UNIVERSITY OF KENT

Output Feedback Sliding Mode Control for Time Delay Systems

by

Xiaoran Han

A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the

Instrumentation, Control and Embedded Systems Research Group
School of Engineering and Digital Arts

August 2011

Declaration of Authorship

I, XIAORAN HAN, declare that this thesis titled, 'OUTPUT FEEDBACK SLIDING MODE CONTROL FOR TIME DELAY SYSTEMS' and the work presented in it are my own. I confirm that:

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- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: 韩晓然

Date: 13/09/2011

"I have missed more than 9000 shots in my career. I have lost almost 300 games. 26 times I have been trusted to take the game winning shot and missed. I've failed over and over again in my life and that is why I succeed...."

Michael Jordan

UNIVERSITY OF KENT

Abstract

Instrumentation, Control and Embedded Systems Research Group
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This Thesis considers Sliding Mode Control (SMC) for linear systems subjected to uncertainties and delays using output feedback. Delay is a natural phenomenon in many practical systems, the effect of delay can be the potential cause of performance deterioration or even instability. To achieve better control performance, SMC with output feedback is considered for its inherent robustness feature and practicality for implementation. In highlighting the main results, firstly a novel output feedback SMC design is presented which formulates the problem into Linear Matrix Inequalities (LMIs). The efficiency of the design is compared with the the existing literature in pole assignment, eigenstructure assignment and other LMI methods, which either require more constraints on system structures or are computationally less tractable. For systems with time-varying state delay, the method is extended to incorporate the delay effect in the controller synthesis. Both sliding surface and controller design are formulated as LMI problems. For systems with input/output delays and disturbances, the robustness of SMC is degraded with arbitrarily small delay appearing in the high frequency switching component of the controller. To solve the problem singular perturbation method is used to achieve bounded performance which is proportional to the magnitudes of delay, disturbance and switching gain. The applied research has produced two practical implementation studies. Firstly it relates to the pointing control of an autonomous vehicle subjected to external disturbances and friction resulting from the motion of the vehicle crossing rough terrain. The second implementation concerns the attitude control of a flexible spacecraft with respect to roll, pitch and yaw attitude angles.

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Finally I would like to dedicate this thesis to my parents for their never-ending support and to the memory of my grandfather.

Canterbury in January 2011
Xiaoran Han

Contents

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Abbreviations

di	*d*elay *i*ndependent
dd	*d*elay *d*ependent
ARE	**A**lgebraic **R**iccati **E**quation
FDE	**F**unctional **D**ifferential **E**quation
ILMI	**I**terative **L**inear **M**atrix **I**nequality
LKF	**L**yapunov **K**rasovskii **F**unctional
LMI	**L**inear **M**atrix **I**nequality
LTI	**L**inear **T**ime **I**nvariant
NFDE	**N**eutral **F**unctional **D**ifferential **E**quation
ODE	**O**rdinary **D**ifferential **E**quation
OFC	**O**utput **F**eedback **C**ontrol
PDE	**P**artial **D**ifferential **E**quation
RFDE	**R**etarded **F**unctional **D**ifferential **E**quation
SMC	**S**liding **M**ode **C**ontrol
SMOFC	**S**liding **M**ode **O**utput **F**eedback **C**ontrol
TDS	**T**ime **D**elay **S**ystem

Nomenclature

e_p	Position error.
e_r	Position rate error.
f_d	Friction level.
f_{ms} and f_{ls}	Motor coulomb friction and load coulomb friction.
h_{max}	The maximum value of a varying parameter $h(t)$: $h(t) \leq h_{max}$.
u	Control input.
u_{eq}	Equivalent control.
$A^{\perp}$	Orthogonal complement matrix of A.
A_i	The $i-$th row of matrix A.
A^g	Generalized (Pseudo) inverse of matrix A.
A^{-L}	Left pseudo-inverse of matrix A.
A^T	Transpose of matrix A.
$\lambda(A)$	Eigenvalues of matrix A.
$\lambda_{min}(A)$	The minimum eigenvalue of matrix A.
$\lambda_{max}(A)$	The maximum eigenvalue of matrix A.
I_m	Identity matrix of dimension m.
J_1	Elevation inertia on load one.
J_m	Motor inertia.
K_m and D_m	Stiffness and damping between motor and load.
K_{12} and D_{12}	Stiffness and damping between load one and load two.
N	Gearbox ratio.
$P > 0$	A symmetric and positive definite matrix.
$\det(A)$	Determinant of matrix A.
$\text{diag}(\mu)$	Diagonal matrix with component μ in its diagonal.
$\dim(A)$	Dimensions of a matrix A.
$\text{grad}\, s$	Gradient of a function s.
$\text{rank}(A)$	Rank of a matrix A.

$\mathrm{Re}(a)$	The real part of a complex number a.
$\sup \|\cdot\|$	The supremum of a bounded continuous functions.
$\mathrm{sign}\,\cdot$	sign function of a variable.
$\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$	A set of real numbers, n component real vectors and n by n real matrices.
$\mathbb{Z}$	A set of integers.
$\mathscr{C}$	Space of continuous function.
$\mathbb{C}$, $\mathbb{C}_{0,1}$	Set of complex numbers and an unit circle in the complex plane.
$\mathscr{C}_{n,h}$ or $\mathscr{C}([-h,\ 0],\ \mathbb{R}^n)$	Space of continuous vector functions mapping the interval $[-h,\ 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence.
$\mathscr{N}$	Null space of a matrix.
$\mathscr{R}$	Range space of a matrix.
θ_m	Motor position.
$\dot{\theta}_b$	Breech velocity.
$\dot{\theta}_l$	Muzzle velocity.
$\dot{\theta}_p$	Pitch rate disturbance.
τ_{am} and τ_{al}	Applied torque to a motor and a load.
ω_{1m} and ω_{1l}	Motor friction and load friction.
$sign(\dot{\theta}_b - \dot{\theta}_p)$	Load friction disturbances.
$sign(\dot{\theta}_m - \dot{\theta}_p)$	Motor friction disturbances.
$\lvert\cdot\rvert$	Absolute value of a variable.
$\|\cdot\|$	Norm of a vector or a matrix.
$\varnothing$	Empty set.
$[0]$	Zero matrix.
$*$	Symmetric elements of a symmetric matrix.

Chapter 1

Introduction

System performance will suffer from the presence of uncertainty and external disturbances. A reflection of this fact can be found in Harold Black's retrospective on his invention of the feedback amplifier (Black, [11], 1977). At one point, he describes the operating procedure for his newly invented feedforward amplifier: "... every hour on the hour–twenty four hours a day–somebody had to adjust the filament current to its correct value. In doing this, they were permitted plus or minus 0.5 to 1 dB variation in the amplifier gain, whereas, for my purpose the gain had to be absolutely perfect. In addition, every six hours it became necessary to adjust the battery voltage, because the amplifier gain would be out of hand. There were other complications too...". Despite his subsequent discovery of the feedback principle and the tireless efforts of many researchers, the problem of plant variability and uncertainty is still with us.

Systems that can tolerate plant variability and uncertainty are called robust–Black's original feedforward amplifier was not robust. Such plant variability and uncertainty arise because it is difficult to model the dynamics of a plant accurately and therefore most mathematical models contain a moderate to high degree of uncertainty associated with neglected dynamics. The subject of robust control began to receive worldwide attention in the late 1970's when it was found that linear quadratic optimal control, state feedback through observers, and other prevailing methods of control system synthesis such as adaptive control, lacked any guarantees of stability or performance under uncertainty. Thus, the issue of robustness becomes significant given increasing performance expectations. As a result subsequent development of robust control techniques, such as H^{∞} infinity loop-shaping (Zames, [149], 1981), (Skogestad and Postlethwaite, [130], 1996) or Sliding Mode Control (SMC) (Utkin, [139], 1977), (Edwards and Spurgeon, [30], 1998) took place.

While unmodeled uncertainties and external disturbance have been the main issue for robust design, *time delay*, or in another words *aftereffect*, i.e. the future states depend not only on the present, but also on the past history, which exists in many physical systems, is often ignored in the control design phase. The existence of delay can be the potential cause of performance deterioration or even instability. Many actual systems have the property of aftereffect. It is believed to

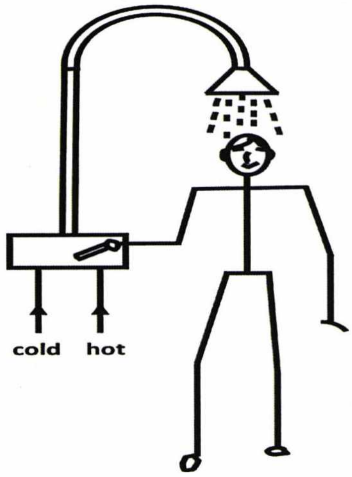


FIGURE 1.1: Qualitative model of water temperature regulation by a showering person

occur in mechanics, control theory, physics, chemistry, biology, medicine, economics, information theory and so on (Kolmanovskii and Myshkis, [98], 1999). For example in a combustion engine, the torque generated by the engine to rotate the crankshaft is delayed by 50 ms to 150 ms due to engine cycle delays, resulting from, e.g., fuel–air mixing, ignition delay, and cylinder pressure force propagation (Zhong, [151], 2006). In economic systems, delays appear in a natural way since decisions and effects (investment policy, commodity markets evolution: price fluctuations trade cycles) are separated by some time interval. In communication systems, data transmission is always accompanied by a *non-zero time interval* between the initiation- and the delivery-time of a message or signal. In modeling immune dynamics between cells, time delays are incorporated to account for the progression of cells through different stages. It is well known that the presence of delays makes system analysis and control design much more complicated. So how can the states of a process at previous moments of time influence directly the present state of evolution of this process?

Suppose a showering person wishing to achieve the desired value T_d of water temperature T by rotation of the mixer handle for cold and hot water Figure 1.1. Assume that the change ΔT in water temperature at the mixer output is proportional to the angle $\Delta\alpha$ of rotation of the mixer, with coefficient k. Let $T_m(t)$ denote the water temperature in the mixer output, and h the constant time needed by the water to go from the mixer output to the tip of the person's head. Assume that the rate of rotation of the handle is proportional to the deviation in water temperature from T_d perceived by the person, with coefficient γ. This γ depends on the person's temperament, and is larger for an energetic person than it is for a phlegmatic one. Because at time t the person feels the water temperature leaving the mixer at time $t-h$, we can find $\dot{\alpha}(t) = -\gamma[T_m(t-h) - T_d]$. This implies an equation for the temperature T_m:

$$\dot{T}_m(t) = -k\gamma[T_m(t-h) - T_d] \tag{1.1}$$

This is a typical representation of dynamical systems with delay. Research in developing techniques for analysis of such class of systems has emerged from classical results (Hale and Verduyn Lunel, [77], 1993) to more recent developments, see the review paper (Richard, [118], 2003) or book (Gu, et al., [73], 2003) and other significant results (Fridman, [42], 2001), (Fridman and Shaked, [54], 2003), (Moon, et al., [110], 2001) and (He, et al., [81], 2007). Also there are comprehensive treatment of robust control of such systems in frequency domain (Gu, et al., [73], 2003), (Q. Zhong, [151], 2006), (Niculescu, [113], 2001).

1.1 Motivation for output feedback sliding mode control of systems with delay

Sliding mode control (SMC) is a discontinuous control action where the primary function of each of the feedback channels is to switch between two distinctively different system structures (or components) such that a new type of system motion, called the sliding mode, exists in a manifold (Utkin, et al., [137], 1999), (Edwards and Spurgeon, [30], 1998). Superb performance of the variable structure system can be achieved, including insensitivity to parameter variations, and complete rejection of a class of disturbances. The dynamics of the system when on the sliding surface is of reduced order. The discontinuous nature of the control action may easily be implemented by conventional power converters with "on-off" as the only admissible operation mode. Due to these properties, the intensity of the research has been maintained at a high level, and sliding mode control has been proved to be applicable to a wide range of problems in robotics, electric drives and generators, process control, vehicle and motion control, for example.

While SMC and the associated distinctive robust properties are being recognized in the international domain, e.g., special sessions in international control conferences, monographs, much of the research has been developed based on full state availability. In many practical situations this is however, not generally the case as it may be prohibitively expensive, and indeed, sometimes impossible to measure all the state variables. One approach to solve this problem is to implement the controller with an observer, where the observer provides state estimates for use by the controller. However, the implementation of the controller/observer is more involved and the theoretical frameworks to ensure stability across a range of practical operation of the plant may be challenging. A more straightforward approach is to use only the subset of state information that is available, i.e. the measured output, within the control design paradigm. Despite a long term effort on solving the output feedback problem, (Davison and Wang, [23], 1975), (Kimura, [93], 1977), (Edwards and Spurgeon, [29], 1995), (Choi, [19], 2002), (P.G. Park, et al. [116], 2007), it remains as one of the open problems in control theory (Blondel and Tsitsiklis, [12], 1997), (Bernstein, [10], 1992) and (Syrmos, [63], 1997). The existing methods are either too restrictive or computationally inefficient.

When time delay is taken into account, the problem becomes even more complicated (Richard, et al., [119], 2001). The application of SMC to the problem of systems with time-delay is not new, but the literature is limited. It is a far from trivial problem generically, involving the combination of delay phenomenon with relay actuators which has the potential to induce oscillations around the sliding surface during the sliding mode (Fridman, et al., [60], 2003), (Gouaisbaut, et al., [71], 2002).

A study on SMC with output feedback for systems with delay presents a theoretically challenging and practically meaningful task. One of the objectives of this thesis is to develop a novel, less restrictive yet computationally efficient SMC scheme using only output information, which can be easily extended to incorporate delay effects. Another objective is to tackle the notorious problem, the switching delay problem due to input/output delay that could be caused by digital control or sensor measurement (Fridman, et al., [52], 2004), (Gao and Chen, [64], 2008). Research in this scenario has been ongoing (Gouaisbaut, et al., [71], 2002), (Fridman, et al., [58], 2002), (Barton, et al., [4], 2005), (Boiko, [13], 2009), nevertheless it remains as one of the most practical problems in the implementation of SMC, which requires fast switching response, or theoretically infinite switching to maintain an ideal sliding motion.

1.2 Challenges in the design of output feedback SMC with time delay

In the design of output feedback SMC, one problem is to solve the existence of a stable sliding surface, i.e. the design of a switching surface in the output vector space which is usually of lower order than the state vector space. Many methods are available for design, including eigenvalue assignment and eigenvector techniques (Zak and Hui, [148], 1993), (El-Khazali and Decarlo, [36], 1992), canonical-form based approach (Edwards and Spurgeon, [29], 1995), bilinear matrix inequality approach (Cao, et al., [16], 1998), (Choi, [19], 2002), or linear matrix inequalities method (LMI) (Crusius and Trofino, [20], 1999) and (Shaked, [124], 2003). It is important to note that all the work described above does not consider the existence problem involving time delay. LMIs have been thought to be an efficient tool for control design for time delay systems, (Emilia, [42], 2001), (Dugard and Verriest, [26], 1997), (Kolmanovskii, et al., [95], 1999), however full state availability is again assumed. It is generally a nonconvex problem to formulate the output feedback problem into LMIs with a reduced number of state measurements.

Another problem is the reachability design, i.e. synthesis of a control law only using the output vector, because the derivative of the sliding surface is always related to the unmeasured states. This is usually solved by the ease of using simulations to find out, through trial and error, the switching gain so that an ideal sliding surface is achieved. However this is not a systematic design strategy and is less efficient. Such a design drawback in selecting the controller becomes especially obvious when there is a switching delay where the resulting oscillations in the closed-loop is proportional to the magnitude of the switching gain. Therefore a systematic approach to compute all the control elements taking into account the delays is desired for practical design.

1.3 Thesis structure and contributions

Chapter 2 introduces the most basic and elementary concepts in SMC such as existence of solutions, attractivity to the sliding manifold, equivalent control as a solution to the sliding mode equation and dynamics of the sliding mode equation. Different approaches are presented. Using order reduction or Lyapunov method the sliding mode can be induced in either each individual sliding surfaces or between intersection of all the surfaces. An important issue arisen from implementation perspective, so called "chattering", caused by the neglected dynamics in system modelling and delay in the actuator, is discussed. It seeks to bring a general appreciation of the subject on SMC.

Chapter 3 firstly recalls the problems in the context of general static output feedback (SOF) design. Despite that existing approaches such as eigenvalue, eigenstructure assignment have been successful in some applications of systems with particular structures, their design methodologies are either too restrictive or computationally complicated. This leads to the conclusion that the problem of output feedback control still remains to be open. Then existing techniques of sliding surface design using output information are presented. Different conditions are derived using the eigenvalue assignment, eigenstructure assignment and LMIs. These methods also assume certain structure of the system to be present, therefore the class of their applications are restricted. Problems of control design are demonstrated between different control structures and their limitations are discussed.

Chapter 4 introduces the representation of the delay system as a functional differential equation and compares with the ordinary differential equation representation of delay-free system. It suggests that the delay system is a type of infinite dimensional system with characteristic roots spanning in the entire range of the delayed duration. Properties of the delay system are demonstrated in terms of existence of solutions, forward, backward continuation of the solutions and their smoothness. Stability of the delay system is analyzed by means of characteristic roots, Lyapunov Krasovskii Functionals and Razumikin approach which bring the stability analysis into a form of LMIs.

Subject of Chapter 5 concerns with the delay effect on SMC and reviews the stage of the existing literature on the subject. Firstly, an output feedback approach is articulated for systems with state delay. For matched constant state delays, a particular type of delays appearing through the input channels, an equivalent control method reduces the system into a delay-free system where conventional output feedback designs are applicable. For unmatched constant delays a method of stability degree is applied in a form of inequality incorporating the properties of the delay system. Secondly, the Lyapunov-Krasovskii functionals and the Razumikhin method are used to derive stability conditions in the form of LMIs for systems with constant and time-varying delays. Synthesis of the controller is formulated from regular form transformation and original form based perspectives. Finally, the effect of input/output delay in SMC is considered, where oscillations of solutions other than asymptotic stability is attainable because of finite switching. The characteristics of the oscillations are analyzed by several authors.

In Chapter 6 the problem of output feedback SMC for linear time-invariant systems is considered. Different from the existing results and those using iterative LMIs, bilinear LMIs or set of LMI constraints, a novel method is proposed for design of the switching surface incorporating only one single LMI. The proposed method is computationally more efficient and less conservative. A solution can now be easily obtained without introducing additional compensators due to the restrictions on the system structures as required before. The content of this Chapter has been published in (Han, et al., [79], 2008).

Chapter 7 shows that the novel output feedback design for SMC in the previous chapter can be extended to systems with time-varying state delays. A time-varying state delay is considered where conditions for both existence of sliding surface and reachability of system trajectories to the defined surface are derived using LMIs. The LMI conditions are delay-dependent on a class of reduced order dynamics in the reaching phase and independent in the sliding function. The output feedback scheme can be easily extended for compensator design. This facilitates a constructive design of output feedback SMC for a rather general class of time delay systems. Results in the chapter has been published in (Han, et al., [80], 2009).

Chapter 8 develops a systematic output feedback design where all the design parameters, including the switching gain, are derived from LMIs in spite of the presence of state delay and unmatched disturbance. In contrast, the existing approaches usually assume large enough gain for the switching control, the value of which is not explicitly given. The usefulness of the proposed design will become significant in what its subsequent chapter is going to develop for input delay system, where the switching control design is crucial in stability analysis and minimization of bounded solutions. The method allows uncertainties to be considered in all the blocks of the system matrices, unlike those using equivalent control where uncertainties are considered only in a particular subsystem. A case study relating to the control of an autonomous vehicle is given where the delay is considered to be present in the motor response to the electrical command. Results from the chapter has been published in (Han, et al., [78], 2010).

SMC in the presence of a small and unavoidable input delay that may be present in controller implementation is studied in Chapter 9. Linear systems with bounded matched disturbances and uncertain system matrices are considered, where the presence of the input delay in the SMC will produce oscillations or potentially even unbounded solutions. Without apriori knowledge of the bounds on the state-dependent terms as required by the existing methods, the design objective is to achieve ultimate boundedness of the closed-loop system with a bound proportional to the delay, the disturbance bounds and the switching gain. This is a non-trivial problem because the relay gain depends on the state bound, whereas the latter bound depends on the relay gain. A controller with a linear gain proportional to the scalar $\frac{1}{\mu}$ is proposed, which for small enough $\mu > 0$ produces a closed-loop *singularly perturbed system* and yields the desired ultimate bound. An LMI-based solution for the evaluation of the design parameters and of the ultimate bound is derived. The superiority of the proposed SMC over conventional methodologies that ignore the input delay within the design phase is demonstrated through application examples. Primary result of the chapter for single input system has been published in (Fridman, et al., [51], 2010). This chapter presents the methodology for multi-input system in presence of both state and input delays. The controller is shown to be applicable for a nonlinear model of spacecraft position control where measurement delay is present due to digital sampling.

Finally, a brief review of the works discussed in all the previous chapters will be given in Chapter 10. Possible extensions of the main results developed in the thesis to solve other control problems are suggested in Chapter 11.

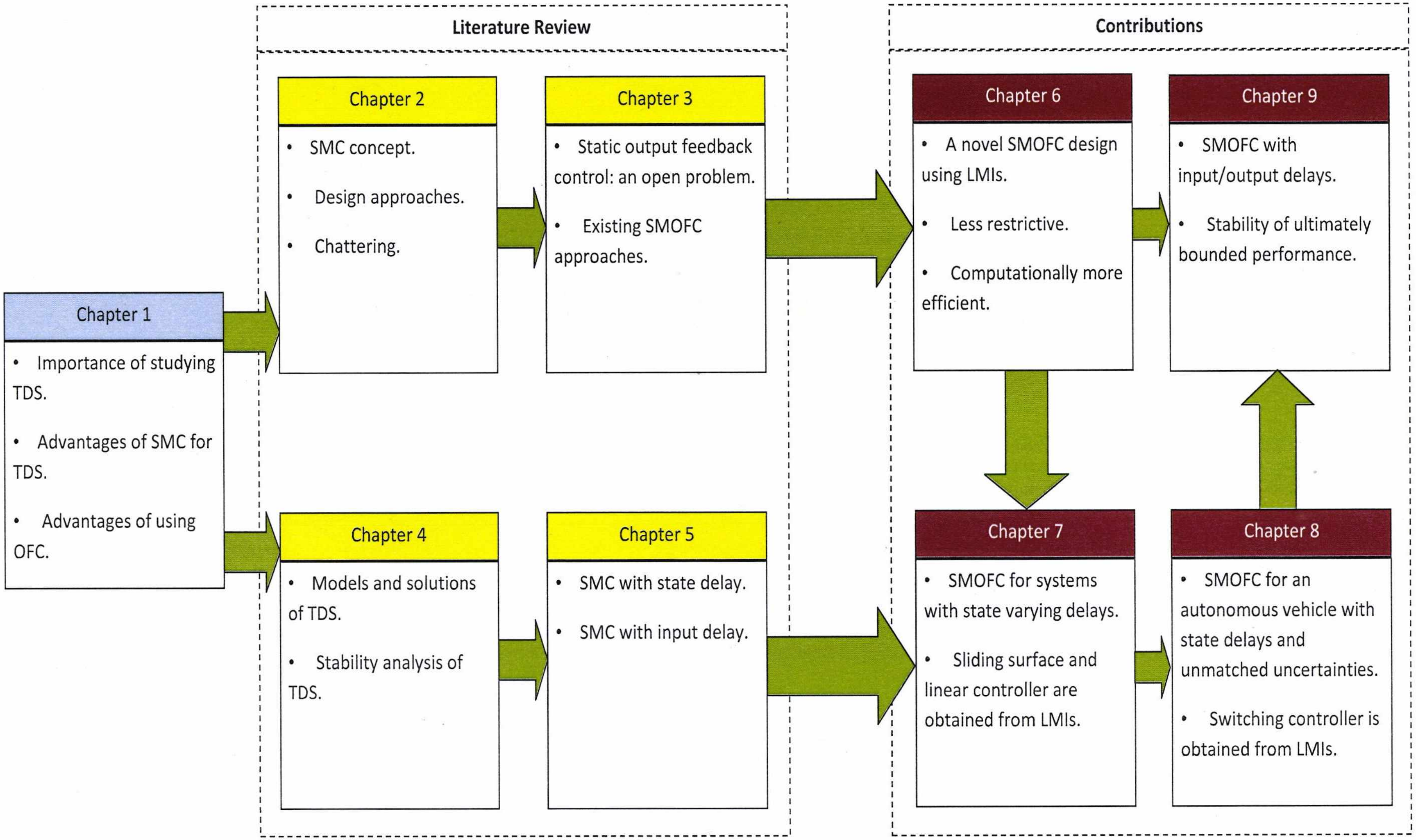


FIGURE 1.2: Chapter block diagram

Chapter 2

An Overview of Classical Sliding Mode Control

2.1 Introduction

Sliding Mode Control (SMC) is a discontinuous control action, which is recognized as an efficient tool for robust control of complex high-order nonlinear dynamic plant operating under uncertain conditions. It was initiated in the former Soviet Union about 50 years ago. Since then the control method has attracted a great deal of attention within the international control community. It has been utilized in the design of robust regulators, model-reference systems, adaptive schemes, tracking systems, state observers, fault detection schemes, and time delay systems. The ideas have successfully been applied to problems as diverse as automatic flight control, control of electric motors, chemical processes, helicopter stability augmentation systems, space systems and robots. Two reasons for its popularity are firstly, low sensitivity to plant variations and disturbances. Secondly, simplicity from the reduced order control design. The main purpose of this chapter is to introduce the most basic and elementary concepts such as existence, attractivity, equivalent control and dynamics in the sliding mode. In Section 2.2, the existence of a sliding mode is firstly illustrated using graphical examples. A mathematical description of sliding modes follows to enhance the readability. In Section 2.3 a number of methods for sliding surface design are described. Reachability conditions to the sliding surface are demonstrated using a simple example. Section 2.4 addresses practical implementation issues of SMC such as chattering caused by unmodeled actuator dynamics and delays.

2.2 "Sliding Mode" Concept

The conventional example to demonstrate sliding modes in terms of the state-space method is a second-order time-invariant relay system (Utkin, et al., [137], 1999 and [138], 2009)

$$\begin{aligned} &\ddot{x} + a_2\dot{x} + a_1 x = u + f(t), \\ u = -M\,\mathrm{sign}(s), \qquad &s = \dot{x} + cx \end{aligned} \tag{2.1}$$

where M, a_1, a_2, c are constant parameters and $f(t)$ is a bounded disturbance. The system behaviour may be analyzed in the phase plane $(x, \dot{x})$, shown in Figure 2.1. The control u undergoes discontinuties at the switching line $s = 0$ and the state trajectories are characterized by two families. The first family corresponds to $s > 0$ and $u = -M$ (upper semiplane); the second family corresponds to $s < 0$ and $u = M$ (lower semiplane). Within the sector from m to n on the switching line, the state trajectories are oriented towards the line. This means that the state trajectory will belong to the switching line for $t > t_1$. Thus the control design paradigm is

- The motion of the state trajectories is towards the switching line $s = 0$.
- State trajectories reach a point where they cannot leave the switching line due to the nature of the variable structure control and thus belong to the switching line $s = 0$, i.e. exhibit a *sliding mode*.
- After the sliding mode starts, further motion is governed by $s = cx + \dot{x} = 0$, i.e. *sliding mode equation*.

Once in the *sliding mode*, the system motion is

- governed by a 1^{st} order equation, i.e. with a *reduced order*.
- solution of the *sliding mode equation* $x(t) = x(t_1)e^{-c(t-t_1)}$ which is selected by the designer and depends neither on the plant parameters nor the disturbance, but only on c, i.e. *invariance*.

2.2.1 Existence of a sliding mode

Consider nonlinear differential equations in an arbitrary n-dimensional state space with an m-dimensional vector of control actions:

$$\dot{x} = f(x, t, u) \tag{2.2}$$

with $x \in \mathbb{R}^n$, $f \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, t denotes the time. The control is selected as a discontinuous function of the state. For example, each component of the control u_i may undergo discontinuities on some

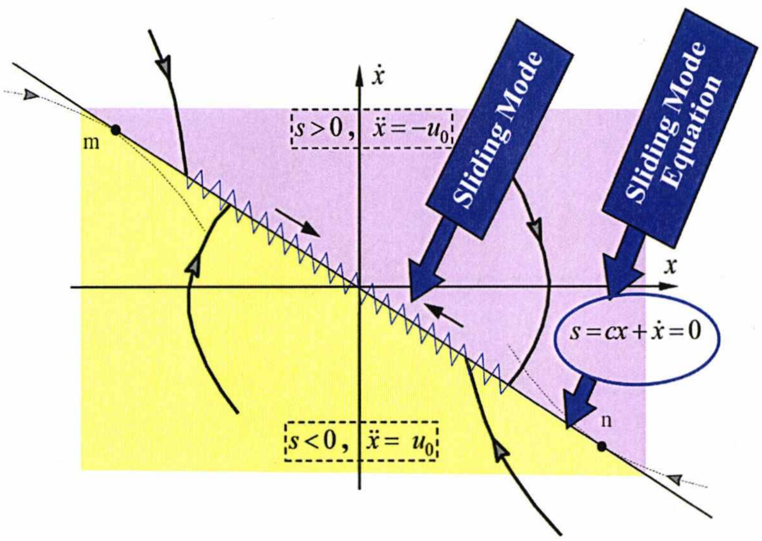


FIGURE 2.1: State plane of the second-order relay system (Utkin, et al., [138], 2009)

nonlinear surface $s_i(x)=0$ in the state space

$$u=\begin{cases} u_i^+(x,t) & \text{if } s_i(x)>0 \\ u_i^-(x,t) & \text{if } s_i(x)<0 \end{cases} \qquad (i=1,...,m) \tag{2.3}$$

where $u_i^+(x,t)$ and $u_i^-(x,t)$ are continuous state functions with $u_i^+(x,t)\neq u_i^-(x,t)$, the $s_i(x)$ are continuous state functions. A sliding mode may thus occur on the intersection of m surfaces $s_i(x)=0 \ (i=1,...,m)$ and the order of the sliding motion equations is $n-m$.

Sliding modes on individual surfaces

Similar to the scalar case (Utkin, et al., [137], 1999), the state trajectories in multi-dimensional sliding mode are oriented towards the discontinuity surface in its vicinity, or the variable describing deviation from the surface and its time-derivative should have opposite signs. Consider the following example with a two-dimensional control vector

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3+f_1(t)+u_1 \\ \dot{x}_3 &= f_2(t)+u_2 \end{aligned}$$

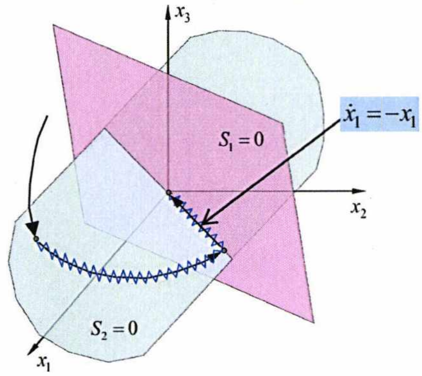


FIGURE 2.2: Two dimensional sliding mode (Utkin, et al., [138], 2009)

where $f_1(t)$ and $f_2(t)$ are unknown bounded disturbances with a known range of variation. The components of the control undergo discontinuities in two planes of the three-dimensional state

$$u_1 = -M_1 \operatorname{sign} s_1, \quad s_1 = x_1 + x_2,$$
$$u_2 = -M_2 \operatorname{sign} s_2, \quad s_2 = x_1 + x_2 + x_3,$$

where M_1, M_2 are positive constant values. If $M_2 > |x_2 + x_3 + f_1(t) + f_2(t)| + M_1$, then the values s_2 and $\dot{s}_2 = x_2 + x_3 + u_1 - M_2 \operatorname{sign} s_2$ have different signs. Hence the plane $s_2 = 0$ is reached after a finite time interval and then sliding mode with state trajectories in this plane will start, as shown in Figure 2.2. For this motion $x_3 = -x_1 - x_2$ and the sliding mode is governed by the second order equation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2 + f_1(t) + u_1$$

Again, for $M_1 > |x_1 + f_1(t)|$, the values s_1 and $\dot{s}_1 = x_1 - M_1 \operatorname{sign}(s_1)$ have different signs and after a finite time interval the state will reach the intersection of the planes $s_1 = 0$ and $s_2 = 0$. The further motion will be in this manifold (straight line formed by the intersection of the two planes), its first order equation may be derived by substituting $-x_1$ for x_2 (since $s_1 = 0$) into the first equation to obtain $\dot{x}_1 = -x_1$. The two-dimensional sliding mode is asymptotically stable, its order is reduced by two when compared with that of the original system and the motion does not depend on the disturbances $f_1(t)$ and $f_2(t)$.

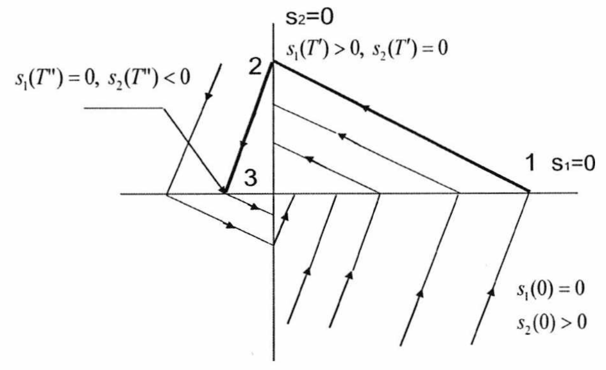


FIGURE 2.3: Sliding mode in the system with two-dimensional control (Utkin, et al., [138], 2009)

Sliding mode on the intersection surfaces

For the general case when the problem of enforcing a sliding mode in the intersection of a set of discontinuity surfaces can not be reduced to sequential treatment of scalar subproblems, a sliding mode may exist in the intersection of the discontinuity surfaces although it does not exist on each of them taken separately. This may be illustrated by the following example (Utkin, et al., [137], 1999)

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= -x_3 + u_1 - 2u_2 \\ \dot{x}_3 &= -x_3 + 2u_1 + u_2 \\ u_1 &= -\operatorname{sign}(s_1), \qquad s_1 = x_1 + x_2 \\ u_2 &= -\operatorname{sign}(s_2), \qquad s_2 = x_1 + x_3 \end{aligned} \tag{2.4}$$

The analysis of the condition for a sliding mode to exist in the intersection of the discontinuity surfaces may be performed in terms of motion projection onto the subspace (s_1, s_2)

$$\begin{aligned} \dot{s}_1 &= -\operatorname{sign}(s_1) + 2\operatorname{sign}(s_2) \\ \dot{s}_2 &= -2\operatorname{sign}(s_1) - \operatorname{sign}(s_2) \end{aligned} \tag{2.5}$$

The state trajectories are straight lines in the state plane (s_1, s_2), Figure 2.3. It is clear from the diagram (Utkin, et al., [137], 1999) that, for any point on $s_1 = 0$ or $s_2 = 0$, the state trajectories are not oriented towards the line. Therefore a sliding mode does not exist on any of the switching lines taken separately. At the same time, the trajectories converge to the intersection of them - the origin in the subspace (s_1, s_2). This section shows that the conditions for a two-dimensional sliding mode to exist cannot be derived from an analysis of scalar cases. Moreover, a sliding mode may exist in the intersection of discontinuity surfaces although it does not exist on each of the surfaces taken separately.

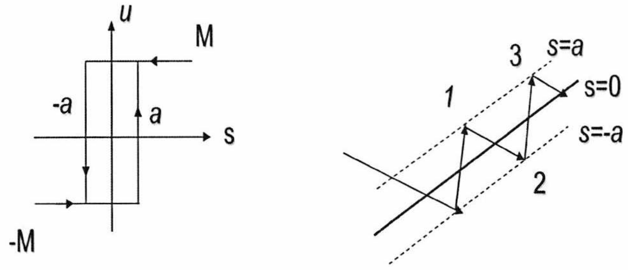


FIGURE 2.4: Relay with hysteresis

2.2.2 Mathematical description of sliding modes

In describing sliding modes mathematically, discontinuous systems are not subject to the conventional theory of differential equations dealing with continuous state functions.[1] The conventional theory does not answer even the fundamental questions as to whether the solution exists and is unique. Common approaches to deal with this solution problem are to employ different methods to replace the original problem by a similar one for which familiar methods are applicable (Utkin et al., [138], 2009). In the practical situation, an ideal sliding mode does not appear due to model imperfections, delay and hysteresis. The effect of such non-ideal factors makes the discontinuity point isolated in time. Thus system solutions can be obtained using standard mathematical tools. Assuming the limit of the solutions exists with small parameters tending to zero, then the limit is taken as the solution describing the ideal sliding mode.

Solution of sliding mode using hysteresis analysis

Consider the system

$$\dot{x} = Ax + bu, \tag{2.6}$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^{n\times 1}$ are constant matrices, $u = M\,\mathrm{sign}\,s$, M is a scalar positive constant value, $s = cx$, $c = (c_1, c_2, \ldots, c_n) = const$. If a relay device is implemented with a hysteresis loop with the width 2Δ, Figure 2.4, then the state trajectories oscillate in a $\Delta-$ vicinity of the switching plane. The value of Δ is assumed to be small such that the state trajectories may be approximated by straight lines with constant state velocity vectors $Ax + bM$ and $Ax - bM$ in the

[1]The most conventional method to derive the existence and uniqueness of the solution to a differential equation consists of functions $f(x)$ satisfying the *Lipschitz* condition $\|f(x_1) - f(x_2)\| < L\|x_1 - x_2\|$ with some positive number L, referred to as the *Lipschitz constant*, for any x_1 and x_2. The condition implies that the function does not grow faster than some linear one which is not the case for discontinuous functions if x_1 and x_2 are close to a discontinuity point, i.e. $\mathrm{sign}(0)$ is not defined.

vicinity of some point x on the plane $s(x) = 0$. Calculate times Δt_1 intervals and Δt_2 and increments Δx_1 and Δx_2 in the state vector for transitions from point 1 to point 2 and from point 2 to point 3, then the average state velocity within the time interval $\Delta t = \Delta t_1 + \Delta t_2$ is found as (Utkin, et al., [137], 1999)

$$\dot{x}_{av} = \frac{\Delta x_1 + \Delta x_2}{\Delta t} = Ax - (cb)^{-1}bcAx$$

Once the width of the hysteresis loop Δ approaches to zero, then

$$\lim_{\Delta \to 0,\ s(x)=0} \dot{x}_{av} = (I_n - (cb)^{-1}bc)Ax, \tag{2.7}$$

with initial state $s[x(0)] = 0$. Hence

$$\dot{s} = c(I_n - (cb)^{-1}bc)Ax = 0$$

i.e. the state trajectories of the sliding mode are oriented along the switching plane. Thus the motion equation has been obtained through regularization using a relay with hysteresis.

Generalization of hysteresis result

This result may be easily interpreted in terms of the relative time intervals for the control input to take each of two extreme values. Consider an arbitrary order system with scalar control

$$\dot{x} = f(x,u) \quad x, f \in \mathbb{R}^n, \quad u(x) \in \mathbb{R}$$

$$u(x) = \begin{cases} u^+(x) & \text{if } s(x) > 0 \\ u^-(x) & \text{if } s(x) < 0 \end{cases} \tag{2.8}$$

the components of vector f, scalar functions $u^+(x)$, $u^-(x)$ and $s(x)$ are continuous and smooth, and $u^+(x) \neq u^-(x)$. It is assumed that a sliding mode occurs on the surface $s(x) = 0$. Let the discontinuous control be implemented with some unspecified imperfections; the control is known to take one of the two extreme values, $u^+(x)$ or $u^-(x)$, and the discontinuity points are isolated in time. As a result, the solution exists in the conventional sense and it does not matter whether a small hysteresis, time delay or time constants are neglected in the ideal model. The state velocity vectors $f^+ = f(x,u^+)$ and $f^- = f(x,u^-)$ are assumed to be constant for some point x on the surface $s(x) = 0$ within a short time interval $(t, t + \triangle t)$. Let the time interval $\triangle t$ consist of two sets of intervals $\triangle t_1$ and $\triangle t_2$ such that $\triangle t = \triangle t_1 + \triangle t_2$, $u = u^+$ for the time from the set $\triangle t_1$ and $u = u^-$ for the time from the set $\triangle t_2$. Then the increment of the state vector after time interval $\triangle t$ is found as

$$\triangle x = f^+ \triangle t_1 + f^- \triangle t_2$$

and the average state velocity as

$$\dot{x}_{av} = \frac{\triangle x}{\triangle t} = \mu f^+ + (1-\mu) f^-$$

where $\mu = \triangle t_1 / \triangle t$ is the relative time for the control to take the value u^+ and $(1-\mu)$ is the relative time it takes the value u^-, $0 \leq \mu \leq 1$. To obtain the vector $\dot{x}$, the time $\triangle t$ should tend to zero. However it is not needed to perform this limit procedure as in the above assumption that state velocity vectors are constant within the time interval $\triangle t$, and therefore the equation

$$\dot{x} = \mu f^+ + (1-\mu) f^- \tag{2.9}$$

represents the motion during the sliding mode. Choosing

$$\mu = \frac{\text{grad}\, s \cdot f^-}{\text{grad}\, s \cdot (f^- - f^+)}$$

the sliding mode equation

$$\dot{x} = f_{sm}, \quad f_{sm} = \frac{(\text{grad}\, s \cdot f^-)}{(\text{grad}\, s) \cdot (f^- - f^+)} f^+ - \frac{(\text{grad}\, s \cdot f^+)}{(\text{grad}\, s) \cdot (f^- - f^+)} f^-, \tag{2.10}$$

as shown in Figure2.5 represents the sliding motion with initial condition $s[x(0)] = 0$. Since the state trajectories during the sliding mode are on the surface $s(x) = 0$, the parameter μ is selected such that the state velocity vector of the system (2.9) is in a tangential plane to this surface. As expected, direct substitution of grad$s = c$, $f^+ = Ax + bu^+$ and $f^- = Ax + bu^-$ into (2.10) results in the sliding mode equation (2.7) for the linear system (2.6) with the discontinuity plane $s(x) = cx = 0$ via hysteresis analysis.

Remark 2.1. This hysteresis and its generalization method for deriving the sliding mode equation is considered by Utkin, et al., [137], (1999) as a physical interpretation of the famous *Filippov* method. The method is intended for solution continuation at a discontinuity surface for differential equations with discontinuous right-hand sides (Fillippov, [41], 1988). According to this method, the ends of all state velocity vectors in the vicinity of a point on a discontinuity surface should be complemented by a minimal convex set and the state velocity vector of the sliding motion should belong to this set. In the case discussed above, the ends of vectors f^+ and f^-, and the minimal convex set is the straight line connecting their ends. The equation of this line is exactly the right-hand side of equation (2.9). The intersection of the line with the tangential plane defines the state velocity vector in the sliding mode, or the right-hand side of the sliding mode equation.

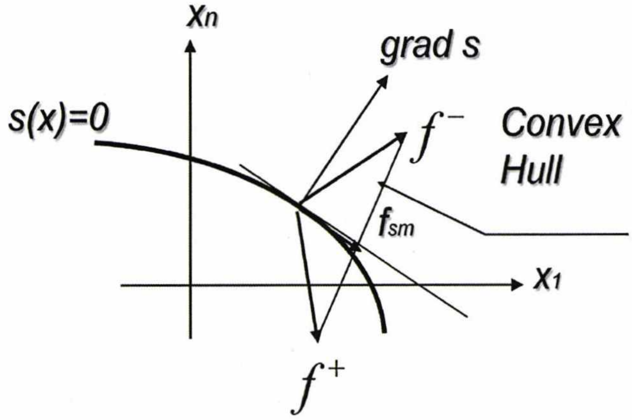


FIGURE 2.5: Sliding mode equation by Filippov's method (Utkin, et al., [138], 2009)

2.2.3 Equivalent control method

Since on the sliding surface $s(x) = 0$, therefore $ds/dt = \dot{s} = 0$, i.e. the time derivative of vector s on the state trajectories of (2.2) is equal to zero:

$$\dot{s}(x) = G \cdot f(x,u) = 0 \tag{2.11}$$

where $G = (\partial s/\partial x)$ is $m \times n$ matrix with gradients of functions $s_i(x)$ as rows. Let a solution to the algebraic equation (2.11) exist, of which a solution of u_{eq} will confine the system (2.2) and thus

$$\dot{x} = f(x, u_{eq}) \tag{2.12}$$

along the sliding manifold $s(x) = 0$, where u_{eq} is referred as *"Equivalent Control"*. The equivalent control can be seen as a replacement of the applied discontinuous control on the intersection of the switching surfaces $s(x) = 0$. For example, solution of the scalar system (2.8) can be found as the intersection of the tangential plane and the locus $f(x,u)$ with control u running from u^- to u^+, Figure 2.6. The intersection point defines the equivalent control u_{eq} and the right-hand side $f(x,u_{eq})$ in the sliding mode equation (2.7).

Remark 2.2. The right-hand side $f(x,u_{eq})$ of the motion equation resulting from the *equivalent control* method in Figure 2.6 does not coincide with f_{sm} in Figure2.5 using *Fillippov's* method. They are equal if the scalar control is linear with respect to control $f(x,u) = f_0(x) + b(x)u$. Then the locus of $f(x,u_{eq})$ in Figure 2.6 coincides with the minimal convex set, i.e. the straight line connecting the end of vectors f^+ and f^- in Figure2.5, of *Fillippov's* method. The discrepancy reflects the fact that different ways of regularization lead to different sliding mode equations in systems with nonlinear functions of control input in motion equations (Utkin, [136], 1992).

Equation (2.11) of the equivalent control method for system (2.12) is of form

$$\dot{s} = Gf + GBu_{eq} = 0 \tag{2.13}$$

where $B \in \mathbb{R}^m$ is an input distribution matrix. Assuming that the matrix GB is nonsingular for any x, the equivalent control $u_{eq}(x)$ is found as the solution to (2.13)

$$u_{eq}(x) = -(G(x)B(x))^{-1}G(x)f(x)$$

and substitute u_{eq} into (2.8) to yield the sliding mode equation as

$$\dot{x} = f(x) - B(x)(G(x)B(x))^{-1}G(x)f(x) \tag{2.14}$$

Equation (2.14) is taken as the equation of the sliding mode on the manifold $s(x) = 0$. It is interesting to note that the sliding motion (2.14) obtained using the equivalent control is the same as that obtained using hysteresis regularization (2.7).

Remark 2.3. The motion in the sliding mode is an ideal description. However, in reality, various imperfections make the state oscillate in some vicinity of the intersection and components of the control are switched at finite frequency, alternately taking the values $u_i^+(x)$ and $u_i^-(x)$. These oscillations have high frequency and slow components. The high frequency is filtered out by the plant under control while its motion in the sliding mode is determined by the slow component. On the other hand, sliding mode equations are obtained by substitution of the equivalent control for the real control. It is reasonable to assume that the equivalent control is close to the slow component of the real control which may be derived by filtering out the high-frequency component using a lowpass filter.

2.3 Design approaches

To this end it can be stated that the sliding mode control design approach involves two independent subproblems of lower dimension:

- Design of the desired dynamics for a system of $(n-m)$th order by proper choice of a sliding manifold $s = 0$;
- Enforcing a sliding motion in this manifold, which is equivalent to a stability problem for the mth order manifold $s = 0$.

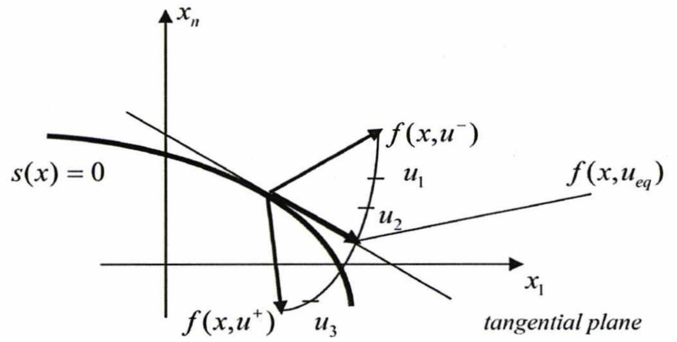


FIGURE 2.6: Sliding mode equation by Equivalent control method (Utkin, et al., [138], 2009)

The motion in the sliding mode is *invariant* to 'uncertainties' in the system which satisfy so called 'matching conditions'

$$h(x,t) \in \mathscr{R}[B], \quad \text{i.e.} \quad h(x,t) = B(x,t)\gamma(x,t), \tag{2.15}$$

where $\|\gamma(x,t)\| \leq \gamma_0(x,t)$ is known. The vector $h(x,t)$ characterizes all disturbance factors in the motion equation

$$\dot{x} = f(x,t) + B(x,t)u + h(x,t) \tag{2.16}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, whose influence on the control process can be rejected. This is because the disturbances act in control space, then there exists control u_h such that $Bu_h = -h$ and hence the system is invariant to $h(x,t)$. But control u_h would hardly be implementable since the disturbances may be inaccessible for measurement. As it has been established the sliding mode equation in any manifold does not depend on control. Similarly, via the equivalent control method, it can be shown that sliding mode is independent on $h(x,t)$ as well, therefore condition (2.15) is the invariance condition for sliding mode control. It is important that for the design of an invariant system there is no need to measure vector h. To ensure sliding mode existence, only an upper estimate of h (a number of function) is needed.

2.3.1 Decoupling

Consider the affine systems

$$\begin{aligned} \dot{x} &= f(x,t) + B(x,t)u, \\ u(x) &= \begin{cases} u^+(x,t) & \text{if } s(x) > 0 \\ u^-(x,t) & \text{if } s(x) < 0 \end{cases} \qquad s^T(x) = [s_1(x), \ldots, s_m(x)]. \end{aligned} \tag{2.17}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. To obtain the sliding mode equation using equivalent control, GB needs to be full rank. Then substitution of

$$u_{eq}(x,t) = -(G(x)B(x,t))^{-1}G(x)f(x,t)$$

to (2.17) yields

$$\begin{aligned} \dot{x} &= f_{sm}(x,t) \\ f_{sm}(x,t) &= f(x,t) - B(x,t)(G(x)B(x,t))^{-1}G(x)f(x,t) \end{aligned} \tag{2.18}$$

Define

$$x^T = [x_1^T \; x_2^T], \;\; x_1 \in \mathbb{R}^{n-m}, \;\; x_2 = s_0(x_1) \text{ for } s(x) = 0$$

then substitution of x_2 into x_1 yield the sliding mode equation

$$\dot{x}_1 = f_{1sm}(x_1, s_0(x_1), t), \tag{2.19}$$

where $f_{sm}^T(x,t) = [f_{1sm}^T(x_1,x_2,t) \;\; f_{2sm}^T(x_1,x_2,t)]$. Features of this method of design are

- The design problem is not a conventional one since the right-hand sides in (2.18) and (2.19) depend not only on the discontinuity manifold equation but also on the gradient matrix G as well.
- If a class of function $s(x)$ is pre-selected, for instance linear functions or in the form of finite series, then both $s(x)$, G and, as a result, the right-hand sides in (2.19) are not independent from the set of parameters to be selected when designing the desired dynamics of sliding motion.

2.3.2 Regular form

The two-stage design procedure - selection of a switching manifold and then finding a control that enforces the sliding mode in this manifold - becomes simpler for systems in so-called *regular form*. The regular form for an affine system (2.16) consists of two blocks

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,x_2,t), & x_1 &\in \mathbb{R}^{n-m} \\ \dot{x}_2 &= f_2(x_1,x_2,t) + B_2(x_1,x_2,t)(u + \gamma(x,t)), & x_2 &\in \mathbb{R}^m, \;\; det(B_2) \neq 0 \end{aligned} \tag{2.20}$$

The first block does not depend on the control, and the dimension of the second block coincides with the dimension of the control. The design is performed in two stages as previously stated. First, an m-dimensional state vector x_2 is considered as the 'control' of the first block and designed as a function of the state x_1 of the first block corresponding to some performance criterion

$$x_2 = -s_0(x_1)$$

Then the discontinuous control should be designed to enforce a sliding mode on the manifold

$$s(x_1, x_2) = x_2 + s_0(x_1) = 0 \tag{2.21}$$

(the design problem of the mth order with m-dimensional control). After a finite time interval, a sliding mode commences on the manifold (2.21) and the system will exhibit the desired behaviour governed by $\dot{x}_1 = f_1[x_1, -s_0(x_1), t]$. Note that the motion is of a reduced order and depends neither on the function $f_2(x_1, x_2, t)$ nor on function $B_2(x_1, x_2, t)$ and $\gamma(x,t)$ in the second equation of the original system (2.20). Regular form based design exhibits the following characteristics

- In contrast to (2.18) and (2.19), the sliding mode equation does not depend on the gradient matrix G, which makes the design problem at the first stage a conventional one; design of m- dimensional control x_2 in $(n-m)-$dimensional system with state vector x_1.
- Calculation of the equivalent control to find the sliding mode equation is not needed.
- GB full rank is also needed.
- Sliding mode is invariant with respect to functions f_2 and B_2 in the second block.

2.3.3 Unit control

The Lyapunov approach of unit control relies on finding the control for which the time-derivative of the *Lyapunov* function is negative along the trajectories of the system. Consider system (2.16), and choose a *Lyapunov* function candidate $V(x) > 0$. Denote

$$W_0 = dV/dt|_{h=0,u=0} = \text{grad}(V)^T f < 0 \tag{2.22}$$

where $\text{grad}(V)^T = [\frac{\partial V}{\partial x_1} \cdots \frac{\partial V}{\partial x_1}]$. Then

$$W = dV/dt = W_0 + \text{grad}(V)^T B(u+\gamma) \tag{2.23}$$

Choose the discontinuous control as

$$u = -\rho(x,t)\frac{B^T \text{grad}(V)}{\|B^T \text{grad}(V)\|} \tag{2.24}$$

where $\rho(x,t) > \gamma_0(x,t)$, with

$$\|B^T \text{grad}(V)\|^2 = (\text{grad}(V)^T B)(B^T \text{grad}(V))$$

then substituting (2.24) into (2.23) yields the derivative of the *Lyapunov* function

$$\begin{aligned} W &= W_0 - \rho(x,t)\|B^T \operatorname{grad}(V)\| + \operatorname{grad}(V)^T B\gamma(x,t) \\ &< W_0 - \|B^T \operatorname{grad}(V)\|[\rho(x,t) - \gamma_0(x,t)] \\ &< 0 \end{aligned}$$

This implies that the perturbed system with control (2.24) is asymptotically stable. Features of this method are

- Control (2.24) undergoes discontinuities in $(n-m)-$ dimensional manifold $s(x) = B^T \operatorname{grad}(V) = 0$ rather than in any individual control component. This is the main factor that distinguishes between the control method here and previous ones.
- The disturbance $h(x,t)$ is rejected on the sliding manifold $s(x) = 0$ where the control (2.24) is discontinuous. In this case the *equivalent control* is taken as $-\gamma(x,t)$.

2.3.4 Reachability

In the sliding mode, the motion of the system is independent of the control. But it is obvious that the control must be designed such that it drives the trajectories to the switching surface and maintain it on this surface once it has been reached. A condition for the attractiveness of the sliding surface can be expressed by the condition

$$s\dot{s} < 0 \tag{2.25}$$

which is called the reachability condition (Itkis, [90], 1976). The following example illustrates the design concept.

EXAMPLE 2.1. Consider a dc-motor modelled by the following transfer function

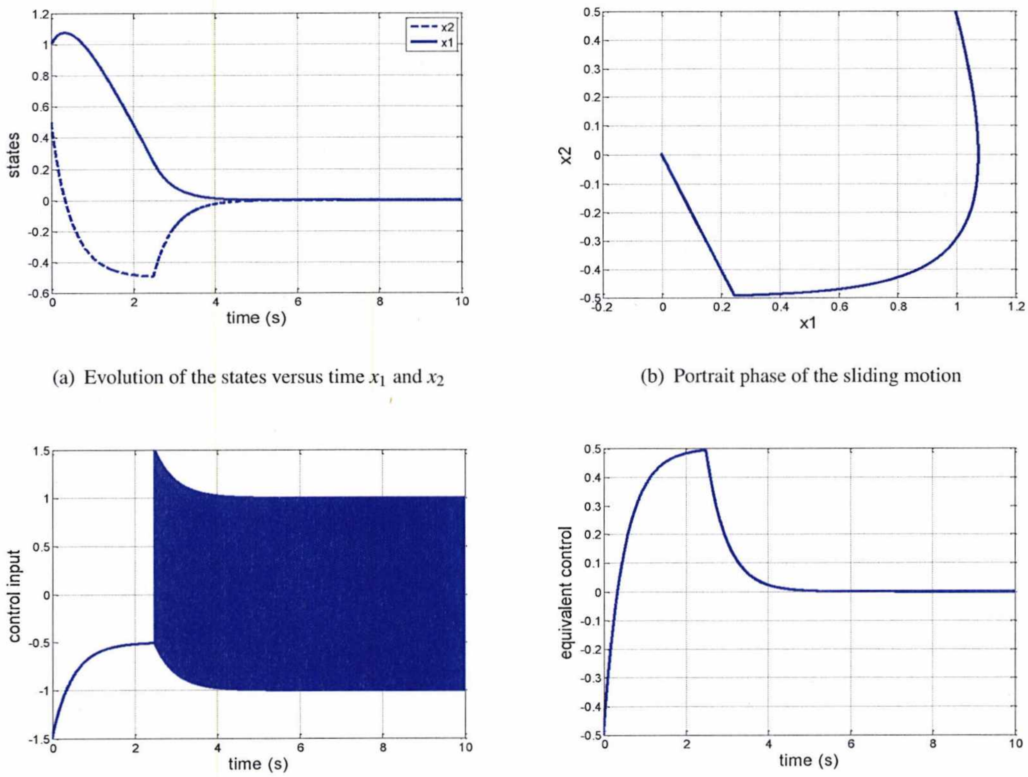
$$Y(p) = \frac{1}{p(p+1)}U(p)$$

Which can be described in state-space form as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 + u \\ y = x_1 \end{cases} \tag{2.26}$$

Assume that the sliding surface is designed as

$$s = x_2 + \alpha x_1 = 0$$



(a) Evolution of the states versus time x_1 and x_2

(b) Portrait phase of the sliding motion

(c) Discontinuous control

(d) Equivalent control

FIGURE 2.7: Sliding mode control (2.27) of system (2.26)

where $\alpha > 0$. Thus

$$\dot{s} = (\alpha - 1)x_2 + u$$

Using the control law $u = -k \operatorname{sign} s$, $k > 0$, the reachability condition is satisfied in the domain

$$\Omega = \{x \, : \, |(a-1)x_2| < k\}$$

since

$$s\dot{s} < |s|(|(\alpha - 1)x_2| - k) < 0$$

Note that condition (2.25) is not sufficient to ensure finite time convergence to the surface. In the latter example, the control

$$u = (1-\alpha)x_2 - ks$$

provides $\dot{s} = -ks$, but the convergence to $s = 0$ is only asymptotic since

$$s(t) = s(0)e^{-kt}$$

where $s(0)$ is the initial value of s. Condition (2.25) is often replaced by the so-called η-reachability condition

$$s\dot{s} \leq -\eta|s|$$

where η is a positive number, which will ensure finite time convergence to $s = 0$. Since by integration

$$|s(t)| - |s(0)| \leq -\eta t$$

this shows that the time required to reach the surface, starting from initial condition $s(0)$, is bounded by

$$t_e = \frac{s(0)}{\eta}$$

In practice, the control law is generally formulated as $u = u_e + u_d$ where u_e is the equivalent control and where u_d is the discontinuous part, ensuring a finite time convergence to the chosen surface. Example (2.1) is simulated using the following control law

$$u = (1-\alpha)x_2 - k\,\mathrm{sign}\, s \tag{2.27}$$

where the term $(1-\alpha)x_2$ represents the equivalent control (since $\dot{s} = 0$ implies $u + (\alpha - 1)x_2 = 0$). Note that the η-reachability condition is satisfied. Figure (2.7(a))- (2.7(d)) show obviously that the sliding motion takes place after about 1 sec. After this time, the dynamics of the system is represented by the reduced order system given by the chosen surface, i.e.

$$\dot{x}_1 = -\alpha x_1 = x_2$$

and the control switches at high frequency. In Figure(2.7(d)) it is seen that the equivalent control, in sliding motion, represents the mean value of the control u. The phase portrait, in Figure(2.7(b)), illustrates the two steps of the dynamic behaviour: first, a parabolic trajectory before the surface is reached (which is called the reaching phase) and then sliding along the designed line $s = 0$ ($x_2 = -\alpha x_1$) to the origin.

