$$

$$\omega_{abi} = \frac{1}{2} e^I_i F_{abI}$$

$$\omega_{a5\mu} = \frac{1}{2} e^c_\mu f_{ac}$$

$$\omega_{aI\mu} = \frac{1}{2} e^c_\mu F_{acI}$$

$$\omega_{IJ\mu} = \frac{1}{2} e^K_\mu C_{IJK}$$

$$\omega_{IJi} = -\frac{1}{2} e^K_i C_{IJK}$$

$$\omega_{a55} = 0$$

$$\omega_{a5i} = 0$$

$$\omega_{aI5} = 0$$

$$\omega_{aIj} = 0$$

$$\omega_{5I\mu} = 0$$

$$\omega_{5I5} = 0$$

$$\omega_{5Ij} = 0$$

$$\omega_{IJ5} = 0.$$

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