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Uniform Finite Time Stabilisation of Non-smooth and Variable Structure Systems with Resets

A Thesis Submitted to The University
of Kent
For The Degree of Doctor of Philosophy
In Electronic Engineering

By
Harshal Oza
December 2012

Abstract

This thesis studies uniform finite time stabilisation of uncertain variable structure and non-smooth systems with resets. Control of unilaterally constrained systems is a challenging area that requires an understanding of the underlying mechanics that give rise to reset or jumps while synthesizing stabilizing controllers. Discontinuous systems with resets are studied in various disciplines. Resets in states are hard nonlinearities. This thesis bridges non-smooth Lyapunov analysis, the quasi-homogeneity of differential inclusions and uniform finite time stability for a class of impact mechanical systems. Robust control synthesis based on second order sliding mode is undertaken in the presence of both impacts with finite accumulation time and persisting disturbances. Unlike existing work described in the literature, the Lyapunov analysis does not depend on the jumps in the state while also establishing proofs of uniform finite time stability. Orbital stabilization of fully actuated mechanical systems is established in the case of persisting impacts with an *a priori* guarantee of finite time convergence between the periodic impacts.

The distinguishing features of second order sliding mode controllers are their simplicity and robustness. Increasing research interest in the area has been complemented by recent advances in Lyapunov based frameworks which highlight the finite time convergence property. This thesis computes the upper bound on the finite settling time of a second order sliding mode controller. Different to the latest advances in the area, a key contribution of this thesis is the theoretical proof of the fact that finite settling time of a second order sliding mode controller tends to zero when gains tend to infinity. This insight of the limiting behaviour forms the basis for solving the converse problem of finding an explicit a priori tuning formula for the gain parameters of the controller when an arbitrary finite settling time is given. These results play a central role in the analysis of impact mechanical systems. Another key contribution of the thesis is that it extends the above results on variable structure systems with and without resets to non-smooth systems arising from continuous finite time controllers while proving uniform finite time stability.

Finally, two applications are presented. The first application applies the above theoretical developments to the problem of orbital stabilization of a fully actuated seven link biped robot which is a nonlinear system with periodic impacts. The tuning of the controller gains leads to finite time convergence of the tracking errors between impacts while being robust to disturbances. The second application reports the outcome of an experiment with a continuous finite time controller.

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CHAPTER 1

Introduction

Feedback control of dynamical systems has been one of the most rigorously studied topics in the second half of the twentieth century. It utilizes mathematical insight to solve intricate problems of stability and performance for dynamical systems. Of principal interest is stability analysis and robust control synthesis for uncertain systems. Classical control methods (e.g. frequency domain methods) and modern control theory (e.g. the state space approach) have established a rich literature on the fundamental task of stabilizing the system and/or obtaining desired performance from a given system. One of the central themes in feedback control design is the analysis of the robustness of a particular control synthesis to external disturbances and to modelling uncertainties. This philosophy of robust control theory is naturally applicable to many engineering applications where the knowledge of the nominal dynamics is often available whereas the aim of the feedback control is to ensure that the performance of the real system under non-nominal conditions remains satisfactory.

Amongst various robust control frameworks, the traditional and second order sliding mode (SOSM) controllers have proved to be a popular method for robust control of uncertain nonlinear systems [1–4]. Sliding mode controllers belong to the family of variable structure controllers [1]. Sliding mode controllers render a defined manifold of the state-space attractive such that, once on the manifold, the system dynamics are asymptotically stable. Inherent invariance properties to a class of disturbances, namely, 'matched disturbances' [1] make sliding mode control attractive. Stabilizing controllers for systems represented by linear and nonlinear differential equations have been established using a sliding mode control framework [5]. The problem of finding a mathematical solution for differential equations with discontinuous right hand sides [6] plays a crucial role in the defining the solutions of the systems employing sliding mode control. The closed-loop dynamics are not only non-Lipschitz but also discontinuous.

In general, discontinuous right hand sides give rise to non-unique solutions in forward time. A distinct feature of sliding mode controllers is that robustness to disturbances with a uniform persistent bound is guaranteed, a robustness property which naturally makes sliding mode controllers a very popular tool in engineering applications [7, 8] and which remains the main driver for theoretical research in control of non-linear discontinuous systems [9, 10].

A less robust class of non-Lipschitzian controllers than the aforementioned discontinuous controllers are the class of continuous finite time controllers. Since early study [11], continuous finite time stabilisation methods have attracted considerable theoretical and practical interest [12–15]. The principal theoretical undercurrent is that the overall closed-loop dynamics are continuous but non-Lipschitz. Although a significant portion of the study of stability and control of dynamical systems often starts from the assumption of Lipschitz continuity of the vector field [16], it is not uncommon to find non-Lipschitz dynamics in real life scenario such as seismic waves approaching the surface of the earth and turbulence (Navier-Stokes equations) amongst others [17]. Non-Lipschitz systems do not possess unique solutions in forward time in general [13] in contrast to the guaranteed existence of a unique solution in the case of Lipschitz dynamics [16]. It was established in references [11] and [13] that it is the non-Lipschitzian dynamics that lead to finite time convergence of solutions of a continuous closed-loop system to the stable equilibrium. Furthermore, the non-Lipschitz continuous finite time controllers possess better robustness properties than their Lipschitz counterparts [13, Th. 5.2, Th.5.3]. However, continuous finite time controllers are less robust when compared to discontinuous controllers as is evident from the fact that the trajectories of the closed-loop system are only ultimately bounded under the continuous finite time synthesis [13, Th. 5.2] when persisting disturbances appear. In contrast, finite time convergence of the trajectories is obtained in the case of second and higher order sliding mode controllers [4, 18] despite the persisting disturbances.

This thesis studies uniform finite time stability of non-smooth continuous and variable structure systems in a plane with and without resets. Finite time stability is perceived in the sense that the trajectories of a planar system converge to the origin in finite time [4, 9, 13] rather than diminish to a bounded set over a finite time interval without actually converging to the origin [19].

Research on discontinuous systems is currently stimulating considerable interest [20, 21]. Discontinuous systems are studied in such diverse research fields as electrical circuit theory [22], mechanical engineering (non-smooth mechanics) [23] and control of hybrid systems [24]. Many different frameworks therefore exist to describe various classes of discontinuous systems, e.g., differential inclusions [6] and measure differential inclusions [25], complementarity systems [23] and [25], variational inequalities [23, Chapter 3] and vibro-impact modelling [26] to name a few. The topic of analysis and robust control for systems with reset events in the state is theoretically challenging and practically relevant. The discontinuity in the vector field, however, may be caused by either the introduction of a discontinuous feedback control law or by the interaction of the open loop system dynamics with its environment. For example, the discontinuity may be caused by the introduction of a discontinuous controller such as a variable structure controller. There are examples of systems where the interaction with the environment causes the right hand side of the dynamical system to take a discontinuous form. Systems with unilateral constraints [22, 23] present a large class of discontinuous systems. Unilaterally constrained systems have state jumps or reset events occurring in an infinitesimally small time giving rise to discontinuity of the vector field. Another form of discontinuity in the open-loop dynamics is the well-known Coulumb friction which involves discontinuity around the point of zero velocity [20].

Resets represent hard nonlinearity in the closed-loop dynamics. This form of discontinuity occurs when a state variable undergoes a jump in its value in an infinitesimally small time. Such systems occur in various disciplines of feedback control [23, 24]. The practical need for studying such systems, and proposing a synthesis framework, has been demonstrated by recent applications to biped robots (see for example [27] and [28]), where the generalized velocities of a robot inherently undergo a reset when the swing leg collides with the ground. Studying systems with resets also finds natural application in the area of hybrid systems [29]. Developing a clear understanding of Lyapunov based stability and robustness properties of a closed-loop system when resets with (or without) a finite accumulation point is a challenging and interesting area of study (see reference [30] and references therein for hybrid systems and [31, 32] for the area of biped robots).

There is plentiful research work pertaining to Lyapunov based stability analysis tools which study asymptotic stability of systems with impulse effects [33–35]. Work in the area of tracking control of impact mechanical systems can be found in references [36–38] pertaining to various forms of Lyapunov stability. The stability of unilaterally constrained systems are studied via analysis of Lyapunov functions in continuous and discrete event phases. It is evident from the aforementioned references that the analysis results obtained are only limited to asymptotic stability [39, 40]. The formal study of finite time stability and stabilisation of unilaterally constrained systems is a more open problem. Uniform finite time stability and stabilisation of unilaterally constrained nonautonomous systems is a new direction of research.

It is known, due to recent advances in Lyapunov theory [9, 13, 41, 42], that the aforementioned non-Lipschitzian controllers (both non-smooth [11, 12] and variable structure controllers [2, 4, 5]) offer a more robust alternative to the Lipschitzian (or linear) controllers due to the superior robustness properties of the former in the stabilisation of systems in the presence of disturbances with persisting or uniformly decaying bounds in the right hand side of the underlying differential equations. It is then natural to expect the same superior robustness properties while studying the uniform finite time stabilisation and uniform finite time stability of unilaterally constrained systems. This is the principal motivation for all the results presented here.

There are three main motivations to study *uniform* stability of dynamical systems. The first scenario where uniformness is of importance is when the right hand side of the differential equation does not involve resets but is a function of time t. These systems are formally known as nonautonomous systems [16]. As with all other stability concepts [43, Section 5], uniform finite time stability does not follow directly from finite time stability for non-Lipschitz systems [44] and is an open problem in the continuous finite time stabilisation literature [11–14, 44–46] even for a simple perturbed double integrator when time varying disturbances appear on the right hand side. The section of the thesis pertaining to the continuous finite time stabilisation is strongly motivated by this open problem.

The second scenario where uniformness is important is when uniformity with respect to disturbances as well as initial data (state and time) is considered. Since this thesis intends to study uniform finite time stability in the case of non-smooth controllers with and without resets in the presence of time varying disturbances with uniformly decaying bound, uniform finite time stability becomes an important problem, a paradigm that is in its infancy even when there are no resets [44]. Also, since the thesis will study variable structure controllers with resets in the presence of time varying discontinuous disturbances with a persisting bound, existing Lyapunov approaches [9, 41, 42] for proving uniform finite time stability do not apply due to the fact that Fillipov's solutions [6] are not defined for such systems. A central theme in the intended Lyapunov analysis is the study of the uniformity with respect to the initial data as well as with respect to the time varying disturbances. For continuous non-autonomous systems, the first challenge of any Lyapunov based asymptotic stability study is often the proof of uniformity of the convergence of the trajectories to the origin with respect to the initial time [16, Th. 4.9]. In other words, the asymptotic convergence of the trajectories does not depend on the initial time and a uniform (or similar) convergence can be obtained for all initial time instants. The second challenge is the uniformity of stability with respect to time varying disturbances. In the area of continuous finite time stability, uniformity of Lyapunov based stability proofs has not been studied rigorously with respect to time varying disturbances. Although general results on Lyapunov theorems [44] exist, explicit identification of a Lyapunov function and computation of the settling time in the presence of (possibly variable structure) time-varying disturbances has never been undertaken in the area of uniform continuous finite time controllers. This is the motivation for studying robust continuous finite time stabilisation and identifying Lyapunov functions to achieve uniform finite time stability with respect to both initial data (state and time) and with respect to time varying (possibly discontinuous) disturbances.

The third scenario where uniformness is of importance is when the right hand side of the differential equation does not depend on time t but is discontinuous where hard non-linearities like resets appear. Solutions of these systems are understood in terms of measure differential equations [47]. The closed-loop autonomous discontinuous systems result from the application of the sliding mode control of the open-loop linear time invariant systems when no external disturbances appear. It is well-known that asymptotic stability substantiated by strong Lyapunov functions implies uniform asymptotic stability for nonautonomous systems [16, Section 4.5] [43, Section 5]. However, this does not generally hold true for discontinuous right hand sides with resets. This is because the standard arguments, that an arbitrary region independent of the initial time t_0 can be defined, do not hold true when, for example, resets occur at the initial time. Furthermore, the discontinuous variable structure controllers are capable of rejecting uniformly bounded time varying disturbances [2][9, Th. 3.2]. Proving uniformity of the resulting non-autonomous closed-loop system is not trivial and has to be substantiated by strong Lyapunov functions even when there is no reset in any of the states. This has been the motivation of recent research work to identify smooth and non-smooth Lyapunov functions explicitly to achieve uniform asymptotic and uniform finite time stability of discontinuous autonomous systems [9, 41, 42]. Extending this uniform finite time stability framework for variable structure systems with resets in the presence of time varying disturbances is a strong motivation.

Resets can be viewed as giving rise to hybrid systems. Hard non-linearities such as resets are often studied in the area of hybrid systems. The literature on hybrid systems is vast and the subject is continually attracting a lot of interest [24]. A natural question as to why a study of resets with and without accumulation point is needed can be posed due to the rich literature on the Lyapunov stability of hybrid systems [29, 48–54]. The above references can be subdivided into those results which formulate the hybrid system as a time invariant one (e.g. [48]) and those which formulate the hybrid systems as a time varying one (e.g. [53, 54]). Both classes of problems inherently impose conditions on the post jump values of the Lyapunov functions [53, Th. 11], [54, Th. 3.1] in addition to the standard conditions for the continuous phase of the dynamics [16, Th. 4.9]. This thesis is motivated by two aspects in the case of systems with resets which are distinctly different in comparison to the methodology used in both the above subclasses of hybrid systems. In the case of time invariant systems with Zeno behaviour of the impacts (such as that studied in [48]), time varying disturbances are not studied while studying uniform Lyapunov stability [48, Th. 3.3]. This thesis will study planar non-autonomous systems arising from a combination of impacts having a finite accumulation point (Zeno behaviour) with time varying discontinuous disturbances. Studying uniform finite time stability of systems with resets when resets have a finite accumulation point and when time varying disturbances appear is a novel theoretical direction especially when the same can be studied without analysing the jumps.

In the case of the time varying hybrid systems literature [53, Th. 11], [54, Th. 3.1], finite time stability is not proven and can be identified as an open problem. From the view point of hybrid systems, this thesis has strong theoretical motivation in that it seeks to prove finite time stability and to compute upper bounds on the finite settling time of a class of unilaterally constrained planar systems (a class of hybrid systems) in the presence of time varying disturbances when resets have a finite accumulation point. Also, finite time orbital stabilisation of unilaterally constrained planar systems is studied when the resets do not have a finite accumulation point.

The beginning of the thesis emphasises developing constructive tuning rules to achieve a pre-specified settling time. Growing interest in the area of SOSM calls for straightforward tuning rules. Several results exist on the problem of proving the finite nature of the settling time given finite gains of a SOSM but the converse problem of finding a tuning formula for the gains given a finite settling time is absent. Ubiquitous controllers such as PID controllers enjoy automatic tuning algorithms to achieve various performance criteria a priori [55]. Amongst other examples, linear systems analysis enjoys mathematical tools like Laplace transforms and state-space solutions to judge if the settling time of exponentially stable systems tend to zero in the limit when the gains of the state feedback matrix tend to infinity. Although, this feature is something natural to expect from a robust controller like SOSM, interestingly, it has never been proven theoretically in the existing works that increasing the gains of a SOSM leads to a decrease in the settling time. This thesis is driven by the need to prove this intuitive result mathematically. The rigorous literature in the area of SOSM gives tuning rules for the gains of a SOSM controller sufficient only to ensure finite time convergence and not for an arbitrary reduction in the finite settling time when increasing the gains of the controller. The study presented in the thesis is thus also motivated by this desire to achieve an arbitrary reduction in the settling time. Such tuning guidelines, as will be shown later, are supported by the proof of the limiting behaviour of the settling time when the gains of a SOSM controller tend to infinity. This result will also form the basis for achieving similar results when the impacts are present.

In the central chapters of the thesis, the application of SOSM to achieve uniform finite time stabilisation and finite time orbital stabilisation of systems with resets is motivated by the above theoretical results obtained for the case without resets. Towards the end of the thesis, orbital stabilisation is studied. It is important to nullify any error caused by the hard nonlinearities when the finite time orbital stabilisation is to be achieved successfully. Frameworks for tracking control of systems with recurring impacts finds natural application in tracking control of biped robots [15, 32]. The control of biped robots poses two challenges. The first challenge is to stabilize the robot around its stable walking gait. This is known as the swing phase or the continuous dynamics. The second challenge is to account for the impulse effects arising from the collision of the feet with a surface while walking. The main motivation for finite time orbital stabilisation lies in formulating a robust SOSM synthesis such that the system exhibits a robust performance in the presence of disturbances in both the continuous and impulse phases of fully actuated systems. It is then natural to anticipate that the combination of the finite time convergence guarantee stemming from the tuning rules coupled with the robustness features of SOSM will pave the way for successfully negotiating the instabilities caused by disturbances both in the continuous phase as well as at the time of resets in velocity for systems such as biped robots. A strong motivation exists to perform this study as the literature does not have results which formulate a robust synthesis in the presence of disturbances in the continuous and the impact phases of the class of systems being studied while also giving constructive tuning guidelines.

1.1 Contributions and thesis organisation

The thesis contributes the following four main results:

1. The theoretical proof of the fact that the settling time of a second order sliding mode controller approaches zero in the limit as gains of the controller tend to infinity.

2. Identification of three new Lyapunov functions (one function for the variable structure controller in the presence of resets and two functions for the non-smooth controller both in the presence and absence of resets) is carried out to prove equiuniform exponential stability of the resulting closed-loop planar systems.

The key contribution is the proof of equiuniform finite time stability of non-smooth and variable structure planar systems with and without resets in the presence of time varying variable structure disturbances.

3. Output feedback: tuning rules for the gains for the second order sliding mode controller and observer.

4. Orbital stabilisation of fully actuated unilaterally constrained uncertain planar systems in finite time using second order sliding mode synthesis.

This thesis is organised as follows.

Chapter 2 gives an overview of continuous time sliding mode control (SMC). In particular, the robustness properties of the sliding mode controllers are discussed. Next, a literature survey of second order sliding mode (SOSM) control is presented. The survey on this topic concludes by references to Lyapunov approaches for SOSM. The chapter also gives a detailed literature review of continuous finite time stabilisation before concluding the chapter by presenting a literature review on the control of unilaterally constrained systems.

Chapter 3 presents a novel homogeneity approach to obtain an upper bound on the settling time for a robust second order sliding mode controller. The stability analysis will be substantiated by a globally radially unbounded non-smooth Lyapunov function. The proposed method will be applied to the 'twisting' controller and will be based on a combination of global exponential stability and global finite time stability of switched systems. The homogeneity regions will be established and graphically illustrated. Tuning rules for the twisting controller will also be presented. It will be shown that the proposed framework does not depend on the differential inequality of the Lyapunov function and hence provides a new methodology for obtaining the upper bound on the settling time for exponentially stable homogeneous systems. This chapter includes the results of a recently published conference publication [56].

Chapter 4 will present a switched control synthesis utilising a robust second-order sliding mode controller to achieve global uniform finite time stability. The framework will be based on straightforward step-by-step application of a classical linear feedback design and the well-known twisting controller. The underlying philosophy is to utilize globally exponentially stable linear feedback so that the trajectories enter an arbitrarily defined domain of attraction in finite time and then switch to the twisting controller so that the trajectories settle at the origin in finite time. The proposed method will be applied to the perturbed double integrator to obtain an upper bound on the settling time of the closed-loop system in a full-state feedback setting and in the presence of time varying disturbances with a persisting bound. Tuning rules to achieve the desired settling time will be explicitly derived without recourse to the differential inequality of the Lyapunov function. This chapter will contribute the first result as outlined above thereby summarising the results of the recent publication [57].

Chapter 5 studies uniform finite time stabilisation of a unilaterally constrained double integrator. The variable structure twisting controller will be utilised in a full state feedback setting for uniform finite time stabilization of a perturbed double integrator in the presence of both a unilateral constraint and uniformly bounded persisting and time varying disturbances. The unilateral constraint involves rigid body inelastic impacts causing jumps in one of the state variables. Firstly, a non-smooth state transformation will be employed to transform the unilaterally constrained system into a jump-free system. The transformed system will be shown to be a switched homogeneous system with negative homogeneity degree where the solutions are well defined. Secondly, a non-smooth Lyapunov function will be identified to establish uniform asymptotic stability of the transformed system. This is the first of the three new Lyapunov functions mentioned above. The global uniform finite time stability with respect to the initial data and with respect to persisting disturbances will then be proved by utilising the homogeneity principle of switched systems. The proposed results will bridge non-smooth Lyapunov analysis, quasi-homogeneity and finite time stability for a class of impact mechanical systems. This chapter will present the part of the second main contribution outlined above (uniform finite time stabilisation using variable structure controllers) while giving a highly detailed account of the recently published articles [58, 59].

Chapter 6 investigates the problems of chapter 4 and 5 but with a continuous non-smooth homogeneous controller. A new Lyapunov function is identified for each of the following problems: (i) uniform continuous finite time stabilisation of a double integrator perturbed by a time varying discontinuous disturbance with a uniformly decaying bound and (ii) the problem in (i) when a unilateral constraint is present as studied in chapter 5. These two new Lyapunov functions are the last two of the three mentioned above. After establishing the uniform exponential stability, the methodology is the same for both objectives. Uniform finite time stability is established with respect to initial conditions and with respect to disturbances for each of the above two cases by extending the quasi-homogeneity principle of switched system. Existing finite time stabilisation approaches will also be discussed to properly motivate the above study. This chapter will present the part of the second main contribution outlined above (uniform finite time stabilisation using continuous finite time controllers). In the context of the discussion in the previous section on uniform continuous finite time stability, this chapter presents a strong contribution in the area of non-smooth controllers when applied to planar systems with and without resets alike.

Chapter 7 establishes the tuning rules similar to Chapter 4 to guarantee that an arbitrarily specified settling time is attained for the output feedback case. The estimate of one of the states will be given by a second order sliding mode (super-twisting) observer. This will be utilised in conjunction with the measured state in the 'twisting' controller. The results of Chapter 4 on finite time convergence will be used to construct a new combined Lyapunov function for the output feedback system. It will be shown that the trajectory of the closed-loop system can escape from the stable equilibrium under the output feedback. In turn, a finite upper bound on the closed-loop system trajectories will be computed explicitly under the output feedback synthesis, thereby proving that the trajectories cannot escape to infinity. After the finite settling time instant of the observer, the twisting controller essentially coincides with the state feedback case for which the results of Chapter 4 become applicable. Furthermore, tuning rules for the observer will be developed to ensure that the observer error converges to zero before the closed-loop trajectories of the output feedback system escapes beyond an arbitrarily specified region. Tuning rules for the twisting controller as established by chapter 4 will be adapted to ensure finite time convergence of every trajectory starting from the boundary of the aforementioned arbitrarily specified region. This chapter presents the third main contribution outlined above that relates to the output feedback case. Also, this chapter incorporates developments of a recent publication [60].

Chapter 8 gives a theoretical result on orbital stabilisation of systems with recurring resets that do not have a finite accumulation time. Tuning rules will be presented for the case when a second order sliding mode controller is utilized to achieve finite time tracking for a class of unilaterally constrained planar systems in the presence of external disturbances in continuous and discrete-event phases. Rigid body inelastic impacts are incorporated at the unstable equilibrium. A new concept of finite time impact attenuation will be discussed and the corresponding Lyapunov based proof will be presented for the aforementioned tracking problem. This chapter thus presents the fourth contribution of the thesis outlined above. This part of the chapter partly elaborates the recently accepted publication [61]. This chapter will also present two applications of the theoretical results presented in the earlier chapters. First application is the finite time tracking control of a a seven link fully actuated biped robot. The tuning rules of chapter 4 are adapted for the biped model. The numerical simulation will demonstrate the direct application of the orbital stabilsation thereby underlining the applicability and importance of the theory developed in Chapter 4. The second application will also be presented which is a recently developed analogue finite time controller which is based on the theoretical results of reference [11] and Chapter 6.

Chapter 9 presents some conclusions for the research carried out in the thesis while summarising the contributions as well as the potential future directions identified on course of the investigation carried out in the earlier chapters.

This chapter presented an introduction to the thesis. A detailed literature review of second order sliding mode controllers, continuous finite time controllers and unilaterally constrained planar systems is presented in the next chapter. The next chapter also presents mathematical preliminaries that will be used throughout the thesis extensively.

CHAPTER 2

Variable structure and non-smooth controllers

The previous chapter asserts that there are two alternatives for finite time stabilisation of systems with resets, namely, discontinuous variable structure controllers and continuous non-Lipschitz controllers. This chapter reviews the basic principles of both alternatives. The literature in the area of variable structure control, sliding mode control and continuous finite time stabilisation is thoroughly reviewed, including coverage of the robustness aspects of these non-linear controllers. These topics underpin the theoretical foundation of the results presented in this thesis. Since finite time convergence properties of non-smooth and variable structure controllers are sought to stabilise unilatrally constrained planar systems, relevant existing work on control of unilaterally constrained systems is also discussed.

The chapter is organised as follows. Section 2.1 introduces variable structure and sliding mode controllers. An introduction to second order sliding mode controllers is also given. Section 2.2 gives details of continuous finite time stabilisation. Section 2.3 describes unilaterally constrained systems and their control. Finally, Section 2.4 introduces basic definitions, including that of uniformity of stability.

2.1 Theory of variable structure control (VSC)

Variable structure control is a robust nonlinear control technique. The origins of variable structure control theory can be found in the literature of the former Soviet Union. A comprehensive review is given in the reference [62]. This reference gives an account of historical events that underpinned the development of sliding mode controllers as they are known today.

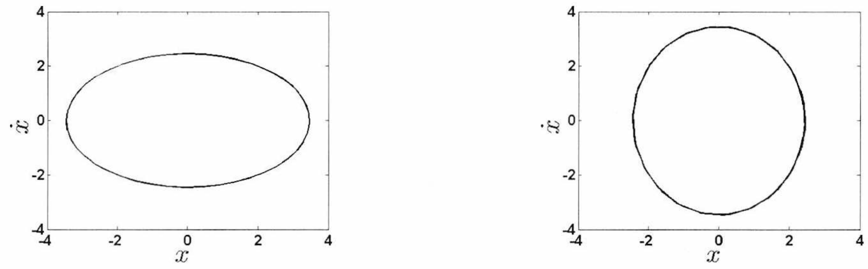

FIGURE 2.1: Two marginally stable structures

The basic concept of variable structure control can be understood by the following classic example appearing in various books and monographs [2, 5] which was originally studied by S. Emel'yanov (see the discussion in [62, Section 2]). Consider the simple double integrator:

$$\ddot{x} = u(x, \dot{x}) \tag{2.1}$$

This is a controllable system. Designing the control input

$$u(x, \dot{x}) = -k\,x, \tag{2.2}$$

which is a linear controller that uses only the first state variable x_1, produces a closed-loop system which is marginally stable and the trajectories starting from any point in the phase-plane $(x, \dot{x})$ exhibit orbital motion as shown in figure 2.1. The first plot is produced with $k = k_1 = -0.5$ and the second with $k = k_2 = -2$. It can be seen that the individual structures are not asymptotically stable. However, the closed-loop system becomes asymptotically stable as shown in figure 2.2 when the structure of the control is switched between the two based on the switching condition $x\dot{x}$ as follows:

$$k = \begin{cases} k_1, & x\dot{x} > 0; \\ k_2, & x\dot{x} < 0. \end{cases} \tag{2.3}$$

The variable structure control produces an asymptotically stable system even when the individual structures are unstable [62, Section 2]. The main feature in the above example is that the feedback gain switches depending on the systems state. The above example of a variable structure system produces trajectories that converge to the origin asymptotically as opposed to being confined to any manifold of codimension one of the state space while converging to the origin asymptotically. This later kind of motion is known as the *sliding mode*. Variable structure systems have been studied with ever increasing interest since this early research work [62, Section 2].

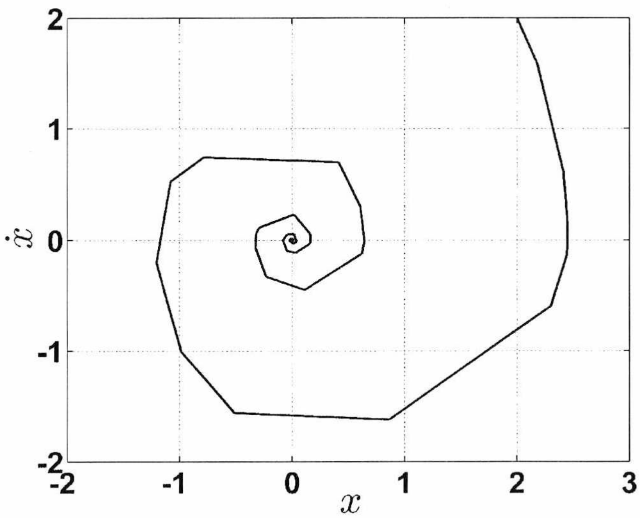

FIGURE 2.2: Asymptotically stable variable structure system

2.1.1 Sliding mode control

The switching logic in the above example can be modified to produce closed-loop trajectories which converge to a straight line in the state-space so that the trajectories move along the straight line after reaching there in finite time. For example, selecting the switching line $s = cx_1 + x_2$ to alter the structure of the feedback and utilising the following variable structure control

$$u(x, \dot{x}) = \begin{cases} -cx_2 - \rho, & s > 0; \\ -cx_2 + \rho, & s < 0, \end{cases} \tag{2.4}$$

generates a sliding mode on the switching line $s = \dot{s} = 0$ of codimension one in the closed-loop system for all $\rho > 0$. The motion of the system on the sliding surface can be obtained as follows:

$$\begin{aligned} \dot{s} = 0 \Rightarrow \dot{x}_2 = -cx_2, \quad & \dot{x}_1 = -cx_1 \\ \Rightarrow x_2(t) = e^{-c(t-t_0)}x_2(t_0), \quad x_1(t) \quad & = e^{-c(t-t_0)}x_1(t_0), \end{aligned} \tag{2.5}$$

where t_0 is the time when the sliding motion starts. The response of the closed-loop system in the phase-plane and the evolution of the states against time are shown in figures 2.3, and 2.4, respectively. A significant motivation to study sliding mode control is that the performance of the closed-loop system depends only on user defined parameters. For example, the closed-loop system (2.1), (2.4) exhibits an ideal sliding motion (2.5) which is dependent only on the parameter c as seen in figure 2.5.

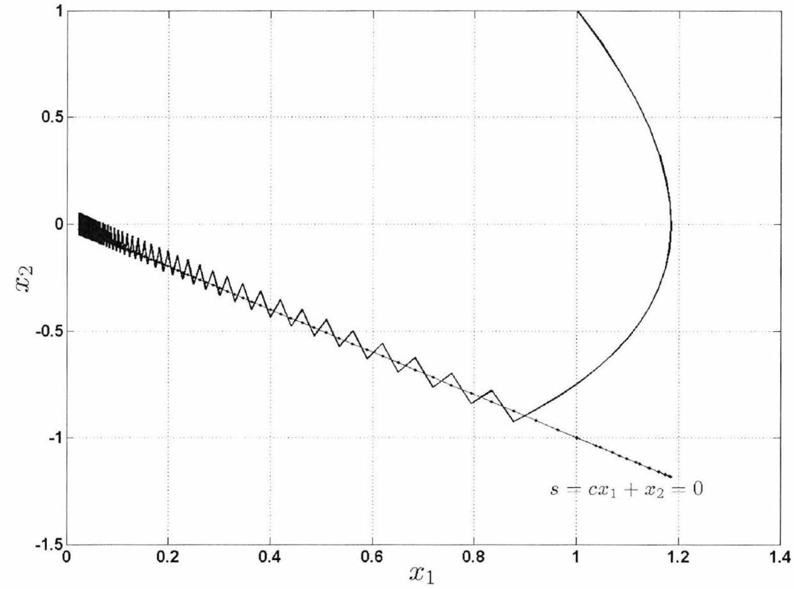

FIGURE 2.3: Sliding mode on $s(t) = 0$

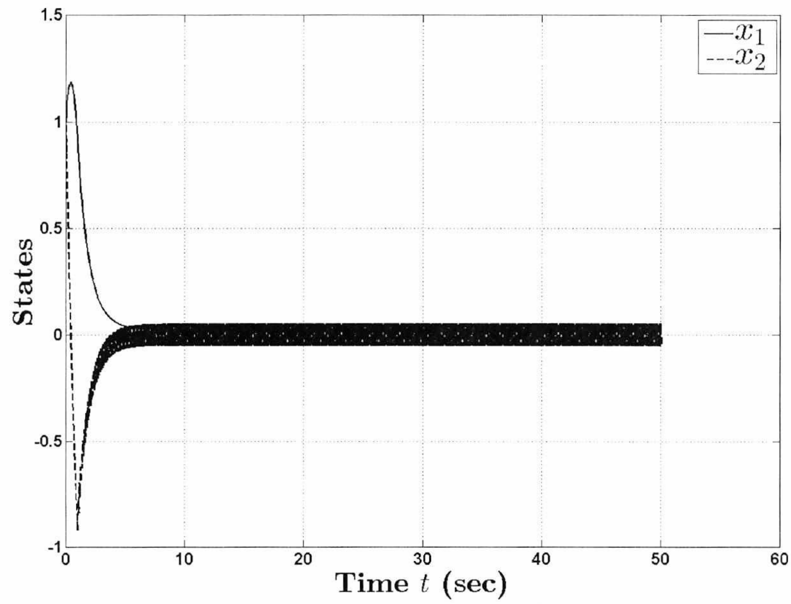

FIGURE 2.4: System states

The motion of the closed-loop system in the above example can be divided into phases. The first phase is generally called the *reaching* phase [2] where $s(t)$ converges from a non-zero initial value to zero. The rest of the motion is called the *sliding mode* [2]. The reaching conditions can be defined as follows [62]:

$$\lim_{s\to 0^+} \dot{s} < 0, \ \lim_{s\to 0^-} \dot{s} > 0 \tag{2.6}$$

One specific reaching condition satisfying (2.6), named as the *η-reachability condition* [63], is [2]:

$$s\dot{s} < -\eta|s| \tag{2.7}$$

It can be seen from the definition $s = cx_1 + x_2$ that the following expressions

$$\begin{aligned} s\dot{s} &= s(cx_2 + u) \\ &= s(-\rho(\text{sign}(s))) \\ &= -\rho|s| \quad < -\eta|s| \end{aligned} \tag{2.8}$$

hold true for all scalars $\eta \in (0, \rho)$.

Robustness and invariance

A further advantage of sliding mode control is that the performance is specified entirely by user defined parameters even in the presence of a class of unknown but bounded parameter variations and disturbances.

Let the system equations (2.1) be modified as

$$\ddot{x} = u(x, \dot{x}) + \omega(t), \tag{2.9}$$

where $\omega(t)$ is a time varying disturbance. Let the following assumption hold true:

$$\text{ess} \sup_{t\geq 0} |\omega(t)| < M, \tag{2.10}$$

where M is a positive scalar known *a priori*. The above assumption means that the only information available about the disturbance $\omega(t)$ is its upper bound. The reachability condition (2.7) can be verified to check if there exists a sliding motion as follows:

$$\begin{aligned} s\dot{s} &= s(cx_2 + u + \omega(t)) \\ &= s(-\rho(\text{sign}(s)) + \omega(t)) \\ &< -(\rho - M)|s| \quad < -\eta|s| \end{aligned} \tag{2.11}$$

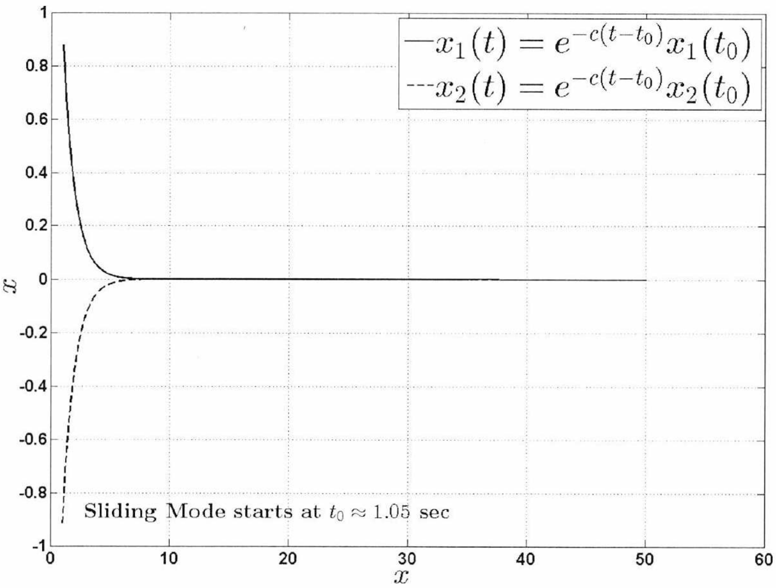

FIGURE 2.5: System states in ideal sliding motion (2.5)

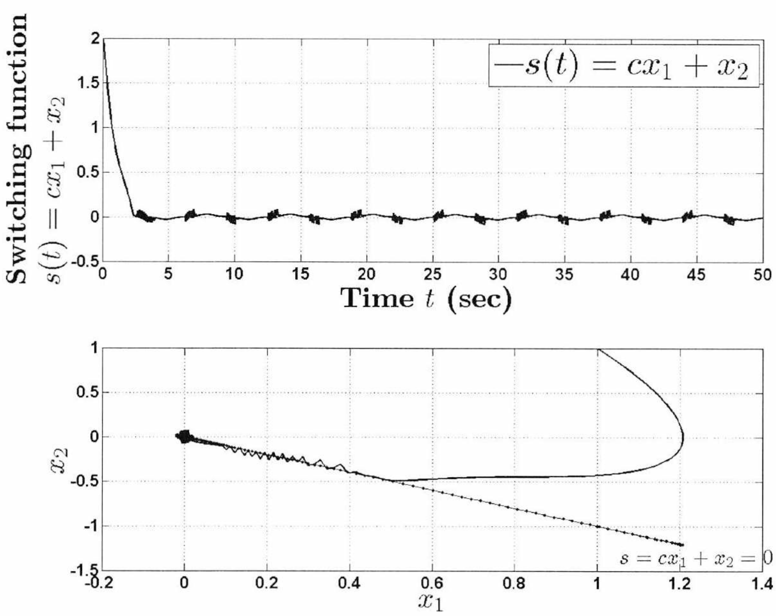

FIGURE 2.6: Robustness of the sliding mode

The last inequality of (2.11) holds true only for those values of ρ and η such that the inequality $\rho > M, \eta \in (0, \rho - M)$ is specified.

Figure 2.6 shows the time history of the sliding surface $s(t)$ and the phase plane plot of the closed-loop system (2.9), (2.4) with a disturbance of the form $\omega(t) = 1.5\sin(t)$. The parameter $\rho = 2$ is chosen. It can be seen than $|\omega(t)| \leq 1.5 = M$. The sliding function s reaches zero as shown in the first plot of Figure 2.6. The ideal and actual state trajectories during the sliding mode are shown in Figure 2.7. The closed-loop system is dependent on the user defined parameter c even in the presence of the disturbance $\omega(t)$. This property of sliding mode control is known as invariance to matched uncertainty, where the matched uncertainty is implicit in the control channels.

More formally, for the general n dimensional system

$$\dot{x} = f(x,t) + B(x,t)u + \omega(x,t), \tag{2.12}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, \omega(x,t) \in \mathbb{R}^m$, matrix $B(x,t)$ is the input matrix and $f(x,t)$ are the dynamics of the system, the sliding mode is insensitive or invariant to uncertainties $\omega(x,t)$ satisfying the *matching* condition $\omega(x,t) \in range(B)$ [64]. The matching condition means that the disturbance vector $\omega(x,t)$ can be represented as a linear combination of the columns of the matrix B (e.g. $\omega(x,t) = \Omega\bar{\omega}(x,t)$ for some matrix Ω such that $range(\Omega) \subset range(B)$ and $\bar{\omega}$ is a non-linear bounded function [2, Section 3.4]).

The following salient properties of the sliding mode are evident from the above example:

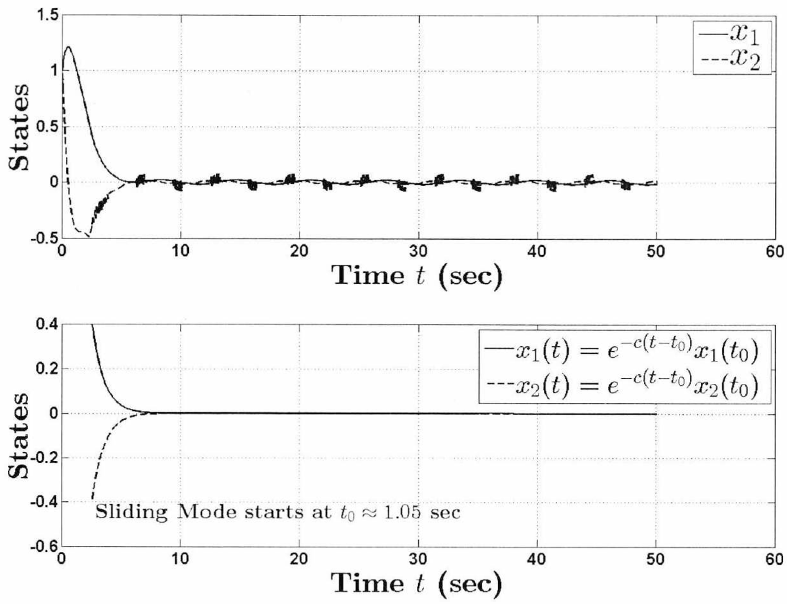

FIGURE 2.7: System states during the sliding motion

1. The sliding mode, when it exists, gives rise to a class of variable structure controllers. The resulting closed-loop system dynamics are dependent on the user defined parameters.
2. The sliding motion is of a reduced order when compared to the dimension of the original system.
3. Motion in the sliding mode is invariant to 'matching' disturbances/uncertainties.

2.1.1.1 Design Methodology

There are two stages in determining a sliding mode controller. The first ensures that the dynamics of the closed-loop system on the sliding surface is stable. This is termed as the *existence* problem [2]. The second relates to the design of a control u that renders the sliding surface designed in the first step attractive. The second step is called the *reachability* problem [2].

Switching function design

For the purposes of illustrating the main design principles, linear systems will be considered. Let the open-loop system be given by

$$\dot{x} = Ax(t) + Bu(t), \tag{2.13}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. It will be assumed that $\text{rank}(B) = m$ and (A, B) is a controllable pair. Let the sliding function be defined as follows:

$$s(t) = Sx(t). \tag{2.14}$$

Hence, the existence problem concerns the design of an appropriate matrix S. System (2.13) can be transformed into regular form

$$\dot{z}_1 = A_{11}z_1(t) + A_{12}z_2(t) \tag{2.15}$$

$$\dot{z}_2 = A_{21}z_1(t) + A_{22}z_2(t) + B_2u(t) \tag{2.16}$$

via a change of coordinates defined by an orthogonal transformation matrix T_r, which may be obtained by QR-decomposition, so that

$$z(t) = T_r x(t). \tag{2.17}$$

The sliding function in the new coordinates is given by

$$s(t) = S_1 z_1(t) + S_2 z_2(t). \tag{2.18}$$

The relationship between the transformed system matrices and the original pair (A, B) can be obtained as follows:

$$T_r A T_r^{\mathrm{T}} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T_r B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{2.19}$$

Similarly, the transformed version of the switching function $s(t)$ in (2.14) can be modified as follows:

$$ST_r = \begin{bmatrix} S_1 & S_2 \end{bmatrix} \tag{2.20}$$

As discussed in the previous subsection, the identity $s(t) = 0$ holds true when a sliding motion occurs. Assume that there exists a sliding mode control $u(s(t))$ such that the ideal slidig mode occurs after a finite time. Then, the following can be obtained from (2.18):

$$S_1 z_1(t) + S_2 z_2(t) = 0. \tag{2.21}$$

Solving for $z_2(t)$, the following linear combination from (2.21)

$$z_2(t) = -S_2^{-1} S_1 z_1(t) \tag{2.22}$$

produces the dynamics

$$\dot{z}_1 = A_{11} z_1(t) + A_{12} z_2(t) \tag{2.23}$$

$$\dot{z}_2 = -S_2^{-1} S_1 z_1(t). \tag{2.24}$$

during the sliding mode. The above two equations can be seen from the perspective that z_2 acts as a pseudo-control input for the first differential equation. Substituting the algebraic relationship (2.24) in (2.23), the following results:

$$\dot{z}_1 = (A_{11} - A_{12} S_2^{-1} S_1) z_1(t) \tag{2.25}$$

Hence, the poles of the sliding mode dynamics are given by the eigenvalues of the matrix $A_{11} - A_{12} S_2^{-1} S_1$. In other words, the *existence problem* becomes that of designing the gain matrix $S_2^{-1} S_1$ appropriately to produce asymptotically stable closed-loop system. Numerous variants of state-feedback control design methods have been adapted to solve this problem. See, for example, references [2, Section 4.7], [65] for a comprehensive review. Methods of robust eigenstructure assignment [66] can be used to force the eigenvalues of the matrix $(A_{11} - A_{12}S_2^{-1}S_1)$ (and of the sliding motion) to lie in the left half of the complex plane while minimising the effects of any parameter variations that do not lie in the range of B. Methods of direct eigen-structure assignment can also be used to place the poles of the sliding motion arbitrarily in the left half of the complex plane [67].

Design Methodology: Control synthesis

Having designed a suitable sliding surface, the next step is that of solving the *reachability problem* such that every trajectory of the system converges in finite time to the sliding surface $s(t) = 0$.

Several articles exist giving various solutions to the reachability problem starting from the pioneering work by [1] (see section [2, Section 3.9] for a review of main historical developments on the topic). One method is briefly described here.

Consider the system (2.13) with the sliding surface designed using one of the aforementioned methods. The convergence of $s(t)$ to zero can be obtained by employing *the reaching law approach* [65] as follows:

$$\dot{s} = -q\ \text{sign}(s) - ks \tag{2.26}$$

where the scalars q, k are positive and dictates the rate at which the sliding variables $s_i(t), i = 1, 2, \ldots, m$ converges to zero, and $s = [\ s_1\ \ s_2\ \ \ldots\ \ s_n\ ]^{\mathrm{T}}$ and $\text{sign}(s) = [\ \text{sign}(s_1)\ \ \text{sign}(s_2)\ \ \ldots\ \ \text{sign}(s_n)\ ]^{\mathrm{T}}$. In order to obtain an expression for a control u that produces such a reaching to the sliding mode, equate the right hand side obtained by differentiating (2.14) and (2.26) as follows:

$$\dot{s} = S\dot{x} = SAx(t) + SBu(t) = -q\text{sign}(s) - ks \tag{2.27}$$

Then, assuming that the matrix $(SB)^{-1}$ is nonsingular and using the final two terms in (2.27) gives the following:

$$u(t) = -(SB)^{-1}(SAx(t) + q\text{sign}(s) + ks) \tag{2.28}$$

The proof of finite time convergence can be obtained by establishing that the following η- reachability condition holds:

$$
\begin{aligned}
s^{\mathrm{T}}\dot{s} &= s^{\mathrm{T}}(SAx(t) + SBu(t)) \\
&= s^{\mathrm{T}}(SAx(t) + SB(-(SB)^{-1}(SAx(t) + q\mathrm{sign}(s) + ks))) \\
&= -s^{\mathrm{T}}(q\mathrm{sign}(s) - ks) \\
&= -q|s| - ks^2 \\
&< 0
\end{aligned}
\tag{2.29}
$$

This completes both the existence reachability problems for the nominal system. It is important to note that above design holds true for uncertain systems. Consider the uncertain version of the system (2.13) as follows:

$$\dot{x} = Ax(t) + B(x)u(t) + \delta(t), \tag{2.30}$$

where the bounded uncertainty $\delta(t) \in \mathbb{R}^{n\times 1}$ satisfies the matching condition as explained earlier. The only information assumed by the design of the sliding mode control is the upper bound on $\delta(t)$. It is possible to express such a matched uncertainty as $\delta(t) = B\omega(t)$ where $\omega(t) \in \mathbb{R}^{m\times 1}$ is a non-linear function bounded by $|\omega(t)| < M$. Utilising the new model in the analysis of the existence problem in (2.29) produces

$$
\begin{aligned}
s\dot{s} &= s^{\mathrm{T}}(SAx(t) + SBu(t) + SB\omega(t)) \\
&= s^{\mathrm{T}}(SAx(t) + SB(-(SB)^{-1}(SAx(t) + q\mathrm{sign}(s) + ks)) + SB\omega(t)) \\
&= -s^{\mathrm{T}}(q\mathrm{sign}(s) - ks - SB\omega(t)) \\
&= -q|s| - ks^2 - s(SB)\omega(t) \\
&\leq -q|s| - ks^2 - |s||(SB)||\omega(t)| \\
&\leq -|s|(q - M|SB|) - ks^2 \\
&< 0 \quad \text{if } q > M|SB|.
\end{aligned}
\tag{2.31}
$$

Hence, the sliding surface design using the nominal dynamics produces an asymptotically stable system that performs as per the solution to the existence problem. Furthermore, the system is *invariant* to the matched disturbances when the gain q of the sliding mode is suitably selected using the known matrices S, B and the upperbound M.

The sliding mode control above is known as traditional sliding mode control or first order sliding mode control. The order here simply means the co-dimension of the sliding dynamics when the study of single input systems is being carried out. To exemplify this point, it can be seen from the dimension $m = 1$ of the control $u(t)$ that the sliding dynamics (2.21) are of the order $n-1$ when the system contains n states. The co-dimension of the sliding mode is thus one.

More importantly, $s(x)$ is continuous and $\dot{s}$ is discontinuous due to the $\text{sign}(s)$ term in the control law. Such a discontinuous control assumes the ability to switch infinitely fast as required by the existence of the *ideal* sliding mode. However, such a motion may not be possible in practice due to the imperfections in implementing the switch ($\text{sign}(s)$). An imperfect relay or switch may contain a dead-zone or hysteresis which consequently prevents ideal sliding behaviour [2, 5]. Although some control applications such as electric drives naturally require a switching control input, the imperfections in switching lead to the so-called *chattering* phenomenon.

Chattering is effectively a high frequency oscillation. The existence of unmodelled actuator and sensor dynamics can also produce chattering. Reference [68] analyses the motion of trajectories in the presence of such parasitic unmodelled dynamics. Ideal sliding cannot occur due to either the inability of the actuator to act infinitely fast or the inability of the sensor to measure the 'true' state 'instantaneously'.

There have been various attempts to tackle the spurious oscillations in the sliding mode systems. The most intuitive solution was to introduce an arbitrary boundary layer around the sliding surface [2, Section 3.7] to remove the discontinuity from the control. For example, a control law

$$u = -\rho \text{sat}(s), \tag{2.32}$$

where,

$$\text{sat} = \begin{cases} 1, & s > \delta; \\ \frac{1}{\delta}, & |s| \leq \delta; \\ -1, & s < \delta. \end{cases} \tag{2.33}$$

introduces a continuous control inside a boundary layer of width $\delta > 0$. The two main consequences of introducing a boundary layer are that ideal sliding will not occur and the system will not be completely invariant to matched uncertainties. The system trajectories, however, remain ultimately bounded.

Amongst the various available methods to establish a continuous control while also preserving the main features (robustness and reduced order motion) of the sliding mode, second order sliding mode controllers have proved key [4]. As mentioned in Chapter 1, second order sliding mode controllers offer a potential solution for finite time convergence to the origin.

2.1.2 Second and higher order sliding mode control

It can be seen from the above discussion that the switching surface s is a continuous function of the state and its derivative, $\dot{s} = f(x) + g(x)u$, is a discontinuous function due to the discontinuous control, where $f(x), g(x)$ are smooth functions. In other words, the relative degree of the switching function $s(t)$ with respect to the control u is one. The relative degree r is the number of times the function s must be differentiated with respect to time in order to have the input u explicitly appear on the right hand side.

Having observed the relationship between the relative degree of s and the discontinuous nature of $\dot{s}$, the main question posed in the second order sliding mode control philosophy is whether or not it is possible to enforce the sliding mode $s = \dot{s} = 0$ with a continuous function $\dot{s}$ and with a discontinuous control operating on the right hand side of the function $\ddot{s}$.

Literature on second order sliding modes is rich. Since A. Levant's work in [4], there have been numerous results on the analysis of second order sliding mode controllers. A sub-optimal second order controller was given in [69] which removed the need for measuring or estimating the derivative of the switching function s (see (2.41)). A combination of linear and twisting controller can be found in [9]. Various versions of the so-called 'super-twisting' controllers have also appeared such as the combination of linear feedback terms and the standard super-twisting controller [14, 41].

In recent years, this topic is receiving new interest in terms of studying the existence of the corresponding strict and non-strict Lyapunov functions. Reference [9] provided the first weak global and strong semi-global Lyapunov functions (see [16, Section 12.1] for concepts of global and semi-global stabilisation of non-linear systems). Various results have subsequently appeared. Lyapunov function design for second order systems via solution of a first order partial differential equation is given in [70] for twisting and in [71] for the super-twisting controllers. A strict Lyapunov function for super-twisting observer first appeared in [41] which has been followed by a very recent results by the same author [42, 72]. The adaptive gain super-twisting controller can be found in [73, 74].

Three cases are of particular interest. The next section describes the basic principles of second and higher order sliding mode synthesis in each of the three cases. Time varying systems will be considered from this point onwards in the thesis.

Planar nonlinear systems with single input

Consider the planar single input single output system of the following form [3]

$$\dot{x} = f(x,t) + g(x,t)u \tag{2.34}$$

where $x \in \mathbb{R}^2, u \in \mathbb{R}$ and functions $f(x,t), g(x,t)$ are assumed sufficiently smooth. It is assumed that a switching function $s(x)$ has been designed using one of the design principles described in the literature for system (2.34). The existence of the sliding mode $s(x) = \dot{s}(x) = 0$ of order one requires that the control u can be designed such that

$$\dot{s} = \frac{\partial s}{\partial x}(f(x,t) + g(x,t)u) = 0. \tag{2.35}$$

In other words, the control

$$u = -(\frac{\partial s}{\partial x}g(x,t))^{-1}f(x,t) - \rho\text{sign}(s) - ks^2 = 0 \tag{2.36}$$

with an arbitrary $\rho > 0, k > 0$ (when using the reaching law approach of the previous section) can be designed to render the sliding surface $s(x) = 0$ attractive in finite time. This is only possible when $(\frac{\partial s}{\partial x}g(x,t))^{-1} \neq 0$. Hence, the relative degree of s with respect to u is equivalent to non-singularity of $(\frac{\partial s}{\partial x}g(x,t))^{-1}$ [75]. It is assumed that the term $(\frac{\partial s}{\partial x}g(x,t))^{-1}$ is non-singular.

A continuous control can be designed by introducing an integrator in the dynamics to increase the relative degree of the resulting system with respect to the physical input as follows:

$$\begin{aligned} \dot{x} &= f(x,t) + g(x,t)u \\ \dot{u} &= v \end{aligned} \tag{2.37}$$

where v is a discontinuous function employing sign(s). Since the relative degree of s with respect to u is one, naturally the relative degree of s with respect to v is two. Proceeding as before, the analysis of the sliding dynamics proceeds as follows:

$$\begin{aligned} \dot{s} &= \frac{\partial s}{\partial x}(f(x,t) + g(x,t)) \\ \ddot{s} &= F(x,\dot{x},t,u) + \frac{\partial s}{\partial x}gv \end{aligned} \tag{2.38}$$

where,

$$F(x,\dot{x},t,u) = \frac{\partial^2 s}{\partial x^2}f^2 + \frac{\partial s}{\partial x}\dot{f} + \frac{\partial^2 s}{\partial x^2}fgu + \frac{\partial s}{\partial x}\dot{g}u \tag{2.39}$$

and the argument (x,t) is dropped by slight abuse of notation for brevity. It is assumed that $\frac{\partial s}{\partial x}g(x,t) = 1$ to clearly illustrate the basic design principles of the second order sliding mode synthesis. Then (2.38) can be re-written as follows:

$$\begin{aligned} \dot{s}_1 &= s_2 \\ \dot{s}_2 &= F(x, \dot{x}, t, u) + v \end{aligned} \tag{2.40}$$

where a new notation of states $s_1 = s, s_2 = \dot{s}$ is being used. Assuming that $F(x, \dot{x}, v)$ is bounded (which essentially means local nature of the analysis), the following so-called 'twisting' control given by A.Levant [4]

$$v = -\mu_1 \text{sign}(s_1) - \mu_2 \text{sign}(s_2) \tag{2.41}$$

causes the states $s_1 = s$ and $s_2 = \dot{s}$ to converge to zero in finite time *without producing any sliding motion in the state space* when μ_1 and μ_2 are chosen such that $0 < \mu_1 - M < \mu_2$ where $\sup_{t \geq 0} |F(x, \dot{x}, u)| < M$.

Recalling the definitions of *real sliding* and *ideal sliding* [1, Section III], ideal sliding mode refers to the case when the system trajectories move exactly on the manifold $s = 0$. Note that due to the identity $s = 0$, the expression $\dot{s} = 0$ also holds true. Such a motion is only possible when the considered system model and the control implementation does not have imperfections like hysteresis and time delay amongst others. The *real sliding mode*, on the other hand, refers to the case when these imperfections cause the actual motion of the system trajectories stay in the vicinity of sliding mode and not identically on the sliding mode. In this context, the *ideal sliding* is caused only at the origin $s = \dot{s} = 0$ by the SOSM controller (2.41) which is of co-dimension two. Hence, this controller is an example of a second order sliding mode for planar uncertain systems with a scalar control. This is different from the traditional (or first order) ideal sliding mode control (defined in [1, Section III]) of planar systems where a part of the state space contains the line $s = 0$ where the trajectories converge and then reach the origin asymptotically. Although the first order ideal sliding mode control also possesses $s = \dot{s} = 0$ in finite time, the motion in the sliding mode is of co-dimension one even in the case of a planar system with single input. This is the major difference between the first order and second order sliding mode control.

It is important to note the salient features of controller (2.41). The trajectories of the closed-loop system (2.37), (2.41) converge to the origin $(x, \dot{x}) = 0$ in finite time since $s(x)$ does so, too without generating any sliding in $\mathbb{R} \backslash \{0, 0\}$. The controller u is continuous. The discontinuous action v acts on $\ddot{s}$ and not on $\dot{s}$. Most importantly, the controller stabilizes the system despite the fact that the uncertain term $F(x, \dot{x}, t, u)$ can be time varying and non-vanishing.

In the context of the discussion in the previous subsection on matched uncertainties, the traditional sliding mode control would have required only the functions f, g to be bounded whereas the twisting controller requires the derivative of these functions to be bounded. This is the price of employing a continuous control u. A large class of mechanical systems, however, have dynamics of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x,t) + \omega(x,t) + g(x,t)u. \end{aligned} \tag{2.42}$$

where $\omega(x,t)$ is the external disturbance, x_1 is position and x_2 is velocity. Then, designing a twisting controller $u = -g^{-1}(\mu_1 \text{sign}(x_1) + \mu_2 \text{sign}(x_2))$ as above while using x_1, x_2 as states will render the twisting control superior to the traditional sliding mode controller since both can reject bounded uncertainty $f(x,t) + \omega(t)$ when assuming $|g|$ is not singular and bounded (see [76, Section 3] for validity of such an assumption) but the traditional sliding mode control cannot ensure finite time convergence of both the states to origin.

Another type of second order sliding mode controller proposed by A.Levant for a general planar system can reject vanishing disturbances in the dynamics of s_1 in (2.40).

$$\begin{aligned} \dot{s}_1 &= s_2 \\ \dot{s}_2 &= F(x,\dot{x},u) - \mu_1 |s|^{\frac{1}{2}} \text{sign}(s) + u_1 \\ \dot{u}_1 &= -\mu_2 \text{sign}(s) \end{aligned} \tag{2.43}$$

This controller $u = -\mu_1 |s|^{\frac{1}{2}} \text{sign}(s) + u_1$ is the so-called 'super-twisting' controller [4]. It has been recently shown [41] that the super-twisting controller can generate a sliding mode of second order in uncertain planar systems of the form

$$\begin{aligned} \dot{s}_1 &= s_2 + F_1(t) \\ \dot{s}_2 &= F_2(x,\dot{x},u) - \mu_1 |s|^{\frac{1}{2}} \text{sign}(s) + u_1 \\ \dot{u}_1 &= -\mu_2 \text{sign}(s) \end{aligned} \tag{2.44}$$

where the bounds on the uncertainty that can be successfully rejected are [41, Th. 3]

$$|F_1| = M|\dot{s}|^{\frac{1}{2}}, \quad |F_2| = M, \tag{2.45}$$

where $M > 0$ is a scalar.

Systems with n states and single input

One consequence of the discussion in the case of planar mechanical systems is that sliding surface design may not be needed. However, in the case of a general n^{th} order system, a second order sliding mode controller will still need a switching surface s. Consider the same system (2.34) with $x \in \mathbb{R}^n, u \in \mathbb{R}$. Then designing the sliding surface $s(x)$ as described in the previous sections and proceeding in a similar way as the planar case will give a continuous controller that stabilises the system asymptotically (and not in finite time) after reaching the sliding surface in a same way as a traditional sliding mode controller would do.

These results are for the case when the sliding variable has relative degree one with respect to the control. When the relative degree of the switching function is two with respect to the control, the second order sliding mode control will generate a traditional sliding mode on the switching surface and will give a reduced order motion of the order $n-2$. The difference here is that the discontinuous control will act in the system and an additional integrator cannot be introduced as was done above (see (2.43)).

It is timely at this juncture to point out that all the results in the thesis pertaining to second order sliding mode controllers are directly applicable to systems in the form (2.42). The results of the thesis are also applicable to the asymptotic stability analysis of controllable systems with n states and with relative degree one and/or two when the analysis of the sliding mode is carried out in the state-space $(s, \dot{s})$.

Systems with n states, multiple inputs and multiple outputs

The general case of non-linear systems with relative degree greater than two is not a valid candidate for the application of second order sliding mode control. The concept of higher order sliding mode control (HOSM) relates to the case when the relative degree of the selected switching function of the system is more than two with respect to the control input. Literature in this area is rigorous. An introduction to the topic can be found in [77, 78] and references therein. Various definitions can be found in [77]. The higher order sliding mode for a single input system is defined as the set of points x such that

$$s = \frac{\mathrm{d}}{\mathrm{d}t}s = \ldots = \frac{\mathrm{d}^{r-1}}{\mathrm{d}t^{r-1}}s = 0 \tag{2.46}$$

with all the functions $\frac{\mathrm{d}^j}{\mathrm{d}t^j}s, 0 \leq j \leq r-1$ continuous and the discontinuous control appears on the right hand side of $\frac{\mathrm{d}^r}{\mathrm{d}t^r}s$ to enforce the finite time stabilisation of trajectories of the closed-loop system (2.34) on the set (2.46) where r is the relative degree of the system.

The literature reviewed thus far in Section 2.1.2 represents the state-of-the-art in the second order sliding mode controllers. In the current literature there is no theoretical proof of a simple intuitive expectation that the settling time approaches zero when gains tend to infinity. As mentioned in Chapter 1, such a result is available in linear classical control theory and it finds natural applications in automatic tuning algorithms [55]. Chapter 4 gives the proof that the finite settling time approaches zero when gains of a second order sliding mode controller approach infinity. This result also forms the basis for studying finite time stability of unilaterally constrained variable structure and non-smooth systems which are discussed in the following sections.

As mentioned in Chapter 1, the concept of finite time stability is perceived in this thesis in the sense that the trajectories of a planar system converges to the origin in finite time [4, 9, 13] rather than diminish to a bounded set over a finite time interval without actually converging to the origin [19]. Having reviewed the literature on discontinuous sliding mode controllers that produce finite time stable systems, the next section discusses advances in the area of finite time stabilisation of systems which employ continuous controllers.

2.2 Continuous finite time stabilisation

Continuous controllers that drive the trajectories of the system to the origin in finite time have been an active area of research. Haimo [11] first posed the question whether it is possible to achieve finite time convergence of the trajectories of a planar system without using the well-known result of discontinuous time optimal control (known as Fuller's phenomenon) [79]. Time optimal control has long been known for studying controllers that produce trajectories convergent to the origin in finite time. Examples of such open-loop time optimal controllers include Fuller's controller [79] and energy optimal controllers [80]. The main disadvantage of open-loop controllers is that they do not produce asymptotic stability (and render the system unstable) if there is any uncertainty or disturbance in the system.

A closed-loop continuous time optimal controller can be found in [81] which combines a continuous optimal state feedback with a time varying feedback that drives the trajectories to zero in finite time. This is a feedback form but any uncertainty or disturbances appearing in the system would render the trajectories only asymptotically stable and not finite time stable. The same is true with the continuous observer (with a closed-loop structure) that converges in finite time [82].

The published literature shows that Haimo's work [11] was the first effort in establishing a continuous finite time stable feedback controller. This work showed that the resulting closed-loop systems are necessarily non-Lipschitz. The robustness to certain classes of disturbances was also proven. Finite time stability of continuous homogeneous systems were studied in [83] where a continuous homogeneous controller for a double integrator system was proposed. Another finite time stable continuous controller was established in [12]. The stability analysis explicitly identified a strict Lyapunov function that satisfied a differential inequality. It is noteworthy that no robustness analysis was performed in the references [12, 83]. Lyapunov and converse Lyapunov theorems were studied in [13] which defined the Lyapunov based finite time stability and established that the continuity properties of the settling time function is a necessary condition for finite time stability. The authors discussed the robustness properties of finite time stable systems and showed that Lipschitz systems are less robust than the finite time non-Lipschitz systems [13, Th 5.2]. A continuous finite time observer was proposed in [46] which was supported by a strict Lyapunov function satisfying the conditions established in [13]. This observer was combined with a finite time stable controller. The output feedback synthesis was shown to be robust to a certain class of disturbances.

Since these early contributions, finite time stabilisation using continuous controllers has attracted ever growing interest. It was shown in [84] that geometric homogeneity of the vector field of the closed-loop system leads to finite time stability of the origin if the equilibrium of the system is asymptotically stable and the degree of homogeneity is negative[1]. Existence of a $\mathcal{C}^1$-smooth Lyapunov function was also given [84, Section 7]. It is noteworthy that this result may be seen as presenting a weak robustness result in the sense that the perturbed system is also finite time stable under the assumption of homogeneity of the disturbance [84, Th. 7.4]. Reference [87, Th. 6] presented proof of existence of a smooth Lyapunov function without assuming continuity of the settling time. Continuous finite time stabilisation of non-linear systems of dimension n with m control inputs also remains an active area. An important result was given in [88, Th. 3.1] which proposed a homogeneous controller that finite time stabilises the nonlinear autonomous system.

All the references discussed above either prove finite time stability for homogeneous systems which are autonomous (i.e. the vector field does not depend on time) or give ultimate boundedness when a time varying perturbation appears. Finite time stability of time varying (non-autonomous) systems have also been studied in recent years. A homogeneous feedback was proposed in [89] which stabilised a class of non-linear

[1]See pioneering works [85, 86] and references therein for detailed analysis of geometric homogeneity and asymptotic stabilisation of continuous homogeneous systems

systems. The control was shown to be robust to uncertain terms with a bound that is a multiplication of the 1-norm of the state vector and certain $\mathcal{C}^1$ functions. Reference [90] proposed an adaptive finite time control of a class of non-linear systems with parametric uncertainties. These results were based on the aforementioned reference [88, Th. 3.1]. Although, the above articles allowed the right hand side to be a function of time or some parameter θ, uniform finite time stability was not achieved with respect to initial time and disturbances. Non-singular real-valued continuous robust terminal sliding mode controllers were established in [91]. Although, the authors define these controllers as exhibiting terminal sliding modes, they are simply continuous finite time stable controllers. The basis for this argument is the fact that these are not variable structure controllers [91, Th. 1]. Furthermore, the main result [91, Th. 1] clearly shows the fact that the controllers do not render the trajectories convergent to the sliding surface when persisting disturbances appear, something quite natural for other variable structure controllers like the non-singular terminal sliders [92] or the traditional sliding mode controllers.

Reference [9] was the first article where *uniform* finite time stability was discussed. Formal Lyapunov and converse Lyapunov theorems for uniform finite time stabilisation of non-autonomous continuous non-linear systems were given in reference [44]. This article extended some of the results of [13] on continuity of settling time to the time varying systems. This article related to establishing general Lyapuov theorems for continuous vector field and did not identify explicit Lyapunov functions for the specific case of homogeneous controllers. Uniform finite time stability of a class of non-linear autonomous systems was obtained in the reference [93, Corollary 2.24, Example 5.5] using the concept of so called 'homogeneity in the bi-limit'. However, in the presence of time varying and vanishing continuous disturbances, the subsequent results only provided global asymptotic stability and not finite time stability [93, Example 5.6]. Most recent results on finite time stability of time varying systems can be found in the reference [94] which does not give proof of uniform finite time stability as defined in [44].

It can be seen from the above literature that Lyapunov functions have not been identified to prove *uniform* finite time stability even for a simple perturbed double integrator system when time varying discontinuous (possibly variable structure) disturbances appear. As mentioned in Chapter 1 one of the main aims of the thesis is to apply the resulting uniform finite time stable controller to unilaterally constrained systems. Chapter 6 studies this topic.

2.3 Unilaterally constrained systems

The previous two subsections detailed the state-of-the-art of discontinuous and continuous finite time stabilisation. The trajectories of the resulting closed-loop variable structure systems were understood in the sense of Filippov's definition [6] and were absolutely continuous. The purpose of this subsection is to give a literature review of the class of control problems where the trajectories of the controlled system undergo a jump. The trajectories are thus discontinuous. These solutions are understood in the sense of measure differential inclusions [47, 95]. Such discontinuous systems occur in various areas such as electrical circuit theory [22], mechanical engineering (non-smooth mechanics) [23] and control of hybrid systems [24]. The literature review in this section is largely dedicated to *control* of impact mechanical systems arising from *unilateral constraints*. Unilaterally constrained mechanical systems occur when the position of the particle is only allowed to evolve in a limited part of the state-space. An introduction to the subject can be found in [23]. From the physicists' viewpoint, it is a theoretically very challenging and interesting subject to study how the motion of a unilaterally constrained particle evolves. Of principal interest to researchers in non-smooth mechanics are the two non-linear phenomena, namely, the nature of contact and the process of collision.

The central chapters of this thesis concern the control of such unilaterally constrained planar systems using non-smooth and variable structure controllers. Such a control problem is relevant in two practically relevant scenarioes. The first scenario is when a vector relative degree two mechanical system (e.g. fully actuated n-link robot) is to be stabilised on the constraint surface. Such a motion can be found, for example, in surface contour machining processes [96, Section VII]. When stabilisation on the constrained surface is to be done via the externally designed control input u, the feedback law and stability analysis have to deal with jumps or *resets* arising from the collision with the constraint surface. The second scenario is when the impacts are recurrent, as in control of biped robots, when the resets give rise to orbits. The control problem then becomes that of ensuring a stable orbit [15].

Controllability and stabilisability of complementarity dynamical systems were studied in reference [40] where a finite accumulation of impacts was considered. Tracking control of frictionless Lagrangian systems with unilateral constraints was studied in [36]. Conditions for weak and strong stability and existence of controllers were developed based on Lyapunov analysis. The key result was to analyse the decay in the post-jump values of the Lyapunov function which essentially is caused by the jumps in generalised velcoity [36, Claim 3]. Reference [39] extended the Krakovskii-LaSalle invariance principle to a class of unilaterally constrained dynamical systems. A general framework for tracking control of biped robots, which are unilaterally constrained mechanical systems, was studied in [97]. A switched controller was proposed in [37] to track a suitably designed trajectory for multi-constraint systems where rigorous Lyapunov analysis along with trajectory planning methods were provided.

The above mentioned references study asymptotic stability and stabilisation. In the context of finite time stability of unilaterally constrained systems, reference [98] studies finite time stabilisation via impulsive control. Results on robust finite time stabilization of linear impulsive systems can be found in [99]. This does not encompass the finite accumulation of impacts.

It can be seen from the above references on mechanical systems as well as from the relevant Lyapunov based stability results of impulsive systems [23], [34], [33], [100], [101], [35], [102], [36], [21] that rigorous Lyapunov analysis has not been carried out for finite time stabilisation and stability of unilaterally constrained systems. As can be seen from the literature review on finite time stabilisation using variable structure and non-smooth controllers, there is a distinct opportunity to apply these methods to the unilaterally constrained systems due to the superior robustness properties than asymptotic stabilisation. Chapter 5 studies this topic with variable structure controllers and Chapter 6 studies this topic with continuous finite time stable controllers.

2.4 Mathematical preliminaries

This thesis will establish uniform finite time stabilisation of variable structure and continuous non-smooth systems with unilateral constraints that gives rise to resets. It is important to clearly outline what is meant by each of the mathematical terms that will be used by all the chapters that follow the above literature review. The sole purpose of this sub-section is to provide the mathematical definitions that will have a recurring presence throughout the thesis.

Consider the discontinuous dynamical system

$$\dot{x} = \phi(x, t) \tag{2.47}$$

where $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is the state vector, $t \in \mathbb{R}$ is the time variable and function $\phi(x, t)$ is piece-wise continuous. The function $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is piece-wise continuous iff $\mathbb{R}^{n+1}$ is partitioned into a finite number of domains $G_j \in \mathbb{R}^{n+1}, j = 1, ..., N$, with disjoint interiors and boundaries ∂G_j of measure zero such that ϕ is continuous within each of these domains and for all $j = 1, ..., N$ it has a finite limit $\phi^j(x,t)$ as the argument $(x^*, t^*) \in G_j$ approaches a boundary point $(x,t) \in \partial G_j$.