2.4 Chattering

The afore mentioned ideal sliding mode is only used as an analytical description of the method. In practical implementation of the controller, a phenomenon called "chattering" around the sliding manifold exists, which results in low control accuracy, high heat losses in electrical power circuits and high wear of moving mechanical parts (Utkin, et al., [138], 2009). Two main causes have been identified which are

- Fast dynamics in the control loop which were neglected in the system model, are often excited by the inherent switching of sliding mode controllers.

- Delay and digital implementations in micro-controllers with fixed sampling rates may lead to discretization chatter.

This section serves to analyze firstly the cause of the chattering due to the unmodeled dynamics in the control loop and secondly introduce methods for reducing chattering.

2.4.1 Chattering caused by unmodeled actuator dynamics

Problem analysis

Consider the following first order plant with second order "unmodeled" actuator dynamics as

$$\dot{x}(t) = ax(t) + d(x,t) + bw(t) \tag{2.28}$$

where $a^- \leq a \leq a^+$ and $0 < b^- \leq b \leq b^+$ are unknown parameters within known bounds, $w(t)$ is the control variable and $|d(x,t)| \leq d^+$ is a disturbance (Utkin, et al., [138], 2009). The control variable $w(t)$ is the output of an 'unmodeled' actuator with stable dynamics

$$w(t) = \frac{\omega^2}{p^2 + 2\omega p + \omega^2} u(t) = \frac{1}{(\mu p + 1)^2} u(t) \tag{2.29}$$

where $u(t)$ is the actual control input to plant (2.28) and p denotes the *Laplace* variable. In (2.29), $\omega > 0$ is the unknown actuator bandwidth with $\omega \gg a$. The small time constant $\mu = \frac{1}{\omega} > 0$ in (2.29) symbolizes that the actuator dynamics are assumed to be significantly faster than the system dynamics (2.28).

The state $x(t)$ of system (2.28) is controlled to track a desired trajectory $x_d(t)$ with a known amplitude bound $|x_d(t)| \leq x_d^+$ and a known bound on the rate of change $|\dot{x}_d(t)| \leq v_d^+$. Setting $a = 0.5,\ b = 1, d(t) = 0.2sin(10t) + 0.3cos(20t) \leq 0.5, \omega = 50$, thus $\mu = 0.02$. The limit on the available control signal is $|u(t)| \leq 2.01$ and the desired trajectory $x_d(t) = sin(t)$, i.e. $x_d^+ = 1$ and $v_d^+ = 1$.

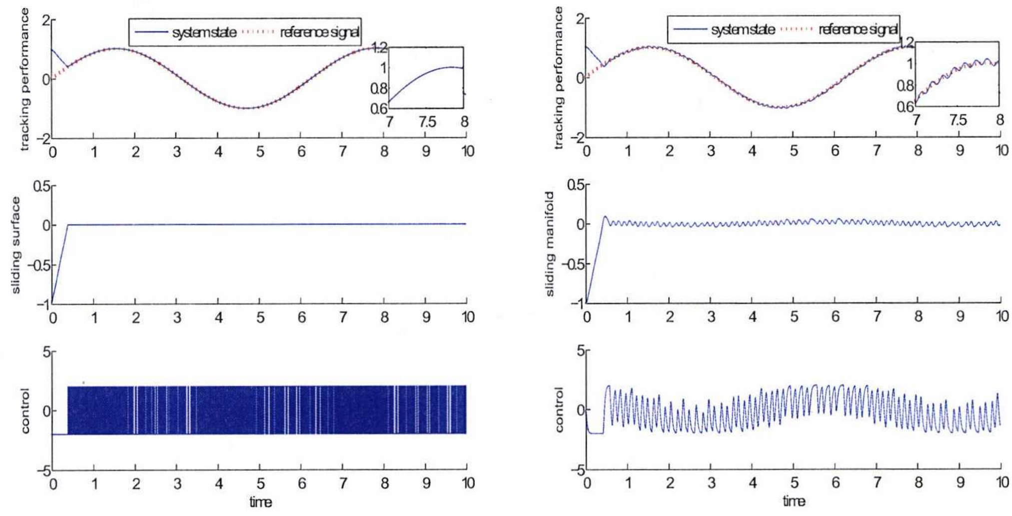
Ideal sliding mode

Neglecting actuator dynamics (2.29) by setting $w(t) = u(t)$, define the sliding variable as

$$s(t) = x_d(t) - x(t) \tag{2.30}$$

and choose the control

$$w(t) = M \, \mathrm{sign}\, s(t) \tag{2.31}$$



(a) Ideal sliding mode without chattering

(b) Sliding mode with chattering caused by unmodelled dynamics

FIGURE 2.8: Sliding mode control of system (2.28)

The Lyapunov function candidate is chosen as

$$V(t) = \frac{1}{2b} s^2(t) \tag{2.32}$$

whose derivative along (2.28) and (2.31) is

$$\dot{V}(t) = \frac{1}{b} s(t)\dot{s}(t) = g(x, x_d, t)s(t) - M|s(t)| \tag{2.33}$$

where $g(x,x_d,t) = \frac{\dot{x}_d(t) - ax(t) - d(t)}{b}$, i.e. $|g(x,x_d,t)| \leq g^+ = \frac{v_d^+ + a^+ x_d^+ + d^+}{b^-}$. Choosing $M \geq g^+ + \frac{\xi}{\sqrt{2b^-}}$ with scalar $\xi > 0$, then

$$\dot{V}(t) \leq -\xi V^{\frac{1}{2}}(t) \tag{2.34}$$

i.e. convergence of system state (2.28) to (2.30) within finite time despite the parametric uncertainty in a and b and unknown disturbance $d(x,t)$ is proved. Figure2.8(a) shows system (2.28) under control (2.31) with $M = 2.01$. From initial condition $x(0) = 1$, a sliding mode occurs in less than $1sec$, $x(t)$ coincides exactly with desired $x_d(t)$, and control $w(t)$ is switched at very high frequency.

Existence of Chattering

To qualitatively illustrate the influence of unmodeled dynamics on the system behaviour, consider the simplest case $a = 0,\ d(x,t) = 0,\ b = 1,\ x_d(t) = 0$ in (2.28) and (2.30). Then

$$\dot{x}^* = -M\,\text{sign}(x), \quad \mu^2\ddot{x} + 2\mu\dot{x} + x = x^*. \tag{2.35}$$

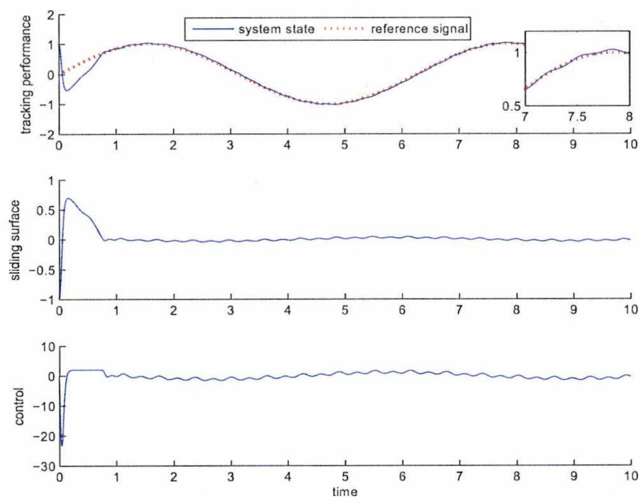


FIGURE 2.9: Boundary layer for chattering reduction

are the motion equations. A *sliding mode cannot occur* in the system since $\dot{x}$ is a continuous time function and cannot have sign opposite to x in the vicinity of the point $x = 0$ where the control undergoes discontinuities. Another way to show the existence of chattering is by using a Lyaponuv method. Rewriting the motion equation (2.35) in the form

$$\begin{aligned}\dot{x} &= w\\ \dot{w} &= v\\ \dot{v} &= -\tfrac{2}{\mu}v - \tfrac{1}{\mu^2}w + \tfrac{1}{\mu^2}u\end{aligned} \tag{2.36}$$

For the control $u = -M\,\text{sign}(x)$, the sign-varying Lyapunov function

$$V = xv - 0.5w^2 \tag{2.37}$$

has a negative time-derivative

$$\dot{V} = x(-\frac{2}{\mu}v - \frac{1}{\mu^2}w + \frac{1}{\mu^2}u) \tag{2.38}$$

for small magnitudes of v and w. This means that the motion is unstable in an $\varepsilon(\mu)-$ order vicinity of the manifold $s(x) = x = 0$, while all the trajectories converge to this vicinity. This instability explains why chattering may be generated in systems with discontinuous control in the presence of unmodelled dynamics. Figure2.8(b) shows the chattering behaviour of system (2.28) under control (2.31), but with actuator dynamics (2.29). Output $x(t)$ is seen to oscillate around the desired $x_d(t)$, sliding motion is not attained and the control is of finite switching frequency.

Solutions

The boundary layer solution, proposed by Slotine, [131], (1983) seeks to avoid control discontinuities and switching action in the control loop. The discontinuous control is replaced by a saturation function which approximates the sign(s) term in the boundary layer of the manifold $s(t) = 0$.

$$u(t) = \begin{cases} M \operatorname{sign} s(t) & \text{for } |s(t)| > \varepsilon \\ \frac{M}{\varepsilon} s(t) & \text{for } |s(t)| \leq \varepsilon \end{cases} \tag{2.39}$$

where ε denotes the width of the layer. The advantage of the method is that within the boundary layer, the continuous control can be designed to neglect the actuator dynamics. The discontinuous control only takes place outside the boundary layer. However the invariance property of sliding mode control is only partially preserved in the sense that the system trajectories are confined to a $\delta(\varepsilon)-$vicinity of the sliding manifold $s(t) = 0$. In the presence of disturbance, only bounded solutions can be achieved. Figure 2.9 shows reduction of chattering of system (2.28) under control (2.39) with actuator dynamics (2.29), where $\varepsilon = 0.05$. A comprehensive introduction for chattering reduction in the presence of unmodelled actuator dynamics using other methods, i.e. observer, disturbance compensation, can be found in (Young, et al., [147], 1999).

The boundary layer approach substitutes the discontinuity of a sliding mode controller by a saturation function and yields slow oscillatory motion around the sliding manifold instead of true sliding along the manifold. Effectively, the sliding mode methodology is utilized to design a continuous high-gain controller.

2.4.2 Chattering caused by time delay

Problem analysis

In (Fridman, et al., [59], 1996) it is shown that even in the simplest one-dimensional delayed relay control system only oscillatory solutions can occur.

EXAMPLE 2.2. The equation

$$\dot{x}(t) = -\operatorname{sign}[x(t-1)] \tag{2.40}$$

has a 4-periodic solution

$$g_0(t) = \begin{cases} t, & \text{for } -1 \leq t \leq 1 \\ 2-t, & \text{for } 1 \leq t \leq 3 \end{cases}$$
$$g_0(t+4k) = g_0(t) \quad k \in \mathbb{Z}$$

Since

$$\dot{g}_0(t) = -\operatorname{sign}[g_0(t-1-4n)]$$

t can be substituted by $(4n+1)t$ to obtain

$$\frac{1}{4n+1}[g_0((4n+1)t)]' = -\operatorname{sign}[\frac{1}{4n+1}g_0((4n+1)t)]$$

Thus, a $4/(4n+1)$-periodic solution to (2.40) is

$$g_n(t) = \frac{1}{4n+1} g_0((4n+1)t), \quad t \in \mathbb{R}$$

for each integer, $n \geq 1$. This means that there exists a countable set of periodic solutions, or the so-called *steady modes*.

Existence of Chattering

Relay delay controllers can stabilize the amplitude of chattering, suppressing the effect of uncertainty in the time delay even in the case when the time delay is variable. Consider the stabilization problem for the simplest unstable system

$$\dot{x} = kx, \quad (x \in \mathbb{R}, \ k > 0) \tag{2.41}$$

by means of a delay relay control law of the form $u = -\operatorname{sign}[x(t-\tau)]$, where τ is the time delay. In this case system equation has the form

$$\dot{x}(t) = -\operatorname{sign}[x(t-\tau)] + kx \tag{2.42}$$

To compute the constant $A > 0$ for which the system (2.42) with initial function

$$\phi(\theta) = A, \quad \theta \in [-\tau, 0] \tag{2.43}$$

has a stable periodic solution for $t > 0$, the state equation prior to the switching instant is considered initially

$$x(t) = \frac{1}{k} + (A - \frac{1}{k})e^{kt}$$

The function $x(t)$ could change its sign if and only if the condition

$$A - \frac{1}{k} < 0$$

holds. In this case an equation can be rewritten for $\gamma-$, which is the root of equation $x(\gamma) = 0$ in the form $e^{k\gamma} = \frac{1}{1-kA}$. From the periodicity of $x(t)$ the equation for the switching moment of the control law in the form $x(\gamma+\tau) = -A$ can be obtained. Then

$$\frac{1}{k} + (A - \frac{1}{k})e^{k\gamma}e^{k\tau} = -A,$$

and consequently $A = (e^{k\tau-1})/k$. This means that a sufficient condition for existence of the periodic solution is

$$0 < k\tau < ln2 \tag{2.44}$$

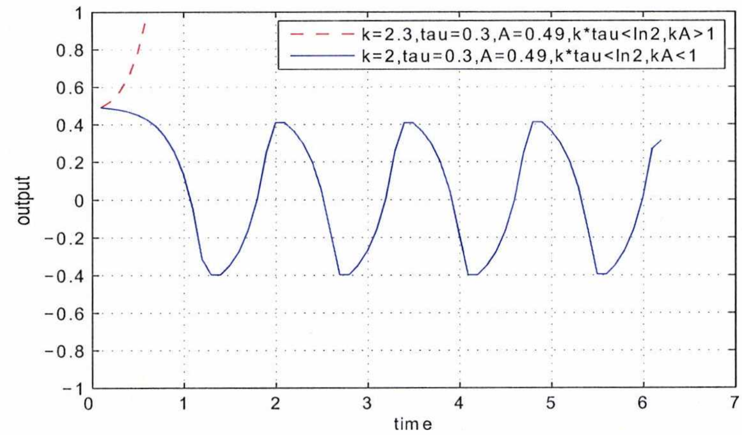


FIGURE 2.10: stable and unstable oscillation in respect to system properties

This implies that for any positive feedback coefficient k, a time delay τ can be chosen, for which there exists a zero frequency stable periodic steady mode of (2.42). Moreover the equation (2.42) has a countable set of steady modes in the interior of the strip $|x| < (e^{k\tau-1})/k$. System (2.42) has unstable solutions $x = \pm 1/k$, and unbounded solutions in the regions $|x| > 1/k$.

This means that the Cauchy problem (2.42) has a bounded solution if for any $t \in [0, \tau]$, $k|x_\phi(t)| < 1$. This means that if $\phi(0) > 0$, then

$$k|x_\phi(t)| = |-1 + (k\phi(0) + 1)e^{kt}| < 1$$

The simulation of stable and unstable oscillation with respect to system properties is plotted in Figure 2.10. A stable periodic solution is obtained if and only if both conditions $0 < k\tau < ln2$ and $k|x_\phi(t)| < 1$ are satisfied and $|x| < A$, otherwise the system has unbounded solutions.

Theorem 2.1. (Fridman, et al., [59], 1996) If condition (2.42) holds and $|\phi(0)| < \frac{2-e^{k\tau}}{ke^{k\tau}}$, then the solution $x_\phi(t)$ of (2.42) is bounded.

It is important to note that

- The condition (2.44) is a sufficient and necessary condition for relay delayed stabilization.
- The size of the *domain of stabilization* is proportional to the control gain.

Fridman, et al., [59], (1996) proposed the following algorithm for controlling the amplitude of the motion: since after a finite time all solutions coincide with the periodic solution, one can extrapolate the next zero for the periodic solution, and reduce the control gain near to the periodic solution zero. This algorithm requires the knowledge of the sign of the state variable with delay and only requires the *stabilization condition (2.44)* to hold. This algorithm is valid for any constant delay satisfying condition (2.44) and does not depend on the delay value.

2.5 Conclusion

The basic concepts of sliding mode control have been reviewed. Notions of sliding surface and ideal sliding mode are introduced. A sliding mode may exist at the intersection of the discontinuity surfaces although it does not exist on each of the surfaces taken separately. It is shown the sliding motion equation can be obtained using hysteresis method, which leads to the concept of equivalent control. The main benefits of sliding mode control lie in the invariance properties and the ability to decouple high dimensional problems into sub-tasks of lower dimensionality. Several design procedures have been shown, which take advantage of the order reduction property to simplify the design. Imperfections in switching devices and delays induce a high-frequency motion called chattering, which prevent an ideal sliding mode from occurring in practice. Some solutions for reducing the chattering are introduced, among which the boundary layer approach is demonstrated to effectively implement a high gain continuous control. Specifically, time delay in relay control causes oscillations of finite frequencies around the sliding manifold. This motivates the main direction of the proposed research which is to study the delay effects on the closed-loop performance when sliding mode control is used.

State feedback sliding mode control has been an intensive research area where many results are available. However in many practical situations, not all the states are measurable. Therefore an output feedback approach presents a more applicable and realistic choice for control design. In the next chapter problems in output feedback sliding mode controller design will be looked at. It will be shown that certain structural requirements are needed to formulate constructive output feedback sliding mode control design approaches. The limitations of the existing approaches due to their restrictions and inefficiency motivates the design of a more efficient SMOFC scheme to be presented in Chapter 6.

Chapter 3

Static Output Feedback Sliding Mode Control

3.1 Introduction

Most of the early theoretical developments in the area of sliding mode control for nominal linear systems with bounded uncertainty assume that all the internal plant states were accessible to the control law. Based on this assumption, many different techniques have been proposed to address the two main design issues in any sliding mode scheme, namely the selection of a sliding surface which generates a stable reduced order motion satisfying the specifications imposed by the designer, and secondly the synthesis of a suitable control law so that the closed-loop system trajectories are forced onto and subsequently remain on the sliding surface. In this situation, no restrictive assumptions need to be imposed on the nominal linear system beyond Kalman controllability. Unfortunately, the assumption that all the states are available to the control law is limiting from a practical viewpoint. In practice, either an observer must be designed to provide an estimate of the internal unmeasured states or the design methods must be modified to make allowance for the fact that only a subset of the state are available for use in the control law. The latter has the advantage in that the static Output Feedback Control (OFC) case incurs less computational/hardware overheads than an observer-based approach. In section 3.2, problems in general static OFC are presented. This serves to introduce the static OFC sliding mode control design problem as given in section 3.3. In section 3.4, solutions of the existence problem are considered based on eigenvalue assignment, eigenstructure assignment and linear matrix inequalities. Section 3.6 shows control law design of four different forms and section 3.7 concludes the chapter.

3.2 The general static output feedback problem

In this section, problems in general static output feedback control are formulated. Some sufficient conditions for pole assignment and eigenstructure assignment are presented.

The static OFC problem is one of the most important open questions in control engineering; see for example the surveys by Blondel and Tsitsiklis, [12], (1997), Bernstein, [10], (1992) and Syrmos, [63], (1997). Simply stated, given a linear time-invariant system, the static OFC problem is to find a static output feedback control so that the closed-loop system has some desirable characteristics, or to determine that such a feedback does not exist. Consider the time-invariant plant described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{3.1}$$

under the influence of a static output feedback injection of the form

$$u(t) = Fy(t) \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$ is the $n-$ dimensional state vector, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$. The closed-loop system is

$$\dot{x} = (A + BFC)x(t) \tag{3.3}$$

The problem of output feedback requires the selection of a constant feedback gain matrix F to achieve various closed-loop properties. It is well known that the system (3.1) is stabilizable via state feedback, where $C = I$, if and only if there exist matrices $P > 0$ and K, of compatible dimensions, such that

$$P(A+BK) + (A+BK)^T P < 0 \tag{3.4}$$

Multiplying (3.4) from both sides by $W = P^{-1}$ then

$$(A+BK)W + W(A+BK)^T < 0 \tag{3.5}$$

Defining $L = KW$ the last equation becomes

$$AW + WA^T + BL + L^T B^T < 0 \tag{3.6}$$

In fact, it is a well-known result (Boyd, et al., [15], 1994) that the LMI (3.6) is feasible in the variables (W, L) if and only if the pair (A, B) is stabilizable, and in this case the state feedback

$$u = LW^{-1}x$$

stabilizes the system (3.1). To find a solution to this problem or to declare the problem unfeasible, if solutions do not exist within a given precision, is a simple task that can be easily carried out with efficient algorithms (Boyd, et al., [15], 1994). For the static output feedback problem, (3.5) translates into a requirement to find a solution of F so that the inequality

$$(A+BFC)W+W(A+BFC)^T<0, \qquad W>0. \tag{3.7}$$

holds. The problem of numerically solving the above matrix inequalities, for W and F, is in fact a very difficult one because it is not convex in general. Syrmos, [63], (1997) concluded that the problem of static OFC is still open despite the availability of many approaches and numerical algorithms. This statement is justified by the fact that no testable necessary and sufficient conditions exist to verify the stability of a given system using static output feedback, and that numerical algorithms cannot be shown to be convergent in general. Moreover, recent results from the theory of computational complexity suggest that numerical algorithms that work well on small-sized problems may fail as the problem size increases.

For the solvability of the static OFC problem Abdallah, et al., [1], (1991) showed that minimum-phase (the finite zeros are stable) and relative degree conditions are necessary and sufficient for a square system (i.e. same number of inputs and outputs) to be strictly positive-real using static output feedback. One of a few systematic approaches that have been developed is the pole placement method. The method seeks to select the gain matrix F in (3.2) such that the poles of the closed-loop system (3.3) are placed at desired locations. In (Herman and Martin, [85], 1977) a necessary and sufficient condition for generic pole assignability with a complex gain matrix F was established as

$$mp>n$$

However, simple counter-examples show that this is only necessary for the case of real F Willems and Hesselink, [142], (1978). Davison and Wang, [23], (1975) and Kimura, [93], (1977) showed that under the conditions $(A,\ B,\ C)$ is minimal with B and C of full rank, then $\min(n,\ m+p-1)$ poles are assignable generically (i.e. for almost all A, B and C). This translates into the sufficient condition for generic pole assignability that

$$m+p\geq n+1 \tag{3.8}$$

In deriving the condition (3.8) Davison and Wang, [23], (1975) provide an explicit formula for construction of F in terms of various matrices constructed from $(A,\ B,\ C)$ and the desired poles. Kimura [93], (1977) used a different approach which relates closely to the eigenstructure assignment techniques. The eigenstructure assignment method for output feedback has been mainly developed based on some features preserved from the state feedback case. While the pole-placement problem for multivariable systems maybe complicated, Moore, [111], (1976) firstly showed that for state feedback, the problem of assigning both eigenvalues and eigenvectors has a straightforward solution. Given a symmetric set of desired closed-loop poles μ_i, $i=1,\dots q$, vectors v_i and u_i are found such that

$$\begin{bmatrix} \mu_i I-A & B \end{bmatrix}\begin{bmatrix} v_i \\ u_i \end{bmatrix}=0 \tag{3.9}$$

Then a state feedback gain K defined by

$$K\begin{bmatrix} v_1 & \cdots & v_q \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_q \end{bmatrix} \tag{3.10}$$

results in the closed-loop structure

$$[\mu_i I - (A - BK)]v_i = 0, \quad [i = 1, \ldots, q] \tag{3.11}$$

so that the v_i are assigned as the closed-loop eigenvectors for eigenvalues μ_i.

There is some freedom in the choice of the v_i, but for a real K to exist they must satisfy

- $v_i \in (\mu_i I - A)^{-1}\mathscr{R}[B]$.
- $v_i = v_j^*$ when $\mu_i = mu_j^*$, (where "$*$" denotes complex conjugation).
- v_i is a linearly independent set.

The integer q may be taken equal to n, but any uncontrollable poles must be included in mu_i, with the associated v_i satisfying $w_i^T v_i \neq 0$, where w_i is the left eigenvector associated with μ_i. Note that (3.9) maybe written as the generalized Lyapunov equation

$$VJ - AV = -BU \tag{3.12}$$

where

$$V = [v_1 \ \ldots \ v_q], \quad U = [u_1 \ \ldots \ u_q], \quad J = \text{diag}(\mu_i)$$

Then (3.10) reduces to

$$KV = U$$

In the case of output feedback, Bengtsson and Lindahl [7], (1974) assumes that a state feedback K which places both eigenvalues and eigenvectors has been selected for (3.10) by some procedure. Then, a method is given to find an output feedback gain F that preserves some of the poles of $(A - BK)$ in (3.11). Although eigenvector assignment was not specifically addressed, the technique involves in fact preserving the eigenvectors v_i associated with the modes μ_i, $i = 1, \ldots, q$. Indeed, although

$$FC = -K$$

may have no solution F, the reduced equation

$$FCV = -KV$$

may have a solution, so that (3.11) becomes

$$[\mu_i I - (A - BFC)]v_i = 0, \quad [i = 1, \ldots, q]$$

In (Srinathkumar, [133], 1978), the technique of (Moore, [111], 1976) was extended to output feedback, essentially by replacing (3.10) with

$$FCV = U$$

From that work, it is clear that

$$\max(m, p)$$

poles are assignable by this method. The algorithm given assigns $p-1$ poles, and an additional procedure was given to assign a total of

$$\min(n, m+p-1)$$

poles generically. The limitation of the design approaches is that although a given number of poles is generically assignable by the above approaches, nothing is known of the remaining closed-loop poles, which may be unstable (Syrmos, [63], 1997).

It is clear that given a controllable-observable system $(A,\ B,\ C)$, then for "almost all" $(B,\ C)$ pairs, $\min(n,\ m+p-1)$ poles can be assigned arbitrarily close to the specified values by using output feedback. In solving the problem of simultaneous assignment of eigenvalues and eigenvectors using output feedback, it is required that the eigenvalues of the closed-loop system are distinct and different from any eigenvalues of the open-loop system. It is clear that the problem of static output feedback is still open. No organized design approach exists except for generic pole assignment which provides algorithms for design. Unfortunately, the generic pole assignment problem is too restrictive and the decision methods are computationally inefficient. The following sections will present the static OFC problem as encountered in sliding mode control design.

3.3 The sliding mode static output feedback control problem

Consider a class of uncertain dynamic systems modeled by the following equations

$$\dot{x}(t) = Ax(t) + B\big(u(t) + f(t,x,u)\big), \quad y(t) = Cx(t) \tag{3.13}$$

where $x(t) \in \mathbb{R}^n$ is the $n-$ dimensional state vector, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $f(t,x,u)$ is some unknown bounded function and constant matrices A, B and C are of appropriate dimensions. The vector $f(t,x,u)$ represents the lumped nonlinearities and/or uncertainties of the system. Initially the intention will be to explore when the static output feedback sliding mode control can be employed. A control will be sought which induces an ideal sliding motion on the surface

$$\mathcal{S} = \{x \in \mathbb{R}^n \ : \ s(t) = FCx(t) = 0\} \tag{3.14}$$

where F is an $m \times p$ real parameter matrix to be designed (White, [141], 1990; Edwards [28], 2001). To facilitate the following analysis, assuming $f(t,x,u) = 0$, El-Ghezawi, et al, [34], (1983), Zak and Hui, [148], (1993) showed that the dynamics of the system (3.13) and (3.14) is determined by the zeros of the system triple (A, B, FC). This can be seen as follows. An equivalent control is derived by setting

$$\dot{s}(t) = 0$$

with respect to (3.13) for all $t \geq t_0$, where t_0 is the time at which switching surface is reached. Therefore

$$\dot{s}(t) = FCAx(t) + FCBu(t) = 0 \tag{3.15}$$

This results in the equivalent control law

$$u_{eq}(t) = -(FCB)^{-1}FCAx(t) \tag{3.16}$$

which exists uniquely if FCB is nonsingular. This implies that F is chosen to have full row rank and

$$\mathscr{N}[F] \cap \mathscr{R}[CB] = \varnothing$$

Assuming rank $(CB) = m$ since F is a design parameter, the restriction that FCB must be nonsingular can be satisfied. Substituting (3.16) into (3.13) the behaviour of the system in the sliding mode is governed by

$$\begin{cases} \dot{x}(t) = [I_n - B(FCB)^{-1}FC]Ax(t) = A_{eq}x(t) \\ FCx(t) = 0 \end{cases} \tag{3.17}$$

The eigenvalues of A_{eq} whose corresponding eigenvectors v_i satisfy

$$FCv_i = 0$$

are system zeros (El-Ghezawi, [34], 1983). The dynamics of the system (3.13) and (3.14) is determined by the zeros of the system triple $(A,\ B,\ FC)$ denoted by $Z_{A,B,FC}$. Switching surface design can be viewed as choosing an output matrix FC so that the system (3.13) and (3.14) has a desired set of system zero locations which in turn govern the dynamics of the system when sliding along $FCx(t) = 0$.

If $p = m$, then F is a nonsingular transformation of the switching surface $Cx(t) = 0$; hence the switching surface dynamics $FCx(t) = 0$ is invariant with respect to F (DeCarlo, et al., [24], 1988). This means output regulation requires that all the system zeros be located in the open left half complex plane. On the other hand, if $p > m$ then it can be seen that the switching surface dynamics depends on $\mathscr{N}[F]$. For singular cases when $det(CB) = 0$, then the equivalent control is either not unique or does not exist (Utkin, [139], 1977). When the equivalent control does not exist sliding modes cannot appear, and the state leaves the intersection of the discontinuity surfaces.

It has been shown in this section that the dynamics of the static output sliding mode control is governed by the zeros of the system triple $(A,\ B,\ FC)$. Conditions for a unique equivalent control to exist are presented. When there are an equal number of inputs and outputs, i.e. $m = p$, the dynamics on the sliding surface (3.14) are shown to be independent of the choice of output gain matrix F. The next section reviews some conditions which must hold if a sliding surface dependent upon the plant outputs alone is to be developed. It will be shown that the system zeros are the eigenvalues of a reduced order matrix which determines the dynamics in the sliding mode.

3.4 Sliding surface design techniques

This section describes some procedures for sliding mode output feedback control (SMOFC) for a class of multivariable linear time-invariant systems. The equivalent control method of (Utkin, [136], 1992) in the output feedback mode yields a reduced-order system exhibiting output feedback equivalent dynamics. Using the Kimura-Davison sufficient conditions mentioned before for pole-placement, some sufficient conditions are derived to assign the eigenvalues of the reduced-order system. The observability and controllability of the reduced system is discussed. Switching surface design algorithms based on eigenstructure assignment and linear matrix inequalities techniques are further discussed.

3.4.1 Pole placement

In (El-Khazali and Decarlo, [37], 1995) a sliding surface was chosen as in (3.14). Assume system (3.13) exhibits the following regular form (Utkin, [136], 1992)

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ B_2\end{bmatrix}u\\ y(t) &= \begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\end{aligned} \tag{3.18}$$

where $B_2 \in \mathbb{R}^{m\times m}$ is nonsingular, $x_1(t) \in \mathbb{R}^{n-m}$, $x_2(t) \in \mathbb{R}^m$. The dynamics of the system (3.13) when on the sliding surface (3.14) is

$$\begin{aligned}\dot{x}_1(t) &= (A_{11} - A_{12}KC_1)x_1(t)\\ y(t) &= (I - C_2K)C_1x_1(t)\end{aligned} \tag{3.19}$$

where $K = (FC_2)^{-1}F$. Switching surface design in (Zak and Hui, [148], 1993) and (El-Ghezawi, et al., [34], 1982) then reduces to choosing the desired poles $\{\lambda_1, \cdots, \lambda_{n-m}\}$ of the reduced order system (3.19), which are known to correspond to the system zeros of the plant $(A,\ B,\ FC)$ in (3.13) and (3.14). Prior to satisfying the 'Kimura-Davison' conditions (3.8), the sliding function gain F

in (El-Khazali and Decarlo, [37], 1995) was chosen with the assumption

$$KC_2 = I_m \tag{3.20}$$

where $C_2 \in \mathbb{R}^{p\times m}$ in (3.19) is full rank for assignability of the closed-loop poles. Under this assumption there exists a matrix $\Gamma \in \mathbb{R}^{m\times(p-m)}$ so that the poles of (3.19) as given by

$$\lambda(\hat{A}_{11} - A_{12}\Gamma\hat{C}_1) \tag{3.21}$$

can be assigned arbitrarily, where $\hat{A}_{11} = A_{11} - A_{12}C_2^{-L}C_1$, $\hat{C}_1 = M^T C_1$, where $C_2^{-L} \in \mathbb{R}^{m\times p}$ is a left pseudo-inverse of C_2 with $M \in \mathbb{R}^{p\times(p-m)}$ a full rank left annihilator of C_2. The reduced order control gain in (3.19) is then constructed as

$$K = C_2^{-L} + \Gamma M^T \tag{3.22}$$

The structural constraint in (3.20) and (3.22) limits the available feedback. This limitation was realised by El-Khazali and Decarlo, [37], (1995) who explored the freedom remaining to assign eigenvalues to the reduced order system (3.19). However the design freedom is based on assumptions that $(A,\ B,\ C)$ is both observable and controllable, $(\hat{C}_1,\ A_{11})$ observable and A_{12} full rank.

The approach by Edwards and Spurgeon [29], (1995) has much more in common with El-Khazali and Decarlo, [37], (1995) in that both require transformation to regular form. The approach by Edwards and Spurgeon, [29], (1995) relies on establishing a classical output feedback pole placement problem for a given subsystem obtained from the original state space matrices. If $\text{rank}(CB) = m$, it can be shown that there exists a coordinate system in which the triple (A, B, C) has the structure

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & T \end{bmatrix} \tag{3.23}$$

where $T \in \mathbb{R}^{p\times p}$ is orthogonal. The difference of this transformation (3.23) from (3.18) in (El-Khazali and Decarlo, [37], 1995) is that the sub-block A_{11} in (Edwards and Spurgeon, [29], 1995) has the following particular structure

$$A_{11} = \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ 0 & \tilde{A}_{11} \end{bmatrix} \tag{3.24}$$

where the eigenvalues of $A_{11}^0 \in \mathbb{R}^{r\times r}$ are the invariant zeros of the system. The reduced order dynamics of the sliding motion is governed by

$$\lambda(A_{11} - A_{12}KC_1) = \lambda(A_{11}^0) \cup \lambda(\tilde{A}_{11} - \tilde{B}_1 K\tilde{C}_1) \tag{3.25}$$

where

$$\tilde{C}_1 := \begin{bmatrix} 0_{(p-m)\times(n-p-r)} & I_{p-m} \end{bmatrix}$$

$\tilde{B}_1$ represents the last $n-p-r$ rows of the matrix A_{12} in (3.23). If the invariant zeros all lie in the open left half plane and the triple $(A,\ B,\ C)$ is controllable and observable, i.e. $(\tilde{A}_{11},\ \tilde{B}_1,\ \tilde{C}_1)$ are also controllable and observable, given Kimura-Davison conditions are satisfied for $(\tilde{A}_{11},\ \tilde{B}_1,\ \tilde{C}_1)$ then the reduced order dynamics (3.25) is output feedback pole assignable.

Remark 3.1. Partition the output matrix in (3.18) so that

$$\begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & T_1|T_2 \end{bmatrix}$$

where

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} = T$$

and T is defined in (3.23). Then it can be shown $(\hat{C}_1,\ \hat{A}_{11})$ in (3.21), which is required to be completely observable by El-Khazali and Decarlo, [37], (1995), is identical to $(\tilde{C}_1,\ \tilde{A}_{11})$ in (3.25) (Edwards and Spurgeon, [29], 1995) which is not completely observable due to the stable invariant zeros. Therefore the result by El-Khazali and Decarlo, [37], (1995) does not apply to systems with invariant zeros while result by Edwards and Spurgeon, [29], (1995) does. The attractive feature of the approach by Edwards and Spurgeon, [29], (1995) is that a standard classical output feedback problem appears. Whilst solving this is still very much an on-going area of research (Syrmos, et al., [63], 1997), many existing software/design packages which address this problem can be employed directly.

To conclude, in solving the existence problem for a stable sliding surface design, pole assignment method is used prior to satisfying Kimura-Davison conditions to achieve a prescribed spectrum for a restricted class of output feedback matrices. The designs discussed all assume the system can be transformed into regular form. Even if the system meets the desired structure, some design techniques will only terminate satisfactorily for a specific class of switching surface.

3.4.2 Eigenstrucuture assignment

For multi-input systems, feedback control laws can yield identical eigenvalues while yielding radically different eigenvectors (Srinathkumar, [133], 1978). The eigenvectors determine the influence of each eigenvalue on each state variable response during transient period. Eigenstructure assignment, which assigns both eigenvalues and eigenvectors simultaneously, has been used for sliding surface design, which does not require to first determine the reduced order output feedback matrix K in (3.25).

In (El-Khazali and Decarlo, [37], 1995) an eigenstructure assignment approach is given involving assigning the zeros of the equivalent system (3.17) or the reduced order subsystem

$$\lambda_i v_i = (A_{11} - A_{12} K C_1) v_i, \qquad \text{for} \quad i = 1, \ldots, n-m. \tag{3.26}$$

where v_i denotes the $n-m$ eigenvectors cooresponding to the $n-m$ zeros of the triple (A,B,FC) in (3.17), associated with λ_i. Once all the eigenvectors $\mathscr{V}_d$ are chosen properly such that $\mathscr{N}[C] \subset \mathscr{V}_d$ then there always exist a full row rank matrix F such that

$$FC\mathscr{V}_d = [0]$$

Zak and Hui, [148], (1993) proposed another eigenstructure design of the system (3.13) without the need to attain the regular form structure. Assumptions made on the system (3.13) are as follows:

- There exists a known nonnegative scalar function $\rho(\cdot) : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$ such that $\|f(t,x,u)\| \leq \rho(t,y(t))$.
- The pair $(A,\ B)$ is controllable and the pair $(A,\ C)$ is observable with the matrices B and C being full rank.
- $p \geq m$, that is, the number of output channels is greater than or equal to the number of inputs, and rank $(CB) = m$.

Necessary and sufficient conditions were given such that designing a suitable sliding matrix is equivalent to finding a matrix $W \in \mathbb{R}^{n\times(n-m)}$ of full rank such that:

1. $\mathscr{R}[W] \cap \mathscr{R}[B] = 0$.
2. $\mathscr{R}[AW - WJ] \subset \mathscr{R}[B]$.
3. $\text{rank}(CW) = p - m$.

where matrix $J = \text{diag}\{\lambda_1, \ldots, \lambda_{n-m}\}$ represents the $n-m$ distinct eigenvalues of (3.17).

Lemma 3.2. *(Zak and Hui, [148], 1993) Suppose the nominal system is both controllable and observable, then there exists a matrix $S \in \mathbb{R}^{m\times n}$, where $S = FC$ so that*

(i) The system $\dot{x} = Ax + Bu$ restricted to the surface $Sx = 0$ has $n-m$ prescribed distinct, nonzero, real eigenvalues $\lambda_1, \ldots, \lambda_{n-m}$,

(ii) SB is nonsingular,

if and only if there exist full rank matrices $W \in \mathbb{R}^{n\times(n-m)}$, $W^g \in \mathbb{R}^{(n-m)\times n}$ so that

(iii) $W^gW = I_{n-m}$, $\quad W^gB = 0$ and $\quad W^gAW = diag(\lambda_1, \ldots, \lambda_{n-m})$.

Since J is full rank then there exists a full rank matrix W so that

$$[I_n - B(SB)^{-1}S]AW = WJ \tag{3.27}$$

Thus,

$$SWJ = S[I_n - B(SB)^{-1}S]AW = 0 \quad \text{i.e.} \quad SW = 0$$

since J is nonsingular. Given condition (ii), then

$$\mathscr{R}[B] \cap \mathscr{R}[W] = 0$$

Since B, W has full rank, the matrix $[B \quad W]$ is invertible with its inverse $\text{col}\{B^g,\ W^g\}$ where W^g and B^g denote the generalized inverses of W and B, respectively with the properties

$$B^gB = I_m, \quad B^gW = 0, \quad W^gB = 0, \quad \text{and} \quad W^gW = I_{n-m}$$

Premultiply (3.27) by W^g to obtain $W^gAW = J$ condition (iii) is proved. Since

$$W^gB = 0 \quad \text{and} \quad \mathscr{R}[AW - WJ] \subset \mathscr{R}[B]$$

it follows that

$$W^gAW = J$$

The above discussion gives precise conditions for the existence of a stable switching surface. It is reasonable to assume that if a switching surface does not exist for the state feedback case, then solution to the existence problem with output feedback would be equally impossible. Therefore Zak and Hui, [148], (1993) assume that a state switching surface can be designed on which the nominal system has the desired eigenvalues. It is shown C, S exist such that $FC = S$ is solvable if and only if

$$\text{rank}(CW) = p - m$$

For $S = FC$ to be full rank, F must also be full rank m since

$$F(CW) = SW = 0$$

therefore

$$\text{rank}(CW) \leq \dim \mathscr{N}[F] = p - m$$

On the other hand, by Sylvester's inequality (F.R. Gantmacher, [63], 1959)

$$\text{rank}(CW) \geq \text{rank}\, C + \text{rank}\, W - n = p + n - m - n = p - m \tag{3.28}$$

Thus, $\text{rank}(CW) = p - m$.

The approach by Zak and Hui, [148], (1993) introduced a sliding surface design without the need to acquire regular forms. However, the problem of computing the matrix W, which is in itself nontrivial, is not addressed. Another problem of their design is that the uncertainties in the system must be bounded by a known function of outputs which excludes some possible uncertainties in the A matrix. K. Shyu, et al., [126], (2000) extended the sliding surface design technique developed by Zak and Hui, [148], (1993) to systems with mismatched uncertainties appearing in block A. Consider the following uncertain system

$$\dot{x}(t) = (A+\Delta A)x(t) + B(u(t) + f(t,x,u)) \tag{3.29}$$

$$y(t) = Cx(t) \tag{3.30}$$

where $\|f(t,x,u)\| \leq k_f + k_m\|x\|$, k_f, k_m are constants. It is assumed that the pair $(A,\ B)$ is controllable, S exists (Zak and Hui, [148], 1993) and the equation $S = FC$ is solvable. System (3.29) and (3.30) is transformed with the transformation matrix $\tilde{M} = \text{col}\{W^g,\ \ S\}$ where $S = B^g$ is selected, into the following form

$$\dot{z}(t) = W^g(A+\Delta A)Wz(t) + W^g(A+\Delta A)Bs(t) \tag{3.31}$$

$$\dot{s}(t) = S(A+\Delta A)Wz(t) + S(A+\Delta A)Bs(t) + u(t) + f(\cdot) \tag{3.32}$$

where $z(t) = W^g x(t)$ and $s(t) = Sx(t)$. Then the stability of the sliding mode dynamics is proven by taking the Lyapunov function, $V = z^T Pz$ corresponding to (3.31) where the uncertainties are bounded by $\|\Delta A\| \leq k_a$ with $k_a < -\lambda_{min}(J)/(\|PW^g\|\|W\|)$, $\lambda_{min}(J)$ denotes the minimum eigenvalue of J in (3.27). Using the equivalent control method, sliding surface can be written

$$s(t) = \dot{s}(t) = 0$$

select

$$P = \text{diag}\{\lambda_{min}(J)/\lambda_1,\ \lambda_{min}(J)/\lambda_2,\ \ldots,\ \lambda_{min}(J)/\lambda_{n-m}\} \tag{3.33}$$

and use the fact $W^g AW = J$, then it follows

$$\begin{aligned}\dot{V} &= z^T(J^T P + PJ)z + 2z^T PW^g \Delta AWz \\ &\leq z^T(J^T P + PJ)z + 2\|\Delta A\|\|z\|^2\|PW^g\|\|W\|\end{aligned} \tag{3.34}$$

It follows from (3.33) that

$$J^T P + PJ = \text{diag}\{2\lambda_{min}(J),\ 2\lambda_{min}(J),\ \ldots,\ 2\lambda_{min}(J)\}$$

Then it is obtained

$$\dot{V} \leq 2\lambda_{min}(J)\|z\|^2 + 2k_a\|z\|^2\|PW^g\|\|W\|$$

So, the dynamics of the sliding mode is stable for the uncertain system (3.30) if

$$k_a < -\lambda_{min}(J)(\|PW^g\|\|W\|)$$

The approach by K. Shyu, et al., [126], (2000) allows mismatched uncertainties to be incorporated in the system matrices and transformation to regular form is not needed, but again the procedure to compute the matrix W is not given.

Eigenstructure assignment for sliding surface design has been proposed. The approach in (Zak and Hui, [148], 1993) does not require the system to adopt regular form and the sliding surface is designed without the need to first construct the control gain matrix. However the uncertainties in the system must be bounded by a known function of outputs which excludes controller design with uncertainties in the state matrix A. Shyu, et al., [126], (2000) showed the results in (Zak and Hui, [148], 1993) can be extended to systems with uncertainties in A. Despite the attractive features of these approaches, no efficient constructive procedure has been developed for controller design using this methodology.

3.5 Linear matrix inequalities for sliding surface design

Linear matrix inequalities (LMIs) have emerged as a powerful formulation and design technique for a variety of linear control problems (Boyd, [15], 1994), (LMIs) (Choi, [19], 2002) and (P.G. Park, et al. [116], 2007). Since solving LMIs is a convex optimization problem, such formulations offer numerically tractable means for tackling problems that lack analytical solution. An LMI can be defined as a constraint of the form:

$$A(x) \triangleq A_0 + x_1 A_1 + \ldots + x_m A_m < 0 \tag{3.35}$$

where $x = [x_1, \ldots, x_m]^T$ is a vector of unknown scalars, A_i are given symmetric matrices. The convexity of LMI 3.35 on x implies that

$$\text{if } A(x) < 0,\ B(y) < 0, \quad \text{then } A(z) < 0,\ \forall z = \theta x + (1-\theta) y,\ 0 < \theta < 1. \tag{3.36}$$

Scherer, et al. [122], (1997) gave an overview of an LMI approach to a multi-objective synthesis of linear output feedback controllers. The design objectives can be a mix of H_∞ performance, H_2 performance, passivity, asymptotic disturbance rejection, time-domain constraints, and constraints on the closed-loop pole location. In addition, these objectives can be specified in different channels of the closed loop system. When all objectives are formulated in terms of a common Lyapunov function, controller design amounts to solving a system of linear matrix inequalities.

In order to find a stable sliding surface under output feedback, (Edwards and Spurgeon, [32], 2003) proposed is to find a symmetric positive definite matrix $\tilde{P}$ for the reduced order system (3.25) so that the following inequality

$$\tilde{P}(\tilde{A}_{11} - \tilde{B}_1 K \tilde{C}_1) + (\tilde{A}_{11} - \tilde{B}_1 K \tilde{C}_1)^T \tilde{P} < 0 \tag{3.37}$$

holds. The inequality is of the same structure as (3.7); it is not convex in terms of $\tilde{P}_1$ and K. Many different LMI approaches have been investigated for the solution of (3.37) (Cao, [16], 1998), (El Ghaoui, [66], 1997). Benton, [9], (1998) advocate synthesizing a symmetric $\tilde{P}$ such that the following LMIs, which are convex in $\tilde{P}$

$$\tilde{P}(\tilde{A}_{11}+\tilde{B}_1K_{sf})+(\tilde{A}_{11}+\tilde{B}_1K_{sf})\tilde{P} \quad < \quad 0, \tag{3.38}$$

$$\tilde{P}_1\tilde{A}_{11}+\tilde{A}_{11}^T\tilde{P}-\sigma\tilde{C}_1^T\tilde{C}_1 \quad < \quad 0 \tag{3.39}$$

hold for some $\sigma > 0$. In inequality (3.38), the gain $K_{sf} := -\tilde{B}_1^T P_{1,\,are} = -K\tilde{C}_1$ where $P_{1,\,are}$ is the stabilising solution to the algebraic Ricatti equation:

$$P_{1,\,are}\tilde{A}_{11}+\tilde{A}_{11}^TP_{1,\,are}-P_{1,\,are}\tilde{B}_1\tilde{B}_1^TP_{1,\,are}=-Q \tag{3.40}$$

where $Q=\varepsilon I$ and $\varepsilon > 0$ is a small design scalar. The given constrained LMIs are difficult to solve and the solution method of (Benton, [9], 1998) is conservative in the sense that infeasibility of the inequalities (3.38), (3.39) with respect to $\tilde{P}$ and σ does not imply (3.37) is infeasible.

Choi, [19], (2002) considered output feedback control of the following system with mismatched uncertainties, where constrained LMIs are developed in regular form.

$$\begin{aligned} \dot{x}(t) &= [A+\Delta A(t)]x(t)+[B+\Delta B(t)]u(t)+f(x,t) \\ y(t) &= Cx(t) \end{aligned} \tag{3.41}$$

where $\|\Delta A(t)\| \leq a$ with a a known nonnegative constant, $\Delta B(t) = BE(t)$ with $\|E(t)\| \leq \psi < 1$. Defining the sliding surface as (3.14), let $\phi \in \mathbb{R}^{n\times(n-m)}$ be any full rank matrix such that

$$B^T\phi = 0 \quad \text{and} \quad \phi^T\phi = I, \quad \text{then} \quad \phi\phi^T+B(B^TB)^{-1}B^T = I \tag{3.42}$$

By using the equality (3.42) the uncertain system (3.41) is rewritten as

$$\begin{aligned} \dot{x}(t) &= [A+\phi\phi^T\Delta A(t)]x(t)+B[u(t)+(B^TB)^{-1}B^T\Delta A(t)x(t)+E(t)u(t)+f(x,t)] \\ y(t) &= Cx(t) \end{aligned} \tag{3.43}$$

The regular form approach is used to partition the original states x in (3.41) into x_1, x_2, where

$$\dot{x}_1 = (\phi^TA\phi-\phi^TABFC\phi)x_1$$

governs the reduced order dynamics of the sliding mode.

When $\phi^T\Delta A(t) = 0$, the system includes only matched uncertainties. The problem is to find a positive-definite matrix X_0 such that

$$(\phi^TA\phi-\phi^TABK)X_0+X_0(\phi^TA\phi-\phi^TABK)^T < 0$$

where $K = FC\phi$. It has been shown that the sliding surface $FCx = 0$ is stable if the following constrained LMI has a solution pair (X, F)

$$\phi^T(AX + XA^T)\phi < 0, \qquad X > 0, \;\; B^T = GCX. \tag{3.44}$$

When $\phi^T \Delta A(t) \neq 0$, i.e. mismatched state space model uncertainty, the following constrained LMI guarantees a stable sliding mode

$$\begin{bmatrix} \phi & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} AX + XA^T + aI & aX \\ aX & -aI \end{bmatrix} \begin{bmatrix} \phi & 0 \\ 0 & I \end{bmatrix} < 0 \\ X > 0, \;\; B^T = FCX. \tag{3.45}$$

Choi, [19], (2002) gave an algorithm to construct the solution pair (X, F), however the solvability of the constrained LMIs (3.44), (3.45) is difficult.

Another approach which neither requires any coordinate transformation nor involves solving a OFC problem is introduced in (Xiang, [145], 2006). Solutions are obtained through the Iterative Linear Matrix Inequality method (ILMI). Consider the uncertain system (3.29) and (3.30), where $\Delta A = DRE$, R is unknown but bounded by $\|R\| \leq 1$ and D, E are known matrices of appropriate dimensions. It is assumed that the matrix pair $(A,\ B)$ is controllable and $rank(CB) = m$. Defining the sliding surface as (3.14) then system (3.29) and (3.30) can be transformed by

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} B^{\perp T} \\ (B^T B)^{-1} B^T \end{bmatrix} x \tag{3.46}$$

where $B^{\perp}$ denotes the orthogonal complement matrix of full column rank of matrix B, i.e. $B^{\perp T} B^{\perp} = I$ and $MM^{\perp} = 0$, into

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} B^{\perp T}(A + \Delta A)B^{\perp} & B^{\perp T}(A + \Delta A)B \\ (B^T B)^{-1} B^T (A + \Delta A) B^{\perp} & (B^T B)^{-1} B^T (A + \Delta A) B \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} (u + f) \tag{3.47}$$

The sliding surface in the new coordinates is

$$s = FCB^{\perp} z_1 + FCBz_2 \tag{3.48}$$

Thus, on the sliding surface the dynamic of the system becomes

$$\dot{z}_1 = B^{\perp T}(A + \Delta A)\Theta z_1 \tag{3.49}$$

where $\Theta = B^{\perp} - B(FCB)^{-1} FCB^{\perp}$. It was shown that there exists a sliding surface matrix F such that the reduced-order system (3.49) is asymptotically stable for all admissible R if and only if there exist symmetric matrices Q_1 and Q_2, and a positive scalar μ such that the following matrix inequalities hold.

$$
\begin{aligned}
&Q = B^{\perp} Q_1 B^{\perp T} + C^T Q_2 C > 0 \\
&QA + A^T Q + \mu^{-1} QDD^T Q + \mu E^T E - QBB^T Q + \alpha I < 0
\end{aligned}
\tag{3.50}
$$

Although (3.50) consists of two simple inequalities, it is not easy to solve because of the negative sign term in the inequality, (3.50) can not be simplified to LMIs. The problem has been dealt with using the ILMI approach. Similar approach can be found in (Cao et al., [16], 1998). Even though only the original system was involved in the design, the feasibility of the ILMI algorithm depends on the selection of initial values for given design parameters and the LMIs in the synthesis methodology are relatively complex.

Eigenvalue assignment methods have been considered for SMOFC in (El-Khazali and Decarlo, [37], 1995), (Edwards and Spurgeon, [29], 1995) which only assign the prescribed spectrum of the sliding mode dynamics to a restricted class of systems. An eigenstructure assignment method is considered in (El-Khazali and Decarlo, [37], 1995), (Zak and Hui, [148], 1993) and (Shyu, et al., [126], 2000). The method yields a sliding surface design without the need to first construct the control gain matrix. However computational efficiency for the solution is low and no systematic procedure is available. Edwards and Spurgeon, [32], (2003), Choi, [19], (2002) and Xiang, [145], (2006) considered solving SMOFC problem in the framework of LMIs and ILMIs. However the constraint LMIs are difficult to solve. As shown in (Edwards and Spurgeon, [32], 2003) all the above methods for the existence problem are, in fact, equivalent to a OFC problem which is consequently more difficult to solve and still represents an open problem.

3.6 Control law development

Output feedback is much simpler to implement than either full state feedback or estimated state feedback; however, satisfying the reaching conditions is not a straight forward problem in the domain of output feedback in sliding mode control. The problem can be stated as follows; once a stable sliding surface has been designed, the next step is to determine a switched control based on output information only that drives the system's output trajectories from (3.13) onto the sliding surface (3.14). As mentioned before the equivalent control $u_{eq} = -(FCB)^{-1}FCAx$ in (3.16) alone would cause the system's output trajectories (3.13) where $f(\cdot) = 0$ to slide onto a hyperplane parallel to the switching surface. However in the context of output feedback it is shown to be impossible to design u_{eq} depending on the output variables (R.EL-Khazali and R. DeCarlo, [37], 1995). If there exists a real matrix $G \in \mathbb{R}^{p\times p}$ such that $CA = GC$ then one can write

$$u_{eq} = -(FCB)^{-1}FGCx = -(FCB)^{-1}FGy$$

and the equivalent control in this case is an explicit output feedback control (Zak and Hui, [148], 1993). A necessary and sufficient condition for the existence of G was given in (R.EL-Khazali and

R. DeCarlo, [37], 1995), in which it was shown there exists $G \in \mathbb{R}^{p\times p}$ such that

$$CA = GC$$

if and only if

$$AV_0 \subset V_0 \tag{3.51}$$

where $V_0 = \mathcal{N}[C]$. Let $V_0 = [v_1, \ldots, v_{n-p}]$ be a matrix whose columns are bases for V_0, then

$$CA[C^T \;\; V_0] = [CAC^T \;\; 0]$$

Thus (3.51) is solvable for $G = CAC^T(CC^T)^{-1}$. Suppose $GC = CA$ then

$$GCV_0 = CAV_0 = [0]$$

This implies that $\mathcal{R}[AV_0] \subset V_0$ or $AV_0 \subset V_0$. So the satisfaction of the condition (3.51) implies that V_0 is A-invariant, i.e. for G to exist such that $GC = CA$, the pair $(C,\ A)$ must be unobservable. Since $(C,\ A)$ is observable by hypothesis, no G exists such that $GC = CA$, i.e. the equivalent control can not depend explicitly on the output variables. In deriving a suitable control law for system (3.14) with disturbance $f(\cdot)$ non-zero, a common control structure is

$$u(t) = u_l(t) + u_n(t) \tag{3.52}$$

where u_l is a linear control, u_n is a discontinuous switched component with respect to the sliding function. The coordinate change $z \to T_1 x$ is convenient for system (3.23), where

$$T_1 = \begin{bmatrix} I_{n-m} & 0 \\ KC_1 & I_m \end{bmatrix}$$

Then the system triple $(\bar{A},\ \bar{B}, F\bar{C})$ has the property that

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad F\bar{C} = \begin{bmatrix} 0 & I_m \end{bmatrix} \tag{3.53}$$

where $\bar{A}_{11} = A_{11} - A_{12}KC_1$, $\bar{A}_{21} = \begin{bmatrix} \bar{A}_{211} & \bar{A}_{212} \end{bmatrix}$ and $\bar{C} = [0 \;\; \bar{T}]$. Several different control laws from the literature will now be analyzed which determine the degrees of freedom in selecting the controller to satisfy the reaching condition

$$\dot{s}^T s < 0$$

To assure the attractiveness of the sliding surface, it is sufficient that the following holds

$$s^T \dot{s} = s^T FC\dot{x} = s^T[FCAx + FCB(u + f(t,x,u))] = s^T[FCAx + \bar{u} + (FCB)f(t,x,u)] < 0 \tag{3.54}$$

where $\bar{u} = FCBu$, for system (3.13), (3.14). To synthesize an output feedback control in the presence of the state dependant term $FCAx$ in (3.54), Zak and Hui, [148], (1993) proposed a control law based on the structural constraint

$$FCA = MC \quad \text{so that} \quad FCAx = MCx = My \tag{3.55}$$

where $M \in \mathbb{R}^{m\times p}$ is a design matrix. Then (3.54) will take the form

$$s^T \dot{s} = s^T[My + \bar{u} + (FCB)f(t,x,u)] < 0$$

Let $(M)_i$ and $(FCB)_i$ denote the $i-$th rows of the matrices M and FCB, respectively. If the entries $\bar{u}_i^+$ and $\bar{u}_i^-$ are chosen to satisfy

$$\begin{aligned} \bar{u}_i^+ &< -(M)_i y - (FCB)_i f(t,x,u) \quad \text{if } s_i > 0, \\ \bar{u}_i^- &< -(M)_i y - (FCB)_i f(t,x,u) \quad \text{if } s_i < 0, \end{aligned} \tag{3.56}$$

then the sufficient condition for the existence and reachability of the sliding mode are satisfied. The existence of M can be seen as follows. Let the row space of FC be spanned by a set of m left eigenvectors of A labeled $v_1, \ldots, v_m$. Assume there exists a nonsingular matrix N such that

$$NFCA = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} A = \Lambda \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = \Lambda NFC \tag{3.57}$$

where $\Lambda = diag\{\lambda_1, \ldots, \lambda_m\}$. Then

$$FCA = N^{-1}\Lambda NFC = N^{-1}\Lambda NFC$$

where $M = N^{-1}\Lambda NF$.

