Del Pezzo fibrations and rank 3 Cox rings

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Abstract

One possible output of the minimal model program is a Mori fibre space. These varieties in 3 dimensions are Fano varieties, del Pezzo fibrations and conic bundles. Uniqueness of this output, the so-called rigidity of a Mori fibre space, is a question which arises naturally. In many cases, it has been proven for a general Fano 3-fold to be rigid. Del Pezzo fibrations over the rational curve have been studied in higher degrees and consequently it is known that if $\deg > 3$ then the del Pezzo fibration is nonrigid.

The goal of this thesis is to study rigidity and nonrigidity of low degree del Pezzo fibrations. We give a construction of these objects and classify the nonrigid ones whose link to the other model is obtained by the ambient space. This, in particular, provides many examples of nonrigid degree 2 del Pezzo fibrations which are not necessarily smooth. It is known that the study of rigidity for degree 3 del Pezzo fibrations is subject to consideration of Corti–Kollár stability condition. A first attempt to generalise this stability notion for lower degree fibrations is given in this thesis. The relation between these stability conditions and Sarkisov program is also studied in an explicit way. This requires techniques of working with rank 3 Cox rings which we develop. In particular, the notion of well-formedness for Cox rings is introduced as a generalisation of well-formedness of weighted projective spaces. We also construct families of cubic surface fibration in dimension 4 and study their nonrigidity in a similar way.

Contents

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Declaration

Aside from Chapters 1 and 2 that have an expository nature, I declare that, to the best of my knowledge, the material contained in this thesis is original work of the author except where otherwise indicated.

Chapter 1

Introduction

1.1 In a nutshell

Throughout this thesis, we study the birational geometry of some classes of Mori fibre spaces in dimensions 3 and 4 whose generic fibre is a low degree del Pezzo surface. The aim is to detect conditions under which these varieties, as Mori fibre spaces, are birationally rigid or not. Theorem 1.1.1 below provides many examples of nonrigid varieties in a constructive way. The first impression coming from this result is that the Gorenstein property has nothing to do with the rigidity conditions for a Mori fibre space. This is because many of these examples are non-Gorenstein and they behave exactly as the Gorenstein ones.

Theorem 1.1.1 (Main Theorem 1). *Let X be a degree 2 del Pezzo fibration over $\mathbb{P}^1$ embedded in $\mathcal{F}$, where $\mathcal{F}$ is a $\mathbb{P}(1,1,1,2)$ bundle over $\mathbb{P}^1$. Then the 2-ray game played on $\mathcal{F}/\{pt\}$ restricts to a type* III *or* IV *Sarkisov link on X if X belongs to one of the families in Table 4.1. Furthermore we calculate the link from a general member in each of these cases and identify the Mori fibre spaces that they link to.*

Chapter 5 is a first attempt to set up a notion of stability for degree 2 del Pezzo fibrations, following Corti-Kollár stability for cubic fibrations. Our approach enables us to construct an example of degree 2 del Pezzo fibration X with a cE_6 isolated singular point such that $-K_X \notin \mathrm{Int}(\mathrm{Mob}(X))$ but the type I Sarkisov link of X, started by blowing up this singular point, gives a square birational map to another model whose natural 2-ray game results in a link to a Fano 3-fold.

On the other hand, this example shows that Conjecture 5.1.1, which is stated mainly for smooth cases by Grinenko, does not hold in general, not even for Gorenstein varieties. Our approach to stability and the evidence coming from this example suggests that the conjecture can be rewritten as

Conjecture 1.1.2. *Let X be a degree 1, 2 or 3 del Pezzo fibration over $\mathbb{P}^1$ and suppose X is stable. Then X is birationally rigid if and only if $-K_X \notin \mathrm{Int}(\mathrm{Mob}(X))$.*

However, we believe that this approach to stability is not universal and a more sensible set up exists. We discuss this issue in more detail in §5.3.

In order to carry out our calculations with Type I and II Sarkisov links we develop some techniques of working with rank 3 Cox rings. In particular, this allows us to factorise explicitly Kollár's stabilisation process for cubic fibrations through Sarkisov links. Our study of higher rank Cox rings also leads to the notion of *well-formedness* for Cox rings as a generalisation of well-formedness of weighted projective spaces.

The following theorem is an observation from a similar approach as in Theorem 1.1.1 to cubic fibrations in dimension 4.

Theorem 1.1.3 (Main Theorem 2). *Let X be a degree 3 del Pezzo surface fibration over $\mathbb{P}^2$ embedded in $\mathcal{F}$, where $\mathcal{F}$ is a split $\mathbb{P}^3$ bundle over $\mathbb{P}^2$. Then the Type* III *or* IV *2-ray game of X is the restriction of that of the ambient space, and furthermore, a complete list of nonrigid families with links obtained in this way is given in Tables 6.1 and 6.2.*

1.2 Overview

We begin by explaining where the problem of rigidity stands and why one should be interested in it. Then we outline briefly what has been done regarding this problem and how our approach tackles some yet missing pieces of the theory. After that, we summarise the structure of this thesis and explain the connections between different parts.

1.2.1 Outline of the problem

We only consider complex projective algebraic varieties. A fundamental question in the theory of classification of algebraic varieties is:

Given a class of birational algebraic varieties, can we find a good representative in that class, which in some sense has the simplest structure among all of them ?

The answer to this question in dimension one is somehow simple as there is only one smooth curve birational to a given irreducible curve, and so we choose that.

Minimal models

For surfaces, the same approach fails immediately as the blow up operation provides infinitely many smooth surfaces birational to a given one. Instead, the theory of minimal models of surfaces, due to Italian algebraic geometers in the beginning of twentieth century, provides the answer. The idea is to find, where possible, a unique model among all smooth ones in a class. This is done by taking any smooth model, and contracting a rational curve with self intersection -1 in it; the key point is that the variety obtained by this contraction is smooth. In most cases the variety obtained at the end of this process is unique. The case where it is not unique are *ruled surfaces* and *del Pezzo surfaces.* The relations between these models is well understood.

One would like to run the same algorithm in higher dimensions, but it does not follow as simply as in the 2-dimensional case. In fact contracting a -1-curve could result in singular spaces. In the early 1980's, the work of Mori and Reid suggested that this theory can possibly be generalised for three dimensions if one allows the 3-folds to have some mild singularities, the so-called *terminal singularities.* After contributions of many mathematicians, it was proved in 1988 by Mori that minimal models of 3-folds exist. Mori's approach is called the Minimal Model Program; MMP for short. The idea is roughly this:

Let X be a 3-fold with at worst terminal singularities.

Step 1. Find a rational curve C on the 3-fold X whose intersection against K_X, the canonical divisor of X, is negative and that is *extremal* in a precise sense. If there is no such curve, then X is a minimal model. Otherwise there is a map $\varphi\colon X \to S$ to a normal projective variety S, which contracts only $[C]$, the numerical class of C, that is the support of all curves C' with the property that

$$C \cdot D = C' \cdot D \quad \text{for all divisors } D \quad .$$

Step 2. If $\dim [C] = 1$, then there exists a map $X \dashrightarrow X^+$ which factorises as $X \to S \leftarrow X^+$ and replaces C by another curve C^+ with $K_{X^+} \cdot C^+ > 0$; furthermore this map is an isomorphism everywhere else. This operation is called a *flip*.
If $\dim [C] = 2$, then replace X by S and go back to Step 1.
If $\dim [C] = 3$, then $\dim S < \dim X$, the morphism φ is a fibration and X/S is called a *Mori fibre space*.

After the work of Mori, the research focus of the subject has been divided into two directions:

1. Minimal model growing up: Generalising techniques and results of minimal model program for higher dimensions.

2. Minimal model growing down: Focusing on explicit geometry of minimal models of 3-folds and relations among them.

The first problem has received more attention after it was proved that the canonical ring of an algebraic variety of general type in any dimension is finitely generated [BCHM10]. A different and simpler proof of this finite generation statement has been recently given by Lazić [Laz08], [Laz09], which implies existence of minimal models in any dimension [CL10].

However, our concern in this thesis is with the second question. The main consideration is the birational relations between Mori fibre spaces. The number of different minimal structures that a Mori fibre space can admit is called its pliability. A precise definition is given in 2.3.2. A Mori fibre space is called *birationally rigid* if it has pliability 1.

What is known about rigidity

A Mori fibre space X/S belongs to one of the following classes, depending on the dimension of S.

(1) $\dim S = 0$ and X is a Fano variety.

(2) $\dim S = 1$ and the generic fibre of φ is a del Pezzo surface; X is called a *del Pezzo fibration* which we denote by dP_n, where n is the degree of the del Pezzo surface that is the generic fibre.

(3) $\dim S = 2$ and the generic fibre of φ is a rational curve; X in this case is called a conic bundle.

It was proved in [CPR00] that a general Fano 3-fold in the famous list of 95 families has pliability one, in other words, a general index one Fano 3-fold hypersurface is birationally rigid. In [CM04], an example of a non-rigid Fano variety with pliability 2 is given.

It is known that a del Pezzo fibration of degree ≥ 5 over $\mathbb{P}^1$ is rational, hence is nonrigid as it is birational to $\mathbb{P}^3$ and it is known that there are infinitely many distinct Mori fibre spaces birational to $\mathbb{P}^3$.

A complete description of rationality for smooth degree 4 del Pezzo fibration can be derived from [Ale87] and [Shr06]. Therefore the question of rigidity only concerns low degree del Pezzo fibrations. Rigidity and non-rigidity of degree 3 del Pezzo fibrations over $\mathbb{P}^1$ have been studied in [BCZ04]. In addition to the question of rigidity, the necessary conditions for these varieties to be rational was proved by Cheltsov in [Che08].

In a series of papers [Gria, Grib, Gric, Gri00a, Gri00b, Gri01a, Gri01b], Grinenko studies the rigidity properties of smooth del Pezzo fibrations of degree 1 and 2. The following statement is the main observation of his work.

Let X be a del Pezzo fibration of degree 1, 2 or 3 over $\mathbb{P}^1$. Then for many classes

$$X \textit{ is birationally rigid if and only if } -K_X \notin \operatorname{Int}(\operatorname{Mob}(X)).$$

The point for this thesis (and [BCZ04]) is that in many concrete situations this is a simple and natural condition to calculate and check. We do many of these in Chapters 4 and 6.

Grinenko proves this statement for the class of smooth degree 1 fibration and gives the only if proof for the two other cases in the smooth case; he leaves the rest as conjectures. He also constructs many examples of dP_2 and dP_1 fibrations, all of which are smooth. The Example 4.4.4 in [BCZ04], shows that this statement is not true if X is not smooth for dP_3 fibrations. We spell this out in Example 5.1.7 in details. The crucial point in this example is that X is *unstable* in the sense of Corti–Kollár. It suggests that one must have stability conditions in mind, when dealing with rigidity of dP_3 fibrations. However, there is as yet no notion of stability for degree 1 or 2 del Pezzo fibrations. On the other hand, there has been no serious study of nonsmooth dP_2 and dP_1 fibrations, perhaps because they are very likely to be singular as they are naturally embedded in singular spaces.

1.2.2 Outline of the thesis

Our aim in this thesis is to study dP_2 fibrations (not necessarily smooth) and their rigidity and nonrigidity. This requires construction of families of dP_2 fibration, especially those which admit some singularities.

In Chapter 4 we give a construction of such families and analyse their type III or IV Sarkisov links in the framework of [BCZ04] to hunt nonrigid ones. As a consequence, a list of non-rigid families is given in Table 4.1.

Chapter 5 is a first attempt to define a suitable stability condition for degree 2 del Pezzo fibrations, which applies to study rigidity of these varieties following the approaches of [Cor96,Kol97,BCZ04]. As a consequence, it is shown that the statement of Grinenko for rigidity is not true in general, even for Gorenstein dP_2 fibrations but requires semistability conditions.

In Chapter 5, we also provide machinery to give explicit factorisation of Corti–Kollár stabiliser maps through Sarkisov links of type I or II. This requires a good understanding of rank 3 Cox rings. In Chapter 3 we study toric varieties with rank 3 Cox rings. These form the ambient space to work out the type I and II Sarkisov links in Chapter 5. In particular, the notion of *well-formedness* for toric varieties is introduced as a generalisation of well-formedness of weighted projective spaces.

Finally in Chapter 6, we study nonrigidity of degree 3 del Pezzo fibrations over $\mathbb{P}^2$. These form a class of Mori fibre spaces in dimension 4. This is the first attempt of its kind in dimension 4, that we know of.

Chapter 2

Preliminaries

This chapter is aimed to provide background materials needed for the purpose of this thesis. Section 2.1 is about toric geometry and Cox rings. We use its ingredients in Chapter 3, where we treat a Cox ring as a generalisation of weighted projective spaces and study blow ups in special cases that we need later. Section 2.2 we briefly go through techniques of the theory of minimal models that we need for the rest of this thesis.

2.1 Toric Varieties

We give a brief outline of the toric geometry we need. This includes basic definitions and construction of toric varieties using fans. This is well-known material, and we have taken our presentation from the comprehensive texts [Ful93], [CLS] and the article by Danilov in [Dan78]. For a less technical but rather short and complete approach to toric geometry we refer to [Cox03]. In [Cox95], Cox introduces *the homogeneous coordinate ring* of a toric variety, an alternative language for toric geometry, in which construction of toric varieties is given in terms of geometric invariant theory. This notion has been a major subject of study in the past two decades and was well adapted and generalized to bigger classes of varieties, the so-called Mori dream spaces, in, for example, [BH03], [BH07], [HK00], [LV09a], [STV07], [SX10] and the survey [LV09b] by Laface and Velasco on Cox rings. Later in this section, we discuss the GIT construction of a toric variety from its fan.

Definition 2.1.1. *A toric variety is a normal algebraic variety T with an open subvariety $T_0 \subset T$ isomorphic to the torus $(\mathbb{C}^*)^n$, such that the action of the torus on itself can be extended to a regular action on T.*

2.1.1 Fan of a toric variety

Let $N \cong \mathbb{Z}^r$ be a lattice and $M = \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^r$ be its dual lattice. Associate real vector spaces to N and M by $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. A rational polyhedral cone in N is a set $\sigma := \{\lambda_1 v_1 + \cdots + \lambda_k v_k \in N_{\mathbb{R}} \mid \lambda_1, \ldots, \lambda_k \in \mathbb{R}^{\geq 0}, \, v_1, \ldots, v_k \in N\}$.
A *face* of σ is the intersection $\sigma \cap l = 0$, where l is a linear form which is nonnegative on σ. It is denoted by $\tau \preccurlyeq \sigma$ if τ is a face of σ. Note that if $\sigma \in N$ is a rational polyhedral cone, then $\sigma^\vee \in M$ is also a rational polyhedral cone.

Definition 2.1.2. *A fan Δ in N is a collection of cones which is closed under $\preccurlyeq$ and satisfies the condition:*

$$\text{if } \sigma_1, \sigma_2 \in \Delta, \text{ then } \sigma_1 \cap \sigma_2 \preccurlyeq \sigma_1, \sigma_2 \quad .$$

Very briefly, one constructs a toric variety $T(\Delta)$ from a fan Δ as follows:

(1) For $\sigma \in \Delta$ let $U_\sigma := \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$.

(2) If $\sigma \in \Delta$ and $\tau \preccurlyeq \sigma$, let $(U_\sigma)_\tau$ be the spectrum of the localization $\mathbb{C}[\sigma^\vee \cap M]_v$ where $v \in \sigma^\vee \cap M$ is a supporting hyperplane for τ.
Note that the inclusion $\mathbb{C}[\tau^\vee \cap M] \hookrightarrow \mathbb{C}[\sigma^\vee \cap M]_v$ induces a morphism $(U_\sigma)_\tau \to U_\tau$.

(3) For σ_1, σ_2 in Δ, define $(U_{\sigma_1})_{\sigma_2} := (U_{\sigma_1})_{\sigma_1 \cap \sigma_2}$.

(4) Define $T(\Delta)$ to be the scheme obtained by glueing the schemes U_σ, where the overlap is define by the isomorphisms $(U_{\sigma_1})_{\sigma_2} \to U_{\sigma_1 \cap \sigma_2} \to (U_{\sigma_2})_{\sigma_1}$.

This defines $T(\Delta)$ as a scheme, which is separated, integral and normal. See [Ful93] §1.4 and §2.1. .

The toric structure is easy to see:
$T_0 = \mathrm{Spec}(\mathbb{C}[M]) = (\mathbb{C}^*)^n$ is an open subvariety of $T(\Delta)$ and the action of the torus T_0 on itself extends to a regular action on T given locally by:

$$\varphi\colon \mathbb{C}[\sigma^\vee \cap M] \longrightarrow \mathbb{C}[\sigma^\vee \cap M] \otimes_{\mathbb{C}} \mathbb{C}[M]$$

$$m \mapsto m \otimes m \quad .$$

Definition 2.1.3. A *ray* is a 1-dimensional cone in Δ. We denote the set of all rays in Δ by $\Delta(1)$.

Remark 2.1.4. Throughout this thesis, we assume $\Delta(1)$ spans $N_{\mathbb{R}}$ for all toric varieties.

Definition 2.1.5. Let X be a normal, irreducible variety.

(a) A *Weil divisor* on X is a finite formal sum

$$D = \sum m_i D_i \quad ,$$

where D_i are distinct irreducible divisors of X and $m_i \in \mathbb{Z}$. The set of all Weil divisors on X is denoted by $\mathrm{Div}(X)$.

(b) Two Weil divisors D_1 and D_2 are *linearly equivalent*, denoted by $D_1 \sim D_2$, if there exists a $f \in \mathbb{C}(X)^*$ such that $\mathrm{div}(f) = D_1 - D_2$. A Weil divisor D is called a *principal divisor* if $D \sim 0$.

(c) The group of Weil divisors on X modulo principal divisors is called the *divisor class group* of X and is denoted by $\mathrm{Cl}(X)$.

(d) A Weil divisor D on X is *Cartier* if it is locally principal. The set of all Cartier divisors on X is denoted by $\mathrm{CDiv}(X)$.

(e) The group of Cartier divisors on X modulo equivalence relation is called the Picard group and is denoted by $\mathrm{Pic}(X)$.

