METHODS OF SYMMETRY REDUCTION AND THEIR APPLICATION

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Abstract

In this thesis methods of symmetry reduction are applied to several physically relevant partial differential equations.

The first chapter serves to acquaint the reader with the symmetry methods used in this thesis. In particular the classical method of Lie, an extension of it by Bluman and Cole [1969], known as the nonclassical method, and the direct method of Clarkson and Kruskal [1989] are described. Other known extensions of these methods are outlined, including potential symmetries, introduced by Bluman, Kumei and Reid [1988]. Also described are the tools used in practice to perform the calculations. The remainder of the thesis is split into two parts.

In Part One the classical and nonclassical methods are applied to three classes of scalar equation: a generalised Boussinesq equation, a class of third order equations and a class of fourth order equations. Many symmetry reductions and exact solutions are found.

In Part Two each of the classical, nonclassical and direct methods are applied to various systems of partial differential equations. These include shallow water wave systems, six representations of the Boussinesq equation and a reaction-diffusion equation written as a system. In Chapters Five and Six both the actual application of these methods and their results is compared and contrasted. In such applications, remarkable phenomena can occur, in both the nonclassical and direct methods. In particular it is shown that the application of the direct method to systems of equations is not as conceptually straightforward as previously thought, and a way of completing the calculations of the nonclassical method via hodograph transformations is introduced. In Chapter Seven it is shown how more symmetry reductions may be found via nonclassical potential symmetries, which are a new extension on the idea of potential symmetries.

In the final chapter the relationship between the nonclassical and direct methods is investigated in the light of the previous chapters. The thesis is concluded with some general remarks on its findings and on possible future work.

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Chapter One : General Introduction

1.1 Introduction and Historical Perspective

The equations that govern physics are usually nonlinear partial differential equations and consequently are difficult to solve explicitly. Approximate solutions may be found by perturbation, asymptotics and numerical methods with much success; however there is much current interest in finding *exact* solutions to such nonlinear partial differential equations. The methods of symmetry reduction provide a way of finding these solutions.

A *symmetry reduction* (also known as a *similarity reduction*) of a partial differential equation is a transformation of its independent and dependent variables such that the number of variables occurring in the transformed equation is (usually) at least one less than in the original equation. As an example, consider the equation

$$u_t = u_{xx} - u(1-u)^2, \tag{1.1}$$

where subscripts denote partial derivatives. If $u(x,t) = w(z)$, where $z = x - ct$, then substituting into (1.1) we find $w(z)$ satisfies the following ordinary differential equation

$$-cw' = w'' - w(1-w)^2, \tag{1.2}$$

where primes denote differentiation with respect to z. This is a symmetry reduction (often called a travelling wave reduction) of equation (1.1) and we have moved from three variables (x,t,u) to two (z,w).

Inherent in the philosophy behind symmetry analysis is the reasoning that the reduced equation (cf. (1.2)) or system of equations is simpler to solve than the original. Hence ultimately exact, special solutions may be found. This is not unreasonable: we move from trying to solve partial differential equations for which finding the general solution is often impossible, to ultimately trying to solve ordinary differential equations for which many different methods of solution exist. Also the reduced system may be recognised as a well known system, or, as happens in Chapter Seven, the search for symmetry reductions of one equation suggests a transformation onto another, well known equation.

It is known that some solutions of some partial differential equations asymptotically tend to solutions of lower dimensional equations obtained by symmetry reduction (cf. Barenblatt [1979]); it is not surprising then that many exact solutions found by symmetry analysis have been interpreted physically. Further, Galaktionov [1990] has used symmetry reductions to study "blow-up" of solutions of nonlinear heat equations.

The beauty of exact solutions is that one can see precisely the effect varying the parameters of the equation has on the solution. Furthermore, explicit solutions (such as those found by symmetry methods) can play an important role in the design and testing of numerical integrators; the solutions provide an important practical check on the accuracy and reliability of such integrators (cf. Ames, Postell and Adams [1992], Shokin [1983]).

The theory of symmetry reductions starts with Lie in the mid- to late nineteenth century, and his desire to relate transformations of a differential equation (which form a group) to information about the equation's integration. (A history of Lie's early work can be found in Hawkins [1994], and references therein.) He succeeded, and found a way of systematically integrating ordinary differential equations by using group-theoretic techniques.

He then extended the theory to encompass partial differential equations, in the shape of a systematic way of finding symmetry reductions. It is this that we shall refer to as the *classical method*, which is described in §1.2. Although the classical method is entirely algorithmic it involves fairly trivial but lengthy calculations. Perhaps it is for this reason and the fact that Lie himself didn't apply it to any physically relevant partial differential equations that it faded into obscurity. The following brief history shows how it and other symmetry methods (relevant to this thesis) have come to prominence today. (A more thorough account is given in Olver [1993, chap. 3].)

It was some half a century later that Birkhoff [1950] resuscitated Lie's work and also Ovsiannikov and co-workers rederived much of it, successfully applying it to many physically important equations.

The next significant landmark is when Bluman and Cole [1969] observed that Lie's method could be naturally generalised, and applied what we shall refer to as the *nonclassical method* to the linear heat equation

$$u_t = u_{xx}. \tag{1.3}$$

In fact they were unable to find any explicit solutions other than those that the classical method gave when they applied their method to (1.3), but they remarked that *"For other equations the non-classical solutions have been shown to be more general"*. Not only was it seemingly no more productive than the classical method, it was certainly more difficult to apply: we shall see that irrespective of the candidate equation the classical method entails solving *linear* partial differential equations, whilst for the nonclassical method these equations become *nonlinear*. Thus the nonclassical method, which we describe in §1.3, like the classical method before it, failed to immediately attract the attention it deserved.

Despite admitting *"In practice, ..., the determining equations for* [the nonclassical method] *may be too difficult to explicitly solve"*, Olver and Rosenau [1986, 1987] generalised the nonclassical method once to include the notion of *weak symmetries*, and then further but along a different vein with the concept of a differential equation with *side conditions* or *differential constraints*. As we shall see in §1.5.2, where it is described in more detail, the latter generalisation embraced many (all?) known methods of symmetry reduction, but the price it paid for generality was impracticability and certainly any sense of an algorithm was lost. They concluded that *"the unifying theme behind finding special solutions to partial differential equations is not, as is commonly supposed, group theory, but rather the more analytical subject of overdetermined systems of partial differential equations"*.

A revival of interest in the nonclassical method came from the development of a new and direct method for finding symmetry reductions by Clarkson and Kruskal [1989]. Motivated by the knowledge that there existed reductions of the Boussinesq equation

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0, \tag{1.4}$$

that could not be found by the classical method (Olver and Rosenau [1987]) they developed what is now known as the *direct method*. This method, unlike Lie's or Bluman and Cole's, used no group theory, but instead an ansatz is made on the form of the reduction. More specifically, applying the method to a scalar partial differential equation like (1.4), one assumes the solution takes the form

$$u(x,t) = F(x,t,w(z(x,t))),$$

and requires that $w(z)$ satisfies an ordinary differential equation. True to form to Olver and Rosenau's conclusions mentioned earlier the method necessitates solving (nonlinear) overdetermined partial differential equations. This framework allowed them to find the known reductions of the Boussinesq equation, and many more! Following the success of the direct method it was applied to many physically interesting equations including those with more independent variables (cf. Clarkson and Winternitz [1991], Lou and Ni [1991] and Clarkson and Hood [1992]) and to systems, with more dependent (and independent) variables (cf. Clarkson [1992], Clarkson and Hood [1993,1994], Hood [1993], Lou [1992] and Lou and Ruan [1993]). A detailed description of the application of the direct method to scalar equations (with two independent variables) is given in §1.4.

We note that the direct method of Clarkson and Kruskal [1989] bears a resemblance to the *Free Parameter Method* of Hansen [1964], except that in the latter boundary conditions play an important role in finding a reduction. The attentive reader will have noticed that no mention of boundary conditions (or initial conditions) has been made until now, and this discourse will be brief. The symmetry methods that are used take no consideration of them in this thesis, though it should be noted that it may be possible to incorporate them (see, for instance Bluman and Kumei [1989]).

With the development of the direct method came interest in the relationship between it and other methods, indeed Clarkson and Kruskal raised the question of equivalence in their original paper. The main results are discussed here, but a more detailed account is given in §1.5.4. Levi and Winternitz [1989] showed the equivalence of the direct method and the nonclassical method for the Boussinesq equation by applying the nonclassical method and finding the same symmetry reductions. However for the Fitzhugh-Nagumo equation

$$u_t = u_{xx} + u(1-u)(u-a),$$

Nucci and Clarkson [1992] showed that there existed nonclassical reductions which the direct method could not find. Subsequently Olver [1994] (see also Arrigo, Broadbridge and Hill [1993], Pucci [1992]) proved the precise nature of the relationship between the two methods, basically that the nonclassical method is more general. More recently Ludlow [1995] provided an example where the classical method gave more reductions than the direct method, dispelling the common belief that the classical method was a special case of the direct method.

Meanwhile, Galaktionov [1990] successfully extended the direct method by using a generalisation of the separation of variables technique with his *Nonlinear Separation Method*, from which Olver [1994] (theoretically) generalised it further and compared it with the method of differential constraints (Olver and Rosenau [1986]). Further, extensions of the direct method have also been given by Burdé [1994], Estévez [1992] and Hood [1995], each of which differs slightly from it but remains an ansatz-based approach. Burdé [1996] has also proposed another generalisation, which takes ideas from both the nonclassical and direct methods. This is described in §1.5.2.

The classical method was generalised from a different perspective by Bluman, Kumei and Reid [1988]. Following the success of Bluman and Kumei [1989], who found more symmetry reductions of the wave equation

$$c^2(x)u_{xx} - u_{tt} = 0,$$

they proposed an algorithm for finding such new reductions, based on rewriting the equation as a *potential* system. Applying the classical method to the potential system gives rise to *potential symmetries*, which we describe in §1.5.1. More recently Bluman and

Shtelen [1995] proposed a natural generalisation of this idea, using the nonclassical rather than the classical method, which is described in the same subsection.

Another area of mathematics relevant to this thesis that also has its roots in the late nineteenth century, is Painlevé analysis. In order to answer a problem posed by Picard [1887], that of classifying certain types of second order ordinary differential equations whose solutions have desirable singularity structure, Painlevé found it necessary to create a method to analyse these singularity structures. This method is known as Painlevé's α-method. Equations whose solutions have such singularity structures are now said to be of *Painlevé-type*, and experience dating back to Kowalevski [1889a,b] tells us that often only equations of Painlevé-type may be solved explicitly. The determination of whether equations are of Painlevé-type, without first knowing their solutions, has gained considerable interest recently since their link with inverse scattering was proposed, via the *Painlevé conjecture* (Ablowitz, Ramani and Segur [1978, 1980] and Hastings and McLeod [1980]). An algorithm based on Kowalevski's work, known as the ARS algorithm (Ablowitz, Ramani and Segur [1980a]) to test ordinary differential equations, and an extension by Weiss, Tabor and Carnevale [1983] to test partial differential equations have been determined more recently. These are described, along with Painlevé's α-method, in §1.6.

Since the determining equations of many of the methods of symmetry reduction form overdetermined systems, in §1.7 we describe the theory of differential Gröbner bases, which is well suited to dealing with such systems. Whilst algorithms to solve linear overdetermined systems have been known for some time (e.g. Janet [1929]), it is only recently that Mansfield and Fackerell [1992] have developed algorithms to cope with fully nonlinear overdetermined systems. These algorithms have been implemented in the MAPLE package `diffgrob2`, which we use throughout this thesis to great effect.

Another application of symbolic manipulation systems (such as MAPLE) is in the generation of the determining equations of the classical and nonclassical methods. Since the calculations can be lengthy, unmanageably so for large systems, and yet are entirely algorithmic, they are ideally suited to such computer calculation. Whilst there are many examples of packages written to produce the determining equations of the classical method, few are written that cope with the requirements of the nonclassical method; an excellent review article is given by Hereman [1994]. In §1.8 we describe the MACSYMA package `symmgrp.max` which satisfies our needs of generating both classical and nonclassical determining equations, and which has been extensively tested.

The remaining sections of this chapter are devoted to a lengthier and more detailed exposition of the introduction thus far, so as to acclimatise the reader to the other chapters of this thesis.

This thesis is split into two parts. Part One deals with scalar equations only, which depend on arbitrary parameters (functions and constants). Since we know the nonclassical method is more general than the direct method for scalar equations, it is preferred in this Part. The classical method is also applied and the results compared with those of the nonclassical method. In Part Two systems of equations are considered, for three reasons. Firstly, since the relationship between the nonclassical and direct methods has only been proven for scalar equations, we apply both these methods as well as the classical method to our systems to try to establish some relations. Secondly, from a potential symmetries point of view, to see if we obtain the same reductions from considering a system rather than its scalar counterpart, and thirdly to compare and contrast the differences in the actual application of the methods. All the details, including some remarkable results, can be found in the chapters themselves.

The thesis is concluded with a discussion on the extension of Olver's proof (Olver [1994]), on the relationship between the direct and nonclassical methods, to systems of equations, some general remarks concerning the work in this thesis and also on possible future work.

1.2 Classical Lie Method

Previous to the discoveries of Lie there only existed *ad hoc* procedures for integrating nonlinear ordinary differential equations. For instance, consider the so-called homogeneous equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y^2}{xy}. \tag{1.5}$$

It is well known that the transformation $y(x) = xv(x)$ leads to a "separation" of the variables, i.e. substituting for y yields

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x},$$

which may be integrated with respect to x, and hence we find the general solution

$$y(x) = x(\ln x^2 + c)^{1/2},$$

where c is an arbitrary constant. Now consider the transformation $x^* = \alpha x$, $y^* = \alpha y$ for some constant α. Substituting into equation (1.5) gives

$$\frac{\mathrm{d}y^*}{\mathrm{d}x^*} = \frac{{x^*}^2 + {y^*}^2}{x^* y^*},$$

i.e. $y^*(x^*)$ satisfies the same equation as $y(x)$. The transformation $x^* = \alpha x$, $y^* = \alpha y$ is said to leave (1.5) *invariant*, and $v = y/x$ is an *invariant* of the transformation since

$$v(x^*, y^*) = \frac{y^*}{x^*} = \frac{\alpha y}{\alpha x} = \frac{y}{x} = v(x, y). \tag{1.6}$$

The theory that Lie proposed, which crucially uses transformations that leave the differential equation invariant, embraced all such *ad hoc* techniques. Not only did it explain why such transformations work, but also showed how to find them. (At higher order the use of these invariants reduces the order of the ordinary differential equation by one.)

However our concern is with the theory of partial differential equations, to which Lie extended his theory, in the shape of a systematic way of finding symmetry reductions. Recall (1.1) and its travelling wave reduction. The transformation $u^* = u$, $x^* = x + \alpha c$, $t^* = t + \alpha$ leaves (1.1) invariant since $u^*(x^*, t^*)$ satisfies

$$u^*_{t^*} = u^*_{x^*x^*} - u^*(1 - u^*)^2,$$

and the invariants of the transformation are $w = u^* = u$ and $z = x^* - ct^* = x - ct$. By writing w as a function of z, we find the travelling wave reduction of (1.1).

Lie's method, the classical method of this thesis, is well known today and has lent itself to many applications, not just finding symmetry reductions. As well as those mentioned in §1.1, one is able to derive new solutions from old ones, to linearise partial differential equations (see §1.5.1), to convert boundary value problems to initial value problems (see Rogers and Ames [1989]), and the method provides a classification of differential equations (see for instance Chapters Two through Four). For other applications and more references see Clarkson [1995].

The aim of the remainder of this section is to provide the ideas behind the classical method, the notation which is used today and the algorithm itself. This is then illustrated in detail with an example, and we end with some remarks. We note that the classical method has been described by many authors, for instance Bluman and Cole [1974], Bluman and Kumei [1989], Hill [1992], Olver [1993], Ovsiannikov [1982], Stephani [1989], who all give more detailed accounts. It is because of this that detail is kept to a minimum in this explanation, however we make the following remark.

Remark 1.2(i). It is assumed that the equations under consideration are smooth functions of all their arguments and that the system comprising of these equations is of maximal rank, and is locally solvable. (For definitions of these properties and examples of their importance see Olver [1993].)

1.2.1 Symmetry Groups and Prolongation.

The principle idea in the classical method is to find the symmetry group of a system of partial differential equations, which we define as

Definition 1.2.1. A *symmetry group*, or *symmetry* of a system of partial differential equations is a group of transformations which maps any solution to another solution of the system.

In this thesis we study a specific group of transformations, one parameter (local) Lie groups of point transformations (and see Remark 1.2(iii) later).

Definition 1.2.2. A *one-parameter (local) Lie group of point transformations* is a transformation of the form

$$\boldsymbol{x}^* = \boldsymbol{X}(\boldsymbol{x}, \boldsymbol{u}; \varepsilon), \qquad \boldsymbol{u}^* = \boldsymbol{U}(\boldsymbol{x}, \boldsymbol{u}; \varepsilon), \tag{1.7}$$

where $\boldsymbol{x} = (x_1, \ldots, x_p) \in \mathbb{C}^p$ and $\boldsymbol{u} = (u_1, \ldots, u_q) \in \mathbb{C}^q$ are the independent and dependent variables, and ε is the (continuous) group parameter. The property of associativity must hold, as well as

(i) the value $\varepsilon = 0$ characterises the identity transformation,

(ii) the transformation is closed: if $\boldsymbol{x}^{**} = \boldsymbol{X}(\boldsymbol{x}^*, \boldsymbol{u}^*; \delta)$, $\boldsymbol{u}^{**} = \boldsymbol{U}(\boldsymbol{x}^*, \boldsymbol{u}^*; \delta)$ then $\boldsymbol{x}^{**} = \boldsymbol{X}(\boldsymbol{x}, \boldsymbol{u}; \delta + \varepsilon)$, $\boldsymbol{u}^{**} = \boldsymbol{U}(\boldsymbol{x}, \boldsymbol{u}; \delta + \varepsilon)$ and in particular $\delta = -\varepsilon$ characterises the inverse.

By expanding (1.7) in a Taylor series about $\varepsilon = 0$ we have

$$\boldsymbol{x}^* = \boldsymbol{x} + \varepsilon \boldsymbol{\xi}(\boldsymbol{x}, \boldsymbol{u}) + O(\varepsilon^2), \tag{1.8i}$$

$$\boldsymbol{u}^* = \boldsymbol{u} + \varepsilon \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{u}) + O(\varepsilon^2), \tag{1.8ii}$$

(because of property (i) above), where

$$\boldsymbol{\xi}(\boldsymbol{x}, \boldsymbol{u}) = \left.\frac{\partial \boldsymbol{X}}{\partial \varepsilon}\right|_{\varepsilon=0}, \qquad \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{u}) = \left.\frac{\partial \boldsymbol{U}}{\partial \varepsilon}\right|_{\varepsilon=0},$$

are called the *infinitesimals* of the one-parameter Lie group of transformations. The transformation (1.7) is said to be in *global form*, whereas (1.8) is said to be in *infinitesimal form*. One of the fundamental theorems of Lie links these two forms: he proved that given the infinitesimal form (1.8) one can reconstruct the global form (1.7) by integrating

$$\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\varepsilon} = \boldsymbol{\xi}(\boldsymbol{x}^*, \boldsymbol{u}^*), \qquad \frac{\mathrm{d}\boldsymbol{u}^*}{\mathrm{d}\varepsilon} = \boldsymbol{\phi}(\boldsymbol{x}^*, \boldsymbol{u}^*),$$

subject to the initial conditions $\boldsymbol{x}^* = \boldsymbol{x}$, $\boldsymbol{u}^* = \boldsymbol{u}$ at $\varepsilon = 0$. This remarkable result implies that all the information about the transformation (1.7) is embodied in (1.8) and in particular in the infinitesimals $(\boldsymbol{\xi}, \boldsymbol{\phi})$.

We now introduce some notation which whilst lengthy will assist us in explaining the classical method algorithm.

Associated with the group of transformations (1.8) is the operator

$$\mathbf{v} \equiv \sum_{j=1}^{p} \xi_j(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial x_j} + \sum_{i=1}^{q} \phi_i(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial u_i}, \tag{1.9}$$

which is known as the *infinitesimal generator* or *vector field* of (1.8).

The *kth prolongation* of this vector field is defined to be

$$\mathrm{pr}^{(k)}\mathbf{v} \equiv \mathbf{v} + \sum_{i=1}^{q}\sum_{j_1=1}^{p} \phi_i^{[j_1]}\frac{\partial}{\partial u_{i,j_1}} + \sum_{i=1}^{q}\sum_{j_1,j_2=1}^{p} \phi_i^{[j_1,j_2]}\frac{\partial}{\partial u_{i,j_1j_2}} + \ldots + \sum_{i=1}^{q}\sum_{j_1,j_2,\ldots,j_k=1}^{p} \phi_i^{[j_1,j_2,\ldots,j_k]}\frac{\partial}{\partial u_{i,j_1j_2\ldots j_k}}, \tag{1.10}$$

where we use the convention

$$u_{i,j_1j_2\ldots j_r} \equiv \frac{\partial^r u_i}{\partial x_{j_1}\partial x_{j_2}\ldots\partial x_{j_r}}, \tag{1.11}$$

and the $\phi_i^{[j_1,j_2,\ldots,j_r]}$ are called the *associated infinitesimals*. These associated infinitesimals arise from considering how the derivatives of $\boldsymbol{u}$ transform under (1.8). By repeated application of the chain rule we find

$$u^*_{i,j_1j_2\ldots j_r} = u_{i,j_1j_2\ldots j_r} + \varepsilon\phi_i^{[j_1,j_2,\ldots,j_r]}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(r)}) + O(\varepsilon^2),$$

where $\phi_i^{[j_1,j_2,\ldots,j_r]}$ is defined recursively by

$$\phi_i^{[j_1,j_2,\ldots,j_r]}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(r)}) \equiv \frac{\mathrm{D}\phi_i^{[j_1,j_2,\ldots,j_{r-1}]}}{\mathrm{D}x_{j_r}} - \sum_{l=1}^{p}\left(\frac{\mathrm{D}\xi_l}{\mathrm{D}x_{j_r}}\right)\frac{\partial^r u_i}{\partial x_{j_1}\partial x_{j_2}\ldots\partial x_{j_{r-1}}\partial x_l}, \tag{1.12}$$

for $r \in \{2,3,\ldots,n\}$ and

$$\phi_i^{[j_1]}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(1)}) \equiv \frac{\mathrm{D}\phi_i}{\mathrm{D}x_{j_1}} - \sum_{l=1}^{p}\left(\frac{\mathrm{D}\xi_l}{\mathrm{D}x_{j_1}}\right)\frac{\partial u_i}{\partial x_l}. \tag{1.13}$$

The operator D_{x_j}, when applied to a function of the form $F(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(r)})$, has the form

$$\mathrm{D}_{x_j} \equiv \frac{\mathrm{D}}{\mathrm{D}x_j} \equiv \frac{\partial}{\partial x_j} + \sum_{s=1}^{q}\frac{\partial u_s}{\partial x_j}\frac{\partial}{\partial u_s} + \sum_{s=1}^{q}\sum_{j_1=1}^{p}\frac{\partial u_{s,j_1}}{\partial x_j}\frac{\partial}{\partial u_{s,j_1}} + \ldots + \sum_{s=1}^{q}\sum_{j_1,j_2,\ldots,j_r=1}^{p}\frac{\partial u_{s,j_1j_2\ldots j_r}}{\partial x_j}\frac{\partial}{\partial u_{s,j_1j_2\ldots j_r}} \tag{1.14}$$

(again invoking (1.11) for convenience) and is called the *total derivative operator*. We should also mention that $\boldsymbol{u}^{(n)}$ represents all the partial derivatives of the form (1.11) with $i \in \{1,\ldots,q\}$, $j_l \in \{1,\ldots,p\}$, $l \in \{1,\ldots,r\}$ and $r \in \{1,\ldots,n\}$.

1.2.2 Finding the infinitesimals.

Now we turn our attention to the system of equations we will be interested in, a system of m equations

$$\boldsymbol{\Delta}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(n)}) = 0, \tag{1.15}$$

where $\boldsymbol{\Delta} = (\Delta_1, \Delta_2, \ldots, \Delta_m)$ of order n with $p(\geq 2)$ independent and q dependent variables. If we apply the transformation (1.8) to system (1.15)* (i.e. to (1.15) with our variables $(\boldsymbol{x}, \boldsymbol{u})$ replaced by $(\boldsymbol{x}^*, \boldsymbol{u}^*)$) we find by Taylor's theorem

$$\boldsymbol{\Delta}(\boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{u}^{*(n)}) = \boldsymbol{\Delta}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(n)}) + \varepsilon \mathrm{pr}^{(n)}\mathbf{v}(\boldsymbol{\Delta}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(n)})) + O(\varepsilon^2),$$

and the need for the prolongation formula becomes apparent. We now appeal to the following theorem (a proof of which may be found in e.g. Olver [1993])

Theorem 1.2.1. *A one-parameter Lie group of transformations (1.8) with vector field (1.9) is a symmetry group of the system (1.15) if and only if*

$$\mathrm{pr}^{(n)}\mathbf{v}(\boldsymbol{\Delta}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(n)})) = 0 \qquad \textit{whenever} \qquad \boldsymbol{\Delta}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(n)}) = 0. \tag{1.16}$$

Since the crux of the classical method is to find the symmetry group of a system of equations, this theorem is precisely what we need. In fact (1.16) may be thought of as the exact requirement of the classical method. We are now in a position to write down the first part of the classical method as applied to (1.15) in an algorithmic form, which we split into two main steps.

Step One: Generate the *determining equations.*

(a) Apply the nth prolongation to the system (1.15).

(b) Choose m derivatives of $\boldsymbol{u}$ from (1.15). These derivatives, which we denote v_i for $i = 1, \ldots, m$, must be chosen so that no v_i is a derivative of another. We must be able to write (1.15) in the form

$$v_i = F_i(\boldsymbol{x}, \boldsymbol{w}), \qquad i = 1, \ldots, m,$$

where $\boldsymbol{w}$ represents the remaining components of $\boldsymbol{u}^{(n)}$, and also be able to express any derivative of a v_i without having to reintroduce it, its derivatives or any of the other v_j or their derivatives. (It is often the case that choosing the highest derivative terms of (1.15) by some compatible ordering (see §1.7) will result in an acceptable set of v_i.)

(c) Use the fact that $\boldsymbol{u}$ must satisfy (1.15) (by Theorem 1.2.1) to remove the v_i and their derivatives from the result of (a).

(d) Equate the coefficients of like derivatives of $\boldsymbol{u}$ to zero (since the expression must hold for all solutions of $\boldsymbol{u}$, the partial derivatives of $\boldsymbol{u}$ must be independent, except for the conditions imposed by the given system). The equations thus found, which will form a linear, overdetermined, homogeneous system of partial differential equations with dependent variables $(\boldsymbol{\xi}, \boldsymbol{\phi})$ and independent variables $(\boldsymbol{x}, \boldsymbol{u})$, are what we call the *determining equations.*

As one can imagine by the (slightly horrendous looking) formulae, the size of the calculations increase as p, q and particularly n increase. However, since this step is entirely algorithmic it is ideally suited to symbolic manipulation programs, and indeed many have been developed. A survey of the packages available is given by Hereman [1994], and we describe in some detail in §1.8 the package `symmgrp.max`, written by Champagne, Hereman and Winternitz [1991].

Step Two: Solve the determining equations.

The determining equations may usually be solved using only elementary methods, though their size, particularly for larger systems, can become an obstacle.

We describe in §1.7 the MAPLE package `diffgrob2` which was written by Mansfield [1993] to cope with such overdetermined systems, by reducing them to a simpler system (see §1.7 for a more precise summary of `diffgrob2`'s capabilities). Whilst the determining equations are linear in $(\boldsymbol{\xi}, \boldsymbol{\phi})$, if the original system contains arbitrary functions of $(\boldsymbol{x}, \boldsymbol{u})$ then the determining equations are effectively nonlinear (see for instance Chapter Two). This makes the solution of the determining equations more difficult, though `diffgrob2` is still able to cope with such difficulties.

1.2.3 Classical symmetry reductions.

In order to find our symmetry reductions once we have found the symmetry group, we want to know the *invariants* of our transformation and in particular solutions $\boldsymbol{u} = \boldsymbol{\Theta}(\boldsymbol{x})$ of (1.15) that are invariant (called *invariant solutions*). If we recall (1.6) we described v as an *invariant* of the transformation $x^* = \alpha x$, $y^* = \alpha y$ since $v(x^*, y^*) = v(x, y)$. To generalise this we say a function $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u})$ is *invariant* under the transformation (1.8) if $\boldsymbol{F}(\boldsymbol{x}^*, \boldsymbol{u}^*) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u})$. A necessary and sufficient condition for $\boldsymbol{F}$ to be an invariant function of (1.8) is

$$\mathbf{v}(\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u})) \equiv 0.$$

where $\mathbf{v}$ is the operator (1.9). The following theorem holds

Theorem 1.2.2. *Consider the set σ of solutions of (1.15)*

$$\sigma = \{(\boldsymbol{x}, \boldsymbol{u}) : \boldsymbol{\Theta}(\boldsymbol{x}) - \boldsymbol{u} = 0\}.$$

A one-parameter Lie group of transformations (1.8), which is a symmetry group of (1.15), with vector field (1.9) leaves σ invariant if and only if

$$\mathbf{v}(\boldsymbol{\Theta}(\boldsymbol{x}) - \boldsymbol{u}) = 0 \qquad \textit{whenever} \qquad \boldsymbol{\Theta}(\boldsymbol{x}) - \boldsymbol{u} = 0.$$

From this we can construct the *invariant surface conditions*, $\boldsymbol{\psi} = (\psi_1, \psi_2, \ldots, \psi_q)$ where

$$\psi_s \equiv \sum_{i=1}^{p} \xi_i(\boldsymbol{x}, \boldsymbol{u}) u_{s,x_i} - \phi_s(\boldsymbol{x}, \boldsymbol{u}) = 0, \qquad s = 1, 2, \ldots, q, \tag{1.17}$$

which as quasilinear first order partial differential equations may be solved via the method of characteristics, i.e. we solve

$$\frac{\mathrm{d}x_1}{\xi_1} = \frac{\mathrm{d}x_2}{\xi_2} = \dots \frac{\mathrm{d}x_p}{\xi_p} = \frac{\mathrm{d}u_1}{\phi_1} = \frac{\mathrm{d}u_2}{\phi_2} = \dots \frac{\mathrm{d}u_q}{\phi_q},$$

to give $p+q-1$ integral surfaces (or invariants) of the form

$$\Lambda_r(\boldsymbol{x}, \boldsymbol{u}) = c_r, \qquad r = 1, 2, \dots, p+q-1,$$

for constants c_r. (These are invariants of the symmetry group of (1.15) as $\mathbf{v}(\Lambda_r(\boldsymbol{x}, \boldsymbol{u})) = 0$.) We now choose q of these invariants to be the new dependent symmetry variables

$$\boldsymbol{U} = \widetilde{\boldsymbol{H}}(\boldsymbol{x}, \boldsymbol{u}), \tag{1.18}$$

and the remaining $p-1$ to be the new independent symmetry variables

$$\boldsymbol{z} = \boldsymbol{Z}(\boldsymbol{x}, \boldsymbol{u}). \tag{1.19}$$

By design (1.18) may be solved for $\boldsymbol{u}$ using the implicit function theorem, to obtain the general solution of (1.17) in the form

$$\boldsymbol{u} = \boldsymbol{H}(\boldsymbol{U}(\boldsymbol{z}), \boldsymbol{x}). \tag{1.20}$$

Substituting (1.20) into (1.15) one obtains the symmetry system

$$\widetilde{\boldsymbol{\Delta}}(\boldsymbol{z}, \boldsymbol{U}, \boldsymbol{U}^{(n)}) = 0, \tag{1.21}$$

which only depends on the new $p+q-1$ symmetry variables. We conclude that the final step is

Step Three: Solve the invariant surface conditions to find the symmetry reduction.

We solve the invariant surface condition by the method of characteristics (though note Remark 1.2(iv) at the end of this section) to find the new symmetry variables $(\boldsymbol{z}, \boldsymbol{U})$, and hence the symmetry system (1.21) by the procedures outlined above. We say that $\boldsymbol{u}$ given by (1.20), where $\boldsymbol{z}$ is given by (1.19), and $\boldsymbol{U}$ satisfies (1.21) is a symmetry reduction of system (1.15).

Example 1.2.1. To see the classical method put into practice, consider equation (1.1),

$$\Delta \equiv \quad u_t - u_{xx} + u(1-u)^2 = 0.$$

Our variables are $\boldsymbol{x} = (x, t)$ and $\boldsymbol{u} = u$ and we construct the one-parameter Lie group of transformations given by

$$x^* = x + \varepsilon\xi(x, t, u) + O(\varepsilon^2), \tag{1.22i}$$

$$t^* = t + \varepsilon\tau(x, t, u) + O(\varepsilon^2), \tag{1.22ii}$$

$$u^* = u + \varepsilon\phi(x, t, u) + O(\varepsilon^2), \tag{1.22iii}$$

with vector field

$$\mathbf{v} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}.$$

Step One

(a) We apply the second prolongation

$$\mathrm{pr}^{(2)}\mathbf{v} = \mathbf{v} + \phi^{[1]}\frac{\partial}{\partial u_x} + \phi^{[2]}\frac{\partial}{\partial u_t} + \phi^{[1,1]}\frac{\partial}{\partial u_{xx}} + \phi^{[1,2]}\frac{\partial}{\partial u_{xt}} + \phi^{[2,2]}\frac{\partial}{\partial u_{tt}},$$

to Δ to yield

$$\mathrm{pr}^{(2)}\mathbf{v}(\Delta) = \phi(1 - 4u + 3u^2) + \phi^{[2]} - \phi^{[1,1]}. \tag{1.23}$$

Hence we need to know the associated infinitesimals $\phi^{[2]}$ and $\phi^{[1,1]}$. From (1.13) we have

$$\begin{aligned}\phi^{[1]} &= \mathrm{D}_x\phi - (\mathrm{D}_x\xi)u_x - (\mathrm{D}_x\tau)u_t,\\ &= \phi_x + \phi_u u_x - (\xi_x + \xi_u u_x)u_x - (\tau_x + \tau_u u_x)u_t,\end{aligned} \tag{1.24}$$

and similarly

$$\phi^{[2]} = \phi_t + \phi_u u_t - (\xi_t + \xi_u u_t)u_x - (\tau_t + \tau_u u_t)u_t. \tag{1.25}$$

We then find $\phi^{[1,1]}$ using the recursion formula (1.12), and (1.24) which yields

$$\begin{aligned}\phi^{[1,1]} =& \mathrm{D}_x\phi^{[1]} - (\mathrm{D}_x\xi)u_{xx} - (\mathrm{D}_x\tau)u_{xt},\\ =& \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3\\ &- \tau_{uu}u_x^2 u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt},\end{aligned} \tag{1.26}$$

after collecting terms, and hence we have $\mathrm{pr}^{(2)}\mathbf{v}(\Delta)$ from (1.23), (1.25) and (1.26).

(b) For this and most scalar equations the choice of derivative, v_i, to substitute back for is quite simple: either u_{xx} or u_t may be chosen here. We choose u_{xx}, simply because there are no differential consequences of it in (1.23) so no further differentiation is required; hence we have

$$u_{xx} = u_t + u(1 - u)^2. \tag{1.27}$$

(c) Replacing occurrences of u_{xx} in (1.23) with (1.27) yields

$$\begin{aligned}&[\phi(1 - 4u + 3u^2) + \phi_t - \phi_{xx} + (2\xi_x - \phi_u)u(1 - u)^2]\\ &\quad + [\xi_{xx} - 2\phi_{xu} - \xi_t + 3\xi_u u(1 - u)^2]u_x + [\tau_{xx} - \tau_t + 2\xi_x + \tau_u u(1 - u)^2]u_t + [2\xi_{xu}\\ &\quad - \phi_{uu}]u_x^2 + [2\tau_{xu} + 2\xi_u]u_x u_t + \xi_{uu}u_x^3 + \tau_{uu}u_x^2 u_t + 2\tau_x u_{xt} + 2\tau_u u_x u_{xt} = 0.\end{aligned} \tag{1.28}$$

(d) By setting like derivatives of u to zero we obtain a system of nine linear, overdetermined partial differential equations with (ξ, τ, ϕ) as the dependent and (x, t, u)

the independent variables:

$$u_x u_{xt}: \quad \tau_u = 0, \tag{1.29i}$$

$$u_{xt}: \quad \tau_x = 0, \tag{1.29ii}$$

$$u_x^2 u_t: \quad \tau_{uu} = 0, \tag{1.29iii}$$

$$u_x^3: \quad \xi_{uu} = 0, \tag{1.29iv}$$

$$u_x u_t: \quad \tau_{xu} + \xi_u = 0, \tag{1.29v}$$

$$u_x^2: \quad 2\xi_{xu} - \phi_{uu} = 0, \tag{1.29vi}$$

$$u_t: \quad \tau_{xx} - \tau_t + 2\xi_x + \tau_u u(1-u)^2 = 0, \tag{1.29vii}$$

$$u_x: \quad \xi_{xx} - 2\phi_{xu} - \xi_t + 3\xi_u u(1-u)^2 = 0, \tag{1.29viii}$$

$$1: \quad \phi(1 - 4u + 3u^2) + \phi_t - \phi_{xx} + (2\xi_x - \phi_u)u(1-u)^2 = 0. \tag{1.29ix}$$

Step Two

Equations (1.29i) and (1.29ii) tell us that $\tau = 2f(t)$, an arbitrary function of t only. Equation (1.29vii) gives an expression for ξ_x which we may integrate with respect to x, and use the fact that (1.29v) now tells us $\xi_u = 0$ to yield $\xi = \frac{\mathrm{d}f}{\mathrm{d}t}x + g(t)$, where $g(t)$ is another arbitrary function of t. Since $\xi_u = 0$, equation (1.29vi) tells us $\phi_{uu} = 0$ so we write $\phi = h(x,t)u + k(x,t)$. The only equations that are not satisfied now are (1.29viii) and (1.29ix). Equation (1.29viii) yields a single equation, whereas equation (1.29ix) is now a polynomial in u of degree three, and thus gives four equations, since the coefficients of powers of u must be zero. We have the system

$$2h_x + \frac{\mathrm{d}^2 f}{\mathrm{d}t^2}x + \frac{\mathrm{d}g}{\mathrm{d}t} = 0, \tag{1.30i}$$

$$h + \frac{\mathrm{d}f}{\mathrm{d}t} = 0, \tag{1.30ii}$$

$$3k - 2h - 4\frac{\mathrm{d}f}{\mathrm{d}t} = 0, \tag{1.30iii}$$

$$h_t - h_{xx} - 4k + 2\frac{\mathrm{d}f}{\mathrm{d}t} = 0, \tag{1.30iv}$$

$$k + k_t - k_{xx} = 0. \tag{1.30v}$$

Equation (1.30ii) gives us $h_x = 0$ (as well as an expression for h) and hence equating powers of x to zero in (1.30i) yields $\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = \frac{\mathrm{d}g}{\mathrm{d}t} = 0$. Equation (1.30iii) gives an expression for k, which when substituted into (1.30iv) together with our expression for h, simply leaves $\frac{\mathrm{d}f}{\mathrm{d}t} = 0$. Notice then that $h = k = 0$, and since $\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}g}{\mathrm{d}t} = 0$, both ξ and τ are constant. In summary, the general solution of (1.29) is

$$\xi = c_1, \qquad \tau = c_2, \qquad \phi = 0, \tag{1.31}$$

where c_1, c_2 are arbitrary constants.

Step Three

To find the symmetry reduction associated with (1.31) we must solve the invariant surface condition

$$c_1 u_x + c_2 u_t = 0, \tag{1.32}$$

or equivalently

$$\frac{\mathrm{d}x}{c_1} = \frac{\mathrm{d}t}{c_2} = \frac{\mathrm{d}u}{0}.$$

Note that if $c_1 = c_2 = 0$, i.e. $\xi = \tau = \phi = 0$ the invariant surface condition gives us no information. By solving (1.32) we have the following (classical) travelling wave reduction

Reduction 1.2.1.

$$u(x,t) = U(z), \qquad z = c_2 x - c_1 t,$$

where $U(z)$ satisfies

$$\widetilde{\Delta} \equiv \quad c_2^2 U'' + c_1 U' - U(1-U)^2 = 0.$$

We conclude this section with the following three remarks.

Remark 1.2(ii). It is possible to set $c_2 = 0$ (with $c_1 \neq 0$) and $c_1 = 0$ (with $c_2 \neq 0$) in Reduction 1.2.1 to give spatially independent and time independent reductions respectively. However in this thesis, when we are faced with the infinitesimals (1.31), we will assume $c_2 \neq 0$ (and then set $c_2 = 1$ without loss of generality since we can divide (1.32) through by c_2 and rename $c_1 \equiv c_1 c_2$) to give the (strict) travelling wave reduction (whilst implicitly assuming that both these special reductions are possible).

Remark 1.2(iii). We have seen how point transformations are characterised by vector fields of the form (1.9). However point transformations are not the only types of local transformations. For a local transformation in general $(\boldsymbol{\xi}, \boldsymbol{\phi})$ may depend on derivatives of $\boldsymbol{u}$. In particular for a *contact transformation*, $q = 1$ and $(\boldsymbol{\xi}, \boldsymbol{\phi})$ depend on $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(1)})$, and a local transformation which is neither a point transformation nor a contact transformation is called a *Lie-Bäcklund transformation* (or *generalised transformation*). A *nonlocal transformation* is a continuous transformation which is not characterised by such vector fields; for instance it might contain integrals of $\boldsymbol{u}$.

Remark 1.2(iv). Whilst Step Three of the classical method advocates solving the invariant surface conditions by the method of characteristics to give the new dependent and independent symmetry variables, these may only be solved in principle. Also, if the infinitesimals $\boldsymbol{\xi}$ depend on $\boldsymbol{u}$ then it may be the case that the new independent symmetry variables also depend on $\boldsymbol{u}$ and thus give implicit reductions, which are more difficult to interpret. However to find symmetry reductions one is simply solving the system comprising of the original system (1.15) and the invariant surface conditions simultaneously. In particular (see Bluman and Kumei [1989]) one could use the invariant surface conditions to remove all derivatives of x_j say (we can assume $\xi_j \neq 0$ without loss of generality), from (1.15) to create a system of m partial differential equations in only $p-1$ independent variables (since x_j only appears as a parameter). If this reduced system may now be solved, the invariant surface conditions then give the functional dependence of x_j. This approach thus gives an alternative to the method of characteristics approach, so may combat the difficulties described above. It is particularly useful for systems with $p = 2$ independent variables since this procedure then yields a system of ordinary differential equations.

1.3 Nonclassical Method

Perhaps the easiest way to demonstrate the nonclassical method† is to point out a slight anomaly in the classical method. In Step One (d) of the classical method one equates the coefficients of like derivatives of $\boldsymbol{u}$ in

$$\mathrm{pr}^{(n)}\mathbf{v}(\boldsymbol{\Delta}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(n)}))|_{\boldsymbol{\Delta}=0} = 0, \tag{1.33}$$

(cf. (1.28)) to zero (i.e. after removing the dependencies of $\boldsymbol{u}$ imposed by the given system) since one assumes that the derivatives of $\boldsymbol{u}$ left are independent. Later, in Step Three, one must solve the invariant surface conditions which are of the form

$$\psi_s \equiv \sum_{i=1}^{p} \xi_i(\boldsymbol{x}, \boldsymbol{u}) u_{s,x_i} - \phi_s(\boldsymbol{x}, \boldsymbol{u}) = 0, \qquad s = 1, 2, \ldots, q. \tag{1.17}$$

Clearly the derivatives u_{s,x_i} for $s = 1, 2, \ldots, q$ *are* related, by the invariant surface conditions so our assumption that they are independent is naive. Bluman and Cole [1969] realised that to find symmetry reductions one is only interested in solutions that satisfy the invariant surface conditions. By including the relations that the invariant surface conditions impose on the derivatives in (1.33), and only then setting the coefficients of like derivatives of $\boldsymbol{u}$ to zero, one obtains the nonclassical determining equations.

Example 1.3.1. Consider again equation (1.1). After Step One (c) of the classical method, which has seen the removal of u_{xx} using (1.27) we have

$$\begin{aligned}
&[\phi(1-4u+3u^2) + \phi_t - \phi_{xx} + (2\xi_x - \phi_u)u(1-u)^2] \\
&\quad + [\xi_{xx} - 2\phi_{xu} - \xi_t + 3\xi_u u(1-u)^2]u_x + [\tau_{xx} - \tau_t + 2\xi_x + \tau_u u(1-u)^2]u_t + [2\xi_{xu} \\
&\quad - \phi_{uu}]u_x^2 + [2\tau_{xu} + 2\xi_u]u_x u_t + \xi_{uu}u_x^3 + \tau_{uu}u_x^2 u_t + 2\tau_x u_{xt} + 2\tau_u u_x u_{xt} = 0.
\end{aligned} \tag{1.28}$$

† The nonclassical method is sometimes referred to as the "method of partial symmetries of the first type" (Vorob'ev [1991]) or the "method of conditional symmetries" (Levi and Winternitz [1989])

There are p cases to consider (see later) and when $p = 2$ these amount to τ being zero or non-zero. Firstly we assume $\tau \neq 0$, and set $\tau = 1$ without loss of generality (since we may divide the invariant surface condition $\xi u_x + \tau u_t - \phi = 0$ by τ and rename ξ, ϕ accordingly). Hence the invariant surface condition gives

$$u_t = \phi - \xi u_x, \tag{1.34}$$

with which we replace occurrences of u_t (and derivatives of u_t if necessary) in (1.28) to yield

$$\begin{aligned}[]&[\phi(1-4u+3u^2) + \phi_t - \phi_{xx} + (2\xi_x - \phi_u)u(1-u)^2 + 2\xi_x\phi] + [\xi_{xx} - 2\phi_{xu} - \xi_t \\ &\quad + 3\xi_u u(1-u)^2 - 2\xi\xi_x + 2\xi_u\phi]u_x + [2\xi_{xu} - \phi_{uu} - 2\xi\xi_u]u_x^2 + \xi_{uu}u_x^3 = 0. \end{aligned} \tag{1.35}$$

Now setting the coefficients of different powers of u_x to zero gives us the nonclassical determining equations in the generic ($\tau \neq 0$) case

$$\xi_{uu} = 0, \tag{1.36i}$$

$$2\xi_{xu} - \phi_{uu} - 2\xi\xi_u = 0, \tag{1.36ii}$$

$$\xi_{xx} - 2\phi_{xu} - \xi_t + 3\xi_u u(1-u)^2 - 2\xi\xi_x + 2\xi_u\phi = 0, \tag{1.36iii}$$

$$\phi(1-4u+3u^2) + \phi_t - \phi_{xx} + (2\xi_x - \phi_u)u(1-u)^2 + 2\xi_x\phi = 0. \tag{1.36iv}$$

Secondly, if $\tau = 0$, we may set $\xi = 1$ without loss of generality so that the invariant surface condition yields

$$u_x = \phi. \tag{1.37}$$

Replacing occurrences of u_x in (1.28) with (1.37) yields a single equation for $\phi(x, t, u)$

$$\phi(1-3u)(1-u) + \phi_t - \phi_{xx} - \phi_u u(1-u)^2 - 2\phi\phi_{xu} - \phi^2\phi_{uu} = 0. \tag{1.38}$$

We leave the solution of these determining equations for later in the section.

A slightly different interpretation of the nonclassical method is given by Levi and Winternitz [1989] and Olver and Rosenau [1986], who append the invariant surface conditions to the system (1.15) under consideration and apply the classical method to this enlarged system. Crucially it can be shown that (see for instance Clarkson and Mansfield [1994c])

$$\mathrm{pr}^{(1)}\mathbf{v}(\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(1)})) \equiv 0 \qquad \text{whenever} \qquad \boldsymbol{\psi} = 0,$$

i.e. this condition is identically satisfied for all solutions of the invariant surface conditions. (Further one can show that the relevant prolongation of any derivative of $\boldsymbol{\psi}$ also vanishes on the solution set of $\boldsymbol{\psi}$.) Hence for the nonclassical method approached this way, one is really requiring

$$\mathrm{pr}^{(n)}\mathbf{v}(\boldsymbol{\Delta}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(n)}))|_{\boldsymbol{\Delta}=0, \boldsymbol{\psi}=0} = 0, \tag{1.39}$$

thus this approach is equivalent to that of Bluman and Cole [1969] outlined above. Indeed (1.39) gives a more precise statement of the nonclassical method with regards to how one generates the determining equations.

1.3.1 The Clarkson-Mansfield algorithm for finding nonclassical determining equations I.

Since one must now not only choose derivatives of (1.15) to substitute back for (cf. Step One (b) of the classical method), but also use the invariant surface conditions, care must be taken in the process. As an example, consider the following due to Clarkson and Mansfield [1994c].

Example 1.3.2. Consider the equation

$$\Delta \equiv \quad u_{tt} - u_{xx} = 0,$$

in the generic ($\tau = 1$) case of the nonclassical method. Applying the second prolongation yields

$$\begin{aligned} \phi^{[2,2]} - \phi^{[1,1]} \equiv \quad & \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 \\ & + (\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_x u_{xx} - [\phi_{tt} + 2\phi_{tu}u_t - \xi_{tt}u_x + \phi_{uu}u_t^2 \\ & - 2\xi_{tu}u_t u_x - \xi_{uu}u_t^2 u_x + \phi_u u_{tt} - 2\xi_t u_{xt} - \xi_u u_x u_{tt} - 2\xi_u u_t u_{xt}]. \quad (1.40) \end{aligned}$$

Now recall that the invariant surface condition is

$$\psi \equiv \quad \xi u_x + u_t - \phi = 0.$$

If one tries to remove t derivatives using the invariant surface condition then use (1.40) to remove u_{xx} terms, u_{tt} terms are introduced back into the equation. Eliminating the u_{tt} term (with the invariant surface condition) introduces a u_{xt} term, whose removal again introduces a u_{xx} term. Clearly we have the beginnings of an infinite loop, and a slightly different choice of derivatives for substitution is required. However the presence of this phenomena does not bode well for symbolic manipulation packages. In fact it prompted Clarkson and Mansfield [1994c] to adapt the way the nonclassical method could be applied again, to make it more amenable to computer calculation.

For our system (1.15), with p independent variables there are p cases to consider: for $1 \leq k \leq p$ we successively set $\xi_k = 1$ and $\xi_{k+1} = \ldots = \xi_p = 0$. The algorithm in the kth case reads

Step One:

Eliminate all the differential consequences of $u_{s,k}$ for $s = 1, 2, \ldots, q$ from (1.15) using the invariant surface conditions. Note that since $\xi_k = 1$ the invariant surface conditions may be written in the form

$$u_{s,k} = \phi_s - \sum_{j=1}^{k-1} \xi_j u_{s,j}, \qquad \text{for} \quad s = 1, 2, \ldots, q.$$

Step Two:

Apply the classical method to this new system (in which there are really $p-1$ independent variables since x_k appears only as a parameter).

Example 1.3.3. To see this alternative algorithm of Clarkson and Mansfield [1994c] put into practice, consider again our prototypic equation (1.1)

$$\Delta \equiv u_t - u_{xx} + u(1-u)^2 = 0,$$

together with the invariant surface condition given by

$$\psi \equiv \quad \xi u_x + \tau u_t - \phi = 0.$$

For the sake of comparison with equations (1.38) and (1.36) respectively, we only generate the determining equations using this algorithm. After this all the different view points coincide – we must solve the determining equations, then solve the invariant surface conditions.

<u>Case One:</u> $\xi = 1$, $\tau = 0$.

Step One:

Remove u_{xx} from (1.1) using $u_x = \phi$, i.e.

$$\widehat{\Delta} \equiv \quad u_t - [\phi_x + \phi_u \phi] + u(1-u)^2 = 0. \tag{1.41}$$

Step Two:

Applying the classical method, one first takes the first prolongation of (1.41):

$$\begin{aligned} \mathrm{pr}^{(1)}\mathbf{v}(\widehat{\Delta}) =& \phi^{[2]} - [\phi_{xx} + 2\phi\phi_{xu} + \phi_x\phi_u + \phi\phi_u^2 + \phi^2\phi_{uu}] + \phi(1-3u)(1-u), \\ =& \phi_t + \phi_u u_t - [\phi_{xx} + 2\phi\phi_{xu} + \phi_x\phi_u + \phi\phi_u^2 + \phi^2\phi_{uu}] + \phi(1-3u)(1-u). \end{aligned} \tag{1.42}$$

The only derivative term we can possibly choose from (1.41) is u_t, and solving (1.41) accordingly yields

$$u_t = \phi_x + \phi_u\phi - u(1-u)^2. \tag{1.43}$$

Removing the u_t derivative from (1.42) using (1.43) then yields

$$\phi_t - u(1-u)^2\phi_u - \phi_{xx} - 2\phi\phi_{xu} - \phi^2\phi_{uu} + \phi(1-3u)(1-u) = 0,$$

which is the same as (1.38).

<u>*Case Two:*</u> $\tau = 1$.

Step One:

Remove u_t from (1.1) using $u_t = \phi - \xi u_x$ i.e.

$$\widehat{\Delta} \equiv \quad [\phi - \xi u_x] - u_{xx} + u(1-u)^2 = 0. \tag{1.44}$$

Step Two:

Firstly we take the second prolongation of (1.44),

$$\begin{aligned}
\mathrm{pr}^{(2)}\mathbf{v}(\widehat{\Delta}) =& [\xi\phi_x + \phi_t + \phi\phi_u - \xi\xi_x u_x - \xi_t u_x - \phi\xi_u - \xi\phi^{[1]}] - \phi^{[1,1]} + \phi(1-3u)(1-u), \\
=& [\xi\phi_x + \phi_t + \phi\phi_u - \xi\xi_x u_x - \xi_t u_x - \phi\xi_u u_x - \xi(\phi_x + \phi_u u_x - \xi_x u_x - \xi_u u_x^2)] \\
& - [\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 + (\phi_u - 2\xi_x)u_{xx} \\
& - 3\xi_u u_x u_{xx}] + \phi(1-3u)(1-u).
\end{aligned} \tag{1.45}$$

We again have only one acceptable choice of derivative term: from (1.44) we choose u_{xx} to give

$$u_{xx} = \phi - \xi u_x + u(1-u)^2, \tag{1.46}$$

and replacing occurrences of u_{xx} in (1.45) using (1.46) yields

$$\begin{aligned}
&[\phi(1-4u+3u^2) + \phi_t - \phi_{xx} + (2\xi_x - \phi_u)u(1-u)^2 + 2\xi_x\phi] + [\xi_{xx} - 2\phi_{xu} - \xi_t \\
&\quad + 3\xi_u u(1-u)^2 - 2\xi\xi_x + 2\xi_u\phi]u_x + [2\xi_{xu} - \phi_{uu} - 2\xi\xi_u]u_x^2 + \xi_{uu}u_x^3 = 0,
\end{aligned} \tag{1.47}$$

which is the same as (1.35) and hence the determining equations are the same as (1.36).

1.3.2 Nonclassical symmetry reductions.

Once the determining equations of the nonclassical method have been generated, as implied in Step Two of the algorithm due to Clarkson and Mansfield above, we solve them and find our nonclassical symmetry reductions in exactly the same way as in the classical method. (The symmetry variables are still of the form (1.20), (1.19) and the symmetry system is still of the form (1.21).) However the main difference between the methods is the fact that the determining equations in the nonclassical method are now *nonlinear*. Another, less significant observation is that there are fewer equations to solve.

Whilst they are nonlinear, they are in general overdetermined, which helps greatly in their solution. (The main exception is the "$\tau = 0$" case, which may only provide a single equation (cf. (1.38)) which is often more complex than the original equation. However one may be able to find simple e.g. polynomial solutions of this equation, which often lead to more interesting reductions and exact solutions.) The MAPLE package **`diffgrob2`** of Mansfield [1993] is now particularly helpful, if not essential certainly when considering large systems of determining equations. For nonlinear systems it is often more useful to use `diffgrob2` interactively, whilst following the algorithms given in its accompanying manual (which are described in §1.7).

For our simple, second order equation (1.1) however the solution of the determining equations can still be carried out manually. First we proceed with the solution of (1.36). Equations (1.36i) and (1.36ii) give

$$\begin{aligned}\xi &= f(x,t)u + g(x,t),\\ \phi &= -\tfrac{1}{3}f^2u^3 + (f_x - fg)u^2 + h(x,t)u + k(x,t),\end{aligned}$$

where f, g, h and k are arbitrary functions of (x,t). Substituting these expressions for ξ, ϕ into (1.36iii) gives a polynomial in u of degree three, whose coefficients must be zero. This gives a further four equations

$$f(9 - 2f^2) = 0, \tag{1.48i}$$

$$3f + gf^2 = 0, \tag{1.48ii}$$

$$2fg_x + 3f + 2fh = 0, \tag{1.48iii}$$

$$g_{xx} - 2h_x - g_t - 2gg_x + 2fk = 0. \tag{1.48iv}$$

The calculation must now be split into two, when $f = 0$ and when $9 - 2f^2 = 0$.

(•) $f = 0$. Substituting $\xi = g(x,t)$, $\phi = h(x,t)u + k(x,t)$ into (1.36iv) yields four more equations after taking coefficients of powers of u to be zero,

$$h + g_x = 0, \tag{1.49i}$$

$$3k - 2h - 4g_x = 0, \tag{1.49ii}$$

$$h - 4k + 2g_x + 2g_xh = 0, \tag{1.49iii}$$

$$k + g_t - g_{xx} + 2g_xk = 0. \tag{1.49iv}$$

Equations (1.49i) and (1.49ii) give h and then k as a multiple of g_x, so (1.49iii) reads $g_x(1 + 3g_x) = 0$. If $g_x = -\frac{1}{3}$, equation (1.49iv) gives a contradiction, whilst if $g_x = 0$ we find $g_t = 0$ and all the equations are satisfied. We have found the classical infinitesimals $\xi = c_1$, $\phi = 0$.

(••) $9 - 2f^2 = 0$. From (1.48ii) we have $g = -3/f$, from (1.48iii) we then have $h = -\frac{3}{2}$ and finally from (1.48iv) we find $k = 0$. We find that equation (1.36iv) is satisfied without placing further conditions on ξ, ϕ, so summarising we have

$$\xi = \pm\tfrac{1}{2}\sqrt{2}\,(3u - 2), \qquad \phi = -\tfrac{3}{2}u(1-u)^2. \tag{1.50}$$

We have already seen what the classical travelling wave reduction is like (cf. reduction 1.2.1), so we concentrate on the reduction that the infinitesimals (1.50) will give. Solving the invariant surface condition by the method of characteristics gives the new symmetry variables as

$$U = 2\ln u - 2\ln(u-1) + \frac{1}{u-1} \mp \tfrac{3}{2}\sqrt{2}\,x, \qquad z = \ln u - \ln(u-1) - \frac{1}{u-1} + \tfrac{3}{2}t,$$

from which we have

$$u(x,t) = \frac{6 + 2U(z) \pm 3\sqrt{2}\,x + 6t - 4z}{2U(z) \pm 3\sqrt{2}\,x + 6t - 4z}, \qquad z(x,t) = \ln u - \ln(u-1) - \frac{1}{u-1} + \tfrac{3}{2}t. \tag{1.51i,ii}$$

We find u_t by differentiating (1.51i) with respect to t, and then solving the expression found (algebraically) for u_t. Similarly for u_x, and u_{xx} is found by differentiating our expression for u_x by x and then eliminating new occurrences of u_x. Having found the necessary derivatives we substitute them into (1.1) and by replacing occurrences of t using (1.51i) we yield an ordinary differential equation. We have found the following (strictly nonclassical) reduction,

Reduction 1.3.1. $u(x,t)$ is given by (1.51) where $U(z)$ satisfies

$$9U'' - (U')^3 + 3U' + 2 = 0. \tag{1.52}$$

This has two simple solutions. If $U' = 2$ then $U(z) = 2z + a_1$, where a_1 is an arbitrary constant. This leads to the special solution

$$u(x,t) = \frac{6 \pm 3\sqrt{2}\,x + 6t + 2a_1}{\pm 3\sqrt{2}\,x + 6t + 2a_1}, \tag{1.53}$$

which is in fact a special case of the travelling wave reduction 1.2.1. If $U' = -1$ then $U(z) = -z + 3a_2$, where a_2 is an arbitrary constant. Then (1.51) may be solved to give the following expression for u,

$$6a_2 - 3t \pm 3\sqrt{2}\,x - 6\ln u + 6\ln(u-1) = 0,$$

which upon exponentiating and rearranging yields

$$u(x,t) = \frac{1}{1 - a_3 \exp\{\frac{1}{2}(\mp\sqrt{2}\,x + t)\}}, \tag{1.54}$$

where $a_3 = \mathrm{e}^{-a_2}$. This is another special case of the travelling wave reduction 1.2.1. In the general case we can easily integrate equation (1.52) once to give

$$\ln(U' - 2) - \ln(U' + 1) + \frac{3}{U' + 1} = z + c_0,$$

where c_0 is an arbitrary constant, but further integration is difficult. However even if it were possible and an explicit expression for $U(z)$ could be found, we would still only have an implicit expression from (1.51) for $u(x,t)$ which it may not be possible to solve for an explicit expression. We therefore turn to Remark 1.2(iv) for an alternative procedure. The invariant surface condition reads

$$\pm\tfrac{1}{2}\sqrt{2}\,(3u-2)u_x + u_t + \tfrac{3}{2}u(1-u)^2 = 0, \tag{1.55}$$

which may be used to eliminate u_t in (1.1) thus

$$u_{xx} \pm \tfrac{1}{2}\sqrt{2}\,(3u-2)u_x + \tfrac{1}{2}u(1-u)^2 = 0.$$

This is a member of the Riccati-chain (cf. Ames [1968], Nucci [1992]) which may be linearised by the transformation $u = \pm\sqrt{2}\,(\ln\eta)_x$ to yield

$$2\eta_{xxx} \mp 2\sqrt{2}\,\eta_{xx} + \eta_x = 0.$$

This equation is easily solved, and requiring that (1.55) also holds yields the following exact solution of (1.1)

$$u(x,t) = \frac{c_2(\pm x + \sqrt{2}\,t + \sqrt{2}\,) + c_3}{c_1\exp\{\tfrac{1}{2}(\mp\sqrt{2}\,x - t)\} + c_2(\pm x + \sqrt{2}\,t) + c_3}, \tag{1.56}$$

where c_1, c_2 and c_3 are arbitrary constants. This exact solution cannot be found from the classical travelling wave reduction 1.2.1 since it cannot be written in the form $u(x,t) = U(z)$ for a single independent variable z (we would require $u(x,t) = U(z_1, z_2)$ for $z_1 = \pm x + \sqrt{2}\,t$ and $z_2 = \mp\sqrt{2}\,x - t$).

To justify the comments made above, that in the $\tau = 0$ case simple solutions of (1.38) lead to more interesting solutions of (1.1), we look for solutions of (1.38) in the form

$$\phi = au^2 + bu + c, \tag{1.57}$$

for a, b and c arbitrary constants. Note that it is easy to show that no solutions that are polynomial in u of degree three or above exist for (1.38), though it may well be the case that other solutions to (1.38) do exist. Also we note that for an autonomous equation like (1.1) we will always find the (rather trivial) spatially independent wave reduction, i.e. $u_x = 0$, by realising that $\phi = 0$ is a solution of (1.38). Substituting (1.57) into (1.38) yields, after equating coefficients of powers of u to zero, the following (algebraic) equations

$$a(2a^2 - 1) = 0, \tag{1.58i}$$

$$b(2a^2 - 1) = 0, \tag{1.58ii}$$

$$4a^2c - 3c + 2ab^2 + 2b + a = 0, \tag{1.58iii}$$

$$c(ab + 1) = 0, \tag{1.58iv}$$

$$c(2ac - 1) = 0. \tag{1.58v}$$

From (1.58i) we see that either $a = 0$ or $2a^2 - 1 = 0$. If $a = 0$ then (1.58ii) implies $b = 0$ and then $c = 0$ also (which gives the spatially independent wave reduction mentioned above). If $2a^2 - 1 = 0$ then from (1.58v) we have that either $c = 0$ or $2ac - 1 = 0$. If $c = 0$ then (1.58iii) gives $b = -1/2a$, whilst if $2ac - 1 = 0$, then (1.58iv) gives $b = -1/a$ and we find (1.58iii) is identically satisfied. Thus we have two separate solutions for ϕ.

(i) $\phi = \pm\frac{1}{2}\sqrt{2}\,u(u-1)$. Solving the invariant surface condition (1.37) and substituting into (1.1) gives the following reduction

Reduction 1.3.2.

$$u(x,t) = \frac{1}{1 - U(z)\exp(\pm\frac{1}{2}\sqrt{2}\,x)}, \qquad z = t,$$

where $U(z)$ satisfies $2U' - U = 0$. This leads to the exact solution

$$u(x,t) = \frac{1}{1 - c_4 \exp\{\frac{1}{2}(\pm\sqrt{2}\,x + t)\}},$$

where c_4 is an arbitrary constant, which is equivalent to (1.54).

(ii) $\phi = \pm\frac{1}{2}\sqrt{2}\,(u-1)^2$. For this solution of ϕ we have the following reduction

Reduction 1.3.3.

$$u(x,t) = \frac{\pm\sqrt{2}\,x + U(z) - 2}{\pm\sqrt{2}\,x + U(z)}, \qquad z = t,$$

where $U(z)$ satisfies $U' = -2$. This gives the exact solution

$$u(x,t) = \frac{\pm\sqrt{2}\,x - 2t + c_5 - 2}{\pm\sqrt{2}\,x - 2t + c_5},$$

where c_5 is an arbitrary constant, which is equivalent to (1.53).

Both of these exact solutions are special cases of both the $\tau \neq 0$ exact solution (1.56) and the classical travelling wave reduction 1.2.1. However, as we will see in later chapters of this thesis, in general the $\tau = 0$ reductions are distinct from both classical and $\tau \neq 0$ reductions. More generally, each of the p cases that one faces in the nonclassical method are in general distinct from one another.

We conclude this section, which has seen a presentation of the nonclassical method with examples, with three remarks.

Remark 1.3(i). That the nonclassical method is more general than the classical method is plain to see in our example of (1.1). To see that this is always so recall that the classical method may be described as finding solutions of (1.15) such that

$$\mathrm{pr}^{(n)}\mathbf{v}(\boldsymbol{\Delta}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(n)}))|_{\boldsymbol{\Delta}=0} = 0, \tag{1.33}$$

and the nonclassical method such that

$$\mathrm{pr}^{(n)}\mathbf{v}(\boldsymbol{\Delta}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(n)}))|_{\boldsymbol{\Delta}=0,\boldsymbol{\psi}=0}=0, \tag{1.39}$$

for invariant surface conditions $\boldsymbol{\psi}$. If (1.39) holds then so does (1.33) so the nonclassical method will always find the solutions that the classical method finds. However, if (1.33) holds, (1.39) does not necessarily hold so there *may* be times when the classical method will not find solutions that the nonclassical method finds. We use the word may, because there are well known examples of when the classical and nonclassical methods yield the same results. The celebrated Korteweg-de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1.59}$$

is one such example.

Remark 1.3(ii). The transformation (1.8) with nonclassical infinitesimals no longer maps solutions to solutions. In the notation of Olver and Rosenau [1987] such transformations form *weak symmetry groups* (see §1.5.2).

Remark 1.3(iii). One of the conditions for the classical method to be applied successfully to a system (1.15) (cf. Remark 1.2(i)) is that the system is *locally solvable* (see Olver [1993] for a definition). We simply state here that a system may not be locally solvable if it has compatibility conditions (see §1.7) and that by including the invariant surface conditions in the system (as one does in the nonclassical method) in general the new system will have compatibility conditions. This is another reason why Clarkson and Mansfield [1994c] adapted the nonclassical method algorithm, and it deals with such difficulties. In fact a truer statement of their algorithm (which we will discuss in §1.7.6 after some theory is introduced) copes with systems that are not locally solvable, by incorporating the compatibility conditions into the calculation.

1.4 Direct Method

The direct method of Clarkson and Kruskal [1989] evolved from their desire to systematically find known reductions of the Boussinesq equation

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0, \tag{1.60}$$

that could not be found using the classical method of Lie. In particular many authors (Nishitani and Tajiri [1982], Olver and Rosenau [1987], Quispel, Nijhoff and Capel [1982], Rosenau and Schwarzmeier [1986]) noted that (1.60) possesses the *accelerating wave* reduction

$$u(x,t) = U(z) - 4c_1^2t^2, \qquad z = x + c_1t^2,$$

where $U(z)$ satisfies

$$U''' + UU' + 2c_1 U = 8c_1^2 z + c_2,$$

where c_1, c_2 are arbitrary constants, but only by apparently *ad hoc* techniques. The classical method finds two canonical reductions, whereas the direct method provided six, including the classical reductions and the accelerating wave reduction above (see Chapter Two).

The direct method for an equation like (1.60), in one dependent and two independent variables, can be stated quite simply: seek a solution of such a partial differential equation in the form

$$u(x,t) = F(x,t,U(z)), \qquad (F_U \neq 0), \tag{1.61}$$

where $z = z(x,t)$, and require that $U(z)$ satisfies an ordinary differential equation. This imposes conditions on $F(x,t,U)$ and $z(x,t)$ and their derivatives in the form of an overdetermined system of equations, whose solution yields the desired reduction. In practice there are a number of subtleties to the method which make it manageable, which are demonstrated best with an example.

Example 1.4.1. Consider equation (1.1), and assume its solution has the form (1.61). On substituting into (1.1) we have

$$F_U z_x^2 U'' + F_{UU} z_x^2 (U')^2 + [2F_{xU} z_x + F_U z_{xx} - F_U z_t]U' + [F_{xx} - F_t + F(1-F)^2] = 0, \tag{1.62}$$

which we require to be an ordinary differential equation. For (1.62) to be an ordinary differential equation, the ratios of coefficients of different derivatives of U must be functions of z, U. As with the nonclassical method there are two cases to consider, namely $z_x \not\equiv 0$ and $z_x \equiv 0$. Assume first that $z_x \not\equiv 0$, and by (1.61) we have $F_U \neq 0$ (so that the ansatz depends explicitly on U). The ratio of coefficients of U'' and $(U')^2$ is

$$\frac{F_{UU}}{F_U} = \frac{\Gamma_{UU}(z,U)}{\Gamma_U(z,U)}, \tag{1.63}$$

where the right hand side simply denotes our intention for ratios to be functions of z, U; $\Gamma(z,U)$ is to be determined. Integrating (1.63) twice with respect to U yields

$$F(x,t,U(z)) = A(x,t)\Gamma(z,U) + B(x,t),$$

for some arbitrary functions of integration A, B. In order to fix the ordinary differential equation that we will generate, we set $\Gamma(z,U) \equiv U(z)$ without loss of generality, since we could transform back via $U \to \Gamma^{-1}(z,U)$. This is simply the most convenient form, but if we so desired we could instead let $\Gamma(z,U) \equiv U^2(z)$ or any other function of $U(z)$. With F linear in U, (1.62) now has the form

$$\begin{aligned} &Az_x^2 U'' + [Az_{xx} + 2A_x z_x - Az_t]U' + A^3U^3 + [3BA^2 - 2A^2]U^2 \\ &\quad + [A_{xx} - A_t + A - 4AB + 3AB^2]U + [B_{xx} - B_t + B(1-B)^2] = 0. \end{aligned} \tag{1.64}$$

We use the coefficient of U'', which we know to be non-zero, to be the *normalising coefficient*, so that all other coefficients should look like $Az_x^2\Gamma_i(z)$ for $\Gamma_i(z)$ to be determined. Hence the determining equations for the direct method in this case are

$$Az_x^2\Gamma_1(z) = Az_{xx} + 2A_x z_x - Az_t, \tag{1.65i}$$

$$Az_x^2\Gamma_2(z) = A^3, \tag{1.65ii}$$

$$Az_x^2\Gamma_3(z) = 3BA^2 - 2A^2, \tag{1.65iii}$$

$$Az_x^2\Gamma_4(z) = A_{xx} - A_t + A - 4AB + 3AB^2, \tag{1.65iv}$$

$$Az_x^2\Gamma_5(z) = B_{xx} - B_t + B(1-B)^2, \tag{1.65v}$$

where $\Gamma_1(z), \ldots, \Gamma_5(z)$ are to be determined. Before we proceed further with the calculation, we make the following remark, which describes the freedoms in the method, which make it manageable.

Remark 1.4(i). If the ansatz (1.61) may be simplified to a linear one, namely

$$u(x,t) = A(x,t)U(z) + B(x,t), \tag{1.66}$$

there are three freedoms which can and should be applied which assist greatly in keeping the calculations manageable. These may each be applied once without loss of generality during the calculation, though once the reduction has been finalised it *may* be possible to apply them again *a posteriori* without loss of generality, which then constitutes a tidying up process. The freedoms are

Freedom (a). If $B(x,t)$ has the form $B(x,t) = B_0(x,t) + A(x,t)\Omega(z)$, then we may set $\Omega(z) = 0$ (by translating $U(z) \to U(z) - \Omega(z)$).

Freedom (b). If $A(x,t)$ has the form $A(x,t) = A_0(x,t)\Omega(z)$, then we may set $\Omega(z) = 1$ (by scaling $U(z) \to U(z)/\Omega(z)$).

Freedom (c). If $z(x,t)$ is determined by an equation of the form $\Omega(z) = z_0(x,t)$, where $\Omega(z)$ is any invertible function, then we may take $\Omega(z) = z$ (by substituting $z \to \Omega^{-1}(z)$).

By rearranging (1.65ii) and taking square roots we have

$$A = z_x\Gamma_2^{1/2}(z),$$

so by freedom (b) we may set $\Gamma_2^{1/2}(z) = 1$ and therefore $A(x,t) = z_x$. Now (1.65iii) yields, after rearranging

$$B = \tfrac{1}{3}z_x\Gamma_3(z) + \tfrac{2}{3},$$

and we may apply freedom (a) to set $\Gamma_3(z) = 0$, hence $B(x,t) = \frac{2}{3}$. Equation (1.65v) yields

$$z_x\Gamma_5^{1/3}(z) = \frac{2^{1/3}}{3},$$

which we may integrate with respect to x to yield

$$\Gamma_6(z) = x + \sigma(t),$$

where $\Gamma_6'(z) = 3(\Gamma_5(z)/2)^{1/3}$. By freedom (c) we may set $\Gamma_6(z) = z$, hence $z = x + \sigma(t)$, and $\Gamma_5(z) = 2/27$. Equation (1.65i) now reads

$$\Gamma_1(z) = -\frac{d\sigma}{dt}, \qquad (1.67)$$

and since we know z is linear in x, $\Gamma_1(z)$ must be constant: let $\Gamma_1(z) = c_1$. Integrating (1.67) yields $\sigma(t) = -c_1 t + c_2$. The only remaining equation, (1.65iv) now simply gives $\Gamma_4(z) = -\frac{1}{3}$. To summarise we have the reduction

Reduction 1.4.1.

$$u(x,t) = U(z) + \tfrac{2}{3}, \qquad z(x,t) = x - c_1 t + c_2,$$

where $U(z)$ satisfies

$$U'' + c_1 U' + U^3 - \tfrac{1}{3}U + \tfrac{2}{27} = 0.$$

If we so desired we could do some *a posteriori* tidying up, and we would realise this reduction is really no different to the (classical) travelling wave reduction 1.2.1.

In the other canonical case, when $z_x \equiv 0$, we may set $z = t$ without loss of generality (essentially by applying freedom (c)). Equation (1.62) now has the form

$$-F_U U' + [F_{xx} - F_t + F(1-F)^2] = 0. \qquad (1.68)$$

As this equation gives only one (complicated) ratio, it is not possible to deduce the form of F. In such cases it is usual to assume F has the linear form (1.66), where $z = t$. Freedoms (a) and (b) still apply, and freedom (c) has already been used. Substituting the linear ansatz into (1.68) yields the determining equations (taking the coefficient of U' to be the normalising coefficient)

$$A\Gamma_1(z) = A^3, \qquad (1.69\text{i})$$

$$A\Gamma_2(z) = 3BA^2 - 2A^2, \qquad (1.69\text{ii})$$

$$A\Gamma_3(z) = A_{xx} - A_t + A - 4AB + 3AB^2, \qquad (1.69\text{iii})$$

$$A\Gamma_4(z) = B_{xx} - B_t + B(1-B)^2, \qquad (1.69\text{iv})$$

where $\Gamma_1(z), \ldots, \Gamma_4(z)$ are to be determined. Equation (1.69i) gives $A = 1$ after using freedom (b), then from (1.69ii) we are able to find $B = 0$ after using freedom (a). Equations (1.69iii) and (1.69iv) now simply define $\Gamma_3(z) = 1$, $\Gamma_4(z) = 0$. We have found the (classical) spatially independent wave reduction, which is a special case of reduction 1.2.1.

The direct method was first applied to an equation with more than two independent variables by Clarkson and Winternitz [1991] in their study of the Kadomtsev-Petviashvili (KP) equation (Kadomtsev and Petviashvili [1970])

$$(u_t + uu_x + u_{xxx})_x \pm u_{yy} = 0.$$

They considered both reductions to partial differential equations (and then to ordinary differential equations) and reductions directly to ordinary differential equations, though commented that the two approaches were equivalent. Applying the direct method to a partial differential equation with p independent variables may be summarised as looking for solutions of the form

$$u(\boldsymbol{x}) = F(\boldsymbol{x}, U(\boldsymbol{z})),$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_p)$ and $\boldsymbol{z} = (z_1(\boldsymbol{x}), z_2(\boldsymbol{x}), \ldots, z_r(\boldsymbol{x}))$ for $1 \leq r \leq p-1$, and requiring that U satisfies a partial differential equation (or ordinary differential equation) in r independent variables

$$\widetilde{\Delta}(\boldsymbol{z}, U, U^{(n)}) = 0.$$

The value of r is decided *a priori*, though of course one may apply the method $p-1$ times to obtain all possible reductions.

The direct method has also been applied successfully to systems of equations (cf. (1.15) for $m > 1$) by many authors (e.g. Clarkson [1992], Clarkson and Hood [1993,1994], Hood [1993], Lou [1992] and Lou and Ruan [1993]) who have considered ansätze of the form

$$u_i(\boldsymbol{x}) = F_i(\boldsymbol{x}, U_i(\boldsymbol{z})), \qquad i = 1, 2, \ldots, q.$$

It may have been the case for the systems considered that these ansätze were sufficient, but to be sure of finding all reductions one should use

$$u_i(\boldsymbol{x}) = F_i(\boldsymbol{x}, \boldsymbol{U}(\boldsymbol{z})), \qquad i = 1, 2, \ldots, q,$$

for $\boldsymbol{U} = (U_1, U_2, \ldots, U_q)$. Examples of where this is necessary are given in Chapters Five and Six. The investigations of these chapters suggest that the application of the direct method to systems of equations is not as obvious as one might expect.

We note that, unlike for the classical and nonclassical methods, it is not clear how to use the direct method on equations that contain arbitrary functions of the dependent variables.

1.5 Extensions and permutations of symmetry methods and their comparison

1.5.1 Potential symmetries.

In §1.2 we considered only point transformations, i.e. those that depended on the dependent and independent variables only, and hence found *point symmetries* of a system. In Remark 1.2(iii) we suggested that other types of transformation exist, and in particular we mentioned nonlocal transformations (which give rise to *nonlocal symmetries*).

Whilst there have been some heuristic approaches to finding nonlocal symmetries (cf. Akhatov, Gazizov and Ibragimov [1991], Kapcov [1982], Konopelchenko and Mokhnachev [1980], Pukhnachev [1987]), other algorithmic approaches have also been proposed. Krasil'shchik and Vinogradov [1984] (see also Vinogradov and Krasil'shchik [1984], Krasil'shchik and Vinogradov [1989]) provided a framework to find nonlocal symmetries, but were unable to exhibit any non-trivial examples. However Bluman and Kumei [1987] found nonlocal symmetries, and used them to find new solutions of the wave equation

$$c^2(x)u_{xx} - u_{tt} = 0.$$

Soon after this Bluman, Kumei and Reid [1988] gave an algorithmic method for finding such nonlocal symmetries which had similarities with the work of Krasil'shchik and Vinogradov. The nonlocal symmetries they found were later given the name *potential symmetries* and a chapter of Bluman and Kumei [1989] is dedicated to them. Whilst in this thesis we concentrate on finding more symmetry reductions through potential symmetries, they may also be used to find mappings which linearise the original system (cf. Bluman and Kumei [1989], Bluman [1993a,c]) or to derive conservation laws (cf. Anco and Bluman [1996]).

The method to find potential symmetries for our system (1.15) with $q = m$ is as follows:

Suppose that a partial differential equation of the system (1.15), without loss of generality Δ_m, is a conservation law, so (1.15) is the system

$$\Delta_i(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(n)}) = 0, \qquad i = 1, 2, \ldots, m-1, \tag{1.70a}$$

$$\sum_{j=1}^{p} \mathrm{D}_{x_j} \Lambda_j(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(n-1)}) = 0. \tag{1.70b}$$

Through (1.70b) we can introduce $p-1$ new *potential* (or *auxiliary*) dependent variables $\boldsymbol{v} = (v_1, v_2, \ldots, v_{p-1})$ and form a *potential* (or *auxiliary*) system $\boldsymbol{\Delta}_{POT}$ of $m+p-1$ partial differential equations given by

$$\begin{aligned}
&\Lambda_1(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(n-1)}) = \frac{\partial v_1}{\partial x_2},\\
&\Lambda_l(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(n-1)}) = (-1)^{l-1}\left[\frac{\partial v_l}{\partial x_{l+1}} + \frac{\partial v_{l-1}}{\partial x_{l-1}}\right], \qquad 1 < l < p,\\
&\Lambda_p(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(n-1)}) = (-1)^{p-1}\frac{\partial v_{p-1}}{\partial x_{p-1}},\\
&\Delta_i(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(n)}) = 0, \qquad i = 1,2,\ldots,m-1.
\end{aligned} \tag{1.71}$$

Apply the classical method to this potential system to find its point symmetries, by considering the transformation

$$\boldsymbol{x}^* = \boldsymbol{x} + \varepsilon\boldsymbol{\xi}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v}) + O(\varepsilon^2), \tag{1.72i}$$

$$\boldsymbol{u}^* = \boldsymbol{u} + \varepsilon\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v}) + O(\varepsilon^2), \tag{1.72ii}$$

$$\boldsymbol{v}^* = \boldsymbol{v} + \varepsilon\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v}) + O(\varepsilon^2). \tag{1.72iii}$$

We note that if $(\boldsymbol{u}(\boldsymbol{x}),\boldsymbol{v}(\boldsymbol{x}))$ solves $\boldsymbol{\Delta}_{POT}$ then $\boldsymbol{u}(\boldsymbol{x})$ solves $\boldsymbol{\Delta}$. If $\boldsymbol{u}(\boldsymbol{x})$ solves $\boldsymbol{\Delta}$ then there exists some $\boldsymbol{v}(\boldsymbol{x})$ such that $(\boldsymbol{u}(\boldsymbol{x}),\boldsymbol{v}(\boldsymbol{x}))$ solves $\boldsymbol{\Delta}_{POT}$ (but $\boldsymbol{v}(\boldsymbol{x})$ is not unique).

Definition 1.5.1. A *potential symmetry* of $\boldsymbol{\Delta}$ is a point symmetry of $\boldsymbol{\Delta}_{POT}$ that does not project onto a point symmetry of $\boldsymbol{\Delta}$.

In order to determine when $\boldsymbol{\Delta}_{POT}$ yields a potential symmetry we appeal to the following theorem

Theorem 1.5.1. *Suppose $(\boldsymbol{\xi},\boldsymbol{\phi},\boldsymbol{\eta})$ give a point symmetry of $\boldsymbol{\Delta}_{POT}$. Then this induces a potential symmetry of $\boldsymbol{\Delta}$ if and only if $(\boldsymbol{\xi},\boldsymbol{\phi})$ depend essentially on $\boldsymbol{v}$; otherwise they project onto a point symmetry of $\boldsymbol{\Delta}$.*

The beauty of the method of Bluman, Kumei and Reid [1988] to find nonlocal symmetries is that it uses the classical method, which is not only well-known but entirely algorithmic and consequently many symbolic manipulation packages exist that assist in the calculations (for instance, see §1.8).

Whilst it may be possible to "gain" (potential) symmetries it is also possible to "lose" them: a point symmetry of $\boldsymbol{\Delta}$ could yield a nonlocal symmetry of $\boldsymbol{\Delta}_{POT}$ and hence the classical method applied to $\boldsymbol{\Delta}_{POT}$ will not find it. However Bluman [1993b] tackles this in the following way: rename $\boldsymbol{v} = \boldsymbol{v}^{\{1\}}$, $\boldsymbol{\Delta}_{POT} = \boldsymbol{\Delta}_{POT}^{\{1\}}$, then if one of the partial differential equations of $\boldsymbol{\Delta}_{POT}^{\{1\}}$ is a conservation law a further $p-1$ potential variables $\boldsymbol{v}^{\{2\}}$ may be introduced to form the auxiliary system $\boldsymbol{\Delta}_{POT}^{\{2\}}$. This process may be continued, provided sufficient conservation laws exist, to $\boldsymbol{\Delta}_{POT}^{\{J\}}$. He then proposes the following two conjectures, which he claims hold for many examples, including those given by Akhatov, Gazizov and Ibragimov [1991] in the framework of potential symmetries:

Conjecture 1.5.1. *A process of obtaining a chain of auxiliary systems* $\boldsymbol{\Delta}^{\{1\}}_{POT}, \boldsymbol{\Delta}^{\{2\}}_{POT}, \ldots, \boldsymbol{\Delta}^{\{N\}}_{POT}$ *through use of equivalent conservation laws, terminates at some finite* N *when either*

(i) $\boldsymbol{\Delta}^{\{N\}}_{POT}$ *has no equivalent conservation laws; or*

(ii) $\boldsymbol{\Delta}^{\{N\}}_{POT}$ *has an infinite number of equivalent conservation laws. In this case* $\boldsymbol{\Delta}^{\{N\}}_{POT}$ *is likely to be linearisable.*

Conjecture 1.5.2. *If* $\boldsymbol{\Delta}^{\{N\}}_{POT}$ *has no equivalent conservation laws then the point symmetries of* $\boldsymbol{\Delta}^{\{N\}}_{POT}$ *yield, through projections, all point symmetries of any subsystem of* $\boldsymbol{\Delta}^{\{N\}}_{POT}$ *(which includes* $\boldsymbol{\Delta}$*).*

(For a discussion of subsystems see Bluman [1993a], and for equivalent conservation laws, which are essentially conservation laws that are worthwhile considering, see Bluman [1993b].) The system $\boldsymbol{\Delta}^{\{N\}}_{POT}$ of Conjecture 1.5.2 is called a *grand* potential system, and recovers all lost symmetries.

A natural extension of this theory is simply to apply the nonclassical method, rather than the classical method, to our potential system $\boldsymbol{\Delta}_{POT}$. This is suggested by Clarkson [1995], Priestley and Clarkson [1995] and Bluman and Shtelen [1995], though as we discuss in a moment Bluman and Shtelen [1995] extend this idea further. Nonclassical symmetries of $\boldsymbol{\Delta}_{POT}$ that are not classical symmetries of $\boldsymbol{\Delta}$ or $\boldsymbol{\Delta}_{POT}$, nor nonclassical symmetries of $\boldsymbol{\Delta}$ are termed *nonclassical potential symmetries.* It should also be possible to find nonclassical potential symmetries from higher order systems, $\boldsymbol{\Delta}^{\{J\}}_{POT}$ say, though it is an open problem to determine whether it is possible for a higher order system, $\boldsymbol{\Delta}^{\{K\}}_{POT}$ for $K > J$, to admit these symmetries via the classical method (including $J = 1$). This is discussed in Chapter Seven.

Whilst the extension is obvious, there appear to be no examples of nonclassical potential symmetries in the literature. Examples are promised in a forthcoming paper by Bluman and Levi [1996], and it is also the subject of investigation in Chapters Five through Seven. Indeed in Chapter Seven we are able to exhibit nonclassical potential symmetries.

Bluman and Shtelen [1995] extend the nonclassical method as applied to potential systems further. They consider a scalar equation

$$\Delta(x, t, u, u^{(n)}) = 0, \tag{1.73}$$

with potential system

$$\Delta_{POT}(x, t, u, u^{(n)}, v, v^{(1)}) = 0, \tag{1.74}$$

and note that if any of the sets of partial differential equations

$$u_t + \xi_1(x, t, u, v)u_x - \phi(x, t, u, v) = 0,$$

$$v_t + \xi_2(x, t, u, v)v_x - \eta(x, t, u, v) = 0,$$

with $\xi_1 \not\equiv \xi_2$,

$$u_t + \xi(x,t,u,v)u_x - \phi(x,t,u,v) = 0,$$
$$v_x - \eta(x,t,u,v) = 0,$$

with $\xi \neq 0$, or

$$u_x - \phi(x,t,u,v) = 0,$$
$$v_t + \xi(x,t,u,v)v_x - \eta(x,t,u,v) = 0,$$

with $\xi \neq 0$, is compatible with the potential system (1.74), then the solutions found will not be of a form considered before. In fact they link this work with *nonclassical Lie-Bäcklund symmetries*, but it is mentioned here only for completeness.

1.5.2 Generalisations of the Nonclassical Method.

We have already seen two ways in which the nonclassical method may be generalised, via potential systems in the previous subsection. However the main generalisations of the nonclassical method come from Olver and Rosenau in two papers that appeared in relatively quick succession, but recently Burdé [1996] has proposed another generalisation.

We describe first the notion of a *weak symmetry group* introduced in Olver and Rosenau [1987]. The symmetry groups obtained by the classical method are termed *strong symmetry groups* and as we have seen map solutions of (1.15) to other solutions of (1.15). By foregoing this criterion, and simply requiring that solutions of (1.15) invariant under the group of transformations (1.7), or equivalently (1.8), are found from a reduced system of differential equations involving a fewer number of independent variables than the original system (1.15), a larger class of solutions may be found. They called groups that satisfied this property *weak symmetry groups*, and by Remark 1.3(ii) we see that the nonclassical symmetry groups fall into this category. Consider the following example

Example 1.5.1. Consider equation (1.1) and the group of transformations defined by (1.8) with

$$\boldsymbol{\xi} \equiv (\xi, \tau) = (\tfrac{1}{2}x - c_1 t^{3/2}, t), \qquad \boldsymbol{\phi} \equiv \phi = 0, \tag{1.75}$$

whose invariant surface condition is

$$(\tfrac{1}{2}x - c_1 t^{3/2})u_x + t u_t = 0. \tag{1.76}$$

By solving the invariant surface condition we find invariant solutions of our group of transformations are

$$u = U(z), \qquad z = xt^{-1/2} + c_1 t.$$

Substituting this into (1.1) yields

$$-\tfrac{1}{2}zU't^{-1} + \tfrac{3}{2}c_1U' = U''t^{-1} - U(1-U)^2. \tag{1.77}$$

By equating powers of t to zero we have a set of two compatible ordinary differential equations for $U(z)$,

$$U'' + \tfrac{1}{2}zU' = 0, \qquad \tfrac{3}{2}c_1U' + U(1-U)^2 = 0,$$

which has solution either $U = 0$, or $U = 1$. We note by comparison with earlier sections that the infinitesimals (1.75) are not classical nor nonclassical, and the equation found after substitution (1.77) is certainly not an ordinary differential equation. Despite this the solutions we have found are invariant under our group given by (1.8) with (1.75). Whilst the solutions we have found are somewhat trivial, and indeed may be found by both the classical and nonclassical methods, Olver and Rosenau [1987] showed by example that this need not be the case.

The main difficulty with the method is knowing which symmetry groups will give us compatible ordinary differential equations to solve, and when these will give more interesting solutions. One possible tactic described by Olver and Rosenau [1987] is to specify the group by external symmetry considerations; for instance by considering the physical problem that the system models or by considering groups that preserve any boundary conditions in the problem.

The second generalisation of the nonclassical method, which is also more general than the idea of weak symmetry groups, is the concept of a system of partial differential equations having *side conditions*. One chooses these side conditions, which take the form of partial differential equations, so that the new overdetermined system, consisting of the original system and the side conditions, is compatible. In the above example our side condition is the equation (1.76), and generally for the classical and nonclassical methods the side condition takes the form of the invariant surface condition. Since the invariants of any group of transformations are found by solving a quasi-linear partial differential equation, this idea also incorporates Olver and Rosenau's idea of weak symmetry groups. However more general side conditions may be considered; for instance if we wanted to find (multiplicative) separable solutions of (1.1), the side condition would be

$$uu_{xt} - u_xu_t = 0. \tag{1.78}$$

An application of differential Gröbner bases (see Example 1.7.4 in §1.7) shows that only solutions that are either time independent or spatially independent are compatible with (1.1) and (1.78).

The difficulty is how to determine which side conditions will be compatible, however as Olver and Rosenau [1986] now point out, *"the key question becomes not which groups are* *relevant to a given system of partial differential equations, but rather which side conditions are admissible, thereby providing genuine solutions of the system?"*

Recently Galaktionov and Posashkov [1996] have used these ideas to find explicit solutions of the one-dimensional quasilinear heat equation

$$u_t = \phi(u)u_{xx} + f(u),$$

which is a generalisation of (1.1). They used a side condition of the form

$$u_t = h_2(u)u_x^2 + h_1(u)u_x + h_0(u).$$

The generalisation by Burdé [1996] actually takes ideas from both the nonclassical and direct methods, and is outlined here for a scalar equation with two independent variables. Initially one uses the method of Clarkson and Mansfield [1994c] to remove, say, the t-derivatives from the equation using the invariant surface condition. Then one takes the nth prolongation of this equation, but the coefficients of different derivatives of u are *not* set to zero to form the determining equations, as one does in the nonclassical method. Instead, one now assumes the symmetry reduction has the form (1.66), the linear ansatz of the direct method. In particular

$$z = z(x,t), \qquad U = \zeta(x,t,u) = \frac{u}{A(x,t)} - \frac{B(x,t)}{A(x,t)}. \tag{1.79}$$

By requiring that the symmetry variables (z, U) are invariant under the group transformation imposes the conditions

$$\xi z_x + \tau z_t = 0,$$
$$\xi\zeta_x + \tau\zeta_t + \phi\zeta_u = 0.$$

These may be solved algebraically for (ξ, ϕ) (set $\tau = 1$), in terms of (A, B, z) by using (1.79). Now by removing u-derivatives from the prolonged equation derived earlier, using (1.66) as one does in the direct method, and occurrences of (ξ, ϕ) using our new expressions, we obtain an expression of the form

$$\sum_{j=1}^{k} C_j(x,t)E_j[U] = 0, \tag{1.80}$$

where the $E_j[U]$ are monomials. Requiring that the $C_j(x,t) = 0$ is equivalent to the direct method. However by requiring (1.80) to be an ordinary differential equation, Burdé [1996] gained new reductions.