Since no solution procedure was given in (Zak and Hui, [148], 1993) to compute the matrix N the structural constraint (3.55) is therefore hard to satisfy. In addition, the uncertainties $f(\cdot)$ in system (3.13) must be bounded by a known function of outputs which excludes some possible uncertainties in the A matrix. If (3.55) is not satisfied then only local stability can be achieved.

Kwan, [102], (1996) proposed a method to estimate the unmeasured states so that the structural constraint (3.55) can be released. However the method is only applicable for Single-Input-Single-Output (SISO) systems and the resulting controller becomes a dynamical one. This can be seen as follows. Differentiating the sliding variable (3.14) with respect to the system in regular form similar to (3.53), then

$$\dot{s} = FC\dot{x} = Mz_1 + Ns + B_2 u \tag{3.58}$$

where M is a matrix of appropriate dimension, N, B_2 are scalars, depends on the unmeasured states in z_1. To avoid the measurement of the states, Zak and Hui, [148], (1993) proposed the structure constraint (3.55) . The dynamic controller in (Kwan, [102], 1996) is a function of

$$\dot{w}(t) = -\lambda_m w(t) + \gamma |s(t)|$$

$u(t,x) \rightarrow u(t,y,w)$, where λ_m is the smallest eigenvalue of a reduced order sliding mode dynamics, γ is a positive number, which bounds $z_1(t)$ from any initial condition onwards. Therefore the measurement of z_1 is avoided.

Since the work originally proposed in (Zak and Hui, [148], 1993) is to choose a sliding surface $s(t) = Fy(t) = 0$ such that the poles of the sliding dynamics are assignable, the fact that the dynamic controller given in (Kwan, [102], 1996) is only applicable for SISO systems means there is no degree of freedom for choosing these poles through F. Indeed, since y (and also $s(t)$) is a scalar, F can not be but a trivial 1×1 matrix and, essentially, $s(t) = y(t) = Cx(t)$. This implies that the sliding mode dynamics is just the zero-dynamics of the system. Therefore, the poles of the sliding mode dynamics can not be changed and are simply the zeros of the system. In (Kwan, [103], 2001) the result for SISO problems is generalized to Multi-Input-Multi-Output (MIMO) systems which allows pole assignment and guarantees global stability.

In (Heck and Ferri, [83], 1989), (El-Khazali and Decarlo, [36], 1992), (Bag, et al., [2], 1997) another form of the linear control u_l is suggested as

$$u_l(t) = -(FCB)^{-1} FCA\bar{N}y(t) \tag{3.59}$$

for system (3.13) where $\bar{N} \in \mathbb{R}^{n \times p}$, instead of the u_{eq} in (3.16). It is argued that to induce a sliding motion, $\bar{N}$ should be chosen so that the inequality

$$x^T(t)C^T F^T FCA(I - \bar{N}C)x(t) \leq 0 \tag{3.60}$$

Heck, et al., [84], (1995) showed the solution for (3.60) can be found by applying two numerical methods, which are the Cutting Plane Method (Kelly, [91], 1960) and Interior Point Methods (Boyd, [14], 1993). The design is formulated in a way that the existing convex optimization techniques used in the solution of LMIs can be used to solve (3.60). The advantage of the design is that a low gain controller can be produced with which the sliding surface is reached infinite time.

In (Edwards and Spurgeon, [31], 2000) it is argued that a necessary and sufficient condition for (3.60) to be satisfied is that $\bar{A}_{211} = 0$ when system (3.13) is put in regular form (3.53). In this case an appropriate choice of the controller design parameter is $\bar{N}^T = [0 \;\; T]$, where T is from (3.23). Generally, $\bar{A}_{211}$ will depend on the choice of the sliding surface, and so a further constraint on the choice of F is imposed. As a result of this constraint on the system structure it was shown in (Edwards and Spurgeon, [31], 2000) that if the triple (A, B, C) has transmission zeros it is not possible to select the design parameters in control law (3.59) to make the switching surface globally attractive. Therefore the control law (3.59) contradicts the usual design approach in which the switching function is chosen independently of the controller structure.

Based on the existence of the sliding mode, Edwards and Spurgeon, [29], (1995) proposed the linear part of the control as

$$u_l = -\gamma F y, \qquad \text{where} \quad \gamma > 0 \tag{3.61}$$

For a large enough scalar γ it can be shown that a sliding motion is obtained in finite time from any initial condition. However the reachability condition $\dot{s}^T s < 0$, where $s(t) = Fy(t)$, only holds in a compact domain around the origin. Outside this domain the controller behaves as a variable structure controller with the property that it forces the state trajectories into the invariant domain (sometimes refered to as the 'sliding patch') in finite time. Inside this domain the reachability condition $\dot{s}^T s < 0$ holds and so sliding occurs in finite time.

A similar control structure was proposed in (Edwards, et al., [28], 2001) as follows

$$u_l = Gy \tag{3.62}$$

where $G = [G_1 \quad G_2]\bar{T}^{-1}$, $G_1 \in \mathbb{R}^{m\times(p-m)}$, $G_2 \in \mathbb{R}^{m\times m}$, $\bar{T}$ is a component of $\bar{C}$ in (3.53). In this control structure the sliding patch also exists, however the problem was formulated in terms of LMI where additional requirements can be incorporated within the LMI such that the eigenvalues of the transformed closed-loop system can be placed within standard convex regions of the complex plane such as circles, strips and cones (Chilali and Gahinet, [18], 1996), (Gahinet, et al., [62], 1995).

So far, numerical methods to design control laws based on output information only, which ensure a sliding mode is attainted, have been explored. Two types of control law designs are introduced in (Zak and Hui, [148], 1993), (Kwan, [102], 1996) and (Heck and Ferri, [83], 1989) for SMOFC, which ensure the reachability of the sliding surface from any initial condition. Some structural constraints required is seen to be conservative and difficult to satisfy in terms of tractable solution procedure. A dynamical controller is developed by Kwan, [102], (1996) which simplifies the design by introducing an additional filter to avoid the measurement of the states. However the dynamical controller is more complex to implement than OFC. Another two controllers are considered in (Edwards and Spurgeon, [29], 1995) and (Edwards, et al., [28], 2001) with a simple structure. However the reachability condition only holds in a compact domain around the sliding surface. Outside this domain the controller behaves as a variable structure controller. Some of these methods are only applicable to certain classes of linear systems. Even if the system has the required structure, then the numerical methods will only terminate satisfactorily for a specific class of switching surface. Other controllers have a simpler structure, however the controllers tend to be high gain control.

3.7 Conclusion

Static output feedback still remains one of the most important open problems in linear system theory despite the availability of many approaches and numerical algorithms. This chapter has reviewed some existing design approaches involving eigenvalue assignment, eigenstructure assignment, LMIs for solving the the existence problem in the case of static output feedback sliding mode control. The methods are only applicable for a certain class of systems. No efficient procedure is available to synthesize the solution. In designing a control law to satisfy the reachability problem, different approaches are available. In order for these numerical schemes to terminate satisfactorily, at best, a certain structural constraint must be satisfied and a particular choice of sliding surface must be made. Other controller designs posse simple structures, however the controllers tend to be high gain.

As it has been demonstrated in Chapter 2 that delay, which exists in many practical situations, affects sliding mode control performance contributing to the chattering and instability. In order to know what delay can change to the system behaviours and correspondingly design an efficient controller, understanding of the most basic characteristics of such delay systems is necessary as demonstrated in the next chapter. Common methods for stability analysis of delay systems are included which will be shown useful for the application of sliding mode controller design to such systems.

Chapter 4

A Delay: What Does It Change?

4.1 Introduction

In the mathematical description of a real physical or biological process, one generally assumes that the future behaviours of the considered process only depend on the present (in the usual sense) state, and therefore can be sufficiently described by ordinary differential equations. This is satisfactory for a large class of practical systems. Due to the existence of some 'time-delay' elements, such as material or information transport heredity (Gopalsamy, [68], 1992; Kuang, [101], 1993; MacDonald, [108], 1989) or computation times, the above-mentioned description is no longer sufficiently accurate for many systems. Roughly speaking, any *interconnection* of physical systems handling and transferring material, energy or information is subject to delays. In addition, several modelling methods introduce delays so as to simplify the complexity of the models. Although approximations can be applied to estimate the delay in practice, this generally lead to unsatisfactory analysis and simulation, as well as poor performance of the resulting design, due to the lack of effective analysis and control design tools targeted at such systems.

In a mathematical framework, such systems may be described in several ways: for example, differential equations on abstract (Bensoussan, et al., [8], 1992; Curtain and Zwart, [21], 1995) or functional spaces (Hale and Lunel, [77], 1993; Kolmanovskii and Myshkis, [97], 1992). In system and feedback theory, one encounters infinite-dimensional, n dimensional or behavioural representations, with advantages and inconveniences in handling *structure* or *control* problems. With respect to the delays, one can have constant or time-varying, discrete or distributed, finite or infinite, state-dependent or not.

This chapter will cover the basic concepts of such delay systems and introduce tools for their stability analysis. The study will prove useful in understanding the resulting system behaviours as well as the design of the controller. Section 4.2 introduces definitions of existence and uniqueness of solution for a class of delay systems known as retarded functional differential equation (RFDE).

In section 4.3, forward, backward continuation and smoothness of the solution for RFDE are discussed. Section 4.4 briefly introduces the idea of another type of delay systems represented by neutral functional differential equation (NFDE). In section 4.5 characteristic roots of RFED and NFDE is explained as a means to analyze the stability of the system. Methods for determining the stability of a delay system are demonstrated in terms of Razumikhin and Lyapunov Krasovskii approaches.

4.2 Models and solutions

A classical hypothesis in the modeling of physical processes is to assume that the future behaviour of the deterministic system can be summed up in its present *state* only. In the case of Ordinary Differential Equation (ODE), the state is an n-vector $x(t)$ moving in Euclidean space $\mathbb{R}^n$. Now, if one has to take into account an irreducible influence of the past, leading to the introduction of a deviated time-argument, then the *state* cannot anymore be a vector $x(t)$ defined at a discrete value of time t. In this section, delay systems that belong to RFDE are known to have infinite linearly independent solutions through characteristic equation analysis. If a function is continuous and satisfies a local Lipschitz condition in the delayed variable, then the local existence and uniqueness of the solution can be proved. Consider the difference equation

$$x(t) = Ax(t-1) + Bx(t-2) \tag{4.1}$$

where A and B are constants. By introducing an additional variable $y(t) = x(t-1)$, this scalar equation (4.1) is equivalent to the two-dimensional equation

$$z(t) = Cz(t-1), \quad z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad C = \begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix} \tag{4.2}$$

To obtain a solution of equation (4.2) defined for all $t > 0$, one must specify a 2-vector function ϕ on $[-1,\ 0]$. For any $\theta \in [-1,\ 0]$, the solution $z(t)$ of equation (4.2) is given by

$$z(t) = C^{t-\theta}\phi(\theta), \quad t = \theta,\ \theta+1,\ ...,\ \theta+k,\ ...$$

Thus the behaviour of the solution is determined by the eigenvalues and eigenvectors of the matrix C (Hale and Lunel, [77], 1993). The eigenvalues of C are the roots of the characteristic equation

$$\rho^2 - A\rho - B = 0 \tag{4.3}$$

The characteristic equation (4.3) can be obtained by seeking nontrivial solutions of Equation (4.1) of the form $x(t) = \rho^t c$, where c is a nonzero constant. In this form, equation (4.1) seems to be no more complicated than the ODE of the type $\dot{x} = Ax$ since it is very similar to a linear map of the plane into itself. However, the analysis of this equation is very sensitive to the numbers *1* and *2* of the equation (4.1) on the right-hand side. Consider the equation

$$x(t) = Ax(t-r) + Bx(t-s) \tag{4.4}$$

where r/s is irrational, $s > r > 0$, the problem is completely different. In contrast to the case $r = 1,\ s = 2$, one cannot obtain any solution of equation (4.4) by specifying initial values only at $x(-r),\ x(-s)$. The problem is basically infinite-dimensional and one sees that a reasonable initial-value problem for equation (4.4) is to specify an initial function on $[-s,\ 0]$ and use equation (4.4) of the form $x(t) = \rho^t c$, where $c \neq 0$ is constant. The resulting equation is

$$\rho^s - A\rho^{s-r} - B = 0 \tag{4.5}$$

This equation for r/s irrational has infinitely many solutions. Therefore, it is not obvious that the solutions of equation (4.5) can be obtained as linear combinations of the characteristic functions. Even without discussing the question of representation of solutions in series, it is not even obvious that the asymptotic stability behavior of the solutions of equation (4.4) is determined by the solutions of the characteristic equation (4.5). Both of these problems have a positive solution and one approach is through the Laplace transform. A generalization of equation (4.4) would be the equation

$$x(t) = \int_{-\infty}^{0} d[\mu(\theta)]x(t+\theta)$$

where μ is a function of bounded variation (Hale and Lunel, [77], 1993).

4.2.1 The notion of state in RFDE

In (Krasovskii, [100], 1962), consider the general Linear time-invariant non-homogenous system

$$\dot{x}(t) = \int_{-h}^{0} d[\mu(\theta)]x(t+\theta) + f(t), \quad x(t) \in \mathscr{C}, \quad t \geq t_0 \tag{4.6}$$

for an appropriate function f, initial time t_0 (eventually 0) and initial condition (continuous vector-valued function $\phi,\ \phi : [-h,\ 0] \mapsto \mathbb{R}^n,\ \|\phi\|_{\mathscr{C}} = \sup \|\phi(\theta)\|$) :

$$x(t_0 + \theta) = \phi(\theta), \quad \forall \theta \in [-h,\ 0] \tag{4.7}$$

Thus, the state of (4.6)-(4.7) at time "t" is the "piece of the trajectories" x between time $t-h$ and t, or, equivalently the *element* x_t in the space of continuous functions defined on $[-h,\ 0]$ and taking values in $\mathbb{R}^n(\mathscr{C}([-h,\ 0], \mathbb{R}^n))$

$$x_t(\theta) = x(t+\theta), \quad -h \leq \theta \leq 0$$

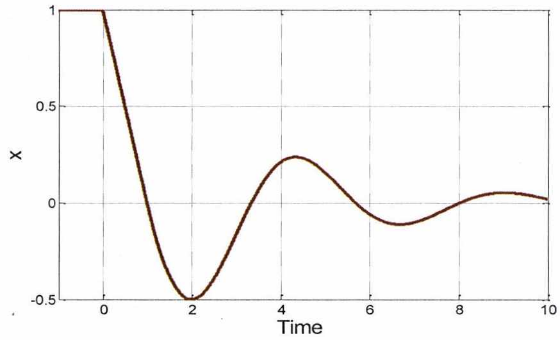


FIGURE 4.1: System $\dot{x}(t) = -x(t-h)$ with $h = 1$, $\phi = 1$

Since for any t, the "piece of trajectories" x_t is defined on $[-h,\ 0]$, it seems natural to use:

$$\|x_t\|_{\mathscr{C}} = \sup_{\theta \in [-h,\ 0]} \|x(t+\theta)\|$$

as an appropriate norm for (Lyapunov's) stability definitions and results.

4.2.2 Solution concept

Fridman, [47], (2010) considered the simple delay equation:

$$\dot{x}(t) = -x(t-h), \ \geq 0 \tag{4.8}$$

In order to define its solution for $t \in [0,h]$, the right-hand side $x(t-h)$ needs to be defined for $t \in [0,h]$, which results in the initial value function

$$x(s) = \phi(s), \quad s \in [-h, 0] \tag{4.9}$$

instead of the initial value $x(0)$ for ODE with $h = 0$. In order to find a solution to this problem, the *step method* initiated by Bellman and Cooke, [6], (1963) was used. First, find a solution on $t \in [0,h]$ by solving

$$t \in [0,h], \ \dot{x}(t) = -\phi(t-h), \ x(0) = \phi(0) \tag{4.10}$$

Then continue this procedure for $t \in [h, 2h]$, $t \in [2h, 3h], \ldots$. The resulting solutions for $h = 1$ and for the initial functions $\phi \equiv 1$ and $\phi = 0.5t$ are given in Figure 4.1. Note that the step method can be applied for solving the initial value problem for general time-delay systems with constant delay. As it is shown in Figure 4.1, rather than an otherwise exponential decaying trajectory of system (4.8) without delay, i.e. $\dot{x}(t) = -x(t)$, oscillating behaviour present due to the delay. Therefore, in

Time Delay System (TDS), a proper state is a function

$$x(t+\phi) = x_t(\phi), \quad \phi \in [-h, 0] \tag{4.11}$$

corresponding to the past time-interval $[t-h, t]$, which leads to an infinite-dimensional system.

4.3 Forward and backward solution of RFDE

In ordinary differential equations with a continuous vector field, one can prove the existence of a solution through a point (δ, x_0) defined on an interval $[\delta - a,\ \delta + a]$, $a > 0$; i.e. the solution exists to the right and left of the initial t-value. For RFDE, this is not necessarily the case. In the first case, existence to the right of the initial t-value, the *forward continuation of a solution through* (δ, θ) is proved using the *step-by-step* method. The solution in the forward delayed time interval can be constructed as the solution of an ODE prior to satisfying a necessary condition that the initial condition is well defined on $[t_0 - h,\ t_0]$. However, in the second case, general results on backward continuation are very difficult to prove although the ideas are relatively simple. Smoothness of the forward continuation is shown by successively stepping through a specified delayed time interval.

4.3.1 Solution to the linear equation

Consider the linear system

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h) \tag{4.12}$$

where $A(t)$ is a quadratic matrix whose elements are continuous functions of t defined for $t \geq 0$. The *adjoint system* corresponding to (4.12) is the system with the advanced argument (Halanay, [74], 1996)

$$\dot{y}(t) = -y(t)A(t) - y(t+h)B(t+h) \tag{4.13}$$

where y is a row vector. Let $x(t)$ and $y(t)$ be arbitrary solutions of the systems (4.12) and (4.13) respectively, denote by (y, x) the function

$$(y,x) = y(t)x(t) + \int_0^t y(t+\alpha)B(t+\alpha)x(t-\alpha-h)d\alpha \tag{4.14}$$

Then $(d/dt)(y,x) \equiv 0$, hence (y,x) =const. Indeed,

$$(y,x) = y(t)x(t) + \int_t^{t+h} y(\alpha)B(\alpha)x(\alpha-h)d\alpha$$

and

$$\begin{aligned}\frac{d}{dt}(y,x) &= \dot{y}(t)x(t)+y(t)\dot{x}(t)+y(t+h)B(t+h)x(t)-y(t)B(t)x(t-h)\\ &= -y(t)A(t)x(t)-y(t+h)B(t+h)x(t)+y(t)A(t)x(t)\\ &+ y(t)B(t)x(t-h)+y(t+h)B(t+h)x(t)-y(t)B(t)x(t-h)\\ &= 0\end{aligned}$$

Consider the nonhomogeneous system

$$\dot{x}(t) = A(t)x(t)+B(t)x(t-h)+f(t) \tag{4.15}$$

with the initial condition

$$x(t_0+\theta) = \phi(\theta),\ \ \theta\in[-h,0],\quad \phi\in\mathscr{C} \tag{4.16}$$

Let $Y(t,\alpha)$ be a matrix which satisfies system (4.13) for $\alpha<t$ (as a function of α) and $Y(t,t)=E$, $Y(t,\alpha)\equiv 0$ for $\alpha>t$. This matrix is easily constructed through the "step-by-step" method (Bellman and Cooke, [6], 1963). Indeed, for $t-h<\alpha<t$, note that

$$\frac{\partial}{\partial\alpha}Y(t,\alpha) = -Y(t,\alpha)A(\alpha),\quad Y(t,t)=E$$

since for $\alpha>t-h$, it follows that $\alpha+h>t$ and $Y(t,\alpha+h)\equiv 0$. It follows from here that in $(t-h,\ t]$ the matrix $Y(t,\alpha)$ is determined by a system of ordinary equations. Further, for $t-2h<\alpha<t-h$, this leads to

$$\frac{\partial}{\partial\alpha}Y(t,\alpha) = -Y(t,\alpha)A(\alpha)-U(\alpha+h)B(\alpha+h),\quad Y(t,t-h)=U(t-h) \tag{4.17}$$

where $U(\alpha)$ denotes the matrix constructed in the preceding step. The procedure continues in the same way and leads to the desired matrix $Y(t,\alpha)$. Multiply (4.15) from the left by the matrix $Y(t,\alpha)$ constructed in this way and integrate with respect to α from t_0 to t. It is obtained that

$$\begin{aligned}\int_{t_0}^{t}Y(t,\alpha)\dot{x}(\alpha)d\alpha &= \int_{t_0}^{t}Y(t,\alpha)A(\alpha)x(\alpha)d\alpha+\int_{t_0}^{t}Y(t,\alpha)B(\alpha)x(\alpha-h)d\alpha\\ &+ \int_{t_0}^{t}Y(t,\alpha)f(\alpha)d\alpha\end{aligned} \tag{4.18}$$

Further, integrating by parts in the left hand part of (4.18)

$$\begin{aligned}&Y(t,t)x(t)-Y(t,t_0)x(t_0)-\int_{t_0}^{t}\frac{\partial}{\partial\alpha}Y(t,\alpha)x(\alpha)d\alpha\\ &= \int_{t_0}^{t}Y(t,\alpha)A(\alpha)x(\alpha)d\alpha+\int_{t_0}^{t}Y(t,\alpha)B(\alpha)x(\alpha-h)d\alpha+\int_{t_0}^{t}Y(t,\alpha)f(\alpha)d\alpha\end{aligned}$$

Taking into account (4.17) it is obtained that

$$\begin{aligned}
x(t) &= Y(t,t_0)x(t_0) - \int_{t_0}^{t} Y(t,\alpha)A(\alpha)x(\alpha)d\alpha - \int_{t_0}^{t} Y(t,\alpha+h)B(\alpha+h)x(\alpha)d\alpha \\
&+ \int_{t_0}^{t} Y(t,\alpha)A(\alpha)x(\alpha)d\alpha + \int_{t_0}^{t} Y(t,\alpha)B(\alpha)x(\alpha-h)d\alpha + \int_{t_0}^{t} Y(t,\alpha)f(\alpha)d\alpha \\
&= Y(t,t_0)x(t_0) - \int_{t_0}^{t} Y(t,\alpha+h)B(\alpha+h)x(\alpha)d\alpha \\
&+ \int_{t_0-h}^{t-h} Y(t,\alpha+h)B(\alpha+h)x(\alpha)d\alpha + \int_{t_0}^{t} Y(t,\alpha)f(\alpha)d\alpha \\
&= Y(t,t_0)x(t_0) + \int_{t_0-h}^{t_0} Y(t,\alpha+h)B(\alpha+h)x(\alpha)d\alpha \\
&- \int_{t-h}^{t} Y(t,\alpha+h)B(\alpha+h)x(\alpha)d\alpha + \int_{t_0}^{t} Y(t,\alpha)f(\alpha)d\alpha
\end{aligned}$$

However, $Y(t,\alpha+h) \equiv 0$ for $t-h < \alpha \leq t$ and it follows that

$$x(t) = Y(t,t_0)x(t_0) + \int_{t_0-h}^{t_0} Y(t,\alpha+h)B(\alpha+h)x(\alpha)d\alpha + \int_{t_0}^{t} Y(t,\alpha)f(\alpha)d\alpha$$

From this formula it is obvious that if $X(t,t_0)$ is the solution of system (4.12) which verifies the conditions $X(t_0,t_0) = E$, $X(t_0,t) \equiv 0$ for $t < t_0$, then $X(t_0,t) \equiv Y(t,t_0)$. The formula below is obtained

$$x(t) = X(t,t_0)\phi(0) + \int_{t_0-h}^{t_0} X(\alpha+h,t)B(\alpha+h)\phi(\alpha-t_0)d\alpha + \int_{t_0}^{t} X(\alpha,t)f(\alpha)d\alpha \tag{4.19}$$

Using (4.19) the results for the scalar case example can be easily extended as follows (Hale, [76], 1971). The "minimum" amount of initial data *necessary* to have a solution $x(t)$ of the equation (4.19) is a *function* defined on the whole interval $[t_0-h,\ t_0]$ at t_0. A standard integration method (Runge-Kutta, Adams) can be used to construct the solution $x(t_0,\phi)$ on $[t_0,\ t_0+h)$ under the assumption that $[t_0,\ t_0+h)$ is included in the maximum interval of existence. In conclusion, using the initial condition defined on $[t_0-h,\ t_0]$, the solution has been constructed on the interval $[t_0,\ t_0+h)$ as the solution of an appropriate ODE.

Thus by iteration, the procedure, known as the *step method*, can continue to any delay-interval $[t_0+kh,\ t_0+(k+1)h)$ (with k a positive integer) included in the maximum interval of existence $[t_0,\ t_x]$, where $t_x > 0$ ($+\infty$ eventually) (Niculescu, [113], 2001). The use of the *step-by-step* method shows that the resulting solution $x(t)$ is a succession of polynomial functions of t, of increasing degree at each interval $[kh,\ (k+1)h]$. Having presented the forward continuation of RFDE, some sufficient conditions for existence to the left of the initial t-value for equation

$$\dot{x} = f(t,x_t), \quad t \geq t_0 \tag{4.20}$$

with initial condition

$$x(t_0+\theta) = \phi(\theta), \quad \theta \in [-\tau,0], \quad \phi \in \mathscr{C} \tag{4.21}$$

will be stated.

4.3.2 Backward continuation

Definition 4.1. (Hale and Lunel, [77], 1993) Suppose $\Omega \subseteq \mathbb{R} \times \mathscr{C}$ is open and $f \in \mathscr{C}(\Omega, \mathbb{R}^n)$. A function $x \in \mathscr{C}([\delta - h - \alpha,\ \delta], \mathbb{R}^n), \alpha > 0$, is a *solution* of equation (4.20) on $[\delta - h - \alpha,\ \delta]$ through $(\delta,\ \phi) \in \Omega$ if $x_\delta = \phi$ and for any $\delta_1 \in [\delta - \alpha,\ \delta]$, $(\delta_1, x_{\delta_1}) \in \Omega$ and x is a solution of equation (4.20) on $[\delta_1 - h,\ \delta]$ through (δ_1, x_{δ_1}). Such a solution is some times referred to as the *backward continuation of a solution through* (δ, ϕ).

Definition 4.1 is very natural and says only that a function defined on $[\delta - h - \alpha,\ \delta]$ is a solution of equation (4.20) on this interval if it has the property that it will satisfy the equation in the forward direction of t, no matter where the initial time is chosen. General results on backward continuation are very difficult to prove although the ideas are relatively simple. To motivate the definitions to follow, consider a simple example

$$\dot{x}(t) = a(t)x(t-1) \tag{4.22}$$

If $\delta = 0$, ϕ is a given function in $\mathscr{C}$ and there exists a backward continuation of a solution through $(0, \phi)$, then it is necessary that ϕ be continuously differentiable on a small interval $(-\varepsilon,\ 0]$ and $\dot{\phi}(0) = a(0)\phi(-1)$. Conversely, if this condition is satisfied and $a(t) \neq 0$ for $t \in (-\varepsilon,\ 0]$, then one can define

$$x(t-1) = \frac{1}{a(t)}\dot{\phi}(t), \quad t \in (-\varepsilon,\ 0]$$

and x will be a solution of equation (4.22) on $(-h - \varepsilon,\ 0]$ with $x_0 = \phi$. Therefore, there is a backward continuation through $(0, \phi)$.

Remark 4.1. In the example (4.22), $a(t) \neq 0$ for $t \in (-\varepsilon,\ 0]$; i.e. the evolution of the system $x(t)$ actually used the information specified at $x(t-1)$. In the general case, it is very difficult to precisely describe the manner in which $f(t, \phi)$ varies with $\phi(-h)$, other additional properties need to be imposed on t and ϕ (Hale and Lunel, [77], 1993).

Similar to ODEs, there is a solution of (4.20) through $(\delta, \phi) \in \Omega$ if f is continuous ($f \in \mathscr{C}(\Omega, \mathbb{R}^n)$). If in addition $f(t, \phi)$ is Lipschitz in its second argument ϕ in each compact set in Ω, then the solution is unique. But, in contrast to ODEs, for FDEs with arbitrary smooth right-hand sides it may occur that originally different solutions coincide after some time. Consider the following example

$$\dot{x}(t) = -x(t-h)[1 - x^2(t)] \tag{4.23}$$

Equation (4.23) has the solution $x(t) = 1$ for all t in $(-\infty,\ \infty)$. Furthermore, if $h = 1,\ \delta = 0$, and $\phi \in \mathscr{C}$, then there is a unique solution $x(0, \phi)$ of equation (4.23) through $(0, \phi)$ that depends continuously on ϕ. If $-1 \leq \phi(0) \leq 1$, these solutions are actually defined on $[-1,\ \infty)$. On the other hand, if $\phi \in \mathscr{C},\ \phi(0) = 1$, then $x(0, \phi)(t) = 1$ for all $t \geq 0$. Therefore, for all such initial values, $x_t(0, \phi)$, $t \leq 1$, is the constant function 1. A translation of a subspace of $\mathscr{C}$ of codimension

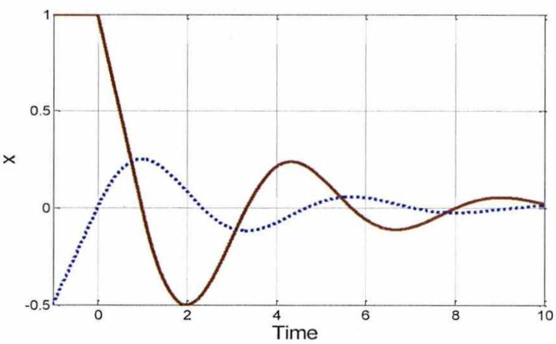


FIGURE 4.2: System $\dot{x}(t) = -x(t-h)$ with $h = 1,\ \phi = 1(plain)$ or $0.5t$ (dotted)

one is mapped into a point for all $t \geq 1$. A special class of RFDEs can be considered: equation (4.20) is said to be *autonomous* (or time-invariant) if $f(t,\phi) = g(\phi)$, where $g(\cdot)$ does not depend on t; *linear* if $f(t,\phi) = L(t)\phi + h(t)$, where operator $L(t)$ is linear; *linear homogeneous* if $h = 0$; *linear time-invariant* if $f(t,\phi) = L\phi$. Then the following properties hold

- The backward continuation on $[-\infty,\ 0]$ of a linear autonomous RFDE is unique.
- There may exist two distinct backward continuations on $[-\infty,\ 0]$ for an autonomous RFDE.
- However, if $f : \mathscr{C} \to \mathbb{R}^n$ is an analytic functional, the bounded backward continuation on $[-\infty,\ 0]$ is unique.

4.3.3 Smoothness in RFDE

The smoothness property of the solutions should be considered as it follows from the step-by-step procedure. Consider the first delay-interval, and more specifically on the point t_0. Due to the form of the differential equation (4.12), and since, in general, the initial condition ϕ is chosen arbitrarily, one can say that:

$$\dot{x}(t_0^+) \neq \dot{\phi}(t_0^-) = \dot{x}(t_0^-) \tag{4.24}$$

i.e. there is a discontinuity in the first derivative of the solution $x(t)$ at $t = t_0$. This can be explained by the definition of $\dot{x}(t_0^+)$ as:

$$\dot{x}(t_0^+) = lim_{h_1 \to 0^+} f(t_0 + h_1, x(t_0 + h_1), \phi(t_0 - h + h_1))$$

Extending to the next delay-intervals $[t_0 + kh,\ t_0 + (k+1)h),\ k > 1$, the solution becomes smoother and smoother from one delay-interval to the next one at the points $t_0 + kh,\ k > 1$ as seen in Figure 4.2. In (Hale, [76], 1971) that

$$y^k(t) = x^k(\delta^k + t) - \tilde{\phi}^k(\delta^k + t)$$

where $\tilde{\phi} \in \mathscr{C}([\delta - r,\ \infty), \mathbb{R}^n)$ can be shown to belong to a compact set K of $\mathscr{C}([-r,\ \alpha), \mathbb{R}^n)$. Therefore, there is a subsequence such that y^k converges uniformly to y^0. Translating these remarks back into x^k gives the existence and uniqueness for x^0 in the interval $[\delta^0 - r,\ \delta^0 + \alpha]$. Smoothness is thus proved by successively stepping through intervals of length α. This result is also shown in (Kolmanovskii and Myshkis, [97], 1992), in which the smoothness property is said to make RFDE to "resemble" a *parabolic partial differential equation (PDE).*

4.4 Neutral systems

Neutral systems are also delay systems, but involve the same order of highest derivative for some components of $x(t)$ at both time t and past time(s) $t' < t$, which implies an increased mathematical complexity. Neutral systems are represented by

$$\dot{x}(t) = f(x_t, t, \dot{x}_t, \phi_t) \tag{4.25}$$

or, in Hale's form (Hale and Lunel, [76], 1993):

$$D\dot{x}_t = \frac{dDx_t}{dt} = f(x_t, t, \phi_t) \tag{4.26}$$

where $D : \mathscr{C} \to \mathbb{R}^n$ is a regular operator (this avoids implicit systems) with deviating argument in time, as for instance

$$Dx_t = x(t) - Fx(t-h) \tag{4.27}$$

where F is a constant matrix. The solutions of *retarded* systems have their differentiability degree smoothed with increasing time (Figure 4.2). This property of "solution smoothing" is no longer true for *neutral* systems: due to the implied difference-equation involving $\dot{x}(t)$ which makes the solution of (4.26) never have more derivatives than the initial function ϕ and therefore $\dot{x}(t_0^+) \neq \dot{x}(t_0^-)$ for all $t \in [-h,\ \infty)$. A sufficient condition for the solution x to have continuous derivative for all $t \geq -h$ is given by the so-called *sewing condition* (Hale, [76], 1971), that is

$$\dot{\phi}(0) = F\dot{\phi}(-h) + f(x_0, 0, \phi_0)$$

As stated by Niculescu, [113], (2001), this discontinuity property, together with other properties, make NFDE "resemble" PDEs of *hyperbolic type.*

4.5 Advances in the theory of stability

In this section, characteristic roots of systems subject to delay are studied as a means to analyze the stability of the system. Necessary and sufficient conditions for the stability of the linear time-invariant RFDE and NFDE are given in terms of the locations of the characteristic roots. Stability analysis based on the Lyapunov Krasovskii Functional and the Razumikin Theorem is briefly reviewed which has been a topic of considerable interest in the recent literature due to the availability of numerical tools like Linear Matrix Inequalities (LMIs).

4.5.1 The characteristic roots of retarded and neutral linear FDEs

Delays are known to have complex effects on stability (Kolmanovskii, et al., [99], 1999). In spite of the situation where delays cause instability, they may also have a stabilizing effect (Fridman, [44], 2006), (Kharitonov, [92], 2005): the well-known example $\ddot{y}(t)+y(t)-y(t-h)=0$ is unstable for $h=0$, but asymptotically stable for $h=1$. This control technique has been applied to sliding mode control (Seuret, et al., [123], 2009).

4.5.1.1 Characteristic roots of RFDE

In the RFDE case (4.12), the necessary and sufficient condition for its asymptotic stability is a straightforward generalization of the ODE theory, since it requires the characteristic equation to have no zeros in the closed right half-plane. Consider the system

$$\begin{aligned} &a_0\dot{x}(t)+b_0x(t)+b_1x(t+\theta)=0, \quad -h\leq\theta\leq 0, \quad a_0\neq 0,\\ &x(t)=\phi(\theta), \quad 0\leq t\leq h \end{aligned} \tag{4.28}$$

where $\phi(\theta)$ is any preassigned, real, continuous function. It can be shown (Bellman and Cooke, [6], 1963) that under suitable conditions the solution $x(t)$ of (4.28) is given by

$$x(t)=\lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}e^{ts}h^{-1}(s)p(s)ds, \quad t>0, \tag{4.29}$$

where

$$\begin{aligned} &h(s)=a_0s+b_0+b_1e^{-hs}\\ &p(s)=a_0g(h)e^{-hs}+(a_0s+b_0)\int_0^h g(t_1)e^{-st_1}dt_1 \end{aligned} \tag{4.30}$$

provided $g\in\mathscr{C}[0,\ h]$. The expression on the right of (4.29) is a contour integral taken along the vertical line joining the points $c-iT$ and $c+iT$ in the complex plane with T a positive number. Considering the location of the zeros of the characteristic function $h(s)$ in (4.30), the zeros of $h(s)$ have the following properties:

- It is symmetric with respect to the real axis.
- It lies entirely in the left half-plane.
- It is similar to an exponential curve for large $|s|$.
- As $|s|\to\infty$ along the curve, the curve becomes more and more nearly parallel to the imaginary axis, and $\mathrm{Re}(s)\to-\infty$.

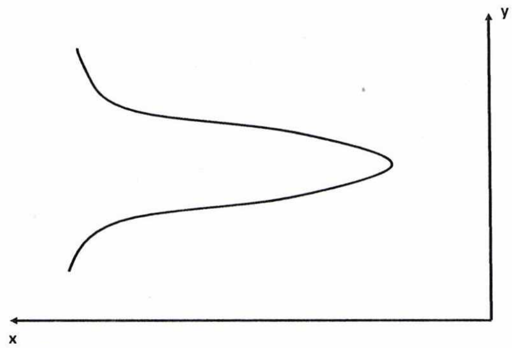


FIGURE 4.3: General appearance of the curve of the zeros of $h(s)$ for large $|s|$

The general appearance of the curve is suggested in Figure 4.3.

4.5.1.2 Characteristic roots of NFDE

Recall the neutral system represented by equations (4.26) and (4.27), where the difference operator $F \neq 0$. The situation is different from the RFDE case because there can appear an infinite number of unstable roots. In (Bellman, [6], 1963), the following first-order equation of neutral type is considered

$$\begin{aligned} &a_0\dot{x}(t) + a_1\dot{x}(t-h) + b_0x(t) + b_1x(t-h) = f(t), \quad a_0 \neq 0, \ \ a_1 \neq 0 \\ &x(t) = \phi(\theta), \ \ 0 \leq t \leq h \end{aligned} \tag{4.31}$$

and the characteristic function associated with the solution of (4.31) is

$$h(s) = a_0 s + a_1 s e^{-hs} + b_0 + b_1 e^{-hs} \tag{4.32}$$

The zeros of (4.32) are shown to lie asymptotically along a vertical line in the complex plane and a sufficient condition for asymptotic stability is that the real parts of the characteristic roots of (4.32) are negative. The relationship between the F-operator and the stability of the system (4.26) and (4.27) is strong, the system can be shown to be strongly stabilizable (when subjected to small variations in the delays) if the operator F in (4.26) and (4.27) is exponentially stable for all values of the delays (Hale, [75], 2002). To facilitate stability analysis of (4.26) and (4.27), assume the difference operator

$$F\phi = \phi(0) - \sum_{j=1}^{M} A_j \phi(-h_j) \tag{4.33}$$

where the coefficients A_1, A_2, ..., A_M are matrices of appropriate dimension and ϕ is the initial function of the system (4.26) and (4.27) at time $\theta = [-h, \ 0]$.

Proposition 4.2. *(Hale, [75], 2002) A necessary and sufficient condition for the asymptotic stability of*

$$x(t) - \sum_{j=1}^{M} a_j x(t-h_j) = 0 \tag{4.34}$$

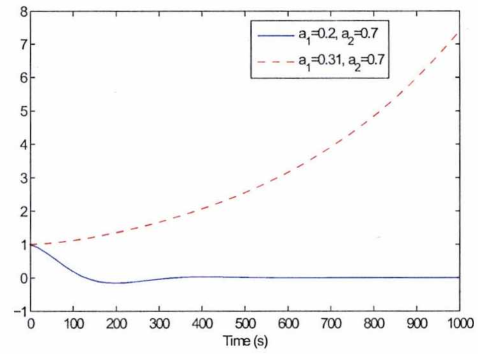


FIGURE 4.4: Response of system (4.36) with $h_1 = 2s$, $h_2 = 5s$

is

$$\sum_{j=1}^{M} |a_j| < 1 \tag{4.35}$$

where $a_j = A_j$ in (4.33) for scalar case and $h = (h_1, h_2, \ldots, r_M)$ are rationally independent.

EXAMPLE 4.1. Consider the system

$$x(t) - a_1 x(t-h_1) - a_2 x(t-h_2) = 0 \tag{4.36}$$

where h_1 and h_2 are rationally independent. According to the assertion above, this equation is not exponentially stable, see Figure 4.4, if and only if

$$|a_1| + |a_2| \geq 1 \tag{4.37}$$

Given (4.37), a feedback control can be applied to stabilize the system (4.36)

$$x(t) - (a_1 - f_1)x(t-h_1) - (a_2 + f_2)x(t-h_2) = 0 \tag{4.38}$$

The closed-loop system (4.38) is exponentially stable if and only if

$$|a_1 + f_1| + |a_2 + f_2| < 1$$

Since $|a_1| + |a_2| \geq 1$, there must be $|f_1| + |f_2| > 0$. Suppose now that the feedback control cannot be applied instantaneously and that there is a small time delay in the feedback

$$x(t) - a_1 x(t-h_1) - a_2 x(t-h_2) - f_1 x(t-h_1-\varepsilon_1) - f_2 x(t-h_2-\varepsilon_2) = 0 \tag{4.39}$$

It can be seen that although this equation is exponentially stable for $\varepsilon_1 = \varepsilon_2 = 0$, there is a sequence $(\varepsilon_1^j, \varepsilon_2^j)$ tending to zero so that (4.39) is exponentially unstable. To prove the claim, choose ε_1 and ε_2 such that h_1, h_2, $h_1 + \varepsilon_1$ and $h_2 + \varepsilon_2$ are rationally independent. It follows from Corollary 4.2 that (4.39) is always exponentially unstable, since

$$|a_1| + |f_1| + |a_2| + |f_2| > |a_1| + |a_2| \geq 1$$

If the difference operator D in (4.26) and (4.27) is stable, then the properties of the NFDE are similar to the RFDE. The system is stable if and only if the characteristic roots are in the left hand plane.

4.5.2 The Razumikhin-type approach

For the retarded or neutral class of system, checking eigenvalue conditions is much harder than for ODEs. Therefore numerous stability approaches have been investigated for example, *Matrix pencils, Norm, measure.* A brief presentation of these methods can be found in the paper (Richard, [117], 1998) and a more complete one in the monographs (Hale and Verduyn-lunel, [77], 1993; Niculescu, [113], 2001). There are general results for stability conditions which are *delay independent (di)* conditions. However sharper results can be expected from *delay-dependent stability (dd)* conditions. This is because the robustness of *di* properties is of course counterbalanced by very conservative conditions. In engineering practice, information on the delay range is generally available and *dd* criteria are likely to give better performance. The most investigated generalization of Lyapunov's second method involves *functionals* $V(t,x_t)$ depending on x_t instead of classical positive definite *functions.* One way to interpret the solution of the considered functional differential equation is as an *evolution in a function space* (Lyapunov-Krasovskii functional) (LKF), another one is as *evolution in an Euclidian space* (Lyapunov-Razumikhin function). Note however that even for a Lyapunov-Razumikhin function candidate, the corresponding derivative is always a functional. This section will illustrate the Lyapunov-Razumikhin function approach in its application to *di* and *dd* stability analysis. *Model transformation* for deriving sufficient conditions for simple delay-dependent stability and the additional dynamics it introduces into the original system are briefly presented.

Theorem 4.1. **Razumikhin Theorem** Suppose $f : \mathbb{R} \times \mathscr{C} \to \mathbb{R}^n$ in (4.20) takes $\mathbb{R}\times$ (bounded sets of $\mathscr{C}$) into bounded sets of $\mathbb{R}^n$, and $u, v, w : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$, v strictly increasing. If there exists a continuously differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that

$$u(\|x\|) \leq V(t,x) \leq v(\|x\|) \tag{4.40}$$

for $x \in \mathbb{R}$ and $x \in \mathbb{R}^n$ and the derivative of V along the solution $x(t)$ of (4.20) satisfies

$$\dot{V}(t,x(t)) \leq -w(\|x(t)\|) \quad \text{whenever} \quad V(t+\theta,x(t+\theta)) \leq V(t,x(t)) \tag{4.41}$$

for $\theta \in [-h,\ 0]$, then the system (4.20) is uniformly stable. If, in addition, $w(s) > 0$ for $s > 0$, and there exists a continuous nondecreasing function $p(s) > s$ for $s > 0$ such that condition (4.41) is strengthened to

$$\dot{V}(t,x(t)) \leq -w(\|x(t)\|) \quad \text{if} \quad V(t+\theta,x(t+\theta)) \leq p(V(t,x(t))) \tag{4.42}$$

for $\theta \in [-h, 0]$, then the system (4.20) is uniformly asymptotically stable. If in addition $lim_{s\to\infty}u(s) = \infty$, then the system (4.20) is globally uniformly asymptotically stable.

4.5.2.1 Delay-independent-Razumikhin Theorem

As an example of the *di* condition using the Razumikhin theorem, consider the system

$$\dot{x}(t) = -a(t)x(t) - b(t)x(t+\theta(t)) \tag{4.43}$$

where a, b and θ are bounded continuous functions on $\mathbb{R}$ with $|b(t)| \leq a(t)$, $\theta \in [-h, 0]$, for all $t \in \mathbb{R}$. If $V(x) = x^2/2$, then

$$\begin{aligned} \dot{V}(x(t)) &= -a(t)x^2(t) - b(t)x(t)x(t+\theta(t)) \\ &\leq -a(t)x^2(t) + |b(t)||x(t)||x(t+\theta(t))| \\ &\leq -[a(t) - |b(t)|]x^2(t) \\ &\leq 0 \end{aligned}$$

if $|x(t)| \geq |x(t+\theta(t))|$. Since $V(x) = x^2/2$, it has been shown that $\dot{V}(x(t)) \leq 0$ if $V(x(t)) \geq V(x(t+\theta(t)))$. Theorem 4.1 implies the solution $x = 0$ of equation (4.43) is uniformly stable. If, in addition, $a(t) \geq \delta > 0$, and there is a k, $0 \leq k < 1$ such that $|b(t)| \leq k\delta$, then the solution $x = 0$ of equation (4.43) is uniformly asymptotically stable. In fact, choose $p(s) = q^2 s$ in (4.42) for some constant $q > 1$. If $V(x) = x^2/2$ as before, then

$$\dot{V}(x(t)) \leq -(1-qk)\delta x^2(t)$$

if $p(V(x(t))) > V(x(t+\theta(t)))$. Since $k < 1$, there is a $q > 1$ such that $1 - qk > 0$ and Theorem (4.1) implies the uniform asymptotic stability of the solution $x = 0$.

Remark 4.3. The condition $|b(t)| \leq k\delta$ imposes a restriction on the magnitude of the parameter b. Roughly speaking, the larger $b(t)$ is, the more difficult it is to satisfy the above Theorem.

4.5.2.2 Delay-dependent-Razumikhin Theorem

As discussed above, a stable system can deteriorate and eventually lose stability as the premultiplier $b(t)$ of the delay term in (4.43) grows from zero. In order to reflect the relation of $|b(t)|$ with respect to the size of delay θ, a *model transformation* based Razumikhin approach is a convenient way to derive (dd) type results. Consider again the system

$$\dot{x}(t) = Ax(t) + Bx(t-h) \tag{4.44}$$

with the initial condition $x_0 = \phi$, where $\phi \in \mathscr{C}([-h,\ 0], \mathbb{R}^n)$. Since

$$\begin{aligned} x(t-h) &= x(t) - \int_{-h}^{0} \dot{x}(t+\theta)d\theta \\ &= x(t) - \int_{-h}^{0} [Ax(t+\theta) + Bx(t-h+\theta)]d\theta, \end{aligned} \tag{4.45}$$

for $t \geq h$, system (4.44) can be written as

$$\dot{x}(t) = (A+B)x(t) + \int_{-h}^{0} [-BAx(t+\theta) - BBx(t-h+\theta)]d\theta \tag{4.46}$$

for arbitrary continuous initial data $\psi \in [-2h,\ 0]$. If the zero solution of equation (4.46) is asymptotically stable (Hale, [77], 1993), then the zero solution of equation (4.44) is asymptotically stable since equation (4.44) is a special case of equation (4.46) with continuous initial data $\psi \in [-2h,\ 0]$. As an example, consider the equation

$$\dot{x}(t) = -bx(t-h) \tag{4.47}$$

where $h > 0$ and the auxiliary problem on $[-2h,\ 0]$ is given by

$$\dot{x}(t) = -bx(t) - b^2 \int_{t-2h}^{t-h} x(s)ds$$

If $V(x) = x^2/2$, then, for any constant $q > 1$,

$$\dot{V} = -bx^2(t) - b^2 \int_{t-2h}^{t-h} x(t)x(s)ds \leq -b(1-qbh)x^2(t)$$

if $V(x(\xi)) < q^2 V(x(t))$, $t-2h \leq \xi \leq t$. Consequently, if $bh < 1$, then there is a $q > 1$ such that $qbh < 1$ and Theorem 4.1 implies asymptotic stability.

Remark 4.4. The *dd* stability condition using the method of *explicit model transformation* introduces *additional dynamics* into the original system yielding spurious poles of the transformed system. It is shown in (Gu, [73], 2003) that the transformed system described by (4.46) with initial condition $\psi \in [-2h,\ 0]$ is equivalent to the following system

$$\begin{cases} \dot{y}(t) = Ay(t) + By(t-h) + z(t) \\ z(t) = B\int_{-h}^{0} z(t+\theta)d\theta \end{cases} \tag{4.48}$$

and the initial condition

$$\begin{cases} y(\theta) = \varphi(\theta), \\ z(\theta) = \phi(\theta), \end{cases}$$

for $-h \leq \theta \leq 0$. Corresponding to a positive real eigenvalue λ_i of B, there is an additional pole on the imaginary axis if and only if $h = \frac{1}{\lambda_i}$. No additional poles corresponding to a negative real eigenvalue λ_i of B will reach the imaginary axis for any finite delay (Gu, [73], 2003).

4.5.3 The Krasovskii method

Theorem 4.2. **(Lyapunov-Krasovskii Stability Theorem)** Suppose $f : \mathbb{R} \times \mathscr{C} \to \mathbb{R}^n$ in (4.20) maps $\mathbb{R}\times$ (bounded sets in $\mathscr{C}$) into a bounded sets in $\mathbb{R}^n$, and that $u, v, w : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions, where additionally $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there exists a continuous differentiable functional $V : \mathbb{R} \times \mathscr{C} \to \mathbb{R}$ such that

$$u(\|\phi(0)\|) \leq V(t,\phi) \leq v(\|\phi\|_c)$$

and

$$\dot{V}(t,\phi) \leq -w(\|\phi(0)\|)$$

then the trivial solution of (4.20) is uniformly stable. If $w(s) > 0$ for $s > 0$, then it is uniformly asymptotically stable. If in addition, $lim_{s\to\infty}u(s) = \infty$, then it is globally uniformly asymptotically stable.

Consider again the system (4.12). A simple stability criterion can be obtained from the following Lyapunov-Krasovskii functional

$$V(x_t) = x^T(t)Px(t) + \int_{t-r}^{t} x^T(\xi)Sx(\xi)d\xi,$$

where the matrices P and S are symmetric and positive definite. It can be easily calculated that the derivative of V along the system trajectory is

$$\dot{V}(x_t) = \begin{bmatrix} x^T(t) & x^T(t-r) \end{bmatrix} \begin{bmatrix} PA_0 + A_0^T P + S & PA_1 \\ A_1^T P & -S \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-r) \end{bmatrix} \tag{4.49}$$

It is clear that $\dot{V}(x_t) \leq -\varepsilon\|x(t)\|^2$ for some sufficiently small $\varepsilon > 0$ if the matrix in expression (4.49) is negative definite.

Proposition 4.5. *(Boyd et al., [15], 1994) System (4.12) is asymptotically stable if there exist real symmetric matrices $P > 0$ and S, such that*

$$\begin{bmatrix} PA_0 + A_0^T P + S & PA_1 \\ A_1^T P & -S \end{bmatrix} < 0$$

is satisfied.

Obviously, setting $A_1 = 0$ makes the link with the Lyapunov equation results for ODEs. However this sufficient condition is far from being necessary. As stated by Richard, [118], (2003) many generalizations have been proposed involving the various terms:

$$\begin{aligned}
V_1(x(t)) &= x^T(t)Px(t), \\
V_2(x_t) &= x^T(t)\int_{-h}^{0} Qx(t+\theta)d\theta, \\
V_3(x_t) &= \int_{-h}^{0} x^T(t+\theta)Sx(t+\theta)d\theta, \\
V_4(x_t) &= \int_{-h}^{0}\int_{t+\theta}^{t} x^T(\theta)Rx(\theta)d\theta ds, \\
V_5(x_t) &= x^T(t)\int_{-h}^{0} P(\eta)x(t+\eta)d\eta, \\
V_6(x_t) &= \int_{-h}^{0}\int_{-h}^{0} x^T(t+\eta)P(\eta,\theta)x(t+\theta)d\eta d\theta
\end{aligned} \tag{4.50}$$

Roughly speaking, V_2, V_3 are used for the delay-independent stability of discrete delays and V_4 for distributed delays or discrete-delay dependent stability. V_5 and V_6 appear, in a general form, in *necessary and sufficient* schemes: (Infante and Castelan, [89], 1978) for linear retarded systems with discrete delays, (Huang, [88], 1989) for distributed ones, and (Louisell, [107], 1991) for varying delays. But, the general computation of the time-varying matrices in V_5 and V_6 comes up against computational problems and the result cannot be applied for robust stability purposes. To avoid such computational limitations, Gu, [72], (1999) introduced more particular forms of V_5, V_6, with piecewise-constant functions $P(\cdot)$, leading to the so-called *discretization scheme*, which effectively reduces the choice of LKF V into choosing a finite number of parameters. One can thus get interesting compromises between the reduction of the conservatism and the computational effort.

The choice of an appropriate Lyapunov-Krasovskii functional is crucial in widening the conservatism of the stability criteria. Some transformations of the original system have been used for stability analysis of retarded type systems (Kolmanovskii and Richard, [96], 1999). The conservatism of approaches based on these transformations is two-fold: the transformed system is not equivalent to the original one and bounds should be obtained for certain terms. In (Fridman, [42], 2001), a descriptor model transformation and a corresponding Lyapunov-Krasovskii Functional are introduced for stability analysis of systems with delays. Delay-dependent/delay-independent stability criteria are derived for linear retarded and neutral type systems. The method is less conservative than other existing criteria (for retarded type systems and neutral systems with discrete delays) since they are based on an equivalent model transformation and require bounds for fewer terms. The idea can be demonstrated by considering the neutral system (4.26), (4.27) written as

$$\dot{x}(t) - F\dot{x}(t-h) = Ax(t-h), \;\; x(t) = \phi(t), \;\; t \in [-h, 0] \tag{4.51}$$

Taking the descriptor form with respect to the system, therefore

$$\dot{x}(t) = y(t), \;\; y(t) = Fy(t-h) + Ax(t-h) \tag{4.52}$$

The latter can be represented in the form of descriptor system with distributed delay in the "fast variable" y:

$$\dot{x}(t) = y(t), \ 0 = -y(t) + Fy(t-h) + Ax(t) - \int_{t-h}^{t} y(s)ds \tag{4.53}$$

Define $E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$, $P_1 = P_1^T > 0$, then

$$\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = x^T P_1 x \tag{4.54}$$

and, hence,

$$\frac{d}{dt} \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 2x^T(t) P_1 \dot{x}(t) = 2 \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} \tag{4.55}$$

Due to (4.53) the latter relations imply that

$$\begin{aligned} & \tfrac{d}{dt} \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ = 2 & \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} y(t) \\ -y(t) + Fy(t-h) + Ax(t) - \int_{t-h}^{t} y(s)ds \end{bmatrix} \end{aligned} \tag{4.56}$$

Thus using the descriptor method, (4.56) allows the derivative of the state $\dot{x}$ to be included as a part of the Lyapunov Krasovskii Functionals. The method also applies to retarded type systems by taking $F = 0$ in (4.51).

Remark 4.6. A summary of many of these techniques for forming LKF can be found in (Niculescu, [113], 2001). Most of these results relies on the construction on a Lyapunov Razumikhin function which allows fast variations of the delay but leads to some conservatism on the upper bound of the time delay. Following the descriptor approach by Fridman, [42], (2001), Fridman and Shaked, [54], (2003) for the first time, treated time-varying delay without any constraints on the delay-derivative (all the previous Lyapunov Krasovskii functional were applicable only to the slowly varying delays, where the delay-derivative is less than 1). He, et al., [82], (2007) uses not only the time-varying-delayed state $x(t+h(t))$ but also the delay-upper-bounded state $x(t+\bar{h})$, where $\bar{h}$ is the upper bound of the varying delay $h(t)$, when constructing LKF for exploiting all possible information for the relationship among a current state $x(t)$, an exactly delayed state $x(t+h(t))$, a marginally delayed state $x(t+\bar{h})$, and the derivative of the state $\dot{x}(t)$. The work is seen to greatly reduce the conservativeness of the computation.