For $\rho \in \Delta(1)$, let D_ρ be the irreducible, T_0 invariant Weil divisor of T correspond to ρ. If we denote the free abelian group of T_0-invariant divisors of T by $\mathbb{Z}^{\Delta(1)}$, every $D \in \mathbb{Z}^{\Delta(1)}$ is of the form

$$D = \sum_\rho m_\rho D_\rho \qquad .$$

2.1.2 Cox construction of toric varieties

Let $\mathrm{Cl}(T)$ be the divisor class group of T and $\mathrm{CDiv}(T) \subset \mathbb{Z}^{\Delta(1)}$ be the group of Cartier divisors on T. Define the map $\varphi\colon M \longrightarrow \mathbb{Z}^{\Delta(1)}$ by $m \mapsto D_m := \sum_\rho \langle m, n_\rho \rangle D_\rho$, where n_ρ is the unique generator of $\rho \cap N$. We have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, equal] & \mathrm{CDiv}(T) \arrow[r] \arrow[d, hook] & \mathrm{Pic}(T) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r, "\varphi"] & \mathbb{Z}^{\Delta(1)} \arrow[r] & \mathrm{Cl}(T) \arrow[r] & 0 \end{tikzcd} \qquad . \tag{2.1}$$

In particular, for any divisor $D \in \mathbb{Z}^{\Delta(1)}$, there is a $[D] \in \mathrm{Cl}(T)$.
Let $\mathrm{Cox}(T) := \mathbb{C}[x_\rho \mid \rho \in \Delta(1)]$. We have the following correspondence between monomials in $\mathrm{Cox}(T)$ and divisors in $\mathbb{Z}^{\Delta(1)}$:

$$x^D := \prod_\rho x_\rho^{m_\rho} \quad \leftrightsquigarrow \quad D = \sum_\rho m_\rho D_\rho$$

The second short exact sequence in 2.1 defines a grading on $\mathrm{Cox}(T)$ by $\deg(x^D) = [D] \in \mathrm{Cl}(X)$. Therefore we can write $\mathrm{Cox}(T)$ as

$$\mathrm{Cox}(T) = \bigoplus_{\alpha \in \mathrm{Cl}(T)} S_\alpha$$

Proposition 2.1.6 ([Cox95], §1.1). *(i) If $\alpha = [D] \in \mathrm{Cl}(X)$, then there is an isomorphism $\varphi_D : S_\alpha \cong H^0(T, \mathcal{O}_T(D))$.*

(ii) If $\alpha = [D]$ and $\beta = [E]$, then there is a commutative diagram:

$$
\begin{tikzcd}
S_\alpha \otimes S_\beta \arrow[r] \arrow[d] & S_{\alpha+\beta} \arrow[d] \\
H^0(T, \mathcal{O}_T(D)) \otimes H^0(T, \mathcal{O}_T(E)) \arrow[r] & H^0(T, \mathcal{O}_T(D+E))
\end{tikzcd}
$$

It is known that the group $G := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(T), \mathbb{C}^*) = (\mathbb{C}^*)^{d-n} \times \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(T)_{tor}, \mathbb{Q}/\mathbb{Z})$, where $d = |\Delta(1)|$. Applying $\mathrm{Hom}(-, \mathbb{C}^*)$ to the second short exact sequence in 2.1 will lead to:

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Delta(1)} \longrightarrow T_0 \longrightarrow 1$$

The restriction of the action of $(\mathbb{C}^*)^{\Delta(1)}$ on $\mathbb{C}^{\Delta(1)}$ to its subgroup G will define the following action:

$$G \curvearrowright \mathbb{C}^{\Delta(1)} : \quad g.t = (g[D_\rho])t_\rho$$

Note that this action induces an action of G on $\mathrm{Cox}(T) = \mathbb{C}[\mathbb{C}^{\Delta(1)}]$ and the grading obtained by this action is the same grading as the grading defined by $\mathrm{Cl}(T)$. This is somehow obvious, as $\mathrm{Cl}(T)$ forms the character group of G.

Definition 2.1.7. *The irrelevant ideal of T, is defined to be $I_T = \langle x^{\hat{\sigma}} \,|\, \sigma \in \Delta \rangle$, where $\hat{\sigma} = \sum_{\rho \notin \sigma(1)} D_\rho$.*

Theorem 2.1.8 ([Cox95], §2.1). *Let T be a toric variety determined by the fan Δ, and $Z = V(I_T) \subset \mathbb{C}^{\Delta(1)}$. Then:*

(i) The set $\mathbb{C}^{\Delta(1)} - Z$ is G-invariant.

(ii) T is isomorphic to the categorical quotient $(\mathbb{C}^{\Delta(1)} - Z)/G$.

(iii) The quotient $(\mathbb{C}^{\Delta(1)} - Z)/G$ is geometric iff Δ is simplicial.

Remark 2.1.9. *If Δ is simplicial, then the Picard group and the divisor class group coincide. This is because simpliciality of Δ implies $\mathrm{Div}(T) \cong \mathbb{Z}^{\Delta(1)}$ and therefore applying this to the short exact sequence 2.1 shows $\mathrm{Pic}(T) \cong \mathrm{Cl}(T)$. This immediately implies $\mathrm{Cl}(T)$ is torsion free and $\mathrm{Pic}(T) \cong \mathbb{Z}^r$, where $r = d - n$. The rank of the toric variety T (or rank of G) is defined to be r.*

2.1.3 GIT and Cox ring

In Subsection 2.1.2, it was shown that a toric variety can be recovered from the action of G on $\mathbb{C}^d$ provided that the irrelevant ideal of T is given. It was also explained how to construct the irrelevant ideal I_T when the top dimensional cones of Δ are given. In other words, we have the following one to one correspondence between the two languages of toric geometry:

Fan Δ		Cox data
$\Delta(1) = \{\rho_1, \ldots, \rho_n\}$	$\leftrightsquigarrow$	$\mathrm{Cox}(T) = \mathbb{C}[x_1, \ldots, x_n]$
Cones in Δ	$\leftrightsquigarrow$	The irrelevant ideal I_T

In what follows, we show how a toric variety can be obtained from geometric invariant theory (GIT) techniques using an ample divisor of T (see [Dol03], §6 and §7 for an introduction to the techniques used in this part). This idea leads to the generalisation of Cox rings for a bigger class of varieties.
A full characterisation of ample divisors on toric varieties is given in [CLS] §6.1.

Consider the action of G on $\mathbb{C}^d$. By linearisation of this action we mean extending the action of G to the trivial line bundle $L = \mathbb{C} \times \mathbb{C}^d$ such that the following diagram is commutative:

$$\begin{tikzcd} G \times \mathbb{C} \times \mathbb{C}^d \arrow[r, "\pi"] \arrow[d, "G\curvearrowright"'] & G \times \mathbb{C}^d \arrow[d, "\curvearrowleft G"] \\ \mathbb{C} \times \mathbb{C}^d \arrow[r, "\pi"] & \mathbb{C}^d \end{tikzcd}$$

Such an action is of the form

$$G \times \mathbb{C} \times \mathbb{C}^d \longrightarrow \mathbb{C} \times \mathbb{C}^d$$

$$(g, \lambda, t) \mapsto (g.\lambda, g.t)$$

where $g.(t_1, \ldots, t_d) = (g([D_{\rho_1}])t_1, \ldots, g([D_{\rho_d}])t_d)$.
Note that $g : \mathrm{Cl}(T) \to \mathbb{C}^*$. Therefore the extension of the action depends on the definition of $g.\lambda$ and this is determined by a choice of a $D \in \mathrm{Cl}(T)$ and defines $g.\lambda = g([D])\lambda$.
Let $\mathcal{L}_D$ denote the line bundle correspond to D. It is clear that there is a bijection between G-invariant sections of $\mathcal{L}_D$ and the D-graded component of $\mathrm{Cox}(T)$. If we denote $R_D = \bigoplus_{i=0}^{\infty} \mathrm{Cox}(T)_{iD}$, then the following isomorphism arises naturally:

$$(\bigoplus_{j=0}^{\infty} H^0(\mathbb{C}^d, \mathcal{L}_D^j))^G \cong R_D$$

The scheme $\mathrm{Proj}(R_D)$ denotes the GIT quotient of $\mathbb{C}^d$ by G.

Definition 2.1.10. *A point $x \in \mathbb{C}^d$ is semistable if there is an $s \in H^0(\mathbb{C}^d, \mathcal{L}_D^j)$ for some j, such that $s(x) \neq 0$. The set of all semistable points in $\mathbb{C}^d$ correspond to D is denoted by $(\mathbb{C}^d)_D^{ss}$.*

Proposition 2.1.11. $\mathrm{Proj}(R_D) \cong (\mathbb{C}^d)_D^{ss} /\!/ G$.

Theorem 2.1.12. *Let D be an ample divisor of T and Δ correspond to the fan obtained by D. Then $(\mathbb{C}^d)_D^{ss} = \mathbb{C}^d - Z$, where Z is the subvariety determined by the irrelevant ideal I_T. In other words,* $\mathrm{Proj}(R_D) \cong (\mathbb{C}^d - Z)/G$.

2.2 Minimal Model Program

In this section, we give a brief overview of materials and techniques of minimal model program that we need in order to pursue our calculations in the rest of this thesis. These materials are mainly taken from [Cor00, KM98, Mat02].

We begin by defining various cones which are central in MMP.

Definition 2.2.1. Let X be a normal, projective variety.

(i) Two Cartier divisors D_1 and D_2 on X are *numerically equivalent*, denoted by $D_1 \sim_{\text{num}} D_2$, if

$$D_1 \cdot C = D_2 \cdot C \quad \text{for every irreducible curve } C \subset X \quad .$$

(ii) A Cartier divisor D is called *numerically effective*, nef for short, if $D \cdot C \geq 0$ for all curves $C \subset X$.

(iii) The *Néron-Severi* group of X, denoted by $\mathrm{N}^1(X)$, is the group

$$\mathrm{N}^1(X) = \mathrm{CDiv}(X)/\sim_{\text{num}} \quad .$$

We denote by $\mathrm{N}^1(X)_{\mathbb{R}}$ the vector space $\mathrm{N}^1(X) \otimes \mathbb{R}$.

(iv) A 1-*cycle* is a formal (finite) combination of irreducible, reduced and proper curves $C = \Sigma a_i C_i$. A 1-cycle is called *effective* if all $a_i \geq 0$.

(v) Two 1-cycles C, C' are called *numerically equivalent* if $D \cdot C = D \cdot C'$ for any Cartier divisor D. The set of 1-cycles with real coefficients modulo numerical equivalence form a $\mathbb{R}$-vector space; denoted by $\mathrm{N}_1(X)$. The class of a 1-cycle is denoted by $[C]$. We have a perfect pairing

$$N_1(X) \times N^1_{\mathbb{R}}(X) \to \mathbb{R} \quad .$$

(vi) Define $\mathrm{NE}(X) = \{\sum a_i [C] \mid 0 \leq a_i \in \mathbb{R}\} \subset \mathrm{N}_1(X)$, where C_i are irreducible curves on X. Denote the closure of $\mathrm{NE}(X)$ in $N_1(X)$ by $\overline{\mathrm{NE}}(X)$.

(vii) A 1-dimensional subspace $R \subset \overline{\mathrm{NE}}(X)$ is called an *extremal ray* if $u, v \in \overline{\mathrm{NE}}(X)$ with $u + v \in R$ implies $u, v \in R$.

Remark 2.2.2. The group $\mathrm{N}^1(X)$ is a free abelian group of finite rank (See [Laz04] Proposition 1.1.16). The rank of $\mathrm{N}^1(X)$ is the *Picard number* of X and is denoted by $\rho(X)$.

Definition 2.2.3. A normal projective variety X is called $\mathbb{Q}$-factorial if for any Weil divisor D, there is positive integer m such that mD is Cartier.

Theorem 2.2.4. *Let X be a normal, $\mathbb{Q}$-factorial projective 3-fold. Suppose $R \subset \overline{\mathrm{NE}}(X)$ is an extremal ray such that $R.K_X < 0$. Then there exists a morphism $\varphi\colon X \to Z$, called an extremal morphism, to a normal projective variety Z with* $\dim Z \leq 3$ *such that*

(1) $\varphi_*\mathcal{O}_X = \mathcal{O}_Z$,

(2) $\varphi(C_i)$ *is a point for all* $C_i \in R$,

(3) φ *is an isomorphism from* $X - \bigcup C_i$ *to* $Z - \varphi(\bigcup C_i)$.

Definition 2.2.5. The morphism φ in Theorem 2.2.4 is called the *extremal morphism*. We call $\bigcup C_i$ the exceptional locus of φ and denote it by $\mathrm{Exc}(\varphi)$. An extremal morphism $\varphi\colon X \to Z$ is called

(1) A *divisorial contraction*, or extremal morphism of divisorial type, if $\mathrm{Exc}(\varphi)$ is an irreducible divisor.

(2) An extremal morphism of *fibre type* if $\dim X > \dim Z$.

(3) A *small contraction* if $\dim \mathrm{Exc}(\varphi) = 1$.

2.2.1 Terminal singularities

Before we state the algorithm of minimal model program we introduce the type of singularities that we consider.

Definition 2.2.6. A normal variety X of dimension n has only *terminal singularities* if

(i) the canonical divisor K_X is $\mathbb{Q}$-Cartier, that is, there exists $m \in \mathbb{N}$ such that mK_X is Cartier;

(ii) there exists a projective birational morphism $f\colon Y \to X$ from a nonsingular variety Y such that in the ramification formula

$$K_Y = f^*K_X + \sum a_i E_i$$

all the coefficients for the exceptional divisors are strictly positive, that is $a_i > 0$ for all E_i exceptional.

A typical example of a terminal singularity is the following.

Example 2.2.7. Let $\mathbb{Z}_2$ act on $\mathbb{C}^3$ by $x_i \mapsto -x_i$, where x_1, x_2 and x_3 are the coordinates on $\mathbb{C}^3$. The quotient $\mathbb{C}^3/\mathbb{Z}_2$ is singular at the origin, we denote this type of singularity by $\frac{1}{2}(1,1,1)$.This is a typical terminal quotient singularity. In fact if $\mathbb{Z}_r$ acts on $\mathbb{C}^3$ by $x_i \mapsto \varepsilon^{a_i} x_i$, where ε is a rth root of unity and $a_i \in \mathbb{Z}$, then the quotient is terminal if and only if $(a_1, a_2, a_3) = (1, -1, a)$, up to a change of basis in $\mathbb{C}^3$, for some integer a coprime to r.

Another example of terminal singularity is the origin in the hypersurface of $\mathbb{C}^4$ with polynomial $x^2 + y^3 + z^4 + t^k$ for some integer $k \geq 4$. We say that this singular point is a cE_6 singularity, because the hyperplane section $(t = 0)$ is a 2-dimensional singularity which is one of the famous Du Val singularities; in this case it is a E_6 Du Val singularity. For a great discussion and classification of terminal singularities see [Rei87].

Now we are ready to run the MMP.

MMP. Let X be a $\mathbb{Q}$-factorial 3-fold with only terminal singularities. Put $X = X_0$ and run the following program.

Step 1. If K_{X_i} is nef then X_i is a *minimal model* and we stop. If K_{X_i} is not nef, then there exists a rational curve C with $C \cdot K_{X_i} < 0$. Consider the extremal morphism $\varphi\colon X_i \to X_{i+1}$ as in Theorem 2.2.4 with the extremal ray $[C]$ and go to Step 2.

Step 2. If φ is a divisorial contraction then go to Step 1. If it is of fibre type stop and call X_i a Mori fibre space. If φ is a small contraction then go to Step 3.

Step 3. Consider the flip $f\colon X_i \dashrightarrow Y_i$ as in Theorem 2.2.8 below. Put $X_{i+1} = Y_i$ and go back to Step 1.

Fact: This process works and terminates.

Theorem 2.2.8. *Let X be a normal, projective $\mathbb{Q}$-factorial 3-fold with only terminal singularities. Suppose $\varphi\colon X \to Y$ is a small contraction. Then there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "\varphi"'] & & X^+ \arrow[dl, "\varphi^+"] \\ & Y & \end{tikzcd}$$

where $\varphi^+\colon X^+ \to Y$ contracts a 1-dimensional locus $\bigcup C_i^+$ with $C_i^+ \cdot K_{X^+} > 0$. The map $f\colon X \dashrightarrow X^+$ is an isomorphism away from the contracted loci. The birational map f is called a flip.

A typical example of a 3-fold flip, perhaps the easiest, is the *Francia flip.* The construction of Francia flip by Cox rings is the following. Let $\mathbb{C}^*$ act on $\mathbb{C}^4$ by coordinates

$$\lambda \cdot (x_1, x_2, x_3, x_4) \mapsto (\lambda^2 x_1, \lambda x_2, \lambda^{-1} x_3, \lambda^{-1} x_4) \quad .$$

This action gives a $\mathbb{Z}$-grading to

$$R = \mathbb{C}[x_1, \dots, x_4] = \bigoplus_{n \in \mathbb{Z}} R_n \quad .$$

Consider 3 graded algebras

$$R_0, \qquad R^+ = \bigoplus_{n \geq 0} R_n, \qquad R^- = \bigoplus_{n \leq 0} R_n \quad .$$

From these one can construct $Y = \operatorname{Spec} R_0$, $X = \operatorname{Proj}_Y R^+$ and $X^+ = \operatorname{Proj}_Y R^-$ such that they fit into the diagram of the flip.

2.3 Mori fibre spaces

One output of MMP is a Mori fibre space (Mfs for short); this is when the extremal contraction $\varphi\colon X \to Z$ is of fibre type. These varieties are naturally one of the following depending on the dimension of Z.

(i) Fano 3-folds when $\dim Z = 0$.

(ii) Del Pezzo fibrations when $\dim Z = 1$.

(iii) Conic bundles when $\dim Z = 2$.

We are interested in uniqueness of this output.

Definition 2.3.1. [[Cor00], Definition 1.2] The Sarkisov category is the category whose objects are Mori fibre spaces and morphisms are birational maps. Let $X \to S$ and $X' \to S'$ be Mori fibre spaces.

(1) A morphism in the Sarkisov category, that is, a birational map $f\colon X \dashrightarrow X'$, is *square* if it fits into a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d] & X' \arrow[d] \\ S \arrow[r, dashed, "g"] & S' \end{tikzcd}$$

where g is a birational map (hence $\mathbb{C}(S) \cong \mathbb{C}(S')$) and if, in addition, the induced birational map of generic fibres $f_L\colon X_L \dashrightarrow X'_L$ is biregular. In this case we say that X/S and X'/S' are *square birational.*

(2) A Sarkisov isomorphism is a birational map $f\colon X \dashrightarrow X'$ which is biregular and square.

(3) A Mori fibre space $X \to S$ is *birationally rigid* if, given any birational map $\varphi\colon X \dashrightarrow X'$ to another Mori fibre space $X' \to S'$, there exists a birational selfmap $\alpha\colon X \dashrightarrow X$ such that the composition $\varphi \circ \alpha\colon X \dashrightarrow X'$ is square.

Definition 2.3.2. ([CR00] 4.6.) We define the *pliability* of a Mori fibre space $X \to S$ as the set

$$\mathcal{P}(X/S) = \{\text{Mfs } Y \to T \mid X \text{ is birational to } Y\}/\text{square equivalence}.$$

We sometimes abuse the term pliability to mean the cardinality of this set.