1.5.3 Generalisations of the Direct Method.

Soon after the appearance of the direct method, Galaktionov [1990] proposed a different kind of ansatz based method in his *method of nonlinear separation.* In studying the equation

$$u_t = u_{xx} + u_x^2 + u^2, \tag{1.81}$$

he supposed that the solution takes the form

$$u(x,t) = \phi(t)[\psi(t) + \theta(x)], \tag{1.82}$$

and required that when this is substituted into (1.81) the equation "separates", i.e. substituting (1.82) into (1.81) yields

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}(\phi\psi) - \phi^2\psi^2\right] + \left[\frac{\mathrm{d}\phi}{\mathrm{d}t} - 2\phi\psi\right]\theta - \phi\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - \phi^2\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + \theta^2\right] = 0.$$

We require

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = c_1\theta + c_2, \qquad \left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + \theta^2 = c_3\theta + c_4, \tag{1.83}$$

and hence

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}(\phi\psi) - \phi^2\psi^2 - c_2\phi - c_4\phi^2\right] + \theta\left[\frac{\mathrm{d}\phi}{\mathrm{d}t} - 2\phi\psi - c_1\phi - c_3\phi^2\right] = 0, \tag{1.84}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. We have an overdetermined system to solve for $\theta(x)$ in (1.83), and an exactly determined system of coupled ordinary differential equations to solve for $\phi(t)$, $\psi(t)$ from (1.84). Using this technique he was able to study the "blow-up" of the solutions.

Whilst the ansatz of Galaktionov [1990] (1.82) is less general than the linear ansatz in the $z_x \equiv 0$ case of the direct method

$$u(x,t) = A(x,t)U(t) + B(x,t),$$

because the condition that Galaktionov demands is different, namely that the equation separates, rather than it satisfies an ordinary differential equation, the class of solutions is different.

In general the method of Galaktionov [1990] as applied to a single equation (1.73) could be thought of as assuming the solution has the form

$$u(x,t) = F(x,t,U_1(z),U_2(z)), \qquad z = z(x,t),$$

and requiring that $U_1(z)$ and $U_2(z)$ satisfy a system of two ordinary differential equations on substitution into (1.73). Within this scenario the direct method is seen to be the special case when $U_2(z)$ satisfies $U_2 = 0$.

Olver [1994] naturally generalised this further theoretically by allowing the solution of (1.73) to have the form

$$u(x,t) = F(x,t,U_1(z),U_2(z),\ldots,U_k(z)),$$

where on substitution into (1.73) the $U_i(z)$ for $i = 1,2,\ldots,k$ satisfy a system of k ordinary differential equations. He calls this the higher order method of Galaktionov, though we note that this generalisation was only written down in order to compare it with the method of Olver and Rosenau [1986] of finding special solutions via side conditions, which we discuss in the next subsection.

A different approach was proposed by Estévez [1992] for equations of the form (1.73): rather than increase the number of new dependent symmetry variables she simply allowed the single dependent symmetry variable to satisfy more than one ordinary differential equation, i.e. a system of compatible ordinary differential equations. With this new approach she was able to find more exact solutions than by simply using the direct method. For instance (cf. Estévez [1992]), consider again equation (1.1). If we assume $u(x,t)$ is linear in the new dependent symmetry variables we have

$$u(x,t) = A(x,t)U(z) + B(x,t),$$

and substituting into (1.1) yields, as in the direct method calculation of Example 1.4.1, equation (1.64). If we set $A = z_x$ (cf. Example 1.4.1.) and require

$$3z_{xx} - z_t = \pm\sqrt{2}\,(2-3B)z_x, \tag{1.85i}$$

$$z_{xxx} - z_{xt} = (3B-1)(B-1)z_x, \tag{1.85ii}$$

$$B_{xx} - B_t - B(1-B)^2 = 0, \tag{1.85iii}$$

then (1.64) reduces to

$$z_x(U'' - U^3) + (2-3B)(\pm\sqrt{2}\,U' + U^2) = 0. \tag{1.86}$$

Rather than require that (1.86) be an ordinary differential equation we require that $U(z)$ satisfies the overdetermined system of equations

$$U'' - U^3 = 0, \qquad \pm\sqrt{2}\,U' + U^2 = 0,$$

which have the common solution

$$U(z) = \pm\sqrt{2}\,/(z - z_0),$$

for z_0 an arbitrary constant. By taking either of the constant solutions of (1.85iii), i.e. $B = 0$ or $B = 1$, and solving (1.85i,ii) we are able to reproduce the exact solution (1.56), which could not be found via the direct method.

A similar approach is given by Burdé [1994], and was used to find solutions of the boundary layer equations

$$uu_x + vu_r = U\frac{\mathrm{d}U}{\mathrm{d}x} + \nu\left(u_{rr} + \frac{1}{r}u_r\right),$$
$$u_x + v_r + \frac{v}{r} = 0,$$

where $U(x)$ is an arbitrary function and ν an arbitrary constant. He found that the fairly restrictive conditions that the direct method imposed could be relaxed, and found genuinely new solutions. Once again one requires that rather than satisfy a single ordinary differential equation, the new symmetry variable may satisfy an overdetermined system.

The extension of the direct method by Hood [1995] allows more dependent *and* more independent symmetry variables. In studying Burgers' equation

$$u_t + uu_x + u_{xx} = 0, \tag{1.87}$$

he assumed that the solutions took the form

$$u(x,t) = A(x,t)p(\xi) + B(x,t)q(\zeta) + C(x,t),$$

where $\xi = \xi(x,t)$ and $\zeta = \zeta(x,t)$, and required that $p(\xi)$ and $q(\zeta)$ satisfy ordinary differential equations. In the case $\xi \equiv \zeta$ this method is the same as Galaktionov's. If $\xi \not\equiv \zeta$, since the independent symmetry variables are different, the ordinary differential equations that $p(\xi)$ and $q(\zeta)$ satisfy must not be coupled. In practice one of p and q must have a fairly simple form (e.g. rational) in order to cope with the inevitable presence of coupled terms, however genuinely new solutions of (1.87) were found.

The main difficulty in each of the generalisations of the direct method mentioned here, though less so for Galaktionov's method, is the lack of an algorithm, particularly when choosing how to split up the candidate symmetry equation into a system of ordinary differential equations. Indeed there will be many ways to do this, though certainly they will not all be compatible and there seems to be no way to determine how best to do this.

1.5.4 Relationships between symmetry methods.

After Clarkson and Kruskal [1989] found many new reductions of the Boussinesq equation with their direct method, there was much interest in realising the connection with it and other symmetry methods. Indeed Clarkson and Kruskal [1989] themselves hoped *"that a group theoretic explanation of the method will be possible in due course"*. This came quickly from Levi and Winternitz [1989], who used the nonclassical method of Bluman and Cole [1969] on the Boussinesq equation and found precisely the same reductions.

The first indication that the direct method and the nonclassical method gave different reductions came from Nucci and Clarkson [1992] who studied the Fitzhugh-Nagumo equation

$$u_t = u_{xx} + u(1-u)(u-a), \tag{1.88}$$

for a constant such that $0 \leq a \leq 1$. It is no coincidence that our prototype equation (1.1) is (1.88) with $a = 1$, and we have already seen that the nonclassical method finds the exact solution (1.56), which the direct method was unable to find. Crucially the infinitesimal of the independent variable x, namely ξ, depended on u (cf. (1.50)).

Subsequently Olver [1994] (see also Arrigo, Broadbridge and Hill [1993], Pucci [1992]) gave the precise nature of the relationship between the nonclassical method and the direct method in the form of two theorems. He considered the second order partial differential equation

$$\Delta(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0. \tag{1.89}$$

Theorem 1.5.2. *There is a one-to-one correspondence between the ansatz of the direct method*

$$u(x,t) = F(x, t, U(z)), \tag{1.90}$$

with $F_U \neq 0$ and the quasi-linear first order differential constraint

$$\mathbf{v}(u) = \xi(x,t)u_x + \tau(x,t)u_t = \phi(x,t,u). \tag{1.91}$$

Theorem 1.5.3. *The ansatz (1.90) will reduce the partial differential equation (1.89) to a single ordinary differential equation for $U(z)$ if and only if the overdetermined system of partial differential equations defined by (1.89) and (1.91) is compatible.*

Whilst it is clear how the proofs of these theorems may be extended to an equation of higher order, it is not so clear how to extend them to systems of partial differential equations. This will be discussed in detail in §8.1.

In the same paper Olver went on to prove the nature of the relationship between the higher order method of Galaktionov and the method of side conditions (also called the *method of differential constraints*). In particular he proved that a partial differential equation admits a reduction to a system of k ordinary differential equations for k dependent symmetry variables in a single independent symmetry variable if and only if an associated kth order side condition is compatible with the equation.

Another important result comes from Ludlow [1995], who dispelled a commonly held belief, that the classical method is a special case of the direct method. He considered the equation

$$u_x u_{xx} - (auu_x - bu_t)(1 - tu_x)^3 = 0, \tag{1.92}$$

and showed that in the case $a = b = \frac{1}{2}$ it admits the (implicit) solution found by the classical method

$$u(x,t) = c_1 \exp\{x - u(x,t)t\} + c_2 \exp\{-x + u(x,t)t\},$$

where c_1 and c_2 are arbitrary constants. This solution cannot be obtained by the direct method.

Whilst the direct method as it stands cannot find implicit reductions and thus fails to find all the reductions that the nonclassical and sometimes even the classical method finds, it is easily adapted to do so: simply assume the solution has the form (1.61) where now $z = z(x, t, u)$. The fundamental obstacle with this approach is the sheer difficulty in solving the associated determining equations. Also as we saw in §1.3, when the reduction is implicit the best way to find exact solutions may be to solve the equation and the invariant surface condition simultaneously, which cannot be implemented in the direct method even with this new implicit ansatz. For these reasons, though mainly due to its difficulty, the direct method has not been applied with an implicit ansatz.

To end this subsection, we make a few comments on the direct, classical and nonclassical methods, which dominate the work in this thesis. Whilst the nonclassical method has been shown to be more general than the direct method for scalar equations, the direct method should certainly not be discarded for a number of reasons:

- it allows the finding of reductions in a single step, whilst in both the classical and nonclassical methods once the determining equations have been solved one must still carry out the not always trivial task of solving the invariant surface conditions;
- one is able to reduce the number of independent variables by any number ($< p$) in a single step, when at the moment this can only be done with the classical method;
- the occurrence of reductions that cannot be found by the direct method but can by the nonclassical method are rare, and even more rare are those that the classical method finds but not the direct method – indeed equation (1.92) was constructed specifically for the purpose of finding such a reduction;
- the advent of symbolic manipulation packages has made the generation of the determining equations in the classical and nonclassical methods relatively easy, and the solution of the determining equations more tractable. However the direct method does not need such sophisticated tools to be applied, indeed without them it is arguably even more easy to apply than the classical method, particularly for higher order equations.

1.6 Painlevé Tests

Once a method of symmetry reduction has been applied we are often still left with a symmetry system to solve if we are to find exact solutions. If this symmetry system is still a system of partial differential equations then further application of our methods may be necessary, unless perhaps it is recognisable or it is now simple enough to solve by other techniques. If the symmetry system is now a system of ordinary differential equations we are often interested to know whether it is of so-called *Painlevé-type*, which as we discuss below, tells us about the singularity structure of the solutions. Experience dating back to Kowalevski [1889a,b] tells us that often only systems that are of Painlevé-type may be solved explicitly. Also determining whether a symmetry system of ordinary differential equations is of Painlevé-type gives us information on the original system through the *Painlevé conjecture*, described in §1.6.2.

Definition 1.6.1. A system of ordinary differential equations is of *Painlevé-type* or *P-type* if all solutions possess the so-called *Painlevé property*, that is, the only *movable* singularities in any solution are poles. A *movable* singularity is a singularity whose location in the complex plane is determined by the constants of integration, as opposed to *fixed* singularities whose location is determined by the form of the equation.

Towards the end of the nineteenth century work was in progress determining which nonlinear ordinary differential equations exhibited which kind of singularities. In answering a question posed by Picard [1887] as to which equations of the form

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = F\left(z, w, \frac{\mathrm{d}w}{\mathrm{d}z}\right), \tag{1.93}$$

where F is rational in $\mathrm{d}w/\mathrm{d}z$, algebraic in w and analytic in z, are of Painlevé-type, Painlevé and his colleagues not only classified such equations, but found six equations whose solutions could not be expressed in terms of previously known functions. A review of the classification is given in Ince [1956] (with corrections by Cosgrove [1991]), which uses the *α-method* due to Painlevé, which we describe in the next subsection. The six new equations, which are of Painlevé-type, are known as the *Painlevé equations* and their solutions define new transcendental functions; they are

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 6w^2 + z, \tag{PI}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha, \tag{PII}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}, \tag{PIII}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{PIV}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \left[\frac{1}{2w} + \frac{1}{w-1}\right]\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \tag{PV}$$

$$\begin{aligned}\frac{\mathrm{d}^2w}{\mathrm{d}z^2} &= \frac{1}{2}\left[\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right]\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \left[\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right]\frac{\mathrm{d}w}{\mathrm{d}z} \\ &\quad + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right],\end{aligned} \tag{PVI}$$

where α, β, γ and δ are arbitrary constants.

In the remaining subsections we describe tests to determine whether ordinary differential equations are of Painlevé-type, an extension to test partial differential equations also, and further extensions. We also describe the link between Painlevé analysis and *complete integrability*.

1.6.1 Painlevé's α-method.

The method that Painlevé and his colleagues used to classify the equations of the form (1.93) is known as the *α-method.* We describe it briefly here and it is used only as a last resort. More expansive descriptions may be found in Golubov [1953] or Kruskal and Clarkson [1992].

It consists of two parts, firstly building necessary conditions and then verifying, by direct integration or otherwise, that these are also sufficient. To show that an ordinary differential equation is of Painlevé-type using this method is more lengthy than otherwise, since one must show, possibly in a number of cases, that the equations in each case may be reducible to known equations of Painlevé-type.

The first part, however, is described here for a system a first order ordinary differential equations

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}z} = \boldsymbol{F}(z, \boldsymbol{w}),$$

where $\boldsymbol{w} = (w_1, w_2, \ldots, w_m)$ and $\boldsymbol{F} = (F_1, F_2, \ldots, F_m)$ not involving the parameter α. Make the transformation

$$\xi = \alpha^p k + \alpha^q z, \qquad \boldsymbol{u} = \alpha^r \boldsymbol{v} + \alpha^s \boldsymbol{w},$$

for some p, q, r, s, k and $\boldsymbol{v} = (v_1, v_2, \ldots, v_m)$ suitably chosen. We obtain a system of the form

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\xi} = \boldsymbol{f}(\xi, \boldsymbol{u}; \alpha),$$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)$ and $\boldsymbol{f} = (f_1, f_2, \ldots, f_m)$. One then expands $\boldsymbol{u}(\xi)$ in powers of α,

$$\boldsymbol{u}(\xi) = \sum_{j=0}^{\infty} \boldsymbol{u}_j(\xi)\alpha^j,$$

and from a result due to Painlevé, the $\boldsymbol{u}_j(\xi)$ are of Painlevé-type if and only if $\boldsymbol{u}(\xi)$ is of Painlevé-type. Thus one creates systems of equations in the $\boldsymbol{u}_j(\xi)$ from which one determines conditions under which they are of Painlevé-type, which is easier to do than by simply considering the original system.

1.6.2 The ARS algorithm and the Painlevé conjecture.

Based on the work of Kowalevski [1889a,b], Ablowitz, Ramani and Segur [1980a] (ARS) developed an algorithm for determining necessary conditions for whether an ordinary differential equation is of Painlevé-type. We describe it here with the ordinary differential equation (1.2)

$$w'' + cw' - w^3 + 2w^2 - w = 0, \tag{1.2}$$

associated with the travelling wave reduction 1.2.1 of our prototype equation (1.1) being given as an example. For an nth order ordinary differential equation to be of Painlevé-type we require that in some neighbourhood of a movable singularity at $z = z_0$ say, the general solution of the ordinary differential equation can be expressed in terms of a Laurent series

$$w(z) = (z - z_0)^p \sum_{j=0}^{\infty} a_j (z - z_0)^j, \tag{1.94}$$

where $a_0 \neq 0$ and $n - 1$ of the constants a_j are arbitrary.

Step One: Find the dominant behaviour.

Firstly we must find the dominant behaviour of solutions of (1.2) in the neighbourhood of a movable singularity at $z = z_0$. We assume that

$$w(z) \sim a_0 (z - z_0)^p \quad \text{as} \quad z \to z_0, \tag{1.95}$$

where a_0 ($\neq 0$) and p are constants to be determined. This is substituted into (1.2) and the method of dominant balance is used to find all values of p such that two or more terms in the resulting expression are of equal order, and the remaining terms are negligible as $z \to z_0$. Then requiring that the balanced terms cancel out usually determines a_0 for each value of p, though a_0 may be arbitrary. Substituting (1.95) into (1.2) yields

$$a_0 p(p-1)(z - z_0)^{p-2} + c a_0 p (z - z_0)^{p-1} - a_0^3 (z - z_0)^{3p} + 2a_0^2 (z - z_0)^{2p} - a_0 (z - z_0)^p = 0.$$

The only way in which these terms will balance according to our needs is if $p - 2 = 3p$, i.e. $p = -1$. We find for $a_0 \neq 0$ that $a_0^2 - 2 = 0$, so that there are two dominant behaviours, i.e.

$$w(z) = \pm\sqrt{2}\,(z - z_0)^{-1} + o((z - z_0)^{-1}) \quad \text{as} \quad z \to z_0.$$

If any of the values of p turn out to be not integer, the dominant behaviour would have been that of an algebraic branch point, so the equation would not have been of Painlevé-type, and is said to fail the test. Note that in Ramani, Dorizzi and Grammaticos [1982] and Ranada *et al.* [1985] the authors describe how rational values of p may be interpreted and introduce the *weak Painlevé property*. An equation admitting an expansion of the form

$$w(z) = (z - z_0)^{p/q} \sum_{j=0}^{\infty} a_j (z - z_0)^{j/q},$$

for p, q integers, is said to possess the weak Painlevé property.

Step Two: Determine the *resonances*.

Substituting (1.94) into (1.2) and equating powers of $z - z_0$ to zero yields equations of the form

$$Q(j)a_j - G_j(z_0, a_0, a_1, \dots, a_{j-1}) = 0, \qquad j \in \mathbb{Z}^+, \tag{1.96}$$

which will also depend on the parameters in the equation. The roots of the equation $Q(r) = 0$ determine the *resonances*: where a_r is not determined by (1.96). To find the resonances we may substitute

$$w(z) = a_0(z - z_0)^p + \beta(z - z_0)^{p+r},$$

into the dominant terms of (1.2) (w'' and $-w^3$) and equate terms up to order β to zero. Doing so, remembering that $a_0^2 = 2$ and $p = -1$ yields

$$\beta[(r-1)(r-2) - 6](z - z_0)^{r-3} = 0,$$

which factorises to give $(r+1)(r-4) = 0$. The presence of the resonance at $r = -1$ is to be expected, though it is not entirely clear what it represents. Ignoring this value of r our only resonance is at $r = 4$. In general we require $n - 1$ distinct integer resonances (not including $r = -1$) for our equation to pass the test since this allows the presence of n arbitrary constants, z_0 and the $n - 1$ constants a_r from (1.96). Then we know that the general solution is being found. In this test only distinct *positive* integer resonances are allowed but recently Conte, Fordy and Pickering [1993] introduced the *perturbative Painlevé test*, which analyses negative resonances.

Step Three: Find the constants of integration.

We substitute

$$w(z) = (z - z_0)^p \sum_{j=0}^{N} a_j (z - z_0)^j + O((z - z_0)^{N+p+1}),$$

into (1.2), where N is the value of the largest resonance, in order to determine the constants a_j. There is no need to go beyond N since (1.96) determines the a_j for $j > N$. In particular we are interested in the solution of

$$G_r(z_0, a_0, a_1, \dots, a_{r-1}) = 0, \tag{1.97}$$

at the resonance values r, where $Q(r) = 0$ (cf. (1.96)). If (1.97) is identically zero for each r then (1.2) is said to have passed the test. However (1.97) may provide a contradiction, in which case (1.2) does not pass the test, or may induce conditions, called *compatibility* *conditions*, on the parameters in the equation in which cases only the class of equations under these conditions pass the test. For (1.2) we find (recall $a_0^2 = 2$)

$$(z - z_0)^{-2} : \qquad a_1 = 4 - a_0 c, \tag{1.98i}$$

$$(z - z_0)^{-1} : \qquad a_2 = -\frac{a_0(c^2 - 2)}{36}, \tag{1.98ii}$$

$$(z - z_0)^{0} : \qquad a_3 = \frac{3a_0 c - 2a_0 c^3 - 2}{108}, \tag{1.98iii}$$

$$(z - z_0)^{1} : \qquad 0 = c(a_0 c - 1)^2(a_0 c + 2), \tag{1.98iv}$$

since a_4 is arbitrary (since the resonance is at $r = 4$). Equation (1.98iv) tells us that (1.2) only passes the test if $c = 0$, since no other values of c satisfy (1.98iv) for both values of a_0 simultaneously.

In fact since (1.2) is of the form (1.93) studied by Painlevé and his colleagues we may consult the classification found in Ince [1956] to see if it "fits into" one of the 50 canonical equations. The equations listed in Ince [1956] are generalisable by a Möbius transformation

$$W(Z) = \frac{A(z)w(z) + B(z)}{C(z)w(z) + D(z)}, \qquad Z = \phi(z), \tag{1.99}$$

where A, B, C, D and ϕ are analytic functions, so they should not simply be consulted at face value. (Also there are many mistakes, so one should consult Cosgrove [1991].) By careful checking therefore, we find that (1.2) is only of Painlevé-type if $c = 0$, showing that the necessary condition above is also a sufficient one. If $c = 0$ (1.2) may be solved in terms of Jacobi elliptic functions (see Whittaker and Watson [1927])

We note that for first order ordinary differential equations there is often no need to use the ARS algorithm to determine whether they are of Painlevé-type. Fuchs proved that the only first order equation of the form

$$\frac{\mathrm{d}w}{\mathrm{d}z} = G(z, w) = \frac{P(z, w)}{Q(z, w)},$$

where P and Q are polynomials in w whose coefficients are analytic in z, that is of Painlevé-type is the generalised Riccati equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} = p_2(z)w^2 + p_1(z)w + p_0(z), \tag{1.100}$$

with $p_2(z)$, $p_1(z)$ and $p_0(z)$ analytic functions (see Ince [1956]). If $p_2(z) = 0$ this equation is linear, whilst if $p_2(z) \neq 0$ it may be linearised via the transformation

$$w(z) = -\frac{1}{p_2(z)W}\frac{\mathrm{d}W}{\mathrm{d}z}.$$

This result is again generalisable by a Möbius transformation (1.99), so for first order equations we need only check to see if they may be written in the form (1.100) via (1.99) to see if they are of Painlevé-type.

The determination of whether an ordinary differential equation was of Painlevé-type gained renewed interest from its linkage to *Inverse Scattering.* Inverse scattering is the technique introduced by Gardner *et al.* [1967] for solving the KdV equation (1.59). It has since been used to solve the initial value problems for many nonlinear evolution equations and such equations that are solvable by inverse scattering are considered to be *completely integrable.* The method may be thought of as the nonlinear analogue of the Fourier transform method for solving linear equations, and a description of it may be found in e.g. Ablowitz and Clarkson [1991].

Inspired by the observations of Ablowitz and Segur [1977], Ablowitz, Ramani and Segur [1978, 1980] and Hastings and McLeod [1980] formulated the *Painlevé conjecture* or *Painlevé ODE test*:

Every ordinary differential equation which arises as a symmetry reduction of a completely integrable partial differential equation is of Painlevé-type, perhaps after a transformation of variables.

This conjecture, if true, provides a useful *necessary* condition to test whether a partial differential equation might be completely integrable. Weakened versions of this test have been proved by Ablowitz, Ramani and Segur [1980b] and McLeod and Olver [1983]. Whilst such ordinary differential equations needed only to be of Painlevé-type, it was often the case that they were expressible in terms of the Painlevé equations PI–PVI (cf. Chapter Two).

The converse of the Painlevé conjecture was shown not to be true by Clarkson [1989]. He showed that the travelling wave reduction of the modified Benjamin-Bona-Mahoney equation,

$$u_t + u_x + u^2 u_x - u_{xxt} = 0, \tag{1.101}$$

is of Painlevé-type and is the only reduction obtainable by the classical and direct methods. However numerical evidence suggests that (1.101) is not solvable by inverse scattering. How the test may be applied is if an ordinary differential equation arising as a symmetry reduction is not of Painlevé-type, the original partial differential equation may be taken as not solvable by inverse scattering. Indeed we may conclude from the conjecture that (1.1) is not solvable by inverse scattering. This illustrates one of the advantages of applying symmetry methods, in that they can be applied to equations which are not integrable, in any sense of the word.

1.6.3 The Painlevé PDE Test.

In order to test whether a partial differential equation might be completely integrable directly, instead of via the Painlevé conjecture, Weiss, Tabor and Carnevale [1983] devised the so-called *Painlevé PDE test.* With the advent of this test it was no longer necessary to first find a partial differential equation's symmetry reductions, which might not exist, or conversely might be numerous. Further Clarkson [1989] showed, in studying the Symmetric Regularised Long Wave equation

$$u_{tt} + au_{xx} + \tfrac{1}{2}b(u^2)_{xt} + cu_{xxtt} = 0, \tag{1.102}$$

for a, b, c constants, that to reach conclusive results via the Painlevé conjecture on an equation's inability to be solvable by inverse scattering, it may be necessary to use the direct method for finding symmetry reductions, not just the classical method. However in the same paper, Clarkson showed that neither (1.101) nor (1.102) were solvable by inverse scattering directly using the Painlevé PDE test, gaining for equation (1.101) conclusive results where the Painlevé ODE test gave none.

It should be noted that Weiss, Tabor and Carnevale [1983] gave no attempt to prove the relationship between their test and completely integrability, though a partial proof may be inferred from the partial proof of McLeod and Olver [1983]. Whilst it has been shown to be by no means foolproof (cf. Clarkson [1989], Kruskal [1991], Kruskal and Clarkson [1992] and Pogrebkov [1989]), the Painlevé PDE test often gives a good indication of whether a partial differential equation might be completely integrable.

The test is analogous to the ARS algorithm, so is simply described here for our partial differential equation (1.1). A solution of (1.1) is sought in the form

$$u(x,t) = \phi^p(x,t)\sum_{j=0}^{\infty} \widetilde{a}_j(x,t)\phi^j(x,t), \tag{1.103}$$

where ϕ and $\widetilde{a}_j$ are analytic functions and $\phi(x,t) = 0$ is a movable non-characteristic singularity manifold. It has been noted by Kruskal [1983] that one can replace (1.103) by

$$u(x,t) = [x-\psi(t)]^p \sum_{j=0}^{\infty} a_j(t)[x-\psi(t)]^j,$$

where $\psi(t)$ and $a_j(t)$ $(a_0(t) \neq 0)$ are analytic function in the neighbourhood of the singularity manifold $x - \psi(t) = 0$.

At leading order we have

$$u(x,t) \sim a_0(t)[x-\psi(t)]^p,$$

and similar to the analysis for (1.2) we find $p = -1$ and $a_0^2 = 2$. We find the resonances, as previously, to be $r = -1, 4$ so we substitute an expansion of the form

$$u(x,t) = [x-\psi(t)]^{-1}\sum_{j=0}^{4} a_j(t)[x-\psi(t)]^j + O([x-\psi(t)]^5),$$

into (1.1) in order to find the functions $a_j(t)$ for $j = 1, 2, 3$, and the compatibility condition associated with $a_4(t)$. We find, by equating powers of $x - \psi(t)$ to zero

$$a_1(t) = \frac{u_0}{6}\frac{\mathrm{d}\psi}{\mathrm{d}t} + \frac{2}{3}, \quad a_2(t) = \frac{u_0}{18} - \frac{u_0}{36}\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2, \quad a_3(t) = \frac{u_0}{54}\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^3 - \frac{u_0}{24}\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} - \frac{u_0}{36}\frac{\mathrm{d}\psi}{\mathrm{d}t} - \frac{1}{54},$$

together with the compatibility condition

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}\left(6u_0\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}-2u_0\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^3+3u_0\frac{\mathrm{d}\psi}{\mathrm{d}t}+2\right)=0. \tag{1.104}$$

Since we require two arbitrary functions and have only one so far, i.e. $a_4(t)$, the existence of compatibility condition (1.104) which imposes conditions on $\psi(t)$, means that (1.1) does not pass the Painlevé PDE test.

1.7 Differential Gröbner Bases and the MAPLE package `diffgrob2`

In both the classical and nonclassical methods of symmetry reduction the determining equations form an overdetermined system of partial differential equations. In order to solve such systems we use the method of *Differential Gröbner Bases*, which provides a systematic framework for finding *compatibility conditions* of the system. It avoids the problems of infinite loops and yields, as far as is currently possible, what may be thought of as a "triangulation" of the system, from which the solution set may be derived more easily (cf. Clarkson and Mansfield [1994a], Mansfield and Fackerell [1992], Reid [1990, 1991]).

It is only recently that Mansfield and Fackerell [1992] have derived algorithms to cope with polynomially nonlinear systems, which are extensions of methods developed to study linear and special classes of nonlinear partial differential equations (see for instance Reid [1991]). These algorithms, which are necessary to calculate a differential Gröbner basis for systems that are in general nonlinear are implemented in the MAPLE package `diffgrob2` (Mansfield [1993]). These include the Kolchin-Ritt algorithm which we describe in §1.7.4, and in its manual (Mansfield [1993]) is described a Direct Search algorithm which we outline in §1.7.5.

In this section we introduce the notation required to understand the theory of differential Gröbner bases and present some of the theory together with examples and the `diffgrob2` syntax necessary to carry out the procedure. A much more detailed account is available in Mansfield [1993]. At the end of the section the method of Clarkson and Mansfield [1994c] for generating the determining equations of the classical and nonclassical methods is described, as promised, in §1.7.6.

The system of partial differential equations under consideration

$$\boldsymbol{\Delta}=(\Delta_1(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(n)}),\Delta_2(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(n)}),\ldots,\Delta_m(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(n)}))=0,$$

must be able to be regarded as polynomial in some unknown functions $\boldsymbol{u}=(u_1,u_2,\ldots,u_q)$, their derivatives

$$\mathrm{D}^{\boldsymbol{\alpha}}u_j=\frac{\partial^{|\boldsymbol{\alpha}|}u_j}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_p^{\alpha_p}}, \tag{1.105}$$

where $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \ldots + \alpha_p$, and the independent variables $\boldsymbol{x} = (x_1, x_2, \ldots, x_p)$. The set of all such derivatives (1.105) is denoted by $\boldsymbol{u}^{(\infty)}$. When inputting partial differential equations the "equals zero" is implicitly assumed. The operator $\mathrm{D}^{\boldsymbol{\alpha}}$ is in general defined by

$$\mathrm{D}^{\boldsymbol{\alpha}} \equiv \frac{\mathrm{D}^{\alpha_1}}{\mathrm{D}x_1^{\alpha_1}} \frac{\mathrm{D}^{\alpha_2}}{\mathrm{D}x_2^{\alpha_2}} \cdots \frac{\mathrm{D}^{\alpha_p}}{\mathrm{D}x_p^{\alpha_p}}.$$

Whilst the theory of differential Gröbner bases is not developed for arbitrary functions of the variables, nevertheless `diffgrob2` is still able to perform the calculations in a natural way. See for instance Chapter Two, where the determining equations include an arbitrary function of the independent variable u.

1.7.1 Orderings.

A fundamental concept, which all the theory presented here requires, is that of an ordering of the derivative terms, which must be *compatible*. If $x_1 < x_2 < \ldots < x_p$ and $u_1 < u_2 < \ldots < u_q$ defines an ordering on the dependent and independent variables, then a compatible ordering of the derivative terms is such that

(i) $\mathrm{D}^{\boldsymbol{\alpha}} u_j < \mathrm{D}^{\boldsymbol{\beta}} u_k$ implies that $\mathrm{D}^{\gamma}\mathrm{D}^{\boldsymbol{\alpha}} u_j < \mathrm{D}^{\gamma}\mathrm{D}^{\boldsymbol{\beta}} u_k$ and

(ii) $\mathrm{D}^{\gamma} u_j < \mathrm{D}^{\gamma}\mathrm{D}^{\boldsymbol{\alpha}} u_j$ for $|\gamma| \neq 0$.

There are many types of ordering (see Mansfield [1993]), but in this thesis we use only one, a lexicographic ordering given by

$$\begin{aligned} &\mathrm{D}^{\boldsymbol{\alpha}} u_j > \mathrm{D}^{\boldsymbol{\beta}} u_k \\ &\text{if } u_j > u_k, \\ &\text{else } j = k \text{ and } \alpha_1 > \beta_1, \\ &\text{else } \alpha_1 = \beta_1, \ldots, \alpha_i = \beta_i, \alpha_{i+1} > \beta_{i+1} \text{ for some } i \text{ such that } 2 \leq i \leq p. \end{aligned} \tag{ALEX}$$

In `diffgrob2` the string variable **termorder** denotes which ordering is used, so for our purposes **termorder:=alex**. We recreate the ordering $x_1 < x_2 < \ldots < x_p$ and $u_1 < u_2 < \ldots < u_q$ via the list **allvars** where **allvars:=[[x1,x2,...,xp],[u1,u2,...,uq]]**. In the following chapters we use the ordering $t < x < \boldsymbol{u}$ on the independent variables, with the ordering on the dependent variables and the variables $\boldsymbol{u}$ to be decided, though $\boldsymbol{\xi} < \boldsymbol{\phi}$ always.

Orderings on the derivative terms induce in turn partial orderings on the partial differential equations in the system, which we define after introducing some notation.

(i) The *highest derivative term* of a partial differential equation f, denoted HDT(f), is such that HDT(f)>DT(f) for all other derivative terms DT(f) in f.

(ii) The *highest power* of the HDT(f) occurring in f is denoted Hp(f).

(iii) The *highest coefficient* of f, denoted Hcoef(f), is the coefficient of $\mathrm{HDT}(f)^{\mathrm{Hp}(f)}$ in f.

(iv) The *head* of f is $\mathrm{Head}(f)=\mathrm{Hcoef}(f)\mathrm{HDT}(f)^{\mathrm{Hp}(f)}$.

(v) The *highest monomial* of f, denoted Hmon(f), is defined recursively as: if Head(f) has one summand, then Hmon(f)=Head(f), else Hmon(f)=Hmon(Head(f)).

(vi) The *separant* of f, the highest coefficient of $D^{\boldsymbol{\alpha}} f$ for any non-zero $|\boldsymbol{\alpha}|$, is denoted Sep(f).

(vii) The *highest unknown* of f, denoted Hu(f), is the unknown function, u_j say, occurring in HDT(f).

To demonstrate each of these new ideas, consider the following partial differential equation

$$f \equiv \quad u u_{xxyy}^2 + u_y u_{xxyy} + u_x u_{yy} + u_{xxx} + u_{xy},$$

under the lexicographic ordering (ALEX). If $y < x$ then the derivative terms in f are ordered $u < u_x < u_{xxx} < u_y < u_{xy} < u_{yy} < u_{xxyy}$, Hu($f$) $= u$ and

HDT(f)	Hp(f)	Hcoef(f)	Head(f)	Hmon(f)	Sep(f)
u_{xxyy}	2	u	uu_{xxyy}^2	uu_{xxyy}^2	$u_y + 2uu_{xxyy}$

If $x < y$ the derivative terms in f are ordered $u < u_y < u_{yy} < u_x < u_{xy} < u_{xxyy} < u_{xxx}$, Hu($f$) $= u$ and

HDT(f)	Hp(f)	Hcoef(f)	Head(f)	Hmon(f)	Sep(f)
u_{xxx}	1	1	u_{xxx}	u_{xxx}	1

HDT(f), Hp(f), Hcoef(f) and Sep(f) are most significant when it comes to using `diffgrob2` in practice (see later), and these may be found via the procedure **diffparse** with the command

diffparse(f, **allvars**, **termorder**, 'HDT', 'Hp', 'Hcoef', 'dt', 'Sep');

This will return each of the terms in quotes, where 'dt' is a list of all derivative terms and their powers occurring in f. This procedure is particularly useful when the expressions become large.

With the notation in place we can define a partial ordering between partial differential equations. Let f and g be partial differential equations, then $g < f$ if either one of the following conditions is satisfied

(i) HDT(g) $<$ HDT(f);

(ii) HDT(g) $=$ HDT(f) and Hp(g) $<$ Hp(f);

(iii) HDT(g) $=$ HDT(f), Hp(g) $=$ Hp(f) and Hcoef(g) $<$ Hcoef(f);

(iv) HDT(g) $=$ HDT(f), Hp(g) $=$ Hp(f), Hcoef(g) $=$ Hcoef(f) and $[g - \text{Hmon}(g)] < [f - \text{Hmon}(f)]$.

If the summands of f and g differ only by constant coefficients, then f and g are said to be of *equal rank*.

1.7.2 Pseudo-Reduction.

Definition 1.7.1. Let $\mathrm{DT}(f)$ occur in f to some power k with coefficient $\mathrm{Coef}(f,\mathrm{DT}(f)^k)$ and suppose $\mathrm{DT}(f) = \mathrm{D}^{\boldsymbol{\alpha}}\mathrm{HDT}(g)$, for some $\boldsymbol{\alpha}$, where if $|\boldsymbol{\alpha}| = 0$ the condition $k \geq \mathrm{Hp}(g)$ is also required. A *pseudo-reduction* to $\widetilde{f}$ of f with respect to g ($f > g$), denoted $f \to_g \widetilde{f}$ is defined by

$$\widetilde{f} = \begin{cases} \dfrac{\mathrm{Hcoef}(\mathrm{D}^{\boldsymbol{\alpha}}g)f - \mathrm{Coef}(f,\mathrm{DT}^k)\mathrm{DT}^{k-1}\mathrm{D}^{\boldsymbol{\alpha}}g}{\gcd[\mathrm{Hcoef}(\mathrm{D}^{\boldsymbol{\alpha}}g),\mathrm{Coef}(f,\mathrm{DT}^k)]} & \text{if } |\boldsymbol{\alpha}| \neq 0, \\ \dfrac{\mathrm{Hcoef}(g)f - \mathrm{Coef}(f,\mathrm{DT}^k)[\mathrm{HDT}(g)]^{k-\mathrm{Hp}(g)}g}{\gcd[\mathrm{Hcoef}(g),\mathrm{Coef}(f,\mathrm{DT}^k)]} & \text{if } |\boldsymbol{\alpha}| = 0, \end{cases}$$

where gcd denotes the greatest common divisor. The *pseudo-normal form* of f with respect to a set $G = \{g_1, g_2, \dots g_s\}$, denoted $\mathrm{normal}^p(f, G)$ is obtained when no further pseudo-reduction with respect to any member of G is possible.

Note that in this definition it is implicitly assumed that $\mathrm{Hcoef}(g) \neq 0$ and $\mathrm{Hcoef}(\mathrm{D}^{\boldsymbol{\alpha}}g) \neq 0$ (and thus $\mathrm{Sep}(g) \neq 0$) when evaluated on solutions of the system. Since these may be differential (non-constant) coefficients they may however be zero. Such incidences are called *singular cases* and the possibility of $\mathrm{Hcoef}(g) = 0$ or $\mathrm{Sep}(g) = 0$ occurring must be dealt with separately. Thus the usefulness of **diffparse** is now evident in that it is necessary to check the $\mathrm{Hcoef}(g)$ and $\mathrm{Sep}(g)$ of any g being used to pseudo-reduce any other partial differential equation f.

Note also that if $\mathrm{Hcoef}(g)$ and $\mathrm{Sep}(g)$ are simply functions of $(\boldsymbol{x}, \boldsymbol{u})$ with no differential consequences of $\boldsymbol{u}$ then pseudo-reduction becomes *strict reduction*. Similarly normal^p is replaced by normal. Also we define *algebraic reduction*

Definition 1.7.2. If f and g are two partial differential equations, we say that g *algebraically reduces* f to $\widehat{f}$ at the monomial M, where M is a summand of f, provided $\mathrm{Hmon}(g)|M$, where

$$\widehat{f} = f - \frac{M}{\mathrm{Hmon}(g)}g.$$

Now normal^p is replaced by normal^a. This type of reduction is useful when considering the calculation of determining equations by the method of Clarkson and Mansfield [1994c] (see §1.7.6.).

Example 1.7.1. Consider two partial differential equations

$$f \equiv uu_{xt} - u_x u_t, \qquad g \equiv u_{xx} - u_t - u(1-u)^2,$$

which are the side condition (1.78) and equation (1.1) respectively (cf. §1.5.2). With an (ALEX) ordering $t < x$, we can pseudo-reduce f with respect to g at both the u_t and u_{xt} terms

$$\begin{aligned} \widetilde{f} &= (-1)f - u\mathrm{D}_x g - (-u_x)g \\ &= -(uu_{xt} - u_x u_t) - u(u_{xxx} - u_{tx} - u_x + 4uu_x - 3u^2 u_x) + u_x(u_{xx} - u_t - u(1-u)^2) \\ &= 2u^3 u_x - 2u^2 u_x - uu_{xxx} + u_x u_{xx}. \end{aligned} \tag{1.106}$$

Since $\widetilde{f}$ cannot be pseudo-reduced further by g, $\widetilde{f}$ is the normal$^p(f,\{g\})$. Also note that normal$^p(f,\{g,\widetilde{f}\})=0$. This pseudo-reduction is in fact strict reduction as f is multiplied by (-1).

In `diffgrob2` pseudo-reduction is implemented with the procedure **reduce**, such that the pseudo-reduction of f with respect to $\boldsymbol{G}=\{g_1,g_2,\ldots,g_s\}$ takes the form

$$\textsf{reduce}(f,[\boldsymbol{G}],\textsf{allvars},\textsf{termorder},\text{'}\widetilde{f}\text{'});$$

In the following chapters, since **termorder** will always be **alex**, and **allvars** is usually the same throughout a calculation and is determined at the start, these are omitted from **reduce** in the text. Thus we will simply use **reduce**(f,[$\boldsymbol{G}$],$\widetilde{f}$).

There is another procedure, **reduceall**, which will carry out pseudo-reduction on a system $\boldsymbol{\Delta}$, such that each member of the result, $\boldsymbol{\Delta}_R$, is pseudo-reduced with respect to every other member, i.e. normal$^p(f,\boldsymbol{\Delta}_R\setminus\{f\})=f$ for any $f\in\boldsymbol{\Delta}_R$. This is achieved by successively using **reduce** on the system comprising of $\boldsymbol{\Delta}$ and the results of pseudo-reduction, $\widetilde{f}$, with respect to every other member of this system. The command

$$\textsf{reduceall}(\boldsymbol{\Delta},\textsf{allvars},\textsf{termorder},\text{'}\boldsymbol{\Delta}_R\text{'},\text{'Xset'});$$

achieves this, where the output 'Xset' is the set of all coefficients with which the elements of $\boldsymbol{\Delta}$ are multiplied in their pseudo-reduction. Thus the output is valid up to the elements of Xset being non-zero, and the singular cases, when they are zero, need to be run through again with these elements added to (or incorporated in) $\boldsymbol{\Delta}$.

Example 1.7.2. If we use **reduceall** on $\boldsymbol{\Delta}=(f,g)$ of Example 1.7.1, then $\boldsymbol{\Delta}_R=(\widetilde{f},g)$.

1.7.3 Cross-Differentiation.

Definition 1.7.3. Consider two partial differential equations f_1 and f_2. If $\mathrm{Hu}(f_1)=\mathrm{Hu}(f_2)$ let $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ be the smallest multi-indices possible such that $\mathrm{D}^{\boldsymbol{\alpha}_1}\mathrm{HDT}(f_1)=\mathrm{D}^{\boldsymbol{\alpha}_2}\mathrm{HDT}(f_2)$, then the *differential S polynomial* (diffSpoly) of f_1 and f_2 is defined to be (for $|\boldsymbol{\alpha}_1|$, $|\boldsymbol{\alpha}_2|$ not both zero)

$$\mathrm{diffSpoly}(f_1,f_2)=\begin{cases}\dfrac{\mathrm{Hcoef}(\mathrm{D}^{\boldsymbol{\alpha}_1}f_1)\mathrm{D}^{\boldsymbol{\alpha}_2}f_2-\mathrm{Hcoef}(\mathrm{D}^{\boldsymbol{\alpha}_2}f_2)\mathrm{D}^{\boldsymbol{\alpha}_1}f_1}{\gcd[\mathrm{Hcoef}(\mathrm{D}^{\boldsymbol{\alpha}_1}f_1),\mathrm{Hcoef}(\mathrm{D}^{\boldsymbol{\alpha}_2}f_2)]} & \text{for } |\boldsymbol{\alpha}_1||\boldsymbol{\alpha}_2|\neq 0,\\[2ex] \dfrac{\mathrm{Hcoef}(f_1)\mathrm{HDT}(f_1)^{\mathrm{Hp}(f_1)-1}\mathrm{D}^{\boldsymbol{\alpha}_2}f_2-\mathrm{Hcoef}(\mathrm{D}^{\boldsymbol{\alpha}_2}f_2)f_1}{\gcd[\mathrm{Hcoef}(f_1),\mathrm{Hcoef}(\mathrm{D}^{\boldsymbol{\alpha}_2}f_2)]} & \text{for } |\boldsymbol{\alpha}_1|=0,\end{cases}$$

and similarly for the case $|\boldsymbol{\alpha}_2|=0$, $|\boldsymbol{\alpha}_1|\neq 0$. Else if $|\boldsymbol{\alpha}_1|=|\boldsymbol{\alpha}_2|=0$ so that $\mathrm{HDT}(f_1)=\mathrm{HDT}(f_2)$, or if f_1, f_2 are nonlinear and $\mathrm{Hu}(f_1)\neq\mathrm{Hu}(f_2)$, then

$$\mathrm{diffSpoly}(f_1,f_2)=\frac{\mathrm{Head}(f_2)f_1-\mathrm{Head}(f_1)f_2}{\gcd[\mathrm{Head}(f_1),\mathrm{Head}(f_2)]}.$$

Otherwise if f_1, f_2 are linear and $\mathrm{Hu}(f_1) \neq \mathrm{Hu}(f_2)$, then $\mathrm{diffSpoly}(f_1, f_2) = 0$.

Example 1.7.3. Consider the two partial differential equations

$$f_1 \equiv u_{xx} - u_t - u(1-u)^2, \qquad f_2 \equiv 2u^3u_x - 2u^2u_x - uu_{xxx} + u_xu_{xx},$$

which are g and $\widetilde{f}$ from Example 1.7.1. Again with an (ALEX) ordering $t < x$, we have

$$\begin{aligned} \mathrm{diffSpoly}(f_1, f_2) &= (-1)D_tf_2 - (-u)D_{xxx}f_1 \\ &= uu_{xxxxx} - uu_{xxx} + 12uu_xu_{xx} + 4u^2u_{xxx} - 6uu_x^3 \\ &\quad - 18u^2u_xu_{xx} - 3u^3u_{xxx} + u_tu_{xxx} + 4uu_xu_t \\ &\quad + 2u^2u_{xt} - 6u^2u_xu_t - 2u^3u_{xt} - u_{xt}u_{xx} - u_xu_{xxt}. \end{aligned} \tag{1.107}$$

The procedure in `diffgrob` that finds the differential S polynomial is called **diffSpoly**.

1.7.4 Differential Gröbner Bases and the Kolchin-Ritt algorithm.

Given a system of partial differential equations $\boldsymbol{\Delta}$, the differential ideal generated by $\boldsymbol{\Delta}$ is

$$I(\boldsymbol{\Delta}) = \left\{ \sum_{\boldsymbol{\alpha},i} g_{\boldsymbol{\alpha},i}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(\infty)})\mathrm{D}^{\boldsymbol{\alpha}}\Delta_i \,\middle|\, \Delta_i \in \boldsymbol{\Delta}, \boldsymbol{\alpha} \in \mathbb{N}^p \right\}.$$

Definition 1.7.4. We define a *Differential Gröbner Basis* of $I(\boldsymbol{\Delta})$ as a set of generators $\boldsymbol{G}$ of $I(\boldsymbol{\Delta})$ such that $\mathrm{normal}^p(f, \boldsymbol{G}) = 0$ for every element f of $I(\boldsymbol{\Delta})$.

Thus the ultimate aim is to find the differential Gröbner basis for our system $\boldsymbol{\Delta}$, which in general is simpler to solve than $\boldsymbol{\Delta}$, yet contains all the information in $\boldsymbol{\Delta}$. This is done via pseudo-reduction and the finding of differential S polynomials, though as we shall see in the following chapters, even if we are unable to go so far, a lot of information is provided along the way. This information, in the form of the $\widetilde{f}$ of pseudo-reduction and the differential S polynomials are called *compatibility conditions.* Often the **reduceall** procedure provides enough information for the system to be solved, see for instance Chapters Three and Four. However there is an algorithm which, for linear systems or in general for systems in which the 'Xset' contains only non-zero terms, guarantees the output of a differential Gröbner basis. For all systems however it is very powerful, and is called the *Kolchin-Ritt* algorithm, and is outlined here:

Given a system of partial differential equations $\boldsymbol{\Delta}$ and an ordering **termorder**, the Kolchin-Ritt algorithm outputs a set of equations $\boldsymbol{\Delta}_{KR}$ such that $I(\boldsymbol{\Delta}_{KR}) = I(\boldsymbol{\Delta})$ and $\mathrm{normal}^p(\textbf{diffSpoly}(f_i, f_j), \boldsymbol{\Delta}_{KR}) = 0$ for each pair $f_i, f_j \in \boldsymbol{\Delta}_{KR}$.

This is achieved by taking the **diffSpoly** of each pair of equations in the system consisting of $\boldsymbol{\Delta}$ and any compatibility conditions obtained *en route.* The pseudo-normal form of the

diffSpoly is found with respect to $\boldsymbol{\Delta}$ and any compatibility conditions so far obtained. If the resulting equation is non-zero it is called a compatibility condition. This process continues until no new compatibility conditions are found.

The output, $\boldsymbol{\Delta}_{KR}$, fails to be a differential Gröbner basis if the so-called *S-set*, the set generated by all factors of the Hcoefs and Seps of the output set, contains an element that is necessarily zero on all solutions of the system, in other words, is in the ideal generated by the output set. Typically this occurs when, for some $f \in \boldsymbol{\Delta}_{KR}$, HDT$(f)$ appears inside a factor raised to some power > 1. An example of this can be seen in Mansfield [1996]. However, if one of the equations in $\boldsymbol{\Delta}_{KR}$ factors, then $\boldsymbol{\Delta}_{KR}$ is not the most beneficial output. When we systematically choose these factors to be zero in turn and append them to the system $\boldsymbol{\Delta}$, more information will be gained from the Kolchin-Ritt algorithm; this we will see in the following example.

The Kolchin-Ritt algorithm is implemented in `diffgrob2` with the command

KolRitt($\boldsymbol{\Delta}$, **allvars**, **termorder**, '$\boldsymbol{\Delta}_{KR}$', **info**={'Xset'});

which also implements **orthreduceall** at the beginning and end of the algorithm to minimise the calculations and to simplify the output. The procedure **orthreduceall** is simply **reduceall** except that only strict reduction is allowed.

Example 1.7.4. Consider again equations f and g from Example 1.7.1

$$f \equiv uu_{xt} - u_x u_t, \qquad g \equiv u_{xx} - u_t - u(1-u)^2,$$

with an (ALEX) ordering $t < x$. The **orthreduceall** procedure will give the same output as **reduceall** on $\boldsymbol{\Delta} = (f, g)$, i.e. $\boldsymbol{\Delta}_{OR} = (\widetilde{f}, g)$ (cf. Example 1.7.2). Now rename $g \equiv f_1$ and $\widetilde{f} \equiv f_2$ to be consistent with Example 1.7.3. The next procedure to be carried out in **KolRitt** is that of taking the **diffSpoly** of f_1 and f_2, as seen in Example 1.7.3, with result (1.107). If we now **reduce**((1.107),[f_1,f_2],$k1$) in order to find $k1 = \text{normal}^p((1.107), \{f_1, f_2\})$ the result is, upon factoring

$$k1: \qquad u^2 u_x (2u-1)(u^3 - 2u^2 + u - u_{xx}).$$

The set $\{f_1, f_2, k1\}$ is the output of **KolRitt** in this instance, i.e. $\boldsymbol{\Delta}_{KR} = \{f_1, f_2, k1\}$ since **diffSpoly**(f_1, f_2), **diffSpoly**$(f_1, k1)$ and **diffSpoly**$(f_2, k1)$ all **reduce** to zero with respect to $f_1, f_2, k1$. The output $\boldsymbol{\Delta}_{KR}$ is also a differential Gröbner basis for the set $\{f, g\}$, though as $k1$ factors we can gain more information by including these factors in turn in the original system $\boldsymbol{\Delta} = (f, g)$.

The first factor gives the trivial solution $u = 0$. The second factor, u_x, when included in the system $\boldsymbol{\Delta}$ will give a differential Gröbner basis after the **orthreduceall** step. We are simply left with

$$\{u_x = 0, u_t + u(1-u)^2 = 0\},$$

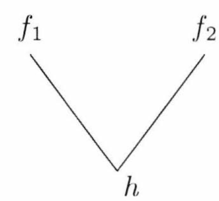
as the differential Gröbner basis. This describes spatially independent solutions. The third factor, $2u - 1$, gives a contradiction when reducing g with respect to it, so is not a solution. Finally the fourth factor, $u(1-u)^2 - u_{xx}$, gives the time independent solution from the differential Gröbner basis

$$\{u_t = 0, u(1-u)^2 - u_{xx} = 0\},$$

found once again by the strict reduction of g with respect to this factor.

1.7.5 A Direct Search strategy and the solution of overdetermined systems in this thesis.

Whilst the termination of the Kolchin-Ritt algorithm has been proved (Mansfield and Fackerell [1992]), the main problem that it encounters is that of "expression swell", in that the length of expressions obtained can become very large and so exceed the memory limit of the available computer. More strategies are available (see Mansfield [1993]) to overcome this, and here we consider the Direct Search strategy, which must be carried out interactively in `diffgrob2`. It is perhaps best described by Figure 1.7.1, where a triangle



represents the differential S polynomial of f_1 and f_2, pseudo-reduced with respect to all the equations in the system, G, i.e. $h = \text{normal}^p(\textsf{\textbf{diffSpoly}}(f_1, f_2), G)$.

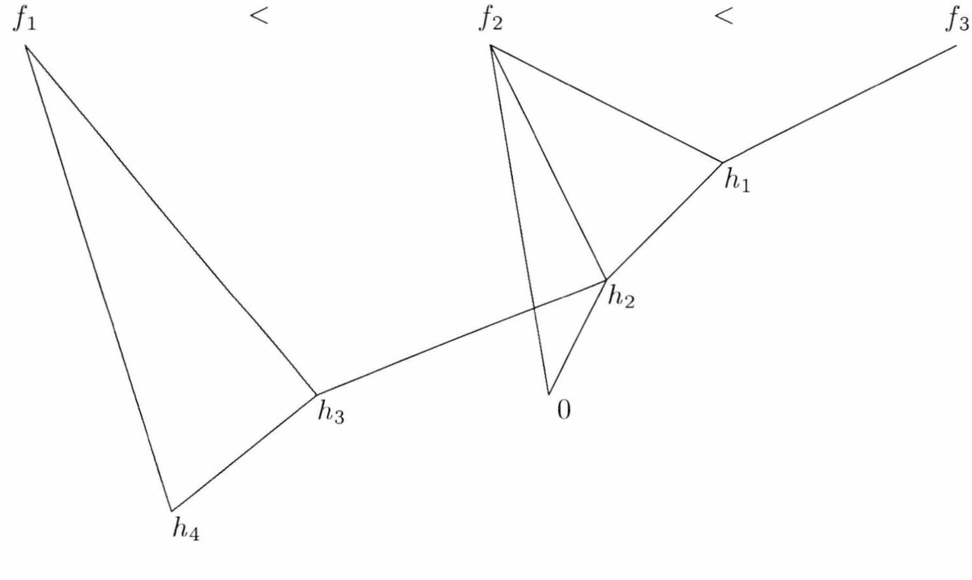


Figure 1.7.1: Schematic representation of a Direct Search strategy

This is really only a template strategy and whilst most of the interactive use of `diffgrob2` in this thesis uses it, we find slight modifications are often more helpful. Indeed one could change the $<$ to $>$, which is then more akin to a Sort strategy (cf. Mansfield [1993]), which is also worthwhile bearing in mind. In fact combinations of the two may also be useful. For instance equations that look long and daunting may pseudo-reduce with respect to lower order equations to something quite simple though certainly not always: in Case 5.3.2 in Chapter Five, which requires the solution of the system of equations in Appendix B, equation (B.1x) quickly gives useful information whilst equations (B.1ix) and (B.1xi) are often the last to be solved. The Direct Search strategy is now illustrated with the following example.

Example 1.7.5. Consider the three equations which come from the nonclassical determining equations in Chapter Two

$$\begin{aligned} f_1 &: \quad \xi_u, \\ f_2 &: \quad k_{uu}\phi_u + k_{uuu}\phi + 2\xi_x k_{uu}, \\ f_3 &: \quad \phi_{uu}, \end{aligned} \tag{1.108}$$

where $k_{uu} \neq 0$, from which we wish to find conditions on $k(u)$. As such, the derivatives of $k(u)$ can be thought of as the coefficients in what is essentially a differential version of finding the echelon form of a matrix. The equations (1.108) fit into the template above with an ordering $x < t < u$ and $\xi < k < \phi$ and note that (ξ, ϕ) depend on (x, t, u). In this ordering $\mathrm{HDT}(f_2) = \phi_u$ and the other HDTs are obvious, as f_1 and f_3 are single term equations. To eliminate ϕ_u we first take the **diffSpoly** of f_2 and f_3 to yield

$$\mathrm{D}_u f_2 - k_{uu} f_3 = 2k_{uuu}\phi_u + k_{uuuu}\phi + 2k_{uuu}\xi_x + 2k_{uu}\xi_{xu}, \tag{1.109}$$

where D is the total derivative operator. We **reduce** this with respect to f_1, f_2 and f_3. Note that f_3 is redundant in this pseudo-reduction as $f_3 >$ (1.109). Now f_2 will remove occurrences of ϕ_u and f_1 the ξ_{xu} term to yield

$$h_1 : \quad 2k_{uu}k_{uuu}\xi_x + (2k_{uuu}^2 - k_{uu}k_{uuuu})\phi,$$

which must be zero on analytic solutions of $\{f_1, f_2, f_3\}$. In order to eliminate ϕ, we take the **diffSpoly** of h_1 and f_2 to yield

$$\begin{aligned} k_{uu}\mathrm{D}_u(h_1) - (2k_{uuu}^2 - k_{uu}k_{uuuu})f_2 =& (4k_{uu}k_{uuu}k_{uuuu} - k_{uu}^2 k_{uuuuu} - 2k_{uuu}^3)\phi \\ & - 2k_{uu}k_{uuu}^2\xi_x + 4k_{uu}^2 k_{uuuu}\xi_x + 2k_{uu}^2 k_{uuu}\xi_{xu}. \end{aligned} \tag{1.110}$$

We **reduce** this with respect to all of f_1, f_2, f_3 and h_1. Here f_2 and f_3 are superfluous in the pseudo-reduction as $f_3 > f_2 >$ (1.110), though this makes no difference to the final result. However h_1 will remove occurrences of ϕ, and again f_1 will remove the ξ_{xu} derivative, to yield

$$h_2 : \quad \xi_x k_{uu}^2(-2k_{uu}k_{uuuu}^2 + k_{uu}k_{uuuuu}k_{uuu} + k_{uuuu}k_{uuu}^2),$$

which gives a condition on $k(u)$ as required.

There are three observations that are noteworthy here. Firstly, whilst this follows the Direct Search strategy the system (1.108) is only part of a larger set of equations, the determining equations (2.11), so this is not a perfect Direct Search on the whole system. Secondly, had we changed to a Sort strategy, in this case the steps and final result would have been the same. Finally, $\text{Hcoef}(h_1) = 2k_{uuu}^2 - k_{uu}k_{uuuu}$ which might be zero, and since we have pseudo-reduced with respect to h_1, the possibility of $\text{Hcoef}(h_1) = 0$ needs to be considered separately. It turns out, however, that this is a special case of the fifth order equation in $k(u)$ which is a factor of h_2, so we may rejoin the general case.

Thus the Direct Search strategy is kept in mind when solving overdetermined systems in this thesis, but we could not claim to hold to it to the letter, and a certain amount of trial and error is inevitable.

Another strategy that is combined with the Direct Search strategy is that of explicitly solving some of the equations, particularly linear ones, or perhaps integrating them if finding the explicit solution is not possible. This is of particular use in solving the determining equations for expressions in the ϕ_i which are often polynomial in the "dependent" variables $\boldsymbol{u}$. Consider a system of determining equations with dependent variables (ξ, ϕ) and independent variables (x, t, u). It is often the case that $\xi_u = \phi_{uu} = 0$, so in this instance rather than maintaining a general u-dependence for (ξ, ϕ) and constantly having to pseudo-reduce with respect to $\{\xi_u, \phi_{uu}\}$, which can take up considerable computer time, we can **reduce** the system with respect to

$$f_4 \equiv \xi(x,t,u) - F(x,t), \qquad f_5 \equiv \phi(x,t,u) - G(x,t)u - H(x,t).$$

Now we may equate coefficients of powers of u to zero, and so possibly gain many equations from a single equation. We now use our Direct Search strategy on this new system. However even if we only knew f_4, and ϕ still had a general u-dependence, so coefficients of powers of u could not be equated to zero, the strategy still cuts down unnecessary differentiation and pseudo-reduction, and so saves time.

Notice that using f_4 can only be advantageous: we have only reduced the number of indeterminates, without increasing the number of determinates. However in general one tries to balance the decrease in indeterminates with the increase in determinates, such as in f_5. Increasing the determinates without a notable decrease in the number of indeterminates is not advantageous. Another balance one must consider is that of increasing the number of determinates via integration generally, versus the decrease in the order of the equation. By decreasing the order of the equation one tends to decrease expression swell. Personally, the choice of increasing the number of determinates is usually preferred. Although this way may take longer, as more determinates must be found, it seems to give more information and therefore the complete solution set is more likely to be found.

An obvious drawback of this type of strategy comes from Example 1.7.5 above. If we **reduce** our system (1.108) with respect to both f_4 and f_5 we are left with only a single equation. In fact we can get round this problem by creating two more equations by differentiating the single equation with respect to u once, then twice, and removing occurrences of H and G between the equations. This gives the same result but is perhaps not as pleasing as Example 1.7.5. If we had only reduced the system with respect to f_4 the working would have been essentially the same as Example 1.7.5, with a small decrease in differentiation.

Finally we mention the problem of spurious cases, which result from equations that factor, and when pseudo-reduction with respect to a partial differential equation f takes place where Hcoef(f) or Sep(f) may be zero (which includes equations that factor). Spurious cases are those that end up having no solution, i.e. they lead to contradictions. Ideally we avoid having to split up the calculation at all, by choosing to pseudo-reduce with respect to equations, f, with guaranteed non-zero Hcoef(f) and Sep(f), but in practice this is often not possible. Indeed it seems that the pay off for being able to solve the systems we encounter is that we must consider many cases to do so, particularly when the equations whose symmetries we are seeking contain arbitrary functions or arbitrary constants (as in Chapters Two through Five). Thus we don't regret the build up of cases, which careful accounting keeps track of, even if they turn out to be spurious.

1.7.6 The Clarkson-Mansfield algorithm for finding nonclassical determining equations II.

Now we have introduced some theory and notation, a truer statement of the algorithm of Clarkson and Mansfield [1994c] for determining the nonclassical determining equations of a system can be given. Note that it uses only algebraic reduction, so rather than finding a differential Gröbner basis in the algorithm, a Gröbner basis is sought, which has a similar definition to its differential counterpart (cf. Definition 1.7.4) except that all calculations are algebraic, so we require normal$^a(f, \boldsymbol{G}) = 0$, rather than normal$^p(f, \boldsymbol{G}) = 0$. For convenience we denote the Gröbner basis of a system $\boldsymbol{\Delta}$, to be GB($\boldsymbol{\Delta}$, **termorder**), which depends on the ordering, **termorder**. For their algorithm the ordering may be a lexicographic or a Bayer-Stillman ordering (Bayer and Stillman [1987]) which is not described here.

Given a system of m equations

$$\boldsymbol{\Delta}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(n)}) = 0,$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_p)$ and $\boldsymbol{u} = (u_1, u_2, \ldots, u_q)$, we assume without loss of generality that $\boldsymbol{\Delta}$ contains all the relevant differentiations of each equation up to order n (cf. Ψ^*

below for the invariant surface conditions). Recall that there are p cases to consider: for $1 \leq k \leq p$ we successively set $\xi_k = 1$ and $\xi_{k+1} = \ldots = \xi_p = 0$. In the kth case the invariant surface conditions read

$$\psi_s \equiv \sum_{j=1}^{k-1} \xi_j u_{s,j} + u_{s,k} - \phi_s, \qquad \text{for } s = 1, 2, \ldots, q.$$

Given also an ordering, **termorder**, the algorithm in the kth case reads

$$\Psi^* := \{\mathrm{D}^{\boldsymbol{\alpha}} \psi_s \mid 1 \leq s \leq q, \boldsymbol{\alpha} \in \mathbb{N}^p, |\boldsymbol{\alpha}| \leq n-1\}$$
$$\mathcal{K} := \{\mathrm{normal}^a(\Delta_j, \Psi^*) \mid \Delta_j \in \boldsymbol{\Delta}\}$$
$$\mathcal{I}nf := \{\mathrm{pr}^{(n)}\mathbf{v}(f) \mid f \in \mathcal{K}\}$$
$$\mathcal{GB} := \mathrm{GB}(\mathcal{K}, \textbf{termorder})$$
$$\mathcal{RI}nf := \{\mathrm{normal}^a(f, \mathcal{GB}) \mid f \in \mathcal{I}nf\}$$
$$\mathcal{D}et\mathcal{E}qns := \{\mathrm{Coef}(f, \boldsymbol{u}^{(n)}) = 0 \mid f \in \mathcal{RI}nf\}$$

where $\mathcal{D}et\mathcal{E}qns$ are the nonclassical determining equations for $\boldsymbol{\Delta}$. The design of $\boldsymbol{\Delta}$ allows this algorithm to cope with compatibility conditions in the original system, so that the true determining equations are calculated in such cases. This algorithm is easily adapted to correspond to a classical method algorithm: simply let $\mathcal{K} := \boldsymbol{\Delta}$.

1.8 The MACSYMA package `symmgrp.max`

The first step of the classical method, that of generating the determining equations, is entirely algorithmic and as a result symbolic manipulation packages have been written to aid the calculations. An excellent survey of the different packages available and a description of their strengths and weaknesses is given by Hereman [1994]. Heuristic procedures have been implemented in some of these programs to try to solve the determining equations, the second step of the classical method, and are largely successful, though not infallible: they are notoriously inadequate at finding special solutions, those other than the general solution, when the determining equations depend on arbitrary functions or arbitrary constants.

Since we prefer the method of differential Gröbner bases to solve the determining equations, which has proved effective in coping with such difficulties (cf. Chapters Two through Five), we use a package that concentrates solely on the first step of the classical method. Written by Champagne, Hereman and Winternitz [1991], it is a MACSYMA package called **`symmgrp.max`**, and has been tested extensively by the authors and many others. Crucial to our needs it may be adapted to generate the determining equations of the nonclassical method by knowledge of the internal syntax, and both arbitrary constants and arbitrary functions may be present in the system under consideration.

Once again a small amount of notation is required: the translation of the original variables into the `symmgrp.max` syntax is $x_i \to$ x[i], $u_j \to$ u[j] and derivatives

$$\frac{\partial^{|\alpha|} u_j}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_p^{\alpha_p}} \to \mathsf{u}[j,[\alpha_1,\alpha_2,\dots,\alpha_p]].$$

The infinitesimals translate as $\xi_i \to$ **eta**i and $\phi_j \to$ **phi**j, and these in the input may be recognised as `symmgrp.max`'s own notation for the infinitesimals. This is the phenomenon that allows `symmgrp.max`, a package initially designed to find classical determining equations, to find nonclassical determining equations also.

To plunge straight in, consider the two files needed to run `symmgrp.max` as a batch job to find the classical determining equations for our prototype equation (1.1)

clfnaone.case:

```
batchload("symmgrp.max")$
writefile("clfnaone.out")$
batch("clfnaone.dat")$
symmetry(1,0,0)$
derivabbrev:true$
printeqn(lode);
save("clfnaone.lsp",lode)$
for j thru p do (x[j]:=concat(x,j))$
for j thru q do (u[j]:=concat(u,j))$
ev(lode)$
clfnaoneode:ev(%,x1=x,x2=t,u1=u)$
grind:true$
stringout("clfnaoneode",clfnaoneode)$
closefile()$
```

clfnaone.dat:

```
p:2$
q:1$
m:1$
parameters:[]$
warnings:true$
sublisteqs:[all]$
subst_deriv_of_vi:true$
info_given:true$
highest_derivatives:all$
depends([eta1,eta2,phi1],[x[1],x[2],u[1]])$
e1:u[1,[0,1]]-u[1,[2,0]]+u[1]*(1-u[1])**2;
v1:u[1,[2,0]];
```

It should be noted that `symmgrp.max` was designed to cope with arbitrarily large systems where some interactive use would be necessary. Thus this brief description of the commands may give an indication of what it is capable of, however as each of the systems that we consider is small enough to run as a batch job, we refer the reader to the authors' detailed description, Champagne, Hereman and Winternitz [1991], to see `symmgrp.max`'s full potential. Many of the commands are simply MACSYMA commands; for details of such commands one can consult the MACSYMA Reference Manual [1988].

Starting with **clfnaone.dat**, which contains mostly `symmgrp.max` commands, we note that the system under consideration in general is still

$$\boldsymbol{\Delta}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(n)}) = 0,$$

with **m** equations, **p** independent variables and **q** dependent variables, hence for (1.1) **m=1**, **p=2** and **q=1**. The **parameters** command allows the user to declare non-zero constants, so they may be removed if they are factors of a determining equation (though note that since the flag for **warnings** is **true** such factors will be listed). One includes arbitrary functions by declaring the variables they depend on with the MACSYMA **depends** command, and then simply putting them in the equations (the e*i*); their derivatives are represented in the usual MACSYMA notation.

The next five commands in **clfnaone.dat** ensure the classical method is applied in full, and must always be included. Slight modifications of their flags or variables allow the calculations to be split into pieces, for instance by considering a subset of the system, or by only taking the determining equations from the highest derivative terms, which are smaller in general. For full details see Champagne, Hereman and Winternitz [1991].

The MACSYMA **depends** command allows us to declare that the infinitesimals depend on all the new independent variables. The equations of the system are called e*i* for i = 1,2,...,m, and the derivatives that are to be substituted back for (the v_i of §1.2) are denoted v*i*, and there must be **m** of them. These v*i* must be chosen in the same manner as the v_i (cf. Step One (b) of the classical method).

The majority of commands in **clfnaone.case** are MACSYMA commands, and include details not central to the actual calculation of the determining equations but the controlling of the output. The first three lines load the package `symmgrp.max`, create a file **clfnaone.out** to send the output to, and read in **clfnaone.dat** respectively. The determining equations are generated via the **symmetry** command which can have **0** or **1** for each of its three arguments (see Champagne, Hereman and Winternitz [1991]). The remainder of the file prints the determining equations via the command **printeqn**, first saves them in internal MACSYMA notation, then saves them in the file **clfnaoneode** in a notation that can easily be translated into `diffgrob2` notation. In fact a tiny amount of editing on the file **clfnaoneode** allows it to be read in by `diffgrob2` and converted to its syntax via the `diffgrob2` command **max2dg**.

Thus entering MACSYMA and typing the command **batch("clfnaone.case");** yields

```
/*******************************************************/
/*      WELCOME TO THE MACSYMA PROGRAM FOR THE         */
/*         CALCULATION OF THE SYMMETRY GROUP           */
/*                  IN BATCH MODE                      */
/*******************************************************/
*** Number of determining equations before simplifications:  9 .  ***
*** Number of determining equations after simplifications:  7 .  ***
*** These determining equations are stored in LODE. ***
```

Notice that `symmgrp.max` has made simplifications. These ensure that the determining equations are free from redundant factors, repetition and trivial differential consequences. The remainder of the screen output is then the determining equations which it displays as

```
                  deta2
Equation 1 :      ----- = 0
                   du
                     1
                  deta2
Equation 2 :      ----- = 0
                   dx
                     1
```

```
                 deta1
Equation 3 :     ----- = 0
                  du
                    1
                  2
                 d phi1
Equation 4 :     ------ = 0
                    2
                  du
                    1

Equation 5 :
deta2     deta1
----- - 2 ----- = 0
 dx        dx
   2         1
Equation 6 :
   2                  2
  d phi1     deta1   d eta1
2 ------- + ----- - ------ = 0
  du   dx      dx       2
    1    1       2    dx
                        1
Equation 7 :
         2
dphi1   d phi1    3 dphi1       2 dphi1       dphi1       2
----- - ------ - u  ----- + 2 u  ----- - u   ----- + 3 u  phi1
 dx        2      1  du         1  du        1  du         1
   2     dx            1              1            1
           1
                                       3 deta1      2 deta1        deta1
        - 4 u  phi1 + phi1 + 2 u  ----- - 4 u  ----- + 2 u  ----- = 0
             1                    1  dx        1  dx        1  dx
                                       1             1            1
```

To generate the nonclassical determining equations for (1.1) we use the method of Clarkson and Mansfield [1994c], described in §1.3.1 and §1.7.6. We only consider the case when $\tau = 1$ as the working when $\tau = 0$ is similar. As discussed in §1.3.1, we simply replace the occurrence of u_t by the invariant surface condition

$$u_t = \phi - \xi u_x.$$

Hence we consider the file **nclfnaone.dat**

```
p:2$
q:1$
m:1$
parameters:[]$
warnings:true$
sublisteqs:[all]$
subst_deriv_of_vi:true$
info_given:true$
highest_derivatives:all$
depends([eta1,phi1],[x[1],x[2],u[1]])$
eta2:1$
e1:phi1-eta1*u[1,[1,0]]-u[1,[2,0]]+u[1]*(1-u[1])**2;
v1:u[1,[2,0]];
```

As discussed before, `symmgrp.max` recognises the infinitesimals **eta1**, **eta2** and **phi1** as its own notation which is why the package may be adapted to generate nonclassical determining equations. The other file **nclfnaone.case** is the same as **clfnaone.case** except with occurrences of the string **clfnaone** replaced by the string **nclfnaone**. This yields

```
/*******************************************************/
/*      WELCOME TO THE MACSYMA PROGRAM FOR THE         */
/*        CALCULATION OF THE SYMMETRY GROUP            */
/*                 IN BATCH MODE                       */
/*******************************************************/
*** Number of determining equations before simplifications:  4 .  ***
*** Number of determining equations after simplifications:  4 .  ***
*** These determining equations are stored in LODE. ***


                  2
                 d eta1
Equation 1 :     ------ = 0
                     2
                  du
                    1
Equation 2 :
 2           2
d phi1      d eta1           deta1
------ - 2 ------- + 2 eta1 ----- = 0
   2        du  dx            du
 du           1   1             1
   1
Equation 3 :
   2                                      2
  d phi1       deta1         deta1    d eta1            deta1      3 deta1
2 ------- - 2 ----- phi1 + ----- - ------ + 2 eta1 ----- - 3 u  -----
  du   dx      du            dx         2            dx         1  du
    1    1       1             2      dx               1              1
                                        1
                                                  2 deta1          deta1
                                           + 6 u  ----- - 3 u  ----- = 0
                                                1   du          1  du
                                                      1              1
Equation 4 :
            2
dphi1    d phi1      3 dphi1      2 dphi1       dphi1     deta1
----- - ------ - u  ----- + 2 u  ----- - u  ----- + 2 ----- phi1
 dx        2          1  du         1  du       1  du        dx
   2     dx              1             1            1          1
           1
          2                                  3 deta1      2 deta1
   + 3 u  phi1  - 4 u  phi1 + phi1 + 2 u  ----- - 4 u  -----
        1                                  1  dx          1  dx
                                               1             1
          deta1
   + 2 u  ----- = 0
        1  dx
             1
```

Whilst in this example the Clarkson-Mansfield approach to generating the determining equations seems to be no more advantageous (see also §1.3.1), for equations which have higher order and mixed t derivatives it is certainly so. For instance the equations in Chapters Three through Five would require a large number of differential consequences of the invariant surface condition to be included in the system. These would each have to be back-substituted for in the correct order, and possibly more than once to get `symmgrp.max` to generate the correct determining equations – it almost doesn't bear thinking about!

Part I

Scalar Equations

Chapter Two : A Generalised Boussinesq Equation

2.1 Introduction

In this chapter we consider a generalised Boussinesq equation

$$u_{tt} = [D(u)]_{xx} + u_{xxxx}, \tag{2.1}$$

where $D(u)$ is an arbitrary sufficiently differentiable function, with the condition that $D_{uu} \neq 0$ to ensure nonlinearity.

The Boussinesq equation which (2.1) generalises is

$$u_{tt} = (\tfrac{1}{2}u^2)_{xx} + u_{xxxx}, \tag{2.2}$$

which is a soliton equation solvable by inverse scattering (see Ablowitz and Haberman [1975], Caudrey [1980,1982], Deift, Tomei and Trubowitz [1982], Zakharov [1974]), originally used by Boussinesq [1871,1872] to describe the propagation of long waves in shallow water. It has been used since to model many other physical phenomena, including one-dimensional nonlinear lattice waves (Zabusky [1967], Toda [1975]), vibrations in a nonlinear string (Zakharov [1974]), and ion sound waves in plasma (Scott [1975]).

The $\tau \neq 0$ nonclassical reductions of the Boussinesq equation (2.2), which we will rederive here, were first found by Clarkson and Kruskal [1989] using the direct method (for $z_x \not\equiv 0$). Later Levi and Winternitz [1989] used the nonclassical method with $\tau \neq 0$ on (2.2) and found the results to be the same as Clarkson and Kruskal's. The $\tau = 0$ results were found by Clarkson [1990] and Lou [1990] by considering the equivalent case in the direct method, namely $z_x \equiv 0$. A comparison of all the direct and nonclassical reductions can be found in Clarkson [1995].

The equation (2.1) itself appears in Rosenau [1986] as a model for propagation of pulses along a transmission line made of a large number of LC-circuits. It is also used as a model to describe vibrations of a single one-dimensional dense lattice (Rosenau [1987,1988]). However Rosenau rejects equation (2.1) in these studies as it is ill-posed for his problems, and assumptions must be made on the nonlinearity of $D(u)$.

Flytzanis, Pnevmatikos and Peyrard [1989] derive the equation

$$u_{tt} - c_0^2 u_{xx} - p(u^2)_{xx} - q(u^3)_{xx} - hu_{xxxx} = 0, \tag{2.3}$$

from a consideration of a one-dimensional monatomic lattice with only nearest neighbour interactions with a cubic-quartic potential. Equation (2.3) can be regenerated by considering the simpler $D(u) = u^3 + cu$ and an appropriate rescaling of x, t, u and translating of u.

In this chapter we find conditions on $D(u)$ such that it allows symmetries, in particular those beyond the (obvious) translational symmetries of the independent variables. We use the classical Lie method (§2.2), and the nonclassical method in both the generic ($\tau \neq 0$) and non-generic ($\tau = 0$) cases (§§2.3,4), to find these symmetries. The MAPLE package `diffgrob2` plays an important role in both the classification of $D(u)$ and the solution of the determining equations (see §1.7 for details). Once the symmetries have been found we find the associated reductions and test whether the ordinary differential equations thus found are of Painlevé-type (see §1.6 for details). In §2.5 we discuss our results.

2.2 Classical symmetries

To apply the classical method we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \tag{2.4i}$$

$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \tag{2.4ii}$$

$$u^* = u + \varepsilon\phi(x,t,u) + O(\varepsilon^2), \tag{2.4iii}$$

where ε is the group parameter. This procedure, which is implemented in `symmgrp.max`, yields a system of linear determining equations in ξ, τ, ϕ, though the presence of our arbitrary function $D(u)$ makes them essentially nonlinear.

$$\tau_u = 0, \tag{2.5i}$$

$$\tau_x = 0, \tag{2.5ii}$$

$$\xi_u = 0, \tag{2.5iii}$$

$$\phi_{uu} = 0, \tag{2.5iv}$$

$$\xi_t = 0, \tag{2.5v}$$

$$2\phi_{xu} - 3\xi_{xx} = 0, \tag{2.5vi}$$

$$2\phi_{tu} - \tau_{tt} = 0, \tag{2.5vii}$$

$$\tau_t - 2\xi_x = 0, \tag{2.5viii}$$

$$D_{uu}\phi_u + D_{uuu}\phi + 2\xi_x D_{uu} = 0, \tag{2.5ix}$$

$$\phi_{tt} - \phi_{xxxx} - D_u\phi_{xx} = 0, \tag{2.5x}$$

$$6\phi_{xxu} + D_{uu}\phi + 2\xi_x D_u - 4\xi_{xxx} = 0, \tag{2.5xi}$$

$$2D_{uu}\phi_x + 4\phi_{xxxu} + 2D_u\phi_{xu} - \xi_{xx}D_u - \xi_{xxxx} = 0. \tag{2.5xii}$$

Applying the Kolchin-Ritt algorithm to system (2.5) with a lexicographic ordering with $\xi < \tau < D < \phi$ yields (amongst others) the equation

$$\xi_x(D_u D_{uu} D_{uuuu} - 2D_u D_{uuu}^2 + D_{uu}^2 D_{uuu}) = 0, \tag{2.6}$$

from which we can classify which $D(u)$ are suitable candidates for classical reductions. Clearly $D_{uuu} = 0$ is one solution, then assuming $D_{uuu} \neq 0$ we divide (2.6) by $D_u D_{uu} D_{uuu}$ to give a logarithmic differential. Solving this equation for $D(u)$ we find there are four canonical cases to consider,

(i) $\xi_x = 0$,

(ii) $D(u) = u^n$, for $n \neq 0, 1$,

(iii) $D(u) = \mathrm{e}^u$,

(iv) $D(u) = \ln u$,

where we have used the invariance of (2.1) under (constant) translations of $D(u)$, and we have scaled and translated u (by constants) as convenient.

Case 2.2.1 $\xi_x = 0$. It is easy to see from (2.5vi,xi) that when $\xi_x = 0$, requiring $D_{uu} \neq 0$ we have $\phi = 0$. Both ξ and τ are then constant, but $D(u)$ remains arbitrary. We have found the travelling wave reduction

Reduction 2.2.1. We choose $\xi = c$, $\tau = 1$ without loss of generality to yield

$$u(x,t) = w(z), \qquad z = x - ct,$$

where $w(z)$ satisfies

$$w'''' + D_{ww}(w')^2 + D_w w'' + c^2 w'' = 0,$$

where $D = D(w)$. This may be integrated twice to yield

$$w'' + D(w) + c^2 w = Az + B, \tag{2.7}$$

where A and B are arbitrary constants. This equation falls into the classification of Painlevé and his colleagues who look for equations that are of Painlevé-type, for algebraic

$D(w)$ (cf. §1.6). From Ince [1956] we see that either $D(w)$ is quadratic in w, then (2.7) may be solved in terms of the first Painlevé equation, PI ($A \neq 0$) or Weierstrass elliptic functions ($A = 0$), or $D(w)$ is cubic in w, in which case we require $A = 0$ and (2.7) may be solved in terms of Jacobi elliptic functions. For non-algebraic $D(w)$ we consider here the two special cases which are non-algebraic above, namely $D(w) = \mathrm{e}^w$ and $D(w) = \ln w$.

If $D(w) = \mathrm{e}^w$ we differentiate (2.7) and make the transformation $w(z) = \ln W(z)$ so that (2.7) becomes rational. However applying the ARS algorithm to this equation we find it is not of Painlevé-type as its resonances are not distinct.

If $D(w) = \ln w$ we again differentiate (2.7), to get a rational equation. There is a difficulty in applying the ARS algorithm as it is not possible to balance the dominant terms. Instead we use Painlevé's α-method, keeping w fixed and transforming $z \to z_0 + \alpha\xi$. It is routine to show that we introduce logarithmic branch points into the expansion $w = \sum w_i \alpha^i$, so that in this instance (2.7) is not of Painlevé-type.

Case 2.2.2 $D(u) = u^n$, for $n \neq 0, 1$. We apply the Kolchin-Ritt algorithm to system (2.5) with $D = u^n$, which yields

$$\xi_u = 0, \qquad \xi_t = 0, \qquad \xi_{xx} = 0, \qquad \tau_u = 0, \qquad \tau_x = 0,$$
$$\tau_t - 2\xi_x = 0, \qquad (n-2)(\phi(n-1) + 2\xi_x u) = 0.$$

We find by reapplying the Kolchin-Ritt algorithm with $n = 2$ that the factor $(n-2)$ is just an artefact of the equations. We have the infinitesimals

$$\xi = c_1 x + c_2, \qquad \tau = 2c_1 t + c_3, \qquad \phi = -\frac{2c_1 u}{n-1}.$$

As well as the travelling wave reduction 2.2.1 above which we are bound to get (setting $c_1 = 0$), we also have the scaling reduction

Reduction 2.2.2. If $c_1 \neq 0$ we set $c_1 = 1$ and $c_2 = c_3 = 0$ without loss of generality, to yield

$$u(x,t) = w(z) t^{-1/(n-1)}, \qquad z = x t^{-1/2},$$

where $w(z)$ satisfies

$$\tfrac{1}{4} z^2 w'' + \frac{3n+1}{4(n-1)} z w' + \frac{n}{(n-1)^2} w = w'''' + n(n-1) w^{n-2} (w')^2 + n w^{n-1} w''. \tag{2.8}$$

In the ARS algorithm, looking for the dominant behaviour of (2.8), we assume that $w(z) \sim a_0 (z - z_0)^p$ as $z \to z_0$, and find $p = -2/(n-1)$. Thus requiring p to be integer (though clearly $p \neq 0$) we find $n = (p-2)/p$. If n is to be integer (remembering $n \neq 0, 1$) we see that $n = -1$, 2 or 3. If $n = 2$ this is a reduction of the Boussinesq equation, and is of Painlevé-type. If $n = 3$ or $n = -1$, equation (2.8) can be integrated with respect to z to yield

(i) If $n = 3$

$$\tfrac{1}{4}z^2w' + \tfrac{3}{4}zw = w''' + 3w^2w' + c_1. \tag{2.9i}$$

Equation (2.9i) has both acceptable leading order behaviour and distinct integer resonances. However the compatibility conditions at the resonance equations require an arbitrary constant to be zero, thus (2.9i) is not of Painlevé-type.

(ii) If $n = -1$

$$\tfrac{1}{4}z^2w^2w' - \tfrac{1}{4}zw^3 = w^2w''' - w' + c_2w^2. \tag{2.9ii}$$

We again have difficulties with the ARS algorithm in balancing dominant terms. Painlevé's α-method, with w fixed and $z \to z_0 + \alpha\xi$, however quickly shows that the solution of (2.9ii) has logarithmic branch points.

If n is not integer, continuing with the ARS algorithm we are able to show that (2.8) has distinct integer resonances, however with a general n it is difficult to finish the algorithm. This is due to the resonances being dependent on n, and also they may be negative which would require perturbative Painlevé analysis (cf. §1.6.2). Therefore we again turn to Painlevé's α-method, using the same transformation as above. We find algebraic branch points thus (2.8) is not of Painlevé-type in this instance.

Case 2.2.3 $D(u) = e^u$. The **KolRitt** procedure simplifies the determining equations (2.5) to the simple system

$$\xi_u = 0, \qquad \xi_t = 0, \qquad \xi_{xx} = 0, \qquad \tau_u = 0, \qquad \tau_x = 0,$$
$$\tau_t - 2\xi_x = 0, \qquad \phi + 2\xi_x = 0.$$

These give the following infinitesimals

$$\xi = c_1x + c_2, \qquad \tau = 2c_1t + c_3, \qquad \phi = -2c_1,$$

which give, as well as the travelling wave reduction 2.2.1, a scaling reduction

Reduction 2.2.3. If $c_1 \neq 0$ we set $c_1 = 1$ and $c_2 = c_3 = 0$ without loss of generality, to yield

$$u(x,t) = w(z) - \ln t, \qquad z = xt^{-1/2},$$

where $w(z)$ satisfies

$$\tfrac{1}{4}z^2w'' + \tfrac{3}{4}zw' + 1 = w'''' + e^w[(w')^2 + w''].$$

By making the transformation $w(z) = \ln W(z)$ we can obtain an equation which is rational in $W(z)$, to which we can apply the ARS algorithm to test if the equation is of Painlevé-type. We find an acceptable leading order behaviour but, whilst the resonances are all integer, they are not distinct. Thus we conclude that the equation is not of Painlevé-type.

Case 2.2.4 $D(u) = \ln u$. Once again we use the **KolRitt** procedure to dramatically simplify the determining equations (2.5),

$$\xi_u = 0, \qquad \xi_t = 0, \qquad \xi_{xx} = 0, \qquad \tau_u = 0, \qquad \tau_x = 0,$$

$$\tau_t - 2\xi_x = 0, \qquad \phi + 2\xi_x u = 0,$$

which give the following infinitesimals

$$\xi = c_1 x + c_2, \qquad \tau = 2c_1 t + c_3, \qquad \phi = -2c_1 u.$$

Not only do these infinitesimals give the travelling wave reduction 2.2.1, but once again we also have a scaling reduction

Reduction 2.2.4. If $c_1 \neq 0$ we set $c_1 = 1$ and $c_2 = c_3 = 0$ without loss of generality, to yield

$$u(x,t) = tw(z), \qquad z = xt^{-1/2},$$

where $w(z)$ satisfies

$$\tfrac{1}{4} z w^2 w' + \tfrac{1}{4} z^2 w^2 w'' = w^2 w'''' + ww'' - (w')^2. \tag{2.10}$$

The ARS algorithm again falls down so we use the α-method with the same transformation as previously. Logarithmic branch points are introduced thus (2.10) is not of Painlevé-type.

2.3 Nonclassical symmetries ($\tau \neq 0$)

In this section we may set $\tau = 1$ without loss of generality. We use the algorithm in Clarkson and Mansfield [1994c], which in this case demands the removal of u_{tt} using the invariant surface condition. The classical Lie method is then applied to this new equation which yields an overdetermined system of nonlinear equations. These determining equations are

$$\xi_u = 0, \tag{2.11i}$$

$$\phi_{uu} = 0, \tag{2.11ii}$$

$$2\phi_{xu} - 3\xi_{xx} = 0, \tag{2.11iii}$$

$$D_{uu}\phi_u + D_{uuu}\phi + 2\xi_x D_{uu} = 0, \tag{2.11iv}$$

$$6\phi_{xxu} + D_{uu}\phi + 2\xi_x D_u - 4\xi_{xxx} - 4\xi^2\xi_x - 2\xi\xi_t = 0, \tag{2.11v}$$

$$\phi_{xxxx} + D_u\phi_{xx} + 4\xi\xi_x\phi_x + 2\xi_t\phi_x - 4\xi_x\phi\phi_u - \phi_{tt} - 2\phi\phi_{tu} - 4\xi_x\phi_t = 0, \tag{2.11vi}$$

$$\begin{aligned} &2D_{uu}\phi_x + 4\phi_{xxxu} + 2D_u\phi_{xu} \\ &\quad + 8\xi\xi_x\phi_u + 2\xi_t\phi_u + 2\xi\phi_{tu} - \xi_{xx}D_u - \xi_{xxxx} - 4\xi\xi_x^2 + 2\xi_t\xi_x + \xi_{tt} = 0. \end{aligned} \tag{2.11vii}$$

Using the Direct Search strategy in the MAPLE package `diffgrob2` on equations (2.11i,ii,iv) gained the following condition (cf. Example 1.7.5)

$$\xi_x D_{uu}^2(-2D_{uu}D_{uuuu}^2 + D_{uu}D_{uuuuu}D_{uuu} + D_{uuuu}D_{uuu}^2) = 0. \tag{2.12}$$

The factor inside the brackets, only involving $D(u)$, can be solved by dividing through by $D_{uu}D_{uuu}D_{uuuu}$ giving a logarithmic differential (assuming $D_{uuuu} \neq 0$; it is clear from (2.12) that $D_{uuuu} = 0$ is one solution). Solving this leads to six separate cases to consider:

$$\begin{aligned}
&\text{(i)} && \xi_x = 0, \\
&\text{(ii)} && D(u) = au^3 + bu^2 + cu + d, \\
&\text{(iii)} && D(u) = (au+b)^n + cu + d, \qquad n \neq 0,1,2,3, \\
&\text{(iv)} && D(u) = ae^{bu} + cu + d, \\
&\text{(v)} && D(u) = \frac{1}{a}\ln(au+b) + cu + d, \\
&\text{(vi)} && D(u) = \frac{1}{a^2}[(au+b)\ln(au+b) - (au+b)] + cu + d,
\end{aligned}$$

where a, b, c and d are arbitrary constants.