For example, the function $\phi(x,t) = \text{sign}(x)$ for $x \in \mathbb{R}$ defines two domains $G_1 = \{x : x \geq 0\}, G_2 = \{x : x \leq 0\}$ in $\mathbb{R}$ with the common boundary of zero measure $\partial G_j = \{x : x = 0\}, j = 1, 2$. Of course, $\phi(x)$ is a piece-wise continuous function since the limits $\lim_{x_1^* \to x^+} \partial G_1 = 1$ and $\lim_{x_2^* \to x^-} \partial G_2 = -1$ are finite $\forall x_1^* \in G_1$, $x_2^* \in G_2, x \in \partial G_j, j = 1, 2$.

Definition 2.1 (Solutions in the sense of Filippov [6]). *Given the differential equation* (2.47), *let the smallest convex closed set* $\Phi(x,t)$ *be introduced for each point* $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ *such that* $\Phi(x,t)$ *contains all the limit points of* $\phi(x^*, t)$ *as* $x^* \to x$, $t =$ *constant, and* $(x^*, t) \in \mathbb{R}^{n+1} \backslash (\cup_{j=1}^N \partial G_j)$. *An absolutely continuous function* $x(\cdot)$ *defined on interval* $\mathbb{I}$ *is said to be a solution of* (2.47) *if it satisfies the differential inclusion*

$$\dot{x} \in \Phi(x,t) \tag{2.48}$$

almost everywhere on interval $\mathbb{I}$.

Solutions to differential equations will be understood as defined in Definition 2.1 throughout the thesis. For the example of $\phi(x,t) = \text{sign}(x)$, the closed convex set Φ is obtained as $\Phi(x,t) = [-1, 1]$. The solution to the differential equation (2.47) exists for arbitrary initial condition $x(t_0)$ on an interval $\mathbb{I} = [t_0, t_1)$ and is non-unique in general [6, Th, 10].

This thesis studies robustness of variable structure and non-smooth systems in order for these controllers to be applied to unilaterally constrained systems. It is important to define the class of perturbations that will be studied. Let the perturbed version of (2.47) be given as follows:

$$\dot{x} = \phi(x,t) + \psi(x,t) \tag{2.49}$$

where, $\psi(x,t)$ is a piece-wise continuous function with components $\psi_1, \psi_2, \ldots, \psi_n$. When studying variable structure systems, the components $\psi_i, i = 1, 2, \ldots, n$ are assumed to be locally uniformly bounded by upperbounds

$$\text{ess sup} \, |\psi_i(x,t)| \leq M_i, \quad i = 1, 2, \ldots, n, \tag{2.50}$$

for almost all $(x,t) \in B_\delta \times \mathbb{R}$, where $B_\delta \subset \mathbb{R}$ is a ball of radius $\delta > 0$, and some positive constants $M_i \geq 0$ fixed *a priori*. When studying a continuous non-smooth vector field $\phi(x,t)$ along with piece-wise continuous perturbations $\psi(x,t)$, the components

$\psi_i, i = 1, 2, \ldots, n$ are assumed to be locally uniformly decaying upperbounds

$$\text{ess sup}\,|\psi_i(x,t)| \leq M_i\bar{\alpha}(\|x\|)), \quad i = 1, 2, \ldots, n, \tag{2.51}$$

where $\bar{\alpha}(\|x(x,t)\|))$ is a continuous positive definite function such that

$$\lim_{\|x\| \to 0} \bar{\alpha}(\|x\|)) \to 0 \tag{2.52}$$

Definition 2.2 (Differential inclusions for uncertain discontinuous systems [9]). *An absolutely continuous function $x(\cdot)$, defined on an interval $\mathbb{I}$, is said to be a solution of the uncertain differential equation* (2.49) *with the rectangular uncertainty constraints* (2.50) *(sectorial constraints* (2.51)*) iff it is a solution of* (2.49) *on the interval $\mathbb{I}$ in the sense of Definition 2.1 for some piece-wise continuous function ψ subject to* (2.50) *(respectively,* (2.51)*).*

An uncertain system (2.49) can be represented as a differential inclusion of the form

$$\dot{x} \in \Phi(x,t) + \Psi(x,t), \tag{2.53}$$

where $\Phi(x,t)$ is the same as defined in Definition 2.1, Ψ is the cartesian product of the intervals $\Psi_i = [-M_i,\ M_i]$ for the uncertainty constraints (2.50), Ψ is cartesian product of the intervals $\Psi = [-M_i\bar{\alpha}(x,t),\ M_i\bar{\alpha}(x,t)]$ for the uncertainty constraints (2.51) with $(x,t) \in \partial G_j, j = 1, 2, \ldots, n$ representing the discontinuity (or limit) points of $\phi(x^*,t)$ as $x^* \to x$ and the set

$$\Phi(x,t) + \Psi(x,t) = \{\phi + \psi : \phi \in \Phi(x,t), \psi \in \Psi(x,t)\}. \tag{2.54}$$

Continuing with an example (see [11]) of a vector field to highlight where the above definition 2.2 is applicable, a planar non-smooth system $\dot{x_1} = x_2, \dot{x}_2 = -|x_1|^{\alpha_1}\text{sign}(x_1) - |x_2|^{\alpha_2}\text{sign}(x_2)$ where α_1, α_2 are positive scalars belonging to the interval $(0,1)$ has the right hand side $\phi_1 = x_2$, $\phi_2 = -|x_1|^{\alpha_1}\text{sign}(x_1) - |x_2|^{\alpha_2}\text{sign}(x_2)$. Consider the uncertainty $\psi_1 = 0, \psi_2(x,t) = M_2|x_2|^{\alpha_2}\text{sign}(x_1)\sin(t)$, $M_2 > 0$. It can be verified that ψ_2 belongs to the class of uncertainties with upperbound (2.50) when $\alpha_2 = 0$ and to the class of uncertainties with upperbound (2.51) when $\alpha_2 \in (0,1)$. In the later case, the inclusion (2.53) can be obtained by considering Ψ as the cartesian product $(0, -M_2|x_2|^{\alpha_2}) \times (0, M_2|x_2|^{\alpha_2})$ on the set of all discontinuity points $\{(x,t) : x_1 = 0\}$.

The main focus of this thesis from the stability analysis viewpoint is on uniform finite time stability with respect to initial time t_0 as well as uncertainty $\psi(x,t)$. It is important to highlight what is meant by uniformity. This is a well studied area for systems with Lipschitz dynamics and many references are available [16, 43, 103] as regards uniformity with respect to initial time. It can be seen from the above references, however, that emphasis is seldom given to uniformity with respect to uncertainty. Definitions [9, Definitons 2.3,2.4,2.5] of (uniform) stability, (uniform) asymptotic stability and (uniform) finite time stability of the inclusion (2.48) for the discontinuous vector field can be seen as the counterparts of the definitions available in the references [16, 43, 44, 103] for similar stability concepts in the case of continuous vector field. Hence, definitions [9, Definitons 2.3,2.4,2.5] are not included here as they only focus on uniformity of stability with respect to initial conditions. The following definitions are inherited from [9, Definitons 2.6,2.7,2.8] which take into account the uniformity of stability with respect to the uncertainty ψ. It should be noted that the word 'equiuniform' appearing in [9] is utilised in the following definitions to refer to uniformity of various stability concepts with respect to the initial conditions as well as the uncertainty ψ.

Suppose that $x = 0$ is an equilibrium point of the uncertain system (2.49), (2.50) (or similarly (2.49), (2.51)), i.e., $x = 0$ is a solution of (2.49) for some function ψ_0, admissible in the sense of either (2.50) or (2.51), and let $x(\cdot, t_0, x^0)$ denote a solution $x(\cdot)$ of (2.49) for some admissible function ψ under the initial conditions $x(t_0) = x^0$.

Definition 2.3 (Equiuniform stability [9]). *The equilibrium point $x = 0$ of the uncertain system* (2.49), (2.50) *(or similarly* (2.49), (2.51)*) is equiuniformly stable iff for each $t_0 \in \mathbb{R}, \epsilon > 0$ there exists $\delta = \delta(\epsilon)$, dependent on ϵ and independent of t_0 and ψ, such that each solution $x(\cdot, t_0, x^0)$ of* (2.49), (2.50) *(or similarly* (2.49), (2.51)*) with the initial data $x^0 \in B_\delta$ exists on the semi-infinite time interval $[t_0, \infty)$ and satisfies the inequality*

$$\|(x(t, t_0, x^0))\| \leq \epsilon \quad \forall\, t \in [t_0, \infty). \tag{2.55}$$

Definition 2.4 (Equiuniform asymptotic stability [9]). *The equilibrium point $x = 0$ of the uncertain system* (2.49), (2.50) *(or similarly* (2.49), (2.51)*) is said to be equiuniformly asymptotically stable if it is equiuniformly stable and the convergence*

$$\lim_{t\to\infty} \|(x(t, t_0, x^0))\| \to 0 \tag{2.56}$$

holds for all the solutions $x(\cdot, t_0, x^0)$ of the uncertain system (2.49), (2.50) *(or similarly* (2.49), (2.51)*) initialized within some B_δ (uniformly in the initial data t_0 and x^0). If this convergence remains in force for each $\delta > 0$, the equilibrium point is said to be globally equiuniformly asymptotically stable.*

Definition 2.5 (Equiuniform finite time stability [9]). *The equilibrium point $x = 0$ of the uncertain system* (2.49), (2.50) *(or similarly* (2.49), (2.51)*) is said to be globally equiuniformly finite time stable if, in addition to the global equiuniform* *asymptotical stability, the limiting relation*

$$x(t, t_0, x^0) = 0 \tag{2.57}$$

holds for all the solutions $x(\cdot, t_0, x^0)$ *and for all* $t \geq t_0 + T(t_0, x^0)$, *where the settling time function*

$$T(t_0, x^0) = \sup_{x(\cdot, t_0, x^0)} \inf\{T \geq 0 : x(t, t_0, x^0) = 0 \quad \textit{for all } t \geq t_0 + T\} \tag{2.58}$$

is such that

$$T(B_\delta) = \sup_{x^0 \in B_\delta, t_0 \in \mathbb{R}} T(t_0, x^0) < \infty \quad \textit{for all } \delta > 0, \tag{2.59}$$

where $\delta = \delta(\epsilon)$ *is independent of* t_0 *and* ψ.

The infimum in (2.58) is to detect the first instant $t = T$ such that $x(t, t_0, x^0) = 0$, $\forall t \geq t_0 + T$ and the suppremum is for taking the worst case trajectory that takes the longest time to arrive at the origin.

The following chapters will also use the concept of geometric homogeneity. The following definitions are inherited from [9, Definitions 2.9, 2.10].

Definition 2.6 (Homogeneity of differential inclusions and equations [9]). *The differential inclusion* (2.48) *(the differential equation* (2.47), *the uncertain systems* (2.49), (2.50) *or the uncertain systems* (2.49), (2.51)*) is called locally homogeneous of degree* $q \in \mathbb{R}$ *with respect to dilation* $(r_1, r_2, \ldots, r_n)$, *where* $r_i > 0, i = 1, 2, \ldots, n$, *if there exist a constant* $c_0 > 0$, *called a lower estimate of the homogeneity parameter, and a ball* $B_\delta \subset \mathbb{R}^n$, *called a homogeneity ball, such that any solution* $x(\cdot)$ *of* (2.48) *(respectively, that of the differential equation* (2.47), *the uncertain systems* (2.49), (2.50) *or the uncertain systems* (2.49), (2.51)*) eveolving within the ball* B_δ, *generates a parameterised set of solutions* $x^c(\cdot)$ *with components*

$$x_i^c(t) = c^{r_i} x_i(c^q t) \tag{2.60}$$

and any parameter $c \geq c_0$.

Definition 2.7 (Homogeneous piece-wise continuous functions [9]). *A piece-wise continuous function* $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ *is called locally homogeneous of degree* $q \in \mathbb{R}$ *with respect to dilation* $(r_1, r_2, \ldots, r_n)$, *where* $r_i > 0, i = 1, 2, \ldots, n$, *and if there exists a constant* $c_0 > 0$ *and a ball* $B_\delta \subset \mathbb{R}^n$ *such that*

$$\phi_i(c^{r_1} x_1, c^{r_2} x_2, \ldots, c^{r_n} x_n, c^{-qt}) = c^{q+r_i} \phi_i(x_1, x_2, \ldots, x_n, t) \tag{2.61}$$

for all $c \geq c_0$ *and almost all* $(x,t) \in B_\delta \times \mathbb{R}$.

The global homogeneity concept for the inclusions (2.48) and the piece-wise continuous functions ϕ can be formally introduced using definitions 2.6 and 2.7 by taking $\delta = \infty$. It should be noted that geometric homogeneity defined in Definition 2.7 is a stronger concept than that appearing in [84] and the references therein since the former additionally covers differential inclusions arising from discontinuous right hand sides.

Definition 2.8 (Quasi-homogeneity principle [20, Th. 4.2]). *Let the following conditions be satisfied:*

1. *a piece-wise continuous function* ϕ *is locally homogeneous of degree* $q \in \mathbb{R}$ *with respect to dilation* $(r_1, r_2, \ldots r_n)$,
2. *components* $\psi_i, i = 1, 2, \ldots, n$ *of a piece-wise continuous function* ψ *are locally uniformly bounded by constants* $M_i \geq 0$,
3. $M_i = 0$ *whenever* $q + r_i > 0$.
4. *the uncertain system* (2.49), (2.50) *is globally equiuniformly asymptotically stable around the origin.*

Then, the uncertain system (2.49), (2.50) *is globally equiuniformly finite time stable around the origin.*

Before concluding this chapter, a few definitions relating to Lyapunov functions are presented. The following three definitions relating to the *weak*, *strong* and *strict* Lyapunov functions can be found in various excellent texts and papers. The reader is referred to the references [43], [16, Ch. 4], [9, 13, 44].

Definition 2.9 (Weak Lyapunov function and uniform stability [16, Th. 4.8]). *Consider the non-autonomous system* (2.47) *with* $\phi(x,t)$ *locally Lipschitz in* x *and piece-wise continuous in* t. *Let* $x = 0$ *be an equilibrium point for* (2.47) *and* $D \subset \mathbb{R}^n$ *be a domain containing* $x = 0$. *Let* $V : [0, \infty) \times D \to \mathbb{R}$ *be a continuously differentiable function such that*

$$\begin{aligned} W_1(x) \leq & V(x,t) \leq W_2(x) \\ \frac{\partial V}{\partial t} + & \frac{\partial V}{\partial x}\phi(x,t) \leq 0 \end{aligned} \tag{2.62}$$

$\forall\, t \geq 0$ *and* $\forall x \in D$, *where* $W_1(x)$ *and* $W_2(x)$ *are continuous positive definite functions on* D. *Then,* $x = 0$ *is uniformly stable.*

Extension of Definition 2.9 to Lipschitz-continuous (or locally Lipschitz) non-smooth function $V(x,t)$ and to piece-wise continuous $\phi(x,t)$ can be found in [20, Th. 3.1, Lemmas 3.1, 3.2].

Definition 2.10 (Strong Lyapunov function and uniform asymptotic stability [16, Th. 4.9]). *Consider the non-autonomous system* (2.47) *with $\phi(x,t)$ locally Lipschitz in x and piece-wise continuous in t. Let $x = 0$ be an equilibrium point for* (2.47) *and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : [0,\infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that*

$$\begin{aligned} W_1(x) \leq & V(x,t) \leq W_2(x) \\ \frac{\partial V}{\partial t} + & \frac{\partial V}{\partial x}\phi(x,t) \leq -W_3(x) \end{aligned} \tag{2.63}$$

$\forall\, t \geq 0$ *and* $\forall x \in D$, *where $W_1(x)$, $W_2(x)$ and $W_3(x)$ are continuous positive definite functions on D. Then, $x = 0$ is uniformly asymptotically stable. If the conditions $D = \mathbb{R}^n$, $W_1(0) = 0$ and $\lim_{\|x\|\to\infty} W_1(x) \to \infty$ hold true, then $x = 0$ is globally uniformly asymptotically stable.*

Definition 2.11 (Strict Lyapunov functions and uniform finite time stability [44, Th. 4.1]). *Consider the non-autonomous system* (2.47) *with $\phi(x,t)$ continuous in x and t. Let $x = 0$ be an equilibrium point for* (2.47) *and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : [0,\infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that*

$$\begin{aligned} W_1(x) \leq & V(x,t) \leq W_2(x) \\ \frac{\partial V}{\partial t} + & \frac{\partial V}{\partial x}\phi(x,t) \leq -k(t)(V(x,t))^{\alpha} \end{aligned} \tag{2.64}$$

$\forall\ t \geq 0$ *and* $\forall x \in D$, *where $W_1(x)$, $W_2(x)$ and $W_3(x)$ are continuous positive definite functions on D, $k : [0,\infty] \to \mathbb{R}^+$ such that $k(t) > 0 \forall t \geq 0$ and $\alpha \in (0,1)$. Then, $x = 0$ is uniformly finite time stable. If the conditions $D = \mathbb{R}^n$, $W_1(0) = 0$ and $\lim_{\|x\|\to\infty} W_1(x) \to \infty$, $\lim_{\|x\|\to\infty} W_2(x) \to \infty$ hold true, then $x = 0$ is globally uniformly finite stable.*

The reader is refereed to [13] where finite time stability of autonomous systems was studied using strict Lyapunov functions. It can be seen from the definitions presented thus far that finite time stability is only meaningful when asymptotic stability is proven first. The thesis also uses the term *semi-global Lyapunov functions* frequently. The following definition of semi-global stabilisation can be found in [16, Section 12.1]:

Definition 2.12 (Semi-global stabilisation [16, Section 12.1]). *If feedback control does not achieve global stabilisation, but can be designed such that any given compact set* $B_r = \{x : \|x\| < r\}$, *with* $0 < r < \infty$ *chosen arbitrarily large, can be included in the region of attraction, the resulting stabilisation is said to be semi-global stabilisation.*

2.5 Conclusion

This chapter presented a comprehensive literature review of second order sliding mode controllers, continuous finite time controllers and control of unilaterally constrained systems. Mathematical preliminaries were also briefly described.

It can be seen from the discussion at the end of Sections 2.1, 2.2 and 2.3 that uniform finite time stability of unilaterally constrained systems as defined in Definition 2.5 is a new and interesting direction. The next two chapters first study the finite time nature of second order sliding mode controllers to establish the main result of achieving arbitrarily less settling time in the case of systems where resets do not appear. The Lyapunov based stability framework for the non-impact case is proposed in such a way that the subsequent chapters can construct the same results for systems with resets by exploiting the applicability of Filippov's solution concept and all the above definitions to systems with resets.

CHAPTER 3

Settling Time Estimate for a Second Order Sliding Mode Controller: A Homogeneity Approach

3.1 Introduction

Second order sliding mode controllers have received considerable attention from researchers within the sliding mode control research community since their formal introduction in [4] and numerous contributions have been made. The principal motivations to study second order sliding mode controllers can be easily identified to be (i) the possibility of having a non-Lipschitz but continuous control and (ii) mitigation of adverse effects of chattering while simultaneously preserving the well-known features of traditional sliding mode control, namely, robustness to persisting disturbances of some class and stabilization via appropriate design of sliding manifold [3, 104]. The main feature of second order sliding mode controllers is that finite time convergence to the origin of a planar controllable system is achieved where the persisting disturbances with a finite upper bound appear via the control channel [4]. Fully actuated controllable mechanical systems with Coulumb friction terms on the right hand side, for example, can be finite time stabilized by the so-called 'twisting' controller which is one type of second order sliding mode controller [105], [20, Ch. 12] [3, Ch. 3].

This branch of sliding mode control, however, has been receiving renewed interest recently relating to identifying strict Lyapunov functions such that the well-known finite time convergence property can be deduced along with traditional Lyapunov stability arguments [3, 41, 42, 70, 71, 106]. The finite time convergence property can be deduced either based on the homogeneity of the differential inclusion [9, 107] or based on the differential inequality of the underlying Lyapunov function (see [3, 41, 42, 70, 71, 106] for the discontinuous case where second order sliding mode controllers are used and

[12, 13] for the case of differential equations with a continuous right hand side). This chapter gives a new Lyapunov based framework such that the differential inequality is not needed. The methodology is based on identifying homogeneity regions such that the finite time of convergence to the origin can be computed. Furthermore, the existing semi-global Lyapunov function is shown to be a globally radially unbounded function with a globally negative definite temporal derivative along the closed-loop trajectory of the system. Existing literature on Lyapunov methods [3, 41, 42, 70, 71, 106] does give settling time estimates but does not give straightforward tuning rules. In this chapter a new method is derived based on the result of finite time stability of homogeneous switched systems [9] to compute a settling time estimate. The main motivation to propose an alternative method of computing the settling time is that explicit tuning guidelines can be established by successfully avoiding any parameter optimization.

The chapter is organized as follows. First a comprehensive review of Lyapunov approaches in the area of second order sliding mode control (SOSM) is given in section 3.2. The problem statement is then motivated in section 3.3. Section 3.4 proves the global nature of an existing semi-global non-smooth Lyapunov function via a mathematical proof of exponential stability and computes the homogeneity regions of the state space. A new method to compute the finite settling time is then established in section 3.5. Section 3.6 gives tuning guidelines to achieve a desired reduction in the settling time. Section 3.7 presents numerical simulation results. Section 3.8 gives a summary of results and motivates further investigations where it is argued why the existing knowledge and the research work proposed in this chapter are important with application to systems with resets in one of the states. Furthermore, a shortcoming of the approach is also discussed which is successfully solved in the next chapter. Some concluding remarks for the chapter are given in 3.9.

3.2 Lyapunov approaches: State of the art in SOSM

A robust second order sliding mode 'twisting' controller to stabilize a planar non-linear system to the origin in finite time was first presented in [4]. A survey of the finite time stability of discontinuous dynamical systems characterized by differential inclusions was carried out in [108]. The first instance where a Lyapunov function for a second order sliding mode controller is presented can be found in [9] where a semi-global strong Lyapunov function enables uniform finite time stability to be established in the presence of non-homogeneous disturbances. The attainment of finite time convergence is mainly due to the homogeneity of the differential inclusions. Recent work [70] formulates the Lyapunov function design problem for the twisting controller in terms of obtaining the solution of a first order partial differential equation. It should be noted that this work leads to an estimate of the settling time using the application of multiple Lyapunov functions. The latest development relating to the twisting controller in terms of providing a strong Lyapunov function that satisfies a differential inequality were recently given by [42, 106].

The twisting controller naturally finds its application to synthesis problems where the relative degree of the sliding function is 2 with respect to the control input [4], [3, Ch. 3]. The other variant of the second order sliding mode controller, the so-called 'super-twisting' algorithm, naturally finds its application in observation of unmeasured state variables. There have been many contributions as far as identifying Lyapunov functions for the super-twisting algorithm is concerned. The first such contribution [41] established uniform finite time stability of the observer by identifying an explicit Lyapunov function and obtained an explicit upper bound on the settling time for the super-twisting algorithm. The problem of Lyapunov function design for the super-twisting controller via generalization of Zubov's method is studied in [71] which is in fact a multiple Lyapunov function approach. Other contributions in this area have been given where the finite time stability of the super-twisting observer is proven via establishing a strict Lyapunov function [14]. More importantly, the Lyapunov function proposed in [14] leads to uniform finite time stability for both the cases, namely (i) when certain parameter of the observer is set at its homogeneity value giving robustness to persisting disturbances and (ii) when the aforementioned parameter is set between the interval $(0, 1)$ giving robustness to only vanishing disturbances. The latest advances in this direction can be found in [109] and [73] where adaptation in gains of the super-twisting controller is presented to minimize the effects of chattering and to reject disturbances with unknown boundaries respectively.

It is interesting to note that most of the references cited above share a unique property, namely, that finite time convergence of the Lyapunov function to the origin of the state space is achieved using an increasing condition on the Lyapunov function given by the differential inequality $\dot{V} \leq -KV^{\alpha}$, where α represents the strong decay rate and $K > 0$ is a constant dependent on system properties. A completely different approach to prove global finite time stability of discontinuous switched systems was established by [9]. The estimate of an upper-bound on the settling time function was computed for quasi-homogeneous differential inclusions based on the definitions of homogeneity ellipsoids in a close vicinity of the origin. However, the definitions of the homogeneity parameters defining the ellipsoids were not identified for the twisting control algorithm. It appears from the above literature that the main difference between the approaches that identify a Lyapunov function $V(x), x \in R^2$ that satisfies the aforementioned differential inequality and those that do not is that the former could not establish uniform finite time stability as defined in Definition 2.5 of the previous chapter[1] (also see [44] for definition of uniform finite time stability for the continuous right hand sides). Although no theoretical proof is presented in this chapter for the above observation, the intuitive reason appears to be the fact that it is more difficult to "guess" a Lyapunov function which satisfies a differential inequality along the closed-loop trajectory while also decaying uniformly in initial data and in disturbances, than the one that does not. The reader is referred to the discussion in Chapter 1 as regards the difference between the finite time stability and uniform finite time stability with respect to initial data and uncertainty (and also to the definition 2.5). This chapter and the rest of the thesis will follow the later method to keep focussed on uniformity of Lyapunov stability while establishing uniform finite time stability.

Robustness Comparison of SOSM with Continuous Finite Time Stabilization

As outlined in the literature review in earlier chapters, the literature on continuous and discontinuous finite time stabilization is rich. One of the early efforts of [11] showed that planar controllable systems, which can be converted in the Brunovsky canonical form

$$\ddot{x} = u(x, \dot{x}), \tag{3.1}$$

can be stabilised in finite time to the origin when a non-Lipschitz but continuous controller of the form

$$u(x, \dot{x}), = -|x|^a \text{sign}(x) - |\dot{x}|^b \text{sign}(\dot{x}) \tag{3.2}$$

with the condition $\frac{\alpha}{2-\alpha} > b, a \in (0, 1)$ is utilised. The same controller but with the condition $\frac{\alpha}{2-\alpha} = b$ also renders the double integrator (3.1) finite time stable [110]. A more rigorous Lyapunov based analysis was given such that the inequality of the form

$$\dot{V} \leq -KV^\alpha \tag{3.3}$$

is satisfied along the closed-loop trajectory of the system (3.1) with a continuous finite time stabilizing controller $u(x_1, x_2)$ defined by

$$u(x, \dot{x}) = -\mu_1 |\phi|^{\frac{\alpha}{2-\alpha}} \text{sign}(\phi) - \mu_2 |\dot{x}|^\alpha \text{sign}(\dot{x}), \tag{3.4}$$

[1]The results in [42] are an exception to this observation as the Lyapunov based finite time stability is uniform due to the proof of the inequality $\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|)$. However, no tuning rules are given for the gain parameters of the controller.

where $\phi = x + \frac{1}{2-\alpha}\text{sign}(\dot{x})|\dot{x}|^{2-\alpha}, \alpha \in (0,1)$. It was shown in [13, Th. 5.2] that finite time controllers exhibit better rejection properties to disturbances than their Lipschitz counterparts in that the ultimate bound on the state under Lipschitz feedback can only be guaranteed to be of the same order of magnitude as the perturbation and not less. It was shown in [14] that vanishing disturbances $\omega(t)$ that satisfy an upper bound of the form

$$\text{ess sup}\,|\omega(t)| \leq |x_2|^{\alpha} \tag{3.5}$$

can be suppressed and asymptotic stability of the perturbed version

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u(x,\dot{x}) + \omega(t) \end{aligned} \tag{3.6}$$

of the system (3.1) can be established, where $x = (x_1, x_2)^{\mathrm{T}} = (x, \dot{x})^{\mathrm{T}}$ is state vector. The work in [46] proposed a finite time observer the analysis of which was supported by a strict Lyapunov function. It should be noted that this observer was not shown robust to parameter uncertainties and measurement noises. Superiority of the super-twisting algorithm for observation of states can be easily seen in the latest research in Lyapunov analysis [14, 41, 71] where it was clearly shown via strict Lyapunov functions that the uncertainty on the right hand side of the first differential equation of (3.6) can have an upper bound (3.5) and that on the right hand side of the second differential equation of (3.6) can have an upper-bound of the form

$$\text{ess sup}\,|\omega(t)| \leq M \tag{3.7}$$

without affecting the finite time stability. Thus the second order sliding mode algorithms are designed to reject a stronger class of disturbances with the upper bound (3.7)[3, 4] as opposed to the bound (3.5).

The second order sliding mode algorithms can be utilised by adding an additional integrator when the relative degree of the sliding variable with respect to the control variable u is one [3]. The second order sliding mode algorithms produce a continuous and non-Lipschitz but more robust algorithms than the aforementioned continuous finite time stabilizing control laws. To support this observation, consider the following non-linear system:

$$\dot{x} = f(x) + g(x)u + \omega \tag{3.8}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}$ and f and g are sufficiently differentiable functions of x. The uncertainty $\omega(t)$ is assumed to be differentiable at least once. Assume that a sliding surface $s(x)$ is designed such that motion of the closed-loop system is stable on the manifold $s(x)$. Also assume that the relative degree of s with respect to u is one.

Consider the first derivative of s during the stable sliding motion as follows:

$$\dot{s} = \frac{\partial s}{\partial x} f(x) + \frac{\partial s}{\partial x} g(x) u \tag{3.9}$$

If it is assumed that $\left(\frac{\partial s}{\partial x} g(x)\right)^{-1} \neq 0$, the following sliding mode controller enforces a sliding motion on $s = 0$

$$u = -\left(\frac{\partial s}{\partial x} g(x)\right)^{-1} \left(\frac{\partial s}{\partial x} f(x) + \rho \text{sign}(s)\right) \tag{3.10}$$

if $\rho > |\frac{\partial s}{\partial x} \omega(t)| = M > 0$ [2, 3] since then $\dot{s} = -(\rho - \frac{\partial s}{\partial x} \omega(t))\text{sign}(s)$ is established. Such a controller will reject the disturbance with an upper bound of the following form:

$$\text{ess sup} \, |\frac{\partial s}{\partial x} \omega(t)| \leq M \tag{3.11}$$

If for technological reasons, a discontinuous controller is not permitted and a continuous finite time stable controller of the form

$$u = -\left(\frac{\partial s}{\partial x} g(x)\right)^{-1} \left(\frac{\partial s}{\partial x} f(x) + \rho |s|^{\alpha} \text{sign}(s)\right) \tag{3.12}$$

is utlised with $\alpha \in (0, 1)$ then only the disturbances ω with an upper bound of the form

$$\sup |\frac{\partial s}{\partial x} \omega(t)| \leq M|s|^{\alpha} \tag{3.13}$$

can be rejected without affecting the finite time convergence to the sliding manifold when $\rho > M$. This claim can be easily checked by considering a Lyapunov function candidate $V = |s|$. The temporal derivative of V along the trajectory (3.9) can be computed with u defined in (3.12) as follows:

$$\begin{aligned} \dot{V} = \dot{s}\text{sign}(s) &= -\rho|s|^{\alpha} + \frac{\partial s}{\partial x} \omega \text{sign}(s) \\ &\leq -(\rho - M)|s|^{\alpha} = -(\rho - M)V^{\alpha} \end{aligned} \tag{3.14}$$

Clearly $s = 0$ is reached in finite time [13, Th. 4.2]. It should be noted that the bound (3.13) is weaker than the bound (3.11). This is the cost of having continuity in control.

Let a second order sliding mode controller be utilised with an additional integrator such that $\dot{u} = \left(\frac{\partial s}{\partial x} g\right)^{-1} v$. The relative degree of the sliding variable s with respect to v becomes two. Then, considering the second derivative of (3.9), the following is obtained:

$$\begin{aligned}\ddot{s} &= F(x,u) + \left(\frac{\partial s}{\partial x}g\right)^{\mathrm{T}}\dot{u}\\ &= F(x,u) + v\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}F(x,u) = &\left(\tfrac{\partial s}{\partial x}\right)^{\mathrm{T}}\dot{f}(x) + \left(\tfrac{\partial^2 s}{\partial x^2}\dot{x}\right)^{\mathrm{T}} f(x) + \left(\tfrac{\partial^2 s}{\partial x^2}\dot{x}\right)^{\mathrm{T}} g(x)u + \left(\tfrac{\partial s}{\partial x}\dot{g}\right)^{\mathrm{T}} u\\ &+ \left(\tfrac{\partial s}{\partial x}\right)^{\mathrm{T}}\dot{\omega} + \left(\tfrac{\partial^2 s}{\partial x^2}\dot{x}\right)^{\mathrm{T}}\omega\end{aligned} \tag{3.16}$$

If it is assumed additionally that $\operatorname{ess\,sup}|F(x,u)| < F^+ < \infty$, the second order sliding mode control law

$$v(s,\dot{s}) = -\mu_1 \mathrm{sign}(\dot{s}) - \mu_2 \mathrm{sign}(s), \tag{3.17}$$

where $0 < F^+ < \mu_1 < \mu_2 - F^+$, guarantees uniform finite time stabilization of the sliding variable s such that $s = \dot{s} = 0$ in finite time [9]. It should be noted that the continuous finite time stabilizing control

$$v(s,\dot{s}) = -\mu_1|s|^{\alpha}\mathrm{sign}(\dot{s}) - \mu_2|\phi(s,\dot{s})|^{\frac{\alpha}{2-\alpha}} \tag{3.18}$$

where $\phi = s + \frac{1}{2-\alpha}|\dot{s}|^{2-\alpha}\mathrm{sign}(\dot{s})$ can only reject vanishing bounds[2]

$$\sup|F(x,u)| < F^+|s|^{\alpha} \tag{3.19}$$

as can be seen from the existing results [84, Th. 7.4]. Clearly, between the two controllers (3.17) and (3.18), (3.17) is more robust as it can reject persisting disturbances (or in terms of ω, once differentiable disturbances with a persistent bound on the derivative $\dot{\omega}$ as given by the right hand side of (3.16)) while maintaining uniform finite time stability. It should be noted that the controller u becomes a smooth controller when $\dot{u} = v$ with (3.18) is used. Whereas, the controller u is continuous but non-Lipschitz when $\dot{u} = v$ with (3.17) is used. Similar analysis on the super-twisting algorithm can be found in [3, Ch. 3].

Having discussed the superior robustness properties, it is important to highlight why the study of finite settling time behaviour and the corresponding tuning rules for the second order sliding mode controllers is relevant practically. Applications such as tracking control of biped robots inherently require stabilization of tracking errors in each joint before the next successive impact occurs when the swing leg touches ground [15]. When a finite time stabilizing controller is used in such applications, an a priori guarantee of achieving the prescribed settling time is of engineering importance to maintain

[2] The choice ess sup is invoked when the disturbances are allowed to be piece-wise continuous and only sup is invoked when the disturbances are restricted to absolutely continuous ones.

overall stability. Thus giving constructive tuning guidelines for a robust second order sliding mode algorithm is theoretically challenging and practically relevant.

3.3 Motivation and problem formulation

The principal motivation and objective of this chapter is to introduce a novel homogeneity based approach which obviates the need for the differential inequality $\dot{V} \leq -kV^{\alpha}$ while obtaining an upper bound on the settling time of the twisting controller. The framework differs significantly from the existing Lyapunov approaches while maintaining similar advantages. The framework is substantiated by exponential stability considerations using a global non-smooth Lyapunov function. The underlying philosophy is to combine the global exponential decay of the state trajectories to the domain of attraction with the finite time stability of quasi-homogeneous inclusion within the domain of attraction. The homogeneity regions are identified which represent the domain of attraction.

The main contribution is twofold. Firstly, the proposed framework, while removing the dependence on differential inequalities, is novel and can inspire a new direction in establishing an upper bound on the settling time of exponentially stable homogeneous systems. Secondly, tuning rules are established for the twisting controller to achieve a desired settling time. In turn, the conservative nature of the settling time estimate becomes insignificant if the gains are appropriately tuned. This is clearly a contribution as the recent work of [106] does not provide straightforward rules to reduce the upper bound and hence cannot achieve a desired improvement in the settling time.

Consider the application of a twisting controller of the form [9]

$$u(x_1, x_2) = -\mu_1 sign(x_2) - \mu_2 sign(x_1) - hx_1 - px_2 \tag{3.20}$$

for stabilizing the following perturbed double integrator system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u(x_1, x_2) + \omega(x_1, x_2, t) \end{aligned} \tag{3.21}$$

The following assumptions are made:

1. An upper bound on the disturbance term $|\omega| < M_1, M_1 > 0$ is known.

2. The controller gains meet the conditions $0 < M_1 < \mu_1 < \mu_2 - M_1$ and let M be such that $0 < M_1 < M < \mu_1 < \mu_2 - M$ holds true. Also $h > \kappa^2, p > \kappa$ for arbitrary $\kappa > 0$.

The objective of this chapter is

1. To establish an upper bound $\mathcal{T}(\mu_1, \mu_2, h, p, M_1, M)$ on the settling time of the system (3.20), (3.21).

2. To establish tuning rules for the controller parameters $\{\mu_1, \mu_2, h, p\}$ to achieve the desired settling time.

3.4 Stability and homogeneity

This section develops the required mathematical details in order to achieve the aims stated in Section 3.3. Firstly, the global exponential stability of the twisting algorithm is established using a global non-smooth Lyapunov function. Secondly, the homogeneity regions are identified step-by-step to facilitate the derivation of finite settling time.

3.4.1 Global Exponential Stability

The following result on global equiuniform asymptotic stability can now be stated (see Definition 2.4 for the definition of equiuniform asymptotic stability):

Theorem 3.1. *Given the assumptions 1 and 2, the closed-loop system (3.20), (3.21) is globally equiuniformly exponentially stable.*

Proof. Consider a non-smooth Lyapunov function candidate [9] as follows:

$$V = \mu_2|x_1| + \tfrac{1}{2}\left(hx_1^2 + x_2^2\right) + \kappa\, x_1\, x_2 \tag{3.22}$$

It should be noted that the outcome of global equiuniform asymptotic stability in [9] was carried out on the premises of semi-global analysis. The negative definite temporal derivative of the Lyapunov function (3.22) was sought only in a compact set $D_R = \{x : \mu_2|x_1| + \tfrac{1}{2}\left(hx_1^2 + x_2^2\right) \leq R\}$. No such assumption is made here to prove global equiuniform exponential stability. Equation (3.22) can be represented in quadratic form as follows:

$$V = \xi_1^{\mathrm{T}}\Psi_1\xi_1 \tag{3.23}$$

where, $\xi_1^{\mathrm{T}} = [\ |x_1|^{\frac{1}{2}} \quad x_1 \quad x_2\ ]$ and

$$\Psi_1 = \frac{1}{2}\begin{bmatrix} 2\mu_2 & 0 & 0 \\ 0 & h & \kappa \\ 0 & \kappa & 1 \end{bmatrix} \tag{3.24}$$

$\Psi_1 > 0$ if $h > \kappa^2$. Hence the following can be deduced:

$$\lambda_{\min}\{\Psi_1\}\,\|\xi_1\|^2 \le V \le \lambda_{\max}\{\Psi_1\}\,\|\xi_1\|^2 \tag{3.25}$$

where $\|\xi_1\|^2 = |x_1| + x_1^2 + x_2^2$. The temporal derivative along the closed-loop system trajectories can be derived as follows:

$$\dot{V} = -(\mu_1 - M_1)|x_2| - (p-\kappa)\,x_2^2 - \kappa\,h\,x_1^2 \; -\kappa(\mu_2 - \mu_1 - M_1)|x_1| - \kappa\,x_1^2 - \kappa p\,x_1\,x_2 \tag{3.26}$$

The quadratic form of (3.26) can be obtained as follows:

$$\dot{V} = -\xi_2^{\mathrm{T}}\Psi_2\xi_2 \tag{3.27}$$

where $\xi_2^{\mathrm{T}} = [\ |x_1|^{\frac{1}{2}} \quad |x_2|^{\frac{1}{2}} \quad x_1 \quad x_2\ ]$ and

$$\Psi_2 = \begin{bmatrix} \kappa(\mu_2 - \mu_1 - M_1) & 0 & 0 & 0 \\ 0 & \mu_1 - M_1 & 0 & 0 \\ 0 & 0 & \kappa\,h & \left(\frac{1}{2}\kappa\,p\right) \\ 0 & 0 & \left(\frac{1}{2}\kappa\,p\right) & p-\kappa \end{bmatrix} \tag{3.28}$$

The matrix $\Psi_2 > 0$ if

$$p > \kappa\,, \quad \kappa^2\,p^2 - 4\kappa\,h\,p + 4\,h\,\kappa^2 < 0 \text{ or } h > \tfrac{p^2\kappa}{4(p-\kappa)} \tag{3.29}$$

The quadratic form (3.27) leads to the following:

$$\dot{V} \le -\lambda_{\min}\{\Psi_2\}\,\|\xi_2\|^2 \tag{3.30}$$

Noting that the inequalities $\|\xi_2\|^2 = |x_1| + |x_2| + x_1^2 + x_2^2 \ge |x_1| + x_1^2 + x_2^2 = \|\xi_1\|^2$ and (3.25) hold true, (3.30) can be re-written as,

$$\begin{gathered} \dot{V} \le -\tfrac{\lambda_{\min}\{\Psi_2\}}{\lambda_{\max}\{\Psi_1\}}V \\ V(x_1(t), x_2(t)) = e^{-K\,(t-t_0)}V(x_1(t_0), x_2(t_0)) \end{gathered} \tag{3.31}$$

where, $K = \frac{\lambda_{\min}\{\Psi_2\}}{\lambda_{\max}\{\Psi_1\}}$. From (3.31) and the definitions of Ψ_1, Ψ_2, the constant K is bounded due to the inequalities $h > \kappa^2$, (3.29) and the assumption (2) imposed on the tuning parameters. It remains to prove that matrices Ψ_1, Ψ_2 are positive definite. It is noted that the quadratic inequality (3.29) leads to the following real interval:

$$\begin{aligned} &p_1 < p < p_2 \quad \text{where,} \\ &p_1 = \frac{4\kappa\, h - \sqrt{16\kappa^2\, h^2 - 16\kappa^4\, h}}{2\kappa^2}, \quad p_2 = \frac{4\kappa\, h + \sqrt{16\kappa^2\, h^2 - 16\kappa^4\, h}}{2\kappa^2} \end{aligned} \tag{3.32}$$

The values p_1, p_2 can further be simplified as follows:

$$\begin{aligned} &p_1 < p < p_2 \quad \text{where,} \quad p_1 = \tfrac{2}{\kappa}\left[h - \sqrt{h(h-\kappa^2)}\right], \\ &p_2 = \tfrac{2}{\kappa}\left[h + \sqrt{h(h-\kappa^2)}\right] \end{aligned} \tag{3.33}$$

If $h > \kappa^2$ holds true then the interval $[p_1, p_2]$ is always real, i.e $[p_1, p_2] \in \mathbb{R}$ and the positive definiteness of Ψ_2 is purely down to the satisfaction of the strict inequality $p_1 < p < p_2$. This is easily achieved since the inequalities $p_1 > 0, p_2 > h, h > \kappa^2$ always hold true for all $h > 1, \kappa < 1$ and hence the choice $p = h, h > 1 > \kappa^2, \kappa < 1$ will always satisfy positive definiteness of both the matrices Ψ_1, Ψ_2 thereby proving the statement of the Theorem 3.1. □

Remark 3.1. The Lyapunov function V is a globally radially unbounded positive definite function which proves global exponential stability due to (3.25), (3.30) and (3.31). Except from the global nature, it is similar to the semi-global Lyapunov function proposed by [9]. The strict Lyapunov function recently proposed by [42, 106] does lead to a settling time estimate and is an even stronger candidate than the function V proposed above. However, no constructive tuning rules for the gain parameters to achieve a desired settling time are currently available.

The Lyapunov function V and the function $\tilde{V} = \mu_2|x_1| + \frac{1}{2}\left(hx_1^2 + x_2^2\right)$ proposed in [9] are equivalent in that the following inequality holds true on a compact set $D_R = \left\{(x_1, x_2) : \tilde{V} \leq R\right\}$:

$$\begin{aligned} &V \leq M_R\tilde{V} \quad \text{where } M_R > \max\left\{\frac{2\mu_2^2+\kappa R}{2\mu_2^2}, 1+\kappa\right\}, \\ &\kappa < \min\left\{1, \frac{2\mu_2^2}{R}, \frac{\mu_2(\mu_1 - M_1 + p\sqrt{2R})}{\mu_2\sqrt{2R}+pR}\right\}. \end{aligned} \tag{3.34}$$

See [9] for further details on semi-global analysis. Hence the restriction $\mu_1 \geq M_1 + \sqrt{2R}$ in addition to the assumptions in section 3.3 suffices to obtain the values $\kappa < 1, M_R > 1 + \kappa$.

3.4.2 Homogeneity Regions

Finite time stability of uncertain switched systems, which are characterised by homogeneous differential equations with discontinuous right hand side, was studied in [9]. The main feature of this quasi-homogeneity study was that the finite time stability was established even when the homogenous differential equation has a uniformly bounded non-homogeneous possibly discontinuous perturbation appearing on the right hand side. From the engineering view-point, the motivation to study such problems stems from the case when second order homogeneous sliding mode controllers are utilised for fully actuated planar non-linear systems. The resulting nominal system may be globally homogeneous enabling the elegant mathematical tools of homogeneity to produce finite time stability. However, it is highly likely that the physical system will have non-homogeneous disturbances where quasi-homogeneity has to be studied. Having discussed this background, it is of interest to establish the regions of state space of the current problem where the aforementioned quasi-homogeneity principle is applicable. The process of identifying the homogeneity regions can be listed as follows:

1. Identify the radius r of the homogeneity ball

$$\mathcal{B}_r = \left\{(x_1, x_2) : \tfrac{x_1^2}{r^2} + \tfrac{x_2^2}{r^2} \leq 1\right\} \tag{3.35}$$

2. Identify the scalar $\delta > 0$ such that the following definition of the homogeneity ellipsoid E_δ holds true:

$$E_\delta = \left\{(x_1, x_2) : \sqrt{\left(\tfrac{x_1}{\delta^{r_1}}\right)^2 + \left(\tfrac{x_2}{\delta^{r_2}}\right)^2} \leq 1\right\} \subseteq \mathcal{B}_r \tag{3.36}$$

   where r_1, r_2 are dilation weights.

3. Identify the scalars $\tilde{R} > 0, \bar{R} > 0$ such that the following expressions of the level sets of the Lyapunov function V holds true in addition to (3.36).

$$\begin{aligned} \Omega_2 &= \left\{(x_1, x_2) : V \leq \bar{R}\right\} \subseteq E_\delta \\ E_\delta \subseteq \Omega_1 &= \left\{(x_1, x_2) : V \leq \tilde{R}\right\} \end{aligned} \tag{3.37}$$

4. Identify $\frac{1}{2}\delta > 0$, corresponding ellipsoid $E_{\frac{1}{2}\delta}$ and level set Ω_3 in a similar way such that the following expressions are satisfied:

$$\begin{aligned} E_{\frac{1}{2}\delta} &= \left\{(x_1, x_2) : \sqrt{\left(\frac{x_1}{(\frac{1}{2}\delta)^{r_1}}\right)^2 + \left(\frac{x_2}{(\frac{1}{2}\delta)^{r_2}}\right)^2} \leq 1\right\} \\ \Omega_3 &= \left\{(x_1, x_2) : V \leq \hat{R}\right\} \subseteq E_{\frac{1}{2}\delta} \end{aligned} \tag{3.38}$$

where $\hat{R} > 0$, which is the supremum of V within the set Ω_3, is also to be identified.

As discussed above, the motivation to achieve the above results is the fact that the estimate of the finite settling time can be obtained once the definitions of the homogeneity ellipsoids E_δ and $E_{\frac{1}{2}\delta}$ are known (See Theorem 3.2 of [9]). The homogeneity degree for the closed-loop system (3.20), (3.21) is $q = -1$ with respect to dilations $r_1 = 2$, $r_2 = 1$. These values will be substituted hereafter. The stated steps can be established as follows:

Step 1: Definition of homogeneity region in terms of 1-norm $\|X\|_1 = |x_1| + |x_2|$

The system (3.20), (3.21) is homogeneous in the sense of Definitions 2.6, 2.7 in a small vicinity of the origin if the following condition holds true:

$$\begin{aligned} h|x_1| + p|x_2| + |\omega(x_1, x_2, t)| \leq M \\ h|x_1| + p|x_2| \leq M - M_1 \end{aligned} \tag{3.39}$$

It is noted here that the summation of linear feedback terms and the uncertainty $h|x_1| + p|x_2| + |\omega(x_1, x_2, t)|$ is treated as nonhomogeneous perturbation. The finite time stability of homogeneous switched systems in the presence of nonhomogeneous perturbations is an established result (see Theorem 4.2 [9] with application to twisting controller). The existence of the parameter M thus represents a bound on the nonhomogeneous perturbations in the right hand side of the switched system (3.20), (3.21). Noting that the inequality

$$h|x_1| + p|x_2| \leq \max\{h, p\}\left(|x_1| + |x_2|\right)$$

always holds true, the requirement (3.39) holds true whenever the following upper bound on $\|X\|_1$ is satisfied:

$$\|X\|_1 = |x_1| + |x_2| \leq \frac{M - M_1}{\max\{h, p\}} \tag{3.40}$$

Step 2: Definition of homogeneity ball $\mathcal{B}_r$ *in terms of 2-norm* $\|X\|_2$

The following is a well-known relationship between the Euclidian norm $\|X\|_2$ and 1-norm $\|X\|_1$ (see [111]):

$$\|X\|_2 \geq \frac{1}{\sqrt{2}}\|X\|_1 \tag{3.41}$$

From (3.41) and (3.40), a conservative bound on the homogeneity radius r of the homogeneity ball $\mathcal{B}_r$ can be obtained as follows:

$$r = \sqrt{|x_1|^2 + |x_2|^2} = \|X\|_2 \leq \tfrac{1}{\sqrt{2}} \tfrac{M-M_1}{\max\{h,p\}} \tag{3.42}$$

The inequalities (3.41) and (3.42), when combined, will always ensure that the inequality (3.40) holds true.

Step 3: Definition of the parameter δ

The aim is to find $\delta > 0$ such that every point (x_1, x_2) contained within the ellipsoid E_δ is also contained within the homogeneity ball $\mathcal{B}_r$. Having computed the homogeneity radius r in *step 2*, if $\delta > 0$ is chosen such that the equalities

$$\begin{aligned} \min\left\{\tfrac{1}{\delta^4}, \tfrac{1}{\delta^2}\right\} = \tfrac{1}{r^2} \Rightarrow \quad & \max\left\{\delta^4, \delta^2\right\} = r^2 \\ \Rightarrow \quad & \max\left\{\delta^2, \delta\right\} = r \end{aligned} \tag{3.43}$$

are satisfied, then due to the fact that the equality

$$\min\left\{\tfrac{1}{\delta^4}, \tfrac{1}{\delta^2}\right\}\left(|x_1|^2 + |x_2|^2\right) = \tfrac{1}{r^2}\left(|x_1|^2 + |x_2|^2\right) \tag{3.44}$$

always holds true, the inequality

$$\begin{aligned} \min\left\{\tfrac{1}{\delta^4}, \tfrac{1}{\delta^2}\right\}\left(|x_1|^2 + |x_2|^2\right) &= \tfrac{1}{r^2}\left(|x_1|^2 + |x_2|^2\right) \\ &\leq \left(\tfrac{x_1}{\delta^2}\right)^2 + \left(\tfrac{x_2}{\delta}\right)^2 \end{aligned} \tag{3.45}$$

also holds true. If the given point $(x_1, x_2) \in E_\delta$, then the inequality

$$\sqrt{\left(\tfrac{x_1}{\delta^2}\right)^2 + \left(\tfrac{x_2}{\delta}\right)^2} \leq 1 \tag{3.46}$$

holds true which, using (3.44), leads to the inequality

$$\frac{1}{r^2}\left(|x_1|^2 + |x_2|^2\right) \leq 1 \tag{3.47}$$

Hence, $(x_1, x_2) \in \mathcal{B}_r$ and the choice (3.43) of δ is indeed valid, which upon further simplification, satisfies:

$$\delta = \min\{r, \sqrt{r}\} \tag{3.48}$$

The aim of computing $\delta > 0$ such that $E_\delta \subseteq \mathcal{B}_r$ is thus achieved.

Step 4: Definition of level sets Ω_1, Ω_2 and the upper bounds $\tilde{R}, \bar{R}$

The first aim is to compute $\bar{R} > 0$ such that the level set Ω_2 satisfies $\Omega_2 \subseteq E_\delta$. Combining the definition of the level set Ω_2 with the inequality (3.34), it suffices that the inequality $\tilde{V} \leq \frac{\bar{R}}{M_R}$ holds true in order that $\Omega_2 \subseteq E_\delta$ is satisfied for any given (x_1, x_2) in a small vicinity of the origin. Hence the following must be satisfied:

$$\frac{\mu_2 M_R}{\bar{R}}|x_1| + \frac{M_R}{2\bar{R}}x_2^2 + \frac{hM_R}{2\bar{R}}x_1^2 \leq 1 \Rightarrow \left(\frac{x_1}{\delta^2}\right)^2 + \left(\frac{x_2}{\delta}\right)^2 \leq 1 \tag{3.49}$$

Having computed the homogeneity ellipsoid parameter δ in *step 3*, if $\bar{R} > 0$ is chosen such that the inequalities

$$\left(\frac{x_1}{\delta^2}\right)^2 \leq \frac{hM_R}{2\bar{R}}x_1^2 \quad , \quad \left(\frac{x_2}{\delta}\right)^2 \leq \frac{M_R}{2\bar{R}}x_2^2 \tag{3.50}$$

are satisfied, then the inequality

$$\left(\frac{x_1}{\delta^2}\right)^2 + \left(\frac{x_2}{\delta}\right)^2 \leq \frac{\mu_2 M_R}{\bar{R}}|x_1| + \frac{M_R}{2\bar{R}}x_2^2 + \frac{hM_R}{2\bar{R}}x_1^2 \tag{3.51}$$

always holds true. For a given point $(x_1, x_2) \in \Omega_1$, the inequality

$$\frac{\mu_2 M_R}{\bar{R}}|x_1| + \frac{M_R}{2\bar{R}}x_2^2 + \frac{hM_R}{2\bar{R}}x_1^2 \leq 1 \tag{3.52}$$

holds true, which using (3.51), leads to the inequality

$$\left(\frac{x_1}{\delta^2}\right)^2 + \left(\frac{x_2}{\delta}\right)^2 \leq 1 \tag{3.53}$$

Hence $(x_1, x_2) \in E_\delta$ and the choice (3.50) of $\bar{R}$ is indeed valid, which upon further simplification, satisfies:

$$\begin{aligned} &\left(\frac{1}{\delta^2}\right)^2 \leq \frac{hM_R}{2\bar{R}} \quad \text{and} \quad \left(\frac{1}{\delta}\right)^2 \leq \frac{M_R}{2\bar{R}} \\ \Rightarrow \quad &\bar{R} = \frac{M_R \delta^2}{2} \min\left\{h\,\delta^2, 1\right\} \end{aligned} \tag{3.54}$$

The second aim is to compute $\tilde{R} > 0$ such that the ellipsoid E_δ satisfies $E_\delta \subseteq \Omega_1$. Combining the definition of the level set Ω_1 with the inequality (3.34), the following expressions must hold true so that $E_\delta \subseteq \Omega_1$ is satisfied for any given (x_1, x_2) in a small vicinity of the origin:

$$\left(\frac{x_1}{\delta^2}\right)^2 + \left(\frac{x_2}{\delta}\right)^2 \leq 1 \Rightarrow \frac{\mu_2 M_R}{\tilde{R}}|x_1| + \frac{M_R}{2\tilde{R}}x_2^2 + \frac{hM_R}{2\tilde{R}}x_1^2 \leq 1 \tag{3.55}$$

In other words, the following must be satisfied:

$$\frac{\mu_2 M_R}{\tilde{R}}|x_1| + \frac{M_R}{2\tilde{R}}x_2^2 + \frac{hM_R}{2\tilde{R}}x_1^2 \leq \left(\frac{x_1}{\delta^2}\right)^2 + \left(\frac{x_2}{\delta}\right)^2 \leq 1 \tag{3.56}$$

Equation (3.56) always holds true if the following is ensured:

$$\begin{aligned} \frac{\mu_2 M_R}{\tilde{R}}\left|x_1\right| \leq (1-\epsilon_1) \\ \frac{M_R}{2\tilde{R}}x_2^2 + \frac{hM_R}{2\tilde{R}}x_1^2 \leq \epsilon_1\left\{\left(\frac{x_1}{\delta^2}\right)^2 + \left(\frac{x_2}{\delta}\right)^2\right\} \end{aligned} \tag{3.57}$$

where, $0 < \epsilon_1 < 1$ is an arbitrary constant. The fact that $(x_1, x_2) \in E_\delta$ leads to $|x_1| \leq \delta^2$ using the first inequality of (3.55). Hence (3.57) can be further simplified to derive formula for $\tilde{R}$ by enforcing the following three sub-conditions:

$$\begin{aligned} \frac{\mu_2 M_R}{\tilde{R}}\left|x_1\right| \leq \frac{\mu_2 M_R}{\tilde{R}}\delta^2 \leq (1-\epsilon_1), \quad \frac{M_R}{2\tilde{R}}x_2^2 \leq \epsilon_1\left(\frac{x_2}{\delta}\right)^2, \\ \frac{hM_R}{2\tilde{R}}x_1^2 \leq \epsilon_1\left(\frac{x_1}{\delta^2}\right)^2 \end{aligned} \tag{3.58}$$

The formula $\tilde{R} = M_R\delta^2 \max\left\{\frac{\mu_2}{1-\epsilon_1}, \frac{h\delta^2}{2\epsilon_1}, \frac{1}{2\epsilon_1}\right\}$ can be deduced from (3.58). The aims of computing $\tilde{R} > 0, \bar{R} > 0$ such that $\Omega_2 \subseteq E_\delta \subseteq \Omega_1$ are thus achieved.

Step 5: Definition of the parameter $\hat{R}$ and level set Ω_3 Similar arguments to those outlined in *Step 4* produces the following formula:

$$\hat{R} = \frac{M_R\delta^2}{8}\min\left\{\frac{h}{4}\delta^2, 1\right\} \tag{3.59}$$

3.5 Settling time estimate

The quasi-homogeneity concept is geometrically depicted in Fig. 3.1. Trajectories of the system (3.20),(3.21) in the phase plane (x_1, x_2) are also shown. The existence of an exponentially decaying global Lyapunov function V is utilized. The point O_1 is the system initial condition which corresponds to the boundary of the level set $\Omega_R = \{(x_1, x_2) : V(x_1, x_2) \leq M_R R\}$ where $R = \tilde{V}(x_1(t_0), x_2(t_0))$. Then, due to the fact that the system decays exponentially towards the origin, it can be deduced that the trajectory enters the homogeneity ball $\mathcal{B}_r$ in finite time, where r is defined by (3.42), and subsequently enters the homogeneity ellipsoid E_δ. This in turn causes the trajectories of the closed-loop systems to satisfy the definition of the level set $\Omega_2 = \left\{(x_1, x_2) : V \leq \tilde{R}\right\} \subseteq E_\delta$ of the Lyapunov function V in finite time. This corresponds to the point O_2. Finally, finite time stability follows from the quasi-homogeneity principle once the system trajectories are inside the ellipsoid E_δ. As a consequence, the settling time of the system is the summation of the following:

$$\mathcal{T}(x_1, x_2) = \mathcal{T}_{O_1 - O_2} + \mathcal{T}_h \tag{3.60}$$

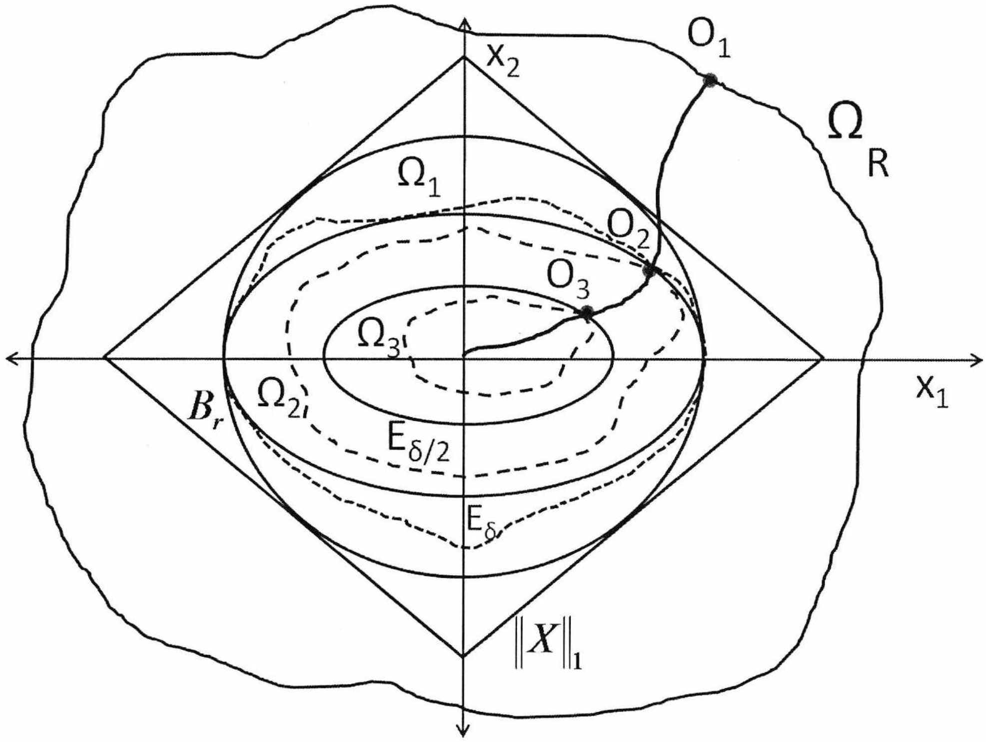

FIGURE 3.1: Quasi-homogeneity concept: homogeneous regions $\|X\|_1$, homogeneity ball $\mathcal{B}_r$, homogeneity ellipsoids (E_δ, $E_{\frac{1}{2}\delta}$) and level sets $\Omega_R, \Omega_1, \Omega_2, \Omega_3$

where $\mathcal{T}_{O_1-O_2}$ is the time taken by the state trajectories of the closed-loop system to attain the level set Ω_2 (point O_2) from the initial condition level set Ω_R (point O_1) and $\mathcal{T}_h$ is finite settling time of the system to attain equilibrium point (0,0) from the homogeneity boundary $E_\delta \subseteq \Omega_1$ which can be readily computed using the expression (3.12) of [9] as follows:

$$\mathcal{T}_h = \frac{c_0{}^q}{1-2^q} \left\{ \sup_{(x_1,x_2)\in E_\delta} \mathcal{T}_{O_2-O_3} \right\} \tag{3.61}$$

where q is homogeneity degree, c_0 is a lower estimate of the homogeneity parameter and $\mathcal{T}_{O_2-O_3}$ is the time taken by the state trajectories of the closed-loop system to travel from the homogeneity boundary $E_\delta \subseteq \Omega_1$ to the boundary $\Omega_3 \subseteq E_{\frac{1}{2}\delta}$ (point O_3). The necessity to use the boundary of the level set Ω_1 in place of Ω_2 stems from the fact that the supremum of $\mathcal{T}_{O_2-O_3}$ has to be taken into consideration while computing the worst possible decay of the Lyapunov function. Hence, the boundary given by Ω_2 has to be utilized to compute $\mathcal{T}_{O_1-O_2}$ and that given by Ω_1 to compute $\mathcal{T}_{O_2-O_3}$ in order to encompass the worst case scenario. Although an overlap of time contributions may occur in the summation (3.60) leading to a conservative result since $\Omega_2 \subseteq \Omega_1$ holds true, the estimate of the settling time thus obtained is a true upper-bound, nevertheless. It can be easily noted that (3.60) is the same as the expression (3.1) of [9]. The terms $\mathcal{T}_{O_1-O_2}$ and $\mathcal{T}_{O_2-O_3}$ can be estimated from the exponential decay (3.31) as follows:

$$\begin{aligned}\mathcal{T}_{O_1-O_2}(\mu_1,\mu_2,h,p,M,M_1,R,\bar{R}) &= t_{O_1} + \tfrac{\ln[M_R R]-\ln[\bar{R}]}{K}\\ \mathcal{T}_{O_2-O_3}(\mu_1,\mu_2,h,p,M,M_1,\tilde{R},\hat{R}) &= \tfrac{\ln[\tilde{R}]-\ln[\hat{R}]}{K}\end{aligned} \quad (3.62)$$

where the substitutions $V(t_{O_1}) = M_R R, V(t_{O_2}) = \bar{R}$ have been utilized corresponding to the level sets Ω_R and Ω_2 at time instants t_{O_1} and t_{O_2} respectively in the first equality and substitutions $V(t_{O_2}) = \tilde{R}, V(t_{O_3}) = \hat{R}$ have been utilized corresponding to the level sets Ω_1 and Ω_3 at time instants t_{O_2} and t_{O_3} respectively in the second equality while utilizing (3.31). Hence (3.60) can be re-written as,

$$\mathcal{T}(\mathcal{T}_{O_1-O_2},\mathcal{T}_h) = t_{O_1} + \tfrac{\ln[M_R R]-\ln[\bar{R}]}{K} + \tfrac{c_0{}^q[\ln(\tilde{R})-\ln(\hat{R})]}{K(1-2^q)} \quad (3.63)$$

Under the stated assumptions, the homogeneity parameters $r, \delta, \tilde{R}, \bar{R}, \hat{R}$ outlined in Section 3.4.2 and in turn the settling time estimate (3.63) can be computed *a priori.*

Remarks on settling time estimate

The estimate of the homogeneity parameter $c \geq c_0$ should be satisfied for the chosen c_0 where c_0 is the lower estimate of the homogeneity parameter. It can be seen from the above development that the closed-loop system is homogeneous once it is inside the ellipsoid E_δ. The identity $\delta\, R_0^{-1} = c$ then leads to $c = 1$ because $R_0 = \delta$ is chosen to facilitate the application of (3.61), where the scalar $R_0 > 0$ represents the largest homogeneity ellipsoid E_{R_0} (see (3.12) of [9] for more details). Hence $c_0 = 1$ is a valid choice.

The settling time estimate consists of the exponential decay summed with the finite settling time. The smaller the homogeneity parameter M the longer it takes for the trajectories to reach the homogeneity regions. Hence, the choice of M should be as large as possible. The interval $M_1 < M < \mu_2 - \mu_1$ is the available range in which M can take values (see Definition 2.2).

The upper bound (3.63) is conservative. The latest advances in the literature also exhibit the conservative nature of the settling time estimate (see [106]).

3.6 Tuning rules

This section establishes constructive tuning rules for the controller parameters μ_1, μ_2, h, p to ensure the desired settling time $\mathcal{T}$ is obtained. The proposed method does not require optimization of gain parameters. The following relations were obtained in previous sections:

$$\begin{gathered}\kappa < 1, M_R > 1+\kappa, p = h, h > 1 > \kappa^2, p > \kappa \\ \mu_1 > M_1 + \sqrt{2R}, 0 < M_1 < M < \mu_1 < \mu_2 - M\end{gathered} \tag{3.64}$$

The following steps can now be performed to obtain the required tuning rules:

step 1: Choose a finite constant M such that $M > M_1 + 1$. As noted earlier in Section 3.4.2, the constant M represents the bound on the non-homogeneous perturbations. In a small vicinity of the origin, parameters μ_1, μ_2 of the variable structure controller enforce finite time stability. The aim is to tune μ_1, μ_2 high enough to satisfy the inequality $0 < M + \mu_1 < \mu_2$. Next, let the following relation be enforced:

$$h > M - M_1 \tag{3.65}$$

Then equalities (3.48) and (3.42) lead to the following formulae:

$$r = \frac{M - M_1}{\sqrt{2h}}, \quad \delta = r \tag{3.66}$$

Step 2: Divide the desired settling time $\mathcal{T}$ into two contributions, i.e. $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ such that $\mathcal{T}_1 = \epsilon_3 \mathcal{T}_2$, where $\epsilon_3 = \rho + (\frac{3}{2} - \rho)e^{-\mathcal{T}}$, $0 \leq \rho \leq \frac{3}{2}$ and $\mathcal{T}_1$ and $\mathcal{T}_2$ represent the exponential decay and homogeneity based finite settling time respectively. Thus the following must hold true in order to achieve the desired settling time $\mathcal{T}$:

$$\frac{\ln[M_R R] - \ln[\bar{R}]}{K} \leq \mathcal{T}_1, \; \frac{c_0{}^q(\ln[\tilde{R}] - \ln[\hat{R}])}{K(1-2^q)} \leq \mathcal{T}_2 \tag{3.67}$$

where $R = \tilde{V}(x_1(t_0), x_2(t_0), t_0)$.

Step 3: Let the following inequality be enforced:

$$h\delta^2 < 1 \quad \text{or using(3.66)}, \; h > \frac{(M - M_1)^2}{2} \tag{3.68}$$

Then from (3.54) and (3.59), it is obtained that,

$$\bar{R} = \frac{hM_R\delta^4}{2} = \frac{M_R(M - M_1)^4}{8\,h^3}, \; \hat{R} = \frac{hM_R\delta^4}{32} = \frac{M_R(M - M_1)^4}{128\,h^3} \tag{3.69}$$

Step 4: Let the following inequality be enforced:

$$\frac{\mu_2}{1 - \epsilon_1} \geq \frac{h\delta^2}{2\epsilon_1} \quad \text{or} \quad h \geq \frac{(M - M_1)^2(1 - \epsilon_1)}{2\epsilon_1\mu_2} \tag{3.70}$$

Also let the following hold true:

$$\frac{\mu_2}{1-\epsilon_1} \geq \frac{1}{2\epsilon_1} \text{ or } \mu_2 \geq \frac{(1-\epsilon_1)}{2\epsilon_1} \tag{3.71}$$

Then from the definition of $\tilde{R}$ in Section 3.4.2, the following can be obtained:

$$\tilde{R} = \frac{M_R \delta^2 \mu_2}{1-\epsilon_1} = \frac{M_R (M-M_1)^2 \mu_2}{h^2(1-\epsilon_1)} \tag{3.72}$$

Step 5: Equation (3.67) may now be invoked. Utilizing the equality in the first inequality of (3.67) facilitates the elimination of the parameter K as follows:

$$K = \frac{\ln[M_R R] - \ln[\bar{R}]}{\mathcal{T}_1} \tag{3.73}$$

Substituting K in the second inequality of (3.67) leads to

$$2\ln(\tilde{R}) - \ln(\hat{R}) \leq \frac{\mathcal{T}_2}{\mathcal{T}_1} \ln(M_R R) - \ln(\bar{R}) \tag{3.74}$$

where, $c_0 = 1, q = -1$ has been utilized. Substituting (3.69) and (3.72) into (3.74), the sub-ordination between h and μ_2 is obtained as follows:

$$\begin{aligned} \left(\frac{64h\mu_2}{(1-\epsilon_1)(M-M_1)}\right)^{\frac{2\mathcal{T}_1}{\mathcal{T}_2}} &\leq \frac{8h^3 R}{(M-M_1)^4} \\ h \geq \left[\frac{8^{(4\mathcal{T}_1-\mathcal{T}_2)}\left(\frac{\mu_2}{1-\epsilon_1}\right)^{2\mathcal{T}_1}(M-M_1)^{(4\mathcal{T}_2-2\mathcal{T}_1)}}{R^{\mathcal{T}_2}}\right]^{\frac{1}{3\mathcal{T}_2-2\mathcal{T}_1}} \end{aligned} \tag{3.75}$$

Hence the parameter μ_2 should be tuned such that the right hand side of (3.75) satisfies the largest lower bound on the parameter h described by the inequalities (3.64), (3.65), (3.68), (3.70). Mathematically, the following needs to be enforced:

$$\begin{aligned} &\left[\frac{8^{(4\mathcal{T}_1-\mathcal{T}_2)}\left(\frac{\mu_2}{1-\epsilon_1}\right)^{2\mathcal{T}_1}(M-M_1)^{(4\mathcal{T}_2-2\mathcal{T}_1)}}{R^{\mathcal{T}_2}}\right]^{\frac{1}{3\mathcal{T}_2-2\mathcal{T}_1}} \\ &\geq \max\left\{\begin{array}{c} 1, (M-M_1), \\ \frac{(M-M_1)^2}{2}, \frac{(M-M_1)^2(1-\epsilon_1)}{2\epsilon_1\mu_2} \end{array}\right\} \end{aligned} \tag{3.76}$$

Solving for μ_2 in all the individual terms of (3.76), the following can be obtained:

$$\mu_2 \geq \bar{\mu}_2 = \max\left\{\begin{array}{c} \left[\frac{R^{\mathcal{T}_2}(1-\epsilon_1)^{2\mathcal{T}_1}}{(M-M_1)^{(4\mathcal{T}_2-2\mathcal{T}_1)}8^{(4\mathcal{T}_1-\mathcal{T}_2)}}\right]^{\frac{1}{2\mathcal{T}_1}}, \\ \left[\frac{R^{\mathcal{T}_2}(1-\epsilon_1)^{2\mathcal{T}_1}}{8^{(4\mathcal{T}_1-\mathcal{T}_2)}(M-M_1)^{\mathcal{T}_2}}\right]^{\frac{1}{2\mathcal{T}_1}} \\ \left[\frac{R^{\mathcal{T}_2}(1-\epsilon_1)^{2\mathcal{T}_1}}{32^{2\mathcal{T}_1}(M-M_1)^{2(\mathcal{T}_1-\mathcal{T}_2)}}\right]^{\frac{1}{2\mathcal{T}_1}} \\ \left[\frac{R^{\mathcal{T}_2}(1-\epsilon_1)^{3\mathcal{T}_2}}{4^{\frac{\mathcal{T}_2+6\mathcal{T}_1}{3\mathcal{T}_2}}(M-M_1)^{(2\mathcal{T}_1-2\mathcal{T}_2)}}\right]^{\frac{1}{3\mathcal{T}_2}} \end{array}\right\} \tag{3.77}$$

where $\bar{\mu}_2$ is introduced for brevity and is utilized in the following. The following tuning rules can be concluded by combining (3.77) and (3.71) with (3.64)[3]:

$$\begin{aligned} \mu_1 &= \max\left\{M, M_1 + \sqrt{2R}\right\}, \\ \mu_2 &= \max\left\{\ \epsilon_2(\mu_1 + M),\ \tfrac{1-\epsilon_1}{2\epsilon_1},\ \bar{\mu}_2\ \right\}, p = h \\ h &= \left[\frac{8^{(4\mathcal{T}_1-\mathcal{T}_2)}\left(\frac{\mu_2}{1-\epsilon_1}\right)^{2\mathcal{T}_1}(M-M_1)^{(4\mathcal{T}_2-2\mathcal{T}_1)}}{R^{\mathcal{T}_2}}\right]^{\frac{1}{3\mathcal{T}_2-2\mathcal{T}_1}} \end{aligned} \tag{3.78}$$

where $\epsilon_2 > 1$ is an arbitrary scalar.