2.4 Sarkisov Program

Sarkisov links are the decompositions of birational maps between Mori fibre spaces into elementary maps. We brief state the nature of these links. A complete description of these is stated in [Cor00], §2.

Remark 2.4.1. In this section, by a flip we mean a flip, flop or antiflip, where flop is roughly a similar surgery as flip with only difference that the contracted curves have trivial intersections with the canonical divisor and an antiflip is the inverse of a flip.

A Sarkisov link between two Mori fibre spaces $X \to S$ and $X' \to S'$ is one of the following birational maps.

Type I. Starts by an extremal blow up of a centre in X then follows by a sequence of forced flips and then a divisorial contraction to X'. This is the following commutative diagram

$$\begin{tikzcd}
& Z \arrow[r, dashed, "\text{flip}"] \arrow[dl, "\text{blow up}"'] & Z' \arrow[dr, "\text{contraction}"] & \\
X \arrow[dr] & & & X' \arrow[dl] \\
& S \arrow[r, equal] & S' &
\end{tikzcd}$$

Type II. Starts by an extremal blow up on X then follows by a sequence of flips to X'. In this case S' has an extremal morphism to a variety T which is birational to S.

$$\begin{tikzcd}
& Z \arrow[r, dashed, "\text{flip}"] \arrow[dl, "\text{blow up}"'] & X' \arrow[d] \\
X \arrow[d] & & S' \arrow[dl, "\text{extremal}"] \\
S \arrow[r, equal] & T &
\end{tikzcd}$$

Type III. Starts by a sequence of flips from X then follows by a divisorial contraction to X', where S has an extremal morphism to T which is birational to S'.

$$\begin{tikzcd}
X \arrow[r, dashed, "\text{flip}"] \arrow[d] & Z \arrow[dr, "\text{divisorial}"] & \\
S \arrow[dr, "\textit{extremal}"'] & & X' \arrow[d] \\
& T \arrow[r, equal] & S'
\end{tikzcd}$$

Type IV. Starts by a sequence of flips to X'. The varieties S and S' have extremal morphisms to the same variety T.

$$\begin{tikzcd}
X \arrow[rr, dashed, "\text{flip}"] \arrow[d] & & X' \arrow[d] \\
S \arrow[dr, "\text{extremal}"'] & & S' \arrow[dl, "\text{extremal}"] \\
& T &
\end{tikzcd}$$

Fact: Any birational map between two Mori fibre spaces factors through a chain of Sarkisov links.

Chapter 3

Rank 2 and rank 3 toric varieties

In this chapter, we construct some tools in toric geometry that we use in other chapters. This, in particular, includes a study of rank 3 Cox rings as a blow up of rank 2 ones. The irrelevant ideal of these varieties is our main concern. The notion of well-formedness is also introduced for higher rank Cox rings. This is a generalisation of well-known well-formedness of weighted projective spaces.

3.1 Well formed Cox rings

In Section 2.1 it was shown how a toric variety can be reconstructed from the GIT. In the rest of this thesis we consider only toric varieties with torsion free divisor class group. It immediately implies that for such a toric variety T the action of G on $\mathrm{Cox}(T)$ has a representing matrix in $\mathcal{M}_{r\times n}(\mathbb{Z})$. We denote this toric variety by $TV(I, A)$, where I is the irrelevant ideal and the action is given by $A \in \mathcal{M}_{r\times n}(\mathbb{Z})$.

Definition 3.1.1. *Let $T = TV(I, A)$ be a toric variety. The rank of T is defined to be* $\operatorname{rank} A$.

When looking at a toric variety in terms of the GIT specified by A and I, one could ask the following question:

Question 3.1.2. Given $TV(I, A)$, can one obtain an isomorphic variety by changing A? If yes, what is the domain of variation and is there a *good* representative for this class of variation?

This question in particular has a well known answer for rank one toric varieties. Note that under our assumptions, a rank one toric variety is nothing but a weighted projective space, namely $\mathbb{P}(a_0, \ldots, a_n)$ for some positive integers $a_0, \ldots, a_n$. These varieties have been studied in [IF00]. It is easy to see ([IF00] §5) that for positive integers α and β

$$\mathbb{P}(a_0, \ldots, a_n) \cong \mathbb{P}(\alpha a_0, \ldots, \alpha a_n) \cong \mathbb{P}(a_0, \beta a_1, \ldots, \beta a_n) \quad .$$

The notion of *well-formedness* for weighted projective spaces plays the role of the good representative as an answer to the question above for rank one toric varieties.

Definition 3.1.3 ([IF00] Definition 5.11). The expression $\mathbb{P}(a_0, \ldots, a_n)$ is *well formed* if $\mathrm{hcf}(a_0, \ldots, \hat{a}_i, \ldots, a_n) = 1$ for each i.

For more details and discussion on well-formedness we refer to [Dol81] and [IF00]. In what follows, we generalise this to higher rank toric varieties.

The following lemma answers the first part of Question 3.1.2 by finding some freedom for A, the matrix of the action of the toric variety $TV(I, A)$.

Lemma 3.1.4. *Let $T = TV(I, A)$ and $B = gA$ for some $g \in \mathrm{GL}(r, \mathbb{Q})$ with integer entries and define T' to be the toric variety $T' = TV(I, B)$. Then T is isomorphic to T'.*

Proof. The varieties T and T' are defined by

$$T = (\mathbb{C}^n - V(I))/G_A \quad , \quad T' = (\mathbb{C}^n - V(I))/G_B \quad ,$$

where $G_A \cong G_B \cong (\mathbb{C}^*)^r$. If we denote $A = (a_{ij})$ and $B = (b_{ij})$, then for $(\lambda_1, \ldots, \lambda_r) \in G_A$ and $(\gamma_1, \ldots, \gamma_r) \in G_B$, the actions are the following:

$$G_A : \qquad (\lambda_1, \ldots, \lambda_r).(x_1, \ldots, x_n) \mapsto (\prod_{i=1}^{r} \lambda_i^{a_{i1}} x_1, \ldots, \prod_{i=1}^{r} \lambda_i^{a_{in}} x_n)$$

$$G_B : \qquad (\gamma_1, \dots, \gamma_r).(x_1, \dots, x_n) \mapsto (\prod_{i=1}^{r} \gamma_i^{a_{i1}} x_1, \dots, \prod_{i=1}^{r} \gamma_i^{a_{in}} x_n)$$

Let (x) and (y) be two vectors in $\mathbb{C}^n$. Let us denote by (x) $\sim_A$ (y) if (x) and (y) are in the same orbit of the action by G_A, and similarly for (x) $\sim_B$ (y). The aim is to show

$$(\mathrm{x}) \sim_A (\mathrm{y}) \quad \text{if and only if} \quad (\mathrm{x}) \sim_B (\mathrm{y}) \quad .$$

If (x) $\sim_B$ (y), then there exists $(\gamma_1, \dots, \gamma_r) \in (\mathbb{C}^*)^r$ such that

$$(y_1, \dots, y_n) = (\prod_{i=1}^{r} \gamma_i^{b_{i1}} x_1, \dots, \prod_{i=1}^{r} \gamma_i^{b_{in}} x_n) \quad .$$

To prove (x) $\sim_A$ (y), we must find $(\lambda_1, \dots, \lambda_r) \in (\mathbb{C}^*)^r$ such that

$$(y_1, \dots, y_n) = (\prod_{i=1}^{r} \lambda_i^{a_{i1}} x_1, \dots, \prod_{i=1}^{r} \lambda_i^{a_{in}} x_n) \quad .$$

This follows from $b_{ij} = \sum_k g_{ik} a_{kj}$, if we put $\lambda_i = \gamma_1^{g_{i1}} \dots \gamma_r^{g_{ir}}$.
Proof for the only if part is very similar and it is done by replacing g by g^{-1}. □

Corollary 3.1.5. If $T = TV(I, A)$ and $\mathrm{rank}(A) = r_0 < r$, then there exists a matrix $A_0 \in \mathcal{M}_{r_0 \times n}(\mathbb{Z})$ such that $T = TV(I, A_0)$.

Remark 3.1.6. By Corollary 3.1.5, without loss of generality, we can always assume that $A \in \mathcal{M}_{r \times n}(\mathbb{Z})$, the defining matrix of $TV(I, A)$, has full rank and hence $\mathrm{rank}(T) = r$.

The result of Lemma 3.1.4, as a partial answer to Question 3.1.2 shows that the expression $TV(I, A)$ is not uniquely determined from A and it varies up to the action of a subset of $\mathrm{GL}(r, \mathbb{Q})$. In fact failure of this set to be a subgroup is the problem of well-formedness. In the rest of this section, we complete our answer to this question by finding a well formed model for $TV(I, A)$ as the representative. Such a model will be unique up to $\mathrm{SL}(r, \mathbb{Q})$.

Definition 3.1.7. Let $M \in \mathcal{M}_{r \times n}(\mathbb{Z})$ be a rank r matrix ($r < n$). Suppose $M_1, \dots, M_s$

are all the $r \times r$ minors of M and let $d_M = \mathrm{hcf}(\det(M_1), \ldots, \det(M_k))$. The matrix M is called *standard* if $d_M = 1$.

Lemma 3.1.8. *For any rank r matrix $M \in \mathcal{M}_{r\times n}(\mathbb{Z})$, there exist matrices $g \in \mathrm{GL}(r, \mathbb{Q}) \cap \mathcal{M}_{r\times r}(\mathbb{Z})$ and $N \in \mathcal{M}_{r\times n}(\mathbb{Z})$ such that $M = gN$ and N is a standard matrix of rank r.*

We try to remove every factor of d_M by multiplying M with a matrix whose inverse is in $\mathrm{GL}(r, \mathbb{Q}) \cap \mathcal{M}_{r\times r}(\mathbb{Z})$. Taking the resulting matrix at the end and applying the reverse process completes the proof.

Proof. If $d_M = 1$, then there is nothing to prove. Assume p is a prime factor of d_M and m is the biggest integer for which $p^m \mid d_M$. If p^k (for some positive k) divides every entry of the first row of M then multiply M with an $r \times r$ diagonal matrix $H = (h_{ij})$ with $h_{ii} = 1$ for $i > 1$ and $h_{11} = \frac{1}{p^k}$. It is obvious that $M^{(1)} = HM \in \mathcal{M}_{r\times n}(\mathbb{Z})$ and $d_M = p^k d_{M^{(1)}}$. If $k = m$ we have managed to remove p^m as it was promised. Now assume $k < m$ and let $M^{(1)} = (a_{ij})$. There is at least one non-zero entry in the first row of $M^{(1)}$ which is not divisible by p. Without loss of generality we can assume this entry is a_{11}. If a_{21} is non-zero and $\mathrm{hcf}(a_{11}, a_{21}) = a$, then there exist integers b and c such that $ba_{11} + ca_{21} = a$. Let H_1 be the following matrix:

$$\begin{pmatrix} b & c & 0 & \cdots & 0 \\ -\frac{a_{21}}{a} & \frac{a_{11}}{a} & & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

The matrix $M^{(2)} = H^1 M^{(1)}$ has the following shape

$$\begin{pmatrix} a & * & \cdots \\ 0 & * & \cdots \\ * & * & \\ \vdots & & \ddots \end{pmatrix}.$$

Obviously $\det(H^1) = 1$ and a is not divisible by p.

By repeating this process for all entries of the first column we can replace them by zero. Now let $M_1^{(2)}$ be the $(r-1) \times (n-1)$ sub-matrix of $M^{(2)}$ obtained by removing the first row and column. Obviously $\det(M^{(2)}) = a.\det(M_1^{(2)})$. This forces p^{m-k} to divide $\det(M_1^{(2)})$.

We can repeat the algorithm above and remove all powers of p from the first row of $M_1^{(2)}$. If there is any factor of p left, we apply the process above to the second column of the new matrix to make its entries equal to zero.

By repeating this algorithm we find a matrix M' for which $d_M = p^m \times d_{M'}$. All these can be done again for a prime factor of $d_{M'}$. After finitely many steps we will have a matrix N with $d_N = 1$. Note that termination of this process is assured by the fact that r is finite and $d_M \in \mathbb{N}$. $\square$

Corollary 3.1.9. *For any toric variety $TV(I, A)$, there exists a standard matrix B such that $TV(I, A) \cong TV(I, B)$.*

Proof. This follows from Lemma 3.1.4 and Lemma 3.1.8. $\square$

Corollary 3.1.10. *For any toric variety, there exists an isomorphic model with no generic stabiliser.*

Proposition 3.1.11. *Let $A \in \mathcal{M}_{r \times n}$ be a matrix of rank r and $TV(I, A)$ the toric variety as before. The following are equivalent.*

(i) $A \colon \mathbb{Z}^n \to \mathbb{Z}^r$ *is surjective.*

(ii) $\wedge^r A \colon \wedge^r \mathbb{Z}^n \to \wedge^r \mathbb{Z}^r \cong \mathbb{Z}$ *is surjective.*

(iii) A *is standard.*

Proof. *(i)* $\Leftrightarrow$ *(ii)* is easy to verify using elementary techniques of multi-linear algebra. The equivalence of *(ii)* and *(iii)* follows from our definition of standard and the fact that *(ii)* holds if and only if there are k integers α_i such that $\sum \alpha_i \det(A_i) = 1$. $\square$

Definition 3.1.12. Let A be an $r \times n$ matrix with integer entries. Suppose A_k is an $r \times (n-1)$ matrix obtained by removing the k-th column of A. The matrix A is called *well formed* if every A_k $(1 \leq k \leq n)$ is standard.

Lemma 3.1.13. *Let $TV(A, I)$ be a toric variety defined by an irrelevant ideal I and an $r \times n$ matrix $A = (a_{ij})$. Assume $q \neq 1$ is a positive integer such that $q \mid a_{1j}$ for $j > 1$ but $q \nmid a_{11}$. Define the matrix $B = (b_{ij})$ by $b_{i1} = q.a_{i1}$ and $b_{ij} = a_{ij}$ for $j > 1$. Then $TV(A, I) \cong TV(B, I)$.*

Proof. By Theorem 2.1.12, given the irrelevant ideal I, there exists an ample divisor D such that

$$T = \operatorname{Proj} R_D \quad ,$$

where D has degree $D = \sum \alpha_i C_i$, where $\alpha_i \in \mathrm{N}_0$ and C_i are the columns of A. Note that we associate D with its degree D and use the same notation for both.

The ring R_{qD} consists of invariant sections of multiples of qD, i.e.

$$R_{qD} = \left(\bigoplus_{j=0}^{\infty} H^0(\mathbb{C}^d, \mathcal{L}_{qD}^j) \right)^G \quad .$$

Let $x_1^{a_1} \ldots x_n^{a_n}$ be a monomial in R_{qD}. Hence there is a positive integer m such that

$$a_1 C_1 + \cdots + a_n C_n = mqD \quad .$$

In particular, $a_1 a_{11} = q.\alpha$ for some nonzero integer α as q divides all a_{1i} for all $j > 1$. This together with the assumption on $q \nmid a_{11}$ implies $q \mid a_1$. Therefore x_1 appears in R_{qD} only with power q. Hence

$$R_{qD} \cong \bigoplus_{j=0}^{\infty} H^0(\operatorname{Spec}(\mathbb{C}[x_1^q, x_2, \ldots, x_n]), \mathcal{L}_D^j)^G \quad .$$

On the other hand, we have

$$TV(B,I) \cong \operatorname{Proj} \bigoplus_{j=0}^{\infty} H^0(\operatorname{Spec}(\mathbb{C}[x_1^q, x_2, \ldots, x_n]), \mathcal{L}_D^j)^G \quad .$$

The algebraic isomorphism between graded rings R_D and R_{qD} finishes the proof. □

Proposition 3.1.14. *For any toric variety $TV(I, A)$ with standard matrix A there exists a well formed matrix B such that $TV(I, A) \cong TV(I, B)$.*

Proof. Assume A_k is not standard for some k. Without loss of generality we can assume $k = 1$. One can run the proof of Lemma 3.1.8 on A_1 to obtain the standard model. The process in the proof of Lemma 3.1.8 starts by factorising p^k from the first row of A_1. Lemma 3.1.13 allows one to multiply the first column of A by p^k and remove this factor from the first row of the new matrix and still get isomorphic varieties. One could complete the process of finding a standard model for A_1 and obtain A' as a standard model with $\det(A) = q.\det(A')$. Repeating our argument at each step of this process assures that replacing A by the matrix

$$A^{(q)} = \left(\begin{array}{c|c} q.a_{11} & \\ \vdots & A' \\ q.a_{r1} & \end{array} \right)$$

will lead to isomorphism $TV(A, i) \cong TV(A^{(q)}, I)$. Repeating this recipe for all non-standard A_k and their factors lead to the well formed matrix B with $TV(A, I) \cong TV(B, I)$.

□

Definition 3.1.15. A toric variety defined by $TV(I, A)$ is called *well formed* if its defining matrix A is well formed.

Theorem 3.1.16. *For any toric variety $TV(I, A)$ there exists a well formed model isomorphic to it.*

Proof. This is an obvious consequence of Corollary 3.1.9 and Proposition 3.1.14. □

Remark 3.1.17. The variety $TV(I, A)$, in general, is a toric Deligne-Mumford stack and if B is the standard matrix for which $TV(I, A) \cong TV(I, B)$, then $TV(I, B)$ is the coarse moduli space of $TV(I, A)$. See [FMN09] for a general theory on this. However, the way we treat well-formedness here is algorithmic and explicit.

So far we have introduced a constructive way for finding well formed models of a given toric variety. In practice it is sometimes easier to find such a model without going through all the steps of the process introduced above. The following is a typical example of such a case.

Example 3.1.18. The matrix in the right hand side is the well formed model of the other matrix, which can be obtained by adding the second row to the first row, i.e. act from the left by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \quad .$$

Then removing a factor of 2 from the first row.

$$\begin{pmatrix} -1 & -1 & -1 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 \end{pmatrix}$$

If we let the irrelevant ideal I be $I = (x_1, x_2, x_3) \cap (x_4, x_5)$, then the variety obtained from this matrix is the weighted blow up of $\mathbb{P}(1, 1, 1, 2)$ at the singularity $\frac{1}{2}(1, 1, 1)$.

3.2 Rank 2 toric varieties

In this section, we consider a special class of rank 2 toric varieties. The goal is to understand their singularities and various blow ups on them to prepare ourselves for studying some Mori fibre spaces, naturally embedded into such varieties. The study of birational geometry of these subvarieties will form the major parts in Chapters 4 and 6 of this thesis.

Definition 3.2.1. A *weighted bundle over* $\mathbb{P}^n$ is a rank 2 toric variety $\mathcal{F} = TV(A, I)$ defined by

(i) $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_m]$.