Case 2.3.1 $\xi_x = 0$. In this case we append $\xi_x = 0$ to the determining equations (2.11) and use the Kolchin-Ritt algorithm to simplify this enlarged system. This yields

$$\xi_u = 0, \tag{2.13i}$$

$$\xi_t\xi(D_{uu}D_{uuuu} - 2D_{uuu}^2) = 0, \tag{2.13ii}$$

$$D_{uu}\phi - 2\xi\xi_t = 0, \tag{2.13iii}$$

$$\phi_{uu} = 0, \tag{2.13iv}$$

$$2\xi_t\phi_u + 2\xi\phi_{tu} + \xi_{tt} = 0, \tag{2.13v}$$

$$\phi_{xu} = 0, \tag{2.13vi}$$

$$\phi_{xxxx} + D_u\phi_{xx} + 2\xi_t\phi_x - \phi_{tt} - \phi\phi_{tu} = 0. \tag{2.13vii}$$

In (2.13ii), if $D_{uu}D_{uuuu} - 2D_{uuu}^2 = 0$ then (2.12) is identically satisfied, thus the solutions of $D_{uu}D_{uuuu} - 2D_{uuu}^2 = 0$ are a subset of the solutions (ii)–(vi) above and therefore this scenario will be considered in the subsequent Cases. With $D_{uu}D_{uuuu} - 2D_{uuu}^2 \neq 0$ we see from (2.13ii) that $\xi_t = 0$, and we have the infinitesimals $\xi = c_1$, $\phi = 0$ for arbitrary $D(u)$. This gives the classical reduction 2.2.1.

Case 2.3.2 $D(u) = au^3 + bu^2 + cu + d$. There are essentially two subcases to consider, when (i) $a = 0$ and when (ii) $a \neq 0$. Other subcases arise but these make our original equation (2.1) linear which we choose not to consider here.

Subcase 2.3.2(i) $D(u) = bu^2 + cu + d$. We can choose $b = \frac{1}{2}$ and $c = d = 0$ without loss of generality, then (2.1) becomes the Boussinesq equation

$$u_{tt} = u_x^2 + uu_{xx} + u_{xxxx}, \tag{2.14}$$

which is a soliton equation solvable by inverse scattering. The symmetries of the Boussinesq equation are well known (see Clarkson and Kruskal [1989], Levi and Winternitz [1989] and Clarkson [1995]), but the calculation is included here for completeness. The determining equations, system (2.11), is simplified enormously using the Kolchin-Ritt algorithm in `diffgrob2` to yield

$$\xi_u = 0, \tag{2.15i}$$

$$4\xi\xi_x^2 - 2\xi_x\xi_t - \xi_{tt} = 0, \tag{2.15ii}$$

$$\xi_{xx} = 0, \tag{2.15iii}$$

$$2\xi_x u + \phi - 4\xi^2\xi_x - 2\xi\xi_t = 0. \tag{2.15iv}$$

We can therefore write ξ in the form $\xi = f(t)x + g(t)$ and substituting this information into (2.15ii) gives us conditions on f and g by equating powers of x to zero

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + 2f\frac{\mathrm{d}f}{\mathrm{d}t} - 4f^3 = 0, \tag{2.16i}$$

$$\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} + 2f\frac{\mathrm{d}g}{\mathrm{d}t} - 4gf^2 = 0. \tag{2.16ii}$$

By making the Cole-Hopf transformation $f(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\ln\psi(t)]$, we find ψ satisfies

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 = c_1\psi^3 + c_2, \tag{2.17}$$

after integrating twice, which is equivalent to the Weierstrass elliptic function equation

$$\left(\frac{\mathrm{d}\wp}{\mathrm{d}t}\right)^2 = 4\wp^3(t + t_0; 0, g_3) - g_3. \tag{2.18}$$

If $f = 0$ then $g = c_5 t + c_6$ is the general solution of (2.16), then $\phi = 2c_5(c_5 t + c_6)$. When $f \neq 0$ we note that $g = f$ is a solution of (2.16ii) so we can write $g = f\eta(t)$ which gives a first order ordinary differential equation in $\frac{\mathrm{d}\eta}{\mathrm{d}t}$, which is solvable and hence we find g in terms of quadratures. Equation (2.15iv) gives us ϕ directly. This yields

$$\xi = f(t)x + g(t),$$

$$\phi = -2fu + 2f\left(2f^2 + \frac{\mathrm{d}f}{\mathrm{d}t}\right)x^2 + 2\left(4f^2 g + 2g\frac{\mathrm{d}f}{\mathrm{d}t} + 2f\frac{\mathrm{d}g}{\mathrm{d}t}\right)x + 2g\left(2gf + \frac{\mathrm{d}g}{\mathrm{d}t}\right),$$

where

$$f(t) = \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\ln\psi(t)], \qquad g(t) = \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\ln\psi(t)]\left(c_3 + c_4\int^t \psi(s)\Big/\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 \mathrm{d}s\right),$$

where $\psi(t)$ satisfies (2.17) and c_1, c_2, c_3 and c_4 are arbitrary constants. There are six canonical reductions

Reduction 2.3.1–6.

1. $u(x,t) = w_1(z)$, $\qquad z = x - c_1 t$,
2. $u(x,t) = w_2(z) + c_2^2 t^2$, $\qquad z = x - \frac{1}{2}c_2 t^2$,
3. $u(x,t) = w_3(z)t^2 + x^2/t^2$, $\qquad z = xt$,
4. $u(x,t) = w_4(z)t^{-1} + (x + 3c_4 t^2)^2/(4t^2)$, $\qquad z = xt^{-1/2} - c_4 t^{3/2}$,
5. $u(x,t) = w_5(z)t^2 + (x + c_5 t^5)^2/t^2$, $\qquad z = xt + c_5 t^6/6$,
6. $u(x,t) = \wp^{-1}\left\{ w_6(z) + \left[\frac{1}{2}z\frac{\mathrm{d}\wp}{\mathrm{d}t} + \frac{c_6}{3g_3}\wp^{3/2}\right]^2 \right\}$, $\qquad z = \wp^{-1/2}\left[x + \frac{c_6}{3g_3}\zeta(t)\right]$,

where $\wp(t;0,g_3)$ is the Weierstrass elliptic function and $\zeta(z)$ is the Weierstrass zeta function defined by the ordinary differential equation

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = -\wp(z), \tag{2.19i}$$

together with the condition

$$\lim_{z\to 0}\left(\zeta(z) - \frac{1}{z}\right) = 0. \tag{2.19ii}$$

Each of the $w_i(z)$ satisfy the equation

$$w_i'''' + w_i w_i'' + (w_i')^2 = F_i(z, w_i), \tag{2.20.i}$$

where the $F_i(z, w_i)$ are

$$F_1 = c_1^2 w_1'', \qquad F_4 = \tfrac{3}{4}zw_4' + \tfrac{3}{2}w_4 + \tfrac{9}{8}z^2,$$

$$F_2 = 2c_2^2 - c_2 w_2', \qquad F_5 = 5c_5 w_5' + 50c_5^2,$$

$$F_3 = 0, \qquad F_6 = \tfrac{9}{8}g_3^2 z^2 - \tfrac{3}{2}g_3 w_6 - \tfrac{3}{4}g_3 z w_6',$$

and (2.20.1,3) are equivalent to the first Painlevé equation, PI, (2.20.2,5) are equivalent to the second Painlevé equation, PII, and (2.20.4,6) are equivalent to the fourth Painlevé equation, PIV.

Subcase 2.3.2(ii) $D(u) = au^3 + bu^2 + cu + d$, and assume $a \neq 0$. We can set $a = 1$, and $b = d = 0$ without loss of generality. The equation we are now considering is

$$u_{tt} = u_{xxxx} + (u^3 + cu)_{xx}, \tag{2.21}$$

which is sometimes called the cubic Boussinesq equation. The Kolchin-Ritt algorithm simplifies the updated system (2.11), to give the following relevant equations

$$\xi_u = 0, \tag{2.22i}$$

$$c\xi\xi_t^2(c - \xi^2) = 0, \tag{2.22ii}$$

$$12\xi\xi_x^2 - \xi_{tt} + 2\xi\xi_{xt} = 0, \tag{2.22iii}$$

$$\xi_x u + \phi = 0. \tag{2.22iv}$$

If we insist $c \neq 0$ then from (2.22ii) we have $\xi_t = 0$, then $\xi_x = 0$ from (2.22iii) and we have the infinitesimals $\xi = c_1$, $\phi = 0$, where c_1 is an arbitrary constant, which is consistent with the rest of the system. These infinitesimals give us the (classical) travelling wave reduction. In the case when $c = 0$, the determining equations now have the form

$$\xi_u = 0, \tag{2.23i}$$

$$\phi_{uu} = 0, \tag{2.23ii}$$

$$2\phi_{xu} - 3\xi_{xx} = 0, \tag{2.23iii}$$

$$\phi_u u + 2\xi_x u + \phi = 0, \tag{2.23iv}$$

$$3\xi_x u^2 + 3\phi u + 3\phi_{xxu} - 2\xi_{xxx} - 2\xi^2\xi_x - \xi\xi_t = 0, \tag{2.23v}$$

$$4\xi_x\phi\phi_u - 3\phi_{xx}u^2 - \phi_{xxxx} - 4\xi\xi_x\phi_x - 2\xi_t\phi_x + \phi_{tt} + 2\phi\phi_{tu} + 4\xi_x\phi_t = 0, \tag{2.23vi}$$

$$\begin{aligned} &6\phi_{xu}u^2 - 3\xi_{xx}u^2 + 12\phi_x u \\ &\quad + 4\phi_{xxxu} + 8\xi\xi_x\phi_u + 2\xi_t\phi_u + 2\xi\phi_{tu} - \xi_{xxxx} - 4\xi\xi_x^2 + 2\xi_t\xi_x + \xi_{tt} = 0. \end{aligned} \tag{2.23vii}$$

With an ordering $t < x < u$ and $\xi < \phi$ we can **reduce**((2.23ii),[(2.23iv)], $k1$), which gives

$$k1: \qquad \phi + \xi_x u = 0.$$

We then **reduce**((2.23iii),[$k1$,(2.23i)], $k2$) which leaves $k2$: $\xi_{xx} = 0$. Finally we **reduce**((2.23v),[$k1$,$k2$,(2.23i)], $k3$) which yields

$$k3: \qquad \xi_t + 2\xi\xi_x = 0.$$

Equations (2.23vi) and (2.23vii) both now reduce to zero, so we have found the differential Gröbner basis, i.e. the system

$$\xi_u = 0, \qquad \xi_{xx} = 0, \qquad \xi_t + 2\xi\xi_x = 0, \qquad \phi + \xi_x u = 0.$$

These infinitesimals only give rise to classical reductions.

<u>*Case 2.3.3* $D(u) = (au+b)^n + cu + d$ *for* $n \neq 0, 1, 2, 3$.</u> Without loss of generality we consider $D(u) = u^n + cu$. Substituting this into the determining equations and using the Kolchin-Ritt algorithm in `diffgrob2` gives a (rather large) system of nine equations. The interesting ones are the first, second, seventh, and ninth equations

$$\xi_u = 0, \tag{2.24i}$$

$$ncu^2\xi\xi_t^2(n-2)(3n+1)(n-1)^3(\xi^2 - c) = 0, \tag{2.24ii}$$

$$nu^2(n-1)(n-2)(3n+1)(n\xi_{xtt} + 4n\xi_x\xi_{xt} - 8\xi_x^3 - \xi_{xtt} - 8\xi_x\xi_{xt}) = 0, \tag{2.24vii}$$

$$n(n-1)(n-2)(n\phi - \phi + 2\xi_x u) = 0. \tag{2.24ix}$$

If $(1+3n)c \neq 0$ it is easy to see first that $\xi_t = 0$, then $\xi_x = 0$ so that $\xi = c_1$, $\phi = 0$ and we have the travelling wave infinitesimals. If $c = 0$ we append the equation $\phi(n-1)+2\xi_x u = 0$ to our determining equations and use **reduceall** on this new system. This yields

$$\xi_u = 0, \qquad (1+3n)\xi_{xx} = 0, \qquad \xi(n-1)(1+3n)(\xi_t + 2\xi\xi_x) = 0, \qquad \phi(n-1) + 2\xi_x u = 0.$$

With a little more effort, using the Direct Search strategy, we can show that the $(1+3n)$ factor is an artefact of the equations whatever value c takes. These infinitesimals give the classical reductions of Case 2.2.2 (for $n \neq 0, 1, 2, 3$).

Case 2.3.4 $D(u) = ae^{bu} + cu + d$. We may consider $D(u) = e^u + cu$ without loss of generality. Substituting this into the determining equations and using the Kolchin-Ritt algorithm gives the following system of seven equations,

$$\xi_u = 0, \tag{2.25i}$$

$$\xi c \xi_t^3(-c+\xi^2)(-c+6\xi^2) = 0, \tag{2.25ii}$$

$$c^2\xi_{tt} - 4\xi^2 c\xi_{tt} + 4\xi^4\xi_{tt} - 8\xi^3\xi_t^2 + 2c\xi\xi_t^2 = 0, \tag{2.25iii}$$

$$2\xi^2\xi_x - c\xi_x + \xi\xi_t = 0, \tag{2.25iv}$$

$$4\xi\xi_x^2 - 2\xi_t\xi_x - \xi_{tt} = 0, \tag{2.25v}$$

$$\xi_{xtt} + 4\xi_x\xi_{xt} = 0, \tag{2.25vi}$$

$$\xi_{xx} = 0, \tag{2.25vii}$$

$$\phi + 2\xi_x = 0. \tag{2.25viii}$$

For the general case $c \neq 0$ this again gives us only the travelling wave infinitesimals. For the case $c = 0$ we use the Direct Search strategy to find the simpler system of equations

$$\xi_u = 0, \qquad \xi_{xx} = 0, \qquad \xi_t + 2\xi\xi_x = 0, \qquad \phi + 2\xi_x = 0.$$

On solving these equations we find the classical results of Case 2.2.3.

Case 2.3.5 $D(u) = \frac{1}{a}\ln(au+b) + cu + d$. We consider $D(u) = \ln u + cu$ without loss of generality. Substituting this into the determining equations and using the Kolchin-Ritt algorithm gives a system of nine equations. The interesting ones are again the first, second, seventh, and ninth equations

$$\xi_u = 0, \tag{2.26i}$$

$$cu^2\xi_t^2\xi(c-\xi^2) = 0, \tag{2.26ii}$$

$$u^2(\xi_{xtt} + 8\xi\xi_{xt} + 8\xi_x^3) = 0, \tag{2.26vii}$$

$$\phi - 2\xi_x u = 0. \tag{2.26ix}$$

If $c \neq 0$ equation (2.26ii) implies that $\xi_t = 0$, then we see that $\xi_x = 0$ from (2.26vii). These infinitesimals give the classical travelling wave reduction. In the special case when $c = 0$ the Direct Search strategy yields the system

$$\xi_u = 0, \qquad \xi_{xx} = 0, \qquad \xi_t + 2\xi\xi_x = 0, \qquad \phi - 2\xi_x u = 0,$$

which yield the classical results of Case 2.2.4.

Case 2.3.6 $D(u) = \frac{1}{a^2}[(au+b)\ln(au+b) - (au+b)] + cu + d$. We consider without loss of generality $D(u) = u\ln u + cu$. Our determining equations, simplified by **KolRitt**, can be solved by looking at only four of them

$$\xi_u = 0, \tag{2.27i}$$

$$\xi_t^3 \xi u(7 - 16\xi^2 + 16\xi^4) = 0, \tag{2.27ii}$$

$$\xi_x = 0, \tag{2.27iii}$$

$$\phi - 2\xi\xi_t u = 0. \tag{2.27iv}$$

The solution of these equations is $\xi = c_1$, $\phi = 0$ which leads to the travelling wave reduction only.

2.4 Nonclassical symmetries ($\tau = 0$)

We may set $\xi = 1$ without loss of generality so that the invariant surface condition reads $u_x = \phi(x,t,u)$. We remove u_x, u_{xx} and u_{xxxx} using the invariant surface condition (as $[D(u)]_{xx} = D_{uu}u_x^2 + D_u u_{xx}$), and apply the classical Lie method to this new equation. The determining equations are

$$\phi_{uu} = 0, \tag{2.28i}$$

$$\phi_{tu} = 0, \tag{2.28ii}$$

$$\begin{aligned}&\phi_{xxxx} + 4\phi_{xu}\phi_{xx} + D_u\phi_{xx} + 6\phi_{xxu}\phi_x + 4\phi\phi_{xxxu} + 6\phi\phi_u\phi_{xxu} + 8\phi\phi_{xu}^2 + 4\phi\phi_u^2\phi_{xu}\\ &+ 2D_u\phi\phi_{xu} + 2D_{uu}\phi^2\phi_u + 4\phi_u\phi_{xu}\phi_x + 3D_{uu}\phi\phi_x - \phi_{tt} + D_{uuu}\phi^3 = 0.\end{aligned} \tag{2.28iii}$$

We use the Direct Search strategy to get some conditions to simplify our problem. We find an equation of the form

$$\phi_t D_{uu}(D_{uu}D_{uuuuu}D_{uuu} - 2D_{uu}D_{uuuu}^2 + D_{uuuu}D_{uuu}^2)\Delta(u,D) = 0,$$

where $\Delta(u,D)$ is an ordinary differential equation in $D(u)$, consisting of approximately 150 terms. If $\phi_t = 0$ we are not much better off than before; (2.28iii) is still a linear equation for $D(u)$ but is just as intractable to solve. Also, assuming $\Delta(u,D) = 0$ is of little help as an equation of this size is somewhat meaningless. We proceed by using the other condition on $D(u)$ we have found, which is in fact the equation we used to classify $D(u)$ in the previous section, equation (2.12). This gives us six canonical cases to consider,

$$\begin{aligned}
&\text{(i)} && D(u) = \tfrac{1}{2}u^2, \\
&\text{(ii)} && D(u) = u^3 + cu, \\
&\text{(iii)} && D(u) = u^n + cu, \qquad \text{for } n \neq 0, 1, 2, 3, \\
&\text{(iv)} && D(u) = e^u + cu, \\
&\text{(v)} && D(u) = \ln u + cu, \\
&\text{(vi)} && D(u) = u \ln u + cu.
\end{aligned}$$

In each case (2.28i,ii) gives us that

$$\phi = A(x)u + B(x,t), \tag{2.29}$$

which we substitute into (2.28iii) and equate independent functions of u to zero. (These independent functions are often powers of u, but may also involve terms like $\ln u$ or e^u.) In the remainder of this section for convenience we will denote the derivatives of A with respect to x with subscripts despite the fact that they are not strictly partial derivatives.

Note that if $\phi = 0$ we are left with the equations

$$u_{tt} = 0, \qquad u_x = 0,$$

to solve, which give the trivial solution $u(x,t) = c_1 t + c_2$.

<u>*Case 2.4.1* $D(u) = \frac{1}{2}u^2$.</u> We have a system of three equations to solve for A and B,

$$A_{xx} + 5AA_x + 2A^3 = 0, \tag{2.30i}$$

$$\begin{aligned}
&A_{xxxx} + 10A_{xx}A_x + B_{xx} + 4AA_{xxx} \\
&+ 6A^2A_{xx} + 12AA_x^2 + 4A^3A_x + 5BA_x + 4A^2B + 3AB_x = 0,
\end{aligned} \tag{2.30ii}$$

$$\begin{aligned}
&B_{xxxx} + 4A_xB_{xx} + 4BA_{xxx} + 6BAA_{xx} + 8BA_x^2 \\
&+ 4BA^2A_x + 6A_{xx}B_x + 2B^2A - B_{tt} + 4AA_xB_x + 3BB_x = 0.
\end{aligned} \tag{2.30iii}$$

We can simplify equation (2.30i) by making the transformation $A = \frac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}x}\ln[\psi(x)]$ then the new equation in ψ can be integrated twice to yield

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)^2 = c_1\psi^{3/2} + c_2. \tag{2.31}$$

If $c_1 \neq 0$, $c_2 = 0$ we can integrate again to give $A = 2/(x + c_3)$. Substituting this information into (2.30ii) we get an Euler equation

$$(x+\alpha)^2B_{xx} + 6(x+\alpha)B_x + 6B = 0,$$

which has solution

$$B(x,t) = \frac{\eta(t)}{(x+c_3)^3} + \frac{\rho(t)}{(x+c_3)^2}.$$

Substituting this into (2.30iii) gives $\rho(t) = 0$ and $\eta(t)$ may take the values 0 or 48. Thus we have the infinitesimals

$$\phi = \frac{2u}{(x+c_3)} + \frac{48\mu}{(x+c_3)^3},$$

where $\mu = 0$ or 1. This gives us the following reduction

Reduction 2.4.1. $D(u) = \frac{1}{2}u^2$

$$u(x,t) = w(t)(x+c_3)^2 - \frac{12\mu}{(x+c_3)^2},$$

where $w(t)$ satisfies

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - 6w^2 = 0.$$

This equation is equivalent to the Weierstrass elliptic function equation.

If $c_1 = c_2 = 0$ we have the trivial solution of (2.30i), namely $A = 0$. In this case (2.30ii) gives $B_{xx} = 0$ so we write $B(x,t)$ in the form $B(x,t) = 2f(t)x + g(t)$ and substitute this into (2.30iii) giving, on equating powers of x to zero, a system of equations to solve for f and g

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 6f^2 = 0, \qquad \frac{\mathrm{d}^2 g}{\mathrm{d}t^2} - 6gf = 0. \tag{2.32i,ii}$$

If $f = 0$ then $g(t) = c_4 t + c_5$, which leads us to the nonclassical reduction

Reduction 2.4.2. $D(u) = \frac{1}{2}u^2$

$$u(x,t) = w(t) + (c_4 t + c_5)x,$$

where $w(t)$ satisfies

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = (c_4 t + c_5)^2,$$

which is easily solved to yield the exact solution

$$u(x,t) = \frac{c_4 t^4}{4} + \frac{c_4 c_5 t^3}{3} + \frac{c_5^2 t^2}{2} + c_6 t + c_7 + (c_4 t + c_5)x.$$

If $f \neq 0$ then equation (2.32i) has solution $f(t) = \wp(t + t_0; 0, g_3)$, and because (2.32ii) has one solution $g = f$, we find the general solution as

$$g(t) = f(t)\left(c_8 + c_9 \int^t \frac{\mathrm{d}s}{f^2(s)}\right).$$

We then have the nonclassical reduction

Reduction 2.4.3. $D(u) = \frac{1}{2}u^2$

$$u(x,t) = w(t) + f(t)x^2 + g(t)x,$$

where $w(t)$ satisfies the inhomogeneous Lamé equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - 2\wp(t+t_0;0,g_3)w = g^2(t), \tag{2.33}$$

and $f(t)$, $g(t)$ are as described above. The homogeneous part of (2.33) has solution (see Ince [1956], p. 379)

$$w(t) = c_{10}w_1(t+t_0) + c_{11}w_2(t+t_0),$$

where $w_1(t), w_2(t)$ are the independent functions

$$w_1(t) = \exp\{-t\zeta(a)\}\frac{\sigma(t+a)}{\sigma(t)}, \qquad w_2(t) = \exp\{t\zeta(a)\}\frac{\sigma(t-a)}{\sigma(t)},$$

where $\zeta(z)$ is the Weierstrass zeta function defined by (2.19), $\sigma(z)$ is the Weierstrass sigma function defined by the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln\sigma(z) = \zeta(z), \tag{2.34i}$$

together with the condition

$$\lim_{z\to 0}\left(\frac{\sigma(z)}{z}\right) = 1, \tag{2.34ii}$$

and a is a zero of the Weierstrass elliptic function i.e. $\wp(a) = 0$. We find the particular integral of (2.33) using the method of variation of parameters, to give the general solution

$$w(t) = c_{10}w_1(t+t_0)+c_{11}w_2(t+t_0)+\frac{1}{W(a)}\int^{t+t_0}\left[w_1(s)w_2(t+t_0)-w_1(t+t_0)w_2(s)\right]g^2(s)\,\mathrm{d}s,$$

where $W(a)$ is the non-zero Wronskian

$$W(a) = w_1\frac{\mathrm{d}w_2}{\mathrm{d}t} - \frac{\mathrm{d}w_1}{\mathrm{d}t}w_2 = -\sigma^2(a)\frac{\mathrm{d}\wp}{\mathrm{d}t}(a),$$

which can be verified using the following addition theorems

$$\zeta(s\pm t) = \zeta(s) \pm \zeta(t) + \frac{1}{2}\left[\frac{\wp'(s)\mp\wp'(t)}{\wp(s)-\wp(t)}\right],$$
$$\sigma(s+t)\sigma(s-t) = -\sigma^2(s)\sigma^2(t)[\wp(s)-\wp(t)]$$

(see Whittaker and Watson [1927], p. 451).

We now have to show what happens if $c_2 \neq 0$ in (2.29). First notice that by making the transformation $A = 2\frac{\mathrm{d}}{\mathrm{d}x}\ln[\psi(x)]$ we can simplify (2.30i) and integrate it twice to yield

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)^2 = a_1\psi^{-6} + a_2, \tag{2.35}$$

where a_1, a_2 are arbitrary constants. If $a_2 = 0$ we get $A = 1/(2x + c_{12})$, which on substituting into (2.30ii) gives an inhomogeneous Euler type equation to solve for $B(x, t)$. The result is substituted into (2.30iii) and found to be inconsistent with this last equation. If we make two further transformations, $\psi = P^2(x)$ in (2.31), and $\psi = P^{1/2}(x)$ in (2.35), then

$$A(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln[P(x)], \tag{2.36}$$

in both cases and both equations become of the form

$$P^2 \left(\frac{\mathrm{d}P}{\mathrm{d}x} \right)^2 = 4a_3 P^3 - a_4, \tag{2.37}$$

which has solution (for which we have set $a_3 = 1$ without loss of generality)

$$P(x) = \wp(\xi; 0, a_4), \qquad \frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{1}{\wp(\xi; 0, a_4)},$$

where $\wp(\xi; g_2, g_3)$ is the Weierstrass elliptic function satisfying

$$\left(\frac{\mathrm{d}\wp}{\mathrm{d}\xi} \right)^2 = 4\wp^3 - g_2\wp - g_3. \tag{2.38}$$

The difference between the two transformations is that from (2.31) a_3, a_4 are a rescaling of c_1, c_2 respectively, whereas from (2.35) they are a rescaling of a_2, a_1 respectively. This means that unless A is of this form for $a_3 a_4 \neq 0$ then either it is not a solution of the system or it has been discussed previously. To show that there are no solutions of this form, we firstly simplify the calculations by showing that if A is given by (2.36) then B is a function of x only.

We take the **diffSpoly** of (2.30ii,iii), seeking to remove t-derivatives first, then **reduce** the result with respect to (2.30iii) which leaves the result as an ordinary differential equation for B with t a parameter. Successive applications of **reduce** to this ordinary differential equation and equation (2.30ii) leaves a polynomial in B of degree four whose coefficients are expressions in A and its derivatives i.e. strictly functions of x. Then unless each of these coefficients is zero, B is necessarily also a function of x only. If we look at the coefficient of x^4,

$$a_5 A^{10} + a_6 A^8 A_x + a_7 A^6 A_x^2 + a_8 A^4 A_x^3 + a_9 A^2 A_x^4 + a_{10} A_x^5,$$

where $a_5, ..., a_{10}$ are known constants, and only the first derivatives of A are present because second derivatives and higher can be removed using (2.30i). Using the transformation (2.36) together with (2.37) yields

$$a_3 \frac{\mathrm{d}P}{\mathrm{d}x} (b_1 a_3^3 P^9 + b_2 a_3^2 a_4 P^6 + b_3 a_3 a_4^2 P^3 + b_4 a_4^3), \tag{2.39}$$

where the $b_1, ..., b_4$ are known constants and a_3, a_4 are as seen in (2.37). Since $a_3 \frac{\mathrm{d}P}{\mathrm{d}x} \neq 0$ to ensure we are not repeating previous work, and because P does not satisfy an algebraic relation, expression (2.39) is non-zero; hence by arguments above $B_t = 0$.

With $B_t = 0$ both (2.30ii) and (2.30iii) are ordinary differential equations for B which, as previously, we may successively **reduce** to a polynomial for B. In fact we can go further than this and **reduce** our expressions to a single equation involving A only (being careful to check that setting the highest coefficient and separant of each expression to zero doesn't yield any solutions). We may remove second and higher order derivatives of A using (2.30i), then transform this using (2.36) and (2.37) into

$$\frac{\mathrm{d}P}{\mathrm{d}x} \sum_{n=0}^{N} \tilde{c}_n a_3^n a_4^{N-n} P^{3n} = 0, \tag{2.40}$$

where the $\tilde{c}_n$ are known integers (not all zero). This is similar to (2.39) and indeed by the same arguments, namely the fact that P doesn't satisfy an algebraic relation and we have no desire to repeat earlier solutions, equation (2.40) represents a contradiction, and we are done.

Case 2.4.2 $D(u) = u^3 + cu$. The system we must solve in this Case is

$$A_{xx} + 8AA_x + 6A^3 = 0, \tag{2.41i}$$

$$8BA_x + 6AB_x + 14A^2B + B_{xx} = 0, \tag{2.41ii}$$

$$\begin{aligned} &18BB_x + 30B^2A + A_{xxxx} + cA_{xx} + 10A_{xx}A_x + 4AA_{xxx} \\ &\quad + 6A^2A_{xx} + 12AA_x^2 + 4A^3A_x + 2cAA_x = 0, \end{aligned} \tag{2.41iii}$$

$$\begin{aligned} &6B^3 + B_{xxxx} + 4A_xB_{xx} + cB_{xx} + 6A_{xx}B_x + 4AA_xB_x \\ &\quad + 4BA_{xxx} + 6BAA_{xx} + 8BA_x^2 + 4BA^2A_x + 2cBA_x - B_{tt} = 0. \end{aligned} \tag{2.41iv}$$

It makes sense to try to solve (2.41i) for $A(x)$ first. Writing $A(x) = R\frac{\mathrm{d}}{\mathrm{d}x}\ln[\psi(x)]$ and substituting into (2.41i) we get

$$R\psi^2\frac{\mathrm{d}^3\psi}{\mathrm{d}x^3} + R(8R-3)\psi\frac{\mathrm{d}\psi}{\mathrm{d}x}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + 3R(3R-1)(R-1)\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)^3 = 0.$$

Choosing $R = \frac{1}{3}$ or $R = 1$ eliminates the last term leaving an equation that we can integrate to a first order equation, i.e. if $R = 1$,

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)^2 = c_1\psi^{-4} + c_2,$$

and if $R = \frac{1}{3}$,

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)^2 = c_3\psi^{4/3} + c_4.$$

If $c_2 = 0$ we find $A = 1/(3x + c_5)$, and if $c_4 = 0$ we find $A = 1/(x + c_6)$. We now consider four special cases

$$\begin{aligned}
&\text{(i)} && A = 0,\\
&\text{(ii)} && A = \frac{1}{x + c_6},\\
&\text{(iii)} && A = \frac{1}{3x + c_5},\\
&\text{(iv)} && A \text{ none of the above.}
\end{aligned}$$

Subcase 2.4.2(i) If $A = 0$ then from (2.41ii,iii) we must have $B_x = 0$. There is also a condition on $B(t)$ from (2.41iv),

$$\frac{\mathrm{d}^2 B}{\mathrm{d}t^2} - 6B^3 = 0.$$

We may integrate this with respect to t to yield

$$\left(\frac{\mathrm{d}B}{\mathrm{d}t}\right)^2 = 3B^4 - \frac{c_7^4}{3}. \tag{2.42}$$

If $c_7 = 0$, we have $B = 1/(c_8 - \sqrt{3}\,t)$, else $B(t)$ is solvable in terms of Jacobi elliptic functions. Solving the invariant surface condition gives the following nonclassical reduction

Reduction 2.4.4. $D(u) = u^3 + cu$

$$u(x,t) = w(t) + xB(t),$$

where $B(t)$ satisfies (2.42) and $w(t)$ satisfies

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - 6B^2(t)w = 0. \tag{2.43}$$

If $B = 1/(c_8 - \sqrt{3}\,t)$, solving (2.43) yields the exact (canonical) solution

$$u(x,t) = c_9 t^2 - \frac{x}{\sqrt{3}\,t},$$

which is not a special case of any classical reduction.

If $c_7 \neq 0$ then we set $c_7 = 1$ without loss of generality, and then $B = \mathrm{i}\,\mathrm{sn}(t;\mathrm{i})/\sqrt{3}$ where $\mathrm{sn}(t;k)$ is the Jacobi elliptic function with modulus k, and (2.43) takes the Jacobian form of the Lamé equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + 2\mathrm{sn}^2(t;\mathrm{i})w = 0. \tag{2.44}$$

The Jacobi and Weierstrass elliptic functions are related by (see Whittaker and Watson [1927])

$$\wp(y; g_2, g_3) = e_3 + (e_2 - e_3)\mathrm{sn}^2(t;k),$$

where $t = y\sqrt{e_1 - e_3} + \mathrm{i}K'$, where $2\mathrm{i}K'$ is the imaginary period of $\mathrm{sn}(t)$, the e_1, e_2 and e_3 are roots of the cubic $4t^3 - g_2 t - g_3 = 0$, and $(e_1 - e_3)k^2 = e_2 - e_3$. Equation (2.44) may thus be transformed to the better known Lamé equation written in terms of Weierstrass elliptic functions

$$\frac{\mathrm{d}^2 w}{\mathrm{d}y^2} - \{h + 2\wp(y)\}w = 0, \tag{2.45}$$

where $h = -2e_3$ is constant. Equation (2.45) has solutions

$$w_1(y) = \exp\{-y\zeta(a)\}\frac{\sigma(y+a)}{\sigma(y)}, \qquad w_2(y) = \exp\{y\zeta(a)\}\frac{\sigma(y-a)}{\sigma(y)},$$

where $\wp(a) = h = -2e_3$, and $\zeta(z)$ and $\sigma(z)$ are the Weierstrass zeta and sigma functions respectively, defined by (2.19) and (2.34) respectively. Since $h \neq e_1, e_2, e_3$ then $w_1(y), w_2(y)$ are in general distinct so the general solution is their linear combination.

Alternatively we can note that $w = B$ is one solution of (2.43) so we can find the general solution in terms of quadratures. This yields the exact (canonical) solution

$$u(x,t) = B(t)\left(x + c_9\int^t \frac{\mathrm{d}s}{B^2(s)}\right), \tag{2.46}$$

where $B(t)$ satisfies (2.42).

Subcase 2.4.2(ii) If $A = 1/(x + c_6)$ it is easy to show using the Direct Search strategy that for system (2.30) to be satisfied it is necessary for $B = 0$. We note that if $B = 0$, then if $A \neq 0$, we again use a Direct Search on system (2.30) and can show that necessarily $A = 1/(x + c_6)$. Thus we have infinitesimal

$$\phi = \frac{u}{x + c_6},$$

which gives the nonclassical reduction

Reduction 2.4.5. $D(u) = u^3 + cu$

$$u(x,t) = w(t)(x + c_6),$$

where $w(t)$ satisfies

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - 6w^3 = 0. \tag{2.47}$$

This is in fact a special case of the exact solution (2.46), when $c_9 = 0$.

Subcase 2.4.2(iii) If $A = 1/(3x + c_5)$, equation (2.30ii) becomes an Euler-type ordinary differential equation for B with solution

$$B(x,t) = \eta(t)(3x + c_5)^{-5/3} + \rho(t)(3x + c_5)^{2/3},$$

where $\eta(t)$ and $\rho(t)$ are arbitrary functions. However for no values of $\eta(t)$, $\rho(t)$ or c can we make this combination of A, B consistent with the remaining equations (2.30iii,iv); there are no solutions in this Subcase.

Subcase 2.4.2(iv) We can differentiate (2.41iii) with respect to t to give, on simplifying

$$3B_tB_x + 3BB_{xt} + 10BB_tA = 0. \tag{2.48}$$

For $A \neq 0$, A must satisfy (2.41i) so we again write A in the form $A(x) = R\frac{\mathrm{d}}{\mathrm{d}x}\ln[\psi(x)]$ (for $R = 1$ or $R = \frac{1}{3}$). If $B_t \neq 0$ we can write

$$-\frac{10R}{3}\frac{\mathrm{d}}{\mathrm{d}x}\ln[\psi(x)] = \frac{B_x}{B} + \frac{B_{xt}}{B_t},$$

which we can integrate with respect to x, to yield

$$-\frac{10R}{3}\ln\psi = \ln B + \ln B_t - \ln f(t),$$

where $f(t)$ is an arbitrary function of integration. We can exponentiate and integrate with respect to t

$$\int^B s\,\mathrm{d}s = \psi^{-10R/3}(x)\int^t f(s)\,\mathrm{d}s,$$

$$\text{i.e.}\qquad B^2 = g(t)\psi^{-10R/3}(x) + h(x), \tag{2.49}$$

where $g(t) = 2\int^t f(s)\,\mathrm{d}s$ and $h(x)$ is an arbitrary function of integration. However this is only a solution of (2.48), not of (2.41iii). We substitute this solution into (2.41iii) together with our other information, and find a linear first order ordinary differential equation to solve for $h(x)$. This has solution

$$h(x) = \frac{80c_1c_2}{7}\psi^{-8} + \frac{cc_1}{2}\psi^{-6} + \frac{160c_1^2}{13}\psi^{-12},$$

for $R = 1$, and for $R = \frac{1}{3}$,

$$h(x) = \frac{cc_4}{18}\psi^{-2} + \frac{80c_3c_4}{567}\psi^{-8/3} + \frac{160c_4^2}{1053}\psi^{-4}.$$

Note that the arbitrary function of integration is consumed by $g(t)$. Substituting our expressions for A and B into (2.41ii), for each value of R, along with our relations on $\psi(x)$ and $h(x)$, gives an equation in $\psi(x)$ and $g(t)$ which is a quadratic in $g(t)$. For the coefficients of this quadratic to be zero we find $c_1 = c_2 = c_3 = c_4 = 0$ which means $A = 0$, in contradiction to our assumption otherwise. If they are not zero, solving the quadratic for $g(t)$ gives $g(t) = k(x)$ for some function $k(x)$, so $g(t)$ is at best a constant. From (2.49) this implies that $B_t = 0$, yielding another contradiction. Hence we have shown that if we require $AB_t \neq 0$ then there are no solutions.

Thus we are left with solving (2.41) where A and B are functions of x only (and not one of the Subcases already considered). Firstly we **reduce**((2.41iii),[(2.41i)], $k1$), then **reduce**((2.41ii),[(2.41i),$k1$], $k2$) and $k2$ is a quartic in B. We obtain another quartic in B by **reduce**((2.41iiii),[(2.41i)], $k3$), **reduce**($k3$,[(2.41ii)], $k4$), **reduce**($k4$,[(2.41i)], $k5$),

reduce($k5$,[$k1$], $k6$), $k6$ being the required quartic. We can now use **reduce** to remove the B's from $k6$ and $k2$ to get an expression of the form

$$\sum_{n=0}^{N} a_i A^{2n} A_x^{N-n} = 0,$$

which we can factor as

$$a_N \prod_{n=0}^{N} (A^2 + b_n A_x) = 0,$$

for some $b_n \in \mathbb{C}$ and a_N is a (known) non-zero constant. For a candidate equation $A^2 + b_j A_x = 0$, to be consistent with (2.30i) it is necessary for b_j to take the values 1 or $\frac{1}{3}$ (or $A = 0$) which are precisely the Subcases 2.4.2(ii) and (iii) above. Thus there are no more solutions.

Case 2.4.3 $D(u) = u^n + cu$ for $n \neq 0, 1, 2, 3$. We initially split this in two, when $n = 4$ and when $n \neq 4$. Firstly we will consider the case when $n = 4$. The determining equations, once we write $\phi = A(x)u + B(x,t)$, yield

$$A_{xx} + 11AA_x + 12A^3 = 0, \tag{2.50i}$$

$$11BA_x + 9AB_x + 30A^2B + B_{xx} = 0, \tag{2.50ii}$$

$$3BB_x + 8B^2A = 0, \tag{2.50iii}$$

$$24B^3 + A_{xxxx} + cA_{xx} + 10A_{xx}A_x + 4AA_{xxx} + 6A^2A_{xx} + 12AA_x^2 + 4A^3A_x + 2cAA_x = 0, \tag{2.50iv}$$

$$B_{xxxx} + 4A_xB_{xx} + cB_{xx} + 6A_{xx}B_x + 4AA_xB_x + 4BA_{xxx} + 6BAA_{xx} + 8BA_x^2 + 4BA^2A_x + 2cBA_x - B_{tt} = 0. \tag{2.50v}$$

We **reduce**((2.50ii),[(2.50iii)], $k1$) to yield

$$B^3(75A_x - 118A^2) = 0.$$

The equation $75A_x - 118A^2 = 0$ is not consistent with (2.50i), unless $A = 0$ which implies (from (2.50iv)) that $B = 0$. Thus we take $B = 0$, then the $n = 4$ case fits into the pattern for general n, namely $B = 0$ and $A(x)$ satisfies the overdetermined system of ordinary differential equations

$$A_{xx} + (3n-1)AA_x + n(n-1)A^3 = 0, \tag{2.51i}$$

$$A_{xxxx} + cA_{xx} + 10A_{xx}A_x + 4AA_{xxx} + 6A^2A_{xx} + 12AA_x^2 + 4A^3A_x + 2cAA_x = 0. \tag{2.51ii}$$

We use successive reduction of these two equations to reduce the order and the degree of the highest derivative term until we get a polynomial in A alone (noting along the way the special cases when the highest coefficient and/or the separant are zero). Since we do not want $A = 0$, each coefficient of our polynomial in A must be zero. There are three scenarios in which the system (2.51) is compatible (not including values of n which we have considered previously ($n = 0, 1, 2, 3$)):

(i) if $n = \frac{1}{2}$, $c = 0$, we have $A = 2/(x + c_1)$ which leads to the nonclassical reduction

Reduction 2.4.6. $D(u) = u^{1/2}$

$$u(x,t) = w(t)(x + c_1)^2,$$

where $w(t)$ satisfies $\dfrac{\mathrm{d}^2 w}{\mathrm{d}t^2} = 0$, which gives the (canonical) exact solution

$$u(x,t) = c_2 x^2 t.$$

This solution is a special solution of the classical scaling reduction 2.2.2.

(ii) if $n = \frac{1}{3}$, $c = 0$, then $A = 3/(x + c_3)$ which gives the nonclassical reduction

Reduction 2.4.7. $D(u) = u^{1/3}$

$$u(x,t) = w(t)(x + c_3)^3,$$

where $w(t)$ satisfies $\dfrac{\mathrm{d}^2 w}{\mathrm{d}t^2} = 0$. This gives the exact (canonical) solution

$$u(x,t) = c_4 x^3 t.$$

This is *not* a special solution of the classical scaling reduction 2.2.2 (though $u(x,t) = c_5 x^3$ is in this case).

(iii) if $n = \frac{5}{3}$, $c = 0$, then our infinitesimal ϕ is $\phi = 3u/(x + c_6)$ which on solving the invariant surface condition gives the following nonclassical reduction

Reduction 2.4.8. $D(u) = u^{5/3}$. We set $c_6 = 0$ without loss of generality to yield

$$u(x,t) = w(t)x^3,$$

where $w(t)$ satisfies

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - 20w^{5/3} = 0. \tag{2.52}$$

Making the transformation $w = W^3$ yields the rational equation

$$3W\frac{\mathrm{d}^2 W}{\mathrm{d}t^2} + 6\left(\frac{\mathrm{d}W}{\mathrm{d}t}\right)^2 - 20W^4 = 0, \tag{2.53}$$

to which we apply the ARS algorithm. There is a dominant behaviour like $W(z) \sim \pm\left(\frac{3}{5}\right)^{1/2}(z - z_0)^{-1}$ with z_0 arbitrary, which in fact continues to give the correct number of arbitrary constants without any branch points. However, balancing the first two terms in (2.53) yields behaviour like $W(z) \sim a_0(z-z_0)^{1/3}$, so we conclude that (2.53) is not of Painlevé-type.

We can integrate (2.52) once with respect to t to yield

$$\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2 = 15w^{8/3} + 2c_7. \tag{2.54}$$

If $c_7 = 0$ we find the exact (canonical) solution

$$u(x,t) = \frac{3\sqrt{15}\,x^3}{25t^3},$$

which is a special solution of reduction 2.2.2, whilst if $c_7 \neq 0$ the reduction is genuinely nonclassical, and (2.54) is solvable by quadratures.

<u>*Case 2.4.4* $D(u) = e^u + cu$.</u> The determining equations give us that $A(x) = 0$ and we have a system of two equations to solve for $B(x,t)$,

$$3BB_x + B_{xx} + B^3 = 0, \tag{2.55i}$$

$$B_{xxxx} + cB_{xx} - B_{tt} = 0. \tag{2.55ii}$$

To solve this system we first solve (2.55i) by making the Cole-Hopf transformation $B = \dfrac{\mathrm{d}}{\mathrm{d}x}[\ln\psi(x)]$ which helps find the solution as

$$B(x,t) = \frac{2F(t)x + G(t)}{F(t)x^2 + G(t)x + H(t)},$$

where $F(t)$, $G(t)$ and $H(t)$ are arbitrary functions. Substituting this into (2.55ii) we get an eighth order polynomial in x to solve, whose coefficients are nonlinear differential equations in F, G and H,

Order x^8

$$2F_t^2F^2G - F_{tt}F^3G - 2F_tG_tF^3 + G_{tt}F^4,$$

Order x^7

$$\begin{aligned}&-4F_tH_tF^3 + 4cF^5 - 2G_t^2F^3 + 2F^4H_{tt} + 3G_{tt}F^3G\\&-3F_{tt}F^2G^2 - 2F_tG_tF^2G - 2F_{tt}F^3H + 4F_t^2FG^2 + 4F_t^2F^2H,\end{aligned}$$

Order x^6

$$\begin{aligned}&2F_tG_tF^2H + 2F_tG_tFG^2 + 2G_{tt}F^3H\\&-8F_tH_tF^2G + 12F_t^2FGH - 4G_t^2F^2G + 3G_{tt}F^2G^2 - 9F_{tt}F^2GH\\&-3F_{tt}FG^3 - 6G_tH_tF^3 + 7F^3H_{tt}G + 14cF^4G + 2F_t^2G^3,\end{aligned}$$

Order x^5

$$48F^5 - F_{tt}G^4 - 4F_tH_tF^2H$$
$$- 4F_tH_tFG^2 + G_{tt}FG^3 + 6F^3H_{tt}H - 6F_{tt}F^2H^2 + 3G_{tt}F^2GH$$
$$+ 8F_t^2FH^2 - 14G_tH_tF^2G - 12F_{tt}FG^2H + 9F^2H_{tt}G^2 - 2G_t^2F^2H - 2G_t^2FG^2$$
$$- 4F^3H_t^2 + 22cF^3G^2 + 12F_tG_tFGH - 4cF^4H + 8F_t^2G^2H + 2F_tG_tG^3,$$

Order x^4

$$10F_t^2GH^2 + 5FH_{tt}G^3 - 10G_tH_tF^2H - 15F_{tt}FGH^2 + 120F^4G - 5F_{tt}G^3H$$
$$- 10cF^3GH - 10F^2H_t^2G + 10F_tG_tG^2H + 15F^2H_{tt}GH$$
$$- 10G_tH_tFG^2 + 20cF^2G^3 + 10F_tG_tFH^2,$$

Order x^3

$$4F_tH_tFH^2 + 240F^3G^2 + 10cFG^4$$
$$+ G^4H_{tt} + 6F^2H_{tt}H^2 - 2G_tH_tG^3 + 4F_tH_tG^2H - 6F_{tt}FH^3 - 3G_{tt}FGH^2$$
$$+ 12FH_{tt}G^2H - 20cF^3H^2 + 14F_tG_tGH^2 - 12G_tH_tFGH + 4F_t^2H^3 + 2G_t^2G^2H$$
$$- 480F^4H + 2G_t^2FH^2 - 8F^2H_t^2H - G_{tt}G^3H - 9F_{tt}G^2H^2 - 8FH_t^2G^2,$$

Order x^2

$$- 7F_{tt}GH^3 + 8F_tH_tGH^2 + 10cFG^3H - 720F^3GH - 3G_{tt}G^2H^2$$
$$+ 9FH_{tt}GH^2 - 2G_tH_tFH^2 - 2G_{tt}FH^3 + 4G_t^2GH^2 + 2cG^5 + 3G^3H_{tt}H$$
$$- 2G_tH_tG^2H + 240F^2G^3 - 30cF^2GH^2 + 6F_tG_tH^3 - 12FH_t^2GH - 2G^3H_t^2,$$

Order x^1

$$240F^3H^2 - 6cFG^2H^2 - 3G_{tt}GH^3 - 480F^2G^2H - 2F_{tt}H^4 - 12cF^2H^3$$
$$+ 4cG^4H + 120FG^4 + 4F_tH_tH^3 - 4G^2H_t^2H + 2FH_{tt}H^3 - 4FH_t^2H^2$$
$$+ 3G^2H_{tt}H^2 + 2G_t^2H^3 + 2G_tH_tGH^2,$$

Order x^0

$$- 120FG^3H + 2G_tH_tH^3 + 120F^2GH^2$$
$$+ 2cG^3H^2 - 6cGFH^3 - G_{tt}H^4 + GH_{tt}H^3 + 24G^5 - 2GH_t^2H^2,$$

We can reduce the equations of coefficients of $x^1, ..., x^7$ with respect to that of x^8 then x^0 to get seven first order equations in F, G and H ($k1$,...,$k7$). We reduce $k2$,...,$k7$ with respect to $k1$ to give $l1$,...,$l6$. These are of the form

$$l1: \quad H^3(FG_t - GF_t)(FGG_t - 2H_tF^2 + 2FHF_t - F_tG^2) = 0,$$
$$li: \quad Hf_i(F, G, H, F_t, G_t, H_t) = 0 \quad \text{for } i = 2, ..., 6.$$

We now reduce *l3*,...,*l6* with respect to *l2* to give $m1$,...,$m4$

$$
\begin{aligned}
m1: &\quad F^4G(2HF - 5G^2)(4HF - G^2)(HF - G^2) = 0,\\
m2: &\quad -F^4G(4HF - G^2) = 0,\\
m3: &\quad -F^3G(16HF - 7G^2)(4HF - G^2)(F^2H^2 + G^4 + 7FG^2H) = 0,\\
m4: &\quad GF^3(HF + 2G^2)(4HF - G^2)(4F^2H^2 + 7G^4 - 20FG^2H) = 0.
\end{aligned}
$$

Thus we have five cases to consider

(i) $H = 0$,

(ii) $H \neq 0, F = 0$,

(iii) $H \neq 0, G = 0$,

(iv) $F, G, H \neq 0, G^2 = 4FH, FG_t - GF_t = 0$,

(v) $F, G, H \neq 0, G^2 = 4FH, FGG_t - 2H_tF^2 + 2FHF_t - F_tG^2 = 0$.

In each case we get the trivial solution $F = G = 0$, i.e. $B(x,t) = 0$.

Case 2.4.5 $D(u) = \ln u + cu$. Our equations for $A(x)$ and $B(x,t)$ imply that $B(x,t) = 0$ which leaves two equations in $A(x)$:

$$2cAA_x + 6A^2A_{xx} + 4AA_{xxx} + 4A^3A_x + 12AA_x^2 + 10A_xA_{xx} + cA_{xx} + A_{xxxx} = 0, \quad (2.56\text{i})$$

$$A_{xx} - AA_x = 0. \quad (2.56\text{ii})$$

We **reduce** (2.56i) with respect to (2.56ii) which yields

$$AA_x(c + 5A^2 + 10A_x) = 0. \quad (2.56\text{iii})$$

This equation and equation (2.56ii) are only compatible if $A_x = 0$, i.e. $A = c_1$. This is consistent with all three equations, and gives the infinitesimal $\phi = c_1 u$. This leads to the nonclassical reduction

Reduction 2.4.9. $D(u) = \ln u + cu$

$$u(x,t) = w(t)\exp(c_1 x),$$

where $w(t)$ satisfies the linear equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - c_1^2(c + c_1^2)w = 0.$$

If $c + c_1^2 = 0$ then $w(t)$ is linear in t, whilst if $c + c_1^2 \neq 0$ this gives the exact solution

$$u(x,t) = \exp(c_1 x)\left[c_2 \exp\left(c_1\sqrt{c + c_1^2}\; t\right) + c_3 \exp\left(-c_1\sqrt{c + c_1^2}\; t\right)\right].$$

Case 2.4.6 $D(u) = u \ln u + cu$. Again our equations for $A(x)$ and $B(x,t)$ imply that $B(x,t) = 0$ which leaves two equations in $A(x)$:

$$2cAA_x + 6A^2A_{xx} + 4AA_{xxx}$$
$$+ 4A^3A_x + 12AA_x^2 + 10A_xA_{xx} + cA_{xx} + A_{xxxx} + A^3 + 5AA_x + A_{xx} = 0, \quad (2.57\text{i})$$
$$A_{xx} + 2AA_x = 0. \quad (2.57\text{ii})$$

We **reduce** (2.57i) with respect to (2.57ii) which yields

$$A(A^2 + 3A_x) = 0. \quad (2.57\text{iii})$$

Equation (2.57iii) is only compatible with (2.57ii) if $A = 0$, which gives the trivial solution outlined in the preamble to this section.

2.5 Discussion

In this chapter we have classified equation (2.1) by the types of classical and nonclassical symmetry reductions it possesses. What is perhaps slightly unusual is the lack of nonclassical reductions in the $\tau \neq 0$ case, unless (2.1) reduces to the Boussinesq equation which by contrast has a wide range of reductions. It is clear that in this respect the Boussinesq equation is special.

We have tested the ordinary differential equations arising from our symmetry reductions to see whether they are of Painlevé-type, and have found that only the reductions of the Boussinesq equation are. By virtue of the Painlevé conjecture (cf. §1.6.2), we can conclude that only the Boussinesq equation in the class of equations we have studied may be solvable by inverse scattering, and from previous studies (see §2.1) we know that it is.

Chapter Three:
A class of Nonlinear Third Order Partial Differential Equations

3.1 Introduction

In this chapter we are concerned with symmetry reductions of the nonlinear third order partial differential equation given by

$$\Delta \equiv u_t - \epsilon u_{xxt} + 2\kappa u_x - u u_{xxx} - \alpha u u_x - \beta u_x u_{xx} = 0, \tag{3.1}$$

where ϵ, κ, α and β are arbitrary constants. The majority of the work in this chapter is to appear in Clarkson, Mansfield and Priestley [1996]. Three special cases of (3.1) have appeared recently in the literature. Up to some rescalings, these are: (i), the Fornberg-Whitham equation (Fornberg and Whitham [1978] and Whitham [1967,1974]), for the parameters $\epsilon = 1$, $\alpha = -1$, $\beta = 3$ and $\kappa = \frac{1}{2}$, (ii), the Rosenau-Hyman equation (Rosenau and Hyman [1993]) for the parameters $\epsilon = 0$, $\alpha = 1$, $\beta = 3$ and $\kappa = 0$, and (iii), the Fuchssteiner-Fokas-Camassa-Holm equation (Camassa and Holm [1993], Camassa, Holm and Hyman [1994], Fuchssteiner [1981] and Fuchssteiner and Fokas [1981]) for the parameters $\epsilon = 1$, $\alpha = -3$ and $\beta = 2$.

The Fornberg-Whitham (FW) equation

$$u_t - u_{xxt} + u_x = u u_{xxx} - u u_x + 3 u_x u_{xx}, \tag{3.2}$$

was used to look at qualitative behaviours of wave-breaking (see Whitham [1967]). It admits a wave of greatest height, as a peaked limiting form of the travelling wave solution (Fornberg and Whitham [1978]),

$$u(x,t) = \tfrac{4}{3} \exp\left(-\tfrac{1}{2}|x - \tfrac{4}{3}t|\right). \tag{3.3}$$

The Rosenau-Hyman (RH) equation

$$u_t = uu_{xxx} + uu_x + 3u_xu_{xx}, \tag{3.4}$$

models the effect of nonlinear dispersion in the formation of patterns in liquid drops (Rosenau and Hyman [1993]). It also has an unusual solitary wave solution, known as a "compacton",

$$u(x,t) = \begin{cases} -\frac{8}{3}c\cos^2\{\frac{1}{4}(x-ct)\}, & \text{if } |x-ct| \leq 2\pi, \\ 0, & \text{if } |x-ct| > 2\pi. \end{cases}$$

These waves interact producing a ripple of low amplitude compacton-anticompacton pairs.

The Fuchssteiner-Fokas-Camassa-Holm (FFCH) equation

$$u_t - u_{xxt} + 2\kappa u_x = uu_{xxx} - 3uu_x + 2u_xu_{xx}, \tag{3.5}$$

first arose in the work of Fuchssteiner and Fokas (see Fuchssteiner [1981], Fuchssteiner and Fokas [1981]) using a bi-Hamiltonian approach; we remark that it is only implicitly written in Fuchssteiner and Fokas [1981] – see equations (26e) and (30) in this paper – though is explicitly written down in Fuchssteiner [1981]. It has recently been rederived by Camassa and Holm [1993] from physical considerations as a model for dispersive shallow water waves. In the case $\kappa = 0$, it admits an unusual solitary wave solution

$$u(x,t) = c\exp\left(-|x-ct|\right), \tag{3.6}$$

where c is an arbitrary constant, which is called a "peakon". A Lax-pair has been found by Camassa and Holm [1993] and a bi-Hamiltonian structure by Fuchssteiner and Fokas [1981] for the FFCH equation (3.5) and so it appears to be completely integrable. Recently the FFCH equation (3.5) has attracted considerable attention. In addition to the aforementioned, other studies include those by Camassa, Holm and Hyman [1994], Cooper and Shepard [1994], Fokas [1994a,b], Fokas and Santini [1994], Fuchssteiner [1993], Gilson and Pickering [1995], Marinakis and Bountis [1995] and Olver and Rosenau [1996].

The FFCH equation (3.5) may be thought of as an integrable modification of the regularized long wave (RLW) equation (Peregrine [1966])

$$u_{xxt} + uu_x - u_t - u_x = 0, \tag{3.7}$$

sometimes known as the Benjamin-Bona-Mahoney equation (cf. Benjamin, Bona and Mahoney [1972]). However, in contrast to (3.5), the RLW equation (3.7) is thought *not* to be solvable by inverse scattering (cf. McLeod and Olver [1983]); its solitary wave solutions interact inelastically (cf. Makhankov [1978]) and only has finitely many local conservation laws (Olver [1994]). However physically it has more desirable properties than the celebrated Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + 6uu_x = 0, \tag{3.8}$$

which was the first equation to be solved by inverse scattering by Gardner *et al.* [1967]. We remark that two other integrable variants of the RLW equation (3.7) are

$$u_{xxt} + 2uu_t - u_x\partial_x^{-1}u_t - u_t - u_x = 0, \tag{3.9}$$

where $\left(\partial_x^{-1}f\right)(x) = \int_x^\infty f(y)\,\mathrm{d}y$, which was introduced by Ablowitz *et al.* [1974], and

$$u_{xxt} + uu_t - u_x\partial_x^{-1}u_t - u_t - u_x = 0, \tag{3.10}$$

which was discussed by Hirota and Satsuma [1976] and are both special cases of the shallow water wave systems we study in Chapter Five. Also Fuchssteiner [1993] has shown that the FFCH equation is related by a hodograph transformation, then a Bäcklund transformation to equation (3.9). We also note that (3.5), with $\kappa = \frac{1}{2}$, (3.7), (3.9) and (3.10) all have the same linear dispersion relation $\omega(k) = -k/(1+k^2)$ for the complex exponential $u(x,t) \sim \exp\{\mathrm{i}[kx + \omega(k)t]\}$.

Recently, Gilson and Pickering [1995] have shown that no equation in the entire class of equations (3.1) will satisfy the necessary conditions of either the Painlevé PDE test or the Painlevé ODE test (see §1.6 for details) to be solvable by inverse scattering. However, the integrable FFCH equation (3.5) does possess the "weak Painlevé" property (cf. §1.6), as does the FW equation (3.2).

All these special travelling wave solutions are essentially exponential solutions, or sums of exponential solutions, and thus would suggest some sort of linearity in the differential equation. This is discussed by Gilson and Pickering [1995], who show that (3.1), with $\alpha \neq 0$ and $\beta(1+\beta) \neq 0$, can be written as

$$(\beta u_x + u\partial_x + \epsilon\partial_t)\left(u_{xx} - \mu^2 u - 2\kappa/\beta\right) = 0, \tag{3.11}$$

where $\partial_x \equiv \partial/\partial x$, $\partial_t \equiv \partial/\partial t$ and $\mu^2 = -\alpha/(1+\beta)$, provided that $\epsilon\alpha + \beta + 1 = 0$, which includes the FFCH equation (3.5). For the travelling wave reduction,

$$u(x,t) = w(z), \qquad z = x - ct,$$

the resulting ordinary differential equation is

$$(2\kappa - c)w' + \epsilon c w''' - ww''' - \alpha ww' - \beta w'w'' = 0, \tag{3.12}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$, which also may be factorised as

$$\left[\beta w' + (w - \epsilon c)\frac{\mathrm{d}}{\mathrm{d}z}\right]\left(w'' - \mu^2 w + \gamma\right) = 0, \qquad (3.13),$$

provided that

$$\mu^2 = -\frac{\alpha}{1+\beta}, \qquad \beta(1+\beta)\gamma - 2\kappa(1+\beta) + c(1+\beta+\alpha\epsilon) = 0.$$

This includes all three special cases (3.2)–(3.5); since $\beta(1+\beta)$ is strictly non-zero in these three cases then a suitable γ can always be found.

Furthermore, if $1+\beta+\alpha\epsilon = 0$ and $\epsilon \neq 0$, then (3.1) with $\kappa = 0$ possesses the "peakon" solution

$$u(x,t) = \epsilon c \exp\left(-\epsilon^{-1/2}|x-ct|\right),$$

where c is an arbitrary constant. More generally, if $\alpha/(1+\beta) < 0$, $1+\beta+\alpha\epsilon \neq 0$ and $\kappa \neq 0$, then (3.1) possesses the solution

$$u(x,t) = \epsilon c \exp\left\{-\left(-\frac{\alpha}{1+\beta}\right)^{1/2}|x-ct|\right\}, \qquad c = \frac{2(1+\beta)\kappa}{1+\beta+\alpha\epsilon}.$$

An alternative justification of these solutions is given in §3.2. If $\alpha/(1+\beta) > 0$, $\beta \neq -1$ and $\alpha\beta \neq 0$, then (3.1) possesses the "compacton" solution

$$u(x,t) = \frac{2[2(1+\beta)\kappa - (1+\beta+\alpha\epsilon)c]}{\alpha\beta}\cos^2\left\{\tfrac{1}{2}\left(\frac{\alpha}{1+\beta}\right)^{1/2}(x-ct)\right\},$$

where c is an arbitrary constant.

In the following sections we shall consider the cases $\epsilon = 0$ and $\epsilon \neq 0$, when we set $\epsilon = 1$ without loss of generality, separately because the presence or lack of the corresponding third order term is significant. In §3.2 we find the classical Lie group of symmetries and associated reductions of (3.1). In §3.3 we discuss the nonclassical symmetries and reductions of (3.1) in the generic case. In §3.4 we consider special cases of the the nonclassical method in the so-called $\tau = 0$; in full generality this case generates a single equation which is considerably more complex than our original equation! In §3.5 we discuss our results.

3.2 Classical symmetries

To apply the classical method we consider the one-parameter Lie group of infinitesimal transformations in (x,t,u) given by

$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \tag{3.14i}$$

$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \tag{3.14ii}$$

$$u^* = u + \varepsilon\phi(x,t,u) + O(\varepsilon^2), \tag{3.14iii}$$

where ε is the group parameter. This procedure yields a system of linear determining equations. There are two cases to consider.

<u>*Case 3.2.1* $\epsilon = 0$</u>. In this case using the MACSYMA package `symmgrp.max` we obtain the following system of ten determining equations

$$\tau_u = 0, \tag{3.15i}$$

$$\tau_x = 0, \tag{3.15ii}$$

$$\xi_u = 0, \tag{3.15iii}$$

$$u\phi_{uuu} + \beta\phi_{uu} = 0, \tag{3.15iv}$$

$$3u^2\phi_{uu} + \beta u\phi_u - \beta\phi = 0, \tag{3.15v}$$

$$3u\phi_{xu} - 3u\xi_{xx} + \beta\phi_x = 0, \tag{3.15vi}$$

$$3u\phi_{xuu} + 2\beta\phi_{xu} - \beta\xi_{xx} = 0, \tag{3.15vii}$$

$$\tau_t u - 3\xi_x u + \phi = 0, \tag{3.15viii}$$

$$\phi_{xxx}u + (\alpha u - 2\kappa)\phi_x - \phi_t = 0, \tag{3.15ix}$$

$$3u^2\phi_{xxu} + \beta u\phi_{xx} + 2\kappa\phi - u^2\xi_{xxx} + (2\alpha u^2 - 4\kappa u)\xi_x + u\xi_t = 0. \tag{3.15x}$$

Next applying the **reduceall** algorithm in the MAPLE package `diffgrob2` to this system yields

$$(2+\beta)\xi_{xx} = 0, \qquad (2+\beta)[\alpha u\xi_{xt} + \xi_{tt} - 2\kappa\xi_{xt}] = 0, \qquad \xi_u = 0,$$
$$\tau_x = 0, \qquad 2\alpha u\xi_x + 2\kappa\xi_x + \xi_t - 2\kappa\tau_t = 0, \qquad \tau_u = 0,$$
$$(2+\beta)[2\kappa\phi + (2\alpha u^2 - 4\kappa u)\xi_x + u\xi_t] = 0.$$

This is simple enough to solve; there is no need to do the full Kochin-Ritt algorithm in this case. The factors which are listed by `diffgrob2` as having multiplied the system (3.15) during the application of **reduceall** are $\beta + 2$ and κ. Thus initially we have special cases when $\beta = -2$ and $\kappa = 0$, and combinations thereof. It transpires that the special case $\beta = -2$ is purely an artefact, but when considering the special case when $\kappa = 0$, we find another special case, when $\alpha = 0$. For the two special cases (a) $\kappa = 0$, $\alpha \neq 0$, and (b) $\kappa = \alpha = 0$, applying the **reduceall** algorithm of `diffgrob2` to (3.15) yields

(a) $\kappa = 0, \alpha \neq 0$

$$2\alpha u\xi_x + \xi_t = 0, \qquad \xi_{tt} = 0, \qquad \xi_u = 0,$$
$$\tau_x = 0, \qquad \tau_{tt} = 0, \qquad \tau_u = 0,$$
$$2\alpha\phi + 2\alpha u\tau_t + 3\xi_t.$$

(b) $\kappa = \alpha = 0$

$$\xi_{xx} = 0, \qquad \xi_t = 0, \qquad \xi_u = 0,$$
$$\tau_x = 0, \qquad \tau_{tt} = 0, \qquad \tau_u = 0,$$
$$\phi - 3u\xi_x + u\tau_t = 0$$

Hence we obtain the infinitesimals listed in table 3.2.1.

Table 3.2.1

Parameters	ξ	τ	ϕ	
$\alpha \neq 0$	$2\kappa c_3 t + c_1$	$c_3 t + c_2$	$-c_3 u$	(3.16)
$\alpha = 0$	$c_4 x + 2\kappa(c_3 - c_4)t + c_1$	$c_3 t + c_2$	$(3c_4 - c_3)u$	(3.17)

where c_1, c_2, c_3, c_4 are arbitrary constants.

Solving the invariant surface condition yields four different canonical reductions

Reduction 3.2.1. α and κ arbitrary. If $c_3 = c_4 = 0$ in (3.16–3.17) we may set $c_1 = c$ and $c_2 = 1$ and we obtain the travelling wave reduction

$$u(x,t) = w(z), \qquad z = x - ct,$$

where $w(z)$ satisfies

$$ww''' + \alpha ww' + \beta w'w'' + (c - 2\kappa)w' = 0.$$

This can be integrated to yield

$$ww'' + \tfrac{1}{2}(\beta - 1)(w')^2 + \tfrac{1}{2}\alpha w^2 + (c - 2\kappa)w = A,$$

where A is an arbitrary constant. Multiplying this by $w^{\beta-2}w'$ and integrating again yields

$$(w')^2 + \frac{\alpha}{1+\beta}w^2 + \frac{2(2\kappa - c)}{\beta}w = \frac{2A}{\beta - 1} + Bw^{1-\beta}, \tag{3.18}$$

where B is an arbitrary constant, for $\beta \neq -1, 0, 1$. Generally if $\beta \neq -1, 0, 1$, then (3.18) is solvable using quadratures, though for certain special values of the parameters there are explicit solutions. For example (i), if $\beta = -2$ or $\beta = -3$, then (3.18) is solvable in terms of Weierstrass or Jacobi elliptic functions, respectively, (ii), if $B = 0$, then (3.18) is solvable in term of trigonometric functions, and (iii), if $c = 2\kappa$ and $\beta = 3$, then $w(z)$ can be expressed in terms of trigonometric functions via the transformation $w(z) = v^{1/2}$.

In the special cases $\beta = -1, 0, 1$ we obtain the equations

$$(w')^2 + \alpha w^2 \ln w + 2(2\kappa - c)w = Bw^2 - A,$$

$$(w')^2 + \alpha w^2 + 2(2\kappa - c)w \ln w = Bw - 2A,$$

$$(w')^2 + \alpha w^2 + 2(2\kappa - c)w = B - A\ln w,$$

respectively, with A and B arbitrary constants. If the coefficient of $\ln w$ in these equations is zero, then $w(z)$ is expressible in terms of elementary functions, otherwise in terms of quadratures.

Reduction 3.2.2. α and κ arbitrary. If $c_4 = 0$ in (3.17) and if $c_3 \neq 0$ in (3.16-3.17), then we may set $c_3 = 1$, $c_1 = c$ and $c_2 = 0$, without loss of generality, and obtain the reduction

$$u(x,t) = t^{-1}w(z), \qquad z = x - c\ln t - 2\kappa t, \tag{3.19}$$

where $w(z)$ satisfies

$$ww''' + \beta w'w'' + \alpha ww' + cw' + w = 0.$$

Reduction 3.2.3. $\alpha = 0$, κ arbitrary. If $c_3 \neq 0$ and $c_4 \neq 0$ in (3.17), we may set $c_4 = m + \frac{1}{3}$, $c_3 = 1$ and $c_1 = c_2 = 0$, without loss of generality, and obtain the reduction

$$u(x,t) = w(z)t^{3m}, \qquad z = (x - 2\kappa t)t^{-m-1/3},$$

where $w(z)$ satisfies

$$ww''' + \beta w'w'' + (m + \tfrac{1}{3})zw' - 3mw = 0.$$

Reduction 3.2.4. $\alpha = 0$, κ arbitrary. If $c_3 = 0$ and $c_4 \neq 0$ in (3.17), we may set $c_4 = m$, $c_1 = 2\kappa$ and $c_2 = 1$, without loss of generality, and obtain the reduction

$$u(x,t) = w(z)\mathrm{e}^{3mt}, \qquad z = (x - 2\kappa t)\mathrm{e}^{-mt},$$

where $w(z)$ satisfies

$$ww''' + \beta w'w'' + mzw' - 3mw = 0.$$

Case 3.2.2 $\epsilon = 1$. In this case we obtain the following system of eleven determining equations:

$$\tau_u = 0, \qquad (3.20\text{i})$$

$$\tau_x = 0, \qquad (3.20\text{ii})$$

$$\xi_u = 0, \qquad (3.20\text{iii})$$

$$\phi_{uu} = 0, \qquad (3.20\text{iv})$$

$$2\phi_{xu} - \xi_{xx} = 0, \qquad (3.20\text{v})$$

$$\beta(u\phi_u - \phi + \xi_t) = 0, \qquad (3.20\text{vi})$$

$$\phi + u\tau_t - u\xi_x - \xi_t = 0, \qquad (3.20\text{vii})$$

$$3u\phi_{xu} + \phi_{tu} + \beta\phi_x - 3u\xi_{xx} - 2\xi_{xt} = 0, \qquad (3.20\text{viii})$$

$$u\phi_{xxu} + \phi + u\tau_t - 3\xi_x u - \xi_t = 0, \qquad (3.20\text{ix})$$

$$u\phi_{xxx} + \phi_{xxt} - \phi_t + (\alpha u - 2\kappa)\phi_x = 0, \qquad (3.20\text{x})$$

$$\begin{aligned} &3u^2\phi_{xxu} + 2u\phi_{xtu} + \beta u\phi_{xx} \\ &\quad + 2\kappa\phi - u^2\xi_{xxx} - u\xi_{xxt} + (2\alpha u^2 - 4\kappa)\xi_x + [(\alpha+1)u - 2\kappa]\xi_t = 0, \end{aligned} \qquad (3.20\text{xi})$$

As in the previous case, we apply the **reduceall** algorithm in the MAPLE package `diffgrob2`, to this system, which yields

$$\begin{aligned} &\xi_x = 0, \qquad (\alpha+1)\xi_{tt} = 0, \qquad \xi_u = 0, \\ &\tau_x = 0, \qquad 2\kappa\tau_t - (\alpha+1)\xi_t = 0, \qquad \tau_u = 0, \\ &2\kappa\phi = [2\kappa - (\alpha+1)u]\xi_t. \end{aligned}$$

This shows that there are two special values of the parameters, namely $\alpha = -1$ and $\kappa = 0$. For the three special cases (a) $\alpha = -1$, $\kappa \neq 0$, (b) $\alpha \neq -1$, $\kappa = 0$ and (c) $\alpha = -1$, $\kappa = 0$, applying the **reduceall** algorithm of `diffgrob2` to (3.20) yields

$$\text{(a)}\quad \alpha = -1,\ \kappa \neq 0 \qquad \begin{array}{lll} \xi_x = 0, & \xi_{tt} = 0, & \xi_u = 0, \\ \tau_x = 0, & \tau_t = 0, & \tau_u = 0, \\ \phi = \xi_t. & & \end{array}$$

$$\text{(b)}\quad \alpha \neq -1,\ \kappa = 0 \qquad \begin{array}{lll} \xi_x = 0, & \xi_t = 0, & \xi_u = 0, \\ \tau_x = 0, & \tau_{tt} = 0, & \tau_u = 0, \\ \phi = -u\tau_t. & & \end{array}$$

$$\text{(c)}\quad \alpha = -1,\ \kappa = 0 \qquad \begin{array}{lll} \xi_x = 0, & \xi_{tt} = 0, & \xi_u = 0, \\ \tau_x = 0, & \tau_{tt} = 0, & \tau_u = 0, \\ \phi - \xi_t + u\tau_t = 0. & & \end{array}$$

Hence we obtain the infinitesimals as recorded in table 3.2.2.

Table 3.2.2

Parameters	ξ	τ	ϕ	
$\kappa \neq 0$	$c_3 t + c_1$	$\dfrac{(1+\alpha)c_3 t}{2\kappa} + c_2$	$c_3\left[1 - \dfrac{(1+\alpha)u}{2\kappa}\right]$	(3.21)
$\alpha \neq -1, \kappa = 0$	c_1	$c_4 t + c_2$	$-c_4 u$	(3.22)
$\alpha = -1, \kappa = 0$	$c_3 t + c_1$	$c_4 t + c_2$	$c_3 - c_4 u$	(3.23)

where c_1, c_2, c_3, c_4 are arbitrary constants.

There are four canonical reductions.

Reduction 3.2.5. α and κ arbitrary. If in (3.21–3.23) $c_3 = c_4 = 0$, we may set $c_1 = c$ and $c_2 = 1$ without loss of generality. Thus we obtain the reduction

$$u(x,t) = w(z), \qquad z = x - ct,$$

where $w(z)$ satisfies

$$ww''' - cw''' + \beta w'w'' + \alpha ww' = (2\kappa - c)w'.$$

This can be integrated with the help of the transformation $w \to W + c$ to yield

$$WW'' + \tfrac{1}{2}(\beta - 1)(W')^2 + \tfrac{1}{2}\alpha W^2 = [2\kappa - (1+\alpha)c]W + A,$$

where A is an arbitrary constant. Then multiplying through by $W^{\beta-2}W'$ and integrating again yields

$$(W')^2 + \frac{\alpha W^2}{\beta+1} = \frac{2[2\kappa - (1+\alpha)c]W}{\beta} + \frac{2A}{\beta - 1} + BW^{1-\beta}, \tag{3.24}$$

provided that $\beta \neq -1, 0, -1$. Generally if $\beta \neq -1, 0, 1$, then (3.24) is solvable using quadratures, though for certain special values of the parameters, there are explicit solutions. For example (i), if $\beta = -2$ or $\beta = -3$, then (3.24) is solvable in terms of Weierstrass or Jacobi elliptic functions, respectively, (ii) if $B = 0$, then (3.24) is solvable in term of trigonometric functions, and (iii) $(1+\alpha)c = 2\kappa$ and $\beta = 3$, then $W(z)$ can be expressed in terms of trigonometric functions via the transformation $W(z) = v^{1/2}$.

In the special cases $\beta = -1, 0, 1$ we obtain the following equations,

$$(W')^2 + \alpha W^2 \ln W = -2[2\kappa - (1+\alpha)c]W - A + BW^2,$$
$$(W')^2 + \alpha W^2 = 2[2\kappa - (1+\alpha)c]W \ln W - 2A + BW,$$
$$2(W')^2 + \alpha W^2 = 4[2\kappa - (1+\alpha)c]W + 4A \ln W + B,$$

respectively, where A and B are arbitrary constants. If the coefficient of $\ln W$ in these equations is zero, then $W(z)$ is expressible in terms of elementary functions, otherwise in terms of quadratures.

We note that equation (3.24) may be written in the form

$$\begin{aligned} -(w-\epsilon c)^{\beta-1}(w')^2 &= -\frac{C}{\beta-1}(w-\epsilon c)^{\beta-1} - D \qquad (3.25)\\ &+\frac{\alpha}{\beta-1}w^2\left[(w-\epsilon c)^{\beta-1} + \frac{2}{\alpha}[c(\beta+1+\epsilon\alpha) - 2\kappa(\beta+1)]\sum_{n=0}^{\beta-2}\frac{(-\epsilon c)^{\beta-n-2}}{n+2}\binom{\beta-2}{n}w^n\right], \end{aligned}$$

where

$$D = B + \frac{2(-\epsilon c)^{\beta}[c(\beta+1+\epsilon\alpha) - 2\kappa(\beta+1)]}{\beta(\beta-1)(\beta+1)},$$

for $\beta \geq 2$ an integer, and ϵ has been reintroduced into the equation. The constant B is as found in (3.24) and A in (3.24) is related to C by $C = 2A + \epsilon c(\epsilon c\alpha - 4\kappa + 2c)$. Requiring that w and its derivatives tend to zero as $z \to \pm\infty$ forces us to set $D = C = 0$. In the work of Camassa and Holm [1993] the condition that $c(\beta+1+\epsilon\alpha) - 2\kappa(\beta+1) \to 0$ is equivalent to their $\kappa \to 0$ which induces the peakon solution, and certainly when $c(\beta+1+\epsilon\alpha) - 2\kappa(\beta+1) = 0$ equation (3.25) becomes

$$(w-\epsilon c)^{\beta-1}\left((w')^2 + \frac{\alpha}{\beta-1}w^2\right) = 0.$$

We obtain the peakon solution of §3.1 as the composition of two exponential solutions with a discontinuity at the peak. At this discontinuity we assume the peak has amplitude $w = \epsilon c$. This argument is consistent with the peaked solutions (3.3) and (3.6) (see Gilson and Pickering [1995]).

Reduction 3.2.6. $\alpha \neq -1$, κ arbitrary. If $c_3 \neq 0$ in (3.21), we may set $c_3 = 1$, $c_2 = 0$ and $c_1 = 2\kappa c/(1+\alpha)$, without loss of generality. Thus we obtain the reduction

$$u(x,t) = \frac{w(z)+c}{t} + \frac{2\kappa}{1+\alpha}, \qquad z = x - \frac{2\kappa t}{1+\alpha} - c\ln t, \qquad (3.26)$$

where $w(z)$ satisfies

$$ww''' + \beta w'w'' - w'' + \alpha ww' + (\alpha+1)cw' + w + c = 0. \qquad (3.27)$$

If $c_4 \neq 0$ in (3.22) we may set $c_4 = 1$, $c_1 = c$ and $c_2 = 0$ to obtain the reduction (3.26) with $\kappa = 0$.

Reduction 3.2.7. $\alpha = -1$, κ arbitrary. If $c_4 = 0$ in (3.23) and if $c_3 \neq 0$ in (3.21) and (3.23), then we set $c_3 = m$, $c_1 = 0$ and $c_2 = 1$, without loss of generality. Thus we obtain the reduction

$$u(x,t) = w(z) + mt, \qquad z = x - \tfrac{1}{2}mt^2, \qquad (3.28)$$

where $w(z)$ satisfies

$$ww''' + \beta w'w'' - ww' - 2\kappa w' - m = 0,$$

which may be integrated to yield

$$ww'' + \tfrac{1}{2}(\beta-1)(w')^2 - \tfrac{1}{2}w^2 - 2\kappa w - mz = A, \qquad (3.29)$$

where A is an arbitrary constant.

Reduction 3.2.8. $\alpha = -1$, $\kappa = 0$. If $c_4 \neq 0$ in (3.23) we may set $c_3 = m$, $c_4 = 1$, $c_1 = c$ and $c_2 = 0$, without loss of generality. Thus we obtain the reduction

$$u(x,t) = \frac{w(z)+c}{t} + m, \qquad z = x - mt - c\ln t, \qquad (3.30)$$

where $w(z)$ satisfies

$$ww''' + \beta w'w'' - w'' - ww' + w + c = 0. \qquad (3.31)$$

3.3 Nonclassical symmetries ($\tau \neq 0$)

To apply the nonclassical method we advocate the algorithm described in Clarkson and Mansfield [1994c] for calculating the determining equations, which avoids difficulties arising from using differential consequences of the invariant surface condition.

In the canonical case when $\tau \neq 0$ we set $\tau = 1$ without loss of generality. We proceed by eliminating u_t and u_{xxt} in (3.1) using the invariant surface condition which yields

$$\begin{aligned}\epsilon\xi u_{xxx} - uu_{xxx} + 3\epsilon\xi_u u_x u_{xx} - \beta u_x u_{xx} - \epsilon\phi_u u_{xx} + 2\epsilon\xi_x u_{xx} + \epsilon\xi_{uu}u_x^3 - \epsilon\phi_{uu}u_x^2 \\ + 2\epsilon\xi_{xu}u_x^2 - \alpha uu_x - 2\epsilon\phi_{xu}u_x + 2\kappa u_x + \epsilon\xi_{xx}u_x - \epsilon\phi_{xx} + \phi - \xi u_x = 0. \qquad (3.32)\end{aligned}$$

We note that this equation now involves the infinitesimals ξ and ϕ that are to be determined. Then we apply the classical Lie algorithm to (3.32) using the third prolongation $\mathrm{pr}^{(3)}\mathbf{v}$ and eliminating u_{xxx} using (3.32). It should be noted that the coefficient of u_{xxx} is $(\epsilon\xi - u)$. Therefore, if this is zero the removal of u_{xxx} using (3.32) is invalid and so the next highest derivative term, u_{xx}, should be used instead. We note again that this has a coefficient that may be zero, so that in the subcase $\epsilon\xi = u$ one again must calculate the determining equations in each scenario separately. Continuing in this fashion, there is a cascade of subcases to be considered. In the remainder of this section, we consider these subcases in turn. First, however, we discuss the case given by $\epsilon = 0$.

Case 3.3.1 $\epsilon = 0$. The first determining equation gives $\xi_u = 0$, and substituting this into the other seven determining equations yields

$$\phi_{uuu}u + \beta\phi_{uu} = 0, \tag{3.33ii}$$
$$3\phi_{xuu}u + 2\beta\phi_{xu} - \beta\xi_{xx} = 0, \tag{3.33iii}$$
$$3\phi_{uu}u^2 + \beta\phi_u u - \beta\phi = 0, \tag{3.33iv}$$
$$3\phi_{xu}u - 3\xi_{xx}u + \beta\phi_x = 0, \tag{3.33v}$$
$$\phi_t u - \phi_{xxx}u^2 - \alpha\phi_x u^2 + 2\kappa\phi_x u + 3\xi_x\phi u - \phi^2 = 0, \tag{3.33vi}$$
$$3\phi_{xxu}u^2 - \xi_{xxx}u^2 + 2\alpha\xi_x u^2 + \beta\phi_{xx}u - 4\xi_x\kappa u + 3\xi\xi_x u + \xi_t u + 2\kappa\phi - \xi\phi = 0. \tag{3.33vii}$$

It is quite straightforward to solve these equations and so we obtain the following infinitesimals: (a), if $\alpha \neq 0$

$$\text{(i)} \qquad \xi = 2\kappa + \frac{c_1}{t + c_2}, \qquad \phi = \frac{-u}{t + c_2},$$
$$\text{(ii)} \qquad \xi = c_1, \qquad \phi = 0,$$

and (b), if $\alpha = 0$

$$\text{(i)} \qquad \xi = \frac{(c_1 + 1)x + 2\kappa(2c_1 - 1)t + c_2}{3(c_1 t + c_3)}, \qquad \phi = \frac{u}{c_1 t + c_3},$$
$$\text{(ii)} \qquad \xi = \frac{x + 4\kappa t + c_1}{3t + c_2}, \qquad \phi = 0,$$

These are all equivalent to classical infinitesimals. Hence in this case there are no new nonclassical symmetries.

Case 3.3.2 $\epsilon = 1$. As discussed in the preamble to this section, we must consider, in addition to the general case, each of the singular cases of the determining equations.

Subcase 3.3.2(i) $\xi \neq u$. We can remove factors of $(\xi - u)$ from the determining equations, and we find that the second determining equation reads $\xi_u = 0$. Reducing the remaining eight determining equations with respect to this, only the last six are non-zero:

$$3\phi_{uu}u^2 - 6\xi\phi_{uu}u + \beta\phi_u u + 3\xi^2\phi_{uu} - \beta\xi\phi_u - \beta\phi + \beta\xi\xi_x + \beta\xi_t = 0, \tag{3.34iv}$$

$$\phi_{uuu}u - \xi\phi_{uuu} + \beta\phi_{uu} = 0, \tag{3.34v}$$

$$\begin{aligned}&\xi_x\phi_u u - \beta\xi\phi_x + \phi\phi_{uu}u + \beta\phi_x u - \xi\phi\phi_{uu} - 5\xi\phi_{xu}u + 4\xi\xi_{xx}u + \phi_{tu}u\\ &\quad - \phi\phi_u + \xi_t\phi_u - \xi\phi_{tu} - \xi^2\xi_{xx} + 3\phi_{xu}u^2 - 3\xi_{xx}u^2 - 2\xi_x^2 u - 2\xi_{xt}u\\ &\quad + 2\xi^2\phi_{xu} + 2\xi_x\phi - 2\xi_t\xi_x + 2\xi\xi_{xt} = 0,\end{aligned} \tag{3.34vi}$$

$$\begin{aligned}&2\xi\kappa\phi_x - \phi_t u + \alpha\phi_x u^2 - 2\kappa\phi_x u - \alpha\xi\phi_x u + 2\phi_{xu}\phi_x u\\ &\quad + \phi\phi_{xxu}u - 2\xi\phi_{xu}\phi_x - \xi_{xx}\phi_x u - \xi\phi\phi_{xxu} - 3\xi_x\phi u + 2\xi\xi_x\phi + \xi\xi_{xx}\phi_x + \xi_x\phi_{xx}u\\ &\quad - \xi\phi_{xxx}u - \xi_t\phi + \phi^2 - \phi\phi_{xx} + \phi_{xxx}u^2 + \phi_{xxt}u - \xi\phi_{xxt} + \xi_t\phi_{xx} + \xi\phi_t = 0,\end{aligned} \tag{3.34vii}$$

$$\begin{aligned}&2\beta\phi_{xu}u - \xi_x\phi_{uu}u - \beta\xi_{xx}u + 2\phi_u\phi_{uu}u + \beta\xi\xi_{xx} - 5\xi\phi_{xuu}u - \xi\phi\phi_{uuu} + \phi\phi_{uuu}u\\ &\quad + \phi_{tuu}u - \phi\phi_{uu} + \xi_t\phi_{uu} - \xi\phi_{tuu} + 3\phi_{xuu}u^2 + 2\xi^2\phi_{xuu} - 2\xi\phi_u\phi_{uu}\\ &\quad + 2\xi\xi_x\phi_{uu} - 2\beta\xi\phi_{xu} = 0,\end{aligned} \tag{3.34viii}$$

$$\begin{aligned}&4\xi_x\kappa u - 2\phi_{uu}\phi_x u - \beta\phi_{xx}u - \xi\xi_{xx}\phi_u - 2\kappa\phi - 2\phi\phi_{xuu}u - 2\phi_{xtu}u + \xi\phi + \xi_{xxt}u\\ &\quad + \xi_{xxx}u^2 + \xi_t\xi_{xx} - 3\phi_{xxu}u^2 + 2\phi\phi_{xu} - \xi\xi_{xxt} + 2\xi^2\xi_x - \xi^2\phi_{xxu} - 2\xi_t\phi_{xu} - \alpha\xi_t u\\ &\quad + \alpha\xi\phi + \xi_{xx}\phi_u u - 2\xi\xi_x\kappa - \xi\xi_{xxx}u - 3\xi\xi_x u + 2\xi\phi_{uu}\phi_x + \xi\xi_x\xi_{xx}\\ &\quad + 4\xi\phi_{xxu}u + \beta\xi\phi_{xx} + 2\xi\phi\phi_{xuu} + 2\xi\phi_u\phi_{xu} - 2\xi\xi_x\phi_{xu} - \alpha\xi_x u^2\\ &\quad + \alpha\xi\xi_x u + 2\xi\phi_{xtu} + 2\xi_t\kappa - \xi_t u - \xi_{xx}\phi - 2\phi_u\phi_{xu}u = 0.\end{aligned} \tag{3.34ix}$$

Reducing (3.34v) with respect to (3.34iv) yields

$$(\beta - 3)\left[(u-\xi)\phi_u - \phi + \xi\xi_x + \xi_t\right] = 0.$$

If $\beta = 3$, then one finds via another route that the expression in the second bracket is necessarily zero. The equation for ϕ can be solved to give

$$\phi = F(x,t)(u-\xi) + \xi\xi_x + \xi_t.$$

When this is substituted into the remaining equations we can then take coefficients of powers of u to be zero, and our problem is then solved. The complete solution set is

(a), if $\alpha \neq -1$

$$\text{(i)} \qquad \xi = c_1, \qquad \phi = 0, \tag{3.35}$$

$$\text{(ii)} \qquad \xi = \frac{2\kappa}{(1+\alpha)} - \frac{c_1}{t+c_2}, \qquad \phi = \frac{2\kappa - (1+\alpha)u}{(1+\alpha)(t+c_2)}, \tag{3.36}$$

(b), if $\alpha = -1$

$$\xi = c_1 t + c_2, \qquad \phi = c_1, \tag{3.37}$$

(c), if $\alpha = -1$ and $\kappa = 0$

$$\xi = c_1 - \frac{c_3}{t + c_2}, \qquad \phi = \frac{c_1 - u}{t + c_2}, \tag{3.38}$$

and (d), if $\beta = -1$ and $\alpha = 0$,

$$\xi = c_1 x - 2c_1 \kappa t + c_2, \qquad \phi = 3c_1 u - 2c_1^2 x + 4c_1^2 \kappa t - 2c_1 c_2 - 2c_1 \kappa, \tag{3.39}$$

The infinitesimals (3.35)–(3.38) give rise to classical reductions, but (3.39) gives the following new nonclassical reduction.

Reduction 3.3.1. If in (3.39), we require $c_1 \neq 0$ we may set $c_2 = 0$, without loss of generality, then we obtain

$$u(x,t) = w(z) \exp(3c_1 t) + c_1 z \exp(c_1 t) + 2\kappa, \qquad z = (x - 2\kappa t - 2\kappa/c_1) \exp(-c_1 t),$$

where $w(z)$ satisfies

$$ww''' - w'w'' + c_1 z w' - 3c_1 w = 0.$$

Subcase 3.3.2(ii) $\xi = u$, not both $\beta = 3$ and $\phi_u = 0$. We generate five determining equations, the first of which is $(\beta - 3)(\beta - 1)\phi_{uu} = 0$. If $\beta = 3$ we find, in contradiction to our assumptions, that $\phi_u = 0$. If $\phi_{uu} = 0$, we write ϕ as a linear function of u, substitute this into the remaining four determining equations and take coefficients of powers of u to be zero. The only solution to these equations is when $\phi = 0$, provided that $\kappa = 0$ and $\alpha = -1$; β remains arbitrary (though $\beta \neq 3$). The invariant surface condition and (3.1) are then solved to give the simple exact solution

$$u(x,t) = \frac{x + c_1}{t + c_2}, \tag{3.40}$$

where c_1 and c_2 are arbitrary constants.

The only remaining case to be considered is when $\beta = 1$, and we may assume that $\phi_{uu} \neq 0$, since taking $\phi_{uu} = 0$ yields the exact solution (3.40) above. In this instance the remaining four determining equations are

$$12\kappa - 2\phi_{xuu}u - 6\alpha u - 6u - 2\phi\phi_{uuu} - 3\phi_u\phi_{uu} - 4\phi_{xu} - 2\phi_{tuu} = 0,$$

$$\begin{aligned}&\phi_{xu}\phi_{xx}u - \phi_u\phi_{xxx}u - \alpha\phi_u\phi_x u - \phi\phi_{xu}u - 2\phi_x\phi_{xx} + \phi\phi_{uu}\phi_{xx} + \phi_{tu}\phi_{xx} - 2\phi_u\phi_{xu}\phi_x \\ &\quad + 2\kappa\phi_u\phi_x + 2\phi\phi_x - \phi^2\phi_{uu} - \phi\phi_u\phi_{xxu} - \phi_{xxt}\phi_u + \phi_t\phi_u - \phi\phi_{tu} = 0,\end{aligned}$$

$$\begin{aligned}&\phi_u\phi_{xuu}u + 4\alpha\phi_u u - \phi\phi_{uu}^2 - \phi_{tu}\phi_{uu} + \phi\phi_u\phi_{uuu} + 6\phi - 4\phi_{xx} + \phi_{tuu}\phi_u - 4\phi_{xxu}u \\ &\quad + 4\phi_u u - 2\phi_{uu}\phi_x - 4\phi\phi_{xuu} + 2\phi_u^2\phi_{uu} - 8\kappa\phi_u - 2\alpha\phi - \phi_{xu}\phi_{uu}u - 4\phi_{xtu} = 0,\end{aligned}$$

$$\begin{aligned}&\phi\phi_{uu}u + \alpha\phi_{xu}u^2 - 2\phi_u\phi_{uu}\phi_x - 2\phi\phi_u\phi_{xuu} - 2\phi_u\phi_{xxu}u + \alpha\phi_{tu}u + 2\phi\phi_{xu}\phi_{uu} \\ &\quad - 2\kappa\phi\phi_{uu} + 2\phi_{tu}\phi_{xu} + \alpha\phi\phi_{uu}u + 2\phi_{xxx}u - 2\phi_x u + 2\phi_{xu}^2 u - 3\phi_u\phi_{xx} \\ &\quad + 2\phi\phi_{xxu} - 2\phi_u^2\phi_{xu} - 2\phi_{xtu}\phi_u + 4\phi\phi_u - 2\kappa\phi_{tu} - \alpha\phi\phi_u - 2\kappa\phi_{xu}u \\ &\quad + \phi_{tu}u + \phi_{xu}u^2 + 2\phi_{xxt} - 2\phi_t = 0.\end{aligned}$$

Using the procedures in the package `diffgrob2` with an ordering designed to eliminate first derivatives with respect to t, then derivatives with respect to x, one can obtain several equations for derivatives of ϕ with respect to u only. One can then continue to produce lower order and lower degree equations in the u-derivatives of ϕ, using repeated cross-differentiation and reductions. For example, the Direct Search strategy described in §1.7.5 may be used. This process suffers from expression swell. No termination of this process was observed by us within the computer memory available, and the expressions obtained contained thousands of summands! One of three results appear likely. Firstly, the process terminates with the highest derivative term being ϕ itself, yielding ϕ to be a function of u alone (note that x and t do not appear explicitly in any of the determining equations). Inserting this into the determining equations, one must have that ϕ is constant, a contradiction to our standing assumption in this subcase. Secondly, the process may terminate with an inconsistency, and thirdly, the process may terminate but with such a large expression that the result is useless.