The above tuning rules are obtained for some finite region around the origin. The parameter ϵ_3 can take any value in the interval $[0, \frac{3}{2}]$. The equality $\epsilon_3 = \rho + (\frac{3}{2} - \rho)e^{-\mathcal{T}}$ leads to infinite gain h when the specified settling time tends to zero. Such a definition is inspired from the expectation that infinitely high linear gains should lead to infinitely small settling time. Conversely, if the specified settling time is very large ($\mathcal{T} \to \infty$), the expression $\epsilon_3 \to \rho$ holds. The parameter ρ lies in the interval $[0, \frac{3}{2}]$. The left boundary $\rho = 0$ causes the expression $\mu_2 = \infty$ to hold whereas the right boundary $\rho = \frac{3}{2}$ causes the expression $h = \infty$ to hold true. It is at the disposal of the designer to select an appropriate value to suit the needs of the system. The value $\rho = \frac{1}{2}$, for example, may offer a good trade-off.

3.7 Simulation

Example 1: Perturbed Double Integrator or planar controllable systems in Brunovsky form

This section presents the numerical simulation for the problem studied in the previous sections. Fig. 3.2 shows the simulation for the specification of the settling time $\mathcal{T} = 2\,sec$ with the disturbance bound $M_1 = 1$. The chattering at the origin is due to numerical integration. The following initial conditions are assumed: $x_1(0) = 0, x_2(0) = 2$. Let the following choices be made according to the tuning rules: $M = 2, \epsilon_1 = 0.4, \epsilon_2 = 1.1, \rho = 0.5, \epsilon_3 = 0.635, \mathcal{T}_1 = 0.777\,sec, \mathcal{T}_2 = 1.223\,sec$. Then the selection $\mu_1 = 3$ and (3.78) leads to the following controller parameters: $\mu_2 = 5.5, h = p = 21.77$.

Example 2: Third order system: Reduction of the order of the closed-loop response by 2

[3] The desired settling time is obtained by imposing conditions on the gains μ_1, μ_2 and h. The equality $p = h$ is specified solely for the ease of tuning and is not a necessary condition.

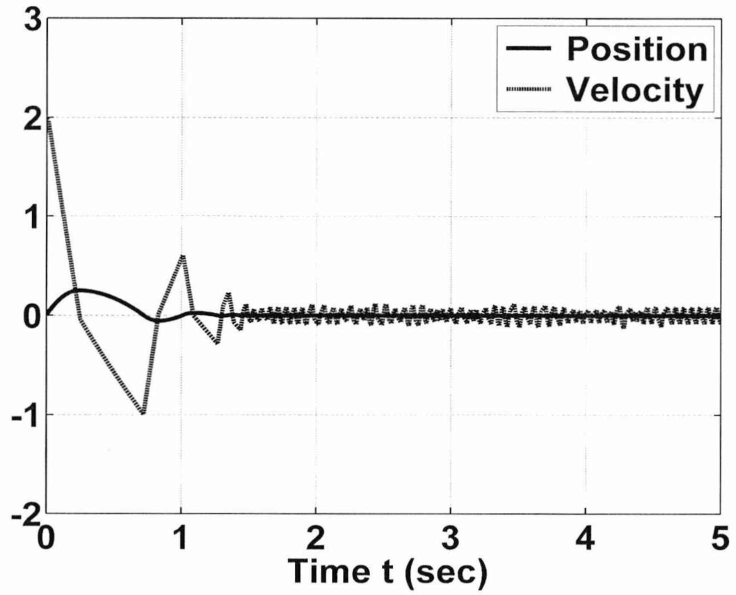

FIGURE 3.2: Settling Time of the system (3.20), (3.21).

The following academic example is considered from the literature [112, Section 3]:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -2x_1 - x_2 + x_3 + 5 + \omega(t) + u \end{aligned} \tag{3.79}$$

where $x_i, i = 1, 2, 3$ are system states. It should be noted that this system is open loop unstable. Let the term $\omega(t) = N_1 sin(10\,t)$ be treated as uncertainty. It can be seen that $\omega(t)$ is upper bounded by the function

$$|\omega(t)| \leq N_1, \tag{3.80}$$

where $N_1 > 0$.

As discussed in [112, Section 3], the traditional sliding mode can be enforced on the sliding surface $s(x) = 0$ where the sliding function

$$s(x) = \ddot{x}_1 + 4\dot{x}_1 + 4x_1 = x_3 + 4x_2 + 4x_1 \tag{3.81}$$

with a corresponding sliding mode controller $u = \rho\text{sign}(s) + 2x_1 + x_2 - x_3 - 5$ with $\rho > N_1$ produces a closed-loop system response given by a reduced order sliding motion of the order two with two close-loop poles located at -2.

As compared to (3.81), the sliding sliding surface

$$s(x) = \dot{x}_1 + cx_1 = x_2 + cx_1, \quad c > 0 \tag{3.82}$$

produces a closed-loop system response given by a reduced order sliding motion of the first order $\dot{x}_1 = -e^{-ct}x_1(0)$ with a pole located at $-c$ when the controller u is designed using the following second order sliding mode controller (see (3.20)):

$$u(s) = -hs - p\dot{s} - \mu_1 \text{sign}(\dot{s}) - \mu_2 \text{sign}(s) + 2x_1 + x_2 - (c+1)x_3 - 5. \tag{3.83}$$

Then, the following reaching phase dynamics results:

$$\begin{aligned} \dot{s} &= \dot{x}_2 + c\dot{x}_1 \\ \Rightarrow \ddot{s} &= \dot{x}_3 + cx_3 = \omega(t) - hs - p\dot{s} - \mu_1 \text{sign}(\dot{s}) - \mu_2 \text{sign}(s) \end{aligned} \tag{3.84}$$

The dynamics (3.84) are finite time stable by Theorem 3.1 if the parameter M_1 of assumption 1 is fixed as $M_1 > N_1$.

Let a settling time specification of 10 sec be given for attaining $s = \dot{s} = 0$. It should be noted that the meaning of settling time is different from example 1 since the attainment of $s = 0$ is being viewed as the objective as opposed to finite time stabilization of the state variables. Once the system slides on the sliding manifold $s = 0$, the system states asymptotically approach the origin. The following initial conditions are assumed: $x_1(0) = x_2(0) = x_3(0) = 1$. This defines the sliding variable initial conditions as $s(0) = \dot{s}(0) = 1.5$. Let the following choices be made according to the tuning rules: $M = 1, \epsilon_1 = 0.4, \epsilon_2 = 1.1, \rho = 0.5, \epsilon_3 = 0.5, \mathcal{T}_1 = 3.34\,\text{sec}, \mathcal{T}_2 = 6.67\,\text{sec}$. Then the selection $\mu_1 = 1.1$ and the same tuning rules (3.78) can be applied with the uncertainty bound set at $N_1 = 0$ which results in $\mu_2 = 2.31, h = p = 7.85$.

Figure 3.3 shows reaching dynamics of the sliding variable s and its derivative $\dot{s}$ for the aforementioned tuning. The phase plane plot in the plane $(s, \dot{s})$ is given in Figure 3.4. It can be seen that the sliding variable s and its derivative $\dot{s}$ go to zero in less time than 3 sec.

Next, it is of interest to investigate the effect of uncertainty $\omega = 1.95\sin(10\,t)$ with a non-zero bound $N_1 = 1.95$. The tuning rules (3.78) with $M_1 = 2, \mu_1 = 3.3$ and with all other parameters unchanged result in $\mu_2 = 6.93, h = p = 13.59$. The resulting response after the attainment of sliding motion (after approximately $t = 2.45\,\text{sec}$) is given in figure 3.5. It can easily be seen that once on the sliding surface, the closed-loop system mimics a first order dynamics $x_1(t) = e^{-ct}x_1(\bar{t}), \bar{t} = 2.45$ which is obviously independent of the uncertainty ω and is dependent only on the parameter $c > 0$.

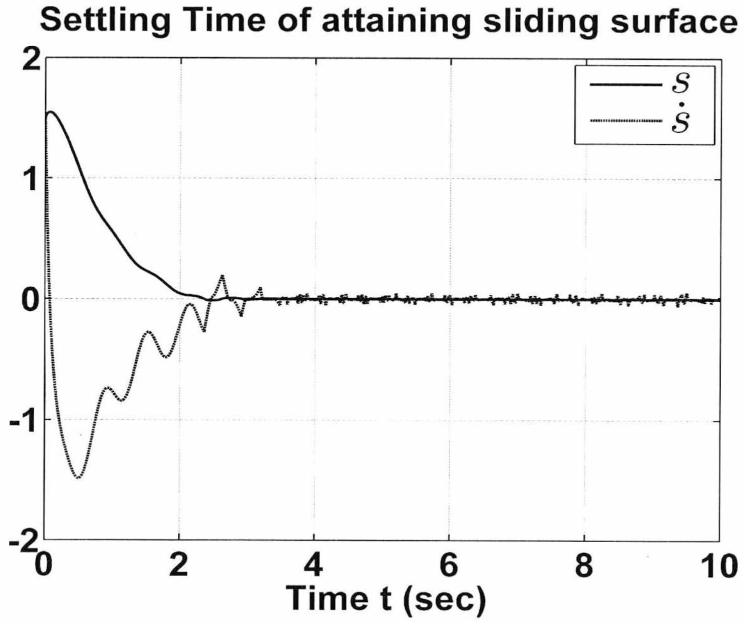

FIGURE 3.3: Settling Time of the of the system (3.83), (3.84).

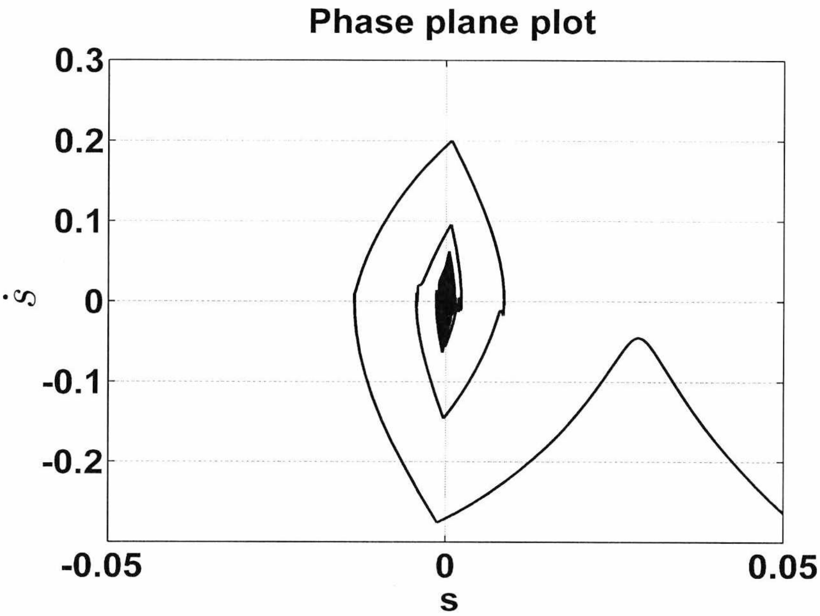

FIGURE 3.4: Phase plane plot in the plane $(s, \dot{s})$ for the system (3.83), (3.84).

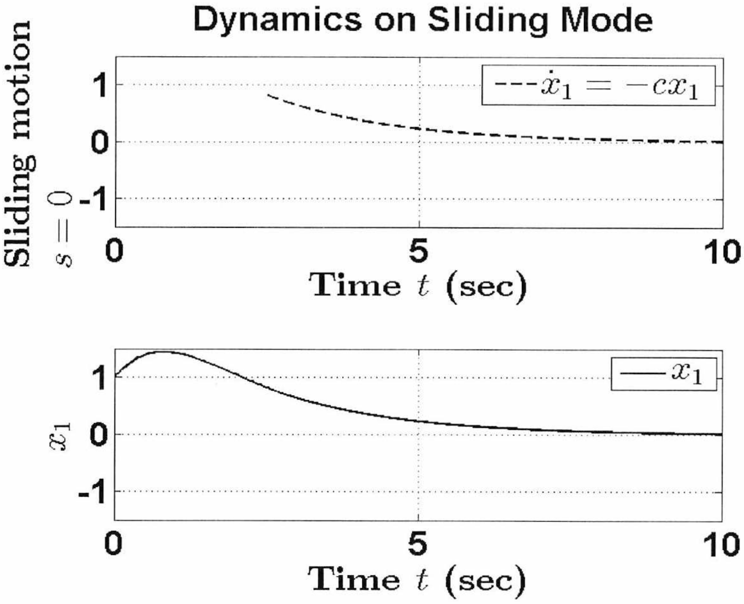

FIGURE 3.5: Sliding motion of the first order of system (3.83), (3.84).

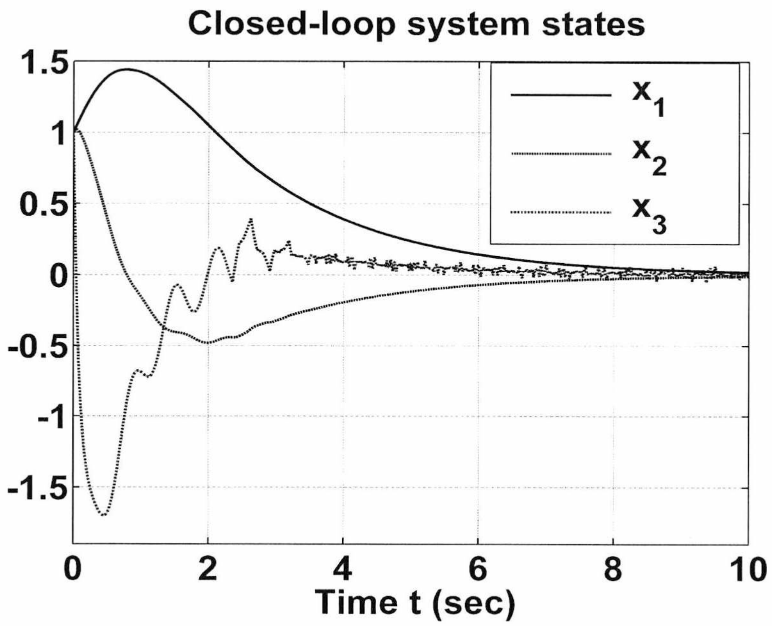

FIGURE 3.6: Asymptotic stability of the closed-loop system (3.79), (3.83).

Furthermore, figure 3.6 shows asymptotic stability of all the states of (3.79) under the feedback (3.83).

3.8 Contributions and further investigations

3.8.1 A Unique Contribution

It can be seen from sections 3.4 and from references [9, 42] that uniform asymptotic and uniform Lyapunov stability are well established results. The following three problems relating to finite time stability are being studied in this thesis:

1. The proof of finite time convergence either via a strict Lyapunov inequality (such as that obtained in [12]) or via the homogeneity of differential inclusions (such as that proposed in [9]).
2. The estimate of the time of convergence of the closed-loop trajectory of planar systems to the origin.
3. The opposite to the problem of item 2 above, namely, finding the gain parameters of the controller *a priori* to achieve a prescribed reduction in the settling time which is specified *a priori.*

The work in [9], which is the first to prove uniform finite time stability for the problem statement in section 3.3 solves the first two of the problems mentioned above via the homogeneity route. The recent work in [42] also solves the same two problems by showing existence of a strict Lyapunov function that decays in finite time to zero while also satisfying the stability criterion for the uniform asymptotic and the uniform finite time stability as defined in the previous chapter. The work in the aforementioned references [9, 42, 106] do not solve the third problem. The work in this chapter is the first instance in the literature which incorporates results of a recent publication [56] (that was proposed prior to [42]) where tuning rules for the so called 'twisting' controller are given while also (i) proving the global uniform asymptotic stability and keeping the global uniform finite time stability intact and (ii) giving the estimate of finite time convergence. This is a clear contribution.

3.8.2 Does settling time approach zero as gains approach infinity?

An observation is in order before moving on to the next chapter. It is not evident from the explicit formula (3.62) of the settling time estimate and from the proofs in all the existing literature on the 'twisting' controller that the expression

$$\lim_{\mu_1 \to \infty, \mu_2 \to \infty} \mathcal{T}(\mathcal{T}_{O_1 - O_2}, \mathcal{T}_h) \to 0 \tag{3.85}$$

holds true. Although the tuning rules presented in this chapter successfully achieve this for the reverse problem of selecting the parameters μ_1, μ_2 when an arbitrary settling time specification is given, the settling time function $\mathcal{T}(\mathcal{T}_{O_1-O_2}, \mathcal{T}_h)$ does not reveal this feature by itself. This behaviour, while studying planar controllable systems, can be found in classical and modern control paradigms and it is only natural to expect the same and extend it to the second order sliding mode controllers. The engineering community, for example, would welcome such a result since the theoretical guarantee of a limiting behaviour such as (3.85) means the ease of usage of controllers to achieve the desired settling time (see [113, Section 7], for example, for remarks on ease of use and tuning of the alternatives to the ubiquitous PID controllers). This is the topic of the next chapter which gives an alternative Lyapunov based approach to the one presented in this chapter while (i) computing an upper bound on the finite settling time and establishing the corresponding tuning rules such that (3.85) holds true and (ii) keeping the proofs of uniform asymptotic stability and uniform finite time stability intact. The next chapter thus seeks to strengthen the result developed in the current chapter.

3.8.3 Discussion on unilateral constraints

Consider the following system in place of (3.20), (3.21):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u(x_1, x_2) + \omega(t) \\ x_1 &\geq 0 \\ x_2(t_k^+) &= -\bar{e}\, x_2(t_k^-) \qquad \text{if} \qquad x_2(t_k^-) < 0,\ x_1(t_k) = 0, \end{aligned} \tag{3.86}$$

Such systems are more formally known as unilaterally constrained systems, the stability study and mechanics of which are mature areas of science [23]. Control of unilaterally constrained systems is one of the central topics of this thesis. The results presented in the previous subsections are of great importance while (i) synthesizing finite time stabilizing control laws, (ii) identifying Lyapunov functions and (iii) revealing the connection between homogeneity and finite time stability of a class of unilaterally constrained systems. Although a great deal of work exists which considers asymptotic stability of unilaterally constrained systems, finite time stability of such systems is a novel concept. It will be clear during the study that the homogeneity of differential equations and the quasi-homogeneity approach presented in this chapter will inherently play a central role in deriving similar results. This is the topic of Chapter 5 which formally presents Lyapunov based proofs to the above effect.

3.9 Conclusion

A novel homogeneity approach leading to both an upper bound on the settling time of the 'twisting' controller and the corresponding tuning rules has been developed. More importantly, the upper bound is obtained without recourse to the differential inequality of the Lyapunov function. The results appear to be superior to existing methods which do not provide tuning guidelines to achieve a specific reduction in settling time. These results may be of particular interest to practising control engineers in implementing a twisting controller. From the theoretical viewpoint, the new philosophy will inspire a similar estimate of the settling time for exponentially stable homogeneous systems. A possible further direction of research can be the study of the above problem when actuator saturation is considered.

As discussed in the section 3.8, a stronger result that preserves all the features of this chapter along with an additional guarantee (3.85) relating to the settling time estimate and the corresponding tuning is presented in the next chapter [4].

[4] The next chapter incorporates developments of a recent publication [57] that is parallel work to that proposed in [42]. The former is stronger of the two as both present a conservative estimate of the settling time while proving uniform finite time stability but straightforward tuning rules and theoretical guarantee of (3.85) is only available in [57].

CHAPTER 4

Global Uniform Finite Time Stabilization and A Priori Tuning

It is of interest to investigate if a control synthesis is easy to use and if it is straightforward to tune the free parameters [113, Secion 7] as well as to study the performance and robustness characteristics. Proportional integral derivative controllers, for example, enjoy numerous automatic tuning algorithms for tuning the gains for a given performance specification which may be in the form of a settling time or percentage overshoot constraint [55, 114]. The second order sliding mode controller utilised in the previous chapter is straightforward to use as it only requires the sign of the states in the case of planar systems. The end users of more complex nonlinear controllers such as second order sliding mode controllers would certainly benefit from access to straightforward tuning rules for achieving a prescribed finite settling time. Such tuning rules which attain a pre-specified settling time cannot be found in the latest literature for both continuous finite time stabilization [12, 44, 94, 115] and discontinuous finite time stabilization [42, 70, 106] of planar systems.

It is interesting to note that the problem of finding controller gains when a finite settling time is given is not a new paradigm. Consider the area of optimal control [80, 116]. The following example is known as the *Fuller phenomenon* [79]. Minimizing the integral

$$\int_0^\infty x^2(t)\mathrm{d}t \tag{4.1}$$

on the trajectory of

$$\ddot{x} = u(x, \dot{x}) \tag{4.2}$$

under the input constraint

$$|u| \leq 1 \quad \forall t \geq 0 \tag{4.3}$$

generates an optimal controller

$$u(x,\dot{x}) = \begin{cases} 1, & \text{if } s(x,\dot{x}) < 0; \\ -1, & \text{if } s(x,\dot{x}) > 0. \end{cases} \tag{4.4}$$

where,

$$s(x,\dot{x}) = x + c\,\dot{x}^2 \operatorname{sign}(\dot{x}) = x + c\,\dot{x}\,|\dot{x}| \tag{4.5}$$

and c is a constant. The optimal trajectories do not generate a sliding mode on the $s = 0$ curve. The trajectory crosses the curve $s = 0$ at countably many points. The time interval between successive switches on the curve $s = 0$ decreases while following a geometric progression [20, Example 1.3] thereby producing a finite accumulation point otherwise known as *Zeno behaviour*. This finite accumulation point is nothing but the finite settling time when viewed from the perspective of achieving convergence of the trajectory to reach the origin in finite time.

Indeed, an open-loop minimum energy control [80]

$$u(x_1, x_2, t) = a + b\,t \tag{4.6}$$

where

$$a = -\frac{2}{t_f^2}(3x_{10} + 2x_{20}t_f), \quad b = \frac{6}{t_f^3}(2x_{10} + x_{20}t_f), \tag{4.7}$$

transfers the state of the system $\ddot{x} = u$ from the initial conditions $x_1(0) = x_{10},\ x_2(0) = x_{20}$ to the origin in a given time t_f while also minimizing the integral cost

$$J(u) = \int_0^{t_f} u(\tau)^2\, \mathrm{d}\tau. \tag{4.8}$$

It can easily be seen from (4.7) that the gains $a \to \infty, b \to \infty$ as $t_f \to 0$. In other words, specification of a smaller settling time leads to higher gains.

It is intuitive to expect that higher gains lead to a smaller settling time in the case of controllable planar systems such as a double integrator. This expectation is indeed appropriate not only in the case of the aforementioned open-loop control but also in the case of closed-loop feedback control. For example, the well-known Ackerman's pole placement procedure for linear controllable systems [117] leads to a controller of the form $u = -Kx$ where the gain matrix K is determined by specifying the poles of the desired characteristic polynomial of the closed-loop system. It holds true theoretically that placing the poles farther to the left hand side of the Laplace plane corresponds to a direct increase in the gain values of K and also to a direct reduction in the decay rate of the trajectories of the states.

This chapter seeks to establish similar results pertaining to the settling time estimate and tuning for the so called 'Twisting' controller which is a second order sliding mode feedback controller [4].

Motivation

Section 3.2 presented the state of the art in Lyapunov approaches for second order sliding mode algorithms. The previous chapter presented a homogeneity approach to establish a settling time estimate and corresponding tuning rules. As discussed in 3.8.2, neither the results of the previous chapter nor the literature establish the desired result of attaining a desired reduction in settling time by increasing the gains. The results presented in this chapter give an explicit formula for the settling time based on integration of the trajectory of the closed-loop system such that the aforementioned intuitive expectation is shown to hold true mathematically. As noted in Section 3.9, this chapter incorporates the results of a recent publication [57] that is parallel work to that proposed in [42]. The former is the stronger of the two as both present a conservative estimate of the settling time while proving uniform finite time stability but straightforward tuning rules and a theoretical guarantee of the desired result that the settling time tends to zero as the gains of the second order sliding mode controller tend to infinity (see (3.85)) is only available in [57].

Existing methods do not provide sufficient and constructive guidelines [9, 42] for tuning the twisting controller. The existing literature inevitably involves optimization of the gain parameters and hence cannot achieve a desired improvement in the finite settling time [106]. It is interesting to note that simultaneous application of a linear plus twisting controller appears to be unable to achieve a desired improvement in the settling time as evident from the recent literature (see [9] and [106]). Obtaining straightforward tuning rules to achieve a desired reduction in the settling time can easily be identified as an open problem as evident from the latest research work on twisting controllers (see [106]). This is the principal motivation of studying an alternative method in this chapter.

Objectives

The objectives of this chapter are as follows:

1. To introduce a novel methodology based on a non-smooth Lyapunov function so that the upper bound on the settling time of the well-known twisting controller is estimated.

2. To derive explicit and constructive tuning rules to achieve arbitrarily small settling time.

3. The settling time and the tuning rules are to be achieved so that the theoretical results fulfill the intuitive expectation, namely, that arbitrarily large linear feedback gains must correspond to an arbitrarily steep decay of the closed-loop trajectories. Such an expectation is justified in cases when feedback control strategies render fully controllable, minimum phase, linear time invariant systems stable in the absence of actuator saturation.

4. The second intuitive expectation is also to be addressed, namely, that increasing the gains of the twisting controller leads to a decrease in the finite settling time of the closed-loop system.

Hence the proposed framework, motivated by the need for more constructive and straightforward tuning guidelines fulfilling the stated objectives, will clearly contribute to the field of second order sliding mode control algorithms.

Methodology

A novel switched control synthesis is presented in this chapter. The proposed framework utilizes a step-by-step application of a classical linear feedback and the twisting controller. Global asymptotic stability is attained using linear feedback which renders the closed-loop system convergent to an arbitrarily large vicinity of the origin in finite time. A switch from the completely linear feedback control law to the well-known twisting controller is then introduced so that the trajectories reach the origin in finite time. The tuning rules are then developed for arbitrary initial conditions. It is noted that the use of linear feedback is not necessary. The proposed method of deriving a finite settling time estimate for the twisting controller is valid even for the case when there is no linear feedback. However, use of linear feedback is always preferable to avoid excessively large variable structure gains in the twisting controller. The analysis of chattering is not within the scope of this chapter and the reader is advised of the latest advances (see [118]) in the literature for implementation issues using digital controllers.

A unique contribution

The main contribution of this chapter is twofold. Firstly, similar to the previous chapter a novel approach to estimate the upper bound on the finite settling time is obtained in the absence of a differential inequality of the form $\dot{V} \leq -kV^{\alpha}$. Secondly, tuning rules are established which guarantee attainment of arbitrarily small settling time while facilitating *apriori* tuning of the linear feedback gains and the twisting controller gains. It is shown that the feedback gains of the twisting control law do not need to be unnecessarily high, thereby avoiding excessive chattering. This is because the relative contribution of linear and twisting control laws in achieving the desired settling time can be chosen arbitrarily by the user. Hence, the proposed state feedback framework renders the resulting control algorithm attractive to practising engineers.

The rest of the chapter is organized as follows. The problem formulation is presented next followed by Section 4.1 which details the proposed control synthesis. The estimate of the upper bound on the reaching time for the robust linear state feedback feedback law for a linear double integrator is proposed in Section 4.2. Section 4.3 establishes the finite settling time for the twisting controller in the presence of uncertainty. The tuning rules that guarantee a specified settling time are established in section 4.4. Section 4.4 also presents simulation results.

System description and problem statement

Let the open-loop system dynamics be given as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u(x_1, x_2) + \omega(t) \end{aligned} \tag{4.9}$$

where $x = (x_1, x_2)^{\mathrm{T}}$ is the state of the system, $\omega(t)$ is a uniformly bounded disturbance and u is a control input. The following assumptions are made:

1. All the states are available for feedback.
2. An upper bound $M > 0$ on the disturbance term $|\omega| \leq M$ is known *a priori*.

The control aim is to establish the following:

1. To synthesize a finite time stabilizing control law $u(x_1, x_2)$.

2. To establish an upper bound $\mathcal{T}_s$ on the settling time of the closed-loop system (4.9) such that $x_1(t) = x_2(t) = 0$ for all $t \geq \mathcal{T}_s$.

3. To establish tuning rules for the parameters of the controller to achieve the desired settling time.

4.1 Switched control synthesis

The following variable structure feedback control law is proposed:

$$u(x_1, x_2) = \begin{cases} Lx, & x \notin \Gamma_R; \\ \Phi(x), & x \in \Gamma_R. \end{cases} \tag{4.10}$$

where

$$\begin{aligned} L &= \begin{bmatrix} -l_1 & -l_2 \end{bmatrix}, \\ \Phi(x) &= -\mu_2 \operatorname{sign}(x_1) - \mu_1 \operatorname{sign}(x_2), \\ \Gamma_R &= \{x : V(x_1, x_2) \leq R\}, \end{aligned} \tag{4.11}$$

and

$$V(x_1, x_2) = \mu_2 |x_1| + \frac{1}{2} x_2^2. \tag{4.12}$$

The parameters $l_1, l_2, \mu_1, \mu_2, R$ are positive scalars. The condition

$$0 < M < \mu_1 < \mu_2 - M \tag{4.13}$$

is required to enforce finite time stability of the closed-loop system to the origin where M is a positive scalar. Furthermore, $R > 0$ can be chosen arbitrarily small or large. The control law $\Phi(x)$ is in fact the well-known twisting controller. The feedback gains l_1, l_2 represent the traditional state feedback. Figure 4.1 graphically depicts the new switched control law and the finite time stability of the origin. The point O_1 shows the arbitrary initial condition.

The underlying philosophy of the above switched control synthesis is the successive application of the classical linear state feedback law and the second-order sliding mode controller. Such a philosophy can be found in the existing literature. Work by [119], for example, utilizes a similar synthesis for achieving finite time stability of second order non-linear systems. The presented tuning rules differ from those presented in this chapter in that the tuning rules were primarily iterative and not as straightforward as those derived in this chapter. The proposed control law (4.10) simplifies the computation of the settling time estimate, thereby yielding tuning rules for the feedback parameters. As will be shown in section 4.4, the resulting tuning rules guarantee the attainment of a specified settling time. Since the open-loop system (4.9) is controllable, there exist feedback gains l_1, l_2 such that the trajectories of the unperturbed closed-loop system exponentially decay to the origin with a desired rate. In turn, while being perturbed by an admissible uniformly bounded disturbance $\omega(t)$, the system states $x_1(t), x_2(t)$ are steered, as $t \to \infty$, to a ball $B_r = \{(x,x_2) : x_1^2 + x_2^2 \leq r^2\}$ with an arbitrarily small radius r, which is determined by the gains l_1, l_2. It can also be deduced that the trajectories of the properly-tuned closed-loop system enter the level set Γ_R of the function $V(x_1, x_2)$ in finite time (at some point O_2) with a desired exponential decay rate provided that B_r is small enough to be inside Γ_R so that the relation $B_r \cap \partial\Gamma_R = \emptyset$ holds true where $\partial\Gamma_R = \{(x_1, x) : \mu_2|x_1| + \frac{1}{2}x_2^2 = R\}$ stands for the boundary of the level set Γ_R.

A switch from the linear state feedback control law to the twisting controller is then introduced at the time instant when the trajectory reaches the level set Γ_R. It can be shown that the function $V(x_1, x_2)$ is indeed a valid Lyapunov function when a twisting controller is utilized for feedback (see [9]). Moreover, the trajectories of the closed-loop system can never leave the level set Γ_R of the Lyapunov function $V(x_1, x_2)$ once they are inside Γ_R. The fact that the temporal derivative of the Lyapunov function satisfies $\dot{V} \leq 0 \; \forall x \in \Gamma_R$ along the trajectories of the closed-loop system **is the main motivation** for choosing the level set Γ_R for switching from the linear to the twisting controller. The finite time stability to the origin using the twisting controller is well-known from [4], [9]. The settling time estimate has also been derived by [106] by showing the existence of a strong Lyapunov function. Hence the proposed switched control synthesis renders the origin of the state-space finite time stable. However, a different approach is utilized in this chapter to facilitate the settling time estimate derivation, on one hand, and to straightforwardly tune the controller parameters that would ensure the desired settling time, on the other hand.

Remark 4.1. Real systems may have small switching delays and parametric uncertainties. These small system defects may provoke some stable sliding modes on the switching set Γ_R, where a switch from the linear feedback to the twisting controller takes place. However, as noted earlier in the section on motivation, the results presented in the following sections on finite settling time and the tuning algorithm are valid even in the absence of the linear feedback; in this case the aforementioned problem does not arise.

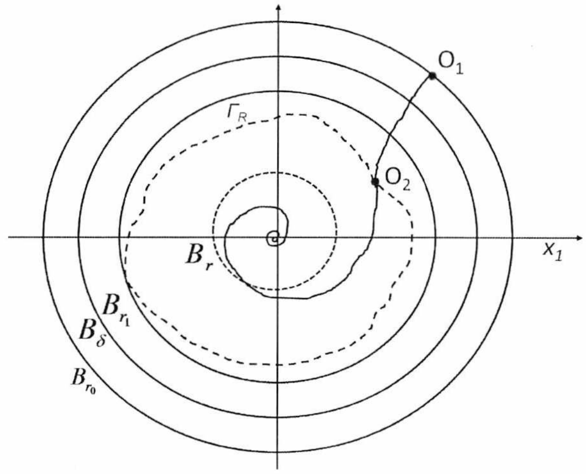

FIGURE 4.1: Concept of finite settling time by utilizing step-by-step application of the linear and the twisting controller, Level set Γ_R, ball B_r, the outer ball B_{r_1} such that $\Gamma_R \subset B_{r_1}$ and the initial condition region B_{r_0}.

4.2 Reaching time estimation with respect to the level set Γ_R using a linear feedback controller

The well-known method of finding the explicit solution of the second order linear time invariant system in the given canonical form is utilized and briefly outlined in this section. In turn, the closed-loop system is globally asymptotically convergent to the ball B_r. The unperturbed closed-loop system (4.9) with the feedback law $u(x_1, x_2) = Lx$ can be described as follows:

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -l_1\, x_1 - l_2\, x_2 \end{aligned} \tag{4.14}$$

Letting

$$l_1 = \alpha\lambda^2, \quad l_2 = (\alpha + 1)\,\lambda, \quad \alpha > 1, \lambda > 0$$

the eigenvalues

$$\lambda_1 = -\lambda, \quad \lambda_2 = -\alpha\,\lambda$$

are imposed on the closed-loop system. Then the general solution of (4.14) is given by

$$\begin{aligned} x_1(t) &= \left[\alpha x_{1_0} + \frac{1}{\lambda} x_{2_0}\right] e^{-\lambda\, t} - \left[x_{1_0} + \frac{1}{\lambda} x_{2_0}\right] e^{-\alpha\,\lambda\, t} \\ x_2(t) &= -\left[\alpha\lambda\, x_{1_0} + x_{2_0}\right] e^{-\lambda\, t} + \alpha\left[\lambda\, x_{1_0} + x_{2_0}\right] e^{-\alpha\,\lambda\, t} \end{aligned} \tag{4.15}$$

where $x_1^0 = x_1(0)$, $x_2^0 = x_2(0)$ are the initial conditions of the system (4.14) defining an initial condition ball $B_{r0} = \sqrt{\left(x_1^0\right)^2 + \left(x_2^0\right)^2}$. Thus the reaching time $\mathcal{T}_1$ for the level set

$$\Gamma_{R_1} = \{(x_1, x_2) : V(x_1, x_2) \leq R_1\}$$

of the Lyapunov function

$$V(x_1, x_2) = \mu_2 |x_1| + \frac{1}{2} x_2^2$$

is estimated on the solutions (4.15), initialized within the ball B_{r_0}, as follows:

$$\begin{aligned}
V(x_1, x_2) &\leq \mu_2 |\alpha x_{1_0} + \frac{1}{\lambda} x_{2_0}| \, e^{-\lambda t} + \mu_2 |x_{1_0} + \frac{1}{\lambda} x_{2_0}| \, e^{-\alpha\lambda t} \\
&+ \frac{1}{2}\left[-(\alpha\lambda\, x_{1_0} + x_{2_0})\, e^{-\lambda t} + \alpha(\lambda\, x_{1_0} + x_{2_0})\, e^{-\alpha\lambda t}\right]^2 \\
&\leq \mu_2 |\alpha x_{1_0} + \frac{1}{\lambda} x_{2_0}| \, e^{-\lambda t} + \mu_2 |x_{1_0} + \frac{1}{\lambda} x_{2_0}| \, e^{-\alpha\lambda t} \\
&+ (\alpha\lambda\, x_{1_0} + x_{2_0})^2 e^{-2\lambda t} + \frac{1}{2}\alpha^2(\lambda\, x_{1_0} + x_{2_0})^2 e^{-2\alpha\lambda t} \\
&\leq \left[\mu_2(\alpha+1)|x_{1_0}| + \frac{2\mu_2}{\lambda}|x_{2_0}| + 2\alpha^2\lambda^2 x_{1_0}^2 + (\alpha^2+1)x_{2_0}^2\right] e^{-\lambda t} \leq R_1
\end{aligned} \tag{4.16}$$

While deriving (4.16), the corresponding gain μ_2 of the twisting controller is viewed as a parameter and the well-known inequality $\frac{1}{2}(a+b)^2 \leq a^2 + b^2$ is employed. The upper bound on the reaching time $\mathcal{T}_1$ is a solution of the transcendental inequality

$$\left[\mu_2(\alpha+1)|x_{1_0}| + \frac{2\mu_2}{\lambda}|x_{2_0}| + 2\alpha^2\lambda^2 x_{1_0}^2 + (\alpha^2+1)x_{2_0}^2\right] e^{-\lambda \mathcal{T}_1} \leq R_1 \tag{4.17}$$

Taking into account $\lim\limits_{\lambda\to\infty} \lambda^2 e^{-\lambda t} = 0$ for all $t > 0$, it follows that $\mathcal{T}_1(\lambda)$, viewed as a function of λ, escapes to zero for all admissible parameters $x_{1_0}, x_{2_0}, \alpha, \mu_2, R_1$ as λ goes to infinity. The upper bound on the reaching time $\mathcal{T}_1(\lambda)$ is thus given by,

$$\mathcal{T}_1 \leq \frac{1}{\lambda} \ln \left[\frac{\mu_2(\alpha+1)|x_{1_0}| + \frac{2\mu_2}{\lambda}|x_{2_0}| + 2\alpha^2\lambda^2 x_{1_0}^2 + (\alpha^2+1)x_{2_0}^2}{R_1}\right] \tag{4.18}$$

It should be noted that (4.15) defines the state transition matrix of (4.14) in the form

$$e^{At} = \begin{bmatrix} \alpha e^{-\lambda t} - e^{-\alpha\lambda t} & \frac{1}{\lambda}(e^{-\lambda t} - e^{-\alpha\lambda t}) \\ -\alpha\lambda e^{-\lambda t} + \alpha\lambda e^{-\alpha\lambda t} & -e^{-\lambda t} + \alpha e^{-\alpha\lambda t} \end{bmatrix} \tag{4.19}$$

Consider now the perturbed version

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -l_1\, x_1 - l_2\, x_2 + \omega(t)
\end{aligned} \tag{4.20}$$

of system (4.14) where the external disturbance is upper bounded

$$|\omega(t)| \leq M \tag{4.21}$$

for almost all $t \geq 0$ by some positive constant M. Then the solution of (4.20), initialized at the origin $x_1(0) = 0, x_2(0) = 0$, is given by

$$x(t) = \int_0^t e^{A(t-\tau)}\Omega(\tau)d\tau$$

where $\Omega = \begin{bmatrix} 0 & \omega \end{bmatrix}^{\mathrm{T}}$. Thus,

$$\begin{aligned} x_1(t) &= \int_0^t \frac{1}{\lambda}\left[e^{-\lambda(t-\tau)} - e^{-\alpha\lambda(t-\tau)}\right]\Omega(\tau)d\tau, \\ x_2(t) &= \int_0^t \left[-e^{-\lambda(t-\tau)} + \alpha e^{-\alpha\lambda(t-\tau)}\right]\Omega(\tau)d\tau \end{aligned} \tag{4.22}$$

The Lyapunov function

$$V(x_1, x_2) = \mu_2|x_1| + \frac{1}{2}x_2^2$$

is now estimated on the solutions (4.22):

$$\begin{aligned} V(x_1, x_2) &\leq \mu_2 M \int_0^t \frac{1}{\lambda}\left[e^{-\lambda(t-\tau)} + e^{-\alpha\lambda(t-\tau)}\right]d\tau + (\alpha+1)^2M^2\left(\int_0^t e^{-\lambda(t-\tau)}d\tau\right)^2 \\ &\leq \frac{2\mu_2 M + (\alpha+1)^2M^2}{\lambda^2} \end{aligned} \tag{4.23}$$

It can be noted that the upper-bound $\frac{2\mu_2 M+(\alpha+1)^2M^2}{\lambda^2}$ in (4.23) escapes to zero as λ goes to infinity.

Clearly relations (4.16) and (4.23), coupled together, ensure that the perturbed system (4.20) with nontrivial initial conditions enters the level set $\Gamma_R = \{(x_1, x_2) : V(x_1, x_2) \leq R\}$ with $R = R_1 + r$ and

$$r \geq \frac{2\mu_2 M + (\alpha+1)^2M^2}{\lambda^2} \tag{4.24}$$

in the same reaching time $\mathcal{T}_1(\lambda)$ and $\mathcal{T}_1(\lambda) \to 0$ as $\lambda \to \infty$.

4.2.1 Definition of B_{r_1}, B_{r_2} with respect to the level set Γ_R and the radii r_1, r_2

The next aim is to define the scalar $r_1 > 0$ such that the expression $\Gamma_R \subset B_{r_1}$ holds. In other words, the following is required:

$$\frac{\mu_2|x_1|}{R} + \frac{x_2^2}{2R} \leq 1 \Rightarrow \left(\frac{x_1}{r_1}\right)^2 + \left(\frac{x_2}{r_1}\right)^2 \leq 1 \tag{4.25}$$

Let the following inequalities be imposed:

$$\left(\frac{x_1}{r_1}\right)^2 \leq \frac{\mu_2|x_1|}{R}, \quad \left(\frac{x_2}{r_1}\right)^2 \leq \frac{x_2^2}{2R} \tag{4.26}$$

Then the expression $(x_1, x_2) \in B_{r_1}$ holds true for every given point $(x_1, x_2) \in \Gamma_R$ in the state space. Note that the following always holds true for all $x \in \Gamma_R$:

$$|x_1| \leq \frac{R}{\mu_2} \tag{4.27}$$

The first inequality of (4.26) can be simplified as follows:

$$|x_1| \left(\frac{1}{r_1}\right)^2 \leq \frac{\mu_2}{R} \tag{4.28}$$

Utilizing the relationship (4.27), the following conservative requirement can be formulated to render sufficiently large value for the radius r_1:

$$|x_1| \left(\frac{1}{r_1}\right)^2 \leq \frac{R}{\mu_2} \left(\frac{1}{r_1}\right)^2 \leq \frac{\mu_2}{R} \tag{4.29}$$

Hence, the following upper-bound on r_1 suffices to satisfy the first inequality of (4.26).

$$r_1 \geq \frac{R}{\mu_2} \tag{4.30}$$

Similarly, the second inequality of (4.26) leads to $r_1 \geq \sqrt{2R}$. Hence the following estimate of the parameter r_1 is obtained:

$$r_1 = \max\left\{\frac{R}{\mu_2}, \sqrt{2R}\right\} \tag{4.31}$$

Next, the definition of the scalar $r_2 > 0$ is to be obtained such that the expression $B_{r_2} \subset \Gamma_R$ holds. In other words, the following is required:

$$\left(\frac{x_1}{r_2}\right)^2 + \left(\frac{x_2}{r_2}\right)^2 \leq 1 \Rightarrow \frac{\mu_2|x_1|}{R} + \frac{x_2^2}{2R} \leq 1 \tag{4.32}$$

It can be noted that for all $x \in B_{r_2}$ the following holds true:

$$|x_1| \leq r_2 \Rightarrow \frac{\mu_2|x_1|}{R} \leq \frac{\mu_2 r_2}{R} \tag{4.33}$$

Let the following inequality be satisfied:

$$r_2 \leq \frac{\rho R}{\mu_2} \tag{4.34}$$

where $0 < \rho < 1$ is an arbitrary scalar. Then the following inequality results:

$$\frac{\mu_2|x_1|}{R} \leq \frac{\mu_2 r_2}{R} \leq \rho \tag{4.35}$$

Furthermore, let the following inequality be satisfied:

$$r_2 \leq \sqrt{2R(1-\rho)} \tag{4.36}$$

Then the following inequality results:

$$\frac{x_2^2}{2R} \leq (1-\rho)\frac{x_2^2}{r_2^2} \tag{4.37}$$

Hence, by combining (4.35) and (4.37), the following is obtained:

$$r_2 = \min\left\{\frac{\rho R}{\mu_2}, \sqrt{2R(1-\rho)}\right\}, \tag{4.38}$$

Thus the following holds true for all $x \in B_{r_2}$:

$$\frac{\mu_2|x_1|}{R} + \frac{x_2^2}{2R} \leq \rho + (1-\rho)\frac{x_2^2}{r_2^2} \leq 1 \tag{4.39}$$

It can be noted that the inequality (4.39) is obtained from the fact that $\frac{x_2^2}{r_2^2} \leq 1 \quad \forall x \in B_{r_2}$. The aim $B_{r_2} \subset \Gamma_R$ is thus achieved.

Remark 4.2. It is assumed that the inequality $r_0 > r_1 > 0$ holds true. Otherwise, the twisting controller is used without application of the linear controller. Section 4.3 derives a finite settling time estimate to the origin starting from the radius r_1. Hence the settling time estimate derived holds true for any arbitrary initial condition.

Remark 4.3. It is possible to guarantee that the relation $r_2 < R$ holds true by imposing a tuning rule on the gain parameter μ_2. Subject to the following condition

$$\mu_2 > \rho\sqrt{\frac{R}{2(1-\rho)}}, \tag{4.40}$$

the following holds true from (4.38):

$$r_2 = \frac{\rho R}{\mu_2}. \tag{4.41}$$

Then $r_2 < R$ is guaranteed if the condition $\mu_2 > \rho$ is satisfied. Summarizing, the first tuning rule on μ_2 is obtained as follows:

$$\mu_2 > \rho \ \max\left\{\sqrt{\frac{R}{2(1-\rho)}},\ 1\right\}. \tag{4.42}$$

Remark 4.4. The reason why (4.29) is conservative can be seen in the outcome equation (4.30). Even small values of the state $|x_1|$ are replaced by the upper bound $\frac{R}{\mu_2}$ in the equation (4.29). Hence the lower bound of the scalar r_1 is conservatively determined by the bound $\frac{R}{\mu_2}$. It should be noted that smaller values of r_1 may suffice if it was not for the conservatism of equation (4.29). The solution to reduce the conservatism is either to choose the switching boundary R small or the gain parameter μ_2 very high. Both approaches have advantages and disadvantages. Choosing R very small may cause the linear feedback gain to increase when dealing with fixed finite settling time requirement. This means that the switch from the linear to the twisting controller occurs very close to the origin, resulting in low magnitude chattering. On the other hand, choosing μ_2 very high overcomes the disadvantage of the previous choice at the cost of high magnitude chattering. It is recommended that the designer strikes a judicious balance between linear and twisting controller gains to trade-off the above conflicting outcomes. This is always a plausible solution because the choice of switching parameter R can be made arbitrarily.

4.3 Settling time estimate of 'Twisting' controller using convolution integral

A finite upper bound on the settling time of the closed-loop system (4.9), (4.10) with the application of twisting controller is presented in this section. The time taken by the trajectories to travel from the point O_4 on the vertical positive semi-axis $e_1^+ = \{x \in \mathbb{R}^2 : x_1 = 0, x_2 > 0\}$ on the ball B_δ to the point O_3 on the e_1^+ axis is computed (see Figure 4.2). When the trajectories are initialized on the positive vertical axis at O_4, the factor by which it gets close to the origin after one revolution can be computed.

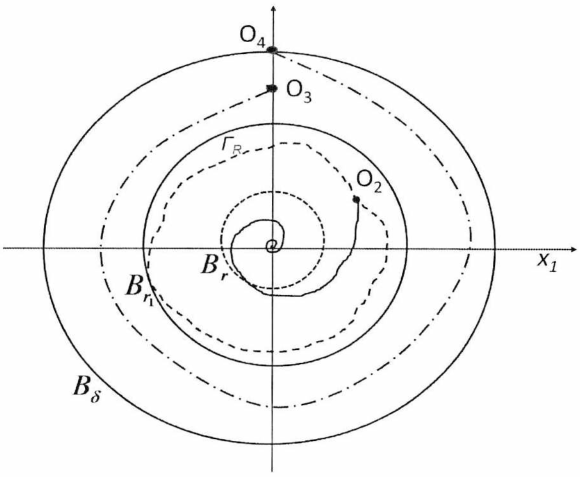

FIGURE 4.2: Finite settling time for the twisting controller

The value of the intercept (point O_3) on positive vertical semi-axis after one iteration should be greater than the radius r_1 of the ball B_{r_1} containing the level set Γ_R. Such a value of δ will ensure that the settling time estimate will be more conservative than the one computed with the initialization on the level set.

The motivation for such a choice of initialization of the trajectories on the ball B_δ stems from the fact that the trajectory, containing O_2 on the level set Γ_R, starting from any arbitrary point below O_4 on the e_1^+ axis cannot intersect the trajectory starting from the point O_4. The basis for this is the fact that different trajectories have no intersections because otherwise they would coincide with each other due to the uniqueness of the solution of the second order linear double integrator system driven by the twisting controller (see [120] and [4]). The approach utilized in the following is a two step process. Firstly, a comparison system, the trajectory of which encompasses the actual system, is defined. Secondly, the comparison system is then initialized on the positive vertical semi-axis e_1^+ with the coordinates $(0, \delta)$. Then the finite settling time is computed for the comparison system subject to the condition $\delta\Psi > r_1$ where Ψ is the factor by which the trajectory gets closer to the origin after one revolution at point O_3. Such an estimate is conservative as it encompasses the actual system trajectory.

Consider the closed-loop system,

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\mu_1 \text{sign}(x_2) - \mu_2 \text{sign}(x_1) + \omega(t)
\end{aligned}
\tag{4.43}
$$

where $\omega(t)$ is a bounded uncertainty which satisfies the upper bound (4.21). The solutions of the closed-loop system (4.43) are understood in the sense of Filippov

[6]. The method of considering a comparison system for such a second order system has been detailed by [120] (also see work by [4]). The same method is inherited in the following and briefly outlined. The right hand side of (4.43) can be obtained as follows:

$$\begin{aligned}\phi_1 &= x_2\\ \phi_2 &= -\mu_1\text{sign}(x_2) - \mu_2\text{sign}(x_1) + \omega(t)\end{aligned} \tag{4.44}$$

Let a comparison system corresponding to (4.43) be given as follows:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\left[\mu_1 - M\right]\text{sign}(x_2) - \mu_2\text{sign}(x_1)\end{aligned} \tag{4.45}$$

In turn, the right hand side of the comparison system (4.45) is obtained as:

$$\begin{aligned}\phi_1^c &= x_2\\ \phi_2^c &= -\left[\mu_1 - M\right]\text{sign}(x_2) - \mu_2\text{sign}(x_1)\end{aligned} \tag{4.46}$$

The comparison system (4.46) and the plant (4.44) are related by the following identity:

$$\begin{aligned}\phi_1 &= \phi_1^c\\ \phi_2 &= \phi_2^c + \Delta\phi\end{aligned} \tag{4.47}$$

where

$$\Delta\phi = -M\,\text{sign}(x_2) + \omega \tag{4.48}$$

It is trivial to note that,

$$\begin{aligned}\Delta\phi \leq 0 \qquad &\text{if } x \in (G_1 \cup G_4)\\ \Delta\phi \geq 0 \qquad &\text{if } x \in (G_2 \cup G_3)\end{aligned} \tag{4.49}$$

where

$$\begin{aligned}G_1 &= \left\{x \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\right\}, \quad G_2 = \left\{x \in \mathbb{R}^2 : x_1 > 0, x_2 < 0\right\}\\ G_3 &= \left\{x \in \mathbb{R}^2 : x_1 < 0, x_2 < 0\right\}, \quad G_4 = \left\{x \in \mathbb{R}^2 : x_1 < 0, x_2 > 0\right\}\end{aligned} \tag{4.50}$$

The idea behind the analysis presented in (4.44) through (4.49) is that of analysing the motion of the trajectory that represents the "worst case" scenario. Reference [120] presented this analysis in detail. The main concept can be seen in Fig. 4.3. For example, the inequality $\Delta\phi \leq 0$ when $x \in G_1$ of (4.50) simply means that the growth of x_2 is more negative in quadrant G_1 than that of x_2^c. In other words, the distance of the point (x_1^c, x_2^c) can evolve farther away from the origin than (x_1, x_2) in quadrant

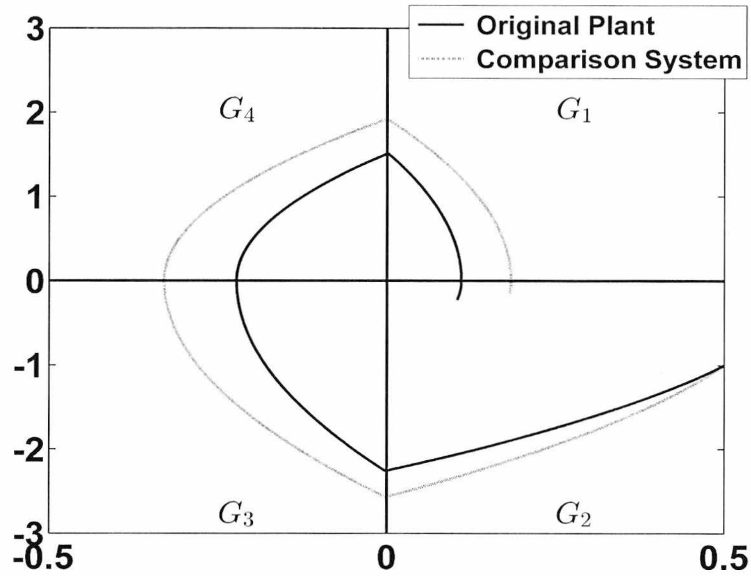

FIGURE 4.3: Phase-plane plot of the closed-loop system (4.43) and that of its comparison system (4.45)

G_1. Also, in terms of the slope of the vector field, the phase-plane of the comparison system contains less steep slope in the quadrant G_1 since $-(\mu_1 - M) - \mu_2$ is always less negative than $-(\mu_1 - \omega) - \mu_2$ resulting in a flatter trajectory (x_1^c, x_2^c) than (x_1, x_2) in G_1. This causes the trajectory (x_1^c, x_2^c) to intercept the semi-axis $x_1^c > 0, x_2^c = 0$ at a point farther to the right of the point where (x_1, x_2) intercepts the semi-axis $x_1 > 0, x_2 = 0$ with the principal axes $x_2 = 0, x_2^c = 0$ coinciding. Similar analysis can be extended for each of the four quadrants.

Remark 4.5. The above method of considering the comparison system that encompasses the actual trajectory is sometimes termed *majorisation* in the literature [4, 120]. Also, the trajectory of the comparison system is called the *majorant curve*.

Hence, the inclusions (4.46) are easily seen to be more conservative dynamics than the original system (4.44) in the sense that the solutions $(x_1(t), x_2(t))$ of the original system (4.43) are bounded by the solutions $(x_1^c(t), x_2^c(t))$ of comparison system (4.45). Hence it suffices for the purpose of estimating the finite settling time to consider system (4.45) which can be represented in the matrix vector notation as follows:

$$\dot{x}(t) = A\,x(t) + B\,u(t) \tag{4.51}$$

where

$$x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$u(t) = -(\mu_1 - M)\mathrm{sign}(x_2) - \mu_2 \mathrm{sign}(x_1) \tag{4.52}$$

The motion in the state space can be obtained using the convolution integral as follows:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}B\,u(\tau)d\tau \tag{4.53}$$

Since the control switches on the axes $x_1 = 0, x_2 = 0$, the integral (4.54) is required to be computed in each quadrant as follows:

$$\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} \begin{bmatrix} 0 \\ -\mu_1 - \mu_2 + M \end{bmatrix} d\tau \quad, x \in G_1; \\
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} \begin{bmatrix} 0 \\ \mu_1 - \mu_2 - M \end{bmatrix} d\tau \quad, x \in G_2; \\
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} \begin{bmatrix} 0 \\ \mu_1 + \mu_2 - M \end{bmatrix} d\tau \quad, x \in G_3; \\
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} \begin{bmatrix} 0 \\ -\mu_1 + \mu_2 + M \end{bmatrix} d\tau \quad, x \in G_4.
\end{aligned} \tag{4.54}$$

It is noted that using such integrals to define the solutions of the comparison system (4.45) is mathematically correct as the control law never generates a sliding mode on the switching lines $x_1 = 0$ and $x_2 = 0$. Hence the solutions always cross the switching lines $x_1 = 0, x_2 = 0$ except at the origin $(x_1, x_2) = 0$ (see [120], [9] and [121]).

Step 1: Time to travel from the point $O_4 \in B_\delta$ to point O_3 on the $e_1^+ = \{x \in \mathbb{R}^2 : x_1 = 0, x_2 > 0\}$ axis:

Case 1: $sign(x_1) = 1, sign(x_2) = 1$: Noting that the initial condition is assumed to lie in the first quadrant for a conservative estimate, equation (4.54) takes the following form:

$$x(t) = e^{At}\begin{bmatrix} 0 \\ \delta \end{bmatrix} + \left[\int_0^t e^{A(t-\tau)}d\tau\right]\begin{bmatrix} 0 \\ -\mu_1 - \mu_2 + M \end{bmatrix} \tag{4.55}$$

where $\delta = x_2(t_0)$ represents the projection on the e_1^+ axis. The matrix exponential in (4.55) can be computed as follows:

$$e^{At} = I + At + \frac{At^2}{2!} + \dots \tag{4.56}$$

Since $A^n = 0, \forall n \geq 2$, eq.(4.56) leads to the following:

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad \int\limits_0^t e^{-A\tau}d\tau = \begin{bmatrix} t & \frac{-t^2}{2} \\ 0 & t \end{bmatrix} \tag{4.57}$$

Further simplification produces:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \delta\, t - \frac{1}{2}t^2(\mu_1 + \mu_2 - M) \\ \delta - t(\mu_1 + \mu_2 - M) \end{bmatrix} \tag{4.58}$$

The time of interception on the $x_2 = 0$ axis can be obtained using the expression of $x_2(t)$ the last inequality of (4.58) as follows:

$$t_{x_2=0} = t_1 = \frac{\delta}{\mu_1 + \mu_2 - M} \tag{4.59}$$

The point of interception on the $x_2 = 0$ axis can be obtained using the expression of $x_1(t)$ in the last inequality of (4.58) as follows:

$$x_1(t_1) = \frac{\delta^2}{2(\mu_1 + \mu_2 - M)} \tag{4.60}$$

Case 2: $sign(x_1) = 1, sign(x_2) = -1$: Repeating the same computation for the second quadrant, equation (4.54) takes the following form:

$$x(t) = e^{A\,t} \begin{bmatrix} x_1(t_1) \\ 0 \end{bmatrix} + \left[\int_0^t e^{A\,(t-\tau)} d\tau \right] \begin{bmatrix} 0 \\ \mu_1 - \mu_2 - M \end{bmatrix} \tag{4.61}$$

Further simplification leads to the following time estimate $t = t_2$ when the trajectory intercepts on the $x_1 = 0$ axis and the intercept $x_2(t_2)$:

$$\begin{aligned} t_2 &= \frac{\delta}{\sqrt{(\mu_2 + \mu_1 - M)(\mu_2 - \mu_1 + M)}} \\ x_2(t_2) &= -\frac{\delta\sqrt{(\mu_2 - \mu_1 + M)}}{\sqrt{(\mu_2 + \mu_1 - M)}} \end{aligned} \tag{4.62}$$

Case 3: $sign(x_1) = -1, sign(x_2) = -1$: Repeating the same computation for the third quadrant, equation (4.54) takes the following form:

$$x(t) = e^{A\,t} \begin{bmatrix} 0 \\ x_2(t_2) \end{bmatrix} + \left[\int_0^t e^{A\,(t-\tau)} d\tau \right] \begin{bmatrix} 0 \\ \mu_1 + \mu_2 - M \end{bmatrix} \tag{4.63}$$

Further simplification leads to the following time estimate $t = t_3$ when the trajectory intercepts on the $x_2 = 0$ axis and the intercept $x_1(t_3)$:

$$\begin{aligned} t_3 &= \frac{\delta}{(\mu_2 + \mu_1 - M)} \frac{\sqrt{(\mu_2 - \mu_1 + M)}}{\sqrt{(\mu_2 + \mu_1 - M)}} \\ x_1(t_3) &= -\frac{\delta^2(\mu_2 - \mu_1 + M)}{2(\mu_2 + \mu_1 - M)^2} \end{aligned} \tag{4.64}$$

Case 4: $sign(x_1) = -1, sign(x_2) = 1$: Repeating the same computation for the third quadrant, equation (4.54) takes the following form:

$$x(t) \le e^{At}\begin{bmatrix} x_1(t_3) \\ 0 \end{bmatrix} + \left[\int_0^t e^{A(t-\tau)}d\tau\right]\begin{bmatrix} 0 \\ \mu_2 - \mu_1 + M \end{bmatrix} \tag{4.65}$$

Further simplification leads to the following time estimate $t = t_4$ when the trajectory intercepts on the $x_1 = 0$ axis and the intercept $x_2(t_4)$:

$$\begin{aligned} t_4 &= \frac{\delta}{(\mu_2 + \mu_1 - M)} \\ x_2(t_4) &= \frac{\delta(\mu_2 - \mu_1 + M)}{(\mu_2 + \mu_1 - M)} \end{aligned} \tag{4.66}$$

Hence the time T_2 taken by the trajectory to travel from the point O_4 on the ball B_δ to some point O_3 on the semi-axis e_1^+ is obtained using (4.59), (4.62), (4.64) and (4.66) as follows:

$$\begin{aligned} T_1 &= t_1 + t_2 + t_3 + t_4 \\ &= \frac{\delta}{\mu_2}\Delta \end{aligned} \tag{4.67}$$

where,

$$\eta = \frac{\mu_1 - M}{\mu_2}, \quad \Delta = \left[\frac{2}{1+\eta} + \frac{1}{\sqrt{(1+\eta)(1-\eta)}} + \frac{\sqrt{1-\eta}}{(1+\eta)\sqrt{1+\eta}}\right] \tag{4.68}$$

Step 2: Time to travel from the point O_3 on the $e_1^+ = \{x \in \mathbb{R}^2 : x_1 = 0, x_2 > 0\}$ axis to the origin:

It can be seen that the time T_1 taken by one revolution depends on the initial condition δ, gain parameters (μ_1, μ_2) and the bound M on the uncertainty. Hence the time T_1 and time taken by the subsequent revolutions can be computed *apriori*. Furthermore, as shown by the last equality of (4.66), the closed-loop trajectory decays closer to the origin by a factor Ψ of the initial condition δ where,

$$\Psi = \frac{1-\eta}{1+\eta} \tag{4.69}$$

The computation of trajectories for four quadrants can be repeated with the initial condition set at $x_2(t_4)$ to obtain the intersection of the trajectory with the $x_1 = 0$ axis at the end of the second revolution as follows:

$$x(T_2) = x_2(t_4)\,\Psi = \delta\Psi^2 \tag{4.70}$$

where T_2 is the time at which the second revolution is completed. Noting that one revolution takes $\frac{\Delta}{\mu_2}$ multiplied by the initial value on the vertical axis (see (4.67)), the total time taken for two revolutions is estimated as follows:

$$T_2 = T_1 + \frac{x_2(t_4)}{\mu_2}\Delta \tag{4.71}$$

where $x_2(t_4)$ is treated as the new initial condition at the start of the second revolution. Equation (4.71) can be simplified using (4.66) and (4.67) as follows:

$$T_2 = \frac{\delta}{\mu_2}\Delta\,[1+\Psi] \tag{4.72}$$

Similar arguments lead to the computation corresponding to the third revolution as follows:

$$\begin{aligned} x(T_3) &= x_2(T_2)\,\Psi = \delta\Psi^3 \\ T_3 &= T_2 + \frac{x_2(T_2)}{\mu_2}\Delta = \frac{\delta}{\mu_2}\Delta\left[1+\Psi+\Psi^2\right] \end{aligned} \tag{4.73}$$

Proceeding further and generalizing the above results for the n^{th} revolution, it is obtained that,

$$\begin{aligned} x(T_n) &= x_2(T_{n-1})\,\Psi = \delta\Psi^n \\ T_n &= T_{n-1} + \frac{x_2(T_{n-1})}{\mu_2}\Delta = \frac{\delta}{\mu_2}\Delta\left[1+\Psi+\Psi^2+\ldots+\Psi^{n-1}\right] \end{aligned} \tag{4.74}$$

The number of revolutions tends to ∞ as time t tends to ∞. Hence the following holds true:

$$\lim_{t\to\infty} T_n = \lim_{n\to\infty} T_n = \lim_{n\to\infty} \frac{\delta}{\mu_2}\Delta\left[1+\Psi+\Psi^2+\ldots+\Psi^{n-1}\right] \tag{4.75}$$

As can be noted from the definition (4.69) of Ψ that the inequality $0 < \Psi < 1$ always holds true because $\mu_2 > \mu_1 + M$. Hence the infinite series in (4.75) can be represented as follows:

$$1+\Psi+\Psi^2+\ldots+\Psi^{n-1} = \frac{1-\Psi^n}{1-\Psi} \tag{4.76}$$

In turn, the time T_n taken by an infinite number of revolutions represents the second segment of the settling time $\mathcal{T}_s$ in question which can be obtained as follows:

$$\begin{aligned} \mathcal{T}_2 = \lim_{n\to\infty} T_n &= \lim_{n\to\infty} \frac{\delta}{\mu_2}\Delta\left[\frac{1-\Psi^n}{1-\Psi}\right] \\ &= \frac{\delta\Delta}{\mu_2(1-\Psi)} \lim_{n\to\infty} (1-\Psi^n) \\ &= \frac{\delta\Delta}{\mu_2(1-\Psi)} < \infty \end{aligned} \tag{4.77}$$

Relation (4.77) thus always gives a finite upper bound $\mathcal{T}_2$ on the settling time estimate of the twisting algorithm for finite values of gains μ_1, μ_2. The following can be obtained from (4.68) and (4.69):

$$\Delta = \frac{2\left[\sqrt{1-\eta^2}+1\right]}{(1+\eta)\left[\sqrt{1-\eta^2}\right]}, \quad 1-\Psi = \frac{2\eta}{1+\eta} \tag{4.78}$$

By substituting Δ and $(1-\Psi)$ obtained from (4.78) and (4.69) respectively, the last equality of (4.77) can be further simplified as follows:

$$\mathcal{T}_2 = \frac{\delta\Delta}{\mu_2(1-\Psi)} = \frac{\delta\left[\sqrt{1-\eta^2}+1\right]}{\mu_2\eta\sqrt{1-\eta^2}} = \frac{\delta\left[\sqrt{1-\eta^2}+1\right]}{(\mu_1-M)\sqrt{1-\eta^2}} \tag{4.79}$$

Then, letting

$$\mu_2 > \beta\mu_1 \tag{4.80}$$

for some $\beta > 1$, the following is obtained from the definition of η and Ψ in (4.68) and (4.69) respectively:

$$\lim_{\mu_1\to\infty} \eta \leq \frac{1}{\beta}, \quad \lim_{\mu_1\to\infty} \Psi \geq \frac{1-\beta^{-1}}{1+\beta^{-1}} = \frac{\beta-1}{\beta+1} \tag{4.81}$$

In order to prove the existence of a finite δ in the limit $\mu_1 \to \infty$, it is necessary to establish the following Lemma.

Lemma 4.1. *There always exists a finite δ in the limit $\mu_1 \to \infty$.*

Proof. Let $\mu_2 > \beta\mu_1$ hold true for some $\beta > 1$ to satisfy the requirement (4.13). Then there exists another scalar $\beta_1 > \beta$ such that $\mu_2 = \beta_1\mu_1$. Then, the definition of η in (4.68) can be analysed as follows:

$$\begin{aligned} \eta &= \frac{\mu_1 - M}{\beta_1\mu_1} = \frac{1}{\beta_1} - \frac{M}{\mu_1} \\ \Rightarrow \lim_{\mu_1\to\infty} \eta &= \frac{1}{\beta_1} \\ &\leq \frac{1}{\beta} \end{aligned} \tag{4.82}$$

The analysis of this section started with the requirement $\delta > \frac{r_1}{\Psi}$. In the limit $\mu_1, \mu_2 \to \infty$, a finite r_1 is defined solely by a finite R due to the definition of r_1 in (4.31). Since

R is a user's choice and is finite, the following inequality holds true from (4.81):

$$\lim_{\mu_1 \to \infty} \frac{r_1}{\Psi} \leq \frac{r_1(\beta+1)}{\beta-1} < \infty \tag{4.83}$$

The statement of Lemma 4.1 is proven due to the finite upper bound (4.83) of $\frac{r_1}{\Psi}$ as an arbitrary δ can always be chosen such that the inequality $0 < \frac{r_1}{\Psi} < \frac{r_1(\beta+1)}{\beta-1} < \delta < \infty$ holds true in the limit $\mu_1 \to \infty$. □

Hence a finite $\delta > \frac{r_1(\beta+1)}{\beta-1} > \frac{r_1}{\Psi}$ can always be chosen[1] even as $\mu_1 \to \infty$. Proof of the existence of a finite δ is thus complete.

It follows that

$$\lim_{\mu_1 \to \infty} T_2 = \lim_{\mu_1 \to \infty} \frac{\delta\left[\sqrt{1-\eta^2}+1\right]}{(\mu_1 - M)\sqrt{1-\eta^2}} \leq \lim_{\mu_1 \to \infty} \frac{2\delta}{(\mu_1 - M)\sqrt{1-\frac{1}{\beta^2}}} = 0. \tag{4.84}$$

Thus, the intuitive expectation and in turn the final objective is achieved, namely, that increasing the gains of the twisting controller causes the settling time to decrease. Tuning rules to achieve an arbitrarily specified settling time are discussed in the next section.

4.3.1 Finite settling time of the origin under the switched control synthesis

The closed-loop system (4.9), (4.1) is finite time stable in the presence of the persisting disturbances $\omega(t)$ with uniformly bounded magnitude $|\omega(t)| < M$. The finite settling time can be represented using the results (4.18) and (4.77) of the Sections 4.2 and 4.3 respectively as follows:

$$T_s(T_1, T_2) = \begin{cases} T_1 + T_2, & 0 < \delta < r_0; \\ T_2, & 0 < r_0 < \delta. \end{cases} \tag{4.85}$$

where $r_0 = \sqrt{x_1^2(t_0) + x_2^2(t_0)}$ is the Euclidian norm of the system initial condition, the estimates T_1 and T_2 are given by (4.18) and (4.77) respectively. The second aim detailed in Section 4 is thus achieved.

[1]See Lemma 4.1.