(ii) The irrelevant ideal of $\mathcal{F}$ is $I = (x_0, \ldots, x_n) \cap (y_0, \ldots, y_m)$.

(iii) and the $(\mathbb{C}^*)^2$ action on $\mathbb{C}^{n+m+2}$ is given by

$$A = \begin{pmatrix} 1 & \ldots & 1 & -\omega_0 & -\omega_1 & \ldots & -\omega_m \\ 0 & \ldots & 0 & 1 & a_1 & \ldots & a_m \end{pmatrix} \quad ,$$

where ω_i are non-negative integers and $\mathbb{P}(1, a_1, \ldots, a_m)$ is a weighted projective space.

The following lemma is an easy consequence of our assumptions.

Lemma 3.2.2. *The weighted bundle $\mathcal{F}$ defined in Definition 3.2.1 is well formed if and only if the weighted projective space $\mathbb{P}(1, a_1, \ldots, a_m)$ is well formed.*

Remark 3.2.3. Without loss of generality we assume that any weighted bundle $\mathcal{F}$ that appears in this thesis is well formed unless otherwise stated.

The following lemma constructs the fan associated to the weighted bundle in Definition 3.2.1.

Proposition 3.2.4. *Let $\beta_1, \ldots, \beta_m, \alpha_1, \ldots, \alpha_n$ be the standard basis of $\mathbb{Z}^{n+m}$. Suppose α_0 and β_0 are the following vectors in $\mathbb{Z}^{n+m}$.*

$$\beta_0 = -\sum_{i=1}^{m} a_i \beta_i \quad , \quad \alpha_0 = -\sum_{j=1}^{n} \alpha_j + \sum_{i=0}^{m} \omega_i \beta_i \quad ,$$

where ω_i are non-negative integers.

Let $\sigma_{rs} = \left\langle \beta_0, \ldots, \hat{\beta}_r, \ldots, \beta_m, \alpha_0, \ldots, \hat{\alpha}_s, \ldots, \alpha_n \right\rangle$ be the cone in $\mathbb{Z}^{n+m}$ generated by $\beta_0, \ldots, \hat{\beta}_r, \ldots, \beta_m, \alpha_0, \ldots, \hat{\alpha}_s, \ldots, \alpha_n$, where α_s and β_r are omitted. If we denote Σ for the *fan in* $\mathbb{Z}^{n+m}$ *generated by maximal cones* σ_{rs} *for all* $0 \leq r \leq n$ *and* $0 \leq s \leq m$*, then* $\mathcal{F} \cong TV(\Sigma)$ *.*

Proof. We compute the GIT construction of this fan following the recipe of Cox given in [Cox03] §10.

By assumption of the lemma, rays $\alpha_0, \dots, \alpha_n, \beta_0, \dots, \beta_m$ in $N = \mathbb{Z}^{m+n}$ form $\Delta(1)$, the set of 1-dimensional cones in Σ. Let us associate the variables $x_0, \dots, x_n, y_0, \dots, y_m$ to these rays. For a given maximal cone σ, define x^σ to be the product of all variables not coming from edges of σ. But maximal cones in Σ are exactly σ_{rs}, which immediately implies $x^{\sigma_{rs}} = x_s y_r$. The irrelevant ideal is given by

$$I = (x^\sigma \mid \sigma \in \Sigma \text{ is a maximal cone}) = (x_s y_r \mid 0 \leq s \leq n \text{ and } 0 \leq r \leq r) \quad .$$

It is clear that the primary decomposition of this ideal is $I = (x_0, \dots, x_n) \cap (y_0, \dots, y_m)$.

In order to describe the GIT construction of $TV(\Sigma)$ we must find the group G such that

$$TV(\Sigma \cong (\mathrm{Spec}[x_0, \dots, x_n, y_0, \dots, y_m] - V(I))/G \quad .$$

The group $G \subset (\mathbb{C}^*)^{m+n+2}$ is defined by

$$G = \{(\mu_0, \dots, \mu_n, \lambda_0, \dots, \lambda_m) \in (\mathbb{C}^*)^{m+n+2} \mid \prod_{i=0}^{n} \mu_i^{\langle e_k, \alpha_i \rangle} . \prod_{j=0}^{m} \lambda_j^{\langle e_k, \beta_j \rangle} = 1, \text{ for all } k\} \quad ,$$

where $e_1, \dots, e_{m+n}$ form the standard basis of $\mathbb{Z}^{m+n}$. But the standard basis of $\mathbb{Z}^{m+n}$, by assumption, is $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$.

Computing this set implies that $(\mu_0, \dots, \mu_n, \lambda_0, \dots, \lambda_m) \in G$ if and only if

$$\mu_i . \mu_0^{\langle \alpha_0, \alpha_i \rangle} . \lambda_0^{\langle \beta_0, \alpha_i \rangle} = 1 \quad \text{and} \quad \lambda_j . \mu_0^{\langle \alpha_0, \beta_j \rangle} . \lambda_0^{\langle \beta_0, \beta_j \rangle} = 1 \quad .$$

In other words, λ_0 and μ_0 determine all other λ_j and μ_i. Therefore the group G is isomorphic to $(\mathbb{C}^*)^2$ and the action on coordinate variables is defined by

$$((\mu,\lambda).x_0) = \mu x_0 \qquad ((\mu,\lambda).x_i) = \mu^{-\langle \alpha_0,\alpha_i\rangle}\lambda^{-\langle \beta_0,\alpha_i\rangle}x_i \quad ,$$

$$((\mu,\lambda).y_0) = \lambda y_0 \qquad ((\mu,\lambda).y_j) = \mu^{-\langle \alpha_0,\beta_j\rangle}\lambda^{-\langle \beta_0,\beta_j\rangle}y_j \quad .$$

In other words, $(\mathbb{C}^*)^2$ acts on $\mathbb{C}[x_0,\dots,x_n,y_0,\dots,y_m]$ by the matrix

$$A = \begin{pmatrix} 1 & \dots & 1 & 0 & \omega_0 a_1 - \omega_1 & \dots & \omega_0 a_m - \omega_m \\ 0 & \dots & 0 & 1 & a_1 & \dots & a_m \end{pmatrix} \quad .$$

We have shown so far that $TV(\Sigma) \cong TV(A,I)$. Multiplying A on the left by the matrix

$$\begin{pmatrix} 1 & -\omega_0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \quad ,$$

together with Lemma 3.1.4 proves that $\mathcal{F} \cong TV(\Sigma)$. □

In [Rei] Chapter 2, Reid gives a detailed analysis of rational scrolls, which, in our setting, are the weighted bundles over $\mathbb{P}^1$, with no weights! In fact they form the smooth cases of weighted bundles.

Let the rational scroll $\mathbb{F}(a_1,\dots,a_n)$ be defined by the action

$$\begin{pmatrix} 1 & 1 & 0 & -a_1 & \dots & -a_n \\ 0 & 0 & 1 & 1 & \dots & 1 \end{pmatrix}$$

on $\mathbb{C}[t_1,t_2,x_0,\dots,x_n]$ and irrelevant ideal $I = (t_1,t_2)\cap(x_0,\dots,x_n)$. Such scroll is smooth and is covered by $2(n+1)$ affine patches. Each of these patches can be computed by

$$\mathcal{U}_{ij} = \operatorname{Spec}\mathbb{C}[t_1,t_2,x_0,\dots,x_n,t_i^{-1},x_j^{-1}]^{\mathbb{C}^*\times\mathbb{C}^*} \quad .$$

Lemma 3.2.5. *The rational scroll $\mathbb{F}(a_1,\dots,a_n)$ is smooth.*

Proof. We only need to show that each of the affine patches covering $\mathbb{F}$ is smooth. Without loss of generality we can assume $i = 1$ and $a_1 \leq \cdots \leq a_n$. Computing the invariants shows that

$$\mathcal{U}_{1j} = \operatorname{Spec}\mathbb{C}[t_1, t_2, x_0, \ldots, x_n, t_1^{-1}, x_j^{-1}]^{\mathbb{C}^*\times\mathbb{C}^*} = \operatorname{Spec}\mathbb{C}[\frac{t_2}{t_1}, \frac{x_0}{x_j t_1^{a_j}}, \frac{x_1}{x_j t_1^{a_j - a_1}}, \ldots, \frac{x_n t_2^{a_n - a_j}}{x_j}] \quad .$$

In the ring in the right hand side above, each x_i appears only in one term and with power one, of course, except in the first term. The parameters t_1 and t_2, either in nominator or denominator, appear to fix the degree to be $(0,0)$. It is clear that the right hand side above is isomorphic to $\mathbb{C}^{n+1}$. □

Proposition 3.2.6. *A well formed weighted bundle $\mathcal{F}$, defined in Definition 3.2.1, is covered by $(n+1)(m+1)$ patches, each of them isomorphic to a quotient of $\mathbb{C}^{n+m}$ by a cyclic group $\mathbb{Z}_r$, for some positive integer r.*

Proof. Similar to Lemma 3.2.5, we construct the patches $\mathcal{U}_{ij}$ for $0 \leq i \leq n$ and $0 \leq j \leq m$. Note that in the toric level, $\mathcal{U}_{ij} = (x_i y_j \neq 0)$ corresponds to the maximal cone σ_{ij} as in Proposition 3.2.4.

$$\mathcal{U}_{ij} = \operatorname{Spec}\mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_m, x_i^{-1}, y_j^{-1}]^{\mathbb{C}^*\times\mathbb{C}^*}$$

Computing the invariants gives

$$\mathcal{U}_{ij} = \operatorname{Spec}\mathbb{C}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{y_0^{a_j}}{y_j}.x_i^{\omega_0 a_j - \omega_j}, \ldots] \quad .$$

Again powers of x_i appear to make each term invariant under the action of the first coordinate of $(\mathbb{C}^*)^2$ and each y_k comes with a power that is the first number which is 0 modulo a_j. In other words, these invariants are exactly the same as those for the action of $\mathbb{Z}_{a_j}$ on $\mathbb{C}[x_0, \ldots, \hat{x}_i, \ldots, x_n, y_0, \ldots, \hat{y}_j, \ldots, y_m]$ by $\frac{1}{a_j}(0, \ldots, 0, 1, a_1, \ldots, a_n)$. □

3.3 Blow ups of rank 2 toric varieties

In this section we construct special toric varieties with rank 3, obtained by blowing up some centres in a rank 2 toric variety. Then we try to understand the nature of the maps from these varieties to the rank 2 ones. Obviously such a map in the level of fans is obtained by only adding an extra ray, together with the appropriate cone subdivision. Our ideas will be illustrated with many examples, either in this chapter or later in Chapter 5, when we apply these methods to realise stable models of some Mori fibrations.

We start by an example, which shows how the second Hirzebruch surface is obtained as the blow up of the $\frac{1}{2}(1,1)$ point on $\mathbb{P}(1,1,2)$ in our setting. Essentially everything is known and easy.

Example 3.3.1 (Motivation). Consider the second Hirzebruch surface $\mathcal{F}(2)$ defined by the quotient

$$\mathcal{F}(2) = (\operatorname{Spec}\operatorname{Cox}(\mathcal{F}(2)) - V(I))/(\mathbb{C}^*)^2 \qquad ,$$

where

(i) $\operatorname{Cox}(\mathcal{F}(2)) = \mathbb{C}[u,v,x,y]$,

(ii) the irrelevant ideal is $I = (u,v) \cap (x,y)$ and

(iii) the action of $(\mathbb{C}^*)^2$ is

$$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix} .$$

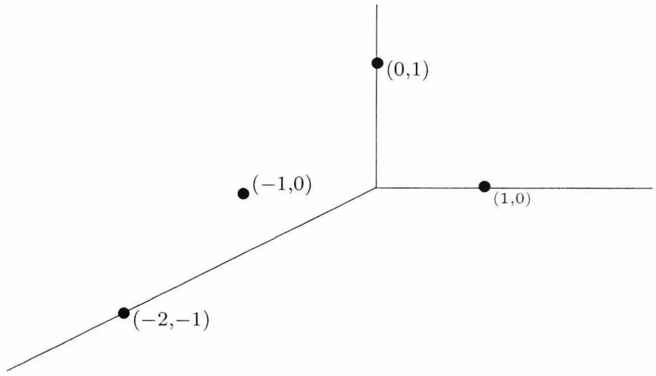
Let the surface $\mathbb{P} = \mathbb{P}(1,1,2)$ be defined, in a rather unusual way, as a rank 1 toric variety with

(i) $\operatorname{Cox}(\mathbb{P}) = \mathbb{C}[a,b,c]$,

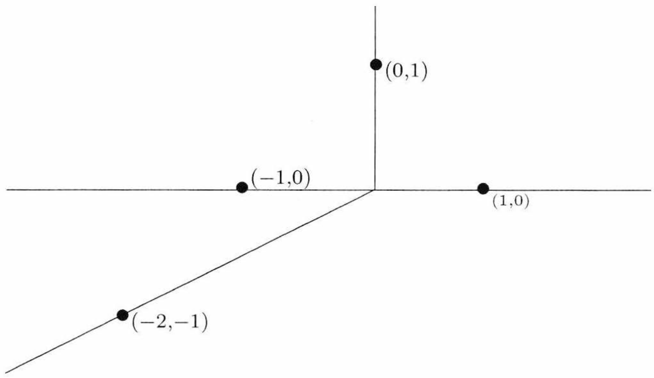
(ii) the irrelevant ideal $J = (a,b,c)$ and

(iii) the action of $\mathbb{C}^*$ by $(1,1,2)$.

The fan of $\mathbb{P}$ in $N = \mathbb{Z}^2$ is:



The singularity at the $\frac{1}{2}(1,1)$ corresponds to the lattice point $(-1,0)$ in the fan above. This singularity can be removed by adding the ray through this point in the lattice and doing the subdivision. The new fan will be:



By Proposition 3.2.4, this fan is the fan of $\mathcal{F}(2)$.

In fact, one can consider this blow up map without going through the fans and just by looking at the algebras

$$\varphi\colon \operatorname{Cox}(\mathbb{P}) = \mathbb{C}[a,b,c] \to \operatorname{Cox}(\mathcal{F}(2)) = \mathbb{C}[u,v,x,y] \qquad ,$$

defined by

$$a \mapsto uy^{\frac{1}{2}} \qquad b \mapsto vy^{\frac{1}{2}} \qquad c \mapsto x \qquad .$$

This expression makes sense by Theorem 1.1 in [BB10]. This, of course, corresponds to the ray structure of the fan, but says nothing about the subdivision of the old cone into new ones. In other words, this map gives an expression between the Cox covers, which is compatible with the maps between the quotients and leaves no information on how the irrelevant ideal is changing. Another way of viewing this algebraic expression is by taking the $\mathbb{Q}$-divisor D to be $\frac{1}{2}(x=0)$ and computing the ring

$$R_D = \bigoplus_{m\geq 0} \mathrm{H}^0(\mathrm{Spec}(\mathrm{Cox}(\mathcal{F})), \mathcal{L}_D^{\otimes m})^{(\mathbb{C}^*)^2} \qquad .$$

Computing this ring shows $R_D \cong \mathrm{Cox}(\mathbb{P})$.

The aim is to consider similar maps for blow ups of the weighted bundle $\mathcal{F}$ over $\mathbb{P}^1$, and analyse the changes to the irrelevant ideal through the blow ups. We have shown in Proposition 3.2.6 that each germ $p_{rs} \in \mathcal{U}_{rs}$ defined by coordinates $x_i = y_j = 0$, for all $i \neq r$ and $j \neq s$, has a cyclic quotient singularity of type $\frac{1}{a_j}(0, 1, \ldots, a_m)$. Of course this singularity is not isolated and one would attempt to resolve this by blowing up the whole singular locus. But this is not our aim! Instead we blow up **a closed point**, which is exactly the point that corresponds to the origin at this germ. The reason for doing this blow up is that, later in Chapter 5, we want to consider the blow up of some subvarieties of $\mathcal{F}$ only at this particular point. We do this by considering the blow up of the ambient space at this point and restrict our attention to the subvariety under this blow up. This will be illustrated by an example, but before that let us see some basic facts about the nature of these blow up maps. Once again, note that our point of view does not concern the maps between the Cox rings as [BB10] already takes care of that part. We are interested in seeing what happens to the irrelevant ideal after the blow up.

Fix $k \in \{0, \ldots, m\}$ and let T be a rank 3 toric variety defined by

(i) $\mathrm{Cox}(T) = \mathbb{C}[X_0, X_1, Y_0, \ldots, Y_m, \xi]$,

(ii) the irrelevant ideal

$$J = (X_0, X_1) \cap (Y_0, \dots, Y_m) \cap (\xi, X_1) \cap (\xi, Y_k) \cap (X_0, Y_0, \dots, \hat{Y_k}, \dots, Y_m) \quad \text{and}$$

(iii) the action of $(\mathbb{C}^*)^3$ given by the matrix

$$\begin{pmatrix} 1 & 1 & -\omega_0 & -\omega_1 & \dots & -\omega_{k-i} & -\omega_k & -\omega_{k+1} & \dots & -\omega_m & 0 \\ 0 & 0 & 1 & a_1 & \dots & a_{k-1} & a_k & a_{k+1} & \dots & a_m & 0 \\ b_k & 0 & b_0 & b_1 & \dots & b_{k-1} & 0 & b_{k+1} & \dots & b_m & -a_k \end{pmatrix} \quad ,$$

where $b_0, \dots, b_m$ are strictly positive integers such that

$$b_i \equiv a_i \mod a_k \text{ for } i \neq k \qquad \text{and} \qquad b_k = r a_k \text{ for some positive integer } r \quad .$$

Proposition 3.3.2. *The rank 3 toric variety T constructed above is the blow up of the weighted bundle $\mathcal{F}$ over $\mathbb{P}^1$ in Definition 3.2.1 at the point* $(0:1;0:\dots:0:1)$.