Subcase 3.3.2(iii) $\xi = u$, $\beta = 3$, $\phi_u = 0$ and not both κ and $\alpha+1$ are zero. One determining equation was obtained which was a polynomial in u of degree two whose coefficients are functions of x, t only, so the coefficients of powers of u must be zero. These equations were easily simplified using the procedures in `diffgrob2` to yield,

$$\kappa \neq 0,\ \alpha = -1, \qquad \phi = 0, \tag{3.41}$$

$$\kappa \neq 0,\ \alpha = -1, \qquad \phi = \frac{-2\kappa}{t + c_1}, \tag{3.42}$$

$$\kappa \text{ arbitrary},\ \alpha \neq -1, \qquad \phi = c_1 \exp(\zeta) + c_2 \exp(-\zeta), \qquad \zeta = \mathrm{i}\sqrt{\alpha}\left(x - \frac{2\kappa t}{1+\alpha}\right). \tag{3.43}$$

In (3.41) if we solve (3.1) and the invariant surface condition as a system of equations we find that the only solution is $u(x,t) = c$, a constant.

In (3.42) we can solve (3.1) and the invariant surface condition to give the exact (canonical) solution

$$u(x,t) = -2\kappa + x/t,$$

which cannot be realised by any of the previously found reductions, though it would not appear to be a particularly interesting solution. It is interesting to note that performing the `KolRitt` algorithm of `diffgrob2` on the system comprising the original equation with the invariant surface condition led to a simple calculation for u. By contrast, the usual procedure of solving the invariant surface condition using the method of characteristics and inserting the result into the original equation to obtain the reduction was considerably more difficult due to the implicit nature of the reduction.

In (3.43) we can again solve our problem to yield the exact (canonical) solution

$$u(x,t) = \frac{-2\kappa}{1+\alpha} \pm \left(c_0 + c_1 \mathrm{e}^{\zeta} + c_2 \mathrm{e}^{-\zeta}\right)^{1/2}, \qquad \zeta = \mathrm{i}\sqrt{\alpha}\left(x - \frac{2\kappa t}{1+\alpha}\right),$$

which is a special case of the travelling wave reduction 3.2.5.

Subcase 3.3.2(iv) $\xi = u$, $\beta = 3$, $\phi_u = 0$, $\kappa = 0$, $\alpha = -1$. We are left simply with the determining equation $\phi_{xx} - \phi = 0$, which produces the following infinitesimal,

$$\phi = g(t)\mathrm{e}^{x} + h(t)\mathrm{e}^{-x}, \tag{3.44}$$

where g and h are arbitrary functions. Hence we have to solve the invariant surface condition

$$uu_x + u_t = g(t)\mathrm{e}^{x} + h(t)\mathrm{e}^{-x}. \tag{3.45}$$

It is straightforward to show that every solution of this equation is also a solution of (3.1).

3.4 Nonclassical symmetries ($\tau = 0$)

In the canonical case of the nonclassical method when $\tau = 0$ we set $\xi = 1$ without loss of generality. We proceed by eliminating u_x, u_{xx}, u_{xxx} and u_{xxt} in (3.1) using the invariant surface condition which yields

$$\begin{aligned} u_t - \epsilon\phi\phi_{uu}u_t - \epsilon\phi_{xu}u_t - \epsilon\phi_u^2 u_t - \phi_{xx}u - \phi_u\phi_x u - \phi^2\phi_{uu}u - 2\phi\phi_{xu}u - \phi\phi_u^2 u \\ - \alpha\phi u - \beta\phi\phi_x - \epsilon\phi_t\phi_u - \beta\phi^2\phi_u - \epsilon\phi_{xt} - \epsilon\phi\phi_{tu} + 2\kappa\phi = 0, \end{aligned} \tag{3.46}$$

which involves the infinitesimal ϕ that is to be determined. As in the $\tau \neq 0$ case we apply the classical Lie algorithm to this equation using the first prolongation $\mathrm{pr}^{(1)}\mathbf{v}$ and eliminate u_t using (3.46). Also similar to the $\tau \neq 0$ case is the possible existence of singular solutions if $\epsilon \neq 0$. In fact if $\epsilon = 1$, there is a singular solution if and only if

$$\phi\phi_u + \phi_x - u - 2\kappa/\beta.$$

The nonclassical method generates a single equation of 138 terms when $\epsilon = 1$, which reduces to 25 terms when $\epsilon = 0$. Since it is not possible to solve these equations explicitly (they are more complex than equation (3.1)!) we seek polynomial solutions in u. Whilst we tackle the cases $\epsilon = 0$ and $\epsilon = 1$ separately in practice, it is convenient to express our results for a general ϵ.

Ansatz 1. $\phi = F(x,t)$. We obtain the following exact solutions for (3.1);

(a) If $\alpha = 0$,

$$u(x,t) = H_1(t)x^3 + H_2(t)x^2 + H_3(t)x + w(t),$$

where the $H_i(t)$ are determined by the determining equations and $w(t)$ by substitution into (3.1); they satisfy

$$\begin{aligned} &H_1' - 6(1+3\beta)H_1^2 = 0, \\ &H_2' - 6(1+3\beta)H_1H_2 = -6\kappa H_1, \\ &H_3' - 6(1+\beta)H_1H_3 = 4\beta H_2^2 - 4\kappa H_2 + 6\epsilon H_1', \\ &w' - 6H_1w = 2\beta H_2H_3 - 2\kappa H_3 + 2\epsilon H_2', \end{aligned}$$

where primes denote differentiation with respect to t. This system of equations is solvable, though has many special cases to be considered, so this is not pursued here.

(b) If $\beta = 0$, $\alpha \neq 0$ and $\alpha\epsilon + 1 \neq 0$, we have the (canonical) exact solution

$$\begin{aligned} u(x,t) &= -\frac{x - 2\kappa t}{\alpha t} \\ &+ t^{-1/(\alpha\epsilon+1)} \left\{ c_1 \exp\left[\sqrt{-\alpha}\left(x - \frac{2\kappa t}{\alpha\epsilon + 1}\right)\right] + c_2 \exp\left[-\sqrt{-\alpha}\left(x - \frac{2\kappa t}{\alpha\epsilon + 1}\right)\right]\right\}. \end{aligned}$$

If $c_1 = c_2 = 0$ then we may drop the restriction on β since the solution then holds for β arbitrary.

(c) If $\beta = \alpha\epsilon + 1 = \kappa = 0$, then we have

$$u(x,t) = H_1(t)\mathrm{e}^{Rx} + H_2(t)\mathrm{e}^{-Rx} + c_1,$$

where $\epsilon R^2 = 1$, $H_1(t)$ and $H_2(t)$ are arbitrary functions and c_1 is an arbitrary constant.

Ansatz 2. $\phi = F(x,t)u^2 + G(x,t)u + H(x,t)$. With this ansatz we find the following exact solutions of (3.1) for various parameter values.

(a) If $\beta = 1$, $\kappa = 0$, then

$$u(x,t) = c_1 \exp\left(-\frac{\alpha x^2}{4}\right).$$

(b) If $\beta = -3$,

$$u(x,t) = c_1 \tan\left\{\tfrac{1}{2}\sqrt{\alpha}\,(x - 2\kappa t)\right\} + \epsilon c_2,$$

where $2\kappa - (\alpha\epsilon + 1)c_2 = 0$.

(c) If $R^2(1+\beta) + \alpha = 0$, then providing $\epsilon R^2 \neq 1$ we have the exact solution

$$u(x,t) = c_1 \exp\left\{R\left[x - \frac{(2\kappa + c_3\beta R^2)t}{1 - \epsilon R^2}\right]\right\} + c_2 \exp\left\{-R\left[x - \frac{(2\kappa + c_3\beta R^2)t}{1 - \epsilon R^2}\right]\right\} + c_3,$$

for β arbitrary.

(d) If $\beta = -\frac{1}{2}$, $2 + \alpha\epsilon \neq 0$ and $1 + 2\alpha\epsilon \neq 0$, then

$$\begin{aligned} u(x,t) = c_1 \exp&\left\{\tfrac{1}{2}R\left[x - \frac{(4\kappa - \alpha c_3)t}{2 + \alpha\epsilon}\right]\right\} + c_2 \exp\left\{R\left[x - \frac{(2\kappa + \alpha c_3)t}{1 + 2\alpha\epsilon}\right]\right\} \\ &+ \frac{c_1^2(2 + \alpha\epsilon)}{8(c_3 + \alpha\epsilon c_3 - 2\kappa\epsilon)} \exp\left\{R\left[x - \frac{(4\kappa - \alpha c_3)t}{2 + \alpha\epsilon}\right]\right\} + c_3, \end{aligned}$$

where $R^2 = -2\alpha$ and we require $c_3 + \alpha\epsilon c_3 - 2\kappa\epsilon \neq 0$.

(e) If $\beta = -\frac{1}{2}$ and $2 + \alpha\epsilon = 0$ then we have

$$u(x,t) = 2\epsilon H_1(t) \exp\left[\pm R(x + 2\kappa t)\right] + \left(c_1 \pm 2\sqrt{\epsilon}\int^t H_1^2(s)\,\mathrm{d}s\right) \exp\left[\pm 2R(x + 2\kappa t)\right] - 2\kappa\epsilon,$$

where $\epsilon R^2 = 1$ and $H_1(t)$ is an arbitrary function.

3.5 Discussion

In this chapter we have classified symmetry reductions of the nonlinear third order partial differential equation (3.1), which contains three special cases that have attracted considerable interest recently, using the classical Lie method and the nonclassical method due to Bluman and Cole [1969]. The use of the MAPLE package `diffgrob2` was crucial in this classification procedure. In the classical case it identified the special cases of the parameters for which additional symmetries might occur whilst in the nonclassical case, the use of `diffgrob2` rendered a daunting calculation tractable and thus solvable.

In their recent paper, Gilson and Pickering [1995] discuss the application of the Painlevé tests for integrability due to Ablowitz, Ramani and Segur [1978,1980] and Weiss, Tabor and Carnevale [1983] to equation (3.1). In particular, they investigate the integrability of the ordinary differential equations arising from the travelling-wave reductions 3.2.1 and 3.2.5 above, and particular values of the parameters in special cases of reductions 3.2.2, 3.2.6–3.2.8. It would be interesting to investigate the integrability of some of the ordinary differential equations arising from the other reductions derived in this chapter using the various methods of Painlevé analysis available (see §1.6 for details), though we shall not pursue this further here. Marinakis and Bountis [1995] have also applied Painlevé analysis to the FFCH equation (3.5); an interesting aspect of their analysis is the use of a hodograph transformation. To conclude we remark that the RH equation (3.4) is a quasilinear partial differential equation of the form discussed by Clarkson, Fokas and Ablowitz [1989]. It is routine to apply their algorithm, which involves a hodograph transformation, for applying the Painlevé PDE test to such quasilinear partial differential equations and show that (3.4) does not satisfy the necessary conditions to be solvable by inverse scattering.

Chapter Four :
A class of Nonlinear Fourth Order Partial Differential Equations

4.1 Introduction

Following the work of the previous chapter, in this chapter we are concerned with symmetry reductions of the nonlinear fourth order partial differential equation given by

$$\Delta \equiv \quad u_{tt} - (\kappa u + \gamma u^2)_{xx} - u u_{xxxx} - \mu u_{xxtt} - \alpha u_x u_{xxx} - \beta u_{xx}^2 = 0, \tag{4.1}$$

where α, β, γ, κ and μ are arbitrary constants. Indeed, this equation may be thought of as an alternative to the class of third order equations we studied in the previous chapter

$$u_t - \epsilon u_{xxt} + 2\kappa u_x = u u_{xxx} + \alpha u u_x + \beta u_x u_{xx}. \tag{4.2}$$

This is analogous to the Boussinesq equation (Boussinesq [1871,1872])

$$u_{tt} = u_{xxxx} + \tfrac{1}{2}(u^2)_{xx}, \tag{4.3}$$

which is a soliton equation solvable by inverse scattering (see Ablowitz and Haberman [1975], Caudrey [1980,1982], Deift, Tomei and Trubowitz [1982], Zakharov [1974]) being an alternative to the Korteweg–de Vries (KdV) equation

$$u_t = u_{xxx} + 6 u u_x, \tag{4.4}$$

another soliton equation, the first to be solved by inverse scattering, by Gardner *et al.* [1967].

Two special cases of (4.1) have appeared recently in the literature both of which model the motion of a dense chain (Rosenau [1994]). The first is obtainable via the transformation $(u, x, t) \mapsto (2\varepsilon\alpha_3 u + \varepsilon\alpha_2, x, t)$ with the appropriate change of parameters, to yield

$$u_{tt} = (\alpha_2 u + \alpha_3 u^2)_{xx} + \varepsilon\alpha_2 u_{xxxx} + 2\varepsilon\alpha_3[u u_{xxxx} + 2u_{xx}^2 + 3u_x u_{xxx}], \tag{4.5}$$

with $\varepsilon > 0$. This equation can be thought of as the Boussinesq equation (4.3) appended with a nonlinear dispersion. It admits both conventional solitons and compact solitons which are often called compactons. Compactons are solitary waves with a compact support (Rosenau [1994], Rosenau and Hyman [1993]). The compact structures take the form

$$u(x,t) = \begin{cases} (3c^2 - 2\alpha_2)\cos^2\{(12\varepsilon)^{-1/2}(x-ct)\}/2\alpha_3, & \text{if } |x-ct| \le 2\pi, \\ 0, & \text{if } |x-ct| > 2\pi. \end{cases} \tag{4.6i}$$

or

$$u(x,t) = \begin{cases} A\cos\{(3\varepsilon)^{-1/2}[x - (\tfrac{2}{3}\alpha_3)^{1/2}t]\}, & \text{if } |x-ct| \le 2\pi, \\ 0, & \text{if } |x-ct| > 2\pi. \end{cases} \tag{4.6ii}$$

These are "weak" solutions as they do not possess the necessary smoothness at the edges, however this would appear not to affect the robustness of a compacton (Rosenau [1994]). Numerical experiments by Rosenau and Hyman [1993] seem to show that compactons interact elastically, reemerging with exactly the same coherent shape.

The second special case of equation (4.1) is obtained from the scaling transformation $(u,x,t) \mapsto (2\alpha_3 u/\varepsilon, \sqrt{\varepsilon}\, x, t)$, again with appropriate parameterisation,

$$u_{tt} = (\alpha_2 u + \alpha_3 u^2)_{xx} + \varepsilon u_{xxtt} + 2\varepsilon\alpha_3[u u_{xxxx} + 2u_{xx}^2 + 3u_x u_{xxx}], \tag{4.7}$$

with $\varepsilon > 0$. This equation, unlike (4.5) is well posed. It also admits conventional solitons and allows compactons like

$$u(x,t) = \begin{cases} (4c^2 - 3\alpha_2)\cos^2\{(12\varepsilon)^{-1/2}(x-ct)\}/2\alpha_3, & \text{if } |x-ct| \le 2\pi, \\ 0, & \text{if } |x-ct| > 2\pi, \end{cases} \tag{4.8i}$$

or

$$u(x,t) = \begin{cases} A\cos\{(3\varepsilon)^{-1/2}[x - (\tfrac{3}{2}\alpha_2)^{1/2}t]\}, & \text{if } |x-ct| \le 2\pi, \\ 0, & \text{if } |x-ct| > 2\pi. \end{cases} \tag{4.8ii}$$

These again are weak solutions, and are very similar to the previous solutions: both (4.6ii) and (4.8ii) are solutions with a variable speed linked to the amplitude of the wave, whereas both (4.6i) and (4.8i) are solutions with arbitrary amplitudes, whilst the wave speed is fixed by the parameters of the equation.

The classical method for finding symmetry reductions of partial differential equations is the Lie group method of infinitesimal transformations, which in practice is a three–step procedure (see §1.2 for details). Particular importance has been placed on the second step in this study. It involves heuristic integration procedures which have been implemented in some SM programs and are largely successful, though not infallible. Commonly, the overdetermined systems to be solved are simple, and heuristic integration is both fast and effective. However, there are occasions where heuristics can break down (cf. Mansfield and Clarkson [1996] for further details and examples). If the classical method is applied to a partial differential equation which contains arbitrary parameters, such as (4.1) or more generally, arbitrary functions, heuristics usually yield the general solution yet miss those special cases of the parameters and arbitrary functions where additional symmetries lie. In contrast the method of differential Gröbner bases, which is described in §1.7, has proved effective in coping with such difficulties (cf. Clarkson and Mansfield [1994a], and the two previous chapters).

We use the MAPLE package `diffgrob2` which has implemented in it the Kolchin-Ritt algorithm using pseudo-reduction instead of reduction, and extra algorithms needed to calculate a differential Gröbner basis (as far as possible using the current theory), for those cases where the Kolchin-Ritt algorithm is not sufficient (see Mansfield and Fackerell [1992]). The package was designed to be used interactively as well as algorithmically, and much use is made of this fact here. It has proved useful for solving many fully nonlinear systems (cf. Clarkson and Mansfield [1994a,b,c,1995]).

In the following sections we shall consider the cases $\mu = 0$ and $\mu \neq 0$, when we set $\mu = 1$ without loss of generality, separately because the presence or lack of the corresponding fourth order term is significant. In §4.2 we find the classical Lie group of symmetries and associated reductions of (4.1). In §4.3 we discuss the nonclassical symmetries and reductions of (4.1) in the generic case. In §4.4 we consider special cases of the the nonclassical method in the so-called $\tau = 0$ case; in full generality this case is somewhat intractable. In §4.5 we discuss our results.

4.2 Classical symmetries

To apply the classical method as described in §1.2 we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$x^* = x + \varepsilon\xi(x, t, u) + O(\varepsilon^2), \tag{4.9i}$$

$$t^* = t + \varepsilon\tau(x, t, u) + O(\varepsilon^2), \tag{4.9ii}$$

$$u^* = u + \varepsilon\phi(x, t, u) + O(\varepsilon^2), \tag{4.9iii}$$

where ε is the group parameter. This procedure yields an overdetermined system of linear determining equations. There are two cases to consider, when $\mu = 0$ and when $\mu \neq 0$.

Case 4.2.1 $\mu = 0$. In this case we generate 15 determining equations, using the MACSYMA package `symmgrp.max`.

$$\tau_u = 0, \qquad \tau_x = 0, \qquad \xi_u = 0, \qquad \phi_{uu} = 0, \qquad \xi_t = 0, \tag{4.10i}$$

$$\alpha(\phi_u u - \phi) = 0, \qquad \beta(\phi_u u - \phi) = 0, \qquad 2\phi_{tu} - \tau_{tt} = 0, \tag{4.10ii}$$

$$4\phi_{xu} u - 6\xi_{xx} u + \alpha\phi_x, \qquad 2\tau_t u - 4\xi_x u + \phi = 0, \tag{4.10iii}$$

$$4\beta\phi_{xu} + 3\alpha\phi_{xu} - 2\beta\xi_{xx} - 3\alpha\xi_{xx} = 0, \tag{4.10iv}$$

$$\phi_{tt} - \phi_{xxxx} u - 2\gamma\phi_{xx} u - \kappa\phi_{xx} = 0, \tag{4.10v}$$

$$3\alpha\phi_{xxu}u + 2\gamma\phi_u u + 4\xi_x\gamma u - \alpha\xi_{xxx}u - 2\gamma\phi = 0, \tag{4.10vi}$$

$$6\phi_{xxu}u^2 + 4\xi_x\gamma u^2 - 4\xi_{xxx}u^2 + 2\beta\phi_{xx}u + 2\xi_x\kappa u - \kappa\phi = 0, \tag{4.10vii}$$

$$4\phi_{xxxu}u + 4\gamma\phi_{xu}u - 2\xi_{xx}\gamma u - \xi_{xxxx}u + \alpha\phi_{xxx} + 4\gamma\phi_x + 2\kappa\phi_{xu} - \xi_{xx}\kappa = 0, \tag{4.10viii}$$

and then use **reduceall** in `diffgrob2` to simplify them to the following system

$$\xi_u = 0, \qquad \xi_t = 0, \qquad \gamma(14\beta + 9\alpha)\xi_x = 0, \qquad \tau_u = 0, \tag{4.11i}$$

$$\gamma\kappa(14\beta + 9\alpha)\tau_t = 0, \qquad \tau_x = 0, \qquad \gamma\kappa(14\beta + 9\alpha)\phi = 0. \tag{4.11ii}$$

Thus we have special cases when $\gamma = 0$, $\kappa = 0$ and/or $14\beta + 9\alpha = 0$. The latter condition provides nothing different unless we specialise further and consider the special case when $\alpha = -\frac{5}{2}$ and $\beta = \frac{45}{28}$. We continue to use **reduceall** in `diffgrob2` for the various combinations and it transpires that there are only four combinations which yield different infinitesimals. Where a parameter is not included it is presumed to be arbitrary.

(a) $\kappa = 0$,
$$\xi_u = 0, \quad \xi_t = 0, \quad \xi_x = 0, \quad \tau_u = 0,$$
$$\tau_{tt} = 0, \quad \tau_x = 0, \quad 2\tau_t u + \phi = 0.$$

(b) $\gamma = 0$,
$$\xi_u = 0, \quad \xi_t = 0, \quad \xi_{xx} = 0, \quad \tau_u = 0,$$
$$\xi_x - \tau_t = 0, \quad \tau_x = 0, \quad 2\xi_x u + \phi = 0.$$

(c) $\gamma = \kappa = 0$,
$$\xi_u = 0, \quad \xi_t = 0, \quad \xi_{xx} = 0, \quad \tau_u = 0,$$
$$\tau_{tt} = 0, \quad \tau_x = 0, \quad 2\tau_t u - 4\xi_x u - \phi = 0.$$

(d) $\alpha = -\frac{5}{2}$, $\beta = \frac{45}{28}$, $\gamma = \kappa = 0$,
$$\xi_u = 0, \quad \xi_t = 0, \quad \xi_{xxx} = 0, \quad \tau_u = 0,$$
$$\tau_{tt} = 0, \quad \tau_x = 0, \quad 2\tau_t u - 4\xi_x u - \phi = 0.$$

Hence we obtain the infinitesimals, as listed in table 4.2.1.

Table 4.2.1

Parameters	ξ	τ	ϕ	
	c_1	c_2	0	(4.12)
$\kappa = 0$	c_1	$c_3t + c_2$	$-2c_3u$	(4.13)
$\gamma = 0$	$c_3x + c_1$	$c_3t + c_2$	$2c_3u$	(4.14)
$\gamma = \kappa = 0$	$c_4x + c_1$	$c_3t + c_2$	$(4c_4 - 2c_3)u$	(4.15)
$\alpha = -\frac{5}{2}, \beta = \frac{45}{28}$, $\gamma = \kappa = 0$	$c_5x^2 + c_4x + c_1$	$c_3t + c_2$	$[4(2c_5x + c_4) - 2c_3]u$	(4.16)

where $c_1, ..., c_5$ are arbitrary constants.

Solving the invariant surface condition yields the following seven different canonical reductions:

Reduction 4.2.1. $\alpha, \beta, \gamma, \kappa$ arbitrary. If in (4.12–4.16) $c_3 = c_4 = c_5 = 0$, then we may set $c_2 = 1$ without loss of generality. Thus we obtain the reduction

$$u(x, t) = w(z), \qquad z = x - c_1 t,$$

where $w(z)$ satisfies

$$(\kappa - c_1^2)w'' + 2\gamma[ww'' + (w')^2] + ww'''' + \alpha w'w''' + \beta(w'')^2 = 0.$$

Reduction 4.2.2. α, β, γ arbitrary, $\kappa = 0$. If in (4.13),(4.15) and (4.16) $c_4 = c_5 = 0$, $c_3 \neq 0$, then we may set $c_2 = 0$, $c_3 = 1$ without loss of generality. Thus we obtain the reduction

$$u(x,t) = t^{-2}w(z), \qquad z = x - c_1 \ln t,$$

where $w(z)$ satisfies

$$ww'''' + \alpha w'w''' + \beta(w'')^2 + 2\gamma[ww'' + (w')^2] - c_1^2 w'' - 5c_1 w' - 6w = 0.$$

Reduction 4.2.3. α, β, κ arbitrary, $\gamma = 0$. If in (4.14) $c_3 \neq 0$, then we may set $c_1 = c_2 = 0$, $c_3 = 1$ without loss of generality. Thus we obtain the reduction

$$u(x,t) = t^2 w(z), \qquad z = x/t,$$

where $w(z)$ satisfies

$$ww'''' + \alpha w'w''' + \beta(w'')^2 + \kappa w'' - z^2 w'' + 2zw' - 2w = 0.$$

Reduction 4.2.4. α, β arbitrary, $\kappa = \gamma = 0$. If in (4.15) and (4.16) $c_3 = c_5 = 0$, $c_4 \neq 0$, then we may set $c_1 = 0$, $c_2 = 1$ without loss of generality. Thus we obtain the reduction

$$u(x,t) = w(z)\exp(4c_4 t), \qquad z = x\exp(-c_4 t),$$

where $w(z)$ satisfies

$$ww'''' + \alpha w'w''' + \beta(w'')^2 - c_4^2 z^2 w'' + 7c_4^2 zw' - 16c_4^2 w = 0.$$

Reduction 4.2.5. α, β arbitrary, $\kappa = \gamma = 0$. If in (4.15) and (4.16) $c_5 = 0$, $c_3 c_4 \neq 0$, then we may set $c_1 = c_2 = 0$, $c_3 = 1$ without loss of generality. Thus we obtain the reduction

$$u(x,t) = w(z)t^{4c_4 - 2}, \qquad z = xt^{-c_4},$$

where $w(z)$ satisfies

$$ww'''' + \alpha w'w''' + \beta(w'')^2 - c_4^2 z^2 w'' + (7c_4^2 - 5c_4)zw' - (16c_4^2 - 20c_4 + 6)w = 0.$$

Reduction 4.2.6. $\alpha = -\frac{5}{2}$, $\beta = \frac{45}{28}$, $\gamma = \kappa = 0$. If in (4.16) $c_3 = 0$, $c_5 \neq 0$, then we may set $c_1 = -mc$, $c_2 = 1$, $c_4 = 0$, $c_5 = m/c$, without loss of generality. Thus we obtain the reduction

$$u(x,t) = \frac{w(z)\exp(-8mt)}{[z - \exp(-2mt)]^8}, \qquad z = \left(\frac{x-c}{x+c}\right)\exp(-2mt),$$

where $w(z)$ satisfies

$$28ww'''' - 70w'w''' + 45(w'')^2 - c^4m^2(1792z^2w'' - 12544zw' + 28672w) = 0.$$

Reduction 4.2.7. $\alpha = -\frac{5}{2}$, $\beta = \frac{45}{28}$, $\gamma = \kappa = 0$. If in (4.16) $c_3c_5 \neq 0$, then we may set $c_1 = -mc$, $c_2 = c_4 = 0$, $c_3 = 1$, $c_5 = m/c$, without loss of generality. Thus we obtain the reduction

$$u(x,t) = \frac{w(z)t^{-2(1+4m)}}{(z - t^{-2m})^8}, \qquad z = \left(\frac{x-c}{x+c}\right)t^{-2m},$$

where $w(z)$ satisfies

$$28ww'''' - 70w'w''' + 45(w'')^2 - 1792c^4m^2z^2w''$$
$$+ (12544m^2 - 4480m)c^4zw' - (28672m^2 - 17920m + 2688)c^4w = 0.$$

Case 4.2.2 $\mu \neq 0$. In this case we set $\mu = 1$ without loss of generality and generate 18 determining equations,

$$\tau_u = 0, \quad \tau_x = 0, \quad \xi_u = 0, \quad \xi_t = 0, \quad \phi_{uu} = 0, \tag{4.17i}$$

$$\phi_{xtu} = 0, \quad \alpha(\phi_u u - \phi) = 0, \quad 2\phi_{tu} - \tau_{tt} = 0, \quad \beta(\phi_u u - \phi) = 0, \tag{4.17ii}$$

$$2\phi_{xu} - \xi_{xx} = 0, \quad 4\phi_{xu}u - 6\xi_{xx}u + \alpha\phi_x = 0, \tag{4.17iii}$$

$$2\tau_t u - 2\xi_x u + \phi = 0, \tag{4.17iv}$$

$$4\beta\phi_{xu} + 3\alpha\phi_{xu} - 2\beta\xi_{xx} - 3\alpha\xi_{xx} = 0, \tag{4.17v}$$

$$\phi_{xxu}u + 2\tau_t u - 4\xi_x u + \phi = 0, \tag{4.17vi}$$

$$3\alpha\phi_{xxu}u + 2\gamma\phi_u u + 4\xi_x\gamma u - \alpha\xi_{xxx}u - 2\gamma\phi = 0, \tag{4.17vii}$$

$$\phi_{tt} - \phi_{xxxx}u - 2\gamma\phi_{xx}u - \kappa\phi_{xx} - \phi_{xxtt} = 0, \tag{4.17viii}$$

$$6\phi_{xxu}u^2 + 4\xi_x\gamma u^2 - 4\xi_{xxx}u^2 + 2\beta\phi_{xx}u + \phi_{ttu}u + 2\xi_x\kappa u - \kappa\phi = 0, \tag{4.17ix}$$

$$4\phi_{xxxu}u + 4\gamma\phi_{xu}u - 2\xi_{xx}\gamma u - \xi_{xxxx}u + \alpha\phi_{xxx} + 4\gamma\phi_x + 2\kappa\phi_{xu} - \xi_{xx}\kappa = 0, \tag{4.17x}$$

and then use **reduceall** in `diffgrob2` to simplify them to the following system,

$$\xi_u = 0, \quad \xi_t = 0, \quad \xi_x = 0, \quad \tau_u = 0, \quad \kappa\tau_t = 0, \quad \tau_x = 0, \quad \kappa\phi = 0.$$

Here $\kappa = 0$ is the only special case, yielding the slightly different system

$$\xi_u = 0, \quad \xi_t = 0, \quad \xi_x = 0, \quad \tau_u = 0, \quad \tau_{tt} = 0, \quad \tau_x = 0, \quad \phi + 2\tau_t u = 0.$$

Thus we have two different sets of infinitesimals, and in both cases α, β and γ remain arbitrary. This is summarised in table 4.2.2.

Table 4.2.2

Parameters	ξ	τ	ϕ	
	c_1	c_2	0	(4.18)
$\kappa = 0$	c_1	$c_3t + c_2$	$-2c_3u$	(4.19)

From these we have the following two canonical reductions:

Reduction 4.2.8. $\alpha, \beta, \gamma, \kappa$ arbitrary. If in (4.18) and (4.19) $c_3 = 0$, then we may set $c_1 = c$ and $c_2 = 1$ without loss of generality. Thus we obtain the following reduction

$$u(x,t) = w(z), \qquad z = x - ct,$$

where $w(z)$ satisfies

$$(\kappa - c^2)w'' + 2\gamma[ww'' + (w')^2] + ww'''' + \alpha w'w''' + \beta(w'')^2 + c^2 w'''' = 0.$$

Reduction 4.2.9. α, β, γ arbitrary, $\kappa = 0$. If in (4.19) $c_3 \neq 0$, then we may set $c_2 = 0$, $c_3 = 1$ and $c_1 = c$ without loss of generality. Thus we obtain the following reduction

$$u(x,t) = t^{-2}w(z), \qquad z = x - c\ln t,$$

where $w(z)$ satisfies

$$ww'''' + \alpha w'w''' + \beta(w'')^2 + c^2w'''' + 5cw''' + 2\gamma[ww'' + (w')^2] + (6 - c^2)w'' - 5cw' - 6w = 0.$$

4.2.3 Travelling wave reductions. As was seen in §4.1, special cases of (4.1) admit interesting travelling wave solutions, namely compactons. In this subsection we look for such solitary waves and others, in the framework of (4.1). Starting with compacton–type solutions, we seek solutions of the form

$$u(x,t) = a_2 \cos^n\{a_3(x - a_1 t)\} + a_4, \tag{4.20}$$

where a_1, a_2, a_3, a_4 are constants to be determined. We include the (possibly non-zero) constant a_4 since u is open to translation. The specific form of the translation will put conditions on a_4, which may or may not put further conditions on the other parameters in (4.20) and those in (4.1) (see below). If $n = 1$ we have the solutions, where the absence of a parameter implies it is arbitrary,

(i) $\alpha = 0,\ \beta = -1,\ \gamma = 0, \qquad a_4 = \dfrac{\kappa - a_1^2 - a_1^2 a_2^2 \mu}{a_2^2}.$ (4.21i)

(ii) $\alpha = 1,\ \beta = 0,\ \gamma > 0, \qquad a_1^2(1 + 2\gamma\mu) - \kappa = 0, \qquad a_3^2 = 2\gamma.$ (4.21ii)

(iii) $\beta = \alpha - 1,\ \dfrac{\gamma}{\alpha} > 0,\ \alpha \neq 1, \qquad a_3^2 = \dfrac{2\gamma}{\alpha}, \qquad a_4 = \dfrac{\alpha(\kappa - a_1^2) - 2a_1^2\gamma\mu}{2\gamma(1 - \alpha)}.$ (4.21iii)

These become $n = 2$ solutions via the trigonometric identity $\cos 2\theta = 2\cos^2\theta - 1$. By earlier reasoning the associated compactons are weak solutions of (4.1). When considering more general n we restrict n to be either 3 or ≥ 4 else the fourth derivatives of $u(x,t)$ that we require in (4.1) would have singularities at the edges of the humps; we find

$$\alpha = \frac{2}{n},\ \beta = \frac{2-n}{n},\ \gamma > 0, \qquad a_1^2(1 + 2\gamma\mu) - \kappa = 0, \qquad a_3^2 = \frac{\gamma}{n}, \qquad a_4 = -\mu a_1^2. \tag{4.22}$$

When $n = 3$ or $n = 4$ our compacton would be a weak solution since not all the derivatives of $u(x,t)$ in (4.1) in these instances are continuous at the edges. For $n > 4$ the solutions are strong.

For more usual solitary waves we seek solutions of the form

$$u(x,t) = a_2 \operatorname{sech}^n\{a_3(x - a_1 t)\} + a_4, \tag{4.23}$$

where a_1, a_2, a_3, a_4 are constants to be determined. If $n = 2$ then $\alpha = -1$, $\beta = -2$ and we have solutions

$$\text{(i)} \quad \gamma < 0, \quad a_1^2(1 + 2\gamma\mu) - \kappa = 0, \quad a_3^2 = -\frac{\gamma}{2}, \quad a_4 = -\tfrac{1}{3}(2a_2 + 3a_1^2\mu), \tag{4.24i}$$

$$\text{(ii)} \quad \gamma^2 \neq 4a_3^4, \quad a_2 = \frac{3a_3^2(\kappa - a_1^2 - 2a_1^2\gamma\mu)}{(2a_3^2 - \gamma)(2a_3^2 + \gamma)}, \quad a_4 = -\frac{\kappa - a_1^2 + 4a_1^2 a_3^2\mu}{2(2a_3^2 + \gamma)}, \tag{4.24ii}$$

and for general n, including $n = 2$ ($\gamma > 0$)

$$\alpha = -\frac{2}{n},\ \beta = -\frac{n+2}{n}, \quad a_1^2(1 + 2\gamma\mu) - \kappa = 0, \quad a_3^2 = \frac{\gamma}{n}, \quad a_4 = -\mu a_1^2. \tag{4.25}$$

Now consider the general travelling wave reduction, $u(x,t) = w(z)$, where $z = x - ct$. The new dependent variable, $w(z)$, satisfies

$$(\kappa - c^2)w'' + 2\gamma[ww'' + (w')^2] + \mu c^2 w'''' + ww'''' + \alpha w'w''' + \beta(w'')^2 = 0. \tag{4.26}$$

In the special case $\beta = \alpha - 1$, we can integrate this twice with respect to z to give

$$(\kappa - c^2)w + \gamma w^2 + \mu c^2 w'' + \tfrac{1}{2}(\alpha - 2)(w')^2 + Az + B = 0, \tag{4.27}$$

with A, B the constants of integration. If we assume $A = 0$, then we make the transformation $W(z) = w(z) + \mu c^2$, multiply (4.27) by $W^{\alpha-3}W'$ and integrate with respect to z to yield

$$\frac{\gamma}{\alpha}W^\alpha + \frac{A_1}{\alpha - 1}W^{\alpha-1} + \frac{A_2}{\alpha - 2}W^{\alpha-2} + W^{\alpha-2}(W')^2 + C = 0, \tag{4.28}$$

for $\alpha \neq 0, 1, 2$, where $A_1 = \kappa - c^2 - 2\gamma\mu c^2$ and $A_2 = B - \mu c^2(\kappa - c^2 - \gamma\mu c^2)$. In the special cases $\alpha = 0, 1, 2$ we obtain respectively

$$\gamma \ln W - \frac{A_1}{W} - \frac{2A_2}{W^2} + \frac{(W')^2}{W^2} + C = 0, \tag{4.29i}$$

$$\gamma W + A_1 \ln W - \frac{A_2}{W} + \frac{(W')^2}{W} + C = 0, \tag{4.29ii}$$

$$\frac{\gamma}{2}W^2 + A_1 W + A_2 \ln W + (W')^2 + C = 0, \tag{4.29iii}$$

where C in the above is a constant of integration. For α an integer ≥ 3, (4.28) may be written (back in terms of $w(z)$) as

$$\begin{aligned} -(w + \mu c^2)^{\alpha-2}(w')^2 &= \frac{B}{\alpha - 2}(w + \mu c^2)^{\alpha-2} + D \\ &+ \frac{\gamma}{\alpha}w^2\left[(w + \mu c^2)^{\alpha-2} + \frac{1}{\gamma}(\alpha(\kappa - c^2) - 2\mu\gamma c^2)\sum_{n=0}^{\alpha-3}\frac{(\mu c^2)^{\alpha-3-n}}{(n+2)}\binom{\alpha-3}{n}w^n\right], \end{aligned} \tag{4.30}$$

where

$$D = C - \frac{(\mu c^2)^{\alpha-1}[\alpha(\kappa - c^2) - 2\mu\gamma c^2]}{\alpha(\alpha-1)(\alpha-2)}.$$

If we require that w and its derivatives tend to zero as $z \to \pm\infty$, then $B = D = 0$. If $\alpha = 3$ this equation induces so-called peakons (Camassa and Holm [1993]) as $\alpha(\kappa - c^2) - 2\mu\gamma c^2 \to 0$ (see Camassa, Holm and Hyman [1994], Gilson and Pickering [1995], Kovalev [1995] and Rosenau [1994]). Similarly if $\alpha = 4$ this equation is of the form found in Gilson and Pickering [1995] which induces the 'wave of greatest height' found in Fornberg and Whitham [1978]. Both solutions, in their limit, have a discontinuity in their first derivative at its peak. Note that if $\alpha(\kappa - c^2) - 2\mu\gamma c^2 = 0$, equation (4.30) becomes

$$(w + \mu c^2)^{\alpha-2}\left[(w')^2 + \frac{\gamma}{\alpha}w^2\right] = 0. \tag{4.31}$$

Since $\alpha > 0$ then we require $\gamma < 0$ to give a peakon of the form

$$u(x,t) = \frac{\alpha(c^2 - \kappa)}{2\gamma}\exp\left\{-\left(\frac{-\gamma}{\alpha}\right)^{1/2}|x - ct|\right\}. \tag{4.32}$$

The height of the wave, because of the form of (4.1), is dependent upon the square of the speed, whereas the peakons in the papers by Camassa and Holm [1993] and Fornberg and Whitham [1978] are proportional to the wave speed.

4.3 Nonclassical symmetries ($\tau \neq 0$)

We apply the nonclassical method using the algorithm described by Clarkson and Mansfield [1994c] for calculating the determining equations, which avoids difficulties arising from using differential consequences of the invariant surface condition, which is necessary in the application to (4.1).

In the canonical case when $\tau \neq 0$ we set $\tau = 1$ without loss of generality. We proceed by eliminating u_{tt} and u_{xxtt} in (4.1) using the invariant surface condition which yields

$$\begin{aligned}
&\xi\xi_x u_x + 2u_x^2\xi\xi_u - 2\phi_u\xi u_x + \xi^2 u_{xx} - \phi_x\xi - \xi_t u_x + \phi\phi_u - \phi\xi_u u_x + \phi_t \\
&\quad - \kappa u_{xx} - 2\gamma(uu_{xx} + u_x^2) - uu_{xxxx} - \alpha u_x u_{xxx} - \beta u_{xx}^2 + \mu[2\phi_{xx}\xi_x - 2\phi_{xu}\phi_x - 4\xi_x^2 u_{xx} \\
&\quad - \phi_{tu}u_{xx} - \phi_u\phi_{xx} - \phi_{xxt} + \xi_{xxt}u_x - \phi_{tuu}u_x^2 + \phi_x\xi_{xx} + \xi_{tuu}u_x^3 + \xi_t u_{xxx} - \xi^2 u_{xxxx} \\
&\quad + \phi_{xxx}\xi - \phi_u^2 u_{xx} - \phi\phi_{xxu} - \xi\xi_{xxx}u_x + \phi\xi_{uuu}u_x^3 + \phi\xi_{xxu}u_x + \phi\xi_u u_{xxx} - \phi\phi_{uu}u_{xx} \\
&\quad - \phi\phi_{uuu}u_x^2 + 2\xi_{xt}u_{xx} + 2\xi_{xtu}u_x^2 - 2\phi_{xtu}u_x - 3\xi_x\xi_{xx}u_x - 4\xi_u\xi_{xx}u_x^2 - 4\xi\xi_{xx}u_{xx} \\
&\quad + 2\phi_u\xi_{xx}u_x - 5\xi_{uu}\xi_x u_x^3 - 8\xi_{xu}\xi_x u_x^2 - 15\xi_u\xi_x u_x u_{xx} - 5\xi\xi_x u_{xxx} + 4\phi_u\xi_x u_{xx} \\
&\quad + 4\phi_{uu}\xi_x u_x^2 + 6\phi_{xu}\xi_x u_x - 2\xi\xi_{uuu}u_x^4 - 5\xi\xi_{xuu}u_x^3 + 2\phi\xi_{xuu}u_x^2 - 6\xi_u\xi_{uu}u_x^4 \\
&\quad - 12\xi\xi_{uu}u_x^2 u_{xx} + 3\phi\xi_{uu}u_x u_{xx} + 4\phi_u\xi_{uu}u_x^3 + 3\phi_x\xi_{uu}u_x^2 - 4\xi\xi_{xxu}u_x^2 - 10\xi_u\xi_{xu}u_x^3 \\
&\quad - 15\xi\xi_{xu}u_x u_{xx} + 2\phi\xi_{xu}u_{xx} + 6\phi_u\xi_{xu}u_x^2 + 4\phi_x\xi_{xu}u_x - 12\xi_u^2 u_x^2 u_{xx} - 8\xi\xi_u u_x u_{xxx}
\end{aligned}$$

$$- 6\xi\xi_u u_{xx}^2 + 9\phi_u \xi_u u_x u_{xx} + 3\phi_x \xi_u u_{xx} + 5\phi_{uu}\xi_u u_x^3 + 8\phi_{xu}\xi_u u_x^2 + 3\phi_{xx}\xi_u u_x$$
$$+ 3\xi_{tu} u_x u_{xx} + 2\phi_u \xi u_{xxx} + 6\phi_{uu}\xi u_x u_{xx} + 5\phi_{xu}\xi u_{xx} + 2\phi_{uuu}\xi u_x^3 + 5\phi_{xuu}\xi u_x^2$$
$$+ 4\phi_{xxu}\xi u_x - 3\phi_u\phi_{uu}u_x^2 - 2\phi_{uu}\phi_x u_x - 2\phi\phi_{xuu}u_x - 4\phi_u\phi_{xu}u_x] = 0. \qquad (4.33)$$

We note that this equation now involves the infinitesimals ξ and ϕ that are to be determined. Then we apply the classical Lie algorithm to (4.33) using the fourth prolongation $\mathrm{pr}^{(4)}\mathbf{v}$ and eliminate u_{xxxx} using (4.33). It should be noted that the coefficient of u_{xxxx} is $(\mu\xi^2 + u)$. Therefore, if this is zero the removal of u_{xxxx} using (4.33) is invalid and so the next highest derivative term, u_{xxx}, should be used instead. We note again that this has a coefficient that may be zero so that in the case $\mu \neq 0$ and $\mu\xi^2 + u = 0$ one needs to calculate the determining equations for the subcases when this is zero and non-zero separately. Continuing in this fashion, there is a cascade of subcases to be considered. In the remainder of this section, we consider these subcases in turn. First, however, we discuss the case given by $\mu = 0$.

Case 4.3.1 $\mu = 0$. In this case we generate the following 12 determining equations.

$$\xi_u = 0, \qquad (4.34\text{i})$$
$$\phi_{uuuu}u + \alpha\phi_{uuu} = 0, \qquad (4.34\text{ii})$$
$$4\phi_{xuuu}u + 3\alpha\phi_{xuu} = 0, \qquad (4.34\text{iii})$$
$$6\phi_{uuu}u + 2\beta\phi_{uu} + 3\alpha\phi_{uu} = 0, \qquad (4.34\text{iv})$$
$$4\phi_{uu}u^2 + \alpha\phi_u u - \alpha\phi = 0, \qquad (4.34\text{v})$$
$$4\phi_{xu}u - 6\xi_{xx}u + \alpha\phi_x = 0, \qquad (4.34\text{vi})$$
$$3\phi_{uu}u^2 + \beta\phi_u u - \beta\phi = 0, \qquad (4.34\text{vii})$$
$$12\phi_{xuu}u + 4\beta\phi_{xu} + 3\alpha\phi_{xu} - 2\beta\xi_{xx} - 3\alpha\xi_{xx} = 0, \qquad (4.34\text{viii})$$
$$6\phi_{xxuu}u^2 + 2\gamma\phi_{uu}u^2$$
$$+ \kappa\phi_{uu}u - \xi^2\phi_{uu}u + 3\alpha\phi_{xxu}u + 2\gamma\phi_u u + 4\xi_x\gamma u - \alpha\xi_{xxx}u - 2\gamma\phi = 0, \qquad (4.34\text{ix})$$
$$6\phi_{xxu}u^2 + 4\xi_x\gamma u^2 - 4\xi_{xxx}u^2 + 2\beta\phi_{xx}u + 2\xi_x\kappa u - 4\xi^2\xi_x u - 2\xi\xi_t u - \kappa\phi + \xi^2\phi = 0, \qquad (4.34\text{x})$$
$$\phi_{tt}u - \phi_{xxxx}u^2 - 2\gamma\phi_{xx}u^2 - \kappa\phi_{xx}u - 4\xi\xi_x\phi_x u$$
$$- 2\xi_t\phi_x u + \phi^2\phi_{uu}u + 4\xi_x\phi\phi_u u + 2\phi\phi_{tu}u + 4\xi_x\phi_t u + \xi\phi\phi_x - \phi^2\phi_u - \phi\phi_t = 0, \qquad (4.34\text{xi})$$
$$4\phi_{xxxu}u^2 + 4\gamma\phi_{xu}u^2 - 2\xi_{xx}\gamma u^2 - \xi_{xxxx}u^2 + \alpha\phi_{xxx}u + 4\gamma\phi_x u$$
$$+ 2\xi\phi\phi_{uu}u + 2\kappa\phi_{xu}u + 8\xi\xi_x\phi_u u + 2\xi_t\phi_u u + 2\xi\phi_{tu}u - \xi_{xx}\kappa u$$
$$- 4\xi\xi_x^2 u + 2\xi_t\xi_x u + \xi_{tt}u - 2\xi\phi\phi_u + \xi\xi_x\phi - \xi_t\phi = 0. \qquad (4.34\text{xii})$$

As guaranteed by the nonclassical method, we get all the classical reductions, but we also have some infinitesimals that lead to nonclassical reductions, namely those described in table 4.3.1 (where primes denote differentiation with respect to t).

Table 4.3.1

Parameters	ξ	ϕ	
$\kappa = 0$	0	$g(t)u$ where $g'' + gg' - g^3 = 0$	(4.35)
$\alpha = \beta = \gamma = 0$	$\pm\sqrt{\kappa}$	$c_3y^3 + c_2y^2 + c_1y + c_0 \quad (y = x \pm \sqrt{\kappa}\, t)$	(4.36)
$\alpha = \beta =$	0	$-g'(t)u/g(t) + g(t)(c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)$	
$\gamma = \kappa = 0$		where $g^2g''' - 4gg'g'' + 2(g')^3 + 24c_4g^4 = 0$	(4.37)

From these we obtain three canonical reductions.

Reduction 4.3.1. α, β, γ arbitrary, $\kappa = 0$. In (4.35) we solve the equation for $g(t)$ by writing $g(t) = \frac{\mathrm{d}}{\mathrm{d}t}[\ln\psi(t)]$ then $\psi(t)$ satisfies

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 = 4c_1\psi^3 + c_2, \tag{4.38}$$

where c_1 and c_2 are arbitrary constants; $c_1 = c_2 = 0$ is not allowed since $g(t) \not\equiv 0$ to obtain more than classical reductions. Hence we obtain the following reduction

$$u(x,t) = w(x)\psi(t),$$

where $w(x)$ satisfies

$$w\frac{\mathrm{d}^4w}{\mathrm{d}x^4} + \alpha\frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}^3w}{\mathrm{d}x^3} + \beta\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)^2 + 2\gamma\left[w\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right] - 6c_1w = 0.$$

There are three cases to consider in the solution of (4.38).

(i) If $c_1 = 0$, we may assume that $\psi(t) = t$ without loss of generality.

(ii) If $c_2 = 0$, then $\psi = [c_2(t + c_3)^2]^{-1}$ and we may set $c_2 = 1, c_3 = 0$ without loss of generality.

(iii) If $c_1c_2 \neq 0$ we may set $c_1 = 1, c_2 = -g_3$ without loss of generality so that $\psi(t)$ is any solution of the Weierstrass elliptic function equation

$$\left(\frac{\mathrm{d}\wp}{\mathrm{d}t}\right)^2 = 4\wp^3(t; 0, g_3) - g_3. \tag{4.39}$$

Reduction 4.3.2. κ arbitrary, $\alpha = \beta = \gamma = 0$. From (4.36) we get the following reduction

$$u(x,t) = w(z) \pm \frac{c_3}{8\sqrt{\kappa}}y^4 \pm \frac{c_2}{6\sqrt{\kappa}}y^3 \pm \frac{c_1}{4\sqrt{\kappa}}y^2 + c_0t, \qquad y = x \pm \sqrt{\kappa}\, t, \qquad z = x \mp \sqrt{\kappa}\, t,$$

where $w(z)$ satisfies

$$\sqrt{\kappa}\, w'''' \pm 3c_3 = 0.$$

This gives us the exact solution

$$u(x,t) = \mp\frac{c_3}{8\sqrt{\kappa}}z^4 + c_4z^3 + c_5z^2 + c_6z + c_7 \pm \frac{c_3}{8\sqrt{\kappa}}y^4 \pm \frac{c_2}{6\sqrt{\kappa}}y^3 \pm \frac{c_1}{4\sqrt{\kappa}}y^2 + c_0t.$$

Reduction 4.3.3. $\alpha = \beta = \gamma = \kappa = 0$. In (4.37) we integrate our equation for $g(t)$ up to an expression with quadratures

$$g\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} - 2\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2 + 24c_4 g\int^t g^2(s)\,\mathrm{d}s + 24c_5 g = 0. \tag{4.40}$$

We get the following reduction

$$u(x,t) = g^{-1}(t)\left[w(x) + (c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)\int^t g^2(s)\,\mathrm{d}s\right],$$

where $w(x)$ satisfies

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - 24c_5 = 0.$$

This is easily solved to give the solution

$$u(x,t) = g^{-1}(t)[c_5x^4 + c_6x^3 + c_7x^2 + c_8x + c_9 + (c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)\int^t g^2(s)\,\mathrm{d}s],$$

where $g(t)$ satisfies (4.40).

Case 4.3.2 $\mu \neq 0$. As discussed earlier in this section, we must consider, in addition to the general case, each of the singular cases of the determining equations. Without loss of generality we set $\mu = 1$.

Subcase 4.3.2(i) $\xi^2 + u \neq 0$. In this the generic case we generate 12 determining equations, far larger than system (4.34) – see appendix A for details. The reason for the increase in size of this system is obvious when one looks at equation (4.33) and the coefficient of μ therein. As expected we have all the classical reductions, however we also have many infinitesimals that lead to nonclassical reductions. These are presented in table 4.3.2, where primes denote differentiation with respect to t.

Table 4.3.2

Parameters	ξ	ϕ	
$\kappa = 0$	0	$g(t)u$ where $g'' + gg' - g^3 = 0$	(4.41)
$1 + 2\gamma = 0$	$c_1t + c_2$	$-2c_1(c_1t + c_2)$	(4.42)
$\kappa = 1 + 2\gamma = 0$	$c_2(t + c_1)^2$	$u(t + c_1)^{-1} - 3c_2^2(t + c_1)^3$	(4.43)
$\alpha = \beta = \gamma = 0$	$\pm\sqrt{\kappa}$	$c_3y^3 + c_2y^2 + c_1y + c_0 \quad (y = x \pm \sqrt{\kappa}\,t)$	(4.44)
$\alpha = -\frac{3}{2}, \beta = 2, \gamma = 0$	$\pm\frac{1}{2}\sqrt{\kappa}\,(x + c_1)$	$\pm 2\sqrt{\kappa}\,u \pm \frac{1}{4}\kappa^{3/2}(x + c_1)^2$	(4.45)
$\alpha = \beta = \gamma = \kappa = 0$	0	$c_3x^3 + c_2x^2 + c_1x + c_0$	(4.46)
$\alpha = \beta = \gamma = \kappa = 0$	0	$(u + c_3x^3 + c_2x^2 + c_1x + c_0)(t + c_4)^{-1}$	(4.47)

From these we obtain six reductions.

Reduction 4.3.4. α, β, γ arbitrary, $\kappa = 0$. In (4.41) we solve the equation for $g(t)$ by writing $g(t) = \frac{\mathrm{d}}{\mathrm{d}t}[\ln \psi(t)]$ then $\psi(t)$ satisfies

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 = 4c_1\psi^3 + c_2, \tag{4.48}$$

though $c_1 = c_2 = 0$ is not allowed to preserve $g(t) \not\equiv 0$ and our desire for a nonclassical reduction. We obtain the following reduction

$$u(x,t) = w(x)\psi(t),$$

where $w(x)$ satisfies

$$w\frac{\mathrm{d}^4w}{\mathrm{d}x^4} + \alpha\frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}^3w}{\mathrm{d}x^3} + \beta\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)^2 + 2\gamma\left[w\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right] + 6c_1\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2} - w\right) = 0.$$

There are three cases to consider in the solution of (4.48). These are identical to cases (i)–(iii) in reduction 4.3.1.

Note that in the special case $\frac{\mathrm{d}^2w}{\mathrm{d}x^2} - w = 0$, we are able to lift the restrictions on $\psi(t)$ so that it is arbitrary, if $\beta + 1 + 2\gamma = \alpha + 2\gamma = 0$. This yields the exact solution

$$u(x,t) = \psi(t)(c_3\mathrm{e}^x + c_4\mathrm{e}^{-x})$$

where $\psi(t)$ is arbitrary, $\kappa = 0$, $\alpha = -2\gamma$ and $\beta = -1 - 2\gamma$.

Reduction 4.3.5. α, β, κ arbitrary, $1 + 2\gamma = 0$. In (4.42) we assume $c_1 \neq 0$ otherwise we get a classical reduction, and then may set $c_2 = 0$ without loss of generality. Thus we obtain the following accelerating wave reduction

$$u(x,t) = w(z) - c_1^2t^2, \qquad z = x - \tfrac{1}{2}c_1t^2,$$

where $w(z)$ satisfies

$$ww'''' + \alpha w'w''' + \beta(w'')^2 - c_1w''' - ww'' + \kappa w'' - (w')^2 + c_1w' + 2c_1^2 = 0.$$

Reduction 4.3.6. α, β arbitrary, $1 + 2\gamma = \kappa = 0$. From (4.43) the following holds for arbitrary c_2, and we may set $c_1 = 0$ without loss of generality. Thus we obtain the reduction

$$u(x,t) = w(z)t - c_2^2t^4, \qquad z = x - \tfrac{1}{3}c_2t^3,$$

where $w(z)$ satisfies

$$ww'''' + \alpha w'w''' + \beta(w'')^2 - 4c_2w''' - ww'' - (w')^2 + 4c_2w' + 12c_2^2 = 0.$$

Reduction 4.3.7. κ arbitrary, $\alpha = \beta = \gamma = 0$. From (4.44) we get the following reduction

$$u(x,t) = w(z) \pm \frac{c_3}{8\sqrt{\kappa}}y^4 \pm \frac{c_2}{6\sqrt{\kappa}}y^3 \pm \frac{c_1}{4\sqrt{\kappa}}y^2 + c_0 t, \qquad y = x \pm \sqrt{\kappa}\, t, \qquad z = x \mp \sqrt{\kappa}\, t,$$

where $w(z)$ satisfies

$$\sqrt{\kappa}\,\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} \pm 3c_3 = 0.$$

This gives us the exact solution

$$u(x,t) = \mp\frac{c_3}{8\sqrt{\kappa}}z^4 + c_4 z^3 + c_5 z^2 + c_6 z + c_7 \pm \frac{c_3}{8\sqrt{\kappa}}y^4 \pm \frac{c_2}{6\sqrt{\kappa}}y^3 \pm \frac{c_1}{4\sqrt{\kappa}}y^2 + c_0 t.$$

Reduction 4.3.8. κ arbitrary, $\alpha = -\frac{3}{2}$, $\beta = 2$, $\gamma = 0$. In (4.45) we may set $c_1 = 0$ without loss of generality. Thus we obtain the following reduction

$$u(x,t) = w(z)x^4 - \tfrac{1}{4}\kappa x^2, \qquad z = \ln x \mp \tfrac{1}{2}\sqrt{\kappa}\, t,$$

where $w(z)$ satisfies

$$\begin{aligned} &4ww'''' - 6w'w''' + 8(w'')^2 \\ &\quad + 16ww''' + 58w'w'' + 116ww'' - \kappa w'' + 236(w')^2 + 776ww' + 672w^2 = 0. \end{aligned}$$

Reduction 4.3.9. $\alpha = \beta = \gamma = \kappa = 0$. From (4.46) and from (4.47) ($c_4 = 0$ without loss of generality) we get the following reductions

$$u(x,t) = w(x) + (c_3x^3 + c_2x^2 + c_1x + c_0)t$$

and

$$u(x,t) = w(x)t - (c_3x^3 + c_2x^2 + c_1x + c_0)$$

respectively. In both cases $w(x)$ satisfies

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = 0.$$

These reductions have a common exact solution, namely

$$u(x,t) = P_3(x)t + Q_3(x),$$

where P_3 and Q_3 are any third order polynomials in x with constant coefficients.

Subcase 4.3.2(ii) $\xi^2 + u = 0$, not both $\alpha = 4$ and $2\xi\phi_u + \xi_u\phi = 0$. The determining equations quickly lead us to require that both $\alpha = 4$ and $2\xi\phi_u + \xi_u\phi = 0$, which is a contradiction.

Subcase 4.3.2(iii) $\xi^2 + u = 0$, $\alpha = 4$, $\phi = H(x,t)u^{-1/4}$, $\beta \neq 3$. Equation (4.33) is now quadratic in u_{xx}, so when `symmgrp.max` solves for an explicit expression in u_{xx} (in order to perform the part (c) of Step One of the classical Lie algorithm – see §1.2.2), it introduces square roots. In such cases, and whenever irrational or transcendental functions appear, through computation or because of their presence in the original partial differential equation, the output is not necessarily the determining equations i.e. it may contain derivatives of the dependent variables in an irrational or transcendental way (see Champagne, Hereman and Winternitz [1991], §3.5). The user then has the task of composing the final list of determining equations. In this subcase four equations are generated by `symmgrp.max`. The first is of the form

$$\left(P_a(x,t,u,u_x)\right)^{1/2} = P_b(x,t,u) \tag{4.49}$$

where P_a is quadratic in u_x and polynomial in fractional powers of u. P_b is polynomial in fractional powers of u. Since our infinitesimal ξ is known and ϕ is known up to an arbitrary function of (x,t), by squaring both sides of (4.49) to leave a polynomial expression in u_x and fractional powers of u, the coefficients of different powers of u and u_x in this new equation must be zero. We find that $\gamma = -\frac{1}{2}$, $\kappa = 0$ and $\phi = 0$. The remaining three equations generated by `symmgrp.max` are then also zero and we are done. The invariant surface condition becomes

$$\pm \mathrm{i}\sqrt{u}\, u_x + u_t = 0,$$

which may be solved implicitly to yield the solution

$$u(x,t) = w(z), \qquad z = x \mp \mathrm{i}\sqrt{u}\, t.$$

However, substituting into our original equation gives $w' = 0$, i.e. $u(x,t)$ is a constant.

Subcase 4.3.2(iv) $\xi^2 + u = 0$, $\phi = H(x,t)u^{-1/4}$, $\alpha = 4$, $\beta = 3$, not all of $H, \kappa, 1+2\gamma = 0$. For the determining equations to be satisfied, each of $H, \kappa, 1+2\gamma$ must be zero, in contradiction to our assumption.

Subcase 4.3.2(v) $\xi^2 + u = 0$, $\phi = 0$, $\alpha = 4$, $\beta = 3$, $\gamma = -\frac{1}{2}$, $\kappa = 0$. Under these conditions equation (4.33) which we apply the classical method to is identically zero. Therefore any solution of the invariant surface condition is also a solution of (4.1). Hence we get the following reduction

Reduction 4.3.10. $\alpha = 4$, $\beta = 3$, $\gamma = -\frac{1}{2}$, $\kappa = 0$. The invariant surface condition is

$$\pm \mathrm{i}\sqrt{u}\, u_x + u_t = 0,$$

which may be solved implicitly to yield

$$u(x,t) = w(z), \qquad z = x \mp \mathrm{i}\sqrt{u}\, t,$$

where $w(z)$ is arbitrary.

4.4 Nonclassical symmetries ($\tau = 0$)

In the canonical case of the nonclassical method when $\tau = 0$ we set $\xi = 1$ without loss of generality. We proceed by eliminating $u_x, u_{xx}, u_{xxx}, u_{xxxx}$ and u_{xxtt} in (4.1) using the invariant surface condition which yields

$$\begin{aligned}
&u_{tt} - \kappa(\phi_x + \phi\phi_u) - 2\gamma(u\phi_x + u\phi\phi_u + \phi^2) - u(\phi_{xxx} \\
&\quad + \phi_u\phi_{xx} + \phi_u^2\phi_x + \phi\phi_u^3 + 4\phi_u\phi^2\phi_{uu} + 5\phi_u\phi\phi_{xu} + 3\phi\phi_{uu}\phi_x + \phi^3\phi_{uuu} + 3\phi^2\phi_{xuu} \\
&\quad + 3\phi\phi_{xxu} + 3\phi_{xu}\phi_x) - \mu(\phi\phi_{uu}u_{tt} + \phi\phi_{uuu}u_t^2 + 2\phi\phi_{tuu}u_t + \phi\phi_{ttu} + \phi_u^2 u_{tt} + 3\phi_u\phi_{uu}u_t^2 \\
&\quad + 4\phi_u\phi_{tu}u_t + \phi_u\phi_{tt} + \phi_{xu}u_{tt} + \phi_{xuu}u_t^2 + 2\phi_{xtu}u_t + 2\phi_t\phi_{uu}u_t + 2\phi_t\phi_{tu} + \phi_{xtt}) \\
&\quad - \alpha\phi(\phi_{xx} + \phi_u\phi_x + \phi\phi_u^2 + \phi^2\phi_{uu} + 2\phi\phi_{xu}) - \beta(\phi_x + \phi\phi_u)^2 = 0, \qquad (4.50)
\end{aligned}$$

which involves the infinitesimal ϕ that is to be determined. As in the $\tau \neq 0$ case we apply the classical Lie algorithm to this equation using the second prolongation $\mathrm{pr}^{(2)}\mathbf{v}$ and eliminate u_{tt} using (4.50). Similar to the nonclassical method in the generic case $\tau \neq 0$, when $\mu \neq 0$ the coefficient of the highest derivative term, u_{tt} is not necessarily zero, thus singular cases are induced. As in the previous section we consider the cases $\mu = 0$ and $\mu \neq 0$ separately, though we collate these separate workings when considering the exact solutions of 4.4.3.

<u>*Case 4.4.1* $\mu = 0$.</u> Generating the determining equations, again using `symmgrp.max`, yields three equations, the first two being $\phi_{uu} = 0$ and $\phi_{tu} = 0$. Hence we look for solutions like $\phi = A(x)u + B(x,t)$ in the third. Taking coefficients of powers of u to be zero yields a system of three equations in A, B. (Note that since A is a function of x only, subscripts of A are not strictly partial derivatives.)

$$\begin{aligned}
&\alpha AA_{xxx} + 2\beta A^2A_{xx} + A_{xxxx} + \alpha A_xA_{xx} + 5\beta AA_x^2 + 6\alpha AA_x^2 + 10A_{xx}A_x \\
&\quad + 2\gamma A_{xx} + 5AA_{xxx} + \alpha A^5 + 10A^2A_{xx} + 10A^3A_x + A^5 + \beta A^5 + 15AA_x^2 \\
&\quad + 4\gamma A^3 + 2\beta A_xA_{xx} + 10\gamma AA_x + 4\alpha A^2A_{xx} + 6\beta A^3A_x + 7\alpha A^3A_x = 0, \qquad (4.51\text{i})
\end{aligned}$$

$$\begin{aligned}
&5\alpha BA_x^2 + 2\beta A^2B_{xx} + \alpha BA_{xxx} + 2\beta BA^4 \\
&\quad + 13BAA_{xx} + 2\alpha A^3B_x + 10\gamma BA_x + 7AA_xB_x + 2\beta A_xB_{xx} + \alpha A_xB_{xx} + 2\kappa AA_x \\
&\quad + \alpha A^2B_{xx} + 15BA^2A_x + \alpha B_xA_{xx} + 6\gamma AB_x + 8\gamma BA^2 + 2\beta A^3B_x + 2\beta B_xA_{xx} \\
&\quad + 2\alpha BA^4 + \alpha AB_{xxx} + B_{xxxx} + 6A_{xx}B_x + 2\gamma B_{xx} + 5BA_{xxx} + AB_{xxx} \\
&\quad + \kappa A_{xx} + 2BA^4 + A^2B_{xx} + A^3B_x + 4A_xB_{xx} + 11BA_x^2 + 4\beta BA_x^2 + 6\beta AB_xA_x \\
&\quad + 7\alpha AB_xA_x + 7\alpha BAA_{xx} + 2\beta BAA_{xx} + 10\beta BA^2A_x + 12\alpha BA^2A_x = 0, \qquad (4.51\text{ii})
\end{aligned}$$

$$\beta AB_x^2 + 4\gamma B^2 A + 2\beta B_x B_{xx} + 5B^2 AA_x + \alpha AB_x^2 + 6\gamma BB_x + 2\kappa BA_x + BAB_{xx}$$
$$+ \alpha B^2 A^3 + \beta B^2 A^3 + \alpha B_x B_{xx} + 3\alpha B^2 A_{xx} + \alpha BB_{xxx} + BA^2 B_x + 3BA_x B_x$$
$$- B_{tt} + BB_{xxx} + \kappa B_{xx} + B^2 A^3 + 3B^2 A_{xx} + 2\beta BAB_{xx} + \alpha BAB_{xx} + 4\beta BA_x B_x$$
$$+ 5\alpha BA_x B_x + 2\beta BA^2 B_x + 2\alpha BA^2 B_x + 4\beta B^2 AA_x + 5\alpha B^2 AA_x = 0. \tag{4.51iii}$$

We try to solve this system using the `diffgrob2` package interactively, however the expression swell is too great to obtain meaningful output. Thus we proceed by making ansätze on the form of $A(x)$, solve (4.51ii) (a linear equation in $B(x,t)$) then finally (4.51iii) gives the full picture. Many solutions have been found as (4.51i) lends itself to many ansätze through choices of parameter values. We present some in 4.4.3.

Case 4.4.2 $\mu \neq 0$. The nonclassical method, when the coefficient of u_{tt} is non-zero, generates a system of three determining equations. However, far from being single-term equations like those in the above Case, the first two contain 41 and 57 terms respectively, and the third 329. The intractability of finding all solutions is obvious. To find some, we allude to our previous Case and look for $\phi = A(x)u + B(x,t)$, and note we set $\mu = 1$ without loss of generality. Three equations then remain, similar to (4.51) which we tackle in the same vein as previously. Some solutions are presented in 4.4.3.

As mentioned in the start of this section, singular solutions may exist, when the coefficient of u_{tt} equals zero, i.e. when

$$1 - \phi\phi_{uu} - \phi_u^2 - \phi_{ux} = 0. \tag{4.52}$$

This may be integrated with respect to u to give

$$u - \phi\phi_u - \phi_x = H(x,t). \tag{4.53}$$

If ϕ satisfies (4.52) then the coefficients of u_t^2 and u_t in (4.50) are both zero. Since no u-derivatives now exist in (4.50) what is left must also be zero, i.e.

$$(2\gamma+\alpha)\phi^2 - \alpha H_x\phi + (2\gamma+\beta+1)u^2 + [\kappa - (2\gamma+2\beta+1)H - H_{xx}]u + \beta H^2 - \kappa H - H_{tt} = 0. \tag{4.54}$$

Thus we need to solve (4.53) and (4.54). The obvious way to proceed is to substitute our expression for ϕ in (4.54) into (4.53). Singular cases present themselves when $\alpha = -2\gamma$ and also when $\alpha H_x = 0$. Once we have found $\phi(x,t,u)$, the related exact solution is given by solving the invariant surface condition, with no further restrictions on the solution. The following are distinct from each other and from the exact solutions in 4.4.3.

(a) $\gamma = -\frac{1}{2}$, $\alpha = 1$, $\beta = \kappa = 0$ and $\phi(x,t,u)$ is given by the relation

$$u - \phi\phi_u - \phi_x = c_1 t + c_2.$$

For instance, if $\phi(x,t,u)$ is linear in u we have the exact solution

$$u(x,t) = w(t)\cosh[x + A(t)] + c_1 t + c_2 + B(t)\sinh[x + A(t)],$$

where $w(t)$, $A(t)$ and $B(t)$ are arbitrary functions.

(b) $\alpha = -2\gamma$, $\beta = -1 - 2\gamma$, $\gamma \neq -\frac{1}{2}$ and $\phi(x,t,u)$ is given by the relation

$$u - \phi\phi_u - \phi_x = -\frac{\kappa}{1 + 2\gamma}.$$

For instance, if $\phi(x,t,u)$ is linear in u we have the exact solution

$$u(x,t) = w(t)\cosh[x + A(t)] - \frac{\kappa}{1 + 2\gamma} + B(t)\sinh[x + A(t)],$$

where $w(t)$, $A(t)$ and $B(t)$ are arbitrary functions.

(c) $\alpha = -2\gamma$, $\beta = -1 - 2\gamma$, and $\phi = \pm u - \kappa x \mp (-\frac{1}{2}\kappa t^2 + c_1 t + c_2) \pm \kappa$. Then

$$u(x,t) = w(t)\exp(\pm x) \pm \kappa x + (-\tfrac{1}{2}\kappa t^2 + c_1 t + c_2),$$

where $w(t)$ is an arbitrary function.

(d) $\gamma = -\frac{1}{2}$, $\alpha = 1$, $\beta = 0$, and

$$\phi = \frac{1}{H_x}(\kappa u - H_{xx}u - \kappa H - H_{tt}),$$

where $H_x(x,t) \neq 0$ and also $H(x,t)$ satisfies the system

$$2\kappa H_{xx} + H_x^2 - \kappa^2 = 0,$$
$$(\kappa^2 - H_x^2)H_{tt} - 2\kappa H_{xt}^2 + \kappa^2(\kappa^2 - H_x^2) = 0.$$

We have assumed that $\kappa^2 - H_x^2 \neq 0$, for a different solution to (c). This yields

$$u(x,t) = [w(t) - 2\kappa x]\sinh[\tfrac{1}{2}(x + c_1 t + c_2)]\cosh[\tfrac{1}{2}(x + c_1 t + c_2)] - c_1^2$$
$$+ \cosh^2[\tfrac{1}{2}(x + c_1 t + c_2)][4\kappa\ln(\cosh[\tfrac{1}{2}(x + c_1 t + c_2)]) - \kappa^2 t^2 + 2c_3 t + 2c_4 - 2\kappa + 2c_1^2]$$

where $w(t)$ is an arbitrary function.

(e) $\beta = -1 - \alpha - 4\gamma$, $\gamma \neq \frac{1}{2}$, $\alpha + 2\gamma \neq 0$, $\phi = \pm\left(u + \frac{\kappa}{1 + 2\gamma}\right)$. Then

$$u(x,t) = w(t)\exp(\pm x) - \frac{\kappa}{1 + 2\gamma}$$

where $w(t)$ is an arbitrary function.

(f) $\gamma = 0$, $\alpha = -2$, $\beta = 1$, $\phi(x,t,u)$ satisfies

$$-2\phi^2 + 2\phi H_x + 2u^2 + 2\kappa u - 2Hu + \kappa^2 + H^2 = 0,$$

and $H(x,t)$ satisfies

$$H_{xx} + \kappa + H = 0, \qquad H_{tt} + \kappa^2 + \kappa H = 0.$$

Then

$$u(x,t) = \tfrac{1}{2}(A^2+B^2)^{1/2}\sinh[\pm x + w(t)] - \kappa + \tfrac{1}{2}(A\sin x + B\cos x),$$

where $A(t)$, $B(t)$ satisfy

$$\frac{\mathrm{d}^2A}{\mathrm{d}t^2} + \kappa A = 0, \qquad \frac{\mathrm{d}^2B}{\mathrm{d}t^2} + \kappa B = 0$$

and $w(t)$ is an arbitrary function.

(g) $\gamma = 0$, $\beta = -1-\alpha$, $\kappa = 0$, $\alpha = (c_1-1)^2/c_1$ where $c_1 \neq 0, 1$ and

$$\phi = u + \frac{c_2t+c_3}{c_1-1}\exp(-c_1x).$$

Then

$$u(x,t) = \begin{cases} w_1(t)\mathrm{e}^x - \frac{1}{2}(c_2t+c_3)x\mathrm{e}^x & \text{if } c_1 = -1, \\ w_2(t)\mathrm{e}^x + (1-c_1^2)^{-1}(c_2t+c_3)\mathrm{e}^{-c_1x} & \text{if } c_1 \neq -1, \end{cases}$$

where $w_1(t)$, $w_2(t)$ are arbitrary functions.

(h) $\gamma = 0$, $\beta = -1-\alpha$ and $c_1^2 + \alpha^2c_1 + 2c_1 + 4\alpha c_1 + 1 = 0$. Also we insist that $\alpha \neq 0, -2$ and $c_1 \neq 0, 1$, then $\phi(x,t,u)$ satisfies

$$\alpha\phi^2 - \alpha H_x\phi - \alpha u^2 + u[\kappa + H(1+2\alpha) - H_{xx}] - \kappa H - (\alpha+1)H^2 - H_{tt} = 0,$$

where $H(x,t)$ satisfies the system of equations

$$H_{tt} + c_1\kappa H = 0,$$
$$(\alpha+2)H_x \pm (1-c_1)(H+\kappa) = 0.$$

Then

$$u(x,t) = \left[\frac{1+2\alpha+c_1}{2\alpha}w(t) + g(x,t)\right]\exp\left\{\frac{(c_1-1)x}{\alpha+2}\right\} - \kappa,$$

where $w(t)$ satisfies $\dfrac{\mathrm{d}^2w}{\mathrm{d}t^2} + c_1\kappa w = 0$ and $g(x,t)$ satisfies

$$\alpha g_x^2 + \frac{2\alpha(c_1-1)}{\alpha+2}gg_x - \alpha(c_1+1)g^2 + (\alpha c_1 + c_1 + 1)w(t)g_x - c_1(1+\alpha+c_1)w(t)g + \frac{c_1(c_1-1)(\alpha+2)}{4\alpha}w^2(t) = 0.$$

which is effectively an autonomous ordinary differential equation with t a parameter, so the first integral may be written down.

<u>4.4.3 Exact solutions</u>. We present some exact solutions which Cases 4.4.1 and 4.4.2 have unearthed, though we describe them here for general μ. The infinitesimal $\phi(x,t,u)$ is given, possibly up to satisfying some equations, followed by the solution, found by solving the invariant surface condition. All of the unknown functions in the sequel are functions of t only, hence we use primes to denote differentiation with respect to t.

Subcase 4.4.3(i) $\gamma = 0$ and $\phi = \dfrac{u}{x} + H_1(t)x + 3H_2(t)x^3 + \dfrac{H_3(t)}{x} + H_4(t)x^{2-\alpha}$. Solving the invariant surface condition gives

$$u(x,t) = \begin{cases} xw(t) + H_1(t)x^2 + H_2(t)x^4 - H_3(t) + \dfrac{H_4(t)x^{3-\alpha}}{2-\alpha} & \text{if } \alpha \neq 2, \\ x\tilde{w}(t) + H_1(t)x^2 + H_2(t)x^4 - H_3(t) + H_4(t)x\ln x & \text{if } \alpha = 2. \end{cases}$$

Various types of solution are found, as seen in table 4.4.1. The $H_i(t)$ are determined by the determining equations, $w(t)$ by substituting back into (4.1).

Table 4.4.1

Parameters	$H_i(t), w(t)$ satisfy	
$\alpha = \frac{1}{2}, \beta = -\frac{1}{8}$	$H_1 = 4\kappa, \quad H_2 = H_3 = H_4'' = 0$ $32w'' = -5H_4^2$	(4.55)
$\alpha = \frac{1}{2}, \kappa = 0$	$H_1 = \mu H_2 = H_3 = 0$ $H_2'' - 72(1+2\beta)H_2^2 = 0$ $16H_4'' - 3(480\beta + 303)H_2H_4 = 0$ $8w'' - 288H_2w = (50\beta + 5)H_4^2$	(4.56)
$\beta = (\alpha^2 - \alpha)/(3-\alpha)$, $\kappa = 0, \alpha \neq 3$	$H_1 = \mu H_2 = H_3 = 0$ $(\alpha - 3)H_2'' + 24(2\alpha+3)(\alpha+1)H_2^2 = 0$ $H_4'' + 3(\alpha+2)(\alpha+1)(\alpha^2 - \alpha - 4)H_2H_4 = 0$ $w'' - 24(\alpha+1)H_2w = 0$ $(\tilde{w}'' - 72H_2\tilde{w} = 90H_2H_4)$	(4.57)
$\alpha = -2, \beta = \frac{6}{5}$	$H_1 = \frac{5}{6}\kappa, \quad \mu H_2 = H_4'' = 0$ $5H_2'' - 24H_2^2 = 0$ $5H_3'' - 120H_2H_3 = -25\kappa^2$ $w'' + 24H_2w = -30H_3H_4$	(4.58)

Subcase 4.4.3(ii) We assume $\phi = B(x,t)$ and find the following

(a) $\gamma = 0$, and $B(x,t) = 4H_1(t)x^3 + 3H_2(t)x^2 + 2H_3(t)x + H_4(t)$, where

$$H_1'' - 24(6\beta + 4\alpha + 1)H_1^2 = 0, \tag{4.59i}$$

$$H_2'' - 24(6\beta + 4\alpha + 1)H_1H_2 = 0, \tag{4.59ii}$$

$$H_3'' - 24(2\beta + 2\alpha + 1)H_1H_3 = 18(2\beta + \alpha)H_2^2 + 12\kappa H_1 + 12\mu H_1'', \tag{4.59iii}$$

$$H_4'' - 24(\alpha + 1)H_1H_4 = 12(2\beta + \alpha)H_2H_3 + 6\kappa H_2 + 6\mu H_2''. \tag{4.59iv}$$

Then

$$u(x,t) = w(t) + H_1(t)x^4 + H_2(t)x^3 + H_3(t)x^2 + H_4(t)x,$$

where $w(t)$ satisfies

$$w'' - 24H_1 w = 2\kappa H_3 + 6\alpha H_2 H_4 + 4\beta H_3^2 + 2\mu H_3''. \tag{4.59v}$$

(b) $B(x,t) = H_1(t) + 2H_2(t)x$

$$H_2'' - 12\gamma H_2^2 = 0, \tag{4.60i}$$

$$H_1'' - 12\gamma H_2 H_1 = 0. \tag{4.60ii}$$

Then

$$u(x,t) = w(t) + H_1(t)x + H_2(t)x^2,$$

where $w(t)$ satisfies

$$w'' - 4\gamma H_2 w = 2\kappa H_2 + 4(6\gamma\mu + \beta)H_2^2 + 2\gamma H_1^2. \tag{4.60iii}$$

(c) $\beta = 1 - \alpha$, $\quad B(x,t) = cH_1(t)e^{cx} + cH_2(t)e^{-cx} + H_4(t)$, where $c^2 = -2\gamma$.

$$H_4'' = 0, \tag{4.61i}$$

$$(1 + 2\gamma\mu)H_1'' - 2\gamma c(2-\alpha)H_4 H_1 + 2\gamma\kappa H_1 = 0, \tag{4.61ii}$$

$$(1 + 2\gamma\mu)H_2'' + 2\gamma c(2-\alpha)H_4 H_2 + 2\gamma\kappa H_2 = 0. \tag{4.61iii}$$

Then

$$u(x,t) = w(t) + H_1(t)e^{cx} - H_2(t)e^{-cx} + H_4(t)x,$$

where $w(t)$ satisfies

$$w'' = 2\gamma H_4^2 - 16\gamma^2(1-\alpha)H_1 H_2. \tag{4.61iv}$$

(d) $\alpha = 2$, $\beta = -1$, and $B(x,t) = cH_1(t)e^{cx} + cH_2(t)e^{-cx} + 2H_3(t)x + H_4(t)$, where $c^2 = -2\gamma$.

$$H_3'' - 12\gamma H_3^2 = 0, \tag{4.62i}$$

$$H_4'' - 12\gamma H_3 H_4 = 0, \tag{4.62ii}$$

$$(1 + 2\gamma\mu)H_1'' - 12\gamma H_3 H_1 + 2\kappa\gamma H_1 = 0, \tag{4.62iii}$$

$$(1 + 2\gamma\mu)H_2'' - 12\gamma H_3 H_2 + 2\kappa\gamma H_2 = 0. \tag{4.62iv}$$

Then

$$u(x,t) = w(t) + H_1(t)e^{cx} - H_2(t)e^{-cx} + H_3(t)x^2 + H_4(t)x,$$

where $w(t)$ satisfies

$$w'' - 4\gamma H_3 w = 2\kappa H_3 + 2\gamma H_4^2 + 4(6\gamma\mu - 1)H_3^2 + 16\gamma^2 H_1 H_2. \tag{4.62v}$$

Subcase 4.4.3(iii) $\gamma = \beta + \alpha + 1 = 0$ and

$$\phi = R(u + H_1(t) + H_2(t)e^{Rx} + H_3(t)e^{m_+ x} + H_4(t)e^{m_- x}),$$

where R is a non-zero constant ($\mu R^2 \neq 1$), and $m_\pm = -\frac{1}{2}R(2 + \alpha \pm n)$ where $n = \sqrt{\alpha(\alpha + 4)}$. Solving the invariant surface condition yields

$$\begin{aligned} u(x,t) =& w(t)e^{Rx} - H_1(t) + RH_2(t)e^{Rx} \\ & - \frac{2H_3(t)}{4+\alpha+n}\exp\left\{-\tfrac{1}{2}Rx(2+\alpha+n)\right\} - \frac{2H_4(t)}{4+\alpha-n}\exp\left\{-\tfrac{1}{2}Rx(2+\alpha-n)\right\}. \end{aligned}$$

The solutions are represented in table 4.4.2

Table 4.4.2

Parameters	$H_i(t), w(t)$ satisfy	
$\alpha = -4$, $\beta = 3$	$H_3 = H_4 = H_1'' = 0$ $(1 - \mu R^2)H_2'' + R^2(R^2 H_1 - \kappa)H_2 = 0$ $(1 - \mu R^2)w'' + R^2(R^2 H_1 - \kappa)w = 2\kappa R^2 H_2 - 4R^4 H_1 H_2 + 2\mu R^2 H_2''$	(4.63)
α arbitrary $j = \frac{7}{2} \pm \frac{1}{2}$, $i = \frac{7}{2} \mp \frac{1}{2}$	$H_2 = H_j = H_1'' = 0$ $(4 - \mu(2 + \alpha \pm n)^2 R^2)H_i'' + R^2 H_i \times$ $[R^2 H_1((\alpha^2 + 4\alpha + 2)(2 + \alpha \pm n)^2 - 4) - \kappa(2 + \alpha \pm n)^2] = 0$ $(1 - \mu R^2)w'' + R^2(R^2 H_1 - \kappa)w = 0$	(4.64)
$\alpha = -3$, $\beta = 2$	$H_2 = H_1'' = 0$ $(2 + \mu R^2(1 + i\sqrt{3}))H_3'' - H_1 H_3 R^4(1 - i\sqrt{3}) + \kappa H_3 R^2(1 + i\sqrt{3}) = 0$ $(2 + \mu R^2(1 - i\sqrt{3}))H_4'' - H_1 H_4 R^4(1 + i\sqrt{3}) + \kappa H_4 R^2(1 - i\sqrt{3}) = 0$ $(1 - \mu R^2)w'' + R^2(R^2 H_1 - \kappa)w = 6H_3 H_4 R^4$	(4.65)
$\alpha = -1$, $\beta = 0$	$H_2 = H_1'' = 0$ $(2 + \mu R^2(1 - i\sqrt{3}))H_3'' - H_1 H_3 R^4(1 + i\sqrt{3}) + \kappa H_3 R^2(1 - i\sqrt{3}) = 0$ $(2 + \mu R^2(1 + i\sqrt{3}))H_4'' - H_1 H_4 R^4(1 - i\sqrt{3}) + \kappa H_4 R^2(1 + i\sqrt{3}) = 0$ $(1 - \mu R^2)w'' + R^2(R^2 H_1 - \kappa)w = 0$	(4.66)

The equations that the various $H_i(t)$ satisfy in this subsection are all solvable, and the order in which a list of equations should be solved is from the top down. The only nonlinear equations all have either polynomial solutions (sometimes only in special cases of the parameters) or are equivalent to the Weierstrass elliptic function equation (4.39). The homogeneous part of any linear equation is either of Euler-type, is equivalent to the Airy equation (see Abramowitz and Stegun [1965]),

$$H''(t) + tH(t) = 0,$$

or is equivalent to the Lamé equation (see Ince [1956]),

$$H''(t) - \{k + n(n+1)\wp(t)\}H(t) = 0, \tag{4.67}$$

where $\wp(t)$ satisfies the Weierstrass elliptic function equation (4.39) and k, n are constants. The particular integral of any non-homogeneous linear equation may always be found, up to quadratures, using the method of variation of parameters.

For instance consider the solution of Subcase 4.4.3(ii) part (b) above. There are essentially two separate cases to consider, either (•) $\gamma = 0$ or (••) $\gamma \neq 0$.

(•) $\gamma = 0$. The functions $H_1(t)$ and $H_2(t)$ are trivially found from (4.60i,ii) to be $H_1(t) = c_1 t + c_2$ and $H_2(t) = c_3 t + c_4$, then (4.60iii) becomes

$$w'' = 2\kappa(c_3 t + c_4) + 4\beta(c_3 t + c_4)^2,$$

which may be integrated twice to yield the exact solution

$$u(x,t) = \begin{cases} \dfrac{\kappa}{3c_3^2}(c_3 t + c_4)^3 + \dfrac{\beta}{3c_3^2}(c_3 t + c_4)^4 & \\ \qquad + c_5 t + c_6 + (c_1 t + c_2)x + (c_3 t + c_4)x^2 & \text{if } c_3 \neq 0, \\ (\kappa c_4 + 2\beta c_4^2)t^2 + c_5 t + c_6 + (c_1 t + c_2)x + c_4 x^2 & \text{if } c_3 = 0. \end{cases}$$

(••) $\gamma \neq 0$. Equation (4.60i) may be transformed into the Weierstrass elliptic function equation (4.39), hence $H_2(t)$ has solution $H_2(t) = \wp(t + t_0; 0, g_3)/2\gamma$. Now $H_1(t)$ satisfies the Lamé equation

$$H_1'' - 6\wp(t + t_0; 0, g_3)H_1 = 0,$$

which has general solution

$$H_1(t) = c_1\wp(t + t_0; 0, g_3) + c_2\wp(t + t_0; 0, g_3)\int^{t+t_0} \frac{\mathrm{d}s}{\wp^2(s; 0, g_3)},$$

where c_1 and c_2 are arbitrary constants. Now $w(t)$ satisfies the inhomogeneous Lamé equation

$$w'' - 2\wp(t + t_0; 0, g_3)w = Q(t), \tag{4.68}$$

where $Q(t) = 2\kappa H_2 + 4(6\gamma\mu + \beta)H_2^2 + 2\gamma H_1^2$, with $H_1(t)$ and $H_2(t)$ as above. The general solution of the homogeneous part of this Lamé equation is given by

$$w_{CF}(t) = c_3 w_1(t + t_0) + c_4 w_2(t + t_0),$$

where c_3 and c_4 are arbitrary constants,

$$w_1(t) = \exp\{-t\zeta(a)\}\frac{\sigma(t+a)}{\sigma(t)}, \qquad w_2(t) = \exp\{t\zeta(a)\}\frac{\sigma(t-a)}{\sigma(t)},$$

in which $\zeta(z)$, $\sigma(z)$ are the Weierstrass zeta and sigma functions defined by the differential equations

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = -\wp(z), \qquad \frac{\mathrm{d}}{\mathrm{d}z}\ln\sigma(z) = \zeta(z),$$

together with the conditions

$$\lim_{z\to 0}\left(\zeta(z)-\frac{1}{z}\right)=0,\qquad \lim_{z\to 0}\left(\frac{\sigma(z)}{z}\right)=1,$$

respectively (Whittaker and Watson [1927]), and a is any solution of the transcendental equation

$$\wp(a)=0,$$

i.e., a is a zero of the Weierstrass elliptic function (cf. Ince [1956], p.379). Hence the general solution of (4.68) is given by

$$\begin{aligned} w(t) &= c_3 w_1(t+t_0) + c_4 w_2(t+t_0) \\ &\quad + \frac{1}{W(a)}\int^{t+t_0} \left[w_1(s)w_2(t+t_0) - w_1(t+t_0)w_2(s)\right] Q(s)\,\mathrm{d}s, \end{aligned} \tag{4.69a}$$

where $W(a)$ is the non-zero Wronskian

$$W(a) = w_1 w_2' - w_1' w_2 = -\sigma^2(a)\wp'(a), \tag{4.69b}$$

and $Q(t)$ is defined above. We remark that in order to verify that (4.69) is a solution of (4.68) one uses the following addition theorems for Weierstrass elliptic, zeta and sigma functions

$$\begin{aligned} \zeta(s\pm t) &= \zeta(s)\pm\zeta(t)+\frac{1}{2}\left[\frac{\wp'(s)\mp\wp'(t)}{\wp(s)-\wp(t)}\right], \\ \sigma(s+t)\sigma(s-t) &= -\sigma^2(s)\sigma^2(t)[\wp(s)-\wp(t)] \end{aligned}$$

(see Whittaker and Watson [1927], p.451).

4.5 Discussion

This chapter has seen a classification of symmetry reductions of (4.1) using the classical Lie method and the nonclassical method due to Bluman and Cole. The presence of arbitrary parameters in (4.1) has led to a large variety of reductions using both symmetry methods for various combinations of these parameters. The use of the MAPLE package `diffgrob2` was crucial in this classification procedure. In the classical case it identified the special values of the parameters for which additional symmetries might occur. In the generic nonclassical case the flexibility of `diffgrob2` allowed the fully nonlinear determining equations to be solved completely, whilst in the so–called $\tau=0$ case it allowed the salvage of many reductions from a somewhat intractable calculation.

An interesting aspect of the results in this chapter is that the class of reductions given by the nonclassical method, which are not obtainable using the classical Lie method, were much more plentiful and richer than the analogous results for the third order equation (4.2) we studied in the previous chapter.

An interesting problem this chapter throws open is whether (4.1) is integrable, or perhaps more realistically for which values of the parameters is (4.1) integrable. Effectively, in finding the symmetry reductions of (4.1), we have provided a first step in using the Painlevé conjecture (see §1.6.2). However the presence of so many reductions makes this a lengthy task and so the PDE test due to Weiss, Tabor and Carnevale [1983] is a more inviting prospect (see §1.6.3). It is likely though that extensions of this test, namely "weak Painlevé analysis" (see Ramani, Dorizzi and Grammaticos [1982], Ranada *et al.* [1985]) and "perturbative Painlevé analysis" (see Conte, Fordy and Pickering [1993]) will be necessary (for instance see Gilson and Pickering [1995]). An outline of these methods of Painlevé analysis is given in §1.6, but we shall not pursue this further here.