4.4 Tuning

This section derives constructive tuning rules to define the gain parameters l_1, l_2, μ_1, μ_2 such that an arbitrarily chosen desired settling time can be achieved. In addition, the parameter R, which defines the boundary of switching between the linear and the twisting control laws, is an independent parameter to be chosen by the user. Let the specified settling time be denoted as $\mathcal{T}_s > 0$. Hence, in order to ensure that the closed-loop system (4.43) settles to the origin in specified time $\mathcal{T}_s$, the following must hold true:

$$\mathcal{T}_1 + \mathcal{T}_2 \leq \mathcal{T}_s \tag{4.86}$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are defined in (4.18) and (4.77) respectively. The proposed tuning strategy is that of dividing the desired settling time $\mathcal{T}_s$ into two time segments which can be arbitrarily chosen by the user. Then each time segment is considered to be the permissible upper bound on $\mathcal{T}_1$ and $\mathcal{T}_2$ which then leads to explicit tuning rules for the gain parameters of both the linear and twisting controller. Mathematically, the arbitrary allotment of the settling time can be obtained as follows:

$$\begin{aligned} \mathcal{T}_{s_1} &= \begin{cases} \epsilon_1 \mathcal{T}_s, & \text{if } r_0 > \delta; \\ 0, & \text{otherwise.} \end{cases} \\ \mathcal{T}_{s_2} &= \begin{cases} (1-\epsilon_1)\mathcal{T}_s, & \text{if } r_0 > \delta; \\ \epsilon_1\ \mathcal{T}_s, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.87}$$

where the scalar $\epsilon_1 \in (0,1)$ can be chosen by the user arbitrarily. Observing (4.18) and (4.79), the objective is to find the tuning of the twisting and linear gains such that the following inequalities hold true:

$$\begin{aligned} \frac{1}{\lambda}\ln\left[\frac{\mu_2(\alpha+1)|x_{1_0}| + \frac{2\mu_2}{\lambda}|x_{2_0}| + 2\alpha^2\lambda^2 x_{1_0}^2 + (\alpha^2+1)x_{2_0}^2}{R_1}\right] &\leq \mathcal{T}_{s_1}, \\ \frac{2\delta}{(\mu_1 - M)\left[\sqrt{1-\eta^2}\right]} &\leq \mathcal{T}_{s_2}. \end{aligned} \tag{4.88}$$

Then (4.86) is always ensured. The application of twisting controller is invoked without linear feedback if the condition $r_0 < \delta$ holds true due to user's choice of R. As mentioned in Section 4.2, gain μ_2 of twisting is seen as a parameter in the formula of $\mathcal{T}_1$. Hence the tuning rules for the parameters μ_1, μ_2 are obtained first in the following.

The requirements on the gains can be summarized from the previous sections as follows:

$$r_1 = \max\left\{\frac{R}{\mu_2}, \sqrt{2R}\right\}, \quad r_2 = \frac{\rho R}{\mu_2}, \quad \delta\,\Psi > r_1,$$
$$\mu_2 > \max\left\{\mu_1 + M, \quad \rho\sqrt{\frac{R}{2(1-\rho)}}, \quad \rho\right\} \tag{4.89}$$

where the scalar $\rho \in (0,1)$ is at user's disposal. It is assumed in the following that the initial condition of the states x_1^0, x_2^0 and the uniform upper-bound M are known. The step-by-step tuning procedure is now detailed:

Step 1: Choose an arbitrary $R > 0$. Let the following condition be imposed on the choice of μ_2:

$$\mu_2 \geq \sqrt{\frac{R}{2}} \tag{4.90}$$

Then, r_1 can be computed by combing (4.31) and (4.90) as follows:

$$r_1 = \sqrt{2\,R} \tag{4.91}$$

Step 2: Next, select η as follows:

$$0 < \eta \leq \frac{1}{\beta} \tag{4.92}$$

where $\beta > 1$ is a tuning variable. The above is motivated by the fact that the variable η thus defined does not allow Ψ to equal either to unity or zero. It can be noted that the following holds true:

$$\frac{1-\eta}{1+\eta} = \Psi \geq \frac{\beta-1}{\beta+1} \tag{4.93}$$

Step 3: Select the scalar δ according to

$$\delta > \begin{cases} \frac{r_1(\beta+1)}{\beta-1}, & \text{if } r_0 > \frac{r_1(\beta+1)}{\beta-1}; \\ \frac{r_0(\beta+1)}{\beta-1}, & \text{if } r_0 < \frac{r_1(\beta+1)}{\beta-1}. \end{cases} \tag{4.94}$$

Then due to (4.94) the requirement $\delta\Psi > r_1$ is always satisfied.

Step 4: If $r_0 < \frac{r_1(\beta+1)}{\beta-1}$, the twisting controller tuning is invoked without linear feedback. Compute μ_1, μ_2 using the next step and apply the twisting controller. If $r_0 > \frac{r_1(\beta+1)}{\beta-1}$, go to *step 6*.

Step 5: Tuning for the twisting controller is carried out in this step using results of Section 4.3. The following requirement can be formulated from (4.79) and the second inequality of (4.88):

$$\frac{2\delta}{(\mu_1 - M)\sqrt{1-\eta^2}} \leq \mathcal{T}_{s2} \tag{4.95}$$

In turn, the following tuning rules on μ_1 and μ_2 can be obtained by combining (4.89) and (4.95):

$$\begin{aligned} \mu_1 &> \max\left\{ M, \quad \frac{2\delta}{\mathcal{T}_{s2}\sqrt{1-\eta^2}} + M \right\} \\ \mu_2 &> \max\left\{ \sqrt{\frac{R}{2}}, \quad \frac{\mu_1 - M}{\eta}, \quad \mu_1 + M, \quad \rho\sqrt{\frac{R}{2(1-\rho)}}, \quad \rho, \quad \beta\,\mu_1 \right\} \end{aligned} \tag{4.96}$$

Remark 4.6. From the tuning viewpoint, higher values of β (and in turn, lower values of η) may be required for very small settling time requirements. This is mainly due to the fact that the behaviour of the settling time estimate (4.79) is fully known only in the limit $\mu_1 \to \infty$. Relationship of $\mathcal{T}_2$ with respect to μ_1 and β is highly nonlinear especially in the vicinity of $\mathcal{T}_2 = 0$ (see (4.79)). Hence (4.79) does not clarify as to the rate at which the settling time can be reduced. Nevertheless, due to (4.84), it can be deduced that there always exists a finite scalar β such that the second inequality of (4.88) is satisfied. In this sense, β is seen as a tuning variable.

Step 6: This step performs the tuning for the linear gains

$$l_1 = \alpha\,\lambda^2, \quad l_2 = (\alpha+1)\,\lambda$$

where $\alpha > 1$ is an arbitrarily chosen scalar and the scalar λ can be obtained as follows: Perform the tuning procedure of *Step 5* to obtain the parameters μ_1, μ_2. Choose an arbitrary scalar $\rho \in (0,1)$. Then choose an arbitrary scalar $r > 0$ such that $0 < r < r_2$ holds true where r_2 is computed from (4.38) using the above ρ. Next, compute $R_1 = R - r$ (it should be noted from Remark 4.3 that the relation $r < r_2 < R$ is guaranteed). To ensure that (4.24) holds true let us choose λ according to

$$\lambda \geq \sqrt{\frac{2\mu_2 M + (\alpha+1)^2 M^2}{r}}. \tag{4.97}$$

Next, utilizing the first inequality in (4.88), the following transcendental equation can be obtained:

$$\frac{1}{\gamma}\ln\left[\frac{\mu_2(\alpha+1)|x_{1_0}| + \frac{2\mu_2}{\gamma}|x_{2_0}| + 2\alpha^2\gamma^2 x_{1_0}^2 + (\alpha^2+1)x_{2_0}^2}{R_1}\right] - \mathcal{T}_{s1} = 0 \tag{4.98}$$

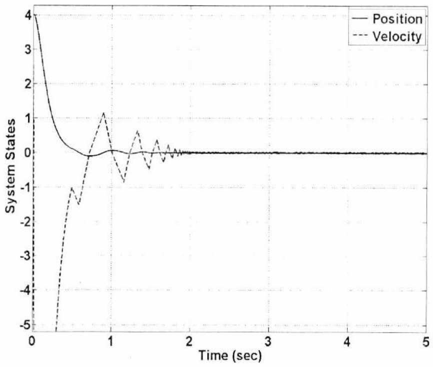

FIGURE 4.4: Numerical simulation for the specification of $\mathcal{T}_s = 10\,sec$, $M = 1$.

It can be noted from (4.18) and (4.23) that there always exists a solution γ such that (4.98) is satisfied. The basis for this is that $\lim_{\gamma \to \infty} \gamma^2 e^{-\gamma t} = 0$. A valid solution γ satisfying (4.98) can be obtained, for example, using numerical optimization routines. Hence the tuning rule for λ can be obtained as follows:

$$\lambda = \max\left\{\sqrt{\frac{2\mu_2 M + (\alpha+1)^2 M^2}{r}},\quad \gamma\right\} \tag{4.99}$$

Then, (4.97) gives the tuning rules for the linear gains l_1, l_2. The third aim detailed in Section 4 is thus achieved.

Figure (4.4) shows the numerical simulation for a specification of $\mathcal{T}_s = 10\,sec$ with the initial condition $x_0^1 = x_0^2 = 4$ and the upper-bound $M = 1$. The aforementioned tuning steps have been carried out rendering the tuning variables as follows:

$$\rho = 0.99,\ r = 0.173,\ R = 1.5,\ \epsilon_1 = 0.3,\ \eta = 0.45,\ \beta = 2.11,\ \alpha = 1.01. \tag{4.100}$$

The tuning rules (4.96) and (4.99) then compute the controller parameters as follows: $\mu_1 = 2.47, \mu_2 = 8.57, l_1 = 123.6, l_2 = 22.243$. A switch from the linear controller to the twisting controller can be seen at $t = 0.5\,sec$.

4.5 Remarks on geometric homogeneity and time collapse

The previous chapter proposed a geometric homogeneity based approach to estimate an upper bound on the finite settling time for a second order sliding mode controller. Geometric homogeneity have been used several times in the literature as a tool to establish stability [9, 84–86]. Early results [85, 86] studied asymptotic stabilization and existence of homogeneous Lyapunov functions when different dilations of the state vector and that of time give rise to homogeneity of the continuous vector field. The extension to a similar study for establishing finite time stability was carried out in [84] where it was shown that geometric homogeneity of vector field of a dynamical system leads to finite time stability if the system is asymptotically stable and has a negative degree of homogeneity.

The results in the previous sections of this chapter can be seen as a special case of a more general geometric homogeneity result [9] pertaining to the finite time stability of switched systems. However, the result in [9] did not provide the estimate of the settling time such that the intuitive expectation is fulfilled, namely, that increasing the gains of the feedback controller reduces the settling time. The results established in earlier sections not only prove the same but also provide tuning rules to achieve pre-specified settling time.

It is interesting to note that the geometric series (4.76) and (4.77) encountered as a natural outcome of the mathematical analysis are indeed in agreement with the earlier results on homogeneity of discontinuous vector field [4], [9, Th. 3.1] that inherently involve a geometric series producing finite time convergence of trajectory due to the negative degree of homogeneity.

Uniform Finite Time Stability

The Definition 2.5 of equiuniform finite time convergence given in Chapter 2 requires that the stability (that may be substantiated by Lyapunov functions) be uniform in the initial conditions of the state and time and the uncertainty. It also means that the parameter ϵ mentioned in the aforementioned definition should be independent of the initial time t_0. As discussed in Section 3.8.2, it is worth noting that the results presented in earlier sections of this chapter satisfy these requirements. The first reason is that the underlying global uniform asymptotic and exponential stability is guaranteed by combining the Lyapunov function based analysis presented in the previous chapter with the semi-global analysis in [9] (also see Remark 3.1). In particular, the inequalities

$$\begin{aligned} L_R V(x_1, x_2) \leq \tilde{V}(x_1, x_2) \leq M_R V(x_1, x_2) \\ V(x_1, x_2) \leq L_R^{-1} M_R R e^{-K_R(t-t_0)} \end{aligned} \tag{4.101}$$

hold true within the compact set $\Gamma_R = \{(x_1, x_2) : V \leq R\}$, $\tilde{V}(x_1, x_2) = \mu_2|x_1| + \frac{1}{2}x_2^2 + \kappa x_1 x_2$, $V(x_1, x_2) = \mu_2|x_1| + \frac{1}{2}x_2^2$ and K_R, L_R, M_R are positive scalars suitably fixed a priori [9, Th. 4]. The compact set Γ_R utilised in the switched synthesis (4.10), which is based on the same function V appearing in (4.12), is the same as that defined in proof of [9, Th. 4.2]. Hence, the definition of equiuniform finite time stability is satisfied due to the very fact that the twisting controller is invoked inside Γ_R. The second reason is that the computation of the transcendental solutions of the integrals (4.54) was carried out without depending explicitly on initial time t_0. More specifically, the point O_1 of figure 4.1 and point O_4 of figure 4.2 do not depend on any fixed value of the initial time t_0. A unique δ thus exists for all finite initial conditions $x_2(t_0)$ for all t_0 because the region B_{r_1} encompasses Γ_R, i.e., $\Gamma_R \subseteq B_{r_1}$ (see section 4.2.1, lemma 4.1 and figure 4.1).

4.6 Remarks on conservatism of settling time estimate

The previous chapter and this chapter presented two methods of computing the upper bound on the settling time of the perturbed double integrator when a twisting controller is used. As mentioned earlier in the thesis, the work presented in [106] also addressed this problem. More recently, the work in [42] solved this problem via identifying strict Lyapunov function. All the results, reported in the thesis or otherwise, exhibit some degree of conservatism in the upper bound of the settling time estimate. This is because the computation of the settling time is based on either the decay rate of the Lyapunov function or the level sets of the same. It is of interest to investigate the conservatism of the upper bound presented in this chapter with that proposed in parallel or existing works. The most recent reference [42] is chosen for this comparison.

Reference [42] considers the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a(x,t) + b(x,t)u, \end{aligned} \tag{4.102}$$

where $x = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2, u \in \mathbb{R}$ and $|a(x,t)| < M$ is bounded uncertainty. If it is assumed that $b(x,t) = 1$ then system (4.102) coincides with (4) when $\omega \triangleq a(x,t)$ and twisting controller is used. It should be noted that the gains need to be selected according to the following condition:

$$\mu_1 + p > \mu_2 - M > \mu_1 > M > \frac{2^{\frac{3}{2}}}{3}\pi_2 \tag{4.103}$$

for some $p > 0, \pi_2 > 0$.

The upper bound on the settling time function established in [42] is as follows:

$$T(x_0) = \frac{3}{\gamma} V^{\frac{1}{3}}(x_0) \tag{4.104}$$

where,

$$
\begin{aligned}
&x_0 = x(t_0), \quad V(x) = \left(\pi_1|x_1| + \frac{1}{2}x_2^2\right)^{\frac{3}{2}} + \pi_2 x_1 x_2, \\
&\pi_2 > 0, \quad p > 0, \quad \pi_1 = p + 2M + \frac{2}{3}\sqrt{2}\pi_2, \\
&\gamma = \min\left\{ \frac{\alpha_1}{\left(\sqrt{2}\pi_1^{\frac{3}{2}} + \frac{2}{3}\pi_2 c^{\frac{3}{2}}\right)^{\frac{2}{3}}}, \frac{\alpha_2}{\left(\sqrt{2}(\frac{1}{2})^{\frac{3}{2}} + \frac{1}{3}\pi_2 c^{-3}\right)^{\frac{2}{3}}} \right\} \\
&\alpha_1 \triangleq \pi_2(\mu_2 - \mu_1 - M) \\
&\alpha_2 \triangleq \frac{3}{2\sqrt{2}}\left(\mu_1 - |\pi_1 - \mu_2| - M - \frac{2}{3}2^{\frac{1}{2}}\pi_2\right), \\
&c > 0
\end{aligned}
\tag{4.105}
$$

Let the initial condition $x_0 = (0.5, -1)$ be considered. Let the upper bound M on the uncertainty ω be $M = 1$. This defines η and Ψ (See Section 4.3). Let $R = 0.5$ be selected. Let $\delta = 1.1\frac{r_1}{\Psi}$ be chosen in order to satisfy the requirement $\delta\Psi > r_1$. Then formula (4.77) leads to the estimate of the upper bound on the settling time $\mathcal{T}_2 = 3.73$ when $\mu_1 = 2, \mu_2 = 4$ are chosen. Furthermore, $\mathcal{T}_1 = 2.05$ can be computed from (4.18) where $\alpha = 2, l_1 = 2, l_2 = 3, R_1 = 2.5$ have been chosen. Hence, the total settling time according to the formula (4.85) is obtained as $\mathcal{T}_1 + \mathcal{T}_2 = 5.78$.

Next, let the upper bound on the settling time given by (4.104) be considered. Choice $\mu_1 = 2, \mu_2 = 4, p = 2, \pi_2 = 0.2$ is a valid one that satisfies (4.103) for $M = 1$. Then $\pi_1 = 4.2$ follows from the definition given in (4.105). The scalars $\alpha_1 = 0.2$ and $\alpha_2 = 0.66$ also follow from the definition given in (4.105). Choosing $c = 1$ and noting that $\gamma = 0.04, V(x_0) = 4.08$ leads to the upper bound $T(x_0) = 127.4$.

The actual settling time of the system is just less than $2.5sec$ as shown in Fig. 4.5. It can be seen that the upper bound $T(x_0) = 127.4$ obtained via the results of [42] is significantly more conservative than $\mathcal{T}_1 + \mathcal{T}_2 = 5.78$ obtained via the results presented in this chapter. This holds true at least for the initial conditions in question.

4.7 Planar systems with unilateral constraints and resets

Robust discontinuous uniform finite time stabilization has been a significant theme in this thesis. The planar systems considered thus far can be described by a discontinuous vector field. The solutions of the closed-loop systems are understood in the sense of

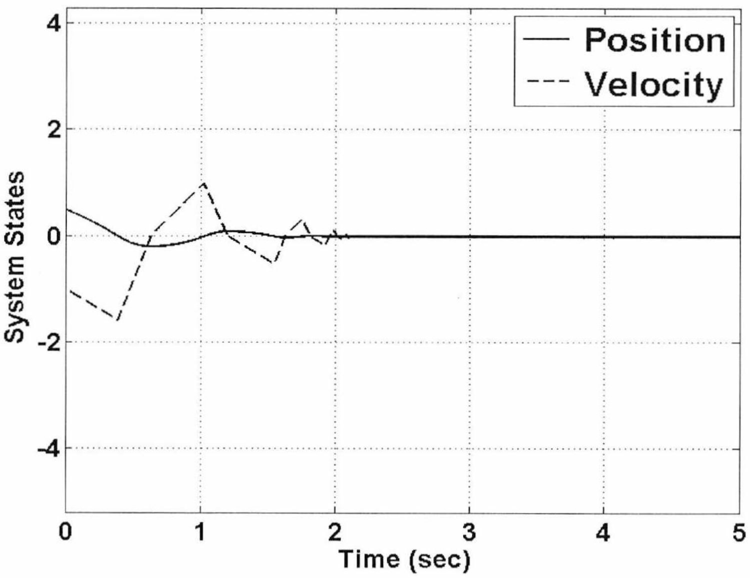

FIGURE 4.5: Actual setting time of the closed-loop system (4.43)

Fillipov's definition [6]. The solutions are absolutely continuous and in general non-unique in forward time.

The focus of the investigation in the next chapter shifts to the study of unilaterally constrained planar systems. More specifically, the focus is on studying the same techniques of homogeneity and non-smooth Lyapunov functions to establish uniform finite time stability and to estimate a finite upper bound on the settling time for unilaterally constrained planar systems.

Section 3.8.3 described a class of unilaterally constrained systems. Systems with unilateral constraints occur when one of the states cannot evolve beyond a physical constraint and typically have discontinuous solutions [23]. Discontinuous systems with jumps in the states, which are a class of complementarity systems, are studied in various disciplines within the sciences. References [23, 40, 95] provide a comprehensive study of solutions, stability and control of non-smooth mechanical systems with discontinuity and collisions and [122] studies complementarity systems in economics that allow for discontinuous solutions. As far as non-smooth mechanics of the motion of a particle is concerned, a unilateral constraint on position induces a jump in its velocity [23, Ch. 1]. A classical example where a unilateral constraint gives rise to jump in velocity is the well-known 'Bouncing Ball' example. For example, robotics applications in mechanical engineering where the position of a particle is hindered by the constraint surface is a typical example of such systems [96]. Control of biped robots, a very active area of research [15, 27, 28], inherently involves jumps in angular velocities of all the joints [97].

As evident from the above references, the practical significance of the study of stability and control of systems with jumps is very relevant to a large class of mechanical systems. It can be seen from the aforementioned literature review that asymptotic stability in the presence of impacts is an active area of research. However, finite time stability in the presence of impacts is less studied. Impacts due to collision present a hard non-linearity within the whole control problem and the challenge of stabilizing the trajectory to the origin arises mainly due to the fact that impacts take the trajectories away from the origin in systems such as (3.86) where a unilateral constraint appears on the position. Hence, discontinuity of solutions and instability due to the hard non-linearity induced by state jumps pose a theoretically challenging problem.

The nonlinearity posed by impacts can be thought of as belonging to two types of systems: (i) Problems where the impacts have a finite accumulation point (see [23, 36] for detailed discussion on the so called 'Zeno' behaviour caused by the accumulation of impacts) and (ii) where the impacts do not have an accumulation point. In this thesis, the first problem is studied in the form of a regulator problem and the second problem is studied in the form of a tracking problem. The second problem inherently involves a definition of stability that is based either on the boundedness of solutions or on the periodicity of solutions [27, 36].

As far as the problems with an accumulation of impacts are concerned, the problem of defining the solutions of non-smooth systems in the closed-loop is theoretically challenging [23, Section 1.3.1(c)]. The state $x(t)$ of unilaterally constrained planar systems such as that given in (3.86) becomes a discontinuous function of time due to impacts. The flow remains continuous only between the impacts. Existing Lyapunov approaches can be found in [23, 101] which inevitably involve analysis of jumps in the corresponding Lyapunov function. From the perspective of formulating a control synthesis, the problem of finite time stabilization of such systems is a novel concept which has not been studied in the published literature (see [23] and references therein). Such a study is relevant to a class of mechanical systems with unilateral constraints [23, 97].

Out of the two aforementioned problems with impacts, the next chapter solves the theoretically challenging and practically relevant problem of uniform finite time stabilization of the unilaterally constrained perturbed double integrator to the origin in the presence of impacts which have a finite accumulation time [2]. The problem of stabilizing the trajectories when the impacts do not have a finite accumulation and persist for all $t \in [0, \infty]$ is studied in Chapter 7. Since the control of joints of fully actuated

[2]As noted in [123], "one can argue that physical systems do not exhibit Zeno behaviour. However, modelling abstraction can often lead to Zeno models of physical systems. Since abstraction is crucial for handling complex systems, understanding when it leads to Zeno hybrid systems is important". This is indeed the case with unilaterally constrained system (3.86) not only in understanding when 'Zeno' motion occurs but also from the viewpoint of achieving stability.

robots can be formulated as the control of a perturbed double integrator [23, Ch. 8], the results in the next chapter cover a large class of mechanical systems. The aforementioned finite time stabilization is achieved by extending the results of the previous chapter on the Lyapunov analysis for planar controllable systems and geometric results of this chapter to the case of unilaterally constrained systems.

4.8 Conclusion

A novel switched control synthesis has been developed. An upper bound on the settling time of the 'twisting' controller and the corresponding tuning rules are also established. The upper bound is obtained in the absence of the differential inequality of the Lyapunov function. Explicit tuning rules, which enable desired reduction in settling time, are obtained due to the several straightforward steps. The results appear to be superior to existing methods that do not provide tuning guidelines to achieve a specific reduction in settling time. These results may be of particular interest to practising control engineers in implementing a twisting controller.

The next chapter utilises the above results which are based on geometric analysis and combines them with the Lyapunov and homogeneity results of the previous chapter to solve the problem discussed in Section 4.7.

CHAPTER 5

Uniform Finite Time Stabilization of a Unilaterally Constrained Perturbed Double Integrator

5.1 Systems with resets

The study of discontinuous systems has received considerable interest amongst control theorists and practitioners [21]. Discontinuous systems are studied in very different research fields such as economics, electrical circuit theory, mechanical engineering, biosciences, systems and control theory. Many different frameworks therefore exist to describe various classes of discontinuous systems, for example, differential inclusions [6], measure differential inclusions [25] and complementarity systems [23] amongst others. Discontinuities appear either due to the nature of the system dynamics or due to the application of discontinuous feedback control. A survey article on discontinuous dynamical systems can be found in [21], which discusses various kinds of discontinuities, their respective solution concepts together with the stability tools available in the literature.

This chapter is concerned with the study of systems with resets or subject to re-initialization of the states. Such systems are studied in various branches of the control sciences. In the study of hybrid systems, for example, systems with resets are studied as a combination of the two phases: (i) continuous time dynamics and (ii) discrete event dynamics caused by state jumps (see recent studies [49, 124] and references therein for a rigorous survey of solutions, stability theory and Lyapunov based frameworks for such systems). Systems with resets are also studied in complementarity systems arising from systems with collisions as studied in the area of non-smooth mechanics [23]. The solution concept for these can be found in [25, 47, 95]. The relevance of systems with impulse effects to the area of robotics is studied in [23, Ch. 8]. The tracking control

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of robotic systems when impacts appear as a hard nonlinearity is both a practically and theoretically challenging problem. Tracking control of biped robots [27, 36], for example, presents a befitting example of systems with impulse effects in velocity.

The main focus of this chapter is on mechanical systems with resets in velocity. The well-known twisting controller, which is a second order sliding mode controller [4], is utilised for finite time stabilization of a double integrator with a unilateral constraint surface as well as with a disturbance term in the velocity dynamics. The disturbance can account for discontinuous terms such as Coulomb friction. The velocity undergoes an instantaneous *reset* when this inelastic collision occurs. It is assumed in this chapter that the restitution or reset map relating the velocities just before and after the time of impact is fully known. However, no such assumption is made on the time of impact. The method of Zhuravlev's non-smooth transformation [125] is utilised to first transform the system into a variable-structure system without jumps (also see references cited in [23, Ch. 1, Sec. 1.4]). Within the domain of engineering applications, such a transformation is very useful in the analysis of vibro-impact systems [125], [126]. The resulting transformed system turns out to be a switched homogeneous system with a negative homogeneity degree [9] where the solutions are well-defined in the sense of Filippov's definition [6], an attribute absent in the case of the original jump system (see [95] for solutions concept of systems with jumps and friction). As an immediate consequence, the resulting transformed system turns out to be a valid candidate for stability analysis via smooth and non-smooth Lyapunov functions [23, Section 1.4]. Next, a non-smooth Lyapunov function is identified to prove global equiuniform asymptotic stability. In turn, the quasi-homogeneity principle [9] is shown to be applicable to the transformed system which, while being locally homogeneous with negative homogeneity degree, is shown to be finite time stable.

A brief review of the literature on control of systems with resets

The monograph [23] details methods for specifying solutions, together with the Lyapunov stability framework and control synthesis for non-smooth mechanical systems with friction and collision. The rigorous theoretical developments in the theory of non-smooth mechanics have been accompanied by applications such as biped robotics [15, 36, 97] thereby emphasizing the practical importance of the developed theory. An introduction and a detailed study of non-smooth Lyapunov functions in the context of discontinuous systems can be found in [20].

Stability studies for systems relevant to this chapter can be traced back to early results, see references [34] for example. The results in [33] and [100] studied the boundedness of solutions and the second method of Lyapunov for stability of linear impulsive systems. A multiple Lyapunov function approach can be found in [101] for switched and hybrid systems. It is noted however, that the Zeno behaviours produced by infinite number of switches and infinite number of impulses accumulating in finite time are not covered by this work [1]. Stability analysis for systems with impulse effects can be found in [35] where conditions on jumps in Lyapunov functions were given for uniform asymptotic stability of the system. Results on Lyapunov stability and dissipativity theory were given in [102] for a class of hybrid systems which covered accumulation of impacts (or the Zeno mode).

Results on asymptotic stability and the invariance principle for unilaterally constrained systems can be found in [36, 39]. Results on robust finite time stabilization of linear impulsive systems can be found in [99], which does not encompass the finite accumulation of impacts. Unilaterally constrained systems can also be seen as hybrid systems. The reader is referred to the latest advances in the literature on stability of hybrid systems [50, 124]. However, more references on hybrid systems are not included here as the study of hybrid systems is not one of the principal topics of this thesis.

It can be seen from the above literature review that uniform finite time stabilization in the presence of accumulating impacts is a new and open problem.

The rest of the chapter is outlined as follows. The problem statement is presented in Section 5.2. Section 5.3 presents arguments about the challenge of the proposed problem statement as well as about the novelty of the presented results. It also provides clear motivation to study the problem defined in Section 5.2. Sections 5.4 and 5.5 contain the main results, namely, the proof of finite time stability and computation of the upper bound on the settling time respectively. More importantly, these two sections extend the Lyapunov framework of Chapter 3 and geometric results of Chapter 4 to cover variable structure systems with resets. Section 5.6 presents numerical simulations. Section 5.7 outlines some future directions of study which are worth pursuing not only due to the underlying theoretical challenge but also due to the practical relevance. Section 5.8 includes discussion on extending the results of the thesis thus far to non-smooth systems, a topic of investigation for the next chapter. Finally, some concluding remarks for the chapter are given in Section 5.9.

[1] In the previous two chapters, the infinite number of switches were covered at the origin, leading to a geometric series.

5.2 Problem statement

The mathematical statement of the problem is given first. The next section follows the problem statement by clearly providing details of the underlying motivation and challenge as well as the unique contribution. Consider the following open loop system [23, Ch. 1]:

$$\dot{x}_1 = x_2 \tag{5.1a}$$

$$\dot{x}_2 = u(x_1, x_2) + \omega(t) \tag{5.1b}$$

$$x_1 \geq 0 \tag{5.1c}$$

$$x_2(t_k^+) = -\bar{e}\, x_2(t_k^-) \quad \text{if} \quad x_2(t_k^-) < 0,\ x_1(t_k) = 0, \tag{5.1d}$$

where x_1, x_2 are the position and the velocity respectively, u is the control input, $\omega(t)$ is a piece-wise continuous disturbance [9, Sec. 2], [6], t_k is the time instant of the k^{th} jump where the velocity undergoes a reset or jump, $\bar{e}$ represents the loss of energy and $x_2(t_k^+)$ and $x_2(t_k^-)$ represent right and left limits respectively of x_2 at the jump time t_k. The equalities (5.1a) and (5.1b) represent the continuous dynamics without jumps in the velocity. The inequality (5.1c) represents the unilateral constraint on the position x_1 which evolves in a domain with a boundary [23]. It is assumed that the jump event occurs instantaneously within an infinitesimally small time and hence mathematically can be represented by *Newton's restitution rule* [23], [15] given by (5.1d) where it is assumed that $\bar{e} \in (0, 1)$. The twisting control law [4] in (x_1, x_2) coordinates is given as follows:

$$u(x_1, x_2) = -\mu_1 \operatorname{sign}(x_2) - \mu_2 \operatorname{sign}(x_1) \tag{5.2}$$

where, $\mu_2 > \mu_1 > 0$. It should be noted that the above control law undergoes a jump whenever the state x_2 undergoes a jump (this is similar to the existing literature [15]). The disturbance ω is assumed to admit a uniform upper bound

$$ess\sup_{t \geq 0} |\omega(x, t)| \leq M \tag{5.3}$$

on its magnitude such that

$$0 < M < \mu_1 < \mu_2. \tag{5.4}$$

It should be noted that the solutions of the closed-loop system (5.1),(5.2), which involve switched terms along with impact, can be defined using existing methods (see [25], [23], [95] and [47] for the solution concept via differential inclusions with both friction and collisions terms on the right hand side).

The aim of this chapter is to i) prove finite time stability and ii) to establish a finite upper bound on the settling time $\mathcal{T}$ of the closed-loop system (5.1), (5.2).

It should be noted that the existing finite time stabilization approaches [4, 9, 12, 14, 106] do not apply to the case of jumps in the velocity dynamics. The motivation to achieve finite time stability is twofold. Firstly, the effect of the jump in velocity when $x_1 = 0$ is a destabilizing one for the double integrator. This is contrary, for example, to the self-stabilizing nature of a bouncing ball where impact with the ground stabilizes the motion [23], with loss of energy at the time of impact. Hence proving finite time stability is a theoretical challenge due to the complexity of the definition of the solutions of the closed-loop system (5.1), (5.2) [95].

Relevance to mechanical systems

Although limited, system (5.1) is not too restrictive as such planar controllable systems may occur when non-linear constraints of the form $F(x) = 0$ are transformed into $x_1 = 0$. The reader is referred to [96, Section III] where the problem of control of constrained robots described by $M(q)\ddot{q} + F(q, \dot{q})\dot{q} = f + u$, where f denotes the generalized constraint forces due to a constraint surface $\phi(q) = 0$, is transformed into $\ddot{x}_1 = u + \omega(t, x) + T^{\mathrm{T}}(x)f$ with a constraint $x_1 \geq 0$. The matrix $T(x)$ is a non-singular transformation and $f = 0$ when there is no contact with the surface. In terms of compact notations of complementarity theory [23], $0 \leq f \perp \phi(q) \geq 0$ holds true. Then, the results presented in this chapter are directly applicable to such planar controllable systems with the aim of finite time stabilization on the constraint surface $x_1 = 0$. Therefore, the proposed results do not merely mimic the bouncing ball like behaviour, but rather they achieve uniform finite time stability in the presence of friction and impacts for a large class of impact mechanical systems. Secondly, the proposed stabilization results can form the basis for similar developments for nonlinear systems with resets in velocity such as biped robots when the joints undergo rebounds of impacts. Thus the problem formulation is both theoretically challenging and practically relevant. It should be noted that the latest advances in the literature of numerical schemes [118] aid the implementation of discontinuous control laws.

5.3 A novel concept: finite time stability in the presence of resets

Finite time stability [13] between two successive impacts when the impacts do not have an accumulation point is relevant to the area of biped robots [27]. The stability of a nonlinear system in the presence of such a finite time convergence of tracking errors between the impacts is better understood in terms of periodic orbits [28]. Such a finite time convergence is fundamentally different from that obtained in the presence of accumulation of impacts otherwise known as 'Zeno' behaviour [23, 29]. This chapter achieves finite time stability of unilaterally constrained systems with the accumulation of impacts.

The existing work [36] can be utilised by showing that the limit of post jump values of the Lyapunov function goes to zero. The problem presented in Section 5.2 can be analysed in a manner very similar to Chapter 4 where integration of the trajectories is carried out and the corresponding Lyapunov function jumps are computed. This must be followed by the computation of the limit of the post reset Lyapunov jumps. Although this is possible theoretically, there is an inherent theoretical challenge due to the way the solutions are defined. For example, the Fillipov's solution concept is no longer feasible for the jump system. Alternative methods [47, 95] should be employed first to clearly define the solutions of the closed-loop system. Then, the next challenge is to see if the convolution method of integration carried out in the previous chapter holds in this case. It is therefore not a straightforward problem to solve, even for a simple jump system (or a simple unilaterally constrained system), such as (5.1),(5.2).

In view of the above discussion, this chapter utilises an existing method of non-smooth transformation [125] to transform the system with resets to a system without resets. This conversion is essential to the subsequent result of equiuniform finite time stability since the trajectory of the transformed system can be defined completely by Fillippov's solutions, a tool not available in the case of the original jump system. The main consequence of this transformation is that all the methods established in the previous two chapters, namely, Lyapunov and geometric analysis of homogeneity of uncertain switched systems, become applicable to the class of systems being studied in this chapter. Interestingly, hope for establishing various Lyapunov frameworks for such a transformed system was expressed in the literature [23, Remark 1.15]. The problem of finite time stability, however, has never been studied even for transformed systems generated by non-smooth transformation. The relevance of the investigation presented in this chapter to engineering applications has been discussed in Section 5.2. This chapter studies this theoretically novel and practically relevant problem.

Theoretical motivation

The theoretical motivation to propose this new framework to replace the existing Lyapunov methods for discontinuous systems [9], [4], [106] is that the later do not apply to the case of jumps in the velocity dynamics. In contrast, a class of non-smooth semi-global Lyapunov functions is shown to exist in this chapter to prove equiuniform asymptotic stability of the transformed system in order to take advantage of the existing finite time stability results [9]. In turn, attainment of global equiuniform finite time stability becomes possible. From a practical viewpoint, the motivation stems from the applicability of the proposed method to the analysis and control of mechanical systems with jumps in velocity such as biped robots [15]. The proposed theory not only covers individual resets but also encompasses the proof of finite time stability under the influence of infinite rebounds of impulses, otherwise known as the so-called 'Zeno mode'. In the context of cyclic gait control of a biped robot, the practical significance of this proof is that the stability of all the actuated joints under the influence of infinite rebounds at each step is substantiated by mathematically transparent Lyapunov theory.

A unique contribution

The main theoretical contributions of this chapter are threefold. Firstly, although results exist for asymptotic stabilization of continuous and discrete dynamics [38], finite time stabilization in the presence of velocity jumps is a novel concept. Whereas the existing stability results using Lyapunov methods [23], [36], [101], [21], [34], [33], [100], [35], [102] inevitably involve analysis of jumps of the Lyapunov function and their respective limits in asymptotic time, the proposed method proves the finite time stability of the origin of a perturbed double integrator in the presence of jumps in velocity without having to analyse jumps in the proposed non-smooth Lyapunov function. Secondly, finite time stability is proved using the homogeneity principle for the switched system thereby obviating the need for obtaining a differential inequality of the Lyapunov function. In turn, the quasi-homogeneity principle [9] is extended to the case where jumps in velocity are present. Finally, the 'twisting' controller [4] is shown to stabilize the unilaterally constrained perturbed double integrator in finite time. Furthermore, an upper bound on the settling time is also computed. These contributions bridge three streams, namely, non-smooth Lyapunov analysis, the homogeneity principle and finite time stability for a class of impact mechanical systems.

This chapter presents one of the main results of the thesis. It sets an analytical benchmark for future studies pertaining to the stability of a class of unilaterally constrained systems by providing rigorous Lyapunov based proofs and geometric homogeneity results. For example, convergence analysis of advanced methods such as [30] may be able to utilise the proposed proof while verifying the results of stabilization of the system trajectory locally on a constraint surface. Furthermore, the results presented in this chapter allow for the so called 'Zeno' mode in constrained planar systems where the impacts have a finite accumulation point.

It should be noted that unlike some existing methods in the hybrid systems literature [50, Section V.C] [52, Example 6.8] that provide uniform asymptotic stability results, the ideal Zeno solution is not required to be eliminated by temporal regularization. Instead, the Zeno solution is automatically encompassed due to the non-smooth transformation paving the way for non-smooth Lyapunov theory [20] to be applied to the problem. More importantly, the uniform finite time stability (as defined in [44], for example,) is achieved for the unilaterally constrained double integrator in the presence of friction and bounded disturbance on the right hand side of the velocity equation as well as in the presence of the Zeno behaviour of the solution, a robustness feature not available in the hybrid systems literature. Furthermore, equiuniform stability (with respect to uncertainty) is a different concept than that given in [44] in that the later only discusses uniformity with respect to the initial time and state. It is interesting to note that the problem of capturing the accumulation of impacts numerically is quite difficult to solve when event based numerical schemes are used [22, Ch. 1]. This is discussed in more detail in Section 5.7.

A remark on 'Zeno' mode

Unlike the existing literature on hybrid systems [29], this chapter does not regularize the Zeno motion temporally or dynamically. For the bouncing ball example, ideal Zeno motion occurs when the ground and the ball are assumed to be rigid and non-deforming and each impact with the ground is assumed to be purely non-elastic [23, 48]. In the context of the bouncing ball analogy, the temporal regularization as described in [29, Section 4] and subsequently utilised in [50, Section V.C] means that the impact takes a small but (not infinitesimally small) time $\epsilon > 0$ and the dynamic regularization means that the impact is elastic but is more like that with a highly stiff spring. No such regularization is employed in this chapter and ideal Zeno modes due to non-elastic impacts are allowed giving rise to instantaneous jumps in zero time[2] Furthermore, existing Lyapunov approaches on the study of 'uniformly small ordinary time' [48] that leads to finite time stabilization results and computation of finite settling time inherently differ in that the jumps in the corresponding Lyapunov function [48, Th. 3.3, Example 3.4] is always needed to be analysed at the time instance of a reset. More importantly, the decrease in successive jumps also have to belong to class $\mathcal{K}_\infty$ (see [48, Th. 3.3]). This chapter does not need such an assumption while proving robust finite time stability and computing the finite settling time without analysing the jumps

[2]The measure of time is in fact better represented by Dirac measure [23] and Dirac distribution.

in the Lyapunov function due to the jump-free system produced by the non-smooth transformation.

5.4 Global equiuniform finite time stability

This chapter employs Zhuravlev-Ivanov's method of non-smooth transformation [125, 127], [23, Sec. 1.4.2] to transform the impact system (5.1) into a jump-free system. Let the non-smooth coordinate transformation be defined as follows:

$$\begin{aligned} x_1 &= |s|, \quad x_2 = R\,v\,\mathrm{sign}(s), \\ R &= 1 - k\,\mathrm{sign}(s\,v), \quad k = \frac{1-\bar{e}}{1+\bar{e}}. \end{aligned} \tag{5.5}$$

The variable structure transformed system

$$\begin{aligned} \dot{s} &= R\,v \\ \dot{v} &= R^{-1}\mathrm{sign}(s)\,(u(|s|, Rv\mathrm{sign}(s)) + \omega(|s|, Rv\mathrm{sign}(s), t)) \end{aligned} \tag{5.6}$$

is then obtained by employing (5.5) and using the dynamics (5.1a), (5.1b) (see [23, Ch. 1] and remarks on non-smooth transformation (5.5) at the end of this subsection for the mechanics viewpoint and detailed derivation). By combining (5.2) and (5.5), the controller (5.2) can be represented in the transformed coordinates as follows:

$$u(|s|, R\,v\,\mathrm{sign}(s)) = -\mu_1\,\mathrm{sign}(sv) - \mu_2 \tag{5.7}$$

Substituting (5.7) into (5.6), the closed-loop system in the new coordinate frame can be obtained as follows:

$$\begin{aligned} \dot{s} &= R\,v \\ \dot{v} &= -\mu_1\,R^{-1}\,\mathrm{sign}(v) - \mu_2\,R^{-1}\,\mathrm{sign}(s) + R^{-1}\mathrm{sign}(s)\omega(t) \end{aligned} \tag{5.8}$$

It should be noted from the definition of the transformation (5.5) that the origin $s = v = 0$ of the system (5.8) corresponds to the origin $x_1 = x_2 = 0$ of the system (5.1),(5.2). Although the transformation (5.5) is not invertible, it is important to note that one starts from the closed-loop system (5.8) and that the original dynamics can be recovered via (5.5). Due to the boundedness of the solutions in both coordinate systems (see Remark 5.2) and due to the unique correspondence between the two coordinate systems at the origin, stability analysis can be performed in the transformed coordinates. The limitation of the non-smooth transformation approach is that it holds true only when the constraint surface is of co-dimension one [23]. The unique advantage of the method is that the solutions of (5.8) are well defined in the sense of Filipov [6]. Furthermore, such a formulation admits both friction and jump phenomena, while guaranteeing existence of solution. The formulation (5.8) also captures the infinite rebounds [38] (the so-called Zeno behavior) once the system stabilizes to the origin and in turn on the constraint surface.

Remarks on non-smooth transformation

Zhuravlev [125] first proposed the non-smooth transformation (5.5) to avoid the jumps in the solutions. The concept was later utilised in [127] to study periodic solutions and stability of planar vibro-impact systems. All the theorems of this work assumed differentiability of planar vector field $f(x)$ where $\ddot{x} = f(x)$. However, the differentiability was needed by the proof of asymptotic stability and not by the definition of non-smooth transformation itself (see [127, Th. 1]).

The main idea of the transformation (5.5) is to use the mirror of the original trajectory $x_1(t)$ with the mirror placed on the constraint $x_1 = 0$ [23, Ch. 1]. The following analysis is included here from [23, Ch. 1] and from [127, Section 1] for completeness. Consider the definitions of transformation $x = |s| = s\ \mathrm{sign}(s)$ for the position and $\dot{x} = R\ v\ \mathrm{sign}(s)$ for the velocity. Let the dynamical equations of planar system $\ddot{x} = f(t, x, \dot{s}) + \sigma_{\dot{x}}(t_k)\delta_{t_k}$ be analysed in transformed coordinates, where $\sigma_{\dot{x}}(t_k)$ represents a jump in the velocity at time t_k. The definition $x = |s| = s\ \mathrm{sign}(s)$ dictates that $\dot{s} = \frac{\mathrm{d}}{\mathrm{d}t}\{s(t)\ \mathrm{sign}(s(t))\} + \sigma_x(t_j)\delta_{t_j}$, where t_j specifies an instant when the sign of $s(t)$ undergoes a change, $\sigma_{(\cdot)}$ represents a jump in the quantity, $\delta_{(\cdot)}$ denotes Dirac delta measure and $\{\dot{F}\}$ represents the derivative of any function $F(t)$ calculated ignoring the points of discontinuity and which is not defined at the points of discontinuity. From the fact that $\sigma_x(t_j^+) = s(t_j^+)\ \mathrm{sign}(s(t_j^+)) - s(t_j^-)\ \mathrm{sign}(s(t_j^-))$ and since $s(t)$ is continuous (due to the fact that $x(t)$ is), it follows that

$$\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\{s(t)\ \mathrm{sign}(s(t))\}\ \mathrm{sign}(s) \triangleq R\ v\ \mathrm{sign}(s), \tag{5.9}$$

where the definition of the new coordinate $\dot{x} \triangleq R\ v\ \mathrm{sign}(s)$ is used. Since, there is no jump in x, $\sigma_x = \sigma_s = 0$. Furthermore, $\frac{\mathrm{d}}{\mathrm{d}t}\{s\ \mathrm{sign}(s)\} = \dot{s}\ \mathrm{sign}(s)$ ignoring the points of discontinuity (i.e. $s = 0$). Hence, (5.9) produces the first equation in (5.5) by canceling $\mathrm{sign}(s)$ on both sides, namely $\dot{s} = R\ v$. As to the velocity equation, it can be obtained that

$$\ddot{x} = f(t, x, \dot{x}) + \sigma_{\dot{x}}(t_k)\delta_{t_k} = \frac{\mathrm{d}}{\mathrm{d}t}\{R\ v\ \mathrm{sign}(s)\} + \sigma_{R\ v\ \mathrm{sign}(s)}(t_j)\delta_{t_j}, \tag{5.10}$$

where t_j denotes generically an instant where $\dot{x} = R\ v\ \text{sign}(s)$ may be discontinuous. An inspection of (5.5) shows that this can occur when either of $s(t)$ or $v(t)$ crosses zero. In the case when $v(t_j) = 0$, it follows that $\sigma_{\dot{x}}(t_j) = \sigma_{R\,v\,\text{sign}(s)}(t_j) = 0$. The following can be obtained from the last term of (5.10):

$$\ddot{x} = R\ \dot{v}\ \text{sign}(s) \tag{5.11}$$

In other words, the jump $\sigma_{\dot{x}}(t_k)$ has to equal zero, since $\sigma_{R\,v\,\text{sign}(s)}(t_j)$ equals zero due to $v(j) = 0$, for the equality (5.10) to hold true. Substituting $\ddot{x} = f$ from (5.10) into (5.11) and simplifying, $\dot{v} = R^{-1}\ \text{sign}(s)\ f$ is obtained as given in the second equation in (5.5). However, if the trajectory intersects the v-axis with $v \neq 0$, then the change of sign is due to s and the jump can be computed as $\sigma_{\dot{x}}(t_j) = 2\ v(t_j)\ \text{sign}(s(t_j^+)) = \sigma_{\dot{x}}(t_k)$, where t_k and t_j coincide. The quantity $2\ v(t_j)\ \text{sign}(s(t_j^+))$ is obtained as follows. The jump occurs only when the variable s changes sign [127, Section 1]. When s changes its sign from positive to negative with $v < 0$ on $s = 0$ axis (i.e. from fourth quadrant of (s, v) plane to third quadrant) at time $t = t_j$, the following can be obtained:

$$\begin{aligned}
v(t_j^+) &= \lim_{t\to t_j^+} v(t_j) = \lim_{t\to t_j^-} v(t_j) = v(t_j^-)\\
\lim_{t\to t_j^+} \text{sign}(s(t_j)) &= -\lim_{t\to t_j^-} \text{sign}(s(t_j)) = -1\\
R(t_j^+) &= \lim_{t\to t_j^+} (1 - k\text{sign}(s(t_j^+))\text{sign}(v(t_j^+))) = 1 - k\\
R(t_j^-) &= \lim_{t\to t_j^-} (1 - k\text{sign}(s(t_j^-))\text{sign}(v(t_j^-))) = 1 + k\\
\sigma_{R\,v\,\text{sign}(s)}(t_j) &= R(t_j^+)\ v(t_j^+)\text{sign}(s(t_j^+)) - R(t_j^-)\ v(t_j^-)\text{sign}(s(t_j^-))\\
\Rightarrow \sigma_{R\,v\,\text{sign}(s)}(t_j) &= -2\,v(t_j^+) = 2\,v(t_j^+)\text{sign}(s(t_j^+))
\end{aligned} \tag{5.12}$$

The same analysis can be carried out when s changes sign from negative to positive with $v > 0$ and $\sigma_{R\,v\,\text{sign}(s)}(t_j) = 2\,v(t_j^+)\text{sign}(s(t_j^+))$ can be obtained.

The coincidence of t_k and t_j above is due to the fact that there is no jump in s or x for all times and the fact that the contact occurs on constraint $x = 0$ when s changes sign with a non-zero v. In this case $\sigma_{\dot{x}}(t_k) = \sigma_{R\,v\,\text{sign}(s)}(t_j) = 2\ v(t_j)\ \text{sign}(s(t_j^+))$ holds true and the same second equation of (5.5) is arrived due to cancelation of equal jump terms on both sides of (5.10). Substituting $\ddot{x}$ from (5.10) into (5.11) and simplifying again produces $\dot{v}$ as given in the second equation in (5.5).

Hence, starting from the transformed system in (5.5), it follows that no impact occurs if the (s, v) trajectory crosses the s-axis (i.e. $\{(s, v) : v = 0\}$). If it crosses the v-axis, then this occurs when $s = 0$ (i.e. $x = 0$, the constraint is attained), and an impact occurs with a magnitude $2\ v(t_j)\ \text{sign}(s(t_j^+))$.

Before beginning with the proof of finite time stability, a summary of the stabilization characteristics is given. The following four facts are summarised from the analysis of the previous section:

1. The point $(0,0) = (x_1, x_2)$ is the equilibrium of the dynamical system (5.1), (5.2). This is because $\dot{x}_1 = 0, \dot{x_2} = 0$ is obtained when, for example, $\omega(t) = 0$ and $(x_1(0), x_2(0)) = (0,0)$. Similarly, the point $(0,0) = (s,v)$ is the equilibrium of the dynamical system (5.6), (5.7). This is because

$$\dot{s} = R\, v = 0, \quad \dot{x_2} = R^{-1}\, \text{sign}(s)\; u(|s|, R\, v\, \text{sign}(s)) = 0$$

is obtained, for example, when $\omega(t) = 0$ and $(s(0), v(0)) = (0,0)$. Hence, the trivial solution $(0,0)$ is a unique solution and is the equilibrium point for both dynamical systems.

2. The non-smooth transformation (5.5) is such that the coordinates (s,v) cannot be retrieved from the (x_1, x_2) as it is a singular transformation. However, the point $(0,0)$ is an exception in that it is uniquely transformed from (x_1, x_2) system to the (s,v) representation and vice versa. This claim can be verified as follows: $v = 0 \Rightarrow \dot{x} = R\, v\, \text{sign}(s) = 0$. Also, the expression $\dot{x} = 0 \Rightarrow v = (R\, \text{sign}(s))^{-1}\, \dot{x} = 0$ holds true. Furthermore, expressions $x = 0 \Rightarrow s = \pm x = 0$ and $s = 0 \Rightarrow x = |s| = 0$ hold true.

3. The boundedness of (s,v) guarantees boundedness of (x_1, x_2) as noted in Remark 5.2.

4. There exists a semi-global Lyapunov function proving equiuniform finite time stability of (5.6), (5.7), thereby proving equiuniform finite time stability of (5.1), (5.2) due to the aforementioned points 1, 2 and 3.

The finite time stability results can now be presented for the transformed closed-loop system (5.8).

Lemma 5.1. *Let the dynamical system be given by (5.8). Also assume $\bar{e} \in (0,1)$, then the following is true:*

$$sign(sv)\; sign(R - R^{-1}) = -1 \tag{5.13}$$

Proof. The parameter R is defined in (5.5). For the case when $\text{sign}(sv) = -1$, R can be computed as $R = 1 - k\text{sign}(sv) = 1 + k = \frac{2}{1+\bar{e}}$. Hence $R - R^{-1} = \frac{(\bar{e}+3)(1-\bar{e})}{2(1+\bar{e})}$. Noting that $\bar{e} \in (0,1)$, it is indeed clear that $\text{sign}(R - R^{-1}) = 1$. Hence the result sign($sv$) sign($R - R^{-1}$) $= -1$. For the case when sign(sv) $= 1$, R can be computed as $R = 1 - k\text{sign}(sv) = 1 - k = \frac{2\bar{e}}{1+\bar{e}}$. Hence $R - R^{-1} = \frac{(3\bar{e}+1)(\bar{e}-1)}{2\bar{e}(1+\bar{e})}$. Noting that $\bar{e} \in (0, 1)$, it is indeed clear that sign($R - R^{-1}$) $= -1$. □

The transformed system (5.8) is not a standard double integrator system and is not a candidate for the application of existing methods [4] because R causes discontinuity in the right hand side of the first equation of (5.8). Hence, proving finite time stability is not as trivial as merely combining the existing controller and existing non-smooth transformation technique. Furthermore, finite time stability of (5.8) has never been studied.

It is of interest to note that the discontinuity and in turn Filippov's inclusion [6] in (5.8) is caused by the fact that R switches between two positive values on sets $\{(s, v) : s = 0\}, \{(s, v) : v = 0\}$ of Lebesgue measure zero. Let the two values of R be defined as follows:

$$R = \begin{cases} R_1 = \frac{2}{1+\bar{e}}, & \text{if sign}(sv) = -1; \\ R_2 = \frac{2\bar{e}}{1+\bar{e}}, & \text{if sign}(sv) = 1. \end{cases} \tag{5.14}$$

Then, it is trivial to note that given $\bar{e} \in (0, 1)$, the following is true from the computations in Lemma 5.1:

$$\begin{aligned} &R_1 > R_2 > 0\ ,\ R_1^{-1} < R_2^{-1}, |R_1 - R_1^{-1}| < |R_2 - R_2^{-1}| \\ &|R_1 - R_1^{-1}| = \frac{3 + \bar{e}}{2}|k|\ ,\ |R_2 - R_2^{-1}| = \frac{3\bar{e} + 1}{2\bar{e}}|k| \end{aligned} \tag{5.15}$$

The following is the first main result that establishes equiuniform finite time time stabilization.

Theorem 5.1. *Given $M = 0$, the impact system (5.1), (5.2) and its transformed version (5.6), (5.7) are globally equiuniformly finite time stable.*

Proof. Lyapunov stability analysis can be performed in the transformed coordinates since both the set of expressions (5.1), (5.2) and (5.6), (5.7) represent the same system. Let a Lyapunov function candidate be given as follows:

$$V(s, v) = \mu_2\, |s| + \frac{1}{2}\, v^2 \tag{5.16}$$

By computing the temporal derivative of this function along the system trajectories in (5.6), (5.7) with $M = 0$, it is obtained that,

$$\dot{V} \leq \mu_2|v|\, |R - R^{-1}|\text{sign}(sv)\, \text{sign}(R - R^{-1}) - \mu_1 R^{-1}|v| \tag{5.17}$$

From Lemma 5.1, equation (5.17) can be simplified as,

$$\dot{V} \le -\mu_2|v|\,|R - R^{-1}| - \mu_1\, R^{-1}\,|v| \tag{5.18}$$

It can be verified that $R^{-1} > 0$ for either sign of sign($s\,v$) since $\bar{e} \in (0,1)$. Since the equilibrium point $s = v = 0$ is the only trajectory of (5.8) on the invariance manifold $v = 0$ where $\dot{V}(s,v) = 0$, the differential inclusion (5.6), (5.7) is globally uniformly asymptotically stable by applying the invariance principle [128], [129]. Moreover, the system described in (5.6), (5.7) is globally homogeneous of the negative degree $q = -1$ with respect to dilation $r = (2,1)$ and is globally uniformly finite time stable according to [9, Theorem 3.1]. □

It can be observed that the Lyapunov function $V(s,v)$ defined in (5.16) is inspired from and is similar in structure to that proposed in [9, Th. 4.1]. However, Lemma 5.1 is needed to prove the negative definiteness of $\dot{V}$ appearing in (5.17) since the right hand side of (5.17) involves the switching element $R - R^{-1}$. The closed loop system (5.8) is a globally homogeneous system if $\omega(t) = 0\ \forall t \ge 0$. Consider next the case when M takes a nonzero value. The discontinuous control law (5.7) can reject any disturbance ω with a uniform upper bound (5.3). Thus, the following result can be stated.

Theorem 5.2. *The closed-loop impact system (5.1), (5.2) and its transformed version (5.6), (5.7) are globally equiuniformly finite time stable, regardless of whichever disturbance ω, satisfying condition (5.3) with $M < \mu_1 < \mu_2 - M$, affects the system.*

Proof. The proof is constructed in several steps.
1. Global Asymptotic Stability
Under the conditions (5.3), (5.4) of this theorem, the time derivative of the Lyapunov function (5.16), computed along the trajectories of (5.6), (5.7) is estimated as follows:

$$\begin{aligned}\dot{V} &= \mu_2|v|\,|R - R^{-1}|\text{sign}(sv)\,\text{sign}(R - R^{-1}) - \mu_1\, R^{-1}\,|v| + R^{-1}\,|v|\text{sign}(sv)\,\omega \\ &\le -\mu_2|v|\,|R - R^{-1}| - (\mu_1 - M)R^{-1}|v|\end{aligned} \tag{5.19}$$

The first term in the last inequality follows from Lemma 5.1. Since $M < \mu_1$ by conditions (5.3), (5.4) of this theorem, the global asymptotic stability of (5.6), (5.7) is then established by applying the invariance principle [128], [129].
2. Semiglobal Strong Lyapunov Functions

The goal of this step is to show the existence of a parameterized family of local Lyapunov functions $V_{\tilde{R}}(s,v), \tilde{R} > 0$ such that each $V_{\tilde{R}}(s,v)$ is well-posed on the corresponding compact set

$$D_{\tilde{R}} = \{(s,v) \in \mathbb{R}^2 : V(s,v) \leq \tilde{R}\}. \tag{5.20}$$

In other words, $V_{\tilde{R}}(s,v)$ is to be positive definite on $D_{\tilde{R}}$ and its derivative, computed along the trajectories of the uncertain system (5.6), (5.7) with initial conditions within $D_{\tilde{R}}$, is to be negative definite in the sense that,

$$\dot{V}_{\tilde{R}}(s,v) \leq -W_{\tilde{R}}(s,v) \tag{5.21}$$

for all $(s,v) \in D_{\tilde{R}}$ and some $W_{\tilde{R}}(s,v)$, positive definite on $D_{\tilde{R}}$. A parameterized family of Lyapunov functions $V_{\tilde{R}}(s,v), \tilde{R} > 0$, with the properties defined above are constructed by combining the Lyapunov function V of (5.16), whose time derivative along the system motion is only negative semi-definite, with the indefinite function $U(s,v) = s\,v$:

$$V_{\tilde{R}}(s,v) = V(s,v) + \kappa_{\tilde{R}} U(s,v) = \mu_2\,|s| + \frac{1}{2}v^2 + \kappa_{\tilde{R}}\,s\,v \tag{5.22}$$

where the weight parameter $\kappa_{\tilde{R}}$ is chosen small enough namely,

$$\kappa_{\tilde{R}} < min\left\{1,\ \frac{2{\mu_2}^2}{\tilde{R}},\ \frac{\mu_2|R_1 - R_1^{-1}| + R_1^{-1}(\mu_1 - M)}{R_1\sqrt{2\tilde{R}}}\right\} \tag{5.23}$$

and R_1 is defined in (5.14). It can be noted from (5.20) that the following inequalities hold true:

$$|s| \leq \tfrac{\tilde{R}}{\mu_2}, \quad |v| \leq \sqrt{2\tilde{R}}. \tag{5.24}$$

Hence, the Lyapunov function (5.22) is positive definite on compact set (5.20) for all $(s,v) \in D_{\tilde{R}} \backslash \{0,0\}$ and $\kappa_{\tilde{R}} > 0$ satisfying (5.23) as shown below:

$$\begin{aligned} V_{\tilde{R}}(s,v) &= \mu_2\,|s| + \frac{1}{2}\,v^2 + \kappa_{\tilde{R}}\,s\,v \geq \mu_2|s| + \frac{1}{2}\,v^2 - \frac{1}{2}\kappa_{\tilde{R}}s^2 - \frac{1}{2}\kappa_{\tilde{R}}\,v^2 \\ &\geq \left(\mu_2 - \frac{\kappa_{\tilde{R}}\tilde{R}}{2\mu_2}\right)|s| + \frac{1}{2}(1 - \kappa_{\tilde{R}})\,v^2 \quad > 0 \end{aligned} \tag{5.25}$$

The time derivative of the indefinite function $U(s,v)$ along the trajectories of the uncertain system (5.6), (5.7) is obtained as follows:

$$\begin{aligned}\dot{U}(s,v) &= R\,v^2 + s\left(-\mu_1\, R^{-1}\,\text{sign}(v) - \mu_2\, R^{-1}\ \text{sign}(s) + R^{-1}\, sign(s)\omega\right)\\ &= R\,v^2 - \mu_1 R^{-1}|s|\text{sign}(sv) - \mu_2\, R^{-1}\ |s| + R^{-1}|s|\ \omega \\ &\le R\,v^2 - R^{-1}|s|\,(\mu_2 - \mu_1 - M)\end{aligned} \tag{5.26}$$

Then, by combining (5.19) and (5.26) the time derivative of (5.22) can be obtained as follows:

$$\dot{V}_{\tilde{R}} \le -\mu_2|v|\ |R - R^{-1}|\ - (\mu_1 - M)\ R^{-1}\ |v| + \kappa_{\tilde{R}} R\,v^2 - \kappa_{\tilde{R}} R^{-1}|s|\,(\mu_2 - \mu_1 - M) \tag{5.27}$$

The parameter R in (5.27) is a state function and keeps switching between the two values as shown in Lemma 5.1. This corresponds to the fact that the rate of decay of the Lyapunov function (5.25) switches depending on R. Considering the slowest decay, a conservative estimate of the upper bound (5.27) can be readily obtained using Lemma 5.1 and Eq.(5.15) as follows:

$$\dot{V}_{\tilde{R}} \le -\mu_2|v|\ |R_1 - R_1^{-1}|\ - (\mu_1 - M)R_1^{-1}|v| + \kappa_{\tilde{R}} R_1\, v^2 - \kappa_{\tilde{R}} R_1^{-1}|s|\,(\mu_2 - \mu_1 - M) \tag{5.28}$$

Noting that, due to (5.19), all possible solutions of the uncertain system (5.6), (5.7), initialized at $t_0 \in \mathbb{R}$ within the compact set (5.20), are a priori estimated by

$$\sup_{t\in[t_0,\infty)} V(s,v) \le \tilde{R}, \tag{5.29}$$

and that (5.24) holds true within the compact set (5.20), (5.28) can be re-written as follows:

$$\begin{aligned}\dot{V}_{\tilde{R}} \le &-\left(\mu_2\,|R_1 - R_1^{-1}|\ + (\mu_1 - M)\,R_1^{-1} - \kappa_{\tilde{R}} R_1\sqrt{2\tilde{R}}\right)|v|\\ &- \kappa_{\tilde{R}} R_1^{-1}(\mu_2 - \mu_1 - M)|s| \le -c_{\tilde{R}}\,[|s| + |v|]\end{aligned} \tag{5.30}$$

where

$$c_{\tilde{R}} = \min\left\{\ \kappa_{\tilde{R}} R_1^{-1}(\mu_2 - \mu_1 - M),\ \mu_2|R_1 - R_1^{-1}| + (\mu_1 - M)R_1^{-1} - \kappa_{\tilde{R}} R_1\sqrt{2\tilde{R}}\ \right\} \tag{5.31}$$

It follows from (5.23) that $c_{\tilde{R}} > 0$. Hence (5.30) results in

$$\dot{V}_{\tilde{R}} \le -K_{\tilde{R}} V_{\tilde{R}}(s,v) \tag{5.32}$$

where

$$K_{\tilde{R}} = c_{\tilde{R}} \left[\max \left\{ \frac{2\mu_2^2 + \kappa_{\tilde{R}} \tilde{R}}{2\mu_2}, \sqrt{\frac{\tilde{R}}{2}}(1 + \kappa_{\tilde{R}}) \right\} \right]^{-1} > 0$$

and the upper estimate

$$V_{\tilde{R}} \leq \frac{2\mu_2^2 + \kappa_{\tilde{R}} \tilde{R}}{2\mu_2} |s| + \sqrt{\frac{\tilde{R}}{2}}(1 + \kappa_{\tilde{R}})|v|$$

of the Lyapunov function (5.25) on compact set (5.20) has been used. Hence the desired uniform negative definiteness (5.21) is obtained with $W_{\tilde{R}}(s,v) = K_{\tilde{R}} V_{\tilde{R}}(s,v)$.

3. Global Equiuniform Asymptotic Stability

Since the inequality (5.32) holds on the solutions of the uncertain system (5.6), (5.7), initialized within the compact set (5.20), the function $V_{\tilde{R}}(s,v)$ decays exponentially

$$V_{\tilde{R}}(s(t), v(t)) \leq V_{\tilde{R}}(s(t_0), v(t_0)) e^{-K_{\tilde{R}}(t-t_0)} \tag{5.33}$$

on these solutions with decay rate $K_{\tilde{R}}$ which depends on the gain parameters μ_1, μ_2, bound M on disturbance ω and the system property R_1. On the compact set (5.20), the following inequality holds (see (5.25)):

$$L_{\tilde{R}} V(s,v) \leq V_{\tilde{R}}(s,v) \leq M_{\tilde{R}} V(s,v) \tag{5.34}$$

for all $(s,v) \in D_{\tilde{R}}$ and positive constants $L_{\tilde{R}}, M_{\tilde{R}}$, satisfying

$$L_{\tilde{R}} < \min \left\{ \frac{2\mu_2^2 - \tilde{R}\kappa_{\tilde{R}}}{2\mu_2^2}, 1 - \kappa_{\tilde{R}} \right\}, \quad M_{\tilde{R}} > \max \left\{ \frac{2\mu_2^2 + \tilde{R}\kappa_{\tilde{R}}}{2\mu_2^2}, 1 + \kappa_{\tilde{R}} \right\} \tag{5.35}$$

The above inequalities (5.33) and (5.34) ensure that the function $V(s,v)$ decays exponentially

$$V(s(t), v(t)) \leq L_{\tilde{R}}^{-1} M_{\tilde{R}} V(s(t_0), v(t_0)) e^{-K_{\tilde{R}}(t-t_0)} \leq L_{\tilde{R}}^{-1} M_{\tilde{R}} \tilde{R}\, e^{-K_{\tilde{R}}(t-t_0)} \tag{5.36}$$

on the solutions of (5.6), (5.7) uniformly in ω and the initial data, located within an arbitrarily large set (5.20). This proves that the uncertain system (5.6), (5.7) is globally equiuniformly asymptotically stable around the origin $(s,v) = (0,0)$.

4. Global Equiuniform Finite Time Stability.

Due to (5.4), the piece-wise continuous [6], [9] uncertainty $R_1^{-1}\omega(t)sign(s)$ in the right hand side of the system (5.6), (5.7) is locally uniformly bounded by $R_1^{-1}M$ whereas the remaining part of the feedback is globally homogeneous with homogeneity degree $q = -1$ with respect to dilation $r = (r_1, r_2) = (2, 1)$. Noting that $q + r_2 \leq 0$, the globally equiuniformly asymptotically stable system (5.6), (5.7) and in turn the original impact system (5.1), (5.2) are globally equiuniformly finite time stable according to [9, Theorem 3.2]. □

Remark 5.1. Given the bound M on the uncertainty ω and fixed values of μ_1, μ_2, an arbitrarily large $\tilde{R}$ can be chosen such that the expression $V(s(t_0), v(t_0)) \leq \tilde{R}$ holds true. In other words, global finite time stability follows from Theorem 5.2 as an arbitrarily large $\tilde{R}$ always exists. The only restriction of the whole formulation is that $\tilde{R}$ should be finite [9] (the initial condition set (5.20) must be known). In the context of control of mechanical systems such as biped robots, the initial condition region is known to be finite and hence the aforementioned condition is not a major restriction. Furthermore, the scalar κ is inversely proportional to the parameter $\tilde{R}$.

Remark 5.2. The singularity of the transformation (5.5) means that the initial conditions s^0, v^0 cannot be retrieved from the actual initial conditions x_1^0, x_2^0 [23]. It is interesting to note that this is not a limitation. It can be proved using the following analysis. Within the compact set (5.20), the following is obtained by utilizing the inequalities (5.24):

$$|x_1| = |s| \leq \frac{\tilde{R}}{\mu_2}, \quad |x_2| = |Rv\text{sign}(s)| = |\sqrt{R^2v^2}| \leq R\sqrt{2\tilde{R}} \tag{5.37}$$

Hence, despite the singularity of the transformation (5.5), the settling time estimate as well as the tuning guidelines (developed in the following sections) based on the transformed system are applicable to the original system. The reason is twofold. Firstly, the mapping from (s, v) to (x_1, x_2) or vice versa is unique at the origin. Secondly, as shown by (5.37), while the transformed system coordinates (s, v) settle from their respective upper bounds $(\frac{\tilde{R}}{\mu_2}, \sqrt{2\tilde{R}})$ to the origin due to finite time stabilization, the original system coordinates (x_1, x_2) settle from their respective upper bounds $(\frac{\tilde{R}}{\mu_2}, R\sqrt{2\tilde{R}})$ to the origin.

Remark 5.3. The results obtained in this section are applicable to second order feedback linearizable non-linear systems with relative degree 2 [75]. Let the nonlinear system be given as follows:

$$\dot{x} = f(x) + g(x)u \tag{5.38a}$$

$$h(x) = x_1 \tag{5.38b}$$

$$\dot{x}(t_k^+) = \bar{e}\dot{x}(t_k^-) \qquad \text{if } \dot{x} < 0, x = 0 \tag{5.38c}$$

where, $u \in \mathbb{R}$ is control input, $x \in \mathbb{R}^2$ is state vector, $h(x)$ is output and the Lie-derivative $\mathcal{L}_g\mathcal{L}_f^{-1}h(x)$ takes non-zero values due to the assumption of feedback linearisability [75]. Such systems may occur, for example, in the field of robotics where $\mathcal{L}_g\mathcal{L}_f^{-1}h(x)$ can take non-zero values. Also, full-state feedback control synthesis is possible as it is realistic to assume the availability of both position and velocity in the case of a broad class of mechanical systems. It is well-known [75] that the control law

$$u(x) = \frac{1}{\mathcal{L}_g\mathcal{L}_f^{-1}h(x)}\left(-\mathcal{L}_f{}^2h(x) + \nu(y)\right) \tag{5.39}$$

transforms the continuous part ($t \notin \{t_k\}$) of the dynamics of system (5.38) into the following double integrator:

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= \nu(y_1, y_2) \end{aligned} \tag{5.40}$$

It remains to check how the impact map (5.38c) is affected in the process of transformation. It should be noted that the new coordinates are given as $y_1 = h(x) = x$ and $y_2 = \mathcal{L}_f h(x) = \dot{x}$. Hence the same impact map as that given by (5.38c) holds true and the control law (5.39) with $\nu(y) = -\mu_1\text{sign}(y_2) - \mu_2\text{sign}(y_1)$ results in the feedback-linearized closed-loop system

$$\begin{aligned} \dot{y}_1 &= y_2 & & \\ \dot{y}_2 &= -\mu_1 \,\text{sign}(y_2) - \mu_2 \,\text{sign}(y_1) & & \text{if } y_1 \neq 0 \\ \dot{y}_2(t_k^+) &= \bar{e}\, \dot{y}_2(t_k^-) & & \text{if } y_2 < 0, y_1 = 0, \end{aligned} \tag{5.41}$$

which is the same as that described by (5.1), (5.2) thereby proving finite time stability of the system (5.41) and in turn finite time stability of a class of nonlinear systems (5.38) in the presence of jumps in the velocity.

5.5 Settling time estimate

A finite upper bound on the settling time of the closed-loop system (5.6), (5.7) is computed in this section. This section utilises the same geometric analysis as that carried out in Section 4.3. As described in section 5.3, the possibility to perform a similar geometric analysis for the transformed system highlights the importance of the non-smooth transformation and that of the finite time stability tools developed in the previous chapters (see section 4.3) to establish similar results for the systems with resets.