Proof. By Proposition 3.2.4, the fan associated to $\mathcal{F}$ consists of 1-dimensional cones β_0, β_1 and $\alpha_0 \dots, \alpha_m$in $N = \mathbb{Z}^{m+1}$ with $2m+2$ maximal cones

$$\sigma_{0i} = \langle \beta_1, \alpha_0 \dots, \hat{\alpha_i}, \dots, \alpha_m \rangle \text{ and } \sigma_{1j} = \langle \beta_0, \alpha_0 \dots, \hat{\alpha_j}, \dots, \alpha_m \rangle \text{ for } 0 \leq i, j \leq m \quad .$$

The last row of the defining matrix of T is clearly adding a new ray in the cone σ_{0k}. The fact that the generator of this ray is an integral vector in N is guaranteed by the conditions imposed on b_i. This implies that T is the blow up of F at a point if it has the correct irrelevant ideal. We complete the proof by showing the irrelevant ideal of the Cox ring of this toric blow up is precisely the ideal of T. This is done by taking the subdivision of σ_{0k} and computing the irrelevant ideal of the new fan using the method of [Cox03], as in the proof of Proposition 3.2.4. The fan of this blow up of Σ consists of rays $\beta_0, \beta_1, \alpha_0, \dots, \alpha_m, \gamma$ and maximal cones

$$\sigma'_{ki} = \langle \beta_0, \alpha_0 \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_k, \dots, \alpha_m, \gamma \rangle \text{ for } i \neq k \text{ and } \sigma'_k = \langle \alpha_0 \dots, \hat{\alpha}_k, \dots, \alpha_m, \gamma \rangle$$

coming from the subdivision of σ_{0k} together with the remaining cones σ_{ij}. If we associate the new variable ξ to the ray γ and $X_0, X1$ to β_0, β_1 and Y_i to α_i, then the irrelevant ideal of this toric variety is the ideal generated by

$$X_1 \cdot Y_i \cdot Y_k \text{ for all } i \neq k, \quad X_0 \cdot X_1 \cdot Y_k, \quad X_1 \cdot Y_i \cdot \xi \text{ for all } i \neq k, \quad X_0 \cdot Y_i \cdot \xi \text{ for all } i \quad .$$

The primary decomposition of this ideal is the irrelevant ideal of T. □

3.3.1 Some invariants

In order to use the weighted bundles in the rest of this thesis, it is essential to compute the Picard number and the canonical class of these varieties. These are explored in this subsection and we do not claim to have done anything original on this matter.

Applying the following theorem of Goto and Watanabe computes the canonical class of a toric variety in terms of the degree of generators of its Cox ring.

Theorem 3.3.3 ([GW78] Corollary 2.2.6.). *Suppose that $R = k[x_1, \dots, x_n]$ is a $\mathbb{Z}^n$-graded polynomial ring and assume that each x_i is a homogeneous element of R. Then $K_R = R(-e)$, where $e = \sum \deg x_i$.*

Corollary 3.3.4. *Let $T = TV(A, I)$ be a toric variety. Then the degree of the anticanonical class of T is the vector which is obtained by adding up the columns of the matrix A.*

Remark 3.3.5. *Of course someone with a bit of experience in toric geometry can notice that this formula is exactly the same as the usual way of computing the anticanonical class of a toric divisor by summing up the torus invariant divisor. In fact, these divisors correspond to the columns of the matrix A, hence everything is obvious.*

The recipe to compute the Picard number of a toric variety in our case follows the following theorem.

Theorem 3.3.6 ([Ful93] §3.4.). *Let T be a toric variety obtaind by a simplicial fan $\Sigma \subset \mathbb{Z}^n$. If the number of 1-dimensional cones in σ is d, then the Picard number of T is $\rho_T = d - n$.*

Corollary 3.3.7. *The Picard number of a rank r toric variety is r.*

Chapter 4

Degree 2 del Pezzo surface fibrations over $\mathbb{P}^1$

In this chapter, we construct families of degree 2 del Pezzo surface fibrations over $\mathbb{P}^1$ embedded in rank 2 toric varieties. Then we study their birational geometry in terms of Sarkisov links by looking at the geometry of the ambient toric variety. A complete list of nonrigid families is presented in Table 4.1 for those with a 2-ray link obtained by restriction of the 2-ray game of the ambient space.

4.1 Hypersurfaces in weighted bundles

Before starting our calculations, we recall the definition of weighted bundles from Chapter 3.

Definition 4.1.1. A *weighted bundle over* $\mathbb{P}^n$ is a rank 2 toric variety $\mathcal{F} = TV(A, I)$ defined by

(i) $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_m]$.

(ii) The irrelevant ideal of $\mathcal{F}$ is $I = (x_0, \dots, x_n) \cap (y_0, \dots, y_m)$.

(iii) and the $(\mathbb{C}^*)^2$ action on $\mathbb{C}^{n+m+2}$ is given by

$$A = \begin{pmatrix} 1 & \dots & 1 & -\omega_0 & -\omega_1 & \dots & -\omega_m \\ 0 & \dots & 0 & 1 & a_1 & \dots & a_m \end{pmatrix},$$

where ω_i are non-negative integers and $\mathbb{P}(1, a_1, \dots, a_m)$ is a weighted projective space.

Definition 4.1.2. (a) Let T be a rank 2 toric variety. Suppose t is a generating variable in the Cox ring of T and that the action of the $(\mathbb{C}^*)^2$ on t is given by $t \mapsto \lambda^a \mu^b t$, where $(a, b) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. We say that the number $\frac{a}{b}$ is the *ratio weight* of the variable t. Note that the ratio weight could be a rational number, $\infty = \frac{|a|}{0}$ or $-\infty = \frac{-|a|}{0}$.

(a) Let T be a rank 2 toric variety with $\mathrm{Cox}(T) = \mathbb{C}[t_1, \dots, t_k]$. Define a total order $\preceq$ on $\{t_0, \dots, t_k\}$ by $t_i \preceq t_j$ if and only if the ratio weight of t_j is less than or equal to the ratio weight of t_i. Note that we allow $-\infty$ and ∞ in their own right. If the ratio weight of t_i is strictly bigger than the one for t_j, we write $t_i \prec t_j$.

Remark 4.1.3. Note that the order $\preceq$ above is induced by the usual order in the set of extended real numbers in the **reverse** direction.

Without loss of generality we can assume the variables of the $\mathrm{Cox}(\mathcal{F})$ in Definition 4.1.1 are in order with respect to $\preceq$. Let $Y_0, \dots, Y_r$ be the partition of $y_0, \dots, y_m$ such that variables contained in each Y_i have the same ratio weight and that Y_i is nonempty and contains all variables with that ratio weight. Furthermore we assume that they are in order with $Y_i \prec Y_{i+1}$, meaning the ratio weight of the variable in Y_i is strictly bigger than the ratio weight of variables in Y_{i+1}. Note that this last condition makes $Y_0, \dots, Y_r$ a unique partition of $y_0, \dots, y_m$.

Consider the ideal $I_j = (x_0, \dots, x_n, Y_0, \dots, Y_{j-1}) \cap (Y_j, \dots, Y_r) \subset \mathrm{Cox}(\mathcal{F})$. Let $\mathcal{F}_j$ be the rank two toric variety defined by $TV(A, I_j)$, i.e.

$$\mathcal{F}_j = (\mathbb{C}^{n+m+2} \setminus V(I_j)) /\!\!/ (\mathbb{C}^*)^2$$

in particular $\mathcal{F}_0 = \mathcal{F}$. The following theorem can be derived directly from [BZ10], Theorem 4.1.

Theorem 4.1.4 ([BZ10], Theorem 4.1). *Let $\mathcal{F}/\mathbb{P}^n$ be a weighted bundle as before. Then the 2-ray link of $\mathcal{F}$ is given by one of the following:*

(1) If $|Y_r| = 1$, i.e. the set Y_r has only one element, then

$$\begin{tikzcd} & \mathcal{F}_0 \arrow[r, "\Psi_1"] \arrow[dl, "\Phi"'] & \mathcal{F}_1 \arrow[r, dashed, "\Psi_2"] & \cdots \arrow[r, "\Psi_{r-1}"] & \mathcal{F}_{r-1} \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1 & & & & & \mathcal{F}_r \end{tikzcd}$$

where $\mathcal{F}_0 = \mathcal{F}$, Ψ_i are isomorphisms in codimension one and Φ' is a divisorial contraction.

(2) If $|Y_r| > 1$, then

$$\begin{tikzcd} & \mathcal{F}_0 \arrow[r, "\Psi_1"] \arrow[dl, "\Phi"'] & \mathcal{F}_1 \arrow[r, dashed, "\Psi_2"] & \cdots \arrow[r, "\Psi_r"] & \mathcal{F}_r \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1 & & & & & \mathbb{P} \end{tikzcd}$$

where $\mathcal{F}_0 = \mathcal{F}$, Ψ_i are isomorphisms in codimension one, Φ' is a fibration and $\mathbb{P} = \mathbb{P}(a_{r_1}, \ldots, a_{r_k})$, where $a_{r_1}, \ldots, a_{r_k}$ are the denominators of the ratio weights of the variables in Y_r.

Definition 4.1.5. Let $\mathcal{F}/\mathbb{P}^n$ be a weighted bundle as in Definition 4.1.1, and $\mathcal{F}_i$ be the varieties appearing in its 2-ray link of Theorem 4.1.4. Let $\overline{X}\colon (f = 0) \subset \mathbb{C}^{n+m+2}$ be a hypersurface in $\mathbb{C}^{n+m+2}$, the Cox cover of $\mathcal{F}$, defined by $f \in \mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_m]$. Assume f is irreducible, reduced and homogeneous with respect to the action of $(\mathbb{C}^*)^2$. Define $X_i \subset \mathcal{F}_i$ to be

$$X_i = (\overline{X} \backslash V(I_i))/(\mathbb{C}^*)^2$$

and let ψ_i (respectively φ, φ') be the restriction of Ψ_i (respectively Φ, Φ') to X_{i-1}. Then we say X_0 has an *$\mathcal{F}$-link* if

(i) ψ_i are isomorphisms in codimension one (possibly isomorphisms).

(ii) φ and φ' are extremal contractions (see Definition 2.2.5).

In other words, X_0 has an $\mathcal{F}$-link if the 2-ray game of X_0 is obtained by the restriction of the 2-ray game of $\mathcal{F}_0$ (although some φ_i may be isomorphisms and hence redundant from the game). If in addition, each X_i is $\mathbb{Q}$-factorial with terminal singularities, then we say X_0 has an *$\mathcal{F}$-Sarkisov link*.

4.2 Sarkisov links from general $dP_2/\mathbb{P}^1$ hypersurfaces

For the rest of this chapter, we consider weighted bundles over $\mathbb{P}^1$ with fibre $\mathbb{P}(1,1,1,2)$; these are a natural place to embed 3-fold degree 2 del Pezzo fibrations.

Definition 4.2.1. A 3-fold X is a *degree 2 del Pezzo fibration over* $\mathbb{P}^1$ (denoted by dP_2 fibration, or simply $dP_2/\mathbb{P}^1$) if X has an extremal contraction of fibre type $\varphi\colon X \to \mathbb{P}^1$ such that

(a) X has at worst terminal singularities and is $\mathbb{Q}$-factorial.

(b) The nonsingular fibres of φ are del Pezzo surfaces of degree two.

Let $\mathcal{F}$ be a rank two toric variety defined by $\mathcal{F} = TV(I, A)$, where $I \subset \mathbb{C}[u, v, x, y, z, t]$ is the irrelevant ideal $I = (u, v) \cap (x, y, z, t)$ and A is the representing matrix of the action of $\mathbb{C}^* \times \mathbb{C}^*$ given by

$$A = \begin{pmatrix} 1 & 1 & -\alpha & -\beta & -\gamma & -\delta \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} . \tag{4.1}$$

Remark 4.2.2. Up to the action of $\mathrm{SL}(2, \mathbb{Z})$, any matrix of type (4.1) can be written uniquely in one of the following forms:

$$(i) \qquad A = \begin{pmatrix} 1 & 1 & 0 & -a & -b & -c \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad 0 < c,\, 0 \le a \le b$$

$$(ii) \qquad A = \begin{pmatrix} 1 & 1 & -a & -b & -c & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad 0 \le a \le b \le c$$

$$(iii) \qquad A = \begin{pmatrix} 1 & 1 & -a & -b & -c & -1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad 0 < a \le b \le c \quad .$$

By Proposition 3.3.7, the Picard group of $\mathcal{F}$ is isomorphic to $\mathbb{Z}^2$. Let L and M be Weil divisors of $\mathcal{F}$ with weights $(1,0)$ and $(0,1)$. For example in the case (i) above, L has section $u \in H^0(\mathcal{F}, L)$ and M has $x \in H^0(\mathcal{F}, M)$. A simple toric singularity analysis, following Proposition 3.2.6, shows that $\mathcal{F}$ is smooth away from the curve $\Gamma_t = (x = y = z = 0)$. The curve Γ_t is a rational curve with singularity of transverse type $\frac{1}{2}(1,1,1)$ along it.

Let $D = 4M - eL \in \mathrm{Div}(\mathcal{F})$ be a divisor in $\mathcal{F}$ and $X = (f = 0) \subset \mathcal{F}$ be the hypersurface of $\mathcal{F}$ defined by a general $f \in H^0(\mathcal{F}, D)$. We say that $X \subset \mathcal{F}$ has bi-degree $(-e, 4)$; this is denoted by

$$\begin{pmatrix} -e \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & -\alpha & -\beta & -\gamma & -\delta \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \quad .$$

The goal is to find conditions on X and $\mathcal{F}$ such that X is a Mori fibre space, whose generic fibre is a del Pezzo surface of degree 2, that has an $\mathcal{F}$-Sarkisov link to another Mori fibre space.

4.2.1 The main result

Theorem 4.2.3. *Consider a hypersurface $X \subset \mathcal{F}$ with*

$$\begin{pmatrix} -e \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & -\alpha & -\beta & -\gamma & -\delta \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix},$$

where the weights α, β, γ are normalised with $\gamma \geq \beta \geq \alpha \geq 0$ and $\delta \geq 0$. Suppose the Type III *or* IV *2-ray game of $\mathcal{F}$ restricts to a Sarkisov link for X. Then the weights $\alpha, \beta, \gamma, \delta, e$ are among those appearing in the left-hand column of Table 4.1.*

Moreover, if X is a general hypersurface of type $(\alpha, \beta, \gamma, \delta; e)$ from table 4.1, then X is nonrigid and a Sarkisov link to another Mori fibre space is described in the remaining columns of Table 4.1, as explained in §4.3.1 below.

No.	$(\alpha, \beta, \gamma, \delta; e)$	ψ_1	ψ_2	φ'	new model
1	$(0,0,0,0;-1)$	n/a	n/a	contraction	$\mathbb{P}(1,1,1,2)$
2	$(0,0,0,1;0)$	n/a	n/a	contraction	$Y_4 \subset \mathbb{P}(1,1,1,2,2)$
3	$(0,0,1,0;0)$	n/a	n/a	contraction to a line	$Y_4 \subset \mathbb{P}(1,1,1,1,2)$
4	$(0,1,1,0;0)$	flop of $2 \times \mathbb{P}^1$	n/a	fibration	dP_2 fibration
5	$(0,0,1,1;0)$	flop of 4 disjoint $\mathbb{P}^1$	n/a	divisorial contraction to a point	$Y_4 \subset \mathbb{P}^4$
6	$(1,1,1,1;2)$	$\cong$	n/a	fibration	conic bundle with discriminant $\Delta_8 \subset \mathbb{P}^2$
7	$(0,1,1,1;1)$	flop	flip	fibration	dP_3 fibration
8	$(0,1,1,2;2)$	flop	n/a	fibration	conic bundle over $\mathbb{P}(1,1,2)$ with disc. $\Delta_{10} \subset \mathbb{P}(1,1,2)$
9	$(0,1,2,1;2)$	flop	$\cong$	contraction	$Y_6 \subset \mathbb{P}(1,1,1,2,3)$
10	$(0,1,1,3;3)$	flop	n/a	contraction	$Y_6 \subset \mathbb{P}(1,1,2,2,3)$
11	$(0,2,2,1;2)$	anti-flip	$\cong$	fibration	dP_1 fibration
12	$(0,1,2,3;3)$	anti-flip	flop	contraction	$Y_5 \subset \mathbb{P}(1,1,1,1,2)$
13	$(0,1,2,4;4)$	anti-flip	$\cong$	fibration	dP_2 fibration over $\mathbb{P}(1,2)$

Table 4.1: Data of Type III and IV links from general degree 2 del Pezzo hypersurface fibrations

4.3 General hypersurfaces

In this section, we prove the constructive part, the second part, of the Theorem 4.2.3 in one direction by explicitly describing the birational link for a general hypersurface in each family in Theorem 4.2.3 and then we show in subsection 4.3.3 that these hypersurfaces are indeed $dP_2/\mathbb{P}^1$. These links are provided from the restriction of the natural 2-ray game of the ambient toric variety $\mathcal{F}$ to X.

4.3.1 Geometry of the links

In order to be consistent with the notation of Theorem 4.1.4, in each case we rewrite the defining numerical system, normalised by the order $\preceq$, and give the numerical system of the rank 2 variety at the end of each link. However, the order of cases below does not follow the order in Table 4.1. We found it more convenient to analyse cases together, when they have similar structures at the end of their links.

Links to conic bundles

Family 6. $u = v \prec t \prec x = y = z$

$$\begin{pmatrix} -2 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 \\ 0 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -2 & -2 & 0 & 1 & 1 & 1 \end{pmatrix}$$

The 2-ray game of $\mathcal{F}$ starts by Ψ_1, which is a flip of type $(2, 2, -1, -1, -1)$ in the neighbourhood $(t \neq 1)$ of the flipping curve $\mathbb{P}^1_{u:v}$. The second and final step of the 2-ray game is a $\mathbb{P}^2$ fibration to $\mathbb{P}^2_{x:y:z}$. Considering X of bi-degree $(-2, 4)$, the Newton polygon of X is

deg of u, v coefficient	
0	t^2
1	$tx^2 \quad txy \quad \ldots \quad tyz \quad tz^2$
2	$x^4 \quad x^3y \quad \ldots \quad yz^3 \quad z^4$.

This means that f, the defining polynomial of X, includes terms of the form t^2 and $l(u,v)tx^2$ and $q(u,v)x^4$, where $l(u,v)$ is a general linear form in u,v and $q(u,v)$ is a general quadratic. It is also useful for us to describe f as the product of the following matrices:

$$\begin{pmatrix} u & v & t \end{pmatrix} \begin{pmatrix} *_4 & *_4 & *_2 \\ *_4 & *_4 & *_2 \\ *_2 & *_2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ t \end{pmatrix} , \tag{4.2}$$

where by $*_k$ we mean a general homogeneous polynomial of degree k in variables x, y, z.

Having the monomial $t^2 \in f$ ensures that X does not intersect with the singular locus of $\mathcal{F}$ as $\mathrm{Sing}(\mathcal{F}) = \Gamma_t$. Having this key monomial also shows that ψ_1, the restriction of Ψ_1 to X, is an isomorphism on X. The restriction of Φ' to X defines a fibration to $\mathbb{P}^2_{x:y:z}$ with fibres being conic curves. The discriminant of this conic is the determinant of the 3×3 matrix in (4.2). The degree of the discriminant in this case is 8.