Part II

Systems of Equations

Chapter Five : Shallow water wave systems

5.1 Introduction

The shallow water wave equation

$$v_{xxt} + \alpha v v_t - \beta v_x \partial_x^{-1} v_t - v_t - v_x = 0, \tag{5.1}$$

where $(\partial_x^{-1} f)(x) = \int_x^\infty f(y)\,\mathrm{d}y$, and α and β are arbitrary, non-zero, constants, can be derived from the classical shallow water theory in the so-called Boussinesq approximation (Espinosa and Fujioka [1994]). Two special cases of (5.1) have attracted some attention in the literature, namely $\alpha = 2\beta$ and $\alpha = \beta$.

Both these special cases are solvable by inverse scattering. Their scattering problems are (i) a second order problem similar to that for the KdV equation if $\alpha = 2\beta$ (Ablowitz *et al.* [1974]), and (ii) a third order problem similar to that for the Boussinesq equation if $\alpha = \beta$ (which has been comprehensively studied by Deift, Tomei and Trubowitz [1982]).

Ablowitz *et al.* [1974] remark that when $\alpha = 2\beta$ (5.1) reduces in the long wave, small amplitude limit to the KdV equation, and also that it responds feebly to short waves, as does the regularized long wave equation (Peregrine [1966])

$$v_{xxt} + v v_x - v_t - v_x = 0, \tag{5.2}$$

sometimes called the Benjamin-Bona-Mahoney equation (cf. Benjamin, Bona and Mahoney [1972]). Indeed equation (5.1) with $\alpha = 2\beta$ may be thought of as the first negative flow of the KdV hierarchy.

N-soliton solutions have been found for both these special cases by Hirota and Satsuma [1976] using Hirota's bilinear technique (Hirota [1980]). Further Hietarinta [1990] showed that (5.1) can be written in Hirota's bilinear form if and only if either $\alpha = 2\beta$ or $\alpha = \beta$.

The equation may also be written in potential form

$$u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0 \tag{5.3}$$

(where $v = u_x$) and this is the form studied by Clarkson and Mansfield [1994b,1995]. A complete catalogue of classical and nonclassical symmetry reductions is given, and in Clarkson and Mansfield [1994b] equation (5.3) is shown to satisfy the necessary conditions of the Painlevé tests (see §1.6) to be completely integrable if and only if either $\alpha = 2\beta$ or $\alpha = \beta$.

Now in order to apply our symmetry methods to the shallow water wave equation (5.1) we must write it as an analytic equation or system. One such adaptation is that previously studied by Clarkson and Mansfield [1994b,1995], equation (5.3). A further four variants are studied here:
(i), the system with two dependent variables

$$v_{xxt} + \alpha v v_t + \beta w v_x - v_t - v_x = 0, \tag{5.4a}$$

$$w_x = v_t, \tag{5.4b}$$

(ii), the system with three dependent variables

$$v = u_x, \tag{5.5a}$$

$$w = u_t, \tag{5.5b}$$

$$v_{xxt} + \alpha v v_t + \beta w v_x - v_t - v_x = 0, \tag{5.5c}$$

(iii), the system derived from a conservation law

$$\psi_x = u_{xxx} + \tfrac{1}{2}(\alpha - \beta)u_x^2 - u_x, \tag{5.6a}$$

$$\psi_t = u_x - \beta u_t u_x, \tag{5.6b}$$

and (iv), a system with two dependent variables only slightly different from system (5.4), and including the dependent variable u,

$$v_{xxt} + \alpha v v_t + \beta u_t v_x - v_t - v_x = 0, \tag{5.7a}$$

$$v = u_x. \tag{5.7b}$$

Some of the work of this chapter has appeared in Priestley and Clarkson [1995], and also is to appear in Clarkson and Priestley [1996].

In §5.2 we apply the classical method to these five variants, in §5.3 the nonclassical method in the generic ($\tau \neq 0$) case, and in §5.4 the direct method. The discussion in §5.5 then highlights the importance of each section and compares the calculations in terms of the differences between the methods and also between the different variants within a method. In particular we witness unusual events in the nonclassical method and realise that the extension of the direct method to systems of equations is not as obvious as one might expect.

5.2 Classical symmetries

In applying the classical method to equations (5.3–5.7) we consider the (relevant subgroup of the) following Lie group of infinitesimal transformations

$$x^* = x + \varepsilon\xi(x, t, \boldsymbol{u}) + O(\varepsilon^2), \tag{5.8i}$$

$$t^* = t + \varepsilon\tau(x, t, \boldsymbol{u}) + O(\varepsilon^2), \tag{5.8ii}$$

$$u^* = u + \varepsilon\phi_1(x, t, \boldsymbol{u}) + O(\varepsilon^2), \tag{5.8iii}$$

$$v^* = v + \varepsilon\phi_2(x, t, \boldsymbol{u}) + O(\varepsilon^2), \tag{5.8iv}$$

$$w^* = w + \varepsilon\phi_3(x, t, \boldsymbol{u}) + O(\varepsilon^2), \tag{5.8v}$$

$$\psi^* = \psi + \varepsilon\phi_4(x, t, \boldsymbol{u}) + O(\varepsilon^2), \tag{5.8vi}$$

where ε is the group parameter, and $\boldsymbol{u}$ consists of an appropriate subset of the set (u, v, w, ψ) dependent on the system involved.

The determining equations are generated as described previously using the MACSYMA package `symmgrp.max`, then simplified using the Kolchin-Ritt algorithm in the MAPLE package `diffgrob2` which yields a triangulation of the system. We simply note the sizes of the determining equations here in the following table

Table 5.2.1

Equation	No. of determining equations
(5.3)	14 linear equations
(5.4)	20 linear equations
(5.5)	24 linear equations
(5.6)	17 linear equations
(5.7)	17 linear equations

The triangulations that `diffgrob2` produces are trivial to solve. The factor $(\alpha - \beta)$ is present sometimes but is only an artefact of the system and does not give further solutions. The infinitesimals that satisfy these determining equations are

$$\xi = c_1 x + c_2, \tag{5.9i}$$

$$\tau = f(t), \tag{5.9ii}$$

$$\phi_1 = -c_1 u + \frac{f(t) + c_1 t}{\beta} + \frac{2c_1 x}{\alpha} + c_3, \tag{5.9iii}$$

$$\phi_2 = -2c_1\left(v - \frac{1}{\alpha}\right), \tag{5.9iv}$$

$$\phi_3 = -\left(\frac{\mathrm{d}f}{\mathrm{d}t} + c_1\right)\left(w - \frac{1}{\beta}\right), \tag{5.9v}$$

$$\phi_4 = -3c_1\psi + \frac{2c_1(t - x - \beta u)}{\alpha} + c_4, \tag{5.9vi}$$

where $f(t)$ is an arbitrary function. We have two canonical reductions.

Reduction 5.2.1. If $c_1 \neq 0$ then we may set $c_1 = 1$, $c_2 = c_3 = c_4 = 0$ and $\dfrac{1}{f(t)} = -\dfrac{\mathrm{d}}{\mathrm{d}t}[\ln g(t)]$, without loss of generality. Hence

$$\begin{aligned} u(x,t) &= U(z)g(t) + \frac{t}{\beta} + \frac{x}{\alpha}, \\ v(x,t) &= g^2(t)V(z) + \frac{1}{\alpha}, \\ w(x,t) &= g_t(t)W(z) + \frac{1}{\beta}, \\ \psi(x,t) &= \Psi(z)g^3(t) - \frac{\beta}{\alpha}U(z)g(t) - \frac{(\alpha+\beta)x}{2\alpha^2}, \end{aligned}$$

where $z = xg(t)$, $U(z)$ satisfies

$$z\frac{\mathrm{d}^4U}{\mathrm{d}z^4} + 4\frac{\mathrm{d}^3U}{\mathrm{d}z^3} + 2\alpha\left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 + \beta U\frac{\mathrm{d}^2U}{\mathrm{d}z^2} + (\alpha+\beta)z\frac{\mathrm{d}U}{\mathrm{d}z}\frac{\mathrm{d}^2U}{\mathrm{d}z^2} = 0, \tag{5.10}$$

and $V(z)$, $W(z)$ and $\Psi(z)$ are given by

$$V = \frac{\mathrm{d}U}{\mathrm{d}z}, \qquad W = U + z\frac{\mathrm{d}U}{\mathrm{d}z}, \qquad \Psi = -\tfrac{1}{3}z\frac{\mathrm{d}^3U}{\mathrm{d}z^3} - \tfrac{1}{6}(\alpha+\beta)z\left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 - \tfrac{1}{3}\beta U\frac{\mathrm{d}U}{\mathrm{d}z}.$$

Clarkson and Mansfield [1994b] apply the ARS algorithm (see §1.6.2) to (5.10) and show it is of Painlevé-type only if (i), $\alpha = 2\beta$ or (ii), $\alpha = \beta$. These two special cases of (5.10) are solvable in terms of solutions of the third Painlevé equation, PIII (see §1.6 and Clarkson and Mansfield [1994b] for more details).

Reduction 5.2.2. If $c_1 = 0$ we set $c_2 = 1$ and $\dfrac{1}{f(t)} = \dfrac{\mathrm{d}g}{\mathrm{d}t}$, without loss of generality

$$\begin{aligned} u(x,t) &= U(z) + c_3g(t) + \frac{t}{\beta}, \\ v(x,t) &= V(z), \\ w(x,t) &= \frac{\mathrm{d}g}{\mathrm{d}t}W(z) + \frac{1}{\beta}, \\ \psi(x,t) &= \Psi(z) + c_4g(t), \end{aligned}$$

where $z = x - g(t)$ and $U(z)$ satisfies

$$\frac{\mathrm{d}^3U}{\mathrm{d}z^3} + \tfrac{1}{2}(\alpha+\beta)\left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 - (1+\beta c_3)\frac{\mathrm{d}U}{\mathrm{d}z} = c_4, \tag{5.11}$$

where c_4 is an arbitrary constant, present because we have integrated this equation once. Then $V(z)$, $W(z)$ and $\Psi(z)$ are given by

$$V = \frac{\mathrm{d}U}{\mathrm{d}z}, \qquad W = c_3 - \frac{\mathrm{d}U}{\mathrm{d}z}, \qquad \Psi_z = c_4 + \beta c_3\frac{\mathrm{d}U}{\mathrm{d}z} - \beta\left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2.$$

If $\alpha+\beta\neq 0$, then (5.11) is equivalent to the Weierstrass elliptic function equation else it is a linear equation.

5.3 Nonclassical symmetries ($\tau \neq 0$)

We now apply the nonclassical method to the systems (5.3)–(5.7) in the generic case $\tau\neq 0$, in which we set $\tau=1$ without loss of generality. We ignore the $\tau=0$ case as the calculations are intractable (because of the lack of higher order t-derivatives in the systems). The number of determining equations and lines of output for these five systems is given in the following table.

Table 5.3.1

Equation	Output	No. of determining equations	
(5.3)	67 lines	8 equations	[3 linear, 5 nonlinear]
(5.4)	583 lines	11 equations	[all nonlinear]
(5.5)	1136 lines	13 equations	[all nonlinear]
(5.6)	181 lines	15 equations	[4 linear, 11 nonlinear]
(5.7)	166 lines	9 equations	[5 linear, 4 nonlinear]

Case 5.3.1 Equation (5.3). Here we summarize the results found in Clarkson and Mansfield [1994b,1995] for comparison with the other systems. There are eight determining equations, three linear and five nonlinear. With a small amount of manipulation it can be shown that there are three cases to consider, (a) $\xi_x\neq 0$, which yields the classical reductions, (b) $\xi_x=0$, with $\alpha=\beta$ and (c) $\xi_x=0$, with $\alpha=-\beta$.

Subcase 5.3.1(i) $\alpha=\beta,\ \xi_x=0$ In this case we have the infinitesimals

$$\xi=\frac{\mathrm{d}f}{\mathrm{d}t},\qquad \phi_1=2P(\zeta)\frac{\mathrm{d}f}{\mathrm{d}t}+\frac{1}{\alpha}, \tag{5.12}$$

where $\zeta=x+f(t)$, $f(t)$ is an arbitrary function and $P(\zeta)$ satisfies

$$\frac{\mathrm{d}^2P}{\mathrm{d}\zeta^2}+\alpha P^2-P=\lambda\zeta+c_1, \tag{5.13}$$

with λ and c_1 arbitrary constants. If $\lambda\neq 0$, then this equation is equivalent to the first Painlevé equation, PI, else it is equivalent to the Weierstrass elliptic function equation (2.38). Solving the invariant surface condition yields the nonclassical reduction

Reduction 5.3.1.

$$u(x,t)=p(\zeta)+q(z)+\frac{t}{\alpha}, \tag{5.14}$$

where $\zeta = x + f(t)$, $z = x - f(t)$, $f(t)$ is an arbitrary function and $P(\zeta) = \dfrac{\mathrm{d}p}{\mathrm{d}\zeta}$ satisfies (5.13) and $Q(z) = \dfrac{\mathrm{d}q}{\mathrm{d}z}$ satisfies

$$\frac{\mathrm{d}^2Q}{\mathrm{d}z^2} + \alpha Q^2 - Q = \lambda z + c_2, \tag{5.15}$$

where c_2 is an arbitrary constant; λ is (effectively) a separation constant.

In particular, if $\lambda = c_1 = c_2 = 0$ (we set $\alpha = 1$ without loss of generality) then equations (5.13) and (5.15) posses the special solutions $P(\zeta) = \frac{3}{2}\operatorname{sech}^2\left(\frac{1}{2}\zeta\right)$ and $Q(z) = \frac{3}{2}\operatorname{sech}^2\left(\frac{1}{2}z\right)$, respectively. Hence the exact solution of (5.3) with $\alpha = \beta = 1$ given by

$$u(x,t) = 3\tanh\left\{\tfrac{1}{2}[x + f(t)]\right\} + 3\tanh\left\{\tfrac{1}{2}[x - f(t)]\right\} + t,$$

is obtained. This is one of the simplest, nontrivial family of solutions of (5.3) with $\alpha = \beta = 1$, obtainable using this reduction, and has a rich variety of qualitative behaviours. This is due to the freedom in the choice of the arbitrary function $f(t)$. One can choose $f_1(t)$ and $f_2(t)$ such that $|f_1(t) - f_2(t)|$ is exponentially small as $t \to -\infty$, yet $f_1(t)$ and $f_2(t)$ are quite different as $t \to +\infty$, so that as $t \to -\infty$ the two solutions are essentially the same, yet as $t \to +\infty$ they are radically different. By a judicious choice of $f(t)$ one can obtain a plethora of different solutions (see the figures in Clarkson and Mansfield [1994b,1995]). It was suggested that to solve (5.3) numerically could pose some fundamental difficulties, as exponentially small perturbations in the initial data cause massive disturbances in the overall solution.

Subcase 5.3.1(ii) $\alpha = -\beta$, $\xi_x = 0$ In this case we have the infinitesimals

$$\xi = \frac{\mathrm{d}f}{\mathrm{d}t}, \qquad \phi_1 = \frac{\mathrm{d}f}{\mathrm{d}t}\eta(z) - \frac{1}{\alpha},$$

where $z = x - f(t)$ and $\eta(z)$ satisfies

$$\frac{\mathrm{d}^4\eta}{\mathrm{d}z^4} - \frac{\mathrm{d}^2\eta}{\mathrm{d}z^2} + \alpha\left[\eta\frac{\mathrm{d}^2\eta}{\mathrm{d}z^2} - \left(\frac{\mathrm{d}\eta}{\mathrm{d}z}\right)^2\right] = 0, \tag{5.16}$$

which is not of Painlevé-type as, whilst it has an acceptable leading order behaviour, does not have a sufficient number of integer resonances. Solving the invariant surface condition yields the nonclassical reduction

Reduction 5.3.2.

$$u(x,t) = U(z) + f(t)\eta(z) - \frac{t}{\alpha}, \tag{5.17}$$

where $U(z)$ satisfies the linear equation

$$\frac{\mathrm{d}^4U}{\mathrm{d}z^4} + (\alpha\eta - 1)\frac{\mathrm{d}^2U}{\mathrm{d}z^2} - \alpha\frac{\mathrm{d}\eta}{\mathrm{d}z}\frac{\mathrm{d}U}{\mathrm{d}z} = \frac{\mathrm{d}^3\eta}{\mathrm{d}z^3} - \frac{\mathrm{d}\eta}{\mathrm{d}z}. \tag{5.18}$$

We note here that the algorithm of Clarkson and Mansfield [1994c], which in this case advocates the removal of u_t, u_{xt} and u_{xxxt} in (5.3), yields an equation of the following form

$$\xi u_{xxxx} + \Delta(x, t, u, u_x, u_{xx}, u_{xxx}) = 0, \tag{5.19}$$

to which we apply the classical method. Since the removal of the highest derivative term assumes $\xi \neq 0$ we should also consider the case when $\xi = 0$, which invalidates the use of u_{xxxx} as the highest derivative term. In fact we have three cases to consider here, as the possibility of the coefficients of consequent highest derivative terms being zero must also be considered.

Subcase 5.3.1(iii) $\xi = 0$, $\phi_{1,u} \neq 0$ In this case we generate six determining equations

$$\phi_1\phi_{1,u}\phi_{1,uuu} - \phi_1\phi_{1,uu}^2 + 2\phi_{1,u}^2\phi_{1,uu} - \phi_{1,tu}\phi_{1,uu} + \phi_{1,tuu}\phi_{1,u} = 0, \tag{5.20i}$$

$$\begin{aligned}&\phi_1\phi_{1,u}\phi_{1,uuuu} - \phi_1\phi_{1,uu}\phi_{1,uuu}\\ &\quad + 3\phi_{1,u}^2\phi_{1,uuu} - \phi_{1,tu}\phi_{1,uuu} + 3\phi_{1,u}\phi_{1,uu}^2 + \phi_{1,tuuu}\phi_{1,u} = 0,\end{aligned} \tag{5.20ii}$$

$$\begin{aligned}&3\phi_{1,u}\phi_{1,uuu}\phi_{1,x} + 3\phi_1\phi_{1,u}\phi_{1,xuuu} - 3\phi_1\phi_{1,uu}\phi_{1,xuu} + 6\phi_{1,u}^2\phi_{1,xuu} - 3\phi_{1,tu}\phi_{1,xuu}\\ &\quad + 9\phi_{1,u}\phi_{1,xu}\phi_{1,uu} + \beta\phi_1\phi_{1,u}\phi_{1,uu} - \phi_{1,u}\phi_{1,uu} + \alpha\phi_{1,u}^3 + 3\phi_{1,xtuu}\phi_{1,u} = 0,\end{aligned} \tag{5.20iii}$$

$$\begin{aligned}&3\phi_{1,u}\phi_{1,uu}\phi_{1,x} + 3\phi_1\phi_{1,u}\phi_{1,xuu} - 3\phi_1\phi_{1,xu}\phi_{1,uu} - \beta\phi_1^2\phi_{1,uu} + \phi_1\phi_{1,uu} + 3\phi_{1,u}^2\phi_{1,xu}\\ &\quad - 3\phi_{1,tu}\phi_{1,xu} + \beta\phi_1\phi_{1,u}^2 + 3\phi_{1,xtu}\phi_{1,u} + \beta\phi_{1,t}\phi_{1,u} - \beta\phi_1\phi_{1,tu} + \phi_{1,tu} = 0,\end{aligned} \tag{5.20iv}$$

$$\begin{aligned}&3\phi_{1,u}\phi_{1,xu}\phi_{1,xx} - \phi_1\phi_{1,uu}\phi_{1,xxx} - \phi_{1,tu}\phi_{1,xxx}\\ &\quad + \beta\phi_1\phi_{1,u}\phi_{1,xx} - \phi_{1,u}\phi_{1,xx} + \alpha\phi_{1,u}\phi_{1,x}^2 + \phi_1\phi_{1,uu}\phi_{1,x} + 3\phi_{1,u}\phi_{1,xxu}\phi_{1,x}\\ &\quad + \phi_{1,tu}\phi_{1,x} + \phi_1\phi_{1,u}\phi_{1,xxxu} - \phi_1\phi_{1,u}\phi_{1,xu} + \phi_{1,xxxt}\phi_{1,u} - \phi_{1,xt}\phi_{1,u} = 0,\end{aligned} \tag{5.20v}$$

$$\begin{aligned}&3\phi_{1,u}\phi_{1,uu}\phi_{1,xx} + 6\phi_{1,u}\phi_{1,xuu}\phi_{1,x} - \alpha\phi_1\phi_{1,uu}\phi_{1,x} + 2\alpha\phi_{1,u}^2\phi_{1,x} - \alpha\phi_{1,tu}\phi_{1,x}\\ &\quad + 3\phi_1\phi_{1,u}\phi_{1,xxuu} - 3\phi_1\phi_{1,xxu}\phi_{1,uu} + 3\phi_{1,u}^2\phi_{1,xxu} - 3\phi_{1,tu}\phi_{1,xxu} + 6\phi_{1,u}\phi_{1,xu}^2\\ &\quad + 2\beta\phi_1\phi_{1,u}\phi_{1,xu} + \alpha\phi_1\phi_{1,u}\phi_{1,xu} - 2\phi_{1,u}\phi_{1,xu} + \alpha\phi_{1,xt}\phi_{1,u} + 3\phi_{1,xxtu}\phi_{1,u} = 0.\end{aligned} \tag{5.20vi}$$

If we **reduce**((5.20iii),[(5.20i),(5.20iv)], $k1$), $k1$ simply tells us that $\alpha = \beta$. Equation (5.20i) may be integrated twice with respect to u to yield

$$\phi_1\phi_{1,u} + \phi_{1,t} = H(x,t)\phi_1 + R(x,t), \tag{5.21}$$

where $H(x,t), R(x,t)$ are arbitrary functions of integration. If we **reduce**((5.20iv),[(5.21)], $k2$), our equation $k2$ may also be integrated with respect to u, and yields

$$(6H_x - 2 + \alpha\phi_1)\phi_1 + 2u(\alpha R + H) = 2K(x,t), \tag{5.22}$$

where $K(x,t)$ is another arbitrary function. Notice that we may assume $3H_x - 1 + \alpha\phi_1 \neq 0$ and $\alpha R + H \neq 0$ else $\phi_{1,u} = 0$. We can now **reduce**((5.21),[(5.22)], $k3$) to yield an equation of the form

$$k3: \qquad (m_1u + m_2)\phi_1 + (m_3u + m_4) = 0,$$

where the m_i are functions of x, t only. The analysis now splits into two, dependent upon whether the highest coefficient of $k3$ is zero or non-zero.

If the highest coefficient is zero then all the $m_i = 0$. We can **reduce**(m3,[m1], $k4$) which leaves

$$k4: \qquad 2(\alpha R + H) + 3(H_{xt} - HH_x) = 0.$$

Notice that now we may assume $H_{xt} - HH_x \neq 0$ else $\alpha R + H = 0$. If we now **reduce** (5.20vi) (with $\alpha = \beta$) progressively with respect to (5.21) then (5.22) and finally $k4$, we are left with an equation of the form $k3$ (replace the m_i by $\widetilde{m}_i$ and call it $\widetilde{k}3$). We note that $\widetilde{m}_1 = 6\alpha H_x(HH_x - H_{xt}) \neq 0$ hence we may **reduce**((5.22),[$\widetilde{k}3$], $k5$) since the highest coefficient of $\widetilde{k}3$ is non-zero. The resultant equation, $k5$, is a fourth order polynomial in u whose coefficients must be zero. One of the coefficients is

$$72\alpha^3 H_x^2 (H_{xt} - H_x H)^2 (\alpha R + H),$$

thus we have a contradiction.

Next, assuming the highest coefficient of $k3$ is non-zero, we may immediately **reduce**((5.22),[$k3$], $k6$) to leave $k6$, a polynomial in u. Requiring its coefficients to be zero leaves (from one of them) $m_1 = 0$. Clearly now $m_2 \neq 0$ and we may assume $\phi_{1,uu} = 0$. Now **reduce**((5.20),[$\phi_{1,uu}$], $k7$) leaves $\phi_{1,u} = 0$ contradicting our original assumption. There are no solutions in this subcase.

Subcase 5.3.1(iv) $\xi = 0$, $\phi_{1,u} = 0$, $1 - \beta\phi_1 \neq 0$ In this case we generate only two determining equations,

$$\beta\phi_{1,t}\phi_{1,x} - \beta\phi_1\phi_{1,xt} + \phi_{1,xt} = 0, \tag{5.23i}$$

$$\begin{aligned} &\beta\phi_{1,t}\phi_{1,xxx} - \beta^2\phi_1^2\phi_{1,xx} + 2\beta\phi_1\phi_{1,xx} - \phi_{1,xx} - \alpha\beta\phi_1\phi_{1,x}^2 + \alpha\phi_{1,x}^2 \\ &\quad - \beta\phi_{1,t}\phi_{1,x} - \beta\phi_1\phi_{1,xxxt} + \phi_{1,xxxt} + \beta\phi_1\phi_{1,xt} - \phi_{1,xt} = 0. \end{aligned} \tag{5.23ii}$$

If we **reduce**((5.23ii),[(5.23i)], $k8$) then ignoring the non-zero factor $1 - \beta\phi_1$, $k8$ looks like

$$k8: \qquad \beta\phi_1\phi_{1,xx} - \phi_{1,xx} + \alpha\phi_{1,x}^2 = 0.$$

Equation (5.23i) and $k8$ are compatible and together reduce (5.23ii) to zero. They have solution, either $\phi_1 = h(t)$ or

$$\phi_1 = \begin{cases} k(t)\mathrm{e}^{c_1 x} + \dfrac{1}{\beta} & \text{if } \alpha + \beta = 0, \\ (c_1 x + c_2)^{\beta/(\alpha+\beta)} + \dfrac{1}{\beta} & \text{otherwise,} \end{cases}$$

where $k(t)$ is an arbitrary function, from which we have three reductions. The first is a classical infinitesimal but the other two are not. They give rise to the following reductions

Reduction 5.3.3. If $\alpha + \beta = 0$

$$u(x,t) = U(x) + K(t)\mathrm{e}^{c_1 x} + \frac{t}{\beta},$$

where $\dfrac{\mathrm{d}K}{\mathrm{d}t} = k(t)$ and $U(x)$ satisfies

$$\alpha\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \alpha c_1\frac{\mathrm{d}U}{\mathrm{d}x} = c_1^3 - c_1,$$

which leads to the exact solution

$$u(x,t) = (c_3 + K(t))\mathrm{e}^{c_1 x} + \frac{(1-c_1^2)x}{\alpha} + \frac{t}{\beta} + c_2. \tag{5.24}$$

Reduction 5.3.4. If $\alpha + \beta \neq 0$

$$u(x,t) = U(x) + \left[(c_1 x + c_2)^{\beta/(\alpha+\beta)} + \frac{1}{\beta}\right]t,$$

where $U(x)$ satisfies

$$(\alpha+\beta)^3(c_1x+c_2)^3\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + \alpha c_1(\alpha+\beta)^2(c_1x+c_2)^2\frac{\mathrm{d}U}{\mathrm{d}x} = c_1(\alpha+\beta)^2(c_1x+c_2)^2 - \alpha c_1^3(2\alpha+\beta),$$

which leads to the exact solution

$$u(x,t) = \begin{cases} (c_4+t)(c_1x+c_2)^{\beta/(\alpha+\beta)} + \dfrac{t}{\beta} + \dfrac{x}{\alpha} + c_5 - \dfrac{\alpha(2\alpha+\beta)c_1}{(\alpha+\beta)^2(\alpha+2\beta)(c_1x+c_2)}, \\ \left[t + c_4 + \dfrac{12c_1}{\alpha}\ln(c_1x+c_2)\right]\Big/(c_1x+c_2) + \dfrac{t}{\beta} + \dfrac{x}{\alpha} + c_5, \end{cases} \tag{5.25}$$

the latter holding if $\alpha + 2\beta = 0$. Both of these exact solutions (5.24) and (5.25) are actually special solutions of classical reductions. Curiously (5.24) is a special case of reduction 5.2.2, whilst (5.25) is a special case of reduction 5.2.1. An easy way to see that these exact solutions are classical is that they satisfy the invariant surface condition with classical infinitesimals (ξ, τ, ϕ_1) given by (5.9), with the constants chosen appropriately.

Subcase 5.3.1(v) $\xi = 0$, $1 - \beta\phi_1 = 0$ Equation (5.20) is identically zero in this instance, and we proceed straight to solving the invariant surface condition. The reduction thus found is classical.

Case 5.3.2 System (5.4). Applying the nonclassical method to (5.4) yields a system of eleven determining equations, all of which are nonlinear. As can be seen from table 5.3.1 it is a large system (the biggest equation has 166 summands!) so it is relegated to Appendix B. The use of symbolic manipulation programs, in particular `diffgrob2` in MAPLE is now essential to make the calculations tractable.

The first equation is

$$\xi\xi_w - \xi_v = 0, \tag{B.1i}$$

and in this preamble we will assume $\xi_v\xi_w \neq 0$ with an ordering $t < x < v < w$ and $\xi < \phi_2 < \phi_3$. If we **reduce**((B.1vi),[(B.1i),(B.1iii)], $k1$) and then use $k1$ to **reduce**((B.1viii),[(B.1i),(B.1iii),$k1$], $k2$) we have in $k2$ a linear equation in ϕ_2, namely

$$k2: \qquad \xi_w\phi_2 - \xi\phi_{2,w} + \phi_{2,v} + \xi_x = 0.$$

We may solve this equation together with (B.1i) using the method of characteristics to give

$$w + \xi v = F(z, x, t),$$

$$\phi_2 = F_x + (v - F_z)\Phi_2(z, x, t),$$

where $z = \xi$. Now since

$$\xi_x = -\frac{F_x}{F_z - v}, \qquad \xi_w = \frac{1}{F_z - v},$$

we may rewrite ϕ_2 as

$$\phi_2 = -\frac{\xi_x + \Phi_2(\xi, x, t)}{\xi_w}. \tag{5.26}$$

We now begin the second step of the nonclassical method, so that v, w once again become dependent variables, x, t independent variables. Recall that the system comprising the original system and the invariant surface conditions is

$$v_{xxt} + \alpha v v_t + \beta w v_x - v_t - v_x = 0, \tag{5.4a}$$

$$w_x = v_t, \tag{5.4b}$$

$$\xi(x, t, v, w)v_x + v_t = \phi_2(x, t, v, w), \tag{5.27i}$$

$$\xi(x, t, v, w)w_x + w_t = \phi_3(x, t, v, w). \tag{5.27ii}$$

We use (B.1i), (5.26) and (5.4b) to leave (5.27i) in the form

$$\xi_v v_x + \xi_w w_x + \xi_x + \Phi_2(\xi, x, t) = 0.$$

We may write this in the form

$$\mathrm{D}_x G(\xi, x, t) = 0,$$

where D_x is the total derivative operator (1.14), $\Phi_2 = \dfrac{G_x}{G_\xi}$, and we may assert that $G_\xi \neq 0$. Integrating this expression gives

$$G(\xi, x, t) = k(t), \tag{5.28}$$

where $k(t)$ is an arbitrary function of integration. (Note that k is a function of t only because v, w are now dependent variables, x, t independent variables.) Equation (5.28) does not infer that ξ is a function of x, t only as this contradicts our assumption to the contrary, but tells us that, as dependent variables, v, w are related by (5.28). However, (5.28) does infer that any solution of the determining equations where ξ is dependent upon v, w may be represented by a solution where ξ is independent of v, w.

This is an extremely important step in finding the complete solution of system (B.1). Attempting to find the solution for $\xi_v \xi_w \neq 0$, even using `diffgrob2`, proved to be lengthy and so far has not been completed due to massive expression swell in the most general cases.

Henceforth we may assume that $\xi_v = \xi_w = 0$, which immediately makes four equations identically zero, so that we are left with seven. They are all still nonlinear. We first **reduce**((B.1x),[(B.1iii),(B.1vi)], $k3$) where $k3$, after factorisation yields

$$k3 = k3_1 k3_2 : \qquad (\phi_{2,v} - \xi\phi_{2,w} + \xi_x)(\xi\phi_{2,ww} - \phi_{2,vw}) = 0.$$

Assuming $k3_2 = 0$ we can integrate with respect to w (since ξ is a function of x, t only). Also we can get a similar expression that can be integrated with respect to v, from (B.1vi) and $k3_2$, and comparing we see

$$\xi\phi_{2,w} - \phi_{2,v} = h(x, t), \tag{5.29}$$

$h(x, t)$ being an arbitrary function of integration. Note that $k3_1 = 0$ is the special case when $h = \xi_x$, so we proceed with the general (5.29). We **reduce**((B.1vii),[(B.1iii),(5.29)], $k4$) to yield

$$k4 : \qquad \xi^2(\xi_x - h)\phi_{2,w} - 2\xi^2 h_x - \xi^2\xi_{xx} - 2\xi_t\xi_x + 2\xi\xi_{xt} + \xi h_t - \xi_t h = 0.$$

The highest derivative term is $\phi_{2,w}$ according to our ordering, so we must consider the case $h = \xi_x$ separately after all. We now split the calculation into five distinct cases, the first two when $h = \xi_x$, the last three when $h \neq \xi_x$.

Subcase 5.3.2(i) $\xi_x = h(x, t) = 0$ With this information (B.1xi) becomes

$$\beta\xi\phi_3 - \alpha\xi^2\phi_2 - \beta\xi_t v + \xi_t = 0, \tag{5.30}$$

and hence removing ϕ_3 and its derivatives from (B.1iv) yields

$$\alpha\xi^2\phi_{2,x} - \beta\xi\phi_{2,t} + \beta\xi_t\phi_2 + (1 - \beta v)\xi_t\phi_{2,v} = 0. \tag{5.31}$$

This is a first order linear partial differential equation which can be solved together with (5.29) by the method of characteristics to give ϕ_2 and hence from (5.30) we have ϕ_3 as well. Substituting these expressions for ϕ_2 and ϕ_3 into the remaining equations we have infinitesimals which give the following invariant surface conditions when $\beta = \pm\alpha$

$$\frac{\mathrm{d}f}{\mathrm{d}t}v_x + v_t = \frac{\mathrm{d}f}{\mathrm{d}t}\Phi(\theta, y), \tag{5.32i}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t}w_x + w_t = \frac{\alpha}{\beta}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2 \Phi(\theta, y) + \left(w - \frac{1}{\beta}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{\mathrm{d}f}{\mathrm{d}t}\right), \tag{5.32ii}$$

where

$$\theta = \left(w + \frac{\mathrm{d}f}{\mathrm{d}t}v - \frac{1}{\beta}\right)\Big/\frac{\mathrm{d}f}{\mathrm{d}t}, \qquad y = \frac{\beta x}{\alpha} + f(t),$$

and $\Phi(\theta, y)$ satisfies

$$\alpha\theta\Phi_y \pm \alpha\theta\Phi\Phi_\theta + \Phi_{yyy} \pm \Phi_\theta\Phi_{yy} + 3\Phi\Phi_y\Phi_{\theta\theta} \pm 3\Phi_{\theta y}\Phi_y + \Phi_\theta^2\Phi_y - \Phi_y \pm \Phi^3\Phi_{\theta\theta\theta}$$
$$+ 3\Phi^2\Phi_{\theta\theta y} \pm 4\Phi^2\Phi_\theta\Phi_{\theta\theta} \pm 3\Phi\Phi_{\theta yy} + 5\Phi\Phi_\theta\Phi_{\theta y} \pm \Phi\Phi_\theta^3 \mp \Phi\Phi_\theta + \alpha\Phi^2 = 0. \tag{5.33$\pm$}$$

At first sight, this equation appears to be difficult to solve in full generality, indeed it is more complex than our original equation. However by using the associated invariant surface conditions (5.32) we can make progress. Remember that in (5.33) the variables v, w are dependent rather than independent variables. If we define

$$\Theta(x,t) = \left(w + \frac{\mathrm{d}f}{\mathrm{d}t}v - \frac{1}{\beta}\right)\Big/\frac{\mathrm{d}f}{\mathrm{d}t},$$

then (5.32) yield

$$\frac{\mathrm{d}f}{\mathrm{d}t}\Theta_x + \Theta_t = \left(1 + \frac{\alpha}{\beta}\right)\frac{\mathrm{d}f}{\mathrm{d}t}\Phi(\theta, y). \tag{5.34}$$

In the case when $\beta = -\alpha$, equation (5.34) is easily solved to give

$$\Theta(x,t) = p(z), \qquad z = x - f(t).$$

Our two earlier variables θ and y are now equivalent to the new variable z defined above so we let $\Phi(\theta, y) = \frac{\mathrm{d}\eta}{\mathrm{d}z}$ (for convenience). The invariant surface conditions (5.32) are now in a form that can be solved and give the following reduction

Reduction 5.3.5. $\beta = -\alpha$

$$v(x,t) = V(z) + f(t)\frac{\mathrm{d}\eta}{\mathrm{d}z},$$
$$w(x,t) = \frac{\mathrm{d}f}{\mathrm{d}t}W(z) - f(t)\frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}\eta}{\mathrm{d}z} - \frac{1}{\alpha},$$

where $z = x - f(t)$, $\eta(z)$ satisfies (5.16), $W(z) = \eta(z) - V(z)$ and $V(z)$ satisfies the linear equation

$$\frac{\mathrm{d}^3V}{\mathrm{d}z^3} + (\alpha\eta - 1)\frac{\mathrm{d}V}{\mathrm{d}z} - \alpha\frac{\mathrm{d}\eta}{\mathrm{d}z}V = \frac{\mathrm{d}^3\eta}{\mathrm{d}z^3} - \frac{\mathrm{d}\eta}{\mathrm{d}z}. \tag{5.35}$$

Whilst the reduction holds, $\eta(z)$ should be pre-determined by the infinitesimals or should in some sense satisfy (5.33–). To show that this is the case, notice that $\Theta(x,t)$ has the same form as θ, though the former is a dependent and the latter an independent variable. We make the hodograph transformation

$$\Phi(\theta, y) = \Omega_s(s, y), \qquad \theta = \Omega(s, y), \tag{5.36}$$

to equation (5.33–), which implements this role reversal to leave a (large) equation for $\Omega(s, y)$. Since $\Phi(\theta, y)$ becomes a function of one variable only we let $y = s$, then $\Omega = \Omega(s)$. The large equation for Ω then simplifies enormously to

$$\frac{\mathrm{d}^4\Omega}{\mathrm{d}s^4} + \alpha\Omega\frac{\mathrm{d}^2\Omega}{\mathrm{d}s^2} - \frac{\mathrm{d}^2\Omega}{\mathrm{d}s^2} - \alpha\left(\frac{\mathrm{d}\Omega}{\mathrm{d}s}\right)^2 = 0. \tag{5.37}$$

We note that since $s = y = z = x - f(t)$ then it is not difficult to show that $\dfrac{\mathrm{d}\eta}{\mathrm{d}z} \equiv \dfrac{\mathrm{d}\Omega}{\mathrm{d}s}$ as required.

There are three special cases of reduction 5.3.5:

(a) if $\eta(z)$ is constant, then we have the classical reduction 5.2.2;

(b) if $V(z) = \eta(z)$, then (5.35) is identically zero, as is $W(z)$, and $\eta(z)$ satisfies (5.16);

(c) if $\eta(z) = c_1 \exp(c_2 z) + (1 - c_2^2)/\alpha$, where we may assume that c_1, c_2 are non-zero constants, then equation (5.16) is satisfied and (5.35) becomes

$$\frac{\mathrm{d}^3V}{\mathrm{d}z^3} + [\alpha c_1 \exp(c_2 z) - c_2^2]\frac{\mathrm{d}V}{\mathrm{d}z} - \alpha c_1 c_2 \exp(c_2 z)V = c_1 c_2 (c_2^2 - 1)\exp(c_2 z). \tag{5.38}$$

Since $V = \exp(c_2 z)$ is a solution of the homogeneous equation we look for solutions of (5.38) in the form $V(z) = g(z)\exp(c_2 z)$. Making the transformation $m(\vartheta) = \dfrac{\mathrm{d}g}{\mathrm{d}z}$ with $\vartheta = \exp(c_2 z)$ yields the Bessel-like equation

$$\vartheta^2\frac{\mathrm{d}^2m}{\mathrm{d}\vartheta^2} + 4\vartheta\frac{\mathrm{d}m}{\mathrm{d}\vartheta} + \left(2 + \frac{\alpha c_1}{c_2^2}\vartheta\right)m = \frac{c_1}{c_2}(c_2^2 - 1). \tag{5.39}$$

Then making the transformation $m(\vartheta) = l(\vartheta)/\vartheta^2$ yields

$$\frac{\mathrm{d}^2l}{\mathrm{d}\vartheta^2} + \frac{\alpha c_1}{c_2^2}\frac{l}{\vartheta} = \frac{c_1}{c_2}(c_2^2 - 1), \tag{5.40}$$

which has general solution (cf. Abramowitz and Stegun [1965])

$$l(\vartheta) = c_3\sqrt{\vartheta}J_1\left(\frac{2\sqrt{\alpha c_1}}{c_2}\sqrt{\vartheta}\right) + c_4\sqrt{\vartheta}\,Y_1\left(\frac{2\sqrt{\alpha c_1}}{c_2}\sqrt{\vartheta}\right) + \frac{c_2}{\alpha}(c_2^2 - 1)\vartheta,$$

where $J_1(z)$ and $Y_1(z)$ are the usual Bessel functions. Hence

$$\begin{aligned}
v(x,t) =& c_5\exp(c_2 z)\int^z \exp(-3c_2\rho/2)J_1\left(\frac{2\sqrt{\alpha c_1}}{c_2}\exp(c_2\rho/2)\right)\mathrm{d}\rho + [c_7 + c_1 c_2 f]\exp(c_2 z)\\
&+ c_6\exp(c_2 z)\int^z \exp(-3c_2\rho/2)Y_1\left(\frac{2\sqrt{\alpha c_1}}{c_2}\exp(c_2\rho/2)\right)\mathrm{d}\rho - \frac{(c_2^2 - 1)}{\alpha},\\
w(x,t) =& -\frac{\mathrm{d}f}{\mathrm{d}t}\exp(c_2 z)\left[c_5\int^z \exp(-3c_2\rho/2)J_1\left(\frac{2\sqrt{\alpha c_1}}{c_2}\exp(c_2\rho/2)\right)\mathrm{d}\rho + c_7\right.\\
&\left.+ c_6\int^z \exp(-3c_2\rho/2)Y_1\left(\frac{2\sqrt{\alpha c_1}}{c_2}\exp(c_2\rho/2)\right)\mathrm{d}\rho + c_1(c_2 f - 1)\right] - \frac{1}{\alpha},
\end{aligned}$$

where c_{3-7} are arbitrary constants, and $z = x - f(t)$.

If $\beta = \alpha$ then the right hand side of (5.34) is no longer zero but yields

$$\frac{\mathrm{d}f}{\mathrm{d}t}\Theta_x + \Theta_t = 2\frac{\mathrm{d}f}{\mathrm{d}t}\Phi(\theta, y). \tag{5.41}$$

First we consider the simplified case $\Phi_y = 0$. Again we make a hodograph-type transformation

$$\Phi(\theta) = 2\frac{\mathrm{d}\Omega}{\mathrm{d}s}, \qquad \theta = 2\Omega(s). \tag{5.42}$$

Thus (5.33+) becomes

$$\frac{\mathrm{d}^4\Omega}{\mathrm{d}s^4} + 2\alpha\Omega\frac{\mathrm{d}^2\Omega}{\mathrm{d}s^2} - \frac{\mathrm{d}^2\Omega}{\mathrm{d}s^2} + 2\alpha\left(\frac{\mathrm{d}\Omega}{\mathrm{d}s}\right)^2 = 0, \tag{5.43}$$

which can be integrated twice to yield either the first Painlevé equation, PI, or the Weierstrass elliptic function equation (2.38), depending on the choice of the constants of integration. However knowing this doesn't make (5.41) much easier to solve. To progress we note, as before, that $\Theta(x,t)$ has the same form as θ. Therefore from (5.41) and by the chain rule and using (5.42) we have

$$\frac{\mathrm{d}\Omega}{\mathrm{d}s}\left(\frac{\mathrm{d}f}{\mathrm{d}t}s_x + s_t - 2\frac{\mathrm{d}f}{\mathrm{d}t}\right) = 0. \tag{5.44}$$

If $\dfrac{\mathrm{d}\Omega}{\mathrm{d}s} = 0$ we obtain a classical reduction, whilst assuming that $\dfrac{\mathrm{d}\Omega}{\mathrm{d}s} \neq 0$ gives

$$s = 2f(t) + G(z), \qquad z = x - f(t). \tag{5.45}$$

Now we are able to solve the invariant surface conditions (5.32), since our z in (5.45) is a characteristic direction in both of equations (5.32). These yield

$$\begin{aligned} v(x,t) &= \Omega(s) + Q(z), \\ w(x,t) &= \frac{\mathrm{d}f}{\mathrm{d}t}[\Omega(s) + H(z)] + \frac{1}{\alpha}, \end{aligned} \tag{5.46}$$

where $z = x - f(t)$. Substituting this into (5.4b) gives

$$\frac{\mathrm{d}H}{\mathrm{d}z} + \frac{\mathrm{d}Q}{\mathrm{d}z} = 2\left(1 - \frac{\mathrm{d}G}{\mathrm{d}z}\right)\frac{\mathrm{d}\Omega}{\mathrm{d}s}. \tag{5.47}$$

There are two possibilities. First if $\dfrac{\mathrm{d}G}{\mathrm{d}z} \neq 1$ we divide by $1 - \dfrac{\mathrm{d}G}{\mathrm{d}z}$ to yield

$$\left(\frac{\mathrm{d}H}{\mathrm{d}z} + \frac{\mathrm{d}Q}{\mathrm{d}z}\right) \Big/ \left(1 - \frac{\mathrm{d}G}{\mathrm{d}z}\right) = 2\frac{\mathrm{d}\Omega}{\mathrm{d}s} = \lambda, \tag{5.48}$$

where λ is a separation constant, since now s and z are independent. Putting $\dfrac{\mathrm{d}\Omega}{\mathrm{d}s} = \tfrac{1}{2}\lambda$ into (5.43) gives $\dfrac{\mathrm{d}\Omega}{\mathrm{d}s} = 0$, a classical reduction. Second if $\dfrac{\mathrm{d}G}{\mathrm{d}z} = 1$ then $s = x + f(t)$

and $\frac{\mathrm{d}H}{\mathrm{d}z}+\frac{\mathrm{d}Q}{\mathrm{d}z}=0$. If we substitute our values of v, w from (5.46) into (5.4a) we find it is necessary to integrate the last expression (with respect to z) to yield $H(z)=-Q(z)$, then $\Omega(s)$ satisfies (5.43). For continuity we let $\Omega(s)\equiv P(\zeta)$, and we have the following reduction

Reduction 5.3.6. $\beta=\alpha$

$$\begin{aligned} v(x,t) &= P(\zeta)+Q(z),\\ w(x,t) &= \frac{\mathrm{d}f}{\mathrm{d}t}[P(\zeta)-Q(z)]+\frac{1}{\alpha}, \end{aligned} \tag{5.49}$$

where $z=x-f(t)$, $\zeta=x+f(t)$, and $P(\zeta)$ and $Q(z)$ satisfy (5.13) and (5.15) respectively. This is the analogue of reduction 5.3.1.

The general case $\Phi=\Phi(\theta,y)$ follows a similar path to the special case just considered. We make a slightly different hodograph transformation than previously, namely

$$\Phi(\theta,y)=2[\Omega_s+\Omega_y],\qquad \theta=2\Omega(s,y), \tag{5.50}$$

which, when using the chain rule, transforms (5.41) to the following equation,

$$[\Omega_s+\Omega_y]\left(\frac{\mathrm{d}f}{\mathrm{d}t}s_x+s_t-2\frac{\mathrm{d}f}{\mathrm{d}t}\right)=0. \tag{5.51}$$

If $\Omega_s+\Omega_y=0$ then we have a classical reduction, whilst assuming $\Omega_s+\Omega_y\neq 0$, leads, as previously, to s being given by (5.45). We may now solve the invariant surface conditions (5.32) so that $v(x,t)$, $w(x,t)$ are given by (5.46), which when substituted into (5.4b) gives (5.47) (though note that now Ω is a function of s and y so we've differentiated partially with respect to s). If $\frac{\mathrm{d}G}{\mathrm{d}z}=1$ then $s=y=x+f(t)$ hence Ω is a function of s only, and we return to the special case. If $\frac{\mathrm{d}G}{\mathrm{d}z}\neq 1$ then we yield (5.48) since s and y are independent of z (again Ω is a function of s and y), and hence $\Omega(s,y)=\frac{1}{2}\lambda s+M(y)$. Substituting this into the transformed (5.33+) doesn't help much. However substituting (5.46) with $\Omega(s,y)=\frac{1}{2}\lambda s+M(y)$ into (5.4a) and requiring the resulting equation to be an ordinary differential equation gives us that either $v(x,t)$ is constant or we reproduce the reduction found in the special case.

Subcase 5.3.2(ii) $h(x,t)=\xi_x$, $h\xi_x\neq 0$ Recall we have equation $k3_1$,

$$k3_1: \qquad \phi_{2,v}-\xi\phi_{2,w}+\xi_x=0,$$

which simplifies both (B.1iii) and $k4$ respectively thus

$$\xi\phi_{3,w}-\phi_{3,v}-\xi_t=0, \tag{5.52i}$$

$$\xi^2\xi_{xx}+\xi_x\xi_t-\xi\xi_{xt}=0, \tag{5.52ii}$$

and (B.1xi) becomes, similar to the previous subcase, an expression linear in both ϕ_2 and ϕ_3 (call it (5.52iii)). If we **reduce**((5.52i),[(5.52iii),$k3_1$], $k5$) we get the condition $\alpha+\beta=0$. Progressing further in this manner we **reduce**((B.1ix),[$k3_1$,(5.52i),(5.52ii)], $k6$), but we are getting into difficulties with lengthy equations, so instead we solve $k3_1$ by the method of characteristics,

$$\phi_2 = \frac{\xi_x}{\xi} w + \Upsilon(z,x,t), \qquad z = w + \xi v,$$

and hence ϕ_3 from (5.52iii). Substituting these new forms of ϕ_2, ϕ_3 into $k6$, we replace w by $z - \xi v$ and equate coefficients of powers of v to zero. The coefficient of v^2 is

$$2\xi\xi_{xx} + \xi_x^2 = 0. \tag{5.53}$$

If we take the **diffSpoly** of (5.53) and (5.52ii) then **reduce** the result with respect to both of these equations we are left with $\xi_x = 0$, in contrast to our assumptions. Thus there are no reductions in this subcase.

Subcase 5.3.2(iii) $\xi_x = 0, h \neq 0$ We can integrate $k4$ with respect to w to give ϕ_2 up to an arbitrary function in (x,t,v). Equation (5.29) gives us ϕ_2's w-dependence, and we are left with

$$\phi_2 = \frac{(w+\xi v)}{\xi^2 h}(\xi h_t - \xi_t h - 2\xi^2 h_x) - hv + a(x,t),$$

where $a(x,t)$ is our arbitrary function. We substitute all this into our original determining equations to see what we have left. There remains four equations; from (B.1iii) and (B.1iv) we have

$$\xi\phi_{3,w} - \phi_{3,v} - \xi_t + \xi h = 0, \tag{5.54i}$$

$$\phi_{3,x} + \phi_2\phi_{3,v} - \phi_{2,v}\phi_3 - \phi_{2,t} = 0 \tag{5.54ii}$$

(ϕ_2 has been left unevaluated in (5.54ii) for convenience), and (B.1ix) and (B.1xi) both give quite lengthy equations. We now use the **diffSpoly** procedure on (5.54i,ii) and **reduce** the result with respect to both (5.54i) and (5.54ii), to give us the compatibility condition between the two equations. This gives us an expression for $\phi_{3,w}$, which, in a way similar to that above for ϕ_2 gives us ϕ_3 in terms of another arbitrary function of x, t,

$$\phi_3 = \frac{(3\xi^2 h_x - \xi h^2 + \xi_t h)}{\xi h} w + \frac{3\xi^2 h_x}{h} v + b(x,t).$$

We now substitute all this into the remaining equations and equate coefficients of powers of v, w to zero, which yields

$$\alpha + \beta = 0, \qquad h = h(t), \qquad a(x,t) = \frac{h}{\alpha\xi}, \qquad b(x,t) = \frac{1}{\alpha}\left(\frac{2\xi_t}{\xi} - \frac{h_t}{h}\right),$$

together with a condition on h and ξ,

$$\xi h^2 + \xi_t h - \xi h_t = 0. \tag{5.55}$$

Hence we have infinitesimals,

$$\xi = \frac{\mathrm{d}f}{\mathrm{d}t}, \qquad \phi_2 = h\left(w + \frac{1}{\alpha}\right) \Big/ \frac{\mathrm{d}f}{\mathrm{d}t}, \qquad \phi_3 = \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln \frac{\mathrm{d}f}{\mathrm{d}t}\right) - h\right]\left(w + \frac{1}{\alpha}\right),$$

where $h(t)$, $f(t)$ satisfy (5.55) and $\beta = -\alpha$. This leads us to the special case (b) of reduction 5.3.5.

Subcase 5.3.2(iv) $h = 0$, $\xi_x \neq 0$ There are no infinitesimals that satisfy both the determining equations and the conditions imposed in this subcase. However the process to show this involves a large number of subcases and thus needs a lengthy exposition. Since all the results are negative anyway, for reasons of conciseness we do not describe the process here.

Subcase 5.3.2(v) $h \neq \xi_x$, $h\xi_x \neq 0$ As in Subcase 5.3.2(iii) we can find ϕ_2 from (5.29) and $k4$ up to an arbitrary function in (x, t). Using similar ideas we can also then find $\phi_{3,w}$, but instead of then using this subcase's equivalent to (5.54) to find ϕ_3 in terms of another arbitrary function, we can use (B.1xi) which has been filled with all the information so far found to give us an expression for ϕ_3 in terms of the same arbitrary function that dictates ϕ_2's form. With ϕ_2 and ϕ_3 found, we substitute them into (B.1iii), and factorise to give

$$\xi^5(h - 2\xi_x)(h - \xi_x)^2(\alpha + \beta) = 0. \tag{5.56}$$

This leaves us with two further cases to consider, namely when $h = 2\xi_x$ and when $\beta = -\alpha$. However, we find that $\beta = -\alpha$ is a spurious special case that is no different from the generic one, i.e. $h = 2\xi_x$. We change our expressions for ϕ_2 and ϕ_3 accordingly, and substitute for them in (B.1xi), which factors to gives us two options, either

$$5\xi^2\xi_{xx} + 4\xi_t\xi_x - 4\xi\xi_{xt} = 0, \tag{5.57i}$$

$$\text{or,} \qquad (4\alpha + 3\beta)\xi_x\xi_t + (5\alpha + 6\beta)\xi^2\xi_{xx} - (4\alpha + 3\beta)\xi\xi_{xt} = 0. \tag{5.57ii}$$

Starting with (5.57ii), we substitute ϕ_2 and ϕ_3 into (B.1iv) and **reduce** the result with respect to (5.57ii) (using an ordering which eliminates t-derivatives first) to give

$$(4\alpha + 3\beta)(3\beta - 2\alpha)(9(\alpha + \beta)\xi\xi_{xx}^2 - 9(\alpha + \beta)\xi\xi_x\xi_{xxx} - (10\alpha + 12\beta)\xi_x^2\xi_{xx}) = 0. \tag{5.58}$$

Thus there are three more cases to consider from (5.58), and choosing the last bracket to be zero, we use **diffSpoly** and **reduce** on this bracket and (5.57ii) to find a compatibility condition

$$(\alpha + \beta)(5\alpha + 6\beta)\xi_{xx} = 0, \tag{5.59}$$

which gives three more cases! However working through each case in turn, including (5.57i), we find that conditions on α, β are special cases of the general case $\xi_{xx} = 0$, then we find $\xi = f(t)x$ (up to a translational symmetry in x), which yields the classical reduction 5.2.1.

Case 5.3.3 System (5.5). There is a difficulty in applying the nonclassical method to system (5.5) using the algorithm of Clarkson and Mansfield [1994c] since the removal of t-derivatives leaves (5.5a,b) in the form

$$\begin{aligned} v &= u_x, \\ w &= \phi_1 - \xi u_x. \end{aligned} \tag{5.60}$$

The only derivative term in each equation is u_x, so we cannot choose a sufficient number of derivative terms to substitute back for (the vi, in the notation of `symmgrp.max`, see §1.8). We can rewrite (5.5) as

$$v = u_x, \tag{5.61a}$$

$$w_x = v_t, \tag{5.61b}$$

$$v_{xxt} + \alpha v v_t + \beta w v_x - v_t - v_x = 0, \tag{5.61c}$$

and apply the nonclassical method to this system (the vi are now u_x, w_x, v_{xxx}) but the corresponding system of determining equations is enormous (see table 5.3.1). Alternatively we can assert from (5.60) that $w = \phi_1 - \xi v$, remembering that this expression holds for u, v, w dependent variables (i.e. $w(x,t) = \phi_1(x,t,u(x,t),v(x,t),w(x,t)) - \xi(x,t,u(x,t),v(x,t),w(x,t))v(x,t)$). With w in this form (5.5) becomes (5.7), though when applying the nonclassical method the infinitesimals now depend on w also. Twelve determining equations are generated comprising the nine determining equations found in Case 5.3.5 and the three single-term equations

$$\xi_w = \phi_{1,w} = \phi_{2,w} = 0.$$

There are no ϕ_3 terms. This leads us to conclude that this case is equivalent to Case 5.3.5.

Case 5.3.4 System (5.6). We generate 15 determining equations. Equations two and three look like

$$2\beta\phi_{1,v} + 3\xi_u = 0, \tag{5.62i}$$

$$2\beta\phi_{1,v} + \xi_u = 0, \tag{5.62ii}$$

from which we deduce that $\xi_u = \phi_{1,v} = 0$. Under these conditions only nine equations remain

$$\xi_v = 0, \tag{5.63i}$$

$$\phi_{1,uu} = 0, \tag{5.63ii}$$

$$\phi_{1,xu} - \xi_{xx} = 0, \tag{5.63iii}$$

$$\phi_{4,v} - 2\phi_{1,u} + \xi_x = 0, \tag{5.63iv}$$

$$2\beta\phi_{4,v} - 3\beta\phi_{1,u} + 3\beta\xi_x + \alpha\phi_{1,u} + \alpha\xi_x = 0, \tag{5.63v}$$

$$\xi\phi_1\phi_{4,u} + \xi\phi_{4,t} - \xi_t\phi_4 + \beta\xi\phi_1\phi_{1,x} - \xi\phi_{1,x} = 0, \tag{5.63vi}$$

$$2\xi_x\phi_4 + \phi_4\phi_{4,v} - \phi_{1,u}\phi_4 + \xi\phi_{4,x} - \xi\phi_{1,xxx} + \xi\phi_{1,x} = 0, \tag{5.63vii}$$

$$\begin{aligned}&\beta\xi\phi_1\phi_{4,v} - \xi\phi_{4,v} + \xi^2\phi_{4,u} + \beta\xi^2\phi_{1,x}\\ &\quad - 2\beta\xi\phi_1\phi_{1,u} + \xi\phi_{1,u} - \beta\xi\phi_{1,t} + \beta\xi\xi_x\phi_1 + \beta\xi_t\phi_1 - \xi\xi_x - \xi_t = 0,\end{aligned} \tag{5.63viii}$$

$$\begin{aligned}&2\xi\xi_x - \xi\alpha\phi_{1,x} - 3\xi\phi_{1,xxu} + \xi\xi_{xxx}\\ &\quad + \xi\beta\phi_{1,x} + \xi\phi_{4,u} + 2\beta\xi_x\phi_1 - \phi_{4,v} - \beta\phi_1\phi_{1,u} - 2\xi_x + \beta\phi_1\phi_{4,v} + \phi_{1,u} = 0.\end{aligned} \tag{5.63ix}$$

There are a number of separate cases to follow, and within these we find both classical reductions, but also the following infinitesimals, when $\beta = \alpha$

$$\xi = \frac{\mathrm{d}f}{\mathrm{d}t}, \qquad \phi_1 = 2\frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}p}{\mathrm{d}\zeta} + \frac{1}{\alpha}, \qquad \phi_4 = 2\frac{\mathrm{d}f}{\mathrm{d}t}\left(\frac{\mathrm{d}^3p}{\mathrm{d}\zeta^3} - \frac{\mathrm{d}p}{\mathrm{d}\zeta} + \mu - \lambda f(t)\right),$$

where μ, λ are arbitrary constants, $f(t)$ is an arbitrary function, $\zeta = x + f(t)$ and $P(\zeta) = \dfrac{\mathrm{d}p}{\mathrm{d}\zeta}$ satisfies (5.13). This then gives rise to the following nonclassical reduction (after renaming $2\mu = c_2 - c_1$)

Reduction 5.3.7.

$$u(x,t) = p(\zeta) + q(z) + \frac{t}{\alpha},$$
$$\psi(x,t) = \Psi(z) + \frac{\mathrm{d}^2p}{\mathrm{d}\zeta^2} - p(\zeta) + (c_2 - c_1)f(t) - \lambda f^2(t),$$

where $z = x - f(t)$, $Q(z) = \dfrac{\mathrm{d}q}{\mathrm{d}z}$ satisfies (5.15) and $\Psi(z) = \dfrac{\mathrm{d}^2q}{\mathrm{d}z^2} - q(z)$. This is the analogue of reduction 5.3.1.

Case 5.3.5 System (5.7). We generate nine determining equations, five of which are linear, the remainder being nonlinear and slightly more lengthy. The first five are

$$\xi_v = 0, \tag{5.64i}$$

$$\phi_{1,v} = 0, \tag{5.64ii}$$

$$\phi_{2,vvv} = 0, \tag{5.64iii}$$

$$3\phi_{2,vv} - 4\xi_u = 0, \tag{5.64iv}$$

$$\phi_2 - \phi_{1,x} - v\phi_{1,u} + v\xi_x + v^2\xi_u = 0. \tag{5.64v}$$

From these five equations we can deduce that

$$\xi = f(x,t), \qquad \phi_1 = k(x,t,u), \qquad \phi_2 = [k_u - f_x]v + k_x. \tag{5.65}$$

Substituting this into the sixth equation simply leaves $k_{uu} = 0$, hence $\phi_{1,uu} = 0$ and $\phi_{2,uu} = \phi_{2,uv} = 0$. If we now use this information in the remaining three equations the system we must solve looks like

$$\xi_v = 0, \qquad \xi_u = 0, \qquad \phi_{1,v} = 0, \tag{5.66i,ii,iii}$$

$$\phi_{2,vv} = 0, \qquad \phi_{1,uu} = 0, \qquad \phi_{2,uu} = 0, \tag{5.66iv,v,vi}$$

$$\phi_{2,uv} = 0, \qquad \phi_2 - \phi_{1,x} - v\phi_{1,u} + v\xi_x = 0, \tag{5.66vii,viii}$$

$$\xi^2\xi_{xx} - 2\xi^2\phi_{2,xv} - \xi^2\phi_{2,u} - 2\xi\xi_{xt} + \xi\phi_{2,tv} - \xi_t\phi_{2,v} + 2\xi_t\xi_x = 0, \tag{5.66ix}$$

$$\begin{aligned}
&\xi\xi_x - \xi\phi_{2,tu} - \xi_t\xi_{xx}\\
&\quad + \xi_t\phi_{2,u} + \xi\xi_{xxt} + \xi^2\phi_{2,xxv} + 2\xi^2\phi_{2,xu} - 2\xi\phi_{2,xtv} - 2\xi^2\xi_x + \beta\xi_t\phi_1 - 2\xi\phi_{2,v}\phi_{2,xv}\\
&\quad - \beta\xi\phi_{1,t} + 2\xi\xi_x\phi_{2,u} + 2\xi_t\phi_{2,xv} + \xi\xi_{xx}\phi_{2,v} - \xi\phi_{1,u}\phi_{2,u} + \alpha\xi^2\phi_2 - \xi\xi_x\xi_{xx}\\
&\quad + 2\xi\xi_x\phi_{2,xv} + \beta\xi^2\phi_{1,u}v + \beta\xi^2\xi_x v - \xi_t + 2\alpha\xi^2\xi_x v - \beta\xi\xi_x\phi_1 - \beta\xi\phi_1\phi_{1,u} = 0,
\end{aligned} \tag{5.66x}$$

$$\begin{aligned}
&\xi^2\phi_{2,u}v - \xi^2\phi_{2,xxu}v - \xi\xi_{xx}\phi_{2,x} + \xi\phi_2\phi_{2,xxv} + \xi\phi_1\phi_{2,xxu} + \xi\phi_{1,xx}\phi_{2,u} - 2\xi_t\phi_{2,xu}v\\
&\quad + 2\xi\phi_{2,xtu}v - \xi_t\phi_{2,xx} - \xi\phi_{2,x} + \xi_t\phi_2 + \xi\phi_{2,xxt} - \xi\phi_{2,t} - \alpha\xi^2\phi_{2,u}v^2 - \beta\xi^2\phi_{2,u}v^2\\
&\quad - \beta\xi^2\phi_{2,x}v + \beta\xi\phi_1\phi_{2,x} + \alpha\xi\phi_1\phi_{2,u}v + \beta\xi\phi_1\phi_{2,u}v + \alpha\xi\phi_{2,t}v - \alpha\xi_t\phi_2 v\\
&\quad - 2\xi\xi_x\phi_2 + 2\alpha\xi\xi_x\phi_2 v - 2\xi\xi_{xx}\phi_{2,u}v + 2\xi\phi_{1,xu}\phi_{2,u}v - 2\xi\xi_x\phi_{2,xu}v + 2\xi\phi_{1,u}\phi_{2,xu}v\\
&\quad + 2\xi\phi_{2,u}\phi_{2,xv}v - \xi\phi_1\phi_{2,u} + 2\xi\phi_{1,x}\phi_{2,xu} + \alpha\xi\phi_2^2 - \xi\phi_{2,u}v + 2\xi\phi_{2,xv}\phi_{2,x} = 0.
\end{aligned} \tag{5.66xi}$$

We proceed by writing $k(x,t,u) = g(x,t)u + h(x,t)$, substituting our expressions from (5.65) into the system (5.66) and taking coefficients of powers of u and v to be zero. The first eight are then identically zero, but the last three give us 10 equations. We find both the classical reductions as we'd expect, but also two nonclassical reductions. The first is given by the following set of infinitesimals, when $\beta = \alpha$

$$\xi = \frac{\mathrm{d}f}{\mathrm{d}t}, \qquad \phi_1 = 2\frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}p}{\mathrm{d}\zeta} + \frac{1}{\alpha}, \qquad \phi_2 = 2\frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}^2p}{\mathrm{d}\zeta^2},$$

where $\zeta = x + f(t)$, $f(t)$ is an arbitrary function and $P(\zeta) = \dfrac{\mathrm{d}p}{\mathrm{d}\zeta}$ satisfies (5.13). Hence we have the following reduction

Reduction 5.3.8. $\beta = \alpha$

$$u(x,t) = p(\zeta) + q(z) + \frac{t}{\alpha},$$
$$v(x,t) = V(z) + P(\zeta),$$

where $z = x - f(t)$, $Q(z) = \dfrac{\mathrm{d}q}{\mathrm{d}z}$ satisfies (5.15) and $V(z) = Q(z)$. This is the analogue of reduction 5.3.1. The second reduction arises from these infinitesimals, when $\beta = -\alpha$

$$\xi = \frac{\mathrm{d}f}{\mathrm{d}t}, \qquad \phi_1 = \frac{\mathrm{d}f}{\mathrm{d}t}\eta(z) - \frac{1}{\alpha}, \qquad \phi_2 = \frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}\eta}{\mathrm{d}z},$$

where $\eta(z)$ satisfies (5.16). Hence we obtain the nonclassical reduction

Reduction 5.3.9. $\beta = -\alpha$

$$u(x,t) = U(z) + f(t)\eta(z) - \frac{t}{\alpha},$$
$$v(x,t) = V(z) + f(t)\frac{\mathrm{d}\eta}{\mathrm{d}z},$$

where $z = x - f(t)$, $U(z)$ satisfies (5.18) and $V(z) = \dfrac{\mathrm{d}U}{\mathrm{d}z}$. This is the analogue of reduction 5.3.2.

5.4 Direct Method

For scalar equations the ansatz

$$u(x,t) = H(x,t,U(z)), \tag{5.67}$$

where $z = z(x,t)$, may often be easily simplified to $u(x,t) = A(x,t)U(z) + B(x,t)$ without loss of generality (the Harry-Dym equation being a notable exception (cf. Clarkson and Kruskal [1989])). In the analysis of *systems* of partial differential equations (here we only consider systems with two independent variables) it is not always so easy to ascertain a simplification of the original ansätze, though the need for some simplification is obvious enough!

One reason is that the ansatz necessary to find one reduction may be more complicated than is necessary for another. For instance to find the classical reduction 5.2.1 for (5.6) $\psi(x,t)$ must certainly be linear in *both* $\Psi(z)$ *and* $U(z)$, whilst to find reduction 5.2.2, $\psi(x,t)$ need only be linear in $\Psi(z)$. This difference will only be evident deep into the calculation.

There are three freedoms that can, and should, be employed to reduce the complexity of the calculations without loss of generality. We describe the freedoms for a system of partial differential equations in two independent variables (x,t) and n dependent variables. We assume that the ansätze are of the form

$$\boldsymbol{u}(x,t) = \boldsymbol{H}(x,t,\boldsymbol{U}(z)), \tag{5.68}$$

where $z = z(x,t)$, $\boldsymbol{u} = (u_1,...,u_n)$, $\boldsymbol{H} = (H_1,...,H_n)$ and $\boldsymbol{U} = (U_1,...,U_n)$. The first two freedoms are described for a general dependent symmetry variable $U_m(z)$ and apply to each of the dependent symmetry variables.

Freedom (a). (*Translating*). If, for every $u_j(x,t)$ such that $\dfrac{\partial u_j}{\partial U_m} \neq 0$, $u_j(x,t)$ may be written in the form

$$u_j(x,t) = \widetilde{H}_j(x,t,\boldsymbol{U}_m^*(z)) + A_j(x,t)[U_m(z) + \Omega(z,\boldsymbol{U}_m^*)], \tag{5.69}$$

where $\boldsymbol{U}_m^* = (U_1, U_2, \ldots, U_{m-1}, U_{m+1}, \ldots, U_n)$, then we may set $\Omega(z, \boldsymbol{U}_m^*) \equiv 0$. This is allowed since we can translate $U_m(z) \to U_m(z) - \Omega(z, \boldsymbol{U}_m^*)$, or alternatively we set $\widehat{U}_m(z) = U_m(z) + \Omega(z, \boldsymbol{U}_m^*)$ and then rename $\widehat{U}_m(z) = U_m(z)$.

Freedom (b). (*Scaling*). If, for every $u_j(x,t)$ such that $\dfrac{\partial u_j}{\partial U_m} \neq 0$, $u_j(x,t)$ may be written in the form

$$u_j(x,t) = \widetilde{H}_j(x, t, \boldsymbol{U}_m^*(z)) + A_j(x,t) U_m(z) \Omega(z, \boldsymbol{U}_m^*), \tag{5.70}$$

where $\boldsymbol{U}_m^*$ is as above, then we may set $\Omega(z, \boldsymbol{U}_m^*) \equiv 1$. This is allowed since we can scale $U_m(z) \to U_m(z)/\Omega(z, \boldsymbol{U}_m^*)$, or alternatively we set $\widehat{U}_m(z) = U_m(z)\Omega(z, \boldsymbol{U}_m^*)$ and then rename $\widehat{U}_m(z) = U_m(z)$.

Freedom (c). (*Inverting*). If $z(x,t)$ is determined by an equation of the form $\Omega(z) = z_0(x,t)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$ (by substituting $z \to \Omega^{-1}(z)$).

The freedoms (a) and (b), of translation and scaling, may be applied once without loss of generality during the calculation for each dependent symmetry variable, and freedom (c) may also be applied once without loss of generality. However here we make some remarks.

Remark 5.4(i). These freedoms may be applied *a posteriori* if they do not cause a loss of generality; they simply tidy up the reduction. This could be called the freedom of hindsight!

Remark 5.4(ii). It may be possible to use freedoms (a) and (b) without such strong conditions on the form of the $u_j(x,t)$. That is if a particular $u_j(x,t)$ can not be written in the required form, (5.69) or (5.70), but parts of it are yet to be determined then it may be possible to exercise the freedom, thus affecting the ultimate form of these undetermined parts (when they are determined) but without losing generality. Examples of this are given in the following calculations (and also in the next chapter).

Remark 5.4(iii). We will use the upper case Greek alphabet (without subscripts) to represent undetermined functions, and such that under any operation the result is given the same letter. Once we give these letters a subscript or use the lower case Greek alphabet we keep track of our operations.

Remark 5.4(iv). We do not consider the singular cases $z_x = 0$ or $z_t = 0$ in this calculation i.e. we assume $z_x z_t \neq 0$.

Remark 5.4(v). The application of the direct method to systems of equations is not entirely clear. The steps carried out to obtain the reductions in this section are explained but the method is formalised only in the next section, in the light of this one.

We now apply the direct method to our various systems.

Case 5.4.1 Equation (5.3). We assume the solution of (5.3) takes the form

$$u(x,t) = H(x,t,U(z)), \tag{5.71}$$

where $z = z(x,t)$, and require that when this is substituted into (5.3), the result is an ordinary differential equation. On substitution we get

$$H_U z_x^3 z_t U'''' + H_{UU} z_x^3 z_t U'U''' + ... = 0.$$

If $z_x z_t \neq 0$ we divide by the coefficient of U'''', and require the new coefficient of $U'U'''$ to be a function of (z,U). This gives

$$\frac{H_{UU}}{H_U} = \frac{\Gamma_{UU}(z,U)}{\Gamma_U(z,U)},$$

which may be integrated twice with respect to U to yield

$$H(x,t,U(z)) = A(x,t)\Gamma(z,U) + B(x,t),$$

and without loss of generality we can assume $\Gamma(z,U) = U(z)$. With this ansatz equation (5.3) reads

$$\begin{aligned}
&Az_x^3 z_t U'''' + [3Az_t z_x z_{xx} + A_t z_x^3 + 3Az_x^2 z_{xt} + 3A_x z_t z_x^2]U''' + (\alpha+\beta)A^2 z_t z_x^2 U'U'' \\
&+ [\alpha AA_x z_t z_x + \beta AA_t z_x^2]UU'' + [Az_t z_{xxx} + 3A_t z_x z_{xx} + 3Az_{xt} z_{xx} + 3A_x z_t z_{xx} \\
&+ 3A_{xt} z_x^2 - Az_x^2 + 3Az_x z_{xxt} + 6A_x z_x z_{xt} + 3A_{xx} z_t z_x - Az_t z_x + \alpha AB_x z_t z_x \\
&+ \beta AB_t z_x^2]U'' + [\alpha A^2 z_x z_{xt} + \beta A^2 z_t z_{xx} + \alpha AA_t z_x^2 + (\alpha+2\beta)AA_x z_t z_x](U')^2 \\
&+ [\beta AA_t z_{xx} + (\alpha+2\beta)A_t A_x z_x + \alpha AA_{xt} z_x + \alpha AA_x z_{xt} + \beta AA_{xx} z_t + \alpha A_x^2 z_t]UU' \\
&+ [A_t z_{xxx} + 3A_{xt} z_{xx} - Az_{xx} - 2A_x z_x + 3A_{xxt} z_x - A_t z_x + Az_{xxxt} \\
&+ 3A_x z_{xxt} + 3A_{xx} z_{xt} - Az_{xt} + A_{xxx} z_t - A_x z_t + \beta AB_t z_{xx} + 2\beta A_x B_t z_x \\
&+ \alpha A_t B_x z_x + \alpha AB_{xt} z_x + \alpha AB_x z_{xt} + \beta AB_{xx} z_t + \alpha A_x B_x z_t]U' \\
&+ [\alpha A_x A_{xt} + \beta A_t A_{xx}]U^2 + [A_{xxxt} - A_{xx} - A_{xt} + \alpha A_x B_{xt} + \alpha A_{xt} B_x \\
&+ \beta A_t B_{xx} + \beta A_{xx} B_t]U + [B_{xxxt} + \alpha B_x B_{xt} + \beta B_t B_{xx} - B_{xt} - B_{xx}] = 0.
\end{aligned} \tag{5.72}$$

We use the coefficient of U'''' to be the normalising coefficient, so that the coefficient of $U'U''$ yields

$$Az_x^3 z_t \Gamma(z) = (\alpha+\beta)A^2 z_x^2 z_t, \tag{5.73}$$

which splits the calculation into two.

Subcase 5.4.1(i) $\alpha + \beta \neq 0$ In (5.73) we can use freedom (b) to yield $A = z_x$. The coefficients of UU'', U''' and $(U')^2$ yield

$$z_x^2 z_t \Gamma_1(z) = \alpha z_t z_{xx} + \beta z_x z_{xt}, \tag{5.74i}$$

$$z_x^2 z_t \Gamma_2(z) = 3z_t z_{xx} + 2z_x z_{xt}, \tag{5.74ii}$$

$$z_x^2 z_t \Gamma_3(z) = (\alpha+3\beta)z_t z_{xx} + 2\alpha z_x z_{xt}. \tag{5.74iii}$$

We can remove the z_{xt} term from (5.74i) using (5.74ii) to yield

$$z_x^2 z_t(2\Gamma_1(z) - \beta\Gamma_2(z)) = (2\alpha - 3\beta)z_t z_{xx}.$$

If $2\alpha - 3\beta \neq 0$ we can integrate this twice with respect to x to yield, after using freedom (c), $z = \theta(t)x + \sigma(t)$. If $2\alpha - 3\beta = 0$, then equations (5.74) are all essentially the same, and give, on integrating once with respect to x,

$$\Gamma_4(z) = \delta(t)z_x^3 z_t^2. \tag{5.75}$$

The arbitrary function $\delta(t)$ represents the function of integration and is non-zero (note that we have to use exponentiation to achieve (5.75)); also $\Gamma_4(z) \neq 0$. This equation doesn't lend itself to further solution, so we turn to the coefficients of UU' and U^2 which yield

$$z_x^4 z_t\Gamma_5(z) = 12z_x z_{xx} z_{xt} + 3z_t z_{xx}^2 + 3z_x^2 z_{xxt} + 2z_x z_t z_{xxx}, \tag{5.76i}$$

$$z_x^4 z_t\Gamma_6(z) = 2z_{xt} z_{xxx} + 3z_{xx} z_{xxt}. \tag{5.76ii}$$

Using `diffgrob2`, we **reduce**((5.76i),[(5.75)], $k1$) and **reduce**((5.76ii),[(5.75)], $k2$). The highest coefficient of $k1$ is non-zero so we **reduce**($k2$,[$k1$], $k3$) and also take the **diffSpoly** of (5.75) and $k1$, then **reduce** the result with respect to both these equations (call the result $k4$). Both $k3$ and $k4$ have z_{xx} as their highest derivative term, and the highest coefficient of $k3$ is non-zero. Thus their (algebraic) reduction is allowed, without worrying about whether the separant is zero. The result of this reduction is an equation of the form

$$\Gamma_7(z)z_{xx} + \Gamma_8(z)z_x^2 = 0, \tag{5.77}$$

where $\Gamma_7(z)$, $\Gamma_8(z)$ are functions of $\Gamma_4(z)$, $\Gamma_5(z)$ and $\Gamma_6(z)$ which may be zero. If $\Gamma_7(z) = 0$ then $\Gamma_8(z)$ must also be zero and we **reduce**($k3$,[$\Gamma_7(z), \Gamma_8(z)$], $k5$) seeking to remove $\Gamma_5(z)$ and $\Gamma_6(z)$ before $\Gamma_4(z)$:

$k5$: $$\Gamma_4' z_x^2 - 5\Gamma_4 z_{xx} = 0.$$

We can integrate this twice with respect to x to give $z = \theta(t)x + \sigma(t)$, after using freedom (c). Similarly if $\Gamma_7(z) \neq 0$, we are able to integrate (5.77) directly, to yield the same result. All this work shows that $2\alpha - 3\beta = 0$ is just a spurious case, and we continue simply with $z = \theta(t)x + \sigma(t)$. Equation (5.74i) now yields

$$\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}x + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)\Gamma_1(z) = \beta\frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

Either $\Gamma_1(z) \equiv 0$ and then $z = x + \sigma(t)$, or $\Gamma_1(z) \equiv \beta/(c_1 z + c_2)$ and it transpires that we may set $\sigma(t) = 0$ i.e. $z = \theta(t)x - c_2$. In the latter option we find the classical reduction

5.2.1, whilst the former gives rise to more than just classical results. Thus continuing with $z = x + \sigma(t)$, the remaining coefficients yield

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_9(z) = \alpha B_x \frac{\mathrm{d}\sigma}{\mathrm{d}t} + \beta B_t - 1, \tag{5.78i}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_{10}(z) = \alpha B_{xt} + \beta \frac{\mathrm{d}\sigma}{\mathrm{d}t} B_{xx}, \tag{5.78ii}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_{11}(z) = B_{xxxt} + \alpha B_x B_{xt} + \beta B_t B_{xx} - B_{xt} - B_{xx}. \tag{5.78iii}$$

Solving (5.78i) for $B(x,t)$ using the method of characteristics yields either $B(x,t) = p(\zeta) + \Gamma(z) + t/\beta$ if $\Gamma_9(z)$ is not constant, or $B(x,t) = c_1\sigma(t) + p_1(\zeta) + t/\beta$, if $\Gamma_9(z) = c_1\beta$, constant, where $\zeta = x - \beta\sigma(t)/\alpha$. If we choose the first expression for B, we would set $\Gamma(z) = 0$, apparently without loss of generality, by freedom (a), but this is now different to the second expression, and generality appears to have been lost. We will continue with the second expression but will show later that generality would not have been lost. Equation (5.78ii) yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_{10}(z) = \left(\frac{\beta^2 - \alpha^2}{\beta}\right) \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}^2 p}{\mathrm{d}\zeta^2}.$$

If $\beta - \alpha \neq 0$ then $p(\zeta)$ must be quadratic in ζ, and it is not difficult to show that we only get the classical reduction 5.2.2. With $\beta = \alpha$, $p(\zeta)$ is still arbitrary and the only equation left is (5.78iii). This equation dictates the form of $p(\zeta)$ in the shape of an ordinary differential equation. We may now take an *a posteriori* translation of $p(\zeta)$ and $U(z)$, namely $p(\zeta) \to p(\zeta) - \frac{1}{2}c_1\zeta$ and $U(z) \to U(z) + \frac{1}{2}c_1 z$ which effectively removes the $c_1\sigma(t)$ term from $B(x,t)$ (so generality would not have been lost). We have the following reduction

Reduction 5.4.1. $\alpha - \beta = 0$

$$u(x,t) = U(z) + p(\zeta) + \frac{t}{\alpha},$$

where $z = x + \sigma(t)$, $\zeta = x - \sigma(t)$ and $P(\zeta) = \dfrac{\mathrm{d}p}{\mathrm{d}\zeta}$ satisfies (5.13). The ordinary differential equation that $Q(z) = U'(z)$ satisfies is (5.15). This is the analogue of reduction 5.3.1.

Subcase 5.4.1(ii) $\alpha + \beta = 0$ We are unable to find an expression for $A(x,t)$ from (5.73), so instead we consider the coefficients of U''', $(U')^2$, UU'' and UU':

$$Az_x^3 z_t \Gamma_1(z) = A_t z_x^3 + 3Az_x^2 z_{xt} + 3A_x z_x^2 z_t + 3Az_x z_t z_{xx}, \tag{5.79i}$$

$$Az_x^3 z_t \Gamma_2(z) = A^2 z_x z_{xt} - A^2 z_t z_{xx} - AA_x z_x z_t + AA_t z_x^2, \tag{5.79ii}$$

$$Az_x^3 z_t \Gamma_3(z) = AA_x z_x z_t - AA_t z_x^2, \tag{5.79iii}$$

$$Az_x^3 z_t \Gamma_4(z) = AA_t z_{xx} + A_t A_x z_x - AA_{xt} z_x - AA_x z_{xt} + AA_{xx} z_t - A_x^2 z_t. \tag{5.79iv}$$

Adding (5.79ii) and (5.79iii) yields

$$Az_x^3 z_t(\Gamma_2(z) + \Gamma_3(z)) = A^2[z_x z_{xt} - z_t z_{xx}]. \tag{5.80}$$

If the right hand side of this is zero, we can integrate with respect to x to yield $z_t = \Sigma(t)z_x$. Treating this as a linear partial differential equation we use the method of characteristics to find the solution $z = \Gamma(x + \Sigma(t))$, then by freedom (c), $z = x + \sigma(t)$. We can also solve (5.79iii) as a linear partial differential equation now which gives $A(x,t) = -\sigma(t)\Gamma_3(z) + \Gamma(z)$. By freedom (b) we may set $\Gamma(z) = 1$ without loss of generality. Equations (5.79i) and (5.79iv) now read

$$(1 - \sigma\Gamma_3)\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_1 = -4\sigma\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_3' - \Gamma_3\frac{\mathrm{d}\sigma}{\mathrm{d}t}, \tag{5.81i}$$

$$(1 - \sigma\Gamma_3)\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_4 = \Gamma_3'\frac{\mathrm{d}\sigma}{\mathrm{d}t}. \tag{5.81ii}$$

Since $\frac{\mathrm{d}\sigma}{\mathrm{d}t} \neq 0$ the expressions $\sigma\frac{\mathrm{d}\sigma}{\mathrm{d}t}$ and $\frac{\mathrm{d}\sigma}{\mathrm{d}t}$ are independent, thus (5.81) spawns the following system of equations

$$\Gamma_3\Gamma_4 = 0, \qquad \Gamma_4 - \Gamma_3' = 0, \qquad \Gamma_1 + \Gamma_3 = 0, \qquad 4\Gamma_3' - \Gamma_1\Gamma_3 = 0.$$

It is not difficult to see that the only solution of these equations is $\Gamma_1 = \Gamma_3 = \Gamma_4 = 0$.

Remark 5.4(vi). Recall that, before applying freedom (b) above, we had found $A(x,t) = -\sigma(t)\Gamma_3(z)+\Gamma(z)$. Another, different application of freedom (b) allows us to set $\Gamma_3(z) = 1$, apparently without loss of generality. The system that equations (5.79i) and (5.79iv) now create is

$$\Gamma\Gamma_1 - 4\Gamma' + 1 = 0, \qquad \Gamma_1 = 0, \qquad \Gamma\Gamma_4 + \Gamma' = 0, \qquad \Gamma_4 = 0,$$

for which no consistent solution exists. It is now clear that we require $\Gamma_3(z) = 0$, and so using freedom (b) to set $\Gamma_3(z) = 1$ is the wrong step to take. However this cannot be known *a priori*, and freedom (b) has been applied *with* loss of generality. A rethink of freedom (b) is needed which is addressed in the next chapter.

Thus continuing with $\Gamma(z) = 1$, $\Gamma_3(z) = 0$, the coefficient of U'' is now

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_5(z) = \alpha\frac{\mathrm{d}\sigma}{\mathrm{d}t}B_x - \alpha B_t - 1,$$

where $\Gamma_5(z)$ has consumed all functions of z. This has solution

$$B(x,t) = -\frac{\sigma(t)\Gamma_5(z)}{\alpha} - \frac{t}{\alpha} + \Gamma_6(z).$$

We can set $\Gamma_6(z) = 0$ by freedom (a) and we rename $\Gamma_5(z) = \alpha\eta(z)$. The coefficient without U or its derivatives then gives an equation which $\eta(z)$ must satisfy, and we have found the nonclassical reduction

Reduction 5.4.2. $\alpha + \beta = 0$

$$u(x,t) = U(z) - \sigma(t)\eta(z) - \frac{t}{\alpha},$$

where $z = x + \sigma(t)$, $U(z)$ satisfies (5.18) and $\eta(z)$ satisfies (5.16). This is the analogue of reduction 5.3.2.

Considering now the case when the right hand side of (5.80) is non-zero, we let $\Gamma_7(z) = \Gamma_2(z) + \Gamma_3(z) \neq 0$ so that

$$A(x,t) = \frac{\Gamma_7(z) z_x^3 z_t}{z_x z_{xt} - z_t z_{xx}},$$

and substitute this expression into equations (5.79). We use `diffgrob2` to achieve this and write the equations in polynomial form, so that we can use the procedures effected in it. We **reduce**((5.79i),[(5.79iii)], $k6$) seeking to remove t-derivatives first, remembering that equations (5.79) are now partial differential equations with z as the dependent variable. Taking the **diffSpoly** of (5.79iii) and $k6$, then reducing the result with respect to both equations, leaves $k7$. The highest coefficient of $k7$ is zero if $13\Gamma_3(z) - 27\Gamma_7(z) = 0$, and assuming it is, $k7$ becomes

$$14\Gamma_7(z) z_{xx} - (\Gamma_1(z)\Gamma_7(z) - 4\Gamma_7'(z)) z_x^2 = 0, \tag{5.82}$$

and recall that $\Gamma_7(z) \neq 0$. If $\Gamma_1(z)\Gamma_7(z) - 4\Gamma_7'(z) = 0$ then $z_{xx} = 0$, whilst if $\Gamma_1(z)\Gamma_7(z) - 4\Gamma_7'(z) \neq 0$ we can integrate (5.82) twice with respect to x and achieve the same result, $z_{xx} = 0$ (after using freedom (c)).

If $z_{xx} = 0$, irrespective of any conditions on $\Gamma_1(z)$, $\Gamma_3(z)$ or $\Gamma_7(z)$, $k6$ becomes

$$z_{xt}[(\Gamma_3(z) - 7\Gamma_7(z)) z_{xt} + (\Gamma_1(z)\Gamma_7(z) - 4\Gamma_7'(z)) z_x z_t] = 0, \tag{5.83}$$

and note that $z_{xt} \neq 0$ since $z_x z_{xt} - z_t z_{xx} \neq 0$. If $\Gamma_3(z) - 7\Gamma_7(z) = 0$ then (5.79iii) is only zero if

$$z_x z_t z_{xtt} - z_x z_{tt} z_{xt} - 8 z_t z_{xt}^2 = 0,$$

but this is compatible with $z_{xx} = 0$ only if $z_{xt} = 0$, so we must have $\Gamma_3(z) - 7\Gamma_7(z) \neq 0$. Equation (5.83) is then effectively

$$\frac{z_{xt}}{z_t} = \Gamma(z) z_x, \tag{5.84}$$

where $\Gamma(z)$ must be non-zero, and since $z_{xx} = 0$ we write $z = \theta(t)x + \sigma(t)$. Equation (5.84) then becomes

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \theta(t)\Gamma(z)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}x + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right).$$

We must set $\Gamma(z) = 1/(c_1 z + c_2)$ and find $c_1 = 1$ and $\sigma(t) = c_3\theta(t) - c_2$. We can set $c_3 = 0$ because of the translational freedom in x, hence $z = \theta(t)x - c_2$. If we are able to show, then, that $z_{xx} = 0$ we can infer that $z = \theta(t)x - c_2$.

Now assume that $13\Gamma_3(z) - 27\Gamma_7(z) \neq 0$, so that we may **reduce**($k6$,[$k7$], $k8$) and immediately check whether the highest coefficient of $k8$ is zero. If it is then $k8$ reduces to zero, but under these conditions $k7$ becomes $\Gamma_7(z)(z_x z_{xt} - z_t z_{xx}) = 0$ which cannot be true. Thus we may assume that the highest coefficient of $k8$ is non-zero and **reduce**((5.79iv),[$k7$,$k8$], $k9$). We also take the **diffSpoly** of $k7$ and $k8$ then **reduce** the result with respect to both equations (call the final result $k10$), then both $k9$ and $k10$ are of the form

$$(\Gamma_7(z)z_{xx} + \widetilde{\Gamma}_1(z)z_x^2)(\widetilde{\Gamma}_2(z)z_{xx}^2 + \widetilde{\Gamma}_3(z)z_x^2 z_{xx} + \widetilde{\Gamma}_4(z)z_x^4) = 0.$$

The $\widetilde{\Gamma}_i(z)$ where $i = 1, ..., 4$ are different for each equation, and it is this difference that we exploit.

First note that whatever value $\widetilde{\Gamma}_1(z)$ takes, if the first bracket is zero then $z_{xx} = 0$, perhaps after applying freedom (c) (hence $z = \theta(t)x$). Then assuming the second bracket is zero, either $\widetilde{\Gamma}_2(z) = \widetilde{\Gamma}_3(z) = \widetilde{\Gamma}_4(z) = 0$ or it can be solved as a quadratic equation to yield

$$\widetilde{\Gamma}_5(z)z_{xx} + \widetilde{\Gamma}_6(z)z_x^2 = 0 \tag{5.85}$$

(where $\widetilde{\Gamma}_5(z)$ is non-zero if $\widetilde{\Gamma}_2(z) \neq 0$). We note that the zeros of $\widetilde{\Gamma}_2(z)$ in $k9$ are different from those in $k10$, so for one of these equations it is not possible to have $\widetilde{\Gamma}_2(z) = \widetilde{\Gamma}_3(z) = \widetilde{\Gamma}_4(z) = 0$ and we can get an equation of the form (5.85) with $\widetilde{\Gamma}_5(z) \neq 0$. Hence $z_{xx} = 0$, hence $z = \theta(t)x - c_2$. With $z = \theta(t)x - c_2$ we can use freedom (b) to let $A(x,t) = \theta(t)$, and the remaining equations lead us to the classical reduction 5.2.1.