4.6 Conclusion

Time delay systems can be viewed as a type of FDE, which makes it more difficult to solve the system equations when compared to ODE. A FDE with infinitely many solutions has been studied and a step-by-step method is introduced to solve the FDE as an ODE. While under certain conditions, solution of forward continuation and its smoothness of FDE can be proved, general results on backward continuation are very difficult to derive which leads to other additional properties being imposed on the FDE. Neutral systems do not have the "solution smoothing" property, when compared to the RFDE. This is due to the implied difference-equation involving $\dot{x}(t)$, whereby the trajectory may "replicate" any irregularity of the initial condition $\phi(t)$, even if the f and D in (4.26) satisfy many smoothness properties. For the stability of RFDE, necessary and sufficient conditions for the asymptotic stability of the linear time-invariant system is a straightforward generalization of the ODE theory, since it requires that the roots of the characteristic equation do not lie in the right half plane. However there exist examples with all the characteristic roots in the left hand plane but the solution is not stable. The so called *delay-independent* stability problems have been investigated using the Lyapunov-Razumikhin condition due to its simplicity. On the other hand, for *delay-dependent* stability problems, the Lyapunov-Krasovskii condition is attractive owing to the structural advantage, exposing the delay information more easily when obtaining the stability criterion. In deriving stability criteria, choice of an appropriate LKF is the key-point. For time-varying delays Krasovskii conditions are less restrictive than the Razumikhin conditions for small enough slowly varying delays. However, till now only the Rasumikhin method provides delay-independent conditions for systems with fast-varying delays. It is known that the general form of this functional (complete) leads to a complicated system of partial differential equations (Zavarei and Jamshidi, [150], 1987). That is why many authors considered special (reduced) forms of LKF and thus derived simpler (but more conservative) sufficient conditions. Among the latter there are delay-independent and delay-dependent conditions. However necessary condition for the application of the reduced LKF is asymptotic stability of the nondelayed system. If the latter conditions do not hold, the complete LKF should be applied in the case of constant delay.

In the next chapter, work in the field of sliding mode control for time delay systems will be reviewed where the analysis of the delay systems treated in this chapter will be seen to be constructive in understanding the closed-loop dynamics. The Lyaponov methods introduced have been favourably applied for control design in much of the existing works.

Chapter 5

Review of SMC for time delay systems

5.1 Introduction

Sliding Mode Control (SMC) as described in Chapter 2 has found wide applications to automotive systems, chemical processes, electrical motor control, etc. due to its ability to handle nonlinear systems and its robustness to parameter uncertainties and external disturbances. Most of the research has focused on systems without time delays. However, time delay, which is common in practical applications, is often the source of performance deterioration or even instability, as demonstrated in Chapter 4. The combination of delay phenomenon with relay actuators makes the situation much more complex, see for instance (Fridman, et al., [57], 1996) and the survey paper (Richard, et al., [119], 2001): designing a sliding mode controller without taking delays into account may lead to unstable or chaotic behaviors or, at least, produce undesirable chattering behaviours (Dambrine, et al., [22], 1998), (Fridman, et al., [57], 1996). Concerning robust stabilization of linear time delay systems with either constant or time-varying parameter uncertainties, existing methods are mainly in the time-domain and based on Krasovskii's approach and the Razumhikin approach, see Chapter 4, where the results are expressed in terms of Ricatti equations (Dugard and Verriest, [26], 1997), (Kolmanovskii and Richard, [96], 1999) or, equivalently, of LMIs (Dugard and Verriest, [26], 1997), (Kolmanovskii, et al., [95], 1999) or on the comparison approach in terms of matrix norms and measures (Dugard and Verriest, [26], 1997), (Bartholoméüs, et al., [3], 1997). Both allow one to deal with time-varying delays, whereas the frequency domain and complex plane methods (generally leading to diophantine polynomial equations) need the delays to be constant. The resulting control laws are of the continuous (often memoryless) feedback type. The importance of study in SMC for time delay systems lies in the fact that in practical situations ideal sliding surface normally does not exist due to unmodelled dynamics and time delays, see Chapter 2, hence designing a SMC taking into account possible delay effect is practically meaningful and theoretically challenging, with the potential to enhance the overall performance.

This chapter describes some research work exploring time domain approaches on the stability of delay systems using SMC, and summarises the approaches commonly adopted to treat such systems. The problems mainly fall in two areas, i.e. SMC with state delay and with input delay. In section 5.2 SMC with constant or time-varying state delay is studied in the state and output feedback context. Lyapunov Krasovskii and Razumikhin approaches which have been mainly studied for delay systems are applied for SMC design either with or without transformation to regular form. In section 5.3, effect of input delay on SMC is investigated. The fact that SMC is a discontinuous control will be destroyed by the presence of delay. A predictor-based approach for SMC design is demonstrated and limitations of the method are discussed.

5.2 State delay

In this section, an output feedback approach is articulated for systems with state delay. Stability analysis is performed using Lyapunov function and the concept of stability degree. Using equivalent control, systems with matched perturbation and constant time delay can be reduced to a delay-free system, while in the unmatched case only bounded solutions can be obtained. The use of Lyapunov Krasovskii functionals and Lyapunov Razumikhin functions formulated as LMIs in SMC state feedback, which is efficient in dealing with time delay systems, are presented and various work considering regular form-based and non regular form-based approaches is reviewed.

5.2.1 An output feedback design

El-Khazali, [35], (1998) introduced a variable structure output feedback control for uncertain state delay systems. The system is considered as

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-\tau) + f_0(x(t),t) + f_1(x(t-\tau),t) + Bu(t), \quad t>0 \\ y(t) &= Cx(t), \quad x(t) = \phi(t), \quad t \in [-\tau, 0] \end{aligned} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$. It is assumed the triple $(C,\ A_0,\ B)$ is complete and CB full rank. The nonlinear uncertainties are bounded and satisfy

$$\|f_0(x(t),t)\| \leq \alpha_0\|x(t)\|, \quad \|f_1(x(t-\tau),t)\| \leq \alpha_1\|x(t-\tau)\|$$

where α_0 and α_1 are positive, known constants. Assuming A_0 is stable in (5.1) stability of the delay system without control

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + f_0(x(t),t) + f_1(x(t-\tau),t), \quad t>0 \tag{5.2}$$

is assessed using the stability criterion:

Theorem 5.1. *(Shyu and Yan, [127], 1993) Let $x(t)$ be the solution of system (5.2). If $z(t) = e^{\beta t}x(t)$, and $z(t) = 0$ is asymptotically stable, then system (5.2) has a stability degree β. If A_0 is stable, then the system is robustly stable with stability degree*

$$\alpha_0 + \gamma\alpha_1 e^{\beta\tau} < \frac{\lambda_{min}(Q) - 2\gamma e^{\beta\tau}\|A_1^T P\|}{2\|P\|}$$

where $\lambda_{min}(Q)$ denotes the minimum eigenvalue of Q and P is a positive-definite matrix solution to the Lyapunov equation

$$(A_0 + \beta I)^T P + P(A_0 + \beta I) = -Q$$

for a positive-definite symmetric matrix, Q, and where

$$\gamma \triangleq \sqrt{\lambda_{max}(P)/\lambda_{min}(P)}.$$

The switching function is designed separately for the case of *matched uncertainties and delay* and the case of *unmatched uncertainties and delay*. In the matched case, $\mathscr{R}[A]_1 \subseteq \mathscr{R}[B]$ and $\mathscr{R}[f_0 + f_1] \subseteq \mathscr{R}[B]$. Defining a switching surface $\sigma = Sy(t) = 0$ where $S \in \mathbb{R}^{m\times p}$, the method of equivalent control yields an *internal state-equivalent system* (El-Khazali and DeCarlo, [37], 1995) of the form

$$\begin{aligned} \dot{x}_{eq}(t) &= \mathbf{P}\{A_0 x_{eq}(t) + A_1 x_{eq}(t-\tau) + (w_0 + w_1)\} \\ &\triangleq A_{0eq}x_{eq}(t) + A_{1eq}x_{eq}(t-\tau) + \mathbf{P}(w_0 + w_1) \end{aligned} \tag{5.3}$$

where $rank[\mathbf{P}] = n - m$ and $\mathbf{P} \triangleq [I_n - B(SCB)^{-1}SC]$ is a projector which projects out all variables that lie in the column-space of the input matrix, B. Then for matched perturbations, the equivalent system described by (5.3) yields

$$\dot{x}_{eq}(t) = A_{0eq}x_{eq}(t), \quad \sigma = SCx_{eq}(t) \tag{5.4}$$

Therefore, the infinite-dimensional system is reduced to a finite-dimensional one, and the problem reduces to a delay-free system. Conventional output feedback tools can thus be used for the design, see Chapter 3.

For the unmatched case, transforming (5.3) into the regular form yields

$$\begin{bmatrix} \dot{x}_{eq1}(t) \\ \dot{x}_{eq2}(t) \end{bmatrix} = \begin{bmatrix} A_{01} & A_{02} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{eq1}(t) \\ x_{eq2}(t) \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{eq1}(t-\tau) \\ x_{eq2}(t-\tau) \end{bmatrix} + \begin{bmatrix} v_{eq} \\ 0 \end{bmatrix} \tag{5.5}$$

and $\sigma = Sy(t) = 0$ yields

$$\sigma = S\begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_{eq1}(t) \\ x_{eq2}(t) \end{bmatrix} = 0 \tag{5.6}$$

The reduced-order system dynamics is given by

$$\dot{x}_{eq1} = A_{0r}x_{eq1}(t) + A_{1r}x_{eq1}(t-\tau) + v_{eq} \tag{5.7}$$

where $A_{0r},\ A_{1r} \in \mathbb{R}^{n-m}$ depend on S and v_{eq} is bounded by $x_{eq1}(t)$ and $x_{eq1}(t-\tau)$. The switching matrix S can be chosen such that the standard pole assignment method can assign the $n-m$ eigenvalues of A_{0r} in the left-half plane. Then to achieve robustness in the sliding mode, Theorem 5.1 can be applied into equation (5.7) for stability analysis.

Note the switching function $\sigma = Sy(t) = 0$ is only sufficient to derive a stable sliding motion if the system satisfies certain observability conditions in the output feedback design. When these observability conditions are violated, then an integral switching function is proposed which assigns all n- eigenvalues of the nominal system. Consider the switching function with integral terms

$$\Omega(t) = \sigma(t) - \int_0^t F_0 y(\theta)d\theta - \delta \int_0^t F_1 y(\theta-\tau)d\theta = 0, \quad \text{where } \sigma = Sy(t) \tag{5.8}$$

where $F_0,\ F_1 \in \mathbb{R}^{m\times p}$ are two real matrices to be determined. The use of such form of switching function to system (5.1) yields a state feedback equivalent system of the form

$$\dot{x}_{eq}(t) = \tilde{A}_0 x_{eq}(t) + \tilde{A}_1 x_{eq}(t-\tau) + (\tilde{f}_0 + \tilde{f}_1) \tag{5.9}$$

where $\tilde{A}_0,\ \tilde{A}_1 \in \mathbb{R}^n$ and $\tilde{f}_0,\ \tilde{f}_1$ are bounded. In the switching function (5.8), the matrix S can be determined to transform the system (5.9) into an observable system, the use of F_0 assigns the n-poles of the non-delay term in the closed loop and the integral term F_1 is used to minimize the effect of delay variables in the sliding mode. A control law can then be designed as a function of the switching variable (5.8) to achieve stability performance.

The work by El-khazali, [35], (1998) exhibits asymptotic stability for matched delay and perturbations and bounded stability when they are unmatched. Stability with delay is derived using Lyapunov functions and the inequality method, which can be conservative as compared to other design tools such as Linear Matrix Inequalities (LMIs). The integral sliding mode was introduced to overcome some conditions imposed on the system such as the Kimura-Davison condition and the observability of the reduced order system. However the drawback is that such surfaces do not yield a reduction of system order in the sliding mode, increasing the complexity of the controller design. In the following section, Lyapunov Krasovskii Functionals (LKF) and the Razumikhin method are used for stability analysis of time delay systems using SMC. The problems are formulated as LMIs which is numerically efficient using the standard Matlab tool box (Gahinet, et al., [62], 1995).

5.2.2 Lyapunov-Krasovskii functionals and Razumikhin functions

In this section Lyapunov-Krasovskii functionals and the Razumikhin method are demonstrated in the context of SMC design. Both constant and time-varying time delays are considered. Synthesis of the controller is formulated from regular form transformation and original form based perspectives.

Regular form based approach

As mentioned in Chapter 2 the use of the regular form allows to simplify the design procedure by considering existence problem and reachability problem separately. In the context of SMC for delay systems, SMC with state delay is considered in (Gouaisbaut, et al., [70], 2002), (Orlov, et al., [115], 2003), (Xia and Jia, [144], 2003) and (Fridman, et al., [50], 2003), where regular-form based approach was employed in deriving the stability of the reduced order system using LKF. In control law design for a known constant delay, Gouaisbaut, et al., [70], (2002), Xia and Jia, [144], (2003) assumes both delay-free states and delayed states are available. For the case of unknown bounded delay, a model transformation method along with Razumikhin's approach was used in (Gouaisbaut, et al., [70], 2002) in deriving stability conditions of the reduced order system in the form of LMIs. An upper bound on the delayed state is required in deriving the condition for reachability to the sliding surface (Gouaisbaut, et al., [70], 2002), (Xia and Jia, [144], 2003), (Fridman, et al., [50], 2003). To briefly describe the approach, consider the following delay system (Gouaisbaut, et al., [70], 2002)

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-h(t)) + Bu(t) + f_1(t,x_t),\ t>0, \\ x(t) &= \phi(t), \quad \text{for } t\in[-h_{max},0],\quad 0\le h(t)\le h_{max} \end{aligned} \tag{5.10}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, $f_1\in\mathbb{R}^k$. It is assumed that the pair $(A+A_d,\ B)$ is controllable. The disturbance is assumed to be matched, i.e. $f_1(x_t,t) = Bf(x_t,t)$ where $\|f\|\le\psi(x_t)$, $\psi(x_t)$ is a known functional and B is full rank: $rank(B)=m$. Suppose the delay is constant. The sliding surface is chosen as

$$s(x) = Sx = B^T X^{-1} x = 0 \tag{5.11}$$

where $S\in\mathbb{R}^{m\times n}$ and X is a matrix to be chosen. The control is of a form

$$u(t) = -(SB)^{-1}\big(SAx(t) + SA_d x(t-h) - \Lambda s(x) + m_0\frac{Ps(x)}{\|Ps(x)\|}\big), \tag{5.12}$$

where $\Lambda\in\mathbb{R}^{m\times m}$ satisfies $\Lambda^T P + P\Lambda = -I$ and P is a positive definite matrix. The switching gain $m_0 = m_1 + \|SB\|\psi(x_t)$ where $m_1>0$ is a real number. Define a positive function as

$$V(t) = s^T(x(t))Ps(x(t)) \tag{5.13}$$

with respect to the sliding surface (5.11). Its derivative along the trajectories of (5.10) is

$$\begin{aligned} \dot{V}(t) &= s^T(x)(\Lambda^T P + P\Lambda)s(x) + 2s^T(x)P(SBf - m_0\frac{Ps(x)}{\|Ps(x)\|}) \\ &= -s^T(x)s(x) + 2s^T(x)P(SBf - m_0\frac{Ps(x)}{\|Ps(x)\|}) \\ &\le 2(\|SB\|\psi(x_t) - m_0)\|Ps(x)\| \\ &\le -2m_1\|Ps(x)\| \le -2m_1\sqrt{\lambda_{min}(P)}\sqrt{V(t)} \end{aligned} \tag{5.14}$$

where $\lambda_{min}(P)$ is the smallest eigenvalue of the matrix P. Hence the reachability to the sliding surface is proved. Performing a nonsingular transformation the reduced order dynamics governing the sliding motion leads to

$$\dot{z}_1(t) = \hat{A}_{11}z_1(t) + \hat{A}_{d11}z_1(t-h) \tag{5.15}$$

where $\hat{A}_{11} = \tilde{B}^T AX\tilde{B}(\tilde{B}^T X\tilde{B})^{-1}$, $\hat{A}_{d11} = \tilde{B}^T A_d X\tilde{B}(\tilde{B}^T X\tilde{B})^{-1}$ and $\tilde{B}^T B = 0$. Take the derivative of the LKF

$$V(t) = y^T(t)Py(t) + \int_{t-h}^{t}\int_{s}^{t} z_1^T(w)\hat{A}_{d11}^T PQ^{-1}P\hat{A}_{d11}z_1(w)dwds \tag{5.16}$$

where $y(t) = z_1(t) + \int_{t-h}^{t}\hat{A}_{d11}z_1(w)dw$, P, $Q \in \mathbb{R}^{n\times n}$ are positive definite matrices, then (5.15) is stable if the following LMIs are feasible.

$$\begin{bmatrix} h^{-1}[X(A+A_d)^T + (A+A_d)X] & XA_d^T & X(A+A_d)^T \\ * & -X & 0 \\ * & * & -X \end{bmatrix} - \sigma \begin{bmatrix} BB^T & 0 & 0 \\ * & BB^T & 0 \\ * & * & BB^T \end{bmatrix} < 0 \tag{5.17}$$

where $\sigma \in \mathbb{R}$. Thus the reachability and existence problems for constant delay have been solved. The control (5.12) requires knowledge of both non-delayed and delayed states. The assumption of a known bound on the state-dependent terms for switching gain design can be restrictive. In the existence design, LKF is used to form LMIs to ensure for stability of the reduced order system. In the following, the Razumikhin's approach based on the model transformation on the reduced order sliding mode dynamics is given for stability design with unknown bounded delay.

When the delay in (5.10) is time-varying, a control of the form

$$u(t) = -(SB)^{-1}\left(SAx(t) - \Lambda s(x) + m_0\frac{Ps(x)}{\|Ps(x)\|}\right) \tag{5.18}$$

where $m_0 = m_1 + \|SB\| \sup_{s\in[-h_{max},0]} \psi(x_t(s)) + \|SA_d\| \sup_{s\in[-h_{max},0]} \|x(t+s)\|$ will drive the system trajectories to the sliding surface (5.11) (Gouaisbaut, et al., [70], 2002). Here the control requires the supreme bound of the delayed states from initial time. The existence problem for the reduced order system (5.15) can be solved using the Leibnitz-Newton formula

$$\dot{z}_1(t) = (\hat{A}_{11} + \hat{A}_{d11})z_1(t) - \int_{t-h}^{t}\hat{A}_{d11}\dot{z}_1(v)dv \tag{5.19}$$

which produces

$$\dot{z}_1(t) = (\hat{A}_{11} + \hat{A}_{d11})z_1(t) - \int_{t-h}^{t}\hat{A}_{d11}\hat{A}_{11}z_1(v)dv - \int_{t-h}^{t}\hat{A}_{d11}\hat{A}_{d11}z_1(v-h)dv \tag{5.20}$$

Choosing a Lyapunov function

$$V(t) = z_1^T(t)Pz_1(t)$$

where $P > 0$, its derivative along the transformed system (5.20) is

$$\dot{V} = z_1^T(t)(Z^T P + PZ)z_1(t) - 2\int_{t-h}^{t} z_1^T(t)P\hat{A}_{d11}\hat{A}_{11}z_1(v)dv - 2\int_{t-h}^{t} z_1^T(t)P\hat{A}_{d11}^2 z_1(v-h)dv \quad (5.21)$$

where $Z = \hat{A}_{11} + \hat{A}_{d11}$. Following Razumikhin's approach

$$V(x(t+\theta)) < qV(x(t))$$

with $q > 1$ for $\theta \in [-h_{max}, 0]$, and using the inequality

$$x^T y \leq x^T X^{-1} x + y^T X y$$

where vectors $x,\ y \in \mathbb{R}^n$ and $X \in \mathbb{R}^{n\times n}$ is positive-definite, it follows that

$$\dot{V}(t) < z_1^T(t)Nz_1(t) \quad (5.22)$$

where $N = Z^T P + PZ + h\alpha^{-1}P\hat{A}_{d11}\hat{A}_{11}S\hat{A}_{11}^T\hat{A}_{d11}^T P + qh(\alpha+\beta)P + h\beta^{-1}P\hat{A}_{d11}^2 S\hat{A}_{d11}^{2^T} P$. Condition (5.22) for which the reduced order system (5.20) is asymptotically stable is reformulated as the following LMIs

$$\begin{gathered}\begin{bmatrix} \alpha X & AX \\ * & X \end{bmatrix} - \gamma_1 \begin{bmatrix} BB^T & 0 \\ * & BB^T \end{bmatrix} > 0, \\ \begin{bmatrix} \alpha X & A_d X \\ * & X \end{bmatrix} - \gamma_2 \begin{bmatrix} BB^T & 0 \\ * & BB^T \end{bmatrix} > 0, \\ \begin{bmatrix} h^{-1}(H^T + H) + q(\alpha+\beta)X & A_d X \\ * & -\frac{1}{2}X \end{bmatrix} - \sigma \begin{bmatrix} BB^T & 0 \\ 0 & BB^T \end{bmatrix} < 0 \end{gathered} \quad (5.23)$$

where $H = (A + A_d)X,\ \sigma \in \mathbb{R},\ \alpha,\ \beta \in \mathbb{R}^+$. Hence, Razumikhin's approach has been used in deriving a delay dependant stability condition for the time-varying delay system. Results are formulated in an LMI framework. This approach may lead to conservative results for the case of time-varying delay because the delay term, which is estimated as a function of the non-delay term, is not fully exploited, as compared with the Lyapunov Krosovskii approach (Fridman and Shaked, [54], 2003), (Moon, et al., [110], 2001) and (Shao, [125], 2009).

Non regular form based approach

Li and Decarlo, [105], (2003) considered SMC design for delay systems with uncertainties, where transformation to regular form is not needed. Stability of the sliding motion and synthesis of a controller were derived with respect to the full order system. The system was described as

$$\dot{x}(t) = (A_0 + \Delta A_0(t))x(t) + \sum_{i=1}^{2}(A_i + \Delta A_i(t))x(t-h_i) + (B + \Delta B(t))u(t) + \Delta f(t) \quad (5.24)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. The matrices $\Delta A_i(t)$, $i = 0,1,2$, and $\Delta B(t)$ are unknown time-varying system parameter uncertainties, $\Delta f(t) = Bf(t)$, with $\|f(t)\| \leq \delta_f$. h_i, $i = 1,2$ are time delays, $x_t(\theta) = x(t+\theta)$ for $\theta \in [-h,0]$, $h = \max(h_i)$. It is assumed that

- $(A_0,\ B)$ is stabilizable, i.e. there exist matrix K such that $\bar{A}_0 = A_0 - BK$ is stable.
- $\Delta B(t)$ is matched, i.e. there exists a matrix $D(t) \in \mathbb{R}^{m\times m}$ such that $\Delta B(t) = BD(t)$ with $\|D(t)\| \leq \delta_b$. B and $B + \Delta B(t)$ are assumed to be of full column rank for all t.
- (Structured perturbation) $\Delta A_i(t) = H_i F_i(t) E_i$ for $i = 0,1$ and 2, where H_i and E_i are known constant matrices with appropriate dimension and F_i is unknown, but satisfies $F_i^T(t) F_i \leq I$.

The sliding surface is defined as

$$\sigma(x,t) = \Gamma B^T P x(t) = 0 \tag{5.25}$$

where $\Gamma \in \mathbb{R}^{m\times m}$ is a positive definite diagonal matrix, and $P \in \mathbb{R}^{n\times n}$ is a positive definite matrix to be defined. Then an existence condition for a stable sliding motion is derived based on the system (5.24) in the form of algebraic Riccati equations (AREs). Choose a LKF

$$V(x,t) = x^T(t)Px(t) + \int_{t-h_1}^{t} x^T(\alpha)P_1x(\alpha)d\alpha + \int_{t-h_2}^{t} x^T(\beta)P_2x(\beta)d\beta$$

and rewrite system (5.24) as

$$\dot{x}(t) = (\bar{A}_0 + \Delta A_0(t))x(t) + \sum_{i=1}^{2}(A_i + \Delta A_i(t))x(t-h_i) + B[(I_m + D(t))\tilde{u}(t) + f(t) - D(t)Kx(t)]$$

where $\tilde{u}(t) = u(t) + Kx(t)$, $\tilde{A}_i = A_i + \Delta A_i(t)$, $i = 1,2$. Differentiating $V(x,t)$

$$\begin{aligned} \dot{V}(x,t) &= x^T(t)\left[P(\bar{A}_0 + \Delta A_0) + (\bar{A}_0 + \Delta A_0)^T P + \textstyle\sum_{i=1}^{2} P_i\right]x(t) + 2x^T(t)P[\textstyle\sum_{i=1}^{2}\tilde{A}_i x(t-h_i)] \\ &- \textstyle\sum_{i=1}^{2} x^T(t-h_i)P_i x(t-h_i) + 2x^T(t)PB[(I_m + D(t))\tilde{u}(t) + f(t) - D(t)Kx(t)] \end{aligned} \tag{5.26}$$

On the sliding surface, $\sigma(x,t)^T = x(t)^T PB = 0$, this expression reduces to a quadratic form

$$\begin{aligned} \dot{V}(x,t) &= x^T(t)\left[P(\bar{A}_0 + \Delta A_0) + (\bar{A}_0 + \Delta A_0)^T P + \textstyle\sum_{i=1}^{2} P_i\right]x(t) \\ &+ 2x^T(t)P[\textstyle\sum_{i=1}^{2}\tilde{A}_i x(t-h_i)] - \textstyle\sum_{i=1}^{2} x^T(t-h_i)P_i x(t-h_i) \end{aligned} \tag{5.27}$$

It is shown that equation (5.27) is negative definite if

$$P(\bar{A}_0 + \Delta A_0) + (\bar{A}_0 + \Delta A_0)^T P + \sum_{i=1}^{2}[P_i + P\tilde{A}_i P_i^{-1}\tilde{A}_i^T P] < 0 \tag{5.28}$$

A set of ARE conditions can be derived that guarantee inequality (5.28) holds and the AREs are transformed to their equivalent LMIs using the Schur complement.

In the reachability design, a controller in the form

$$u(t) = -Kx(t) + u_{eq,nom}(t) + u_{comp}(t) + u_n(t) \tag{5.29}$$

is shown to drive the uncertain system (5.24) to the sliding surface (5.25) in finite time. The equivalent control term

$$u_{eq,nom}(t) = -(B^TPB)^{-1}[B^TP\bar{A}_0x(t) + \sum_{i=1}^{2} B^TPA_ix(t-h_i)] \tag{5.30}$$

is used to overcome the known terms $B^TP\bar{A}_0x(t)$ and $\sum_{i=1}^{2} B^TPA_ix(t-h_i)$. The component $u_{comp}(t)$ is designed to overcome the structured parameter uncertainties $\Delta A_i(t)$, $i = 0,1,2$, and is of the form

$$u_{comp}(t) = -\frac{\|\sigma(x,t)^TB^TP\|B^TPB\sigma(x,t)}{\|B^TPB\sigma(x,t)\|^2}[f_0\|x(t)\| + \sum_{i=1}^{2} f_i\|x(t-h_i)\|]$$

where $f_i \geq \|H_i\|\|E_i\| + \delta_b\|B\|\|(B^TPB)^{-1}B^TPA_i\|$ for $i = 0,1,2$. The component

$$u_n(t) = -(B^TPB)^{-1}d(x)\,\mathrm{sign}(\sigma(x,t))$$

where $d(x) \geq \|B^TPB\|(\beta\|x(t)\| + \kappa) + \alpha$ with $\alpha,\ \beta,\ \kappa > 0$ functions rejecting the external disturbance $\Delta f(t)$. Using the control law (5.29), the derivative of the Lyapunov function $\dot{V}(\sigma(x,t)) \leq -\alpha\|\sigma(x,t)\|$ where $V(\sigma(x,t)) = 0.5\sigma^T(x,t)\sigma(x,t)$.

Since a priori knowledge of the upper bounds on β and κ is needed in the above control design, an adaptive estimation algorithm was used to eliminate the need for the explicit knowledge of the bound. System (5.24) with control (5.29) is asymptotically stable if

$$d(x) \geq \|B^TPB\|(\hat{\beta}\|x(t)\| + \hat{\kappa}) + \alpha$$

where

$$\dot{\hat{\beta}} = \gamma_1\|\sigma(x,t)\|\|B^TPB\|\|x(t)\|, \quad \dot{\hat{\kappa}} = \gamma_2\|\sigma(x,t)\|\|B^TPB\|$$

where $\gamma_i > 0$ for $i = 1$ and 2. The γ_i control the rate of convergence, i.e. larger γ_i yields faster convergence. The adaptive control reduces the switching gain by reducing the adaptation parameters γ_i. This leads to reduction of chattering in practical applications. However it was shown that the adaptive control cannot guarantee finite time stability. In the case where the delay is unknown, then the equivalent control (5.30) should no longer depend on $x(t-h_i)$, and an additional assumption on the bound of the delay term was introduced as $\|x(t-h_i)\| \leq \phi_i\|x(t)\| + \varphi_i$ for some scalars $\phi_i, \varphi_i > 0$ in the reachability condition derivation.

In this section SMC for state delay system is considered. An equivalent control for matched uncertainties and constant delay reduces the system to a delay free system. The choice of integral switching function allows an output feedback controller to be designed when certain system structures and observability conditions are not met. LKF and Razumikhin methods have been the main tools for stability analysis of these time delay systems. Control synthesis of SMC in terms of regular form transformation and without transformation are demonstrated. For sufficiently small state delay, asymptotic stability can be achieved.

5.3 Input delay

While SMC for the state delay case can achieve asymptotic stability, only bounded solutions can be obtained in the presence of input delay in SMC due to the delayed switching effect. This section shows the delay effects on relay control by demonstrating the oscillation and bifurcations in such systems. Steady modes and stability of the oscillations are analyzed by various authors. A predictor based approach which transforms the delay system into a delay free system is demonstrated. The limitations of such approaches in the uncertain case are discussed.

5.3.1 Behaviour analysis on sliding mode

The effect of delay on SMC was investigated in (Gouaisbaut, et al., [71], 2002). It was shown the presence of delay can induce oscillations around the design surface and possible behavioural changes (bifurcations) arising in such relay delay systems. Bounded solutions were obtained which estimate that the solutions starting from a bounded initial condition will enter into another bounded region. To demonstrate the idea consider the system

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t), \ \ \forall_i = 1,\dots(n-1) \\ \dot{x}_n(t) &= f(t,x) + g(t,x)u(t-\tau) \\ y(t) &= x_1(t) \end{aligned} \tag{5.31}$$

where $|f(t,x)| < M$ and $\tau > 0$ is a constant delay. In the normal design procedure without the delay τ, a linear sliding manifold $s(x)$ is selected as

$$s(x) = \sum_{i=1}^{n} a_i x_i(t), \ \ a_n = 1 \tag{5.32}$$

with the a_i coefficients determined to ensure $a_0 + a_1 x(t) + \ldots + x^n(t)$ is a Hurwitz polynomial. Reachability of the corresponding state sliding motion is satisfied by ensuring $\dot{s}(t) = -k\,\mathrm{sign}\,s(t)$, where $k > 0$

$$\begin{aligned} u(t) &= u_{eq}(t,x(t)) - \tfrac{k}{g}\,\mathrm{sign}\,s(t) \\ u_{eq}(t,x(t)) &= -\tfrac{1}{g}\left(\textstyle\sum_{i=1}^{n-1} a_i x_{i+1}(t) + f(t,x(t))\right) \end{aligned} \tag{5.33}$$

In the presence of delay the control law (5.33) will become

$$u(t-\tau) = u_{eq}(t,x(t-\tau)) - \frac{k}{g}\,\mathrm{sign}\,s(t-\tau) \tag{5.34}$$

The derivative of the switching function (5.32) with the control (5.34) becomes

$$\dot{s}(t) = \Sigma_{i=1}^{n-1} a_i x_{i+1}(t) - \Sigma_{i=1}^{n-1} a_i x_{i+1}(t-\tau) + f(t,x(t)) - f(t,x(t-\tau)) - k\,\mathrm{sign}\,s(t-\tau) \quad (5.35)$$

i.e.

$$\dot{s}(t) = g\Delta_t^{t-\tau}u_{eq} - k\,\mathrm{sign}\,s(t-\tau), \quad \text{where } \Delta_t^{t-\tau}u_{eq} = u_{eq}(t,x(t-\tau)) - u_{eq}(t,x(t)) \quad (5.36)$$

The solutions of the switching function (5.32) are derived by taking the derivative of the Lyapunov function $V(x(t)) = \frac{1}{2}s^2(x(t))$. Assuming $|\Delta_t^{t-\tau}u_{eq}| < M\tau$ for sufficiently small τ

$$\dot{V}(x(t)) < (gM\tau - k)\sqrt{V(x(t-\tau))} + \tau(k^2 + g^2M^2) \quad (5.37)$$

holds. Therefore

$$gM\tau < k \quad (5.38)$$

is a necessary condition for bounded stability. The bounded region of the switching function is obtained as

$$\mathscr{R}_\infty = \{x \in \mathbb{R}^n \; : \; s^2(x(t)) < 2v_\infty\} \quad (5.39)$$

where $v_\infty^2 = \frac{\tau^2(k^2+g^2M^2)^2}{(k-gM\tau)^2}$. Local stability analysis shows those states starting from an initial condition bounded within the region

$$\mathscr{I}_0 = \{x \in \mathbb{R}^n \; : \; |s^2(x) - 2v_\infty| < v_\infty^2 \frac{2v_\infty - \alpha\tau}{2v_\infty + \alpha\tau}\} \quad (5.40)$$

will converge to the bounded domain (5.39) if the following condition holds

$$\sqrt{2}(k^2 + g^2M^2) > (k - gM\tau) > 0 \quad (5.41)$$

Example 5.1. Consider

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_1(t)x_2(t) + u(t-\tau) \end{aligned} \quad (5.42)$$

Define the switching function as $s(x) = x_2 + 2x_1$ and choose the control

$$u(t-\tau) = -x_1(t-\tau)x_2(t-\tau) - 2x_2(t-\tau) - k\,\mathrm{sign}\,s(t-\tau) \quad (5.43)$$

where $k = 10$. The time delay $0.1s$ has been neglected in the control design. Choosing $M \approx 30$ in (5.31), the condition (5.38) is satisfied such that solutions of system (5.42) are boundedly stable. For the given initial conditions, $v_\infty \approx 14.3$ is obtained. It can be checked that the condition (5.41) is valid. Then for solutions starting from the set

$$\mathscr{I}_0 = \{x \in \mathbb{R}^n \; : \; |s^2(x) - \frac{200}{7}| < 193\} \quad (5.44)$$

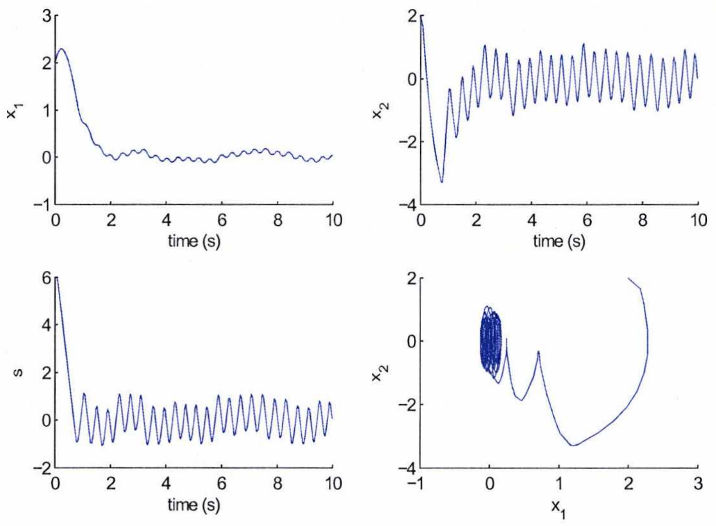


FIGURE 5.1: system (5.42) with control (5.43), designed without care of a delay $\tau = 0.1s$. (Figures are generated using Simulink ode1(Euler) solver with fixed-step size $0.0001s$)

(as M does not increase) reach the set

$$\mathscr{R}_\infty = \{x \in \mathbb{R}^n \; : \; s^2(x) < 28.6\} \tag{5.45}$$

Simulation results are shown in Figure 5.1. It can be seen that x_1 and x_2 have one oscillation frequency leading to a limit cycle. It has been demonstrated that as the parameters k and τ change, bifurcations occur.

As demonstrated in the above example the presence of relay delay within a SMC induces oscillations around the design surface. Possible behavioural changes (bifurcations) arise in such relay delay systems.

5.3.2 Steady modes and stability

Some work (Fridman, et al., [58], 2002), (Barton, et al., [4], 2005) has studied one-dimensional prototype examples of the form (5.46) and found that this type of system typically admits periodic orbits that switch back and forth between the two vector fields. Moreover, they have classified all the possible dynamics of system (5.46) completely and studied the behaviour with respect to perturbations, including periodic forcing.

An algorithm for controlling the amplitudes of the motion is proposed in (Fridman, et al., [58], 2002). Since after finite time all solutions coincide with the periodic solution, one can extrapolate the next zero for the periodic solution, and reduce the control gain near to the periodic solution zero. The algorithm assumes constant delay and requires the knowledge of the sign of the state variable with the delay, which is obtained using an observer. Consider the problem

$$\begin{aligned} \dot{x}(t) &= -\operatorname{sign} x(t-1) + F(x(t),t), \quad t \geq 0 \\ |F(x,t)| &\leq p < 1, \\ x(t) &= \varphi(t),\ t \in [-1,0], \quad \varphi \in \mathscr{C}[-1,0] \end{aligned} \tag{5.46}$$

for which there exists a unique continuous solution $x_\varphi(t)$, $t \in [-1,\infty)$ (Kolmanovskii and Myshkis, [97], 1992). It has been shown that any motion of the system (5.46) turns into a steady mode after a period of time, with a motion of constant frequency. The oscillation of finite frequency is a function of the delay, switching gain and disturbance. Hence, for the delay relay control system

$$\dot{x} = -kx + F(t,x), \quad x \in \mathbb{R}, \quad k > 0 \tag{5.47}$$

where $|F(t,x)| \leq \varepsilon$ represents a bounded perturbation, a control law of the form

$$u(t) = -\lambda \operatorname{sign} x(t-\gamma)$$

where λ is the switching gain and $\gamma > \varepsilon$ is assumed to yield a stable bounded solution of the closed-loop system

$$\dot{x} = -kx + F(t,x) - \lambda \operatorname{sign} x(t-\gamma) \tag{5.48}$$

Then the amplitude is estimated as

$$|x(\gamma)| \leq \int_0^\gamma e^{-k(\gamma-\tau)}(|\lambda| + |F(\tau,x(\tau))|)d\tau \leq \frac{\gamma+\varepsilon}{k}(1-e^{-k\gamma}) \leq \gamma(\lambda+\varepsilon) \tag{5.49}$$

i.e. the delay $\gamma \to 0$, $|x(\gamma)| \to 0$. Since the oscillation amplitude is also a function of the switching gain, Fridman, et al. [58], (2002) showed that a smaller region of attraction of system (5.47) can be achieved by the following adaptive control law

$$\begin{aligned} \dot{x}(t) &= F(x,t) + u(t) \\ u(t) &= \alpha(t) \operatorname{sign} x(t-1) \end{aligned} \tag{5.50}$$

Assuming the following conditions hold

$$F(0,t) \equiv 0, \quad \tfrac{\partial F}{\partial x}(x,t) \leq k < \ln 2, \quad t \in \mathbb{R}, \quad |x| < \alpha/k \tag{5.51}$$

then all the solutions of the equation

$$\dot{x}(t) = F(x,t) + \alpha \operatorname{sign} x(t-1)$$

where the initial conditions start from the bound

$$|x(0)| = |\varphi(0)| < \alpha(2\exp(-k) - 1)/k \tag{5.52}$$

enter into the following bound

$$|x_\varphi(t)| \leq \frac{\alpha}{k}(e^k - 1), \;\; |\dot{x}_\varphi(t)| \leq \alpha e^k \tag{5.53}$$

Using this relationship and an observer which is designed to estimate the zeros of $x(t)$ and signs of $x(t)$ with the delay 1, the switching gain $\alpha(t)$ can be decreased. Hence the bound on the solution decreases. Moreover as the step of the minimization tends to infinity, asymptotic stability can be achieved.

Fridman, et al. [60], (2003) showed local stability of the system (5.48) for the simplest scalar case and the result was extended to single-input single-output (SISO) and multi-input multi-output (MIMO) controllable systems. In (Fridman, et al., [61], 2004) non-local stabilization of the amplitudes of the oscillations for time-varying delay was demonstrated which requires knowledge of the solution amplitudes at the delayed time moment. This suggests that for any initial condition $\varphi(t) : |\varphi(0)| \leq \mathbb{R}$ in (5.48) there exists an instant of time $T > 0$, such that for any $t > T$, a control law can be designed to ensure the bounded solution $|x(t)| < \varepsilon$, $0 < t < \infty$, where $\varepsilon > 0$ is selected apriori. The algorithm first finds conditions ensuring the magnitude of the steady oscillations r_∞ is less than the magnitude of the initial conditions r_0: $\frac{r_\infty}{r_0} < 1$. First assume the size of stabilization domain for each value of relay delayed control gain is the amplitude of the initial conditions for the next step, then finally by decreasing the control gain, the system enters into a smaller neighbourhood of zero. The proposed control law requires knowledge of the amplitude of the solutions at the delayed time instant; the upper bound of the time delay; upper bound of initial conditions and the size of the desired neighbourhood of the zero solution.

The approaches proposed for stability of relay delay systems by Fridman, et al., [58], (2002), [61], (2004) are attractive as they suggest algorithms to reduce steady state oscillation in the presence of relay delay. However an observer is either used, which introduces additional dynamics, or solution amplitudes at the delayed time moment need to be known introducing complexities into the controller design.

In order to understand the possible dynamics and bifurcations in relay delay equations, Barton, et al., [4], (2005) proposed a combination of numerical and analytical methods for a particular type of model using relay control, formulated as a piecewise-constant delay differential equation (DDE). Numerical solutions of a related equation, where the discontinuities of the original DDE are smoothed out, are used to guide the construction of explicit solutions of the original DDE. On the other hand, the construction of explicit solutions provides initial data for numerical continuation of the smoothed equation. The cameo model for a relay controller of an externally forced system with delayed feedback, namely the non-smooth equation is described as

$$\dot{x} = \operatorname{sign}\left(f_A^T(t) - x(t-1)\right) \tag{5.54}$$

where $f_A^T(t) = Af(t/T)$, and f is a unbiased square-wave function with forcing period T. To apply numerical continuation of the system (5.54), the following smoothed approximation of the function (5.54) is given as

$$\dot{x} = \tanh\left(\frac{f_A^T(t) - x(t-1)}{\varepsilon}\right) \tag{5.55}$$

where $f_A^T = Ay(\frac{2(1+\delta)t}{T})$ and $y(t)$ is the solution of

$$\delta\dot{y} = -y(t) - (1+\lambda)y(t-1) + by^3(t-1) \tag{5.56}$$

The DDE (5.54) is known to have square-wave-like solutions for appropriate parameter selections.

It is worth noting that the equation (5.54) is a generalization of the system (5.46) considered in (Fridman, et al., [58], 2002), where $F(x(t),t) = \dot{f}_A^T$. The problem considered in (Fridman, et al., [58], 2002) does not encompass forcing functions $F(x(t),t)$ that have unbounded or discontinuous derivatives, such as the squarewave considered in (Barton, et al., [4], 2005). The approach of considering a system for which solutions can be constructed explicitly is also taken in (Bayer and Heiden, [5], 1998) and (Norbury and Wilson, [114], 2000). Norbury and Wilson, [114], (2000) considered a period-T forced linear delay model with saturation, where period-T solutions denote the solution period that is equal to the forcing period-T. For sufficiently large forcing amplitude, they constructed a period-T forced, linear delay model with saturation. Period-T solutions were constructed, but due to the more complicated form of the model sharp existence boundaries are not derived. Bayer and Heiden, [5], (1998) studied a model for delayed relay control in the form of a second order DDE (without forcing). They use analytical techniques to construct explicit solutions and numerical simulation of the initial value problem to investigate the stability of the solutions.

The continuation results obtained in (Barton, et al., [4], 2005) are robust with respect to the degree of smoothing applied to the non-smooth equation. The numerical continuation of a suitably smoothed delay equation may prove useful when dealing with a general piecewise-smooth delay equation, for which explicit solutions cannot be constructed. One of the applications of this solution construction method can be to study the dynamics and bifurcations of a DDE. A key problem is to find suitable solutions from which to start a bifurcation study, using numerical continuation.

In (Sieber, [128], 2006) forward evolution $E(t,\cdot)$ for a general n-dimensional system of differential equations of the form

$$\dot{x}(t) = \begin{cases} f_1(x(t)) & \text{if } g(x(t-\tau)) < 0 \\ f_2(x(t)) & \text{if } g(x(t-\tau)) \geq 0 \end{cases} \tag{5.57}$$

is studied, which maps an initial value $x_0 \in \mathscr{C}([-\tau,0],\ \mathbb{R}^n)$ to its time-T image $E(T,x_0) \in \mathscr{C}([-\tau,0],\ \mathbb{R}^n)$. The method of steps (Diekmann, et al., [25], 1995), (Hale and Verduyn Lunel, [77], 1993) was used. The past $x_0(t)(t \in [-\tau,0])$ is treated as an inhomogeneity, the ensuing ODE is solved for all times up to τ, and then the history is shifted before the process is repeated. It was shown that the evolution $E(\cdot,x_0)$ does not depend on the complete shape of $x_0 \in \mathscr{C}([-\tau,0],\mathbb{R}^n)$ but only on the

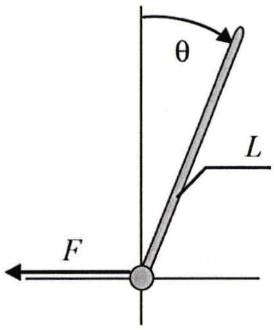


FIGURE 5.2: Sketch of the setup for the controlled inverted pendulum on a cart

position of $x_0(0) \in \mathbb{R}^n$ (the *headpoint* of x_0) and the finitely many switching times $t_1 - \tau, \ldots, t_\mu - \tau$ in the interval $[-\tau, 0]$. It suggests that the dynamics of delayed relay systems such as (5.57) are governed by only finitely many coordinates despite the infinite-dimensionality of the underlying phase space.

Sieber, [128], (2006) compared the results obtained between linear control and relay control subject to delay. Consider the dynamical system

$$mL\ddot{\theta} = mg\sin\theta + F\cos\theta \tag{5.58}$$

where m is the mass of the bob in the pendulum, L is the radius of the pendulum moving in a circle, g is the acceleration due to gravity, θ is the inclination angle of the pendulum, and the force F is applied as a feedback to the cart with the goal of stabilizing the unstable upright position $\theta = 0$ as in Figure 5.2. The parameters of equation (5.58) can chosen such that the equation becomes

$$\ddot{\theta} = \sin\theta + F\cos\theta \tag{5.59}$$

Time has been re-scaled to units of $\sqrt{2L/(3g)}$ in (5.59). There exists a fixed reaction time in the application of the feedback force $F(\theta, \dot{\theta})$, i.e. delay τ in the arguments of F, which increases for decreasing L.

The problem was formulated as follows; given a potentially large delay $\tau > 0$, find a function $F : \mathbb{R}^2 \to \mathbb{R}$ such that the feedback law $F(\theta(t-\tau), \dot{\theta}(t-\tau))$ is able to stabilize the system (5.59) to the upright position $\theta = 0$. It is well known that for linear F this is impossible as soon as the delay τ exceeds a certain critical value τ_c. The critical delay $\tau_c = \sqrt{2}$ is derived in the textbook by Stépán [134], (1989) for the classical PD control law $F(\theta(t-\tau), \dot{\theta}(t-\tau)) = -a\theta(t-\tau) - b\dot{\theta}(t-\tau)$. Sieber and Krauskopf, [129], (2005) found that even if one accepts small stable oscillations around the upright position as successful balancing, the restriction on the delay cannot be relaxed substantially beyond the critical value obtained from the linear theory. Such relaxation is however possible for any given delay with a relay delay control

$$F = -\varepsilon \operatorname{sign}[p(\theta(t-\tau), \dot{\theta}(t-\tau))]$$

where p is a smooth or piecewise affine function, if one accepts small oscillations as successful balancing (Sieber, [128], 2006).

5.3.3 A predictor-based design

Another approach to analyse SMC in the presence of input delay is developed in the predictor space (Li and Yurkovich, [106], 2001), (Fiagbedzi and Pearson, [38], 1986), (Richard, et al., [119], 2001), (Roh and Oh, [120], 1999). The delay compensation techniques that can cope with an arbitrarily large delay rely on dynamic feedback where the feedback depends on the predictor, obtained by a real-time solution of a functional equation. Under a predictor-based controller, therefore, a time-delay system can be transformed into a delay-free system in which the delay is eliminated from the closed-loop system. The approach allows eigenvalue assignment problems to be solved without any restriction on the time delay and spectral properties of open-loop system. In (Roh and Oh, [120], 1999) the following linear input-delay system with uncertainties described by

$$\dot{x}(t) = Ax(t) + Bu(t-\tau) + f_0(x(t),t) + f_1(x(t-\tau),t) \tag{5.60}$$

is considered, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $\tau \in ([0,\ \infty), \mathbb{R})$ are the state vector, the input vector and the known delay time, respectively, and A and B are constant matrices with appropriate dimensions. The uncertainties, $f_0(x(t),t)$ and $f_1(x(t-\tau),t)$ represent the nonlinear parameter perturbations with respect to the current state and the delayed state of the system respectively. The following assumptions are made;

$$\begin{aligned} &f_0(x(t),t) = Be_0(x(t),t), \quad f_1(x(t-\tau),t) = Be_1(x(t-\tau),t), \\ &\|e_0(x(t),t)\| \leq \rho_0\|x(t)\| + k, \quad \|e_0(x(t-\tau),t)\| \leq \rho_1\|x(t-\tau)\| \end{aligned} \tag{5.61}$$

for $\rho_0,\ \rho_1,\ k > 0$. A predictor variable is considered as

$$\bar{x}(t) = e^{A\tau}x(t) + \int_{-\tau}^{0} e^{-A\theta}Bu(t+\theta)d\theta \tag{5.62}$$

and the sliding surface is defined as

$$\sigma(\bar{x}) = S\bar{x} = 0 \tag{5.63}$$

Since the sliding function is governed by the state at the next delay interval, the effect of input delay on the sliding function will be compensated. Choosing $\dot{\sigma} = 0$, an equivalent control is obtained as

$$u(t) = u_{eq} + u_N \qquad \begin{aligned} u_{eq} &= -[SB]^{-1} SA\bar{x} \\ u_N &= \begin{cases} -\frac{(SB)^{-1}\sigma S e^{A\tau} B}{\|\sigma\|}\hat{\delta}(x,t) & \text{if } \|\sigma\| \neq 0, \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{5.64}$$

where $\hat{\delta}(x,t) = \rho\|x\| + k + \beta$, for $\rho = \rho_0 + \rho_1 q$, $q > 1$, $\beta > 0$, is the upper bound on the norm of the total uncertainty of the system. The sliding function with the control (5.64) is described as

$$\dot{\sigma} = SBu_N + (Se^{A\tau}B)\{e_0(x(t),t) + e_1(x(t-\tau),t)\} \tag{5.65}$$

Roh and Oh, [120], (1999) asserted that the control law (5.64) will asymptotically stabilize the dynamics (5.65), i.e. a sliding mode always exists. The proof was based on using the Lyapunov function $V(\sigma,t) = \frac{1}{2}\sigma^T\sigma$ such that $\dot{V} \leq -\beta\|\sigma^T\|\|Se^{A\tau}B\| < 0$. However Nguang, [112], (2001) pointed out the fundamental error of (Roh and Oh, [120], 1999) is that of not realizing that $e^{A\tau}f_0(x(t),t)$ and $e^{A\tau}f_1(x(t-\tau),t)$ are no longer matched uncertainties. More precisely, $\mathscr{R}[e^{A\tau}B]$ is not necessarily included in $\mathscr{R}[B]$. Their assertion was derived based on the the fact that $e^{A\tau}f_0(x(t),t)$ and $e^{A\tau}f_1(x(t-\tau),t)$ are still matched uncertainties; however, this is not true in general.

To correct this problem Roh and Oh, [121], (2000) proposed a switching gain adaptation scheme as

$$\hat{\delta}(x,t) = \hat{\rho}(x,t)\|x\| + \hat{k}(x,t) \tag{5.66}$$

where $\hat{\rho}(x,t)$ and $\hat{k}(x,t)$ are adaptation parameters for ρ and k, respectively. They can be obtained by

$$\begin{aligned} \hat{\rho}(x,t) &= \hat{\rho}_{t_0} + \xi_\rho^{-1}\int_{t_0}^{t}\|B^TS^T\sigma\|\|x\|dt \\ \hat{k}(x,t) &= \hat{k}_{t_0} + \xi_k^{-1}\int_{t_0}^{t}\|B^TS^T\sigma\|dt \end{aligned} \tag{5.67}$$

where $\hat{\rho}_{t_0}$ and $\hat{k}_{t_0}$ are the initial values of $\hat{\rho}(x,t)$ and $\hat{k}(x,t)$, respectively. ξ_ρ and ξ_k are adaptation gains with positive constant values. By choosing appropriate $\{\hat{\rho}_{t_0}, \hat{k}_{t_0}\}$ and $\{\xi_\rho, \xi_k\}$, the rate of parameter adaptation can be adjusted. Using such predictor based SMC with an adaptive switching gain, asymptotic stability of the sliding surface was achieved in the presence of delay and uncertainties. Another advantage of using the adaptive term is the *a priori* knowledge of the upper bound on the uncertainties, which is required in conventional SMC without adaptive gain, is no longer needed.

Even though methods based on functional predictors can be globally asymptotically stable at the linear level, they can have exponentially large transients if the initial condition is too far from the equilibrium. See also (Mondié and Michiels, [109], 2003) for a survey on implementation problems of functional predictors and how to overcome them. In the case of small delays, polynomial forward prediction, such as used in substructuring (Wallace, et al., [140], 2005) and (Horiuchi and

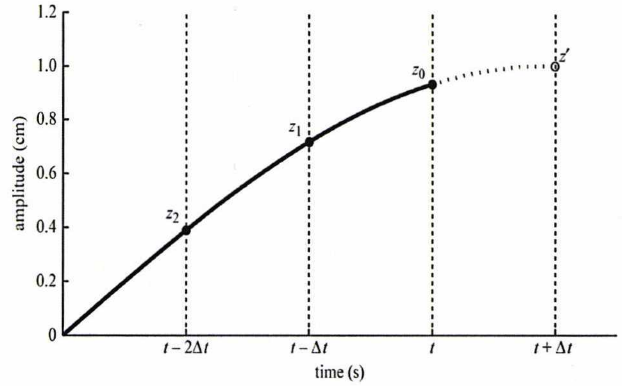


FIGURE 5.3: Single time-step prediction of $n = 3$ control points for a $N = 2$ order polynomial.

Konno, [86], 2001) in mechanical and civil engineering, is often successful and easier to implement in real-time. These algorithms are based on using predefined coefficients, a_i, for an Nth order polynomial fit of a number of control points, following the equation

$$z' = \sum_{i=0}^{N} a_i z_i \tag{5.68}$$

where x_0 is the present state and x_i are the previously calculated states at previous $\tau \times i$ instants. Figure 5.3 shows the forward predicted point $\bar{x}$ being obtained by extrapolating the polynomial function over the present displacement x_0 and N previous calculated values, thus making the number of control points used, $n = N + 1$.

Note that the prediction in Figure 5.3, which is by extrapolation on a third-order polynomial, is based on the assumption that the variation of acceleration is linear with respect to time. Wallace, et al., [140], (2005) improved the result by Horiuchi and Konno, [86], (2001), which is shown to be more generic in terms of flexibility and accuracy.

5.4 Conclusion

SMC with state delay and input delay is reviewed. A stability degree approach, which uses the Lyapunov method for output feedback SMC with state delay, is introduced. For a matched known constant delay, the output feedback system can be reduced to a delay-free system using the equivalent control approach. A sliding surface design with integral terms is shown to minimize the effect of delay and release the constraints on observability as well as Kimura-Davison conditions. Lyapunov Krasovskii and Razumikhin methods, which are widely studied for delay systems, are applied to SMC design with constant and time-varying delay. Results on SMC with delay are demonstrated in terms of LMIs, where both transformation of the original system to regular form as well as methods which deal directly with the original system representation have been considered. Provided the state delay system dynamics are stable on the sliding surface, a control law with a sufficiently large gain will produce asymptotic stability. However, in the input delay case, the situation is much more complicated, involving oscillations and bifurcations in the closed-loop behaviours as illustrated in an example. Study on the one dimensional relay delay system is shown to have a periodic steady state solution with constant frequency. The oscillation is a function of the delay, disturbance and the switching gain. Control algorithms to mitigate the effect of such oscillations are suggested to minimize the delay effect. Explicit solution construction shows the dynamics of delayed relay systems is governed by only finitely many coordinates despite the infinite-dimensionality of the underlying phase space. A predictor based control design for systems with constant delay is able to achieve asymptotic stability. However this leads to a memory based controller which is not robust for even the case of matched uncertainties. This chapter has demonstrated the effects of delay on SMC which potentially affect system stability and certainly affect system performance.

The existing research has mainly focused on SMC of delay systems with state feedback. An output feedback SMC for a delay system, see Chapter 3, where research has been scarce, would pose a challenging task even for systems without delay. The existing output feedback design techniques, e.g. eigenvalue assignment, eigen-structure assignment, cannot be directly applied to systems with time delay. In the next chapter a novel output feedback SMC formulation for non-delay systems will be demonstrated. The method is based on the descriptor approach (Fridman, [42], 2001) and leads to a solution in terms of linear matrix inequalities. When compared to existing methods, the proposed method is efficient and less conservative than other results, giving a feasible solution when the Kimura-Davison conditions are not satisfied.

Chapter 6

A Novel Output Feedback Design of Sliding Surface for Delay-free Systems: An LMI Approach

6.1 Introduction

Works in this chapter seeks to develop a novel output feedback SMC scheme which is more computationally efficient and tractable than those discussed in Chapter 3, and can be extended to incorporate delay effect into the controller design phase. The eigenvalue assignment and eigenvector assignment methods in Chapter 3 are design tools for systems without delays, it is rather difficult to exploit these methods for control design of systems with delay. The most efficient tool for stability analysis with delay has been using Lyapunov Krasovskii Functionals or Razumikhin method expressed as form of LMIs, as see chapter 5. Therefore, an LMI-based output feedback controller design will be considered as a framework before extending the methodology to systems with delay.

In the existing results in output feedback SMC design using LMIs, iterative LMI approaches have been exploited to solve the static output feedback problem using a bilinear matrix inequality formulation, see (Cao, et al., [16], 1998), (Choi, [19], 2002), (Huang and Nguang, [87], 2006). In (Edwards, [27], 2004) where the regular form was not used for synthesization of the control law, LMIs were derived for switching function design whilst minimizing the cost function associated with the control. Sufficient conditions for static output feedback controller design using LMIs have also been sought. Although only sufficient, the solutions have the advantage of being linear and, hence, easily tractable using standard optimization techniques, see, (Crusius and Trofino, [20], 1999), (Shaked, [124], 2003). As well within the existence problem, LMI methods have also been considered within the context of developing a sliding mode control strategy which solves the reachability problem for a given sliding surface. For example, LMI methods which yield reachability conditions for designing static sliding mode output feedback controllers were presented in (Edwards, et al., [28], 2001).

In this chapter the descriptor approach introduced in Chapter 4 is applied to derive LMIs for the solution of the sliding mode control output feedback problem. An example from the literature illustrates the efficiency of the method. In section 6.2 the problem formulation is described. A solution to the existence problem is presented in section 6.3. Section 6.4 shows the formulation for the reachability problem and an example is demonstrated to show the efficiency of the method.