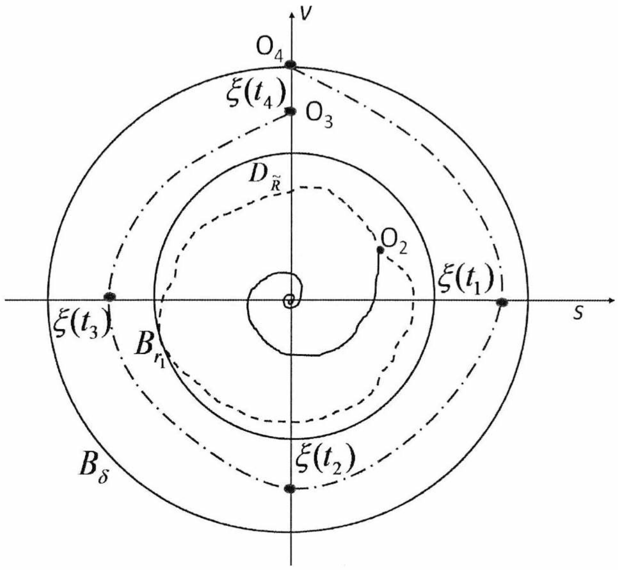

FIGURE 5.1: Schematic of finite settling time behavior of the transformed system (5.6),(5.7)

The concept is graphically depicted in Figure 5.1. When the trajectories are initialized on the positive vertical semi-axis $e_1^+ = \{x \in \mathbb{R}^2 : x_1 = 0, x_2 > 0\}$ at O_4, the factor by which it gets closer to the origin after one revolution can be computed. The value of the intercept (point O_3) on the positive vertical semi-axis after one revolution should be greater than the radius r_1 of the ball B_{r_1} containing the level set $D_{\tilde{R}}$ defined in (5.20) (see Section 5.5.1 for the definition of r_1 to render the relation $D_{\tilde{R}} \subset B_{r_1}$ to hold true).

The choice of δ, such that $\delta\Psi > r_1$ where Ψ is the factor by which the trajectory gets closer to the origin after one revolution at point O_3, will ensure that the settling time estimate will be more conservative than the one computed with the initialization on the level set $D_{\tilde{R}}$ (point O_2). The motivation for such a choice of initialization of the trajectories on the ball B_δ stems from the fact that the trajectory, containing O_2 on the level set $D_{\tilde{R}}$, starting from any arbitrary point below O_4 (see Figure 5.1) on the e_1^+ axis cannot intersect the trajectory starting from the point O_4. The basis for this is the fact that the solutions of (5.8) are unique everywhere. In fact, the solution does not remain on the axes $s = 0, v = 0$ for finite time and always crosses the axes except at the origin [9, Th. 4.1]. Hence, different trajectories have no intersections because otherwise they would coincide with each other outside the origin due to the uniqueness of the solution.

The approach utilized in the following is a two step process. Firstly, a comparison system [120], the trajectory of which encompasses the actual system, is defined as shown in Figure 5.1 (see the early work on the majorant curves [4] for the twisting controller; also see Remark 4.5). Secondly, the comparison system is then initialized on the positive vertical semi-axis e_1^+ with the coordinates $(0, \delta)$. Then the finite settling time is computed for the comparison system subject to the condition $\delta\Psi > r_1$.

5.5.1 Definition of the radius r_1 such that $D_{\tilde{R}} \subset B_{r_1}$

As described in the first paragraph of Section 5.5, it is necessary for the computation of δ to compute the radius r_1 such that the expression $D_{\tilde{R}} \subset B_{r_1}$ holds true. The following is required:

$$\frac{\mu_2|s|}{\tilde{R}} + \frac{v^2}{2\tilde{R}} \leq 1 \Rightarrow \left(\frac{s}{r_1}\right)^2 + \left(\frac{v}{r_1}\right)^2 \leq 1 \tag{5.42}$$

Impose the following inequalities:

$$\left(\frac{s}{r_1}\right)^2 \leq \frac{\mu_2|s|}{\tilde{R}}, \quad \left(\frac{v}{r_1}\right)^2 \leq \frac{v^2}{2\tilde{R}} \tag{5.43}$$

Then the expression $(s, v) \in B_{r_1}$ holds true for every given point $(s, v) \in D_{\tilde{R}}$ in the (s, v) state space. Note that the following always holds true for all $(s, v) \in D_{\tilde{R}}$:

$$|s| \leq \frac{\tilde{R}}{\mu_2} \tag{5.44}$$

The first inequality of (5.43) can be simplified as follows:

$$|s|\left(\frac{1}{r_1}\right)^2 \leq \frac{\mu_2}{\tilde{R}} \tag{5.45}$$

Utilizing the relationship (5.44), the following more conservative requirement can be formulated from (5.45):

$$|s|\left(\frac{1}{r_1}\right)^2 \leq \frac{\tilde{R}}{\mu_2}\left(\frac{1}{r_1}\right)^2 \leq \frac{\mu_2}{\tilde{R}} \tag{5.46}$$

Hence, the upper-bound $r_1 \geq \frac{\tilde{R}}{\mu_2}$ on r_1, obtained from (5.46), suffices to satisfy the first inequality of (5.43).

Similarly, the second inequality of (5.43) leads to $r_1 \geq \sqrt{2\tilde{R}}$, combining which with $r_1 \geq \frac{\tilde{R}}{\mu_2}$, the following estimate of the parameter r_1 is obtained:

$$r_1 = \max\left\{\frac{\tilde{R}}{\mu_2}, \sqrt{2\tilde{R}}\right\} \tag{5.47}$$

5.5.2 Finite Settling time

Let the right hand side of (5.8) be written as follows:

$$\begin{aligned} \phi_1 &= R\, v, \\ \phi_2 &= -\mu_1 R^{-1}\text{sign}(v) - \mu_2 R^{-1}\text{sign}(s) + R^{-1}\,\text{sign}(s)\omega(t) \end{aligned} \tag{5.48}$$

Let a comparison system corresponding to (5.8) be given as follows:

$$\begin{aligned} \dot{s} &= R\, v \\ \dot{v} &= -\left[\mu_1 - M\right] R^{-1}\text{sign}(v) - \mu_2 R^{-1}\text{sign}(s) \end{aligned} \tag{5.49}$$

In turn, the right hand side

$$\begin{aligned} \phi_1^c &= R\, v \\ \phi_2^c &= -\left[\mu_1 - M\right] R^{-1}\text{sign}(v) + \mu_2 R^{-1}\text{sign}(s) \end{aligned} \tag{5.50}$$

of the comparison system (5.49) relates to (5.48) as

$$\begin{aligned} \phi_1 &= \phi_1^c \\ \phi_2 &= \phi_2^c + \Delta\phi \end{aligned} \tag{5.51}$$

where

$$\Delta\phi = -MR^{-1}\,\text{sign}(v) + R^{-1}\text{sign}(s)\omega. \tag{5.52}$$

It is trivial to note that,

$$\Delta\phi \begin{cases} \leq 0, & \text{if } (s,v) \in (G_1 \cup G_4); \\ \geq 0, & \text{if } (s,v) \in (G_2 \cup G_3). \end{cases} \tag{5.53}$$

where

$$\begin{aligned} G_1 &= \{(s,v) : s > 0, v > 0\}, \quad G_2 = \{(s,v) : s > 0, v < 0\} \\ G_3 &= \{(s,v) : s < 0, v < 0\}, \quad G_4 = \{(s,v) : s < 0, v > 0\} \end{aligned} \tag{5.54}$$

By virtue of (5.51), (5.53), the motion of the plant (5.8) is dominated by that of (5.49) subject to the same initial condition. In other words the solutions $(s(t), v(t))$ of the system (5.8) and the solutions $(s^c(t), v^c(t))$ of the comparison system (5.49) rotate around the origin and in each region $G_i, i = 1,2,3,4$ the plot of the trajectory $(s(t), v(t))$ is bounded by the plot of the trajectory $(s^c(t), v^c(t))$ and the switching lines $s = 0, v = 0$. Hence it suffices for the purpose of estimating the finite settling time to consider system

(5.49) which can be represented in the matrix vector notation as follows:

$$\dot{\zeta}(t) = A\,\zeta(t) + B\,u, \tag{5.55}$$

where

$$\zeta = \begin{bmatrix} s & v \end{bmatrix}^{\mathrm{T}}, A = \begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5.56}$$

and

$$u = -(\mu_1 - M)\,\mathrm{sign}(v)R^{-1} - \mu_2\,\mathrm{sign}(s)\,R^{-1}\,. \tag{5.57}$$

The motion in the state space can be obtained using the convolution integral as follows:

$$\zeta(t) = e^{A\,t}\,\zeta^0 + \int_0^t e^{A\,(t-\tau)}\;B\,u\quad \mathrm{d}\tau \tag{5.58}$$

$\zeta^0 = [s(t_0) \quad v(t_0)]^{\mathrm{T}}$ is initial condition. It should be noted that R is a time varying discontinuous function of state ζ. The integral (5.58) holds for time-invariant systems. However, this integral can be used for defining the solutions $\zeta(t)$ of the time varying system (5.55) in each quadrant $G_i, i = 1,2,3,4$ defined in (5.54) since R remains constant in a given quadrant. Hence, the same procedure can be repeated as that utilised in Section 4.3. Since the control switches on the axes $\zeta_1 = s = 0, \zeta_2 = v = 0$, the integral (5.55) is required to be computed in each quadrant utilizing Bu as follows:

$$B\,u = \begin{bmatrix} 0 \\ -(\mu_1 - M)\,\mathrm{sign}(v)\,R^{-1} - \mu_2\,\mathrm{sign}(s)\,R^{-1} \end{bmatrix} \tag{5.59}$$

It is noted that using such integrals to define the solutions of the comparison system (5.49) is mathematically correct as the control law never generates a sliding mode on the switching lines $\zeta_1 = 0$ and $\zeta_2 = 0$. Hence the solution always crosses these switching lines except at the origin [4], [120]. The matrix exponential in (5.58) can be computed as follows:

$$e^{A\,t} = I + At + \frac{At^2}{2!} + \ldots \tag{5.60}$$

Since $A^n = 0, \forall n \geq 2$, (5.60) leads to the following:

$$\begin{aligned} e^{A\,t} &= I + At = \begin{bmatrix} 1 & Rt \\ 0 & 1 \end{bmatrix}, \\ \int_0^t e^{-A\tau} d\tau &= \begin{bmatrix} t & \frac{-Rt^2}{2} \\ 0 & t \end{bmatrix}. \end{aligned} \tag{5.61}$$

Utilizing (5.58), (5.59) and (5.61), the following can be obtained (see Section 4.3):

$$t_1 = \frac{\delta R}{\mu_2(1+\eta)}, \quad t_2 = \frac{\delta R}{\mu_2\sqrt{(1+\eta)(1-\eta)}}, \quad t_3 = \frac{\delta R\sqrt{1-\eta}}{\mu_2(1+\eta)\sqrt{1+\eta}}, \quad t_4 = t_1 \tag{5.62}$$

where $\eta = \frac{\mu_1 - M}{\mu_2}$ and t_1 is time taken by the trajectory to travel from the semi-axis $\{\zeta \in \mathbb{R}^2 : \zeta_1 = 0, \zeta_2 > 0\}$ to the semi-axis $\{\zeta \in \mathbb{R}^2 : \zeta_1 > 0, \zeta_2 = 0\}$ and so on. Furthermore, the interception of the trajectory on the positive and negative semi-axes can be obtained as follows:

$$\begin{aligned} \zeta_1(t_1) &= \frac{(\delta R)^2}{2\mu_2(1+\eta)}, & \zeta_2(t_2) &= -\frac{\delta\sqrt{1-\eta}}{\sqrt{1+\eta}}, \\ \zeta_1(t_3) &= -\frac{(R\delta)^2(1-\eta)}{2\mu_2(1+\eta)^2}, & \zeta_2(t_4) &= \frac{\delta(1-\eta)}{1+\eta}, \end{aligned} \tag{5.63}$$

where the intercepts $\zeta_1(t_1), \zeta_2(t_2), \zeta_1(t_3), \zeta_2(t_4)$ are depicted in the Figure 5.1. Hence the time T_1 taken by the trajectory to travel from the point O_4 on the ball B_δ to some point O_3 on the semi-axis e_1^+ is obtained using (5.62) as follows:

$$T_1 = t_1 + t_2 + t_3 + t_4 = \frac{R\Delta}{\mu_2}\delta \leq \frac{R_1\Delta}{\mu_2}\delta \tag{5.64}$$

where $\Delta = \frac{2}{1+\eta} + \frac{1}{\sqrt{(1+\eta)(1-\eta)}} + \frac{\sqrt{1-\eta}}{(1+\eta)\sqrt{1+\eta}}$. It can be seen that the time T_1 taken by one revolution depends on the initial condition δ, gain parameters (μ_1, μ_2), system property R_1 and the bound M on the uncertainty. Hence the time T_1 and time taken by the subsequent revolutions can be computed *apriori*. Furthermore, as shown by the last equality of (5.63), the closed-loop trajectory decays closer to the origin by a factor Ψ of the initial condition δ where $\Psi = \frac{1-\eta}{1+\eta} < 1$. A similar computation can be repeated with the initial condition set at $\zeta_2(t_4)$ to obtain the next intersection of the trajectory with the semi-axis $e_1^+ = 0$ at the end of the second revolution, namely $\zeta_2(T_2) = \zeta_2(t_4)\,\Psi = \delta\Psi^2$, where T_2 is the time at which the second revolution is completed. Noting that one revolution takes $\frac{R\Delta}{\mu_2}$ multiplied by the initial value on the vertical axis (see (5.64)), the total time taken for two revolutions is estimated as follows:

$$T_2 = T_1 + \frac{R\Delta}{\mu_2}\zeta_2(t_4) \leq \frac{R_1\Delta}{\mu_2}\delta\left[1 + \Psi\right] \tag{5.65}$$

where the last equality of (5.63) and the right hand side of (5.64) are utilized. Noting that the number of revolutions $n \to \infty$ as time $t \to \infty$, above steps can be repeated and the following generalization for the n^{th} revolution can be obtained:

$$\lim_{t\to\infty} T_n = \lim_{n\to\infty} T_n \leq \lim_{n\to\infty} \frac{R_1\Delta}{\mu_2}\delta\left[1 + \Psi + \Psi^2 + \ldots + \Psi^{n-1}\right] \tag{5.66}$$

Noting that the inequality $0 < \Psi < 1$ always holds true, the infinite series in (5.66) can be represented by a convergent geometric series. In turn, the upper-bound on the settling time $\mathcal{T}_s$ of the system (5.8) can be obtained *apriori* as follows:

$$\mathcal{T}_s = \lim_{n\to\infty} T_n \le \lim_{n\to\infty} \frac{R_1\Delta}{\mu_2}\delta\left[\frac{1-\Psi^n}{1-\Psi}\right] \le \frac{R_1\Delta\delta}{\mu_2(1-\Psi)} < \infty \tag{5.67}$$

It can be seen from (5.66) and (5.67) that the result on switched planar uncertain systems developed Section 4.3 is proven to hold true for a system with friction and collision terms on the right hand side.

5.6 Numerical simulation

This section presents numerical simulation results illustrating finite time stabilization of unilaterally constrained planar systems. The numerical simulation result is presented in Figure 5.2 which gives a comparison between the system (5.1), (5.2) with $\mu_1 = 1, \mu_2 = 2, M = 0.5, \bar{e} = 0.9$ and the transformed system (5.8). Appropriate initial conditions $x_1(t_0) = 2, x_2(t_0) = 1$ and $s(t_0) = 2, v(t_0) = [1-k]^{-1}$ are used. The jump in velocity occurs when s changes sign [23]. The simulation is carried out using the event based Runge-Kutta method and it is inherently prone to exhibit departure from the physical behaviour for both the discontinuities in the system (5.1), (5.2), namely, the 'sign' function and the jump. The system settles in less than 7 *sec* which is less than the upper-bound 23.6865 *sec* computed using (5.67). The phase plane plot of the jump system can be seen in Figure 5.3 and that of (5.8) containing no impacts can be seen in Figure 5.4 for the aforementioned initial conditions. It is easily observed that the phase plane plot in Figure 5.4 closely resembles the phase plot of the twisting controller [4, 9].

5.7 Some open problems

The results in this chapter have assumed an idealistic impact model where the reset in velocity occurs instantaneously. An interesting and challenging open problem is to study finite time stabilization in the presence of a slip as in the case of a border collision bifurcation at the time of impact. Within the framework of *grazing bifurcations*, such impact systems are said to have *grazing velocities* [23, Problem 5.2], [130, 131] and are more formally studied within mechanical engineering where the trajectory of the system comes in contact with the constraint surface tangentially [130] or with zero velocity [131]. Although, this is an active research area of non-smooth mechanics, the investigation of whether finite time stabilization can play a significant role or not is a

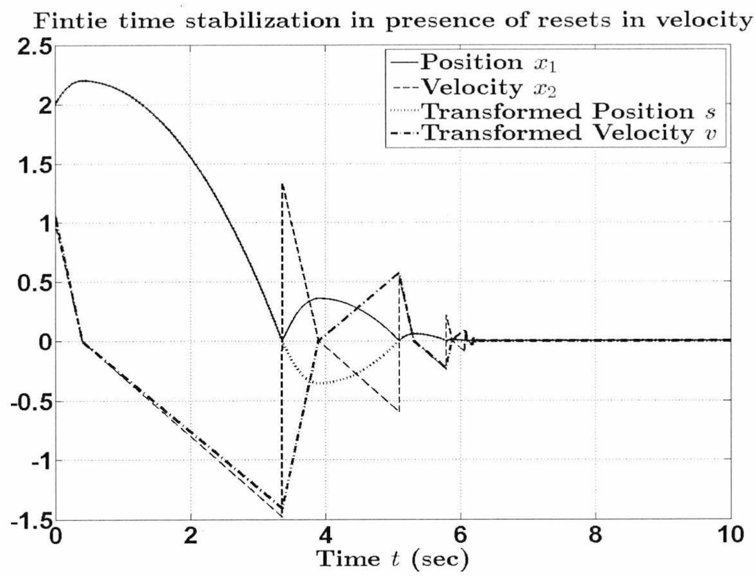

FIGURE 5.2: Finite time stabilization of the unilaterally constrained system (5.1), (5.2) and that of its transformed version (5.8).

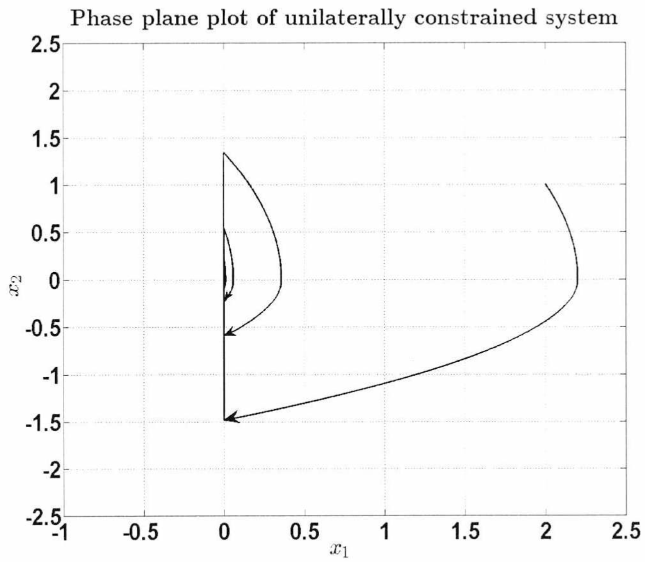

FIGURE 5.3: Phase plane plot of the unilaterally constrained system (5.1), (5.2).

practically relevant topic since it has direct applications in terms of both behavioural and stability analysis of advanced hybrid mechanical systems such as biped robots [130] and that of most commonly used systems such as rotating shafts with bearings [132].

Another open problem is extending the proof of finite time stability when measurement of only the position is available and an observer is constructed to estimate velocity, which in turn is to be utilised by output feedback. There are two theoretical

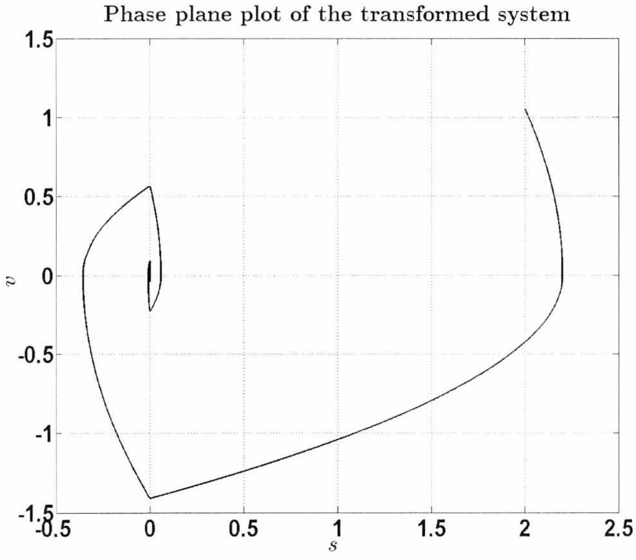

FIGURE 5.4: Phase plane plot of the system (5.8) obtained via non-smooth transformation.

challenges here. First, an observer for the velocity has to be finite time stable since the corresponding finite time state feedback controller needs the 'true state' information for stabilizing the system in finite time even when there are no impacts. Furthermore, it is required that the observer stabilizes faster than the controller otherwise the overall system can lead to instability. Such an output feedback scenario when there are no impacts has been studied, for example, in the reference [46] in the case of continuous homogeneous systems and in [14, 42] in the case of discontinuous systems amongst others. This requirement is similar to the classical separation principle and has always been an area of investigation when an observed state is used by the control synthesis. It will be shown in Chapter 7, which incorporates the revised results of a recent publication [60], that the system trajectory can escape to instability unless it is ensured that the observer settles at the origin quicker than the controller. Second challenge is that the observer also has to 'see through' the Zeno mode arising due to the accumulation of impacts. As the impacts go on accumulating near Zeno motion, the time interval between two successive impacts goes to zero with a finite limit. When there is no uncertainty in the impact parameter $\bar{e}$, an impacting observer may be constructed. Even in this ideal case, impact times have to be known precisely for introducing a reset in the observer. However, when the parameter $\bar{e}$ is not known completely, it is difficult to nullify the post impact error in the observed and real velocities in the presence of the accumulation of impacts. This remains an open problem in the area of finite time stabilization. Two possible directions are (i) to achieve ultimate boundedness in the presence of uncertainty at the time of impact and (ii) to achieve finite time stabilization as presented in 5.2 but with the feedback (5.2) using observed velocity $\hat{x}_2$ and with the assumption that impact times are also known in addition to accurate knowledge of $\bar{e}$.

The third problem is a very natural extension of the results obtained in this chapter to the time discretization of dynamics with accumulating impacts. The system (5.1), (5.2) with accumulating impacts possesses a Zeno mode. It is well-known in the area of numerical integration of discontinuous systems that event based explicit numerical schemes are inherently prone to spurious oscillations close to the discontinuity surface [22, Section 1.2.3.1]. Such oscillations are known as 'numerical-chatter' in the area of sliding mode control [2, 3], which appears even in the absence of uncertainty. The recent results on numerical integration of variable structure systems [118, 133] using Euler's implicit method appear to offer an advanced alternative such that the discrete-time chatter is completely removed for first and second order sliding mode systems at least when there is no disturbance $\omega(t)$. Conversion of the jump system (5.1), (5.2) to a variable structure system (5.8) was key to the Lyapunov analysis presented in this chapter. It is then a very interesting problem to study whether the closed-loop variable structure system (5.8) lends itself as a valid candidate of the latest numerical schemes [118, 133], solving in the process a very challenging open problem of 'going thorough' the Zeno mode of impacts in discrete time and capturing the analytical solution numerically without recourse to any approximation such as temporal or dynamic regularization [29].

5.8 Extension to non-smooth systems

This chapter presented finite time stability of systems with reset using a second order sliding mode controller which is a variable structure controller. It builds on the frameworks of homogeneity and Lyapunov stability developed previous in chapters. Finite time stability is obtained in the presence of disturbances $\omega(t)$ with a persisting bound $|\omega(t)| \leq M_0$ in the continuous phase of motion. As discussed in Section 3.2, if the robustness requirement is relaxed to $|\omega(t)| \leq M_0|x_2|^{\alpha}$, the finite time stability of a double integrator can be established by using existing continuous finite time stabilization schemes [11, 12]. However, these methods do not ensure uniform or equiuniform finite time stability.

The previous two chapters utilised and extended the semi-global Lyapunov framework [9] extensively. Furthermore, the second order sliding mode controller utilised was a homogeneous controller [4]. It can be seen from references from the continuous finite time stabilization literature [11, 12, 14, 44–46, 115] that attaining uniform or equiuniform finite time stabilization of a perturbed double integrator is an open problem when a continuous controller is used. It is then a very interesting problem to investigate whether the homogeneity and Lyapunov frameworks developed thus far in the thesis can be extended to the non-smooth systems with continuous homogeneous right hand sides [85, 86] generated by homogeneous continuous non-Lipschitz controllers [84]. The main motivation for such an extension is in achieving *equiuniform* finite time stabilization.

Considering (i) the rich literature on the subject of continuous finite time controllers [134], (ii) the fact that it remains an active area of research [94, 135] and (iii) that uniform finite time stability is an open problem even for a perturbed double integrator (see [14] for asymtotic stability), establishing equiuniform finite time stability of non-smooth planar systems with and without impacts is an important and challenging problem. This is the topic of the next chapter which establishes uniform finite time stability not only for a perturbed double integrator (4.9) but also for the impact mechanical system (5.1) using a continuous control u.

5.9 Conclusions

Robust finite time stabilization has been considered for a unilaterally constrained perturbed double integrator. A non-smooth state transformation is employed to generate a jump-free system which is then shown to be finite time stable. The theoretical contribution of the presented work lies in achieving finite time stabilization of a class of impact mechanical systems with accumulation of impacts without the need to analyze jumps in the Lyapunov function explicitly. A finite upper-bound on the settling time is also estimated. Deriving similar results for nonlinear impact systems and deriving numerical schemes for such systems are seen as the future immediate goals. From a practical viewpoint, the results will motivate a similar development for nonlinear impact mechanical systems such as biped robots.

The next chapter extends the results obtained thus far in the thesis, which are based on geometric homogeneity and Lyapunov functions, to solve the problem discussed in Section 5.8.

CHAPTER 6

Uniform Continuous Finite Time Stabilization of Planar Systems

Finite time continuous controllers

Chapter 4 presented a brief review of Fuller's phenomenon [79]. There is no sliding motion in Fuller's time optimal controller when it is used for stabilising planar controllable systems. Instead, the decreasing geometric progression of the switching times lead to a finite convergence time and the switching curve is given by two half segments of parabolas [116]. The trajectory of a planar closed-loop system exhibits *Zeno behaviour* after reaching the origin when a time optimal controller designed using Fuller's phenomenon is employed. It is important to note that such motion at the origin is of co-dimension two for planar systems and is referred to as a second order sliding mode [4] in the sliding mode community when a class of robust feedback controllers are under study. A so-called 'sub-optimal' approach can be found in [69] for the second order sliding mode controller. A near time optimal control using sliding mode technique can be found in [136].

All the above controllers are discontinuous controllers. It is interesting to study whether finite time convergence can be attained when a continuous bounded controller is used. This was the question posed in [11]. A significant motivation for this is the desire to remove the discontinuity from the controller and avoid the high gain nature of the control at the origin. This is an interesting problem as the resulting controller is necessarily non-Lipschitz [11]. It is well-known that a continuous finite time controller has better robustness properties [11], [13, Th. 5.2] than its Lipschitz counterpart. Since the early result in [11], the topic of finite time continuous controllers has always been an active area of research as evident from the literature review presented in the next section.

This chapter studies *equiuniform* continuous finite time stabilisation. It is of interest to study if the Lyapunov and quasi-homogeneity results of [9] can be extended to achieve uniform continuous finite time stabilization of a perturbed double integrator with and without resets. The flow of this chapter is similar to previous chapters. In particular, section 6.3 is similar to Chapter 4 since it achieves *equiuniform continuous* finite time stabilisation of a perturbed double integrator such as (4.9) without resets and section 6.4 is similar to Chapter 5 since it achieves *equiuniform continuous* finite time stabilization when resets in velocity are present due to a unilateral constraint.

It is well-known that geometric homogeneity leads to finite time stability of an uncertain continuous system if it is asymptotic stable and if the degree of homogeneity is negative [84]. Due to geometric homogeneity, the same finite time stability result holds true for uncertain switched systems with negative homogeneity degree but with an additional condition of uniform asymptotic stability [9]. Both areas have been studied separately as the underlying mathematical apparatus was different. However, the result on switched systems [9] is a stronger one as it presents a control synthesis which is robust to disturbances with persisting bounds. Hence, it is a valid candidate for rejecting disturbances which are piece-wise continuous as defined by Fillipov's definition of solutions [6]. With the above background on geometric homogeneity and finite time stability, this chapter extends the discontinuous case to the continuous one. The main motivation is to prove robustness of continuous controllers for a larger class of uncertainties while also achieving uniform finite time stability, a relatively new concept, which is in its infancy in the finite time stabilisation paradigms [44] and which is not guaranteed by existing approaches [11, 12].

The proposed theoretical development considers a perturbed double integrator. An existing finite time stabilizing, continuous, bounded homogeneous controller [14], [83] is utilized. However, the result on finite time stability of switched systems [9] is employed in place of a continuous approach [13] in order to extend the class of perturbations that can be successfully suppressed in finite time. Equiuniform asymptotic stability of the closed-loop system is achieved by identifying a class of semi-global strong $\mathcal{C}^1$ Lyapunov functions. Equiuniform finite time stability then follows from the quasi-homogeneity principle which is extended for a continuous vector field. An explicit upper bound on the settling time is then computed using the homogeneity regions.

The main objectives of this chapter are (i) to achieve equiuniform continuous finite time stabilization of planar controllable systems with piece-wise continuous, nonhomogeneous disturbances and (ii) to achieve (i) for systems with unilateral constraints.

A brief literature review dedicated to continuous finite time stabilization is given next before the problem statement is formalized.

6.1 The state of the Art in continuous finite time stabilisation

Continuous finite time stabilization of linear and non-linear control systems has been an active area of research. Introduction of continuous finite time controllers [11] revealed the non-Lipschitzian nature of the closed-loop dynamics of planar finite time stable systems. Lyapunov and converse Lyapunov theorems were subsequently established and the continuity properties of the settling time function were studied [12, 13]. The literature shows that the aforementioned theoretical results find applications in robotics [11, 14, 15, 88] as well as in aerospace engineering [110, 137]. A survey of finite time stability and stabilization is available in [108].

Stability of non-linear homogeneous systems is also a well studied area. Earlier results on asymptotic stabilisation [85, 86] of continuous homogeneous systems are based on the definition of a class of dilations where each state is dilated with a different weight [85]. It was established in [84] that geometric homogeneity leads to finite time stability if the homogeneity degree of the asymptotically stable continuous homogeneous system is negative. A result on output feedback synthesis which combines a continuous finite time observer with the continuous finite time controller can be found in [46]. Finite time stability and uniform finite time stability of nonlinear time varying systems was studied in [44]. Uniform finite time stability results were established in [9] for discontinuous homogeneous systems with a negative homogeneity degree but with an additional requirement of uniform asymptotic stability. A settling time estimate and tuning for the planar discontinuous case with rectangular disturbances has been established in Chapter 4 which is a special case of the more general result [9].

Continuous finite time stability and homogeneity have been rigorously studied and several results exist. Continuous finite time stable controllers proposed in [11] for a double integrator system $\ddot{x} = u$ take the form

$$u(x, \dot{x}) = -|x|^b \text{sign}(x) - |\dot{x}|^\alpha \text{sign}(\dot{x}) \tag{6.1}$$

with the condition $\frac{\alpha}{2-\alpha} > b$ whereas those proposed in [83, Sec. 4] take the form

$$u(x, \dot{x}) = -|x|^b \text{sign}(x) - |\dot{x}|^\alpha \text{sign}(\dot{x}) \tag{6.2}$$

with the equality $\frac{\alpha}{2-\alpha} = b$. The difference between the above two controller formulations is that the former has a phase-plane plot that has a closed-loop trajectory approaching the origin tangentially to the asymptote $x_1 = 0$ whereas the later has the closed-loop trajectories approaching the origin while also rotating infinitely many times around the origin. The robustness to a class of continuous disturbances was established in [11] when the controller (6.1) is utilized. Asymptotic stability and robustness to piece-wise continuous disturbances was recently established in [14] when the controller

$$u(x, \dot{x}) = -\mu_2|x|^b \text{sign}(x) - \mu_1|\dot{x}|^\alpha \text{sign}(\dot{x}) \tag{6.3}$$

is used with $\alpha \in [0, 1)$ and $\mu_2 > \mu_1 > 0$. Another class of finite time controllers can be found in [12] which take the form

$$u(x, \dot{x}) = -\mu_1|\phi|^{\frac{\alpha}{2-\alpha}} \text{sign}(\phi) - \mu_2|\dot{x}|^\alpha \text{sign}(\dot{x}) \tag{6.4}$$

where $\phi = x + \frac{1}{2-\alpha}\text{sign}(\dot{x})|\dot{x}|^{2-\alpha}$ is the manifold which the closed-loop trajectory first reaches in finite time, then slides on the same and eventually reaches the origin in finite time. This synthesis, while being supported by a strict homogeneous Lyapunov function [12], prescribes better rejection of continuous disturbances than that achieved by Lipschitz controllers [13, Th. 5.2]. An output feedback finite time stabilization result can be found in [46] which also establishes robustness to a class of continuous homogeneous disturbances. A detailed survey of various control strategies for a double integrator can be found in [138].

Several results also exist on continuous finite time stabilization of linear and nonlinear systems of higher dimension than two (see [84] and references therein for linear controllable systems and [45, 87, 89, 91, 94, 115, 135, 139, 140] for nonlinear systems). In all the above references, the robustness to disturbances is obtained by considering the vector field as one of the following three cases: (i) A summation of more than one homogeneous vector field [84], (ii) a summation of a continuous homogeneous vector field and a continuous homogeneous perturbation (see, for example, [46, Lemma 3]) or (iii) a non-homogeneous vector field with continuous non-homogeneous disturbance (see, for example, [55]). In this chapter, robustness of the controller (6.3) is established in the presence of *piece-wise continuous* non-homogeneous disturbances $\omega(x, \dot{x})$ that admit a uniform upper-bound

$$ess \sup_{t \geq 0} |\omega(x, \dot{x}, t)| \leq M_0|\dot{x}|^\alpha. \tag{6.5}$$

This is established for $\alpha \in (\frac{2}{3}, 1)$ while preserving finite time stability of a controllable planar system, a robustness feature not enjoyed by the aforementioned references[1].

Motivation

The first objective is to achieve equiuniform finite time stabilization for the case without jumps in velocity. The theoretical motivation to propose a new Lyapunov and homogeneity framework for planar continuous homogeneous vector fields is to give *uniform* finite time stability with respect to uncertainty and initial data alike while giving a more robust synthesis than the existing methods which utilize the link between homogeneity [86], [85] and finite time stability [84] and computing the settling time for the class of homogeneous controllers (6.3) in the presence of non-homogeneous perturbations. Equiuniform finite time stability is a stronger feature than finite time stability and requires the Lyapunov stability to be uniform with respect to the initial time and disturbances (see [44, Remark 3.1] and Definition 2.5). The result presented in the following sections of this chapter achieves this for the class of controllers (6.3). This is the main motivation and a key contribution of this chapter.

The method proposed in [12] relies on the homogeneity property of the strict Lyapunov function and that of its derivative. It is known that every controllable linear system admits a class of homogeneous finite time stabilizing controller accompanied by existence of the differential inequality of a Lyapunov function (see [84, Sec. 7,8] and references therein). However, construction of a strict Lyapunov function is required to find an explicit formula for both the upper bound on the settling time and the tuning rules for the given homogeneous controller. For example, the homogeneous controller [83, Sec. 4] was shown to render the unperturbed double integrator finite time stable only via the homogeneity route. Finite time stability as well as the existence of a Lyapunov function can be inferred from the existing results [84] but an explicit formula for the settling time is not available for this class of controllers (the effect of a class of perturbations was explicitly included in the asymptotic Lyapunov stability analysis recently [14]). Furthermore, the proposed results can motivate similar developments for even arbitrarily higher order controllable systems in the presence of piece-wise continuous perturbations, which is also an interesting problem from the engineering viewpoint. It should be noted that the proposed method requires uniform asymptotic stability of the origin, a condition stronger than that required by existing results [84].

[1] The continuous terminal slider proposed in [91] makes no assumption on the continuity of lumped disturbances. However, the corresponding synthesis leads to finite time stability only if $\|\omega\| = 0$. In the presence of nonzero ω and with the upper bound of the form $\|\omega\| \leq b_0 + b_1\|x\| + b_2\|x\|^2$, the states are rendered only ultimately bounded. An upper bound of the form (6.5) is not discussed.

As to the second objective, a double integrator is considered with a unilateral constraint surface. This is the same problem as the previous chapter (see the discussion in section 5.3) but with an aim to establish similar results in the case when a continuous homogeneous controller (6.3) is used. The velocity undergoes an instantaneous *reset* when the inelastic collision occurs. Similar to Chapter 5, it is assumed in this chapter that the restitution or reset map relating the velocities just before and after the time of impact is fully known. However, no such assumption is made on the time of impact. The continuous homogeneous controller (6.3) is considered for finite time stabilization of the closed-loop system in the presence of resets in velocity. The method of Zhuravlev-Ivanov non-smooth transformation [125, 127] is utilised to first transform the system into a variable-structure system without jumps. The discontinuous finite time stabilization using the quasi-homogeneity principle of switched systems has been established in Chapter 5 using the above transformation approach. Within the engineering applications, such a transformation is very useful in the analysis of vibro-impact systems [125, 126]. The resulting transformed system turns out to be a switched homogeneous system with a negative homogeneity degree [9] whose solutions are well-defined in the sense of Filippov's definition [6], an attribute absent in the case of the original jump system (see [95] for solutions concept of systems with jumps and friction). It is important to note that the use of finite time stability of switched systems [9] is the only suitable method for the proposed synthesis due to the switched nature of the transformed system despite the continuous controller (6.3). This is because all the existing references on continuous homogeneous systems [11, 12, 14, 46] require continuity of the vector field, a condition unavoidably violated at the time of jumps.

The theoretical motivation to propose this new framework in place of the existing continuous finite time stabilisation approaches [11, 12, 14, 46] stems from the fact that the existing homogeneity frameworks [12, 46, 86] are not applicable to systems with jumps because the requirements of continuity of the vector field and of the Lyapunov function are not satisfied at the time of jumps in velocity. It also turns out that the Lyapunov stability framework proposed in [12] is not applicable to the transformed system. The reader is referred to section 6.4.1 for the theoretical proof of this claim. In contrast, a class of C^1 semi-global Lyapunov functions is shown to exist in Section 6.4 to prove equiuniform asymptotic stability of the transformed system in order to take advantage of the existing results [9] for a continuous synthesis (6.3). In turn, attainment of equiuniform finite time stability and derivation of explicit formulae of the upper bound on the settling time becomes possible. Also note that the existing Lyapunov approaches for discontinuous systems [4, 9, 106] do not apply to the case of jumps in the velocity dynamics. From a practical viewpoint, the motivation stems from the applicability of the proposed method to the analysis and control of mechanical systems with jumps in velocity such as biped robots when continuous homogeneous controllers like (6.3) are utilised.

The rest of the chapter is outlined as follows. The problem statement for the first objective is presented in Section 6.2. Section 6.3 establishes equiuniform finite time stability of planar non-smooth systems and computes an explicit formula for the upper bound on the finite settling time. Thus, this subsection achieves the first objective of the chapter. Section 6.4 repeats the same when resets are present due to the unilateral constraint surface thereby achieving the second objective of the chapter. Theoretical contributions of this chapter along with future directions and some open problems in the area of continuous finite time stabilization are discussed in Section 6.5. Section 6.6 motivates the problem of tuning the gains for a finite time output feedback variable structure synthesis which is the topic of the next chapter. Section 6.7 outlines the conclusions of the chapter.

6.2 Problem Statement

Consider the open loop system (4.9). The modified version of the twisting control law [4] in (x_1, x_2) coordinates is given in [14] and in (6.3) with $\alpha \in [0, 1)$ and $\mu_2 > \mu_1 > 0$ (also see [83] for purely continuous case $\alpha \in (0, 1)$) with the gains $\mu_1 = \mu_2 = 1$. As in [14], the piece-wise continuous disturbance ω is assumed to admit a uniform upper bound

$$\underset{t \geq 0}{ess\sup} |\omega(x, t)| \leq M|x_2|^\alpha \tag{6.6}$$

on its magnitude such that

$$0 < M < \mu_1 < \mu_2 - M. \tag{6.7}$$

The aim of this section is to i) prove equiuniform finite time stability and ii) to establish a finite upper bound on the settling time $\mathcal{T}$ of the closed-loop system (4.9), (6.3) for $\alpha \in (\frac{2}{3}, 1)$ in the presence of the disturbance ω with the upper bound (6.6).

Global asymptotic stability of such a perturbed double integrator can be found in [14, Theorem 2]. Global finite time stability for the unperturbed case with the controller (6.3) was established via homogeneity in [83]. Finite time stability for a class of perturbations was established in [84, Th. 7.4] by combining the geometric homogeneity of a vector field with Lyapunov stability result [13]. However, the class of non-homogeneous disturbances (6.6) has not been studied explicitly in the literature while proving finite time stability. It was also established that there always exists a strict Lyapunov function [84, Th. 7.2] for the continuous homogeneous vector-field such as that of the closed-loop system (4.9), (6.3). However, finding a finite upper bound on the settling time for (4.9), (6.3) is still an open problem. This chapter solves this problem. Recalling the discussion in the previous section on motivation, the problem formulation is theoretically challenging and practically relevant.

While finite time stability of the closed-loop system (4.9),(6.3) with $\alpha \in (0,1)$ can be established from either of the two methods [83, Th. 2], [9, Th. 3.2], the result for discontinuous systems [9] is more general as it encompasses the discontinuous case when $\alpha = 0$ [14, Th. 1]. Moreover, existing results [46, 83, 84, 86] encompass only continuous disturbances $\omega(t)$ whereas the proposed method accommodates even the piece-wise continuous ones because the result on switched systems [9] is utilized and extended in Theorem 6.1 below. For example, the results in the next section allow disturbances of the form $|x_2|^\alpha \text{sign}(x_1)$ whereas the existing literature does not allow this due to the discontinuity on the line $x_1 = 0$. Of course, in the presence of continuous disturbances with an upper-bound $|x_2|^\alpha$, it is enough to apply [14, Th.1] to establish global asymptotic stability and in turn [84, Th. 7.4] to establish global (but not uniform) finite time stability of the closed-loop system (4.9), (6.3).

The controller (6.3) does not belong to the class of controllers proposed in [11, Corollary 1]. The phase plane plot of the closed-loop system (4.9),(6.3) with $\alpha \in (0,1)$ can be found in [14] which shows that the trajectories spiral infinitely around the origin without approaching tangentially to the hyperplane $x_1 = 0$ as they move to the origin.

The following Lemma extends the existing result [9, Lem. 2.12] for the present case with uniformly decaying piece-wise continuous disturbances ω and is utilized in the proof of the main result. It should be noted that the unperturbed closed-loop system (4.9),(6.3) with $M = 0$ is globally homogenous of degree $q = -1$ with respect to dilations $(r_1, r_2) = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$ (see [83, Sec. 4]). Furthermore, the definitions 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 given in Section 2.4 and definitions [9, Definitions 2.3, 2.4, 2.5] are invoked hereafter for the solution concept, homogeneity and finite time stability of the closed-loop system (4.9),(6.3).

Lemma 6.1. *Let the function $\omega(x_1, x_2, t)$ be a piece-wise continuous function which is locally uniformly bounded by the upper-bound (6.6). Then, the uncertain differential equation (4.9),(6.3) with the uncertainty constraints (6.6) is locally homogeneous of degree $q = -1$ with respect to the dilation $(r_1, r_2) = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$.*

Proof. Let $x(\cdot) = (x_1(\cdot), x_2(\cdot))^{\mathrm{T}}$ be a solution of (4.9),(6.3) under some piece-wise continuous function ω, satisfying (6.6), and let $x(\cdot)$ evolve within a homogeneity ball

$B_\delta \subset R^2$ where the homogeneity condition 2.7 holds almost everywhere for all $c \geq c_0 > 0$. Then it is straightforward to verify that for arbitrary $c \geq \max(1, c_0)$ the function $x^c(\cdot)$ with components $x_i^c(t) = c^{r_i} x_i(c^q t), i = 1, 2$, is a solution of (4.9),(6.3) with the piece-wise continuous function $\omega(x_1, x_2, t) = \omega^c(x_1, x_2, t)$ which is are as follows:

$$\omega^c(x_1, x_2, t) = c^{q+r_2}\omega(c^{-r_1}x_1, c^{-r_2}x_2, c^q t) \tag{6.8}$$

where, the right hand side represents a parameterized set of uncertainties. The following holds true due to the parameterization (6.8):

$$\begin{aligned} |\omega^c(x_1, x_2, t)| &= |c^{q+r_2}\omega(c^{-r_1}x_1, c^{-r_2}x_2, c^q t)| \\ \Rightarrow \quad |\omega^c(x_1, x_2, t)| &\leq c^{q+r_2}M|c^{-r_2}x_2|^\alpha \quad \leq \quad c^{q+r_2-\alpha r_2}M|x_2|^\alpha \end{aligned} \tag{6.9}$$

Hence, all parameterized disturbance functions represented by the right hand side of (6.8) are admissible in the sense of (6.6) if the following holds true:

$$c^{q+r_2-\alpha r_2} \leq 1 \tag{6.10}$$

From the definitions $r_2 = \frac{1}{1-\alpha}, q = -1$, it is obtained that

$$q + r_2 - \alpha r_2 = 0 \Rightarrow c^{q+r_2-\alpha r_2} \leq 1 \tag{6.11}$$

and that the function $\omega^c(x_1, x_2, t)$ is admissible in the sense of (6.6). Recalling definitions 2.6, 2.7 the solutions $x_1^c(t) = c^{r_1}x_1(c^q t), x_2^c(t) = c^{r_2}x_2(c^q t)$ are solutions of the system (4.9), (6.3) with the piece-wise continuous function $\omega(x_1, x_2, t) = \omega^c(x_1, x_2, t)$ given by (6.8). Thus, any solution of the differential equation (4.9), (6.3) evolving within a homogeneity ball B_δ, generates a parameterized set of solutions $x_1^c(t), x_2^c(t)$ with the parameter c large enough. Hence, (4.9), (6.3) is locally homogeneous of degree $q = -1$ with the dilation $(r_1, r_2) = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$. This proves the statement of Lemma 6.1. □

The importance of Lemma 6.1 lies in the fact that proving uniform asymptotic stability of the perturbed system (4.9), (6.3) in the presence of disturbances $\omega(x_1, x_2, t)$ will render the existing result on finite time stability of switched systems [9, Th 3.1] applicable to the present case. Equiuniform asymptotic stability is proven next by identifying a class of semi-global $\mathcal{C}^1$ Lyapunov functions for a limited range of $\alpha \in (\frac{2}{3}, 1)$.

6.3 Global Uniform Finite Time Stability

This section is a parallel to Chapter 4 as discussed in the first few paragraphs of this chapter since it achieves the *equiuniform continuous* finite time stabilisation of a perturbed double integrator such as (4.9) without resets. The following result is in order.

Theorem 6.1. *Given* $\alpha \in (\frac{2}{3}, 1)$, *the closed-loop system (4.9), (6.3) is globally equiuniformly finite time stable, regardless of whichever disturbance* ω, *satisfying condition (6.6) with* $0 < M < \mu_1 < \mu_2 - M$, *affects the system.*

Proof. The proof is given in several steps.
1. Global Asymptotic Stability Let the following candidate Lyapunov function V be considered [83] [14]:

$$V(x_1, x_2) = \mu_2 \frac{2-\alpha}{2} |x_1|^{\frac{2}{2-\alpha}} + \frac{1}{2} x_2^2 \qquad (6.12)$$

Under the conditions of the theorem, the time derivative of the function $V(x_1, x_2)$, computed along the trajectories of (4.9), (6.3) is estimated as follows [14, Th. 1]:

$$\dot{V} \leq -(\mu_1 - M)|x_2|^{\alpha+1} \qquad (6.13)$$

There is no sliding mode on $x_2 = 0$ where $\dot{V} = 0$ since $x_2 \dot{x}_2 \not< 0$. It should be noted that the Lyapunov function (6.12) is a globally radially unbounded positive definite function and that its derivative remains negative definite for almost all t due to (6.13) and due to the fact that there is no sliding on $x_2 = 0$. Noting that $M < \mu_1$ by a condition of the theorem and that the equilibrium point $x_1 = x_2 = 0$ is the only trajectory of (4.9), (6.3) on the invariance manifold $x_2 = 0$ where $\dot{V}(x_1, x_2) = 0$, the global asymptotic stability of (4.9), (6.3) is then established by applying the invariance principle [128],[129].
2. Semi-global Strong Lyapunov Functions.
The goal of this step is to show the existence of a parameterized family of semi-global Lyapunov functions $V_{\tilde{R}}(x_1, x_2)$, with an *a priori* but arbitrarily given $\tilde{R} > 0$, such that each $V_{\tilde{R}}(x_1, x_2)$ is well-posed on the corresponding compact set

$$D_{\tilde{R}} = \{(x_1, x_2) \in \mathbb{R}^2 : V(x_1, x_2) \leq \tilde{R}\}. \qquad (6.14)$$

In other words, $V_{\tilde{R}}(x_1, x_2)$ is to be positive definite on $D_{\tilde{R}}$ and its derivative, computed along the trajectories of the uncertain system (4.9), (6.3) with initial conditions within

$D_{\tilde{R}}$, is to be negative definite in the sense that,

$$\dot{V}_{\tilde{R}}(x_1, x_2) \le -W_{\tilde{R}}(x_1, x_2) \tag{6.15}$$

for all $(x_1, x_2) \in D_{\tilde{R}}$ and for some $W_{\tilde{R}}(x_1, x_2)$, positive definite on $D_{\tilde{R}}$. A parameterized family of Lyapunov functions $V_{\tilde{R}}(x_1, x_2)$, $\tilde{R} > 0$, with the properties defined above are constructed by combining the Lyapunov function V of (6.12), whose time derivative along the system motion is only negative semi-definite, with the indefinite functions

$$\begin{aligned}
U(x_1, x_2) &= U_1(x_1, x_2) + U_2(x_1, x_2) + U_3(x_1, x_2) \\
U_1(x_1, x_2) &= \kappa_1 |x_1|^{\frac{2\alpha}{2-\alpha}} \operatorname{sign}(x_1)\, |x_2|^{2\alpha}\, x_2 && (6.16) \\
U_2(x_1, x_2) &= \kappa_1\, \kappa_2\, x_1^3\, x_2\, |x_2|^{\alpha}, && (6.17) \\
U_3(x_1, x_2) &= \kappa_1\, \kappa_2\, \kappa_3\, x_1^5\, x_2
\end{aligned}$$

as follows:

$$V_{\tilde{R}}(x_1, x_2) = V(x_1, x_2) + \sum_{i=1}^{3} U_i(x_1, x_2) \tag{6.18}$$

where the positive weight scalars $\kappa_i, i = 1, 2, 3$ are chosen small enough so that,

$$\begin{aligned}
K_1 &= \frac{2\alpha}{2-\alpha} \rho^{\frac{3\alpha-2}{2}} \left(2\tilde{R}\right)^{\frac{1+\alpha}{2}} + (\mu_1 + M)(1+2\alpha)\rho^{\alpha} \left(2\tilde{R}\right)^{\frac{2\alpha-1}{2}} \\
&+ 3\kappa_1\kappa_2\rho^{2-\alpha} \left(2\tilde{R}\right)^{\frac{1}{2}} + 5\kappa_2\kappa_3\rho^{2(2-\alpha)} \left(2\tilde{R}\right)^{\frac{1-\alpha}{2}} \\
\kappa_2 &< \frac{(1+2\alpha)\mu_2}{(1+\alpha)(\mu_1+M)\, \rho^{3(1-\alpha)}} && (6.19) \\
\kappa_3 &< \frac{(1+\alpha)\mu_2}{(\mu_1+M)\, \rho^{\frac{4-3\alpha}{2}}} \\
\kappa_1 &< \min\left\{ \frac{\mu_1 - M}{K_1}, \frac{\mu_2(2-\alpha)}{\kappa_2\rho^{5-3\alpha}\left(1+\kappa_3\rho^{2(2-\alpha)}\right)}, \frac{1}{\rho^{2\alpha} + \left(2\tilde{R}\right)^{2\alpha-1} + \kappa_2\left(\left(2\tilde{R}\right)^{\alpha} + \kappa_3\right)} \right\}
\end{aligned}$$

where,

$$\rho = \frac{2\tilde{R}}{(2-\alpha)\mu_2} \tag{6.20}$$

An *a priori* definition of the scalars κ_i is always possible. This is because for known initial conditions $x^0 \in \mathbb{R}^2$, a given bound M and in fixed values of μ_1, μ_2, there always exists an arbitrarily large $\tilde{R}$ such that $V \le \tilde{R}$ holds true. Then (6.19) gives straightforward formulae for $\kappa_i, i = 1, 2.3$

Remark 6.1. The functions $V_{\tilde{R}}$ and $U_i, i = 1, 2, 3$ are not only continuous but also C^1 smooth for all $x \in \mathbb{R}^2$ for $\alpha \in (\frac{2}{3}, 1)$. Setting $\alpha = 0$ in the following analysis corresponds to the discontinuous case for which the finite time stability has been established following a similar semi-global analysis [14, Th. 4]. It is further noted that the expressions $2\alpha - 1 > 0, 3\alpha - 2 > 0$ hold true due to $\alpha \in (\frac{2}{3}, 1)$ in the derivations below.

Noting that, due to (6.13), all possible solutions of the uncertain system (4.9), (6.3), initialized at $t_0 \in \mathbb{R}$ within the compact set (6.14), are *a priori* estimated by

$$\sup_{t \in [t_0, \infty)} V(x_1, x_2) \leq \tilde{R}, \tag{6.21}$$

the following inequalities hold true:

$$|x_1|^{\frac{2}{2-\alpha}} \leq \rho, \quad |x_2| \leq \sqrt{2\tilde{R}}. \tag{6.22}$$

Let the positive definiteness of the Lyapunov function (6.18) be verified. The following analysis is in order for the indefinite functions $U_i, i = 1, 2, 3$.

$$\begin{aligned} U_1(x_1, x_2) = \kappa_1 |x_1|^{\frac{2\alpha}{2-\alpha}} \operatorname{sign}(x_1) \, |x_2|^{2\alpha} \, x_2 &\geq -\frac{\kappa_1}{2} |x_1|^{\frac{4\alpha}{2-\alpha}} x_2^2 - \frac{\kappa_1}{2} \, |x_2|^{4\alpha} \\ &\geq -\left(\rho^{2\alpha} + \left(2\tilde{R}\right)^{2\alpha - 1}\right) \frac{\kappa_1}{2} x_2^2 \end{aligned} \tag{6.23}$$

where, (6.22) and the trivial inequality $2ab > -(a^2 + b^2), \forall a, b \in \mathbb{R}$ have been utilised. Similarly, U_2 and U_3 can also be analysed as follows:

$$\begin{aligned} U_2(x_1, x_2) = \frac{\kappa_1 \kappa_2}{2} x_1^3 x_2 \, |x_2|^{\alpha} &\geq -\kappa_1 \, \kappa_2 \, \left(x_1^6 + x_2^2 |x_2|^{2\alpha}\right) \\ &\geq -\frac{\kappa_1 \, \kappa_2}{2} \, |x_1|^{\frac{2}{2-\alpha}} \rho^{5-3\alpha} - \frac{\kappa_1 \, \kappa_2}{2} \, \left(2\tilde{R}\right)^{\alpha} x_2^2. \end{aligned} \tag{6.24}$$

$$\begin{aligned} U_3(x_1, x_2) = \frac{\kappa_1 \, \kappa_2 \, \kappa_3}{2} \, x_1^5 \, x_2 &\geq -\kappa_1 \, \kappa_2 \, \kappa_3 x_1^{10} - \kappa_1 \, \kappa_2 \, \kappa_3 x_2^2 \\ &\geq -\frac{\kappa_1 \, \kappa_2 \, \kappa_3}{2} |x_1|^{\frac{2}{2-\alpha}} \rho^{9-5\alpha} - \frac{\kappa_1 \, \kappa_2 \, \kappa_3}{2} x_2^2 \end{aligned} \tag{6.25}$$

Hence, the Lyapunov function (6.18) is positive definite on a compact set (6.14); for all $(x_1, x_2) \in D_{\tilde{R}} \backslash \{0, 0\}$ and $\kappa_i > 0, i = 1, 2, 3$ satisfying (6.19), as shown below:

$$\begin{aligned} V_{\tilde{R}}(x_1, x_2) &= \mu_2 \frac{2 - \alpha}{2} \, |x_1|^{\frac{2}{2-\alpha}} + \frac{1}{2} \, x_2^2 + \sum_{i=1}^{3} U_i(x_1, x_2) \\ &\geq \left(\mu_2 \frac{2 - \alpha}{2} - \frac{\kappa_1 \, \kappa_2}{2} \, \rho^{5-3\alpha} \left(1 + \kappa_3 \rho^{2(2-\alpha)}\right)\right) |x_1|^{\frac{2}{2-\alpha}} \qquad (6.26) \\ &\quad + \left(1 - \kappa_1 \left(\rho^{2\alpha} + \left(2\tilde{R}\right)^{2\alpha-1} + \kappa_2 \left(\left(2\tilde{R}\right)^{\alpha} + \kappa_3\right)\right)\right) \frac{1}{2} x_2^2 \qquad (6.27) \\ &\geq L_{\tilde{R}} V(x_1, x_2) \end{aligned}$$

where,

$$L_{\tilde{R}} < \min \left\{ \begin{array}{l} \frac{\mu_2(2-\alpha)}{2} - \frac{\kappa_1\kappa_2\rho^{5-3\alpha}}{2}\left(1+\kappa_3\rho^{2(2-\alpha)}\right), \\ \qquad 1-\kappa_1\left(\rho^{2\alpha}+\left(2\tilde{R}\right)^{2\alpha-1}+\kappa_2\left(\left(2\tilde{R}\right)^{\alpha}+\kappa_3\right)\right) \end{array} \right\} \tag{6.28}$$

It should be noted that $L_{\tilde{R}} > 0$ due to (6.19) and hence positive definiteness of $V_{\tilde{R}}$ is ensured from (6.26) on $D_{\tilde{R}} \setminus \{0,0\}$. Similarly it can be shown that the following inequality holds true:

$$V_{\tilde{R}}(x_1,x_2) \leq M_{\tilde{R}} V(x_1,x_2) \tag{6.29}$$

where,

$$M_{\tilde{R}} > \max \left\{ \begin{array}{l} \mu_2\frac{2-\alpha}{2} + \frac{\kappa_1\ \kappa_2}{2}\ \rho^{5-3\alpha}\left(1+\kappa_3\rho^{2(2-\alpha)}\right), \\ \qquad 1+\kappa_1\ \kappa_2\left(\left(2\tilde{R}\right)^{\alpha}+\kappa_3\right) \end{array} \right\} \tag{6.30}$$

is a positive scalar. The time derivative of the indefinite function $U_1(x_1,x_2)$ along the trajectories of the uncertain system (4.9), (6.3) is as follows:

$$\begin{aligned} \dot{U}_1(x_1,x_2) &= \kappa_1\frac{2\alpha}{2-\alpha}|x_1|^{\frac{3\alpha-2}{2-\alpha}}|x_2|^{2\alpha+2} + \kappa_1|x_1|^{\frac{2\alpha}{2-\alpha}}\,\mathrm{sign}(x_1)|x_2|^{2\alpha}\dot{x}_2 \\ &\quad +2\alpha\kappa_1|x_1|^{\frac{2\alpha}{2-\alpha}}\mathrm{sign}(x_1)|x_2|^{2\alpha-1}\mathrm{sign}(x_2)x_2\dot{x}_2 \\ &= \kappa_1\frac{2\alpha}{2-\alpha}|x_1|^{\frac{3\alpha-2}{2-\alpha}}|x_2|^{2\alpha+2} - \kappa_1\mu_1|x_1|^{\frac{2\alpha}{2-\alpha}}\mathrm{sign}(x_1)|x_2|^{3\alpha}\mathrm{sign}(x_2) \\ &\quad -\kappa_1\mu_2|x_1|^{\frac{3\alpha}{2-\alpha}}|x_2|^{2\alpha} + \kappa_1\omega|x_1|^{\frac{2\alpha}{2-\alpha}}\,\mathrm{sign}(x_1)|x_2|^{2\alpha} \\ &\quad -2\alpha\kappa_1\mu_1|x_1|^{\frac{2\alpha}{2-\alpha}}\mathrm{sign}(x_1)|x_2|^{3\alpha}\mathrm{sign}(x_2) \\ &\quad -2\alpha\kappa_1\mu_2|x_1|^{\frac{3\alpha}{2-\alpha}}|x_2|^{2\alpha} + 2\alpha\kappa_1\omega|x_1|^{\frac{2\alpha}{2-\alpha}}\,\mathrm{sign}(x_1)|x_2|^{2\alpha} \\ &\leq -\kappa_1\mu_2(1+2\alpha)|x_1|^{\frac{3\alpha}{2-\alpha}}|x_2|^{2\alpha} + \kappa_1\frac{2\alpha}{2-\alpha}|x_1|^{\frac{3\alpha-2}{2-\alpha}}|x_2|^{2\alpha+2} \\ &\quad +\kappa_1(\mu_1+M)(1+2\alpha)|x_1|^{\frac{2\alpha}{2-\alpha}}|x_2|^{3\alpha} \end{aligned} \tag{6.31}$$

The temporal derivative of U_2 along the trajectories of the closed-loop system (4.9), (6.3) is as follows:

$$\begin{aligned} \dot{U}_2 &= 3\kappa_1\ \kappa_2 x_1^2|x_2|^{\alpha+2} + \kappa_1\kappa_2 x_1^3|x_2|^{\alpha}\dot{x}_2 + \alpha\kappa_1\ \kappa_2\ x_1^3\ |x_2|^{\alpha-1}\mathrm{sign}(x_2)x_2\dot{x}_2 \\ &= 3\kappa_1\ \kappa_2\ x_1^2\ |x_2|^{\alpha+2} - \kappa_1\kappa_2\mu_1 x_1^3|x_2|^{2\alpha}\mathrm{sign}(x_2) + \kappa_1\ \kappa_2\ x_1^3\ |x_2|^{\alpha}\ \omega \\ &\quad -\kappa_1\kappa_2\mu_2 x_1^2|x_1|^{\frac{2}{2-\alpha}}|x_2|^{\alpha} - \alpha\kappa_1\ \kappa_2\ \mu_1\ x_1^3\ |x_2|^{2\alpha}\ \mathrm{sign}(x_2) \\ &\quad +\alpha\kappa_1\ \kappa_2\ x_1^3\ |x_2|^{\alpha}\ \omega - \alpha\kappa_1\ \kappa_2\ \mu_2\ x_1^2\ |x_1|^{\frac{2}{2-\alpha}}\ |x_2|^{\alpha} \\ &\leq 3\kappa_1\ \kappa_2\ x_1^2\ |x_2|^{\alpha+2} - \kappa_1\ \kappa_2\ (1+\alpha)\ \mu_2\ x_1^2\ |x_1|^{\frac{2}{2-\alpha}}\ |x_2|^{\alpha} \\ &\quad +\kappa_1\ \kappa_2\ (1+\alpha)\ (\mu_1+M)\ |x_1|^3\ |x_2|^{2\alpha} \end{aligned} \tag{6.32}$$

The temporal derivative of U_3 along the trajectories of the closed-loop system (4.9), (6.3) can be obtained as follows:

$$\begin{aligned} \dot{U}_3 &= 5\,\kappa_1\,\kappa_2\,\kappa_3\,x_1^4\,x_2^2 + \kappa_1\,\kappa_2\,\kappa_3\,x_1^5\,\dot{x}_2 \\ &= 5\,\kappa_1\,\kappa_2\,\kappa_3\,x_1^4\,x_2^2 - \kappa_1\,\kappa_2\,\kappa_3\,\mu_1\,x_1^5\,|x_2|^{\alpha}\mathrm{sign}(x_2) \\ &\quad -\kappa_1\,\kappa_2\,\kappa_3\,\mu_2\,x_1^4\,|x_1|^{\frac{2}{2-\alpha}} + \kappa_1\,\kappa_2\,\kappa_3\,x_1^5\,\omega \\ &\leq 5\,\kappa_1\,\kappa_2\,\kappa_3\,x_1^4\,x_2^2 + \kappa_1\,\kappa_2\,\kappa_3\,(\mu_1 + M)\,|x_1|^5\,|x_2|^{\alpha} \\ &\quad -\kappa_1\,\kappa_2\,\kappa_3\,\mu_2\,x_1^4\,|x_1|^{\frac{2}{2-\alpha}} \end{aligned} \tag{6.33}$$

It should be noted that the inequality

$$|x_2|^{2\alpha} = |x_2||x_2|^{2\alpha-1} \leq |x_2|\left(2\tilde{R}\right)^{\frac{2\alpha-1}{2}} \tag{6.34}$$

holds true due to the condition $\alpha \in (\frac{2}{3}, 1)$ of the theorem. The last inequalities of (6.31), (6.32) and (6.33) are re-written by utilizing (6.22) and (6.34) as follows:

$$\begin{aligned} &\sum_{i=1}^{3} \dot{U}_i(x_1, x_2) \\ &\leq -\kappa_1\kappa_2\left(\,(1+\alpha)\,\mu_2 - \kappa_3(\mu_1 + M)\rho^{\frac{4-3\alpha}{2}}\right)x_1^2\,|x_1|^{\frac{2}{2-\alpha}}\,|x_2|^{\alpha} \\ &\quad -\kappa_1\left(\,\mu_2(1+2\alpha) - \kappa_2\,(1+\alpha)\,(\mu_1 + M)\rho^{3(1-\alpha)}\right)|x_1|^{\frac{3\alpha}{2-\alpha}}\,|x_2|^{2\alpha} \\ &\quad +\kappa_1 K_1 |x_2|^{\alpha+1} \end{aligned} \tag{6.35}$$

where, $K_1 > 0$ is defined in (6.19) and the corresponding upper bound on $|x_1|$ and $|x_2|$ from (6.22) are utilized. It should be noted that $\kappa_i, i = 1, 2, 3$ are all positive constants due to (6.19). Hence, by combining (6.13) and (6.35), the time derivative of (6.18) can be obtained as follows:

$$\begin{aligned} \dot{V}_{\tilde{R}} &\leq -\beta_1 x_1^2\,|x_1|^{\frac{2}{2-\alpha}}\,|x_2|^{\alpha} - \beta_2 |x_1|^{\frac{3\alpha}{2-\alpha}}\,|x_2|^{2\alpha} \\ &\quad - (\mu_1 - M - \kappa_1 K_1)\,|x_2|^{\alpha+1} - \kappa_1\,\kappa_2\,\kappa_3\,\mu_2\,x_1^4\,|x_1|^{\frac{2}{2-\alpha}}, \end{aligned} \tag{6.36}$$

where the constants

$$\begin{aligned} \beta_1 &= \kappa_1\kappa_2\left(\,(1+\alpha)\,\mu_2 - \kappa_3(\mu_1 + M)\rho^{\frac{4-3\alpha}{2}}\right) \\ \beta_2 &= \kappa_1\left(\,\mu_2(1+2\alpha) - \kappa_2\,(1+\alpha)\,(\mu_1 + M)\rho^{3(1-\alpha)}\right), \end{aligned} \tag{6.37}$$

and the expression (6.22) are utilised. It should be noted that the expressions $\beta_1 > 0, \beta_2 > 0$ hold true due to (6.19). Ignoring the negative semi-definite terms with β_1, β_2, (6.36) can be rewritten as follows:

$$\dot{V}_{\tilde{R}} \;\leq\; -\left(\mu_1 - M - \kappa_1 K_1\right)|x_2|^{\alpha+1} - \kappa_1\,\kappa_2\,\kappa_3\,\mu_2\,x_1^4\,|x_1|^{\frac{2}{2-\alpha}} \tag{6.38}$$

Furthermore, the following inequalities hold true within the compact set (6.14):

$$x_2^2 = |x_2|^2 = |x_2|^{\alpha+1}|x_2|^{1-\alpha} \leq |x_2|^{\alpha+1}\left(\sqrt{2\tilde{R}}\right)^{1-\alpha} \quad \Rightarrow \quad -|x_2|^{\alpha+1} \leq -\frac{x_2^2}{\left(\sqrt{2\tilde{R}}\right)^{1-\alpha}} \tag{6.39}$$

Hence, (6.38) can be simplified as follows:

$$\dot{V}_{\tilde{R}} \leq -c_{\tilde{R}}\left(|x_1|^{\frac{10-4\alpha}{2-\alpha}} + x_2^2\right) \tag{6.40}$$

where,

$$c_{\tilde{R}} = \min\left\{\frac{\mu_1 - M - \kappa_1 K_1}{\left(\sqrt{2\tilde{R}}\right)^{1-\alpha}},\quad \kappa_1\,\kappa_2\,\kappa_3\,\mu_2\right\} > 0. \tag{6.41}$$

Case 1: $|x_1| \geq 1$:

The following inequality holds true for $|x_1| \geq 1$:

$$\frac{10-4\alpha}{2-\alpha} \geq \frac{2}{2-\alpha} \Leftrightarrow |x_1|^{\frac{10-4\alpha}{2-\alpha}} \geq |x_1|^{\frac{2}{2-\alpha}} \tag{6.42}$$

Also, the following can be obtained from (6.29):

$$\frac{M_{\tilde{R}}}{2}\max\{1, \mu_2(2-\alpha)\}(|x_1|^{\frac{2}{2-\alpha}} + x_2^2) \geq V_{\tilde{R}}(x_1, x_2) \tag{6.43}$$

Hence, the following inequality is then obtained for $|x_1| \geq 1$ by combining (6.40), (6.42) and (6.43):

$$\dot{V}_{\tilde{R}} \leq -\bar{\kappa}_1 V_{\tilde{R}} \tag{6.44}$$

where,

$$\bar{\kappa}_1 = \frac{2c_{\tilde{R}}}{M_{\tilde{R}}\max\{1, \mu_2(2-\alpha)\}} > 0. \tag{6.45}$$

Case 2: $|x_1| < 1$:

Noting that the following inequalities hold true for $|x_1| < 1$,

$$|x_1|^{\frac{10-4\alpha}{2-\alpha}} > |x_1|^{\frac{2\gamma}{2-\alpha}} \Leftrightarrow \frac{10-4\alpha}{2-\alpha} < \frac{2\gamma}{2-\alpha} \Leftrightarrow \gamma > 5 - 2\alpha, \tag{6.46}$$

and for some $\gamma > 5 - 2\alpha$. Noting that $5 - 2\alpha < \frac{11}{3}$ always holds true due to $\alpha \in (\frac{2}{3}, 1)$, $\gamma \geq \frac{11}{3}$ is a valid choice. In the following, $\gamma = 4$ is chosen. It can be seen that the following equality holds true:

$$\begin{aligned}\left(|x_1|^{\frac{2}{2-\alpha}}+x_2^2\right)^4 &= |x_1|^{\frac{8}{2-\alpha}}+4\,|x_1|^{\frac{6}{2-\alpha}}x_2^2+6\,|x_1|^{\frac{4}{2-\alpha}}x_2^4+4\,|x_1|^{\frac{2}{2-\alpha}}x_2^6+x_2^8 \\ &\le \max\{\rho^{2\alpha-1},K_2\}\left(|x_1|^{\frac{10-4\alpha}{2-\alpha}}+x_2^2\right)\end{aligned} \tag{6.47}$$

where the bounds (6.22) has been utilised resulting in the following definition of K_2:

$$K_2=\max\left\{4\,\rho^3,6\,\rho^2(2\tilde{R}),4\rho(2\tilde{R})^2,(2\tilde{R})^3\right\}>0 \tag{6.48}$$

Note that the following can be obtained from (6.29):

$$\left(\frac{M_{\tilde{R}}}{2}\max\{1,\mu_2(2-\alpha)\}(|x_1|^{\frac{2}{2-\alpha}}+x_2^2)\right)^4\ge\left(V_{\tilde{R}}(x_1,x_2)\right)^4 \tag{6.49}$$

Then, the following can be obtained by combining (6.40), (6.47) and (6.49):

$$\dot{V}_{\tilde{R}}(x_1,x_2)\le-c_{\tilde{R}}\left(|x_1|^{\frac{10-4\alpha}{2-\alpha}}+x_2^2\right)\le-\bar{\kappa}_2\left(V_{\tilde{R}}\right)^4 \tag{6.50}$$

where,

$$\bar{\kappa}_2=\frac{c_{\tilde{R}}}{\left(\frac{M_{\tilde{R}}}{2}\max\{1,\mu_2(2-\alpha)\}\right)^4\max\{\rho^{2\alpha-1},K_2\}}>0. \tag{6.51}$$

Hence, the desired uniform negative definiteness (6.15) is obtained by combining (6.44) and (6.50) as follows:

$$W_{\tilde{R}}(x_1,x_2)=\min\left\{\bar{\kappa}_1V_{\tilde{R}},\bar{\kappa}_2\left(V_{\tilde{R}}\right)^4\right\} \tag{6.52}$$

3. Global Equiuniform Asymptotic Stability

Since the inequality (6.50) holds on the solutions of the uncertain system (4.9), (6.3), initialized within the compact set (6.14), the decay of the function $V_{\tilde{R}}(x_1,x_2)$ can be found by considering the majorant solution $\nu(t)$ of $V_{\tilde{R}}$ as follows:

$$\dot{\nu}(t)=\begin{cases}-\bar{\kappa}_1\nu(t), & \text{if } |x_1|\ge 1;\\ -\bar{\kappa}_2\nu^{\gamma}, & \text{if } |x_1|<1.\end{cases} \tag{6.53}$$

where, $\gamma>5-2\alpha$ is introduced for generality. A more conservative decay than that in (6.53) can be computed. There are two possible sub-cases, namely, $\nu(t)\ge 1$ and $\nu(t)<1$ for each of the cases $|x_1|\ge 1$ and $|x_1|<1$. The following expressions hold true for a positive definite function $\nu(t)$ and a scalar $\gamma > 1$:

$$\begin{aligned} \nu(t)^\gamma &\geq \nu(t) \Rightarrow -\nu(t)^\gamma \leq -\nu(t) \quad \text{if } \nu(t) \geq 1; \\ \nu(t)^\gamma &\leq \nu(t) \Rightarrow -\nu(t) \leq -\nu(t)^\gamma \quad \text{if } \nu(t) < 1. \end{aligned} \tag{6.54}$$

Hence, the decay (6.53) is modified by utilising (6.54) independent of the magnitude of $|x_1|$ and dependent on $\nu(t)$ as follows :

$$\dot{\nu}(t) = \begin{cases} -\bar{\kappa}\nu, & \text{if } \nu(t) \geq 1; \\ -\bar{\kappa}\nu^\gamma, & \text{if } \nu(t) < 1. \end{cases} \tag{6.55}$$

where

$$\bar{\kappa} = \min\{\bar{\kappa}_1, \bar{\kappa}_2\} \quad > 0. \tag{6.56}$$

The solution for the case $\nu(t) < 1$ can be obtained as follows:

$$\int_{\nu_0}^{\nu(t)} \frac{\mathrm{d}\zeta(t)}{\zeta^\gamma} = -\bar{\kappa} \int_{t_1}^{t} \mathrm{d}\tau \tag{6.57}$$

where $\nu_0 = \nu(t_1)$ where t_1 is the time instant when the solution $\nu(t)$ satisfies the condition $\nu(t) = 1$. The general solution of $\nu(t)$ of (6.55) can then be obtained as follows:

$$\nu(t) = \begin{cases} \nu(t_0)\, e^{-\bar{\kappa}(t-t_0)}, & \text{if } \nu(t) \geq 1; \\ \nu(t_1) \left(\frac{1}{\bar{\kappa}(t-t_1)(\gamma-1)\nu_0^{\gamma-1}+1} \right)^{\frac{1}{\gamma-1}}, & \text{if } \nu(t) < 1. \end{cases} \tag{6.58}$$

It is noted that $t_1 = t_0$ if $\nu(t_0) \leq 1$. It can be easily seen that the solution $\nu(t) \rightarrow 0$ as $t \rightarrow \infty$ and that the decay rate depends on the gain parameters μ_1, μ_2 and bound M on the disturbance ω. On the compact set (6.14), the following inequality holds (see (6.26), (6.29)):

$$L_{\tilde{R}} V(x_1, x_2) \leq V_{\tilde{R}}(x_1, x_2) \leq M_{\tilde{R}} V(x_1, x_2) \tag{6.59}$$

for all $(x_1, x_2) \in D_{\tilde{R}}$ and positive constants $L_{\tilde{R}}, M_{\tilde{R}}$. The above inequalities (6.58) and (6.59) ensure that the globally radially unbounded function $V(x_1, x_2)$ decays exponentially

$$V(x_1(t), x_2(t)) \leq \begin{cases} L_{\tilde{R}}^{-1} M_{\tilde{R}} \tilde{R} e^{-\bar{\kappa}(t-t_0)}, & \text{if } V_{\tilde{R}} \geq 1; \\ L_{\tilde{R}}^{-1} M_{\tilde{R}} \tilde{R} \left(\frac{1}{\bar{\kappa}(t-t_1)(\gamma-1)\nu_0^{\gamma-1}+1} \right)^{\frac{1}{\gamma-1}}, & \text{if } V_{\tilde{R}} < 1. \end{cases} \tag{6.60}$$

on the solutions of (4.9), (6.3) uniformly in ω and the initial data, located within an arbitrarily large set (6.14). This proves that the uncertain system (4.9), (6.3) is globally equiuniformly asymptotically stable around the origin $(x_1, x_2) = (0, 0)$.