Case 5.4.2 System (5.4). We assume that the solution of (5.4) is of the form

$$v(x,t) = F(x,t,V(z),W(z)), \tag{5.86a}$$

$$w(x,t) = G(x,t,V(z),W(z)), \tag{5.86b}$$

where $z = z(x,t)$, and require that the result is a system of ordinary differential equations for $V(z)$, $W(z)$. We note that there is no splitting of either equation, in keeping with the principles of the direct method, in contrast to other methods (e.g. Galaktionov [1990] – see §1.5.3). Substituting these ansätze into (5.4a) gives

$$F_V z_x^2 z_t V''' + F_W z_x^2 z_t W''' + ... = 0. \tag{5.87}$$

Without loss of generality we use the coefficient of V''' to be the normalising coefficient, so in (5.87) we require

$$F_V \Gamma(z,V,W) = F_W, \tag{5.88}$$

where we have assumed $z_x z_t \neq 0$, and $\Gamma(z,V,W)$ is a function to be determined. Treating this like a linear first order partial differential equation in F, we find the four characteristic directions $\mu_1 = F$, $\mu_3 = x$, $\mu_4 = t$ and an unknown μ_2. However since μ_2 is ultimately a function of z, because V and W are both functions of z, we may write the solution of (5.88) as $F = \widetilde{F}(\mu_2(z), x, t)$. Now by simply renaming things we may let $F_W = 0$ without loss of generality. Equation (5.4a) now yields

$$F_V z_x^2 z_t V''' + F_{VV} z_x^2 z_t V'V'' + ... = 0. \tag{5.89}$$

We divide by the coefficient of V''' and require that the new coefficient of $V'V''$ be a function of (V, z) i.e. on rearranging

$$\frac{F_{VV}}{F_V} = \frac{\Gamma_{1,VV}(z, V)}{\Gamma_{1,V}(z, V)}.$$

Integrating with respect to V twice gives

$$F(x, t, V(z(x, t))) = A(x, t)\Gamma_1(z, V) + B(x, t), \tag{5.90}$$

and without loss of generality we can assume $\Gamma_1(z, V) = V(z)$. The coefficient of V' in (5.4a) now yields

$$G = z_t z_x \Gamma_2(z, V, W) + D(x, t) - \frac{\alpha A z_t}{\beta z_x} V, \tag{5.91}$$

where $D(x, t)$ is given by

$$A_{xx} z_t + 2A_{xt} z_x + 2A_x z_{xt} + A_t z_{xx} + A z_{xxt} + \alpha A B z_t + \beta A D z_x - A z_x - A z_t = 0.$$

Without loss of generality we assume $\Gamma_2(z, V, W) = W(z)$. System (5.4) now reads (keeping $D(x, t)$ unevaluated for convenience)

$$\begin{aligned} &A z_x^2 z_t V''' + [2A_x z_x z_t + A_t z_x^2 + 2A z_x z_{xt} + A z_{xx} z_t]V'' + [\alpha A A_t - \alpha A A_x z_t / z_x]V^2 \\ &\quad + \beta A z_x^2 z_t W V' + [A_{xxt} + \alpha B A_t + \alpha A B_t + \beta D A_x - A_x - \alpha A B_x z_t / z_x - A_t]V \\ &\quad + \beta z_x z_t A_x W V + \beta z_x z_t B_x W + [B_{xxt} + \alpha B B_t + \beta D B_x - B_x - B_t] = 0, \end{aligned} \tag{5.92i}$$

$$\begin{aligned} &z_x^2 z_t W' + [z_{tx} z_x + z_t z_{xx}]W \\ &\quad - \left[\frac{\alpha + \beta}{\beta}\right] A z_t V' + \frac{\alpha}{\beta}\left[\frac{A z_t z_{xx}}{z_x^2} - \frac{A z_{tx}}{z_x} - \frac{A_x z_t}{z_x} - \frac{\beta A_t}{\alpha}\right] V + [D_x - B_t] = 0. \end{aligned} \tag{5.92ii}$$

Assuming the coefficient of W' to be the normalising coefficient in (5.92ii), the coefficient of V' yields

$$z_x^2 z_t \Gamma(z) = (\alpha + \beta) A z_t, \tag{5.93}$$

hence we split our analysis into two.

Subcase 5.4.2(i) $\alpha + \beta \neq 0$. From (5.93) we have $A = z_x^2$ by freedom (b), then the coefficient of VW in (5.92i) yields

$$z_x^4 z_t \Gamma(z) = z_x^2 z_t z_{xx},$$

which we may integrate with respect to x, and find $z = \theta(t)x + \sigma(t)$, after using freedom (c). The coefficient of V^2 in (5.92i) now yields

$$\theta^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} x + \frac{\mathrm{d}\sigma}{\mathrm{d}t} \right) \Gamma(z) = \theta \frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

$\Gamma(z)$ is necessarily of the form $\Gamma(z) = c_1/(z + c_2)$ and we find that either $z = \theta(t)x - c_2$ if $c_1 \neq 0$ or $z = x + \sigma(t)$ if $c_1 = 0$. The remaining analysis then leads us to the two classical reductions found in section 5.2, after a small amount of tidying up (for instance notice that $w(x,t)$ can be written as independent of $V(z)$).

Subcase 5.4.2(ii) $\alpha + \beta = 0$. The system of equations that $A(x,t)$ must satisfy is

$$Az_x^2 z_t \Gamma_1(z) = 2A_x z_x z_t + A_t z_x^2 + 2Az_x z_{xt} + Az_{xx} z_t, \tag{5.94i}$$

$$z_x^2 z_t \Gamma_2(z) = A_t - \frac{A_x z_t}{z_x}, \tag{5.94ii}$$

$$Az_x^2 z_t \Gamma_3(z) = A_x z_x z_t, \tag{5.94iii}$$

$$z_x^2 z_t \Gamma_4(z) = z_x z_{xt} + z_{xx} z_t, \tag{5.94iv}$$

$$z_x^2 z_t \Gamma_5(z) = A_t - \frac{Az_{xt}}{z_x} + \frac{Az_{xx} z_t}{z_x^2} - \frac{A_x z_t}{z_x}, \tag{5.94v}$$

and by subtracting (5.94ii) from (5.94v) we also have

$$z_x^2 z_t (\Gamma_5(z) - \Gamma_2(z)) = A \left[\frac{z_{xx} z_t}{z_x^2} - \frac{z_{xt}}{z_x} \right]. \tag{5.95}$$

If the right hand side of this is zero, since $A \neq 0$ we have

$$z_t z_{xx} - z_x z_{xt} = 0,$$

which may be integrated with respect to x to yield $z_t = \Sigma(t) z_x$. This may be thought of as a linear first order partial differential equation and solved as such to give $z = \Gamma(x + \Sigma(t))$, hence by freedom (c) $z = x + \sigma(t)$. Equation (5.94ii) may now also be solved by the method of characteristics, to leave $A = \Gamma_6(z) + \sigma(t)\Gamma_2(z)$. Similar to Subcase 5.4.1(ii) freedom (b) allows us to set either $\Gamma_6(z) = 1$ or $\Gamma_2(z) = 1$ and yet if we set $\Gamma_2(z) = 1$ we run into difficulties (cf. Remark 5.4(vi)). We therefore set $\Gamma_6(z) = 1$, and equations (5.94i) and (5.94iii) yield the following system

$$\Gamma_3 = 0, \qquad \Gamma_2\Gamma_3 - \Gamma_2' = 0, \qquad \Gamma_1 - \Gamma_2 = 0, \qquad \Gamma_1\Gamma_2 - 3\Gamma_2' = 0,$$

which has solution $\Gamma_1(z) = \Gamma_2(z) = \Gamma_3(z) = 0$. Turning to the coefficient of W in (5.92i) we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma(z) = \frac{\mathrm{d}\sigma}{\mathrm{d}t} B_x,$$

which may be integrated with respect to x to yield $B(x,t) = \Gamma(z) + \delta(t)$ where $\delta(t)$ is an arbitrary function. We may set $\Gamma(z) = 0$ and the few remaining steps lead us to the classical reduction 5.2.2.

If the right hand side of (5.95) is non-zero, it gives us an expression for $A(x,t)$. We can get another expression for $A(x,t)$ by taking (5.94i) – (5.94ii) – 3(5.94iii) to yield

$$A(z_x^2 z_t \Gamma_1 - 3z_x^2 z_t \Gamma_3 - 2z_x z_{xt} - z_{xx} z_t) = z_x^4 z_t \Gamma_2. \tag{5.96}$$

If $\Gamma_2(z) = 0$ we may use (5.94iv) to remove occurrences of z_{xt} from (5.96) to give

$$z_{xx} + z_x^2(\Gamma_1 - 3\Gamma_3 - 2\Gamma_4) = 0. \tag{5.97}$$

This implies that $z = \theta(t)x + \sigma(t)$ (perhaps after using freedom (c)) independent of the coefficient of z_x^2. Equation (5.94ii) now gives $A = \Gamma(z)$, hence $A = 1$ by freedom (b). However (5.94v) now gives

$$\theta^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} x + \frac{\mathrm{d}\sigma}{\mathrm{d}t} \right) \Gamma_5(z) = -\frac{1}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t},$$

and because $\Gamma_5(z) \neq 0$ (since $\Gamma_5(z) - \Gamma_2(z) \neq 0$) we have $\Gamma_5(z) = 1/(c_1 z + c_2)$. The coefficient of x now gives a contradiction, that $\theta(t)$ is constant (and hence the right hand side of (5.95) is zero).

Hence we assume $\Gamma_2(z) \neq 0$; comparing (5.95) and (5.96) and removing z_{xt} using (5.94iv) we have

$$(\Gamma_5 - 3\Gamma_2) z_{xx} - z_x^2[(\Gamma_1 - 3\Gamma_3 - 2\Gamma_4)(\Gamma_5 - \Gamma_2) + \Gamma_2\Gamma_4] = 0.$$

If $\Gamma_5 - 3\Gamma_2 \neq 0$ this equation, similar to (5.97) gives $z = \theta(t)x + \sigma(t)$. If $\Gamma_5 - 3\Gamma_2 = 0$ we must work a little harder to get our coveted $z_{xx} = 0$: we know (5.94v) – 3(5.94ii) = 0 which yields

$$2A_x z_x z_t - 2A_t z_x^2 - A z_x z_{xt} + A z_{xx} z_t = 0,$$

which when subtracted from (5.94i) yields

$$A z_x^2 z_t \Gamma_1(z) = 3A_t z_x^2 + 3A z_x z_{xt}.$$

Dividing through by $A z_x^2$ and integrating with respect to t we get an expression of the form

$$A = \frac{\rho(t)\Gamma_7(z)}{z_x}.$$

Now (5.94iii) becomes, on rearranging

$$z_{xx} - (\Gamma_7\Gamma_3 - \Gamma_7')z_x^2 = 0,$$

and hence we have $z = \theta(t)x + \sigma(t)$ as required.

With $z = \theta(t)x + \sigma(t)$, equation (5.94iv) now becomes

$$\theta^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} x + \frac{\mathrm{d}\sigma}{\mathrm{d}t} \right) \Gamma_4(z) = \theta \frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

Taking $\Gamma_4(z) = 0$ contradicts our assumption that the right hand side of (5.95) is non-zero, hence we take $\Gamma_4(z) = 1/(c_1 z + c_2)$ and find $z = \theta(t)x - c_2$. Now A can be found from (5.95), as $A = \theta^2(t)$, after using freedom (b) and the remaining analysis leads us to the classical reduction 5.2.1.

However, we are not finished yet. We retain the ansatz $F = A(x,t)V(z) + B(x,t)$ as the following doesn't affect the work required to find this. Equation (5.92ii) now reads

$$G_W W' z_x + (G_V z_x - A z_t)V' + G_x - A_t V - B_t = 0, \tag{5.98}$$

which we write in the form

$$\Gamma_W(z,V,W)W' + \Gamma_V(z,V,W)V' + \Gamma_z(z,V,W) = 0, \tag{5.99}$$

and now it can be integrated to yield $\Gamma(z,V,W) = a_1$, constant, or alternatively $W = \widetilde{\Gamma}(z,V)$. We use this to let $G = G(x,t,V(z))$, and so now (5.98) must be identically zero (since we have effectively solved it). Hence

$$G_V z_x - A z_t = 0, \tag{5.100i}$$

$$G_x - A_t V - B_t = 0. \tag{5.100ii}$$

Equation (5.100i) gives G, on integrating

$$G(x,t,V(z)) = \frac{A z_t}{z_x} V + D(x,t), \tag{5.101}$$

which splits (5.100ii) into two equations,

$$\left(\frac{A z_t}{z_x}\right)_x - A_t = 0, \tag{5.102i}$$

$$D_x - B_t = 0. \tag{5.102ii}$$

Summarising so far, our ansätze are now

$$v(x,t) = A(x,t)V(z) + B(x,t), \qquad w(x,t) = \frac{A z_t}{z_x} V(z) + D(x,t), \tag{5.103a,b}$$

and we have two equations for our unknowns, (5.102). We may still use freedom (b), but must use freedom (a) with caution, making sure that the translation $V(z) \to V(z) - \Omega(z)$, doesn't make us lose generality. Equation (5.4a) now reads

$$\begin{aligned} &A z_x^2 z_t V''' + [2A_x z_x z_t + A_t z_x^2 + 2A z_x z_{xt} + A z_{xx} z_t]V'' \\ &\quad + [A_{xx} z_t + 2A_{xt} z_x + 2A_x z_{xt} + A_t z_{xx} + A z_{xxt} + \alpha A B z_t + \beta A D z_x - A z_x - A z_t]V' \\ &\quad + (\alpha+\beta)A^2 z_t V V' + [A_{xxt} + \alpha B A_t + \alpha A B_t + \beta D A_x + \beta A B_x z_t/z_x - A_x - A_t]V \\ &\quad + [\alpha A A_t + \beta A A_x z_t/z_x]V^2 + [B_{xxt} + \alpha B B_t + \beta D B_x - B_x - B_t] = 0. \end{aligned} \tag{5.104i}$$

The coefficient of VV' yields

$$Az_x^2 z_t \Gamma(z) = (\alpha + \beta) A^2 z_t, \tag{5.105}$$

which splits the analysis into whether $\alpha + \beta$ is zero or non-zero.

Subcase 5.4.2(iii) $\alpha + \beta \neq 0$ Using freedom (b) we have that $A = z_x^2$. Equation (5.102i) then gives us

$$z_{xx} z_t - z_x z_{tx} = 0, \tag{5.106}$$

which may be integrated with respect to x to yield $z_t = \Sigma(t) z_x$. This may be thought of as a linear first order partial differential equation and solved as such to give $z = \Gamma(x + \Sigma(t))$, hence by freedom (c) $z = x + \sigma(t)$; now $A = 1$ and the remaining non-trivial coefficients yield

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_4(z) = \alpha B \frac{\mathrm{d}\sigma}{\mathrm{d}t} + \beta D - 1, \tag{5.107i}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_5(z) = \alpha B_t + \beta \frac{\mathrm{d}\sigma}{\mathrm{d}t} B_x, \tag{5.107ii}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_6(z) = B_{xxt} + \alpha B B_t + \beta D B_x - B_x - B_t, \tag{5.107iii}$$

together with (5.102ii). Equation (5.107ii) is a linear first order partial differential equation so can be solved using the method of characteristics,

$$B(x,t) = \frac{\Gamma_5(z)}{\alpha} + P(\zeta), \qquad \zeta = x - \frac{\beta}{\alpha}\sigma(t),$$

and notice that $\zeta \neq z$ since $\alpha + \beta \neq 0$. Equation (5.107i) gives us D directly,

$$D(x,t) = \frac{1}{\beta}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\left(\Gamma_4 - \Gamma_5 - \alpha P(\zeta)\right) + \frac{1}{\beta}.$$

Substituting these expressions for B, D into (5.102ii) yields

$$(\alpha^2 - \beta^2)\frac{\mathrm{d}P}{\mathrm{d}\zeta} = \alpha\Gamma_4' - \alpha\Gamma_5' - \beta\Gamma_5'.$$

If $\alpha - \beta \neq 0$, then $\frac{\mathrm{d}P}{\mathrm{d}\zeta} = c_2$, constant, which leads to the classical reduction 5.2.2. However if $\alpha - \beta = 0$, then $\Gamma_4' = 2\Gamma_5'$, which upon integration leaves $\Gamma_4 = 2\Gamma_5 + c_3$. We may now use freedom (a) to set $\Gamma_5 = 0$. Equation (5.107iii) now defines $P(\zeta)$, and we may let $c_3 = 0$ without loss of generality, by an *a posteriori* translation of V and P. Thus we have found the nonclassical reduction

Reduction 5.4.3. $\alpha - \beta = 0$.

$$v(x,t) = V(z) + P(\zeta),$$

$$w(x,t) = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\left(V(z) - P(\zeta)\right) + \frac{1}{\alpha},$$

where $z = x + \sigma(t)$, $\zeta = x - \sigma(t)$, $P(\zeta)$ satisfies (5.13) and $Q(z) = V(z)$ satisfies (5.15). This reduction is equivalent to the nonclassical reduction 5.3.1.

Subcase 5.4.2(iv) $\alpha + \beta = 0$ The coefficients of V^2 and V'' yield respectively

$$z_x^2 z_t \Gamma_8(z) = A_t - \frac{A_x z_t}{z_x}, \tag{5.108i}$$

$$A z_x^2 z_t \Gamma_9(z) = 2A_x z_x z_t + A_t z_x^2 + 2A z_x z_{xt} + A z_{xx} z_t. \tag{5.108ii}$$

From (5.108i), (5.102i) we get another relation,

$$z_x^2 z_t \Gamma_8(z) = A \left[\frac{z_{xt}}{z_x} - \frac{z_t z_{xx}}{z_x^2} \right]. \tag{5.109}$$

We must consider the cases when the right hand side of (5.109) is zero and non-zero. If it is zero, remembering $A \neq 0$, this leaves equation (5.106) after rearranging, hence $z = x + \sigma(t)$. Equation (5.102i) then gives a linear first order partial differential equation which when solved gives $A = \Gamma(z)$, thus by freedom (b) we can choose $A = 1$. Equation (5.108ii) is now satisfied if $\Gamma_9(z) = 0$. The remaining non-trivial coefficients are simply system (5.107), remembering that $\beta = -\alpha$. Solving (5.107ii) as a linear first order partial differential equation yields $B = \sigma\Gamma_5(z)/\alpha + \Gamma_7(z)$, and D is given immediately by (5.107i),

$$D(x,t) = \frac{1}{\alpha}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\left(\alpha\Gamma_7 - \Gamma_4 + \sigma\Gamma_5\right) - \frac{1}{\alpha}.$$

We may use freedom (a) to set $\Gamma_7(z) = 0$, then (5.102ii) yields $\Gamma_5 = -\Gamma_4'$. We let $\Gamma_4 = \alpha\eta(z)$ for convenience, and we have

Reduction 5.4.4. $\alpha + \beta = 0$.

$$v(x,t) = V(z) - \sigma(t)\frac{\mathrm{d}\eta}{\mathrm{d}z},$$

$$w(x,t) = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\left(V(z) - \eta(z) - \sigma(t)\frac{\mathrm{d}\eta}{\mathrm{d}z}\right) - \frac{1}{\alpha},$$

where $z = x + \sigma(t)$, $\eta(z)$ satisfies (5.16) and $V(z)$ satisfies (5.35) This is equivalent to the nonclassical reduction 5.3.2.

If the right hand side of (5.109) is non-zero (note that $\Gamma_8(z) \neq 0$), then we proceed slightly differently. From (5.108ii), (5.102i) we get an expression which we can integrate with respect to x to get the relation $A = \Gamma(z)\Sigma(t)/z_t$, hence we choose $A = \sigma(t)/z_t$ by freedom (b). On substituting this into (5.108) and (5.102i) we have that $\Gamma_9(z) = 0$ and

$$\frac{1}{z_t}\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \frac{\sigma z_{tt}}{z_t^2} = -\frac{\sigma z_{xx}}{z_x^2}, \tag{5.110i}$$

$$z_x^2 z_t \Gamma_8(z) = \frac{1}{z_t}\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \frac{\sigma z_{tt}}{z_t^2} + \frac{z_{xt}}{z_t z_x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}, \tag{5.110ii}$$

and we easily glean a third equation

$$z_x^2 z_t \Gamma_8(z) = -\frac{\sigma z_{xx}}{z_x^2} + \frac{z_{xt}}{z_t z_x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}. \tag{5.110iii}$$

We now make use of `diffgrob2` to solve this system of equations (and therefore write them in polynomial form). We use **diffSpoly** on (5.110ii,iii), using an alex ordering which seeks to remove t-derivatives, and call the result $k1$. We successively **reduce**($k1$,[(5.110iii)], $k2$), then **reduce**($k2$,[(5.110ii)], $k3$), and finally **reduce**($k1$,[(5.110iii),$k3$], $k4$) to leave

$k4 \equiv k4_1 k4_2 k4_3$:

$$(\Gamma_8' z_x^2 + 4\Gamma_8 z_{xx})(4\Gamma_8' z_x^2 + \Gamma_8 z_{xx})$$
$$(38 z_x^4 (\Gamma_8')^2 + 79 z_x^2 z_{xx}\Gamma_8\Gamma_8' - 25 z_x^4 \Gamma_8 \Gamma_8'' - 100 z_x^2 \Gamma_8^2 z_{xxx} + 308\Gamma_8^2 z_{xx}^2) = 0.$$

Choosing $k4_1 = 0$, we can integrate with respect to x twice to leave $\Gamma(z) = \Phi(t)x + \Theta(t)$, hence $z = \phi(t)x + \theta(t)$ by freedom (c). In (5.110i) this gives

$$\left[\frac{\mathrm{d}\sigma}{\mathrm{d}t}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}x + \frac{\mathrm{d}\theta}{\mathrm{d}t}\right) - \sigma\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}x + \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right)\right] \Big/ \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}x + \frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = 0,$$

so that either $\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = 0$ or the coefficients of x^1 and x^0 must be zero so that $\sigma = \gamma_1 \frac{\mathrm{d}\phi}{\mathrm{d}t} = \gamma_2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$, where γ_1, γ_2 are non-zero constants. However in both cases, in substituting into (5.110ii) we arrive at a contradiction. Choosing $k4_2 = 0$ gives the same answer as when $k4_1 = 0$, because of the freedoms we can exert, but when $k4_3 = 0$ life is slightly more complicated. We can multiply through the equation by $-z_x^{-102/25}\Gamma_8^{-63/25}/25$ and we have an exact integral, which integrates to

$$4\Gamma_8^{-13/25} z_x^{-77/25} z_{xx} + \Gamma_8^{-38/25}\Gamma_8' z_x^{-27/25} = \Phi(t). \tag{5.111}$$

If $\Phi \equiv 0$ we have a similar situation to that already considered so assume $\Phi \not\equiv 0$. We can integrate (5.111) again with respect to x to leave

$$\Gamma_8^{-13/25} z_x^{-52/25} = \Phi(t)x + \Theta(t).$$

Taking roots, and using our freedom of expression of arbitrary functions gives

$$\Gamma(z) z_x = (\Phi(t)x + \Theta(t))^{-25/52},$$

which once integrated with respect to x leaves

$$\Gamma(z) = \frac{52}{27\Phi(t)}(\Phi(t)x + \Theta(t))^{27/52} + \Delta(t).$$

Again using our freedom in expressing arbitrary functions and also freedom (c) we have $z = (\phi(t)x + \theta(t))^{27/52} + \delta(t)$. Substituting this into (5.110i) leaves

$$\frac{27}{52}(\phi x + \theta)^{-25/52}\left[\frac{\mathrm{d}\sigma}{\mathrm{d}t}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}x + \frac{\mathrm{d}\theta}{\mathrm{d}t}\right) - \sigma\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}x + \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right)\right] + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}\delta}{\mathrm{d}t} - \sigma\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2}$$
$$+ \frac{50\sigma}{52(\phi x + \theta)}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}x + \frac{\mathrm{d}\theta}{\mathrm{d}t}\right) + \frac{25}{27}\left(\frac{\mathrm{d}\delta}{\mathrm{d}t}\right)^2 (\phi x + \theta)^{-27/52} = 0.$$

Since we only have one term in $(\phi x+\theta)^{-27/52}$ we must have $\frac{\mathrm{d}\delta}{\mathrm{d}t}=0$ which means that z can be rescaled to be $z=\phi(t)x+\theta(t)$ which has already been considered.

Case 5.4.3 System (5.5). We assume the general ansätze

$$u=H(x,t,U(z),V(z),W(z)), \tag{5.112a}$$

$$v=F(x,t,U(z),V(z),W(z)), \tag{5.112b}$$

$$w=G(x,t,U(z),V(z),W(z)), \tag{5.112c}$$

where $z=z(x,t)$. Our aim to show that this problem is no different from that of system (5.7), by showing that $H_W=F_W=0$ (or equivalently that H and F depend on two dependent variables only). The presence of $w(x,t)$ in (5.5c) may be replaced using (5.5b) so that we are left with (5.7), and (5.5b) simply gives $w(x,t)$, once $u(x,t)$, $v(x,t)$ are found.

With the ansätze as above equation (5.5a) is

$$F=H_x+H_UU'z_x+H_VV'z_x+H_WW'z_x. \tag{5.113}$$

Assuming that the coefficient of U' is non-zero without loss of generality, we have from W' that

$$H_U\Gamma(z,U,V,W)=H_W,$$

assuming $z_x\neq 0$. By considering the characteristic directions (as we did with the solution of equation (5.88)), we have $H=\widetilde{H}(c_2(z),V(z),x,t)$, i.e. H depends on two dependent variables only so without loss of generality we let $H_W=0$. To show the same for F, we look at (5.5c),

$$F_Vz_x^2z_tV'''+F_Wz_x^2z_tW'''+...=0,$$

and hence we have

$$F_V\Gamma(z,U,V,W)=F_W,$$

assuming $z_xz_t\neq 0$. It is clear that now we have found an equation of this form, our goal is achieved.

Case 5.4.4 System (5.6). We assume the solution of (5.6) is of the general form

$$u=H(x,t,U(z),\Psi(z)), \tag{5.114a}$$

$$\psi=K(x,t,U(z),\Psi(z)), \tag{5.114b}$$

where $z=z(x,t)$. The first part of this Case mirrors the beginnings of Case 5.4.2, in that we use the presence of the highest derivative term in (5.6a), u_{xxx}, to simplify ansatz (5.114a) to

$$u(x,t)=A(x,t)U(z)+B(x,t). \tag{5.115}$$

Now we use the coefficient of $(U')^2$ in (5.6b) as the normalising coefficient, so that the coefficient of Ψ' yields

$$A^2 z_x z_t \Gamma_\Psi(z, U, \Psi) = K_\Psi z_t.$$

Integrating with respect to Ψ yields

$$K = A^2 z_x \Gamma(z, U, \Psi) + \widetilde{K}(x, t, U(z)),$$

and we may set $\Gamma(z, U, \Psi) = \Psi(z)$ without loss of generality. Our system (5.6) with these ansätze reads

$$\begin{aligned}
&A z_x^3 U''' + [3A z_x z_{xx} + 3A_x z_x^2] U'' + \tfrac{1}{2}(\alpha - \beta) A^2 z_x^2 (U')^2 \\
&\quad + [(\alpha - \beta) A A_x z_x U + A z_{xxx} + 3A_x z_{xx} - \widetilde{K}_U z_x + 3A_{xx} z_x - A z_x + (\alpha - \beta) A B_x z_x] U' \\
&\quad - A^2 z_x^2 \Psi' - [A^2 z_{xx} + 2AA_x z_x]\Psi + [\tfrac{1}{2}(\alpha - \beta) A_x^2 U^2 + (\alpha - \beta) A_x B_x U + A_{xxx} U \\
&\quad - A_x U - \widetilde{K}_x + \tfrac{1}{2}(\alpha - \beta) B_x^2 + B_{xxx} - B_x] = 0, \qquad (5.116\text{i}) \\
&A^2 z_x z_t \Psi' + \beta A^2 z_t z_x (U')^2 \\
&\quad [\beta A A_t z_x U + \beta A A_x z_t + \widetilde{K}_U z_t + \beta A B_t z_x - A z_x + \beta A B_x z_t] U' + [2AA_t z_x + A^2 z_{tx}]\Psi \\
&\quad + [\beta A_x A_t U^2 + \beta A_t B_x + \beta A_x B_t - A_x U + \widetilde{K}_t + \beta B_x B_t - B_x] = 0. \qquad (5.116\text{ii})
\end{aligned}$$

The coefficient of Ψ' in (5.116i) then yields (using the coefficient of U''' as the normalising coefficient)

$$A z_x^3 \Gamma(z) = A^2 z_x^2,$$

and hence we have $A = z_x$, by freedom (b). (Note that Remark 5.4(ii) comes into play here: although $\psi(x, t)$ cannot be written in the required form, we may still apply freedom (b) as $\widetilde{K}(x, t, U(z))$ is undetermined.) The coefficient of U'' in (5.116i) then yields

$$z_x^4 \Gamma(z) = z_x^2 z_{xx},$$

which after integrating twice with respect to x and applying freedom (c) yields $z = \theta(t)x + \sigma(t)$. Next we find conditions on z, from the coefficient of Ψ in (5.116ii)

$$\theta(t)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} x + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)\Gamma(z) = \frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

Since z is linear in x, let $\Gamma(z) = c_1/(z + c_2)$. If $c_1 \neq 0$ then $c_1 = 1$ and we may set $\sigma = 0$ without loss of generality since (5.6) is invariant under translations of x. Thus $z = x\theta(t) - c_2$, and we are led to the classical reduction 5.2.1. If $c_1 = 0$, then we may set $\theta(t) = 1$, hence $z = x + \sigma(t)$, but we find more than just classical reductions. The coefficient of U' in (5.116i) yields

$$\Gamma_U(z, U) = \widetilde{K}_U - (\alpha - \beta) B_x + 1,$$

and hence

$$\widetilde{K} = \Gamma(z, U) + [(\alpha - \beta)B_x - 1]U(z) + D(x,t).$$

By freedom (a), of translation of $\Psi(z)$, we may set $\Gamma(z,U) = U(z)$ without loss of generality. The remaining coefficients of system (5.116) yield

$$\Gamma_1(z) = (\alpha - \beta)B_{xx}, \tag{5.117i}$$

$$\Gamma_2(z) = B_{xxx} + \tfrac{1}{2}(\alpha - \beta)B_x^2 - B_x - D_x, \tag{5.117ii}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_3(z) = \alpha\frac{\mathrm{d}\sigma}{\mathrm{d}t}B_x - 1 + \beta B_t, \tag{5.117iii}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_4(z) = (\alpha - \beta)B_{xt}, \tag{5.117iv}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_5(z) = D_t - B_x + \beta B_x B_t. \tag{5.117v}$$

From equations (5.117i) and (5.117iv) we see that we are going to get a special case if $\alpha - \beta = 0$, so we split our calculation in two.

Subcase 5.4.4(i) $\alpha - \beta \neq 0$. We integrate (5.117i) twice with respect to x to yield

$$B(x,t) = \Gamma_6(z) + \rho(t)x + \delta(t),$$

and we set $\Gamma_6(z) = 0$ by freedom (a). (We again refer to Remark 5.4(ii) where in this scenario $D(x,t)$ is undetermined so freedom (a) may be applied without loss of generality.) Substituting this expression for $B(x,t)$ into (5.117iv) we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_4(z) = \frac{\mathrm{d}\rho}{\mathrm{d}t},$$

thus $\Gamma_4(z)$ is necessarily constant and this integrates with respect to t to give $\rho(t) = c_3\sigma(t) + c_4$. We can get another condition for $B(x,t)$ by taking the **diffSpoly** of (5.117ii) and (5.117v) seeking to remove $D(x,t)$. This yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(\Gamma_2' + \Gamma_5') = c_3\frac{\mathrm{d}\sigma}{\mathrm{d}t}(\alpha c_3\sigma + \alpha c_4 - 1),$$

hence we must have $\Gamma_2' + \Gamma_5' = c_5$, constant. Since we are requiring $z_t \neq 0$, then $\dfrac{\mathrm{d}\sigma}{\mathrm{d}t} \neq 0$, and hence

$$c_5 = c_3(\alpha c_3\sigma + \alpha c_4 - 1).$$

If $c_3 \neq 0$ then $\sigma(t)$ is constant, contradicting $z_t \neq 0$, thus $c_3 = 0$ which forces $c_5 = 0$ also. To find an expression for $\delta(t)$ we turn to equation (5.117iii) which yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_3(z) = \alpha c_4\frac{\mathrm{d}\sigma}{\mathrm{d}t} - 1 + \beta\frac{\mathrm{d}\delta}{\mathrm{d}t}.$$

Now $\Gamma_3(z) = c_6$, constant, and integrating with respect to t gives $\delta(t)$:

$$\delta(t) = \frac{1}{\beta}[t + (c_6 - \alpha c_4)\sigma(t)].$$

From (5.117ii) we have

$$\Gamma_2(z) = \tfrac{1}{2}(\alpha - \beta)c_4^2 - c_4 - D_x,$$

which may be integrated with respect to x to yield

$$D(x,t) = \Gamma_7(z) + \left[\tfrac{1}{2}(\alpha - \beta)c_4^2 - c_4\right] x + \mu(t),$$

where $\Gamma_7'(z) = -\Gamma_2(z)$. Finally substituting this into (5.117v) yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_5(z) = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_7'(z) + \frac{\mathrm{d}\mu}{\mathrm{d}t} + c_4(c_6 - \alpha c_4)\sigma,$$

which confirms the earlier working that $\Gamma_2(z) + \Gamma_5(z) = c_7$, constant. This gives $\mu(t)$ on integrating with respect to t, as

$$\mu(t) = (c_7 - c_4c_6 - \alpha c_4^2)\sigma(t) + c_8.$$

Whilst at first glance what we have found may seem new, it is no more than the classical reduction 5.2.2, which some *a posteriori* tidying up will show.

Subcase 5.4.4(ii) $\alpha - \beta = 0$. We solve (5.117iii) for $B(x,t)$ using the method of characteristics. If $\Gamma_3(z)$ is not constant or is zero then B takes the form $B(x,t) = \Gamma(z) + t/\alpha + p(\zeta)$, where $\zeta = x - \sigma(t)$, and we could take $\Gamma(z) = 0$ without loss of generality by freedom (a) (and Remark 5.4(ii) since $D(x,t)$ is undetermined). However if $\Gamma_3(z) = c_3$ constant, then

$$B(x,t) = c_3\sigma(t) + t/\alpha + p(\zeta), \tag{5.118}$$

so it is this (seemingly) more general form we use. It is now clear how to integrate (5.117ii) with respect to x to get this expression for $D(x,t)$:

$$D(x,t) = \frac{\mathrm{d}^2p}{\mathrm{d}\zeta^2} - p(\zeta) + \Gamma_6(z) + \delta(t),$$

where $\delta(t)$ is the arbitrary function of integration. The final equation to be satisfied is (5.117v), which yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}[\Gamma_4(z) + \Gamma_6'(z)] = \frac{\mathrm{d}\delta}{\mathrm{d}t} - \frac{\mathrm{d}\sigma}{\mathrm{d}t}\left[\frac{\mathrm{d}^3p}{\mathrm{d}\zeta^3} + \alpha\left(\frac{\mathrm{d}p}{\mathrm{d}\zeta}\right)^2 - (c_3\alpha + 1)\frac{\mathrm{d}p}{\mathrm{d}\zeta}\right]. \tag{5.119}$$

The most general form that $\Gamma_4(z) + \Gamma_6'(z)$ may take is now $\Gamma_4(z) + \Gamma_6'(z) = -\lambda z - c_4$, and choosing $\delta(t) = -\lambda\sigma^2(t) + c_5\sigma(t)$ leaves (5.119) as an ordinary differential equation

$$\frac{\mathrm{d}^3p}{\mathrm{d}\zeta^3} + \alpha\left(\frac{\mathrm{d}p}{\mathrm{d}\zeta}\right)^2 - (c_3\alpha + 1)\frac{\mathrm{d}p}{\mathrm{d}\zeta} - \lambda\zeta - (c_4 + c_5) = 0. \tag{5.120}$$

With system (5.117) satisfied we may tidy up the reduction, as follows: we translate $p(\zeta) \to p(\zeta) + \frac{1}{2}c_3\zeta$ and $U(z) \to U(z) - \frac{1}{2}c_3 z$, which effectively removes the $c_3\sigma(t)$ term from $B(x,t)$. We then translate $\Psi(z) \to \Psi(z) + \frac{1}{2}c_3 z - \Gamma_6(z)$ to remove the x term in $D(x,t)$ which arises from the translation of $p(\zeta)$, and to remove $\Gamma_6(z)$ from $D(x,t)$. We note that the form of $B(x,t)$ we chose to stick with, (5.118), was really no more general (since setting $c_3 = 0$ doesn't make us lose generality). By simply renaming some constants, we have found the following nonclassical reduction

Reduction 5.4.5. $\alpha - \beta = 0$

$$u(x,t) = U(z) + p(\zeta) + \frac{t}{\alpha},$$

$$\psi(x,t) = \Psi(z) + \frac{\mathrm{d}^2 p}{\mathrm{d}\zeta^2} - p(\zeta) + (c_1 - c_2)\sigma(t) - \lambda\sigma^2(t),$$

where $z = x + \sigma(t)$, $\zeta = x - \sigma(t)$ and $P(\zeta) = \dfrac{\mathrm{d}p}{\mathrm{d}\zeta}$ satisfies (5.13). Also $Q(z) = U'(z)$ satisfies (5.15) and $\Psi(z) = U''(z) - U(z)$. This is the analogue of reduction 5.3.1.

<u>*Case 5.4.5 System (5.7).*</u> We assume the general ansätze

$$u = H(x,t,U(z),V(z)), \tag{5.121a}$$

$$v = F(x,t,U(z),V(z)), \tag{5.121b}$$

and in a similar fashion to Case 5.4.3, show that with a small amount of manipulation this may be simplified greatly. Equation (5.7a) looks like,

$$F_V z_x^2 z_t V''' + F_U z_x^2 z_t U''' + ... = 0,$$

and using similar techniques to those used previously we assume without loss of generality that $F_U = 0$. Similarly we may assume $H_V = 0$ since equation (5.7b) gives us the condition

$$H_U \Gamma(z,U,V) = H_V.$$

By considering the coefficient of $V'V''$ in (5.7a) then gives the simplified ansatz (cf. (5.90) and associated working), $v(x,t) = A(x,t)V(z) + B(x,t)$. Now the coefficient of $U'V'$ in (5.7a) yields

$$Az_t z_x^2 \Gamma(z,U) = \beta A z_t z_x H_U(z,U), \tag{5.122}$$

hence by integrating once with respect to U, we find

$$H(z,U) = z_x \Gamma(z,U) + D(x,t), \tag{5.123}$$

and without loss of generality we set $\Gamma(z,U) = U(z)$. Now (5.7) looks like

$$\begin{aligned} &Az_x^2 z_t V''' + [2A_x z_x z_t + A_t z_x^2 + 2Az_x z_{xt} + Az_{xx} z_t]V'' \\ &\quad + [A_{xx} z_t + 2A_{xt} z_x + 2A_x z_{xt} + A_t z_{xx} + Az_{xxt} + \alpha AB z_t + \beta A D_t z_x - Az_t - Az_x]V' \\ &\quad + [A_{xxt} + \alpha B A_t + \alpha A B_t + \beta D_t A_x - A_t - A_x]V + \alpha A A_t V^2 + \alpha A^2 z_t V V' \\ &\quad + \beta A z_x^2 z_t U'V' + \beta A z_x z_{xt} UV' + \beta A_x z_x z_t U'V + \beta A_x z_{xt} UV + \beta B_x z_{xt} U \\ &\quad + [B_{xxt} + \alpha B B_t + \beta D_t B_x - B_t - B_x] = 0, \end{aligned} \tag{5.124i}$$

$$AV - z_x^2 U' - z_{xx} U + [B - D_x] = 0. \tag{5.124ii}$$

If we let the coefficient of U' in (5.124ii) be the normalising coefficient, then U gives, upon rearranging

$$z_x\Gamma(z) = \frac{z_{xx}}{z_x}, \tag{5.125}$$

which may be integrated twice to yield, after using freedom (c), $z = \theta(t)x+\sigma(t)$. Choosing the coefficient of V''' in (5.124i) to be the normalising coefficient, V'' now yields

$$\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}x + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)\Gamma(z) = \frac{\mathrm{d}\theta}{\mathrm{d}t}. \tag{5.126}$$

Either $\Gamma(z) \equiv 0$, hence $z = x + \sigma(t)$, or $\Gamma(z) \equiv 1/(c_1 z + c_2)$ and we can show that $z = x\theta(t) - c_2$ without loss of generality. The remaining steps lead us to the classical reductions of section 5.2 only.

This is not all we can do though. We can use (5.124ii) to substitute into (5.124i) and require the result to be a ordinary differential equation (whilst (5.124ii) must also be an ordinary differential equation). Equation (5.122) is no longer valid, nor the work after it, but the preceding work is. Equation (5.124ii) now reads

$$AV - H_U U' z_x + B - H_x = 0, \tag{5.127}$$

and we replace occurrences of U' in (5.124i) using (5.127). Notice that we are not simply using (5.7b) to write (5.7a) as (5.3), and no differentiation is involved. The coefficient of V in (5.127) yields (using U' as the normalising coefficient)

$$H_U z_x \Gamma(z,U) = A,$$

and since $Az_x \neq 0$ we can write this in the form

$$H_U = \frac{A}{z_x}\Gamma_U(z,U).$$

We may integrate this with respect to U to give $H = A\Gamma(z,U)/z_x + D(x,t)$, and set $\Gamma(z,U) = U(z)$ without loss of generality. The system we are requiring to be an ordinary differential equation is now

$$\begin{aligned}
&Az_x^2 z_t V''' + [2A_x z_x z_t + A_t z_x^2 + 2Az_x z_{xt} + Az_{xx} z_t]V'' + (\alpha+\beta)A^2 z_t VV' + [A_{xx} z_t \\
&\quad + 2A_{xt} z_x + 2A_x z_{xt} + A_t z_{xx} + Az_{xxt} + \alpha ABz_t + \beta AD_t z_x - Az_t - Az_x + \beta ABz_t \\
&\quad - \beta AD_x z_t]V' + \beta A[(A/z_x)_t z_x - (A/z_x)_x z_t]UV' + [\alpha AA_t + \beta AA_x z_t/z_x]V^2 \\
&\quad + [A_{xxt} + \alpha BA_t + \alpha AB_t + \beta D_t A_x + \beta A_x B z_t/z_x - \beta A_x D_x z_t/z_x + \beta AB_x z_t/z_x \\
&\quad - A_t - A_x]V + \beta A_x[(A/z_x)_t - (A/z_x)_x z_t/z_x]UV + \beta B_x[(A/z_x)_t - (A/z_x)_x z_t/z_x]U \\
&\quad + [B_{xxt} + \alpha BB_t + \beta D_t B_x + \beta BB_x z_t/z_x - \beta D_x B_x z_t/z_x - B_t - B_x] = 0,
\end{aligned} \tag{5.128i}$$

$$AV - AU' - (A/z_x)_x U + [B - D_x] = 0. \tag{5.128ii}$$

The coefficients of V''' and U' are the normalising coefficients. From the coefficient of VV' in (5.128i) we then obtain equation (5.105) which splits our analysis into two.

Subcase 5.4.5(i) $\alpha + \beta \neq 0$ Using freedom (b) (scaling $U(z)$ or $V(z)$), we have $A = z_x^2$. This makes our ansatz for $u(x,t)$ the same as (5.123), and we are able to show that either $z = x + \sigma(t)$ or $z = x\theta(t) - c_2$ in the same way as before, using (5.125) then (5.126). If $z = x\theta(t) - c_2$ we again recover the classical reduction 5.2.1, whilst if $z = x + \sigma(t)$ we find another reduction. Thus assuming $z = x + \sigma(t)$, we use the coefficient of the term in (5.128ii) that contains no powers or derivatives of U, V, to yield $B = D_x + \Gamma(z)$. We may set $\Gamma(z) = 0$ using freedom (a) (translating $V(z)$). The three remaining non-trivial coefficients then yield

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_1(z) = \alpha\frac{\mathrm{d}\sigma}{\mathrm{d}t}D_x + \beta D_t - \frac{\mathrm{d}\sigma}{\mathrm{d}t} - 1, \tag{5.129i}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_2(z) = \alpha B_t + \beta\frac{\mathrm{d}\sigma}{\mathrm{d}t}B_x, \tag{5.129ii}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Gamma_3(z) = B_{xxt} + \alpha BB_t + \beta D_t B_x - B_t - B_x, \tag{5.129iii}$$

where $B = D_x$. Equation (5.129i) may be solved via the method of characteristics to yield, after using freedom (a) (translating $U(z)$), $D(x,t) = p(\zeta) + t/\beta$ where $\zeta = x - \alpha\sigma(t)/\beta$. Equation (5.129ii) then leaves

$$\Gamma_2(z) = (\beta^2 - \alpha^2)\frac{\mathrm{d}^2 p}{\mathrm{d}\zeta^2}.$$

If $\dfrac{\mathrm{d}^2 p}{\mathrm{d}\zeta^2}$ is constant we end up with the classical reduction 5.2.2, whilst if $\alpha - \beta = 0$, $p(\zeta)$ is given by (5.129iii), where $\Gamma_3(z)$ is necessarily constant. In summary we have the nonclassical reduction,

Reduction 5.4.6. $\beta = \alpha$

$$u(x,t) = p(\zeta) + U(z) + \frac{t}{\alpha},$$
$$v(x,t) = V(z) + P(\zeta),$$

where $z = x + \sigma(t)$ and $\zeta = x - \sigma(t)$. $Q(z) = U'(z)$ satisfies (5.15), $P(\zeta) = \dfrac{\mathrm{d}p}{\mathrm{d}\zeta}$ satisfies (5.13) and $V(z) = Q(z)$. This is the analogue of reduction 5.3.1.

Subcase 5.4.5(ii) $\alpha + \beta = 0$ The coefficients of U in (5.128ii), and UV', UV, V^2 and V'' in (5.128i) yield respectively,

$$A\Gamma_1(z) = \frac{A_x}{z_x} - \frac{Az_{xx}}{z_x^2}, \tag{5.130i}$$

$$Az_x^2 z_t \Gamma_2(z) = Az_x\left[\frac{A_t}{z_x} - \frac{Az_{tx}}{z_x^2} + \frac{Az_{xx}z_t}{z_x^3} - \frac{A_x z_t}{z_x^2}\right], \tag{5.130ii}$$

$$Az_x^2 z_t \Gamma_3(z) = A_x \left[\frac{A_t}{z_x} - \frac{Az_{tx}}{z_x^2} + \frac{Az_{xx}z_t}{z_x^3} - \frac{A_x z_t}{z_x^2} \right], \qquad (5.130\text{iii})$$

$$Az_x^2 z_t \Gamma_4(z) = A \left[A_t - \frac{A_x z_t}{z_x} \right], \qquad (5.130\text{iv})$$

$$Az_x^2 z_t \Gamma_5(z) = 2A_x z_x z_t + A_t z_x^2 + 2Az_x z_{xt} + Az_{xx} z_t. \qquad (5.130\text{v})$$

We can also get, from (5.130ii,iv),

$$z_x z_t (\Gamma_2(z) - \Gamma_4(z)) = A \left[\frac{z_t z_{xx}}{z_x^3} - \frac{z_{tx}}{z_x^2} \right]. \qquad (5.130\text{vi})$$

If we assume that the square bracketed expression in (5.130ii,iii) is non-zero, and also that $A_x \neq 0$, then (5.130ii,iii) combine to give

$$A_x \Gamma_2(z) - Az_x \Gamma_3(z) = 0.$$

This may be integrated to yield $A = \Gamma(z)\rho(t)$ and hence using freedom (b), $A = \rho(t)$, but this contradicts $A_x \neq 0$. Assuming then that $A_x = 0$, equation (5.130i) may now be simplified and integrated twice to yield $z = \theta(t)x + \sigma(t)$, upon using freedom (c). Now equation (5.130vi) reads

$$\theta \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} x + \frac{\mathrm{d}\sigma}{\mathrm{d}t} \right) (\Gamma_2(z) - \Gamma_4(z)) = \frac{A}{\theta^2} \frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

Since the right hand side contains no x terms, then $\Gamma_2(z) - \Gamma_4(z) = c_1/(z + c_2)$. If $c_1 = 0$, then $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$, and (5.130iv) may be used to find $A = 1$, after using freedom (b). However, the square bracketed expression in (5.130ii,iii) is now zero, contradicting the earlier assumption that it was not. If $c_1 \neq 0$, then we find $A = \theta(t)$ and we may set $\sigma(t) = 0$ without loss of generality. The remaining calculations lead us to classical reduction 5.2.1.

It remains to consider the case when

$$\frac{A_t}{z_x} - \frac{Az_{tx}}{z_x^2} + \frac{Az_{xx}z_t}{z_x^3} - \frac{A_x z_t}{z_x^2} = 0. \qquad (5.131)$$

Note that now $\Gamma_2(z) = \Gamma_3(z) = 0$. We consider separately the cases when the right hand side of (5.130vi) is zero and non-zero. If it is zero, we are left with equation (5.106), hence $z = x + \sigma(t)$. Equation (5.131) then may be solved as a first order linear partial differential equation to yield $A = \Gamma(z)$, hence $A = 1$ by freedom (b). We use the coefficient of the term in (5.128ii) that contains no powers or derivatives of U, V, to yield $B = D_x + \Gamma(z)$. and set $\Gamma(z) = 0$ using freedom (a) (translating $V(z)$). The three remaining non-trivial coefficients then yield system (5.129), remembering $\beta = -\alpha$. Equation (5.129i) may be solved via the method of characteristics to yield, after using freedom (a) (translating $U(z)$), $D(x,t) = -\sigma(t)\eta(z) - t/\alpha$. Equation (5.129ii) is then identically satisfied, whilst (5.129iii) implies that $\eta(z)$ satisfies (5.16). We have found the nonclassical reduction,

Reduction 5.4.7. $\beta = -\alpha$

$$u(x,t) = U(z) - \sigma(t)\eta(z) - \frac{t}{\alpha},$$
$$v(x,t) = V(z) - \sigma(t)\frac{\mathrm{d}\eta}{\mathrm{d}z},$$

where $z = x + \sigma(t)$, $U(z)$ satisfies (5.18) and $V(z) = U'(z)$. This is the analogue of reduction 5.3.2.

Finally, we assume that the right hand side of (5.130vi) is non-zero, hence $\Gamma_4(z) \neq 0$. With $\Gamma_4(z) \neq 0$, there are no solutions since the system consisting of (5.130iv,v) and (5.131) is now that found to have no solutions in Subcase 5.4.2(iv).

5.5 Discussion

In this chapter we have discussed symmetry reductions using the classical, nonclassical and direct methods for five variants of a shallow water wave equation, namely the scalar equation (5.3) and the systems (5.4)–(5.7).

Each of these methods give the same reductions when applied to the system (5.4) as when applied to the scalar counterpart equation (5.3). What is unusual about the calculation for the system (5.4) is the large increase in complexity in moving from a scalar equation to this system. Whilst for both the system and the scalar equation the determining equations for the classical method are of similar complexity (and are all linear) the nonclassical method paints quite a different picture. For the system there are 11 determining equations, all nonlinear, which constitute 583 lines of computer generated output. Even when we manage to show that it is sufficient to assume $\xi_v = \xi_w = 0$ this reduces to 8 fully nonlinear equations and 117 lines of output. In comparison, the scalar equation (5.3) has only 8 determining equations, 3 linear and 5 nonlinear, which produce 67 lines of output, greatly simplifying the problem at hand.

Coupled with this strange phenomena is the negligible difference in complexity when considering the scalar equation (5.3) and system (5.7), which is only slightly different from (5.4). Despite 181 lines of output being generated for system (5.7), the presence of 5 linear determining equations allows us to quickly simplify the system of determining equations to system (5.66). The difficulty of solution of this system is comparable with that for the determining equations of (5.3).

Another difficulty we observed was that the reductions obtained using the nonclassical method on (5.4) arise in a very unusual manner, and one has to use a hodograph transformation. Indeed we can establish a set of infinitesimals which would give reduction 5.3.6 more naturally by working the method of characteristics backwards, namely

$$\xi = \frac{\mathrm{d}f}{\mathrm{d}t}, \qquad \phi_2 = 2\frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}P}{\mathrm{d}\zeta}, \qquad \phi_3 = \left(w - \frac{1}{\alpha}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{\mathrm{d}f}{\mathrm{d}t}\right) + 2\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2\frac{\mathrm{d}P}{\mathrm{d}\zeta}, \qquad (5.132)$$

where $\zeta = x + f(t)$, $f(t)$ is an arbitrary function, and $P(\zeta)$ satisfies

$$\frac{\mathrm{d}^3P}{\mathrm{d}\zeta^3} + 2\alpha P\frac{\mathrm{d}P}{\mathrm{d}\zeta} - \frac{\mathrm{d}P}{\mathrm{d}\zeta} = c_1, \tag{5.133}$$

with c_1 an arbitrary constant. It is straightforward to show that the infinitesimals (5.132) satisfy the determining equations arising from the nonclassical method for (5.4) if and only if $P(\zeta) = c_2$, a constant, which leaves a classical reduction. Similarly for reduction 5.3.5 the infinitesimals would need to be

$$\xi = \frac{\mathrm{d}f}{\mathrm{d}t}, \qquad \phi_2 = \frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}\eta}{\mathrm{d}z}, \qquad \phi_3 = \left(w + \frac{1}{\alpha}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{\mathrm{d}f}{\mathrm{d}t}\right) - \left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2\frac{\mathrm{d}\eta}{\mathrm{d}z}, \tag{5.134}$$

where $\eta(z)$ satisfies (5.16) but we find that the determining equations imply that $\dfrac{\mathrm{d}\eta}{\mathrm{d}z} = 0$, giving a classical reduction. Consequently we assert that the infinitesimals arising from the nonclassical method for (5.4) which give rise to the nonclassical reductions are "*unnatural*".

The system (5.5) is shown to give equivalent reductions to (5.7), yet if we were to solve system (5.61) the calculations are horrendous. System (5.61) admits both *natural* and *unnatural* infinitesimals for *both* nonclassical reductions, however these could be hard to find if they were not known *a priori*.

The system (5.6) admits (natural) infinitesimals which give rise only to one of the nonclassical reductions, in the case when $\alpha = \beta$. Therefore not only must we be concerned with simplicity of calculation, but different representations of an equation do not necessarily give equivalent results.

The system (5.7) appears to be the simplest representation of the shallow water wave system and the associated calculations are similar in complexity to the scalar equation (5.3). Certainly the procedure of removal of a nonlocal term which gives rise to (5.4) may well *not* be the best solution.

We have applied the direct method due to Clarkson and Kruskal [1989] to the five systems (5.3)–(5.7), and have obtained the same results as using the nonclassical method. However the application of the direct method is not entirely straightforward either. Whilst we revert to using `diffgrob2` to help with the calculations for the scalar equation (5.3), it is clear how to apply the direct method. This is not so for systems (5.4)–(5.7). We must use ansätze in which each dependent variable depends on all the new symmetry variables. In other applications of the direct method to systems (cf. Lou [1992], Lou and Ruan [1993]) this is not always made clear, though a much simplified ansatz may be sufficient. Also there are two peculiarities which were necessary to reproduce the results of section 5.3.

Firstly, once the ansätze for $v(x,t), w(x,t)$ had been substituted into (5.4b) we had to integrate it and use this integrated form in (5.4a) in order for the resultant (5.4a) to be an ordinary differential equation. If (5.4b) were not integrated, neither of the nonclassical reductions could have been found, even if we had initiated the following second peculiarity.

It was necessary to use (5.7b) in (5.7a) to make (5.7a) an ordinary differential equation i.e. we required that (5.7b) was an ordinary differential equation and then substituted it into (5.7a). Only after this substitution did we require (5.7a) to be an ordinary differential equation, and without it we gained only the classical reductions of section 5.2.

We are therefore led to conclude that both of these techniques must be considered in any direct method calculation involving systems. The problem is magnified particularly in larger systems, since only one equation could be an ordinary differential equation and every other need not be except under the influence of this single equation. Similarly with the need to integrate, which one must to be integrated, if any?

Here we try to formalise the direct method as applied to systems and make comments that hopefully simplify matters.

For a system of n equations in n dependent variables with symmetry variable $\boldsymbol{U}(z) = (U_1(z), ..., U_n(z))$ we represent an equation in the system which is not an ordinary differential equation by

$$E[z, \boldsymbol{U}] + \sum_{j=1}^{m} F_j(x,t) E_j[z, \boldsymbol{U}] = 0. \tag{5.135}$$

The $E[z, \boldsymbol{U}]$, $E_j[z, \boldsymbol{U}]$ are ordinary differential equations in which the order and degree of any symmetry variable is necessarily less than or equal to the order and degree of the original equation. The $F_j(x, t)$ are functions of (x, t) but not of z only. Note the dependence of the $E_j[z, \boldsymbol{U}]$ on $\boldsymbol{U}$; if they are not dependent on any of the dependent symmetry variables they will either be an explicit function of z which certainly cannot be zero and we are finished, or they may be an ordinary differential equation in some other dependent variable which is not a symmetry variable (cf. the role $\eta(z)$ and $P(\zeta)$ play in reductions 5.3.2 and 5.3.1 respectively).

In applying the direct method of Clarkson and Kruskal [1989] we have assumed the maximum number of symmetry variables is equal to the number of dependent variables in the original system. This is a principle difference between the direct method and more general ansatz based methods which allow more (see §1.5.3 which describes the methods of Galaktionov [1990] and Olver [1994]). Note that for scalar equations ($n = 1$) the existence of the scenario (5.135) would mean that no symmetry reductions using the direct method were possible. This represents another difference between the direct method and other ansatz based methods which allow more than n symmetry equations (see §1.5.3 for the methods of Estévez [1992] and Burdé [1994]).

For (5.135) to be an ordinary differential equation it is necessary for $E_j[z, \boldsymbol{U}] = 0$ for $j = 1, ..., m$, which we require to come from other equations in the system. Note that in system (5.7), "$E_1[z, \boldsymbol{U}] = 0$" came from (5.7b) whilst in system (5.4) we had $\dfrac{\mathrm{d}}{\mathrm{d}z}$"$E_1[z, \boldsymbol{U}] = 0$" from (5.4b) which we had to integrate first. However note that a single equation in the system might contribute to making more than one $E_j[z, \boldsymbol{U}] = 0$.

The need for an $E_j[z, \boldsymbol{U}] = 0$ to make our equation (5.135) an ordinary differential equation is not known either *a priori* or *a posteriori* (we do not always have the luxury of knowing the reductions in advance!). There is no loss of generality with simple substitution (without integration); if it turns out that the substitution was not needed for more reductions, the full set of reductions will still be found (since if an equation was an ordinary differential equation without substitution it will also be an ordinary differential equation with substitution). However if an equation is integrated this may rule out some reductions, in which integration is not possible (see Case 5.4.2 – after integration it is not possible to retrieve classical reduction 5.2.1).

It is clear that the order of $E_j[z, \boldsymbol{U}] = 0$ must be less than or equal to the order of (5.135). Thus for simple substitution (without integration) a symmetry equation must have order less than or equal to the order of (5.135) if it is to be considered as a candidate for substitution into (5.135). This means that for instance in Case 5.4.5, considering system (5.7) there was no need to consider the case when (5.7a) is substituted into (5.7b).

It is less obvious as to which equations we may need to integrate. Whilst the order of $E_j[z, \boldsymbol{U}] = 0$ must be less than or equal to the order of (5.135), the order of a candidate symmetry equation if initially greater than that of (5.135) may be reduced by integration. Hence the class of candidate symmetry equations to be considered is extended.

I believe that it is no coincidence that integration was necessary in Case 5.4.2, and a hodograph transformation was needed in Case 5.3.2, the equivalent case for the nonclassical method. Also the fact that the obvious infinitesimals, (5.132) and (5.134), failed seems to be linked. Since in the large amount of literature on the nonclassical method, occurrences of the latter two phenomena are rare, if not unique, the same may be true for integration in the direct method, in which case it will often be unnecessary. However this is just speculation and the fact remains that the integration of the symmetry equations must be considered in the direct method if we are to find all possible reductions.

In the nonclassical method, the need for the hodograph transformation makes the calculation more difficult. However it is clear how to apply the nonclassical method to systems and indeed the identification of this difficulty in this chapter surely makes future calculations more amenable. The opposite is true of the findings in this chapter for the direct method. Whilst the direct method can still be attempted manually, with so much stacked against it, however seldom such phenomena occur, one must conclude that in applications to systems of partial differential equations the nonclassical method (with the tools we have at our disposal) is better than the direct method.

Chapter Six : Six representations of the Boussinesq Equation

6.1 Introduction

In this chapter we consider six different systems that have as their compatibility condition, the Boussinesq equation

$$u_{xxxx} + uu_{xx} + u_x^2 + u_{tt} = 0, \tag{6.1}$$

which was also a special case of the generalised Boussinesq equation we studied in Chapter Two (up to a scaling of t). The physical applicability of the Boussinesq equation (6.1) is discussed in Chapter Two, as are its classical and nonclassical symmetries (though these were already well known).

A motivation for studying these systems is provided by Clarkson [1995] who poses the question " *Can any additional reductions of the Boussinesq equation be generated through any of these potential representations?* " The question alludes to both the classical and nonclassical methods of symmetry reduction and the six representations

1. $v_x = -u_t,$ (6.2a)

   $v_t = u_{xxx} + uu_x,$ (6.2b)

2. $v_x = -xu_t,$ (6.3a)

   $v_t = xu_{xxx} - u_{xx} + xuu_x - \frac{1}{2}u^2,$ (6.3b)

3. $v_x = u - tu_t,$ (6.4a)

   $v_t = tu_{xxx} + tuu_x,$ (6.4b)

4. $v_x = u - (x+t)u_t,$ (6.5a)

   $v_t = (x+t)u_{xxx} - u_{xx} + (x+t)uu_x - \frac{1}{2}u^2,$ (6.5b)

5. $v_x = xu - xtu_t,$ (6.6a)

$$v_t = xtu_{xxx} - tu_{xx} + xtuu_x - \tfrac{1}{2}tu^2, \tag{6.6b}$$

6. $$v_x = xu - (1+xt)u_t, \tag{6.7a}$$

$$v_t = (1+xt)u_{xxx} - tu_{xx} + (1+xt)uu_x - \tfrac{1}{2}tu^2, \tag{6.7b}$$

are those that Clarkson is referring to, which were written down by Bluman [1995].

A more general motivation is the search for potential symmetries *per se*, and in particular nonclassical potential symmetries (see §1.5.1). The extension of the nonclassical method to potential systems is formalised by Bluman and Shtelen [1995], who also extend these ideas further (see §1.5.1 for details). Also, the concept of nonclassical potential symmetries is implicitly referred to by Clarkson [1995] in his question above, and Priestley and Clarkson [1995] discuss the existence of nonclassical potential symmetries, when the criteria for (classical) potential symmetries do not carry through to their nonclassical counterparts. (Namely that the existence of potential symmetries is assured if any of the infinitesimals of the non-potential variables depend explicitly on the potential variables, whereas such a condition has no implications when considering nonclassical potential symmetries.) This is also discussed in §6.5, and Chapter Seven focuses also on nonclassical potential symmetries.

In the remaining sections of this chapter we apply the classical method (§6.2), the nonclassical method in the generic ($\tau \neq 0$) case (§6.3) and the direct method in the generic ($z_x \neq 0$) case (§6.4) to our six potential systems. Each section begins with a brief summary of the relevant results for the scalar Boussinesq equation (6.1), and is followed by the findings for the potential systems. In the final section (§6.5) these results are compared and discussed.

We include the calculations of the direct method in this chapter not only for comparison with the results of the nonclassical method but also to clarify some of the observations made in Chapter Five on its application to systems of partial differential equations and to realise their importance.

6.2 Classical symmetries

To apply the classical method we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u, v) given by

$$x^* = x + \varepsilon\xi(x,t,u,v) + O(\varepsilon^2), \tag{6.8i}$$

$$t^* = t + \varepsilon\tau(x,t,u,v) + O(\varepsilon^2), \tag{6.8ii}$$

$$u^* = u + \varepsilon\phi_1(x,t,u,v) + O(\varepsilon^2), \tag{6.8iii}$$

$$v^* = v + \varepsilon\phi_2(x,t,u,v) + O(\varepsilon^2), \tag{6.8iv}$$

where ε is the group parameter. This procedure, which is implemented in `symmgrp.max`, yields a system of linear determining equations for each case. By applying the **KolRitt** procedure in diffgrob2 these linear equations are simplified greatly. It is then routine to find the infinitesimals, which we summarise in table 6.2.1. along with the classical infinitesimals for the scalar Boussinesq equation (6.1).

Table 6.2.1

System	ξ	τ	ϕ_1	ϕ_2
(6.1)	c_1x+c_2	$2c_1t+c_3$	$-2c_1u$	
(6.2)	c_1x+c_2	$2c_1t+c_3$	$-2c_1u$	$-3c_1v+c_4$
(6.3)	c_1x	$2c_1t+c_3$	$-2c_1u$	$-2c_1v+c_4$
(6.4)	c_1x+c_2	$2c_1t$	$-2c_1u$	$-c_1v+c_4$
(6.5)	c_1	$-c_1$	0	c_4
(6.6)	c_1x	$2c_1t$	$-2c_1u$	c_4
(6.7)	0	0	0	c_4

6.3 Nonclassical symmetries ($\tau \neq 0$)

As a reminder, when finding the nonclassical symmetries of the Boussinesq equation the determining equations reduce to solving

$$\xi_u = 0, \tag{6.9i}$$

$$4\xi\xi_x^2 - 2\xi_x\xi_t - \xi_{tt} = 0, \tag{6.9ii}$$

$$\xi_{xx} = 0, \tag{6.9iii}$$

$$2\xi_x u + \phi_1 + 4\xi^2\xi_x + 2\xi\xi_t = 0, \tag{6.9iv}$$

which has solution, either

$$\xi = c_4t + c_5, \qquad \phi_1 = -2c_4(c_4t + c_5), \tag{6.10}$$

or

$$\xi = f(t)x + g(t),$$
$$\phi_1 = -2fu - 2f\left(2f^2 + \frac{\mathrm{d}f}{\mathrm{d}t}\right)x^2 - 2\left(4f^2g + 2g\frac{\mathrm{d}f}{\mathrm{d}t} + 2f\frac{\mathrm{d}g}{\mathrm{d}t}\right)x - 2g\left(2gf + \frac{\mathrm{d}g}{\mathrm{d}t}\right),$$

where

$$f(t) = \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\ln\psi(t)], \qquad g(t) = \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\ln\psi(t)]\left(c_6 + c_7\int^t \psi(s)\Big/\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 \mathrm{d}s\right),$$

and $\psi(t)$ satisfies

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 = c_8\psi^3 + c_9, \tag{6.11}$$

where c_1–c_9 are arbitrary constants. These infinitesimals then yield six canonical reductions (which can be seen in Chapter Two).

Remark 6.3(i). If we find for any of our systems that ξ, ϕ_1 are independent of v, then to find the reduction $u(x,t) = F(x,t,U(z))$ for $z = z(x,t,u)$ we do not need to know ϕ_2 since we may solve the invariant surface condition,

$$\xi u_x + u_t = \phi_1,$$

without this knowledge. We must however find all the compatibility conditions of the determining equations before we can say we know ξ, ϕ_1. Whilst we will not be able to write down the full reduction without knowing ϕ_2, by knowing ξ, ϕ_1 we can compare our results with the scalar Boussinesq symmetries above. Hence if we encounter this situation ($\xi_v = \phi_{1,v} = 0$) in the following, we do not explicitly calculate ϕ_2. (As compensation for not being able to write the reductions down, they are included in the workings of the direct method.)

Case 6.3.1 System (6.2). In this Case (and in fact in the remaining five Cases) using the algorithm of Clarkson and Mansfield [1994c] in the generation of the nonclassical determining equations requires the removal of u_t and v_t using the invariant surface condition. Applying the classical method to this new system yields the following overdetermined system of nonlinear equations.

$$\xi\xi_v + \xi_u = 0, \tag{6.12i}$$

$$\xi^2\xi_{vv} + \xi\xi_v^2 + \xi_u\xi_v + \xi_{uu} + 2\xi\xi_{uv} = 0, \tag{6.12ii}$$

$$\xi\phi_{2,v} + \phi_{2,u} - \xi_v\phi_2 - \xi^2\phi_{1,v} - \xi\phi_{1,u} + \xi\xi_v\phi_1 - \xi\xi_x - \xi_t = 0, \tag{6.12iii}$$

$$\phi_1\phi_{2,v} - \phi_{2,x} - \phi_{1,v}\phi_2 - \xi\phi_1\phi_{1,v} - \phi_1\phi_{1,u} - \phi_{1,t} + \xi_v\phi_1^2 - \xi_x\phi_1 = 0, \tag{6.12iv}$$

$$\begin{aligned}&\xi^3\xi_{vvv} + 4\xi^2\xi_v\xi_{vv} + 3\xi\xi_u\xi_{vv}\\&\quad + \xi\xi_v^3 + \xi_u\xi_v^2 + \xi_{uu}\xi_v + 5\xi\xi_{uv}\xi_v + \xi_{uuu} + 3\xi\xi_{uuv} + 3\xi^2\xi_{uvv} + 3\xi_u\xi_{uv} = 0,\end{aligned} \tag{6.12v}$$

$$\begin{aligned}&3\xi^2\phi_{1,vv} + 7\xi\xi_v\phi_{1,v} + 3\xi_u\phi_{1,v} + 3\phi_{1,uu} + 6\xi\phi_{1,uv}\\&\quad + 4\xi_v\phi_{1,u} + 9\xi\xi_{vv}\phi_1 + 5\xi_v^2\phi_1 + 9\xi_{uv}\phi_1 - 5\xi_v\xi_x - 9\xi\xi_{xv} - 9\xi_{xu} = 0,\end{aligned} \tag{6.12vi}$$

$$\begin{aligned}&3\xi\phi_1\phi_{1,vv} - 3\xi_v\phi_{1,x} - 3\xi\phi_{1,xv} + \xi\phi_{1,v}^2 + \phi_{1,u}\phi_{1,v}\\&\quad + 5\xi_v\phi_1\phi_{1,v} - 2\xi_x\phi_{1,v} - 3\phi_{1,xu} + 3\phi_1\phi_{1,uv} + 3\xi_{vv}\phi_1^2 - 6\xi_{xv}\phi_1 + 3\xi_{xx} = 0,\end{aligned} \tag{6.12vii}$$

$$\begin{aligned}&\phi_1\phi_{1,v}u + 3\xi\xi_x\phi_1 - 2\xi\xi_v\phi_1^2 + 4\phi_1^2\phi_{1,v}\phi_{1,vv} - \xi\phi_1\phi_{1,u} - \xi^2\phi_1\phi_{1,v}\\&\quad + \xi\phi_1\phi_{2,v} - \xi\phi_{1,v}\phi_2 - 3\phi_1\phi_{1,vv}\phi_{1,x} + \phi_{2,t} - \phi_{1,xxx} - 2\xi_v\phi_1\phi_2 - 5\phi_1\phi_{1,v}\phi_{1,xv}\\&\quad - \phi_{1,x}u + \phi_1\phi_{2,u} + \phi_2\phi_{2,v} - \phi_{1,u}\phi_2 + \phi_{1,v}\phi_{1,xx} + \phi_1^3\phi_{1,vvv} - \phi_{1,v}^2\phi_{1,x}\\&\quad + \phi_1\phi_{1,v}^3 + \xi_t\phi_1 + \xi_u\phi_1^2 - 3\phi_1^2\phi_{1,xvv} + 3\phi_1\phi_{1,xxv} + 3\phi_{1,xv}\phi_{1,x} + 3\xi_x\phi_2 = 0,\end{aligned} \tag{6.12viii}$$

$$\begin{aligned}&4\xi^2\xi_{vv}\phi_{1,v} + 5\xi\xi_v\phi_{1,uv} - 5\xi\xi_v\xi_{xv} + 3\xi\xi_u\phi_{1,vv} - 2\xi_{xu}\xi_v - 6\xi\xi_{xuv}\\&\quad + 3\xi_u\phi_{1,uv} + 5\xi\xi_{uv}\phi_{1,v} + 3\xi\xi_v^2\phi_{1,v} + \xi^3\phi_{1,vvv} + \xi_{uu}\phi_{1,v} + \xi_v\phi_{1,uu} - \xi_v^2\xi_x\end{aligned}$$

$$+ 3\xi\phi_{1,uuv} + 3\xi^2\phi_{1,uvv} + 3\xi_{uv}\phi_{1,u} + 3\xi_{uuv}\phi_1 - 3\xi_{uv}\xi_x - 3\xi^2\xi_{xvv} - 3\xi_u\xi_{xv}$$
$$+ \xi_v^3\phi_1 + \xi_v^2\phi_{1,u} + 3\xi_u\xi_{vv}\phi_1 - 3\xi\xi_{vv}\xi_x + 2\xi_u\xi_v\phi_{1,v} + 8\xi\xi_v\xi_{vv}\phi_1 + 6\xi\xi_{uvv}\phi_1$$
$$+ 5\xi_{uv}\xi_v\phi_1 + 3\xi^2\xi_{vvv}\phi_1 + 3\xi\xi_{vv}\phi_{1,u} + \phi_{1,uuu} - 3\xi_{xuu} + 4\xi^2\xi_v\phi_{1,vv} = 0, \tag{6.12ix}$$

$$\phi_1 - 3\xi_x\phi_1\phi_{1,vv} + 3\xi_v\phi_1\phi_{1,v}^2 - 3\xi\xi_v\phi_2 - 2\xi_v\phi_1 u - 3\xi\phi_{1,vv}\phi_{1,x}$$
$$+ 4\xi_{vv}\phi_1^2\phi_{1,v} + 4\xi_v\phi_1^2\phi_{1,vv} - 5\xi_{xv}\phi_1\phi_{1,v} + \xi_v\phi_{1,xx} - 2\xi\xi_u\phi_1 + 5\phi_1\phi_{1,uv}\phi_{1,v}$$
$$- 5\xi_v\phi_1\phi_{1,xv} - 2\phi_{1,xu}\phi_{1,v} + 3\phi_1^2\phi_{1,uvv} - 6\phi_1\phi_{1,xuv} - 3\xi_{xvv}\phi_1^2 - 5\xi\phi_{1,v}\phi_{1,xv}$$
$$- 6\xi\phi_1\phi_{1,xvv} + 3\xi\phi_1^2\phi_{1,vvv} - 3\phi_{1,u}\phi_{1,xv} + 3\xi_{xv}\phi_{1,x} + 3\xi\phi_{1,xxv} + 3\xi_x\phi_{1,xv}$$
$$- 3\xi_{vv}\phi_1\phi_{1,x} - 2\xi_v\phi_{1,v}\phi_{1,x} - 5\xi^2\xi_v\phi_1 + 8\xi\phi_1\phi_{1,v}\phi_{1,vv} - \xi_{xxx} + 3\xi_{xxv}\phi_1 + 3\xi^2\xi_x$$
$$+ 3\phi_{1,xxu} - \xi^3\phi_{1,v} + \xi\phi_{2,u} + \xi^2\phi_{2,v} + \xi\phi_{1,v}^3 + \phi_{1,u}\phi_{1,v}^2 - \xi_x\phi_{1,v}^2 + \xi_{xx}\phi_{1,v}$$
$$- \xi^2\phi_{1,u} + \xi_{vvv}\phi_1^3 + \xi\xi_t + 2\xi_x u - 4\xi_u\phi_2 - 3\phi_{1,uv}\phi_{1,x} + 3\phi_1\phi_{1,u}\phi_{1,vv} = 0, \tag{6.12x}$$

$$3\xi_{vv}\phi_1\phi_{1,u} - 3\xi\xi_v u + 6\xi\phi_1\phi_{1,uvv} - 6\xi\xi_{xvv}\phi_1 - 5\xi_v\xi_{xv}\phi_1 + 3\xi\xi_v\phi_{1,v}^2 + 3\xi_u\phi_1\phi_{1,vv}$$
$$+ 5\xi_{uv}\phi_1\phi_{1,v} + 3\xi_v^2\phi_1\phi_{1,v} + \xi_u\phi_{1,v}^2 + 4\xi^2\phi_{1,v}\phi_{1,vv} - 2\xi_v\xi_x\phi_{1,v} + 3\xi\phi_{1,u}\phi_{1,vv} - 3\xi_u u$$
$$- 3\xi\xi_x\phi_{1,vv} + 5\xi_v\phi_1\phi_{1,uv} + \xi_v\xi_{xx} - 3\xi_{vv}\xi_x\phi_1 - 3\xi_{uv}\phi_{1,x} - 3\xi_u\phi_{1,xv} - 5\xi\xi_v\phi_{1,xv}$$
$$+ 3\xi^2\phi_1\phi_{1,vvv} - 3\xi\xi_{vv}\phi_{1,x} + \phi_{1,uu}\phi_{1,v} - \xi_v^2\phi_{1,x} + 8\xi\xi_v\phi_1\phi_{1,vv} + 4\xi_v\xi_{vv}\phi_1^2 + 3\xi\xi_{vvv}\phi_1^2$$
$$+ 2\xi_v\phi_{1,u}\phi_{1,v} + 5\xi\phi_{1,uv}\phi_{1,v} - 5\xi\xi_{xv}\phi_{1,v} - 3\xi^3\xi_v - 3\xi^2\xi_u - 3\phi_{1,xuu} + 3\xi_{xxu}$$
$$+ 3\xi_{uvv}\phi_1^2 + 3\phi_1\phi_{1,uuv} - 2\xi_v\phi_{1,xu} - 6\xi\phi_{1,xuv} + 3\phi_{1,u}\phi_{1,uv} - 3\xi_x\phi_{1,uv} - 3\xi_{xv}\phi_{1,u}$$
$$- 6\xi_{xuv}\phi_1 + 3\xi_{xv}\xi_x + 3\xi\xi_{xxv} - 2\xi_{xu}\phi_{1,v} - 3\xi^2\phi_{1,xvv} + 8\xi\xi_{vv}\phi_1\phi_{1,v} = 0. \tag{6.12xi}$$

The nonclassical determining equations of the other five systems are similar in both size and content, so for conciseness we do not include them in this thesis. It is due to these similarities that the steps taken to solve each system of determining equations are essentially of the same form. Hence we provide a detailed account of the solution of system (6.12), whereas for the other five systems the detail is minimal.

For the moment we will assume that $\xi_v\xi_u \neq 0$. We **reduce**((6.12vi),[(6.12i)], $k1$) and then use $k1$ to **reduce**((6.12ix),[$k1$,(6.12i)], $k2$), to leave a nonlinear first order equation in ϕ_1. Then reduce((6.12vii),[(6.12i),$k2$], $k3$) and use this to reduce((6.12x),[(6.12i),$k2$,$k3$], $k4$) which leaves ϕ_1 equal to some function of ϕ_2 (and ξ) and its derivatives. To get ϕ_1 in terms of ξ and its derivatives only we reduce((6.12iii),[(6.12i),$k2$,$k4$], $k5$) and then reduce((6.12iii),[(6.12i),$k2$,$k5$], $k6$), which leaves

$k6$: $$\phi_1\xi_v - \xi_x = 0,$$

which can be substituted into $k5$ to give

$$2\xi\xi_v^2\phi_2 + 2\xi\xi_v\xi_t + \xi_x = 0. \tag{6.13}$$

With ϕ_1 and ϕ_2 found we **reduce** all the determining equations with respect to $k6$, (6.13) and (6.12i) which leaves only two conditions that ξ must satisfy,

$$\xi_v^2\xi_{xx} - 2\xi_v\xi_{xv}\xi_x + \xi_{vv}\xi_x^2 = 0, \tag{6.14i}$$

$$8\xi^3\xi_v^3\xi_{tv}\xi_t - 4\xi^3\xi_v^4\xi_{tt} - 4\xi^2\xi_v^3\xi_x^2 - 2\xi^2\xi_v^3\xi_{xt} + 2\xi^2\xi_v^2\xi_{xv}\xi_t$$
$$+ 6\xi^2\xi_v^2\xi_{tv}\xi_x - 4\xi^3\xi_{vv}\xi_v^2\xi_t^2 - \xi_v^2\xi_x^2 - 6\xi^2\xi_{vv}\xi_v\xi_t\xi_x + \xi_v\xi\xi_{xv}\xi_x - 2\xi\xi_{vv}\xi_x^2 = 0. \qquad (6.14\text{ii})$$

The difficulty now is to solve these two equations together with (6.12i). There are a number of ways we can get some of the way there, for instance assuming ξ is separable, but finding the complete solution, and one that may be easily interpreted, seems intractable. Using `diffgrob2` produces large expression swells, and termination of the algorithms thus seems unlikely. Equation (6.14i) bears a resemblance to the Born-Infeld equation, which may be solved using hodograph transformations (cf. Whitham [1974], p. 617). Attempting to solve (6.14i) in this manner is feasible, but the solution (as one would expect) is not easy to interpret. We therefore bypass the problem, by beginning step two of the nonclassical method, solving the invariant surface conditions. To illustrate the following steps, on the right of the general working we will show what happens in the special case $\xi = v/u$, $\phi_1 = \phi_2 = 0$ which satisfies the determining equations. The system we must solve is

$$v_x = -u_t, \qquad (6.2\text{a})$$

$$v_t = u_{xxx} + uu_x, \qquad (6.2\text{b})$$

$$\xi(x,t,u,v)u_x + u_t = \phi_1(x,t,u,v), \qquad (6.15\text{i})$$

$$\xi(x,t,u,v)v_x + v_t = \phi_2(x,t,u,v). \qquad (6.15\text{ii})$$

We use (6.12i), (6.2a) and $k6$ to leave (6.15i) in the following form

$$\xi_u u_x + \xi_v v_x + \xi_x = 0, \qquad -\frac{v}{u^2}u_x + \frac{1}{u}v_x = 0.$$

At this stage u, v are now dependent variables and ξ is some function of u, v (and maybe x, t also). We can write this as

$$\mathrm{D}_x\left(\xi\right) = 0, \qquad \mathrm{D}_x\left(\frac{v}{u}\right) = 0,$$

where D_x is the total derivative (1.14). We can integrate this with respect to x, which leaves an arbitrary function of t, our other independent variable

$$\xi = f_1(t), \qquad \frac{v}{u} = f_2(t). \qquad (6.16\text{i,ii})$$

Remember ξ is a function of u, v (and maybe x, t also). If ξ was known explicitly (6.16i) would give a relation between u and v (e.g. in (6.16ii) we have $v = f_2(t)u$). We then use this relation to give the new invariant surface conditions

$$f_1(t)u_x + u_t = \phi_1, \qquad (6.17\text{i})$$

$$f_1(t)v_x + v_t = \phi_2. \qquad (6.17\text{ii})$$

where ξ has been replaced by $f_1(t)$ because v is related to u from (6.16i) in this way. Notice we are not saying that the infinitesimal ξ is a function of t only, because (i) this contradicts $\xi_u\xi_v \neq 0$, and (ii) v is now related to u (from (6.16i)) and when we think of infinitesimals, u and v are independent. Notice also that ϕ_1, ϕ_2 are unknown still. However, knowing that the invariant surface conditions (when $\xi_u\xi_v \neq 0$) will end up looking like (6.17) we can start from scratch with these conditions. This is equivalent to our infinitesimal ξ being a function of t only in the determining equations, thus we may assume without loss of generality that $\xi_u = \xi_v = 0$. In our example where $\xi = v/u$, $\phi_1 = \phi_2 = 0$ we have seen that $v = f_2(t)u$, which produces the invariant surface conditions

$$f_2(t)u_x + u_t = 0,$$
$$f_2(t)v_x + v_t = 0.$$

Solving these together with the original system (6.2), we arrive at the (classical) travelling wave solution.

Thus assuming $\xi_u = \xi_v = 0$, equation (6.12vi) reduces to

$$\xi^2\phi_{1,vv} + 2\xi\phi_{1,uv} + \phi_{1,uu} = 0, \tag{6.18}$$

and using this additional relation the eleventh equation reduces, on factorisation, to

$$(\xi\phi_{1,vv} + \phi_{1,uv})(\phi_{1,u} + \xi_x + \xi\phi_{1,v}) = 0. \tag{6.19}$$

If we assume the first bracket of (6.19) is zero, and solve together with (6.18) then we get the relation

$$\phi_{1,u} + \xi\phi_{1,v} = h(x,t), \tag{6.20}$$

and assuming the second bracket is zero merely means $h(x,t) = -\xi_x$. Keeping a general h, we proceed by reducing equation (6.12vii) with respect to the information we know to give

$$\xi_x\phi_{1,v} + 3\xi_{xx} - 3h_x + h\phi_{1,v} = 0. \tag{6.21}$$

For $h \neq -\xi_x$, this implies that $\phi_{1,vv} = 0$ and also $\phi_{1,vu} = 0$, so from (6.18) $\phi_{1,uu} = 0$ as well. Then equation (6.12x), when reduced with respect to this information and (6.12iii), becomes essentially

$$\phi_1 + 2\xi_x u + k(x,t) = 0. \tag{6.22}$$

Thus $\phi_{1,v} = 0$ and either (i) $\phi_{1,u} = 0$ and $\xi_x = 0$ or (ii) $\phi_{1,u} = -2\xi_x$ (and $h = -2\xi_x$ from (6.20)).

(i) Equation (6.22) simplifies to yield $\phi_1 = -2\xi\xi_t$. The remaining determining equations now only need to be solved for ϕ_2 which is not entirely necessary in as much as ϕ_2 is not needed explicitly in order to solve the invariant surface conditions for the information we need, but we must still find the remaining compatibility conditions, so that ξ can be found explicitly (cf. Remark 6.3(i)). In doing this we find $\xi_{tt} = 0$ which yields a known nonclassical reduction.

(ii) A further condition can be found, from (6.21) that $\xi_{xx} = 0$ and from (6.22) we have $\phi_1 = -2\xi_x u - 4\xi^2\xi_x - 2\xi\xi_t$. Our remaining equations have (6.9ii) as their compatibility condition, thus we have duplicated system (6.9).

If we assume $h(x,t) = -\xi_x$ then (6.21) tells us $\xi_{xx} = 0$ and the tenth equation is like (6.22) with $k(x,t) = 4\xi^2\xi_x + 2\xi\xi_t$. Comparing (6.22) and (6.20) we see $\xi_x = 0$ in order to keep consistency, and we have returned to option (i) above. In conclusion, we have found all the known nonclassical reductions but no more.

Case 6.3.2 System (6.3). As with the previous Case there are eleven determining equations and the first plays a significant role

$$x\xi\xi_v + \xi_u = 0. \tag{6.23}$$

We initially assume $\xi_u\xi_v \neq 0$, and solve for ϕ_1 and ϕ_2 as before to yield

$$\xi_v\phi_1 x^2 - \xi_x x + \xi = 0, \tag{6.24i}$$

$$2\xi_v^2\xi\phi_2 x^2 + 4\xi_v\xi^3 x + 2\xi_v\xi x u + 2\xi_v\xi\xi_t x^2 + \xi_x x - \xi = 0, \tag{6.24ii}$$

with two lengthy conditions for ξ

$$\begin{aligned}
&4\xi\xi_v^3 xu - \xi^2\xi_{vv} - 3\xi\xi_v^2 - 2x\xi_v\xi\xi_{xv} \\
&\quad + 2x\xi\xi_{vv}\xi_x - \xi_v^2 x^2\xi_{xx} + 3x\xi_v^2\xi_x - x^2\xi_{vv}\xi_x^2 + 2\xi_v\xi_{xv}x^2\xi_x = 0,
\end{aligned} \tag{6.25i}$$

$$\begin{aligned}
&16\xi^7\xi_{vv}x^2\xi_v^2 - 16\xi^5\xi_v^3 x^3\xi_{tv} + 4\xi_v^3\xi^4 x - 8\xi^3\xi_v^3 x^4\xi_{tv}\xi_t + 16\xi^5\xi_{vv}x^2\xi_v^2 u \\
&\quad + 16\xi^4\xi_v^4 x^3\xi_t + 16\xi^5\xi_{vv}x^3\xi_v^2\xi_t + 6\xi^2\xi_{vv}\xi_x x^2 u\xi_v + 6\xi^2\xi_{vv}\xi_x x^3\xi_v\xi_t + 2\xi^2\xi_v^3 x^3\xi_{xt} \\
&\quad + 8\xi_v^4\xi^4 - 2\xi^2\xi_v^2\xi_{xv}x^2 u - 4\xi^3\xi_v^3 x^2\xi_x - 12\xi_v^4 x^2\xi^4 u + 2\xi^3\xi_{vv} \\
&\quad + \xi_v\xi^2\xi_{xv}x - \xi_v\xi\xi_{xv}x^2\xi_x - 2\xi^2\xi_v^2\xi_{xv}x^3\xi_t - \xi_x\xi\xi_v^2 x - 4x^2\xi^4\xi_v^2\xi_{xv} \\
&\quad + 12x^2\xi^4\xi_v\xi_{vv}\xi_x + 8\xi^3\xi_{vv}x^3\xi_v^2\xi_t u - 8\xi^3\xi_v^3 x^3 u\xi_{tv} + 8\xi^4\xi_v^5 x^3 u^2 - 6\xi^3\xi_{vv}x^2\xi_v\xi_t \\
&\quad - 12x\xi_v\xi^5\xi_{vv} - 6\xi^3\xi_{vv}xu\xi_v + 4\xi^3\xi_{vv}x^4\xi_v^2\xi_t^2 + 2\xi_v^3\xi_x x^2 u\xi + 2x^2\xi\xi_{vv}\xi_x^2 \\
&\quad + 12\xi^3\xi_v^4 x^3\xi_x u + 4\xi^3\xi_{vv}x^2 u^2\xi_v^2 - 4x\xi^2\xi_{vv}\xi_x - 16\xi_v^5 x\xi^4 u + 4\xi^3\xi_v^4 x^4\xi_{tt} \\
&\quad + 4x^3\xi_v^3\xi^2\xi_x^2 - 6\xi^2\xi_v^2 x^3\xi_{tv}\xi_x - 8\xi_v^4\xi^3\xi_x x + 6\xi^3\xi_v^2 x^2\xi_{tv} + \xi_v^2\xi_x^2 x^2 = 0.
\end{aligned} \tag{6.25ii}$$

As these conditions for ξ are more difficult to solve than system (6.14) the situation is more desperate if we were trying to find a complete solution. Similar to the previous Case however, we bypass this. We use (6.3a), (6.23) and (6.24i) to leave the invariant surface condition for u in the following form

$$\frac{\xi_u}{x}u_x + \frac{\xi_v}{x}v_x + \frac{\xi_x}{x} - \frac{\xi}{x^2} = 0.$$

Then we can write this as

$$\mathrm{D}_x\left(\frac{\xi}{x}\right)=0,$$

where D_x is the total derivative. We can integrate this with respect to x, which leaves an arbitrary function of t, our other independent variable

$$\frac{\xi}{x}=f(t).$$

We then use this relation to give the new invariant surface conditions

$$xf(t)u_x+u_t=\phi_1, \tag{6.26i}$$

$$xf(t)v_x+v_t=\phi_2, \tag{6.26ii}$$

where ξ has been replaced by $xf(t)$. Knowing that the invariant surface conditions must look like (6.26) when $\xi_u\xi_v\neq 0$ we can assume without loss of generality that $\xi_u=\xi_v=0$.

With $\xi_u=\xi_v=0$ we can find both ϕ_1 explicitly and ξ, together with the compatibility conditions, as previously which yields

$$\xi=f(t)x \qquad \phi_1=-2fu-2x^2\left(f\frac{\mathrm{d}f}{\mathrm{d}t}+2f^3\right), \tag{6.27}$$

where $f(t)$ satisfies

$$2f\frac{\mathrm{d}f}{\mathrm{d}t}-4f^3+\frac{\mathrm{d}^2f}{\mathrm{d}t^2}=0.$$

Recall that for the scalar Boussinesq equation (6.1), $\xi=f(t)x+g(t)$, where f and g satisfied various conditions. In this case we have the same f and $g\equiv 0$, thus no new reductions.

<u>*Case 6.3.3 System (6.4)*</u>. Initially assuming $\xi_u\xi_v\neq 0$ we solve for ϕ_1 and ϕ_2, which yields

$$\xi_v\phi_1t^2-\xi_xt-\xi_vtu-1=0, \tag{6.28i}$$

$$2\xi_v^2\xi\phi_2t^2-2\xi_v^2\xi^2t^2u-\xi_vtu-4\xi_v\xi^2t+2\xi_v\xi\xi_tt^2+1+\xi_xt=0, \tag{6.28ii}$$

with two conditions for ξ, more lengthy than (6.25). We use the same procedure as used in Cases 6.3.1 and 6.3.2 to show that when $\xi_u\xi_v\neq 0$ the determining equations will yield no more than when $\xi=f(t)-x/t$, so we assume $\xi_u=\xi_v=0$. We are now able to show that system (6.9) is compatible with the determining equations in this Case, and thus we have the full nonclassical complement of reductions. However there are no more infinitesimals, beyond the known nonclassical ones.

<u>*Case 6.3.4 System (6.5)*</u>. Assuming $\xi_u\xi_v\neq 0$ we can find ϕ_1 and ϕ_2 in a similar fashion to above, but substituting these expressions into the determining equations leads

to the contradiction $\xi_v = 0$ (and hence $\xi_u = 0$ from the first determining equation $(x+t)\xi\xi_v + \xi_u = 0$). Continuing to solve the determining equations we find only the infinitesimals $\xi = -1$, $\phi_1 = 0$ (and ϕ_2 is constant) which is the same as the classical method for system (6.5).

Case 6.3.5 System (6.6). We again find ϕ_1 and ϕ_2 by firstly assuming $\xi_u\xi_v \neq 0$, and similar to Case 6.3.4 above we find that $\xi_v = 0$. Then because the first determining equation reads $xt\xi\xi_v + \xi_u = 0$, we have $\xi_u = 0$ also. Solving the now simplified system of determining equations we encounter the same situation as Case 6.3.2 above, namely that whereas in the scalar Boussinesq equation we find $\xi = f(t)x + g(t)$, here we have $g(t) \equiv 0$ and ϕ_1 is affected accordingly.

Case 6.3.6 System (6.7). The first determining equation reads $(1+xt)\xi\xi_v + \xi_u = 0$, and similar to the two previous Cases we find $\xi_v = 0$, and hence $\xi_u = 0$. The remaining determining equations then yield the infinitesimals

$$\xi = -\frac{x}{t}, \qquad \phi_1 = \frac{2u}{t} + \frac{6x^2}{t^3},$$

which leads to a known nonclassical reduction.

6.4 Direct Method

In this section we apply the direct method to each of our six potential systems, in order to extend and clarify the workings in §5.4 and the discussion in §5.5, which throw light on the application of the direct method to systems of partial differential equations.