Family 8. $u = v \prec x \prec y = z = t$

$$\begin{pmatrix} -2 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 \\ 2 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 2 \end{pmatrix}$$

Let us describe the birational geometry of the ambient space $\mathcal{F}$. The 2-ray game of $\mathcal{F}$ starts by mapping to $\mathbb{P}^1$ in one side (the given extremal contraction) and anti-flip $(1,1,-1,-1,-2)$ in the other side. This anti-flip can be read by fixing the action of the second component of the $(\mathbb{C}^*)^2$ in the neighbourhood $(x \neq 0)$ by putting $x = 1$. Then the game follows by an extremal contraction of fibre type to $\mathbb{P}(1,1,2)$. To restrict this toric 2-ray game to X, we need to know f, the defining polynomial of

X, which can be seen from the Newton polygon of X,

deg of u, v coefficient	
0	$x^2t \quad xy^2 \quad xyz \quad xz^2$
1	$xyt \quad xzt \quad xy^3 \quad xy^2z \quad xyz^2 \quad xz^3$
2	$y^2t \quad yzt \quad z^2t \quad t^2$.

Here our essential terms in f are x^2t and $q(u,v)t^2$, where $q(u,v)$ is a general quadratic in u, v. Having $q(u,v)t^2 \in f$ means that the singular locus of (a general quasismooth) X is the intersection of X with Γ_t, which in this case is only two points $(q = 0) \cap \Gamma_t$.

The $\mathcal{F}$-Sarkisov link of a general X in this family, starts by an Atiyah flop and follows by a fibration to $\mathbb{P}(1,1,2)$ with conic curve fibres. The flop is the restriction of the $(1,1,-1,-1,-2)$ anti-flip on $\mathcal{F}$. The restriction is a flop because the monomial $x^2t \in f$ allows us to eliminate the variable t in the neighbourhood $(x \neq 0)$.

Similar to the previous case, considering the defining polynomial of X in the form

$$\begin{pmatrix} u & v & t \end{pmatrix} \begin{pmatrix} *_4 & *_4 & *_3 \\ *_4 & *_4 & *_3 \\ *_3 & *_3 & *_2 \end{pmatrix} \begin{pmatrix} u \\ v \\ t \end{pmatrix} \tag{4.3}$$

tells us that the degree of the discriminant of the conic in this case is 10.

Remark 4.3.1. In [MP08], a list of possible singularities that the base variety of a conic bundle can admit is provided. By Theorem 1.2.7. in [MP08], $\mathbb{P}(1,1,2)$ is a legal base since it has only a quotient singularity $\frac{1}{2}(1,1)$, which is Du Val.

Links to del Pezzo fibrations

Family 4. $u = v \prec x = t \prec y = z$

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix} \leftrightsquigarrow \begin{pmatrix} 4 \\ 0 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The 2-ray game of $\mathcal{F}$ in this case is represented by

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi_1^-"] & & \mathcal{F}_1 \arrow[dl, "\Psi_1^+"'] \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1_{u:v} & & \mathcal{G} & & \mathbb{P}^1_{y:z} \end{tikzcd},$$

where the composition map $\Psi_1 = (\Psi_1^+)^{-1} \circ \Psi_1^-$, is a toric 4-fold flop. Both Ψ_1^- and Ψ_1^+ are isomorphism away from $\mathbb{P}^1 \times \mathbb{P}^1$. The first map, Ψ_1^-, contracts the surface $\mathbb{P}^1_{u:v} \times \mathbb{P}_{x:t}(1,2)$ to $\mathbb{P}^1_{x:t}$ and Ψ_1^+ contracts $\mathbb{P}^1_{y:z} \times \mathbb{P}^1_{x:t}$ to the same line. This composition defines Ψ_1 as a toric 4-fold flop. The next step of the 2-ray game, Φ' provides a fibration to $\mathbb{P}^1_{y:z}$ with fibres isomorphic to $\mathbb{P}(1,1,1,2)$.

The defining equation of X has the form $f = g + h$, where $g = g(x,t)$ is a quartic in variables x and t only. This ensures that the restriction of Ψ_1^- contracts two disjoint $\mathbb{P}^1$, defined by $(g = 0) \cap \mathbb{P}^1_{u:v} \times \mathbb{P}_{x:t}(1,2)$ to two points in $\mathbb{P}^1_{x:t}$, namely the solutions of $(g = 0) \subset \mathbb{P}(1,2)$. This argument shows that ψ_1 is formed of a flop $\psi_1 \colon X \to X_1$, which flops two disjoint copies of $\mathbb{P}^1$. At the end of the link, the restriction of Φ' to X_1 provides the extremal contraction of fibre type to $\mathbb{P}^1$ with degree 2 del Pezzo fibres.

Family 7. $u = v \prec x \prec t \prec y = z$

$$\begin{pmatrix} -1 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix} \leftrightsquigarrow \begin{pmatrix} 3 \\ -2 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -2 & -2 & -1 & 0 & 1 & 1 \end{pmatrix}$$

This case is similar to the previous one and the result was already found in [BCZ04]. A full analysis is given in [BCZ04] Family 5, §4.4.2..

Family 11. $u = v \prec x \prec t \prec y = z$

$$\begin{pmatrix} -2 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 6 \\ -2 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 2 & 3 & 0 & 0 \\ -1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

The diagram of the 2-ray game of $\mathcal{F}$ is

$$\begin{tikzcd} & \mathcal{F} \arrow[r, dashed, "\Psi_1"] \arrow[dl, "\Phi"'] & \mathcal{F}_1 \arrow[r, dashed, "\Psi_2"] & \mathcal{F}_2 \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1_{u:v} & & & & \mathbb{P}^1_{y:z} \end{tikzcd},$$

where Ψ_1 is the anti-flip $(1, 1, -1, -2, -2)$ flipping a copy of $\mathbb{P}^1$ to $\mathbb{P}(1, 2, 2)$. In particular, the flipping locus of $\mathcal{F}_1$ has line of singularity of transverse type $\frac{1}{2}(1, 1, 1)$. Note that $\mathcal{F}$ contains a singular line Γ_t, which is preserved by Ψ_1. The second anti-flip Ψ_2, is of type $(2, 2, 1, -3, -3)$, which flips a surface $\mathbb{P}(1, 2, 2)$ (including Γ_t) to a singular curve of transverse type $\frac{1}{3}(1, 2, 2)$. $\Phi' \colon \mathcal{F}_2 \to \mathbb{P}^1$ is a fibration, with $\mathbb{P}(1, 1, 2, 3)$ fibres.

Now we consider the restriction of this game to X. The essential monomials of the defining polynomial of X are t^2 and x^3y. The first monomial, t^2 shows that $\Gamma_t \cap X$ is empty for a general X. In fact, Bertini Theorem implies that X is smooth as the base locus of the linear system D includes only the curve $\Gamma_x = (u_0 : v_0; 1 : 0 : 0 : 0)$, which is guaranteed to be smooth by $x^3y \in f$.

The restriction of Ψ_1 to X is a Francia anti-flip as we can eliminate the variable y in a neighbourhood of the flipping curve using x^3y and implicit function theorem. Note that the variety X_1 has a $\frac{1}{2}(1, 1, 1)$ singularity obtained by this anti-flip. The restriction of Ψ_2 to X_1 is an isomorphism as $t^2 \in f$. And finally, $\varphi' \colon X_1 \to \mathbb{P}^1$ is a Mori fibre space with generic fibre isomorphic to a del Pezzo surface of degree 1.

Family 13. $u = v \prec x \prec y \prec z = t$

$$\begin{pmatrix} -4 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -4 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 4 \\ 0 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 2 \end{pmatrix}$$

A similar argument shows that the general X in this case, after a Francia anti-flip has an extremal contraction of fibre type to $\mathbb{P}(1,2)$, with generic fibre isomorphic to a degree 2 del Pezzo surface.

Links to Fano 3-folds

Family 1. $u = v \prec x = y = z = t$

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

The defining polynomial of a general X in this case is of the form $uf_4(x,y,z,t) = vg_4(x,y,z,t)$, for general degree 4 polynomials f and g in variables x, y, z, t. The 2-ray game of $\mathcal{F}$ is continued by a fibration Φ' to $\mathbb{P}(1,1,1,2)$ with $\mathbb{P}^1$ fibres. The restriction of this map to X provides $\varphi' : X \to \mathbb{P}(1,1,1,2)$, which contracts the divisor $(f = g = 0) \subset X$ to a curve in $\mathbb{P}(1,1,1,2)$, defined by the same set of equations.

Family 2. $u = v \prec x = y = z \prec t$

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \leftrightsquigarrow \begin{pmatrix} 4 \\ 0 \end{pmatrix} \subset \begin{pmatrix} 2 & 2 & 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The 2-ray game of the ambient toric variety is described by

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "\Phi"'] \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1_{u:v} & & \mathbb{P}(1,1,1,2,2) \end{tikzcd},$$

where Φ' is the divisorial contraction defined by the basis of the Riemann-Roch space of the divisor $D_x \sim (x = 0)$. More precisely, the equation of Φ' is

$$\Phi' \colon \mathcal{F} \to \mathbb{P}(1,1,1,2,2)$$

$$(u : v; x : y : z : t) \mapsto (x : y : z : ut : vt) \qquad .$$

It is clear from this equation that the divisor $(t = 0)$ is contracted to the surface $\mathbb{P}^2_{x:y:z}$. Note that this map has no base point, as the locus where all these monomials vanish is precisely the Cox irrelevant ideal of $\mathcal{F}$, i.e. $(u, v) \cap (x, y, z, t)$.

The equation of a general X in this family is of the form $t^2 q(u, v) = f(x, y, z) + \ldots$, where q is a quadratic polynomial in u, v and f is a quartic with variables x, y, z. Such X has two singular points of type $\frac{1}{2}(1, 1, 1)$, which are located at the intersection of X with Γ_t, that is the solutions of $(q = 0) \cap \Gamma_t$. Then X follows the 2-ray game of the ambient space by contracting the divisor $(t = 0)$ to the curve $(f = 0) \subset \mathbb{P}^2_{x:y:z}$ on an index 3 Fano 3-fold defined by $X_4 \subset \mathbb{P}(1, 1, 1, 2, 2)$.

The equation of the Fano 3-fold, the image of X under this map, can be derived explicitly using this coordinate map. For example if the coordinate variables on $\mathbb{P}(1, 1, 1, 2, 2)$ are x, y, z, u', v', then this Fano variety is the hypersurface defined by

$$q(u', v') = f(x, y, z) + \ldots \qquad .$$

This shows that this Fano variety is general in its family.

Corollary 4.3.2. *A general Fano 3-fold hypersurface $Y_4 \subset \mathbb{P}(1, 1, 1, 2, 2)$ is birational to a degree 2 del Pezzo fibration over $\mathbb{P}^1$.*

Family 3. $u = v \prec x = y = t \prec z$

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 4 \\ 0 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Analysis of the link is similar to the previous case with the final divisorial contraction Φ' with equation

$$(u; v; x : y : t : z) \mapsto (uz : vz : x : y : t) \qquad .$$

The image of X under this map is an index 2 Fano hypersurface defined by a quartic in $\mathbb{P}(1,1,1,1,2)$. It is easy to check that this Fano variety is general in the family.

Corollary 4.3.3. *A general Fano 3-fold hypersurface $Y_4 \subset \mathbb{P}(1,1,1,1,2)$ is birational to a degree 2 del Pezzo fibration over $\mathbb{P}^1$.*

Family 5. $u = v \prec x = y \prec t \prec z$

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 4 \\ 0 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ -2 & -2 & -1 & -1 & 0 & 1 \end{pmatrix}$$

The 2-ray game of $\mathcal{F}$ starts by a flop and continues by a divisorial contraction to $\mathbb{P}^4$. The toric flop contracts a copy of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$ and extracts another $\mathbb{P}^1 \times \mathbb{P}^1$. The restriction of this birational map to X flops 4 analytically disjoint copies of $\mathbb{P}^1$, since the defining polynomial of X includes a quartic in the x, y variables.

A general X in this family is singular at two points of type $\frac{1}{2}(1,1,1)$. As usual, these points are the locus where X meets Γ_t. In fact we can assume that the defining polynomial of X is of the form $(u^2 + v^2)t^2 + f(x,y) + \dots$, where f is a general quartic in x, y. The divisorial contraction has the coordinate description

$$(u : v; x : y : t : z) \mapsto (uz^2 : vz^2 : xz : yz : t) \qquad ,$$

which shows that the divisor $(z = 0)$ gets contracted to the point $p_t \in \mathbb{P}^4$. The equation near this point has a local type $u^2 + v^2 + x^4 + y^4$. In other words this point is terminal. In fact this example was already known to be nonrigid. See [CPR00], Example 7.5.1.

Family 9. $u = v \prec x \prec t \prec y \prec z$

$$\begin{pmatrix} -2 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 6 \\ -2 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 2 & 3 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 1 \end{pmatrix}$$

The 2-ray game on the ambient space is

$$\begin{tikzcd}
 & \mathcal{F} \arrow[r, dashed, "\Psi_1"] \arrow[dl, "\Phi"'] & \mathcal{F}_1 \arrow[r, dashed, "\Psi_2"] & \mathcal{F}_2 \arrow[dr, "\Phi'"] & \\
\mathbb{P}^1_{u:v} & & & & \mathbb{P}(1,1,1,2,3)
\end{tikzcd} \quad ,$$

where Ψ_1 is the anti-flip $(1,1,-1,-1,-2)$ and Ψ_2 is the flip $(2,2,1,-1,-3)$. The final contraction is

$$\Phi'\colon (u:v;x:t:y:z) \mapsto (u_0:v_0:y:x_0:z_0) = (uz:vz:y:xz:tz) \quad ,$$

which is the ordinary blow up of the smooth point $p_y \in \mathbb{P}(1,1,1,2,3)$. The Newton polygon of X in this family is described by

deg of u, v coefficient	
0	$t^2 \quad x^3z \quad x^2y^2 \quad xty$
1	$xy^3 \quad x^2yz \quad xtz \quad ty^2$
2	$y^4 \quad xy^2z \quad tyz \quad x^2z^2$
3	$xyz^2 \quad tz^2 \quad y^3z$
4	$xz^3 \quad y^2z^2$
5	yz^3
6	z^4 .

Having the term $t^2 \in f$, the defining polynomial of X, guarantees smoothness of X. The map ψ_1, the restriction of Ψ_1 to X, is an Atiyah flop as the variable z can be eliminated in a neighbourhood of the flopping curve using the monomial x^3z and the implicit function theorem. Similarly, we can observe that ψ_2 is an isomorphism as $t^2 \in f$. The image of X_1 under φ' is an index 2 Fano hypersurface Y defined by a degree 6 polynomial in $\mathbb{P}(1,1,1,2,3)$. One can see that under this map, the divisor $(z=0)$ goes to the point $p_y \in Y$. This point is a cA_1 point as the defining polynomial of Y is

$$t_0^2 + x_0^3 + y^4u_0v_0 + u_0^6 + v_0^6 + \dots \quad .$$

Conversely, a general Fano hypersurface $Y_6 \subset \mathbb{P}(1,1,1,2,3)$ with a cA_1 point is birational to a degree 2 del Pezzo fibration over $\mathbb{P}^1$.

Family 10. $u = v \prec x \prec y = z \prec t$

$$\begin{pmatrix} -3 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -3 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 6 \\ -1 \end{pmatrix} \subset \begin{pmatrix} 2 & 2 & 3 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

The 2-ray game on $\mathcal{F}$ is

$$\begin{tikzcd} & \mathcal{F}_0 \arrow[r, dashed, "\Psi_1"] \arrow[dl, "\Phi"'] & \mathcal{F}_1 \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1 & & & \mathbb{P}(1,1,2,2,3) \end{tikzcd},$$

where Ψ_1 is the anti-flip $(1,1,-1,-1,-3)$. And the final contraction is $\Phi' \colon \mathcal{F}_1 \to \mathbb{P}(1,1,2,2,3)$ defined by

$$(u; v; x : y : z : t) \mapsto (y : z : u_0 : v_0 : x_0) = (y : z : ut : vt : xt) \quad .$$

This map contracts the divisor $(t = 0)$ on $\mathcal{F}_1$ to the line $\mathbb{P}^1_{y:z} \subset \mathbb{P}(1,1,2,2,3)$.

The Newton polygon of a general X in this family is

$\deg S^k(u,v,w)$	
0	$x^2t \quad xy^3 \quad xy^2z \quad xyz^2 \quad xz^3$
1	$y^4 \quad \ldots z^4 \quad xyt \quad xzt$
2	$y^2t \quad yzt \quad z^2t$
3	t^2 .

The coefficient of t^2 in the equation indicates that

$$\mathrm{Sing}(X) = \Gamma_t \cap X = 3 \times \frac{1}{2}(1,1,1) \quad .$$

The map ψ_1, obtained by restricting Ψ_1 to X is a flop $(1,1,-1,-1)$, as we are able to eliminate the variable t near the flopping curve using the monomial x^2t. The map φ' contracts the divisor $(t=0) \subset X_1$ to the line $\mathbb{P}^1_{y:z}$ on an index 3 Fano variety Y defined by a degree 6 polynomial in $\mathbb{P}(1,1,2,2,3)$. The defining polynomial of Y is

$$x_0^2 + g_3(u_0, v_0) + uq_4(y,z) + vq_4'(y,z) + \dots \qquad ,$$

where g_3 is a general cubic in the variables u_0, v_0; q and q' are general quartics in y, z. Hence Y is smooth along $\mathbb{P}^1_{y:z}$ and has only 3 singular points of type $\frac{1}{2}(1,1,1)$, namely at the solutions of $(g_3 = 0)$.

Family 12. $u = v \prec x \prec y \prec t \prec z$

$$\begin{pmatrix} -3 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -1 & -3 & -2 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{pmatrix} \leftrightsquigarrow \begin{pmatrix} 5 \\ -6 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 0 \\ -2 & -2 & -3 & -1 & 0 & 1 \end{pmatrix}$$

The 2-ray game of $\mathcal{F}$ is represented in the diagram:

$$\begin{tikzcd} & \mathcal{F} \arrow[r, "\Psi_1"] \arrow[dl, "\Phi"'] & \mathcal{F}_1 \arrow[r, "\Psi_2"] & \mathcal{F}_2 \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1_{u:v} & & & & \mathbb{P}(1,1,1,1,2) \end{tikzcd} \qquad ,$$

where Ψ_1 is the anti-flip $(1,1,-1,-3,-2)$ and Ψ_2 is the smooth flip $(1,1,1,-1,-1)$. The singular locus of X is characterised by the coefficient of $t^2 \in f$; this is a cubic in u, v, so for X general $\mathrm{Sing}(X) = 3 \times \frac{1}{2}(1,1,1)$. The map ψ_1, the restriction of Ψ_1 to X, is the Francia anti-flip as the variable t can be eliminated in a neighbourhood of $\Gamma_x = (u_0 : v_0; 1 : 0 : 0 : 0)$ using the monomial x^2t. Similarly, using the monomial xy^3, we can eliminate the variable x in a neighbourhood of the flipping locus of Ψ_2 and observe that ψ_2 is an Atiyah flop. The final map φ', contracts the divisor $(z = 0)$ to a point on an index 1 Fano hypersurface defined by a degree 5 polynomial in $\mathbb{P}(1,1,1,1,2)$. Note that this Fano hypersurface is quasi-smooth away from the image of contraction, which is a cD_4 singularity as it is locally defined by $x^2 + u^3 + v^3 + y^4$. It was shown in [CPR00] that a general quasi-smooth Fano hypersurface of this type is birationally rigid.