4. Global Uniform Finite Time Stability.

Due to (6.6), the piece-wise continuous (see Definition 2.1) uncertainty $\omega(x_1, x_2, t)$ in the right hand side of the system (4.9), (6.3) is locally uniformly bounded by $M|x_2|^{\alpha}$ whereas the remaining part of the feedback is globally homogeneous with homogeneity degree $q = -1$ with respect to dilation $(r_1, r_2) = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$. It remains to verify, however, whether the existing finite time stability result [9, Theorem 3.1] is applicable to the continuous case in question. In the presence of piece-wise continuous disturbances $\omega(x_1, x_2, t)$, Lemma 6.1 proves that the closed-loop system (4.9), (6.3) is locally homogeneous of degree $q = -1$ with respect to dilations $(r_1, r_2) = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$. Thus, coupling the homogeneity of the perturbed system (4.9), (6.3) within the arbitrarily large homogeneity ball $B_\delta \subset D_{\tilde{R}}$, where compact set (6.14) is chosen arbitrarily large, with the global equiuniform asymptotic stability of the system (4.9), (6.3), it is obtained that the closed-loop system (4.9), (6.3) is globally equiuniformly finite time stable according to [9, Theorem 3.1]. □

Remark 6.2. It should be noted that the step 4 above can be seen as an extension of [9, Theorem 3.2] with a modification of the second condition $q + r_i \leq 0, M_i > 0$ of [9, Theorem 3.2] while utilizing the local homogeneity of the differential equation. Step 4 above also shows that all other conditions of [9, Theorem 3.2] are precisely met leading to applicability of [9, Theorem 3.1] and in turn equiuniform finite time stability of perturbed system (4.9), (6.3). The condition $q+r_i \leq 0, M_i > 0$ was a requirement in [9, Lemma 2.12, Theorem 3.2] to prove validity of the parameterized set of uncertainties in the presence of a rectangular uncertainty constraint of the type $\operatorname{ess\,sup}_{t\geq 0} |\omega_i(t)| < M_i$. The condition $q + r_2 < 0$ is modified as $q + r_2 + \alpha r_2 \leq 0$ for the present case since there are only uniformly decaying disturbances (6.6) present as against rectangular ones. Also it is noted from Lemma 6.1 that under the parametrization (6.8) of the disturbance $\omega = \omega^c$ and the parameterization $x_1^c(t) = c^{r_1} x_1(c^q t), x_2^c(t) = c^{r_2} x_2(c^q t)$ of solutions, the second equation of (4.9), (6.3) can be re-written as follows:

$$\begin{aligned} \dot{x}_2(c^{r_1}x_1, c^{r_2}x_2, c^{-q}t) &= -c^{q+r_2}\mu_1|x_2|^{\alpha}\operatorname{sign}(x_2) - c^{q+r_2}\mu_2|x_1|^{\frac{\alpha}{2-\alpha}}\operatorname{sign}(x_1) \\ &\quad + c^{q+r_2}\omega(c^{-r_1}x_1, c^{-r_2}x_2, c^q t). \end{aligned} \tag{6.61}$$

The first two terms on the right hand side of (6.61) stem from homogeneity of the nominal system (4.9), (6.3) and the third term stem from the parametrization (6.8). Differential inclusion (6.61) is clearly a locally homogeneous differential equation in the sense of Definitions 2.7, 2.6 due to the validity of (6.8).

Remark 6.3. Given the bound M on the uncertainty ω and fixed values of μ_1, μ_2, an arbitrarily large $\tilde{R}$ can be chosen such that the expression $V(s(t_0), v(t_0)) \leq \tilde{R}$ holds true. In other words, global finite time stability follows from Theorem 6.1 as an arbitrarily large $\tilde{R}$ always exists. The only restriction of the whole formulation is that $\tilde{R}$ should be finite [9] (the initial condition set (6.14) must be known). In the context of control of a large class of mechanical systems, for example, the initial condition region is known to be finite and hence the aforementioned condition is not a major restriction. Furthermore, the scalars $\kappa_i, i = 1, 2, 3$ are inversely proportional to the parameter $\tilde{R}$.

6.3.1 Settling Time Estimate

A finite upper bound on the settling time of the closed-loop system (4.9), (6.3) is computed in this section. The identification of the parameters of the homogeneity regions [9, Theorem 3.2] can lead to an explicit formula for the finite settling time. A method similar to the one developed in Section 3.5 for a discontinuous controller [56] is extended to the continuous case to identify the required parameters of the homogeneity regions.

Quasi-Homogeneity Regions

The process of identifying the parameters of the quasi-homogeneity regions can be listed as follows:

1. Identify the radius r of the homogeneity ball

$$\mathcal{B}_r = \left\{(x_1, x_2) : \frac{x_1^2}{r^2} + \frac{x_2^2}{r^2} \leq 1\right\} \tag{6.62}$$

2. Identify the scalar $\delta > 0$ such that the following definition of the homogeneity ellipsoid E_δ holds true [9]:

$$E_\delta = \left\{(x_1, x_2) : \sqrt{\left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2} \leq 1\right\} \subseteq \mathcal{B}_r \tag{6.63}$$

   where r_1, r_2 are dilation weights.

3. Identify the scalars $R' > 0, \bar{R} > 0$ such that the following expressions of the level sets of the Lyapunov function $V_{\tilde{R}}$ hold true in addition to (6.63).

$$\begin{aligned} \Omega_2 &= \left\{(x_1, x_2) : V_{\tilde{R}}(x_1, x_2) \leq \bar{R}\right\} \subseteq E_\delta \\ E_\delta \subseteq \Omega_1 &= \left\{(x_1, x_2) : V_{\tilde{R}}(x_1, x_2) \leq R'\right\} \end{aligned} \tag{6.64}$$

4. Identify the scalar $\hat{R} > 0$ of the level set Ω_3 corresponding to the ellipsoid $E_{\frac{1}{2}\delta}$ [9] in a similar way such that the following expressions are satisfied:

$$\begin{aligned} E_{\frac{1}{2}\delta} &= \left\{(x_1, x_2) : \sqrt{\left(\frac{x_1}{(\frac{1}{2}\delta)^{r_1}}\right)^2 + \left(\frac{x_2}{(\frac{1}{2}\delta)^{r_2}}\right)^2} \leq 1\right\} \\ \Omega_3 &= \left\{(x_1, x_2) : V_{\tilde{R}}(x_1, x_2) \leq \hat{R}\right\} \subseteq E_{\frac{1}{2}\delta} \end{aligned} \tag{6.65}$$

The motivation to achieve the above results is the fact that the estimate of the finite settling time can be obtained by utilising the exponential decay (6.58) once the definitions of the parameters $r, \delta, \bar{R}, R', \hat{R}$ are obtained (recall that the finite time stability results [9, Theorem 3.2] and Theorem 6.1 apply in the vicinity of the origin defined by the homogeneity ball $\mathcal{B}_r$ and ellipsoids E_δ, $E_{\frac{1}{2}\delta}$). The stated steps can be established as follows:

Step 1: Definition of radius r of the homogeneity ball $\mathcal{B}_r$

Lemma 6.2. *Given a positive scalar $M_0 \in (M(\sqrt{2\tilde{R}})^\alpha, 1 + M(\sqrt{2\tilde{R}})^\alpha)$ and conditions $\mu_1 > \max\{1, M\}, \mu_2 > \mu_1$, the following upper bound on 1-norm $\|X\|_1$ holds true in finite time:*

$$|x_1| + |x_2| \leq \frac{M_0 - M\left(2\tilde{R}\right)^{\frac{\alpha}{2}}}{\mu_2} \tag{6.66}$$

for some arbitrary scalars $\epsilon_1 \in (0,1), \alpha \in (\frac{2}{3}, 1)$.

Proof. It should be noted that this Lemma was not needed in the proof of Theorem 6.1. Theorem 6.1 asserts the fact that the trajectories of the closed-loop system (4.9), (6.3) decay exponentially within the vicinity $D_{\tilde{R}}$ of the origin if the conditions $q + r_2 - \alpha r_2 \leq 0$ and (6.7) are met. Hence, in finite time, the system trajectories enter a region close to origin where the homogeneous part of the second differential equation $-\mu_1|x_2|^\alpha - \mu_2|x_1|^{\frac{\alpha}{2-\alpha}}$ dominates the non-homogeneous part $\omega(t)$. Let this region be denoted as

$$\mu_2(|x_1|^{\frac{\alpha}{2-\alpha}} + |x_2|^\alpha) + M(2\tilde{R})^{\frac{\alpha}{2}} < M_0 \tag{6.67}$$

with the positive scalar M_0. Since the Lyapunov function $V_{\tilde{R}}$ decays exponentially, the above constant M_0 can be arbitrarily chosen and the trajectories are guaranteed to enter the region (6.67) in finite time. Let the choice be $M_0 \in (M(2\tilde{R})^{\frac{\alpha}{2}}, 1 + M(2\tilde{R})^{\frac{\alpha}{2}})$. Noting that $|\omega(x_1, x_2, t)| \leq M|x_2|^\alpha \leq M(2\tilde{R})^{\frac{\alpha}{2}}$ within the compact set (6.14), the following can be obtained from (6.67):

$$\mu_2|x_1|^{\frac{\alpha}{2-\alpha}} + \mu_1|x_2|^{\alpha} + |\omega(x_1, x_2, t)| \leq \ \mu_2(|x_1|^{\frac{\alpha}{2-\alpha}} + |x_2|^{\alpha}) + M\left(2\tilde{R}\right)^{\frac{\alpha}{2}} \leq M_0 \quad (6.68)$$

Hence, (6.67) is a conservatively large upper bound for the chosen scalar M_0 on the non-homogeneous right hand side $-\mu_1|x_2|^{\alpha} - \mu_2|x_1|^{\frac{\alpha}{2-\alpha}} + \omega(x_1, x_2, t)$. The following is obtained from (6.68):

$$\mu_2(|x_1|^{\frac{\alpha}{2-\alpha}} + |x_2|^{\alpha}) + M(2\tilde{R})^{\frac{\alpha}{2}} < M_0 \Rightarrow |x_1|^{\frac{\alpha}{2-\alpha}} + |x_2|^{\alpha} \leq \frac{M_0 - M(2\tilde{R})^{\frac{\alpha}{2}}}{\mu_2} \quad (6.69)$$

Noting that the bound appearing in the right hand side of (6.69) is always less than unity due to the conditions $\mu_2 > 1, 0 < M_0 - M\left(2\tilde{R}\right)^{\frac{\alpha}{2}} < 1$ of Lemma 6.2, the inequality $|x_1| + |x_2| \leq |x_1|^{\frac{\alpha}{2-\alpha}} + |x_2|^{\alpha}$ also holds true. Hence a conservative estimate of the homogeneity region within the compact set (6.14) in terms of 1-norm can be obtained from (6.69) as follows:

$$|x_1| + |x_2| \leq \frac{M_0 - M\left(2\tilde{R}\right)^{\frac{\alpha}{2}}}{\mu_2} \quad (6.70)$$

where $\tilde{R} = V(x_1(t_0), x_2(t_0))$. □

The above result is conservative in the sense that the computation of the time based on the exponential decay is more if the bound $|\omega(x_1, x_2, t)| \leq M\left(2\tilde{R}\right)^{\frac{\alpha}{2}}$ is utilised in place of $|\omega(x_1, x_2, t)| \leq |x_2|^{\alpha}$. It is recalled here that the uncertainty $\omega(x_1, x_2, t)$ is treated as a nonhomogeneous perturbation. The finite time stability of homogeneous switched systems in the presence of nonhomogeneous perturbations was established in the previous section (see Theorem 6.1). The following is a well-known relationship between the Euclidian norm $\|X\|_2$ and 1-norm $\|X\|_1$ of vector $X = (x_1, x_2)^{\mathrm{T}}$ (see [111]):

$$\|X\|_1 \leq \sqrt{2}\|X\|_2 \quad (6.71)$$

From (6.70) and (6.71), a conservative bound on the homogeneity radius r of the homogeneity ball $\mathcal{B}_r$ can be obtained as follows:

$$r = \sqrt{x_1^2 + x_2^2} \leq \frac{M_0 - M\left(2\tilde{R}\right)^{\frac{\alpha}{2}}}{\sqrt{2}\mu_2} \quad (6.72)$$

The inequalities (6.71) and (6.72), when combined, will always ensure that the inequality (6.70) holds true.

Step 2: Definition of the parameter δ

The aim is to find $\delta > 0$ such that every point (x_1, x_2) contained within the ellipsoid E_δ is also contained within the homogeneity ball $\mathcal{B}_r$. Having computed the homogeneity radius r in *step 1*, if $\delta > 0$ is chosen such that the equalities

$$\min\left\{\frac{1}{\delta^{2r_1}}, \frac{1}{\delta^{2r_2}}\right\} = \frac{1}{r^2} \Rightarrow \max\left\{\delta^{2r_1}, \delta^{2r_2}\right\} = r^2 \Rightarrow \max\left\{\delta^{r_1}, \delta^{r_2}\right\} = r \quad (6.73)$$

are satisfied, then due to the fact that the equality

$$\min\left\{\frac{1}{\delta^{2r_1}}, \frac{1}{\delta^{2r_2}}\right\}\left(x_1^2 + x_2^2\right) = \frac{1}{r^2}\left(x_1^2 + x_2^2\right) \quad (6.74)$$

always holds true, the inequality

$$\min\left\{\frac{1}{\delta^{2r_1}}, \frac{1}{\delta^{2r_2}}\right\}\left(x_1^2 + x_2^2\right) = \frac{1}{r^2}\left(x_1^2 + x_2^2\right) \le \left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2 \quad (6.75)$$

also holds true. If the given point $(x_1, x_2) \in E_\delta$, then the inequality

$$\sqrt{\left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2} \le 1 \quad (6.76)$$

holds true which, using (6.75), leads to the inequality

$$\frac{1}{r^2}\left(x_1^2 + x_2^2\right) \le 1 \quad (6.77)$$

Hence $(x_1, x_2) \in \mathcal{B}_r$ and the choice (6.73) of δ is indeed valid, which upon further simplification, satisfies:

$$\delta = \min\left\{r^{\frac{1}{r_1}}, r^{\frac{1}{r_2}}\right\} \quad (6.78)$$

The aim of computing $\delta > 0$ such that $E_\delta \subseteq \mathcal{B}_r$ is thus achieved.

Step 3: Definition of scalars $\bar{R}, R'$ of the level sets Ω_1, Ω_2

The first aim is to compute $\bar{R} > 0$ such that the level set Ω_2 satisfies $\Omega_2 \subseteq E_\delta$. Combining the definition of the level set Ω_2 with the inequality (6.29), it suffices that the inequality $V \le \frac{\bar{R}}{M_{\tilde{R}}}$ holds true in order that $\Omega_2 \subseteq E_\delta$ is satisfied for any given (x_1, x_2) in a small vicinity of the origin. Hence the following must be satisfied:

$$\frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2\bar{R}}|x_1|^{\frac{2}{2-\alpha}} + \frac{M_{\tilde{R}}}{2\bar{R}}x_2^2 \le 1 \Rightarrow \left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2 \le 1 \quad (6.79)$$

Having computed the homogeneity ellipsoid parameter δ in *step 2*, if $\bar{R} > 0$ is chosen such that the inequalities

$$\left(\frac{x_1}{\delta^{r_1}}\right)^2 \leq \frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2\bar{R}}|x_1|^{\frac{2}{2-\alpha}} \quad , \quad \left(\frac{x_2}{\delta^{r_2}}\right)^2 \leq \frac{M_{\tilde{R}}}{2\bar{R}}x_2^2 \tag{6.80}$$

are satisfied, then the inequality

$$\left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2 \leq \frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2\bar{R}}|x_1|^{\frac{2}{2-\alpha}} + \frac{M_{\tilde{R}}}{2\bar{R}}x_2^2 \tag{6.81}$$

always holds true. For a given point $(x_1, x_2) \in \Omega_1$, the inequality

$$\frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2\bar{R}}|x_1|^{\frac{2}{2-\alpha}} + \frac{M_{\tilde{R}}}{2\bar{R}}x_2^2 \leq 1 \tag{6.82}$$

holds true, which using (6.81), leads to the inequality

$$\left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2 \leq 1 \tag{6.83}$$

Hence $(x_1, x_2) \in E_\delta$ and the choice (6.80) of $\bar{R}$ is indeed valid. Noting from (6.22) that $x_1^2 < |x_1|^{\frac{2}{2-\alpha}}\rho^{1-\alpha}$, requirement (6.80) can be reformulated as follows:

$$\begin{aligned} \left(\frac{x_1}{\delta^{r_1}}\right)^2 &\leq \frac{|x_1|^{\frac{2}{2-\alpha}}}{\delta^{2r_1}}\left(\frac{2\tilde{R}}{(2-\alpha)\mu_2}\right)^{1-\alpha} \leq \frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2\bar{R}}|x_1|^{\frac{2}{2-\alpha}}, \\ \left(\frac{x_2}{\delta^{r_2}}\right)^2 &\leq \frac{M_{\tilde{R}}}{2\bar{R}}x_2^2. \end{aligned} \tag{6.84}$$

The above inequalities (6.84) result in the following definition of $\bar{R}$:

$$\bar{R} = \frac{M_{\tilde{R}}}{2}\min\left\{\delta^{2r_1}\mu_2^{2-\alpha}\frac{(2-\alpha)^{2-\alpha}}{\left(2\tilde{R}\right)^{1-\alpha}}, \quad \delta^{2r_2}\right\} \tag{6.85}$$

The second aim is to compute $R' > 0$ such that the expression $E_\delta \subseteq \Omega_1$ is satisfied. Combining the definition of the level set Ω_1 with the inequality (6.29), it suffices that the inequality $V \leq \frac{R'}{M_{\tilde{R}}}$ holds true in order that $E_\delta \subseteq \Omega_1$ is satisfied for any given (x_1, x_2) in a small vicinity of the origin. Hence the following must be satisfied::

$$\left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2 \leq 1 \Rightarrow \frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2R'}|x_1|^{\frac{2}{2-\alpha}} + \frac{M_{\tilde{R}}}{2R'}x_2^2 \leq 1 \tag{6.86}$$

If the inequality

$$\frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2R'}|x_1|^{\frac{2}{2-\alpha}} + \frac{M_{\tilde{R}}}{2R'}x_2^2 \leq \left(\frac{x_1}{\delta^{r_1}}\right)^2 + \left(\frac{x_2}{\delta^{r_2}}\right)^2 \tag{6.87}$$

holds true then (6.86) always holds true for all $(x_1, x_2) \in E_\delta$. Equation (6.87) always holds true if the following is ensured:

$$\frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2R'}|x_1|^{\frac{2}{2-\alpha}} \leq (1-\epsilon_1), \quad \frac{M_{\tilde{R}}}{2R'}x_2^2 \leq \epsilon_1 \left(\frac{x_2}{\delta^{r_2}}\right)^2 \tag{6.88}$$

where, $0 < \epsilon_1 < 1$ is an arbitrary constant. The fact that $(x_1, x_2) \in E_\delta$ leads to $|x_1| \leq \delta^{r_1}$ by definition. Hence (6.88) can be further simplified to derive formula for R' by enforcing the following sub-conditions:

$$\begin{aligned} \frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2R'}|x_1|^{\frac{2}{2-\alpha}} &\leq \frac{\mu_2(2-\alpha)M_{\tilde{R}}}{2R'}\delta^{\frac{2r_1}{2-\alpha}} \leq (1-\epsilon_1), \\ \frac{M_{\tilde{R}}}{2R'}x_2^2 &\leq \epsilon_1 \left(\frac{x_2}{\delta^{r_2}}\right)^2. \end{aligned} \tag{6.89}$$

Hence the following formula

$$R' = \frac{M_{\tilde{R}}}{2}\max\left\{\delta^{\frac{2r_1}{2-\alpha}}\frac{\mu_2(2-\alpha)}{1-\epsilon_1}, \quad \frac{\delta^{2r_2}}{\epsilon_1}\right\} \tag{6.90}$$

can be deduced from (6.89). The aims of computing $R' > 0, \bar{R} > 0$ such that $\Omega_2 \subseteq E_\delta \subseteq \Omega_1$ are thus achieved.

Step 4: Definition of the parameter $\hat{R}$ of the level set Ω_3 Similar arguments to those outlined in *Step 3* produces the following formula[2]:

$$\hat{R} = \frac{M_{\tilde{R}}}{8}\min\left\{\frac{\delta^{2r_1}\left(\mu_2(2-\alpha)\right)^{2-\alpha}}{\left(2\tilde{R}\right)^{1-\alpha}}, \delta^{2r_2}\right\} \tag{6.91}$$

Finite Settling Time Estimate

The quasi-homogeneity concept[3] is geometrically depicted in Fig. 6.1. Trajectories of the system (4.9), (6.3) in the phase plane (x_1, x_2) are also schematically shown. The existence of a uniformly decaying global Lyapunov function $V_{\tilde{R}}$ is utilized (see (6.58)). The point *O_1* is the system initial condition which corresponds to the boundary of the level set $\Omega_{\tilde{R}} = \left\{(x_1, x_2) : V_{\tilde{R}}(x_1, x_2) \leq M_{\tilde{R}}\tilde{R}\right\}$ where $\tilde{R} = V(x_1(t_0), x_2(t_0))$. Then, due to the fact that the system decays exponentially towards the origin, it can be deduced that the trajectory enters the homogeneity ball $\mathcal{B}_r$ in finite time, where r is defined by (6.72), and subsequently enters the homogeneity ellipsoid E_δ. This in

[2] All constants $M_{\tilde{R}}, M_{\bar{R}}, M_{\hat{R}}, M_{\acute{R}}$ corresponding to the semi-global regions $D_{\tilde{R}}, \Omega_i, i = 1, 2, 3$ can be chosen as $M_{\tilde{R}} = M_{\bar{R}} = M_{\hat{R}} = M_{R'}$ since only the lower bound (6.30) is to be satisfied.

[3] This figure is inspired from Section 3.5 of Chapter 3 which studied a switched counterpart $\alpha = 0$ of the control law (6.3).

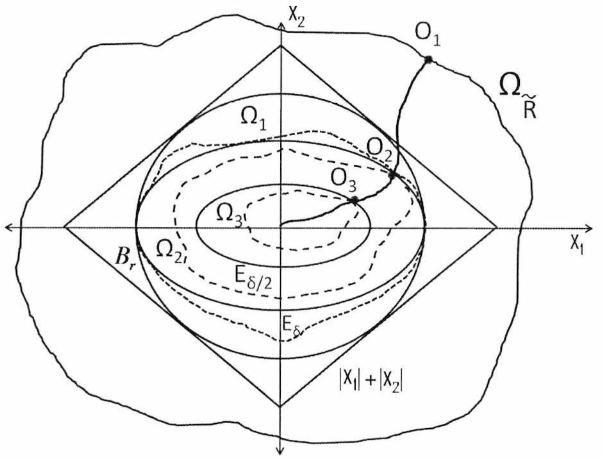

FIGURE 6.1: Quasi-homogeneity concept: homogeneous regions $\|(x_1, x_2)\|_1$, homogeneity ball $\mathcal{B}_r$, homogeneity ellipsoids (E_δ, $E_{\frac{1}{2}\delta}$) and level sets $\Omega_R, \Omega_1, \Omega_2, \Omega_3$

turn causes the trajectories of the closed-loop system to satisfy the definition of the level set $\Omega_2 = \{(x_1, x_2) : V_{\tilde{R}}(x_1, x_2) \leq \bar{R}\} \subseteq E_\delta$ of the Lyapunov function $V_{\tilde{R}}$ in finite time. This corresponds to the point O_2. Finally, finite time stability follows from the quasi-homogeneity principle once the system trajectories are inside the ellipsoid E_δ (see Theorem 6.1 and [9, Th. 3.1]). As a consequence, the settling time of the system is the summation of the following:

$$\mathcal{T} = \mathcal{T}_{O_1 - O_2} + \mathcal{T}_h \tag{6.92}$$

where $\mathcal{T}_{O_1-O_2}$ is the time taken by the state trajectories of the closed-loop system to attain the level set Ω_2 (point O_2) from the initial condition level set $\Omega_{\tilde{R}}$ (point O_1) and $\mathcal{T}_h$ is finite settling time of the system to attain equilibrium point $(0, 0)$ from the homogeneity boundary $E_\delta \subseteq \Omega_1$ which can be readily computed using the expression (3.12) of [9] as follows:

$$\mathcal{T}_h = \frac{c_0{}^q}{1 - 2^q} \left\{ \sup_{(x_1, x_2) \in E_\delta} \mathcal{T}_{O_2 - O_3} \right\} \tag{6.93}$$

where q is homogeneity degree, c_0 is a lower estimate of the homogeneity parameter and $\mathcal{T}_{O_2-O_3}$ is the time taken by the state trajectories of the closed-loop system to travel from the homogeneity boundary $E_\delta \subseteq \Omega_1$ to the boundary $\Omega_3 \subseteq E_{\frac{1}{2}\delta}$ (point O_3). While computing $\mathcal{T}_{O_2-O_3}$, the necessity to use the boundary of the level set Ω_1 in place of Ω_2 stems from the fact that the supremum of $\mathcal{T}_{O_2-O_3}$ has to be taken into consideration while computing the worst possible decay of the Lyapunov function. Hence, the boundary given by Ω_2 has to be utilized to compute $\mathcal{T}_{O_1-O_2}$ and that given by Ω_1 to compute $\mathcal{T}_{O_2-O_3}$ in order to encompass the worst case scenario. Although an overlap of time contributions may occur in the summation (6.92) leading to a conservative result since $\Omega_2 \subseteq \Omega_1$ holds true, the estimate of the settling time thus obtained is a true upper-bound, nevertheless. The terms $\mathcal{T}_{O_1-O_2}$ and $\mathcal{T}_{O_2-O_3}$ can be estimated for $\gamma = 4$ from the decay (6.58) as follows:

$$\mathcal{T}_{O_1-O_2}(\mu_1,\mu_2,M,\tilde{R},\bar{R}) = \begin{cases} t_{O_1} + \bar{t}, & \text{if } M_{\tilde{R}}\tilde{R} \geq 1, \bar{R} > 1; \\ t_{O_1} + t_1 + \frac{1-\bar{R}^3}{3\bar{\kappa}\bar{R}^3}, & \text{if } M_{\tilde{R}}\tilde{R} \geq 1, \bar{R} < 1; \\ t_{O_1} + \frac{\left(\left(\frac{M_{\tilde{R}}\tilde{R}}{\bar{R}}\right)^3 - 1\right)}{3\bar{\kappa}(\bar{R}M_{\tilde{R}})^3}, & \text{if } M_{\tilde{R}}\tilde{R} < 1. \end{cases}$$

$$\mathcal{T}_{O_2-O_3}(\mu_1,\mu_2,M,\tilde{R},\hat{R}) = \begin{cases} \frac{\ln(\frac{R'}{\hat{R}})}{\bar{\kappa}}, & R' > 1, \hat{R} > 1; \\ \frac{\ln R'}{\bar{\kappa}} + \frac{1-\hat{R}^3}{3\bar{\kappa}\hat{R}^3}, & R' > 1, \hat{R} < 1; \\ \frac{\left(\left(\frac{R'}{\hat{R}}\right)^3 - 1\right)}{3\bar{\kappa}(\hat{R})^3}, & R' < 1. \end{cases} \tag{6.94}$$

where $\bar{t}$ and t_1 can be obtained from the first equality of the exponential decay (6.58) as

$$\bar{t} = \frac{\ln \frac{M_{\tilde{R}}\tilde{R}}{\bar{R}}}{\bar{\kappa}}, \quad t_1 = \frac{\ln\left(M_{\tilde{R}}\tilde{R}\right)}{\bar{\kappa}}, \tag{6.95}$$

and the substitutions $V_{\tilde{R}}(t_{O_1}) = M_{\tilde{R}}\tilde{R}, V_{\tilde{R}}(t_{O_2}) = \bar{R}$ have been utilized corresponding to the level sets $\Omega_{\tilde{R}}$ and Ω_2 at time instants t_{O_1} and t_{O_2} respectively in the first equality and substitutions $V_{\tilde{R}}(t_{O_2}) = R', V_{\tilde{R}}(t_{O_3}) = \hat{R}$ have been utilized corresponding to the level sets Ω_1 and Ω_3 at time instants t_{O_2} and t_{O_3} respectively in the second equality while utilizing (6.58).

Under the stated assumptions, the homogeneity parameters $r, \delta, \tilde{R}, \bar{R}, \hat{R}$ outlined in Section 6.3.1 and in turn the settling time estimate (6.92) can be computed *apriori.* It remains to give conditions under which the estimates $\mathcal{T}_{O_1-O_2}, \mathcal{T}_h$ are guaranteed to be positive, or in other words, expressions $M_{\tilde{R}}\tilde{R} > \bar{R}$ and $R' > \hat{R}$ always hold true.

Lemma 6.3. *Given a positive scalar $M_0 \in (M(\sqrt{2\tilde{R}})^\alpha, 1 + M(\sqrt{2\tilde{R}})^\alpha)$, conditions $\mu_1 > \max\{1, M\}, \mu_2 > \max\{\mu_1, \frac{1-\epsilon_1}{(2-\alpha)\epsilon_1}, \frac{2-\alpha}{4}\}$, $\tilde{R} > 1$ and the condition $1 > \epsilon_1 > 1 - \bar{\epsilon}^2 > 0$, with some positive scalar $\bar{\epsilon} \in (0,1)$, the expressions $M_{\tilde{R}}\tilde{R} > \bar{R}$ and $R' > \hat{R}$ always hold true.*

Proof. Due to the choice $M_0 \in (M(\sqrt{2\tilde{R}})^\alpha, 1 + M(\sqrt{2\tilde{R}})^\alpha)$, the inequality

$$(M_0 - M(\sqrt{2\tilde{R}})^\alpha)^2 < 1 \tag{6.96}$$

holds true. Using the condition $\mu_2 > \frac{2-\alpha}{4}$, (6.96) can be modified as follows

$$\frac{(M_0 - M(\sqrt{2\tilde{R}})^{\alpha})^2}{4\mu_2^2}\mu_2(2-\alpha) < 1. \tag{6.97}$$

The inequality $\tilde{R} > 1$ holds true by assumption and hence (6.97) can be re-written as

$$\frac{(M_0 - M(\sqrt{2\tilde{R}})^{\alpha})^{\alpha}}{4\mu_2^2}\mu_2(2-\alpha) < \tilde{R}. \tag{6.98}$$

Rearranging (6.98) and raising the power by $2-\alpha$ results in

$$\left(\frac{M_0 - M(\sqrt{2\tilde{R}})^{\alpha}}{\sqrt{2}\mu_2}\right)^{2(2-\alpha)}(\mu_2(2-\alpha))^{2-\alpha} < 2\tilde{R}^{2-\alpha}. \tag{6.99}$$

Recalling the definition of δ from (6.78), multiplying by $M_{\tilde{R}}$ on both sides, and noting that $\frac{2r_1}{r_2} = 2(2-\alpha)$ produces

$$M_{\tilde{R}}\delta^{\frac{2r_1}{r_2}}\frac{(\mu_2(2-\alpha))^{2-\alpha}}{2\tilde{R}^{1-\alpha}} < 2M_{\tilde{R}}\tilde{R}, \tag{6.100}$$

which in turn, recalling the definition of $\bar{R}$ from (6.85), gives

$$M_{\tilde{R}}\delta^{\frac{2r_1}{r_2}}\frac{(\mu_2(2-\alpha))^{2-\alpha}}{2\tilde{R}^{1-\alpha}} < M_{\tilde{R}}\tilde{R}. \tag{6.101}$$

Using a similar analysis, (6.96) produces

$$\frac{M_{\tilde{R}}}{2}\delta^{2r_2} < M_{\tilde{R}}\tilde{R}. \tag{6.102}$$

Hence, $M_{\tilde{R}}\tilde{R} > \bar{R}$ follows from (6.101), (6.102). The second claim to be proved is $R' > \hat{R}$. First, the following, which stems from the condition $\mu_2 > \frac{(1-\epsilon_1)}{\epsilon_1(2-\alpha)}$, is in order to simplify (6.90):

$$\begin{aligned}\mu_2 > \frac{(1-\epsilon_1)}{\epsilon_1(2-\alpha)} \quad &\Rightarrow \quad \frac{\mu_2(2-\alpha)}{1-\epsilon_1} > \frac{1}{\epsilon_1}\\ &\Rightarrow \quad \delta^{\frac{2r_1}{2-\alpha}-2r_2}\frac{\mu_2(2-\alpha)}{1-\epsilon_1} > \frac{1}{\epsilon_1} \quad \text{because } \frac{2r_1}{2-\alpha} - 2r_2 = 0.\\ &\Rightarrow \quad \delta^{\frac{2r_1}{2-\alpha}}\frac{\mu_2(2-\alpha)}{1-\epsilon_1} > \frac{\delta^{2r_2}}{\epsilon_1}\end{aligned} \tag{6.103}$$

Recalling the definition (6.90) of R', (6.103) produces

$$R' = \frac{M_{\tilde{R}}}{2}\delta^{\frac{2r_1}{2-\alpha}}\frac{\mu_2(2-\alpha)}{1-\epsilon_1}. \tag{6.104}$$

It can be seen from the definition of $\hat{R}$ in (6.91) that the terms inside the $\min\{\cdot\}$ function are less than unity since $\delta < 1$ due to the condition $M_0 \in (M(\sqrt{2\tilde{R}})^{\alpha}, 1 + M(\sqrt{2\tilde{R}})^{\alpha})$. Hence, in order to prove $R' > \hat{R}$, it suffices to prove that the right hand side of (6.104) is greater than $\frac{M_{\tilde{R}}}{8}$. Let the scalar $\bar{\epsilon} \in (0,1)$ be selected small enough such that

$$1 > \frac{(M_0 - M(2\tilde{R})^{\frac{\alpha}{2}})}{\sqrt{2}} > \bar{\epsilon}\mu_2 > 0, \quad \epsilon_1 > 1 - \bar{\epsilon}^2 \Leftrightarrow \frac{\bar{\epsilon}^2}{1-\epsilon_1} > 1 \tag{6.105}$$

holds true. This is always possible for fixed values of μ_2. Then, the following holds from (6.105):

$$\begin{aligned}
\frac{(M_0 - M(2\tilde{R})^{\frac{\alpha}{2}})}{\sqrt{2}} > \bar{\epsilon}\mu_2 \quad &\Rightarrow \quad \frac{(M_0 - M(2\tilde{R})^{\frac{\alpha}{2}})}{\sqrt{2}\mu_2} > \bar{\epsilon} \\
&\Rightarrow \quad \delta > \bar{\epsilon}^{\frac{1}{r_2}} \quad (\text{since (6.78) and } r < 1, r_1 > r_2) \text{ produces } \delta = r^{\frac{1}{r_2}} \\
&\Rightarrow \quad \delta^{\frac{2r_1}{2-\alpha}} > \bar{\epsilon}^{\frac{2r_1}{(2-\alpha)r_2}} \qquad (6.106) \\
&\Rightarrow \quad \delta^{\frac{2r_1}{2-\alpha}} > \bar{\epsilon}^2 \\
&\Rightarrow \quad \delta^{\frac{2r_1}{2-\alpha}} \frac{\mu_2(2-\alpha)}{1-\epsilon_1} > \frac{\mu_2(2-\alpha)}{1-\epsilon_1}\bar{\epsilon}^2 \\
&\Rightarrow \quad \delta^{\frac{2r_1}{2-\alpha}} \frac{\mu_2(2-\alpha)}{1-\epsilon_1} > \frac{\mu_2(2-\alpha)}{1-\epsilon_1}\bar{\epsilon}^2 > 1 \\
&\qquad\qquad \text{since } \mu_2 > 1, 2-\alpha > 1, \tfrac{\bar{\epsilon}^2}{1-\epsilon_1} > 1.
\end{aligned}$$

Noting that $\min\left\{\frac{\delta^{2r_1}(\mu_2(2-\alpha))^{2-\alpha}}{(2\tilde{R})^{1-\alpha}}, \delta^{2r_2}\right\} < 1$, combining the last inequality of (6.106) with (6.104) produces

$$\frac{M_{\tilde{R}}}{2}\frac{\mu_2(2-\alpha)}{1-\epsilon_1}\bar{\epsilon}^2 > \frac{M_{\tilde{R}}}{8}\min\left\{\frac{\delta^{2r_1}(\mu_2(2-\alpha))^{2-\alpha}}{(2\tilde{R})^{1-\alpha}}, \delta^{2r_2}\right\} \Rightarrow R' > \hat{R}. \tag{6.107}$$

Thus, the estimate (6.92) proves to be a positive real constant. □

Lemma 6.3 gives a conservative estimate of the upper bound on the settling time (6.92) even for those initial conditions for which $D_{\tilde{R}}$ can be defined with values of $\tilde{R}$ smaller than unity.

Remark 6.4. The estimate of the homogeneity parameter c should satisfy $c \geq c_0$ for the chosen c_0 where c_0 is the lower estimate of the homogeneity parameter. It can be seen from the above development that the closed-loop system is homogeneous once it is inside the ellipsoid E_δ. The identity $\delta R_0^{-1} = c$ then leads to $c = 1$ because $R_0 = \delta$ is chosen to facilitate the application of (6.93), where the scalar $R_0 > 0$ represents the largest homogeneity ellipsoid E_{R_0} (see (3.12) of [9] for more details). Hence $c_0 = 1$ is a valid choice.

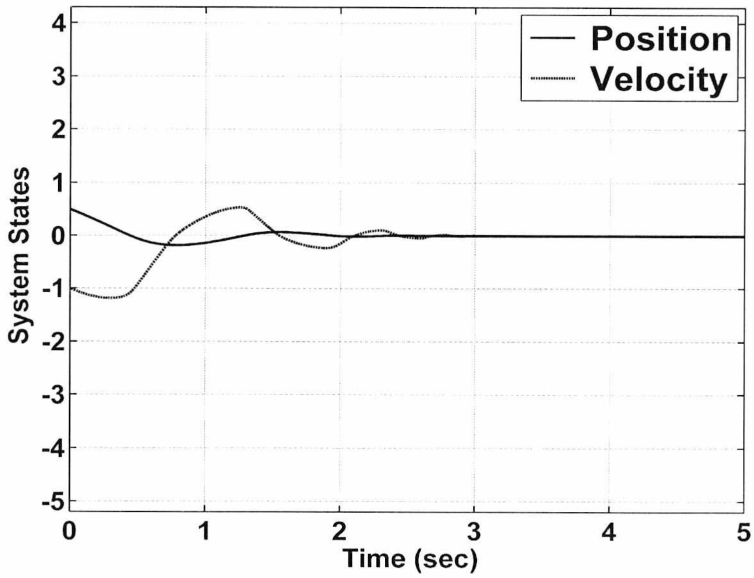

FIGURE 6.2: Finite setting time behaviour of the closed-loop system (4.9), (6.3)

6.3.2 Numerical Simulation

Let a perturbed double integrator (4.9) be considered with the controller (6.3). Let the parameters $\mu_1 = 2, \mu_2 = 4, \alpha = 0.5$ be chosen. Let the parameter M defined in (6.6) be selected as $M = 1$. Let the uncertainty be selected as $\omega(x,t) = M|x_2|^\alpha \bar{a}(t)$, where $\bar{a}(t)$ is a number generated randomly in the interval $(0, 1)$. The resulting response is shown Fig. 6.2 where it is clear that the systems states settle to the origin in finite time (approximately in $3\,sec$) despite the uncertainty $\omega(x_1, x_2, t)$. The phase plane plot of the system can be found in Fig. 6.3.

6.4 Unilateral constraints and resets

The second objective set out in Section 6.2 is studied in this subsection. This section is a parallel to Chapter 5 as discussed in the first few paragraphs of this chapter since it achieves the *equiuniform continuous* finite time stabilisation of a perturbed double integrator such as (4.9) when resets in velocity are present due to a unilateral constraint.

The main focus is on mechanical systems with resets in velocity as studied in Chapter 5. A perturbed double integrator is considered with a unilateral constraint surface. The velocity undergoes an instantaneous *reset* when the inelastic collision occurs. Like Chapter 5, it is assumed in this section that the restitution or reset map relating

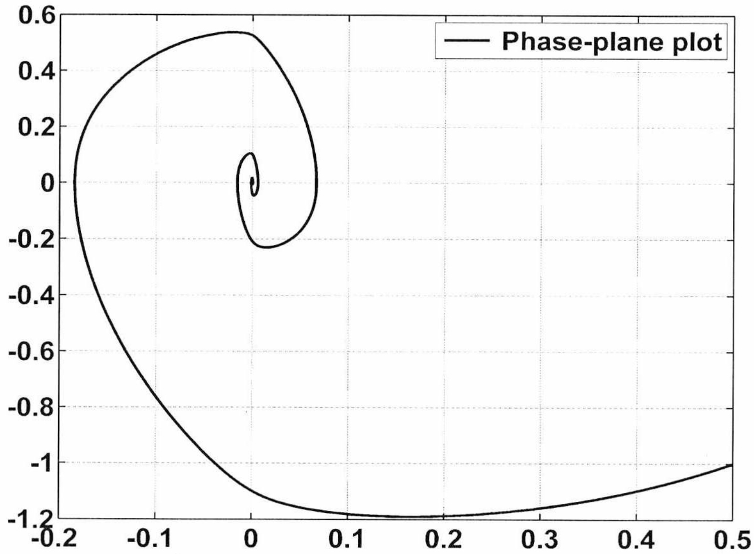

FIGURE 6.3: Phase-plane plot of the closed-loop system (4.9), (6.3)

the velocities just before and after the time of impact is fully known. However, no such assumption is made on the time of impact. The continuous controller (6.3) is considered for uniform finite time stabilization in the presence of the resets in velocity. The method of Zhuravlev-Ivanov non-smooth transformation [125, 127] is utilised to first transform the system into a variable-structure system without jumps (also see references cited in [23, Ch. 1, Sec. 1.4]).

Within engineering applications, such a transformation is very useful in the analysis of vibro-impact systems [125, 126] (also see relevance to mechanical problems described in Section 5.2). The resulting transformed system is a switched homogeneous system with a negative homogeneity degree as defined in Definition 2.7 whose solutions are well-defined in the sense of Filippov's definition (as defined in Definition 2.1), an attribute absent in the case of the original jump system (see [95] for solution concept of systems with jumps and friction). It is important to note that the use of finite time stability of switched systems [9] is the only suitable method for the proposed synthesis due to the switched nature of the transformed system despite the continuous controller (6.3). This is because all the existing references on continuous homogeneous systems [11, 12, 14, 46, 85, 86] require continuity of vector field, a condition unavoidably violated at the time of jumps.

The reader is referred to Section 6.1 for the theoretical and practical motivations to study the problem studied in this section which is formulated next.

Problem Statement

Consider the closed-loop system (5.1) (6.3) in the presence of the disturbance ω with an upper bound (6.6). It is noteworthy that the solutions of the closed-loop system (5.1),(6.3), which involve switched terms along with impact, can be defined using existing methods (see [23], [95], [25] and [47] for solutions concept of differential inclusions with both friction and collisions terms on the right hand side).

The aim of this section is to prove finite time stability of the closed-loop system (5.1), (6.3) in the presence of the disturbance ω that satisfies the constraint (6.6).

Global asymptotic stability of a perturbed double integrator without jumps in velocity has recently been established [14, Theorem 2]. As discussed in Section 6.3, global finite time stability for a class of perturbations can be established by combining the geometric homogeneity [83, 84] results with Lyapunov stability results [13] as there always exists a strict Lyapunov function [84] for the homogeneous vector field (5.1), (6.3) when there is no jump. However, proving equiuniform finite time stability of (5.1), (6.3) is an open problem regardless of whether the jumps in velocity are present or not. Previous section solved this problem for the case when there are no jumps and this section solves the same problem for the case when jumps in velocity are present.

Remark 6.5. While finite time stability of the unperturbed closed-loop system (5.1), (6.3) without jumps and with $\alpha \in (0, 1)$ can be established from either of the two methods [83, Th. 2], [9, Th. 3.2], the result for discontinuous systems [9] is more general as it encompasses the discontinuous case when $\alpha = 0$ [14, Th. 1]. Moreover, existing results [83, 84, 86] encompass only continuous disturbances $\omega(t)$ whereas the proposed method accommodates even the piece-wise continuous ones as the result on discontinuous systems [9] is utilised and extended in Theorem 6.2 below.

6.4.1 Global equiuniform finite time stability

This section employs the method of Zhuravlev-Ivanov non-smooth transformation [125, 127], [23, Sec. 1.4.2] to transform the impact system (5.1) into a jump-free system. The main philosophy of the non-smooth transformation was described in detail in Chapter 5. The variable structure-wise transformed version of the system (5.1), (6.3)

$$\begin{aligned} \dot{s} &= R\, v \\ \dot{v} &= R^{-1}\text{sign}(s)\left(u(|s|, Rv\text{sign}(s)) + \omega(|s|, Rv\text{sign}(s), t)\right) \end{aligned} \tag{6.108}$$

is then obtained by employing (5.5) and using the dynamics (5.1a), (5.1b) (see [23, Ch. 1] and Section 5.4 for a viewpoint from mechanics). By combining (6.3) and (5.5), the controller (6.3) can be represented in the transformed coordinates as follows:

$$u(|s|, R\,v\,\mathrm{sign}(s)) = -\mu_1 |Rv\mathrm{sign}(s)|^{\alpha}\mathrm{sign}(Rv\mathrm{sign}(s)) - \mu_2 \big||s|\big|^{\frac{\alpha}{2-\alpha}}\mathrm{sign}(|s|) \quad (6.109)$$

Substituting (6.109) into (6.108), the closed-loop system in the coordinate frame (s, v) can be obtained as follows:

$$\begin{aligned} \dot{s} &= R\,v \\ \dot{v} &= -\mu_1 R^{\alpha-1}|v|^{\alpha}\mathrm{sign}(v) - \mu_2 R^{-1}|s|^{\frac{\alpha}{2-\alpha}}\mathrm{sign}(s) + R^{-1}\mathrm{sign}(s)\omega(|s|, Rv\mathrm{sign}(s), t) \end{aligned} \quad (6.110)$$

Furthermore, the upper-bound (6.6) can be revised in the new coordinates as follows:

$$ess\sup_{t\geq 0} |\omega(s, v, t)| \leq M|Rv\mathrm{sign}(s)|^{\alpha} = MR^{\alpha}|v|^{\alpha} \quad (6.111)$$

Remark 6.6. As discussed in Section 5.4 in remarks on non-smooth transformation, the origin $s = v = 0$ of the system (6.110) corresponds to the origin $x_1 = x_2 = 0$ of the system (5.1),(6.3). Since the transformation (5.5) is not invertible, it is important to recall that one starts from the closed-loop system (6.110) and that the original dynamics can be recovered via (5.5). The non-smooth transformation approach holds true only when the constraint surface is of co-dimension one [23]. The solutions of (6.110) are well defined in the sense of Filippov's definition of solutions [6]. Furthermore, such a formulation admits both friction and jump phenomena, while guaranteeing existence of the solutions. The formulation (6.110) captures the infinite rebounds [38] (the so-called Zeno behavior as discussed in Chapter 5) once the system stabilizes on the origin and in turn on the constraint surface.

A test of the existing Lyapunov functions

As noted in [14], the proposed controller (6.3) is simpler than the controller proposed in [12]. Recall that the control law in the original coordinates is given in [12] as follows:

$$u(x_1, x_2) = -\mathrm{sign}(x_2)|x_2|^{\alpha} - \mathrm{sign}(\phi_{\alpha}(x_1, x_2))|\phi_{\alpha}(x_1, x_2)|^{\frac{\alpha}{2-\alpha}} \quad (6.112)$$

where, $\phi_{\alpha}(x_1, x_2) = x_1 + \frac{1}{2-\alpha}\mathrm{sign}(x_2)|x_2|^{2-\alpha}$. The closed-loop system is then given by (5.1), (6.112). The following transformed closed-loop system can be obtained by applying the non-smooth coordinate transformation (5.5):

$$\begin{aligned} \dot{s} &= Rv \\ \dot{v} &= R^{-1}\mathrm{sign}(s)\left(u(|s|, Rv\mathrm{sign}(s))\right) \\ &= -R^{\alpha-1}\mathrm{sign}(v)|v|^{\alpha} - R^{-1}\mathrm{sign}(s)\mathrm{sign}(\phi_{\alpha})|\phi_{\alpha}|^{\frac{\alpha}{2-\alpha}}, \end{aligned} \quad (6.113)$$

where $\phi_\alpha(s,v) = |s| + \frac{R^{2-\alpha}}{2-\alpha}\text{sign}(sv)|v|^{2-\alpha}$. Let the Lyapunov function proposed in [12] be defined in the transformed coordinates as follows:

$$V(s,v) = \frac{2-\alpha}{3-\alpha}|\phi_\alpha|^{\frac{3-\alpha}{2-\alpha}} + r_2 v\phi_\alpha + \frac{r_1}{3-\alpha}|v|^{3-\alpha} \tag{6.114}$$

where, $r_1 > 1, r_2 < 1$ are arbitrary scalars. The temporal derivative of (6.114) along the trajectories of the transformed closed-loop system (6.113) can be obtained as follows:

$$\begin{aligned}\dot{V}(s,v) = &-R^{\alpha-1}r_1 v^2 - R^{1-\alpha}|v|^{1-\alpha}|\phi_\alpha|^{\frac{1+\alpha}{2-\alpha}} \\ &- R^{-1}r_2|\phi_\alpha|^{\frac{2}{2-\alpha}}\text{sign}(s) - R^{\alpha-1}r_2\phi_\alpha\text{sign}(v)|v|^\alpha \\ &- (R^{1-\alpha}r_2 + R^{-1}r_1\text{sign}(s))\text{sign}(v\phi_\alpha)|v|^{2-\alpha}|\phi_\alpha|^{\frac{\alpha}{2-\alpha}}\end{aligned} \tag{6.115}$$

Although the homogeneity properties

$$\begin{aligned}V(k^{2-\alpha}s, kv) &= k^{3-\alpha}V(s,v) \\ \dot{V}(k^{2-\alpha}s, kv) &= k^2\dot{V}(s,v)\end{aligned} \tag{6.116}$$

hold true for the transformed system, it is not mathematically correct to restrict the analysis on the closed curve

$$(s,v): \max_{s,v\neq(0,0)} (|\phi_\alpha|^{\frac{1}{2-\alpha}}, |v|) = 1 \tag{6.117}$$

encircling the origin of the closed-loop system (6.113) as it was done in [12]. This is because it is always possible to have initial conditions either starting from or intersecting the semi-axis $(s,v) : v = 0, s < 0$ of the transformed system (6.113) thereby causing (6.115) to take positive values due to an additional 'sign(s)' in the third term on the right hand side of (6.115). This is clearly in contrast to the negative definiteness of the derivative of the Lyapunov function obtained in [12] due to which it was possible to perform analysis only on the closed curve encircling the origin when combined with the homogeneity property (6.116).

Intuitive justification using phase plots

The non-positive definiteness of the temporal derivative of the Lyapunov function of the *transformed system* can be intuitively understood by analysing the phase plane plot of the *original system* (5.1),(6.112). It is clear from figures 6.4 and 6.6 that there can be at most one jump in the system. This is because the controller (6.112) does not let the position trajectory hit the constraint surface (5.1c) if the initial conditions are in the first quadrant $(x_1, x_2) : x_1 \geq 0, x_2 > 0$ (the restitution (5.1d) does not apply even for $x_1 = 0$ since $x_2 > 0$). This is theoretically verified by the fact that the controller (6.112) renders the manifold $\phi_\alpha = 0$ attractive for all initial conditions ([12, Fig. 1])

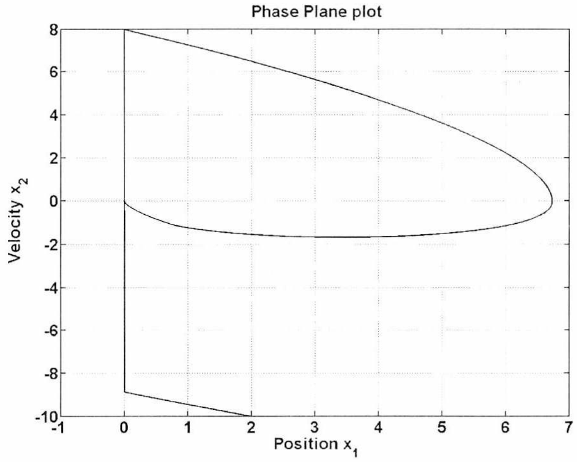

FIGURE 6.4: Phase-Plane plot of the closed-loop system (5.1), (6.112) with initial conditions $(x_1^0, x^0, 2) = (2, -10)$

and the trajectory slides on this manifold reaching to origin in finite time. For the jump system, this means that all the trajectories starting from the initial conditions emanating from the first quadrant $(x_1, x_2) : x_1 \geq 0, x_2 > 0$ never cross the constraint surface (5.1c) and hence there is no accumulation of impacts. In fact, the trajectory 'smoothly' goes to zero in finite time for this set of initial conditions (see post impact behaviour in figure 6.5). This might be perceived as an advantage of this controller at least for the initial conditions starting from the first quadrant $(x_1, x_2) : x_1 \geq 0, x_2 > 0$.

However, there can be initial conditions in the fourth quadrant $(x_1, x_2) : x_1 \geq 0, x_2 < 0$ such that the trajectories intersect the constraint surface (5.1c) ([12, Fig. 1]) with $x_2 < 0$ so that the restitution rule (5.1d) applies. This is a jump situation (which can occur at initial time t_0 as well) causing a jump in the Lyapunov function proposed in [12] as shown in figure 6.6.

It should be noted that the jump occurs only once as now the velocity resets to the first quadrant $(x_1, x_2) : x_1 \geq 0, x_2 > 0$ from where there are no more jumps as discussed above. This reset instant means that the transformed variable s changes sign or in other words (6.115) can take a positive value either before or after the jump event.

Noting that the proof of finite time stability given in [12, Proposition 1] depends on the conditions of continuity, differentiability and existence of the differential inequality (see [141, Theorem 1]), it cannot be applied to the jump system (5.1),(6.112) because the Lyapunov function does not remain continuous and it does not satisfy conditions (ii),(iii) of [141, Theorem 1] at the time of jump. Hence, the main results of [12],[141] do not apply. The results of [12],[141] become applicable if the only jump present is omitted from the analysis. However, such a proof is rather a heuristic one and not a

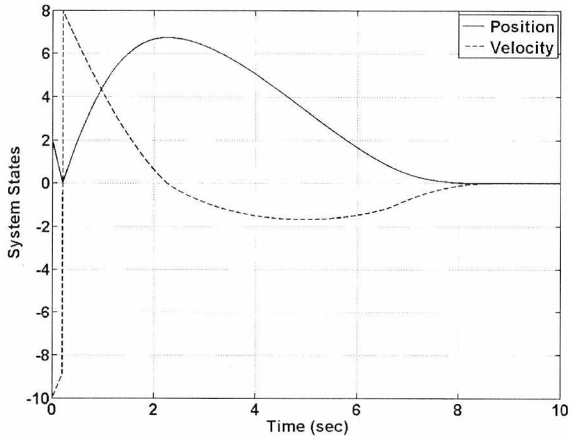

FIGURE 6.5: Time histories of the states of the closed-loop system (5.1), (6.112) with initial conditions $(x_1^0, x_2^0) = (2, -10)$

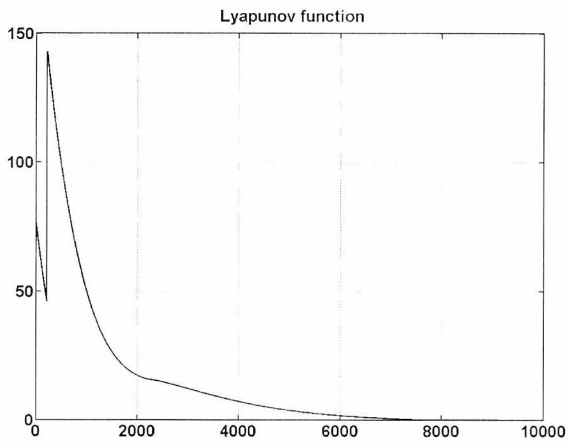

FIGURE 6.6: Jump in the Lyapunov function for the closed-loop system (5.1), (6.112)

mathematical one. Geometrically, it may be possible to apply the settling time analysis for two separate segments of the trajectory on either side of the impact but then the impact time needs to be computed to find the end of the first trajectory which is not a straightforward problem when a non-Lipschitz controller (6.112) is used.

Thus, motivated by the fact that the Lyapunov framework presented in [12] is applicable neither to the original jump-system (5.1),(6.112) due to the violation of continuity requirements at the time of jump nor to the transformed system (6.113) due to the non-negative definiteness of the derivative of the Lyapunov function, the proof of finite time stability of the closed-loop system (5.1), (6.3) is now presented. Lemma 5.1 is recalled from Chapter 5.

It is of interest to note that the discontinuity and in turn Filippov's inclusion [6] in (6.110) is caused by the fact that R switches between two positive values on sets

$\{(s,v) : s = 0\}, \{(s,v) : v = 0\}$ of Lebesgue measure zero. Let the two values of R be defined as follows:

$$R = \begin{cases} R_1 = \frac{2}{1+\bar{e}}, & \text{if } \text{sign}(sv) = -1; \\ \\ R_2 = \frac{2\bar{e}}{1+\bar{e}}, & \text{if } \text{sign}(sv) = 1. \end{cases} \tag{6.118}$$

Then, it is trivial to note that given $\bar{e} \in (0,1)$, the following is true from the computations in Lemma 5.1:

$$\begin{array}{c} R_1 > R_2 > 0 \ , \ R_1^{-1} < R_2^{-1}, |R_1 - R_1^{-1}| < |R_2 - R_2^{-1}| \\ |R_1 - R_1^{-1}| = \frac{3+\bar{e}}{2}|k| \ , \ |R_2 - R_2^{-1}| = \frac{3\bar{e}+1}{2\bar{e}}|k| \end{array} \tag{6.119}$$

Theorem 6.2. *Given $M = 0, \alpha \in (0,1)$, the impact system (5.1), (6.3) and its transformed version (6.108), (6.109) are globally finite time stable.*

Proof. Lyapunov stability analysis can be performed in the transformed coordinates since both the set of expressions (5.1), (6.3) and (6.108), (6.109) represent the same system (see subsection on remarks on the non-smooth transformation (5.5) at the end of Section 5.4). Let a Lyapunov function candidate be given as follows:

$$V(s,v) = \mu_2 \frac{2-\alpha}{2} |s|^{\frac{2}{2-\alpha}} + \frac{1}{2} v^2 \tag{6.120}$$

Note that the function $V(s,v)$ is a globally radially unbounded C^1 smooth positive definite function. By computing the temporal derivative of this function along the system trajectories in (6.108), (6.109) with $M = 0$, it is obtained that,

$$\dot{V} \leq \mu_2 |v| |s|^{\frac{\alpha}{2-\alpha}} |R - R^{-1}| \text{sign}(sv) \text{sign}(R - R^{-1}) - \mu_1 R^{\alpha-1} |v|^{\alpha+1} \tag{6.121}$$

From Lemma 5.1, (6.121) can be simplified using Lemma 5.1 as follows:

$$\dot{V} \leq -\mu_2 |v| |s|^{\frac{\alpha}{2-\alpha}} |R - R^{-1}| - \mu_1 R^{\alpha-1} |v|^{\alpha+1} \tag{6.122}$$

It can be verified that $R^{\alpha-1} > 0$ for either sign of sign$(s\ v)$ since $\bar{e} \in (0,1)$. Furthermore, the trajectory of the closed-loop system (6.108), (6.109) in (s,v) plane never generates a sliding mode on $v = 0$ since $v\dot{v} \not< 0$. Since, the equilibrium point $s = v = 0$ is the only trajectory of (6.110) on the invariance manifold $v = 0$ where $\dot{V}(s,v) = 0$ and since (6.122) holds true for almost all t, the differential inclusion (6.108), (6.109) is globally equiuniformly asymptotically stable by applying the invariance principle [128], [129]. Moreover, the system described in (6.108), (6.109) is globally homogeneous of the negative degree $q = -1$ with respect to dilation $r = \left(\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha}\right)$ and is globally equiuniformly finite time stable according to [9, Theorem 3.1]. □

The closed-loop system (6.110) is a globally homogeneous system if $\omega(x,t) = 0\ \forall t \geq 0$. Consider next the case when M takes a nonzero value. The control law (6.109) can reject any disturbance ω with a uniform upper bound (6.111). Global asymptotic stability was established in [14]. However, to establish finite time stability, *uniform* asymptotic stability is required [9]. The next theorem achieves this objective for the transformed system (6.110) (and equivalently for the jump system (5.1), (6.3)). It should be noted that the proof of the next theorem utilizes the same philosophy as that [9] utilized for a discontinuous version of the control law (6.3). However, the existing result [9] is not applicable directly as the Lyapunov function (6.120) proposed in [9, 14] does not suffice to prove uniform asymptotic stability of (6.110). Moreover, the quasi-homogeneity theorem [9, Theorem 3.2] needs to be extended for the continuous case with jumps as it was done in Lemma 6.1 in Section 6.2. Thus, the following result can be stated.

Theorem 6.3. *Given $\alpha \in (\frac{1}{2}, 1)$, the closed-loop impact system (5.1), (6.3) and its transformed version (6.108), (6.109) are globally equiuniformly finite time stable, regardless of whichever disturbance ω, satisfying condition (6.111) (or equivalently (6.6)) with $M < \mu_1 < \mu_2$, affects the system.*

Proof. the proof is given in several steps.
1. Global Asymptotic Stability Under the conditions of the theorem, the time derivative of the Lyapunov function (6.120), computed along the trajectories of (6.108), (6.109) is estimated as follows:

$$\dot{V} \leq -\mu_2 |v||s|^{\frac{\alpha}{2-\alpha}} |R - R^{-1}| - (\mu_1 - M) R^{\alpha-1} |v|^{\alpha+1} \tag{6.123}$$

The first term in the right hand side of (6.123) follows from Lemma 5.1. Since $M < \mu_1$ by a condition of the theorem, the global asymptotic stability of (6.108), (6.109) is then established by applying the invariance principle [128],[129] as discussed in Theorem 6.2.
2. Semiglobal Strong Lyapunov Functions.