We will apply the direct method to each of the systems (6.2)–(6.7) in turn, by considering the ansätze

$$u(x,t) = F(x,t,U(z),V(z)), \tag{6.29a}$$

$$v(x,t) = G(x,t,U(z),V(z)), \tag{6.29b}$$

where $z = z(x,t)$. For each system we consider first the requirement that each equation in the system is an ordinary differential equation, then the requirement that the equation with lower derivatives (equations (6.2a),...,(6.7a)) be an ordinary differential equation, and the other equation be an ordinary differential equation only after this first equation is substituted into it.

As with previous applications of the direct method we will use the upper case Greek alphabet (without subscripts) to represent undetermined functions, and such that under any operation the result is given the same letter. Once we give these letters a subscript or use the lower case Greek alphabet we keep track of our operations.

We make use of three freedoms, the first two of which are described here for the new symmetry variable $U(z)$ but which apply equally well to $V(z)$. The freedoms may be applied once for each symmetry variable without loss of generality. For convenience of notation in the description of these freedoms we let $(u, v) \to (u_1, u_2)$.

Freedom (a). (*Translating*). If, for every $u_j(x,t)$ such that $\dfrac{\partial u_j}{\partial U} \neq 0$, $u_j(x,t)$ may be written in the form

$$u_j(x,t) = H_j(x,t,V(z)) + A_j(x,t)[U(z) + \Omega(z, V(z))], \tag{6.30}$$

then we may set $\Omega(z, V(z)) \equiv 0$. This is allowed since we can translate $U(z) \to U(z) - \Omega(z, V(z))$, or alternatively we set $\widetilde{U}(z) = U(z) + \Omega(z, V(z))$ and then rename $\widetilde{U}(z) = U(z)$.

Freedom (b). (*Scaling*). If, for every $u_j(x,t)$ such that $\dfrac{\partial u_j}{\partial U} \neq 0$, $u_j(x,t)$ may be written in the form

$$u_j(x,t) = H_j(x,t,V(z)) + A_j(x,t)U(z)\Omega(z, V(z)), \tag{6.31}$$

then we may set $\Omega(z, V(z)) \equiv 1$. This is allowed since we can scale $U(z) \to U(z)/\Omega(z, V(z))$, or alternatively we set $\widetilde{U}(z) = U(z)\Omega(z, V(z))$ and then rename $\widetilde{U}(z) = U(z)$.

Freedom (c). (*Inverting*). If $z(x,t)$ is determined by an equation of the form $\Omega(z) = z_0(x,t)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$ (by substituting $z \to \Omega^{-1}(z)$).

In fact we can often be more relaxed about these freedoms, which will be shown later, in the working (cf. Remark 5.4(ii)). Also we allow ourselves the "freedom of hindsight" which allows us to effectively tidy up the reduction once we have solved our determining equations.

We may start the application of the direct method to systems (6.2)–(6.7) under a general framework, namely

$$v_x = a_t u - a(x,t)u_t, \tag{6.32a}$$

$$v_t = a(x,t)u_{xxx} - a_x u_{xx} + a(x,t)uu_x - \tfrac{1}{2}a_x u^2. \tag{6.32b}$$

(To represent the Boussinesq equation we must have $a_{xx} = a_{tt} = 0$.) Not only do the following steps work for each system, they are unaffected by the substitution process mentioned above. Equation (6.32b) yields

$$F_U z_x^3 U''' + F_V z_x^3 V''' + ... = 0, \tag{6.33}$$

and without loss of generality we use the coefficient of U''' to be the normalising coefficient. Note that here and in the remainder of this section we use primes to denote differentiation with respect to z. Thus from (6.33) we require

$$F_U\Gamma(z,U,V) = F_V, \tag{6.34}$$

assuming $z_x \neq 0$, and $\Gamma(z,U,V)$ is a function to be determined. Treating this like a linear first order partial differential equation in F, we find the four characteristic directions $\mu_1 = F$, $\mu_3 = x$, $\mu_4 = t$ and an unknown μ_2. However since μ_2 is ultimately a function of z, because U and V are both functions of z, we may write the solution of (6.34) as $F = \widetilde{F}(\mu_2(z), x, t)$. Now by renaming $\mu_2 \equiv U$ we see that $F_V = 0$. Equation (6.32b) now reads

$$F_U z_x^3 U''' + F_{UU} z_x^3 U'U'' + ... = 0. \tag{6.35}$$

Thus requiring the coefficient of $U'U''$ to be a function of (z,U) (but not of V, as $F_V = 0$) yields

$$\frac{F_{UU}}{F_U} = \frac{\Gamma_{1,UU}(z,U)}{\Gamma_{1,U}(z,U)}.$$

Integrating with respect to U twice gives

$$F(x,t,U(z)) = A(x,t)\Gamma_1(z,U) + B(x,t), \tag{6.36}$$

and without loss of generality we can assume $\Gamma_1(z,U) = U(z)$. Equation (6.32a) under this new ansatz is

$$G_V z_x V' + (G_U z_x + aAz_t)U' + ... = 0,$$

and assuming the coefficient of V' to be the normalising coefficient gives

$$G_V z_x \Gamma(z,U,V) = G_U z_x + aAz_t.$$

This is similar to equation (6.34) above and may be solved in the same way, to give

$$G(x,t,U(z),V(z)) = -\frac{aAz_t}{z_x}U(z) + \widetilde{G}(x,t,V(z)). \tag{6.37}$$

System (6.32) now reads

$$\widetilde{G}_V z_x V' + \left[aA_t - \left(\frac{aAz_t}{z_x}\right)_x - a_tA\right]U + [\widetilde{G}_x + aB_t - a_tB] = 0, \tag{6.38i}$$

$$\begin{aligned}
&aAz_x^3U''' + [3aA_xz_x^2 + 3aAz_xz_{xx} - a_xAz_x^2]U'' + aA^2z_xUU' \\
&\quad + \left[aAz_{xxx} + 3aA_xz_{xx} + 3aA_{xx}z_x - a_xAz_{xx} - 2a_xA_xz_x + aABz_x + \frac{aAz_t^2}{z_x}\right]U' \\
&\quad - \widetilde{G}_V z_t V' + \left[aA_{xxx} - a_xA_{xx} + aAB_x + aA_xB + \left(\frac{aAz_t}{z_x}\right)_t - a_xAB\right]U \\
&\quad + [aAA_x - \tfrac{1}{2}a_xA^2]U^2 + [aB_{xxx} - a_xB_{xx} + aBB_x - \tfrac{1}{2}a_xB^2 - \widetilde{G}_t] = 0.
\end{aligned} \tag{6.38ii}$$

If we were to effect substitution we would solve for V' in (6.38i) (as this is the normalising coefficient) and substitute for V' in (6.38ii). However with or without substitution we may still use the coefficient of UU' in (6.38ii) to yield

$$aAz_x^3\Gamma(z) = aA^2 z_x,$$

and hence $A = z_x^2$ by freedom (b). The route that the solution of each system follows now splinters, so we look at the individual systems separately.

However first we re-write freedoms (a) and (b), taking into account the fact that it transpires that in each case we are able to simplify $\widetilde{G}(x,t,V(z))$ to the form $\widetilde{G}(x,t,V(z)) = C(x,t)V(z) + D(x,t)$, and that freedom (b) for $U(z)$ has been applied.

Freedom (a)(i). (Translating $V(z)$). Note that $\dfrac{\partial u}{\partial V} = 0$, so either: if $v(x,t)$ is of the form

$$v(x,t) = D_0(x,t) - az_x z_t U(z) + C(x,t)[V(z) + \Omega(z)]$$

(i.e. $D(x,t) = D_0(x,t) + C(x,t)\Omega(z)$) then we may set $\Omega(z) = 0$ without loss of generality; or if $v(x,t)$ is of the form

$$v(x,t) = E_0(x,t)U(z) + D(x,t) + C(x,t)[V(z) + \Omega(z)U(z)] \tag{6.39}$$

(i.e. $-az_x z_t = E_0(x,t) + C(x,t)\Omega(z)$) then we may set $\Omega(z) = 0$ without loss of generality.

Freedom (a)(ii). (Translating $U(z)$). Note that both $u(x,t)$ and $v(x,t)$ depend on $U(z)$. Then if

$$u(x,t) = B_1(x,t) + z_x^2[U(z) + \Omega(z)], \tag{6.40i}$$

$$v(x,t) = C(x,t)V(z) + D_1(x,t) - az_x z_t[U(z) + \Omega(z)] \tag{6.40ii}$$

(i.e. $B(x,t) = B_1(x,t) + z_x^2\Omega(z)$ and $D(x,t) = D_1(x,t) - az_x z_t\Omega(z)$) then we may set $\Omega(z) = 0$ without loss of generality.

Two things are noteworthy here: firstly that if $u(x,t)$ has the form (6.40i) but $v(x,t)$ maintains the form $v(x,t) = C(x,t)V(z) - az_x z_t U(z) + D(x,t)$, then we may still use this freedom if $D(x,t)$ is still undetermined. Similarly the freedom can still be applied if $v(x,t)$ has the form (6.40ii) whilst $u(x,t)$ maintains its original form and $B(x,t)$ is still undetermined. Secondly, we cannot recreate the situation of (6.39) in freedom (a)(ii), i.e. even if

$$u(x,t) = z_x^2 U(z) + B(x,t),$$

$$v(x,t) = C_0(x,t)V(z) - az_x z_t[U(z) + \Omega(z)V(z)] + D(x,t)$$

(i.e. $C(x,t) = C_0(x,t) - az_x z_t\Omega(z)$) and $B(x,t)$ is undetermined we are not allowed to use the freedom. This is because $B(x,t)$ cannot "consume" the symmetry variable $V(z)$ (whereas it could consume some arbitrary function of z, namely $\Omega(z)$). This shows that there are times when the conditions of freedom may be relaxed, and also times when they cannot.

Freedom (b). (Scaling $V(z)$). Note that $\dfrac{\partial u}{\partial V} = 0$, so if $v(x,t)$ is of the form

$$v(x,t) = C_1(x,t)\Omega(z)V(z) + D(x,t)$$

(i.e. $C(x,t) = C_1(x,t)\Omega(z)$) then we may set $\Omega(z) = 1$ without loss of generality.

As promised, for reference, the results for the scalar Boussinesq equation are summarised, as follows (cf. Clarkson and Kruskal [1989])

$$u(x,t) = \theta^2(t)U(z) - \frac{1}{\theta^2(t)}\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2, \qquad z(x,t) = x\theta(t) + \sigma(t),$$

where $\theta(t)$ and $\sigma(t)$ are any solutions of

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = a_1\theta^5, \qquad \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} = (a_1\sigma + a_2)\theta^4, \tag{6.41a,b}$$

and $U(z)$ satisfies

$$U'''' + UU'' + (U')^2 + (a_1 z + a_2)U' + 2a_1 U = 2(a_1 z + a_2)^2. \tag{6.42}$$

Generally this equation may be solved in terms of the fourth Painlevé equation, PIV, if $a_1 = 0$ in terms of the second Painlevé equation, PII, and if $a_1 = a_2 = 0$ in terms of the first Painlevé equation, PI or the Weierstrass elliptic function equation (2.38).

Case 6.4.1 System (6.2). Our ansatz in this Case is

$$u(x,t) = z_x^2 U(z) + B(x,t),$$
$$v(x,t) = -z_x z_t U(z) + \widetilde{G}(x,t,V(z)),$$

i.e. $a = 1$. The coefficient of U^2 in (6.38ii) yields

$$z_x^5\Gamma(z) = z_x^3 z_{xx}, \tag{6.43}$$

hence dividing by z_x^4 and integrating twice with respect to x, with some manipulation of our Greek letters, gives $\Gamma(z) = x\Theta(t) + \Sigma(t)$, thus $z = x\theta(t) + \sigma(t)$ by freedom (c). We now split our analysis into whether we use substitution or not.

Subcase 6.4.1(i) Without substitution. The coefficient of V' in (6.38ii) yields

$$\theta^5\Gamma(z,V) = \widetilde{G}_V\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right).$$

If $z_t = 0$, then we assume $z = x$, and the coefficient of U' gives $B = \Gamma(z)$, which by freedom (a)(ii) and the fact that $\widetilde{G}(x,t,V(z))$ is undetermined gives $B = 0$. Thus $u_t = 0$, so $v_x = 0$ by (6.2a) and by (6.2b) v can only be a linear function of t. This is a special case of the classical travelling wave reduction.

Thus assuming $z_t \neq 0$ we find

$$\widetilde{G} = \theta^5 V(z) \bigg/ \left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t} \right) + D(x,t).$$

Then the coefficient of U in (6.38i) yields

$$\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = \theta^6 \Gamma_1(z) \bigg/ \left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t} \right),$$

which implies that $\Gamma_1(z) = c_1 z + c_2$ as z is linear in x. The coefficient of x then yields $\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = c_1\theta^6$ which gives canonically either $\theta = 1$, or $\theta = t^{-1/2}$. These then yield the classical travelling wave and scaling reductions respectively.

Subcase 6.4.1(ii) With substitution. If we solve for V' in (6.38i) and substitute our expression into (6.38ii), the coefficient with no U terms or V derivatives yields

$$z_x^5 \Gamma(z,V) = \widetilde{G}_t - \frac{z_t}{z_x}\widetilde{G}_x + k(x,t),$$

where $k(x,t)$ is known but lengthy. We may use the method of characteristics (recall that $z = x\theta(t) + \sigma(t)$) to give $\widetilde{G} = C(x,t)\Gamma(z,V) + D(x,t)$ and then take $\Gamma(z,V) = V(z)$ without loss of generality. From the coefficient of V in (6.38ii) we have

$$C_t - \frac{z_t}{z_x}C_x = z_x^5\Gamma_1(z), \tag{6.44}$$

from which we find

$$C = \Gamma_1(z)\int^t \theta^5(s)\,\mathrm{d}s + \Delta(z),$$

by the method of characteristics. Note that $\Gamma_1(z)$ is precisely the $\Gamma_1(z)$ in (6.44), and $\Delta(z)$ is the new "dependent variable" arising from the method of solution (i.e. the constants of the method are $\mu_1 = z$, $\mu_2 = \Delta$). Freedom (b) allows us to set either $\Gamma_1(z) = 1$, or $\Delta(z) = 1$. Assuming neither, the coefficient of U in (6.38i) yields

$$\theta\left(\Gamma_1(z)\int^t \theta^5(s)\,\mathrm{d}s + \Delta(z)\right)\Gamma_2(z) = \theta\frac{\mathrm{d}\theta}{\mathrm{d}t}. \tag{6.45}$$

Now recall that the condition on $\theta(t)$ in the Boussinesq equation is (6.41a). Integrating this condition with respect to t, gives

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = c_1\int^t \theta^5(s)\,\mathrm{d}s + c_2, \tag{6.46}$$

and the reductions are found by setting (i) $c_1 = c_2 = 0$, (ii) $c_1 = 0$, $c_2 \neq 0$, (iii) $c_2 = 0$, $c_1 \neq 0$, and (iv) $c_1 c_2 \neq 0$. In (6.45), (i) can be achieved by setting $\Gamma_2(z) = 0$ and (iv) with $\Gamma_2(z) \neq 0$ and $\Gamma_1(z)\Delta(z) \neq 0$ (and we could set one of $\Gamma_1(z)$, $\Delta(z)$ to be 1). However to achieve both (ii) *and* (iii) requires the ability to set both $\Gamma_1(z)$ and $\Delta(z)$ to be zero (though *not* simultaneously). Therefore if we set either $\Gamma_1(z) = 1$ or $\Delta(z) = 1$ we have lost generality. In order to salvage the situation we write

$$C = \widehat{\Gamma}_1(z)\left(c_1 \int^t \theta^5(s)\,\mathrm{d}s + \widetilde{\Delta}(z)\right) \quad \text{or} \quad C = \widehat{\Delta}(z)\left(\widetilde{\Gamma}_1(z)\int^t \theta^5(s)\,\mathrm{d}s + c_2\right)$$

and we may then set $\widehat{\Gamma}_1(z) = 1$ or $\widehat{\Delta}(z) = 1$ respectively. We may then use (6.45) to show that $\widetilde{\Delta}(z)$ and $\widetilde{\Gamma}_1(z)$ respectively must be constant. Summarising, we require

$$C = c_1 \int^t \theta^5(s)\,\mathrm{d}s + c_2,$$

where c_1 and c_2 are not simultaneously zero, and $\theta(t)$ satisfies

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = c_3\left(c_1 \int^t \theta^5(s)\,\mathrm{d}s + c_2\right), \tag{6.47}$$

which gives us all the freedom we need. The coefficient of U' in (6.38ii) then gives $B(x,t)$ by

$$\theta^5\Gamma(z) = \theta^3 B + \theta\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2, \tag{6.48}$$

and we may set $\Gamma(z) = 0$ without loss of generality by freedom (a)(ii) and the fact that $D(x,t)$ is undetermined. We find $D(x,t)$ from the coefficient without any U, V terms in it, in equation (6.38i), which yields

$$\theta\left(c_1 \int^t \theta^5(s)\,\mathrm{d}s + c_2\right)\Gamma(z) = D_x + B_t,$$

which may be integrated with respect to x to yield

$$D(x,t) = \left(c_1 \int^t \theta^5(s)\,\mathrm{d}s + c_2\right)\Gamma(z) - \int^x B_t(s,t)\,\mathrm{d}s + \lambda(t),$$

where $B(x,t)$ is known fron (6.48), $\lambda(t)$ is an arbitrary function of integration, and we may set $\Gamma(z) = 0$ by freedom (a)(i). From the coefficient of U in (6.38ii) we find

$$\theta^5\Gamma(z) = \theta\left(x\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}\right),$$

and hence writing $\Gamma(z) = c_4 z + c_5$ we have

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = c_4\theta^5, \tag{6.49i}$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} = \theta^4(c_4\sigma + c_5), \tag{6.49ii}$$

which is consistent with (6.47) if $c_4 = c_3c_1$. The remaining coefficients are now functions of z as required, up to finding an expression for $\lambda(t)$ in terms of quadratures. We have the following reduction

Reduction 6.4.1.

$$\begin{aligned}
u(x,t) =&\theta^2(t)U(z) - \frac{1}{\theta^2(t)}\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2,\\
v(x,t) =& \left(c_1\int^t \theta^5(s)\,\mathrm{d}s + c_2\right)V(z) - \theta\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)U(z) - \frac{2}{3\theta^3(t)}\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^3\\
&+ \frac{2}{\theta^2(t)}\left[\frac{x^3}{3}\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{x^2}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right) + x\frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}\right] + \lambda(t)
\end{aligned}$$

where $z(x,t) = x\theta(t) + \sigma(t)$, $U(z)$, $V(z)$ satisfy

$$\begin{aligned}
&V' + c_3U = 0,\\
&\begin{cases}
U''' + (c_1c_3z + c_5)U + UU' - \dfrac{2}{3c_1c_3}(c_1c_3z + c_5)^3 + c_6 - c_1V = 0 & \text{if } c_1c_3 \neq 0\\
U''' + c_5U + UU' - 2c_5^2z + c_6 - c_1V = 0 & \text{if } c_1c_3 = 0
\end{cases}
\end{aligned} \tag{6.50}$$

and $\lambda(t)$ satisfies

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \begin{cases}
2\theta\left(\dfrac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2(c_1c_3\sigma + c_5) - c_6\theta^5 + \dfrac{2\theta^5}{3c_1c_3}(c_1c_3\sigma + c_5)^3 & \text{if } c_1c_3 \neq 0\\
2c_5\theta\left(\dfrac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - c_6\theta^5 + 2c_5^2\sigma\theta^5 & \text{if } c_1c_3 = 0
\end{cases}$$

Removing $V(z)$ from (6.50) we find $U(z)$ satisfies (6.42) where $a_1 = c_1c_3$ and $a_2 = c_5$.

Case 6.4.2 System (6.3). Our ansätze are now

$$\begin{aligned}
u(x,t) &= z_x^2U(z) + B(x,t),\\
v(x,t) &= -xz_xz_tU(z) + \widetilde{G}(x,t,V(z)),
\end{aligned}$$

i.e. $a(x,t) = x$. The coefficient of U^2 in (6.38ii) yields

$$xz_x^5\Gamma(z) = 2xz_x^3z_{xx} - \tfrac{1}{2}z_x^4,$$

thus dividing through by xz_x^4 and integrating with respect to x yields $z_x\Gamma(z) = \Theta(t)x^{1/4}$, after some manipulation of Greek letters. Integrating again with respect to x we have $\Gamma_1(z) = \Theta_1(t)x^{5/4} + \Sigma_1(t)$. The coefficient of U'' in (6.38ii), however yields

$$xz_x^5\Gamma(z) = 9xz_x^3z_{xx} - z_x^4,$$

which by the same process yields $\Gamma_2(z) = \Theta_2(t)x^{10/9} + \Sigma_2(t)$. Comparing expressions for $\Gamma_1(z)$ and $\Gamma_2(z)$, we must have $\Sigma_1(t) = \Sigma_2(t) = 0$ and using freedom (c) gives $z = x\theta(t)$.

Subcase 6.4.2(i) Without substitution. From the coefficient of V' in (6.38ii) we have

$$x\theta^5\Gamma(z,V) = \widetilde{G}_V x\frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

If $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 0$, we set $z = x$, and from the coefficient of U' in (6.38ii) we find $B(x,t) = 0$ after using freedom (a)(ii) and the fact that $\widetilde{G}(x,t,V(z))$ is undetermined. Looking at (6.3a), since $u(x,t) = U(x)$ we must have $v_x = 0$, then (6.3b) implies that $v(x,t)$ must be a linear function of t, since the right hand side of (6.3b) is a function of x. This is a classical reduction. If $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} \neq 0$, we find

$$\widetilde{G} = \theta^5 V(z) \Big/ \frac{\mathrm{d}\theta}{\mathrm{d}t} + D(x,t),$$

then the coefficient of U' in (6.38ii) gives $B(x,t)$ as

$$B(x,t) = -\frac{x^2}{\theta^2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2,$$

after using freedom (a)(ii) and the fact that $D(x,t)$ is undetermined. The coefficients of U and V in (6.38ii) yield respectively

$$\theta^4\Gamma_3(z) = x^2\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}, \tag{6.51i}$$

$$\theta^4\Gamma_4(z) = \theta^5\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \Big/ \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2. \tag{6.51ii}$$

Thus $\Gamma_3(z) = c_1 z^2$, which leaves $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = c_1\theta^5$, and $\Gamma_4(z) = c_2$, hence (6.51ii) yields

$$c_2\theta^6 = \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2.$$

Together these equations for $\theta(t)$ imply that canonically $\theta = t^{-1/2}$ (recalling that $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} \neq 0$), which leads only to the classical scaling reduction.

Subcase 6.4.2(ii) With substitution. Similar to Case 6.4.1 above we find that $\widetilde{G}$ is linear in V by considering the coefficient of (6.38ii) which has no U terms in it (after substituting for V' from (6.38i)), and we write $\widetilde{G} = C(x,t)V(z) + D(x,t)$. The coefficient of V in (6.38ii) then yields

$$C_t - \frac{z_t}{z_x}C_x = xz_x^5\Gamma(z),$$

and hence we find $C = \Gamma(z)\int^t \theta^4(s)\,\mathrm{d}s + \Delta(z)$, and set $\Gamma(z) = 1$ by freedom (b) (and call $\Delta(z) \equiv \delta(z)$). The coefficient of U in (6.38i) is now zero, but the coefficient of V in (6.38i) yields

$$\theta\left(\int^t \theta^4(s)\,\mathrm{d}s + \delta(z)\right)\Gamma(z) = \theta\delta'(z).$$

For $\theta \neq 0$ we require $\delta'(z) = \Gamma(z) = 0$, i.e. $\delta(z) = c_3$. Similar to the subcase without substitution above we find

$$B(x,t) = -\frac{x^2}{\theta^2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2,$$

from the coefficient of U' in (6.38ii). We find $D(x,t)$ from the only untouched coefficient left in (6.38i)

$$\theta\left(\int^t \theta^4(s)\,\mathrm{d}s + c_3\right)\Gamma(z) = D_x + xB_t.$$

integrating this with respect to x we find $D = -\int^x sB_t(s,t)\,\mathrm{d}s + \lambda(t)$ after using freedom (a)(i), where $\lambda(t)$ is an arbitrary function of integration. The coefficient of U in (6.38ii) gives

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = c_4\theta^5,$$

and we find $\lambda(t)$ in (6.38ii) from the coefficient with no U or V terms in it. We have the following reduction

Reduction 6.4.2.

$$u(x,t) = \theta^2(t)U(z) - \frac{x^2}{\theta^2(t)}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2,$$
$$v(x,t) = \left(\int^t \theta^4(s)\,\mathrm{d}s + c_3\right)V(z) - x^2\theta(t)\frac{\mathrm{d}\theta}{\mathrm{d}t}U(z) + \frac{x^4}{2}\left[c_4\theta^3\frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{1}{\theta^3}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^3\right] + \lambda(t),$$

where $z = x\theta(t)$, $U(z)$, $V(z)$ satisfy

$$\begin{aligned} &V' = 0, \\ &zU''' - U'' + zUU' - V - \tfrac{1}{2}U^2 + c_4z^2U - \tfrac{1}{2}c_4^2z^4 + c_6, \end{aligned} \tag{6.52}$$

and $\lambda(t)$ satisfies

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{2}{\theta^2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - c_6\theta^4.$$

Removing $V(z)$ from (6.52) we see that $U(z)$ satisfies (6.42) for $a_1 = c_4$ and $a_2 = 0$.

Case 6.4.3 System (6.4). Our ansätze read

$$u(x,t) = z_x^2U(z) + B(x,t),$$
$$v(x,t) = -tz_xz_tU(z) + \widetilde{G}(x,t,V(z)),$$

so $a(x,t) = t$. The coefficient of U^2 in (6.38ii) gives an equation of the form (6.43), hence we find $z = x\theta(t) + \sigma(t)$. As equation (6.38i) contains no U' terms, with or without substitution we can use the coefficient of U' in (6.38ii) to give

$$t\theta^5\Gamma(z) = t\theta^3B + t\theta\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2, \tag{6.53}$$

and we set $\Gamma(z) = 0$ by freedom (a)(ii) and the fact that $\widetilde{G}(x,t,V(z))$ is undetermined.

Subcase 6.4.3(i) Without substitution. We use the coefficient of V' in (6.38ii) to yield

$$t\theta^5\Gamma(z,V) = \widetilde{G}_V\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right).$$

If $z_t = 0$, we may set $z = x$, hence $A = 1$, and also from above, $B = 0$. The coefficient of V' in (6.38i) helps us find $\widetilde{G} = V(x) + D(x,t)$, the term with no U's or V's in (6.38i) then gives $D = f(t)$ after using freedom (a)(i) and finally we find $f = c_1t^2$ from the term with no U's or V's in (6.38ii). This leaves the time independent solution for $u(x,t)$.

For $z_t \neq 0$, we have found $\widetilde{G}$, as

$$\widetilde{G} = t\theta^5\Gamma(z,V)\Big/\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right) + D(x,t),$$

and we may set $\Gamma(z,V) = V(z)$ without loss of generality. The coefficient of V in (6.38i) then yields

$$t\theta^6\Gamma(z)\Big/\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right) = t\theta^5\frac{\mathrm{d}\theta}{\mathrm{d}t}\Big/\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2.$$

We require $\Gamma(z) = c_1/(z + c_2)$ to yield

$$c_1t\theta^6\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right) = t\theta^5\frac{\mathrm{d}\theta}{\mathrm{d}t}(x\theta + \sigma + c_2). \tag{6.54}$$

This means either $c_1 = 0$ or $c_1 = 1$. If $c_1 = 1$, the remaining part of (6.54) allows us to set $\sigma = 0$, and the remaining coefficients lead us to the classical scaling reduction. However if $c_1 = 0$, then $\theta = 1$ and $A = 1$. The coefficient of U in (6.38i) now yields

$$t\Gamma(z)\Big/\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 1,$$

thus $\Gamma(z) = 2c_3$ and we may let $\sigma = c_3t^2$. We find $D(x,t)$ in the usual way, from the term in (6.38i) with no U's or V's in, which gives $D(x,t) = 4c_3^2t^2x + \lambda(t)$, after using freedom (a). We find $\lambda(t)$ from the same term in (6.38ii) which then gives the following nonclassical reduction

Reduction 6.4.3.

$$u(x,t) = U(z) - 4c_3^2t^2,$$
$$v(x,t) = V(z) - 2c_3t^2U(z) + 4c_3^2t^2x + 2c_3^3t^4,$$

where $z = x + c_3t^2$, and $U(z)$, $V(z)$ satisfy

$$\begin{aligned} &V' = U,\\ &8c_3^2z + 2c_3V' - U''' - UU' - 4c_3U = 0. \end{aligned} \tag{6.55}$$

Removing $V(z)$ from (6.55) we find that $U(z)$ satisfies (6.42) for $a_1 = 0$, $a_2 = 2c_3$.

Subcase 6.4.3(ii) With substitution. We find $\widetilde{G}$ is linear in V, after consideration of the terms with no U's or V's in, so we write $\widetilde{G} = C(x,t)V(z) + D(x,t)$. From the coefficient of V in (6.38ii) we have

$$C_t - \frac{z_t}{z_x} C_x = t z_x^5 \Gamma_1(z),$$

from which we find $C = \Gamma_1(z) \int^t s\theta^5(s)\,\mathrm{d}s + \Delta(z)$. The coefficient of U in (6.38i) then yields

$$\theta \left(\Gamma_1(z) \int^t s\theta^5(s)\,\mathrm{d}s + \Delta(z) \right) \Gamma_2(z) = t\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} - \theta^2, \tag{6.56}$$

and similar to Subcase 6.4.1(ii) we must apply a variant of freedom (b) and instead of setting $\Gamma_1(z) = 1$ we set $\Gamma_1(z) = c_1$, constant. In equation (6.56) we must set $\Delta(z) = c_2$ and $\Gamma_2(z) = c_3$, both constant, so that now $\theta(t)$ satisfies

$$t\frac{\mathrm{d}\theta}{\mathrm{d}t} - \theta = c_3 \left(c_1 \int^t s\theta^5(s)\,\mathrm{d}s + c_2 \right). \tag{6.57}$$

We find $D(x,t) = \int^x B(s,t) - tB_t(s,t)\,\mathrm{d}s + \lambda(t)$ as previously, after using freedom (a)(i). The coefficient of U in (6.38ii) yields

$$t\theta^5 \Gamma(z) = t\theta \left(x\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} \right),$$

from which we set $\Gamma(z) = c_4 z + c_5$, to give

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = c_4\theta^5, \qquad \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} = \theta^4(c_4\sigma + c_5). \tag{6.58i,ii}$$

These conditions are consistent with (6.57) if $c_4 = c_1 c_3$, the remaining coefficients are now functions of z up to finding $\lambda(t)$, and we have the nonclassical reduction

Reduction 6.4.4.

$$\begin{aligned} u(x,t) &= \theta^2(t)U(z) - \frac{1}{\theta^2(t)} \left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t} \right)^2, \\ v(x,t) &= \left(c_1 \int^t s\theta^5(s)\,\mathrm{d}s + c_2 \right) V(z) - t\theta(t) \left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t} \right) U(z) + \lambda(t) \\ &\quad + \frac{x^3}{3} \left[\frac{2t}{\theta^2}\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} - \frac{1}{\theta^2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{2t}{\theta^3}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^3 \right] + \frac{x^2}{2} \left[\frac{2t}{\theta^2} \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\frac{\mathrm{d}\sigma}{\mathrm{d}t} \right) \right. \\ &\quad \left. - \frac{2}{\theta^2}\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \frac{4t}{\theta^3}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 \frac{\mathrm{d}\sigma}{\mathrm{d}t} \right] + x \left[\frac{2t}{\theta^2}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} - \frac{1}{\theta^2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - \frac{2t}{\theta^3}\frac{\mathrm{d}\theta}{\mathrm{d}t}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \right], \end{aligned}$$

where $z(x,t) = x\theta(t) + \sigma(t)$, and $\theta(t)$, $\sigma(t)$ satisfy (6.57), (6.58ii) (with $c_4 = c_1 c_3$). Then $U(z)$, $V(z)$ satisfy system (6.50) and $\lambda(t)$ satisfies

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \begin{cases} \dfrac{2t}{\theta^4}\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\left(\dfrac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^3 - c_6 t\theta^5 + \dfrac{2t\theta^5}{3c_1c_3}(c_1c_3\sigma + c_5)^3 & \text{if } c_1c_3 \neq 0 \\ \dfrac{2t}{\theta^4}\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\left(\dfrac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^3 - c_6 t\theta^5 + 2c_5^2 t\sigma\theta^5 & \text{if } c_1c_3 = 0 \end{cases}$$

Case 6.4.4 System (6.5). In this Case we show the workings without substitution, but neglect to show those with substitution, as the results are no different. We have the ansätze

$$u(x,t) = z_x^2 U(z) + B(x,t),$$
$$v(x,t) = -(x+t)z_x z_t U(z) + \widetilde{G}(x,t,V(z)).$$

As in Case 6.4.2 we compare the coefficients of U^2 and U'' in (6.38ii) to find z. They yield after integrating with respect to x twice, respectively

$$\Gamma_1(z) = (x+t)^{5/4}\Theta_1(t) + \Sigma_1(t),$$
$$\Gamma_2(z) = (x+t)^{10/9}\Theta_2(t) + \Sigma_2(t),$$

hence $\Sigma_1(t) = \Sigma_2(t) = 0$ and $z = (x+t)\theta(t)$ by freedom (c). The coefficient of V' in (6.38ii) gives

$$\widetilde{G} = (x+t)\theta^5\Gamma(z,V) \Big/ \left[(x+t)\frac{\mathrm{d}\theta}{\mathrm{d}t} + \theta\right] + D(x,t),$$

after integration with respect to V, and we set $\Gamma(z,V) = V(z)$ without loss of generality. (Note the denominator of the first term cannot be zero.) Thus the coefficient of U in (6.38i) yields

$$2\theta^2 = \theta^5\Gamma(z) \Big/ \left[(x+t)\frac{\mathrm{d}\theta}{\mathrm{d}t} + \theta\right],$$

thus we choose $\Gamma(z) = c_1 z + c_2$. This yields

$$2\theta^2\left[(x+t)\frac{\mathrm{d}\theta}{\mathrm{d}t} + \theta\right] = \theta^5[c_1(x+t)\theta + c_2].$$

We see that $2\theta^3 = c_2\theta^5$, hence $\theta(t)$ must be constant, so we set $\theta(t) = 1$. This now leads to the classical travelling wave reduction.

Case 6.4.5 System (6.6). Our ansätze are

$$u(x,t) = z_x^2 U(z) + B(x,t),$$
$$v(x,t) = -xtz_x z_t U(z) + \widetilde{G}(x,t,V(z)),$$

thus $a(x,t) = xt$. Similar to both Cases 6.4.2 and 6.4.4, the coefficients of U^2 and U'' in (6.38ii) yield respectively

$$\Gamma_1(z) = (xt)^{5/4}\Theta_1(t) + \Sigma_1(t), \tag{6.59i}$$
$$\Gamma_2(z) = (xt)^{10/9}\Theta_2(t) + \Sigma_2(t), \tag{6.59ii}$$

so $\Sigma_1(t) = \Sigma_2(t) = 0$ and we set $z = x\theta(t)$ without loss of generality, by freedom (c). The coefficient of U' in (6.38ii) yields, irrespective of whether substitution is used or not,

$$xt\theta^5\Gamma(z) = xt\theta^3 B + x^3 t\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2, \tag{6.60}$$

from which we find $B(x,t)$, after setting $\Gamma(z) = 0$ by freedom (a)(ii) and the fact that $\widetilde{G}(x,t,V(z))$ is undetermined.

Subcase 6.4.5(i) Without substitution. As previously we initially look at the coefficient of V' in (6.38ii) which yields

$$x\frac{\mathrm{d}\theta}{\mathrm{d}t}\widetilde{G}_V = xt\theta^5\Gamma(z,V).$$

If $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 0$, we let $z = x$, and then $A = 1$. From (6.60) we see that $B = 0$, and from the coefficient of V' in (6.38i) we find $\widetilde{G} = V(x) + D(x,t)$. The only remaining untouched coefficient in (6.38i) gives $D = f(t)$, after using freedom (a), and finally, the coefficient with no U's or V's in (6.38ii) gives $f = c_1t^2$, thus we have found the time independent solution for $u(x,t)$.

For $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} \neq 0$, we take

$$\widetilde{G} = t\theta^5V(z)\Big/\frac{\mathrm{d}\theta}{\mathrm{d}t} + D(x,t),$$

then the coefficient of U in (6.38i) gives

$$t\theta^6\Gamma(z)\Big/\frac{\mathrm{d}\theta}{\mathrm{d}t} = x\theta^2,$$

and hence we have that $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = c_2t\theta^5$, by setting $\Gamma(z) = c_2z$. With this condition on $\theta(t)$, the coefficient of U in (6.38ii) now yields

$$xt\theta^5\Gamma(z) = 2c_2x^2t\theta^6 + 5c_2^2x^2t^3\theta^{10}.$$

We must set $\Gamma(z) = c_3z$, thus on rearranging

$$t^2\theta^4 = \frac{c_3 - 2c_2}{5c_2^2},$$

and hence canonically $\theta = t^{-1/2}$, which leads to the classical scaling reduction.

Subcase 6.4.5(ii) With substitution. We find, as previously that $\widetilde{G}$ may be written in the form $\widetilde{G} = C(x,t)V(z) + D(x,t)$. From the coefficient of U in (6.38i) we have

$$C\theta\Gamma(z) = x\theta^2,$$

and hence we may choose $C = 1$ without loss of generality, by freedom (b). The coefficient of U in (6.38ii) yields

$$xt\theta^5\Gamma(z) = x^2t\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2},$$

hence we choose $\Gamma(z) = c_1z$ to leave

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = c_1\theta^5. \qquad (6.61)$$

The coefficient of (6.38i) with no U's or V's in yields

$$\theta\Gamma(z) = D_x + xB - xtB_t,$$

so after integrating with respect to x we have

$$D(x,t) = \Gamma(z) + \frac{x^4}{4}\left[2c_1 t\theta^3\frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{2t}{\theta^3}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^3 - \frac{1}{\theta^2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right] + \lambda(t),$$

where $\lambda(t)$ is an arbitrary function of integration and we may set $\Gamma(z) = 0$ by freedom (a)(i). We find $\lambda(t)$ from the coefficient of (6.38ii) with no U's or V's in, which leads to the following reduction

Reduction 6.4.5.

$$u(x,t) = \theta^2(t)U(z) - \frac{x^2}{\theta^2(t)}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2,$$

$$v(x,t) = V(z) - x^2 t\theta(t)\frac{\mathrm{d}\theta}{\mathrm{d}t}U(z) + \frac{x^4}{4}\left[2c_1 t\theta^3\frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{2t}{\theta^3}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^3 - \frac{1}{\theta^2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right] + \lambda(t),$$

where $z = x\theta(t)$, $U(z)$ and $V(z)$ satisfy

$$V' = zU, \tag{6.62i}$$

$$2zU''' + 2zUU' - 2U'' - U^2 - 2c_2 + 2c_1 z^2 U - c_1^2 z^4 = 0, \tag{6.62ii}$$

$\theta(t)$ satisfies (6.61), and $\lambda(t)$ satisfies

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = c_2 t\theta^4(t) + \frac{2t}{\theta^2(t)}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2.$$

By differentiating (6.62ii) with respect to z we find that it satisfies (6.42) for $a_1 = c_1$, $a_2 = 0$.

<u>*Case 6.4.6 System (6.7)*</u>. We have the following ansätze

$$u(x,t) = z_x^2 U(z) + B(x,t),$$

$$v(x,t) = -(1+xt)z_x z_t U(z) + \widetilde{G}(x,t,V(z)),$$

so $a(x,t) = 1 + xt$. The coefficients of U^2 and U'' in (6.38ii) yield equations of the form (6.59) (with xt replaced by $1 + xt$) and hence we find $z = (1+xt)\theta(t)$.

Subcase 6.4.6(i) Without substitution. From the coefficient of V' in (6.38ii) we have

$$(1+xt)t^5\theta^5\Gamma(z,V) = \left[(1+xt)\frac{\mathrm{d}\theta}{\mathrm{d}t} + x\theta\right]\widetilde{G}_V,$$

from which we find $\widetilde{G}$ as

$$\widetilde{G} = (1+xt)t^5\theta^5 V(z) \Big/ \left[(1+xt)\frac{\mathrm{d}\theta}{\mathrm{d}t} + x\theta\right] + D(x,t),$$

since the denominator is non-zero. The coefficient of U in (6.38i) then yields

$$(1+xt)t^6\theta^6\Gamma(z) \Big/ \left[(1+xt)\frac{\mathrm{d}\theta}{\mathrm{d}t} + x\theta\right] = (1-xt)t\theta^2.$$

We must take $\Gamma(z) = c_1 z + c_2$, but successively taking coefficients of powers of x to be zero leads us to $\theta(t) = 0$, thus there are no reductions.

Subcase 6.4.6(ii) With substitution. As in previous subcases with substitution we are able to find $\widetilde{G} = C(x,t)V(z) + D(x,t)$ without loss of generality. The coefficient of U in (6.38i) now yields

$$Ct\theta\Gamma(z) = (1-xt)t\theta^2,$$

and hence we let $C(x,t) = (1-xt)\theta(t)$ by freedom (b). As usual it is the coefficient of U' in (6.38ii) which gives $B(x,t)$; here it takes the form

$$B(x,t) = -\frac{1}{t^2\theta^2}\left[(1+xt)\frac{\mathrm{d}\theta}{\mathrm{d}t} + x\theta\right]^2.$$

The coefficients of V and U in (6.38ii) give respectively

$$\begin{aligned}(1+xt)t^5\theta^5\Gamma_1(z) &= \frac{\mathrm{d}\theta}{\mathrm{d}t},\\ (1+xt)t^5\theta^5\Gamma_2(z) &= (1+xt)t\theta\left[(1+xt)\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2x\frac{\mathrm{d}\theta}{\mathrm{d}t}\right].\end{aligned}$$

We let $\Gamma_1(z) = c_1/z$ and $\Gamma_2(z) = c_2 z + c_3$ and find these equations are only compatible if $c_1 = c_2 = c_3 = 0$, i.e. $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 0$. The coefficient with no U's or V's in (6.38i) now yields

$$D_x + \frac{3x^3}{t^2} + \frac{2x^2}{t^3} = t\Gamma(z),$$

and hence on integrating with respect to x we find without loss of generality, that

$$D(x,t) = -\frac{3x^4}{4t^2} - \frac{2x^3}{3t^3} + \lambda(t),$$

after using freedom (a)(i), and then from the same coefficient in (6.38ii) we find

$$-2t\frac{\mathrm{d}\lambda}{\mathrm{d}t} + 4 = t^6\Gamma(z).$$

We let $\Gamma(z) = -6c_4$, thus $\lambda(t) = 2\ln t + c_4 t^6$. By doing some *a posteriori* tidying up, we can make $C = 1$ and $z = xt$, which gives the following reduction

Reduction 6.4.6.

$$u(x,t) = t^2 U(z) - \frac{x^2}{t^2},$$
$$v(x,t) = V(z) - (1+z)zU(z) - \frac{3x^4}{4t^2} - \frac{2x^3}{3t^3} + 2\ln t + c_4 t^6,$$

where $z = xt$ and $U(z)$, $V(z)$ satisfy

$$V' = (z-1)U, \tag{6.63i}$$
$$2(z+1)U''' + 2(z+1)UU' - 2U'' - U^2 - 12c_4 = 0. \tag{6.63ii}$$

By differentiating (6.63ii) with respect to z we see that it satisfies (6.42) for $a_1 = a_2 = 0$.

6.5 Discussion

In §6.2 we saw that none of the six potential Boussinesq systems gave any potential symmetries, indeed we lost symmetries in all but one case. However because we still only had to solve linear determining equations there was a negligible difference in complexity between the systems and the scalar equation.

The calculation for the nonclassical method, in §6.3 however, did increase in complexity and certainly in Cases 6.3.2 and 6.3.3 the determining equations could not be solved explicitly. It was only with a bit of ingenuity that we could show that there was no need to solve them completely but that instead we could bypass the problem. Coupled with this increase in complexity, we also lost symmetries in the majority of cases, that is most of the systems didn't give the full complement of nonclassical reductions, and certainly we found no new symmetries in any of the cases.

The findings of Priestley and Clarkson [1995] are backed up in this chapter. In Case 6.3.1 we saw how any reduction which arose from infinitesimals for which $\xi_u \xi_v \neq 0$ could be reproduced by infinitesimals for which $\xi_u = \xi_v = 0$. Indeed in the specific case when $\xi = v/u$, $\phi_1 = \phi_2 = 0$ we found these infinitesimals gave rise to the travelling wave reduction. Clearly ξ depends on v, the potential variable, and yet we obtain only a classical reduction. Thus we conclude that if the infinitesimals of the non-potential variable depend on the potential variable, then this has no bearing on whether we have obtained nonclassical potential symmetries.

In applying the direct method to our systems we calculated separately the symmetries that used substitution and those that did not. Notice that without substitution we are always able to find the classical symmetries in this chapter but rarely find any nonclassical symmetries (Cases 6.4.3 and 6.4.5 are the exceptions). By using substitution however we can recover all the symmetries that the nonclassical method finds – such is the importance of using substitution.

We should also mention that no integration is necessary in the direct method, nor were unnatural infinitesimals present in the nonclassical method.

One other problem we must address is the ability to lose generality when applying freedom (b). Not only did it occur in this chapter but also in Chapter Five (cf. Remark 5.4(vi)) when we applied the method to both the scalar equation (5.3) and system (5.4), so it was not just a product of using the direct method on systems of equations. To recap on what the problem is, in Case 6.4.1 we had

$$v(x,t) = \left(\Gamma_1(z)\int^t \theta^5(s)\,\mathrm{d}s + \Delta(z)\right)V(z) - \theta^2(t)U(z) + D(x,t),$$

where $z = \theta(t)x + \sigma(t)$. Later working showed that we required the ability to set both $\Gamma_1(z)$ and $\Delta(z)$ to be zero (though *not* simultaneously), but applying freedom (b) would allow us to set either $\Gamma_1(z) = 1$ or $\Delta(z) = 1$ and we would thus lose generality. A simple, effective solution is to give freedom (b) a caveat so it reads (for scaling $V(z)$ above with $\dfrac{\partial u}{\partial V} = 0$)

Freedom (b). If $v(x,t)$ has the form

$$v(x,t) = D(x,t) + E(x,t)U(z) + C(x,t)\Omega(z)V(z),$$

then we may set $\Omega(z) = 1$, if, when $C(x,t)$ may be written as a sum of terms, the ability to set each of these terms (individually) to zero is unaffected.

In the general setting (6.31), it is the $A_j(x,t)$ (instead of $C(x,t)$) to which the caveat applies. We note that this situation can only occur if at least two summands of $C(x,t)$ (that may not be written as a single function of z) contain different arbitrary functions of z. Also note that there would appear not to be similar difficulties with freedoms (a) or (c).

Chapter Seven : A Reaction–Diffusion Equation

7.1 Introduction

In this chapter we continue the theme of potential symmetries, and in particular nonclassical potential symmetries. We are interested in the reaction–diffusion equation

$$\theta_t = \theta^2\theta_{xx} + 2b\theta^2, \tag{7.1}$$

where b is a non-zero constant, for which Bluman [1993a] has found potential symmetries. Making the transformation $\theta = 1/u$ yields

$$u_t = -\left(\frac{1}{u} + bx^2\right)_{xx}, \tag{7.2}$$

which may be written in potential form as

$$v_x = u, \tag{7.3a}$$

$$u^2 v_t = u_x - 2bxu^2. \tag{7.3b}$$

There is also a higher order system

$$v_x = u, \tag{7.4a}$$

$$uw_t = -(1 + bx^2 u), \tag{7.4b}$$

$$w_x = v, \tag{7.4c}$$

and we may take combinations of (7.2), (7.3) and (7.4) to yield other potential systems (see Bluman [1993a]). In fact in his study of the reaction-diffusion equation (7.1), Bluman [1993a] finds the following invertible mapping via its potential symmetries.

$$z_1 = t, \qquad z_2 = v, \qquad w_1 = x\exp\{b(xv - w)\},$$

$$w_2 = \left(bx^2 + \frac{1}{u}\right)\exp\{b(xv - w)\}, \qquad w_3 = \frac{1}{b}(\exp\{b(xv - w)\} - 1),$$

where w_1, w_2, w_3 satisfy the linear system

$$w_{3,v} = w_1, \qquad w_{3,t} = w_2, \qquad w_{1,v} = w_2.$$

In the following sections we rederive the potential symmetries for each system and find all their reductions for equation (7.2) and system (7.3) and a sample of reductions for (7.4) in §7.2. In §7.3 we apply the nonclassical method to the systems above (though only in the generic $\tau \neq 0$ case). Also in §7.3, as a result of applying the nonclassical method, we find an interesting relation between these reaction–diffusion equations and Burgers' equation

$$u_t = u_{xx} - uu_x, \tag{7.5}$$

which involves the use of a hodograph transformation. In §7.4 we discuss the results found in the previous sections.

7.2 Classical symmetries

To apply the classical method we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u, v, w) given by

$$x^* = x + \varepsilon\xi(x, t, \boldsymbol{u}) + O(\varepsilon^2), \tag{7.6i}$$

$$t^* = t + \varepsilon\tau(x, t, \boldsymbol{u}) + O(\varepsilon^2), \tag{7.6ii}$$

$$u^* = u + \varepsilon\phi_1(x, t, \boldsymbol{u}) + O(\varepsilon^2), \tag{7.6iii}$$

$$v^* = v + \varepsilon\phi_2(x, t, \boldsymbol{u}) + O(\varepsilon^2), \tag{7.6iv}$$

$$w^* = w + \varepsilon\phi_3(x, t, \boldsymbol{u}) + O(\varepsilon^2), \tag{7.6v}$$

where ε is the group parameter, and $\boldsymbol{u}$ consists of an appropriate subset of the set (u, v, w). This procedure, which is implemented in `symmgrp.max`, yields a system of linear determining equations for each case.

<u>*Case 7.2.1 Equation (7.2)*</u>. We obtain the following determining equations

$$\tau_u = 0, \qquad \tau_x = 0, \qquad \xi_u = 0, \qquad \phi_{1,uu} = 0, \qquad 2\phi_{1,xu}u^2 - \xi_{xx}u^2 + \xi_t = 0,$$

$$\tau_t u - 2\xi_x u + 2\phi_1 = 0, \qquad 2b\phi_{1,u}u^2 - \phi_{1,xx}u^2 - 4b\xi_x u^2 + \phi_{1,t} = 0,$$

which may be simplified using **KolRitt** in `diffgrob2` to yield

$$\xi_{xx} = 0, \quad \xi_t = 0, \quad \xi_u = 0, \quad \tau_x = 0, \quad 2\xi_x + \tau_t = 0, \quad \tau_u = 0, \quad 2\xi_x u - \phi_1 = 0.$$

These are then solved trivially to yield infinitesimals

$$\xi = c_1 x + c_2, \qquad \tau = -2c_1 t + c_3, \qquad \phi_1 = -2c_1 u. \tag{7.7}$$

Incidentally the infinitesimals for (7.1) give the same reductions. These are

Reduction 7.2.1. If $c_1 = 0$, we let $c_3 = 1$, $c_2 = c$, to give

$$u(x,t) = U(z), \qquad z = x - ct,$$

where $U(z)$ satisfies

$$UU'' - 2(U')^2 + cU^3U' - 2bU^3 = 0,$$

which on division by U^3 may be integrated with respect to z to yield

$$U' = (2bz + A)U^2 - cU^3, \tag{7.8}$$

where A is the constant of integration. If we assume $c \neq 0$, to achieve solutions that are not time independent, then (7.8) may not be transformed via a Möbius transformation (1.99) onto a generalised Riccati equation, so is not of Painlevé-type (see §1.6.2).

Reduction 7.2.2. If $c_1 \neq 0$, we may set $c_1 = 1$, $c_2 = c_3 = 0$ to give

$$u(x,t) = tU(z), \qquad z = xt^{1/2},$$

where $U(z)$ satisfies

$$zU^3U' - 2UU'' + 4(U')^2 + 2U^4 + 4bU^3 = 0, \tag{7.9}$$

which is not of Painlevé-type as it has a leading order behaviour like $U(z) \sim \pm \mathrm{i} z_0^{-1/2}(z - z_0)^{-1/2}$ and thus algebraic branching. If we try to remove this leading order branching by making the transformation $\widetilde{U} = U^2$, the resulting equation contains a term like $\widetilde{U}^{3/2}$, so this transformation is of no help.

Case 7.2.2 System (7.3). The determining equations are as follows

$$\tau_u = 0,$$
$$\tau_v - \xi_u u^2 = 0,$$
$$\phi_{2,u}u^2 - \xi_u u^3 + \tau_v u + \tau_x = 0,$$
$$\phi_{2,u}u^2 - \xi_u u^3 - \tau_v u - \tau_x = 0,$$
$$4b\xi_u u^3 x + \phi_{2,v}u - \phi_{1,u}u - \tau_t u + \xi_x u + 2\phi_1 = 0,$$
$$2b\phi_{2,u}u^2 x - 2b\xi_u u^3 x - \xi_v u^2 + \phi_{2,v}u - \xi_x u + \phi_{2,x} - \phi_1 = 0,$$
$$4b^2\xi_u u^4 x^2 + 2b\xi_v u^3 x$$
$$\quad - 2b\phi_{1,u}u^2 x + 2b\xi_x u^2 x + 4b\phi_1 ux - \xi_t u^3 + \phi_{2,t}u^2 + 2b\xi u^2 - \phi_{1,v}u - \phi_{1,x} = 0,$$

which are simplified using **KolRitt** in `diffgrob2`

$$\xi_{vv}=0,\qquad \xi_u=0,\qquad 2\xi_v xb-\xi_t=0,\qquad \xi_{xv}=0,\qquad \xi_{xx}=0,\qquad \tau_v=0,$$
$$\tau_u=0,\qquad 2\xi_x+\tau_t=0,\qquad \tau_x=0,\qquad \xi_v u^2+2\xi_x u+\phi_1=0,$$
$$\phi_{2,v}+\xi_x=0,\qquad \phi_{2,u}=0,\qquad 2bx\xi_x-2b\xi-\phi_{2,t}=0,\qquad \phi_{2,x}=0,$$

from which we find

$$\xi=\frac{c_4}{2b}v+(c_4t+c_1)x+c_2, \tag{7.10i}$$
$$\tau=-c_4t^2-2c_1t+c_3, \tag{7.10ii}$$
$$\phi_1=-2(c_4t+c_1)u-\frac{c_4}{2b}u^2, \tag{7.10iii}$$
$$\phi_2=-(c_4t+c_1)v-2bc_2t+c_5. \tag{7.10iv}$$

The dependence of ξ on v ensures these give potential symmetries. If $c_4=0$ we have the classical infinitesimals for equation (7.2), but for $c_4\neq 0$ they give two new reductions.

Reduction 7.2.3. When $c_3c_4\neq 0$ we may set $c_1=c_2=0$ and $c_4=1$ to give

$$u(x,t)=\frac{2b(c_3-t^2)}{t+U(z)},\qquad v(x,t)=\frac{c_5t}{c_3}+(c_3-t^2)^{1/2}V(z),$$

where $z(x,t)$ is defined implicitly by

$$z(x,t)=\left(x+\frac{c_5}{2bc_3}\right)(c_3-t^2)^{1/2}-\frac{t}{2b}\left(v-\frac{c_5t}{c_3}\right)(c_3-t^2)^{-1/2},$$

then $U(z)$, $V(z)$ satisfy

$$UV'-2b=0, \tag{7.11i}$$
$$4b^2z-c_3VV'+U'=0. \tag{7.11ii}$$

Equation (7.11ii) may be integrated with respect to z, then we can use (7.11i) to give

$$V'(4b^2z^2-c_3V^2+A)+4b=0, \tag{7.12}$$

where A is an arbitrary constant. A simple solution of this is given by $V(z)=a_1z$ for $a_1^2=4b^2/c_3$, but this leads only to a spatially independent solution for $u(x,t)$, which may be found from the scaling reduction 7.2.2, for $U(z)=-2b$. In general, remembering that $c_3\neq 0$, equation (7.12) may not be transformed via a Möbius transformation to a generalised Riccati equation and is therefore not of Painlevé-type.

Reduction 7.2.4. When $c_4\neq 0$ and $c_3=0$, we may set $c_1=c_2=0$ and $c_4=1$ without loss of generality, to give

$$u(x,t)=-\frac{2bt^2}{t+U(z)},\qquad v(x,t)=\frac{c_5}{2t}+tV(z), \tag{7.13i,ii}$$

where $z(x,t)$ is again defined implicitly

$$z(x,t) = xt - \frac{c_5}{4bt} + \frac{1}{2b}\left(v - \frac{c_5}{2t}\right).$$

The symmetry variables $U(z)$, $V(z)$ satisfy

$$UV' + 2b = 0, \tag{7.14i}$$

$$4b^2 z + c_5 V' - U' = 0, \tag{7.14ii}$$

which may again be written as a single equation for $V(z)$,

$$V'(2b^2z^2 + c_5 V + A) + 2b = 0, \tag{7.15}$$

where A is an arbitrary constant. If $c_5 \neq 0$ then (7.15) may not be transformed onto a generalised Riccati equation via a Möbius transformation, and is therefore not of Painlevé-type. However for $c_5 = 0$ we have

$$V(z) = \int^z \frac{-2b}{2b^2s^2 + A}\, \mathrm{d}s + c_7.$$

Since z is a function of v, if $A \neq 0$ this equation, together with (7.13ii) gives an implicit expression for v in terms of x, t, after integrating. However if $A = 0$, we may find v explicitly after solving a quadratic for v. We have

$$v(x,t) = \tfrac{1}{2}c_7 t - bxt \pm \tfrac{1}{2}\left[t^2(2bx + c_7)^2 + 8t\right]^{1/2},$$

$$u(x,t) = \frac{\pm bt^2(2bx + c_7)}{\left[t^2(2bx + c_7)^2 + 8t\right]^{1/2}} - bt,$$

finding $u(x,t)$ from (7.3a). To compare this with the classical scaling reduction 7.2.2 we set $c_7 = 0$ (since we can translate x appropriately in reduction 7.2.2 if necessary). We find that this exact solution is a solution of the scaling reduction, for

$$U(z) = \pm b^2 z(b^2z^2 + 2)^{-1/2} - b.$$

Whilst this is a classical solution, it was only found by looking at a potential symmetry.

Note that we could write (7.3) in the form

$$v_x = u, \tag{7.16a}$$

$$v_x^2 v_t = v_{xx} - 2bxv_x^2, \tag{7.16b}$$

and we get the same (potential) symmetries.

Case 7.2.3 System (7.4). We have to solve the following determining equations

$$\tau_u = 0, \tag{7.17i}$$

$$\phi_{3,u} - \xi_u v = 0, \tag{7.17ii}$$

$$\phi_{2,u} - \xi_u u = 0, \tag{7.17iii}$$

$$\tau_w v + \tau_v u + \tau_x = 0, \tag{7.17iv}$$

$$b\tau_v u x^2 - \xi_v uv + \phi_{3,v} u + \tau_v = 0, \tag{7.17v}$$

$$\phi_{2,w} v - \xi_w uv - \xi_v u^2 + \phi_{2,v} u - \xi_x u + \phi_{2,x} - \phi_1 = 0, \tag{7.17vi}$$

$$\begin{aligned} &b^2\tau_w u^2 x^4 - b\xi_w u^2 v x^2 + b\phi_{3,w} u^2 x^2 - b\tau_t u^2 x^2 + 2b\tau_w u x^2 - 2b\xi u^2 x \\ &\quad + \xi_t u^2 v - \xi_w uv - \phi_{3,t} u^2 + \phi_{3,w} u - \tau_t u + \phi_1 + \tau_w = 0, \end{aligned} \tag{7.17vii}$$

$$\begin{aligned} &b\tau_w uvx^2 + b\tau_v u^2 x^2 + b\tau_x u x^2 - \xi_w uv^2 - \xi_v u^2 v + \phi_{3,w} uv - \xi_x uv \\ &\quad + \tau_w v + \phi_{3,v} u^2 + \phi_{3,x} u - \phi_2 u + \tau_v u + \tau_x = 0. \end{aligned} \tag{7.17viii}$$

These are again simplified using the **KolRitt** procedure, to give the larger but simpler system

$$\tau_w = 0, \qquad \tau_v = 0, \qquad \tau_u = 0, \qquad \tau_x = 0, \qquad \tau_{ttt} = 0, \qquad b\xi_w - \xi_{ww} = 0, \tag{7.18i-vi}$$

$$4b\xi_v + \tau_{tt} - 4\xi_{vw} = 0, \qquad \xi_u = 0, \qquad 2b\xi_x + b\tau_t - 2\xi_{xw} = 0, \tag{7.18vii-ix}$$

$$4\xi_{xv} + 8\xi_w + 4b\xi_v v + \tau_{tt} v = 0, \qquad \xi_{xx} + 2b\xi_x v + b\tau_t v + b\xi_w v^2 = 0, \tag{7.18x-xi}$$

$$\xi_{vv} - \xi_t + 2x\xi_v b + x^2\xi_w b = 0, \qquad 2\xi_x - 2\phi_{3,w} + \tau_t + 4\xi_w v = 0, \tag{7.18xii-xiii}$$

$$\xi_v v - \phi_{3,v} = 0, \qquad \phi_{3,u} = 0, \qquad 2\phi_{3,xx} + 2b\xi_x v^2 + b\tau_t v^2 = 0, \tag{7.18xiv-xvi}$$

$$4b\xi_w vx^2 - 2\xi_v - 2\phi_{3,t} - b\tau_t x^2 - 4b\xi x + 2bx^2\xi_x + 2v\xi_{vv} + 4vx\xi_v b = 0, \tag{7.18xvii}$$

$$2\xi_w v^2 + 2\phi_{3,x} - 2\phi_2 + \tau_t v = 0, \tag{7.18xviii}$$

$$2\xi_w uv + 2\xi_v u^2 + 2\xi_x u + 2\phi_1 - \tau_t u = 0. \tag{7.18xix}$$

This is straightforwardly solved to give

$$\xi = \frac{1}{b}\exp\{b(w - vx)\}(bAx - A_v) + \frac{c_4}{2b}v + (c_4 t + c_1)x + c_2, \tag{7.19i}$$

$$\tau = -c_4 t^2 - 2c_1 t + c_3, \tag{7.19ii}$$

$$\phi_1 = \frac{1}{b}\exp\{b(w - vx)\}[bA(bx^2u^2 - u) + A_t u^2 - 2bxu^2 A_v] - 2(c_4 t + c_1)u - \frac{c_4}{2b}u^2, \tag{7.19iii}$$

$$\phi_2 = -(c_4 t + c_1)v - 2bc_2 t + c_5, \tag{7.19iv}$$

$$\phi_3 = \frac{1}{b}\exp\{b(w - vx)\}[A(1 + bvx) - vA_v] + \frac{c_4}{4b}v^2 - 2bc_2 xt - \frac{c_4}{2b}t + c_5 x + c_6, \tag{7.19v}$$

where $A = A(t, v)$ satisfies the linear heat equation

$$A_t = A_{vv}. \tag{7.20}$$

If $A = c_4 = 0$ we regain the original infinitesimals, if $A = 0$ we have found the potential infinitesimals of system (7.3), whilst if $A \neq 0$ we have some more potential symmetries. Finding all the reductions in this instance appears to be extremely difficult, if not impossible. To give some examples of reductions, consider first the case when $A = a$, constant and $c_1 = c_2 = c_4 = 0$ in which we set $c_3 = 1$ without loss of generality. We have

$$\xi = ax\exp\{b(w - vx)\}, \qquad \tau = 1, \qquad \phi_1 = a(bx^2u^2 - u)\exp\{b(w - vx)\},$$
$$\phi_2 = c_5, \qquad \phi_3 = \frac{a}{b}(1 + bvx)\exp\{b(w - vx)\} + c_5x + c_6.$$

Expressions for u and v can be gained using the method of characteristics, then one for w via (7.4b,c) and the invariant surface condition for w, which gives an algebraic equation. The independent symmetry variable may now be found, again with the method of characteristics, to give two reductions

Reduction 7.2.5. $c_6 \neq 0$

$$u(x,t) = \frac{-1}{bx(x + U(z))}, \qquad v(x,t) = V(z) + c_5t, \tag{7.21i,ii}$$
$$w(x,t) = x(V(z) + c_5t) + \frac{1}{b}\ln\left[\frac{b}{a}(bxU(z) - c_5x - c_6)\right], \tag{7.21iii}$$

where z is the implicit dependent variable

$$z(x,t,u) = \exp\{-bc_6t\}\left(c_5 + bx + \frac{1}{ux} + \frac{c_6}{x}\right).$$

Now $U(z)$ and $V(z)$ satisfy

$$bzUV' - 1 = 0, \tag{7.22i}$$
$$c_6bzV' - bzU' + bU - c_5 = 0, \tag{7.22ii}$$

which may be written as a first order equation for $U(z)$

$$-bzUU' + bU^2 - c_5U + c_6 = 0, \tag{7.23}$$

where $V(z)$ is given by (7.22i). Equation (7.23) may be integrated since it is separable to give

$$\frac{c_5}{2b(c_5^2 - 4c_6b)^{1/2}}\ln\left[\frac{2bU - (c_5^2 - 4c_6b)^{1/2} - c_5}{2bU + (c_5^2 - 4c_6b)^{1/2} - c_5}\right] + \frac{1}{2b}\ln(bU^2 - c_5U + c_6) = \frac{1}{b}\ln z + \frac{1}{b}\ln c_7,$$

where c_7 is a non-zero constant. The implicit nature of this equation, coupled with the fact that z is a function of u means that an explicit solution here for $u(x,t)$ is unlikely if $c_5 \neq 0$. For instance setting $c_6 = 3c_5^2/16b$ leaves a quartic for U after exponentiating. A solution of this quartic is $U = c_5/4b$ but this gives a time independent solution, and the other, not complex, solution is

$$U(z) = h(z) + \frac{c_7z^2}{3bh(z)} + \frac{3c_5}{4b},$$

where

$$h^3(z) = \frac{c_7 z^2 (27c_5^2 - 16c_7 b z^2)^{1/2}}{12\sqrt{3}\, b} + \frac{c_7 c_5 z^2}{4b^2}.$$

Whilst $U(z)$ has been found explicitly, the overall expression for $u(x,t)$, given by this and (7.21i) is still implicit. However if $c_5 = 0$ we are able to find

$$U(z) = \left(\frac{c_7 z^2 - c_6}{b} \right)^{1/2},$$

with c_7 an arbitrary constant. If $c_7 = 0$ then we simply have a time independent solution for $u(x,t)$ whereas if $c_7 \neq 0$, we may solve (7.21i) for u, and after some manipulation we find

$$u(x,t) = \frac{\pm[(c_7 b(bx^2 + c_6)\exp\{-2c_6 bt\} - bx^2)/c_6]^{-1/2} - 1}{bx^2 + c_6}. \tag{7.24}$$

This exact solution may not be found by the previous applications of the classical method on the lower order system or the original equation. We may find $V(z)$ from (7.22i) as

$$V(z) = -\frac{1}{\sqrt{bc_6}} \sin^{-1}\left(\frac{\sqrt{c_6}}{\sqrt{c_7}\,|z|} \right),$$

and hence $v(x,t)$ and $w(x,t)$ from (7.21ii) and (7.21iii) respectively.

Reduction 7.2.6. $c_6 = 0$

$$u(x,t) = \frac{-1}{bx(x + U(z))}, \qquad v(x,t) = V(z) + c_5 t, \tag{7.25i,ii}$$

$$w(x,t) = x(V(z) + c_5 t) + \frac{1}{b}\ln\left[\frac{bx}{a}(bU(z) - c_5)\right], \tag{7.25iii}$$

where z is given by

$$z(x,t,u) = t - \frac{u}{b(c_5 xu + bx^2 u + 1)},$$

and $U(z)$, $V(z)$ satisfy

$$V' - 3U + c_5 = 0, \tag{7.26i}$$

$$U' - b^2U^3 + 2bc_5U^2 - c_5^2 U = 0. \tag{7.26ii}$$

Equation (7.26ii) may be integrated with respect to z to give for $c_5 \neq 0$

$$\frac{U}{bU - c_5}\exp\left\{\frac{-c_5}{bU - c_5}\right\} = \exp\{z + c_8\},$$

for c_8 an arbitrary constant, which is again implicit. If $c_5 = 0$ we find

$$U(z) = (c_9 - 2b^2 z)^{-1/2},$$

for c_9 an arbitrary constant, which leads to the exact solution

$$u(x,t) = \frac{1}{bx^2}\left(\pm[(c_9 - 2b^2 t)x^2 + 1]^{-1/2} - 1\right). \tag{7.27}$$

After some manipulation we find that this may be found from the classical scaling reduction 7.2.2 for

$$U(z) = \frac{1}{bz^2}\left[\pm(1-2b^2z^2)^{-1/2}-1\right].$$

It seems unlikely however that this solution of (7.9) would be found under normal circumstances.

Another idea for a reduction comes from when $A(t,v) = F(y) = \mathrm{e}^{-c_5 y} + a_1$ where $y = v - c_5 t$, and $c_1 = c_2 = c_4 = 0$ and we set $c_3 = 1$ without loss of generality. The infinitesimals are now (leaving F unevaluated for convenience)

$$\begin{aligned}
\xi &= \frac{1}{b}\exp\{b(w-vx)\}\left(bFx - \frac{\mathrm{d}F}{\mathrm{d}y}\right), \qquad \tau = 1,\\
\phi_1 &= \frac{1}{b}\exp\{b(w-vx)\}\left(bF(bx^2u^2-u) + \frac{\mathrm{d}^2F}{\mathrm{d}y^2}u^2 - 2b\frac{\mathrm{d}F}{\mathrm{d}y}xu^2\right), \qquad \phi_2 = c_5,\\
\phi_3 &= \frac{1}{b}\exp\{b(w-vx)\}\left(F(1+bvx) - v\frac{\mathrm{d}F}{\mathrm{d}y}\right) + c_5x + c_6.
\end{aligned}$$

The advantage of choosing $A(t,v)$ like this is that y is one of the invariants, which can be seen from using the method of characteristics. The equation

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}v}{c_5},$$

gives $y \equiv V(z)$, so that

$$v(x,t) = V(z) + c_5 t, \tag{7.28}$$

for z to be found. Thus occurrences of F and its derivatives may be treated as constants in integrating using the method of characteristics, hence

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\phi_1}{\xi},$$

becomes a first order Riccati equation for u, which may be linearised. It has solution

$$u(x,t) = -bF^2\Big/\left[b^2F^2x^2 - bF\frac{\mathrm{d}F}{\mathrm{d}y}x - F\frac{\mathrm{d}^2F}{\mathrm{d}y^2} + \left(\frac{\mathrm{d}F}{\mathrm{d}y}\right)^2 + U(z)bF^2\left(bFx - \frac{\mathrm{d}F}{\mathrm{d}y}\right)\right]. \tag{7.29}$$

We find $w(x,t)$ as before, from an algebraic expression, as

$$\begin{aligned}
w(x,t) = x(V(z)+c_5t) + \frac{1}{b}\ln\Bigg\{\frac{1}{F^3}\Bigg[\left(\frac{\mathrm{d}F}{\mathrm{d}y}\right)^2 - F\frac{\mathrm{d}^2F}{\mathrm{d}y^2} - U(z)bF^2\frac{\mathrm{d}F}{\mathrm{d}y}\\
- c_6bF^2 + x\left(U(z)b^2F^3 - c_5bF^2 - bF\frac{\mathrm{d}F}{\mathrm{d}y}\right)\Bigg]\Bigg\}.
\end{aligned} \tag{7.30}$$

Finally we find z from

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}t}{1},$$

removing occurrences of u, v, w using (7.29), (7.28) and (7.30) respectively, to give two different expressions for z: if $c_6 = 0$

$$z(x,t,u) = t - \frac{u}{b(bx^2u + c_5xu + 1)}, \tag{7.31i}$$

or, if $c_6 \neq 0$,

$$z(x,t,u,v) = -bF^2\left(c_6 + \frac{1}{u} + bx^2 + c_5x\right)\exp\{-c_6bt\}\Big/\left(bxF - \frac{\mathrm{d}F}{\mathrm{d}y}\right). \tag{7.31ii}$$

In (7.29), (7.28) and (7.30) we write $F = F(V(z))$, whereas in (7.31) we use $F = F(y)$. Unfortunately things now start to break down.

In the simplest case $c_6 = a_1 = 0$, we find that (7.4a) gives two equations for U, V, namely

$$\mathrm{e}^{c_5V}V' - bU = 0, \tag{7.32i}$$

$$bU^2V' - \mathrm{e}^{c_5V}U' = 0, \tag{7.32ii}$$

which combine to give the single equation for $V(z)$

$$V'' + (V')^3 - c_5(V')^2 = 0. \tag{7.33}$$

A simple solution of this comes when $V' = c_5$, but this leads to a time independent solution only. Thus for $V' \neq c_5$ we may integrate (7.33) once with respect to z to yield

$$\frac{V'}{V'-c_5}\exp\left\{-\frac{c_5}{V'}\right\} = \exp\{z + c_7\},$$

upon exponentiating, which is implicit. Since this is implicit, and also because z depends on u it seems unlikely that we will find any explicit solutions here.

Any attempt to find solutions in the cases when c_6 and a_1 are not necessarily zero is hampered by the sheer size of the calculation. For instance, if $c_6 = 0$, $a_1 \neq 0$, the expression for u_x is a rational function with 96 terms on the numerator and 112 on the denominator, and this still includes explicit occurrences of u and powers of u! Due to the memory limitations of the available computer the calculation could not be finished. Further, since in the simplest case ($c_6 = a_1 = 0$) we are unable to find explicit solutions, unable in fact to obtain even an implicit solution for u or v as functions of x, t only, no further cases are considered here.

7.3 Nonclassical symmetries ($\tau \neq 0$)

In this section we apply the nonclassical method when $\tau \neq 0$ to our various systems and recall that we may set $\tau = 1$ without loss of generality. We use the algorithm of Clarkson and Mansfield [1994c] in practice, which calls for us to remove t-derivatives from our equations using the invariant surface condition, and to apply the classical method to this new system.

Case 7.3.1 Equation (7.2). The determining equations are

$$\xi_{uu}u + 2\xi_u = 0, \tag{7.34i}$$

$$2\xi\xi_u u^4 + \phi_{1,uu}u^2 - 2\xi_{xu}u^2 - 2\phi_{1,u}u + 2\phi_1 = 0, \tag{7.34ii}$$

$$\phi_{1,t}u^2 - 2b\phi_{1,u}u^2 + 2\xi_x\phi_1 u^2 + 4b\xi_x u^2 + 2\phi_1^2 u + 4b\phi_1 u - \phi_{1,xx} = 0, \tag{7.34iii}$$

$$2\xi_u\phi_1 u^3 - 2\xi\xi_x u^3 + 6b\xi_u u^3 - \xi_t u^3 - 2\xi\phi_1 u^2 - 2\phi_{1,xu}u + \xi_{xx}u + 4\phi_{1,x} = 0. \tag{7.34iv}$$

Notice that both (7.34i) and (7.34ii) are linear second order Euler equations, so we can look for solutions of the homogeneous part of each equation in the form u^n, where n is to be determined. We find

$$\xi = \frac{F(x,t)}{u} + G(x,t), \qquad \phi_1 = H(x,t)u^2 + K(x,t)u - 2F^2 u \ln u + 2GFu^2 \ln u - F_x,$$

where $F(x,t)$, $G(x,t)$, $H(x,t)$ and $K(x,t)$ are arbitrary functions of (x,t). Substituting these expressions for ξ and ϕ_1 into (7.34iii) and (7.34iv) we quickly find $F(x,t) = 0$, and then that the infinitesimals must be of the form

$$\xi = \frac{-x + c_1}{2(t + c_2)}, \qquad \phi_1 = \frac{u}{t + c_2},$$

or $\xi = c_3$, $\phi_1 = 0$, which are the classical infinitesimals of (7.2). We note that applying the nonclassical method to equation (7.1) also results in finding only classical infinitesimals.

Case 7.3.2 System (7.3). In this Case we first apply the nonclassical method to the single equation (7.16b) only, which we noted in §7.2 also generates potential symmetries. We allow the infinitesimals (ξ, ϕ_2) to depend on all of (x,t,u,v), and the determining equations are then

$$\xi_u = 0, \tag{7.35i}$$

$$\phi_{2,u} = 0, \tag{7.35ii}$$

$$\phi_{2,xx} = 0, \tag{7.35iii}$$

$$2\xi\phi_{2,v} - 2b\xi_v x - \xi_{vv} + \xi_t = 0, \tag{7.35iv}$$

$$2\phi_{2,xv} - 4b\phi_{2,x}x - 2\phi_2\phi_{2,x} - \xi_{xx} = 0, \tag{7.35v}$$

$$2b\phi_{2,v}x - 2\xi\phi_{2,x} - \phi_{2,vv} + 2\phi_2\phi_{2,v} + \phi_{2,t} + 2\xi_{xv} + 2b\xi = 0. \tag{7.35vi}$$

We solve these determining equations by writing $\phi_2 = F(t,v)x + G(t,v)$ from (7.35ii,iii), where $F(t,v)$, $G(t,v)$ are arbitrary functions of (t,v), then (7.35v) gives us

$$\xi_{xx} = -(2F^2 + 4bF)x + 2F_v - 2FG,$$

which we may integrate twice with respect to x to find ξ, and introduce two more arbitrary functions of (t,v). The subsequent coefficients of powers of x in the two remaining equations then give us the classical infinitesimals (7.10) and also the two different sets of nonclassical infinitesimals

$$\xi = 0, \qquad \phi_2 = -2bx, \tag{7.36i,ii}$$

or

$$\xi = -b^2x^3 - bG(t,v)x^2 + H(t,v)x + K(t,v), \tag{7.37i}$$

$$\phi_2 = bx + G(t,v), \tag{7.37ii}$$

where $G(t,v)$, $H(t,v)$ and $K(t,v)$ satisfy

$$G_{vv} - 2GG_v - G_t - 2H_v = 0, \tag{7.38i}$$

$$H_{vv} - 2HG_v - H_t + 2bK_v = 0, \tag{7.38ii}$$

$$K_{vv} - 2KG_v - K_t = 0. \tag{7.38iii}$$

If we now apply the nonclassical method to (7.16a) we find ϕ_1 explicitly and the condition $\xi_u u - \phi_{2,u} = 0$, which is satisfied by the above infinitesimals. We have

$$\phi_1 = \phi_{2,x} + (\phi_{2,v} - \xi_x)u - \xi_v u^2,$$

hence $\qquad \phi_1 = -2b, \qquad$ (7.36iii)

or $\qquad \phi_1 = b + (G_v + 3b^2x^2 + 2bGx - H)u - (-bG_v x^2 + H_v x + K_v)u^2, \qquad$ (7.37iii)

respectively. However, if we now apply the nonclassical method to the system (7.3) we get only *two* determining equations,

$$\phi_1 - [\phi_{2,x} + (\phi_{2,v} - \xi_x)u - \xi_v u^2 + (\phi_{2,u} - u\xi_u)(\phi_2 u^2 - \xi u^3 + 2u^2bx)] = 0, \tag{7.39i}$$

$$\begin{aligned}
&\xi_x\phi_2u^2 - \phi_{1,u}\phi_2u^2 - \xi\xi_xu^3 + 2\phi_1\phi_2u + 4b\phi_1ux + \xi^2\phi_{2,u}u^5 + \xi_u\phi_2^2u^4 \\
&\quad - \xi\phi_{2,v}u^3 + \xi\phi_{1,u}u^3 - \xi_u\phi_1u^3 + \phi_2\phi_{2,v}u^2 + \phi_1\phi_{2,u}u^2 - 2\xi\phi_1u^2 + 2b\xi u^2 \\
&\quad - \xi_tu^3 + \phi_{2,t}u^2 - \phi_{1,v}u - \phi_{1,x} - \xi\xi_u\phi_2u^5 - \xi\phi_2\phi_{2,u}u^4 + 4b^2\xi_uu^4x^2 \\
&\quad - 2b\xi\xi_uu^5x - 2b\xi\phi_{2,u}u^4x + 4b\xi_u\phi_2u^4x + 2b\xi_vu^3x - 2b\phi_{1,u}u^2x + 2b\xi_xu^2x = 0.
\end{aligned} \tag{7.39ii}$$

The first equation gives ϕ_1 explicitly, and removing all ϕ_1 derivatives from the second equation we get a quasilinear second order pde for ϕ_2, with various ξ derivatives as coefficients,

$$\begin{aligned}
&4u^4b^2x^2\phi_{2,uu} - u^5\xi_{uu}\phi_2^2 - 4u^5\xi b\phi_{2,uu}x \\
&\quad + 4u^3bx\phi_{2,uv} - 4u^4bx\xi_{uv} - 4u^5b^2x^2\xi_{uu} - 2u\phi_2\phi_{2,x} + 2\xi u^2\phi_{2,x} - 2b\xi_uu^3 \\
&\quad + 2b\phi_{2,u}u^2 + 2\phi_2\phi_{2,xu}u^2 - 2u^5\xi\phi_2\phi_{2,uu} - 2\xi_{xu}\phi_2u^3 - 2\xi\phi_{2,xu}u^3 - 2u^4\phi_2\xi_{uv} \\
&\quad + u^4\phi_2^2\phi_{2,uu} - 2u^4\xi\phi_{2,uv} + 2u^3\phi_2\phi_{2,uv} + 2\xi\xi_{xu}u^4 - 2\xi_u\phi_2^2u^4 + 2\xi\phi_{2,v}u^3 \\
&\quad - 2\phi_2\phi_{2,v}u^2 - 2b\xi u^2 - u^7\xi^2\xi_{uu} + \xi_tu^3 - \phi_{2,t}u^2 - 4b\xi_{xu}u^3x + 4\xi\xi_u\phi_2u^5 \\
&\quad - 8b^2\xi_uu^4x^2 + 8b\xi\xi_uu^5x - 8b\xi_u\phi_2u^4x - 2b\xi_vu^3x + \phi_{2,vv}u^2 + \phi_{2,xx} - 4ubx\phi_{2,x} \\
&\quad + 4b\phi_{2,xu}u^2x - 2\xi_{xv}u^2 - \xi_{vv}u^3 + 2u^5\xi\xi_{uv} + 2\phi_{2,xv}u - \xi_{xx}u + u^6\xi^2\phi_{2,uu} - 2u^2bx\phi_{2,v} \\
&\quad + 4u^6\xi b\xi_{uu}x - 2u^6\xi^2\xi_u - 4u^5\phi_2b\xi_{uu}x + 4u^4\phi_2b\phi_{2,uu}x + 2u^6\xi\xi_{uu}\phi_2 = 0.
\end{aligned} \tag{7.40}$$

It is unlikely that we will be able to solve this equation in full generality. Fortunately we notice that if ξ, ϕ_1, ϕ_2 are solutions of the determining equations, then

$$\widehat{\xi} = \xi + F(x,t,u,v), \tag{7.41i}$$

$$\widehat{\phi}_1 = \phi_1 + (\phi_2 u^2 - \xi u^3 + 2u^2 bx)F, \tag{7.41ii}$$

$$\widehat{\phi}_2 = \phi_2 + Fu, \tag{7.41iii}$$

for $F(x,t,u,v)$ an arbitrary function, are also solutions of the determining equations, and correspond to the same reduction. To see this, for solutions of the determining equations we have

$$\xi u_x + u_t = \phi_1, \tag{7.42i}$$

$$\xi v_x + v_t = \phi_2, \tag{7.42ii}$$

together with system (7.3), which represents a specific solution/reduction corresponding to specific infinitesimals ξ, ϕ_1, ϕ_2. In terms of our new variables $\widehat{\xi}, \widehat{\phi}_1, \widehat{\phi}_2$, system (7.42) becomes

$$(\widehat{\xi} - F)u_x + u_t = \widehat{\phi}_1 - [(\widehat{\phi}_2 - Fu)u^2 - (\widehat{\xi} - F)u^3 + 2u^2 bx]F, \tag{7.43i}$$

$$(\widehat{\xi} - F)v_x + v_t = \widehat{\phi}_2 - Fu. \tag{7.43ii}$$

From equations (7.43ii) and (7.3a) we have

$$\widehat{\xi} v_x + v_t = \widehat{\phi}_2,$$

by interchanging u for v_x. From (7.43ii) and (7.3a,b) we find

$$u_x = u^2(\widehat{\phi}_2 - \widehat{\xi} u) + 2u^2 bx.$$

Using this in (7.43i) gives

$$\widehat{\xi} u_x + u_t = \widehat{\phi}_1,$$

as required. What we have is the same reduction for different infinitesimals. As an example, a simple solution of the determining equations is given by (7.36), which corresponds to

$$u_t = -2b, \qquad v_t = -2bx, \qquad v_x = u, \qquad u^2 v_t = u_x - 2bxu^2. \tag{7.44}$$

Solving this system gives $u(x,t) = -2bt + c_1$, $v(x,t) = -2bxt + c_1 x + c_2$, which is in fact also a solution given by the classical scaling reduction 7.2.2, for $U(z) = -2b$. Our transformation gives new infinitesimals

$$\widehat{\xi} = F, \qquad \widehat{\phi}_1 = -2b, \qquad \widehat{\phi}_2 = Fu - 2bx,$$

for arbitrary $F(x,t,u,v)$, corresponding to the invariant surface conditions

$$Fu_x + u_t = -2b, \tag{7.45i}$$

$$Fv_x + v_t = Fu - 2bx, \tag{7.45ii}$$

together with (7.3). Substituting (7.3a) into (7.45ii) gives $v_t = -2bx$, and hence substituting for u_x from (7.3b) into (7.45i) gives $u_t = -2b$. Hence we have reclaimed system (7.44).

The transformation (7.41) gives us the freedom to choose $F(x,t,u,v)$ and hence $\widehat{\xi}$ however we like, in the knowledge that we won't lose any information. For instance we can choose $\widehat{\xi} = 0$ and we only have two infinitesimals to solve for (note that $\widehat{\xi}, \widehat{\phi}_1, \widehat{\phi}_2$ will satisfy the same determining equations as ξ, ϕ_1, ϕ_2). Unfortunately this is still a nontrivial task, as the equation that $\widehat{\phi}_2$ satisfies is

$$\begin{aligned} &2\widehat{\phi}_2\widehat{\phi}_{2,xu}u^2 - 2\widehat{\phi}_2\widehat{\phi}_{2,v}u^2 - \widehat{\phi}_{2,t}u^2 + 2b\widehat{\phi}_{2,u}u^2 \\ &\quad + 4b\widehat{\phi}_{2,xu}u^2x + 2\widehat{\phi}_{2,xv}u + \widehat{\phi}_{2,xx} - 4ubx\widehat{\phi}_{2,x} - 2u\widehat{\phi}_2\widehat{\phi}_{2,x} + 2u^3\widehat{\phi}_2\widehat{\phi}_{2,uv} + u^4\widehat{\phi}_2^2\widehat{\phi}_{2,uu} \\ &\quad - 2u^2bx\widehat{\phi}_{2,v} + 4u^4\widehat{\phi}_2 b\widehat{\phi}_{2,uu}x + 4u^4b^2x^2\widehat{\phi}_{2,uu} + 4u^3bx\widehat{\phi}_{2,uv} + \widehat{\phi}_{2,vv}u^2 = 0. \end{aligned} \tag{7.46}$$

Similarly, one could assume that $\widehat{\phi}_2 = 0$ which leaves the (equally difficult!) equation to solve for $\widehat{\xi}$,

$$\begin{aligned} &4u^2b\widehat{\xi}_{xu}x - 2u^3\widehat{\xi}\widehat{\xi}_{xu} + 2u^2b\widehat{\xi}_u + 2ub\widehat{\xi} - u^2\widehat{\xi}_t \\ &\quad + 8b^2x^2\widehat{\xi}_u u^3 - 8u^4\widehat{\xi}b\widehat{\xi}_u x + 2bx\widehat{\xi}_v u^2 + 2\widehat{\xi}_{xv}u + \widehat{\xi}_{xx} + 2u^5\widehat{\xi}^2\widehat{\xi}_u \\ &\quad + u^6\widehat{\xi}^2\widehat{\xi}_{uu} - 2\widehat{\xi}\widehat{\xi}_{uv}u^4 - 4u^5\widehat{\xi}b\widehat{\xi}_{uu}x + 4u^4b^2x^2\widehat{\xi}_{uu} + 4b\widehat{\xi}_{uv}u^3x + \widehat{\xi}_{vv}u^2 = 0. \end{aligned} \tag{7.47}$$

An alternative is to choose $F = u\xi_u - \phi_{2,u}$. Under these conditions (7.42i,ii) become

$$\widehat{\xi}u_x + u_t = \widehat{\phi}_1 = \widehat{\phi}_{2,x} + (\widehat{\phi}_{2,v} - \widehat{\xi}_x)u - \widehat{\xi}_v u^2,$$

$$\widehat{\xi}v_x + v_t = \widehat{\phi}_2,$$

and we also find the relation $\widehat{\phi}_{2,u} - u\widehat{\xi}_u = 0$ from (7.41). In this case it suffices to consider $\widehat{\xi}, \widehat{\phi}_2$ under this relationship. Firstly assuming $\widehat{\xi}_u = \widehat{\phi}_{2,u} = 0$ we find nonclassical infinitesimals (7.36) and (7.37). Considering $\widehat{\xi}_u\widehat{\phi}_{2,u} \neq 0$ the system of two equations to solve for $\widehat{\xi}, \widehat{\phi}_2$ suffers greatly from expression swell, and trying various ansätze for $\widehat{\xi}$ (e.g. polynomial in u) has so far led to no more reductions.

Hopefully then, the motivation for studying equation (7.16b) first is now clear – the determining equations for (7.3) would appear to be too difficult to solve, even with our transformation (7.41), and yet they appear to offer no more solutions than those already found by considering equation (7.16b).

We therefore turn our attention to the infinitesimals (7.37) and in particular to the solution of system (7.38). We note that the system (7.38) is equivalent to a system found in the process of applying the nonclassical method to Burgers' equation, (7.5). For Burgers' equation we have infinitesimals (Arrigo, Broadbridge and Hill [1993])

$$\xi = -\tfrac{1}{2}u + g(x,t), \tag{7.48i}$$

$$\phi = -\tfrac{1}{4}u^3 + \tfrac{1}{2}g(x,t)u^2 + h(x,t)u + k(x,t), \tag{7.48ii}$$

where $g(x,t)$, $h(x,t)$ and $k(x,t)$ satisfy

$$g_{xx} - 2gg_x - g_t - 2h_x = 0, \tag{7.49i}$$

$$h_{xx} - 2hg_x - h_t - k_x = 0, \tag{7.49ii}$$

$$k_{xx} - 2kg_x - k_t = 0. \tag{7.49iii}$$

Note that $g(x,t) = G(v,t)$, $h(x,t) = H(v,t)$ and $k(x,t) = -2bK(v,t)$. Some exact solutions of (7.49) have been found by Pucci [1992], who considers $g = g(t)$ and finds the general solution of the reduced system. A wider class of solutions is found by Arrigo, Broadbridge and Hill [1993] who also consider the case when $g = g(x)$ and $h = k = 0$ and solve the single ordinary differential equation for $g(x)$. Finally Clarkson and Mansfield [1994c] find more solutions again, by the use of differential Gröbner bases. They find solutions of the form $g = a(t)x + b(t)$ where the equation that $a(t)$ satisfies may be linearised by a Cole-Hopf transformation $a(t) = R\dfrac{\mathrm{d}}{\mathrm{d}t}[\ln\psi(t)]$ for R to be determined. Then $\psi(t)$ satisfies $\dfrac{\mathrm{d}^4\psi}{\mathrm{d}t^4} = 0$ and $b(t)$ satisfies $\dfrac{\mathrm{d}^3}{\mathrm{d}t^3}b(t)\psi(t) = 0$. For our system (7.38) this solution is

$$G(t,v) = [(3a_1t^2 + 2a_2t + a_3)v + 2(a_5t^2 + a_6t + a_7)]/2\psi(t), \tag{7.50i}$$

$$H(t,v) = -[(3a_1t + a_2)v^2 + (4a_5t + 2a_6)v + (6a_1t^2 + 2a_8t + 2a_9)]/4\psi(t), \tag{7.50ii}$$

$$K(t,v) = -[a_1v^3 + 2a_5v^2 + (2a_8 + 6a_1t - 2a_2)v + (4a_5t - 4a_{10})]/8b\psi(t), \tag{7.50iii}$$

where

$$\psi(t) = a_1t^3 + a_2t^2 + a_3t + a_4. \tag{7.50iv}$$

We note that there are no more solutions for which $g(x,t)$ is polynomial in x. As noted by Ludlow [1995] one could apply the nonclassical method to system (7.49) but this is not considered here. However in the final section, §7.4, we do find some more solutions of (7.38), which appear to be new.

The similarity between (7.38) and (7.49) is really too striking to be a coincidence. By noting that v in (7.38) is replaced by x in (7.49) leads us to consider a hodograph transformation of (7.16b). Let

$$t = \rho, \qquad x = \eta(\zeta,\rho), \qquad v = \zeta, \tag{7.51}$$

be the inverse of a pure hodograph transformation, then the derivatives transform as follows (see Clarkson, Fokas and Ablowitz [1989])

$$v_x = \eta_\zeta^{-1}, \qquad v_t = -\eta_\rho \eta_\zeta^{-1}, \qquad v_{xx} = -\eta_{\zeta\zeta}\eta_\zeta^{-3}.$$

Then equation (7.16b) transforms into

$$\eta_\rho = \eta_{\zeta\zeta} + 2b\eta\eta_\zeta,$$

which is Burgers' equation (7.5), up to a scaling of η. Hence we have our explanation.

Since Burgers' equation is better known than the reaction–diffusion equation we are studying, and because it may be transformed onto the linear heat equation

$$\psi_t = \psi_{xx}, \tag{7.52}$$

(by a Cole-Hopf transformation (Cole [1951], Hopf [1950])), in terms of finding exact solutions of (7.1) many may be found by taking solutions of Burgers' equation and using the transformation (7.51) to find (implicit) solutions. This is not pursued further here. However the calculation itself is still of value, and we find some explicit solutions in order to compare them with solutions from §7.2. We take three different avenues.