4.3.2 Mobile cones

The aim is to prove that all varieties listed in Table 4.1 satisfy the conditions of Definition 4.2.1. In fact the only remaining part to check is the Picard number. This is done in 4.3.3. On the other hand, we must prove that this is the complete list; meaning any $dP_2/\mathbb{P}^1$ which does not appear in this list cannot have a link to another Mori fibre space following the 2-ray game of $\mathcal{F}$. In order to pursue these goals, some computations regarding the mobile divisors on X and $\mathcal{F}$ need to be introduced. We show later that these divisors play a key role in the geometry of X. We begin this by recalling some standard definitions and then compute various cones associated with $\mathcal{F}$ and X.

Definition 4.3.4. Let X be a normal, projective variety.

(1) A divisor $D = \sum a_i D_i$ on X is called *effective* $a_i \geq 0$ are integers for all i, where D_i are prime divisors on X and only finitely many $a_i \neq 0$.

(2) The cone generated by effective Cartier divisors in $\mathrm{N}^1(X)_{\mathbb{R}}$ is called the *effective cone* and is denoted by $\mathrm{NE}^1(X)$. The closure of this cone is called the *pseudo-effective cone* and is denoted by $\overline{\mathrm{NE}}^1(X)$.

(3) An effective divisor D is called *mobile* if the base locus of the linear system corresponding to D has codimension strictly bigger than 1. In other words, if the support of this linear system does not contain a divisor.

(4) The closed subcone of $\mathrm{N}^1(X)_{\mathbb{R}}$ generated by all mobile divisors of X is called the *mobile cone* of X and is denoted by $\mathrm{Mob}(X)$. The reader might find this cone with the name *movable cone* in some articles.

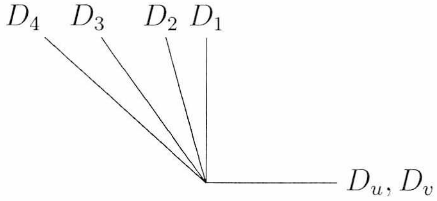
(5) A Cartier divisor d on X is called *semiample* if the linear system $|mD|$ is base point free for some $m \in \mathbb{N}$.

Proposition 4.3.5. *Let $\mathcal{F}$ be the toric variety described in 4.4. Then*

(i) *the pseudo-effective cone of $\mathcal{F}$ is generated by D_u and D_4, and*

(ii) *the mobile cone* $\mathrm{Mob}(\mathcal{F})$ *is generated by D_u and D_3, where D_u, D_v and D_i are divisors defined by $(u = 0)$, $(v = 0)$ and $(x_i = 0)$.*

Proof. By Theorem 3.3.6 we know that the Picard number of $\mathcal{F}$ is $\rho(\mathcal{F}) = 2$. Therefore $\mathrm{N}^1(\mathcal{F})_{\mathbb{R}} \cong \mathbb{R}^2$ and hence we can draw all these cones in the plane:



The rays are labelled by divisors that lie on them away from the origin. Note that the rays correspond to some D_i and D_j might coincide. This is exactly when $x_i = x_j$.

Obviously $\langle D_u, \dots, D_4 \rangle \subset \overline{\mathrm{NE}}^1(\mathcal{F})$. We show that any prime divisor corresponding to a lattice point in the plane outside of this cone is not numerically equivalent to an effective divisor. Any divisor given by a lattice point in $\mathbb{R}^2 - \overline{\mathrm{NE}}^1(\mathcal{F})$ is numerically equivalent to a divisor A, A' or A'', where

$$\begin{aligned} A &= -\mu D_u + \lambda D_4 && \text{for } \mu > 0,\ \lambda \geq 0, \\ A' &= -\mu D_u - \lambda D_4 && \text{for } \mu > 0,\ \lambda > 0, \\ A'' &= \mu D_u - \lambda D_4 && \text{for } \mu \geq 0,\ \lambda > 0. \end{aligned}$$

We show that A cannot be effective. Define a curve $l = (x_1 = x_2 = x_3 = 0) \subset \mathcal{F}$, where without loss of generality $b_4 = 1$. We have

$$A \cdot l = -\mu D_u \cdot l + \lambda D_4 \cdot l = -\mu < 0 \quad .$$

Since A is prime, we must have $l \subset A$. Now consider the family of curves defined by the ideal

$$I_C = (x_1, x_2 + \varphi_{\delta-\beta}(u,v)x_4, x_3\psi_{\delta-\gamma}(u,v)x_4) \quad .$$

For any curve C in this family and any divisor D on $\mathcal{F}$, there exists a positive rational number r such that $r(l \cdot D) = C \cdot D$. Hence The support of this family lies in A. On the other hand, it is easy to see that for any point in D_1 there is a curve C in this family which contains that point. In other words, D_1 is contained in the support of this family and hence $D_1 \subset A$. But A is prime and this is a contradiction.

Proofs for the other two cases, A' and A'' are similar and we do not write them here.

In order to prove (ii), we must show that the cone generated by D_u and D_3 is the $\mathrm{Mob}(\mathcal{F})$. The divisor D_u is mobile as $D_v \in |D_u|$ and hence this linear system is base point free. Any effective divisor $\mathbb{Q}$-linearly equivalent to D_3 is of the form $\lambda D_4 + \mu D_i$ or $\lambda D_4 + \mu D_u$ for some positive integers λ and μ. Therefore $\mathrm{Bs}(D_3) \subset (x_3 = x_4 = 0)$, and hence $|D_3|$ has no fixed component; the fixed part has codimension at least two. This shows that $\langle D_u, D_3 \rangle \subset \mathrm{Mob}(\mathcal{F})$. To complete the proof we must show that any effective divisor in $\overline{\mathrm{NE}}^1(\mathcal{F}) - \mathrm{Mob}(\mathcal{F})$ is not mobile. But any such divisor is numerically equivalent to a divisor of the form $\mu D_3 + \lambda D_4$ for some non-negative integers μ and λ. The fixed part of the linear system of such divisor includes D_4 and hence this divisor cannot be mobile. $\square$

Definition 4.3.6. ([HK00], Definition 1.10) A normal projective variety X is called a *Mori dream space* if

(i) X is $\mathbb{Q}$-factorial and $\mathrm{Pic}(X) = N^1(X)$ is finitely generated.

(ii) there are finitely many birational maps $f_i \colon X \dashrightarrow X_i$ for $1 \leq i \leq k$, which are isomorphisms in codimension one, such that if B is a mobile divisor then there is an index $1 \leq i \leq k$ and a semiample divisor B_i on X_i such that $B = f_i^* B_i$.

The key point of this definition is that it allows one to run MMP on X in a very easy and clear way. If X is a Mori dream space then the pseudo-effective cone $\overline{\mathrm{NE}}^1(X)$ is divided into finitely many rational polyhedra, $R_1, \ldots, R_m$,

$$\overline{\mathrm{NE}}^1(X) = \bigcup_{j=1}^{m} R_j \quad .$$

The mobile cone is a union of $M_1, \ldots, M_k$, some subset of the rational polyhedra $R_1, \ldots, R_m$, and the birational maps $f_1, \ldots, f_k$ defined in 4.3.6 are precisely the maps φ_{B_i} associated to a big mobile divisor B_i belonging to the interior of each polytope M_i. For details see [HK00] Proposition 1.11.

It was proved in [BCHM10] Corollary 1.3.1. that any log Fano variety is a Mori dream space. In particular, a dP_2 fibration is a Mori dream space. The whole idea of defining techniques in this chapter is that we are trying to find dP_2 fibrations $X \subset \mathcal{F}$ whose decomposition of $\mathrm{Mob}(X)$ into $M_1, \ldots, M_k$ coincides with the decomposition of $\mathrm{Mob}(\mathcal{F})$ into such polytopes. In other words, X is embedded into $\mathcal{F}$ and

$$\mathrm{Cox}(X) = \mathrm{Cox}(\mathcal{F})/(f = 0) \quad .$$

Lemma 4.3.7. *Let $X \subset \mathcal{F}$ be a hypersurface of the rank two toric variety in 4.4 defined by a homogeneous polynomial of bi-degree $(\omega, 4)$. If X is a dP_2 fibration then $\sigma = \langle L, X \cap D_3 \rangle$ is a subcone of* $\mathrm{Mob}(X)$.

Proof. Similar to the proof of Proposition 4.3.5 (ii) one can check that $\mathrm{Bs}\,|L|$ is empty and $\mathrm{Bs}\,|D_3|$ has no fixed component. Note that $\mathrm{Bs}\,|D_3|$ is included in the locus $(x_3 = x_4 = 0)$. And this locus must have codimension strictly bigger than 1. Otherwise, if $(x_3 = x_4 = 0)$ defines a divisor on X then Proposition 4.4.7 implies that X is not a dP_2 fibration. □

4.3.3 The Picard group

The aim in this section is to prove $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ for a general X in Table 4.1.

Let us first recall some technical tools that we use in the proof. This includes a version of the Lefschetz hyperplane theorem and a generalised Kodaira vanishing theorem.

Theorem 4.3.8. [Generalised Kodaira vanishing, [KM98] Theorem 2.70.] *Let (X, Δ) be a proper klt pair. Let N be a $\mathbb{Q}$-Cartier Weil divisor on X such that $N \equiv M + \Delta$, where M is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then $H^i(X, \mathcal{O}_X(-N)) = 0$ for $i < \dim X$.*

Remark 4.3.9. Let V and W be algebraic varieties. Recall that any algebraic map

$\pi\colon V \to W$ can be decomposed into finitely many varieties $V_i \subset V$ of varying dimension, on each of which π restricts to a map with constant fibre dimension.

Definition 4.3.10. Define $D(\pi)$, the *measure of deviation* of $\pi : V \to W$, to be

$$D(\pi) = \sup_i\{(\text{the fibre dimension of } \pi \text{ in } V_i) \ - (\text{the codimension of } V_i \text{ in } V)\} \quad .$$

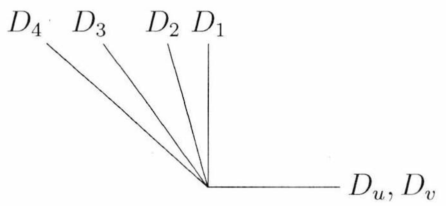
Theorem 4.3.11. [Lefschetz hyperplane theorem, [GM88] §2.2] *Let $\pi\colon V \to \mathbb{C}^N$ be a proper map of a purely n-dimensional (possibly singular) algebraic variety into complex affine space. Then $H_i(V) = 0$ for $i > n + D(\pi)$.*

Lemma 4.3.12. *Let $X \subset \mathcal{F}$ be a hypersurface defined by*

$$\begin{pmatrix} -e \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 0 & 0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} \quad ,$$

where the variables are in order $u = v \prec x_1 \preceq x_2 \preceq x_3 \preceq x_4$ and $\{\beta_1, \beta_2, \beta_3, \beta_4\} = \{1, 1, 1, 2\}$. Suppose $\mathcal{F}_i$ and X_i are birational models of $\mathcal{F}$ and X obtained by small modifications as in Theorem 4.1.4 and Definition 4.1.5. Let $\mathcal{U}_i = \mathcal{F}_i - X_i$ be the complement of each X_i in $\mathcal{F}_i$. Consider the point $x = (-e, 4) \in \mathbb{Z}^2$ and recall from Proposition 4.3.5 that $\mathrm{Mob}(\mathcal{F})$ *is a cone in $\mathbb{R}^2 = \mathbb{Z}^2 \otimes \mathbb{R}$ with the same copy of $\mathbb{Z}^2$. If $X \in$* $\mathrm{Int}(\mathrm{Mob}(\mathcal{F}))$*, then $H_5(\mathcal{U}_i) = H_6(\mathcal{U}_i) = 0$ for some i.*

Proof. Consider the map $\Phi_{|D|}\colon \mathcal{F} \to \mathbb{P}^N$ defined by the linear system of the divisor $D = 4M - eL$ and assume $D \in \mathrm{Mob}(\mathcal{F})$. By Proposition 4.3.5, $\overline{\mathrm{NE}}^1(\mathcal{F})$ has the following decomposition:



,

where the rays are labelled by divisors that lie on them away from the origin.

From geometric invariant theory we have the following characterisation (possibly after taking a positive multiple of D):

(i) $\Phi_{|D|}$ is an embedding of $\mathcal{F}_i$ if $D \in \mathrm{Int}\langle D_i, D_{i+1}\rangle$, where D_i and D_{i+1} do not lie on the same ray.

(ii) $\Phi_{|D|}$ is a small contraction from $\mathcal{F}_i$ if $D = aD_i$ for some positive integer a and $D_i \in \mathrm{Int}(\mathrm{Mob}(\mathcal{F}))$.

(iii) $\Phi_{|D|}$ is an extremal contraction of divisorial or fibre type otherwise.

Suppose $D \in \mathrm{Int}(\mathrm{Mob}(\mathcal{F}))$; in particular it is in one of the cases (i) or (ii) above.

Let $\mathcal{U}_i = \mathcal{F}_i - X_i$, where i is the integer for which (i) or (ii) above is satisfied. Suppose $\varphi\colon \mathcal{U}_i \to \mathbb{C}^N$ be the restriction of $\Phi_{|D|}$ to $\mathcal{U}_i$. The map φ is proper because $\Phi_{|D|}$ is a projective morphism and X_i is the complete preimage of a hyperplane section of the target variety. Since this map contracts at most a 2-dimensional subspace of $\mathcal{F}_i$ and is isomorphism everywhere else, the codimension of every V_j in Definition 4.3.10 is at least 2, while the fibre dimension is at most 2. Hence $D(\varphi) \leq 0$ so by Theorem 4.3.11 we conclude that $\mathrm{H}_5(\mathcal{U}_i) = \mathrm{H}_6(\mathcal{U}_i) = 0$. Note that $\dim_{\mathbb{C}}(\mathcal{U}_i) = 4$ and $\dim_{\mathbb{R}}(\mathcal{U}_i) = 8$. □

Corollary 4.3.13. $H^2_c(\mathcal{U}_i) = H^3_c(\mathcal{U}_i) = 0$.

Proof. This proof follows from Lemma 4.3.12 and Poincaré duality . □

Lemma 4.3.14. *Let $\mathcal{F}$ be the ambient toric variety of any family in Table 4.1 except 1,2 and 3. Then $H^2(\mathcal{F}_i) = \mathbb{Z}^2$ for all models $\mathcal{F}_i$ obtained by flips, flops or antiflips from $\mathcal{F}$.*

Proof. From the short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{\mathcal{F}} \to \mathcal{O}^*_{\mathcal{F}} \to 0$$

one constructs the long exact sequence

$$\cdots \to \mathrm{H}^1(\mathcal{F},\mathbb{Z}) \to \mathrm{H}^1(\mathcal{F},\mathcal{O}_{\mathcal{F}}) \to \mathrm{H}^1(\mathcal{F},\mathcal{O}^*_{\mathcal{F}}) \to \mathrm{H}^2(\mathcal{F},\mathbb{Z}) \to \mathrm{H}^2(\mathcal{F},\mathcal{O}_{\mathcal{F}}) \to \cdots \quad .$$

On the other hand, for any $\mathcal{F}$ in Families 4,…,12 in Table 4.1 there exists a birational model $\mathcal{F}_i$, obtained by some flips (flops or antiflips) for which $-K_{\mathcal{F}_i}$ is nef and big. Applying Theorem 4.3.8 for the pair $(\mathcal{F}_i, 0)$ and divisor $-K_{\mathcal{F}_i}$ gives $\mathrm{H}^j(\mathcal{F}_i, \mathcal{O}_{\mathcal{F}_i}(-K_{\mathcal{F}_i})) = 0$ for all $j < 4$. This vanishing together with Serre duality implies

$$\mathrm{H}^1(\mathcal{F}_i, \mathcal{O}_{\mathcal{F}_i}) = \mathrm{H}^2(\mathcal{F}_i, \mathcal{O}_{\mathcal{F}_i}) = 0 \quad .$$

The fact that $\mathcal{F}_i$ have rational singularities ensures that the vanishing above holds for all models $\mathcal{F}_i$.

By Lemma 3.3.6 $\mathrm{Pic}(\mathcal{F}_i) \cong \mathbb{Z}^2$ for all models $\mathcal{F}_i$ obtained by flips, flops or antiflips from $\mathcal{F}$. Using the fact that $\mathrm{H}^1(\mathcal{F}_i, \mathcal{O}^*_{\mathcal{F}_i}) \cong \mathrm{Pic}(\mathcal{F}_i)$, the exact sequence above, together with the vanishing statements that we proved imply $\mathrm{H}^2(\mathcal{F}_i) \cong \mathbb{Z}^2$. □

Proposition 4.3.15. *Let $X \subset \mathcal{F}$ be a hypersurface defined by $f \in H^0(\mathcal{F}, D)$, where $D = 4M - eL$ and $(-e, 4) \in \mathrm{Int}(\mathrm{Mob}(\mathcal{F}))$. If $\mathcal{F}$ is the abient space of one of the families in Table 4.1 except families 1,2 and 3, then $H^2(X_i) \cong \mathbb{Z}^2$ for $X_i \subset \mathcal{F}_i$, where $\mathcal{F}_i$ is the model specified in Lemma 4.3.12.*

Proof. Consider the following exact sequence:

$$\cdots \to \mathrm{H}^2_c(\mathcal{U}_i) \to \mathrm{H}^2(\mathcal{F}_i) \to \mathrm{H}^2(X_i) \to \mathrm{H}^3_c(\mathcal{U}_i) \to \cdots$$

By Corollary 4.3.13, this exact sequence implies $\mathrm{H}^2(\mathcal{F}_i) \cong \mathrm{H}^2(X_i)$. The proof follows from Lemma 4.3.14. □

Lemma 4.3.16. *For a general X in Table 4.1 $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.*

Proof. For any such X there exists a model X_i obtained by some flips, flops or antiflips from X such that $-K_{X_i}$ is nef and big on X_i. Considering the pair $(X_i, 0)$, which is a klt pair as X_i is terminal, and applying Theorem 4.3.8 gives $\mathrm{H}^j(X_i, \mathcal{O}_X(-K_{X_i})) = 0$ for all $j < 3$. This together with Serre duality implies $\mathrm{H}^1(X_i, \mathcal{O}_{X_i}) = \mathrm{H}^2(X_i, \mathcal{O}_{X_i}) = 0$. The rationality of singularities of X_i allows one to lift this vanishing to all X_k. In particular, $\mathrm{H}^1(X, \mathcal{O}_X) = \mathrm{H}^2(X, \mathcal{O}_X) = 0$. □

Theorem 4.3.17. *Let $X \subset \mathcal{F}$ be a general $dP_2/\mathbb{P}^1$ in one of the families in Table 4.1 then* $\mathrm{Pic}(X) \cong \mathbb{Z}^2$.