6.2 Problem formulation

Consider an uncertain dynamical system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(t,x,u), \\ y(t) &= Cx(t), \end{aligned} \tag{6.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ with $m \le p \le n$. Assume that the nominal linear system (A,B,C) is known and that the input and output matrices B and C are both of full rank. The unknown function f: $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, which represents the system non-linearities plus any model uncertainties in the system, is assumed to satisfy the matching condition

$$f(t,x,u) = B\xi(t,x,u) \tag{6.2}$$

where the bounded function $\xi : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies

$$\|\xi(t,x,u)\| < k_1\|u\| + a(t,y) \tag{6.3}$$

for some known function $a : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}_+$ and positive constant $k_1 < 1$. Initially the intention will be to explore when static output feedback sliding mode control can be employed. A control law will be sought which induces an ideal sliding motion on the surface

$$s = \{x \in \mathbb{R}^n : FCx = 0\} \tag{6.4}$$

for some selected matrix $F \in \mathbb{R}^{m\times p}$ of the form

$$u(t) = -\gamma F y(t) - v_y \tag{6.5}$$

where γ is a design parameter and the discontinuous vector

$$v_y(t) = \begin{cases} -\rho(t,y)\frac{Fy(t)}{\|Fy(t)\|} & \text{if } Fy \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.6}$$

where $\rho(t,y)$ is some positive scalar function of the outputs

$$\rho(t,y) = (k_1\gamma\|Fy\| + \alpha(t,y) + \gamma_2)/(1-k_1)$$

where γ and γ_2 are positive design scalars.

Hyperplane design

It can be shown that if $rank(CB) = m$ there exists a coordinate system in which the triple (A,B,C) in system (6.1) has the structure

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} C = \begin{bmatrix} 0 & T \end{bmatrix} \tag{6.7}$$

where $B_2 \in \mathbb{R}^{m\times m}$ is non-singular and $T \in \mathbb{R}^{p\times p}$ is orthogonal (Edwards and Spurgeon, [29], 1995). Furthermore, $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and the remaining sub-blocks in the system matrix are partitioned accordingly. Let

$$\begin{bmatrix} F_1 & F_2 \end{bmatrix} = FT \tag{6.8}$$

where $F_1 \in \mathbb{R}^{p-m}$ and $F_2 \in \mathbb{R}^m$. As a result

$$FC = \begin{bmatrix} F_1C_1 & F_2 \end{bmatrix} \tag{6.9}$$

where

$$C_1 = \begin{bmatrix} 0_{(p-m)\times(n-p)} & I_{(p-m)} \end{bmatrix} \tag{6.10}$$

Therefore $FCB = F_2B_2$ and the square matrix F_2 is nonsingular. By assumption the uncertainty is matched and therefore the sliding motion is independent of the uncertainty. In addition, because the canonical form in (6.7) can be viewed as a special case of the regular form normally used in sliding mode controller design, the reduced-order sliding motion is governed by a free motion with system matrix

$$A_{11}^s = A_{11} - A_{12}F_2^{-1}F_1C_1 \tag{6.11}$$

which must therefore be stable. If $K \in \mathbb{R}^{m\times(p-m)}$ is defined as $K = F_2^{-1}F_1$ then

$$A_{11}^s = A_{11} - A_{12}KC_1 \tag{6.12}$$

and the problem of hyperplane design is equivalent to a static output feedback problem for the system (A_{11}, A_{12}, C_1), where (A_{11}, A_{12}) is controllable and (A_{11}, C_1) is observable.

6.3 A novel LMI approach to solve existence problem

Theorem 6.1. *Given scalars ε, δ and a matrix $M \in \mathbb{R}^{(p-m)\times(n-p)}$, if there exists a $(n-m)\times(n-m)$ symmetric matrix $\bar{P} > 0$, and matrices $Q_{22} \in \mathbb{R}^{(p-m)\times(p-m)}$, $Q_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$, $Q_{12} \in \mathbb{R}^{(n-p)\times(p-m)}$, $Y \in \mathbb{R}^{m\times(p-m)}$ such that the LMI*

$$\bar{\Theta} = \begin{bmatrix} A_{11}Q_2 - A_{12}[YM \quad \delta Y] + Q_2^T A_{11}^T - [YM \quad \delta Y]^T A_{12}^T & \bar{P} - Q_2 + \varepsilon Q_2^T A_{11}^T - \varepsilon [YM \quad \delta Y]^T A_{12}^T \\ * & -\varepsilon Q_2 - \varepsilon Q_2^T \end{bmatrix} < 0 \tag{6.13}$$

holds, then the reduced order system (6.12) is asymptotically stable. Once K has been synthesized, choose

$$F = F_2 \begin{bmatrix} K & I_m \end{bmatrix} T^T \tag{6.14}$$

Proof. It will be shown that system (6.7) is stable if the reduced order system (6.12) is stable. Consider a Lyapunov function $V = x_1^T P x_1$ with

$$\dot{x}_1 = (A_{11} - A_{12}KC_1)x_1 \tag{6.15}$$

Using the descriptor method in Chapter 4 the right-hand side of the expression

$$0 = 2[x_1^T P_2^T + \dot{x}_1^T P_3^T][-\dot{x}_1 + (A_{11} - A_{12}KC_1)x_1] \tag{6.16}$$

with matrix parameters $P_2, P_3 = \varepsilon P_2 \in \mathbb{R}^{n-m}$ is added into the right hand-side of $\dot{V} = 2\dot{x}_1^T P x_1$. It is necessary to find the conditions that guarantee that

$$\dot{V} = 2\dot{x}_1^T P x_1 + 2[x_1^T P_2^T + \dot{x}_1^T P_3^T][-\dot{x}_1 + (A_{11} - A_{12}KC_1)x_1] < 0 \tag{6.17}$$

Setting $\eta = col\{x_1, \dot{x}_1\}$ it follows that

$$\dot{V} = \eta^T \Theta \eta \leq 0 \tag{6.18}$$

if the matrix inequality

$$\Theta = \begin{bmatrix} P_2^T(A_{11} - A_{12}KC_1) + (A_{11} - A_{12}KC_1)^T P_2 & P - P_2^T + \varepsilon (A_{11} - A_{12}KC_1)^T P_2 \\ * & -\varepsilon P_2 - \varepsilon P_2^T \end{bmatrix} < 0$$

is feasible. Multiplying the latter inequality from the right and the left by $diag\{P_2^{-1}, P_2^{-1}\}$ and its transpose respectively and denoting $Q_2 = P_2^{-1}$, $\bar{P} = Q_2^T P Q_2$, $\Theta < 0$ if and only if

$$\bar{\Theta} = \begin{bmatrix} (A_{11} - A_{12}KC_1)Q_2 + Q_2^T(A_{11} - A_{12}KC_1)^T & \bar{P} - Q_2 + \varepsilon Q_2^T (A_{11} - A_{12}KC_1)^T \\ * & -\varepsilon Q_2 - \varepsilon Q_2^T \end{bmatrix} < 0$$

Choose the LMI variable Q_2 in the following form

$$Q_2 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{22}M & \delta Q_{22} \end{bmatrix} \tag{6.19}$$

where Q_{22} is a $(p-m)\times(p-m)$-decision variable and M is some $(p-m)\times(n-p)$-matrix and δ is a tuning parameter. Then it can be verified that

$$KC_1Q_2 = \begin{bmatrix} KQ_{22}M & \delta KQ_{22} \end{bmatrix}. \tag{6.20}$$

Defining

$$Y = KQ_{22} \tag{6.21}$$

It follows

$$KC_1Q_2 = \begin{bmatrix} YM & \delta Y \end{bmatrix} \tag{6.22}$$

The matrix K can be found by solving the LMI (6.20) with the tuning parameters δ and ε and tuning matrix M. □

The following section develops conditions to ensure that the uncertain system is quadratically stable and an ideal sliding motion is induced on s in finite time.

6.4 Reachability problem

It can be shown (Edwards, et al., [28], 2001) that the following system transformation and control structure exist such that $z(t) = T_1x_r(t)$, where $T_1 = \begin{bmatrix} I_{n-m} & 0 \\ KC_1 & I_m \end{bmatrix}$ so that the system triple $(\bar{A}, \bar{B}, F\bar{C})$ has the property that

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} F\bar{C} = \begin{bmatrix} 0 & F_2 \end{bmatrix} \tag{6.23}$$

where $\bar{A}_{11} = A_{11} - A_{12}KC_1$. Let $\bar{P}$ be a symmetric positive definite matrix partitioned conformably with the matrix in (6.23) so that

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix} \tag{6.24}$$

then the matrix $\bar{P}$ satisfies the structural constraint

$$\bar{P}\bar{B} = \bar{C}F^T \tag{6.25}$$

if the design matrix $F_2 = B_2^T\bar{P}_2$. The matrix $\bar{P}$ can be shown to be a Lyapunov matrix for $A_0 = \bar{A} - \gamma\bar{B}F\bar{C}$ for all $\gamma > \gamma_0$, where γ_0 is defined to ensure $L(\gamma) = \bar{P}A_0 + A_0^T\bar{P}$ is negative definite

(Edwards *et al.* 1995). In the new coordinate system the uncertain system (6.1) can be written as

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}(u(t) + \xi(t,z,u)) \tag{6.26}$$

Proposition 6.2. *(Edwards and Spurgeon, [29], 1995) The variable structure control law (6.5) quadratically stabilizes the uncertain system given in (6.26).*

Proof. Consider as a candidate Lyapunov function the positive definite expression

$$V(z) = z^T \bar{P} z \tag{6.27}$$

Taking derivatives along the system trajectory and using the structural constraint from (6.25) gives

$$\begin{aligned}
\dot{V} &= z^T(\bar{A}^T\bar{P} + \bar{P}\bar{A} - 2\gamma(F\bar{C})^T F\bar{C})z + 2z^T\bar{P}\bar{B}(\xi - v_y) \\
&= z^T L(\gamma)z + 2y^T F^T(\xi - v_y) \\
&\leq z^T L(\gamma)z - 2y^T F^T v_y + 2\|Fy\|\|\xi\| \\
&= z^T L(\gamma)z - 2\rho(t,y)\|Fy\| + 2\|Fy\|\|\xi\| \\
&< z^T L(\gamma)z - 2\|Fy\|(\rho(t,y) - k_1\|u\| - \alpha(t,y))
\end{aligned}$$

But by definition

$$\rho(t,y) = (k_1\gamma\|Fy\| + \alpha(t,y) + \gamma_2)/(1-k_1)$$

and so by rearranging

$$\begin{aligned}
\rho(t,y) &= k_1\rho(t,y) + k_1\gamma\|Fy\| + \alpha(t,y) + \gamma_2 \\
&\geq k_1(\|v_y\| + \gamma\|Fy\|) + \alpha(t,y) + \gamma_2 \\
&\geq k_1\|u\| + \alpha(t,y) + \gamma_2
\end{aligned} \tag{6.28}$$

Using (6.28) in the inequality for the Lyapunov derivative

$$\dot{V} < z^T L(\gamma)z - 2\gamma_2\|Fy\| < 0, \quad \text{if } z \neq 0 \text{ and } \gamma > \gamma_0$$

and therefore the system is quadratically stable. □

Corollary 6.3. *(Edwards and Spurgeon, [29], 1995) An ideal sliding motion takes place on the surface S in the domain*

$$\Omega = \{z \in \mathbb{R}^n \;:\; \|B_2^{-1}A_0^L\|\|z\| < \gamma_2 - \eta\}$$

where matrix A_0^L represents the last m rows of A_0 and η is a small scalar satisfying $0 < \eta < \gamma_2$.

Proof. Substituting from equation (6.5) it follows from (6.26) and (6.4) that

$$\dot{s} = F\bar{C}A_0 z + F_2 B_2(\xi - v_y)$$

Let $V_c : \mathbb{R}^m \to \mathbb{R}$ be defined by

$$V_c(s) = 2s^T(F_2^{-1})^T \bar{P}_2 F_2^{-1} s$$

Then using the fact that $F_2^T = \bar{P}_2 B_2$ it follows that

$$(F_2^{-1})^T \bar{P}_2 F_2^{-1} F \bar{C} A_0 = B_2^{-1} A_0^L$$

Then it can be verified that

$$\begin{aligned} \dot{V}_c &= 2s^T B_2^{-1} A_0^L z + 2s^T(\xi - v_y) \\ &\leq 2\|s\| \|B_2^{-1} A_0^L z\| - 2\gamma_2 \|s\| \\ &< -2\eta \|s\| \end{aligned} \tag{6.29}$$

if $z \in \Omega$. So there exists a t_0 such that $z(t) \in \Omega$ for all $t > t_0$. Consequently (6.29) holds for all $t > t_0$. A sliding motion will thus be attained in finite time. □

Remark 6.4. The proposed method is suitable for static output feedback sliding mode controller design where Kimura-Davison conditions, written as $n \leq m + p - 1$, are not satisfied. No constraints are imposed on the dimensions of the reduced-order triple A_{11}, A_{12}, C_1. This represents a constructive and efficient approach to output feedback based design for a relatively broad class of systems.

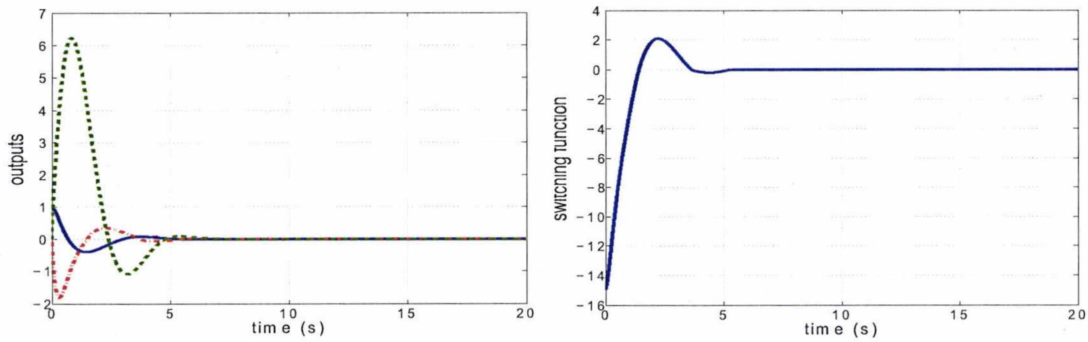
Evidence of the efficiency of the method will be demonstrated by considering an example taken from the literature, (Edwards, et al., [33], 2003)

Example 6.1. Consider the fourth order system

$$A = \begin{bmatrix} -2.724 & -13.808 & 0 & 0 \\ 0.73 & -4.782 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 14.921 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1.355 \\ 0.812 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.30}$$

with system states $[v\ r\ \psi\ Y]^T$, which corresponds to the linearization of the rigid body dynamics of a passenger vehicle. The first state, v, is the lateral velocity, the second state, r, is the yaw rate, the third state, ψ, represents the vehicle orientation and the fourth state, Y, is the lateral deviation from an intended lane position. The input to the system is the angular position of the front wheels relative to the chassis. Transforming the system into the canonical form for design of the switching surface as in equation (6.11) yields

$$A_{11} = \begin{bmatrix} -3.9422 & 0 & 0 \\ 1 & 0 & -14.921 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 12.4066 \\ -1.6687 \\ 1 \end{bmatrix} \tag{6.31}$$



(a) Output Against time

(b) Switching function

FIGURE 6.1: Response of system (6.30) with control matrix (6.33)

The gain from the LMI tool solver with $\delta = 0.2$, $\varepsilon = 1$ and $M = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ yields

$$K = \begin{bmatrix} -0.7579 & 8.1203 \end{bmatrix} \tag{6.32}$$

and hence

$$F = \begin{bmatrix} 4.997 & 0.6154 & 0.4664 \end{bmatrix} \tag{6.33}$$

The sliding mode dynamics have poles at

$$\{-1.7479 + 1.2162i \quad -1.7479 - 1.2162i \quad -9.8313\}$$

Simulation of the states of the system with initial conditions [0 0 1 0] is plotted in Figure 6.1(a). The switching functions are shown in Figure 6.1(b). From the figure it can be seen that the system outputs are stabilized after two seconds; the sliding surface is reached in less than one second and system trajectories remain on it for subsequent time.

6.5 Conclusion

The development of sliding mode schemes for uncertain linear system representations when only output information is available to the controller has been presented. A descriptor Lyapunov functional method for switching function design has been derived which lead to an LMI solution. No additional constraints are imposed on the dimensions or structure of the reduced order triple associated with design of the switching surface. A numerical example shows the effectiveness of the method.

In the next chapter the novel switching surface design for output feedback SMC presented in this chapter will be extended to systems subject to time-varying state delays. The magnitude of the linear gain used to construct the controller is also verified as an appropriate solution to the reachability problem using LMIs. A stability analysis for the full-order time-delay system with discontinuous right-hand side is formulated. The method facilitates the constructive design of sliding-mode static-output-feedback controllers for a rather general class of time-delay systems.

Chapter 7

A Sliding Mode Controller Design for Systems with State Delay

7.1 Introduction

This chapter extends the novel output feedback design for SMC in the previous chapter to systems with state delays. A system with time-varying state delay and matched disturbances is considered which is transformed into regular form where conditions for both existence of sliding surface and reachability of system trajectories to the defined surface are derived using LMIs. The LMI conditions is delay-dependent on a class of reduced order dynamics in the reaching phase and independent in the sliding function, due to the presence of non-linear switching component in the control. The output feedback scheme can be easily extended for compensator design. This facilitates a constructive design of output feedback SMC for a rather general class of time delay systems.

To briefly recall the existing sliding mode control (SMC) techniques for systems with state delay included in Chapter 5, Gouaisbaut, et al., [77], (2002) considered the development of sliding mode controllers for operation in the presence of single or multiple, constant or time-varying state delays. This uses the usual regular form method of solution for matched uncertainties and full state availability is assumed. This problem has also been considered in (Li and DeCarlo, [105], 2003) where a class of uncertain time delay systems with multiple fixed delays in the system states is considered. The paper considers unmatched and time varying parameter uncertainties together with matched and bounded external disturbances, but again full state information is assumed to be available to the controller. In (Fridman and Shaked, [53], 2002) Lyapunov functionals were for the first time introduced for the analysis of time varying delay. In (Fridman, et al., [50], 2003) the descriptor approach in Chapter 4 to stability and control of linear systems with time-varying delays, which is based on the Lyapunov-Krasovskii techniques, was combined with results on the sliding mode control of such systems. The systems under consideration were subjected to norm-bounded uncertainties and uncertain bounded delays and the solution given in terms of linear matrix inequalities. Orlov, et al., [115], (2003) developed a sliding mode control synthesis for a class of uncertain time-delay systems with nonlinear disturbances and unknown delay values whose unperturbed dynamics is linear. The synthesis was based on a new delay-dependent stability criterion. The controller is robust against sufficiently small delay variations and external disturbances.

It is important to emphasize that much of the above literature on sliding mode control of time-delay systems assumes full-state feedback. An output feedback-based approach is worth investigating where the existing developments using state feedback may be examined for their compatibility to solving output feedback problem. In Section 7.2 the problem of output feedback SMC for systems with state time varying delay and matched disturbances is formulated. The existence problem is considered in Section 7.3. In Section 7.4 and 7.5 stability of the full order closed loop system is derived via LMIs and the reachability problem is presented. Compensator based design is demonstrated in Section 7.6. Examples from the literature are used to demonstrate the efficiency of the methods.

7.2 Problem Formulation

Consider an uncertain time-delay system

$$\begin{aligned} \dot{z}(t) &= Az(t) + A_d z(t-\tau(t)) + B(u(t) + \xi(t,z,u)) \\ y(t) &= Cz(t) \end{aligned} \tag{7.1}$$

where $z \in \mathbb{R}^n, u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ with $m \leq p \leq n$. The time-varying delay $\tau(t)$ is supposed to be bounded $0 \leq \tau(t) \leq h$ and it may be either slowly varying (i.e. differentiable delay with $\dot{\tau}(t) \leq d < 1$) or fast varying (piecewise continuous delay). Assume that the nominal linear system (A, A_d, B, C) is known and that the input and output matrices B and C are both of full rank. The unknown function ξ: $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, which represents the system non-linearities plus any model uncertainties, is assumed to satisfy the matching condition with the bound given in (6.3). It can be shown that if $rank(CB) = m$ there exists a coordinate system in which the system (A, A_d, B, C) has the structure

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ A_d = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \ C = \begin{bmatrix} 0 & T \end{bmatrix} \end{aligned} \tag{7.2}$$

where $B_2 \in \mathbb{R}^{m \times m}$ is non-singular and $T \in \mathbb{R}^{p \times p}$ is orthogonal. The system can be represented as

$$\begin{aligned}
\dot{z}_1(t) &= A_{11}z_1(t) + A_{d11}z_1(t-\tau(t)) + A_{12}z_2(t) + A_{d12}z_2(t-\tau(t)) \\
\dot{z}_2(t) &= \textstyle\sum_{i=1}^{2}(A_{2i}z_i(t) + A_{d2i}z_i(t-\tau(t))) + B_2(u(t) + \xi(t,z,u)) \\
y(t) &= Cz(t)
\end{aligned} \tag{7.3}$$

Assume that sliding surface is defined same as (6.4) with respect to z where (6.8), (6.9) and (6.11) hold and define the output feedback control law similar to (6.5), (6.6), the closed-loop system (7.3) and (6.5) can be described by the following equations:

$$\begin{aligned}
\dot{z}_1(t) &= (A_{11} - A_{12}KC_1)z_1(t) + (A_{d11} - A_{d12}KC_1)z_1(t-\tau(t)) \\
\dot{s}(t) &= (A_{21} - \gamma B_2KC_1)z_1(t) + (A_{d21} - \gamma B_2KC_1)z_1(t-\tau(t)) \\
&\quad + (A_{22} - \gamma B_2)z_2(t) + (A_{d22} - \gamma B_2)z_2(t-\tau(t)) + B_2(\xi(t,z,u) - v(t)) \\
y(t) &= Cz(t)
\end{aligned} \tag{7.4}$$

7.3 The Existence Problem

On the sliding manifold $s(t) = 0$, it is well known (Zinober, [152], 1994) that the reduced-order sliding motion is governed by a free motion with system matrix

$$\dot{z}_1(t) = (A_{11} - A_{12}KC_1)z_1(t) + (A_{d11} - A_{d12}KC_1)z_1(t-\tau(t)) \tag{7.5}$$

Consider a Lyapunov-Krasovskii functional

$$\begin{aligned}
V(t) &= z_1^T(t)Pz_1(t) + \textstyle\int_{t-h}^{t} z_1^T(s)Ez_1(s)ds + \int_{t-\tau(t)}^{t} z_1^T(s)Sz_1(s)ds \\
&\quad + h\textstyle\int_{-h}^{0}\int_{t+\theta}^{t} \dot{z}_1^T(s)R\dot{z}_1(s)dsd\theta
\end{aligned} \tag{7.6}$$

where the symmetric matrices $P > 0$ and $E, S, R \geq 0$.

The condition $\dot{V}(t) < 0$ guarantees asymptotic stability of the reduced order system as in (Hale and Lunel, [77], 1993). Differentiating $V(t)$ along (7.5),

$$\begin{aligned}
\dot{V}(t) &= 2z_1^T(t)P\dot{z}_1(t) + h^2\dot{z}_1{}^T(t)R\dot{z}_1(t) - h\textstyle\int_{t-h}^{t}\dot{z}_1{}^T(s)R\dot{z}_1(s)ds \\
&\quad + z_1^T(t)(E+S)z_1(t) - z_1^T(t-h)Ez_1(t-h) - (1-\dot{\tau}(t))z_1^T(t-\tau(t))Sz_1(t-\tau(t))
\end{aligned} \tag{7.7}$$

Further using the identity

$$-h\textstyle\int_{t-h}^{t}\dot{z}_1{}^T(s)R\dot{z}_1(s)ds = -h\int_{t-h}^{t-\tau(t)}\dot{z}_1{}^T(s)R\dot{z}_1(s)ds - h\int_{t-\tau(t)}^{t}\dot{z}_1{}^T(s)R\dot{z}_1(s)ds \tag{7.8}$$

and applying Jensen's inequality (Gu, et al., [73], 2003)

$$\textstyle\int_{t-\tau(t)}^{t}\dot{z}_1{}^T(s)R\dot{z}_1(s)ds \geq \frac{1}{h}\int_{t-\tau(t)}^{t}\dot{z}_1{}^T(s)dsR\int_{t-\tau(t)}^{t}\dot{z}_1(s)ds \tag{7.9}$$

and

$$\int_{t-h}^{t-\tau(t)} \dot{z_1}^T(s)R\dot{z_1}(s)ds \geq \frac{1}{h}\int_{t-h}^{t-\tau(t)} \dot{z_1}^T(s)dsR\int_{t-h}^{t-\tau(t)} \dot{z_1}(s)ds. \quad (7.10)$$

then

$$\begin{aligned}\dot{V}(t) &\leq 2z_1^T(t)P\dot{z_1}^T(t) + h^2\dot{z_1}^T(t)R\dot{z_1}(t)\\ &-(z_1(t)-z_1(t-\tau(t)))^T R(z_1(t)-z_1(t-\tau(t)))\\ &-(z_1(t-\tau(t))-z_1(t-h))^T R(z_1(t-\tau(t))-z_1(t-h))\\ &+z_1^T(t)(E+S)z_1(t)-z_1^T(t-h)Ez_1(t-h)\\ &-(1-d)z_1^T(t-\tau(t))Sz_1(t-\tau(t))\end{aligned} \quad (7.11)$$

Using the descriptor method

$$\begin{aligned}0 \equiv\ & 2(z_1^T(t)P_2^T + \dot{z_1}^T(t)P_3^T)[-\dot{z_1}(t) + (A_{11}-A_{12}KC_1)z_1(t)\\ &+(A_{d11}-A_{d12}KC_1)z_1(t-\tau(t))]\end{aligned} \quad (7.12)$$

with matrix parameters P_2, $P_3 = \varepsilon P_2 \in \mathbb{R}^{n-m}$ is added into the right-hand side of (7.11). Setting $\eta(t) = col\{z_1(t), \dot{z_1}(t), z_1(t-h), z_1(t-\tau(t))\}$, it follows that

$$\dot{V}(t) \leq \eta^T(t)\Theta\eta(t) \leq 0 \quad (7.13)$$

if the matrix inequality

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & 0 & \theta_{14}\\ * & \theta_{22} & 0 & \theta_{24}\\ * & * & \theta_{33} & \theta_{34}\\ * & * & * & \theta_{44}\end{bmatrix} < 0 \quad (7.14)$$

is feasible, where

$$\begin{aligned}\theta_{11} &= P_2^T(A_{11}-A_{12}KC_1) + (A_{11}-A_{12}KC_1)^T P_2 + E + S - R\\ \theta_{12} &= P - P_2^T + \varepsilon(A_{11}-A_{12}KC_1)^T P_2\\ \theta_{14} &= P_2^T(A_{d11}-A_{d12}KC_1) + R\\ \theta_{22} &= -\varepsilon P_2 - \varepsilon P_2^T + h^2 R\\ \theta_{24} &= \varepsilon P_2^T(A_{d11}-A_{d12}KC_1)\\ \theta_{33} &= -(E+R)\\ \theta_{34} &= R\\ \theta_{44} &= -2R-(1-d)S\end{aligned} \quad (7.15)$$

Multiplying the matrix Θ from the right and the left by $diag\{P_2^{-1}, P_2^{-1}, P_2^{-1}, P_2^{-1}\}$ and its transpose respectively and denoting

$$Q_2 = P_2^{-1},\ \widehat{P} = Q_2^T P Q_2,\ \widehat{R} = Q_2^T R Q_2,\ \widehat{E} = Q_2^T E Q_2,\ \widehat{S} = Q_2^T S Q_2$$

it follows $\Theta < 0 \Leftrightarrow \widehat{\Theta} < 0$ where

$$\widehat{\Theta} = \begin{bmatrix} \widehat{\theta}_{11} & \widehat{\theta}_{12} & 0 & \widehat{\theta}_{14} \\ * & \widehat{\theta}_{22} & 0 & \widehat{\theta}_{24} \\ * & * & \widehat{\theta}_{33} & \widehat{\theta}_{34} \\ * & * & * & \widehat{\theta}_{44} \end{bmatrix} < 0 \tag{7.16}$$

and

$$\begin{aligned} \widehat{\theta}_{11} &= (A_{11} - A_{12}KC_1)Q_2 + Q_2^T(A_{11} - A_{12}KC_1)^T + \widehat{E} + \widehat{S} - \widehat{R} \\ \widehat{\theta}_{12} &= \widehat{P} - Q_2 + \varepsilon Q_2^T(A_{11} - A_{12}KC_1)^T \\ \widehat{\theta}_{14} &= (A_{d11} - A_{d12}KC_1)Q_2 + \widehat{R} \\ \widehat{\theta}_{22} &= -\varepsilon Q_2 - \varepsilon Q_2^T + h^2\widehat{R} \\ \widehat{\theta}_{24} &= \varepsilon(A_{d11} - A_{d12}KC_1)Q_2 \\ \widehat{\theta}_{33} &= -\widehat{E} - \widehat{R} \\ \widehat{\theta}_{34} &= \widehat{R} \\ \widehat{\theta}_{44} &= -2\widehat{R} - (1-d)\widehat{S} \end{aligned} \tag{7.17}$$

Suppose Q_2 is defined in (6.19) and (6.20), (6.22) hold, substitute (6.22) into (7.17) to yield

$$\begin{aligned} \widehat{\theta}_{11} &= A_{11}Q_2 - A_{12}[Y \quad \delta Y] + Q_2^T A_{11}^T - [Y\mathscr{M} \quad \delta Y]^T A_{12}^T + \widehat{E} + \widehat{S} - \widehat{R} \\ \widehat{\theta}_{12} &= \widehat{P} - Q_2 + \varepsilon Q_2^T A_{11}^T - \varepsilon[Y\mathscr{M} \quad \delta Y]^T A_{12}^T \\ \widehat{\theta}_{14} &= A_{d11}Q_2 - A_{d12}[Y\mathscr{M} \quad \delta Y] + \widehat{R} \\ \widehat{\theta}_{22} &= -\varepsilon Q_2 - \varepsilon Q_2^T + h^2\widehat{R} \\ \widehat{\theta}_{24} &= \varepsilon A_{d11}Q_2 - \varepsilon A_{d12}[Y\mathscr{M} \quad \delta Y] \\ \widehat{\theta}_{33} &= -\widehat{E} - \widehat{R} \\ \widehat{\theta}_{34} &= \widehat{R} \\ \widehat{\theta}_{44} &= -2\widehat{R} - (1-d)\widehat{S} \end{aligned} \tag{7.18}$$

with the tuning parameters δ, ε and $\mathscr{M}$. The following Lemma may now be stated

Lemma 7.1. *Given a-priori selected tuning parameters ε, δ and $\mathscr{M} \in R^{(p-m)\times(n-p)}$, then (7.16) is an LMI in the decision variables $\widehat{P} > 0$, $\widehat{E} \geq 0$, $\widehat{S} \geq 0$, $\widehat{R} \geq 0$ and matrices $Q_{22} \in \mathbb{R}^{(p-m)\times(p-m)}$, $Q_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$, $Q_{12} \in \mathbb{R}^{(n-p)\times(p-m)}$, $Y \in \mathbb{R}^{m\times(p-m)}$. If a solution to (7.16) exists, which may be readily obtained from available LMI tools, then the reduced order system (7.5) is asymptotically stable for all differentiable delays $0 \leq \tau(t) \leq h$, $\dot{\tau}(t) \leq d < 1$. Moreover, (7.5) is asymptotically stable for all piecewise-continuous delays $0 \leq \tau(t) \leq h$, if the LMI (7.16) is feasible with $\widehat{S} = 0$.*

7.4 Stability of the Full Order Closed Loop System

It can be shown that there exist a coordinate system in which the system triple $(\bar{A}, \bar{A}_d, \bar{B}, F\bar{C})$ has the property that

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \bar{A}_d = \begin{bmatrix} \bar{A}_{d11} & \bar{A}_{d12} \\ \bar{A}_{d21} & \bar{A}_{d22} \end{bmatrix},$$
$$\bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, F\bar{C} = \begin{bmatrix} 0 & F_2 \end{bmatrix} \tag{7.19}$$

where $\bar{A}_{11} = A_{11} - A_{12}KC_1$ and $\bar{A}_{d11} = A_{d11} - A_{d12}KC_1$ and F_2 is a design parameter. Let $\bar{P}$ be given by (6.24), then the matrix $\bar{P}$ satisfies the structural constraint

$$\bar{P}\bar{B} = \bar{C}^T F^T \tag{7.20}$$

if the design matrix $F_2 = B_2^T \bar{P}_2$. The matrix $\bar{P}$ can be shown to be a Lyapunov matrix for

$$\bar{A}_0 = \bar{A} - \gamma\bar{B}F\bar{C} = \bar{A} - \gamma\bar{B}\begin{bmatrix} 0 & F_2 \end{bmatrix} \tag{7.21}$$

for sufficiently large γ (Edwards and Spurgeon, [29], 1995). In the new coordinate system the uncertain system (7.1) can be written as

$$\dot{z}(t) = \bar{A}z(t) + \bar{A}_d z(t-\tau) + \bar{B}(u(t) + \xi(t,z,u)) \tag{7.22}$$

The closed-loop system will have the form

$$\dot{z}(t) = \bar{A}_0 z(t) + \bar{A}_d z(t-\tau) + \bar{B}(\xi(t,y_t) - v_y(t)) \tag{7.23}$$

For large enough $\gamma > 0$, these conditions are delay-independent with respect to the delay in z_2. However, for derivation of this condition using Lyapunov-Krasovskii techniques, it is necessary to consider the case where $\dot{\tau} \leq d < 1$. A stability condition for the full order closed loop system can be derived using the following Lyapunov-Krasovskii functional:

$$\begin{aligned} V(t) = {} & z^T(t)\bar{P}z(t) + \int_{t-h}^{t} z^T(s)\bar{E}z(s)ds + \int_{t-\tau(t)}^{t} z^T(s)\bar{S}z(s)ds \\ & + h\int_{-h}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)\bar{R}\dot{z}(s)dsd\theta \end{aligned} \tag{7.24}$$

where the matrix $\bar{E}, \bar{S} \geq 0$ and $\bar{R} = \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$ as it is desired to determine a stability condition for the time delay system which is delay independent with respect to delay in $z_2(t)$. Differentiating

$V(t)$ along the closed loop trajectories

$$\begin{aligned}\dot{V}(t) \leq\ & 2z^T(t)\bar{P}\dot{z}^T(t) + h^2\dot{z}^T(t)\bar{R}\dot{z}(t)\\ & -(z(t)-z(t-\tau(t)))^T\bar{R}(z(t)-(t-\tau(t)))\\ & -(z(t-\tau(t))-z(t-h))^T\bar{R}(z(t-\tau(t))-z(t-h))\\ & +z^T(t)(\bar{E}+\bar{S})z(t) - z^T(t-h)\bar{E}z(t-h)\\ & -(1-d)z^T(t-\tau(t))\bar{S}z(t-\tau(t))\end{aligned} \tag{7.25}$$

Substitute the right-hand side of equation (7.23) into (7.25). Setting $\varsigma(t) = col\{z(t), z(t-h), z(t-\tau(t))\}$, it follows that

$$\dot{V}(t) \leq \varsigma(t)^T\Phi_h\varsigma(t) + h^2\dot{z}^T(t)\bar{R}\dot{z}(t) + 2z^T\bar{P}\bar{B}(\xi(t,z,u) - v(t)) < 0 \tag{7.26}$$

is satisfied if $\varsigma^T(t)\Phi_h\varsigma(t) + h^2\dot{z}^T(t)\bar{R}\dot{z}(t) < 0$ and $2z^T\bar{P}\bar{B}(\xi(t,z,u) - v(t)) < 0$, where

$$\Phi_h = \begin{bmatrix} \phi_{11} - \bar{R} & 0 & \bar{P}\bar{A}_d + \bar{R}\\ * & -(\bar{E}+\bar{R}) & \bar{R}\\ * & * & -2\bar{R}-(1-d)\bar{S}\end{bmatrix} \tag{7.27}$$

with

$$\phi_{11} = \bar{A}_0^T\bar{P} + \bar{P}\bar{A}_0 + \bar{S} + \bar{E} \tag{7.28}$$

Setting $\rho(t) = col\{z(t), z(t-h), z(t-\tau), \xi(t,z,u) - v(t)\}$

$$\begin{aligned} h^2\dot{z}^T(t)\bar{R}\dot{z}(t) &= h^2[z^T(t)\bar{A}_0 + z^T(t-\tau)\bar{A_d}^T + (\xi(t,z,u) - v(t)^T)\bar{B}^T]\bar{R}\cdot\\ &\quad [\bar{A}_0 z + \bar{A}_d z(t-\tau) + \bar{B}(\xi(t,z,u) - v(t))]\\ &= \rho^T(t)\begin{bmatrix}\bar{A}_0^T\\ 0_n\\ \bar{A_d}^T\\ \bar{B}^T\end{bmatrix}\begin{bmatrix} I_{(n-m)}\\ 0\end{bmatrix} h^2\bar{R}_1 \begin{bmatrix} I_{(n-m)}\\ 0\end{bmatrix}^T \begin{bmatrix}\bar{A}_0^T\\ 0_n\\ \bar{A_d}^T\\ \bar{B}^T\end{bmatrix}^T \rho(t)\end{aligned} \tag{7.29}$$

Using the Schur complement, $\xi^T(t)\Phi_h\xi(t) + h^2\dot{z}^T(t)\bar{R}\dot{z}(t) < 0$ holds if

$$\begin{bmatrix} \Phi_h & \begin{matrix} h\bar{A}_0^T\begin{bmatrix} I_{(n-m)}\\ 0\end{bmatrix}\bar{R}_1\\ 0_{(n,n-m)}\\ h\bar{A_d}^T\begin{bmatrix} I_{(n-m)}\\ 0\end{bmatrix}\bar{R}_1\end{matrix}\\ * \;\; * \;\; * & -\bar{R}_1\end{bmatrix} < 0 \tag{7.30}$$

Inequality (7.30) is an LMI in the decision variables $\bar{P}_1 > 0$, $\bar{E} \geq 0$, $\bar{S} \geq 0$ and $\bar{R}_1 \geq 0$. Equation (7.26) is valid if (7.30) is satisfied and given

$$\begin{aligned} &2z^T \bar{P}\bar{B}(\xi(t,z,u) - v(t)) \\ &= 2y^T F^T(\xi(t,z,u) - v(t)) \\ &\leq -2y^T F^T v(t) + 2\|Fy(t)\|\|\xi(t,z,u)\| \\ &= -2\rho(t,y)\|Fy(t)\| + 2\|Fy(t)\|\|\xi(t,z,u)\| \\ &< -2\|Fy(t)\|(\rho(t,y) - k_1\|u(t)\| - \alpha(t,y)) \end{aligned} \tag{7.31}$$

Then given (6.28) and equation (7.26) if (7.30) is valid, then from (7.31) and (6.28)

$$\dot{V}(t) < -2\gamma_2\|Fy(t)\| < 0 \quad \text{if } z(t) \neq 0 \tag{7.32}$$

and therefore the system is asymptotically stable.

Lemma 7.2. *Given large enough γ, let there exist $n \times n$ matrices $\bar{P}_1 > 0$, $\bar{E} \geq 0$, $\bar{S} \geq 0$, $\bar{R}_1 \geq 0$ from the LMI solver such that LMI (7.30) holds. Given that the design parameters k_1, $\alpha(t,y)$, γ_2 and $\bar{P}_2$ have been selected so that condition (7.32) holds, the closed loop system (7.22) is asymptotically stable for all differentiable delays $0 \leq \tau(t) \leq h$, $\dot{\tau}(t) \leq d \leq 1$.*

7.5 Finite Time Reachability to the Sliding Manifold

Corollary 7.3. *An ideal sliding motion takes place on the surface S if*

$$\|B_2^{-1}\bar{A}_0^L z(t)\| + \|B_2^{-1}\bar{A}_d^L z(t-\tau)\| < \gamma_2 - \eta \tag{7.33}$$

where the matrices $\bar{A}_0^L$ and $\bar{A}_d^L$ represent the last m rows of $\bar{A}_0$ and $\bar{A}_d$ respectively and η is a small scalar satisfying $0 < \eta < \gamma_2$.

Proof.

$$\dot{s}(t) = F\bar{C}\bar{A}_0\bar{z}(t) + F\bar{C}\bar{A}_d z(t-\tau) + F_2 B_2(\xi(t,z,u) - v(t)) \tag{7.34}$$

Let $V_c : \mathbb{R}^m \to \mathscr{R}$ be defined by

$$V_c(s) = s^T(t)(F_2^{-1})^T \bar{P}_2 F_2^{-1} s(t) \tag{7.35}$$

Then using the fact that $F_2^T = \bar{P}_2 B_2$ it follows that

$$\begin{aligned} (F_2^{-1})^T \bar{P}_2 F_2^{-1} F\bar{C}\bar{A}_0 = B_2^{-1}\bar{A}_0^L \\ (F_2^{-1})^T \bar{P}_2 F_2^{-1} F\bar{C}\bar{A}_d = B_2^{-1}\bar{A}_d^L \end{aligned} \tag{7.36}$$

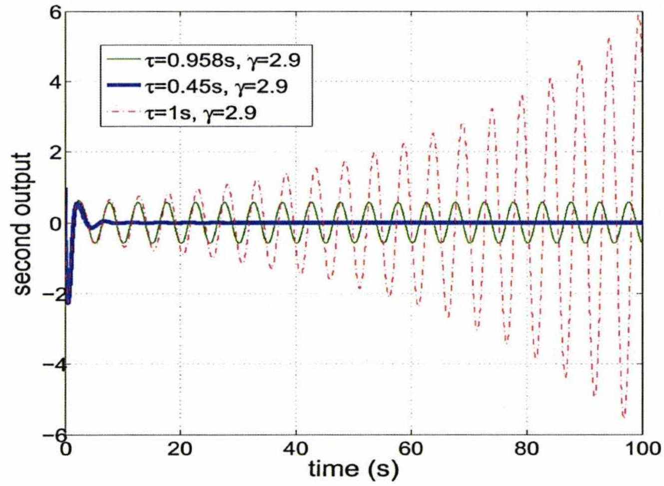


FIGURE 7.1: Output Against time

Then it can be verified that

$$\begin{aligned}\dot{V}_c &= 2s^T(t)B_2^{-1}\bar{A}_0^L z(t) + 2s^T(t)B_2^{-1}\bar{A}_d^L z(t-\tau) + 2s^T(t)(\xi(t,z,u) - v(t)) \\ &\leq 2\|s(t)\|\|B_2^{-1}\bar{A}_0^L z(t)\| + 2\|s(t)\|\|B_2^{-1}\bar{A}_d^L z(t-\tau)\| - 2\gamma_2\|s(t)\| \\ &< -2\eta\|s(t)\|\end{aligned} \tag{7.37}$$

if $z(t)$ and $z(t-\tau) \in \Omega$. It follows that there exists a t_0 such that $z(t)$ and $z(t-\tau) \in \Omega$ for all $t > t_0$. Consequently equation (7.37) holds for all $t > t_0$. A sliding motion will thus be attained in finite time. □

Example 7.1. The following model of a liquid monopropellant rocket motor has been considered in (Feng, et al., [39], 1995). It is assumed that the variable $\kappa = 0.8$ in this case, where $A_d(1,1) = -\kappa$ and $A(1,1) = \kappa - 1$. The outputs have been chosen to be the second and fourth states so that in (7.1)

$$A = \begin{bmatrix} -0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.8 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.38}$$

Clearly the Kimura-Davison conditions are not met. Here the rate at which the delay varies with

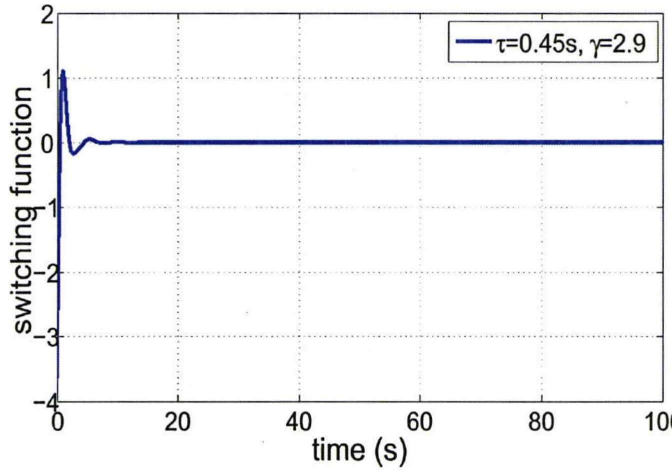


FIGURE 7.2: Switching Function

time has been examined with $d = 0$, a constant delay. The gain from the LMI tool solver with $\delta = 50$, $\varepsilon = 0.5$, $h = 0.45s$, $\bar{P}_2 = 1$, $\gamma = 2.9$ and $M = [5 \quad 0.2]$, yields $F = [-1 \quad -2.0754]$. The poles of the sliding mode dynamics are $\{-2.67, \ -0.4, \ -0.2\}$. A simulation was performed with the initial state values $[1 \ 1 \ 1 \ 1]$. As can be seen the LMI solver gave a feasible result for stability for $h \leq 0.45s$. Due to the conservativeness of our LMI formulation, the closed loop system is asymptotically stable for all $\tau < 0.958s$ before behaving with limit cycle of a particular frequency, and became unstable for $\tau \geq 1s$ as shown in Figure 7.1. The LMI solver gave feasible closed loop stability results for controller gain $\gamma \geq 2.9$ while a choice of $\gamma \geq 1$ in simulation was able to stabilize the system with the compensation of longer settling time. The switching function for $h = 0.45s$, $\gamma = 2.9$ is plotted in Figure 7.2.

7.6 The compensator-based existence problem

For certain system triples (A_{11}, A_{12}, C_1), LMI (7.16) is known to be infeasible. In this case consider a dynamic compensator similar to that of (El-Khazali and DeCarlo, [37], 1995)

$$\dot{z}_c(t) = Hz_c(t) + Dy(t) \tag{7.39}$$

where the matrices $H \in \mathbb{R}^{q \times q}$ and $D \in \mathbb{R}^{q \times p}$ are to be determined. Define a new hyperplane in the augmented state space, formed from the plant and compensator state spaces, as

$$S_c = \{(z(t), z_c(t)) \in \mathbb{R}^{n+q} \ : \ F_c z_c(t) + FCz(t) = 0\} \tag{7.40}$$

where $F_c \in \mathbb{R}^{m\times q}$ and $F \in \mathbb{R}^{m\times p}$. Define $D_1 \in \mathbb{R}^{q\times(p-m)}$ and $D_2 \in \mathbb{R}^{q\times m}$ as

$$\begin{bmatrix} D_1 & D_2 \end{bmatrix} = DT \tag{7.41}$$

then the compensator can be written as

$$\dot{z}_c(t) = Hz_c(t) + D_1C_1z_1(t) + D_2z_2(t) \tag{7.42}$$

where C_1 is defined in (6.11). The sliding motion, obtained by eliminating the coordinates $z_2(t)$, can be written as

$$\begin{aligned}\dot{z}_1(t) &= (A_{11} - A_{12}KC_1)z_1(t) - A_{12}K_cz_c(t) \\ &+ (A_{d11} - A_{d12}KC_1)z_1(t-\tau) - A_{d12}K_cz_c(t-\tau) \\ \dot{z}_c(t) &= (D_1 - D_2K)C_1z_1(t) + (H - D_2K_c)z_c(t)\end{aligned} \tag{7.43}$$

where $K = F_2^{-1}F_1$ and $K_c = F_2^{-1}F_c$, then similar to (Edwards and Spurgeon, [32], 2003) the design problem becomes one of selecting a compensator, represented by the matrices D_1, D_2 and H, and a hyperplane, represented by the matrices K and K_c, so that the system,

$$\begin{aligned}\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_c(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} A_{11} - A_{12}KC_1 & -A_{12}K_c \\ (D_1 - D_2K)C_1 & H - D_2K_c \end{bmatrix}}_{A_c} \begin{bmatrix} z_1(t) \\ z_c(t) \end{bmatrix} \\ &+ \underbrace{\begin{bmatrix} A_{d11} - A_{d12}KC_1 & -A_{d12}K_c \\ 0 & 0 \end{bmatrix}}_{A_{cd}} \begin{bmatrix} z_1(t-\tau) \\ z_c(t-\tau) \end{bmatrix}\end{aligned} \tag{7.44}$$

is stable. To obtain the compensator gains this problem can be shown to be a new output feedback problem with

$$\begin{aligned}A_c &= \underbrace{\begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}}_{A_q} - \underbrace{\begin{bmatrix} A_{12} & 0 \\ D_2 & -I_q \end{bmatrix}}_{B_q} \underbrace{\begin{bmatrix} K & K_c \\ D_1 & H \end{bmatrix}}_{K_q} \underbrace{\begin{bmatrix} C_1 & 0 \\ 0 & I_q \end{bmatrix}}_{C_q} \\ A_{cd} &= \underbrace{\begin{bmatrix} A_{d11} & 0 \\ 0 & 0 \end{bmatrix}}_{A_{qd}} - \underbrace{\begin{bmatrix} A_{d12} & 0 \\ 0 & 0 \end{bmatrix}}_{B_{qd}} \underbrace{\begin{bmatrix} K & K_c \\ D_1 & H \end{bmatrix}}_{K_q} \underbrace{\begin{bmatrix} C_1 & 0 \\ 0 & I_q \end{bmatrix}}_{C_q}\end{aligned} \tag{7.45}$$

The existence problem represented by system (7.44), where A_c and A_{cd} are partitioned as in equation (7.45) and D_2 is a tuning parameter, can be solved as for the non-compensated case (7.5).

Similar to (6.22)

$$\begin{aligned} K_qC_qQ_2 &= K_q \begin{bmatrix} 0_{(p-m+q)\times(n-p)} & I_{p-m+q} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{22}\mathcal{M} & \delta Q_{22} \end{bmatrix} \\ &= \begin{bmatrix} K_qQ_{22}\mathcal{M} & \delta K_qQ_{22} \end{bmatrix} \\ &= \begin{bmatrix} Y\mathcal{M} & \delta Y \end{bmatrix} \end{aligned} \tag{7.46}$$

where $Y = K_qQ_{22}$, $\mathcal{M} \in \mathbb{R}^{(p-m+q)\times(n-p)}$ is a tuning matrix.

Example 7.2. Consider the delay system

$$\begin{aligned} A &= \begin{bmatrix} 0 & 25 & -1 \\ 1 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0 & 0.3 & -0.1 \\ 0 & 0.2 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{7.47}$$

from which

$$A_{11} = \begin{bmatrix} 0 & 25 \\ 1 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

It follows that

$$\lambda(A_{11} - A_{12}KC_1) = \pm\sqrt{(25+K^2)}$$

and so (7.16) is infeasible. Now, consider designing a first order compensator. Choosing $D_2 = 1$ it follows that

$$\begin{aligned} A_q &= \begin{bmatrix} 0 & 25 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_{qd} = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ B_q &= \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}, \quad B_{qd} = \begin{bmatrix} -0.1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ C_q &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Choosing $\delta = 50$, $\varepsilon = 5$, $\mathcal{M} = \begin{bmatrix} 10 & 4 \end{bmatrix}'$ and $d = 0$ (constant delay) with the maximum allowable delay $h = 0.25s$, the LMI tool solver returns

$$K_q = \begin{bmatrix} -25.38 & -0.37 \\ -25.32 & -5.03 \end{bmatrix}$$

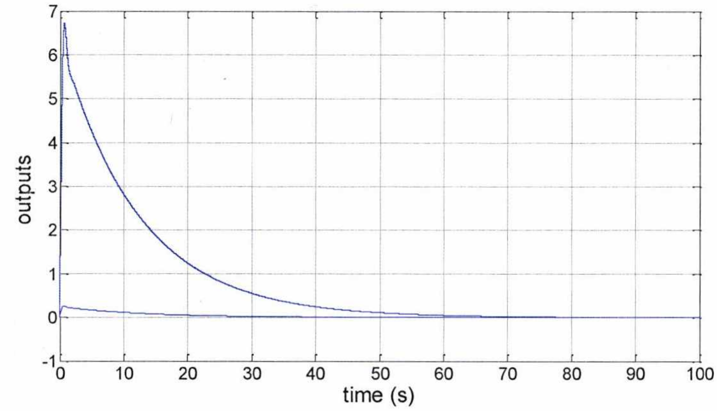


FIGURE 7.3: compensator based controller design $h = 2.5s$

The augmented system with compensator given by

$$A_a = \begin{bmatrix} H & DC \\ 0 & A \end{bmatrix}, \; A_{da} = \begin{bmatrix} 0 & 0 \\ 0 & A_d \end{bmatrix}, \; B_a = \begin{bmatrix} 0 \\ B \end{bmatrix}, \; C_a = \begin{bmatrix} I_q & 0 \\ 0 & C \end{bmatrix}$$

is asymptotically stablised by the controller

$$\begin{bmatrix} F_c & F \end{bmatrix} = \begin{bmatrix} -0.369 & -25.38 & 1 \end{bmatrix}$$

Taking the controller in the form of (6.5), (6.6) where $\gamma = 10$, $\rho = 10$, simulation shows the outputs of the system (7.47) starting from initial conditions $[0.5,\ 0,\ 0]$ eventually converge to zero after 80s, as shown in Figure 7.3. The system trajectories are confined to the predefined switching surface (7.40) rapidly in less than one second as shown in Figure 7.4.

7.7 Conclusion

A descriptor Lyapunov-Krasovskii functional method has been introduced for static output feedback switching function design for systems with state time-varying delays. The delay is assumed bounded with a known upper bound, either slowly or rapidly varying. Also, a novel stability analysis of the full-order closed-loop discontinuous time-delay system has been performed via the Krasovskii method, which is delay-independent in $z_2(t)$ (and thus the delay is restricted to be slowly-varying) and delay-dependent in $z_1(t)$, i.e. in the state of the reduced-order system. The proposed OFC design approach also applies to compensator based design. Examples have shown the effectiveness of the method.

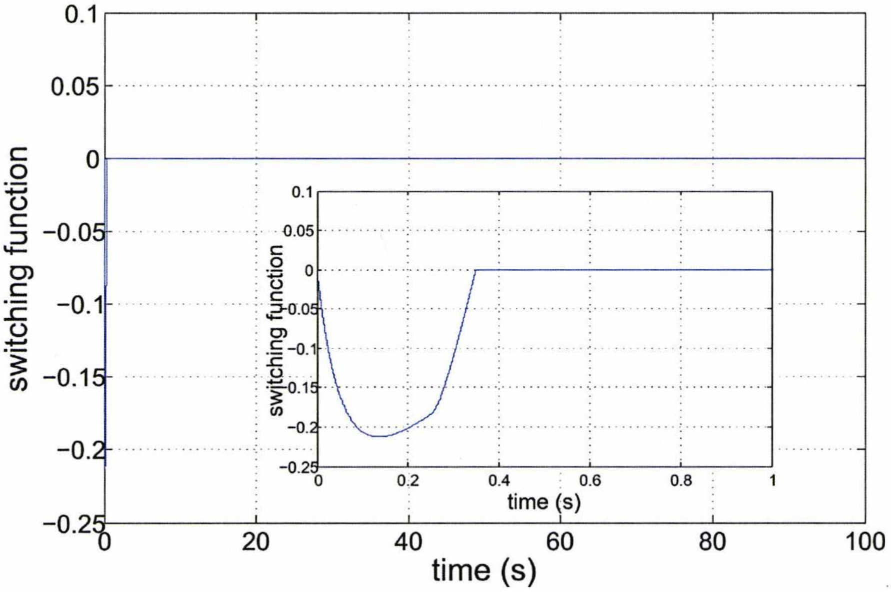


FIGURE 7.4: compensator based controller design $h = 2.5s$

There are however two major limitations of the work. Firstly only matched uncertainties are considered and secondly the switching gain in the controller is chosen large enough through trial-and-error in order to achieve sliding mode in the simulations. For a more generic controller to be developed, unmatched uncertainties should be included into the design phase to enhance the system performances. The next chapter considers the development of constructive sliding mode control strategies based on measured output information only for linear, time delay systems with bounded disturbances that are not necessarily matched. The novel feature of the method is that a systematic approach is given for the design where all the control parameters are computed using linear matrix inequalities, including the switching gain necessary to achieve sliding mode, even in the presence of unmatched uncertainties, disturbances and time delay effects. This eliminates the need to determine the switching control gain and closed-loop performance through simulations. The methodology provides guarantees on the level of closed-loop performance that will be achieved by uncertain systems which experience delay. A case study involving the practical application of the design methodology in the area of autonomous vehicle control will be presented

Chapter 8

Sliding Mode Control of Systems with Disturbances: An application to Autonomous Vehicle Control

8.1 Introduction

In the previous chapter the existence and reachability problems for systems in the presence of matched uncertainty are considered for output feedback sliding mode control of time-delay systems. The delay is assumed to be time-varying and bounded where the upper bound is known. In line with the development of output feedback controllers in the non-delayed case, LMIs are used to select all the parameters of the closed-loop sliding mode controller. However, no explicit calculation of the switching gain in the nonlinear part of the control was given, it was only assumed to be large enough to induce the sliding mode. While asymptotic stability in the presence of matched disturbances can be achieved by sliding mode control, unmatched disturbances usually lead to only bounded stability. For example, in (Fernando and Fridman, [40], 2006) robustness properties of integral sliding-mode controllers are studied where the Euclidean norm of the unmatched perturbation is minimized by selecting a projection matrix.

This chapter develops a means to select all the design parameters, including the switching gain, from LMIs in the presence of state delay with matched and unmatched disturbances. The method is able to deal with polytopic type uncertainties in all blocks of the system matrices. No additional assumption is made on the bound of the uncertain states in the reachability design, as required by other work. It is demonstrated that the state trajectories of the system converge towards a ball with a pre-specified exponential convergence rate. In Section 8.2 the problem of design for systems with unmatched disturbances is formulated. The existence of a stable bounded solution to the sliding motion is presented in Section 8.3 and Section 8.4 shows the formulation of the reachability problem which will ensure that the sliding mode is reached. A problem from the literature is used to provide a tutorial example of how the paradigm can be used to solve both the existence and reachability problems for practical design. A case study relating to the control of an autonomous vehicle is used to further illustrate the design process in Section 8.5.

8.2 Problem formulation

Consider a dynamical system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau(t)) + Bu(t) + B_1 w(t) \\ y(t) &= Cx(t) \\ x(t_0-\tau(t)) &= \phi(\tau(t)) \quad for \quad \tau(t) \in [0,\ h] \end{aligned} \tag{8.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m, w \in \mathbb{R}^k$ and $y \in \mathbb{R}^p$ with $m < p < n$, ϕ is absolutely continuous with square integrable $\dot{\phi}$, h is an upper-bound on the time-delay function $(0 \leq \tau(t) \leq h,\ \forall t \geq 0)$. The time-varying delay may be either slowly varying (i.e. a differentiable delay with $\dot{\tau}(t) \leq d < 1$) or fast varying (piecewise continuous delay). Assume that the nominal linear system (A, A_d, B, B_1, C) is known and that the input and output matrices B and C are both of full rank. The disturbance is assumed to be bounded whereby $\|w(t)\| \leq \Delta$ with a known upper bound $\Delta > 0$. A control strategy will be sought which induces an ideal sliding motion with desirable performance characteristics on the surface given be (6.4).