The goal of this step is to show the existence of a parameterized family of local Lyapunov functions $V_{\tilde{R}}(s,v)$, with an *a-priori* given $\tilde{R} > 0$, such that each $V_{\tilde{R}}(s,v)$ is well-posed on the corresponding compact set

$$D_{\tilde{R}} = \{(s,v) \in \mathbb{R}^2 : V(s,v) \leq \tilde{R}\}. \tag{6.124}$$

In other words, $V_{\tilde{R}}(s,v)$ is to be positive definite on $D_{\tilde{R}}$ and its derivative, computed along the trajectories of the uncertain system (6.108), (6.109) with initial conditions within $D_{\tilde{R}}$, is to be negative definite in the sense that,

$$\dot{V}_{\tilde{R}}(s,v) \leq -W_{\tilde{R}}(s,v) \tag{6.125}$$

for all $(s,v) \in D_{\tilde{R}}$ and for some $W_{\tilde{R}}(s,v)$, positive definite on $D_{\tilde{R}}$. A parameterized family of Lyapunov functions $V_{\tilde{R}}(s,v), \tilde{R} > 0$, with the properties defined above are constructed by combining the Lyapunov function V of (6.120), whose time derivative along the system motion is only negative semi-definite, with the indefinite function

$$\begin{aligned} U(s,v) &= U_1(s,v) + U_2(s,v) \\ &= s\,v|v|^{\alpha} + \kappa_1 \alpha s^3 v, \end{aligned} \tag{6.126}$$

as follows:

$$\begin{aligned} V_{\tilde{R}}(s,v) &= V(s,v) + \kappa_{\tilde{R}} U(s,v) \\ &= \mu_2 \frac{2-\alpha}{2} |s|^{\frac{2}{2-\alpha}} + \frac{1}{2}v^2 + \kappa_{\tilde{R}} \left(sv|v|^{\alpha} + \kappa_1 \alpha s^3 v \right), \end{aligned} \tag{6.127}$$

where the weight parameters $\kappa_{\tilde{R}}, \kappa_1$ are chosen small enough namely,

$$\begin{aligned} \kappa_{\tilde{R}} &< min \left\{ \frac{1}{\left(\sqrt{2\tilde{R}}\right)^{2\alpha} + \kappa_1 \alpha}, \quad \frac{(2-\alpha)\mu_2}{\rho^{(1-\alpha)}\left(1 + \kappa_1 \alpha \rho^{2(2-\alpha)}\right)}, \quad \frac{\mu_2 |R_1 - R_1^{-1}|}{\Theta}, \quad \frac{\mu_1 - M}{R_1 \sqrt{2\tilde{R}}} \right\} \\ \kappa_1 &< \frac{\mu_2 R_1^{-1}(1+\alpha)}{3\alpha(\mu_1 + M) R_1^{\alpha-1} \rho^{\frac{4-3\alpha}{2}} \sqrt{2\tilde{R}}} \end{aligned} \tag{6.128}$$

where,

$$\begin{aligned} \rho &= \frac{2\tilde{R}}{(2-\alpha)\mu_2} \\ \Theta &= (\mu_1 + M)(1+\alpha) R_1^{\alpha-1} (2\tilde{R})^{\frac{2\alpha-1}{2}} \rho^{1-\alpha} + 3\kappa_1 \alpha R_1 \rho^{\frac{4-3\alpha}{2}} \sqrt{2\tilde{R}} \end{aligned} \tag{6.129}$$

and R_1 is defined in (6.118). Noting that, due to (6.123), all possible solutions of the uncertain system (6.108), (6.109), initialized at $t_0 \in \mathbb{R}$ within the compact set (6.124), are *a-priori* estimated by

$$\sup_{t \in [t_0, \infty)} V(s,v) \leq \tilde{R}, \tag{6.130}$$

the following inequalities hold true:

$$|s|^{\frac{2}{2-\alpha}} \leq \rho, \quad |v| \leq \sqrt{2\tilde{R}}. \tag{6.131}$$

The Lyapunov function (6.127) is positive definite on compact set (6.124), for all $(s,v) \in D_{\tilde{R}} \backslash \{0,0\}$ and $\kappa_{\tilde{R}} > 0$ satisfying (6.128) as shown below:

$$\begin{aligned} V_{\tilde{R}}(s,v) &= \mu_2 \frac{2-\alpha}{2} |s|^{\frac{2}{2-\alpha}} + \frac{1}{2} v^2 + \kappa_{\tilde{R}} s\, v|v|^{\alpha} + \kappa_{\tilde{R}} \kappa_1 \alpha s^3 v \\ &\geq \mu_2 \frac{2-\alpha}{2} |s|^{\frac{2}{2-\alpha}} + \frac{1}{2} v^2 - \frac{1}{2}\kappa_{\tilde{R}} s^2 - \frac{1}{2}\kappa_{\tilde{R}} |v|^{2\alpha} v^2 - \frac{1}{2}\kappa_{\tilde{R}}\kappa_1 \alpha s^6 - \frac{1}{2}\kappa_{\tilde{R}}\kappa_1 \alpha v^2 \\ &\geq L_{\tilde{R}} V(s,v) \end{aligned} \tag{6.132}$$

where,

$$L_{\tilde{R}} < \min\left\{ \mu_2 \frac{2-\alpha}{2} - \frac{1}{2}\kappa_{\tilde{R}}\, \rho^{(1-\alpha)} \left(1 + \kappa_1 \alpha \rho^{2(2-\alpha)}\right),\quad 1 - \kappa_{\tilde{R}}(\sqrt{2\tilde{R}}^{2\alpha} + \kappa_1 \alpha) \right\}, \tag{6.133}$$

the trivial inequality $2ab > -(a^2 + b^2), \forall a, b \in \mathbb{R}$, the equalities

$$s^6 = |s|^{\frac{2(1-\alpha)}{2-\alpha}} |s|^{\frac{2}{2-\alpha}} |s|^4, \quad |v|^{2(\alpha+1)} = v^2 |v|^{2\alpha} \tag{6.134}$$

and the bound (6.131) are utilized. It can be noted that $L_{\tilde{R}} > 0$ due to (6.128) and hence positive definiteness of $V_{\tilde{R}}$ is ensured from (6.132) on $D_{\tilde{R}} \setminus \{0,0\}$. Similarly it can be shown that the following inequality holds true:

$$V_{\tilde{R}}(s,v) \leq M_{\tilde{R}} V(s,v) \tag{6.135}$$

where,

$$M_{\tilde{R}} > \max\left\{ \mu_2 \frac{2-\alpha}{2} + \frac{1}{2}\kappa_{\tilde{R}}\, \rho^{(1-\alpha)} \left(1 + \kappa_1 \alpha \rho^{2(2-\alpha)}\right),\quad 1 + \kappa_{\tilde{R}}(\sqrt{2\tilde{R}}^{2\alpha} + \kappa_1 \alpha) \right\} \tag{6.136}$$

The time derivative of the indefinite function $U(s,v)$ along the trajectories of the uncertain system (6.108), (6.109) is obtained as follows:

$$\begin{aligned} \dot{U}_1(s,v) &= R|v|^{\alpha+2} + s|v|^{\alpha}\left(-\mu_1\, R^{\alpha-1}\, |v|^{\alpha}\mathrm{sign}(v)\right) \\ &\quad + s|v|^{\alpha}\left(-\mu_2 R^{-1} |s|^{\frac{\alpha}{2-\alpha}} \mathrm{sign}(s)\right) \\ &\quad + s|v|^{\alpha} R^{-1} \mathrm{sign}(s)\omega(s,v,t) \\ &\qquad + \alpha s v |v|^{\alpha-1}\mathrm{sign}(v)\left(-\mu_1\, R^{\alpha-1}\, |v|^{\alpha}\mathrm{sign}(v)\right) \\ &\qquad + \alpha s v |v|^{\alpha-1}\mathrm{sign}(v)\left(-\mu_2 R^{-1}|s|^{\frac{\alpha}{2-\alpha}}\mathrm{sign}(s)\right) \\ &\qquad + \alpha s v |v|^{\alpha-1}\mathrm{sign}(v) R^{-1}\mathrm{sign}(s)\omega(s,v,t) \\ &\leq R|v|^{\alpha+2} + (\mu_1 + M)(1+\alpha)R^{\alpha-1}|s||v|^{2\alpha} \\ &\quad - \mu_2(1+\alpha)R^{-1}|s|^{\frac{2}{2-\alpha}}|v|^{\alpha} \end{aligned} \tag{6.137}$$

Similarly,

$$\begin{aligned} \dot{U}_2(s,v) &= 3\kappa_1\alpha R s^2 v^2 + \kappa_1\alpha s^3\left(-\mu_1\, R^{\alpha-1}\,|v|^\alpha \mathrm{sign}(v)\right) \\ &+\kappa_1\alpha s^3\left(-\mu_2 R^{-1}|s|^{\frac{\alpha}{2-\alpha}}\mathrm{sign}(s)\right) \\ &+\kappa_1\alpha s^3 R^{-1}\mathrm{sign}(s)\omega(s,v,t) \\ &\leq 3\kappa_1\alpha R s^2 v^2 + \kappa_1\alpha(\mu_1+M)R^{\alpha-1}|s|^3|v|^\alpha \\ &-\kappa_1\alpha R^{-1}\mu_2|s|^{\frac{6-2\alpha}{2-\alpha}} \end{aligned} \tag{6.138}$$

It should be noted that the inequality

$$|s| = |s|^{\frac{2(1-\alpha)}{2-\alpha}}|s|^{\frac{\alpha}{2-\alpha}} \leq \rho^{1-\alpha}|s|^{\frac{\alpha}{2-\alpha}}, \quad |v|^{2\alpha} = |v||v|^{2\alpha-1} \leq |v|\sqrt{2\tilde{R}}^{\,2\alpha-1} \tag{6.139}$$

holds true due to the condition $\alpha \in (\frac{1}{2}, 1)$ of the theorem. The last inequalities of (6.137) and (6.138) are re-written by utilizing (6.131) and (6.139) as follows:

$$\begin{aligned} \dot{U}_1(s,v) &\leq R|v|^{\alpha+1}\sqrt{2\tilde{R}} \\ &+(\mu_1+M)(1+\alpha)R^{\alpha-1}|s|^{\frac{\alpha}{2-\alpha}}|v|\sqrt{2\tilde{R}}^{\,2\alpha-1}\rho^{1-\alpha} \\ &-\mu_2(1+\alpha)R^{-1}|s|^{\frac{2}{2-\alpha}}|v|^\alpha \\ \dot{U}_2(s,v) &\leq 3\kappa_1\alpha R|s|^{\frac{\alpha}{2-\alpha}}|v|\,\rho^{\frac{4-3\alpha}{2}}\sqrt{2\tilde{R}} \\ &+\kappa_1\alpha(\mu_1+M)R^{\alpha-1}|s|^3|v|^\alpha - \kappa_1\alpha\mu_2 R^{-1}|s|^{\frac{6-2\alpha}{2-\alpha}}, \end{aligned} \tag{6.140}$$

where the corresponding upper bound on $|v|$ and $|s|$ from (6.131) are utilized. The parameter R in (6.123) and (6.140) is a state function and keeps switching between the two values as shown in Lemma 5.1. This corresponds to the fact that the rate of decay of the Lyapunov function (6.127) switches depending on R. Hence, by combining (6.123) and (6.140) and by considering the slowest decay by utilizing (6.119), the time derivative of (6.127) can be obtained as follows:

$$\dot{V}_{\tilde{R}} \leq -\beta_1|v||s|^{\frac{\alpha}{2-\alpha}} - \beta_2|v|^{\alpha+1} - \kappa_{\tilde{R}}\beta_3|v|^\alpha|s|^{\frac{2}{2-\alpha}} - \kappa_{\tilde{R}}\kappa_1\alpha\mu_2 R^{-1}|s|^{\frac{6-2\alpha}{2-\alpha}}, \tag{6.141}$$

where

$$\begin{aligned} \beta_1 &= \mu_2|R_1 - R_1^{-1}| - \kappa_{\tilde{R}}\Theta \\ \beta_2 &= (\mu_1 - M)R_1^{\alpha-1} - \kappa_{\tilde{R}}R_1\sqrt{2\tilde{R}} \\ \beta_3 &= \mu_2 R_1^{-1}(1+\alpha) - \kappa_1\alpha(\mu_1+M)R_1^{\alpha-1}\rho^{\frac{4-3\alpha}{2}}\sqrt{2\tilde{R}}, \end{aligned} \tag{6.142}$$

(6.131) and the equality $|s|^3 = |s|^{\frac{2}{2-\alpha}}|s|^{\frac{4-3\alpha}{2-\alpha}}$ are utilised. It should be noted that the expressions $\beta_1 > 0, \beta_2 > 0, \beta_3 > 0$ hold true due to (6.128).

Ignoring the negative semi-definite terms with β_1, β_3, (6.141) can be rewritten as follows:

$$\dot{V}_{\tilde{R}} \leq -\beta_2|v|^{\alpha+1} - \kappa_{\tilde{R}}\kappa_1\alpha\mu_2 R^{-1}|s|^{\frac{6-2\alpha}{2-\alpha}} \tag{6.143}$$

Furthermore, the following inequalities hold true within the compact set (6.124):

$$v^2 = |v|^2 = |v|^{\alpha+1}|v|^{1-\alpha} \leq |v|^{\alpha+1}\left(\sqrt{2\tilde{R}}\right)^{1-\alpha} \Rightarrow -|v|^{\alpha+1} \leq -\frac{v^2}{\left(\sqrt{2\tilde{R}}\right)^{1-\alpha}} \quad (6.144)$$

Hence, (6.143) can be simplified as follows:

$$\dot{V}_{\tilde{R}} \leq -c_{\tilde{R}}\left[|s|^{\frac{6-2\alpha}{2-\alpha}} + v^2\right] \quad (6.145)$$

where,

$$c_{\tilde{R}} = \min\left\{\frac{\beta_2}{\left(\sqrt{2\tilde{R}}\right)^{1-\alpha}},\ \kappa_{\tilde{R}}\kappa_1\alpha\mu_2 R_1^{-1}\right\} > 0 \quad (6.146)$$

Case 1: $|s| \geq 1$:

The following inequality holds true for $|s| \geq 1$:

$$\frac{6-2\alpha}{2-\alpha} \geq \frac{2}{2-\alpha} \Leftrightarrow |s|^{\frac{6-2\alpha}{2-\alpha}} \geq |s|^{\frac{2}{2-\alpha}} \quad (6.147)$$

Also, the following can be obtained from (6.135):

$$\frac{M_{\tilde{R}}}{2}\max\{1, \mu_2(2-\alpha)\}(|s|^{\frac{2}{2-\alpha}} + v^2) \geq V_{\tilde{R}}(s, v) \quad (6.148)$$

Hence, the following inequality is then obtained for $|s| \geq 1$ by combining (6.145), (6.147) and (6.148):

$$\dot{V}_{\tilde{R}} \leq -\bar{\kappa}_1 V_{\tilde{R}} \quad (6.149)$$

where,

$$\bar{\kappa}_1 = \frac{2c_{\tilde{R}}}{M_{\tilde{R}}\max\{1, \mu_2(2-\alpha)\}}. \quad (6.150)$$

Case 2: $|s| < 1$:

Noting that the following inequalities hold true for $|s| < 1$:

$$|s|^{\frac{6-2\alpha}{2-\alpha}} > |s|^{\frac{2\gamma}{2-\alpha}} \Leftrightarrow \frac{6-2\alpha}{2-\alpha} < \frac{2\gamma}{2-\alpha} \Leftrightarrow \gamma > 3-\alpha, \quad (6.151)$$

for some $\gamma > 3-\alpha$. Noting that $3-\alpha < \frac{5}{2}$ always holds true due to $\alpha \in (\frac{1}{2}, 1)$, $\gamma \geq \frac{5}{2}$ is a valid choice. In the following, $\gamma = 3$ is chosen. It can be seen that the following equality holds true:

$$\left(|s|^{\frac{2}{2-\alpha}} + v^2\right)^3 = |s|^{\frac{6}{2-\alpha}} + 3|s|^{\frac{4}{2-\alpha}}v^2 + 3|s|^{\frac{2}{2-\alpha}}v^4 + v^6 \leq \max\{\rho^\alpha, K_1\}\left(|s|^{\frac{6-2\alpha}{2-\alpha}} + v^2\right) \quad (6.152)$$

where the bounds (6.131) has been utilised resulting in the following definition of K_1:

$$K_1 = \max\left\{3\,\rho^2, 3\,\rho(2\tilde{R}), (2\tilde{R})^2\right\} > 0 \tag{6.153}$$

Note that the following can be obtained from (6.135):

$$\left(\frac{M_{\tilde{R}}}{2}\max\{1, \mu_2(2-\alpha)\}(|s|^{\frac{2}{2-\alpha}} + v^2)\right)^3 \geq \left(V_{\tilde{R}}(s,v)\right)^3 \tag{6.154}$$

Then, the following can be obtained by combining (6.145), (6.152) and (6.154):

$$\dot{V}_{\tilde{R}}(s,v) \leq -c_{\tilde{R}}\left(|s|^{\frac{6-2\alpha}{2-\alpha}} + v^2\right) \leq -\bar{\kappa}\left(V_{\tilde{R}}\right)^3 \tag{6.155}$$

where,

$$\bar{\kappa} = \frac{c_{\tilde{R}}}{\left(\frac{M_{\tilde{R}}}{2}\max\{1, \mu_2(2-\alpha)\}\right)^3 \max\{\rho^\alpha, K_1\}} > 0 \tag{6.156}$$

Hence, the desired uniform negative definiteness (6.125) is obtained by combining (6.149) and (6.155) as follows:

$$W_{\tilde{R}}(s,v) = \min\left\{\bar{\kappa}_1 V_{\tilde{R}}, \bar{\kappa}\left(V_{\tilde{R}}\right)^3\right\} \tag{6.157}$$

3. Global Equiuniform Asymptotic Stability

Since the inequality (6.155) holds on the solutions of the uncertain system (6.108), (6.109), initialized within the compact set (6.124), the decay of the function $V_{\tilde{R}}(s,v)$ can be found by considering the majorant solution $\nu(t)$ of $V_{\tilde{R}}$ as follows:

$$\dot{\nu}(t) = \begin{cases} -\bar{\kappa}_1\nu, & |s| \geq 1; \\ -\bar{\kappa}_2\nu^\gamma, & |s| < 1. \end{cases} \tag{6.158}$$

where, $\gamma > 3-\alpha$ is introduced for generality. Similar to Section 6.3, a more conservative decay than that in (6.158) can be computed. There are two possible sub-cases, namely, $\nu(t) \geq 1$ and $\nu(t) < 1$ for each of the cases $|s| \geq 1$ and $|s| < 1$. The following expressions hold true for a positive definite function $\nu(t)$ and a scalar $\gamma > 1$:

$$\begin{aligned} \nu(t)^\gamma &\geq \nu(t) \Rightarrow -\nu(t)^\gamma \leq -\nu(t) \quad \text{if } \nu(t) \geq 1; \\ \nu(t)^\gamma &\leq \nu(t) \Rightarrow -\nu(t) \leq -\nu(t)^\gamma \quad \text{if } \nu(t) < 1. \end{aligned} \tag{6.159}$$

Hence, the decay (6.158) is modified by utilising (6.159) independent of the magnitude of $|s|$ and dependent on $\nu(t)$ as follows :

$$\dot{\nu}(t) = \begin{cases} -\bar{\kappa}\nu, & \text{if } \nu(t) \geq 1; \\ -\bar{\kappa}\nu^{\gamma}, & \text{if } \nu(t) < 1. \end{cases} \tag{6.160}$$

where,

$$\bar{\kappa} = \min\{\bar{\kappa}_1, \bar{\kappa}_2\} \quad > 0. \tag{6.161}$$

The solution for the case $\nu(t) < 1$ can be integrated as follows:

$$\int_{\nu_0}^{\nu(t)} \frac{\mathrm{d}\nu(t)}{\nu^{\gamma}} = -\bar{\kappa} \int_{t_0}^{t} \mathrm{d}t \tag{6.162}$$

where $\nu_0 = \nu(t_0)$. The general solution of $\nu(t)$ of (6.160) can then be obtained by utilising (6.160) and by combining the solutions of both (6.149) and (6.162) as follows:

$$\nu(t) = \begin{cases} \nu(t_0)\, e^{-\bar{\kappa}(t-t_0)}, & \text{if } \nu(t) \geq 1; \\ \nu(t_1) \left(\frac{1}{\bar{\kappa}(t-t_1)(\gamma-1)\nu_0^{\gamma-1}+1} \right)^{\frac{1}{\gamma-1}}, & \text{if } \nu(t) < 1. \end{cases} \tag{6.163}$$

It can be easily seen that the solution $\nu(t) \to 0$ as $t \to \infty$ and that the decay rate depends on the gain parameters μ_1, μ_2, bound M on disturbance ω and the system property R_1. On the compact set (6.124), the following inequality holds (see (6.132), (6.135)):

$$L_{\tilde{R}} V(s,v) \leq V_{\tilde{R}}(s,v) \leq M_{\tilde{R}} V(s,v) \tag{6.164}$$

for all $(s,v) \in D_{\tilde{R}}$ and positive constants $L_{\tilde{R}}, M_{\tilde{R}}$. The above inequalities (6.163) and (6.164) ensure that the globally radially unbounded function $V(x_1, x_2)$ decays exponentially

$$V(s(t), v(t)) \leq \begin{cases} L_{\tilde{R}}^{-1} M_{\tilde{R}} \tilde{R} e^{-\bar{\kappa}(t-t_0)}, & \text{if } V_{\tilde{R}} \geq 1; \\ L_{\tilde{R}}^{-1} M_{\tilde{R}} \tilde{R} \left(\frac{1}{\bar{\kappa}(t-t_1)(\gamma-1)\nu_0^{\gamma-1}+1} \right)^{\frac{1}{\gamma-1}}, & \text{if } V_{\tilde{R}} < 1. \end{cases} \tag{6.165}$$

on the solutions of (6.108), (6.109) uniformly in ω and the initial data, located within an arbitrarily large set (6.124). This proves that the uncertain system (6.108), (6.109) is globally equiuniformly asymptotically stable around the origin $(s, v) = (0, 0)$.

4. Global Equiuniform Finite Time Stability.

Due to (6.111), the piece-wise continuous (see Definition 2.1) uncertainty $R_1^{\alpha-1}\omega(t)\text{sign}(s)$ in the right hand side of the system (6.108), (6.109) is locally uniformly bounded by $R_1^{\alpha-1}M|v|^{\alpha}$ whereas the remaining part of the feedback is globally homogeneous with homogeneity degree $q = -1$ with respect to dilation $r = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$. It remains to verify, however, whether the existing quasi-homogeneity result [9, Theorem 3.2] can be extended to the continuous case in question. Let the piece-wise continuous function $R_1^{\alpha-1}\omega(s,v,t)\text{sign}(s) = R_1^{\alpha-1}\omega^c(s,v,t)\text{sign}(s)$ be defined for an arbitrary $c \geq \max\{1, c_0\}$, where c_0 is lower homogeneity parameter as defined in Definition 2.6, as follows:

$$\omega^c(s,v,t) = c^{q+r_2}\omega(c^{-r_1}s, c^{-r_2}v, c^q t) \tag{6.166}$$

where, the right hand side represents a parameterised set of uncertainties. Then the following holds true:

$$\begin{aligned} |\omega^c(s,v,t)| &= |c^{q+r_2}\omega(c^{-r_1}s, c^{-r_2}v, c^q t)| \\ |\omega^c(s,v,t)| &\leq c^{q+r_2}MR^{\alpha}|c^{-r_2}v|^{\alpha} \\ &\leq c^{q+r_2-\alpha r_2}MR^{\alpha}|v|^{\alpha} \end{aligned} \tag{6.167}$$

Hence all parameterised uncertainties represented by the right hand side of (6.166) are admissible in the sense of (6.111) if the following holds true:

$$c^{q+r_2-\alpha r_2} \leq 1 \tag{6.168}$$

From the definitions $r_2 = \frac{1}{1-\alpha}, q = -1$, it is obtained that

$$q + r_2 - \alpha r_2 = 0 \Rightarrow c^{q+r_2-\alpha r_2} \leq 1 \tag{6.169}$$

Hence, recalling Definitions 2.6, 2.7, the solutions $s^c(t) = c^{r_1}s(c^q t), v^c(t) = c^{r_2}v(c^q t)$ are solutions of the system (6.108), (6.109) with the piece-wise continuous function $\omega(s,v,t) = \omega^c(s,v,t)$. Hence, any solution of the differential equation (6.108), (6.109) evolving within a homogeneity ball B_δ, generates a parameterised set of solutions $s^c(t)$, $v^c(t)$ with the parameter c large enough. Hence, (6.108), (6.109) is locally homogeneous of degree $q = -1$ with the dilation $(r_1, r_2) = (\frac{2-\alpha}{1-\alpha}, \frac{1}{1-\alpha})$. Thus, the globally equiuniformly asymptotically stable system (6.108), (6.109) and in turn the original impact system (5.1), (6.3) are globally equiuniformly finite time stable according to [9, Theorem 3.1]. □

Unification of Lyapunov functions for the discontinuous and continuous controllers

It is interesting to observe that the parameterised classes of Lyapunov functions (6.120) and (6.127) are inspired from (5.16) and (5.22) respectively. In particular, the Lyapunov function given in (6.127) equals that given in (5.22) for the case when $\alpha = 0$. In this case the proof of Theorem 5.2 coincides with that of Theorem 6.3 above. Furthermore, the synthesis given in (6.3) reduces to that given in (5.2) for the case when $\alpha = 0$. Hence, this section can be seen as a generalization of Chapter 5 for the case when impacts have a finite accumulation point. It must be noted however, that the apparatus utlised to prove equiuniform finite time stability is the same, namely, combining a positive definite function with indefinite functions to construct a class of strong semi-global Lyapunov functions [9, 120]. Hence, this thesis presents a common methodology for both the continuous (6.3) and discontinuous controllers (5.2) thereby unifying the Lyapunov analysis of variable structure and non-smooth systems at least in the case of the resets.

Remark 6.7. It should be noted that the step 4 above can be seen as a slight modification of the condition $q+r_i \leq 0, M_i > 0$ of [9, Lemma 2.12, Theorem 3.2] while proving local homogeneity of the differential equation. The condition $q+r_i \leq 0, M_i > 0$ was a requirement in [9, Lemma 2.12, Theorem 3.2] to prove validity of the parameterised set of uncertainties in the sense of a rectangular uncertainty constraint of the type $\sup_{t\geq 0} |\omega_i(t)| < M_i$. The condition $q + r_2 < 0$ can be modified as $q + r_2 + \alpha r_2 \leq 0$ for the present case since there are only sectorial disturbances (6.111) present as against rectangular ones. Also it is noted that under the parameterisation (6.166), the second equation of (6.108), (6.109) can be re-written as follows:

$$\begin{aligned} \dot{v}(c_1^r s, c_2^r v, c^{-q} t) = &-c^{q+r_2}\mu_1 R^{\alpha-1}|v|^{\alpha}\text{sign}(v) - c^{q+r_2}\mu_2 R^{-1}|s|^{\frac{\alpha}{2-\alpha}}\text{sign}(s) \\ &+ c^{q+r_2}R^{-1}\omega(c^{-r_1}s, c^{-r_2}v, c^{q}t), \end{aligned} \tag{6.170}$$

The first two terms on the right hand side of (6.170) stem from homogeneity of the nominal system (6.108), (6.109) and the third term stem from the parametrization (6.166). Differential equation (6.170) is clearly a locally homogeneous differential equation in the sense of definitions 2.6, 2.7 due to the validity of (6.166). All other conditions of [9, Theorem 3.2] are precisely met leading to finite time stability of perturbed system (6.108), (6.109).

Remark 6.8. While the Lyapunov function V is C^2 smooth for $\alpha \in (\frac{1}{2}, 1)$ and it is C^1 smooth for $\alpha \in (0, 1)$, $V_{\tilde{R}}$ is C^1 smooth for all $\alpha \in (0, 1)$. Setting $\alpha = 0$ in the above analysis corresponds to the discontinuous case for which the finite time stability can be derived following a similar semi-global analysis using the same indefinite function (6.126)(see [9] for jump-free case).

Remark 6.9. Remarks 5.1, 5.2 and 5.3 are applicable to the results of this Section as well.

6.5 Contributions and future directions

6.5.1 Theoretical contributions

Absence of resets

In summary, there are four main theoretical contributions of Section 6.3. Firstly, the finite time stability attained in this chapter is *uniform* in the initial data and in the perturbation, a feature that cannot be guaranteed using existing methods. This is because the proposed results are based on equiuniform asymptotic stability substantiated by a class of Lyapunov functions.

Secondly, since the proposed method is founded on the quasi-homogeneity principle of switched systems [9], the non-homogeneous perturbations are allowed to be piece-wise continuous [6] when compared with the continuous ones considered in existing methods [13, 46, 84]. In turn, the quasi-homogeneity principle [9] is extended to planar controllable homogeneous systems with piece-wise continuous inhomogeneous disturbances.

Thirdly, finding a finite upper bound on the settling time is an open problem when a class of continuous homogeneous controller proposed in [14, 83] is utilized in the presence of piece-wise continuous non-homogeneous disturbances. This chapter solves this problem without the need to find a Lyapunov function satisfying a differential inequality. Hence, the proposed framework highlights a novel method (see Section 3.5 for the discontinuous case).

Finally, above results are proven to hold true with the same continuous controller for a unilaterally constrained perturbed double integrator by showing existence of a class of semi-global Lypunov functions but without the need to study jumps in the same. The final result is of theoretical significance as it bridges three streams of nonsmooth Lyapunov analysis, quasi-homogeneity and finite time stability for a class of impact mechanical systems while using a continuous controller. Practically, such a result finds natural application in mechanical systems with impacts which have a finite accumulation time.

Presence of resets with a finite accumulation

The main theoretical contribution of Section 6.4 is threefold. Firstly, the proof of equiuniform finite time stability of the origin of the translational perturbed double integrator is established in the presence of jumps in the velocity using a continuous controller. Whereas the existing Lyapunov approaches [21, 23, 33–36, 100–102] inevitably involve analysis of jumps of the Lyapunov function and their respective limits in asymptotic time, the proposed method proves the finite time stability of the origin of a perturbed double integrator in the presence of jumps in velocity without having to analyse jumps in the proposed C^1 Lyapunov function, a clear contribution.

Secondly, this section extends the quasi-homogeneity principle [9] to the continuous case with sectorial disturbances when the jumps in velocity are present. Since the proposed method is founded on the quasi-homogeneity principle of switched systems [9], the non-homogeneous perturbations are allowed to be piece-wise continuous [6] as against the continuous ones considered in existing methods [13, 46, 84]. Thus the proposed method covers a broader class of perturbations.

Finally, the proposed method enables computation of an explicit formula of the finite upper bound on the settling time as developed in Section 6.3.1, which is not presented in this thesis as it is very similar to Section 6.3.1.

6.5.2 Potential future directions

Some interesting future directions are discussed in this section.

A potential generalization to dimension n

It is of interest to extend the results presented in this chapter in dimension n, i.e. $\frac{\mathrm{d}^n}{\mathrm{d}t^n}x(t) = u$ encompassing piece-wise continuous disturbances. The Lyapunov function V given in (5.16) is homogeneous in the sense of the definition given in the statement of [86, Lemma 2] since for all $c = \max\{1, c_0\}, c_0 > 0$ and $r_1 = \frac{2-\alpha}{1-\alpha}, r_2 = \frac{1}{2-\alpha}, r_3 = \frac{2}{1-\alpha}$ the equality

$$V(c^{r_1}x_1, c^{r_2}x_2) = c^{r_3}V(x_1, x_2) \tag{6.171}$$

holds true, where $r_3 > r_1, r_3 > r_2$. The presented Lyapunov analysis utilized a $\mathcal{C}^1$ smooth homogeneous Lyapunov function (6.171) and combined this with indefinite functions $U_i, i = 1, 2, 3$ to form a equiuniformly exponentially stable Lyapunov function $V_{\tilde{R}}$. It is of interest to investigate if this procedure can be successfully extended to the case of dimension n. It appears plausible due to the rich literature on the subject to carry out step-by-step application of (i) synthesizing a homogeneous controller [139, Th. 3.1] (also see [87]), (ii) constructing a Lyapunov function using [86, Th. 2] to establish global asymptotic stability combined with indefinite functions [120] for purely continuous homogeneous disturbances ω to establish global equiuniform asymptotic stability and (iii) extending the same procedure for piece-wise continuous ones using switched system analysis [9] to prove equiuniform finite time stability.

Open problems in continuous finite time stabilization of planar systems

Another possible strand emanating from this chapter is equiuniform finite time stabilization using an output feedback synthesis when a continuous finite time observer and continuous finite time controller is used. Such an architecture can be found in reference [46]. However, uniform finite time stability is ensured for neither the controller nor the observer. Constructing a equiuniform continuous finite time stable observer is an open problem in itself. Combining such an observer with the equiuniform continuous finite time controller utilised in this chapter can produce equiuniform output feedback finite time synthesis. In this problem, there are two interesting challenges from the theoretical and practical viewpoints. Theoretically, the challenge is in proving equiuniform finite time stability and computing the upperbound on the settling time of the overall closed-loop system. From a practical viewpoint, the challenge is in establishing constructive *a priori* tuning rules, as developed in Chapter 4, for the observer as well as the controller so that the pre-specified finite settling time is achieved. Furthermore, the requirement that the observer should settle before the controller is also a part of this tuning challenge. Inspiration in terms of 'guessing' a uniformly decaying Lyapunov functions for the observer may come from the existing continuous and discontinuous observers [46], [14, 71, 72].

Irrespective of the potential observer developments, the problem of establishing constructive tuning rules to a priori guarantee attainment of a specified finite settling time for the uniform continuous finite time controller presented in this chapter, or any other existing homogeneous controllers, remains to be solved, too.

6.6 A priori tuning for output feedback

It is worth noting that the previous and current chapters studied the equiuniform finite time stabilization problem with an accumulation of impacts. Chapter 8 studies the orbital stability problem outlined in Section 4.7, namely, achieving finite time convergence of the state trajectory to a command trajectory when the impacts do not have a finite accumulation.

The current chapter is the only chapter concerning non-smooth continuous systems. The focus of the investigation shifts back to variable structure systems from the next chapter onwards. It is clear that the thesis presented state feedback based results thus far. It was inherently assumed that both the states are available by measurement. It is interesting to study whether finite time stability and corresponding tuning rules can be established for the case when only one of the state is available for measurement. This is the area of output feedback control [2].

The problem of establishing constructive tuning rules for the gains of the observer and that of the controller when the control synthesis utilises observed states, is not solved in the literature on output feedback Higher Order Sliding Mode Controllers (HOSM) in a constructive manner like Chapter 4. Given recent advances in the literature on Lyapunov stability of finite time second order sliding mode (SOSM) observers [14, 42, 71], it is natural to pose the following question: Is it possible to compute constructive tuning rules for the gains of the output feedback synthesis that utilises a SOSM observer and a SOSM controller? While the proof of finite time stability of the observer in the presence of disturbances with a persisting bound can be found in the recent literature [14, 41, 71, 72], tuning rules to guarantee the attainment of a finite settling time for the output feedback synthesis are not available. The next chapter studies this problem when the synthesis utilises a combination of a uniformly finite time stable second order sliding mode controller and a finite time stable second order sliding mode observer.

6.7 Conclusions

Uniform finite time stability of planar controllable systems is established for a continuous homogeneous controller. The focus was on uniformity of stability with respect to the initial data and uncertainty. The quasi-homogeneity principle of switched systems is extended to the continuous case with uniformly decaying, but piece-wise continuous, disturbances by identifying a class of $\mathcal{C}^1$ smooth, strong Lyapunov functions. In turn, it gives a more robust version of the existing homogeneity and finite time stability results. Furthermore, these results of equiuniform continuous finite time stabilization and quasi-homogeneity principle are also extended to the case of unilaterally constrained systems thereby unifying the methodologies of non-smooth Lyapunov analysis, homogeneity and equiuniform finite time stability of both variable structure and non-smooth systems at least in the case where the resets appear in velocity at the time of impact.

The next chapter extends the geometric homogeneity based results obtained in Chapter 4 on finite time controller by combining the same with the existing result on finite time observer [4, 14] for tuning gains of both the controller and observer thereby solving the problem discussed in Section 6.6.

CHAPTER 7

Tuning Rules for Second Order Sliding Mode Based Output Feedback Synthesis

Previous chapters have studied equiuniform finite stabilisation both using continuous and discontinuous state feedback control. The formulations inherently assumed that all the states are available by measurement. Studying finite time stabilisation with a finite time observer framework is a natural next step, particularly given the rich literature on finite time observers [4, 14, 42, 46, 72]. As with any observer based synthesis, finite time observers also pose the problem of analysing the trajectory of the closed-loop system in that it is of interest to establish how far the trajectories can escape away from origin when the controller uses state estimates for feedback. The literature in this area demonstrates that the overall finite time stability of the closed-loop system is governed by the finite time stability property of the observer error dynamics and that of the controller [14, 42]. In particular, the settling time of the closed-loop system is no more than the sum of the values for the finite settling time of the observer and the controller [14, Th. 6].

The aim of this chapter is to establish tuning rules for the gains of both the observer and the controller to achieve an *a priori* specified settling time for planar systems when a second order sliding mode observer is used to form a second order sliding mode based output feedback controller. The observer is formed by using the super-twisting algorithm [4] to estimate one of the states. The estimated and the measured states are then used by the twisting controller [4] in the presence of external disturbances.

Although a priori tuning rules are not available in the existing literature, there are several relevant previous contributions. The finite time stable output feedback synthesis [46] combines a continuous finite time observer with a continuous finite time stable controller from [12]. While this synthesis closely matches with the contribution of this chapter, there are two major differences. Firstly, the continuous controller cannot reject persisting disturbances completely thereby producing only ultimately bounded solutions [13, Th 5.2]. In contrast, the present work utilises a combination of a discontinuous controller and a discontinuous observer which can reject persisting disturbances. Secondly, no tuning rules are given to ensure attainment of an *a priori* specified settling time. Work on developing a second order sliding mode observer for biped robots [142] does give tuning rules but they are not in terms of an *a priori* settling time. Furthermore, the works in [142] and references therein use a continuous finite time stable controller for which the tuning rules are not available in the literature (note that a high gain version of such a controller was proposed in [15] without tuning rules).

As to the literature in the area of second order sliding mode controllers, the existing tuning rules [4, 9, 41, 71] are sufficient only for achieving finite time convergence and not an arbitrary settling time. Other contributions [143, 144] for a class of non-linear systems give results on attaining an a priori specified settling time by utilising higher order sliding mode techniques based on the integral sliding mode philosophy [145]. Tuning rules for the twisting algorithm are proposed to guarantee a specified settling time. The method utilises finite time convergent optimal control as one component of the control law. Then, an integral sliding surface is designed and the sliding mode is enforced by a second order sliding mode controller, which forms the second component of the control law. The main difference between the output feedback synthesis [143, 144] and the work presented here is that the former proposes the tuning rules by utilising a method based on the theory of open-loop time optimal control whereas the presented work proposes tuning results based on a Lyapunov function for a robust control algorithm in a closed-loop setting.

The underlying theory of the presented work is in fact a special case of a more general output feedback result [18] combining the r^{th} order sliding mode differentiator with an r^{th} order sliding mode controller. Guidelines for appropriately selecting the gain parameters of the differentiator and the controller are stipulated but these do not achieve an arbitrarily specified settling time. It should be noted that chattering due to the discontinuous nature of the twisting controller is not a limitation. Recent advances in numerical schemes for discontinuous systems [118, 133] facilitate a discrete time chattering free implementation rendering ultimately bounded trajectories for the perturbed case, a result which is no worse than the implementation of continuous finite time controllers [12] in the presence of persisting disturbances (see [14, Sec. II] and [13, Th 5.2]). Hence, the development of tuning rules for an output feedback synthesis using second order sliding mode algorithms is the natural next step to further develop existing recent results (see [118, 133] and Chapter 4. The presented tuning results may be seen as a starting point towards achieving sophisticated results such as Ziegler-Nichols tuning rules[113] and auto-tuning of PID control for the second order sliding mode control algorithms.

The presented work proposes a new Lyapunov function by combining the existing Lyapunov function for the observer [14, Th. 5] with that of the twisting controller [9, Th. 4.2]. The tuning for the observer gains is straightforwardly achieved using the differential inequality of the strict Lyapunov function for the observer [14, Th. 5]. However, the tuning for the twisting controller is achieved using an approach based on a non-strict Lyapunov function as studied in Chapter 4.

7.1 Problem Statement

Consider the following perturbed double integrator:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u(x_1, x_2) + \omega(t)\end{aligned} \tag{7.1}$$

where x_1, x_2 are the position and the velocity respectively, u is the control input and $\omega(t)$ is external disturbance. The twisting control law [4] with observed state $\hat{x}_2$ and measured state x_1 is given as follows [14]:

$$u(x_1, x_2) = -\mu_1 \operatorname{sign}(\hat{x}_2) - \mu_2 \operatorname{sign}(x_1) \tag{7.2}$$

where, $\mu_2 > \mu_1 > 0$. The output feedback synthesis (7.2) utilises a super twisting observer [4, 14, 41] of the following form to estimate $\hat{x}_2$:

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + k_1 \left|x_1 - \hat{x}_1\right|^{\epsilon} \operatorname{sign}(x_1 - \hat{x}_1) + k_2 \left(x_1 - \hat{x}_1\right)\\ \dot{\hat{x}}_2 &= u + k_3 \operatorname{sign}(x_1 - \hat{x}_1) + k_4 \left(x_1 - \hat{x}_1\right)\end{aligned} \tag{7.3}$$

where

$$\begin{aligned}&k_1, k_3 > 0, k_2, k_4 \geq 0\\ &\text{if } k_4 = 0 \text{ then } k_2 = 0,\end{aligned} \tag{7.4}$$

and the homogeneity value $\epsilon = \frac{1}{2}$ is the only value that can reject disturbances with a persisting uniform bound (7.6) (see [14, Th. 5]). The observer (7.3) can be re-written with $\epsilon = \frac{1}{2}$ as follows:

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + k_1 \left|x_1 - \hat{x}_1\right|^{\frac{1}{2}} \operatorname{sign}(x_1 - \hat{x}_1) + k_2 \left(x_1 - \hat{x}_1\right)\\ \dot{\hat{x}}_2 &= u + k_3 \operatorname{sign}(x_1 - \hat{x}_1) + k_4 \left(x_1 - \hat{x}_1\right)\end{aligned} \tag{7.5}$$

The following assumptions are made:

1. The disturbance ω is assumed to admit a uniform upper bound

$$ess \sup_{t \geq 0} |\omega(t)| \leq M \tag{7.6}$$

on its magnitude such that

$$0 < M < \mu_1 < \mu_2 - M. \tag{7.7}$$

2. A finite bound on the energy-like function

$$\frac{1}{2}x_1^2(t_0) + \frac{1}{2}x_2^2(t_0) \leq R^2 \tag{7.8}$$

at time $t = t_0$ is *a priori* known and R is arbitrarily chosen by the user, $(x_1^0, x_2^0) = (x_1(t_0), x_2(t_0))$. Furthermore, t_0 is set to zero without loss of generality.

The goal of this chapter is to develop constructive tuning rules for achieving an arbitrarily specified finite settling time for the closed-loop system (7.1), (7.2) and (7.5). The basis for the first assumption can be found in the literature [4, 9]. Note that the second assumption is not a limitation of the presented method, at least for a large class of mechanical systems. Industrial robotics, for example, satisfy this assumption since the upper-bound on the left hand side of (7.8) at time $t = 0$ is generally known in terms of the total energy of the system.

7.2 Lyapunov Analysis

Results based on strict and non-strict Lyapunov functions for finite time stability of the perturbed double integrator using a twisting control law [4] can be found in the literature [106] for the case of full state feedback. An alternative proof of finite time stability for the general output feedback case can also be found in [18]. In the case of planar systems, the existing result gives continuous finite time stabilization for uniformly decaying disturbances [46, Lemma 3]. A recent output feedback result [14] utilises a bounded continuous modification of (7.2) and proves asymptotic stability when x_2 is estimated by utilising (7.5) (see [14, Th. 6]). Both of the above results do not solve the converse problem of establishing constructive tuning rules to achieve arbitrarily specified settling time T^{d}. Furthermore, the case of the discontinuous controller (7.2) considered in [14, Th. 7] proves finite time stability provided that after some finite time instant $t = T_{\mathrm{obsv}}$ the errors $e_1 = x_1 - \hat{x}_1, e_2 = x_2 - \hat{x}_2$ reach the origin (see the result on finite settling time T_{obsv} of the super-twisting observer [14, 41]). It is only under these conditions that this output feedback formulation is shown to be a valid candidate of [9, Th 4.2]. However, the motion before the observer has converged has not been addressed.

The main difference between Theorem 7.1 below and [14, Th 7] is that it contains an additional result providing an upper bound on the magnitude of the states and hence it can be seen as a stronger version of [14, Th. 7]. Another motivation to propose Theorem 7.1 lies in the fact that, in conjunction with the subsequent theorems, the presented results enable the user to have an *a priori* guarantee of achieving the arbitrarily specified settling time for the observer, for the controller and for the output feedback system (7.1), (7.2) and (7.5), a result not available in [14].

Theorem 7.1. *Consider the system (7.1) under the influence of a disturbance* ω *satisfying (7.6). Let (7.1) be driven by observer based dynamic feedback (7.2), (7.5) with controller parameters* μ_1, μ_2 *satisfying (7.7), with parameter* $\epsilon = \frac{1}{2}$ *and with observer parameters* $k_i, i = 1, 2, 3, 4$ *satisfying condition (7.4). Let assumptions* (1), (2) *of Section 7.1 hold true for given* M *and fixed gains* μ_1, μ_2*. Then the following statements hold true:*

1. *The trajectories of the closed-loop system (7.1), (7.2) and (7.5) cannot escape the compact set*
$$\Omega_{\tilde{R}} = \{x_1, x_2 : \mu_2|x_1| + \frac{1}{2}x_2^2 \leq 2\tilde{R}\} \tag{7.9}$$
where,
$$\tilde{R} = \left(v_0^{\frac{1}{2}} + K_1\, T_{obsv}\right)^2, \; K_1 = \sqrt{2}(\mu_1 + M)$$
$$v_0 \geq \mu_2|x_1^0| + \frac{1}{2}x_0^2 + 2\,k_3|e_1^0| + k_4(e_1^0)^2 + \frac{1}{2}(e_2^0)^2 + \frac{1}{2}\left(e_2^0 - k_1|e_1^0|^{\epsilon} sign(e_1^0) - k_2 e_1^0\right)^2 \tag{7.10}$$
with $x_1^0 = x_1(t_0), x_2^0 = x_2(t_0),\; e_1^0 = x_1(t_0) - \hat{x}_1(t_0), e_2^0 = x_2(t_0) - \hat{x}_2(t_0)$.

2. *The finite convergence time of the states* (x_1, x_2) *to reach the origin is given by* $T_{\text{obsv}} + T_{\text{tw}}$ *where* T_{tw} *is the finite settling time of the twisting controller for the full state feedback.*

Proof. Consider the Lyapunov function candidate [9]
$$V_1(x_1, x_2) = \mu_2\,|x_1| + \frac{1}{2}\,x_2^2. \tag{7.11}$$

The following can be obtained by taking temporal derivatives of (7.11) along the trajectories of the closed-loop system (7.1), (7.2) and (7.5):

$$\begin{aligned} \dot{V}_1 &= \mu_2 \operatorname{sign}(x_1) x_2 + x_2 \left(-\mu_1 \operatorname{sign}(\hat{x}_2) - \mu_2 \operatorname{sign}(x_1) + \omega(t)\right) \\ &= -\mu_1 x_2 \operatorname{sign}(\hat{x}_2) + x_2 \omega(t) \end{aligned} \tag{7.12}$$

Clearly the worst case of increase in the candidate function V_1 occurs when $\operatorname{sign}(\hat{x}_2) \neq \operatorname{sign}(x_2)$ and $\omega(t) = M \operatorname{sign} x_2(t)$. With this in mind, the following upper-bound can be obtained:

$$\dot{V}_1 \leq (\mu_1 + M) |x_2| \tag{7.13}$$

It should be noted that the upper-bound (7.13) is a function of the true plant state x_2 and not the observer state $\hat{x}_2$ even though the temporal derivative was taken with observed velocity. It is known from the literature [14] that the Lyapunov function

$$V_2(e_1, e_2) = 2 k_3 |e_1| + k_4 e_1^2 + \frac{1}{2} e_2^2 + \frac{1}{2} \left(e_2 - k_1 |e_1|^{\epsilon} \operatorname{sign}(e_1) - k_2 e_1\right)^2 \tag{7.14}$$

has a negative definite temporal derivative [14, Th. 5]

$$\dot{V}_2 = -\kappa V_2^{\epsilon} \tag{7.15}$$

along the error system,

$$\begin{aligned} \dot{e}_1 &= e_2 - k_1 |e_1|^{\epsilon} \operatorname{sign}(e_1) - k_2 e_1 \\ \dot{e}_2 &= -k_3 \operatorname{sign}(e_1) - k_4 e_1, \end{aligned} \tag{7.16}$$

in the presence of the disturbance (7.7) if (7.4) is satisfied. Expression (7.15) is derived in [14, th. 5] using (7.5) where $\kappa > 0, \epsilon = \frac{1}{2}$ and $e_1 = x_1 - \hat{x}_1, e_1 = x_2 - \hat{x}_2$ are the observer errors.

The super-twisting controller proposed in [4] is homogeneous in the sense of Definition 2.7. The homogeneity of the error dynamics (7.16) under the application of super-twisting observer is ensured only for the specific value $\epsilon = \frac{1}{2}$ [14]. Reference [14] proves the finite time stability of the error dynamics even under the non-homogeneous values, i.e. $\epsilon \neq \frac{1}{2}$. This chapter uses the homogeneous version with $\epsilon = \frac{1}{2}$ for simplicity.

A Lyapunov function candidate $V(x_1, x_2, e_1, e_2)$ can be defined combining V_1 and V_2 to estimate the largest upper-bound on the trajectories as follows:

$$V(x_1, x_2, e_1, e_2) = V_1 + V_2 \tag{7.17}$$

Noting that both the functions V_1, V_2 are positive definite and radially unbounded [9, Th. 4.2] [14], the function V also shares the same attributes. Furthermore the following inequalities hold true:

$$V_1 \leq V, \quad V_2 \leq V \tag{7.18}$$

Taking the temporal derivative of V along (7.1), (7.2) and (7.5) and utilizing the relationships (7.13),(7.15), the following inequality can be obtained:

$$\dot{V} \leq (\mu_1 + M)\,|x_2| - \kappa\, V_2^{\epsilon} \tag{7.19}$$

Ignoring the negative definite term in (7.19), the steepest increase in the Lyapunov function candidate V can be obtained as follows:

$$\dot{V} \leq (\mu_1 + M)\,|x_2| \tag{7.20}$$

Noting from (7.11) that $|x_2| \leq \sqrt{2V_1}$, the following inequality can be obtained:

$$\dot{V} \leq \sqrt{2}(\mu_1 + M)\sqrt{V} \tag{7.21}$$

where the first inequality of (7.18) has been utilized. Hence the steepest increase in the function $V(t), t \in [t_0, T_{\text{obsv}}]$ can be defined by the following equality:

$$\dot{v} = \sqrt{2}(\mu_1 + M)\sqrt{v} \tag{7.22}$$

where t_0 is the initial time and T_{obsv} is finite settling time of the observer [14, 41]. The inequality $V(t) \leq v(t), t \in [t_0, T_{\text{obsv}}]$ follows by the comparison principle [16] for some positive function $v(t)$ if $v(t_0) \geq V(x_1(t_0), x_2(t_0), e_1(t_0), e_2(t_0))$. The non-zero solution of the equality (7.22) can be given as follows:

$$v(t) = \left[v_0^{\frac{1}{2}} + K_1\, t \right]^2, \qquad t \in [t_0, T_{\text{obsv}}] \tag{7.23}$$

where $K_1 = \sqrt{2}(\mu_1 + M)$ and $v_0 = v(t_0)$. The following should be noted:

1. It is clear from (7.23) that $\sup_{t \geq 0} v(t)$ can be obtained by substituting $t = T_{obsv}$.

2. The system (7.1), (7.2) and (7.5) coincides with full state feedback system after $t = T_{\text{obsv}}$, for which the finite settling time is known (see references [106], [9, Th. 4.2] and sections 4.3, 4.3.

3. The Lyapunov function candidate V cannot increase after $t = T_{obsv}$ and has negative semi-definite temporal derivative [9]

$$\dot{V} \leq -(\mu_1 - M)\,|x_2| \tag{7.24}$$

with $V_2 = 0$.

Noting that inequalities (7.18) always hold true, the 1-norms of true plant states x_1, x_2 cannot escape the following finite upper bounds:

$$\sup_{t\geq 0} V(t) \leq \sup_{t\geq 0} v(t) = \left(v_0^{\frac{1}{2}} + K_1\, T_{obsv} \right)^2$$

$$\sup_{t\geq 0} |x_1(t)| \leq \sup_{t\geq 0} \frac{V_1(t)}{\mu_2} \leq \sup_{t\geq 0} \frac{V(t)}{\mu_2} \tag{7.25}$$

$$\sup_{t\geq 0} |x_2| \leq \sup_{t\geq 0} \sqrt{2V_1(t)} \leq \sup_{t\geq 0} \sqrt{2V(t)}$$

Utilising the last two sub-equations of (7.25) it can be concluded that the trajectories of the closed-loop system (7.1), (7.2) and (7.5) cannot leave the region defined by the finite compact set

$$\Omega_{\tilde{R}} = \{x_1, x_2 : \mu_2|x_1| + \frac{1}{2}x_2^2 \leq 2\tilde{R}\}$$
$$\tilde{R} = \sup_{t\geq 0} v(t) = \left(v_0^{\frac{1}{2}} + K_1\, T_{obsv} \right)^2 \tag{7.26}$$

around the origin $\forall\ t \geq 0$. Once the observer errors satisfy $(e_1, e_2) = (0, 0)$ for all $t \geq T_{\text{obsv}}$, the finite time stability of (7.1), (7.2), (7.5) and the corresponding finite upper bound on the settling time follows by applying [14, th. 4.2] within the compact set (7.26). Furthermore, the trajectories of the closed-loop system (7.1), (7.2) and (7.5) remain at the origin for all $t \geq T_{\text{obsv}} + T_{\text{tw}}$. □

7.3 Tuning

This section presents tuning rules for the observer gains and the controller gains to guarantee attainment of an *a priori* specified settling time T^d for the output feedback synthesis. This section has synergy with Section 4.3, which provided constructive tuning rules for the state feedback case, in that it provides constructive tuning rules for the output feedback case.

The approach adopted is that of tuning the controller gains independently using the user specified region $\Omega_{\tilde{R}}$ appearing in (7.26) of the state trajectories with $\tilde{R}$ chosen arbitrarily large by the user with a constraint on its lower bound (see Theorem 7.2). Then the desired settling time T_{obsv} for the observer can be obtained using (7.23) and the initial condition v_0.

It should be noted that due to the second assumption, the variable v_0 is a known quantity. The observer gains are then tuned using the differential inequality (7.15) to achieve the desired settling time constraint T_{obsv}. It should be noted that the energy-like function (7.8) is utilised in the process. Although the state x_2 is not measured, the upper-bound on the left hand side of (7.8) is generally known for a large class of mechanical systems at initial time $t = t_0$. Other forms of energy-like functions can also be considered in a similar way. The controller gains are tuned to ensure finite time convergence to origin starting from any initial condition within $\Omega_{\tilde{R}}$.

In summary, the observer errors settle to zero before the trajectories of the closed-loop system can reach beyond $\Omega_{\tilde{R}}$ and the controller ensures that all trajectories starting within $\Omega_{\tilde{R}}$ remain at the origin $(x_1, x_2) = (0, 0)$ for all $t \geq T^{\text{d}}$.

7.3.1 Controller Gains

The finite settling time for the twisting controller with $x_2 = \hat{x}_2$ has been computed for the open loop system (7.1) with and without jumps in state x_2 as studied in sections 4.3, 4.3 and 5.5. The tuning rules based on the convolution integration method for a comparison system developed in Chapter 4 are employed in this subsection. The detailed proof of Theorem 7.2 below can be obtained from Chapter 4 and is included briefly with an additional tuning rule for completeness of this chapter. First, assumption (7.8) can be utilised to propose appropriate guideline for selecting $\tilde{R}$.

Lemma 7.1. *There always exists a scalar $\hat{R}$ such that $\Omega_{\hat{R}} \subseteq B_{\sqrt{2}R}$ where,*

$$\Omega_{\hat{R}} = \{\mu_2|x_1^0| + \frac{1}{2}x_2^0 \leq \hat{R}\},\ B_{\sqrt{2}R} = \{(x_1^0)^2 + (x_2^0)^2 \leq 2R^2\}. \tag{7.27}$$

Proof. The aim is to define the scalar $\hat{R} > 0$ such that the expression $\Omega_{\hat{R}} \subset B_{\sqrt{2}R}$ holds. In other words, the following is required:

$$\frac{\mu_2|x_1^0|}{\hat{R}} + \frac{(x_2^0)^2}{2\hat{R}} \leq 1 \Rightarrow \frac{1}{2}\left(\frac{x_1^0}{R}\right)^2 + \frac{1}{2}\left(\frac{x_2^0}{R}\right)^2 \leq 1 \tag{7.28}$$

Impose the following inequalities:

$$\frac{1}{2}\left(\frac{x_1^0}{R}\right)^2 \leq \frac{\mu_2|x_1^0|}{\hat{R}}, \quad \frac{1}{2}\left(\frac{x_2^0}{R}\right)^2 \leq \frac{(x_2^0)^2}{2\hat{R}} \tag{7.29}$$

Then the expression $(x_1^0, x_2^0) \in B_R$ holds true for every given point $(x_1^0, x_2^0) \in \Omega_{\hat{R}}$. Note that the following always holds true for all $(x_1^0, x_2^0) \in \Omega_{\hat{R}}$:

$$|x_1^0| \leq \frac{\hat{R}}{\mu_2} \tag{7.30}$$

The first inequality of (7.29) can be simplified as follows:

$$\frac{1}{2}|x_1^0|\left(\frac{1}{R}\right)^2 \leq \frac{\mu_2}{\hat{R}} \tag{7.31}$$

Utilizing the relationship (7.30), the following requirement, which is even more conservative than (7.31), can be formulated:

$$\frac{1}{2}|x_1^0|\left(\frac{1}{R}\right)^2 \leq \frac{1}{2}\frac{\hat{R}}{\mu_2}\left(\frac{1}{R}\right)^2 \leq \frac{\mu_2}{\hat{R}} \tag{7.32}$$

Hence, the following upper-bound on $\hat{R}$, obtained from (7.32), suffices to satisfy the first inequality of (7.29).

$$\hat{R} \leq \sqrt{2}\mu_2 R \tag{7.33}$$

Similarly, the second inequality of (7.29) leads to $R \leq R^2$. Combining this with (7.32), the following estimate of the parameter $\hat{R}$ is obtained:

$$\hat{R} = \min\left\{\sqrt{2}\mu_2 R, R^2\right\} \tag{7.34}$$

Further simplification $\hat{R} = R^2$ is obtained if the following condition is imposed:

$$\mu_2 \geq \frac{R}{\sqrt{2}} \tag{7.35}$$

Hence, the statement of Lemma 7.1 is proven. □

An upper-bound on the function V_1 at $t = t_0$ can be obtained from the definition of $\Omega_{\hat{R}}$ in (7.27) as follows:

$$V_1(x_1(t_0), x_2(t_0)) \leq \hat{R}. \tag{7.36}$$

It is noted from (7.14) that $V_2(e_1^0, e_2^0) = e_2^2(t_0)$ holds true because $e_1 = 0$ can be enforced due to the availability of the state x_1, thereby rendering $V_2(e_1^0, e_2^0)$ independent of the gains $k_i, i = 1, 2, 3, 4$. Since the largest magnitude of x_2 is limited to $\sqrt{2}R$ due to (7.8), the initial estimate $\hat{x}_2(t_0)$ can be chosen from the interval $[-\sqrt{2}R, \sqrt{2}R]$. Hence, the worst case (largest) magnitude of $V_2(e_1^0, e_2^0)$ is given by $8R^2$.

Theorem 7.2. *Consider the system (7.1) under the influence of the disturbance ω satisfying (7.6). Let (7.1) be driven by state feedback (7.2) with $x_2 = \hat{x}_2$ under the influence of disturbance ω satisfying (7.6). Let the assumptions (1), (2) in Section 7.1 hold true. Let the parameter $\tilde{R}$ be chosen such that*

$$\tilde{R} > \hat{R} + 8R^2. \tag{7.37}$$

Then, for some $\eta \in (0,1), \beta > 1 \rho \in (0,1)$, the tuning rules

$$\begin{aligned} &\mu_1 > \max\left\{ M, \quad \frac{2\delta}{T^d\sqrt{1-\eta^2}} + M \right\} \\ &\mu_2 > \max\left\{ \sqrt{\frac{\tilde{R}}{2}}, \quad \frac{R}{\sqrt{2}}, \quad \frac{\mu_1 - M}{\eta}, \quad \mu_1 + M, \quad \rho\max\{\frac{\tilde{R}}{2(1-\rho)}, 1\} \quad \beta\,\mu_1 \right\} \end{aligned} \tag{7.38}$$

guarantee that the actual settling time T_{tw} of the full state feedback system is less than the specified settling time T^{d}.

Proof. Once the observer settles to the origin $(e_1, e_2) = (0, 0)$, the closed-loop system trajectories can be shown to be encompassed by the following comparison system (see Section 4.3):

$$\dot{x}(t) = A\,x(t) + B\,u(t), \tag{7.39}$$

where

$$\begin{aligned} &x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &u(t) = -(\mu_1 - M)\mathrm{sign}(x_2) - \mu_2\mathrm{sign}(x_1), \end{aligned} \tag{7.40}$$

The solution of (7.39) is obtained as follows:

$$x(t) = e^{A\,t}\,x(t_0) + \int_{t_0}^{t} e^{A\,(t-\tau)} B\,u(\tau)d\tau \tag{7.41}$$

where $x(t_0) = [x_1(t_0)\ x_2(t_0)]^{\mathrm{T}}$. The finite settling time T_{tw} of the above system can be obtained by applying the convolution integral method step-by-step as follows (see Section 4.3):

$$T_{\mathrm{tw}} = \frac{\delta\Delta}{\mu_2(1-\Psi)} = \frac{\delta(\sqrt{1-\eta^2}+1)}{(\mu_1 - M)\sqrt{1-\eta^2}} \le \frac{2\delta}{(\mu_1 - M)\sqrt{1-\eta^2}} < \infty \tag{7.42}$$

where the following quantities have been used to compute the second last term on the right.

$$\eta = \frac{\mu_1 - M}{\mu_2} < 1, \quad \Delta = \left[\frac{2}{1+\eta} + \frac{1}{\sqrt{(1+\eta)(1-\eta)}} + \frac{\sqrt{1-\eta}}{(1+\eta)\sqrt{1+\eta}}\right]$$
$$\Psi = \frac{1-\eta}{1+\eta} < 1 \tag{7.43}$$

Also, the scalar δ is chosen such that the inequality $\delta\Psi > r_1$ holds true where the parameter

$$r_1 = \max\left\{\frac{\tilde{R}}{\mu_2}, \sqrt{2\tilde{R}}\right\}, \tag{7.44}$$

which can be derived in a similar manner as that utilised in Section 4.2.1, is the radius of the ball $B_{r_1} \supset \Omega_{\tilde{R}}$ with $\Omega_{\tilde{R}}$ defined in (7.9). The tuning rule $\mu_2 > \frac{R}{\sqrt{2}}$ follows from the proof of Lemma 7.1. The rest of the tuning rules follow from the results of Section 4.4.

Combining (7.7), the existing result from Section 4.4 and (7.35) leads to the tuning rules (7.38). The aim of guaranteeing $T_{\text{tw}} < T^{\text{d}}$ is achieved for all $x \in \Omega_{\tilde{R}}$ if the tuning rules (7.38) are used. Such a tuning rule ensures that $\mu_1 \to \infty, \mu_2 \to \infty$ as $T^{\text{d}} \to 0$ while observing the constraint $\mu_2 \geq \frac{\mu_1 - M}{\eta}$ for selected constants η, β. □

Remark 7.1. The requirement of the lower bound for the parameter $\tilde{R}$ defined in (7.37) is not restrictive. Expression (7.21) of Theorem 7.1 clearly shows that there is a possibility that the states may evolve initially away from the origin. Therefore, it is natural to assume the selection of $\tilde{R}$ such that it represents a larger region than that defined by the semi-global region represented by $\hat{R} + 8R^2$, where $2\sqrt{2}R$ is the maximum error in the velocity estimation at time t_0.

7.3.2 Observer Gains

The previous section presented the tuning rules for ensuring that the state feedback $\Phi(x)$ of (4.1) drives every trajectory starting within $\Omega_{\tilde{R}}$ to the origin in finite time. Theorem 7.1 showed that trajectories cannot leave the region defined by $\Omega_{\tilde{R}}$ and that observer based feedback (7.2) coincides with the state feedback $\Phi(x)$ of (4.1) after the observer dynamics (7.16) settle at the origin in finite time.

It should be noted that the parameter $\tilde{R}$ is fixed by the user's specification. In turn, the usefulness of Theorem 7.2 towards the final goal of finite time output feedback stabilisation holds only if the actual settling time of the observer satisfies an upper bound obtained from the second equality of (7.26) with the aforementioned fixed $\tilde{R}$.

This section first computes this constraint to derive an upper bound on the desired settling time of the observer. Next, this upper bound is utilised in conjunction with the existing result [14, Th. 5] to solve the converse problem of establishing the tuning rules for the gains of the observer to achieve arbitrarily small settling time for the observer error dynamics (7.16). The following result is in order:

Theorem 7.3. *Consider the system (7.1) under the influence of the disturbance ω satisfying (7.6). Let (7.1) be driven by observer based dynamic feedback (7.2), (7.5). Let the assumptions (1), (2) in Section 7.1 hold true. Let the parameter $\tilde{R}$ be chosen arbitrarily subject to (7.37). Then, the tuning rules*

$$\begin{aligned} k_4 &= \beta_2\Big(\frac{\sqrt{v_2^0}}{T}\Big)^{\frac{1}{3}}, \quad k_2 > \sqrt{2k_4} \\ k_1 &> \max\{2M, \frac{2M(2k_4+3k_2^2)}{3(k_2^2-2k_4)}, \frac{\beta_1}{\sqrt{2}}\Big(\frac{\sqrt{v_2^0}}{T}\Big)^{\frac{1}{3}}(2k_4+3k_2^2),\ \}, \\ k_3 &> \max\{\frac{1+k_1}{2}, \frac{\frac{3}{4}k_1^2(k_1-2M)+2M(1+k_1)}{k_1+2M}, \quad \beta_3^2\Big(\frac{\sqrt{v_2^0}}{T}\Big)^{\frac{2}{3}}\} \end{aligned} \tag{7.45}$$

when $k_2 > 0, k_4 > 0$ and the tuning rules

$$\begin{aligned} k_1 &> 2\beta_1\Big(\frac{\sqrt{2v_2^0}}{T}\Big)^{\frac{1}{2}} + 2M, \\ k_3 &= \max\{\frac{1+k_1}{2}, \frac{\frac{3}{4}k_1^2(k_1-2M)+2M(1+k_1)}{k_1+2M}, \quad 8\beta_3^2\frac{\sqrt{2v_2^0}}{T}\} \end{aligned} \tag{7.46}$$

when $k_2 = k_4 = 0$ where,

$$T = \min\{T_{\tilde{R}},\ T^{\mathrm{d}}\}, \quad \beta_1 > 1, \beta_2 > 1, \beta_3 > 1, \quad T_{\tilde{R}} = \frac{\sqrt{\tilde{R}} - \sqrt{v_0}}{K_1} \tag{7.47}$$

guarantee that the trajectories of the closed-loop system cannot escape the compact set $\Omega_{\tilde{R}}$ before the observer errors (e_1, e_2) settle to the origin.

Proof. Let the observer dynamics be initialized with $e_1^0 = x_1^0 - \hat{x}_1^0, e_2^0 = x_2^0 - \hat{x}_2^0$ with $\hat{x}_1^0, \hat{x}_2^0$ arbitrarily chosen. Scalar $\tilde{R}$ is selected as given by (7.37). Then, the constraint on the observer settling time can be obtained from (7.23) as follows:

$$T_{\tilde{R}} = \frac{\sqrt{\tilde{R}} - \sqrt{v_0}}{K_1} = \frac{\sqrt{\tilde{R}} - \sqrt{\hat{R} + 8R^2}}{K_1} \tag{7.48}$$

where the upper bound on the function (7.8) is utilised to obtain the worst case (i.e. least) value for $T_{\tilde{R}}$. Finite time stability using strict Lyapunov functions has been recently established [14, 41]. The settling time of the system (7.1), (7.5) specified with $\epsilon = \frac{1}{2}$ can be obtained using the Lyapunov function (7.14) and its derivative (7.15) and is given as follows [14, Th. 5]:

$$T(e_1^0, e_2^0) \leq \frac{2\sqrt{V_2(e_1^0, e_2^0)}}{\kappa} \tag{7.49}$$

where,

$$\kappa = \gamma_2\sqrt{2k_3}, ,\gamma_2 = \min\{\frac{2(k_1k_3 - M - Mk_1)}{4k_3 + 3k_1^2}, \frac{k_1 - 2M}{4}, \frac{2k_1k_4}{2k_4 + 3k_2^2}\} \tag{7.50}$$

with the condition that

$$\min\{\frac{k_1}{2}, \quad \frac{k_1k_3}{1 + k_1}\} > M. \tag{7.51}$$

Having chosen $R, \tilde{R}, \hat{x}_1^0, \hat{x}_2^0$, the goal is to ensure that the finite settling time of the observer system (7.5) satisfies the following:

$$T_{\text{obsv}} = \frac{2\sqrt{v_2^0}}{\kappa} \leq T = \min\{T_{\tilde{R}}, \quad T^{\text{d}}\} \tag{7.52}$$

where $v_2^0 \geq V_2(e_1^0, e_2^0)$. It is noted from (7.14) that $V_2(e_1^0, e_2^0) = e_2^2(t_0)$ holds true because $e_1 = 0$ can be enforced due to the availability of the state x_1, thereby rendering $V_2(e_1^0, e_2^0)$ independent of the gains $k_i, i = 1, 2, 3, 4$. First, let the following be imposed on the observer gains:

$$\frac{2(k_1k_3 - M - Mk_1)}{4k_3 + 3k_1^2} > \frac{k_1 - 2M}{4} > \frac{2k_1k_4}{2k_4 + 3k_2^2} \tag{7.53}$$

The requirement (7.51) can be simplified as $k_1 > 2M$ if the expression $\frac{k_1k_3}{1+k_1} > \frac{k_1}{2}$ or $k_3 > \frac{1+k_1}{2}$ holds true. Using this, further simplification of the last two terms and the first two terms of the inequality (7.53) leads respectively to the following:

$$\begin{aligned} &k_1 > \max\{2M, \frac{2M(2k_4 + 3k_2^2)}{3(k_2^2 - 2k_4)}\} \text{ with } \quad k_2 > \sqrt{2k_4}, \\ &k_3 \geq \max\{\frac{1 + k_1}{2}, \frac{\frac{3}{4}k_1^2(k_1 - 2M) + 2M(1 + k_1)}{k_1 + 2M}\}. \end{aligned} \tag{7.54}$$

Then, the following holds true due to (7.54):

$$\gamma_2 = \frac{2k_1k_4}{2k_4 + 3k_2^2} \tag{7.55}$$

The following requirement can be obtained by substituting (7.55) into (7.50) and in turn into (7.52):

$$\kappa = \frac{2\sqrt{2}k_1k_4\sqrt{k_3}}{2k_4 + 3k_2^2}, \quad \frac{\sqrt{v_2^0}(2k_4 + 3k_2^2)}{\sqrt{2}k_1k_4\sqrt{k_3}} \leq T \tag{7.56}$$

The second inequality in (7.56) and in turn (7.52) are always guaranteed for the chosen $\tilde{R}$ if the following inequality holds true:

$$\frac{k_1 k_4 \sqrt{k_3}}{2k_4 + 3k_2^2} > \sqrt{\frac{v_2^0}{2}\frac{1}{T}} \tag{7.57}$$

The above requirement can be translated into three sub-conditions as follows. First select k_4 such that the inequality

$$k_4 > \beta_2 (\frac{\sqrt{v_2^0}}{T})^{\frac{1}{3}} \tag{7.58}$$

holds true for some arbitrarily chosen tuning scalar $\beta_2 > 1$. The gain k_2 immediately follows from (7.54). Next, select k_1 such that the inequality

$$k_1 > \frac{\beta_1}{\sqrt{2}} (\frac{\sqrt{v_2^0}}{T})^{\frac{1}{3}} (2k_4 + 3k_2^2) \tag{7.59}$$

holds true for some arbitrarily chosen tuning scalar $\beta_1 > 1$. Finally select k_3 such that the inequality

$$\sqrt{k_3} > \beta_3 (\frac{\sqrt{v_2^0}}{T})^{\frac{1}{3}} \tag{7.60}$$

Then, combining (7.58), (7.59) and (7.60) ensures that the requirement (7.57) and in turn (7.52) are always satisfied. Combining expressions (7.54), (7.58), (7.59) and (7.60) results in the tuning rule (7.45). A similar analysis can lead to tuning rules (7.46) for the case $k_2 = k_4 = 0$ where (7.53) and (7.54) ensure $\gamma = \frac{k_1 - 2M}{4}$. Since the settling time constraint $T_{\tilde{R}}$ in (7.48) is computed using specified $\tilde{R}$, it follows from (7.56) that the trajectories of the closed-loop system (7.1), (7.2), (7.5) cannot escape $\Omega_{\tilde{R}}$ by Theorem 7.1 for the chosen $\tilde{R}$ before $(e_1, e_2) = (0, 0)$. □

Remark 7.2. It should be noted that the tuning algorithm presented above gives liberty to choose parameters $R, \tilde{R}, T^{\mathrm{d}}$. This inherently imposes a constraint on the observer gains in terms of the computed variable $T_{\tilde{R}}$. The gains should be such that the observer errors settle to zero before the closed-loop system trajectories escape the region $\Omega_{\tilde{R}}$. It may happen that the computed constant $T_{\tilde{R}}$ is such that the inequality $T_{\tilde{R}} < T^{\mathrm{d}}$ holds true for some stringent specification of $\tilde{R}$ and hence the need for introducing the term $\min\{T_{\tilde{R}}, T^{\mathrm{d}}\}$ in (7.52). The tuning rules for the observer gains ensure that the gains $k_i \to \infty$, $i = 1, 2, 3, 4$ as $\min\{T_{\tilde{R}}, T^{\mathrm{d}}\} \to 0$. It is noteworthy that this mathematical result satisfies the intuitive expectation mentioned in the section 'Objectives' of Chapter 4, namely, that increasing the gains of the observer leads to a decrease in the finite settling time of the error system (7.16).

The expressions (7.38) and (7.45) thus give tuning rules for the gains μ_1, μ_2 of the controller (7.2) and for the gains $k_i, i = 1, 2, 3, 4$ of the observer (7.5) respectively for arbitrarily chosen parameters $\tilde{R}, R, T^{\mathrm{d}}$ and given values of $x_0^1, V(e_1^0, e_2^0), M$. Hence, the observer errors are guaranteed to settle to zero before the trajectories of the closed-loop system can reach beyond $\Omega_{\tilde{R}}$ and the controller ensures that all trajectories starting from within the set $\Omega_{\tilde{R}}$ remain at the origin $(x_1, x_2) = (0, 0)$ for all $t \geq T^{\mathrm{d}}$. It should also be noted that the observer gain k_3 is a function of controller gain μ_1. This is visible in (7.45) in that $T_{\tilde{R}}$ is a function of μ_1.

7.4 Numerical Simulation

Figure 7.1 shows numerical simulation results for the output feedback closed-loop system (7.1), (7.2) and (7.5) with the following parameters specified *a priori* as per the assumptions in section 7.1:

$$R = 1, \quad M = 1, \quad T^{\mathrm{d}} = 10 \text{ sec.} \tag{7.61}$$

Let the following choices be made:

$$\tilde{R} = 18, \quad \hat{x}_1^0 = 0.1, \quad \hat{x}_2^0 = -1. \tag{7.62}$$

To simulate the physical scenario, the plant initial conditions are chosen as $x_1^0 = 0.1, x_2^0 = 1$. Note that expression (7.8) is satisfied at the initial time $t_0 = 0$. Then, applying the tuning rules (7.38) and (7.45) produces

$$\begin{aligned} r_1 = 6, \quad \delta = 17.6 \\ \mu_1 = 5.44, \quad \mu_2 = 13.15 \\ k_4 = 2.14, \quad k_2 = 3.1, \quad k_1 = 55, \quad k_3 = 13.5. \end{aligned} \tag{7.63}$$

It can be seen from Figure 7.1 that the achieved settling time is less than $0.95 \text{ sec} < T^{\mathrm{d}} = 10$ sec. Furthermore, the Lyapunov function $\sup_{t \geq 0} V(t) < 5.82$ which is less that the chosen upper-bound $\tilde{R} = 18$ (see (7.26)). It can be seen that the Lyapunov function candidate $V_1(x_1, x_2)$ increases until the time $\hat{x}_2$ equals x_2 at approximately $t = 0.15$ sec, after which time V_1 decreases monotonously. The function $V_1(x_1, x_2)$ represents a proper Lyapunov function as soon as $e_2 = 0$ is achieved.

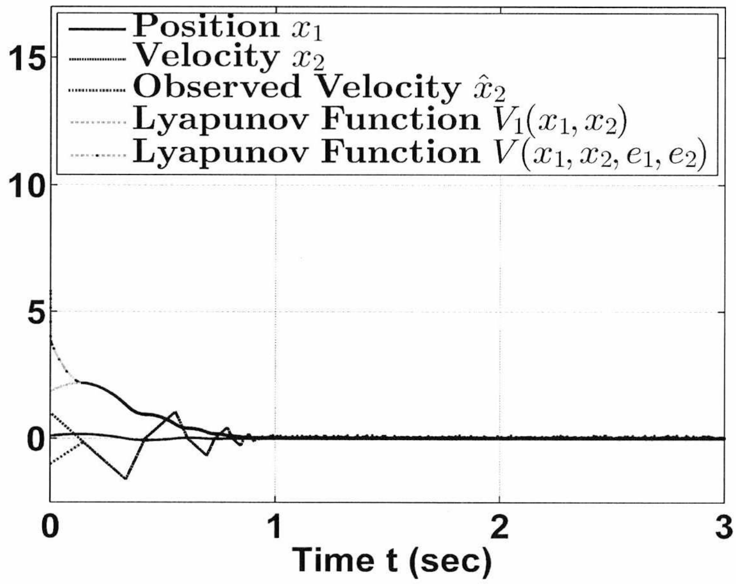

FIGURE 7.1: Finite settling time behaviour of the output feedback system (7.1), (7.2) and (7.5) under the tuning rules (7.38) and (7.45)

7.5 Summary

7.5.1 Contributions

This chapter contains two contributions. Firstly, a finite upper bound on the states of the closed-loop system is computed using the Lyapunov function for the second order sliding mode based output feedback synthesis of planar controllable systems. Furthermore, the user can choose this upper bound arbitrarily larger than the initial condition region. This proves that the closed-loop system cannot escape to infinity.