I. $G = a_1$, $H = a_2$ *and* $K = a_3$. The infinitesimals read

$$\xi = -b^2x^3 - ba_1x^2 + a_2x + a_3, \qquad \phi_1 = b + (3b^2x^2 + 2ba_1x - a_2)u, \qquad \phi_2 = bx + a_1. \tag{7.53}$$

These give rise to the following nonclassical (potential) reduction

Reduction 7.3.1.

$$u(x,t) = \frac{U(z) + bx}{-b^2x^3 - ba_1x^2 + a_2x + a_3}, \qquad v(x,t) = V(z) + \int^x \frac{bs + a_1}{-b^2s^3 - ba_1s^2 + a_2s + a_3}\,\mathrm{d}s,$$

where

$$z(x,t) = t - \int^x \frac{\mathrm{d}s}{-b^2s^3 - ba_1s^2 + a_2s + a_3},$$

and $U(z)$, $V(z)$ satisfy

$$V' + U - a_1 = 0,$$
$$U^2V' + U' + a_2U - a_3b = 0.$$

With regards explicit solutions, the first integral of the single equation for $U(z)$ may be written down

$$\int^{U(z)} \frac{\mathrm{d}s}{s^3 - a_1s^2 - a_2s + a_3b} = z + a_4, \tag{7.54}$$

for a_4 an arbitrary constant. The simplest solution in this case comes when $a_1 = a_2 = a_3 = 0$, which gives the exact solution

$$u(x,t) = \frac{1}{bx^2}\left(\pm[(a_5 - 2t)b^2x^2 + 1]^{-1/2} - 1\right), \tag{7.55}$$

for a_5 an arbitrary constant, which is equivalent to (7.27) which comes from the classical scaling reduction 7.2.2.

Another solution may be found when $a_1 = a_3 = 0$ and $a_2 \neq 0$. We find

$$u(x,t) = \frac{\pm[(x^2 - a_7 b(b^2x^2 - a_2)\exp\{2a_2 t\})/a_2]^{-1/2} - b}{b^2x^2 - a_2}, \tag{7.56}$$

where a_7 is a non-zero constant, and $v(x,t)$ may be found without too much difficulty if required. This is the exact solution (7.24) with slightly different names of constants.

In general, for other combinations of the constants, (7.54) will give an implicit equation for $U(z)$, though notice that z is a function of (x,t) only. Pucci [1992] considered the equivalent infinitesimals in her study of Burgers' equation and found exact solutions in the special cases we have considered.

II. $G = -1/v$, $H = K = 0$. We have

$$\xi = -b^2x^3 + \frac{bx^2}{v}, \qquad \phi_1 = b + \left(\frac{1}{v^2} + 3b^2x^2 - \frac{2bx}{v}\right)u + \frac{bx^2}{v^2}u^2, \qquad \phi_2 = bx - \frac{1}{v}.$$

We find firstly that

$$v(x,t) = \frac{1 + bxV(z)}{bx},$$

and use this to find an expression for z, namely

$$z(x,t,v) = \frac{1}{b^2x^2}\left[\frac{1}{3(bxv-1)} + \frac{1}{2}\right] - t.$$

Whilst we may now use

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\phi_1}{\xi},$$

to find a first order Riccati equation for u, this turns out to be too complicated to solve explicitly, so instead we use (7.3a) to give us an expression for u: this gives

$$u(x,t) = -\frac{1}{bx^2} - \frac{3VV'(1+bxV)}{x(3b^3V^2x^3 + V')}.$$

Finally we find that $V(z)$ must satisfy

$$VV'' + V(V')^3 - 2(V')^2 = 0.$$

Making the transformation $R = 1/V$ we find

$$R^4R'' + (R')^3 = 0,$$

which may be integrated to yield

$$\frac{3R^3R'}{1 + 3a_8R^3} = -1,$$

where a_8 is an arbitrary constant. If $a_8 \neq 0$ integrating this again produces a lengthy implicit expression, whereas if $a_8 = 0$ we find

$$R(z) = (6z - a_9)^{-1/2}. \tag{7.57}$$

Since z is a function of v (7.57) gives an algebraic expression for v, which is the cubic

$$bxv^3 - 3v^2 + (6btx + a_9 bx)v - 6t - a_9 = 0,$$

and hence any root of this gives $v(x,t)$, and at least one root will be real. The unknown $u(x,t)$ is perhaps most easily found from (7.3a) which will give

$$u(x,t) = -\frac{bv^3 + (6bt + a_9 b)v}{3bxv^2 + 6btxv + a_9 bx - 6v}.$$

The exact solution found from the equivalent infinitesimals for Burgers' equation in this case has been considered by Arrigo, Broadbridge and Hill [1993].

III. *G, H, K given by (7.50) where $a_2 = a_3 = a_4 = a_6 = a_7 = a_9 = 0$, so that $a_1 \neq 0$.* In this instance ξ and ϕ_2 become

$$\xi = -b^2\left(x + \frac{v}{2bt}\right)^3 - \frac{3}{2t}\left(x + \frac{v}{2bt}\right) - \frac{a_5 b}{a_1 t}\left(x + \frac{v}{2bt}\right)^2 - \frac{a_5}{2ba_1t^2} - \frac{a_8}{2a_1t^2}\left(x + \frac{v}{2bt}\right) - \frac{a_{10}}{2ba_1t^3}, \tag{7.58i}$$

$$\phi_2 = bx + \frac{3v}{2t} + \frac{a_5}{a_1 t}. \tag{7.58ii}$$

Recall that the invariant surface condition for $v(x,t)$ reads

$$\xi v_x + v_t = \phi_2,$$

then making the transformation $v(x,t) = \rho(x,t) - 2btx$, this invariant surface condition becomes

$$\left[\frac{\rho^3}{8bt^3} + \frac{3\rho}{4bt^2} + \frac{a_5\rho^2}{4a_1bt^3} + \frac{a_5}{2a_1bt^2} + \frac{a_8\rho}{4a_1bt^3} + \frac{a_{10}}{2a_1bt^2}\right]\rho_x - \rho_t = \frac{a_1\rho^3 + 2a_5\rho^2 + 2a_8\rho + 4a_{10}}{4a_1t^2}.$$

Applying the method of characteristics to this, one yields

$$\frac{\mathrm{d}t}{-4a_1t^2} = \frac{\mathrm{d}\rho}{a_1\rho^3 + 2a_5\rho^2 + 2a_8\rho + 4a_{10}},$$

which is prime for integrating as it has separated, but if all the constants are arbitrary this leaves a complicated expression for ρ. However if $a_5 = a_8 = a_{10} = 0$, this may be integrated to leave

$$\rho(x,t) = \left(\frac{2t}{4tV(z) - 1}\right)^{1/2}, \tag{7.59}$$

which may be used to find the other characteristic direction, i.e.

$$z(x,t,\rho) = x + \frac{2t-\rho^2}{2bt\rho}. \tag{7.60}$$

Both (7.59) and (7.60) may be transformed back into expressions involving v. We use (7.3a) to find a complicated expression for $u(x,t)$, which when used in (7.3b) gives us the ordinary differential equation

$$V'' - b^2 = 0.$$

This may be solved to leave $V(z) = \frac{1}{2}b^2z^2 + b_1z + b_2$, for b_1, b_2 arbitrary constants, and since z is a function of v we have to solve an algebraic equation to find v. This is the cubic

$$bv^3 + (2b^2tx - 4b_1t)v^2 + (8b_2bt^2 - 6bt - 8b_1bt^2x)v + (16b_2b^2t^3x - 4b^2t^2x + 8b_1t^2) = 0,$$

which is guaranteed to have at least one real root. Again $u(x,t)$ is most easily found from (7.3a) which yields

$$u(x,t) = \frac{8b_1bt^2v - 2b^2tv^2 - 16b_2b^2t^3 + 4b^2t^2}{4b^2tvx - 8b_1bt^2x + 3bv^2 - 8b_1tv + 8b_2bt^2 - 6bt}.$$

This exact solution has not been considered in any of the previous symmetry studies of Burgers' equation.

Case 7.3.3 System (7.4). The system (7.4) suffers from a lack of derivatives to substitute back for (cf. Case 5.3.3) since w_t will be replaced by $\phi_3 - \xi w_x$ from the invariant surface condition. For this reason and also because the classical infinitesimals are so complicated there is likely to be little gained by looking at the nonclassical symmetries anyway, if there are any, we do not consider the nonclassical symmetries here.

7.4 Discussion

Let us first consider the condition for potential symmetries to exist – we see from §1.5.1 that if any of the infinitesimals of the non-potential variables (i.e. ξ, τ or ϕ_1 here) depend essentially on the potential variables (v or w here) then these give rise to potential symmetries. As discussed in Priestley and Clarkson [1995] and Chapter Six we see that in the nonclassical method this condition has no implications, i.e. we were able to find infinitesimals of the non-potential variables that depended on the potential variables, and yet nonclassical potential symmetries were not procured. However none of the infinitesimals (7.53) depend on v yet these give rise to a nonclassical (potential) reduction. We have found that not only are nonclassical potential symmetries not guaranteed if the requisite infinitesimals *do* depend on the potential variables, but also that nonclassical potential symmetries can exist if the requisite infinitesimals *do not* depend on the potential variables. Moreover, nonclassical potential symmetries can and do exist.

In applying the nonclassical method to system (7.3) we have seen, in (7.55), how classical exact solutions may be obtained from very different (nonclassical potential) symmetries. Similarly the exact solution (7.24) which arises from applying both the classical method to (7.4) and the nonclassical method to (7.3), is obtained from totally different infinitesimals in each instance. Thus an important question is to decide whether the very different infinitesimals (7.37) and (7.19) actually give the same solutions, and if not are one's solutions a subset of the other's, or will both give some distinct solutions.

In addition to the specific question of whether (7.37) and (7.19) give equivalent solutions, a more general question is whether there exist nonclassical potential symmetries whose solutions cannot be obtained by considering higher order potential systems.

We now give an explanation as to why the apparently different infinitesimals give the same reductions, and also give some reasoning as to why we think there do exist nonclassical potential symmetries whose solutions cannot be obtained by considering higher order potential systems.

Firstly we write down the algebraic expression given by the invariant surface condition for w, (7.4b) and (7.4c), namely

$$\frac{1}{b}\exp\{b(w-vx)\} = \left[\frac{c_4v^2}{4b} + (c_4t+c_1)vx + c_2v + (c_4t^2+2c_1t-c_3)\left(\frac{1}{u}+bx^2\right) + 2bc_2xt \right.$$
$$\left. + \frac{c_4t}{2b} - c_5x - c_6\right] \Big/ A(t,v)(-c_4t^2-2c_1t+c_3).$$

By choosing $\widehat{\xi}$ to be ξ/τ, where ξ and τ come from (7.19i) and (7.19ii) respectively, and using this algebraic expression, we find $\widehat{\xi}$ in terms of x,t,u,v only. Similarly we may find $\widehat{\phi}_1 = \phi_1/\tau$ and $\widehat{\phi}_2 = \phi_2/\tau$ from (7.19), again in terms of x,t,u,v only. We can now use the transformation (7.41), which we used to simplify the solution of the nonclassical determining equations in Case 7.3.2, in order to directly compare (7.19) and (7.37), since our new infinitesimals $\widehat{\xi}$, $\widehat{\phi}_1$ and $\widehat{\phi}_2$ are now independent of w. Recall that in §7.3 we showed that the new infinitesimals found by this transformation led to the same reductions as those that the infinitesimals they were generated from gave. With $\widehat{\phi}_2$ as defined above and ϕ_2 from (7.37ii), (7.41iii) gives

$$\frac{c_5-(c_4t+c_1)v-2bc_2t}{-c_4t^2-2c_1t+c_3} = bx + G(t,v) + F(x,t,u,v)u,$$

which defines $F(x,t,u,v)$. With this $F(x,t,u,v)$, (7.41i) now gives a cubic in x, whose coefficients must be zero. This yields the following expressions for G, H, K

$$G(t,v) = \left[(2bc_2t - c_5 + c_1v + c_4vt)A + (c_3 - 2c_1t - c_4t^2)A_v\right]/A(c_4t^2+2c_1t-c_3), \quad (7.61\text{i})$$
$$H(t,v) = \left[(4bc_6 - c_4v^2 - 4bc_2v - 6c_4t - 4c_1)A + (8bc_2t + 4c_1v - 4c_5\right.$$

$$+ 4c_4vt)A_v\big]/4A(c_4t^2 + 2c_1t - c_3), \quad (7.61\text{ii})$$

$$K(t,v) = \big[\,\big(c_4v^2 + 2c_4t + 4c_2bv - 4bc_6\big)\,A_v - (2c_4v + 4c_2b)\,A\big]/4Ab(c_4t^2 + 2c_1t - c_3). \quad (7.61\text{iii})$$

In order for these expressions to satisfy (7.38) we find that $A(t,v)$ satisfies

$$AA_{vvv} - A_vA_{vv} - AA_{tv} + A_tA_v = 0. \quad (7.62)$$

We may divide (7.62) by A^2 and integrate with respect to v, to find

$$A_{vv} - A_t = Af_1(t),$$

for $f_1(t)$ an arbitrary function. Whilst this is different from (7.20), the expressions in (7.61) are invariant under a scaling of A by an arbitrary function of t. Then if we choose $A = f_2(t)M(t,v)$ where $f_2(t)$ satisfies $\dfrac{\mathrm{d}f_2}{\mathrm{d}t} + f_1f_2 = 0$ then $M(t,v)$ satisfies $M_t - M_{vv} = 0$. Thus choosing $f_1 \neq 0$ will give no different expressions in (7.61) than choosing $f_1 = 0$, so we set $f_1 = 0$ without loss of generality, and our $A(t,v)$ from (7.61) satisfies (7.20). In fact this is slightly academic as requiring that the final part of the transformation (7.41ii) is satisfied with G, H, K given by (7.61) yields $A(t,v)$ must satisfy (7.20) directly.

Thus we have shown that there are no solutions of (7.62) under which (7.61) gives infinitesimals that cannot be recreated by (7.19) for some $A(t,v)$ that satisfies (7.20). That is, there are no G, H, K of the form (7.61) under which (7.37) gives more symmetries than (7.19). Conversely, there are no solutions of (7.20) which give infinitesimals (7.19) which (7.61) cannot recreate. Thus no classical symmetries of (7.4) give rise to solutions that the nonclassical symmetries of (7.3) cannot find.

We note, though, that G, H, K are not restricted by the nonclassical determining equations of (7.3) to being of the form (7.61). For instance they could be given by (7.50). By direct comparison of (7.50) with (7.61) these give equations that $A(t,v)$ must satisfy, which by using `diffgrob2` to find the compatibility conditions, reduces to a system of algebraic equations that the c_i of (7.61) and the a_i of (7.50) must satisfy. In order to show that (7.61) can recreate (7.50) we must be able to find c_i in terms of the a_i which solve the system. The general case of all the a_i being non-zero gets very messy, but by only requiring, for instance, that $a_1a_3a_7a_{10} \neq 0$ we find that no combination of the c_i can solve this system. These therefore represent symmetries of (7.3) that the classical symmetries of (7.4) cannot reproduce. Therefore, in this instance, the potential symmetries of (7.4) are a subset of the nonclassical potential symmetries of (7.3).

This way of comparing the infinitesimals also gives the explanation as to why very different infinitesimals give the same solutions. For the exact solution (7.27) we had $A = a$, constant and $c_1 = c_2 = c_4 = c_5 = c_6 = 0$ with $c_3 = 1$, which when substituted into (7.61) gives

$$G = H = K = 0,$$

as used to give (7.55). For (7.24) we had $A = a$, $c_1 = c_2 = c_4 = c_5 = 0$ with $c_3 = 1$ which gives

$$G = 0, \qquad H = -bc_6, \qquad K = 0,$$

again as used to give (7.56). These both confirm our observations of §7.3.

We have now seen that infinitesimals that appear to be different yet give the same solutions are in fact the same. Thus having found infinitesimals of (7.3) that are genuinely different from any infinitesimals of (7.4), it seems likely that they will lead to genuinely different solutions. If this observation is correct, in answer to the two self-posed questions above, firstly the solutions found by applying the classical method to (7.4) are a subset of those found by applying the nonclassical method to (7.3). Secondly, and more generally, there do exist nonclassical potential symmetries whose solutions cannot be found by considering the potential symmetries of higher order systems.

Incidentally the form of G, H and K given by (7.61) lends itself to finding more solutions of (7.38). Whilst some interesting solutions of (7.20) can lead to simpler, known, expressions in (7.61), some new expressions may be found. For instance if $c_1 = c_2 = c_3 = c_5 = c_6 = 0$, then

$$A(t, v) = b_1 t^{1/2} \exp\left\{-\frac{v^2}{4t}\right\},$$

leads to the infinitesimals (7.58). Also note that if $A = b_2 v$ with $c_1 = c_2 = c_4 = c_5 = c_6 = 0$, then the infinitesimals with $G = -1/v$, $H = K = 0$ are recreated. However if $A = b_i \exp\{-b_j^2 t\} \cos\{b_j v\}$, which satisfies (7.20) this gives a genuinely new solution of (7.38). Indeed since (7.20) is linear one could take $A(t, v)$ to be the infinite sum of such functions, and those of the form $b_i \exp\{-b_j^2 t\} \sin\{b_j v\}$ which also satisfy (7.20). Also adding a constant term to this sum (cf. choosing $A(t, v) = \exp\{-c_5(v - c_5 t)\} + a_1$) creates a slightly different look to (7.38) again. Whether these will be helpful in finding exact solutions, though, is an entirely different question.

Chapter Eight : Discussion and Open Problems

8.1 The relationship between the direct and nonclassical methods

In this section we consider the relationship between the direct and nonclassical methods for systems of equations, via a discussion on an extension of the proof of Olver [1994]. The two theorems that encapsulate his work are stated in §1.5.4, and will be reformed here as applied to systems. The working follows closely that of Olver's so that it might be obvious to the reader unfamiliar with it, how the theorems were proven in the scalar case. In fact Olver states *"we will concentrate on the simple case of a single second order partial differential equation in two independent and one dependent variables, but our results can be straightforwardly generalized to arbitrary higher order systems of partial differential equations."* However in the following we will show that the extension to systems of partial differential equations is not straightforward, and perhaps is not true.

Consider a system of two equations

$$\begin{aligned} \Delta_1(x,t,u,u^{(n_1)},v,v^{(n_2)}) &= 0, \\ \Delta_2(x,t,u,u^{(n_3)},v,v^{(n_4)}) &= 0. \end{aligned} \tag{8.1}$$

The direct method of Clarkson and Kruskal [1989] as applied to (8.1) calls for an ansatz of the form

$$\begin{aligned} u(x,t) &= F(x,t,U(z),V(z)) = F(x,t,U(\rho(x,t)),V(\rho(x,t))), \\ v(x,t) &= G(x,t,U(z),V(z)) = G(x,t,U(\rho(x,t)),V(\rho(x,t))), \end{aligned} \tag{8.2}$$

where $z=\rho(x,t)$, which when substituted into (8.1) must form a system of two ordinary differential equations. Let $\mathbf{w}=\tau(x,t)\partial_t+\xi(x,t)\partial_x$ be any vector field such that $\mathbf{w}(\rho)=0$. Applying $\mathbf{w}$ to (8.2) yields

$$\begin{aligned} \tau u_t+\xi u_x &= \tau F_t+\xi F_x = \widetilde{F}(x,t,U,V), \\ \tau v_t+\xi v_x &= \tau G_t+\xi G_x = \widetilde{G}(x,t,U,V). \end{aligned} \tag{8.3}$$

Assuming that the 2×2 Jacobian determinant of F, G with respect to u, v is non-zero and $F_U G_V \neq 0$, we solve (8.2) for U, V, using the implicit function theorem:

$$\begin{aligned} U &= \widehat{F}(x,t,u,v), \\ V &= \widehat{G}(x,t,u,v). \end{aligned} \tag{8.4}$$

(If the Jacobian determinant is zero, then $F_U G_V - F_V G_U = 0$ and by the method of characteristics we can see that F and G must both be functions of x, t and one other variable, i.e. $u = F^*(\mu(x,t,U,V),x,t)$, $v = G^*(\mu(x,t,U,V),x,t)$, or $\mu(x,t,U,V) = \check{F}(x,t,u) = \check{G}(x,t,v)$. Thus there exists an algebraic relation between u and v, which may be used to replace, say v, in (8.1). In other words (8.1) is really only a system in one variable. We impose the condition $F_U G_V \neq 0$ without loss of generality, so that both U and V are represented in the ansatz (8.2).) Substituting (8.4) into the right hand side of (8.3) we find that if u, v have the form (8.2) then they satisfy

$$\begin{aligned} \mathbf{w}(u) &= \tau u_t + \xi u_x = \phi_1(x,t,u,v), \\ \mathbf{w}(v) &= \tau v_t + \xi v_x = \phi_2(x,t,u,v). \end{aligned} \tag{8.5}$$

which are the invariant surface conditions. Conversely suppose u, v satisfy equations of the form (8.5). Assuming $\mathbf{w} \neq 0$ let $z = \rho(x,t)$ be the unique (local) invariant of $\mathbf{w}$ and define $y = \eta(x,t)$ so that $\mathbf{w}(\rho) = 0$, $\mathbf{w}(\eta) = 1$. Thus, the (y,z) coordinates serve to rectify the vector field, $\mathbf{w} = \partial_y$, and (8.5) reduces to the parameterised ordinary differential equations

$$\begin{aligned} u_y &= \psi_1(y,z,u,v), \\ v_y &= \psi_2(y,z,u,v). \end{aligned} \tag{8.6}$$

Fixing y_0, from the existence and uniqueness theorems for an initial value problem for a system of first order ordinary differential equations, the solution of (8.6) exists and is unique. With $u(y_0,z) = U(z)$, $v(y_0,z) = V(z)$, the general solution has the form $u = \check{F}(y,z,U,V)$, $v = \check{G}(y,z,U,V)$. Since z and y are simply functions of x, t, we have regained the direct method ansatz (8.2). We have therefore proved:

Theorem 8.1.1. *There is a one-to-one correspondence between ansätze of the direct method (8.2) with* $F_U G_V \neq 0$ *and non-zero Jacobian, and quasi-linear first order differential constraints (8.5)*

Solutions to (8.5) are just the functions which are invariant under the one-parameter group generated by the vector field

$$\mathbf{v} = \tau(x,t)\partial_t + \xi(x,t)\partial_x + \phi_1(x,t,u,v)\partial_u + \phi_2(x,t,u,v)\partial_v. \tag{8.7}$$

Recall that requiring that the side condition (8.5) is compatible with the original system (8.1), in the sense that the overdetermined system comprising of (8.1) and (8.5) has no compatibility conditions, is equivalent to applying the nonclassical method of Bluman and Cole [1969], with infinitesimal generator $\mathbf{v}$.

We now reformulate the second theorem of Olver [1994] as applied to our system (8.1) and make some natural extensions to the proof. However as we will observe, in no way may this be considered a proof of the theorem we state.

Theorem 8.1.2. *The ansatz (8.2) will reduce the system (8.1) to a pair of ordinary differential equations for $U(z)$, $V(z)$ if and only if the overdetermined system of partial differential equations defined by (8.1), (8.5) is compatible.*

"Proof." Let $y = \eta(x,t)$, $z = \rho(x,t)$, $a = \omega_1(x,t,u,v)$ and $b = \omega_2(x,t,u,v)$ be rectifying coordinates for the vector field $\mathbf{v} = \partial_y$. Then the side conditions take the form $a_y = 0$, $b_y = 0$ in these coordinates, leading to the simplified ansatz $a = a(z)$, $b = b(z)$ which is just (8.2) rewritten in the (y,z,a,b) coordinates. Rewrite the original system in these coordinates,

$$\widehat{\Delta}_1(y,z,a,a^{(k_1)},b,b^{(k_2)}) = 0, \tag{8.8i}$$

$$\widehat{\Delta}_2(y,z,a,a^{(k_3)},b,b^{(k_4)}) = 0. \tag{8.8ii}$$

Now if a,b satisfy the constraints $a_y = b_y = 0$, then the system reduces to

$$\check{\Delta}_1\left(y,z,a,a_z,a_{zz},\ldots,\frac{\partial^{k_1}a}{\partial z^{k_1}},b,b_z,b_{zz},\ldots,\frac{\partial^{k_2}b}{\partial z^{k_2}}\right) = 0, \tag{8.9i}$$

$$\check{\Delta}_2\left(y,z,a,a_z,a_{zz},\ldots,\frac{\partial^{k_3}a}{\partial z^{k_3}},b,b_z,b_{zz},\ldots,\frac{\partial^{k_4}b}{\partial z^{k_4}}\right) = 0, \tag{8.9ii}$$

which will be a system of ordinary differential equations for a,b as functions of z if and only if, apart from an overall factor in either equation, it is independent of y, perhaps after some interplay between the equations (see below). To remove this ambiguity and to help us later, we solve each of $\widehat{\Delta}_1$ and $\widehat{\Delta}_2$ in (8.8) for the highest derivative terms in both the orderings $z < y$, $a < b$ and $z < y$, $b < a$

$$\widehat{\Delta}_1 \equiv \begin{cases} \dfrac{\partial^{k_1}a}{\partial z^{k_1}} = \widetilde{\Delta}_1(y,z,a,\widetilde{a}^{(k_1)},b,b^{(k_2)}), \\ \dfrac{\partial^{k_2}b}{\partial z^{k_2}} = \widetilde{\Delta}_2(y,z,a,a^{(k_1)},b,\widetilde{b}^{(k_2)}), \end{cases} \tag{8.10i}$$

$$\widehat{\Delta}_2 \equiv \begin{cases} \dfrac{\partial^{k_3}a}{\partial z^{k_3}} = \widetilde{\Delta}_3(y,z,a,\widetilde{a}^{(k_3)},b,b^{(k_4)}), \\ \dfrac{\partial^{k_4}b}{\partial z^{k_4}} = \widetilde{\Delta}_4(y,z,a,a^{(k_3)},b,\widetilde{b}^{(k_4)}). \end{cases} \tag{8.10ii}$$

where the $a^{(k_i)}$ and $b^{(k_j)}$ with tildes do not include the derivative term that has been solved for. Now if a,b satisfy $a_y = b_y = 0$, (8.10) becomes

$$\check{\Delta}_1 \equiv \begin{cases} \dfrac{\partial^{k_1}a}{\partial z^{k_1}} = \widetilde{\Delta}_1\left(y,z,a,a_z,a_{zz},\ldots,\dfrac{\partial^{k_1-1}a}{\partial z^{k_1-1}},b,b_z,b_{zz},\ldots,\dfrac{\partial^{k_2}b}{\partial z^{k_2}}\right), \\ \dfrac{\partial^{k_2}b}{\partial z^{k_2}} = \widetilde{\Delta}_2\left(y,z,a,a_z,a_{zz},\ldots,\dfrac{\partial^{k_1}a}{\partial z^{k_1}},b,b_z,b_{zz},\ldots,\dfrac{\partial^{k_2-1}b}{\partial z^{k_2-1}}\right), \end{cases} \tag{8.11i,ii}$$

$$\check{\Delta}_2 \equiv \begin{cases} \dfrac{\partial^{k_3} a}{\partial z^{k_3}} = \widetilde{\Delta}_3 \left(y, z, a, a_z, a_{zz}, \ldots, \dfrac{\partial^{k_3 - 1} a}{\partial z^{k_3 - 1}}, b, b_z, b_{zz}, \ldots, \dfrac{\partial^{k_4} b}{\partial z^{k_4}} \right), \\ \dfrac{\partial^{k_4} b}{\partial z^{k_4}} = \widetilde{\Delta}_4 \left(y, z, a, a_z, a_{zz}, \ldots, \dfrac{\partial^{k_3} a}{\partial z^{k_3}}, b, b_z, b_{zz}, \ldots, \dfrac{\partial^{k_4 - 1} b}{\partial z^{k_4 - 1}} \right). \end{cases} \tag{8.11iii,iv}$$

The reduced system, which contains one equation from each of $\check{\Delta}_1$ and $\check{\Delta}_2$ in (8.11), is equivalent to a system of two ordinary differential equations if and only if it is independent of y. As we have seen in Chapters Five and Six, from a practical point of view this may be achieved by substituting between the equations or by integrating one and then substituting. It is not unfeasible that some differentiation may be needed also.

On the other hand, the compatibility conditions between the side conditions and the equations (8.10) are found by cross-differentiation (taking the diffSpoly). For instance between $\check{\Delta}_1$ and a_y we use (8.11i) to give

$$0 = \frac{\partial^{k_1} a_y}{\partial z^{k_1}} = \mathrm{D}_y \widetilde{\Delta}_1 = \widetilde{\Delta}_{1,y} + \sum_{r \in \{a,b\}} \left(\frac{\partial r}{\partial y} \frac{\partial \widetilde{\Delta}_1}{\partial r} \right.$$
$$\left. + \sum_{j_1 \in \{y,z\}} \frac{\partial r_{j_1}}{\partial y} \frac{\partial \widetilde{\Delta}_1}{\partial r_{j_1}} + \ldots + \sum_{j_1, j_2, \ldots, j_{k_1 - 1} \in \{y,z\}} \frac{\partial r_{j_1 j_2 \cdots j_{k_1 - 1}}}{\partial y} \frac{\partial \widetilde{\Delta}_1}{\partial r_{j_1 j_2 \cdots j_{k_1 - 1}}} \right).$$

By pseudo-reducing with respect to a_y and b_y we see that for normal^p(diffSpoly$(a_y, \check{\Delta}_1)$, $\{a_y, b_y, \check{\Delta}_1\}) = 0$ we require $\widetilde{\Delta}_{1,y} = 0$. Similarly for the other combinations of side condition and $\check{\Delta}_i$ we find also $\widetilde{\Delta}_{2,y} = \widetilde{\Delta}_{3,y} = \widetilde{\Delta}_{4,y} = 0$. However rather than necessarily being identically zero, either $\widetilde{\Delta}_{3,y} \equiv \widetilde{\Delta}_{4,y} \equiv 0$ and the equations $\widetilde{\Delta}_{i,y} = 0$ for $i = 1, 2$ need only be able to be pseudo-reduced to being identically zero by $\check{\Delta}_2$ or vice versa.

The crux of the proof lies with the equivalence of the two conditions, namely that the reduced system is independent of y.

However the main problem is how we get the reduced system to be independent of y, i.e. what operations are allowed? Our study in Chapters Five and Six, particularly Cases 5.3.2 and 5.4.2, give us some indication of what may or may not be allowed.

Firstly in the direct method we had to allow integration of a reduced equation in order to recover a reduction. Conversely in the nonclassical method, the natural infinitesimals were not allowed, yet these were found by going backwards from the direct method reductions, and integration would have been necessary had they been allowed. Instead the nonclassical method circumvents this problem by introducing unnatural infinitesimals, from which we find the reductions using a hodograph transformation. Despite the different approach, integration is still necessary.

In our studies no differentiation of a reduced equation has been necessary, but will this always be so? If differentiation is necessary, it is not too difficult to adapt the direct method to cope with this – the original equation to be differentiated has to give a satisfactory reduced equation (an ordinary differential equation or partial differential equation depending on the original number of independent variables), which is then differentiated and substituted into other equations. Knowing when or if to differentiate becomes a major problem as one cannot really tell *a priori* or *a posteriori* if it is necessary, though it has never been necessary before! It is difficult to say what will happen in the nonclassical method, as similarly it has never been necessary here either. Whether it will differentiate when necessary is discussed below.

We know that both the direct and nonclassical methods allow substitution (without integration or differentiation) and it has occurred a number of times in our examples.

Thus before the above working may be considered a proof of Theorem 8.1.2, two open questions must be answered.

1. Will the nonclassical method cope with differentiation of a reduced equation if it is necessary, or simply will this never be necessary?
2. Will the nonclassical method always be able to induce integration, perhaps by allowing unnatural infinitesimals?

In order to provide some answers to these questions consider the following two examples

Example 8.1.1. Consider the system

$$v_t = u_{xxx} + uu_x - t(u_{xx} + v_{xx}), \tag{8.12a}$$

$$v_x = -u_t, \tag{8.12b}$$

which could be thought of as a perturbed Boussinesq system (cf. system (6.2)). We can apply the nonclassical method to this system to generate a set of nonclassical determining equations. Rather than solving this system completely, we simply note that they do admit the infinitesimals (with $\tau = 1$)

$$\xi = -1, \qquad \phi_1 = 0, \qquad \phi_2 = 0.$$

Solving the invariant surface conditions gives the new symmetry variables

$$u(x,t) = U(z), \qquad v(x,t) = V(z), \qquad z = x + t,$$

and substituting these into (8.12) yields

$$V' = U''' + UU' - t(V'' + U''), \tag{8.13i}$$

$$V' + U' = 0. \tag{8.13ii}$$

We can differentiate (8.13ii) once with respect to z and use the result to give the system of ordinary differential equations,

$$V' = U''' + UU', \tag{8.14i}$$

$$V' + U' = 0. \tag{8.14ii}$$

Clearly in this case differentiation was necessary, and the nonclassical method allowed it.

Example 8.1.2. Consider the system

$$u_{xx} + u_x^3 + v_t^2 - t(u - v) = 0, \tag{8.15a}$$

$$u_{tt} - v_{xx} = 0. \tag{8.15b}$$

If we apply the nonclassical method to this system we find there are no symmetries when $\tau = 1$, and when $\tau = 0$ we are able to find the exact solution

$$u(x,t) = c_2 x + c_3 t + c_4, \qquad v(x,t) = c_2 x + g(t),$$

where $g(t)$ satisfies

$$\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2 + tg = c_3 t^2 + c_4 t - c_2^3.$$

It is possible, however, to substitute

$$u(x,t) = U(z), \qquad v(x,t) = V(z), \qquad z = x + ct,$$

where $c^2 = 1$, into (8.15) to give

$$U'' + (U')^3 + (V')^2 - t(U - V) = 0, \tag{8.16i}$$

$$U'' - V'' = 0. \tag{8.16ii}$$

Whilst this is not a system of ordinary differential equations, (8.16ii) may be integrated twice with respect to z (setting the constants of integration to zero) to yield

$$U'' + (U')^3 + (V')^2 = 0,$$

$$U - V = 0.$$

Now this is a system of ordinary differential equations, and clearly the direct method which allows integration will obtain this reduction, whilst the nonclassical method has not. In a sense this gives a special exact solution rather than a full reduction as we have set the constants of integration to zero.

8.2 Concluding Remarks

In this final section we make some remarks concerning the work in this thesis, on possible further work and mention some open problems.

We initially focus on the relationship between the nonclassical and direct methods, and their application. However first we must clarify what we mean by the direct method. Whilst the nonclassical method algorithm is clear, and the differentiation, substitution and integration that we have seen is either invoked by it or not, this is not so for the direct method. In a sense we are playing catch-up in the direct method: as we discover such new phenomena, we amend the direct method so as to include them. Indeed to paraphrase what one of the creators of the direct method said recently (Kruskal [1996]), *" the direct method is more a philosophy of approach than any particular ansatz"*. This flexibility has brought us reductions that the nonclassical method cannot find, in certain cases.

It may be possible to define the direct method differently and prove the nature of the relationship between this method and the nonclassical method, but even this is perhaps flawed. For instance, an obvious redefinition may be to disallow integration in the direct method, so as to eliminate examples such as Example 8.2.2. However we have seen how integration was necessary in Chapter Five, to recover a reduction that the nonclassical method had found, albeit with "unnatural infinitesimals". Perhaps if we disallow both integration in the direct method and unnatural infinitesimals in the nonclassical method, we may be able to form a relationship, but both methods are now weaker versions of their former selves and the relationship between the full methods remains hazy.

From a computational point of view, this new flexibility in the direct method causes trouble. In Chapter Five we concluded that because of the need to also consider substitution and integration on top of the usual calculation in the direct method, the nonclassical method was better suited to systems of partial differential equations. Now that we have also discovered that differentiation may be necessary, which again has to be checked separately in the direct method whilst appears to be automatically included in the nonclassical method, this conclusion is confirmed.

In defence of the direct method, most of the things stacked against it are rare. Indeed the example of Ludlow [1995] of when the direct method fails to find a classical reduction, and the investigations of this thesis appear to be unique in the literature. Coupled with this many authors have successfully used the direct method on systems to find many new symmetry reductions (e.g. Clarkson [1992], Clarkson and Hood [1993,1994], Hood [1993], Lou [1992] and Lou and Ruan [1993]).

A test to discover *a priori* whether the nonclassical method will find more reductions than the classical method would be very useful. The calculations for the nonclassical method are much harder than for the classical method as the determining equations are nonlinear as opposed to linear, so any test could save a considerable amount of time.

Some comments regarding a similar test, to discover whether observations on the symmetries of a potential system might be related to its scalar (or non-potential) counterpart are now made. These are based on work in Chapters Five, Six and Seven.

(i) If a potential system gives less classical symmetries than its non-potential counterpart, this does not necessarily imply that it will have less nonclassical symmetries (cf. system (6.4)).

(ii) If a potential system gives the same classical symmetries as its non-potential counterpart, this does not necessarily imply that it will have the same nonclassical symmetries, it may have less (cf. system (5.6)).

(iii) If the classical and nonclassical methods give the same symmetries for a non-potential system this does not necessarily imply that they will give the same symmetries for the potential system (cf. system (7.3)).

(iv) If a potential system gives potential symmetries this does not imply that it will also give nonclassical potential symmetries (simply because the nonclassical method does not always find more symmetries than the classical method).

Unfortunately these are all negative responses, so we are unable to reach satisfactory conclusions. A much stronger observation, which is certainly true for the systems we have dealt with is

(v) If a system doesn't admit potential symmetries, then neither will it admit nonclassical potential symmetries.

This is closer to what we require (if true) since if we find no potential symmetries, we need only apply the nonclassical method to the non-potential system as the potential system will give the same symmetries or less. Since in our calculations applying the nonclassical method to the potential system is at least as hard as applying it to the non-potential system and often much harder, not only do we not have to apply the nonclassical method twice, we can apply it, in general, with more ease. Whether statement (v) is true remains an open problem.

The MAPLE package `diffgrob2` by Mansfield [1993] has proved very successful in solving the overdetermined systems in this thesis, and in many other instances (cf. Clarkson and Mansfield [1994a,b,c, 1995]). For the (linear) classical determining equations the procedures implemented in `diffgrob2` may be run without user interaction, up to understanding the output and the possible need for further runs arising from this. For the nonlinear determining equations of the nonclassical method the calculations often have to be carried out interactively. It would be interesting to see if the algorithms implemented in `diffgrob2` could be improved or simply if more algorithms are implemented, whether `diffgrob2` was able to cope with nonlinear systems, without user interaction, more of the time. (For a discussion on what problems need to be addressed in order to improve the algorithms see Mansfield [1993].)

Another task that `diffgrob2` could be adapted to perform is that of integration of its output, i.e. to solve the system completely. `diffgrob2` would be particularly suited to this as it will already have picked out the special cases which heuristic integration procedures often miss. Whilst at the moment its output is often trivial to solve, particularly from classical determining equations, it can also be lengthy. Thus `diffgrob2` could be used to save still more time.

The interest in symmetry methods to find exact solutions to nonlinear partial differential equations has increased recently, primarily with the introduction of the direct method by Clarkson and Kruskal [1989]. New methods are still appearing and the volume of publications is increasing each year. (For instance compare the number of citations of Bluman and Cole [1969] year by year on BIDS (Bath Information and Data Services).) However there is also much new interest in applying these methods to difference equations and differential-difference equations. The lack of an obvious translation of Lie's work to this subject has caused lively debate – see for instance Levi and Winternitz [1991, 1993a,b] and Quispel, Capel and Sahadevan [1992, 1993].

The evident applicability and adaptability of these methods to a wide range of problems (cf. §§1.1, 1.2) will no doubt ensure that they will continue to expand, permute and gain popularity in the future.

Appendix A

In this appendix we list the determining equations that are generated in Subcase 4.3.2(i) i.e. in the generic case when $\xi^2 + u \neq 0$.

$$\xi_u = 0 \tag{A.1i}$$

$$\alpha\xi^2\phi_u - 2\alpha\xi\xi_t + 4\phi_{uu}u^2 - \alpha\phi + 4\xi^4\phi_{uu} + \alpha\phi_u u + 8\xi^2\phi_{uu}u - 2\alpha\xi^2\xi_x = 0 \tag{A.1ii}$$

$$\beta\xi^2\phi_u + 3\phi_{uu}u^2 + 3\xi^4\phi_{uu} + 6\xi^2\phi_{uu}u - \beta\phi + \beta\phi_u u - 2\beta\xi^2\xi_x - 2\beta\xi\xi_t = 0 \tag{A.1iii}$$

$$12\xi^2\phi_{uuu}u + 3\alpha\phi_{uu}u + 6\phi_{uuu}u^2 + 2\beta\phi_{uu}u + 2\beta\xi^2\phi_{uu} + 3\alpha\xi^2\phi_{uu} + 6\xi^4\phi_{uuu} = 0 \tag{A.1iv}$$

$$\begin{aligned} &7\xi^2\xi_{xx}u - 10\xi\xi_x^2u - 2\xi^2\xi_t\phi_u \\ &\quad - 4\xi\xi_{xt}u - \alpha\xi^2\phi_x - 2\xi\phi\phi_u - 4\xi_t\xi_x u - \alpha\phi_x u + 2\xi_t\phi_u u + 2\xi^3\phi\phi_{uu} \\ &\quad + 5\xi\xi_x\phi + 2\xi\phi_{tu}u - 6\xi^2\phi_{xu}u + 4\xi^2\xi_t\xi_x - 2\xi\xi_t^2 + 6\xi_{xx}u^2 + 2\xi^3\phi_{tu} + \xi_{tt}u \\ &\quad + \xi^4\xi_{xx} + \xi^2\xi_{tt} + 4\xi\xi_x\phi_u u - 4\phi_{xu}u^2 + 2\xi\phi\phi_{uu}u - 2\xi^4\phi_{xu} - 4\xi^3\xi_{xt} - \xi_t\phi = 0 \end{aligned} \tag{A.1v}$$

$$\alpha\xi^2\phi_{uuu} + \alpha\phi_{uuu}u + 2\xi^2\phi_{uuuu}u + \xi^4\phi_{uuuu} + \phi_{uuuu}u^2 = 0 \tag{A.1vi}$$

$$\begin{aligned} &3\alpha\xi_{xx}u - 4\beta\xi^2\phi_{xu} - 12\phi_{xuu}u^2 \\ &\quad - 3\alpha\phi_{xu}u + 6\xi\phi_{tuu}u + 3\alpha\xi^2\xi_{xx} - 18\xi^2\phi_{xuu}u + 9\xi_t\phi_{uu}u - 4\beta\phi_{xu}u - 6\xi\phi\phi_{uu} \\ &\quad + 2\beta\xi_{xx}u + 6\xi^3\phi\phi_{uuu} - 15\xi^3\xi_x\phi_{uu} - 3\xi^2\xi_t\phi_{uu} - 3\alpha\xi^2\phi_{xu} + 12\xi^3\phi_u\phi_{uu} \\ &\quad - 3\xi\xi_x\phi_{uu}u + 6\xi\phi\phi_{uuu}u + 12\xi\phi_u\phi_{uu}u + 2\beta\xi^2\xi_{xx} - 6\xi^4\phi_{xuu} + 6\xi^3\phi_{tuu} = 0 \end{aligned} \tag{A.1vii}$$

$$\begin{aligned} &2\xi^3\phi_{xu}\phi_{xx} - \xi^3\xi_{xx}\phi_{xx} + 2\xi\xi_t\phi\phi_{xxu} - 2\xi\xi_t\phi_t \\ &\quad + \xi_{xx}\phi_{xt}u + \xi^3\phi_{xxu}\phi_x + 2\xi_{xxt}\phi_x u + \phi^2\phi_{uu}u - 2\xi_t\phi_x u + 2\xi^2\phi\phi_{tu} - \xi_{xx}\phi\phi_x - \phi_t\phi_{xxu}u \\ &\quad - 2\xi_x\phi_{xxt}u - 2\xi\xi_t\xi_{xx}\phi_x + 2\phi\phi_{tu}u - \kappa\phi_{xx}u - 2\phi_{xt}\phi_{xu}u + 4\xi_{xt}\phi_{xx}u \\ &\quad - 2\phi\phi_{xxtu}u + 4\xi\xi_t\phi_{xu}\phi_x + 2\xi\xi_t\phi_{xxt} - 2\xi^2\phi_{xt}\phi_{xu} + 4\xi_x\phi_t u - 2\xi^2\phi\phi_{xxtu} \\ &\quad - 4\xi^2\phi_{xtu}\phi_x + 4\xi^2\xi_{xt}\phi_{xx} - \xi^2\phi^2\phi_{xxuu} + \phi\phi_u\phi_{xx} - \xi^2\kappa\phi_{xx} + \xi^2\xi_{xx}\phi_{xt} - \xi^2\phi_t\phi_{xxu} \\ &\quad + \xi\phi\phi_x - 2\phi\phi_{xu}^2u - 2\xi^2\phi_{uu}\phi_x^2 - 2\xi_x\phi\phi_{xx} + 2\xi_t\phi_{xxx}u + 2\xi^2\xi_x\phi_{xu}\phi_x + \xi^2\phi^2\phi_{uu} \\ &\quad - \phi^2\phi_{xxuu}u + 2\phi\phi_{xu}\phi_x - \xi^2\phi_{xxxx}u + 2\xi^2\xi_{xxt}\phi_x - 2\xi^3\xi_x\phi_x - 2\xi^2\phi\phi_{xu}^2 - 2\phi_{tu}\phi_{xx}u \\ &\quad + 4\xi_x^2\phi_{xx}u - 2\gamma\phi_{xx}u^2 + 2\xi^2\xi_x\phi_t - 2\xi^2\phi_{tu}\phi_{xx} - \phi\phi_u\phi_{xxu}u - \xi\phi\phi_{xxx} - 2\xi\xi_t\phi\phi_u \\ &\quad - 2\phi_{uu}\phi_x^2u - 4\phi_{xtu}\phi_x u - 2\phi\phi_{uu}\phi_{xx}u - 4\phi\phi_{xuu}\phi_x u - \xi^2\phi_{xxtt} + \xi^2\phi_{tt} - \phi_{xxxx}u^2 \\ &\quad + 2\xi\xi_x\phi_{xxx}u + \xi_{xx}\phi\phi_{xu}u + \xi^2\xi_{xx}\phi_u\phi_x + \xi_x\xi_{xx}\phi_x u - \xi\xi_{xx}\phi_{xx}u + \xi\phi_{xxu}\phi_x u \\ &\quad + \xi_{xx}\phi_u\phi_x u - 2\xi_x\phi\phi_{xxu}u - 2\xi^2\phi\phi_{uu}\phi_{xx} + 4\xi_x\phi\phi_u u + 2\xi\phi_{xu}\phi_{xx}u \\ &\quad - 2\xi_x\phi_u\phi_{xx}u - 2\xi^2\gamma\phi_{xx}u - 2\xi_x\phi_{xu}\phi_x u - \xi^2\xi_x\xi_{xx}\phi_x + 2\xi^2\xi_x\phi\phi_u + 2\xi\xi_t\phi_u\phi_{xx} \\ &\quad - 4\xi\xi_t\xi_x\phi_{xx} - 4\xi^2\phi\phi_{xuu}\phi_x - \phi_{xxtt}u + \phi_{tt}u - \phi^2\phi_u + \phi\phi_{xxt} - \phi\phi_t + \phi^2\phi_{xxu} \\ &\quad - 2\xi^2\phi_u\phi_{xu}\phi_x - \xi^2\phi\phi_u\phi_{xxu} - 4\xi\xi_x\phi_x u + \xi^2\xi_{xx}\phi\phi_{xu} - 2\phi_u\phi_{xu}\phi_x u = 0 \end{aligned} \tag{A.1viii}$$

$$\begin{aligned} &2\xi^3\xi_{xx}\phi_u - 4\xi^2\xi_{xxx}u - 8\xi\xi_t\xi_x^2 - 3\xi^3\xi_x\xi_{xx} + 4\xi\xi_t\xi_{xt} + \phi^2\phi_{uuu}u - \kappa\phi \\ &\quad + 4\xi_{xt}\xi_x u + 2\phi_{tu}\phi_u u - 2\xi\xi_t u - 4\xi\phi_{xtu}u - 4\xi^3\phi\phi_{xuu} + 2\beta\xi^2\phi_{xx} - 4\xi^2\xi_x u + 4\xi_x\gamma u^2 \end{aligned}$$

$$+ 2\xi^2\gamma\phi - 4\xi_{xt}\phi_u u + 2\phi\phi_{tuu}u - 4\xi^2\xi_x\phi_{tu} - 4\xi^2\xi_{xt}\phi_u - 5\xi^3\phi_{uu}\phi_x - 8\xi_x^2\phi_u u$$
$$+ 2\xi_x\phi_u^2 u - 2\xi\xi_t\kappa - 4\xi^3\phi_u\phi_{xu} + 7\xi_t\xi_{xx}u + \xi^2\phi_t\phi_{uu} - 4\xi\xi_{xx}\phi + \phi_t\phi_{uu}u + \xi^2\phi^2\phi_{uuu}$$
$$+ 2\xi\xi_{xxt}u - \xi^2\xi_t\xi_{xx} + 2\xi^2\xi_t\phi_{xu} - 8\xi_t\phi_{xu}u + 6\xi^3\xi_x\phi_{xu} - 2\xi_x\phi_{tu}u - 2\xi\xi_t\phi\phi_{uu}$$
$$+ 2\xi^2\phi_{tu}\phi_u - 2\xi\xi_t\phi_u^2 + 7\xi^2\phi_{xxu}u + 8\xi\xi_t\xi_x\phi_u - 4\xi\phi_u\phi_{xu}u - 2\xi_x\phi\phi_{uu}u - 4\xi\xi_x\phi_{xu}u$$
$$+ 2\beta\phi_{xx}u + 2\xi^2\phi\phi_{tuu} + 2\xi\xi_{xx}\phi_u u + \xi^2\phi_{ttu} + 8\xi_x^3 u - 4\xi^3\phi_{xtu} + 2\xi^3\xi_{xxt} + 5\xi\phi\phi_{xu}$$
$$+ 2\xi_{xt}\phi - 4\xi\xi_t\gamma u + \xi^4\phi_{xxu} + 5\xi\xi_x\xi_{xx}u - 5\xi\phi_{uu}\phi_x u - 4\xi\phi\phi_{xuu}u - 2\xi\xi_t\phi_{tu} + \phi_{ttu}u$$
$$- \phi^2\phi_{uu} + 3\xi^2\phi\phi_u\phi_{uu} + 3\phi\phi_u\phi_{uu}u - \phi\phi_u^2 - \phi\phi_{tu} + 2\xi_x\kappa u - 4\xi^2\xi_x\phi\phi_{uu} + 4\xi_x\phi\phi_u$$
$$+ 6\phi_{xxu}u^2 - 4\xi_{xxx}u^2 - 2\xi^2\xi_{xtt} - 2\xi^4\xi_x - 2\xi_{xtt}u - 4\xi_x^2\phi + \xi^2\phi + 8\xi^2\xi_{xt}\xi_x = 0 \quad \text{(A.1ix)}$$

$$2\xi\phi_{tuuu}u - 5\xi^3\xi_x\phi_{uuu} - 3\alpha\phi_{xuu}u + 2\xi^3\phi\phi_{uuuu} + 6\xi\phi_{uu}^2 u + 3\xi_t\phi_{uuu}u$$
$$+ 6\xi^3\phi_u\phi_{uuu} + 6\xi^3\phi_{uu}^2 + 2\xi^3\phi_{tuuu} - 2\xi\phi\phi_{uuu} - \xi^2\xi_t\phi_{uuu} - 3\alpha\xi^2\phi_{xuu} + 6\xi\phi_u\phi_{uuu}u$$
$$+ 2\xi\phi\phi_{uuuu}u - 6\xi^2\phi_{xuuu}u - \xi\xi_x\phi_{uuu}u - 4\phi_{xuuu}u^2 - 2\xi^4\phi_{xuuu} = 0 \quad \text{(A.1x)}$$

$$4\xi^2\phi_{tu}\phi_{uu} + \xi^2\phi_t\phi_{uuu} - \alpha\xi^2\xi_{xxx} + 4\xi^2\phi_{tuu}\phi_u - 6\xi_{xt}\phi_{uu}u - 2\gamma\phi + 4\phi_{tu}\phi_{uu}u$$
$$+ 2\gamma\phi_u u + \phi^2\phi_{uuuu}u + 2\phi\phi_{tuuu}u - \alpha\xi_{xxx}u + 5\xi\phi\phi_{xuu} + 2\xi^2\xi_t\phi_{xuu} + 4\xi_x\phi\phi_{uu}$$
$$- 4\xi^2\xi_x\phi_{tuu} + 2\xi^2\gamma\phi_u + 4\xi^2\xi_x^2\phi_{uu} - 5\xi^3\phi_{uuu}\phi_x + 2\gamma\phi_{uu}u^2 + \xi^2\kappa\phi_{uu} + 6\xi^3\xi_x\phi_{xuu}$$
$$- 8\xi^3\phi_u\phi_{xuu} - 4\xi^3\phi\phi_{xuuu} - 2\xi\xi_t\phi_{tuu} + 4\phi_{tuu}\phi_u u + \phi_t\phi_{uuu}u - 14\xi^3\phi_{xu}\phi_{uu}$$
$$+ 2\xi^2\phi\phi_{tuuu} - 8\xi_t\phi_{xuu}u - 6\xi^2\xi_{xt}\phi_{uu} + 6\xi^3\xi_{xx}\phi_{uu} + 3\alpha\phi_{xxu}u - 2\xi_x\phi_{tuu}u$$
$$- \xi^2\phi_{uu}u + 4\phi\phi_{uu}^2 u + \kappa\phi_{uu}u + 4\xi_x\gamma u + 4\phi_u^2\phi_{uu}u - 8\xi\phi_u\phi_{xuu}u - 4\xi\phi_{xtuu}u$$
$$+ \xi^2\phi^2\phi_{uuuu} + 7\xi^2\phi_{xxuu}u - 14\xi\phi_{xu}\phi_{uu}u - 2\xi_x\phi_u\phi_{uu}u + 2\xi^2\gamma\phi_{uu}u + 6\xi\xi_{xx}\phi_{uu}u$$
$$+ 4\xi^2\phi_u^2\phi_{uu} - 4\xi\xi_t\gamma - \phi^2\phi_{uuu} - \phi\phi_{tuu} + 6\phi_{xxuu}u^2 - 4\xi^3\phi_{xtuu} - 2\xi\xi_t\phi\phi_{uuu}$$
$$- 4\xi\phi\phi_{xuuu}u - 4\xi_x^2\phi_{uu}u - 5\xi\phi_{uuu}\phi_x u - \xi^4\phi_{uu} - 8\xi^2\xi_x\phi_u\phi_{uu} + 5\phi\phi_u\phi_{uuu}u$$
$$+ 5\xi^2\phi\phi_u\phi_{uuu} + 8\xi\xi_t\xi_x\phi_{uu} - 2\xi_x\phi\phi_{uuu}u + 3\alpha\xi^2\phi_{xxu} + \xi^4\phi_{xxuu} - 3\phi\phi_u\phi_{uu}$$
$$- 4\xi\xi_x\phi_{xuu}u - 4\xi^2\xi_x\phi\phi_{uuu} + \xi^2\phi_{ttuu} + \phi_{ttuu}u + 4\xi^2\phi\phi_{uu}^2 - 6\xi\xi_t\phi_u\phi_{uu} = 0 \quad \text{(A.1xi)}$$

$$2\phi_{xt}\phi_{uu}u + 4\phi_{tuu}\phi_x u + 2\xi_t\xi_x u + 4\xi\phi\phi_{xxu} + 8\xi\xi_x\phi_u u - 2\xi_{xx}\gamma u^2 + 4\phi\phi_{xtuu}u - 6\xi\phi_{xu}^2 u$$
$$+ 6\phi_{tu}\phi_{xu}u - 6\xi^3\phi_{xuu}\phi_x + 2\phi_t\phi_{xuu}u + \xi^2\xi_t\phi_{xxu} + 4\gamma\phi_x u + 4\phi_{xxxu}u^2 + 2\phi_{xttu}u$$
$$+ 6\xi^2\phi\phi_u\phi_{xuu} - 4\xi^2\xi_x\phi\phi_{xuu} + 4\xi^2\phi_{xtu}\phi_u + 4\xi^2\phi\phi_{xtuu} - 2\phi^2\phi_{xuu} + 2\xi_{xx}\phi\phi_u$$
$$+ 2\xi^2\phi_u^2\phi_{xu} - 4\xi^3\phi_{uu}\phi_{xx} + 2\phi_u^2\phi_{xu}u + 5\xi_x^2\xi_{xx}u - 2\xi^2\xi_t\phi_u + 2\xi\phi_{tu}u + \xi_{tt}u + 2\xi^3\phi\phi_{uu}$$
$$- 2\xi^3\phi_u\phi_{xxu} - 4\phi\phi_u\phi_{xu} - 10\xi^2\xi_{xt}\phi_{xu} - 4\xi\xi_t\phi_{xtu} + 4\xi^2\phi_{tuu}\phi_x + 2\xi^2\xi_t\xi_x + 4\gamma\phi_{xu}u^2$$
$$+ 5\xi^2\xi_{xt}\xi_{xx} + \alpha\phi_{xxx}u + \xi\xi_x\phi + 2\xi^2\phi^2\phi_{xuuu} - 4\xi\xi_t\phi\phi_{xuu} + 6\xi^2\phi_{tu}\phi_{xu} + 5\xi\xi_{xx}\phi_{xu}u$$
$$+ 2\kappa\phi_{xu}u - 10\xi_{xt}\phi_{xu}u + 4\xi^2\gamma\phi_x - \xi_{xx}\kappa u - 3\xi_x\xi_{xx}\phi + 2\xi^2\xi_{xxt}\xi_x + 2\xi^2\kappa\phi_{xu}$$
$$+ \xi^3\xi_x\phi_{xxu} + 2\xi^2\phi_{xt}\phi_{uu} + 2\xi^2\xi_x^2\phi_{xu} - \xi\xi_{xxx}\phi - 2\xi^2\xi_{xxt}\phi_u - 2\phi\phi_{uu}\phi_x - \xi^2\xi_{xx}\phi_u^2$$
$$+ 2\xi_t\phi_u u + 2\xi_t\xi_{xxx}u + 5\xi^3\xi_{xx}\phi_{xu} + 2\xi\xi_t\xi_{xxt} - \xi^2\xi_x^2\xi_{xx} + 4\phi\phi_{uuu}\phi_x u + 6\phi_u\phi_{uu}\phi_x u$$
$$- 2\phi\phi_{xtu} + 8\xi^2\phi\phi_{xu}\phi_{uu} - 2\xi_x\xi_{xx}\phi_u u - 3\xi^2\xi_{xx}\phi_{tu} - 2\xi\phi_{xxtu}u - \xi\xi_{xx}^2 u - \xi_{xxtt}u$$

$$
\begin{aligned}
&- 3\xi^2\xi_{xx}\phi\phi_{uu} + 2\xi^3\phi_{tu} - \xi^3\xi_{xx}^2 - 2\xi^3\phi\phi_{xxuu} - \xi^2\xi_{xxxx}u + 2\xi^2\phi_t\phi_{xuu} \\
&- \xi_{xx}\phi_u^2u + 6\xi_x\phi\phi_{xu} - 2\xi_{xxt}\phi_u u - 4\xi^2\xi_x\phi_{xtu} - 2\xi\phi\phi_u - \xi^2\xi_{xx}\kappa \\
&+ 5\xi_{xt}\xi_{xx}u + \alpha\xi^2\phi_{xxx} - 8\xi\xi_t\phi_u\phi_{xu} - 4\xi\xi_x^2u - 6\xi^3\phi_{xu}^2 + 6\phi\phi_u\phi_{xuu}u \\
&+ 8\phi\phi_{xu}\phi_{uu}u - 4\xi^2\xi_x\phi_u\phi_{xu} + \xi_{xxt}\phi - 2\xi^3\phi_{xxtu} - 10\xi_x^2\phi_{xu}u + 4\phi_{xtu}\phi_u u \\
&- 2\xi\xi_t^2 - 4\xi\xi_t\phi_{uu}\phi_x - 3\xi_{xx}\phi_{tu}u + 2\phi^2\phi_{xuuu}u + 12\xi\xi_t\xi_x\phi_{xu} - \xi_{xxxx}u^2 - 2\xi_x\phi_{uu}\phi_x u \\
&- 2\xi^3\xi_x^2 + 2\xi^2\phi_{xttu} + 4\xi^3\xi_x\phi_u - 6\xi\phi_{xuu}\phi_x u + 2\xi\phi\phi_{uu}u - 2\xi\phi_u\phi_{xxu}u - 2\xi\phi\phi_{xxuu}u \\
&+ 4\xi^2\gamma\phi_{xu}u + 2\xi^2\xi_x\xi_{xx}\phi_u - 7\xi_t\phi_{xxu}u - \xi^2\xi_{xxtt} + 4\xi^2\phi_{xxxu}u - 7\xi\xi_x\phi_{xxu}u \\
&+ 4\xi^2\phi\phi_{uuu}\phi_x - 6\xi\xi_t\xi_x\xi_{xx} + \xi^2\xi_{tt} - 4\xi\phi_{uu}\phi_{xx}u - 6\xi^2\xi_x\phi_{uu}\phi_x + 6\xi^2\phi_u\phi_{uu}\phi_x \\
&- 2\xi^2\xi_{xx}\gamma u - 3\xi_{xx}\phi\phi_{uu}u + 2\xi\xi_x\xi_{xxx}u - \xi_t\phi + 4\xi_x\phi_u\phi_{xu}u + 4\xi\xi_t\xi_{xx}\phi_u = 0 \qquad \text{(A.1xii)}
\end{aligned}
$$

Appendix B

In this appendix we list the determining equations generated by the nonclassical method for system (5.4) in Case 5.3.2.

$$\xi\xi_w - \xi_v = 0, \tag{B.1i}$$

$$\xi^2\xi_{ww} + \xi\xi_w^2 - \xi_v\xi_w + \xi_{vv} - 2\xi\xi_{vw} = 0, \tag{B.1ii}$$

$$\xi\phi_{3,w} - \phi_{3,v} - \xi_w\phi_3 + \xi^2\phi_{2,w} - \xi\phi_{2,v} - \xi\xi_w\phi_2 - \xi\xi_x - \xi_t = 0, \tag{B.1iii}$$

$$\phi_{3,x} + \phi_2\phi_{3,w} - \phi_{2,w}\phi_3 + \xi\phi_2\phi_{2,w} - \phi_2\phi_{2,v} - \phi_{2,t} - \xi_w\phi_2^2 - \xi_x\phi_2 = 0, \tag{B.1iv}$$

$$\begin{aligned}&5\xi^2\xi_v\xi_{ww} - \xi^4\xi_{www} - 6\xi^3\xi_w\xi_{ww} - 3\xi^2\xi_w^3 + 5\xi\xi_v\xi_w^2 - 3\xi\xi_{vv}\xi_w + 9\xi^2\xi_{vw}\xi_w \\ &\quad - 2\xi_v^2\xi_w + \xi\xi_{vvv} - 3\xi^2\xi_{vvw} + 2\xi_v\xi_{vv} + 3\xi^3\xi_{vww} - 7\xi\xi_v\xi_{vw} = 0,\end{aligned} \tag{B.1v}$$

$$\begin{aligned}&6\xi^3\phi_{2,vw} - 3\xi^4\phi_{2,ww} - 3\xi^2\phi_{2,vv} - 6\xi^3\xi_{xw} + 3\xi_v^2\phi_2 + 6\xi^2\xi_{xv} \\ &\quad - 3\xi\xi_{tv} + 3\xi^2\xi_{tw} - 6\xi^2\xi_w^2\phi_2 + 3\xi_v\xi_w\phi_3 - 6\xi^3\xi_w\phi_{2,w} - 3\xi\xi_w\phi_{3,v} \\ &\quad + 3\xi^2\xi_{ww}\phi_3 - 2\xi\xi_v\phi_{2,v} + 3\xi^2\xi_w\phi_{3,w} + 3\xi^2\xi_w\phi_{2,v} - 3\xi\xi_t\xi_w - 3\xi\xi_{vw}\phi_3 \\ &\quad + 12\xi^2\xi_{vw}\phi_2 - 3\xi\xi_{vv}\phi_2 - 3\xi\xi_w^2\phi_3 - 6\xi^2\xi_w\xi_x + 5\xi^2\xi_v\phi_{2,w} + \xi\xi_v\xi_x \\ &\quad - 9\xi^3\xi_{ww}\phi_2 + 3\xi_t\xi_v - 2\xi\xi_v\xi_w\phi_2 = 0,\end{aligned} \tag{B.1vi}$$

$$\begin{aligned}&\xi^2\phi_{2,w}\phi_{3,w} + \xi_w\phi_{2,v}\phi_3 + 2\xi\xi_{vw}\phi_2^2 + 2\xi\xi_w\phi_{3,x} + 2\xi\xi_{tw}\phi_2 - 2\xi_w^2\phi_2\phi_3 \\ &\quad + 3\xi\xi_v\phi_{2,x} - 4\xi^2\xi_w\phi_{2,x} - 3\xi^2\xi_{ww}\phi_2^2 - 2\xi_v\xi_x\phi_2 + 2\xi\xi_{xw}\phi_3 - 2\xi^3\phi_{2,xw} \\ &\quad + \xi^2\phi_{2,tw} - 2\xi_t\xi_x - \xi\phi_2\phi_{2,vv} + \xi^2\phi_{2,ww}\phi_3 - 2\xi^2\xi_x\phi_{2,w} - \xi\phi_{2,vw}\phi_3 \\ &\quad - 4\xi^2\xi_{xw}\phi_2 - 2\xi_t\xi_w\phi_2 + 2\xi\xi_{xv}\phi_2 + \xi_v\phi_2\phi_{2,v} - 2\xi_w\xi_x\phi_3 - \xi\xi_t\phi_{2,w} \\ &\quad + 4\xi^2\phi_2\phi_{2,vw} - 2\xi_v\xi_w\phi_2^2 - \xi\phi_{2,w}\phi_{3,v} - 3\xi^3\phi_2\phi_{2,ww} - \xi\xi_w\phi_{2,w}\phi_3 - \xi\phi_{2,tv} \\ &\quad - \xi^2\xi_{xx} + 2\xi\xi_{xt} + 2\xi^2\phi_{2,xv} + \xi_t\phi_{2,v} + 3\xi\xi_v\phi_2\phi_{2,w} + 2\xi\xi_{ww}\phi_2\phi_3 \\ &\quad - 6\xi^2\xi_w\phi_2\phi_{2,w} + 2\xi\xi_w\phi_2\phi_{3,w} - \xi\xi_w\phi_2\phi_{2,v} = 0,\end{aligned} \tag{B.1vii}$$

$$\begin{aligned}&9\xi^2\xi_{vw}\xi_x - 3\xi\xi_{vv}\xi_x - 6\xi^2\xi_w^2\xi_x - 6\xi^2\xi_w^3\phi_2 - 3\xi^4\xi_{www}\phi_2 + 4\xi^3\xi_{ww}\phi_{2,v} \\ &\quad - 5\xi^2\xi_v\phi_{2,vw} + 9\xi^3\xi_{vw}\phi_{2,w} + 2\xi\xi_{vw}\phi_{3,v} - 2\xi^2\xi_{ww}\phi_{3,v} - 2\xi^2\xi_{vw}\phi_{3,w} + \xi^2\xi_{tw}\xi_w \\ &\quad + \xi_t\xi_v\xi_w - \xi\xi_t\xi_w^2 - \xi^2\xi_t\xi_{ww} - \xi_v\xi_{vv}\phi_2 - \xi^2\xi_w\phi_{2,vv} - \xi\xi_v^2\phi_{2,w} \\ &\quad - \xi_{vv}\xi_w\phi_3 + 2\xi\xi_t\xi_{vw} + 2\xi^3\xi_{ww}\phi_{3,w} + \xi^3\xi_w\phi_{3,ww} + \xi\xi_{vvw}\phi_3 + \xi_v\xi_w^2\phi_3 \\ &\quad + 4\xi^2\xi_{xv}\xi_w - 5\xi^2\xi_{vvw}\phi_2 - 2\xi^2\xi_{vww}\phi_3 - 4\xi^4\xi_w\phi_{2,ww} - 6\xi^3\xi_{ww}\xi_x + \xi\xi_{vvv}\phi_2 \\ &\quad + 4\xi^3\xi_v\phi_{2,ww} - 2\xi^2\xi_w\phi_{3,vw} + \xi\xi_v\phi_{2,vv} + 3\xi\xi_{vv}\phi_{2,v} - 4\xi^2\xi_{vv}\phi_{2,w} + \xi^3\xi_{www}\phi_3 \\ &\quad + 4\xi^2\xi_w^2\phi_{2,v} - 6\xi^3\xi_w^2\phi_{2,w} - 5\xi^4\xi_{ww}\phi_{2,w} + 7\xi^3\xi_{vww}\phi_2 + \xi_v^2\xi_w\phi_2 + 5\xi^3\xi_w\phi_{2,vw} \\ &\quad - \xi\xi_w^3\phi_3 - 7\xi^2\xi_{vw}\phi_{2,v} + \xi^2\xi_w^2\phi_{3,w} + \xi\xi_w\phi_{3,vv} - 6\xi^3\xi_w\xi_{xw} - \xi\xi_{tw}\xi_v \\ &\quad - 2\xi\xi_v\xi_{xv} + 6\xi^2\xi_v\xi_w\phi_{2,w} - 2\xi^4\xi_{xww} - 2\xi^2\xi_{xvv} + 4\xi^3\xi_{xvw} + 4\xi^2\xi_v\xi_{xw} \\ &\quad + 6\xi^2\xi_v\xi_{ww}\phi_2 - 3\xi\xi_v\xi_w\phi_{2,v} - 15\xi^3\xi_w\xi_{ww}\phi_2 - 2\xi^2\xi_{tvw} + 5\xi\xi_v\xi_w\xi_x + 16\xi^2\xi_{vw}\xi_w\phi_2 \\ &\quad - 3\xi\xi_v\xi_{vw}\phi_2 - 3\xi\xi_{vv}\xi_w\phi_2 + 4\xi\xi_v\xi_w^2\phi_2 - \xi^5\phi_{2,www} + 2\xi\xi_{vw}\xi_w\phi_3 + \xi\xi_{tvv} + \xi^2\phi_{2,vvv}\end{aligned}$$

$$+\xi^3\xi_{tww}+3\xi^4\phi_{2,vww}-3\xi^3\phi_{2,vvw}-\xi\xi_v\xi_w\phi_{3,w}-\xi\xi_v\xi_{ww}\phi_3-\xi_t\xi_{vv}=0, \tag{B.1viii}$$

$$\begin{aligned}
&2\xi\phi_2\phi_{2,xtw}-\xi\xi_{xx}\phi_{2,x}+\xi\phi_{2,xxw}\phi_3-\xi\xi_w\phi_{2,x}^2-\xi_w\phi_{2,xx}\phi_3+2\xi\phi_{2,xw}\phi_{3,x}\\
&+\xi\phi_2\phi_{2,w}\phi_{2,ww}\phi_3+\xi\xi_w\phi_2\phi_{2,xx}-\xi_t\phi_2\phi_{2,w}^2-\xi^2\phi_2\phi_{2,xxw}-\xi_t\phi_2^2\phi_{2,ww}-\xi_v\phi_2^3\phi_{2,ww}\\
&-2\xi^2\phi_{2,xw}\phi_{2,x}+\xi\phi_{2,tw}\phi_{2,x}-\xi_t\phi_{2,w}\phi_{2,x}-2\xi_t\phi_2\phi_{2,xw}-\xi_v\phi_2\phi_{2,xx}+\xi_w\phi_2\phi_3\\
&-\xi\phi_{2,w}\phi_3-2\xi_v\phi_2^2\phi_{2,xw}+\xi\phi_2^2\phi_{2,tww}-\xi^2\phi_2^3\phi_{2,www}+\xi\phi_{2,w}\phi_{3,xx}+2\xi\phi_2^2\phi_{2,xvw}\\
&-2\xi\xi_x\phi_2-\xi_v\phi_2^2\phi_{2,w}^2-\xi_w\phi_2\phi_{2,w}^2\phi_3+2\xi\phi_{2,xv}\phi_{2,x}+\alpha\xi\phi_2^2+\xi\phi_2^3\phi_{2,vww}\\
&-3\xi\xi_w\phi_2^2-\xi\phi_2\phi_{2,w}+\xi\phi_2\phi_{2,xxv}+\xi\phi_2^2\phi_{2,w}\phi_{3,ww}+\xi\phi_{2,ww}\phi_{2,x}\phi_3-\xi_w\phi_{2,w}\phi_{2,x}\phi_3\\
&-\xi_t\phi_{2,xx}+2\xi\phi_2\phi_{2,ww}\phi_{3,x}+\xi\phi_2^2\phi_{2,www}\phi_3+2\xi\phi_2\phi_{2,w}\phi_{3,xw}+\xi_v\phi_2^2-\xi\phi_{2,t}\\
&+\xi\phi_{2,xxt}-\xi\phi_{2,x}+\xi\phi_2\phi_{2,w}^2\phi_{3,w}+\xi\phi_2\phi_{2,tw}\phi_{2,w}-2\xi_w\phi_2\phi_{2,xw}\phi_3+\beta\xi\phi_{2,x}w\\
&-2\xi^2\phi_2^2\phi_{2,xww}-3\xi^2\phi_2\phi_{2,ww}\phi_{2,x}-2\xi\xi_x\phi_{2,w}\phi_{2,x}+\beta\xi\phi_2\phi_{2,w}w-\xi_w\phi_2^2\phi_{2,ww}\phi_3\\
&-\xi\xi_{ww}\phi_2^2\phi_{2,x}-\xi_v\phi_2\phi_{2,w}\phi_{2,x}-\xi\xi_w\phi_2^3\phi_{2,ww}+3\alpha\xi\xi_w\phi_2^2v+2\alpha\xi\xi_x\phi_2v\\
&-2\xi\xi_x\phi_2\phi_{2,w}^2-3\xi\xi_w\phi_2^2\phi_{2,w}^2+2\xi\phi_2\phi_{2,xv}\phi_{2,w}+3\xi\phi_2^2\phi_{2,vw}\phi_{2,w}-2\xi\xi_{ww}\phi_2^3\phi_{2,w}\\
&-4\xi\xi_{xw}\phi_2^2\phi_{2,w}-2\xi\xi_{xx}\phi_2\phi_{2,w}+2\xi\phi_2^2\phi_{2,ww}\phi_{3,w}+\xi_t\phi_2+2\xi\phi_2\phi_{2,xw}\phi_{3,w}\\
&+2\xi\phi_2\phi_{2,xww}\phi_3-4\xi\xi_w\phi_2\phi_{2,w}\phi_{2,x}-\alpha\xi_v\phi_2^2v-\alpha\xi_t\phi_2v+\xi\phi_{2,w}\phi_{2,x}\phi_{3,w}\\
&-\alpha\xi_w\phi_2\phi_3v+\alpha\xi\phi_{2,t}v+3\xi\phi_2\phi_{2,vw}\phi_{2,x}-2\xi\xi_{xw}\phi_2\phi_{2,x}-3\xi^2\phi_2^2\phi_{2,w}\phi_{2,ww}\\
&-2\xi\xi_x\phi_2^2\phi_{2,ww}-2\xi^2\phi_2\phi_{2,w}\phi_{2,xw}-2\xi\xi_x\phi_2\phi_{2,xw}+\alpha\xi\phi_{2,w}\phi_3v=0,
\end{aligned} \tag{B.1ix}$$

$$\begin{aligned}
&2\xi\phi_{2,v}\phi_{2,vv}-2\xi^2\phi_{2,ww}\phi_{3,v}-2\xi\xi_{xw}\phi_{3,v}-4\xi^3\xi_{xww}\phi_2-2\xi\xi_{xvw}\phi_3-5\xi^3\xi_x\phi_{2,ww}\\
&+2\xi_{xv}\xi_w\phi_3-3\xi^4\phi_{2,w}\phi_{2,ww}-3\xi\xi_w\xi_x^2+6\xi^2\xi_{xvw}\phi_2+2\xi_v\xi_{vw}\phi_2^2+5\xi^2\xi_{vww}\phi_2^2\\
&-3\xi\xi_w^3\phi_2^2-4\xi\xi_{xv}\phi_{2,v}+2\xi\xi_t\phi_{2,vw}+6\xi^2\xi_{xv}\phi_{2,w}-2\xi^2\phi_{2,vv}\phi_{2,w}-3\xi^3\xi_w\phi_{2,w}^2\\
&+2\xi^2\xi_v\phi_{2,xw}+3\xi^3\phi_{2,v}\phi_{2,ww}-3\xi^4\phi_2\phi_{2,www}+7\xi^2\xi_{vw}\phi_{2,x}-3\xi\xi_{vv}\phi_{2,x}+2\xi^2\xi_{xww}\phi_3\\
&+2\xi\phi_{2,vw}\phi_{3,v}-2\xi\xi_w\phi_{3,xv}+2\xi^2\xi_{xw}\phi_{3,w}-2\xi^2\phi_{2,vw}\phi_{3,w}+2\xi^3\phi_{2,ww}\phi_{3,w}\\
&+2\xi^2\xi_w\phi_{3,xw}-2\xi\xi_{vw}\phi_{3,x}+\xi\xi_{tw}\xi_x-\xi_t\xi_w\xi_x-\xi_t\xi_w^2\phi_2-\xi_v\xi_w^2\phi_2^2-\xi\xi_v\phi_{2,tw}\\
&-\xi\xi_{tw}\phi_{2,v}-4\xi\xi_{vv}\phi_2\phi_{2,w}-\xi_w^2\xi_x\phi_3-\xi_w^3\phi_2\phi_3+\xi_w^2\phi_{2,v}\phi_3-\xi_w\phi_{2,vv}\phi_3\\
&+\xi\phi_{2,vvw}\phi_3-\xi\xi_w\phi_{2,v}^2-\xi_v\phi_2\phi_{2,vv}+\xi\phi_2\phi_{2,vvv}+\xi^2\xi_{tw}\phi_{2,w}+\xi_t\xi_v\phi_{2,w}\\
&+\xi_v^2\phi_2\phi_{2,w}+\xi^2\xi_v\phi_{2,w}^2-\xi^2\xi_t\phi_{2,ww}+\xi^2\xi_{ww}\phi_{2,w}\phi_3+\xi_v\xi_w\phi_{2,w}\phi_3+4\xi^3\phi_{2,xvw}\\
&+3\xi^2\xi_v-2\xi^2\phi_{2,tvw}+2\xi_t\xi_{xv}-2\xi\xi_t\xi_{xw}-\xi\xi_v\phi_2\phi_{2,vw}+\xi_v\xi_w\phi_2\phi_{2,v}-2\xi\xi_{tvw}\phi_2\\
&-12\xi^3\xi_{ww}\phi_2\phi_{2,w}+2\xi_t\xi_{vw}\phi_2-\xi\xi_v\phi_{2,ww}\phi_3+\xi\xi_w\xi_x\phi_{3,w}+3\beta\xi\xi_vw-2\xi^2\phi_{2,xvv}\\
&-3\beta\xi^2\xi_ww+\xi\phi_{2,tvv}+\xi^3\phi_{2,tww}+\xi\xi_{tw}\xi_w\phi_2-\xi^2\xi_w\xi_{xx}+3\xi\xi_v\xi_w\phi_{2,x}\\
&-2\xi\xi_w\xi_{xw}\phi_3+2\xi_{vw}\xi_w\phi_2\phi_3+2\xi^2\xi_{xtw}+2\xi^2\xi_{www}\phi_2\phi_3+2\xi\xi_w\phi_{2,vw}\phi_3-2\xi\xi_w^2\phi_{2,w}\phi_3\\
&-2\xi\xi_{ww}\phi_2\phi_{3,v}-2\xi\xi_w\phi_2\phi_{3,vw}-2\xi\xi_{vw}\phi_2\phi_{3,w}+4\xi^2\xi_{ww}\phi_2\phi_{3,w}\\
&+2\xi^2\xi_{ww}\phi_{3,x}+2\xi^2\xi_w\phi_{2,w}\phi_{3,w}+2\xi^2\xi_w\phi_2\phi_{3,ww}-3\xi\xi_v+5\xi^2\xi_{ww}\phi_2\phi_{2,v}\\
&+8\xi^2\xi_w\phi_2\phi_{2,vw}-2\xi\xi_t\xi_w\phi_{2,w}+3\xi\xi_v\xi_x\phi_{2,w}-8\xi^2\xi_w\xi_x\phi_{2,w}-\xi_t\phi_{2,vv}
\end{aligned}$$

$$+ 16\xi^2\xi_{vw}\phi_2\phi_{2,w} - 12\xi^2\xi_w^2\phi_2\phi_{2,w} + 4\xi\xi_v\xi_w\phi_2\phi_{2,w} + 4\xi^2\xi_w\phi_{2,v}\phi_{2,w}$$
$$+ 4\xi^2\xi_v\phi_2\phi_{2,ww} + 4\xi\xi_{xv}\xi_w\phi_2 - 2\xi\xi_t\xi_{ww}\phi_2 - 10\xi^2\xi_w\xi_{xw}\phi_2 + 7\xi\xi_{vw}\xi_x\phi_2 - 6\xi\xi_w^2\xi_x\phi_2$$
$$- 9\xi^2\xi_{ww}\xi_x\phi_2 + 7\xi\xi_{vw}\xi_w\phi_2^2 - \xi^3\xi_{xxw} + 4\xi\xi_w\xi_x\phi_{2,v} - 5\xi\xi_{vw}\phi_2\phi_{2,v} + 4\xi\xi_w^2\phi_2\phi_{2,v}$$
$$- 2\xi^3\xi_w\phi_{2,xw} - 2\xi\xi_x\phi_{2,vv} - 5\xi^2\phi_2\phi_{2,vvw} - 2\xi^2\phi_{2,vww}\phi_3 + 2\xi_v\xi_{xv}\phi_2$$
$$+ \xi^3\phi_{2,www}\phi_3 + 5\xi^3\phi_{2,vw}\phi_{2,w} + \xi^2\xi_w\phi_{2,tw} - 3\xi^3\xi_{www}\phi_2^2 + 4\xi^2\xi_{xw}\phi_{2,v}$$
$$- 2\xi^2\phi_{2,w}\phi_{3,vw} + \xi^3\phi_{2,w}\phi_{3,ww} - 4\xi^2\xi_w^2\phi_{2,x} + \xi_t\xi_w\phi_{2,v} - 2\xi\xi_{xvv}\phi_2 + 2\xi^2\xi_{tww}\phi_2$$
$$+ 7\xi^2\xi_x\phi_{2,vw} - 4\xi^3\xi_{ww}\phi_{2,x} - 5\xi^2\phi_{2,v}\phi_{2,vw} - 2\xi\xi_{vvw}\phi_2^2 - 6\xi^3\xi_{xw}\phi_{2,w} + 4\xi\xi_{xv}\xi_x$$
$$- 6\xi^2\xi_{xw}\xi_x - 2\xi\xi_{xtv} + \xi\phi_{2,w}\phi_{3,vv} - 9\xi^3\xi_w\phi_2\phi_{2,ww} - \xi_v\xi_w\xi_x\phi_2 - \xi\xi_w\phi_{2,v}\phi_{3,w}$$
$$+ 3\xi^2\xi_w - 3\alpha\xi^2\xi_v v - \xi\xi_v\phi_{2,v}\phi_{2,w} - 2\xi\xi_{vww}\phi_2\phi_3 + 3\alpha\xi^3\xi_w v - 12\xi^2\xi_w\xi_{ww}\phi_2^2$$
$$- \xi\xi_{ww}\phi_{2,v}\phi_3 - 3\xi^3\xi_w - \xi\xi_w\phi_2\phi_{2,vv} - 2\xi^4\phi_{2,xww} - \xi\xi_v\phi_{2,w}\phi_{3,w} + \xi\xi_w^2\phi_2\phi_{3,w}$$
$$+ \xi^2\xi_{xxv} + 7\xi^3\phi_2\phi_{2,vww} - \xi\xi_w\xi_{ww}\phi_2\phi_3 + \xi\xi_{ww}\xi_x\phi_3 = 0, \qquad \text{(B.1x)}$$

$$2\xi_w\phi_{2,xv}\phi_3 - 2\xi\phi_2\phi_{2,tvw} + \xi^2\xi_w\phi_{2,xx} + 2\alpha\xi^2\xi_x v - 2\xi\phi_2\phi_{2,xvv} - 2\xi_v\xi_{xw}\phi_2^2$$
$$+ \xi_t\phi_{2,v}\phi_{2,w} - \xi\xi_t\phi_{2,w}^2 + 2\xi\xi_{ww}\phi_2\phi_{3,x} + \xi\xi_{tw}\phi_{2,x} - \xi_t\xi_w\phi_{2,x} - \xi_w\xi_x\phi_{2,w}\phi_3$$
$$- \xi\xi_v\phi_{2,xx} - 2\xi_w\xi_{xw}\phi_2\phi_3 - \xi_w\xi_{ww}\phi_2^2\phi_3 - \xi\xi_w\xi_{xx}\phi_2 + 2\xi\xi_w\phi_{2,v}\phi_{2,x} - 2\beta\xi\xi_w\phi_2 w$$
$$+ \xi\xi_x\phi_{2,tw} - \xi^2\xi_{www}\phi_2^3 - 4\xi\xi_{xw}\xi_x\phi_2 - \xi_v\xi_x\phi_2\phi_{2,w} + 6\alpha\xi^2\xi_w\phi_2 v + \xi\xi_w\phi_{3,xx}$$
$$+ 8\xi^2\phi_2\phi_{2,vw}\phi_{2,w} + 4\xi^2\phi_2\phi_{2,ww}\phi_{3,w} - 4\alpha\xi\xi_v\phi_2 v - 3\xi\xi_{ww}\xi_x\phi_2^2$$
$$- \xi\xi_w\phi_2\phi_{2,ww}\phi_3 - \beta\xi\phi_3 - 2\xi\xi_w\phi_{2,xw}\phi_3 - 4\xi\xi_w\xi_{xw}\phi_2^2 - 2\xi_w^2\phi_2\phi_{2,w}\phi_3$$
$$- 2\xi\phi_2\phi_{2,vww}\phi_3 + \xi\xi_{ww}\phi_{2,x}\phi_3 + 2\xi_w\phi_2\phi_{2,vw}\phi_3 - 2\xi^3\phi_{2,w}\phi_{2,xw}$$
$$+ 2\xi\xi_{xww}\phi_2\phi_3 + 2\xi\xi_{ww}\phi_2^2\phi_{3,w} + \xi\xi_{www}\phi_2^2\phi_3 - 2\xi\phi_2\phi_{2,vw}\phi_{3,w}$$
$$+ 2\xi\xi_{xw}\phi_2\phi_{3,w} - 2\xi\phi_2\phi_{2,w}\phi_{3,vw} + 2\xi^2\phi_2\phi_{2,www}\phi_3 + 3\xi\xi_w\phi_2^2\phi_{2,vw} + 5\xi\xi_x\phi_2\phi_{2,vw}$$
$$+ 2\xi\xi_{xw}\phi_2\phi_{2,v} - 3\xi\xi_w\xi_{ww}\phi_2^3 - 2\xi\phi_2\phi_{2,ww}\phi_{3,v} - 2\xi\xi_v\phi_2\phi_{2,xw} + \xi\xi_{xxw}\phi_3$$
$$- 2\xi^2\xi_w\phi_2\phi_{2,xw} + \xi^2\phi_{2,w}\phi_{2,ww}\phi_3 - 2\xi\phi_2\phi_{2,vv}\phi_{2,w} - \xi\phi_{2,v}\phi_{2,ww}\phi_3$$
$$- 6\xi^2\xi_w\phi_2\phi_{2,w}^2 - 7\xi^2\xi_x\phi_2\phi_{2,ww} + \xi\xi_w\phi_2\phi_{2,tw} - 2\xi\xi_t\phi_2\phi_{2,ww} + 3\xi^2\phi_2\phi_{2,v}\phi_{2,ww}$$
$$- 6\xi^2\xi_w\phi_2^2\phi_{2,ww} + \beta\xi_w\phi_3 w - 4\xi\xi_w\xi_x\phi_{2,x} - \xi_v\xi_{ww}\phi_2^3 - \xi_t\xi_{ww}\phi_2^2 - \xi_v\xi_{xx}\phi_2$$
$$+ 2\xi^2\phi_{2,w}\phi_{3,xw} - \xi_w^2\phi_{2,x}\phi_3 + 2\xi^2\phi_{2,xw}\phi_{3,w} - 4\xi^2\xi_x\phi_{2,xw} + 2\xi\xi_{xw}\phi_{3,x} - \xi^2\xi_{xxw}\phi_2$$
$$+ \alpha\xi^2\phi_2 - \xi\xi_x\xi_{xx} + 2\xi^2\phi_{2,ww}\phi_{3,x} + \xi\xi_{xxv}\phi_2 - 2\xi^2\xi_{xx}\phi_{2,w} + 2\xi_v\phi_2^2\phi_{2,vw}$$
$$- 2\xi_t\xi_{xw}\phi_2 + 4\xi\xi_v\phi_2 + 2\xi\xi_{xtw}\phi_2 - \xi_w\xi_{xx}\phi_3 - 2\xi\phi_{2,xw}\phi_{3,v} - 2\xi\phi_{2,w}\phi_{3,xv}$$
$$- 6\xi^3\phi_2\phi_{2,w}\phi_{2,ww} + \xi\xi_{tw}\phi_2\phi_{2,w} - 4\xi^2\xi_w\phi_{2,w}\phi_{2,x} + 2\xi_t\phi_2\phi_{2,vw} + \beta\xi_t w - 2\xi\xi_x^2\phi_{2,w}$$
$$+ 2\xi^2\phi_2\phi_{2,tww} + 2\xi\xi_{xvw}\phi_2^2 + 2\xi\xi_w\phi_2 - 2\xi\phi_{2,vw}\phi_{3,x} - 2\xi^2\xi_{xww}\phi_2^2 + 5\xi^2\phi_2^2\phi_{2,vww}$$
$$+ 2\xi^2\phi_{2,v}\phi_{2,xw} + 2\xi^2\phi_{2,xv}\phi_{2,w} + \xi^2\phi_{2,tw}\phi_{2,w} - \xi_t\xi_x\phi_{2,w} - \xi\phi_{2,tw}\phi_{2,v} + \xi\xi_{xx}\phi_{2,v}$$
$$+ 2\xi_t\phi_{2,xv} + \xi\xi_{ww}\phi_2\phi_{2,w}\phi_3 + 5\xi\xi_{vw}\phi_2\phi_{2,x} - 4\xi\xi_w^2\phi_2\phi_{2,x}$$
$$+ 2\xi^2\phi_{2,xww}\phi_3 - 3\xi^3\phi_{2,ww}\phi_{2,x} - 2\xi\phi_{2,vv}\phi_{2,x} + 5\xi^2\phi_{2,vw}\phi_{2,x}$$

$$
\begin{aligned}
&- 3\xi^3\phi_2^2\phi_{2,www} - 4\xi^3\phi_2\phi_{2,xww} - 2\xi^2\xi_x\phi_{2,w}^2 - 5\xi^2\xi_{ww}\phi_2\phi_{2,x} \\
&+ 6\xi\xi_{xv}\phi_2\phi_{2,w} + \xi\xi_w\phi_{2,x}\phi_{3,w} + \xi\xi_x\phi_{2,ww}\phi_3 - \xi\xi_w\phi_{2,w}^2\phi_3 + \xi_w\phi_{2,v}\phi_{2,w}\phi_3 - \xi_v\phi_2 \\
&- \xi^3\phi_{2,xxw} - 3\xi\phi_2\phi_{2,v}\phi_{2,vw} + 2\xi^2\phi_2\phi_{2,w}\phi_{3,ww} - 8\xi\xi_w\xi_x\phi_2\phi_{2,w} - 10\xi^2\xi_{xw}\phi_2\phi_{2,w} \\
&- 2\xi_t\xi_w\phi_2\phi_{2,w} + \xi\xi_x\phi_{2,w}\phi_{3,w} - 2\xi_v\xi_w\phi_2^2\phi_{2,w} + \xi\xi_{ww}\phi_2^2\phi_{2,v} + 2\xi\xi_w\phi_2\phi_{3,xw} - \xi_t \\
&+ 7\xi\xi_{vw}\phi_2^2\phi_{2,w} - \xi\phi_{2,v}\phi_{2,w}\phi_{3,w} + \xi_v\phi_2\phi_{2,v}\phi_{2,w} - 6\xi\xi_w^2\phi_2^2\phi_{2,w} + \beta\xi_v\phi_2 w - \beta\xi\xi_x w \\
&- 9\xi^2\xi_{ww}\phi_2^2\phi_{2,w} + \xi\xi_w\phi_2^2\phi_{3,ww} + 2\xi\xi_x\phi_{2,v}\phi_{2,w} - \xi\xi_v\phi_2^2\phi_{2,ww} + \xi\xi_v\phi_{2,w}\phi_{2,x} \\
&- \xi_v\xi_w\phi_2\phi_{2,x} + 4\xi\xi_w\phi_2\phi_{2,v}\phi_{2,w} + 2\xi\xi_w\phi_2\phi_{2,w}\phi_{3,w} + 2\xi_v\phi_2\phi_{2,xv} + 4\xi\xi_{xv}\phi_{2,x} \\
&+ \xi\xi_{vww}\phi_2^3 - 6\xi^2\xi_w\phi_2 + 2\xi\xi_x\phi_{2,xv} - 2\xi\phi_{2,xvw}\phi_3 + \xi^2\phi_{2,w}^2\phi_{3,w} - 4\xi^2\xi_{xw}\phi_{2,x} \\
&+ \xi\xi_{tww}\phi_2^2 + 6\xi^2\phi_2\phi_{2,xvw} - 2\xi\phi_{2,v}\phi_{2,xv} - 2\xi\xi_t\phi_{2,xw} - 2\xi\phi_2^2\phi_{2,vvw} - \xi_w\phi_3 \\
&+ \xi^2\phi_{2,xxv} - \xi_t\xi_{xx} + 2\xi^2\phi_{2,xtw} - 2\xi\phi_{2,xtv} - 2\xi^2\xi_x + \xi\xi_x + \xi\xi_{xxt} = 0, \qquad \text{(B.1xi)}
\end{aligned}
$$

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