Similar to the structure transformation in (7.2), the system (A, A_d, B, B_1, C) can be transformed as

$$A_r = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, A_{d_r} = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}, B_r = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, B_{1_r} = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, C_r = \begin{bmatrix} 0 & T \end{bmatrix} \tag{8.2}$$

Assume (6.8), (6.9) and (6.11) hold, the reduced-order sliding mode dynamics are governed by the following reduced order system

$$\dot{x}_1(t) \quad = \quad (A_{11} - A_{12}KC_1)x_1(t) + (A_{d11} - A_{d12}KC_1)x_1(t-\tau(t)) + B_{11}w(t) \tag{8.3}$$

The presence of the unmatched uncertainty means that, in general, asymptotic stability cannot be attained by the system (8.3).

8.3 Existence problem

It will be shown that the system (8.2) is exponentially attracted to a bounded region in $\mathbb{R}^n$ if the reduced-order system (8.3) is exponentially attracted to a bounded domain in $\mathbb{R}^{n-m}$. Consider the

Lyapunov-Krasovskii functional below for the exponential stability analysis of (8.3)

$$\begin{aligned} V(t) &= x_1^T(t)Px_1(t) + \int_{t-h}^{t} e^{\alpha(s-t)} x_1^T(s)Ex_1(s)ds + \int_{t-\tau(t)}^{t} e^{\alpha(s-t)} x_1^T(s)Sx_1(s)ds \\ &+ h\int_{-h}^{0}\int_{t+\theta}^{t} e^{\alpha(s-t)} \dot{x}_1^T(s)R\dot{x}_1(s)dsd\theta \end{aligned} \quad (8.4)$$

with $(n-m)\times(n-m)$-matrices $P>0$ and $E\geq 0, S\geq 0, R\geq 0$. To prove exponential stability of the system (8.3) using (8.4), we will use the following lemma.

Lemma 8.1. *(Fridman and Dambrine, [48], 2009) Let* $V:[0,\infty)\rightarrow R^{+}$ *be an absolutely continuous function. If there exist* $\alpha>0$ *and* $b>0$ *such that the derivative of* V *satisfies almost everywhere the inequality*

$$\frac{d}{dt}V(t)+\alpha V(t)-b\|w(t)\|^2\leq 0$$

then it follows that for all $\|w(t)\|\leq\Delta$

$$V(t)\leq e^{-\alpha(t-t_0)}V(t_0)+\tfrac{b}{\alpha}\Delta^2,\quad t\geq t_0$$

Differentiating $V(t)$ from (8.4) yields

$$\begin{aligned} M \leq\ & 2x_1^T(t)P\dot{x}_1(t)+h^2\dot{x}_1{}^T(t)R\dot{x}_1(t) \\ & -he^{-\alpha h}\textstyle\int_{t-h}^{t}\dot{x}_1{}^T(s)R\dot{x}_1(s)ds+x_1^T(t)(E+S)x_1(t) \\ & -x_1^T(t-h)Ex_1(t-h)e^{-\alpha h}+\alpha x_1^T(t)Px_1(t) \\ & -(1-\dot{\tau}(t))x_1^T(t-\tau(t))Sx_1(t-\tau(t))e^{-\alpha\tau(t)} \\ & -bw^T(t)w(t) \end{aligned} \quad (8.5)$$

Suppose equations (7.8), (7.9) hold for the term $-h\int_{t-h}^{t}\dot{x}_1{}^T(s)R\dot{x}_1(s)ds$, then equation (8.5) becomes

$$\begin{aligned} M \leq\ & 2x_1^T(t)P\dot{x}_1{}^T(t)+\alpha x_1^T(t)Px_1(t)+h^2\dot{x}_1{}^T(t)R\dot{x}_1(t) \\ & -[(x_1(t)-x_1(t-\tau(t)))^T R(x_1(t)-x_1(t-\tau(t)))- \\ & (x_1(t-\tau(t))-x_1(t-h))^T R(x_1(t-\tau(t))-x_1(t-h))]e^{-\alpha h} \\ & +x_1^T(t)(E+S)x_1(t)-x_1^T(t-h)Ex_1(t-h)e^{-\alpha h} \\ & -(1-d)x_1^T(t-\tau(t))Sx_1(t-\tau(t))e^{-\alpha h}-bw^T(t)w(t) \end{aligned} \quad (8.6)$$

Using the descriptor method

$$\begin{aligned} 0 \equiv\ & 2(x_1^T(t)P_2^T+\dot{x}_1{}^T(t)P_3^T)[-\dot{x}_1(t)+(A_{11}-A_{12}KC_1) \\ & x_1(t)+(A_{d11}-A_{d12}KC_1)x_1(t-\tau(t))+B_{11}w(t)] \end{aligned} \quad (8.7)$$

where matrix parameters P_2, $P_3=\varepsilon P_2\in\mathbb{R}^{n-m}$ are added to the right-hand side of (8.6). Setting $\eta(t)=col\{x_1(t),\dot{x}_1(t),x_1(t-h),x_1(t-\tau(t)),w(t)\}$, then $M\leq\eta^T(t)\Theta\eta(t)\leq 0$ if the matrix $\Theta<0$. Multiplying matrix Θ from the right and the left by $diag\{P_2^{-1},P_2^{-1},P_2^{-1},P_2^{-1},I\}$ and its transpose respectively and denoting

$$Q_2 = P_2^{-1}, \widehat{P} = Q_2^T P Q_2,\ \widehat{R} = Q_2^T R Q_2,\ \widehat{E} = Q_2^T E Q_2,\ \widehat{S} = Q_2^T S Q_2$$

it follows $\Theta < 0 \Leftrightarrow \widehat{\Theta} < 0$ where

$$\widehat{\Theta} = \begin{bmatrix} \widehat{\theta}_{11} & \widehat{\theta}_{12} & 0 & \widehat{\theta}_{14} & \widehat{\theta}_{15} \\ * & \widehat{\theta}_{22} & 0 & \widehat{\theta}_{24} & \widehat{\theta}_{25} \\ * & * & \widehat{\theta}_{33} & \widehat{\theta}_{34} & 0 \\ * & * & * & \widehat{\theta}_{44} & 0 \\ * & * & * & * & \widehat{\theta}_{55} \end{bmatrix} < 0 \tag{8.8}$$

and

$$\begin{aligned}
&\widehat{\theta}_{11} = (A_{11} - A_{12}KC_1)Q_2 + \alpha\widehat{P} + Q_2^T(A_{11} - A_{12}KC_1)^T + \widehat{E} + \widehat{S} - \widehat{R}e^{-\alpha h} \\
&\widehat{\theta}_{12} = \widehat{P} - Q_2 + \varepsilon Q_2^T(A_{11} - A_{12}KC_1)^T \\
&\widehat{\theta}_{14} = (A_{d11} - A_{d12}KC_1)Q_2 + \widehat{R}e^{-\alpha h} \\
&\widehat{\theta}_{15} = B_{11} \qquad \widehat{\theta}_{22} = -\varepsilon Q_2 - \varepsilon Q_2^T + h^2\widehat{R} \\
&\widehat{\theta}_{24} = \varepsilon(A_{d11} - A_{d12}KC_1)Q_2 \qquad \widehat{\theta}_{25} = \varepsilon B_{11} \\
&\widehat{\theta}_{33} = -(\widehat{E} + \widehat{R})e^{-\alpha h} \qquad \widehat{\theta}_{34} = \widehat{R}e^{-\alpha h} \\
&\widehat{\theta}_{44} = -2e^{-\alpha h}\widehat{R} - (1-d)\widehat{S}e^{-\alpha h} \qquad \widehat{\theta}_{55} = -bI
\end{aligned} \tag{8.9}$$

Given that Q_2 is defined in (6.19) and (6.20), (6.22) hold, substitute (6.22) into (8.9) to yield

$$\begin{aligned}
&\widehat{\theta}_{11} = A_{11}Q_2 - A_{12}[Y \quad \delta Y] + Q_2^T A_{11}^T + \alpha\widehat{P} - [Y\mathscr{M} \quad \delta Y]^T A_{12}^T + \widehat{E} + \widehat{S} - \widehat{R}e^{-\alpha h} \\
&\widehat{\theta}_{12} = \widehat{P} - Q_2 + \varepsilon Q_2^T A_{11}^T - \varepsilon[Y\mathscr{M} \quad \delta Y]^T A_{12}^T \\
&\widehat{\theta}_{14} = A_{d11}Q_2 - A_{d12}[Y\mathscr{M} \quad \delta Y] + \widehat{R}e^{-\alpha h} \\
&\widehat{\theta}_{15} = B_{11} \qquad \widehat{\theta}_{22} = -\varepsilon Q_2 - \varepsilon Q_2^T + h^2\widehat{R} \\
&\widehat{\theta}_{24} = \varepsilon A_{d11}Q_2 - \varepsilon A_{d12}[Y\mathscr{M} \quad \delta Y] \quad \widehat{\theta}_{25} = \varepsilon B_{11} \\
&\widehat{\theta}_{33} = -(\widehat{E} + \widehat{R})e^{-\alpha h} \qquad \widehat{\theta}_{34} = \widehat{R}e^{-\alpha h} \\
&\widehat{\theta}_{44} = -2e^{-\alpha h}\widehat{R} - (1-d)\widehat{S}e^{-\alpha h} \qquad \widehat{\theta}_{55} = -bI
\end{aligned} \tag{8.10}$$

The following Proposition can now be stated:

Proposition 8.2. *Given scalars $h > 0$, $d < 1$, $\alpha > 0$, ε, δ, b and a matrix $\mathscr{M} \in \mathbb{R}^{(p-m)\times(n-p)}$, if there exist $(n-m)\times(n-m)$ matrices $\widehat{P} > 0$, $\widehat{E} \geq 0$, $\widehat{S} \geq 0$, $\widehat{R} \geq 0$ and matrices $Q_{22} \in \mathbb{R}^{(p-m)\times(p-m)}$,*

$Q_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$, $Q_{12} \in \mathbb{R}^{(n-p)\times(p-m)}$, $Y \in \mathbb{R}^{m\times(p-m)}$ such that the LMI (8.8) with matrix entries (8.10) holds, then the reduced order system (8.3), where $K = YQ_{22}^{-1}$, is exponentially attracted to the ellipsoid

$$x_1^T(t)Px_1(t) \leq \frac{b}{\alpha}\Delta^2 \tag{8.11}$$

where $P = Q_2^{-T}\widehat{P}Q_2^{-1}$, for all differentiable delays $0 \leq \tau(t) \leq h$, $\dot{\tau}(t) \leq d < 1$. Moreover, the reduced order dynamics (8.3) is exponentially stable for all piecewise-continuous delays $0 \leq \tau(t) \leq h$, if the LMI (8.8) is feasible with $\widehat{S} = 0$.

Remark 8.3. Since the LMI (8.8) is affine in the system matrices A, A_d and B_1, the results are applicable to the case where these matrices are uncertain. Denote $\Omega = \begin{bmatrix} A & A_d & B_1 \end{bmatrix}$ and assume that $\Omega \in \mathscr{C}o\{\Omega_j,\ j = 1,...N\}$, namely, $\Omega = \sum_{j=1}^N f_j(t)\Omega_j$ for some $0 \leq f_j(t) \leq 1$, $\sum_{j=1}^N f_j(t) = 1$, where the N vertices of the polytope are described by $\Omega_j = \begin{bmatrix} A^{(j)} & A_d^{(j)} & B_1^{(j)} \end{bmatrix}$. One has to solve the LMIs simultaneously for all the N vertices, applying the same decision matrices for all vertices.

Example 8.1. Consider the following simple system taken from (Gouaisbaut, et al., [69], 2004), which is in regular form with polytopic uncertainties and unknown (bounded) perturbations $\beta(t)$ and $f(t)$

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -3 & 2 & 1 \\ 2 & 1+sin(x_3(t)) & 1 \\ 1 & 1 & x_2^2(t)+1 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0.2 \\ -0.2 & -0.5 & 1 \end{bmatrix} x(t-\tau) \\
&+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bar{u}(t) + \begin{bmatrix} \beta(t)x_1(t) + 0.5w(t) \\ -0.5\beta(t)x_1(t-\tau) - 0.5w(t) \\ 0.2\beta(t)x_2(t) + w(t) \end{bmatrix} \\
y(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t)
\end{aligned} \tag{8.12}$$

with $0 \leq \beta(t) \leq 2$ and disturbance $w(t) \in [-1,\ 1]$. The delay is assumed to be time-varying. In order to present (8.12) in the form of (8.1) with uncertain matrices, define the control variable $\bar{u}(t)$ as follows:

$$\bar{u}(t) = u(t) + (x_2^2(t) + 1)x_3(t) \tag{8.13}$$

where $u(t)$ is the sliding mode control variable of the form (8.17). This change is possible because $x_2(t)$ and $x_3(t)$ are measured. Considering next $sin(x_3(t))$ as uncertainty $\gamma(t) = sin(x_3(t)) \in [-1,1]$, the above system is represented as a polytopic system with four vertices defined by $\gamma = \pm 1$, $\beta = 0$ and $\beta = 2$

$$\dot{x}(t) = \textstyle\sum_{j=1}^4 f_j(t)[A^{(j)}x(t) + A_d^{(j)}x(t-\tau)] + B\bar{u}(t) + B_1 w(t) \tag{8.14}$$

where

$$A^{(1)} = \begin{bmatrix} -3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, A^{(2)} = \begin{bmatrix} -3 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, A^{(3)} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1.4 & 0 \end{bmatrix}, A^{(4)} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1.4 & 0 \end{bmatrix},$$
$$A_d^{(1)} = A_d^{(2)} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0.2 \\ -0.2 & -0.5 & 1 \end{bmatrix}, A_d^{(3)} = A_d^{(4)} = \begin{bmatrix} 0.5 & 0 & 0 \\ -1 & 1 & 0.2 \\ -0.2 & -0.5 & 1 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix} \tag{8.15}$$

Note that state-feedback sliding mode control of the above system without unmatched disturbances and polytopic uncertainties $0.2\beta(t)x_2(t)$ in $\dot{x}_3(t)$ was considered in (Gouaisbaut, et al., [69], 2004). The work employs LMI methods for the solution of the existence problem, and is suitable for uncertain systems where the polytopic uncertainties appear only in the subsystem (8.3).The control law is derived based on the assumption that those states varying in the span of the control input are bounded so that a large enough switching gain can induce the sliding motion in finite time. An appropriate switching gain must usually be determined by trial and error.

The advantage of the proposed method is that it facilitates analysis of polytopic uncertainties appearing in the input channel and switching gain is derived from LMIs, ensuring finite reachability onto the sliding surface with a prescribed decay rate.

The initial function is taken as $x(t) = [1,\ 1\ \ -1]^T$ for $t \in [-\tau,\ 0]$. To construct K for the reduced order system (8.3) according to Proposition 8.2, the parameter settings in the LMI (8.8) with entries (8.10) are selected with the delay-upperbound $h = 1s$ and the rate of change of the time-varying delay $\dot{\tau} \le d = 0.1$. For $\delta = 2$, $\varepsilon = 0.3$, $M = 2$ and choosing $\alpha = 0.1$, $b = 0.005$, then it is obtained that the LMI variables

$$\widehat{P} = \begin{bmatrix} 949.4 & 39.5 \\ * & 925.9 \end{bmatrix}, \ Y = 1237.4,\ Q_{22} = 175,\ K = 7.07$$

Once a stable sliding mode dynamics has been designed, the next step is to find a controller which ensures the closed loop-system reaches the prescribed sliding surface in finite time. This will now be considered in general terms.

8.4 Reachability problem

It can be shown there exists a coordinate system so that the system $(\bar{A}, \bar{A}_d, \bar{B}, F\bar{C})$ has the property

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \bar{A}_d = \begin{bmatrix} \bar{A}_{d11} & \bar{A}_{d12} \\ \bar{A}_{d21} & \bar{A}_{d22} \end{bmatrix} \bar{B} = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \bar{B}_1 = \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{12} \end{bmatrix} F\bar{C} = \begin{bmatrix} 0 & I_m \end{bmatrix} \tag{8.16}$$

where $z_1(t) = x_1(t)$, $z_2(t) = s(t)$, $\bar{A}_{11} = A_{11} - A_{12}KC_1$ and $\bar{A}_{d11} = A_{d11} - A_{d12}KC_1$ exhibit the reduced order sliding-mode dynamics. Also, $\bar{C} = [0 \quad \bar{T}]$, where $\bar{T} \in \mathbb{R}^{p\times p}$ is nonsingular. The control law from (Edwards, et al., [28], 2001) is considered as

$$u(t) = -Gy(t) - v_y(t) \tag{8.17}$$

where

$$G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \bar{T}^{-1} \tag{8.18}$$

$$v_y(t) = \begin{cases} \rho \frac{Fy(t)}{\|Fy(t)\|} & \text{if } Fy(t) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{8.19}$$

where $G_1 \in \mathbb{R}^{m\times(p-m)}$, $G_2 \in \mathbb{R}^{m\times m}$, $F = \begin{bmatrix} K & I_m \end{bmatrix} T^{-1}$. The uncertain system (8.1) becomes

$$\dot{z}(t) = \bar{A}z(t) + \bar{A}_d z(t-\tau(t)) + \bar{B}u(t) + \bar{B}_1 w(t) \tag{8.20}$$

Closing the loop in the system (8.20) with the control law (8.17) yields

$$\dot{z}(t) = A_0 z(t) + \bar{A}_d z(t-\tau(t)) - \bar{B}v_y(t) + \bar{B}_1 w(t) \tag{8.21}$$

where $A_0 = \bar{A} - \bar{B}G\bar{C}$. Let $\bar{P}$ be a symmetric positive definite matrix partitioned conformably with (8.16) so that $\bar{P} = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix}$. It follows that $\bar{P}\bar{B} = (F\bar{C})^T P_2$ and from (8.16) $Fy(t) = z_2(t)$. It can be shown that

$$\begin{aligned} \psi &= \bar{P}A_0 + A_0^T \bar{P} \\ &= \begin{bmatrix} \bar{P}_1\bar{A}_{11} + \bar{A}_{11}^T\bar{P}_1 & \bar{P}_1\bar{A}_{12} + (\bar{A}_{21} - G_1C_1)^T\bar{P}_2 \\ * & \bar{P}_2\bar{A}_{22} + \bar{A}_{22}^T\bar{P}_2 - \bar{P}_2G_2 - (\bar{P}_2G_2)^T \end{bmatrix} \\ &= \begin{bmatrix} \bar{P}_1\bar{A}_{11} + \bar{A}_{11}^T\bar{P}_1 & \bar{P}_1\bar{A}_{12} + \bar{A}_{21}^T\bar{P}_2 - (L_1C_1)^T \\ * & \bar{P}_2\bar{A}_{22} + \bar{A}_{22}^T\bar{P}_2 - L_2 - (L_2)^T \end{bmatrix} \end{aligned} \tag{8.22}$$

where $L_1 = \bar{P}_2 G_1$ and $L_2 = \bar{P}_2 G_2$. A stability condition for the full order closed loop system can be derived using the following Lyapunov-Krasovskii functional

$$\begin{aligned} V(t) &= z^T(t)\bar{P}z(t) + \int_{t-h}^{t} e^{\bar{\alpha}(s-t)} z^T(s)\bar{E}z(s)ds + \int_{t-\tau(t)}^{t} e^{\bar{\alpha}(s-t)} z^T(s)\bar{S}z(s)ds \\ &+ h\int_{-h}^{0}\int_{t+\theta}^{t} e^{\bar{\alpha}(s-t)} \dot{z}^T(s)\bar{R}\dot{z}(s)dsd\theta \end{aligned} \tag{8.23}$$

where $\bar{E} \geq 0$, $\bar{S} \geq 0$ and $\bar{R} = \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & 0 \end{bmatrix}$ where $\bar{R}_1 \geq 0$ (as it is desired to determine a stability condition for the time delay system which is delay-independent of $z_2(t)$). Then

$$\begin{aligned} \bar{M} &= \dot{V} + \bar{\alpha}V - \bar{b}w^T(t)w(t) \\ &\leq 2z^T(t)\bar{P}\dot{z}^T(t) + \bar{\alpha}z^T(t)\bar{P}z(t) + h^2\dot{z}^T(t)\bar{R}\dot{z}(t) \\ &- [(z(t) - z(t-\tau(t)))^T\bar{R}(z(t) - z(t-\tau(t))) \\ &+ (z(t-\tau(t)) - z(t-h))^T\bar{R}(z(t-\tau(t)) - z(t-h))]e^{-\bar{\alpha}h} \\ &+ z^T(t)(\bar{E}+\bar{S})z(t) - z^T(t-h)\bar{E}z(t-h)e^{-\bar{\alpha}h} \\ &- (1-d)z^T(t-\tau(t))\bar{S}z(t-\tau(t))e^{-\bar{\alpha}\tau(t)} - \bar{b}w^T(t)w(t) \end{aligned} \tag{8.24}$$

Substitute the right-hand side of equation (8.21) into (8.24). Setting $\varsigma(t) = col\{z(t), z(t-h), z(t-\tau(t)), w(t)\}$, then

$$\dot{V}(t) \leq \varsigma(t)^T\Phi_h\varsigma(t) + h^2\dot{z}^T(t)\bar{R}\dot{z}(t) + 2z^T\bar{P}\bar{B}(\bar{B}_{12}w(t) - v_y(t)) < 0 \tag{8.25}$$

is satisfied if $\varsigma^T(t)\Phi_h\varsigma(t) + h^2\dot{z}^T(t)\bar{R}\dot{z}(t) < 0$ and $2z^T\bar{P}\bar{B}(\bar{B}_{12}w(t) - v_y(t)) < 0$, where

$$\Phi_h = \begin{bmatrix} \phi_{11} & 0 & \bar{P}\bar{A}_d + \bar{R}e^{-\bar{\alpha}h} & \begin{bmatrix} \bar{P}_1\bar{B}_{11} \\ 0 \end{bmatrix} \\ * & \phi_{22} & \bar{R}e^{-\bar{\alpha}h} & 0 \\ * & * & -2e^{-\bar{\alpha}h}\bar{R} - (1-d)\bar{S}e^{-\bar{\alpha}h} & 0 \\ * & * & * & -\bar{b}I \end{bmatrix} \tag{8.26}$$

with

$$\phi_{11} = \psi + \bar{\alpha}\bar{P} + \bar{S} + \bar{E} - \bar{R}e^{-\bar{\alpha}h}; \quad \phi_{22} = -(\bar{E}+\bar{R})e^{-\bar{\alpha}h}$$

Setting $\xi(t) = col\{z(t), z(t-h), z(t-\tau(t)), w(t), v_y(t)\}$ and $\bar{I} = \begin{bmatrix} I_{(n-m)} & 0 \end{bmatrix}^T$, it is obtained that

$$
\begin{aligned}
h^2\dot{z}^T(t)\bar{R}\dot{z}(t) &= [z^T(t)A_0^T + z^T(t-\tau(t))\bar{A}_d^T - v_y^T(t)\bar{B}^T + w^T(t)\bar{B}_1^T]h^2\bar{R}[A_0 z(t) \\
&+ \bar{A}_d z(t-\tau(t)) - \bar{B}v_y(t) + \bar{B}_1 w(t)] \\
&= \xi^T(t)\begin{bmatrix} A_0^T \\ 0 \\ \bar{A}_d^T \\ \bar{B}_1^T \\ \bar{B}^T \end{bmatrix} \bar{I}h^2\bar{R}_1\bar{I}^T \begin{bmatrix} A_0^T \\ 0 \\ \bar{A}_d^T \\ \bar{B}_1^T \\ \bar{B}^T \end{bmatrix}^T \xi(t)
\end{aligned}
\tag{8.27}
$$

Using the Schur complement, $\xi^T(t)\Phi_h\xi(t) + h^2\dot{z}^T(t)\bar{R}\dot{z}(t) < 0$ holds if

$$
\begin{bmatrix}
 & hA_0^T\begin{bmatrix} I_{(n-m)} \\ 0 \end{bmatrix}\bar{R}_1 \\
 & 0 \\
\Phi_h & h\bar{A}_d^T\begin{bmatrix} I_{(n-m)} \\ 0 \end{bmatrix}\bar{R}_1 \\
 & h\bar{B}_1^T\begin{bmatrix} I_{(n-m)} \\ 0 \end{bmatrix}\bar{R}_1 \\
* \quad * \quad * \quad * & -\bar{R}_1
\end{bmatrix} < 0
\tag{8.28}
$$

for some $\bar{\alpha} > 0$, $\bar{b} > 0$ and $0 \leq \tau(t) \leq h$, i.e. to ensure the exponential attractiveness of (8.21) to the ellipsoid $z^T(t)\bar{P}z(t) \leq \frac{\bar{b}}{\bar{\alpha}}\Delta^2$. Given the control structure in (8.18), then

$$
\begin{aligned}
& 2z^T(t)\bar{P}\bar{B}(\bar{B}_{12}w(t) - v_y(t)) \\
& = 2z_2^T(t)\bar{P}_2(\bar{B}_{12}w(t) - v_y(t)) \\
& \leq -2\rho\bar{P}_2\|z_2(t)\| + 2\bar{P}_2\|\bar{B}_{12}\|\|z_2(t)\|\Delta \\
& < 0
\end{aligned}
$$

The latter inequality implies exponential attractivity of the ellipsoid $z^T(t)\bar{P}z(t) \leq \frac{\bar{b}}{\bar{\alpha}}\Delta^2$, thus for $t \to \infty$, $z^T(t-\tau(t))\bar{P}z(t-\tau(t)) \leq \frac{\bar{b}}{\bar{\alpha}}\Delta^2$ holds. The following proposition can now be stated:

Proposition 8.4. *Given scalars $h > 0$, $d < 1$, $\bar{\alpha} > 0$, $\bar{b} > 0$, assume there exist $n \times n$ matrices $\bar{P} = diag\{\bar{P}_1, \bar{P}_2\} > 0$ with $P_2 \in \mathbb{R}^{m \times m}$, $\bar{E} \geq 0$, $\bar{S} \geq 0$, a $(n-m) \times (n-m)$-matrix $\bar{R}_1 \geq 0$, $L_1 \in \mathbb{R}^{m \times (p-m)}$, $L_2 \in \mathbb{R}^{m \times m}$ such that LMI (8.28) is feasible. Then for $\rho > \|\bar{B}_{12}\|\Delta$ the closed-loop system (8.21), where $G_1 = \bar{P}_2^{-1}L_1$, $G_2 = \bar{P}_2^{-1}L_2$, is exponentially attracted to the ellipsoid $z^T(t)\bar{P}z(t) \leq \frac{\bar{b}}{\bar{\alpha}}\Delta^2$ for all $\tau(t) \in [0,h]$. Consequently it also holds that $z^T(t-\tau(t))\bar{P}z(t-\tau(t)) \leq \frac{\bar{b}}{\bar{\alpha}}\Delta^2$ for $t \to \infty$.*

Denote

$$
A_0^L = [0\ I_m]A_0; \quad \bar{A}_d^L = [0\ I_m]\bar{A}_d; \quad \beta = \frac{\bar{b}}{\bar{\alpha}}\Delta^2
\tag{8.29}
$$

Given $\delta_1 > 0,\ \delta_2 > 0$, conditions will now be derived that guarantee the solutions of (8.16) satisfy the bound

$$\|A_0^L z(t)\| < \delta_1, \quad \|A_d^L z(t-\tau(t))\| < \delta_2 \tag{8.30}$$

for $t \rightarrow \infty$. The following inequalities

$$\begin{array}{l} z^T(t)(A_0^L)^T(A_0^L)z(t) \le \delta_1^2 \frac{z^T(t)\bar{P}z(t)}{\beta} \\ z^T(t-\tau(t))(A_d^L)^T(A_d^L)z(t-\tau(t)) \le \delta_2^2 \frac{z^T(t-\tau(t))\bar{P}z(t-\tau(t))}{\beta} \end{array} \tag{8.31}$$

guarantee (8.30). Hence equivalently

$$(A_0^L)^T(A_0^L) \le \frac{\delta_1^2 \bar{P}}{\beta}; \quad (A_d^L)^T(A_d^L) \le \frac{\delta_2^2 \bar{P}}{\beta} \tag{8.32}$$

or by Schur complements

$$\begin{bmatrix} -\frac{\delta_1^2 \bar{P}}{\beta} & (A_0^L)^T \\ * & -I \end{bmatrix} < 0; \quad \begin{bmatrix} -\frac{\delta_2^2 \bar{P}}{\beta} & (A_d^L)^T \\ * & -I \end{bmatrix} < 0 \tag{8.33}$$

Theorem 8.5. *Given scalars $\bar{\alpha} > 0,\ \bar{b} > 0$, let there exist $n \times n$ matrices $\bar{P} = diag\{\bar{P}_1, \bar{P}_2\} > 0$, $\bar{E} \ge 0,\ \bar{S} \ge 0$, a $(n-m) \times (n-m)$-matrix $\bar{R}_1 \ge 0$, $L_1 \in \mathbb{R}^{m \times (p-m)}$, $L_2 \in \mathbb{R}^{m \times m}$ such that the LMI (8.28) is feasible for $0 \le \tau(t) \le h,\ \dot{\tau}(t) \le d < 1$. Let δ_1 and δ_2 satisfy (8.33) with the notation given in (8.29). Then for*

$$\rho > \|B_{12}\|\Delta + \delta_1 + \delta_2 \tag{8.34}$$

an ideal sliding motion takes place on the surface $\mathcal{S}$. The closed loop system (8.17), (8.20) is ultimately bounded by

$$limsup_{t \rightarrow \infty} z^T(t)\bar{P}z(t) \le \frac{\bar{b}}{\bar{\alpha}}\Delta^2$$

Proof. Substituting the control law it follows from (8.20) that

$$\dot{s}(t) = F\bar{C}A_0 z(t) + F\bar{C}\bar{A}_d z((t-\tau(t)) + (\bar{B}_{12}w(t) - v_y(t))$$

Let $V_c : \mathbb{R}^m \rightarrow \mathbb{R}$ be defined by $V_c(s) = s^T(t)\bar{P}_2 s(t)$. It follows that

$$\bar{P}_2 F\bar{C}A_0 = \bar{P}_2 A_0^L; \quad \bar{P}_2 F\bar{C}A_d = \bar{P}_2 A_d^L$$

Starting from initial condition $z(t_0)$, it can be verified that there exists $t_1 > 0$ such that for all $t \ge t_1$,

$$\begin{array}{rcl} \dot{V}_c(s) & = & 2s^T(t)\bar{P}_2 A_0^L z(t) + 2s^T(t)\bar{P}_2 A_d^L z(t-\tau(t)) + 2s^T(t)\bar{P}_2(\bar{B}_{12}w(t) - v_y(t)) \\ & \le & 2\|s(t)\|\|\bar{P}_2\|(\|A_0^L z(t)\| + \|A_d^L z(t-\tau(t))\|) - 2(\delta_1+\delta_2)\|s(t)\|\|\bar{P}_2\| \\ & < & -2\eta\|s(t)\| \end{array} \tag{8.35}$$

where $\eta = \delta_1 + \delta_2 - \|A_0^L z(t)\| - \|A_d^L z(t-\tau(t))\|$. A sliding motion will thus be attained in finite time.

□

Remark 8.6. Since LMIs (8.8), (8.28) and (8.33) are affine in the system matrices A, A_d and B_1, the results are applicable to the case where these matrices are uncertain with polytopic type uncertainties (see Remark 8.3). One has to solve the LMIs simultaneously for all the N vertices, applying the same decision matrices for all vertices. In contrast to the existing methods in the literature (Gouaisbaut, et al., [69], 2004), (Seuret, et al., [123], 2009) polytopic type uncertainties can be incorporated in all the blocks of A, A_d, B_1 and not only in A_{11}, A_{d11} because the switching gain ρ (and not only the sliding surface) is found using LMIs.

Example 8.2. Following on from Example 8.1, where a sliding surface prescribing stable dynamics has been designed for the uncertain system (8.12), then the control law in (8.18) will have the sliding function matrix $F = \begin{bmatrix} 7.07, & 1 \end{bmatrix}$. A control gain G must be designed which will bring the closed-loop system into a bounded region centered at the sliding surface. Setting $\bar{\alpha} = 0.3$, $\bar{b} = 5$ in Proposition 8.4, it is obtained that

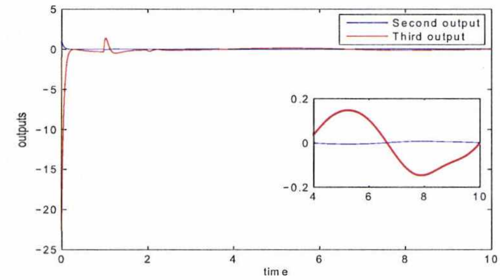
$$\bar{P} = \begin{bmatrix} 22.4 & -12.8 & 0 \\ * & 29 & 0 \\ * & * & 0.68 \end{bmatrix}, \; L_1 = -6.08, \; L_2 = 85.4$$

which gives

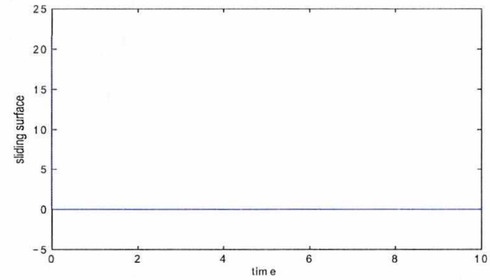
$$G = \begin{bmatrix} -9, & 126.6 \end{bmatrix} \bar{T}^{-1}, \text{ where } \bar{T} = \begin{bmatrix} 1 & 0 \\ -7.07 & 1 \end{bmatrix}$$

Once the state of the closed-loop system has entered the sliding patch $z^T(t)\bar{P}z(t) \leq \frac{\bar{b}}{\bar{\alpha}}\Delta^2$, the switching gain $\rho = 753$ derived from LMI (8.33) will ensure the sliding surface is reached in finite time. Figure 8.2 shows that the sliding surface is reached in finite time and the outputs of the system are stable with ultimate bound $\|y(t)\| \leq 0.2$.

Sliding mode control has the advantage over linear control for its absolute rejection of the matched uncertainties. To verify the statement, suppose there is a change of the matched disturbance at time $10s$ of magnitude from $1 \rightarrow 50$, then sliding surface remains unaffected so that $B_1 = \begin{bmatrix} 0 & 0 & 50 \end{bmatrix}^T$. While keeping the same control parameters obtained so far comparisons between using sliding mode control and only the linear control G for the new closed loop design with only matched disturbances are made in Figure 8.3. As can be seen, the sliding mode control is more robust than the linear control to matched disturbances. The difference between using linear part of the control G alone and SMC with the switching for system with unmatched disturbances can be demonstrated below. For the same uncertain system, suppose the unmatched disturbances are changed from $B_1 = \begin{bmatrix} 0.5, & -0.5, & 1 \end{bmatrix}^T$ to $B_1 = \begin{bmatrix} 2, & -2, & 1 \end{bmatrix}^T$ after the initial $10s$. Using the linear control $G = \begin{bmatrix} 25, & 4 \end{bmatrix}$ alone in the feedback, the responses for the outputs $y(t)$, sliding function $s(t)$, and control input $u(t)$ are plotted in Figure 8.2(a). As can be seen the outputs



(a) Outputs



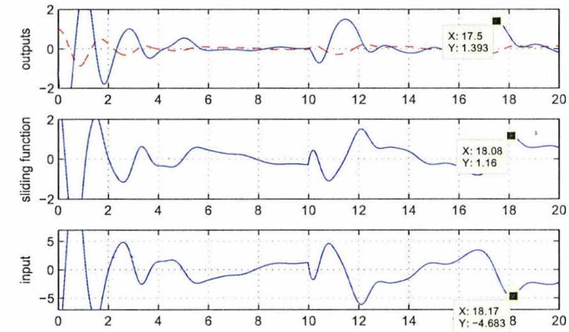
(b) Sliding surface

FIGURE 8.1: Closed-loop response with delay $h = 1s$

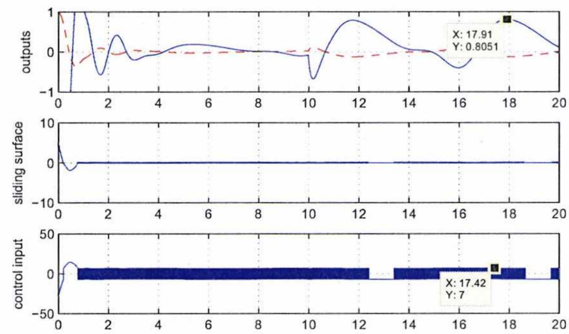
$\|y(t)\| \leq 1.4$, sliding function is bounded as $\|s(t)\| \leq 1.16$ and control input $\|u(t)\| \leq 4.7$. If SMC is used with the same linear gain and a switching gain $\rho = 7$, the system responses are plotted in Figure 8.2(b), where $\|y(t)\| \leq 0.81$, $s(t) = 0$, $\|u(t)\| \leq 7$. Therefore for system with unmatched disturbances, SMC can give an ideal sliding surface rather than a bounded sliding function given by its linear control part. As a result, a smaller bound of the outputs can be obtained. For linear control to yield the similar bound on the outputs as by SMC, the linear gain needs to be increased from $G = \begin{bmatrix} 25, & 4 \end{bmatrix}$ to $\hat{G} = \begin{bmatrix} 134, & 21 \end{bmatrix}$ as seen in Figure 8.2(c), where $\|y(t)\| \leq 0.83$. To conclude, for a linear control to give similar outputs performance in presence of unmatched disturbances, the linear gain needs to be larger, but not substantially larger than the linear part of the SMC. In another words, a linear control design for system with unmatched disturbances can give similar output bound as SMC if the linear control is large enough.

8.5 Application to autonomous vehicle control

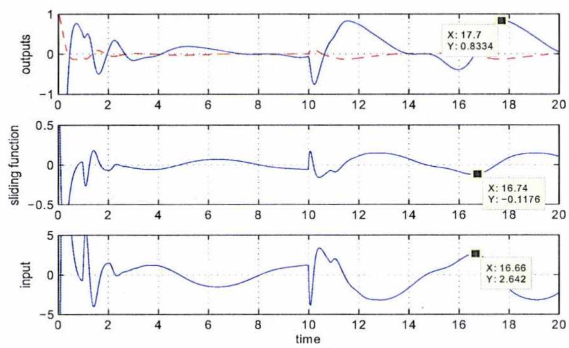
Autonomous vehicles are expected to operate effectively in time-varying and uncertain conditions. Here a case study described in (Yao, et al., [146], 2006) is considered, where the elevation angle of the gun barrel of a vehicle in space should be maintained while the hull of the vehicle is subject to external disturbances resulting from the motion of the vehicle across rough terrain. To meet the high control specifications, sliding mode control has been considered within the application domain for its robustness against friction and disturbances and its ease of implementation for motor



(a) linear control with G



(b) SMC with linear part G



(c) linear control with $\hat{G}$

FIGURE 8.2: Closed-loop response with linear control and SMC

drive control. The existence problem must determine a sliding surface that minimizes the ultimate bound of the reduced-order dynamics in the presence of time-varying state delay and unmatched disturbances relating to frictional effects. A fully nonlinear simulation model of the system is available for controller analysis and testing (Yao, et al., [146], 2006). The model is physically

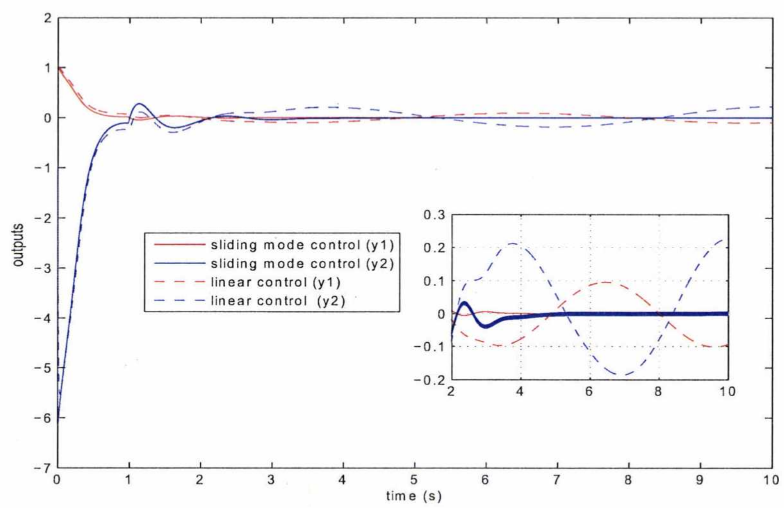


FIGURE 8.3: Comparison between sliding mode control and linear control in the presence of matched disturbance a

based and is known to represent with high fidelity the dynamics and behaviour of the real system:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -\frac{D_m}{J_mN^2} & -\frac{K_m}{J_mN} & \frac{D_m}{J_mN} & 0 & 0 \\ \frac{1}{N} & 0 & -1 & 0 & 0 \\ \frac{D_m}{J_1N} & \frac{K_m}{J_1} & \frac{-D_m-D_{12}}{J_1} & \frac{-K_{12}}{J_1} & \frac{D_{12}}{J_1} \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & \frac{D_{12}}{J_2} & \frac{K_{12}}{J_2} & \frac{-D_{12}}{J_2} \end{bmatrix}}_{A} x(t) + A_d x(t-\tau(t)) + \underbrace{\begin{bmatrix} \frac{K_t}{J_m} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B} u(t)$$

$$+ \underbrace{\begin{bmatrix} -\frac{D_m(N-1)}{J_mN^2} & -\frac{1}{J_m} & 0 & \frac{1}{J_mN^2} & 0 \\ \frac{N-1}{N} & 0 & 0 & 0 & 0 \\ \frac{D_m(N-1)}{J_1N} & 0 & -\frac{1}{J_1} & 0 & \frac{1}{J_1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{B_{11}} w(t) \tag{8.36}$$

where the state vector $x(t) = \begin{bmatrix} \dot{\theta}_m(t), & \theta_{mb}(t), & \dot{\theta}_b(t), & \theta_{bl}(t), & \dot{\theta}_l(t) \end{bmatrix}^T$, $w(t) = \begin{bmatrix} \dot{\theta}_p(t), & \omega_{1m}(t), & \omega_{1l}(t) & F_{mb} sign(\dot{\theta}_m(t) - \dot{\theta}_p(t)), & F_{mb} sign(\dot{\theta}_b(t) - \dot{\theta}_p(t)) \end{bmatrix}^T$. A delay-free case is firstly studied by setting $A_d = 0$ as to benchmark with the control performance achieved by (Yao, et al., [146], 2006). The friction related signals are

$$F_{mb}(t) = K_\theta \left| \tfrac{D_m}{N} \dot{\theta}_m(t) + \tfrac{D_m(N-1)}{N} \dot{\theta}_p(t) + K_m \theta_{mb}(t) \right|$$

$$\omega_{1m}(t) = f_{ms} \cdot sign(\tau_{am}(t) - J_m \ddot{\theta}_p(t))$$

$$\omega_{1l}(t) = f_{ls} \cdot sign(\tau_{a1}(t) - J_1 \ddot{\theta}_p(t))$$

where the second and third state of the disturbance from $w(t)$ are a function of the friction level f_d, where f_d can take values $0,1,2,3$ (Yao, et al., [146], 2006). Note

$\theta_{mb} = \frac{1}{N}\theta_m - \theta_b + (1-\frac{1}{N})\theta_p$;
$\dot{\theta}_b$: breech velocity;
θ_m: motor position;
$\dot{\theta}_l$: muzzle velocity;
$\dot{\theta}_p$: pitch rate disturbance;
J_m: motor inertia;
N: gearbox of ratio;
J_1: elevation inertia on load one;
K_m and D_m: stiffness and damping between the motor and the load;
K_{12} and D_{12}: stiffness and damping between the load one and the load two;
τ_{am} and τ_{al}: applied torque to the motor and the load;
ω_{1m} and ω_{1l}: motor friction and the load friction;
f_{ms} and f_{ls}: motor coulomb friction and the load coulomb friction;
u: control input defined as the voltage input to the power amplifier;
$\dot{\theta}_p$: disturbance input defined as pitch rate disturbance;
ω_{1m} and $sign(\dot{\theta}_m - \dot{\theta}_p)$: motor friction disturbances;
ω_{1l} and $sign(\dot{\theta}_b - \dot{\theta}_p)$: load friction disturbances

The parameter values used in (Yao, et al, [146], 2006) define

$$A = \begin{bmatrix} -338.14 & -2.55\times 10^7 & 50942 & 0 & 0 \\ 0.0066 & 0 & -1 & 0 & 0 \\ 0.66 & 50000 & -110.1 & -15000 & 10 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 7.69 & 11538 & -7.69 \end{bmatrix} \quad B = \begin{bmatrix} 4523.1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$B_{11} = \begin{bmatrix} -50604 & -769 & 0 & 5.1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 99 & 0 & -0.01 & 0 & 0.01 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{8.37}$$

and the disturbance is known to be

$$\begin{aligned} &\|\dot{\theta}_p\| \leq 0.07,\ \ \|\omega_{1m}\| \leq 0.5\times f_d,\ \ \|\omega_{1l}\| \leq 10\times f_d, \\ &\|F_{mb} sign(\dot{\theta}_m - \dot{\theta}_p)\| \leq 4,\ \ \|F_{mb} sign(\dot{\theta}_b - \dot{\theta}_p)\| \leq 4 \end{aligned} \tag{8.38}$$

The vehicle dynamics is augmented with an extra state related to the breech position, where the desired breech position is zero. Denote $C_a = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, the state space representation of the augmented system is given by

$$A_c = \begin{bmatrix} 0 & -C_a \\ 0 & A \end{bmatrix}, \; B_c = \begin{bmatrix} 0 \\ B \end{bmatrix}, C_c = \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} \tag{8.39}$$

Sliding Mode Control Design

A design which does not incorporate knowledge of delay effects is first performed to yield a benchmark level of performance. Firstly it is necessary to construct K for the reduced order system (8.3) according to Proposition 8.2. The parameter settings in LMI (8.8) with entries (8.10) are selected as $A_d = 0$, $h = 0s$. If $\delta = 11.5$, $\varepsilon = 0.0000002$, $M = \begin{bmatrix} 0.15 & 0.015 \\ 0.21 & 0.06 \\ 0.009 & 0.0003 \end{bmatrix}$ and choosing $\alpha = 7.2$, $b = 0.0005$, then it is obtained that

$$K = \begin{bmatrix} 0.9, & 28, & 327 \end{bmatrix} \tag{8.40}$$

The poles of the corresponding reduced order system are

$$\begin{bmatrix} -4.15 \pm j106, & -389.6 \pm j160.7, & -2823.7 \end{bmatrix} \tag{8.41}$$

The control law in (8.18) will have the sliding function matrix

$$F = \begin{bmatrix} -327 & 0.0002 & 28 & 0.9 \end{bmatrix} \tag{8.42}$$

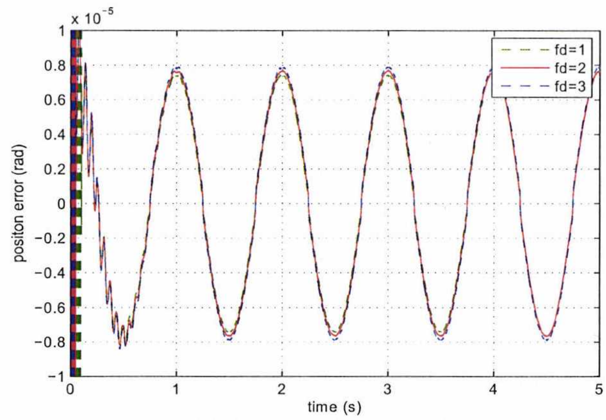
A control G is designed which will bring the closed-loop system into a bounded region centered about the sliding surface. Setting $\bar{\alpha} = 0.8$, $\bar{b} = 3.88 \times 10^{-6}$ in Proposition 8.4, it is obtained that

$$G = \begin{bmatrix} -1.02 \times 10^7, & 7.6, & 910610, & 27834 \end{bmatrix} \tag{8.43}$$

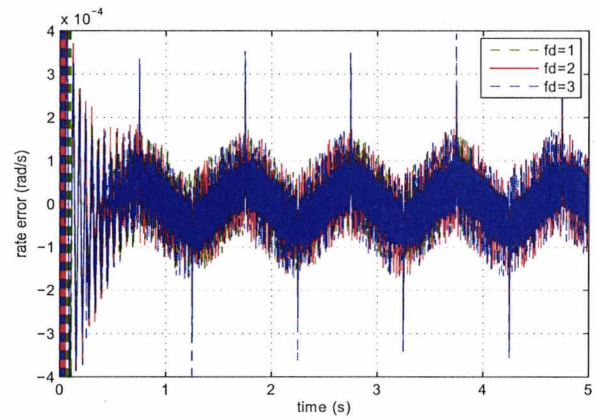
The closed-loop poles of $A_c - B_c G C_c$ are

$$\begin{bmatrix} -31257 & -2810.4 & -390.8 \pm j160.9 & -4.2 \pm j106 \end{bmatrix}$$

The switching gain $\rho = 1561$, which is derived from LMI (8.33), will ensure the sliding surface is reached in finite time. Figure 8.4 shows the position error and rate error for $f_d = 1,2,3$ using the proposed controller. In the original case study (Yao, [146], 2006) an observer was used to estimate the effect of the disturbance and the equivalent control method was used to synthesize the control law, which was augmented with an additional PI control. With this strategy the position and rate errors were $\|e_p(t)\| \leq 0.2 \times 10^{-3} rad$ and $\|e_r(t)\| \leq 0.01 rad/s$ respectively. Setting a fixed sampling frequency of $10kHz$ and choosing $ode3$ solver in the Simulink, $\|e_p(t)\| \leq 0.8 \times 10^{-5} rad$, $\|e_r(t)\| \leq 0.4 \times 10^{-3} rad/s$ was achieved, as seen in Figure 8.4, for $fd = 1,2,3$ with the proposed control scheme. The output feedback sliding mode control approach presented in this chapter has thus improved the tracking accuracy over previous results in (Yao, [146], 2006). The ultimate bound of the outputs is a function of the unmatched disturbance, but it can be seen that the effect of the friction disturbance on the control performance after changing $f_d = 1, \rightarrow 3$ is very small.



(a) position error



(b) rate error

FIGURE 8.4: Closed-loop response without delay

Speed control of a motor in the presence of uncertainties such as friction normally exhibit delays due to the fact that the mechanical response of the motor is slower than the electrical command. The size of the delay will depend upon the physical parameters of the actuator and can vary from milliseconds to several seconds, depending on the application. To take account of such delay effects in the actual system for the control design purpose, it is desirable to introduce a system model incorporating delay into the system model used for design. This will provide a means to analyze the potential delay effect on the stability of the closed-loop system at the design stage.

Assume the delay matrix

$$A_d = \begin{bmatrix} -10 & 20 & 0 & 0 & 0 \\ 0.007 & 0 & 0.1 & 0 & 0 \\ 0 & 20 & -2 & 1 & 0 \\ 0 & 0 & 0.1 & 0 & 0.1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix} \quad \text{and the augmented matrix} \quad A_{d_c} = \begin{bmatrix} 0 & 0 \\ 0 & A_d \end{bmatrix} \tag{8.44}$$

The open loop tests on system (8.37) with the delay matrix A_d in (8.44) shows that the vehicle system with state delay $h \leq 3ms$ yields a breech position error of $\|e_p(t)\| \leq 0.01 rad/s$ as expected from the known system response. The augmented linear system with delay has dynamics close to those of the original plant.

Designing a controller without considering explicitly possible delay effects within the control design process can lead to deterioration of the system performance and sometimes even instability. Suppose there is a constant delay $h = 3ms$ in the system where the values of F, G are taken as in (8.42) and (8.43) respectively, with the delay distribution matrix in (8.44). In this case the position error will increase from $\|e_p(t)\| \leq 0.8 \times 10^{-5} rad$ to $\|e_p(t)\| \leq 1.4 \times 10^{-3} rad$. The closed-loop system becomes unstable for $h \geq 4ms$ when delay effects are not incorporated in the design process.

A controller will now be designed based on a model incorporating delay effects. Firstly to construct K for the reduced order system (8.3) according to Proposition 8.2, the parameter settings in LMI (8.8) with entries (8.10) are selected with the delay-upperbound $h = 10ms$ and the rate of change of the time-varying delay $\dot{\tau} \leq d = 0$. If for $\delta = 3$, $\varepsilon = 0.0013$, $M = \begin{bmatrix} 0.0018 & 0.002 \\ 0.22 & 0.22 \\ 7.54 & 1.1 \end{bmatrix}$ and choosing $\alpha = 0.9$, $b = 0.0048$, then it is obtained that

$$K = \begin{bmatrix} 0.01, & 16, & 16.8 \end{bmatrix} \tag{8.45}$$

The poles of the reduced order system are

$$\begin{bmatrix} -5.23 \pm j92.9, & -71.3 \pm j254.8, & -477.8 \end{bmatrix} \tag{8.46}$$

Thus the control law in (8.18) will have the sliding function matrix

$$F = \begin{bmatrix} -16.8, & 0.0002, & 16, & 0.01 \end{bmatrix}$$

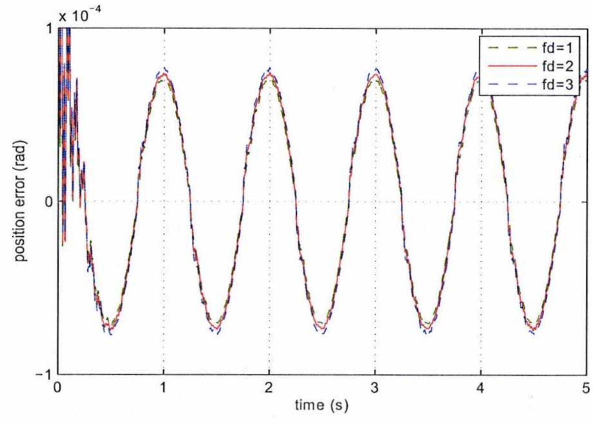
The control G is designed to bring the closed-loop system into a bounded region centered about the sliding surface. Setting $\bar{\alpha} = 0.8$, $\bar{b} = 3.88 \times 10^{-6}$ in Proposition 8.4, it is obtained that

$$G = \begin{bmatrix} -2.02 \times 10^6, & 26.7, & 1.94, & 0.002 \end{bmatrix}$$

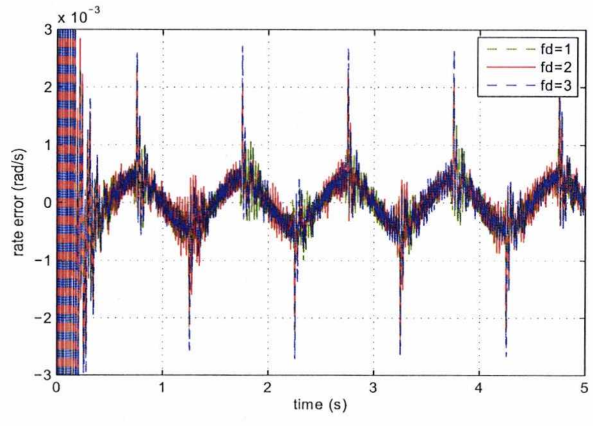
The closed-loop poles of $A_c - B_cGC_c$ are

$$\begin{bmatrix} -120700, & -481.1, & -75.2 \pm j253.2, & -5.43 \pm j92.9 \end{bmatrix}$$

The switching gain $\rho = 68632$, which is derived from LMI (8.33), will ensure the sliding surface is reached in finite time. The initial function was chosen as $x(t_0 - \tau) = 0$ for $\tau \in [0\ h]$ in the simulation. The closed-loop performance is shown in Figure 8.5 for $f_d = 1,2,3$. The position and rate error are kept within the bound $\|e_p(t)\| \leq 0.8 \times 10^{-4} rad$, $\|e_r(t)\| \leq 0.1 \times 10^{-4} rad/s$ in the presence of delay. Despite the effect of the friction disturbance on the nonlinear model which is not fully rejected, the controller is seen to be robust to the disturbance even in the presence of delay. This has demonstrated the efficiency of the proposed control scheme on a system of practical interest.



(a) position error



(b) rate error

FIGURE 8.5: Closed loop response with constant delay $h = 10ms$ present in the plant where the control has been designed based on a time-delay system model

8.6 Conclusion

The development of output feedback based sliding mode control schemes for systems in the presence of state delay and both matched and unmatched disturbances has been presented. A descriptor Lyapunov functional approach has been used for switching function design. The methodology has been implemented using LMIs and can give desirable sliding mode dynamics. The advantage of the method is that for the first time and despite only output feedback being available, not only the switching function is derived from LMIs but also the switching gain required to solve the reachability problem is determined using LMIs. The method allows polytopic uncertainties to be included in all blocks of A, A_d, B_1 and not only in A_{11}, A_{d11} as with other methods. This is novel even for systems without delay. As well as an example incorporating polytopic uncertainties, the methodology has also been applied to a nonlinear autonomous vehicle control problem. Nonlinear simulations show that the gun barrel is maintained at the desired position, despite variation in the vehicle motion caused by friction.

While the study of SMC in the presence of state delay has been ongoing, results on the effect of input delay in SMC are scarce. Is has been demonstrated in Chapter 5 that arbitrary small input delays in SMC lead to oscillations. The system may even have unbounded solutions for higher values of delay. In the next chapter output feedback SMC of systems with bounded matched disturbances will be considered in the presence of a small uncertain time-varying input delay, which can be present in the implementation of feedback control. The design objective is to achieve ultimate boundedness of the closed-loop system response with a bound proportional to the bounds on the delay, switching gain and disturbance. The behaviour of the closed-loop system is described using a singular perturbation method. A Linear Matrix Inequalities (LMIs)-based solution for the evaluation of the design parameters and of the ultimate bound will be derived using Lyapunov-based methods. Since the ultimate bound is directly related to the amplitude of the relay gain in the SMC, the proposed methodology will establish a sufficiently small relay gain to minimize the resulting ultimate bound. An extension to systems with state delay will be provided.

Chapter 9

Sliding Mode Control for Input Delay Systems: A Singular Perturbation Approach

9.1 Introduction

SMC in the presence of state delay has been presented in the previous chapters. Asymptotic stability can be achieved for sufficiently small delay varying within a bound. This chapter considers the presence of varying delay in the feedback loop in SMC. Such delays normally arises due to the time spent on attaining output measurement or the slow response in the actuator input. In this situation ideal sliding motion cannot usually be achieved, which has been shown in Chapter 2. The combination of delay phenomenon with relay actuators induces oscillations of finite frequency, or bifurcations around the critical delay value, around the sliding surface and even instability (Fridman, et al., [49], 2000), (Fridman, et al., [56], 1993), (Levaggi and Punta, [104], 2006), (Fridman, [55], 1993).

Sampled-data control can be considered as a control with a delayed input. When control engineers approach SMC, the choice of sampling rate is an immediate, and extremely critical design decision (Utkin, [136], 1992). The existing work on sampled-data SMC transforms the system to discrete-time. However, this approach is not constructive for uncertain systems, for example. The approach proposed by (Fridman, [46], 2010) considers the sample-data control as a continuous control with fast input varying delay where many existing approaches for continuous system can be applied.