Secondly, tuning rules are developed for both, the observer and the controller to ensure finite settling time of the closed-loop system in the presence of external disturbances. The tuning rules are developed using the largest upper bound on the states specified by the user. The observer gains are tuned to ensure that the observer error converges to zero before the trajectories of the closed-loop system escape the region specified by the user. The controller gains are tuned to guarantee that every trajectory starting within the user specified region remains at the origin after the time instant specified by the settling time specification.

7.5.2 Future work

A future extension of this work can be the development of similar tuning rules for the output feedback case which utilizes arbitrarily higher order sliding mode observers and controllers. As Lyapunov approaches for the higher order sliding mode algorithms are being continually developed, such tuning rules will be of significant relevance to engineering applications. Another potential strand of research can be the development of advanced tuning rules such as auto-tuning and step response tuning to produce commercially competitive second order sliding mode control algorithms.

7.5.3 Orbital stabilisation of planar systems with resets

Recalling the discussion presented in Section 4.7, this thesis intends to study uniform finite time stabilisation for mainly two types of problems when unilaterally constrained planar systems are considered: (i) Problems where the impacts have a finite accumulation point (see [23, 36] for detailed discussion on the so called 'Zeno' behaviour caused by the accumulation of impacts) and (ii) where the impacts do not have an accumulation point. The first problem was studied in Chapter 5 by utilising a variable structure controller and in Chapter 7 by utilising a non-smooth continuous homogeneous controller.

The second type of problem mentioned above is considered in this thesis in terms of a tracking problem where recurring impacts in the state variable x_2 as well as that in its desired value x_2^{d} leads to studying stability of the closed-loop system in terms of 'orbital stabilisation' rather than regulation at the origin. The trajectory mainly possesses two phases, namely, the continuous phase when there are no impacts and the solutions are absolutely continuous in the sense of Filippov's definition [6] and the discontinuous phase when the velocity x_2 undergoes a reset. The practical motivation to study such systems can be found in the area of robotics (see, for example, reference [15] in the area of biped robotics where collision of feet with ground inherently gives rise to impulse effects in velocity). The control problem consists of two parts: (i) Designing an appropriate desired trajectory $(x_1^{\mathrm{d}}, x_2^{\mathrm{d}})$ so that it represents the desired stable periodic orbit and (ii) utilising a finite time synthesis to drive the actual trajectory (x_1, x_2) to the desired $(x_1^{\mathrm{d}}, x_2^{\mathrm{d}})$ one in finite time in the presence of both uncertainty at the time of impact and disturbance in the continuous phase. This thesis does not study the trajectory planning problem mentioned above. The focus is mainly on the second problem mentioned above, namely, *equiuniform* finite time stabilisation of closed-loop system between the impacts to guarantee a priori the attainment of the desired orbit. This is the topic of study for the next chapter which revisits and adapts the tuning rules presented in Chapter 4 to achieve what is defined as 'Finite Time Impact Attenuation'. This concept has been first introduced in and is being published in the publication [61] along with corresponding discrete-time framework.

The next chapter, which is also the penultimate chapter of the thesis, also demonstrates the applicability of the theoretical results presented thus far in this thesis by presenting two applications. The first application relates to the experimental results of the K-RAFT© controller[1]. This experiment highlights the application of Chapter 6 using analogue circuits. The second application relates to the numerical simulation results of finite time tracking for a biped robot with pre-specified settling time using a second order sliding mode synthesis[2]. This application highlights the application of the 'Finite Time Impact Attenuation' concept with the state feedback which inherently relies on both the tuning rules and the uniform finite time stabilisation concepts developed in Chapter 4.

7.6 Conclusion

Constructive tuning rules to achieve an arbitrarily specified settling time are developed for the output feedback synthesis. A second order sliding mode (super-twisting) observer and a second order sliding mode (twisting) controller were utilized. It is shown that the trajectories of the closed-loop system cannot escape to infinity in finite time with appropriate tuning. The user can arbitrarily choose both the settling time and the allowable upper-bound on the closed-loop trajectories.

The next chapter presents the orbital stabilisation studies along with some applications to fully actuated mechanical systems as discussed in Section 7.5.3.

[1]The financial support by Kent Innovation and Enterprise, University of Kent which holds the copy right, is acknowledged by the author who was one of the principal investigators for this research work. The permission by Kent Innovation and Enterprise to include the outcomes of the research on K-RAFT in this thesis is also acknowledged.

[2]The research work contributed by the overseas collaborators Dr. Yannick Aoustin and Dr. Christine Chevallereau on biped robot modelling and simulation is acknowledged. The contribution of the thesis in this application lies mainly in proposing the robust SOSM synthesis given by Chapter 4 along with its tuning adopted for the biped robot under investigation.

CHAPTER 8

Finite Time Orbital Stabilisation and Applications

Dynamical systems with non-smooth vector fields and unilateral constraints have been studied in various disciplines of science and engineering such as electrical engineering and robotics amongst others [23] as noted in previous chapters. Establishing conditions for stability of such systems is a theoretical challenge due to the likely destabilizing effects of the unilateral constraints, resulting in state jumps. In this chapter, orbital stabilisation of planar controllable linear and feedback linearizable non-linear systems with unilateral constraints are studied. Recurring impacts give rise to unstable dynamics. This chapter is concerned with synthesizing robust second order sliding mode controllers to induce stable orbits in the presence of impacts stemming from the unilateral constraints.

Unilaterally constrained systems can be studied as linear or non-linear complementarity systems [23]. The closed-loop dynamics of these systems are strongly affected by hard non-linearities characterized by both the nature of the contact of two bodies as well as collision, with the constraint resulting in restitution of one of the states of a planar system. The existing literature is abundant on the topic of stability of systems with jumps (see [23], [37] and the references therein).

The main focus of this chapter is on finite time tracking of trajectories of the unilaterally constrained system to the desired periodic trajectories by using a second order sliding mode controller together with tuning of the controller parameters. The theoretical motivation to study such a problem stems from several viewpoints. Firstly, asymptotic and finite time stability when impacts have a finite accumulation point have been studied for restitution coefficient in the interval $[0, 1)$ (see [23] and Chapter 5). However, an *a priori* guarantee and tuning to achieve finite time convergence for the tracking of periodic trajectories is not studied for the general case where restitution in the plant state is beyond the interval $[0, 1)$. Furthermore, robust finite time stabilization is theoretically an equally difficult challenge when compared to its asymptotic counterpart [37] provided that the effects of (possibly dissipative) impacts tend to destabilize the system.

Secondly, research on robustness of the controller to disturbances at the time of state jumps is not rigorous. This chapter studies a similar class of systems to that presented in recent research [146] where a finite time observer is proposed to reject disturbances at the time of impacts. The presented result utilises tuning for a second order sliding mode (SOSM) controller [4] such that it is robust to persisting disturbances in both the continuous and discrete-event phases of the trajectories, while requiring only upper bounds on the uncertainties to cause the tracking errors to converge to zero in finite time between successive impacts.

Finally, there exists a strong motivation to extend the study of the aforementioned problem to the discrete time setting. The existing results on numerical methods for sliding mode synthesis [118, 133] can be directly utilised for the tracking control of a unilaterally constrained system in the presence of impacts. A combination of the tuning of a robust SOSM synthesis that is based on the continuous time setting with digital implementation, which matches the behaviour of the discrete time closed-loop system to that of the continuous time system at least in the case of no uncertainty, is theoretically novel for the underlying impact system. This chapter does not study this problem but the reader is referred to a recent result [61] where the aforementioned numerical integration is being published.

From a practical viewpoint, the motivation stems from the need to propose tuning rules and digital implementation of SOSM for the presented class of systems. Following the recent advances in (i) Lyapunov methods for higher order sliding mode controllers and observers [9, 106], (ii) finite time convergence and tuning as developed in Chapter 4 and (iii) numerical schemes [118, 133], a natural next step is to provide similar tuning and digital implementation for the problem of orbital stabilisation. This problem is relevant to the area of biped robots and in general applicable to fully actuated robots with unilateral constraints and thus has practical significance. This chapter establishes the theory of orbital stabilisation in the presence of uncertainty in both the continuous and impact phases of motion leading to what is referred to in the subsequent sections as finite time impact attenuation in continuous time. The discrete time counterpart [1]

[1]Research work carried out by collaborators Dr. Vincent Acary and Dr. Bernard Brogliato in the area of implicit numerical scheme for an SOSM controlled unilaterally constrained system is acknowledged. Reference [61] has this work as its second half. The first half of reference [61] on finite time impact attenuation in continuous time setting is included in this chapter.

is being published elsewhere [61] as mentioned above.

The rest of the chapter is organized as follows. Section 8.1 outlines the problem statement pertaining to the finite time orbital stabilisation of a unilaterally constrained planar system. Section 8.2 introduces the new concept of 'Finite Time Impact Attenuation' to encompass uncertainties in both the continuous and reset phases of motion. Section 8.2 also gives proof of the above new concept for a planar system described in Section 8.1 by utilising a SOSM controller. Next, two applications are presented. Section 8.3 demonstrates the application of the orbital stabilsation in a seven link biped robot thereby underlining the applicability and importance of the theory developed in Chapter 4 and Section 8.2. Section 8.4 gives the second application which is a recently developed analogue finite time controller which is based on the theoretical results of reference [11] and Chapter 6. Section 8.5 summarizes the contributions of this chapter. Section 8.6 draws some conclusions.

8.1 Problem Statement

Let the continuous system dynamics be given by (5.1). Let the velocity restitution (or impact event due to the unilateral constraint) (5.1d) be revised according to the following Newtonian Reset Map:

$$x_2(t_k^+) = -\bar{e}x_2(t_k^-) + \omega_\Delta \quad (x_1, x_2) \in \mathcal{S} \tag{8.1}$$

where the notation $x(t_k^+)$ and $x(t_k^-)$ denote the right limit $\lim_{t \nearrow t_k} x(t_k)$ and the left limit $\lim_{t \searrow t_k} x(t_k)$ respectively, $\bar{e} \in [0, 1)$ is the restitution parameter responsible for the energy absorbtion [23], $t_k, k = 1, 2, \ldots, \infty$ are the time instants at which these impulses occur and ω_Δ characterises the restitution uncertainty. Such an uncertainty ω_Δ represents a lack of complete knowledge of the linear reset map (8.1). Furthermore, the unilateral constraint surface $\mathcal{S}$ is defined by $\mathcal{S} = \{(x_1, x_2) : x_1 = 0\}$.

Let $(x_1^{\rm d}(t), x_2^{\rm d}(t) = \dot{x}_1^{\rm d})$ represent a desired trajectory of the same structure as that of the unilaterally constrained plant, i.e., the desired position is a continuous function of time whereas the desired velocity is piecewise continuous and it undergoes restitution in the opposite direction whenever $x_1^{\rm d}(t) = 0$. The following assumptions are made throughout.

1. The desired trajectory $x_1^{\rm d}(t), x_2^{\rm d}(t)$ is a periodic function of time with period $T^{\rm d}$ such that $x_1(t) \neq 0 \quad \forall t \neq nT^{\rm d}$ and $x_1(nT^{\rm d}) = 0, n = 0, 1, 2, \ldots,$ thereby

causing the periodic velocity restitution $x_2^{\rm d}(nT^{d+}) = -K{\rm sign}(x_2^{\rm d}(nT^{d-}))$ with some positive constant K.

2. An impulse event occurs in infinitesimally small time. The impact event is detected.

3. $\operatorname*{ess\,sup}_{t\geq 0}|\omega| \leq N$ and $\operatorname*{ess\,sup}_{k\geq 0}|\omega_\Delta| \leq M_\Delta$ where N, M_Δ are *a priori* known positive scalars.

4. The upper bound $\tilde{R}$ on the quantity $\max\{|x_1(t_0)|, |x_2(t_0)|, |x_1^{\rm d}(t_0)|, |x_2^{\rm d}(t_0)|\}$, where t_0 is the initial time, is known a priori and is finite so that $\tilde{R} > K + M_\Delta$.

The first assumption ensures that the chosen desired trajectory is periodic as the reset event in desired velocity $x_2^{\rm d}(t)$ causes the post reset velocity $x_2^{\rm d}(nT^{d+})$ to take the same value after every impact for all $n = 0, 1, 2, \ldots$. For example, setting $x_1^{\rm d}(t) = \frac{1}{2}\bar{\alpha}t^2 + \bar{\beta}t + \gamma$, where $\bar{\alpha} = -2\frac{K}{T}, \bar{\beta} = K, \gamma = 0$ are scalars, gives a periodic solution in the state-space $(x_1^{\rm d}, x_2^{\rm d})$. Then, upon using a finite time stable controller to achieve $x_1 = x_1^{\rm d}, x_2 = x_2^{\rm d}$ before the next impact at t_k with settling time $T_{\rm tw} < T^{\rm d}$, the constraint surface becomes $S = \{(x_1, x_2) : x_1 = x_1^{\rm d} = 0\}$. The second assumption is made simply to show the instantaneous nature of the impact event and to enable restitution in the desired velocity. The third assumption represents the upper bound on the uncertainty $\omega(t)$ and ω_Δ respectively. The last assumption dictates that the results presented in the chapter are of a local nature[2]. The following tracking dynamics can be obtained by defining the error variables $e_1(t) = x_1(t) - x_1^{\rm d}(t), e_2(t) = x_2(t) - x_2^{\rm d}(t)$:

$$\begin{array}{ll} \dot{e}_1 = e_2 \quad \dot{e}_2 = u(e) + \omega(t) - \dot{x}_2^{\rm d}(t) & (x_1, x_2) \notin S \\ e_2(t_k^+) = -\bar{e}x_2(t_k^-) + \omega_\Delta - x_2^{\rm d}(t_k^+) & (x_1, x_2) \in S \end{array} \tag{8.2}$$

When $\dot{x}_2^{\rm d}(t)$ is bounded, the first line of (8.2) coincides with the standard perturbed double integrator [4, 9, 14] and the twisting controller drives the errors e_1, e_2 to the origin in arbitrarily small finite time. Hence, it is guaranteed that the discrete event t_k and $nT^{\rm d}$ coincide. When t_k coincides with the impacting time of the desired trajectories *i.e.* $t_k = nT^{\rm d}$, the impact law satisfies $e_2(t_k^+) = -\bar{e}x_2(t_k^-) + \omega_\Delta - K{\rm sign}x_2^{\rm d}(t_k^-)$. The aim of this section is to propose (i) a switched state feedback synthesis and (ii) the corresponding tuning rules to establish the gains of the controller to give an *a priori* guarantee of finite time convergence of the states of system (8.2) to the origin $(e_1, e_2) = (0, 0)$.

[2]Global finite time stabilisation can be achieved as outlined in Chapter 4. However, it suffices to consider only a local domain of the state-space for a large class of engineering applications. For example, such an upper bound is generally known *a priori* for a large class of mechanical systems in terms of the boundedness of total energy at the initial time.

8.2 Finite Time Tracking of Periodic Trajectory

The motivation to introduce the following definition is to accommodate the uncertainty at the time of impacts.

Definition 8.1. *A unilaterally constrained system is said to possess finite time impact attenuation in the closed-loop if for every $\epsilon > 0$, there exists an open neighborhood $\Omega_{\tilde{R}}$ around the origin such that every solution $\phi(t)$ of (8.2) that satisfies $\phi(0) = p$, also satisfies (i) $\phi(t) = 0$ for all $t \in [nT^{\mathrm{d}} + T_{\mathrm{tw}}, (n+1)T^{\mathrm{d}})$ and for some $T_{\mathrm{tw}} < T^{\mathrm{d}}$ and (ii) $|\phi(t)| < \epsilon$ for all $t \in [nT^{\mathrm{d}}, nT^{\mathrm{d}} + T_{\mathrm{tw}})$ for all $p \in \Omega_{\tilde{R}}$, where $n = 0, 1, 2, \ldots, \infty$.*

Remark 8.1. The uncertainty ω_Δ at the time of jumps in the actual plant velocity makes the error states deviate from the desired periodic trajectory at every jump and hence finite time stability cannot in general be proven for all $t \geq 0$. This is the main reason why only boundedness $|\phi(t)| < \epsilon$ can be claimed in the periodic intervals $t \in [nT^{\mathrm{d}}, nT^{\mathrm{d}} + T_{\mathrm{tw}})$ for finite gains. This is the main motivation for the definition of finite time impact attenuation.

It should be noted that the differentiability of the solution $\phi(t)$ is not required unlike the existing definitions [15, Sec. III.A]. As will be shown in this section, this is because it is possible to construct a switched synthesis for which an absolutely continuous solution $\phi(t)$ in the sense of Fillipov [6] is shown to exist for all $t \in [nT^{\mathrm{d}}, (n+1)T^{\mathrm{d}})$ where $n = 0, 1, 2, \ldots, \infty$. Let the switched control synthesis (4.10) be utlised along with corresponding tuning rules (4.96) and (4.99) in the coordinates (e_1, e_2) as follows:

$$u(e_1, e_2) = \begin{cases} Le, & (e_1, e_2) \notin \Omega_R; \\ \Phi(e), & (e_1, e_2) \in \Omega_R, \end{cases} \tag{8.3}$$

The tuning rules (4.96) and (4.99) cannot account for the uncertainty ω_Δ. Let the aforementioned tuning rules be revised as follows:

$$\begin{aligned} &\mu_1 > \max\left\{ N, \quad \frac{2\delta}{T_{s_2}\sqrt{1-\eta^2}} + N \right\} \\ &\mu_2 > \max\left\{ \sqrt{\tfrac{R}{2}}, \ \tfrac{\mu_1 - N}{\eta}, \ \mu_1 + N, \ \rho\sqrt{\tfrac{R}{2(1-\rho)}}, \ \rho, \ \beta\,\mu_1 \right\} \\ &l_1 = \alpha\,\lambda^2, \quad l_2 = (\alpha + 1)\,\lambda \end{aligned} \tag{8.4}$$

where

$$\lambda \geq \sqrt{\tfrac{2\mu_2 N+(\alpha+1)^2N^2}{r}},\ \delta > \begin{cases} \tfrac{r_1(\beta+1)}{\beta-1}, & \text{if } r_0 > \tfrac{r_1(\beta+1)}{\beta-1}; \\ \tfrac{r_0(\beta+1)}{\beta-1}, & \text{if } r_0 < \tfrac{r_1(\beta+1)}{\beta-1}, \end{cases}$$
$$\lambda = \max\left\{\sqrt{\tfrac{2\mu_2 N+(\alpha+1)^2N^2}{r}},\ \kappa\right\}, r_2 = \tfrac{\rho R}{\mu_2}$$
$$\left(\mu_2(\alpha+1)|e_{1_0}| + \tfrac{2\mu_2}{\kappa}|e_{2_0}|\right) e^{-\kappa \mathcal{T}_{s_1}}$$
$$+ \left(2\alpha^2\kappa^2 e_{1_0}^2 + (\alpha^2+1)e_{2_0}^2\right) e^{-\kappa \mathcal{T}_{s_1}} \leq R_1, \tag{8.5}$$

$$\begin{aligned} \mathcal{T}_{s_1} &= \epsilon_1 T_{\text{tw}}, & \mathcal{T}_{s_2} &= (1-\epsilon_1)T_{\text{tw}} & &\text{if } r_0 > \delta; \\ \mathcal{T}_{s_1} &= 0, & \mathcal{T}_{s_2} &= \epsilon_1\, T_{\text{tw}} & &\text{otherwise}; \end{aligned} \tag{8.6}$$

$T_{\text{tw}} < T^{\text{d}}$ is user's choice with T^{d} representing desired settling time, $r_1 = \sqrt{2R}$, $r \in (0, r_2)$, $R_1 = R - r$, $\alpha > 1$ is an arbitrary scalar, $\beta > 1$ is a tuning variable, $\eta \in (0, \frac{1}{\beta})$, R is an arbitrary positive constant representing the switching boundary, $\epsilon_1 \in (0,1), \rho \in (0,1)$ are arbitrary scalars, $r_0 = \sqrt{(e_{1_0})^2 + (e_{2_0})^2}$ is the upper bound on the Euclidian norm of the system initial conditions where $e_{1_0} = e_1(t_0), e_2(t_0) + M_\Delta = e_{2_0}$. It should be noted that there always exists a solution κ such that (8.6) is satisfied (and can be obtained using numerical optimization routines). The basis for this is that $\lim_{\kappa\to\infty} \kappa^2 e^{-\kappa\, t} = 0$ for all $t \geq 0$. The quantity on the left hand side of the last inequality (8.5) can be viewed as a function of κ. The parameter μ_2 is a fixed entity on the left hand side of (8.5) due to the order in which the parameters μ_1, μ_2, l_1, l_2 are tuned. Furthermore, parameter μ_1 is a function of $\mathcal{T}_{s_2}$ and parameters l_1, l_2 are functions of $\mathcal{T}_{s_1}$.

In the following, a switched discontinuous controller is utilised and hence, the vector field of the closed-loop system is discontinuous. Existing results [35] are not applicable and a new proof is required. An *a priori* guarantee stemming from (8.4) is utilised for obtaining finite time convergence to the origin between the impacts. In this case, the stability of the system can at best be defined in the sense of Definition 8.1 due to the uncertainty at the time of the jumps. The following result is in order.

Theorem 8.1. *Assume that the trajectory $x_1^{\text{d}}(t), x_2^{\text{d}}(t)$ is designed such that assumptions (1), (2), (3) and (4) hold true. Then, the closed-loop system (8.2),* (8.3) *possesses finite time impact attenuation if the tuning rules (8.4) are employed for determining the gains l_1, l_2, μ_1, μ_2.*

Proof. Let the neighbourhood $\Omega_{\tilde{R}}$ around the origin be chosen a follows:

$$\Omega_{\tilde{R}} = \{(x_1, x_2, x_1^{\text{d}}, x_2^{\text{d}}) : \max\{|x_1(t_0)|, |x_2(t_0)|, |x_1^{\text{d}}(t_0)|, |x_2^{\text{d}}(t_0)|\} \leq \tilde{R}\} \tag{8.7}$$

The tuning rules (8.4) ensure that the error variables $(e_1(t), e_2(t)) \in \mathbb{R}^2$ settle to the origin $(0,0)$ of the continuous dynamics in less time than T^{d} for all $(x_1, x_2, x_1^{\mathrm{d}}, x_2^{\mathrm{d}}) \in \Omega_{\tilde{R}}$ when the parameters e_{10}, e_{20} are chosen greater than $\tilde{R}$ in (8.4). See Chapter 4 for a detailed mathematical proof. What remains to be proven is the existence of finite time impact attenuation.

The upper bounds N, M_Δ on disturbances $\omega(t), \omega_\Delta$ are sufficient to compute the largest time required by the trajectory to enter the level set Ω_R of the Lyapunov function $V(e_1, e_2) = \mu_2|e_1| + \frac{1}{2}e_2^2$ in finite time when a linear controller is being used. Although the control u, when switched to the second order sliding mode controller, is discontinuous on the axes $e_1(t) = 0, e_2(t) = 0$, the continuous flow can be integrated piece-wise in each of the four quadrants

$$\begin{aligned} G_1 &= \{e : e_1 > 0, e_2 > 0\}, G_2 = \{e : e_1 > 0, e_2 < 0\}, \\ G_3 &= \{e : e_1 < 0, e_2 < 0\}, G_4 = \{e : e_1 < 0, e_2 > 0\}. \end{aligned} \tag{8.8}$$

The solutions of the closed-loop system are given in (8.9) which proves uniform boundedness of the states as a function of time in the neighbourhood B_{r_0} where B_{r_0} is a ball centered at the origin with a radius $r_0 > 0$. See Section 4.3 for a detailed computation of the integrals in (8.9). Let the trajectory be analysed in the period $t \in [nT^{\mathrm{d}}, (n+1)T^{\mathrm{d}})$ for any n. The scalar $e^0 = r_0 = \sqrt{e_{10}^2 + e_{20}^2}$ is the upper bound on the norm of the periodic initial conditions (or equivalently jumps) $e(nT^{\mathrm{d}})$ because $\tilde{R} > K + M_\Delta$.

$$\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} \left[\alpha e_{1_0} + \frac{1}{\lambda} e_{2_0}\right] e^{-\lambda t} - \left[e_{1_0} + \frac{1}{\lambda} e_{2_0}\right] e^{-\alpha\lambda t} \\ \quad + \int_0^t \frac{1}{\lambda}\left[e^{-\lambda(t-\tau)} - e^{-\alpha\lambda(t-\tau)}\right]\omega(\tau)\mathrm{d}\tau \\ \\ -\left[\alpha\lambda e_{1_0} + e_{2_0}\right] e^{-\lambda t} + \alpha\left[\lambda e_{1_0} + e_{2_0}\right] e^{-\alpha\lambda t} \\ \quad + \int_0^t \left[-e^{-\lambda(t-\tau)} + \alpha e^{-\alpha\lambda(t-\tau)}\right]\omega(\tau)\mathrm{d}\tau \end{bmatrix}, & \text{if } e \notin \Omega_R; \\ e^{At}e(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau, & \text{otherwise.} \end{cases} \tag{8.9}$$

where,

$$\begin{aligned} & e = \begin{bmatrix} e_1 & e_2 \end{bmatrix}^{\mathrm{T}}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ & u(t) = -(\mu_1 - N)\mathrm{sign}(e_2) - \mu_2 \mathrm{sign}(e_1) \end{aligned} \tag{8.10}$$

By definition of the desired trajectory $x_2^{\rm d}$, the time $T^{\rm d}$ between the two jumps is known. The switched controller (8.3) is tuned as per (8.4) to obtain an arbitrarily small settling time $T_{\rm tw} < T^{\rm d}$ for an arbitrary initial condition. Since $K < \tilde{R}$, all post-impact errors $e(nT^{d+})$ settle at the origin in less time than $T_{\rm tw}$ due to the fact that the upper bound M_Δ on ω_Δ is embedded in the tuning rules in terms of worst case value of e_{20}. Hence, it holds that for any arbitrary initial condition $e(nT^{d+}) \in \Omega_{\tilde{R}}$ where $n = 0, 1, 2, \ldots, \infty$, the continuous flow (8.9) of the system (8.2), (8.3) between the impacts settles at the point $e_1 = 0, e_2 = 0$ before the next jump event $t = (n+1)T^{d-}$. Then it follows that the jumps in both the actual and the desired velocities occur at the same time $t = t_k = nT^{\rm d}$. Hence, the trajectory of (8.2), (8.3) always passes through the point $(e_1, e_2, t) = (0, 0, nT^{\rm d})$, where $n = 0, 1, 2, \ldots, \infty$.

In the presence of uncertainty ω_Δ, the system (8.2),(8.3) possesses finite time impact attenuation in the sense of Definition 8.1 as there exist finite scalars $\epsilon > 0, \delta > 0, T_{\rm tw} < T^{\rm d}$ such that for all $\|(e_1(nT^{d+}), e_2(nT^{d+}))\| < \epsilon$ and $(e_1(nT^{d+}), e_2(nT^{d+})) \in \Omega_{\tilde{R}}$, the inequalities (i) $(e_1(T_{\rm tw}), e_2(T_{\rm tw})) = (0, 0)$ for all $t \in [nT^{\rm d} + T_{\rm tw}, (n+1)T^{\rm d})$ and (ii) $\|e(t)\| < \delta, \delta = \epsilon + |M_\Delta|$ hold true $\forall t \in [nT^{\rm d}, nT^{\rm d} + T_{\rm tw}]$. □

Remark 8.2. The theorem above proves Lyapunov stability (and not asymptotic stability) of this tracking problem. However, it proves equiuniform finite time stability in the presence of impacts $\forall t \geq 0$ in the case of gains μ_1, μ_2, l_1, l_2 approaching infinity as this gives $T_{\rm tw} \to 0$ despite the uncertainties ω, ω_Δ (see Chapter 4 for theoretical proof).

This subsection marks the end of the theoretical developments of the thesis. Applications of the theory developed thus far are discussed in the next two subsections.

8.3 Finite time tracking for a biped robot

This section demonstrates the first application of variable structure tracking control of a seven-link biped robot. A second order sliding mode controller is utilised to track reference trajectories for all the joints of a fully actuated biped robot. The tuning rules revisited in the previous section for the 'twisting' controller are used to guarantee *a priori* attainment of a prescribed settling time between two successive impacts.

The main motivation for the study is that the existing literature on biped robots and finite time controllers does not have straightforward rules for tuning the controller in the presence of disturbances in the continuous and impact phases. The torque vector is modeled as the control input. Smoothing of the discontinuous controller is carried out by utilising a high gain linear controller inside a boundary layer defined by an arbitrarily small region around the origin thereby avoiding numerical errors in the simulations. The overall accuracy of motion control is dictated by the size of this layer leading to practical stability of the closed-loop system. The tuning rules that hold true for reaching the origin in finite time also hold true for reaching the boundary layer in finite time. The main motivation of this subsection is to provide straightforward and realizable engineering guidelines for the reference trajectory generation and for the tuning[3] of a robust finite time controller for achieving stable walking gait of a biped in the presence of disturbances in both continuous and impact phases. Numerical simulations of a biped robot are shown to support the theoretical results.

The biped robot under consideration is a fully actuated robot. Gait stability is to be ensured by designing the reference trajectories and ensuring that they are tracked in finite time. The main focus of this section is on finite time tracking of the trajectories of the joints of a fully actuated biped robot to the desired periodic trajectories by using a second order sliding mode controller together with tuning of the controller parameters.

There is a strong theoretical motivation to study this application. Firstly, an *a priori* guarantee with appropriate tuning to prescribe finite time convergence for the tracking of periodic trajectories has not been studied for the biped robot when a finite time controller is utilised. Previous work in [15] gives a high gain version of a continuous finite time controller but *a priori* tuning which results in finite gains is an open problem. Secondly, the present chapter utilises the tuning rules developed in Chapter 4 for a SOSM controller which is directly applicable to the biped model, which, as will be shown in the following, is a feedback equivalent of a perturbed double integrator system. The controller is robust to disturbances while requiring only the knowledge of an upper bound on the disturbance to cause the tracking errors to converge to zero in finite time between successive impacts. Following recent advances in (i) Lyapunov methods for higher order sliding mode controllers and observers [9, 42, 70, 106] and (ii) finite time convergence and tuning developed in Chapter 4 there is clearly an opportunity to provide similar engineering guidelines for achieving stable walking gait of a biped in finite time.

The literature on control of biped robots is vast (see [97] for a comprehensive survey). Previous work utilising SOSM includes [32], [142]. Previous results on continuous finite time stabilization for biped robots [15] do not give explicit tuning rules. The main contribution of the presented result is that the tuning rules for the controller are

[3]Research work in the area of trajectory generation and development of a biped model is carried out by collaborators Dr. Yannick Aoustin and Dr. Christine Chevallereau whose support is duly acknowledged. The contribution of the thesis is in demonstrating applicability of the theoretical results developed in Chapter 4 in achieving a stable and robust walking gait of a fully actuated biped robot.

given for the tracking problem with an *a priori* guarantee of attaining a pre-specified settling time. The tuning rules ensure that the tracking errors always converge to the origin before the subsequent impact event occurs thereby guaranteeing a stable walking gait. The method differs from existing contributions such as [143, 144] that depend on open-loop optimal control. The drawback of the presented method is that it is more conservative. Nevertheless, this work gives theoretical starting values for tuning that guarantee finite time tracking of the states in the presence of disturbances during both impact and continuous phases of the biped dynamics.

Furthermore, a boundary layer is proposed for smoothing of the discontinuous 'twisting' controller. This produces an ultimately bounded closed-loop trajectory. Since the joint torque is modeled as the control input, and since it is preferable to avoid chattering of the torque in a biped as it produces a discontinuity in the reaction force that may produce take-off or fall down of the robot, the boundary layer is introduced around the origin as there is no sliding anywhere other than at the origin [14]. It follows that the closed-loop system trajectory enters the closed region around the origin in finite time due to the proposed tuning. Furthermore, it is shown that there is no sliding mode either on the principal axes within the boundary layer or on the boundary of the layer. These features resemble the boundary layer approach in traditional sliding mode synthesis and hence readily provide a natural extension to SOSM but with the accompanying tuning guidelines intact. The combination of this tuning and boundary layer approach renders these most recent advances in SOSM control suitable for industrial engineering applications encompassing a large class of unilaterally constrained mechanical systems such as biped robots.

Problem Statement

Recall the problem statement presented in Section 8.1. The aim of this subsection is to utilise the SOSM state feedback synthesis (8.3) with the corresponding tuning rules (8.4) for the controller gains to give an *a priori* guarantee of finite time convergence of the states of a fully actuated biped robot to the desired trajectory by utilising the equivalence between the error dynamics of (8.2) and that of each actuated joint of a biped. In other words, the aim is to induce finite time impact attenuation as established in Theorem 8.1 in a fully actuated biped robot.

8.3.1 Revision of tuning rules adapted to biped application

The tuning rules (8.4) were revisited for a switched synthesis where a linear state feedback is utilised to first bring the closed-loop trajectory arbitrarily close to the

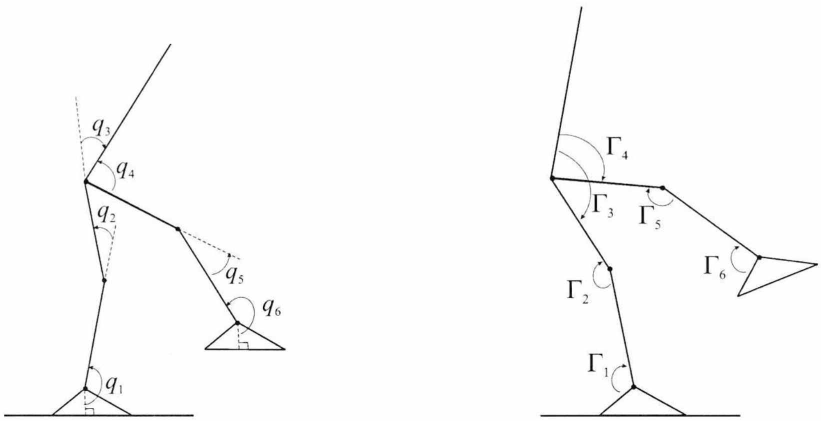

FIGURE 8.1: Seven-link bipedal robot.

origin in finite time and then a twisting controller takes over to drive the error states to the origin in finite time. Since in this section a twisting controller without linear feedback will be used, a slight adjustment in the tuning rules is needed to make the results of Chapter 4 applicable to a purely twisting controller. This can be done, for example, by setting $R = \frac{r_0^2}{2}$. Since $\beta > 1$ holds true by choice, the design rule $r_1 = \sqrt{2R}$ always results in $r_0 < r_1\frac{\beta+1}{\beta-1}$ thereby causing no linear feedback (see Section 4.4). This is equivalent to selecting a deliberately high switching boundary to avoid linear feedback from the initial time.

8.3.2 Biped Model

The bipedal robot considered in this section is walking on a rigid and horizontal surface. It is modeled as a planar biped, which consists of a torso, hips, two legs with knees and feet (see Fig. 8.1). The walking gait takes place in the sagittal plane and is composed of single support phases and impacts. The complete model of the biped robot consists of two parts: the differential equations describing the dynamics of the robot during the swing phase, and an impulse model of the contact event (the impact between the swing leg and the ground is modeled as a contact between two rigid bodies [27]). In the single support phase, the dynamic model, considering an implicit contact of the stance foot with the ground (*i.e.* there is no take off and no sliding during the single support phase), can be written as follows:

$$D(q)\ddot{q} + C(q,\dot{q}) + G(q) = \Gamma \tag{8.11}$$

where $q = (q_1\ q_2\ q_3\ q_4\ q_5\ q_6)^{\mathrm{T}} \in \mathbb{R}^6$ is the vector of the generalized coordinates (see Fig. 8.1), $\Gamma = (\Gamma_1\ \Gamma_2\ \Gamma_3\ \Gamma_4\ \Gamma_5\ \Gamma_6)^{\mathrm{T}} \in \mathbb{R}^6$ is the vector of joint torques [4], D is

[4]Leg 1 is the stance one, leg 2 the swing one.

the symmetric, positive definite 6×6 inertia matrix. As the kinetic energy of the biped is invariant under a rotation of the world frame, q_1 defines the orientation of the biped in the world frame. Terms $C(q, \dot{q})$ and $G(q)$ are the 6×1 matrices of the centrifugal, coriolis and gravity forces respectively. From (8.11), the state-space form can be written as follows:

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ D^{-1}(-C - G + B\Gamma) \end{pmatrix} = f(\mathbf{x}) + g(x_1) \cdot \Gamma \tag{8.12}$$

with $\mathbf{x} = \begin{pmatrix} q^{\mathrm{T}} & \dot{q}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} x_1^{\mathrm{T}} & x_2^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$. The state space is chosen such that $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^{12} = \{\mathbf{x} = [x_1^{\mathrm{T}}\, x_2^{\mathrm{T}}]^{\mathrm{T}} \mid x_1,\ x_2 \in \mathcal{N}\}$, where $\mathcal{N} = \{(\ x_1,\ x_2) \in \mathbb{R}^6 \mid |x_1| < M < \infty\}$ and $\mathcal{M} = (-\pi, \pi)^6$. The model (8.12) is used to define the control with the assumption that a flat contact occurs between the stance foot and the ground. A more general model that includes a unilateral contact between the foot and the ground is used for the simulation where four contact points are considered at the corner of the foot. Various solutions exist to determine the contact of each corner of the feet with the ground. The contact forces between the feet and the ground reaction are calculated using a constraint-based approach. This approach belongs to the family of time-stepping approaches. Let the vector $\mathbf{R} \in \mathbb{R}^8$ be the reaction force vector, which is obtained by stacking the reaction force vectors of the two corners of each foot. Vector $\mathbf{R}_k$ at $t = t_k$ is expressed at each sampling period as a function of the generalized position vector $\mathbf{q^k} \in \mathbb{R}^9$ composed of the variable orientation of each link and the Cartesian coordinates x, y of the hips, the associate velocity vector $\mathbf{q_v^k} \in \mathbb{R}^9$ for the biped and Γ^k with an algebraic equation

$$G(\mathbf{R}_k, \mathbf{q^k}, \mathbf{q_v^k}, \Gamma^k) = 0 \tag{8.13}$$

Let vector v^{k+1} be the Cartesian velocities of the corners in contact with the ground at $t = t_k$. The normal components must be non negative to avoid interpenetration. The identity $v_{in}^{k+1} = 0$ means that the contact remains and the inequality $v_{in}^{k+1} > 0$ means that the contact vanishes. The normal components $r_k^{in} > 0$ of $\mathbf{R}_k$, when contact occurs, are also subject to non negative constraints. These components can avoid interpenetration but they cannot avoid the stance foot take-off. It is clear that the variables v_{in}^{k+1} and r_k^{in} are complementarity quantities:

$$v_{in}^{k+1} \geq 0 \quad \perp \quad r_k^{in} \geq 0 \tag{8.14}$$

Furthermore, the variables v_{in}^{k+1} and r_k^{in} are subject to constraints imposed by friction which leads to a linear complementarity condition. The valid cases of contact for each corner are determined using constrained optimization [147].

8.3.3 Pre-feedback and Reference Trajectory

The walking gait, which is composed of single support phases and impacts, has been defined by $x_1^d(t)$, $x_2^d(t)$ and $\dot{x}_2^d(t)$ satisfying the conditions of contact using an off-line optimization [148]. The objective of the control is that each joint angle follows its reference trajectory. The reference walking minimizes the integral of the norm of the torque vector for a given distance. The walking velocity is selected to be 0.5 m/s. The duration of one step is 0.53s. Since the impact is instantaneous and passive, the control law is defined only during the single support phase. The torque vector Γ is defined based on the dynamic model (8.11) as follows:

$$\Gamma = D(x_1)(\dot{x}_2^d(t) + u) + C(x_1, x_2) + G(x_1) \tag{8.15}$$

where u is defined by (8.3). The pre-feedback (8.15) enables the system (8.11) to be transformed into exactly the same form as the continuous dynamics of (8.2), thereby rendering the tuning rules (8.4) applicable to the biped problem.

8.3.4 Numerical Simulation

8.3.4.1 Boundary Layer Approach For SOSM Controller

A brief mathematical treatment of the boundary layer approach is provided in this section before proceeding with the numerical simulation of the biped. Since, in the following, the torque of each joint of the biped will be modeled as a control input, let the following boundary layer be applied to the twisting control to avoid any discontinuity in the reaction force that may produce undesirable take-off or fall down of the biped robot:

$$\begin{aligned} u(e_1, e_2) &= -\mu_1 \text{sign}(e_2) - \mu_2 \text{sign}(e_1) \quad &\text{if } (e_1, e_2) \notin \Omega_\epsilon \\ u(e_1, e_2) &= -\mu_1 \tfrac{e_2}{|e_2|+\epsilon} - \mu_2 \tfrac{e_1}{|e_1|+\epsilon} \quad &\text{if } (e_1, e_2) \in \Omega_\epsilon \end{aligned} \tag{8.16}$$

where $\epsilon > 0$ is an arbitrarily small parameter to be selected by the user and $\Omega_\epsilon = \{\frac{1}{2}e^{\mathrm{T}}e \leq \epsilon\}$ is the boundary layer. There are a number of properties of the boundary layer definition (8.16) requiring analysis. Firstly, the control is continuous within the boundary layer. Secondly, utilizing the continuous phase (i.e. $x \notin S$) of the system equations (8.2) and control (8.16), it should be noted that there is no sliding mode on the principal axes $e_1 = 0, e_2 = 0$ as well as on the boundary $\partial\Omega_\epsilon = \{(e_1, e_2) :$

$V_\epsilon(e_1, e_2) = \frac{1}{2}e^{\mathrm{T}}e = \epsilon\}$ because the inequalities

$$e_1\dot{e}_1 = e_1e_2 \not< 0, \quad e_2\dot{e}_2 = -\mu_1\frac{e_2^2}{|e_2|+\epsilon} - \mu_2\frac{e_1e_2}{|e_1|+\epsilon} \not< 0, \\ V_\epsilon\dot{V}_\epsilon = e_1e_2 + e_2\dot{e}_2 \not< 0 \tag{8.17}$$

hold true. Furthermore, since the control (8.16) assumes a time-varying high gain linear state feedback within the boundary layer, it can be shown that the closed-loop trajectory decays exponentially to the ideal SOSM at the origin. However, robustness is only guaranteed against vanishing disturbances with an upper bound $\frac{N|e_2|}{|e_2|+\epsilon}$ due to the linear control. This shows that the trajectory of the closed-loop system (8.2), (8.16) with $x \notin \mathcal{S}$ either converges to the boundary layer in finite time from the outside due to the tuning rules (8.4) or remains inside the boundary layer without sliding or chattering. It should be noted that a detailed analysis of the closed-loop system stability with respect to the parasitic dynamics [68] within the boundary layer is not presented here and remains an ongoing area of study. Finally, robustness is lost only within the boundary layer and in the limit as $\epsilon \to 0$ the continuous control resembles the discontinuous one.

8.3.4.2 Robust Walking Cycles

The model (8.12) is utilized in this section to show numerical simulations of a stable walking gait by achieving a finite settling time via the tuning rules (8.4). The desired convergence time for tracking the reference trajectories is defined to be 0.5 seconds.

The robustness of the tracking control (8.15) is verified by introducing a resultant disturbance force F_ω on the hip joint of the biped with projections $F_{x\omega} = 50\,N$ and $F_{y\omega} = 2.5\,N$ in the horizontal and vertical planes respectively. Such a force is used for the duration of $0.07\,sec$ to simulate two disturbance effects. On one hand, the effect of $F_{x\omega}$ represents a disturbance in the continuous phase of the dynamics (8.11) as it starts from $1.08\,sec$ in the first cycle of the biped which belongs to the continuous phase of the trajectory. On the other hand, the combined effect of both the forces $F_{x\omega}$ and $F_{y\omega}$ may represent the uncertainty in the impact map. In the physical sense, the resultant disturbance force F_ω can be perceived to arise an from uneven walking surface which has not been taken into account when modelling the resets in generalized velocity $\dot{q}$ upon impact with the ground (see Fig. 8.2 where unpredictable impacts appear in the right leg when it is the stance leg).

The effect of the aforementioned disturbance force on the hip joint can be studied via the principle of virtual work as follows. Let the hip joint coordinates be given by (x_h, y_h). Let a disturbance force F_ω be applied as mentioned above. Let the effect of the disturbance force F_ω on the dynamics of the generalized coordinates q be denoted by Γ_ω. Hence, the biped model (8.11) can be revised as follows:

$$\ddot{q} = D^{-1}\left(\Gamma + \Gamma_\omega - C(q,\dot{q}) - G(q)\right) \tag{8.18}$$

It can be seen from the above that the quantity Γ_ω appears as a disturbance in the $\ddot{q}$ dynamics. In the following, an *a priori* known upper bound

$$N \triangleq \sup_{t\geq 0} |D^{-1}\Gamma_\omega| = 19.2 \tag{8.19}$$

is utilised by the tuning rules (8.4) to cover the worst effect produced by the disturbance force F_ω while tuning the gains μ_1, μ_2 for each joint. Thus, the modeling information utilised for the control synthesis (8.15) lies in the usage of model matrices D, C, G and the *a priori* known upper bound (8.19).

Next, tuning rules (8.4) are used to produce a twisting controller with gains $\mu_1 = 33.97, \mu_2 = 121.28$. It should be noted here that the tuning rules are conservative because they are based on a Lyapunov function and on a comparison system that encompasses the true trajectories in each quadrant of the planar state-space (see Chapter 4). Hence, the aforementioned values of gains provide a good starting point from where the gains should be reduced to avoid excessive chattering.

Figure 8.2 shows the heights of the feet for eight consecutive steps with gain selection $\mu_1 = 25, \mu_2 = 28$. The corresponding velocities of the feet in the horizontal and vertical direction can be seen in figures 8.3 and 8.4 respectively. Legends 'P1' and 'P3' represent the 'toe' of the right and left foot respectively. Similarly, 'P2' and 'P4' represent the 'heel' of the right and left foot respectively. It is clear from all the plots of Fig. 8.4 that the jump occurs when the foot velocity is negative. This is bouncing ball like behaviour but with a zero post jump velocity.

Periodic orbits in each of the i^{th} joints are depicted in terms of phase-plane plots of $q_i, \dot{q}_i$ in figures 8.5, 8.6 and 8.7. It can be seen that each joint velocity undergoes a jump at the time of collision of the feet with the walking surface and that the actual trajectory follows the reference trajectory closely.

8.3.4.3 Robustness Analysis

The effect of the disturbance force can be seen in several plots. For example, the velocity in the vertical direction for the right foot exhibits severe effects of a disturbance as can be seen from the high amplitude impulse like change just at the end of the first step (see Fig. 8.4). This is the result of the combination of disturbance forces $F_{x\omega}$

and $F_{y\omega}$ which can be perceived in the physical sense as the effect an uneven surface would have at the end of a step. The effect on the biped is a destabilizing one in the continuous phase also. This can be seen in the plot of velocity of the left foot in the vertical direction as it gets affected in its early flight in the next step as shown by Fig. 8.4. The disturbance effect can also be seen in terms of feet height as demonstrated in Fig. 8.2 by a departure from nominal gait at the end of the first step when the right foot takes over as the stance leg.

The biped returns to its nominal or desired gait after the disturbance force disappears as can be seen from figures 8.2, 8.8 and subsequent orbits in figures 8.5, 8.6 and 8.7.

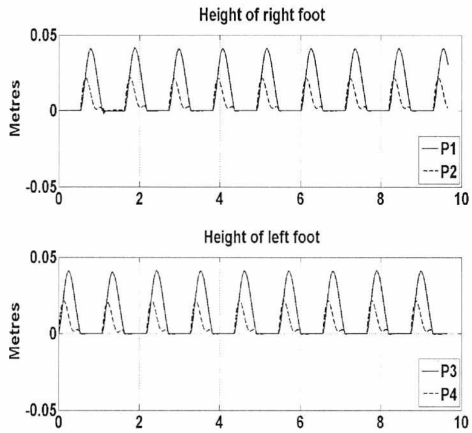

FIGURE 8.2: Feet height in the walking gait with 0.5 sec settling time

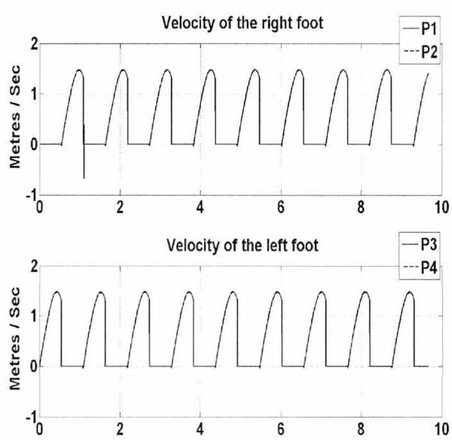

FIGURE 8.3: Feet velocity in horizontal direction in the walking gait with 0.5 sec settling time

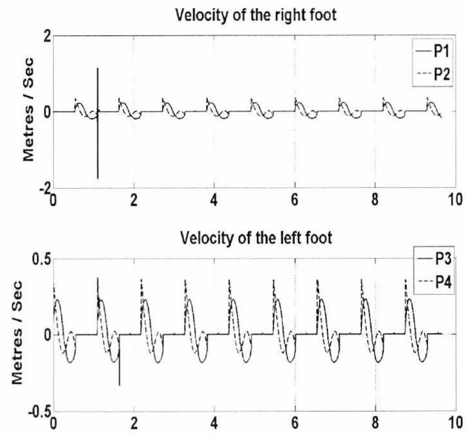

FIGURE 8.4: Feet velocity in vertical direction in the walking gait with 0.5 *sec* settling time

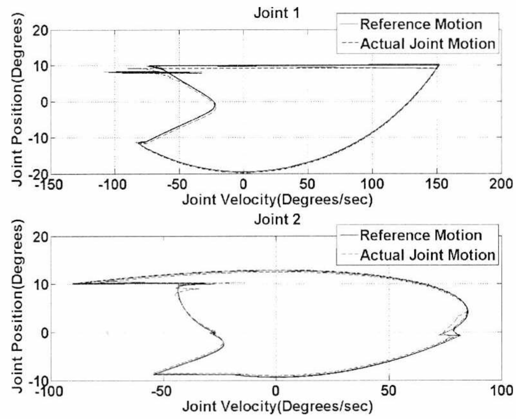

FIGURE 8.5: Periodic orbit in joints 1 and 2 in a walking gait

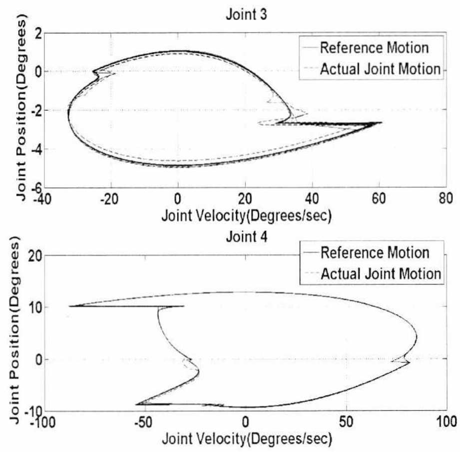

FIGURE 8.6: Periodic orbit in joints 3 and 4 in a walking gait

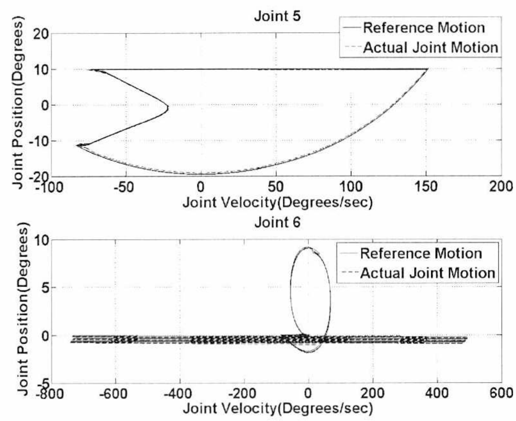

FIGURE 8.7: Periodic orbit in joints 5 and 6 in a walking gait

8.3.4.4 Convergence Time to the Boundary Layer

A further plot is shown in Fig. 8.9 which shows the dynamics of the tracking error $\dot{q}_1 - \dot{q}_1^{\mathrm{d}}$ in joint 1. It can be seen that the error always converges to the boundary layer $|\dot{q}_1 - \dot{q}_1^{\mathrm{d}}| < \epsilon = 0.48$ in less time than 0.5 *sec* while also withstanding the disturbance force F_ω in the same duration of time.

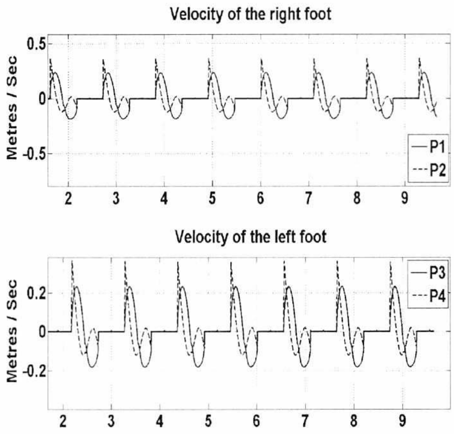

FIGURE 8.8: Feet velocity in vertical direction demonstrating asymptotically stable walking gait despite disturbance

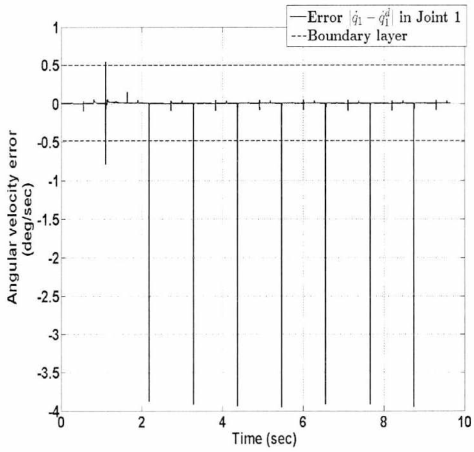

FIGURE 8.9: Convergence time of velocity tracking error $|\dot{q}_1 - \dot{q}_1^{\rm d}|$ to the boundary layer in joint 1

8.4 Experimental results of the 'K-RAFT' continuous finite time controller

The previous section provided a simulation study of the application of a second order sliding mode controller to the control of a biped robot to achieve finite time impact attenuation as described in 8.2. The aim of this section is to demonstrate application of a non-smooth controller (6.3) to finite time stabilization of a planar controllable system.

Finite time tracking control of the speed of a DC motor is experimentally implemented. A novel analogue circuit for a homogeneous finite time controller K-RAFT© [5,6] is developed. The performance of the closed-loop system closely matched that of simulation. Figure 8.10 shows the plot of commanded and actual speed of a DC motor [149] obtained from the experimental implementation of K-RAFT©.

[5]Copyright Kent innovation and Enterprise

[6]The financial support by Kent Innovation and Enterprise, University of Kent which holds the copy right, is acknowledged by the author who was one of the principal investigators for this research work. The permission by Kent Innovation and Enterprise to include the outcomes of the research on K-RAFT in this thesis is also acknowledged.

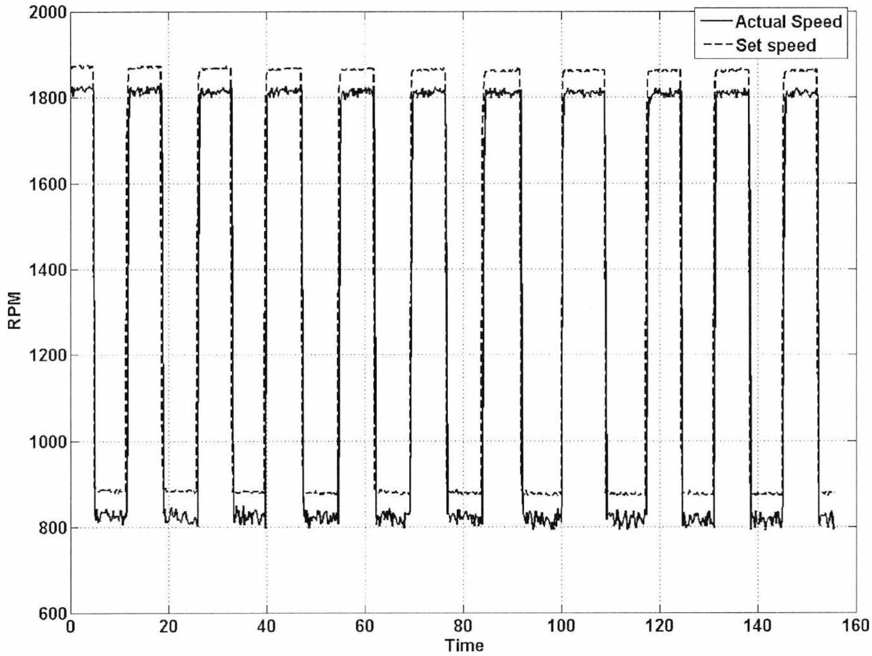

FIGURE 8.10: K-RAFT controller : The above plot shows 2% steady-state error when a series of step commands for speed are applied

8.5 Contributions

One of the main contributions of the chapter is that the tuning rules for the controller are given for the presented tracking problem leading to an orbital stabilisation with an *a priori* guarantee of attaining a pre-specified settling time. The method differs from existing results [143] that depend on open-loop optimal control. The drawback of the presented method is that it is more conservative. Nevertheless, it gives theoretical starting values for tuning that guarantee finite time tracking of the states of the hybrid system in the presence of disturbances in both continuous and discrete event (jump) phases.

The second contribution is the application of the finite time impact attenuation concept established in Section 8.3 to the tracking control of a fully actuated biped robot. Furthermore, this chapter also confirms the significance and applicability of equiuniform continuous finite time stabilisation by way of an experimental evaluation of a new analogue finite time controller as demonstrated by the results included in Section 8.4.

Thus, this chapter consolidates applications of Chapters 4, 6 and Section 8.2 thereby successfully demonstrating the importance of the novel theoretical in the area of equiuniform continuous and discontinuous finite time stabilisation of variable structure and non-smooth systems with and without resets. Although highly theoretical in nature, the new proofs of Lyapunov analysis and tuning rules contributed by the thesis are naturally suitable to engineering applications as this chapter demonstrates.

8.6 Conclusions

This chapter marks the end of the technical results presented in this thesis. First, tuning rules are established when a second order sliding mode controller is utilized to achieve finite time tracking for a class of unilaterally constrained planar systems in the presence of external disturbances in continuous and discrete-event phases.

Two applications are presented in the second part of the chapter. As a first application, a robust second order sliding mode controller is utilised along with its *a priori* tuning rules to achieve a finite settling time for the tracking of error dynamics of a fully actuated biped robot. Biped robots are unilaterally constrained systems with recurring impacts due to the collision of the feet with the ground. The results give straightforward engineering guidelines to achieve stable walking of a biped. Each joint follows its reference trajectory in finite time before the next impact occurs with the ground, thereby producing a stable periodic orbit in this non-linear system. Furthermore, the boundary layer approach makes it possible to produce joint torques without chattering at the origin. Numerical results are presented to show robustness of the non-linear synthesis and ease of use via straightforward tuning. The second application demonstrated experimental results of a continuous finite time controller for finite time tracking control of planar controllable systems without resets.

These results, while being based on the theoretical developments of earlier chapters, demonstrate that the latest advances in the area are appropriate for industrial applications. Potential future directions include the study of theoretical conditions for stability of the system within the boundary layer in the presence of parasitic dynamics and development of similar results for under-actuated bipeds.

The next chapter draws conclusions of the work presented in this thesis and presents a discussion on the future research directions.

CHAPTER 9

Conclusions

The thesis presents four contributions to the domain of stability analysis. The first contribution of the thesis is theoretical proof that the settling time of a second order sliding mode controller approaches zero in the limit as gains of the controller tend to infinity. This result, while being a first in the area of SOSM, leads to the tuning rules for arbitrary reduction of the finite settling time of planar controllable systems when a second order sliding mode controller is employed. This research work adds not only theoretical value to the existing literature but also provides constructive tuning guidelines that are relevant to a large class of engineering applications where finite time stabilisation of uncertain dynamical systems is required.

The second contribution is the identification of three new Lyapunov functions to prove equiuniform finite time stabilisation leading to a rigorous Lyapunov analysis framework . One Lyapunov function is identified for each of the three cases (i) planar non-smooth system without resets, (ii) planar non-smooth system with resets (iii) and variable structure systems with resets in one of the states. The novelty lies in the Lyapunov based proof of equiuniform asymptotic and equiuniform finite time stability as well as the computation of an upper bound on the finite settling time of non-smooth and variable structure systems with and without resets in the presence of time varying variable structure disturbances.

The third contribution is the development of an *a priori* tuning rules for the gains of a second order sliding mode controller and observer to guarantee that application of the resulting output feedback ensures the desired finite settling time is attained.

The fourth contribution is establishing tuning rules for the gains of a second order sliding mode controller for the finite time orbital stabilisation of fully actuated unilaterally constrained uncertain planar systems.

9.1 Summary

This section summarises the results obtained in each of the preceding chapters.

Chapter 3 has proved that the existing class of semi-global Lyapunov functions are in fact global in nature. This coupled with the geometric homogeneity of differential inclusions has lead to global equiuniform finite time stability. A finite upper bound on the settling time has been computed via a new method of identifying homogeneity regions of the state-space thereby removing the need to identify strict Lyapunov functions satisfying a differential inequality. This chapter has also given constructive tuning guidelines for the gains of the second order sliding mode controller based on the computed finite settling time.

Chapter 4 has proposed a switched synthesis for global equiuniform finite time stabilisation. A combination of a linear and a second order sliding mode controller has been utilised. The literature on second order sliding mode controllers [4] has not studied the behaviour of the finite settling time as the gains of the controller tend to infinity. Such a result is available in linear system theory via tools such as the Laplace transform in the case of classical control theory and via analysis of the eigenvalues of closed-loop system matrices for modern control systems. Although intuitively obvious, such a result has been absent even for the simplest case of a perturbed double integrator when the 'twisting' controller is used. Chapter 4 has formally proved that the finite settling time tends to zero when the gains of the so called 'twisting' controller [4] approach infinity. The above theoretical result relating to the limiting behaviour of the finite settling time forms the backbone of the next contribution which is the converse problem of finding the gains of the twisting controller when specification of the desired finite settling time is provided. *A priori* constructive tuning rules are developed which provide simple formulae for the controller gains. These contributions can be seen as steps towards achieving more sophisticated auto-tuning algorithms for the benefit of a larger community of researchers and engineers alike.

Chapter 5 has built on the analysis framework developed in Chapter 4 in order to study unilaterally constrained uncertain planar non-linear systems. This chapter has presented a Lyapunov based proof of finite time stability and the computation of the finite settling time in the case of unilaterally constrained planar systems where the resets arising from the unilateral constraints have a finite accumulation time (zeno behaviour). A non-smooth semi-global strong Lyapunov function has been identified. This is the first of the three Lyapunov functions mentioned above. A key contribution is that the equiuniform finite time stability is achieved by utilising second order sliding mode controllers without having to analyse jumps in the Lyapunov function while also proving robustness to time varying persisting (possibly discontinuous) disturbances. The use of a non-smooth state transformation renders the homogeneity principle of switched systems applicable. This effectively proves equiuniform finite time stability in the presence of hard nonlinearities arising from impacts while also preserving the proof of robustness to Coulomb friction-like disturbances on the right hand side. Proof of robustness to impacts as well as disturbances with persisting bound is theoretically novel.

Chapter 6 has established similar results as Chapters 4 and 5 but using continuous finite time controllers. The main contribution and focus is on robustness properties. This chapter has studied the cases with and without resets in velocity. In the absence of impacts, the new Lyapunov analysis has identified a class of $\mathcal{C}^1$ smooth strong semi-global Lyapunov functions to establish equiuniform asymptotic stability of a class of continuous homogeneous non-smooth finite time controllers. This is the second of the three Lyapunov functions mentioned above. Equiuniform finite time stability follows for the unperturbed dynamics by applying the homogeneity of the vector field. Equiuniform finite time stability of the dynamics in the presence of piece-wise continuous time varying nonhomogeneous disturbances is achieved by extending the principle of homogeneity of nonautonomous switched systems to the continuous dynamics by a suitable parameterisation of the non-homogeneous disturbances. Consequently, the thesis contains preliminary efforts to prove Lyapunov based uniform finite time stability for planar controllable systems while also proving robustness and uniformity with respect to time varying non-homogeneous sectorial disturbances. A key contribution is that the presented results extend the existing literature on homogeneity and finite time stability by both presenting uniform finite time stabilization with respect to the initial data as well as the external disturbance and by dealing with a broader class of nonhomogeneous time varying disturbances for planar controllable systems.

In the presence of resets, Chapter 6 has applied the class of non-smooth continuous controllers to unilaterally constrained systems when the resets have a finite accumulation point. A $\mathcal{C}^1$ smooth Lyapunov function is identified such that it coincides with the Lyapunov function proposed for the case of variable structure controller in Chapter 5. This is the last of the three Lyapunov functions mentioned above. Hence, a unification of the Lyapunov approaches for the unilaterally constrained non-smooth and variable structure systems is presented in this thesis characterised by the underlying theories of equiuniform finite time stability, homogeneity of differential inclusions and non-smooth Lyapunov analysis for switched nonautonomous systems [9].

Chapter 7 has presented the Lyapunov analysis results for the output feedback case where a combination of a second order sliding mode controller and a second order sliding mode observer has been used. The tuning rules developed in Chapter 4 have been extended for the case of output feedback synthesis when a second order sliding mode observer given by the so-called 'super-twisting' observer [4] is utilised to estimate one of the states of the planar system. The second order sliding mode controller has been utilised to form an output feedback control which uses the estimated state and measured state. Tuning rules have been developed essentially to meet two specifications given by the user: (i) The region of the state space beyond which the trajectories of the closed-loop system, if initialised within this region, must not evolve and (ii) finite settling time of the overall output feedback closed-loop system. The tuning rules for the gains of the observer and controller ensure that the observer converges faster than the controller while meeting the first specification with the controller converges within the specified settling time.

The thesis has studied, in Chapter 8, the finite time stability of unilaterally constrained systems without finite accumulation of reset events. Equiuniform finite time orbital stabilisation between impacts is studied giving rise to a new concept of finite time impact attenuation. Lyapunov based analysis results of the overall closed-loop system are given. This chapter has also exemplified the concept of finite time impact attenuation in the numerical experiment of tracking control of a seven link biped robot. The unilaterally constrained dynamics of a seven link fully actuated biped robot were reduced to a stable orbit in finite time despite the uncertainties in continuous and reset phases. This chapter also briefly presented the outcomes of an experimental evaluation of continuous homogeneous finite time stable controller of Chapter 6 implemented using analogue circuits.

9.2 Future research directions

Chapter 5 studied finite time stabilisation of unilaterally constrained planar system in the presence of resets having a finite accumulation point. As mentioned in Section 5.7, an interesting future direction is the study of the finite time stabilisation in the presence of a slip which induces a border collision bifurcation at the time of impact. Within the framework of *grazing bifurcations*, such impact systems are said to have *grazing velocities* [23, Problem 5.2], [130, 131] and are more rigorously studied within mechanical engineering where the trajectory of the system comes in contact with the constraint surface tangentially [130] or with zero velocity [131]. It would be interesting to study how control can affect the overall dynamics. One immediate new direction is to consider mechanical systems with the mass moving in a plane which is operated upon by two actuators operating in perpendicular directions (e.g. an XY-table found in manufacturing industries with Computer Numerical Control (CNC)). When grazing velocities are considered along with Coulumb friction and planar constraint surfaces, the control problem becomes challenging. The constraint surface may not be co-dimension one as assumed in Chapter 5. When the constraint surface is not differentiable, it may give rise to infinitely many post jump velocity vectors in a plane (see [23] for more details). One possible solution may be the use of a second order sliding mode controller in each of the two directions to robustly stabilise such a system on the constraint surface. This may find application in the area of constrained robotics. The robustness of SOSM controllers may help nullify unwanted grazing and/or unwanted post-jump velocity directions and achieve the desired resultant velocity vector in the plane in finite time despite the discontinuity in the constraint surface.

Chapter 6 proposed equiuniform finite time stabilisation of planar controllable systems using continuous non-Lipschitz controllers when there are no resets in the system. A generalisation of this result to the case of equiuniform finite time stability of controllable systems in dimension n is challenging but presents a good opportunity as discussed in Section 6.5.2. The literature shows that following advances in the homogeneity of dynamical systems [85], Rosier [86] first proved the existence of homogeneous Lyapunov functions proving asymptotic stability of non-linear homogeneous systems. This result was extended to finite time stability in [84] using the connection between finite time stability and geometric homogeneity. These results were mainly valid for continuous vector fields.

Chapter 6 has considered a right hand side which is a combination of a continuous vector field generated by a continuous non-Lipschitz homogeneous controller and a non-homogeneous, possibly discontinuous, disturbance. Although, the global Lyapunov functions belong to the class defined by [86, Lemma 2], the corresponding semi-global Lyapunov functions do not belong to the same despite being $\mathcal{C}^1$ smooth (see (6.16)). If this method of applying semi-global functions [120] is to be extended in dimension n to prove equiuniform finite time stability for all the controllable systems, explicit semi-global Lyapunov functions have to be identified (or 'guessed') as was done in Chapter 6. The reason for this is that the existence of even strong Lyapunov functions is not guaranteed (let alone those which follow a strict differential inequality [84]) for dynamical systems with discontinuous right hand side. Converse Lyapunov theorems ensuring existence of only weak Lyapunov functions to give Lagrange stability are available at best [150]. 'Guessing' Lyapunov functions such as that appearing in (6.16) fo n^{th} order system may not be straightforward. One possible systematic way forward may be to extend the existing results of Lyapunov function design for discontinuous right hand side [150] to prove converse Lyapunov theorems guaranteeing the existence of exponentially decaying Lyapunov functions in the presence of variable structure non-homogeneous uncertainties. In this case, geometric homogeneity of switched systems [9] may become applicable thereby opening the window of opportunity for robust equiuniform finite time stabilisation of controllable systems in dimension n.

Another area open for study is equiuniform output feedback finite time stabilisation as discussed in Section 6.5.2 thereby extending the existing framework [46] to encompass more robust observers and controllers.

Furthermore, constructive tuning rules are not available for any of the continuous homogeneous controllers. Deriving constructive tuning guidelines for the continuous controllers reviewed in Chapter 6 in a similar way as was done in Chapter 4 will be of theoretical and practical interest.

Chapter 8 studied the application of biped robots as a natural candidate of the theory proposed in the preceding chapters. The model of the biped considered a fully actuated dynamical system with constant vector relative degree two. It is interesting to extend the finite time convergence guarantee and the corresponding tuning rules to the case when the number of second order sliding mode controllers u is less than the number of generalised coordinates q. Such under-actuated systems have been studied rigorously in the biped literature via the analysis of Hybrid Zero Dynamics [15, 27]. A possible direction is to study a combination of recent advances in reference trajectory design [151] of biped robots, which can be expressed as a function of time and adapted stride-by-stride to render the resulting hybrid zero dynamics stable, with the second order sliding mode controllers. If successful, the above combination will substantiate finite time stability of fully actuated variables via strong Lyapunov functions and that of hybrid zero dynamics via stable eigenvalues of Linearised Poincaré maps [151].

List of Publications

Articles published in journals

1. Harshal Oza, Yury Orlov and Sarah Spurgeon, "*Lyapunov-based settling time estimate and tuning for twisting controller*," IMA Journal of Mathematical Control and Information, vol. 29, no. 4, pp. 471-490, 2012.

   [Online]. Available: http://dx.doi.org/10.1093/imamci/dnr037

2. Harshal Oza, Yury Orlov and Sarah Spurgeon, "*Finite time stabilization of a perturbed double integrator with unilateral constraints*," Mathematics and Computers in Simulation, 2012.

   [Online]. Available: http://dx.doi.org/10.1016/j.matcom.2012.02.011

Articles in conference proceedings

1. Harshal Oza, Yury Orlov and Sarah Spurgeon, "*Settling time estimate for a second order sliding mode controller: A homogeneity approach*," in the proceedings of the 18th IFAC World Congress, pp. 3956-3961, 2011.

   [Online]. Available: http://dx.doi.org/10.3182/20110828-6-IT-1002.00228

2. Harshal Oza, Yury Orlov and Sarah Spurgeon, "*Finite time stabilization of perturbed double integrator with jumps in velocity*," in the proceedings of the 50^{th} IEEE Conference on Decision and Control and European Control Conference (CDC-ECC), 2011, pp. 4610-4615.

   [Online]. Available: http://dx.doi.org/10.1109/CDC.2011.6160482

3. Harshal Oza, Yury Orlov and Sarah Spurgeon, "*Tuning rules for second order sliding mode based output feedback synthesis*," in the 12^{th} International Workshop on Variable Structure Systems (VSS), 2012, pp. 130-135.

   [Online]. Available: http://dx.doi.org/10.1109/VSS.2012.6163490

4. Harshal Oza, Vincent Acary, Yury Orlov, Sarah Spurgeon and Bernard Brogliato, "*Finite time tracking of unilaterally constrained planar systems with pre-specified settling time: second order sliding mode synthesis and chattering-free digital implementation*," in Proceedings of the 51st IEEE Conference on Decision and Control (CDC), 2012, pp. 5471-5476.

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