Fridman, et al., [60], (2003) proposes an algorithm for local stability of a multi-input relay delay control system. In this work, a desired amplitude of the oscillation is predefined and bounds on the initial condition and the time-varying delay are chosen depending on the pre-defined amplitude of the oscillation. The closed-loop system is then guaranteed to exhibit the pre-defined amplitude of oscillation. The result was then extended to semi-global stabilization in (Fridman, et al., [61], 2004) in the sense that for any initial condition, a delayed relay control gain adaptation can be found based on knowledge of the oscillation amplitude of the relay control in each delayed interval, the upper bound of the delay and upper bound on the initial conditions. The oscillation amplitudes are then reduced by decreasing the relay gain once all the solutions are within a desired neighborhood of the origin.

In this chapter, static output-feedback SMC for systems with bounded disturbances is considered under uncertain *time-varying input delays*. The design objective is to achieve ultimate boundedness of the closed-loop system with a bound proportional to the size of the delay and the disturbance. For small enough delay such a controller should have advantages over a corresponding linear controller, because the linear control will produce a bound proportional to the disturbance only.

In the existing results (Fridman, et al., [58], 2002), an apriori constant bound is assumed on the state-dependent terms of the system, which is restrictive. Then the relay gain is chosen to be greater than this bound. The main contribution in this chapter is a general framework for SMC in the presence of input delay *without any apriori knowledge of the bounds on the system states*. The following design difficulty arises, which does not appear in the absence of input delay: the relay gain depends on the ultimate bound on the state, whereas the latter bound depends on the relay gain. To overcome this difficulty, a sliding mode controller is designed with a linear gain proportional to the scalar $\frac{1}{\mu}$, which for small enough $\mu > 0$ produces a closed-loop *singularly perturbed system* and which allows the desired ultimate bound to be achieved for the closed-loop system. The design process seeks to enlarge μ to avoid a high gain control. The resulting ultimate bound is proportional to the size of the delay, disturbance and the switching gain. Therefore trade-offs can be made in the design between the linear and the discontinuous part of the controller in order to minimize the effect of the delay. The result obtained in Chapter 8 on switching gain design using LMIs will play an important role in the present controller development.

The chapter is organized as follows. Section 9.2 presents the problem formulation. The LMI solution for the existence of a sliding manifold is given in Section 9.3. Results on the SMC design and on the resulting ultimate bound are presented in Section 9.4, where a spacecraft control problem with input delay is given to demonstrate the design methodology. An extension of the method to input and state delay is shown in Section 9.6, illustrated by a liquid mono-propellant rocket motor control problem.

9.2 Problem formulation

Consider the following uncertain dynamical system with time-varying input delay $\tau(t)$ and disturbance $w(t)$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\big(u(t-\tau(t)) + w(t)\big) \\ y(t) &= Cx(t) \end{aligned} \tag{9.1}$$

where $x(t) \in \mathbb{R}^n$, $x(t_0) = x_0$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ with $m < p < n$. It is assumed that $u(t) = 0$ for $t < t_0$. The disturbance $w(t) \in \mathbb{R}^m$ is matched with a known bound $\|w(t)\| \leq \Delta$. The time-delay $\tau(t)$ is assumed to be bounded $\tau(t) \leq \tau^*$ and sufficiently small. It is supposed that the delay is either fast varying (without any constraints on the delay derivative) or slowly varying, where the delay-derivative satisfies the bound $\dot{\tau} \leq d < 1$. Assuming B and C are both of full rank, a controller will be designed which for sufficiently large t induces the motion of the closed-loop system in the $\tau^*\Delta$-neighborhood of the surface

$$\mathscr{S} = \{x \in \mathbb{R}^n : z_2(t) = FCx(t) = 0\} \tag{9.2}$$

for some selected matrix $F \in \mathbb{R}^{m\times p}$. The relation $z_2(t) = FCx(t)$ will define a sliding manifold where it is noted that the sliding motion can be achieved only under ideal SMC condition with $\tau = 0$.

Remark 9.1. Since a static output feedback control is designed, the results are applicable to both input delay, τ_i, and output delay, τ_o, where in the closed-loop system the resulting delay is $\tau = \tau_i + \tau_0$.

9.3 Sliding manifold design

It can be shown that if $rank(CB) = m$, there exists a change of coordinates $x_r = T_r x$, where $T_r \in \mathbb{R}^{n\times n}$ is non-singular, in which the system has the regular form

$$\begin{aligned} \dot{x}_r(t) &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x_r(t) + \begin{bmatrix} 0 \\ I_m \end{bmatrix} \left(u(t-\tau(t)) + w(t)\right) \\ y(t) &= \begin{bmatrix} 0 & T \end{bmatrix} x_r(t) \end{aligned} \tag{9.3}$$

where $x_r(t) = col\{x_1(t), x_2(t)\}$, $T \in \mathbb{R}^{p\times p}$ is invertible (Edwards and Spurgeon, [29], 1995). Furthermore, $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and the remaining sub-blocks in the system matrix are partitioned accordingly. Given $K \in \mathbb{R}^{m\times(p-m)}$, let (6.8), (6.9) and (6.11) hold. Defining the sliding manifold as

$$z_2(t) = Fy(t) = x_2(t) + KC_1x_1(t) \tag{9.4}$$

the reduced-order dynamics is governed by the system

$$\dot{x}_1(t) = (A_{11} - A_{12}KC_1)x_1(t) + A_{12}z_2(t) \tag{9.5}$$

with input z_2. The system triple A_{11}, A_{12}, C_1 is assumed to be stabilizable. In the presence of input delay, z_2 in (9.5) will not vanish in finite time. Therefore, a K is sought which not only stabilizes (9.5) (as in the case without delay), but also produces input-to-state stability (with the smallest gain possible). Sufficient conditions for the input-to-state stability of (9.5) are given by the following lemma:

Lemma 9.2. *Given scalars $\alpha > 0$, ε, ε_1, b and a matrix $M \in \mathbb{R}^{(p-m)\times(n-p)}$, if there exists an $(n-m)\times(n-m)$ matrix $P > 0$, and matrices $Q_{22} \in \mathbb{R}^{(p-m)\times(p-m)}$, $Q_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$, $Q_{12} \in \mathbb{R}^{(n-p)\times(p-m)}$, $Y \in \mathbb{R}^{m\times(p-m)}$ such that LMI*

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & A_{12} \\ * & -\varepsilon Q_2 - \varepsilon Q_2^T & \varepsilon A_{12} \\ * & * & -bI_m \end{bmatrix} < 0 \tag{9.6}$$

holds, where $Q_2 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{22}M & \varepsilon_1 Q_{22} \end{bmatrix}$ *and*

$$\begin{aligned} \theta_{11} &= A_{11}Q_2 - A_{12}[YM \quad \varepsilon_1 Y] + \alpha P \\ &\quad + Q_2^T A_{11}^T - [YM \quad \varepsilon_1 Y]^T A_{12}^T, \\ \theta_{12} &= P - Q_2 + \varepsilon Q_2^T A_{11}^T - \varepsilon [YM \quad \varepsilon_1 Y]^T A_{12}^T \end{aligned}$$

then the solution of (9.5) with $K = YQ_{22}^{-1}$ and with the initial condition $x_1(t_0)$ at initial time t_0 is bounded by

$$x_1^T(t)\hat{P}x_1(t) < e^{-\alpha(t-t_0)}x_1^T(t_0)\hat{P}x_1(t_0) + \frac{b}{\alpha}\sup \|z_{2[t_0,t]}\|^2 \tag{9.7}$$

where $\hat{P} = Q_2^{-T} P Q_2^{-1}$.

Proof. The proof follows analogously using the result from Lemma 8.1 and Theorem 6.1. □

Remark 9.3. To minimize the ultimate bound on x_1, the following procedure is adopted from (Fridman and Dambrine, [48], 2009). The $\zeta \in \mathbb{R}$ is minimized subject to LMI (9.6) and

$$\begin{bmatrix} -P & Q_2^T \\ * & -\zeta I_{n-m} \end{bmatrix} < 0 \tag{9.8}$$

The latter LMI is equivalent to $\hat{P} = Q_2^{-T} P Q_2^{-1} > \zeta^{-1} I_{n-m}$ and leads to

$$\limsup_{t\to\infty} \|x_1(t)\|^2 < \zeta \frac{b}{\alpha} \limsup_{t\to\infty} \|z_2(t)\|^2 \tag{9.9}$$

Once K has been found, the sliding manifold is selected as (6.8). In the following, a delayed sliding mode controller is designed which ensures that the closed-loop system is ultimately bounded with the resulting bound proportional to the delay, the disturbance and the switching gain.

9.4 Controller design: a singular perturbation approach

Defining a further change of coordinates by $z = \begin{bmatrix} I_{n-m} & 0 \\ KC_1 & I_m \end{bmatrix} x_r$ in (9.3) it follows that

$$\begin{aligned} \dot{z}_1(t) &= \bar{A}_{11}z_1(t) + \bar{A}_{12}z_2(t) \\ \dot{z}_2(t) &= \bar{A}_{21}z_1(t) + \bar{A}_{22}z_2(t) + u(t-\tau(t)) + w(t) \end{aligned} \tag{9.10}$$

where $z(t) = col\{z_1(t),\ z_2(t)\}$, $z_1(t) = x_1(t)$ and

$$\begin{aligned} \bar{A}_{11} &= A_{11} - A_{12}KC_1, & \bar{A}_{12} &= A_{12}, \\ \bar{A}_{21} &= KC_1\bar{A}_{11} - A_{22}KC_1 + A_{21}, & \bar{A}_{22} &= KC_1\bar{A}_{12} + A_{22} \end{aligned}$$

For $i = 1, ..., m$, denote the i-th component z_2 by z_{2_i}. A control law of the form

$$u(t) = -\frac{F}{\mu}y(t) - (1+\delta)\Delta\left[\ \operatorname{sign} z_{2_1}(t)\ \dots\ \operatorname{sign} z_{2_m}(t)\ \right]^T \tag{9.11}$$

with the tuning parameter $\mu > 0$ will be designed to bring the system (9.10) to the $\tau^*\Delta$-neighborhood of the sliding surface (9.2). The closed-loop system (9.10), (9.11) has the form

$$\dot{z}_1(t) = \bar{A}_{11}z_1(t) + \bar{A}_{12}z_2(t) \tag{9.12}$$

$$\begin{aligned} \mu\dot{z}_2(t) &= \mu\bar{A}_{21}z_1(t) + \mu\bar{A}_{22}z_2(t) - z_2(t-\mu\xi(t)) \\ &+\mu\left[w(t) - (1+\delta)\Delta\left[\ \operatorname{sign} z_{2_1}(t-\mu\xi(t))\ \dots\ \operatorname{sign} z_{2_m}(t-\mu\xi(t))\ \right]^T\right] \end{aligned} \tag{9.13}$$

with the initial condition

$$z(t_0) = z_0, \quad z(t) = 0, \quad t < t_0 \tag{9.14}$$

where $\mu\xi(t) = \tau(t), 0 \leq \xi(t) \leq h$. For small $\mu > 0$ (9.10), (9.11) is a singularly perturbed system. The delay is scaled by μ in order to guarantee robust stability with respect to small enough delay (Fridman, [43], 2002).

Remark 9.4. For $\xi \equiv 0$, a conventional SMC is designed as follows (Edwards and Spurgeon, [29], 1995): find $\mu > 0$ such that the linear controller $u_l(t) = -\frac{F}{\mu}y(t)$ asymptotically stabilizes (9.10) with $w \equiv 0$. Then, for all $\delta > 0$, (9.11) asymptotically stabilizes (9.10) with non-zero $\|w\| \leq \Delta$. Therefore, given $\delta > 0$ the following bound $|[\bar{A}_{21_i}\ \bar{A}_{22_i}]z(t)| \leq \delta\Delta$ is valid for big enough t, which implies finite time convergence of the closed-loop system to $z_2 = 0$. The Lyapunov-based proofs of the stability and of the finite time convergence use the relation $z_{2i}(t)\operatorname{sign} z_{2_i}(t) \geq 0$.

For non-zero $\xi(t)$, the product $z_{2i}(t)\operatorname{sign} z_{2_i}(t-\mu\xi(t))$ may change sign and the closed-loop system (9.10), (9.11) is not asymptotically stable.

Given $h > 0$, the main problem is *the choice of* $\mu > 0$ *and of* $\delta > 0$ (if any) that guarantee the following ultimate bound

$$\limsup_{t\to\infty} |[\bar{A}_{21} \;\; \bar{A}_{22}]z(t)| \leq \delta\Delta, \; \forall\, \xi(t) \in [0,h] \tag{9.15}$$

for solutions of (9.12), (9.13). Matrix inequalities are derived to establish μ and δ via a singular perturbation approach, which guarantees the feasibility of these matrix inequalities for small enough μ. Finally, it will be proved that the closed-loop system is ultimately bounded with bound proportional to $\tau^*\Delta$. For recent results on stability of singularly perturbed systems with small delay, refer to (Chen, et al., [17], 2010), (Glizer, [67], 2009).

9.4.1 Input-to-state stability of a singularly perturbed time-delay system

The closed-loop system (9.12), (9.13) is described as

$$\begin{aligned} \dot{z}_1(t) &= \bar{A}_{11}z_1(t) + \bar{A}_{12}z_2(t) \\ \mu\dot{z}_2(t) &= \mu\bar{A}_{21}z_1(t) + \mu\bar{A}_{22}z_2(t) - z_2(t-\mu\xi(t)) + \mu\bar{w}(t) \end{aligned} \tag{9.16}$$

with the input

$$\bar{w}(t) = w(t) - (1+\delta)\Delta\big[\; \mathrm{sign}\, z_{2_1}(t-\mu\xi(t)) \;\ldots\; \mathrm{sign}\, z_{2_m}(t-\mu\xi(t)) \;\big]^T \tag{9.17}$$

where $\|\bar{w}(t)\| \leq [1+(1+\delta)\sqrt{m}]\Delta$. By using the Lyapunov-Krasovskii method, conditions are derived for the input-to-state stability of (9.16). Let $P_\mu \in \mathbb{R}^{n\times n}$ be a positive definite matrix with the following structure (Kokotovic, et al., [94], 1986)

$$P_\mu = \begin{bmatrix} P_1 & \mu P_2^T \\ * & \mu P_3 \end{bmatrix} > 0 \tag{9.18}$$

where $P_1 \in \mathbb{R}^{n-m}$. For (9.16), choose the Lyapunov-Krasovskii functional of the form

$$\begin{aligned} V_\mu(t) &= z^T(t)P_\mu z(t) + \int_{t-\mu h}^{t} e^{\bar{\alpha}(s-t)} z_2^T(s)Gz_2(s)ds \\ &+ \mu h\int_{-\mu h}^{0}\int_{t+\theta}^{t} e^{\bar{\alpha}(s-t)}\dot{z}_2^T(s)R\dot{z}_2(s)dsd\theta \\ &+ \int_{t-\mu\xi(t)}^{t} e^{\bar{\alpha}(s-t)} z_2^T(s)Sz_2(s)ds \end{aligned} \tag{9.19}$$

where G, R and $S \in \mathbb{R}^m$ are positive matrices.

Lemma 9.5. *Given positive scalars* μ, h, $\bar{\alpha}$ *and* $\bar{b}$, *let there exist* $P_\mu > 0$ *in (9.18) with* $(n-m)\times(n-m)$ *matrix* $P_1 > 0$, $m\times(n-m)$*-matrix* P_2 *and* $m\times m$ *positive matrices* P_3, G, R, S *such that* *the following LMI*

$$\Theta_\mu = \begin{bmatrix} \bar\theta_{11} & \bar\theta_{12} & 0 & -P_2^T & P_2^T & h\mu\bar{A}_{21}^T R \\ * & \bar\theta_{22} & 0 & \bar\theta_{24} & P_3 & h\mu\bar{A}_{22}^T R \\ * & * & \bar\theta_{33} & \bar\theta_{34} & 0 & 0 \\ * & * & * & \bar\theta_{44} & 0 & -hR \\ * & * & * & * & -\bar{b}I_m & hR \\ * & * & * & * & * & -R \end{bmatrix} < 0 \tag{9.20}$$

where

$$\begin{aligned} \bar\theta_{11} &= P_1\bar{A}_{11} + \bar{A}_{11}^T P_1 + \mu P_2^T\bar{A}_{21} + \mu\bar{A}_{21}^T P_2 + \bar\alpha P_1, \\ \bar\theta_{12} &= P_1\bar{A}_{12} + \mu\bar{A}_{21}^T P_3 + \mu\bar{A}_{11}^T P_2^T + \mu P_2^T\bar{A}_{22} + \bar\alpha\mu P_2^T, \\ \bar\theta_{22} &= \mu P_2\bar{A}_{12} + \mu\bar{A}_{12}^T P_2^T + \mu P_3\bar{A}_{22} + \mu\bar{A}_{22}^T P_3 + \bar\alpha\mu P_3 \\ &+ G - e^{-\bar\alpha\mu h}R + S, \\ \bar\theta_{24} &= -P_3 + e^{-\bar\alpha\mu h}R, \\ \bar\theta_{33} &= -e^{-\bar\alpha\mu h}G - e^{-\bar\alpha\mu h}R, \\ \bar\theta_{34} &= e^{-\bar\alpha\mu h}R, \\ \bar\theta_{44} &= -2e^{-\bar\alpha\mu h}R - (1-d)Se^{-\bar\alpha\mu h} \end{aligned} \tag{9.21}$$

is feasible. Then solutions of (9.12)-(9.14) satisfy the bound

$$z^T(t)P_\mu z(t) < e^{-\bar\alpha(t-t_0)}z^T(t_0)P_\mu z(t_0) + \tfrac{\mu^2\bar{b}}{\bar\alpha}\sup\|\bar{w}_{[t_0,t]}\|^2 \tag{9.22}$$

for all $\xi(t) \in [0,\ h]$ with $\mu\dot\xi(t) \le d < 1$ (and thus (9.12)-(9.13) is input-to-state stable). Moreover, solutions of (9.12)-(9.14) satisfy (9.22) for all fast-varying delays $\xi(t) \in [0,\ h]$ if LMI (9.20) is feasible with $S = 0$.

Proof. The following inequality

$$W(t) = \tfrac{d}{dt}V_\mu(t) + \bar\alpha V_\mu(t) - \mu^2\bar{b}\bar{w}^T(t)\bar{w}(t) < 0 \tag{9.23}$$

along the trajectories of (9.12), (9.13) for $\|z_0\|^2 + \sup\|\bar{w}_{[t_0,t]}\|^2 > 0$ guarantees (9.22) (Fridman and Dambrine, [48], 2009). Differentiating V of the structure (9.18), (9.19) along (9.16) it follows that

$$\begin{aligned}
W(t) \leq\ & 2z_1^T(t)P_1[\bar{A}_{11}\ \bar{A}_{12}]z(t) + 2\mu z_2^T(t)P_2[\bar{A}_{11}\ \bar{A}_{12}]z(t) \\
& +2z_1^T(t)P_2^T\left(\mu[\bar{A}_{21}\ \bar{A}_{22}]z(t) - z_2(t-\mu\xi(t)) + \mu\bar{w}(t)\right) \\
& +2z_2^T(t)P_3\left(\mu[\bar{A}_{21}\ \bar{A}_{22}]z(t) - z_2(t-\mu\xi(t)) + \mu\bar{w}(t)\right) \\
& -\mu^2\bar{b}\bar{w}^T(t)\bar{w}(t) + \bar{\alpha}z_1^T(t)P_1z_1(t) + \bar{\alpha}\mu z_2^T(t)P_2z_1(t) \\
& +\bar{\alpha}\mu z_1^T(t)P_2^Tz_2(t) + \bar{\alpha}\mu z_2^T(t)P_3z_2(t) + \mu^2h^2\dot{z}_2^T(t)R\dot{z}_2(t) \\
& -\mu h\int_{t-\mu h}^{t} e^{-\bar{\alpha}\mu h}\dot{z}_2^T(s)R\dot{z}_2(s)ds + z_2^T(t)Gz_2(t) \\
& -e^{-\bar{\alpha}\mu h}z_2^T(t-\mu h)Gz_2(t-\mu h) + z_2^T(t)Sz_2(t) \\
& -(1-d)e^{-\bar{\alpha}\mu\xi(t)}z_2^T(t-\mu\xi(t))Sz_2(t-\mu\xi(t))
\end{aligned} \tag{9.24}$$

Using the identity

$$\begin{aligned}
& -\mu h\int_{t-\mu h}^{t} e^{-\bar{\alpha}\mu h}\dot{z}_2^T(s)R\dot{z}_2(s)ds \\
& = -\mu h\int_{t-\mu h}^{t-\mu\xi(t)} e^{-\bar{\alpha}\mu h}\dot{z}_2^T(s)R\dot{z}_2(s)ds - \mu h\int_{t-\mu\xi(t)}^{t} e^{-\bar{\alpha}\mu h}\dot{z}_2^T(s)R\dot{z}_2(s)ds
\end{aligned}$$

and applying Jensen's inequality

$$\begin{aligned}
& -\mu h\int_{t-\mu h}^{t-\mu\xi(t)} e^{-\bar{\alpha}\mu h}\dot{z}_2^T(s)R\dot{z}_2(s)ds \\
& \leq -e^{-\bar{\alpha}\mu h}\int_{t-\mu h}^{t-\mu\xi(t)}\dot{z}_2^T(s)ds R\int_{t-\mu h}^{t-\mu\xi(t)}\dot{z}_2(s)ds \\
& \leq -e^{-\bar{\alpha}\mu h}[z_2^T(t-\mu\xi(t)) - z_2^T(t-\mu h)]R[z_2(t-\mu\xi(t)) - z_2(t-\mu h)]
\end{aligned}$$

$$\begin{aligned}
& -\mu h\int_{t-\mu\xi(t)}^{t} e^{-\bar{\alpha}\mu h}\dot{z}_2^T(s)R\dot{z}_2(s)ds \\
& \leq -e^{-\bar{\alpha}\mu h}\int_{t-\mu\xi(t)}^{t}\dot{z}_2^T(s)ds R\int_{t-\mu\xi(t)}^{t}\dot{z}_2(s)ds \\
& \leq -e^{-\bar{\alpha}\mu h}[z_2^T(t) - z_2^T(t-\mu\xi(t))]R[z_2(t) - z_2(t-\mu\xi(t))]
\end{aligned}$$

Then, setting $\zeta(t) = col\{z_1(t), z_2(t), z_2(t-\mu h), z_2(t-\mu\xi), \mu\bar{w}(t)\}$ and applying Schur complements to the term $\mu^2h^2\dot{z}_2^T(t)\bar{R}\dot{z}_2(t)$, where $\dot{z}_2(t)$ is substituted by the right-hand site of (9.16), it is established that $W(t) < 0$ if $\Theta_\mu < 0$. □

9.4.2 LMIs for the controller design

Conditions will now be derived that guarantee the bound (9.15) for the solutions of (9.13). Taking into account (9.22) and, thus,

$$\limsup_{t\to\infty} z^T(t)P_\mu z(t) < \tfrac{\mu^2\bar{b}}{\bar{\alpha}}[1+(1+\delta)\sqrt{m}]^2\Delta^2 \tag{9.25}$$

it may be concluded that (9.15) holds if the following inequality is satisfied for $t \to \infty$:

$$\mu^2 z^T(t)[\bar{A}_{21}\ \bar{A}_{22}]^T[\bar{A}_{21}\ \bar{A}_{22}]z(t) < \frac{\bar{\alpha} z^T(t) P_\mu z(t)\delta^2}{\bar{b}[1+(1+\delta)\sqrt{m}]^2}$$

Hence, the inequality

$$\begin{bmatrix} \frac{-\bar{\alpha}\delta^2}{\bar{b}[1+(1+\delta)\sqrt{m}]^2}P_1 & \frac{-\mu\bar{\alpha}\delta^2}{\bar{b}[1+(1+\delta)\sqrt{m}]^2}P_2^T & \mu\bar{A}_{21}^T \\ * & \frac{-\mu\bar{\alpha}\delta^2}{\bar{b}[1+(1+\delta)\sqrt{m}]^2}P_3 & \mu\bar{A}_{22}^T \\ * & * & -I_m \end{bmatrix} < 0 \tag{9.26}$$

guarantees that the solutions of (9.12), (9.13) satisfy the bound (9.15). By Schur complements, (9.26) is feasible if the following matrix inequality is feasible

$$\frac{-\bar{\alpha}\delta^2}{\bar{b}[1+(1+\delta)\sqrt{m}]^2}\begin{bmatrix} P_1 & \mu P_2^T \\ * & \mu P_3 \end{bmatrix} + \mu^2 \begin{bmatrix} \bar{A}_{21}^T \\ \bar{A}_{22}^T \end{bmatrix} \begin{bmatrix} \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} < 0 \tag{9.27}$$

Matrix inequalities (9.18), (9.20) and (9.26) have been derived for finding the parameters μ and δ of the controller (9.11). It will now be shown that if

$$\Theta_0 = \begin{bmatrix} P_1\bar{A}_{11}+\bar{A}_{11}^T P_1 + \bar{\alpha}P_1 & P_1\bar{A}_{12} & 0 & -P_2^T & P_2^T & 0 \\ * & G-R+S & 0 & -P_3+R & P_3 & 0 \\ * & * & -G-R & R & 0 & 0 \\ * & * & * & -2R-(1-d)S & 0 & -hR \\ * & * & * & * & -\bar{b}I_m & hR \\ * & * & * & * & * & -R \end{bmatrix} < 0 \tag{9.28}$$

is feasible, then for all $\delta > 0$ inequalities (9.18), (9.20) and (9.26) are feasible for all small enough μ. Let P_1, P_2, P_3 satisfy $\Theta_0 < 0$. Then for small enough $\mu > 0$, (9.18) and (9.20) are feasible for the same μ – *independent* matrices P_1, P_2, P_3. Hence, given $\delta > 0$, (9.27) is feasible for small enough $\mu > 0$.

It is easily seen that $\Theta_0 < 0$ guarantees exponential stability with decay rate $\bar{\alpha}/2$ of the slow subsystem

$$\dot{z}_s(t) = \bar{A}_{11} z_s(t), \quad z_s(t) \in \mathbb{R}^{n-m} \tag{9.29}$$

and asymptotic stability of the fast subsystem of (9.16)

$$\mu \dot{z}_f(t) = -z_f(t - \mu\xi(t)),\ \xi(t) \in [0,h], \quad z_f(t) \in \mathbb{R}^m \tag{9.30}$$

Since $\bar{A}_{11}$ is Hurwitz, there exists $P_1 > 0$ satisfying $P_1\bar{A}_{11} + \bar{A}_{11}^T P_1 + \bar{\alpha}P_1 < 0$ for small enough $\bar{\alpha} > 0$. Choose next $P_2 = 0$, $G = S = 0$ and $R = P_3 = p_3 I_m$. By using Schur complements, it can be shown that $\Theta_0 < 0$ holds for big enough $p_3 > 0, \bar{b}$ and for small enough h.

The following sufficient conditions for the feasibility of (9.18), (9.20) and (9.26) have been proved:

Proposition 9.6. *(i) Given positive scalars h, $\bar{\alpha}$ and $\bar{b}$, let there exist an $(n-m)\times(n-m)$-matrix $P_1>0$, an $m\times(n-m)$-matrix P_2 and positive $m\times m$-matrices P_3, G, R, S such that LMI (9.28) is feasible. Then, for all $\delta>0$ there exists $\mu(\delta)>0$ such that for all $\mu\in(0,\mu(\delta)]$ LMIs (9.18), (9.20) and (9.26) are feasible and, thus, solutions of (9.12)-(9.13) satisfy the bound (9.15).*

(ii) LMI (9.28) is feasible for small enough h, $\bar{\alpha}$ and big enough $\bar{b}$.

9.4.3 Ultimate boundedness of the closed-loop system

Let $\phi(t,t_0,\mu)$ be the fundamental solution of the equation

$$\mu\dot{\zeta}(t)=-\zeta(t-\mu\xi(t)),\ \zeta(t)\in\mathbb{R}$$

with $\phi(t_0,t_0,\mu)=1$ and $\phi(t,t_0,\mu)=0$ for $t<t_0$. By using the arguments of Lemma 9.5 and choosing

$$\begin{array}{l} V_2=\mu\rho\zeta^2(t)+q\int_{t-\mu h}^{t}e^{\alpha_2(s-t)}\zeta^2(s)ds \\ +\mu hr\int_{-\mu h}^{0}\int_{t+\theta}^{t}e^{\alpha_2(s-t)}\dot{\zeta}^2(s)dsd\theta+\psi\int_{t-\mu\xi(t)}^{t}e^{\alpha_2(s-t)}\zeta^2(s)ds \end{array} \tag{9.31}$$

with positive scalars ρ,q,r,ψ, it can be shown that the feasibility of the μ-independent LMI

$$\begin{bmatrix} \psi+q-r & 0 & -\rho+r & 0 \\ * & -q-r & r & 0 \\ * & * & -(1-d)\psi-2r & hr \\ * & * & * & -r \end{bmatrix}<0 \tag{9.32}$$

yields the following bound

$$|\phi(t,t_0,\mu)|\leq e^{-\frac{\alpha_2(t-t_0)}{\mu}} \tag{9.33}$$

for small enough $\alpha_2>0$ and $\forall\ \mu>0,\ \xi(t)\leq h,\ \mu\dot{\xi}\leq d<1$. Note that (9.32) is feasible for $h\leq 1.414$ if $d=0$ and for $h\leq 1.22$ if d is unknown (i.e. for fast varying delay).

The main result may now be formulated:

Theorem 9.7. *Given positive constants μ, h, $\bar{\alpha}$, $\bar{b}$ and δ let there exist an $(n-m)\times(n-m)$-matrix $P_1>0$, an $m\times(n-m)$-matrix P_2, positive $m\times m$-matrices P_3, G, R, S and positive scalars ρ,q,r,ψ such that LMIs (9.18), (9.20), (9.26) and (9.32) are feasible. Let $z(t)$ be a solution to (9.12)-(9.14). Then every component of $z_2(t)$ satisfies the bound*

$$\limsup_{t\to\infty}|z_{2_i}(t)|\leq 2M_0\mu h,\ M_0=(1+\delta)(1+\sqrt{m})\Delta \tag{9.34}$$

where $i=1,\dots,m$ denotes the i-th component of z_2 for all $\xi(t)\in[0,\ h]$ with $\mu\dot{\xi}\leq d<1$. Moreover, the solution to (9.12)-(9.14) satisfies (9.34) for all fast varying delays $\xi(t)\in[0,\ h]$ if the above LMIs are feasible with $S=0$ and $\psi=0$.

Proof. The i-th component of differential equation (9.13) with the initial condition (9.14) can be represented in the form of an integral equation (Kolmanovskii and Myshkis, [97], 1992)

$$z_{2_i}(t) = \phi(t,t_0,\mu)z_{2_i}(t_0) + \int_{t_0}^{t} \phi(t,s,\mu)\Big[[\bar{A}_{21_i}\ \bar{A}_{22_i}]z(s) + w_i(s) - (1+\delta)\Delta\,\mathrm{sign}\, z_{2_i}(s-\mu\xi(s))\Big]ds \tag{9.35}$$

The feasibility of (9.26) implies the bound (9.15), then the following inequality holds for $t \to \infty$:

$$|[\bar{A}_{21_i}\ \bar{A}_{22_i}]z(s) + w_i(s) - (1+\delta)\Delta\,\mathrm{sign}\, z_{2_i}(s-\mu\xi(s))| < M_0 \tag{9.36}$$

Taking into account (9.33) and (9.36), it is established from (9.35) that for $t \to \infty$

$$\begin{aligned} |z_{2_i}(t+\theta) - z_{2_i}(t)| &\le \Big|\int_{t+\theta}^{t} \phi(t,s,\mu)\Big([\bar{A}_{21_i}\ \bar{A}_{22_i}]z(s) \\ &\quad + w_i(s) - (1+\delta)\Delta\,\mathrm{sign}\, z_{2_i}(s-\mu\xi(s))\Big)ds\Big| \\ &< M_0 \int_{t+\theta}^{t} e^{-\frac{\alpha_2(t-s)}{\mu}} ds < \mu M_0 \frac{1-e^{-2\alpha_2 h}}{\alpha_2} \\ &\le 2M_0\mu h \end{aligned} \tag{9.37}$$

where $\theta \in [-2\mu h,\ 0]$. Therefore,

$$z_{2_i}(t) - 2M_0\mu h < z_{2_i}(t+\theta) < z_{2_i}(t) + 2M_0\mu h \tag{9.38}$$

for $t \to \infty$ and the following implication holds

$$|z_{2_i}(t)| \ge 2M_0\mu h \Rightarrow \mathrm{sign}\, z_{2_i}(t+\theta) = \mathrm{sign}\, z_{2_i}(t) \tag{9.39}$$

for large enough t. Thus, from (9.15), (9.36) and (9.39) for sufficiently large t the following implication follows:

$$\begin{aligned} &|z_{2_i}(t)| \ge 2M_0\mu h \Rightarrow \\ &z_{2_i}^T(t)\big[[\bar{A}_{21_i}\ \bar{A}_{22_i}]z(t+\theta) + w_i(t+\theta) \\ &- (1+\delta)\Delta\,\mathrm{sign}\, z_{2_i}(t+\theta)\big] \\ &< |z_{2_i}(t)|(|[\bar{A}_{21_i}\ \bar{A}_{22_i}]z(t+\theta)| + \Delta) - (1+\delta)\Delta|z_{2_i}(t)| \\ &< 0 \end{aligned} \tag{9.40}$$

It will be shown next that the z_{2_i}-component of the solutions to (9.13) exponentially converges to the ball (9.34). Moreover, for sufficiently large t, whenever $z_{2_i}(t)$ achieves the ball (9.34), it will never leave it. Taking into account (9.40), for sufficiently large t it follows that

$$
\begin{aligned}
|z_{2_i}(t)| &\geq 2M_0\mu h \Rightarrow \\
\frac{d}{dt}\mu z_{2_i}^2(t) &= 2\mu z_{2_i}(t)\dot{z}_{2_i}(t) \\
&= 2z_{2_i}(t)\big[-z_{2_i}(t-\mu\xi(t)) + \mu\big([\bar{A}_{21_i}\ \bar{A}_{22_i}]z(t) \\
&+ w_i(t) - (1+\delta)\Delta \operatorname{sgn} z_{2_i}(t)\big)\big] \\
&\leq -2z_{2_i}(t)\big(z_{2_i}(t) - \int_{t-\mu\xi(t)}^{t} \dot{z}_{2_i}(s)ds\big) \\
&= -2z_{2_i}^2(t) + 2z_{2_i}(t)\Big[\int_{t-\mu\xi(t)}^{t} -\frac{z_{2_i}(s-\mu\xi(t))}{\mu}ds \\
&+ \int_{t-\mu\xi(t)}^{t} [\bar{A}_{21_i}\ \bar{A}_{22_i}]z(s) + w_i(s) \\
&- (1+\delta)\Delta \operatorname{sgn} z_{2_i}(s)ds\Big] \\
&\leq -2z_{2_i}^2(t) - 2\frac{z_{2_i}(t)}{\mu}\int_{t-\mu\xi(t)}^{t} z_{2_i}(s-\mu\xi(t))ds
\end{aligned}
$$

Therefore, given (9.39) holds for large enough t, it follows that

$$
-\int_{t-\mu\xi(t)}^{t} z_{2_i}(t)z_{2_i}(s-\mu\xi(t))ds \leq 0
$$

Hence

$$
|z_{2_i}(t)| \geq 2M_0\mu h \Rightarrow \frac{d}{dt}\mu z_{2_i}^2(t) \leq -2z_{2_i}^2(t) \tag{9.41}
$$

Assume now that for large enough t_1 the z_{2_i} component of the solution to (9.13) is outside the ball (9.34). Then from (9.41) it follows that for all $t \geq t_1$ such that $|z_{2_i}(t)| \geq 2M_0\mu h$ then

$$
z_{2_i}^2(t) \leq e^{-\frac{2}{\mu}(t-t_1)} z_{2_i}^2(t_1) \tag{9.42}
$$

i.e. z_{2_i} exponentially converges to the ball (9.34). Let $t_2 > t_1$ is the time when $|z_{2_i}(t_2)| = 2M_0\mu h$. Then due to (9.41) $z_{2_i}^2(t_2^+) < z_{2_i}^2(t_2)$. Therefore, whenever $z_{2_i}(t)$ attains the ball (9.34), it will never leave it. □

Remark 9.8. Under the conditions of Lemma 9.2 and Theorem 9.7 it follows that solutions of the closed-loop system (9.12), (9.13) satisfy

$$
\begin{aligned}
&\limsup_{t\to\infty} z_1^T(t)\hat{P}z_1(t) \leq 4\tfrac{b}{\alpha} m M_0^2 \mu^2 h^2, \\
&\limsup_{t\to\infty} z_2^T(t)z_2(t) \leq 4mM_0^2\mu^2 h^2
\end{aligned} \tag{9.43}
$$

Remark 9.9. In order to minimize the high gain nature of the control, it is desirable to find a larger μ such that the conditions of Theorem 1 are satisfied. However, increase in μ leads to increase of the switching parameter δ, and thus the increase of the overall bound. Therefore, a trade-off should be made between the minimization of the bound and the limit of the high linear control effort such that the control specifications are satisfied.

Remark 9.10. Since LMIs (9.18), (9.20), (9.26) and (9.32) are affine in the system matrices, the results are applicable to the case where these matrices have polytopic type uncertainties.

Remark 9.11. Consider now (9.1) using the linear controller

$$u_l(t) = -\frac{F}{\mu} y(t) \tag{9.44}$$

Then the closed-loop system (9.10), (9.44) has the form (9.16) with $\bar{w}(t) = w(t)$. Under the conditions of Lemma 2, the solutions of (9.10), (9.44) satisfy the bound

$$z^T(t) P_\mu z(t) < e^{-\bar{\alpha}(t-t_0)} z_0^T P_\mu z_0 + \frac{\mu^2 \bar{b}}{\bar{\alpha}} \Delta^2$$

As expected, the ultimate bound of the solutions under the linear controller does not vanish for $h \to 0$. Therefore, the SMC has advantages over linear control for small delays.

9.5 Application to position control of spacecraft

Consider a dynamic model of a spacecraft with flexible appendages, actuated by gas jets and/or reaction wheels given in (Gennaro, [65], 1998)

$$\begin{aligned} J_T\dot{\omega} + J_R\dot{\Omega} + \delta^T\ddot{\eta} &= -\tilde{\omega}(J_T\omega + J_R\Omega + \delta^T\dot{\eta}) + w(t) \\ J_R(\dot{\omega} + \dot{\Omega}) &= u(t-\tau(t)) \\ \ddot{\eta} + C\dot{\eta} + K\eta &= -\delta\dot{\omega} \end{aligned} \tag{9.45}$$

where $\omega = col\{\dot{\phi},\ \dot{\theta},\ \dot{\psi}\}$ are the spacecraft angular velocities with respect to roll, pitch, and yaw attitude angles. η is the modal coordinate vector that describes the flexible dynamics under the hypothesis of small elastic deformations. $\Omega = col\{\Omega_1,\ \Omega_2,\ \Omega_3\}$ are the reaction wheel relative angular velocities with respect to the main body. $w(t)$ is disturbances bounded by a known bound $\|w(t)\| \leq 10$. With the assumption of small Euler angle rotations, setting $\bar{x}(t) = col\{\phi(t),\ \theta(t),\ \psi(t),\ \eta(t)\}$, the dynamic model in equation (9.45) can be approximated by neglecting the nonlinear coupling terms as

$$\bar{M}\ddot{\bar{x}}(t) + \bar{C}\dot{\bar{x}}(t) + \bar{K}\bar{x}(t) = \bar{B}(u(t-\tau(t)) + \chi(t) + w(t))$$

where

$$\bar{M} = \begin{bmatrix} J_T - J_R & \delta^T \\ \delta & I \end{bmatrix},\ \bar{C} = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix},\ \bar{K} = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix},\ \bar{B} = \begin{bmatrix} I_3 \\ 0 \end{bmatrix}$$

and $\chi(t)$ is a function of the nonlinear coupling terms between attitude and vibrations in (9.45). Setting $x(t) = col\{\phi(t),\ \theta(t),\ \psi(t),\ \eta(t),\ \dot{\phi}(t),\ \dot{\theta}(t),\ \dot{\psi}(t),\ \dot{\eta}(t)\}$, system (9.45) is put in the following form ready for the control design

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\big(u(t-\tau(t)) + \chi(t) + w(t)\big) \\ y(t) &= Cx(t) \end{aligned} \tag{9.46}$$

where

$$A = \begin{bmatrix} 0 & I \\ -\bar{M}^{-1}\bar{K} & -\bar{M}^{-1}\bar{C} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \bar{M}^{-1}\bar{B} \end{bmatrix}, C = \begin{bmatrix} I_3 & 0_{3\times4} & 0_3 & 0_{3\times4} \\ 0_3 & 0_{3\times4} & I_3 & 0_{3\times4} \end{bmatrix}$$

The problem of stabilizing the system (9.46) when subjected to a time-varying delay in the control input and possibly in the output measurements is considered. Such delays normally exist in the actuators in precision pointing control (Tony, [135], 2001), (Wong and Breckenridge, [143], 1995). In section *A* below, simulation results for the closed-loop system (9.45) with the control law derived from (9.11) are shown for a time-varying input delay $\mu\xi(t) \leq 0.01s$. In section *B*, the linear control (9.44) is used for control of the spacecraft. The results are then compared and the advantage of using sliding mode control over linear control is demonstrated. Although the control has been designed for system (9.46) by neglecting the nonlinear term $\chi(t)$, it was reintroduced in the simulation to generate a more realistic simulation model of the spacecraft as given in (9.45) for controller testing. The parameters of the spacecraft are taken from (Gennaro, [65], 1998) with $N = 4$ elastic modes.

9.5.1 Using sliding mode control

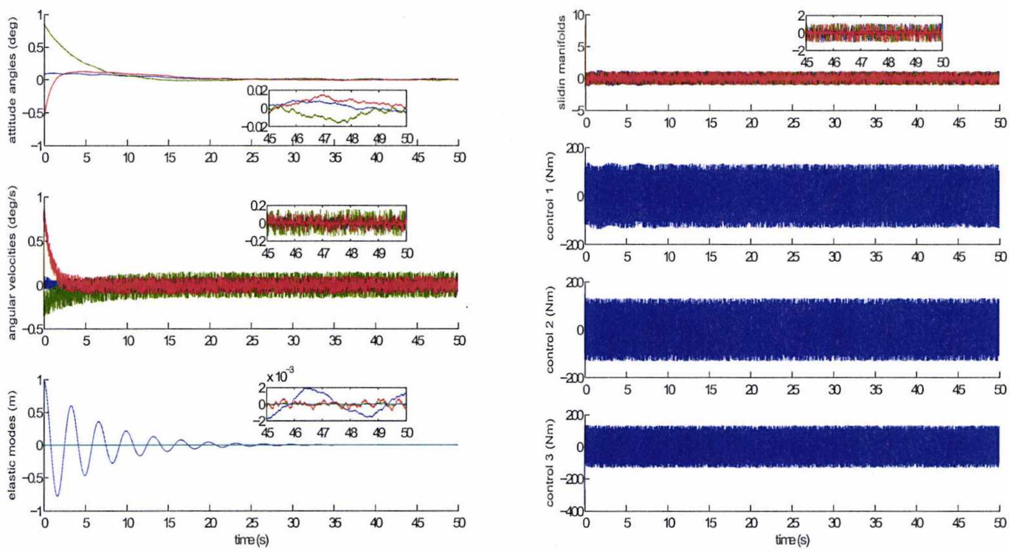
Setting $\alpha = 0.24,\ b = 0.0001,\ \varepsilon = 0.1,\ \varepsilon_1 = 0.8$ and the tuning matrix

$$M = \begin{bmatrix} 0.04 & -0.1 & 0.16 & -0.1 & 0.1 & -0.4 & 0.2 & -0.036 \\ 0.04 & -0.04 & 0.16 & 0.02 & -0.202 & -0.022 & 0.022 & 0.028 \\ -0.2 & 0.022 & 0.02 & -0.02 & 0.202 & -0.038 & -0.206 & 0.022 \end{bmatrix}$$

in LMI (9.6), the reduced order system (9.5) is ultimately bounded with

$$K = \begin{bmatrix} -9.6 & 12.5 & -88 \\ 21.7 & -77.8 & -78 \\ -683.6 & -14.6 & 913 \end{bmatrix}$$

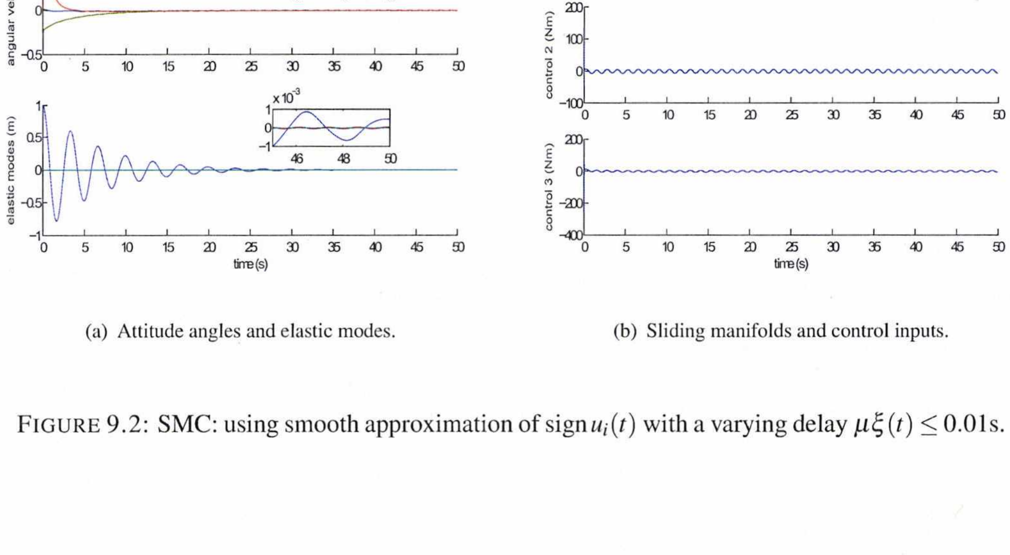
Setting $\bar{\alpha} = 0.158,\ \bar{b} = 0.02,\ \mu = 0.032,\ \mu h = 0.01s,\ d = 1$ in LMI (9.20) the closed-loop system (9.13) is ultimately bounded with F obtained from (6.8). Then the switching gain parameter $\delta = 16$ was obtained from LMI (9.26). In the simulation the initial values were selected as $\phi(0) = 0.09°,\ \theta(0) = 0.86°,\ \psi(0) = -0.57°$ and $\dot{\phi}(0) = \dot{\theta}(0) = \dot{\psi}(0) = 0$. The varying delay was implemented as $\mu\xi(t) = 0.005sin(200t) + 0.005 \leq 0.01s$, where $\mu\dot{\xi}(t) \leq 1$. In Figure 9.1, the switched control $\mathrm{sign}\, u_i(t)$ was used to verify the theoretical bound. In Figure 9.2, the smooth approximation $\frac{u_i(t)}{|u_i(t)|+0.2}$ was used instead of $\mathrm{sign}\, u_i(t)$ for implementation of a practical controller (Spurgeon and Davies, [132], 1993).



(a) Attitude angles and elastic modes.

(b) Sliding manifolds and control inputs.

FIGURE 9.1: SMC: using $\text{sign}\, u_i(t)$ with a varying delay $\mu\xi(t) \leq 0.01$s.



(a) Attitude angles and elastic modes.

(b) Sliding manifolds and control inputs.

FIGURE 9.2: SMC: using smooth approximation of $\text{sign}\, u_i(t)$ with a varying delay $\mu\xi(t) \leq 0.01$s.

In Figure 9.1, smooth control of roll, pitch and yaw angles takes place in about 20 seconds with the bound $\|[\phi(t),\ \theta(t),\ \psi(t)]\| \leq 0.02$ degree and $\|[\dot{\phi}(t), \dot{\theta}(t), \dot{\psi}(t)]\| \leq 0.2$ degree/second. The four elastic modes are stabilized smoothly after 40 seconds with the bound $\|\eta(t)\| \leq 0.002$m. Heavy chattering of the sliding variables and controls were observed due to the control component $\mathrm{sign}\, u_i(t)$. In Figure 9.2, $\|[\phi(t), \theta(t), \psi(t)]\| \leq 0.001$ degree, $\|[\dot{\phi}(t), \dot{\theta}(t), \dot{\psi}(t)]\| \leq 0.002$ degree/second, and $\|\eta(t)\| \leq 0.001$ m were observed with significant reduction in chattering of the sliding variables and controls. This is due to the use of the smooth approximation of the control term $\mathrm{sign}\, u_i(t)$. The maximum control signal is in the third control channel at the level of 350 N.m.

It is seen that the controller design, which neglected nonlinearities, achieves the control specifications. The theoretical bound of the sliding manifold $\|z_2(t)\|_{t\to\infty} \leq 9.3$ given by (9.34) is a reasonable estimate of the actual bound $\|z_2(t)\| \leq 1.8$ in Figure 9.1(b).

9.5.2 Using only the linear part of the control

Sliding mode control has advantages over linear control for sufficiently small input delay. Figure 9.3 shows the closed-loop performance using linear control (9.44). The attitude angles are bounded as $\|[\phi(t),\ \theta(t),\ \psi(t)]\| \leq 0.005$ degree and $\|[\dot{\phi}(t), \dot{\theta}(t), \dot{\psi}(t)]\| \leq 0.02$ degree/second. The four elastic modes are stabilized smoothly after 40 seconds with the bound $\|\eta(t)\| \leq 0.002$m.

While the SMC with switched term function $\mathrm{sign}\, u_i(t)$ does not exhibit advantages over linear control for the size of input delay studied, the SMC using smoothing improves the accuracy of the attitude angle by five times when compared with using only the linear part of the control, Figure 9.1(a), 9.2(a), 9.3.

9.6 Extension to systems with input and state delay

The following uncertain dynamical system is considered with a state time-varying delay $r(t)$, an input time-varying delay $\tau(t)$ and with a matched disturbance $w(t)$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t - r(t)) + B\big(u(t - \tau(t)) + w(t)\big) \\ y(t) &= Cx(t) \end{aligned} \tag{9.47}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ with $m < p < n$. It is assumed that the delays and the disturbance are bounded as follows: $r(t) \in [0,\ r^*]$, $\tau(t) \in [0,\ \tau^*]$ and $\|w(t)\| \leq \Delta$. The delays may be either slowly varying with $\dot{r}(t) \leq d_1 < 1$, $\dot{\tau}(t) \leq d_2 < 1$ or fast varying (without any constraints on the delay derivatives).

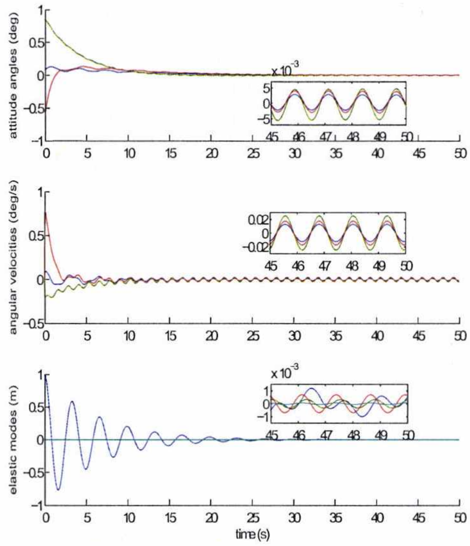


FIGURE 9.3: Linear control: Attitude angles and elastic modes with a varying delay $\mu\xi(t) = \mu\xi(t) \leq 0.01$s.

Assume that the input and output matrices B and C are both of full rank. Then the sliding manifold can be defined by (9.2).

9.6.1 Sliding manifold design

In regular form, the system (9.47) becomes

$$\begin{aligned}\dot{x}_r(t) &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x_r(t) + \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix} x_r(t-r(t)) \\ &+ \begin{bmatrix} 0 \\ I_m \end{bmatrix} \big(u(t-\tau(t)) + w(t)\big) \\ y(t) &= \begin{bmatrix} 0 & T \end{bmatrix} x_r(t) \end{aligned} \tag{9.48}$$

Defining the sliding manifold as in (9.4), the reduced-order system with inputs $z_2(t)$ and $z_2(t-r(t))$ has the form

$$\begin{aligned}\dot{x}_1(t) &= (A_{11} - A_{12}KC_1)x_1(t) + (A_{d11} - A_{d12}KC_1)x_1(t-r(t)) \\ &+ A_{12}z_2(t) + A_{d12}z_2(t-r(t)) \end{aligned} \tag{9.49}$$

where the triple $(A_{11}+A_{d11},\ A_{12}+A_{d12},\ C_1)$ is assumed to be stabilizable. The sliding manifold is found from the following lemma:

Lemma 9.12. *Given tuning parameters $\alpha > 0$, ε, ε_1, b_1, $b_2 > 0$ and a matrix $M \in \mathbb{R}^{(p-m)\times(n-p)}$, let there exist $(n-m)\times(n-m)$ matrices $P > 0$, $G \geq 0$, $S \geq 0$, $R \geq 0$ and matrices $Q_{22} \in \mathbb{R}^{(p-m)\times(p-m)}$,*

$Q_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$, $Q_{12} \in \mathbb{R}^{(n-p)\times(p-m)}$, $Y \in \mathbb{R}^{m\times(p-m)}$, $K = YQ_{22}^{-1}$ *such that LMI*

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & 0 & \theta_{14} & A_{12} & A_{d12} \\ * & \theta_{22} & 0 & \theta_{24} & \varepsilon A_{12} & \varepsilon A_{d12} \\ * & * & \theta_{33} & \theta_{34} & 0 & 0 \\ * & * & * & \theta_{44} & 0 & 0 \\ * & * & * & * & -b_1 I_m & 0 \\ * & * & * & * & * & -b_2 I_m \end{bmatrix} < 0 \tag{9.50}$$

holds, where

$$\begin{aligned}
\theta_{11} &= A_{11}Q_2 - A_{12}[Y \quad \varepsilon_1 Y] + Q_2^T A_{11}^T + \alpha P \\
&\quad - [YM \quad \varepsilon Y]^T A_{12}^T + G + S - Re^{-\alpha r^*}, \\
\theta_{12} &= P - Q_2 + \varepsilon Q_2^T A_{11}^T - \varepsilon [YM \quad \varepsilon_1 Y]^T A_{12}^T, \\
\theta_{14} &= A_{d11}Q_2 - A_{d12}[YM \quad \varepsilon_1 Y] + Re^{-\alpha r^*}, \\
\theta_{22} &= -\varepsilon Q_2 - \varepsilon Q_2^T + r^{*2} R, \\
\theta_{24} &= \varepsilon A_{d11} Q_2 - \varepsilon A_{d12}[YM \quad \varepsilon_1 Y], \\
\theta_{33} &= -(G+R)e^{-\alpha r^*}, \\
\theta_{34} &= Re^{-\alpha r^*}, \\
\theta_{44} &= -2e^{-\alpha r^*}R - (1-d_1)Se^{-\alpha r^*}
\end{aligned}$$

Then for all ultimately bounded z_2, solutions of (9.49) satisfy the following inequality:

$$\limsup_{t\to\infty} x_1^T(t)\hat{P}x_1(t) < \frac{b_1+b_2}{\alpha} \limsup_{t\to\infty} \|z_2(t)\|^2$$

where $\hat{P} = Q_2^{-T} P Q_2^{-1}$ *and* $Q_2 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{22}M & \varepsilon_1 Q_{22} \end{bmatrix}$.

Proof. The proof follows analogously using the result from Lemma 8.1 and proposition 8.2. □

9.6.2 Controller design and the resulting ultimate bound

The change of coordinates $z = \begin{bmatrix} I_{n-m} & 0 \\ KC_1 & I_m \end{bmatrix} x_r$ leads to

$$\begin{aligned}
\dot{z}_1(t) &= \bar{A}_{11}z_1(t) + \bar{A}_{d11}z_1(t-r(t)) + \bar{A}_{12}z_2(t) \\
&\quad + \bar{A}_{d12}z_2(t-r(t)) \\
\dot{z}_2(t) &= \bar{A}_{21}z_1(t) + \bar{A}_{d21}z_1(t-r(t)) + \bar{A}_{22}z_2(t) \\
&\quad + \bar{A}_{d22}z_2(t-r(t)) + u(t-\tau(t)) + w(t)
\end{aligned}$$

The controller is chosen as in (9.11). Then the closed-loop system has the form

$$\begin{aligned}\dot{z}_1(t) &= \bar{A}_{11}z_1(t)+\bar{A}_{d11}z_1(t-r(t))+\bar{A}_{12}z_2(t)\\ &\quad +\bar{A}_{d12}z_2(t-r(t))\\ \mu\dot{z}_2(t) &= \mu\bar{A}_{21}z_1(t)+\mu\bar{A}_{d21}z_1(t-r(t))+\mu\bar{A}_{22}z_2(t)\\ &\quad +\mu\bar{A}_{d22}z_2(t-r(t))-z_2(t-\mu\xi(t))+\bar{w}(t)\end{aligned} \tag{9.51}$$

where $\bar{w}(t)$ is given by (9.17) with $\|\bar{w}(t)\| \leq [1+(1+\delta)\sqrt{m}]\Delta$, $\mu\xi(t)=\tau(t)$, $0\leq\xi(t)\leq h$, $z(t)=col\{z_1(t),\, z_2(t)\}$.

Let P_μ be of the same structure as (9.18), then input-to-state stability of the latter system can be derived using the Lyapunov-Krasovskii functional of the form

$$\begin{aligned}V_\mu(t) &= z^T(t)P_\mu z(t)+\int_{t-r^*}^{t} e^{\bar{\alpha}(s-t)}z_1^T(s)G_1z_1(s)ds+\\ &\int_{t-r(t)}^{t} e^{\bar{\alpha}(s-t)}z_1^T(s)S_1z_1(s)ds+\int_{t-\mu h}^{t} e^{\bar{\alpha}(s-t)}z_2^T(s)G_2z_2(s)ds+\\ &\int_{t-\mu\xi(t)}^{t} e^{\bar{\alpha}(s-t)}z_2^T(s)S_2z_2(s)ds+\int_{t-r(t)}^{t} e^{\bar{\alpha}(s-t)}z_2^T(s)S_3z_2(s)ds\\ &+\int_{t-r^*}^{t} e^{\bar{\alpha}(s-t)}z_2^T(s)S_4z_2(s)ds\\ &+r^*\int_{-r^*}^{0}\int_{t+\theta}^{t} e^{\bar{\alpha}(s-t)}\dot{z}_1^T(s)R_1\dot{z}_1(s)dsd\theta\\ &+\mu h\int_{-\mu h}^{0}\int_{t+\theta}^{t} e^{\bar{\alpha}(s-t)}\dot{z}_2^T(s)R_2\dot{z}_2(s)dsd\theta\\ &+\mu^2r^*\int_{-r^*}^{0}\int_{t+\theta}^{t} e^{\bar{\alpha}(s-t)}\dot{z}_2^T(s)R_3\dot{z}_2(s)dsd\theta\end{aligned} \tag{9.52}$$

with positive matrices G_1, G_2, S_1, S_2, S_3, S_4, R_1, R_2 and R_3. Similar to Lemma 9.5, it is established that

Lemma 9.13. *Given positive scalars r^*, μ, h, $\bar{\alpha}$ and $\bar{b}_1$, let there exist $P_\mu > 0$ in (9.18) with $(n-m)\times(n-m)$ matrix $P_1 > 0$, $m\times(n-m)$-matrix P_2, $m\times m$ positive matrix P_3, $(n-m)\times(n-m)$ positive matrices G_1, S_1, R_1, $m\times m$ positive matrices G_2, S_2, S_3, S_4, R_2 and R_3 such that the LMI*

$$\Theta_\mu = \begin{bmatrix} \bar{\theta}_{1,1} & \bar{\theta}_{1,2} & \cdots & \bar{\theta}_{1,12}\\ * & \bar{\theta}_{2,2} & \cdots & \bar{\theta}_{2,12}\\ * & * & \ddots & \vdots\\ * & * & \cdots & \bar{\theta}_{12,12}\end{bmatrix} < 0 \tag{9.53}$$

with entries from Table 1 is feasible. Then solutions of (9.51) satisfy the bound

$$\limsup_{t\to\infty} z^T(t)P_\mu z(t) < \frac{\mu^2\bar{b}_1\left(1+(1+\delta)\sqrt{m}\right)^2}{\bar{\alpha}}\Delta^2 \tag{9.54}$$

Table 1, Entries of LMI (9.53)	
$\bar{\theta}_{1,1} = \bar{\alpha}P_1 + P_1\bar{A}_{11} + \bar{A}_{11}^T P_1$ $+\mu P_2^T \bar{A}_{21} + \mu \bar{A}_{21} P_2$ $+G_1 + S_1 - R_1 e^{-\bar{\alpha}r^*}$	$\bar{\theta}_{1,2} = P_1\bar{A}_{d11} + R_1 e^{-\bar{\alpha}r^*}$ $+\mu P_2^T \bar{A}_{d21}$
$\bar{\theta}_{1,4} = P_1\bar{A}_{12} + \mu\bar{A}_{21}^T P_3$ $+\mu\bar{A}_{11}^T P_2^T + \mu P_2^T \bar{A}_{22} + \alpha\mu P_2^T$	$\bar{\theta}_{1,5} = P_1\bar{A}_{d12} + \mu P_2^T \bar{A}_{d22}$
$\bar{\theta}_{1,7} = -P_2^T$	$\bar{\theta}_{1,9} = \mu P_2^T$
$\bar{\theta}_{1,10} = r^*\bar{A}_{11}^T R_1$	$\bar{\theta}_{1,11} = \mu h \bar{A}_{21}^T R_2$
$\bar{\theta}_{1,12} = \mu r^* \bar{A}_{21}^T R_3$	$\bar{\theta}_{2,2} = -2R_1 e^{-\bar{\alpha}r^*}$ $-(1-d_1)S_1 e^{-\bar{\alpha}r*}$
$\bar{\theta}_{2,3} = R_1 e^{-\bar{\alpha}r^*}$	$\bar{\theta}_{2,4} = \mu\bar{A}_{d21}^T P_3 + \mu\bar{A}_{d11}^T P_2^T$
$\bar{\theta}_{2,10} = r^*\bar{A}_{d11}^T R_1$	$\bar{\theta}_{2,11} = \mu h \bar{A}_{d21}^T R_2$
$\bar{\theta}_{2,12} = \mu r^* \bar{A}_{d21}^T R_3$	$\bar{\theta}_{3,3} = -e^{-\bar{\alpha}r^*}(R_1 + G_1)$
$\bar{\theta}_{4,4} = \mu P_3 \bar{A}_{22} + \mu\bar{A}_{22}^T P_3$ $+\mu P_2 \bar{A}_{12} + \mu\bar{A}_{12}^T P_2^T + \mu\bar{\alpha}P_3$ $-R_2 e^{-\bar{\alpha}\mu h} - \mu^2 e^{-\bar{\alpha}r^*} R_3$ $+G_2 + S_2 + S_3 + S_4$	$\bar{\theta}_{4,5} = \mu P_3 \bar{A}_{d22} + \mu P_2 \bar{A}_{d12}$ $+\mu^2 e^{-\bar{\alpha}r^*} R_3$
$\bar{\theta}_{4,7} = -P_3 + R_2 e^{-\bar{\alpha}\mu h}$	$\bar{\theta}_{4,9} = \mu P_3$
$\bar{\theta}_{4,10} = r^*\bar{A}_{12}^T R_1$	$\bar{\theta}_{4,11} = \mu h \bar{A}_{22}^T R_2$
$\bar{\theta}_{4,12} = \mu r^* \bar{A}_{22}^T R_3$	$\bar{\theta}_{5,5} = -(1-d_1)S_3 e^{-\bar{\alpha}r^*}$ $-2\mu^2 e^{-\bar{\alpha}r^*} R_3$
$\bar{\theta}_{5,6} = \mu^2 e^{-\bar{\alpha}r^*} R_3$	$\bar{\theta}_{5,10} = r^*\bar{A}_{d12}^T R_1$
$\bar{\theta}_{5,11} = \mu h \bar{A}_{d22}^T R_2$	$\bar{\theta}_{5,12} = \mu r^* \bar{A}_{d22}^T R_3$
$\bar{\theta}_{6,6} = -e^{-\bar{\alpha}r^*}(\mu^2 R_3 + S_4)$	$\bar{\theta}_{7,7} = -2R_2 e^{-\bar{\alpha}\mu h}$ $-(1-d_2)S_2 e^{-\bar{\alpha}\mu h}$
$\bar{\theta}_{7,8} = R_2 e^{-\bar{\alpha}\mu h}$	$\bar{\theta}_{7,11} = -hR_2$
$\bar{\theta}_{7,12} = -r^* R_3$	$\bar{\theta}_{8,8} = -(R_2 + G_2)e^{-\bar{\alpha}\mu h}$
$\bar{\theta}_{9,9} = -\mu^2 \bar{b}_1 I_m$	$\bar{\theta}_{9,11} = \mu h R_2$
$\bar{\theta}_{9,12} = \mu r^* R_3$	$\bar{\theta}_{10,10} = -R_1$
$\bar{\theta}_{11,11} = -R_2$	$\bar{\theta}_{12,12} = -R_3$

for all $r(t) \in [0,\ r^*]$ and $\xi(t) \in [0,\ h]$ with $\dot r(t) \le d_1 < 1$ and $\mu\dot\xi \le d_2 < 1$. Moreover, solutions of (9.51) satisfy (9.54) for all fast-varying delays $r(t) \in [0,\ r^*]$ or $\xi(t) \in [0,\ h]$ if LMI (9.53) is feasible with $S_1 = S_3 = 0$ or $S_2 = 0$ respectively.

Conditions will be derived that guarantee the following bounds

$$\begin{aligned} &\limsup_{t\to\infty} \|[\bar A_{21}\ \bar A_{22}]z(t)\| < \kappa_1\delta\Delta, \\ &\limsup_{t\to\infty} \|[\bar A_{d21}\ \bar A_{d22}]z(t-r(t))\| < \kappa_2\delta\Delta \end{aligned} \tag{9.55}$$

for solutions of (9.51), where $\kappa_1+\kappa_2 \le 1$. Given (9.54) and consequently

$$\limsup_{t\to\infty} z^T(t-r(t))P_\mu z(t-r(t)) < \frac{\mu^2\bar b_1\left(1+(1+\delta)\sqrt{m}\right)^2}{\bar\alpha}\Delta^2 \tag{9.56}$$

are true, (9.55) holds if the following inequality is satisfied

$$\begin{aligned} &z^T(t)[\bar A_{21}\ \bar A_{22}]^T[\bar A_{21}\ \bar A_{22}]z(t) < \frac{\bar\alpha\kappa_1^2\delta^2 z^T(t)P_\mu z(t)}{\mu^2\bar b_1(1+(1+\delta)\sqrt{m})^2}, \\ &z^T(t-r(t))[\bar A_{d21}\ \bar A_{d22}]^T[\bar A_{d21}\ \bar A_{d22}]z(t-r(t)) \\ &\qquad < \frac{\bar\alpha\kappa_2^2\delta^2 z^T(t-r(t))P_\mu z(t-r(t))}{\mu^2\bar b_1(1+(1+\delta)\sqrt{m})^2} \end{aligned} \tag{9.57}$$

for $t\to\infty$. Hence, the inequalities

$$\begin{aligned} &\begin{bmatrix} \frac{-\bar\alpha\kappa_1^2\delta^2P_1}{\bar b_1\left(1+(1+\delta)\sqrt{m}\right)^2} & \frac{-\bar\alpha\mu\kappa_1^2\delta^2P_2^T}{\bar b_1\left(1+(1+\delta)\sqrt{m}\right)^2} & \mu\bar A_{21}^T \\ * & \frac{-\bar\alpha\mu\kappa_1^2\delta^2P_3}{\bar b_1\left(1+(1+\delta)\sqrt{m}\right)^2} & \mu\bar A_{22}^T \\ * & * & -I_m \end{bmatrix} < 0, \\ &\begin{bmatrix} \frac{-\bar\alpha\kappa_2^2\delta^2P_1}{\bar b_1\left(1+(1+\delta)\sqrt{m}\right)^2} & \frac{-\bar\alpha\mu\kappa_2^2\delta^2P_2^T}{\bar b_1\left(1+(1+\delta)\sqrt{m}\right)^2} & \mu\bar A_{d21}^T \\ * & \frac{-\bar\alpha\mu\kappa_2^2\delta^2P_3}{\bar b_1\left(1+(1+\delta)\sqrt{m}\right)^2} & \mu\bar A_{d22}^T \\ * & * & -I_m \end{bmatrix} < 0 \end{aligned} \tag{9.58}$$

guarantee that the solutions of (9.51) satisfy the bound (9.55).

Proposition 9.14. *Given positive constants r^*, μ, h, $\bar\alpha$, $\bar b_1$, κ_1, κ_2, δ let there exist an $(n-m)\times(n-m)$-matrix $P_1 > 0$, an $m\times(n-m)$-matrix P_2 and positive $(n-m)\times(n-m)$ matrices G_1, S_1, R_1, positive $m\times m$ matrices P_3, G_2, S_2, S_3, S_4, R_2 and R_3 such that LMI $\Theta_0 < 0$ is feasible, where Θ_0 is given by (9.53) with $\mu = 0$. Then, for positive scalars κ_1, κ_2, where $\kappa_1+\kappa_2 \le 1$ and all $\delta > 0$, there exists $\mu(\delta) > 0$ such that for all $\mu \in (0, \mu(\delta)]$ LMIs (9.18), (9.53) and (9.58) are feasible and, thus, solutions of (9.51) satisfy the bound (9.55).*

Then similar to Theorem 9.7:

Theorem 9.15. *Given positive constants r^*, μ, h, $\bar\alpha$, $\bar b_1$, κ_1, κ_2, δ let there exist an $(n-m)\times(n-m)$-matrix $P_1 > 0$, an $m\times(n-m)$-matrix P_2 and positive $(n-m)\times(n-m)$ matrices G_1, S_1, R_1,* *positive $m \times m$ matrices P_3, G_2, S_2, S_3, S_4, R_2 and R_3 such that LMIs (9.18), (9.53) and (9.58) are feasible. Then the solutions of (9.51) satisfy the following bound*

$$\limsup_{t \longrightarrow \infty} |z_{2_i}(t)| \leq 2M_0 \mu h \tag{9.59}$$

where $M_0 = (1+\delta)(1+\sqrt{m})\Delta$, for all $r(t) \in [0,\ r^*]$, $\xi(t) \in [0,\ h]$ with $\dot{r}(t) \leq d_1 < 1$ and $\mu\dot{\xi} \leq d_2 < 1$. Moreover, the solutions of (9.51) satisfy (9.59) for all fast varying delays $r(t) \in [0,\ r^*]$ or $\xi(t) \in [0,\ h]$ if the above LMIs are feasible with $S_1 = S_3 = 0$ or $S_2 = 0$ respectively.

Also in the case of state delay, LMIs (9.50), (9.53) and (9.58) are affine in the system matrices, i.e. the results are applicable to the case where these matrices have polytopic type uncertainties.

Example 9.1. Recall the liquid monopropellant rocket motor model considered in Chapter 7 with bounded parameter uncertainties and unknown disturbances. Assume time-varying delays are present in the states due to non-steady flow and in the input due to delayed pressure supply, the system matrices according to (9.47) are

$$\begin{aligned} A &= \begin{bmatrix} 0.2\rho(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ A_d &= \begin{bmatrix} -1-0.2\rho(t) & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{9.60}$$

where $\rho(t) = sin(t)$ and the disturbance $w(t) = 0.2sin(3t)$ with bound $\|w(t)\| \leq 0.2$. The time-varying state delay and input delay were chosen as $r(t) = 0.1sin(t) + 0.1 \leq 0.2s$ and $\tau(t) = 0.05sin(t) + 0.05 \leq 0.1s$ respectively with $\dot{r}(t) \leq 0.1$ and $\dot{\tau}(t) \leq 0.05$. LMI solutions for controller design incorporate matrices A and A_d with two vertices corresponding to $\rho(t)_{max,min} = \pm 1$. The advantages of the proposed control will be demonstrated by comparing it to a conventional control in the literature.

9.6.2.1 Proposed SMC

Setting $r^* = 0.2s$, $d_1 = 0.1$, $\alpha = 0.922$, $b_1 = 0.0002$, $b_2 = 0.00001$, $\varepsilon = 1.5$, $\varepsilon_1 = 3.6$, $M = [4\ 2.4]$ in LMI (9.50) and $\zeta = 151000$ in LMI (9.8), the reduced order system (9.49) was found ultimately bounded with $K = 1.0053$. Setting $\bar{\alpha} = 0.29$, $\bar{b}_1 = 0.1$, $\mu = 0.17$, $\mu h = 0.1s$, $d_2 = 0.05$ in LMI (9.53), solutions were feasible with $F = [1\ 1.0053]$. Setting $\kappa_1 = 0.9999$, $\kappa_2 = 0.0001$ in LMI (9.58), the switching gain was obtained as $\delta = 36$. Thus the controller (9.11) has been fully synthesized. According to (9.59), the controller guarantees that the ultimate bound of the sliding manifold of system (9.60) satisfies $\|z_2(t)\|_{t \to \infty} \leq 2.96$. In the simulation, initial functions were

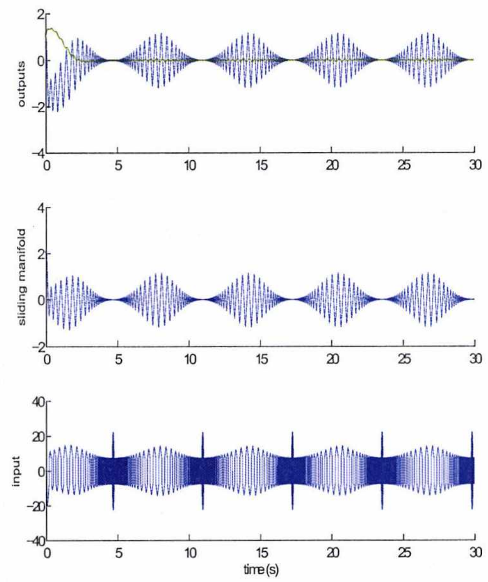


FIGURE 9.4: System (9.60) with control (9.11) in the presence of state and input varying delays.

chosen as [2, 1, −2, 1] for $t \in [-r^*, 0]$. The response of the system is given in Figure 9.4 using the switching component, $\mathrm{sign}\, u(t)$. It is observed that the solutions of the system tend to zero as the varying input delay $\tau(t) = 0$ and grow as the delay increases. The bound on the sliding variable is $\|z_2(t)\| \leq 1.2$ which agrees with the theoretical estimation. The outputs are bounded by $\|y(t)\| \leq 1.2$.

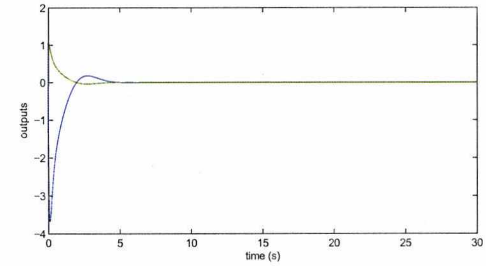
9.6.2.2 Conventional SMC

Control design without considering possible delay may lead to undesirable performance, or even instability. A conventional control from Chapter 8 from the literature was chosen for the system (9.60). The control methodology only guarantees stability with respect to state delay, and is of the form

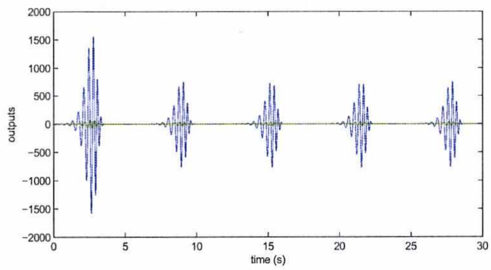
$$u(t) = -Gy(t) - \rho \frac{Fy(t)}{\|Fy(t)\|} \tag{9.61}$$

It was found using LMIs that the closed-loop system was asymptotically stable for state delay $r(t) \leq 0.2s$, with $G = [20\ 74]$, $F = [1\ 4.5]$, $\rho = 5.3$. The response is plotted in Figure 9.5(a). Next, the input delay $\mu\xi(t) \leq 0.1s$ was reintroduced into the closed-loop system and the response is plotted in Figure 9.5(b). The control (9.61) is seen to give a desirable performance in presence of the state delay, but was unable to cope with the additional input delay, the output signals experience large derivations from zero.

Comparing the two control methodologies in the presence of input delay, the proposed control guarantees the ultimate boundedness of the system within a known domain, while the other controller produced unacceptable results.



(a) In the presence of state delay only.



(b) In the presence of both state and input delay.

FIGURE 9.5: System (9.60) with control (9.61).

Remark 9.16. Even without input delay, the new SMC design method has advantages over the previous results. Thus, the matrix P_μ for the analysis of the closed-loop system is full and not diagonal (Edwards and Spurgeon, [29], 1995) and the result in Chapters 6-8. The conservativeness brought by the diagonally structured P_μ was verified by setting $P_2 = 0$ in (9.18) for the above example while keeping all the other tuning parameters in the LMIs unchanged. The feasible input delay in this case was found to be $\mu h = 0.04s$, which is considerably smaller than that obtained using the full P_μ. Hence the full order P_μ leads to less restrictive results. Also, for the first time a static output feedback SMC is designed via Krasovskii method for systems with fast varying state delays. The results in Chapters 6-8, which are delay-independent in switching function, treat only systems with slowly varying delays.

9.7 Conclusion

Sliding mode control with input time-varying delay and with matched bounded disturbances has been studied using a singular perturbation approach. Ultimately bounded solutions of the delayed system are found based on LMI formulations and various Lyapunov-based methods. The ultimate bound is found to be proportional to the delay, the disturbances and the switching gain. The full order structure of the matrix P_μ allows the conservativeness of the numerical solutions to be reduced. In the extension to both state and input delays, the method is applicable to all fast varying delays. The proposed control brings the delay analysis into the design phase which is shown in the examples to have essential advantages when compared with linear control for sufficiently small input delay and other conventional controls where input delay is not considered. The method is applicable to linear systems with polytopic type uncertainties in all blocks of system matrices.

Chapter 10

Discussion and Conclusion

In this thesis, Sliding Mode Control (SMC) using output information for systems with delay has been considered. The robustness property of SMC can be severely impaired due to the presence of delay, producing oscillations or even instability. Design methodologies are needed to take into account the delay effect when designing a controller. Existing results have mainly been concerned with state feedback design. Insufficient attention has been paid to the problem of SMC using output feedback, which is more practical in many applications. The existing results on output feedback SMC are restricted to certain systems with particular structures and are computationally costly. This thesis seeks to develop a novel output feedback SMC scheme which is more computationally efficient and less conservative in synthesizing a feasible controller. The output feedback scheme has been developed to extend readily to the delay case. Linear Matrix Inequalities (LMIs), which are an efficient, powerful numerical tool, have been used extensively.

In Chapter 1 the motivation for robust control is given, including the need to increase system performance under uncertain operating conditions. Time delay is shown to exist in many situations in our practice, which can be undesirable in system operation and control. The motivation for using an output feedback scheme is due to the practicality and flexibility needed for the control implementation. It is expected that increased system performance can be achieved by taking account of delays in the design phase. The challenges firstly lie in building an output feedback design scheme which can be extended to analyze the delay effects. Despite numerous existing contribution in the field, output feedback design still remains one of the fundamental open problems in control theory. The combination of delay and the switching nature of SMC is not naturally synergistic. Work to investigate such effects and improve the performance by other means present a challenging and promising milestone for the research community.

SMC and its properties are demonstrated in Chapter 2. It is a type of discontinuous control where the dynamics of the system once on the sliding surface is of reduced order. The sliding Mode taking place between multiple surfaces can be defined as the discontinuous points shared by all the surfaces. Two ways to reach the sliding mode have been shown. Firstly, each surface is reached sequentially and trajectories evolve until the last surface is reached, i.e. the sliding mode between all the surfaces is completed. In the second case, sliding mode does not take place in each surface but only at the intersection point between the surfaces, i.e. system trajectories move across all the surfaces towards the intersection of the surfaces. Since the sliding motion is discontinuous, its solution cannot be obtained using conventional tools. One way to derive the solution is by considering model imperfections which make the discontinuity points isolated in time. The solution exists as all the small parameters describing the imperfections tend to zero. The equivalent control can be seen as the solution to the sliding motion dynamics replacing the original discontinuous control with a continuous one. Different design approaches are given based on decoupling, regular form transformation and unit vector control. Chattering due to unmodelled actuator dynamics and delays is analyzed. Solutions are proposed to reduce the chattering which are using the boundary layer with saturation, observer or disturbance compensation.

In Chapter 3 general static output feedback control is formulated as a non-convex problem which remains an open problem. Necessary or sufficient conditions are given for eigenvalue and eigenstructure assignment methods, but they are restrictive and the decision methods are computationally inefficient. It is shown that the dynamics of the sliding motion is governed by the zeros of the system which are desired to lie at the left of the complex plane. Conditions for a unique equivalent control to exist are presented. Sufficient conditions are derived for eigenvalue assignment with respect to the reduced order system. However the design technique will only terminate satisfactorily for a specific class of switching surface. Eigenstructure assignment permits the switching surface design without the need to first determine the output feedback gain matrix. The method does not require the system to adopt a regular form. Despite the attractive features, no efficient, constructive procedure has been developed for controller design using this methodology. LMIs have been considered as an efficient, powerful tool to provide a tractable means for tackling the problem, but the existing work either requires iterative LMIs or consists of multiple LMI constraints which are difficult to solve simultaneously. Numerical methods to design control laws based on output information only have been exploited. Two types of control law designs are introduced. Some structural constraints required are conservative and difficult to satisfy using a tractable solution procedure. Other controllers have a simpler structure, however the control tends to be of high gain.

Chapter 4 covers the basic concepts of delay systems and introduces tools for their stability analysis. Delay systems that belong to RFDE are known to have infinite linearly independent solutions through characteristic equation analysis. If a function is continuous and satisfies a local Lipschitz condition in the delayed variable, then the local existence and uniqueness of the solution can be proved. Forward continuation of a solution is proved using the step-by-step method. The solution in the forward delayed time interval can be constructed as the solution of an ODE prior to satisfying a necessary condition that the initial condition is well defined in the past delay interval. Unlike the ODE case, general results on the backward continuation for RFDE are very difficult to prove although the ideas are relatively simple. Neutral systems as a type of delay systems involve the same order of highest derivative for some components of the states at both present time and past time, which introduces an increased mathematical complexity. Characteristic roots of systems subject to delay are studied as a means to analyze the stability of the system. Delay-Independent stability have been investigated using Lyapunov-Razumikhin condition due to its simplicity. On the other hand, for delay-dependent stability the Lyapunov-Krasovskii condition is attractive owing to the structural advantage. Using descriptor method the derivative of highest order in the state can be included to derive less conservative results. For time-varying delays Krasovskii conditions are less restrictive than the Razumikhin conditions for small enough slowly varying delays. However, till now only the Rasumikhin method provides delay-independent conditions for systems with fast-varying delays.

Chapter 5 studies the delay effect on SMC, which exists in the state and the input. Constant and time-varying delays have been considered. An output feedback approach is articulated for systems with state delay. Using the equivalent control, systems with matched perturbation and constant time delay can be reduced to a delay-free system, but in the unmatched case only bounded solutions can be obtained. The use of Lyapunov Krasovskii functionals and Lyapunov Razumikhin functions formulated as LMIs in SMC state feedback, which is efficient in dealing with time delay systems, have been presented. Various contributions considering regular form-based and non regular form-based approaches are reviewed. In deriving the control law knowledge of both non-delayed and delayed states is required by many authors when choosing the switching gain. This leads to restrictive results since the bounds on the state dependent terms are generally not known. To eliminate the need for explicit knowledge of the bound, an adaptive estimation algorithm is used. However the controller becomes a dynamical one and the structure is complex. In the presence of input delay, only bounded solutions can be obtained using SMC. Steady modes and stability analysis of the oscillations are reviewed from the literature. An algorithm for controlling the amplitudes of the motion has been proposed which uses the observer to predict the future behaviour of the state in the next delay interval and reduce the control gain near the periodic solution zero. Since the oscillation amplitude is a function of the switching gain, a smaller region of attraction can be achieved by using an adaptive switching gain. Even though a predictor based control for systems with constant delay produces asymptotic stability, it leads to a controller with memory. The method does not facilitate robust design even for the case of matched uncertainties.

A novel output feedback approach for SMC was proposed in Chapter 6. It follows the conventional regular-form based method to transform the problem of output feedback design in SMC to a general output feedback problem. Compared with other approaches, only a single LMI constraint is involved in the proposed method which does not require the structural restrictions of other work. This leads to a simple but more efficient and less conservative output feedback design. The method can be applied to certain output feedback problems which were previously solved using dynamic compensation. There is however a potential limitation of this method which reduces the efficiency when applied to very large order systems. This is because the LMI formulation depends on choosing a tuning matrix M, where the size of the tuning matrix increases with the system dimensions. For example in the spacecraft model considered in Chapter 9, the tuning matrix is of order three by eight. The difficulty of choosing such a large matrix can cause extra conservativeness in the solution or even make it impossible to find a suitable solution.

In the design of SMC for systems with state time delay in Chapter 7, a descriptor Lyapunov-Krasovskii functional method has been introduced for the switching function design with time-varying delay. The delay is assumed bounded with a known upper bound. Not only the existence problem is solved using LMIs but also the method is used to find the magnitude of the linear gain to construct an appropriate solution to the reachability problem. Under the conditions where output feedback control is not possible, the method is extended to compensator-based design. One limitation of using the Lyapunov Krasovskii method is that only slow varying delay, i.e. the derivative of the delay is less than 1, is considered. In the case of fast varying delay, only the Razumikin approach is considered a suitable tool. The linear part of the controller proposed in the chapter depends on the switching function. In another words, the transient response of the system will depend on the switching function which is chosen to govern the steady state behaviour. Such a controller structure may be limiting since the reaching phase design is coupled with the design of the steady state response. It is shown that an ideal sliding mode can only be achieved provided there exists a switching parameter larger than the state dependent terms and disturbances. This follows the conventional design approach where the drawback is that no explicit knowledge of the switching parameter value is known but it is assumed to be large enough. In practical implementations this assumption may cause chattering to the mechanical components due to the imperfection of modeling, dynamics of the sensors or switching limitations of the devices.

Chapter 8 considers the case of state time-varying delay with both matched and unmatched bounded disturbances. LMIs are formulated to incorporate the effect of the disturbances and the delay so that the closed-loop solutions are guaranteed to be bounded. A systematic approach is given to derive a stabilizing controller where all the parameters of the controller including the switching gain are calculated explicitly to yield the minimization of the bound. This result will prove important in the design of controllers for systems with input delay as follows in the next Chapter. The method allows analysis of polytopic uncertainties included in all blocks of system matrices rather than only in the subsystems as with other equivalent control methods. A nonlinear simulation of an autonomous vehicle is performed where the delay is caused due to the discrepancy between the electrical command generated by the computer to the motor and the mechanical response of the motor to reach to the desired speed. This controller which incorporates the delay effect at the design stage, is seen to produce satisfactory performance where a conventional control that neglects the delay causes instability. The limitations of the aforementioned coupled reaching phase design and steady state design has been relaxed by independent synthesis of the linear control gain matrix and the switching gain matrix.

SMC with time-varying delays in the states and multi-input channels as presented in Chapter 9 employs the singular perturbation method which enables a feasible LMI solution to be obtained. Unlike the existing results on relay control with delay, apriori knowledge of the *bounds on the system states* is not needed. Ultimately bounded solutions of the delayed system are found which are proportional to the size of delay, the disturbances and the switching gain. The switching gain design using LMIs from Chapter 8 is shown to be particularly useful in achieving the minimization of the resulting bound. Previously the Lyapunov function has been chosen in Chapters 7 and 8 to have diagonal structure. This leads to conservatism of the results. In this chapter the limitation has been lifted to allow full order Lyapunov matrix analysis which is shown to tolerate a larger bound on the delay. The proposed controller design produces satisfactory results in its application to a nonlinear model of spacecraft position control where delays are caused by digital sampling. The approach provides a guaranteed solution in the form of LMI constraints for systems with sufficiently small state, input/output delay and matched disturbance. For unmatched disturbances and uncertainties, the method however does not give assurance of the feasibility of a solution.

Chapter 11

Future work

The novel switching surface design for output feedback SMC in chapter 6 may work well for small-sized problems, but the efficiency will reduce as the problem size increases. Convergence of the LMI solution is not guaranteed even if there is a solution. A more efficient design is needed which would require less tuning parameters and possibly no tuning matrix.

The controller structure in Chapter 8 has the characteristics of large control gain. Even though this control structure brings additional freedom allowing linear controller design alone from the switching function, the large control energy generally is undesirable in practice. An improvement would be to take into account of the control effort in the phase of controller synthesis, i.e. selecting a sliding surface and a control gain for a given system in an optimal way such that the performance of the reduced order system is balanced against the control costs required to maintain sliding. In this aspect the work by Edwards, [27], (2004) is seen to be particularly helpful.

Since the switching gain can now be obtained from LMIs in Chapter 8, the method can be incorporated in the framework of adaptive switching gain design. This will reduce the conservativeness when choosing the initial adaptive gain prior to the incomplete knowledge of the system. In case of input/output delay discussed in Chapter 9, this adaptive framework will reduce the oscillations around the sliding surface.

Another important application of the input/output delay approach presented in Chapter 9 is for sampled-data SMC control. In this case sampled output is formulated as a delayed output $y(t_k) = y(t-\tau(t))$, where $\tau(t) = t - t_k,\ t \in [t_k, t_{k+1})$ is a sawtooth delay with $\dot{\tau} = 1$, (Fridman, [52], 2004), (Fridman, [46], 2010). Control design for state delay presented in Chapter 7 and 8 considers the range of time-varying delay being from zero to an upper bound. In practice, the range of delay may vary for which the lower bound is not restricted to be zero. An extension taking into account the lower bound of the delay may considerably reduce the conservativeness (He, [81], 2007), (Gu, et al., [73], 2003), (Fridman, [45], 2006). SMC design in this case will present some challenges if less conservative results are desired with respect to the terms associated with the switching control.

In Chapter 9, the singular perturbation method is used with respect to the linear control component which is dependant on the switching function. Even though the singular perturbation parameter is chosen to be as large as possible to lower the linear control action, it becomes more difficult to enlarge the parameter to a desired level as the size of the system increases. An independent linear controller synthesis from the switching controller, similar to the control structure in Chapter 8, could be a desirable solution since additional design freedom is introduced in the linear controller design. However difficulty arises as the separation of the controller synthesis produces a high gain nature as pointed out in Chapter 3.

The work developed for state delay and input delay in this thesis can be applied to consider observer design when there are delays in the plants. In the normal design procedure, the sliding surface is set to be the error difference between the observer outputs and system outputs, which is therefore forced to zero. In the presence of the output measurement delay, especially unknown time-varying delay, the situation will become more complicated. If there exists a known constant, or known time-varying measurement delay, the delay can be implemented to the observer's outputs. In this case the result in Chapter 9 can be used for observer design. Of course another obvious extension of observer design is to consider systems with uncertainties, which, even without delay, is a challenging problem.

Author's Biography

Xiaoran Han was born in Fushun, China. He received the Advanced Diploma in Electrical and Mechanical engineering from Fushun Mining College, Fushun, China, in 2002, and the M.Sc. degree in Electrical and Electronic Engineering from the University of Leicester, Leicester, U.K., in 2006. He is currently working toward the Ph.D. degree at the University of Kent, Canterbury, U.K. His research interests include sliding-mode control, time-delay systems, network control, modeling of biological systems.

List of publications

Conference publications:

1. X. Han, E. Fridman and S.K. Spurgeon. Sliding mode observer for fault reconstruction under sampled output: a time-delay approach. *Conference on Decision and Control*, Orlando, USA, 2011.

2. E. Fridman, X. Han and S.K. Spurgeon. A singular perturbation approach to sliding mode control in the presence of input delay. *Symposium on Nonlinear Control Systems*, Bologna, Italy, 2010.

3. X. Han, E. Fridman and S.K. Spurgeon. Output feedback sliding mode control of time delay systems with bounded disturbances. *Conference on Decision and Control*, Shanghai, China, 2009.

4. X. Han, E. Fridman and S.K. Spurgeon. Output feedback sliding mode control of systems with bounded disturbances. *European Control Conference*, Budapest, Hungary, 2009.

5. X. Han, E. Fridman and S.K. Spurgeon. On the design of sliding mode static output feedback controllers for systems with time-varying delay. *Variable Structure System*, Antalya, Turkey, 2008.

6. X. Han, E. Fridman, S.K. Spurgeon and C. Edwards. Sliding mode controllers using output information: an LMI approach. *UK Automatic Control Conference*, Manchester, UK, 2008.

Journal publications:

1. X. Han, E. Fridman, S.K. Spurgeon and C. Edwards. On the design of sliding mode static output feedback controllers for systems with state delay. *IEEE Transactions on Industrial Electronics*, 56: 3656-3664, 2009.

2. X. Han, E. Fridman and S.K. Spurgeon. Sliding mode control of uncertain systems in the presence of unmatched disturbances with applications. *International Journal of Control*, 83: 2413-2426, 2010.

3. X. Han, E. Fridman and S.K. Spurgeon. Sliding mode control for input delay systems: a singular perturbation approach. *submitted*.

Bibliography

[1] C. Abdallah, P. Dorato, and S. Karni. SPR design using feedback. In *Conference on Decision and Control*, Brighton, U.K, 1991.

[2] S. Bag, S.K. Spurgeon, and C. Edwards. Output feedback sliding mode design for linear uncertain systems. *Proceeding of IEE*, 144:209–216, 1997.

[3] A.G. Bartholoméüs, M. Dambrine, and J.P. Richard. Stability of perturbed systems with time varying delay. *System and Control Letters*, 31:155–163, 1997.

[4] D.A.W. Barton, B. Krauskopf, and R.E. Wilson. Explicit periodic solutions in a model of a relay controller with delay and forcing. *Nonlinearity*, 18:2637–2656, 2005.

[5] W. Bayer and U. Heiden. Oscillation types and bifurcations of a nonlinear second-order differential difference equation. *Journal of Dynamics and Differential Equations*, 10:303–326, 1998.

[6] R. Bellman and K.L. Cooke. *Differential difference equations*. Academic press, New York, USA, 1963.

[7] G. Bengtsson and S. Lindahl. A design scheme for incomplete state or output feedback with applications to boiler and power system control. *Automatica*, 10:15–30, 1974.

[8] A. Bensoussan, G.D. Prato, M.C. Delfour, and S.K. Mitter. *Representation and control of infinite dimensional systems*. Birkhäuser, Boston, 1992.

[9] R.E. Benton and D. Smith. Static output feedback stabilization with prescribed degree of stability. *IEEE Transactions and Automatic Control*, 43:1493–1496, 1998.

[10] D. Bernstein. Some open problems in matrix theory arising in linear systems and control. *Linear Algebra and its Applications*, 162-164:409–432, 1992.

[11] H.S. Black. Inventing the negative feedback amplifier. *IEEE Spectrum*, pages 55–60, 1977.

[12] V. Blondel and J.N. Tsitsiklis. NP- hardness of some linear control design problems. *SIAM Journal on Control and Optimization*, 35:2118–2127, 1997.

[13] I. Boiko. Oscillations and transfer properties of relay feedback systems with timedelay linear plants. *Automatica*, 45:2897–2902, 2009.

[14] S. Boyd and L. Ghaoui. Method of centers for minimizing generalized eigenvalues. *Linear Algebra and its Applications*, 188-189:63–112, 1993.

[15] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and contorl theory*. Society for Industrial and Applied Mathematics, Philadelphia, 1994.

[16] Y. Cao, J. Lam, and Y. Sun. Static output feedback stabilization: an ILMI approach. *Automatica*, 34:1541–1645, 1998.

[17] W.H. Chen, S.T. Yang, X. Lu, and Y. Shen. Exponential stability and exponential stabilization of singularly perturbed stochastic systems with time-varying delay. *International Journal of Robust and Nonlinear Control*, 20:2021–2044, 2010.

[18] M. Chilali and P. Gahinet. H^{∞} design with pole placement constraints: An LMI approach. *IEEE Transactions and Automatic Control*, 41:358–367, 1996.

[19] H. Choi. Variable structure output feedback control for a class of uncertain dynamic systems. *Automatica*, 38:335–341, 2002.

[20] C.A.R. Crusius and A. Trofino. Sufficient LMI conditions for output feedback control problems. *IEEE Transactions on Automatic Control*, 44:1053–1057, 1999.

[21] R.F. Curtain and H.J. Zwart. *An introduction to infinite-dimensional linear systems theory, Texts in Applied Mathematics*. Springer, New York, 1995.

[22] M. Dambrine, F. Gouaisbaut, W. Perruquetti, and J.P. Richard. Robustness of sliding mode control under delays effects: a case study. *Proceedings on 2nd IEEE-IMACS CESA, Comput. Eng. in Systems Applications*, pages 817–821, 1998.

[23] E. Davison. On pole assignment in linear systems with incomplete state feedback. *IEEE Transactions on Automatic Control*, 15:348–351, 1970.

[24] R. Decaulo, S.H. Zak, and G.P. Matthews. Variable structure control of nonlinear multivariable systems: a tutorial. *IEEE Proceedings*, 76:212–232, 1988.

[25] O. Diekmann, S. van Gils, S.M. Verduyn Lunel, and H.-O. Walther. *Delay Equations*, volume 110 of *Applied Mathematical Sciences*. Springer-Verlag, 1995.

[26] L. Dugard and E.I. Verriest. *Stability and control of time delay systems*, volume 228 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, New York, 1997.

[27] C. Edwards. A practical method for the design of sliding mode controllers using linear matrix inequalities. *Automatica*, 40:1761–1769, 2004.

[28] C. Edwards, A. Akoachere, and S.K. Spurgeon. Sliding mode output feedback controller design using linear matrix inequalities. *IEEE Transactions on Automatic Control*, 46:115–119, 2001.

[29] C. Edwards and S.K. Spurgeon. Slding mode stabilization of uncertain systems using only output information. *International Journal of Control*, 62:1129–1144, 1995.

[30] C. Edwards and S.K. Spurgeon. *Sliding Mode Control: Theory and Application*. Taylor and Francis, 1998.

[31] C. Edwards and S.K. Spurgeon. On the limitations of some vriable structure output feedback controller designs. *Automatica*, 36:743–748, 2000.

[32] C. Edwards and S.K. Spurgeon. Linear matrix inequality methods for designing sliding mode output feedback controllers. *IEE Proceedings - Control Theory and Applications*, 150:539–545, 2003.

[33] C. Edwards, S.K. Spurgeon, and R.G. Hebden. On the design of sliding mode output feedback controllers. *International Journal of Control*, 76:893–905, 2003.

[34] O. El-Ghezawi, S. Billings, and A. Zinober. Variable structure systems and system zeros. *Proceedings of the IEE, Part D*, 130:1–5, 1983.

[35] R. El-khazali. Variable structure robust control of uncertain time-delay systems. *Automatica*, 34:327–332, 1998.

[36] R. El-Khazali and R.A. Decarlo. Variable structure output feedback control. In *American Control Conference*, pages 871–875, Chicago, USA, 1992.

[37] R. El-Khazali and R.A. Decarlo. Output feedback variable structure control design. *Automatica*, 31:805–816, 1995.

[38] Yawvi A. Faigbedzi and Allan E. Pearson. Feedback stabilization of linear autonomous time lag systems. *IEEE Transactions on Automatic Control*, 31:847–855, 1986.

[39] Z. Feng, C. Mian, and G. Weibing. Variable structure control of time-delay systems with a simulation study on stabilizing combustion in liquid propellant rocket motors. *Automatica*, 31:1031–1037, 1995.

[40] C. Fernando and L. Fridman. Analysis and design of integral sliding manifolds for systems with unmatched perturbations. *IEEE Transactions on Automatic Control*, 51:853–858, 2006.

[41] A. Filippov. *Differential Equation with Discontinuous Right-Hand Sides*. Kluwer, Dordrecht, Netherland, 1988.

[42] E. Fridman. New lyapunov krasovskii functional for stability of linear retarded and neutral type systems. *Systems & Control letters*, 43:309–319, 2001.

[43] E. Fridman. Effects of small delays on stability of singularly perturbed systems. *Automatica*, 38:897–902, 2002.

[44] E. Fridman. Descriptor discretized Lyapunov functional method: Analysis and design. *IEEE Transactions on Automatic Control*, 51:890–897, 2006.

[45] E. Fridman. A new lyapunov technique for robust control of systems with uncertain non-small delays. *IMA Journal of Math. Control & Information*, 23:165–179, 2006.

[46] E. Fridman. A refined input delay approach to sampled-data control. *Automatica*, 46:421–427, 2010.

[47] E. Fridman. *The Industrial Electronics Handbook*, volume 5, chapter Time-delay systems. CRC Press, Taylor & Francis, 2011.

[48] E. Fridman and M. Dambrine. Control under quantization, saturation and delay: A LMI approach. *Automatica*, 10:2258–2264, 2009.

[49] E. Fridman, L. Fridman, and E. Shustin. Steady modes in relay control systems with time delay and periodic disturbances. *ASME Journal of Dynamical Systems, Control and Measurement*, 122:732–737, 2000.

[50] E. Fridman, F. Gouaisbaut, Dambrine M, and J.P. Richard. Sliding mode control of systems with time-varying delays via descriptor approach. *International Journal of Systems Science*, 34:553–559, 2003.

[51] E. Fridman, X. Han, and S.K. Spurgeon. A singular perturbation approach to sliding mode contorl in the presence of input delay. In *Symposium on Nonlinear Control Systems*, Bologna, Italy, 2010.

[52] E. Fridman, A. Seuret, and J.P. Richard. Robust sampled-data stabilization of linear systems: an input delay approach. *Automatica*, 40:1441–1446, 2004.

[53] E. Fridman and U. Shaked. An improved stabilization method for linear time-delay systems. *IEEE Transactions on Automatic Control*, 11:1931–1937, 2002.

[54] E. Fridman and U. Shaked. Delay-dependent stability and H^∞ control: constant and time-varying delays. *International Journal of Control*, 76:48–60, 2003.

[55] L. Fridman. The seperation of motions in different speeds discontinuous control systems with delay. *Automation and Remote Control*, 7:1263–1275, 1997.

[56] L. Fridman, E. Fridman, and E. Shustin. Steady modes in a discontinuous control system with time delay. *Pure Mathematics and Applications*, 4:59–73, 1993.

[57] L. Fridman, E. Fridman, and E. Shustin. Steady modes and sliding modes in the relay control systems with time delay. In *Conference on Decision and Control*, pages 4601–4606, Kobe, Japan, 1996.

[58] L. Fridman, E. Fridman, and E. Shustin. *Sliding Mode Control in Engineering*, chapter Steady Modes in Relay Systems with Delay. Marcel Dekker, Inc, 2002.

[59] L. Fridman, E. Fridman, and E.I. Shustin. Steady modes and sliding modes in the relay control systems with time delay. In *Conference on Decision and Control*, Kobe, Japan, 1996.

[60] L. Fridman, V. Strygin, and A. Polyakov. Stabilization of amplitude of osillations via relay delay control. *International Journal of Control*, 76:770–780, 2003.

[61] L. Fridman, V. Strygin, and A. Polyakov. Nonlocal stabilization via delayed relay control rejecting uncertainty in a time delay. *International Journal of Robust Nonlinear Control*, 14:15–37, 2004.

[62] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali. *LMI control toolbox, user guide*. Natick, MA: MathWorks, Inc, 1995.

[63] F.R. Gantmacher. *The theory of Matrices*. Chelsea Publishing Co, New York, USA, 1959.

[64] H. Gao and T. Chen. Network-based H^{∞} output tracking control. *IEEE Transactions on Automatic Control*, 53:655–667, 2008.

[65] S.Di Gennaro. Adaptive robust tracking for flexible spacecraft in presence of disturbance. *Journal of Guidance, Control and Dynamics*, 12:545–568, 1998.

[66] L. El Ghaoui, F. Oustry, and M. Aitrami. A cone complementarity linearization algorithm for static output feedback and related problems. *IEEE Transactions on Automatic Control*, 42:1171–1176, 1997.

[67] V.Y. Glizer. L_2-stabilizability conditions for a class of nonstandard singularly perturbed functional-differential systems. *Dynamics of Continuous, Discrete & Impulsive Systems*, 16:181–213, 2009.

[68] K. Gopalsamy. *Stability and Oscillations in Delay Differential Equations of Population Dynamics*. Kluwer, Dordrecht, Netherland, 1992.

[69] F. Gouaisbaut, Y. Blanco, and J.P. Richard. Robust sliding mode control of non-linear systems with delay: a design via polytopic formulation. *International Journal of Control*, 77:206–215, 2004.

[70] F. Gouaisbaut, M. Dambrine, and J.P. Richard. Robust control of delay systems: A sliding mode control design via LMI. *Systems and Control Letters*, 46:219–230, 2002.

[71] F. Gouaisbaut, W. Perruquetti, and J.-P. Richard. *Sliding Mode Control in Engineering*, chapter Sliding Mode Control for Systems with Time Delay. Marcel Dekker, Inc, 2002.

[72] K. Gu. A generalized discretization scheme of Lyapunov functional in the stability problem of linear uncertain time-delay systems. *International Journal of Robust and Nonlinear Control*, 9:1–14, 1999.

[73] K. Gu, V.L. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Birkhäuser, Boston, USA, 2003.

[74] A. Halanay. *Differential Equation: Stability, Oscillations, Time Lags*. Academic Press, New york, USA, 1966.

[75] J. Hale and S.M. Verduyn Lunel. Strong stabilization of neutral functional differential equations. *IMA Journal of Mathematical Control and Information*, 19:5–23, 2002.

[76] J.K. Hale. *Functional differential equations*. Springer-Verlag, New york, USA, 1971.

[77] J.K. Hale and S.M. Verduyn Lunel. *Introduction to Functional Differential Equations*. Springer-Verlag, New york, USA, 1993.

[78] X. Han, E. Fridman, and S.K. Spurgeon. Sliding mode control of uncertain systems in the presence of unmatched disturbances with applications. *International Journal of Control*, 83:2413–2426, 2010.

[79] X.R. Han, E. Fridman, S.K. Spurgeon, and C. Edwards. Sliding mode controllers using output information: An LMI approach. In *UK Automatic Control Conference*, Manchester, UK, 2008.

[80] X.R. Han, E. Fridman, S.K. Spurgeon, and C. Edwards. On the design of sliding mode static output feedback controllers for systems with state delay. *IEEE Transactions on Industrial Electronics*, 56:3656–3664, 2009.

[81] Y. He, Q.G. Wang, C. Lin, and M. Wu. Delay-range-dependent stability for systems with time-varying delay. *Automatica*, 43:371–376, 2007.

[82] Y. He, Q.G. Wang, C. Lin, and M. Wu. Delay-range-dependent stability for systems with time-varying delay. *Automatica*, 43:371–376, 2007.

[83] B. Heck and A. Ferri. Application of output feedback to variable structure systems. *Journal of Guidance Control and Dynamics*, 12:932–935, 1989.

[84] B. Heck, S.V. Yallapragada, and M.K.H. Fan. Numerical methods to design the reaching phase of output feedback variable structure control. *Automatica*, 12:275–279, 1995.

[85] R. Herman and C. Martin. Applications of algebraic geometry to systems theory: Part 1. *IEEE Transactions on Automatic Control*, 22:19–25, 1977.

[86] T. Horiuchi and T. Konno. A new method for compensating actuator delay in real time hybrid experiments. *Philosophical Transactions of the Royal Society*, A,359:1893–1909, 2001.

[87] D. Huang and S.K. Nguang. Robust H^∞ static output feedback control of fuzzy systems: An ILMI approach. *IEEE Transactions on Systems, Man and Cybernetics*, 36:216–222, 2006.

[88] W. Huang. Generalization of Lyapunov's theorem in a linear delay system. *Journal of Mathematical Analysis and Applications*, 142:83–94, 1989.

[89] E.F. Infante and W.B. Castelan. A lyapunov functional for a matrix difference-differential equation. *Journal of Differential Equations*, 29:439–451, 1978.

[90] U. Itkis. *Control systems of variable structure*. Wiley, New york, USA, 1976.

[91] J.E. Kelly. The cutting plane method for solving convex programs. *Journal of the Society for Industrial and Applied Mathmatics*, 8:703–712, 1960.

[92] V.L. Kharitonov, S. Niculescu, J. Moreno, and W. Michiels. Static output feedback stabilization: necessary conditions for multiple delay controllers. *IEEE Transactions on Automatic Control*, 50:82–87, 2005.

[93] H. Kimura. A further result on the problem of pole assignment by output feedback. *IEEE Transations on Automatic Control*, 22:458–463, 1977.

[94] P. Kokotovic, H. Khalil, and J. O'Reilly. *Singular Perturbation Methods in Control: Analysis and Design*. Academic Press, New York, USA, 1986.

[95] V. Kolmanovskii, S.I. Niculescu, and J.P. Richard. On the Lyapunov Krasovskii functionals for stability analysis of linear delay systems. *International Journal of Control*, 72:374–384, 1999.

[96] V. Kolmanovskii and J.P. Richard. Stability of some linear systems with delay. *IEEE Transactions on Automatic Control*, 5:984–989, 1999.

[97] V.B. Kolmanovskii and A.D. Myshkis. *Applied Theory of functional differential equations*. Kluwer, Dordrecht, Netherland, 1992.

[98] V.B. Kolmanovskii and A.D. Myshkis. *Introduction to the theory and applications of functional differential equations*. Kluwer Academic, Dordrecht, Netherland, 1999.

[99] V.B. Kolmanovskii, S.I. Niculescu, and K. Gu. Delay effects on stability: A survey. In *Conference on Decision and Control*, Phoenix, USA, 1999.

[100] N.N. Krasovskii. On the analytic construction of an optimal control in a system with time-lags (in Russian). *Journal of Applied Mathematics and Mechanics*, 26:50–67, 1962.

[101] Y. Kuang. *Delay Differential Equations with Applications in Population Dynamics*. Academic press, San Diego, USA, 1993.

[102] C.M. Kwan. On variable structure output feedback controllers. *IEEE Transactions on Automatic Control*, 41:1691–1693, 1996.

[103] C.M. Kwan. Further results on variable output feedback controllers. *IEEE Transactions on Automatic Control*, 46:1505–1508, 2001.

[104] L. Levaggi and E. Punta. Analysis of a second-order sliding-mode algorithm in presence of input delays. *IEEE Transactions on Automatic Control*, 51:1325–1332, 2006.

[105] X. Li and R.A. DeCarlo. Robust sliding mode control of uncertain time delay systems. *International Journal of Control*, 76:1296–1305, 2003.

[106] X. Li and S. Yurkovich. Sliding mode control of delayed systems with application to engine idle speed control. *IEEE Transactions on Control Systems Technology*, 9:802 – 810, 2001.

[107] J. Louisell. *A stability analysis for a class of differential-delay equations having time-varying delay*. Lecture Notes in Mathematics. Springer, Berlin, 1991.

[108] N. MacDonald. *Biological Delay Systems: Linear Stability Theory*. Cambridge University Press, Cambridge, UK, 1989.

[109] S. Mondié and W. Michiels. Finite spectrum assignment of unstable time-delay systems with a safe implementation. *IEEE Transactions on Automatic Control*, 12:2207–2212, 2003.

[110] Y.S. Moon, P. Park, and W.H. Kwon. Delay-dependent robust stabilization of uncertain state-delayed systems. *International Journal of Control*, 74:1447–1455, 2001.

[111] B. Moore. On the flexibility offered by state feedback in multivariable control. *IEEE Transactions on Automatic Control*, 21:689–692, 1976.

[112] Sing Kiong Nguang. Comments on "robust stabilization of uncertain input-delay systems by sliding mode control with delay compensation". *Automatica*, 37:1677, 2001.

[113] S.L. Niculescu. *Delay Effects on Stability*. Springer-Verlag, London, UK, 2001.

[114] J. Norbury and R.E. Wilson. Dynamics of constrained differential delay equations. *Journal of Computational and Applied Mathematics*, 125:201–215, 2000.

[115] Y. Orlov, W. Perruquetti, and J.P. Richard. Sliding mode control synthesis of uncertain time-delay systems. *Asian Journal of Control*, 5:568–577, 2003.

[116] P.G. Park, D.J. Choi, and S.G. Kong. Output feedback variable structure control for linear systems with uncertainties and disturbances. *Automatica*, 43:72–79, 2007.

[117] J.P. Richard. Some trends and tools for the study of time delay system. In *Conference on Computational Engineering in Systems Applications*, Tunisia, 1998.

[118] J.P. Richard. Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39:1667–1694, 2003.

[119] J.P. Richard, F. Gouaisbaut, and W. Perruquetti. Sliding mode control in the presence of delay. *Kybernetica*, 37:277–294, 2001.

[120] Y. Roh and J. OH. Robust stabilization of uncertain input-delay systems by sliding mode control with delay compensation. *Automatica*, 35:1861–1865, 1999.

[121] Y. Roh and J. OH. Sliding mode control with uncertainty adaptatioin for uncertain input-delay systems. *International Journal of Control*, 73:1255–1260, 2000.

[122] C. Scherer, P. Gahinet, and M. Chilali. Multi-objective output feedback control via LMI optimization. *IEEE Transactions on Automatic Control*, 42:896–911, 1997.

[123] A. Seuret, C. Edwards, S.K. Spurgeon, and E. Fridman. Static output feedback sliding mode control design via an artificial stabilizing delay. *IEEE Transations on Auutomatic Control*, 54:256–265, 2009.

[124] U. Shaked. An LPD approach to robust H_2 and H^∞ static output feedback design. *IEEE Transactions on Automatic Control*, 48:866–872, 2003.

[125] H. Shao. New delay-dependent stability criteria for systems with interval delay. *Automatica*, 45:744–749, 2009.

[126] K. Shyu, Y. Tsai, and C. Lai. A dynamic output feedback controllers for mismatched uncertain variable structure systems. *Automatica*, 37:775–779, 2000.

[127] K.K. Shyu and J.J. Yan. Robust stability of uncertain time-delay systems and its stabilization by variable structure control. *International Journal of Control*, 57:237–246, 1993.

[128] J. Sieber. Dynamics of delayed relay systems. *Nonlinearity*, 19:2489–2527, 2006.

[129] J. Sieber and B. Krauskopf. Extending the permissible control loop latency for the controlled inverted pendulum. *Dynamical Systems: An International Journal*, 20(2):189–199, 2005.

[130] S. Skogestad and I. Postlethwaite. *Multivariable feedback control: analysis and design*. Wiley, 1996.

[131] J.J. Slotine and S.S. Sastry. Tracking control of nonlinear systems using sliding surfaces with application to robot manipulators. *International Journal of Control*, 38:465–492, 1983.

[132] S.K. Spurgeon and R. Davies. A nonlinear control strategy for robust sliding mode performance in the presence of unmatched uncertainty. *International Journal of Control*, 57:1107–1123, 1993.

[133] S. Srinathkumar. Eigenvalue/eigenvector assignment using output feedback. *IEEE Transactions on Automatic Control*, 23:79–81, 1978.

[134] G. Stépán. *Retarded dynamical systems: stability and characteristic functionn*. Longman Scientific, UK, 1989.

[135] H. Tony. Effect of time delay on the stability of flexible structures with rate feedback control. *Journal of Vibration and Control*, 7:33–49, 2001.

[136] V. Utkin. *Sliding Modes in Control and Optimization*. Springer-Verlag, Berlin, 1992.

[137] V. Utkin, J. Guldner, and J. Shi. *Sliding Mode Control in Electromechanical Systems*. Taylor and Francis, London, UK, 1999.

[138] V. Utkin, J. Guldner, and J. Shi. *Sliding Mode Control in Electromechanical Systems, Second Edition*. CRC Press, Boca Raton, USA, 2009.

[139] Vadim I. Utkin. Variable structure systems with sliding modes. *IEEE Transactions on Automatic Control*, 22:212–222, 1977.

[140] M.I. Wallace, D.J. Wagg, and S.A. Neild. An adaptive polynomial based forward prediction algorithm for multi-actuator real-time dynamic substructuring. *Proceedings of the Royal Society*, 461:3807–3826, 2005.

[141] B.A. White. *Applications of output feedback in variable structure control*. Peter Peregrinus Ltd, London, UK, 1990.

[142] J. Willems and W. Hesselink. Generic properties of the pole placement problem. In *IFAC World Congress*, pages 1725–1729, Helsinki, SF, 1978.

[143] E.C. Wong and W.G. Breckenridge. An attitude control design for the Cassini Spacecraft. *Technical Report by Jet Propulsion Laboratory California Institute of Technology*, 1995.

[144] Y. Xia and Y. Jia. Robust sliding mode control for uncertain time delay systems: an LMI approach. *IEEE Transactions on Automatic Control*, 48:1086–1091, 2003.

[145] J. Xiang, W. Wei, and H. Su. An ILMI approach to robust static output feedback sliding mode control. *International Journal of Control*, 79:959–967, 2006.

[146] L. Yao, S.K. Spurgeon, and C. Edwards. An investigation of novel reconfiguration methods for use in autonomous and semi autonomous vehicles. *SEAS DTC Technical Conference*, 2006.

[147] K.D. Young, V.I. Utkin, and Ü. ÖZGüner. A control engineer's guide to sliding mode control. *IEEE Transactions on Control Systems Technology*, 7:328–342, 1999.

[148] S.H. Zak and S. Hui. On variable structure output feedback controllers for uncertain dynamic systems. *IEEE Transactions on Automatic Control*, 38:1509–1512, 1993.

[149] G. Zames. Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverse. *IEEE Transactions on Automatic Control*, 26:301–320, 1981.

[150] M.M. Zavarei and M. Jamshidi. *Time-Delay Systems, Analysis, Optimization and Applications*. North Holland, Amsterdam, 1987.

[151] Q. Zhong. *Robust Control of Time-delay Systems*. Springer-Verlag, London, UK, 2006.

[152] A.S.I. Zinober. *In Variable Structure and Lyapunov Control*. Springer-Verlag, London, UK, 1994.