Proof. Let X be a general $dP_2/\mathbb{P}^1$ in one of the families of Table 4.1 except families 1, 2 and 3. By Proposition 4.3.15, $\mathrm{H}^2(X_i) \cong \mathbb{Z}^2$ for some model X_i obtained by some flips, flops or antiflips from X. On the other hand, Lemma 4.3.16 implies $\mathrm{H}^1(X_i, \mathcal{O}_{X_i}) = \mathrm{H}^2(X_i, \mathcal{O}_{X_i}) = 0$. Applying this to the exact sequence

$$\cdots \to \mathrm{H}^1(X_i, \mathbb{Z}) \to \mathrm{H}^1(X_i, \mathcal{O}_{X_i}) \to \mathrm{H}^1(X_i, \mathcal{O}^*_{X_i}) \to \mathrm{H}^2(X_i, \mathbb{Z}) \to \mathrm{H}^2(X_i, \mathcal{O}_{X_i}) \to \cdots$$

enables one to see $\mathrm{H}^1(X_i, \mathcal{O}^*_{X_i}) \cong \mathrm{H}^2(X_i, \mathbb{Z})$; hence $\mathrm{Pic}(X_i) \cong \mathbb{Z}^2$. The fact that X_i is isomorphic to X in codimension 1 shows that $\mathrm{Pic}(X) \cong \mathbb{Z}^2$.

In order to finish the proof, we must show that $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ for a general X in families 1, 2 and 3. But we know that any such X is obtained by a blow up of a Fano 3-fold, where the Picard rank is 1 as in each case the hypersurface is general, or $\mathbb{P}(1,1,1,2)$, which completes the proof. □

4.4 Linear systems on $\mathcal{F}$

In this section, we show that any hypersurface $X \subset \mathcal{F}$ under the hypothesis of Theorem 6.4.5, which does not appear in the Table 4.1 either is not a dP_2 fibration (Definition 4.2.1) or does not provide an $\mathcal{F}$-Sarkisov link.

Let us fix a general setting for $\mathcal{F}$ and X. Let $\mathcal{F}$ be the rank two toric variety with Cox ring $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[u, v, x_1, x_2, x_3, x_4]$ and irrelevant ideal $I = (u, v) \cap (x_1, \ldots, x_4)$ with the action of $(\mathbb{C}^*)^2$ defined by

$$\begin{pmatrix} 1 & 1 & -a_1 & -a_2 & -a_3 & -a_4 \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 \end{pmatrix}, \qquad (4.4)$$

where a_i are non-negative integers and $\{b_1, \ldots, b_4\} = \{1, 1, 1, 2\}$ such that the coordinate variables of $\mathrm{Cox}(\mathcal{F})$ are in order $u = v \prec x_1 \preceq x_2 \preceq x_3 \preceq x_4$. Let X be a hypersurface of $\mathcal{F}$ defined by a homogeneous polynomial of bi-degree $(\omega, 4)$ with respect to the action above. We sometimes switch these variable names to our favourite u, v, x, y, z, t when we need to write explicit equations. Otherwise, we keep this notation, as it enables us to consider the order of variables without confusion about the position of the variable t and having to divide into three types described at the beginning of Section 4.2. .

4.4.1 Elimination process

In this subsection, we provide the key tools to eliminate cases which do not occur in Table 4.1.

In the following lemma, we consider the coordinate variables of $\mathcal{F}$ to be u, v, x, y, z, t and the variable t corresponds to the coordinate, which has been acted by $(\lambda^{-\gamma}, \mu^2) \in (\mathbb{C}^*)^2$.

Lemma 4.4.1. *If X is taken as a hypersurface in $\mathcal{F}$, it fails to be terminal if any of the following holds:*

1. *$\mathcal{F}$ is of type (i), and $e > 2c$.*
2. *$\mathcal{F}$ is of type (ii), and $e > 0$.*
3. *$\mathcal{F}$ is of type (iii), and $e > 2$.*

Proof. In any of these cases, whenever t appears in a term of f, it is multiplied by a nonconstant polynomial in x, y, z, which implies $\Gamma_t \subset X$. We recall that the curve Γ_t is defined as $\Gamma_t = (x = y = z = 0) \subset X$. Therefore X has a line of singularity, but 3-fold terminal singularities are isolated by [Rei80]. □

We are interested in cases that $\sigma = \mathrm{Mob}(X)$. In particular, these are the cases when the type III and IV 2-ray game of X follows the one from $\mathcal{F}$. The following lemma helps us to eliminate cases when X fails to follow such link at the beginning of the game.

Theorem 4.4.2. *Let $X \subset \mathcal{F}$ be defined as in 4.4. If X is not obtained by one of the following, then either it is not a dP_2 fibration or the first step of its 2-ray game cannot be obtained by the restriction of the one from $\mathcal{F}$.*

(i) $a_1 = a_2 = a_3 = a_4 = 0$ *and* $\omega = 1$.

(ii) $a_1 = a_2 = a_3 = 0$, $a_4 = 1$ *and* $\omega = 0$.

(iii) $a_1 = a_2 = 0$, $a_3a_4 \neq 0$ *and* $\omega = 0$.

(iv) $x_1 \prec x_2, x_3, x_4$ *and there is a monomial with only variables* x_1, x_2, x_3, x_4 *in the defining equation of* X.

Proof. Assume x_1, x_2, x_3, x_4 have equal ratio weight, i.e. $x_1 = x_2 = x_3 = x_4$. Then there is no Ψ_i and the 2-ray game of $\mathcal{F}$ is followed by a fibration to $\mathbb{P}(1,1,1,2)$. Without loss of generality we can assume this common weight is zero. In other words, by adding a multiple of the second row of the matrix A to the first row we can assume $X \subset \mathcal{F}$ is defined by

$$\begin{pmatrix} \omega \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}.$$

If $\omega = 0$, then $X \cong \mathbb{P}^1 \times dP_2$. If we denote the generic fibre by S, then $\mathrm{H}^1(S, \mathcal{O}_S) = 0$ together with Exercise 12.6 in Chapter III [Har] implies that $\mathrm{Pic}(X) = \mathrm{Pic}(S) \times \mathrm{Pic}(\mathbb{P}^1)$. And hence $\rho_X > 2$ and therefore X is not a Mori fibre space. If $\omega = 1$, then the equation of X has the form $uf = vg$ for f, g degree 4 homogeneous polynomials in $\mathbb{P}(1,1,1,2)$. It shows that X is the blow up of $\mathbb{P}(1,1,1,2)$ along a curve defined by $(f = g = 0)$. This was done by restricting Φ' to X, which shows the 2-ray game of X comes from $\mathcal{F}$. This case was given as Family 1 in Table 4.1.
If $\omega > 1$, then X is generically an ω-cover of $\mathbb{P}(1,1,1,2)$, which fails to be a dP_2 fibration.

To move onto the next case, suppose the ratio weight of x_1, x_2, x_3 is equal and normalised to zero and different from that of x_4. In other words, $x_1 = x_2 = x_3 \prec x_4$ and

$X \subset \mathcal{F}$ is defined by

$$\begin{pmatrix} \omega \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -a \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 \end{pmatrix} ,$$

for a positive integer a. In this case, the 2-ray game of $\mathcal{F}$ is followed by a divisorial contraction to $\mathbb{P} = \operatorname{Proj} \bigoplus_k \operatorname{Cox}(\mathcal{F})_{(0,k)}$, with exceptional divisor $(x_4 = 0)$. If $\omega < 0$, then X is reducible and hence not a dP_2 fibration.

If $\omega = 0$ and $a = 1$, then φ' is a divisorial contraction from X, which is case (ii). This forms Family 2 and Family 3 in Table 4.1. The failure of case $\omega = 0$ and $a > 1$ is proved in Lemma 4.4.6 below.

The interesting case is when $\omega > 0$. In this situation the image of restriction of the contraction on $\mathcal{F}$ to X is a surface, hence this map does not define the 2-ray game of X. This means that X does not have an $\mathcal{F}$-Sarkisov link. But when $b_4 = \omega = a = 1$, we show in Example 4.4.3 that X is non-rigid. Note that this case does not appear in Table 4.1 as the 2-ray game is given by a different ambient space. Apart from this special case, if X forms a dP_2 fibration, we expect it to be non-rigid. For a discussion (no proofs) on the rigidity of this type of fibrations we refer to the next chapter.

For part (iii), assume $a_1 = a_2 = 0$ and $x_1, x_2 \prec x_3, x_4$. In this case, the 2-ray game of $\mathcal{F}$ is continued by an anti-flip (or flop), which contracts $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$ and extracts a copy of $\mathbb{P}^1 \times \mathbb{P}(a_3, a_4)$. If $\omega = 0$, then the restriction of this operation to X will be a finite number (2 or 4) of disjoint anti-flips (or flops) of type $(1, 1, -a_3, -a_4)$. This is the case mentioned in (iii).

If $\omega < 0$, then the Picard number of X is bigger than two, which is proved in Proposition 4.4.7. This shows that X is not a dP_2 fibration.

If $\omega > 0$, then the restriction of the ambient anti-flip (flop) defines an small contraction in one side and an isomorphism in the other side, which clearly does not read the 2-ray game of X.

Assume $x_1 \prec x_2, x_3, x_4$. In this case the 2-ray game of $\mathcal{F}$ at the level of Ψ_1 can be read as a flip (flop or anti-flip) of type $(\alpha, \alpha, -\beta_1, -\beta_2, -\beta_3)$. It is obvious that this will restrict to a 3-fold flip (flop or anti-flip) on X if the extracted surface, $\mathbb{P}(\beta_1, \beta_2, \beta_3)$ with coordinate variables x_2, x_3, x_4, intersected with X defines a curve. This will be valid only if this surface is not a subvariety of X. This means the defining polynomial of X must have at least one monomial with only x_i variables. Note that if a term of the form x_1^k appears in the equation, X will pass this step of the 2-ray game isomorphically and nothing contradicts our statements. $\square$

Example 4.4.3. Let $X \subset \mathcal{F}$ be defined in the usual way by

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -a \\ 0 & 0 & 1 & 1 & 2 & 1 \end{pmatrix},$$

where $a > 0$ is an integer. It was shown in the proof of Theorem 4.4.2 that such X does not have an $\mathcal{F}$-link. Here we show that X can be embedded into another scroll $\mathcal{F}'$ such that X has an $\mathcal{F}'$-Sarkisov link to another Mori fibre space.

Let us fix the variables of $\mathcal{F}$ in order by u, v, x, y, t, z as usual. The defining polynomial of X is of the form $uf = vg$ for some bi-degree $(0, 4)$ polynomials f, g. Now we apply unprojection operations of [PR04]. Let s be a rational function defined by

$$s = \frac{f}{v} = \frac{g}{u}$$

with bi-degree $(-1, 4)$. Then treat it as a variable in equations $us = g$ and $vs = f$. This enables us to embed X into the scroll $\mathcal{F}'$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 & -a \\ 0 & 0 & 1 & 1 & 2 & 4 & 1 \end{pmatrix},$$

where the variables in order are $u = v \prec x = y = t \prec s \prec z$. The variety X is embedded into $\mathcal{F}'$ as the complete intersection of two hypersurfaces $us = g$ and $vs = f$.

$\mathcal{F}'$ is a 5-fold toric variety of rank 2 whose 2-ray game starts by an anti-flip (or flop) of type $(1, 1, -1, -a)$ over a surface $\mathbb{P}(1, 1, 2)$. Meaning, it contracts a copy of $\mathbb{P}^1 \times \mathbb{P}(a, a, 2)$ to $\mathbb{P}(1, 1, 1)$ in one side and extracts a copy of $\mathbb{P}(1, a) \times \mathbb{P}(1, 1, 2)$ in the other side. The restriction of this map to X defines an anti-flip (or flop), consisting 2 disjoint anti-flip (or flop) of type $(1, 1, -1, -a)$. Then it has a divisorial contraction to a codimension 2 Fano 3-fold of index one defined by $Y_{4,4} \subset \mathbb{P}(1, 1, 1, 1, 2, 3)$.

The key point in this example is that the $\sigma \subset \mathrm{Mob}(X)$ but they are not equal. However, as $-K_X$ is still in the pseudo-effective cone, we managed to find another embedding of X for which $\mathrm{Mob}(X)$ is the restriction of that of the ambient space. This allowed us to read $-K_X \in \mathrm{Int}(\mathrm{Mob}(X))$.

However, it is a fair point to mention here that the tie between the anticanonical class of X and the mobile cone of X plays a major role in rigidity of X. This will shape the idea of the next chapter. Lemma 4.4.4 below is a key eliminating tool for us, which is entirely based on this idea.

Before stating the lemma, we say a few words about the anticanonical classes of $\mathcal{F}$ and X. By Theorem 3.3.3 the anticanonical divisor of $\mathcal{F}$ has bi-degree $(2 - \sum a_i, \sum b_i)$. By adjunction we have

$$-K_X = (-K_{\mathcal{F}} - X)|_X$$

and hence the anticanonical divisor of X has bi-degree $(2 - \sum a_i - \omega, 1)$.

Lemma 4.4.4. *Let X be a hypersurface of $\mathcal{F}$, as in the assumption of Theorem 4.4.2, satisfying conditions of Theorem 4.4.2 and Lemma 4.4.1, which has an $\mathcal{F}$-link. If $-K_X \sim mD_3 - nD_u$ for a positive integer m and a non-negative integer n, then the last morphism of the 2-ray game of X is not an extremal contraction.*

Proof. The proof is given case by case, depending on the ratio weights of the variables. In each case we find a curve inside the exceptional locus of φ', which has positive intersection against the anticanonical class. This shows that the last morphism of the 2-ray game is not an extremal contraction.

Case I $x_2 \prec x_3 \preceq x_4$

Let $C = (x_1 = x_4 = f = 0) \subset \mathrm{Exc}(\varphi')$, where f is the defining polynomial of X. Note that the irrelevant ideal of the domain variety of φ' is defined by $(u, v, x_1, x_2) \cap (x_3, x_4)$. Therefore $D_3 \cdot C = 0$, which implies

$$-K \cdot C = 0 - nD_u \cdot (x_1 = x_4 = f = 0) \leq 0$$

Case II $x_1 \prec x_2 = x_3 \preceq x_4$

Let $C = (x_2 = x_4 = f = 0)$. As the irrelevant ideal in this case is $(u, v, x_1) \cap (x_2, x_3, x_4)$, similar argument shows

$$-K \cdot C = 0 - nD_u \cdot (x_2 = x_4 = f = 0) \leq 0$$

Case III $x_1 = x_2 = x_3 \prec x_4$

The irrelevant ideal in this case is $(u, v) \cap (x_1, x_2, x_3, x_4)$. Without loss of generality we can assume that X is defined by

$$\begin{pmatrix} \omega \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -a \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 \end{pmatrix},$$

where a is a positive integer. Theorem 4.4.2 together with Lemma 4.4.6 implies $\omega = 0$ and $a = 1$. As we consider general X, in this case there is nothing to prove, in fact m and n in the assumption cannot be found.

□

Remark 4.4.5. Note that Lemma 4.4.4 implies that in order to have an $\mathcal{F}$-link from X, it is necessary for the ratio weight of $-K_X$ to be strictly less than that of the coordinate variable x_3. This is simply saying that $-K_X \in \mathrm{Int}(\mathrm{Mob}(X))$.

Lemma 4.4.6. *Let $X \subset \mathcal{F}$ be defined by*

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -a \\ 0 & 0 & 1 & 1 & 2 & 1 \end{pmatrix},$$

with variables in order $u = v \prec x = y = t \prec z$ with $a \in \mathbb{Z}$, $a \geq 1$. If the integer a is strictly bigger than 1, then the image of the last morphism of the 2-ray game of X is not terminal.

Proof. If $a > 1$, then the image of $\mathcal{F}$ under the last morphism of its 2-ray game is defined by the quotient of $\mathbb{P}(1,1,1,1,2)$ by the action of $\frac{1}{a}(1,1,0,0,0)$. In particular, this variety has a singular locus of dimension 2. Hence the image of X under this map has non-isolated singularities (along a crve) and therefore is not terminal. □

Proposition 4.4.7. *Let $X \subset \mathcal{F}$ be defined as before. If $D = (x_3 = x_4 = 0) \subset X$ forms a divisor on X, i.e. if the defining polynomial of X is of the form $x_3 f = x_4 g$, then ρ_X, the Picard number of X, is at least 3.*

Proof. As in the assumption, let the defining polynomial of X be $x_3 f = x_4 g$ for non-constant polynomials f, g. Let $M \sim (x_1 = 0)$ and $L \sim (u = 0)$ be two other divisors on X. We show that D, M and L are linearly independent and hence $\mathrm{Pic}(X)$ has at least three generators. To do so, we find three curves inside X and compute their intersections with these divisors. These number form a 3×3 matrix. If the rank of this matrix is bigger than 3, we have shown that these divisors are linearly independent.

Consider three curves $C_1, C_2, C_3 \subset X$ defined by

$$C_1 = (u = x_3 = x_4 = 0) \qquad C_2 = (x_1 = x_3 = x + 4 = 0) \qquad C_3 = ((v = x_2 = 0) \cap X)$$

Computing intersection numbers gives:

$$\begin{pmatrix} L \cdot C_1 & L \cdot C_2 & L \cdot C_3 \\ M \cdot C_1 & M \cdot C_2 & M \cdot C_3 \\ D \cdot C_1 & D \cdot C_2 & D \cdot C_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & * & 0 \\ * & * & 1 \end{pmatrix},$$

where $*$ denotes some numbers that we have no interest in computing them. Which shows that this matrix has full rank and hence $\rho_X > 2$. □

A typical example of a variety concerned in Proposition 4.4.7 has following shape:

$$X \in \begin{pmatrix} -1 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Before we start the next section let us recall that $\mathcal{F}$ is said to be of type (i), (ii) or (iii) if the corresponding action of $(\mathbb{C}^*)^2$ has the following representations. Note that an easy argument shows that any $\mathcal{F}$ considered in this chapter has a unique representation in one of these types.

$$(i) \qquad A = \begin{pmatrix} 1 & 1 & 0 & -a & -b & -c \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad 0 < c,\, 0 \le a \le b$$

$$(ii) \qquad A = \begin{pmatrix} 1 & 1 & -a & -b & -c & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad 0 \le a \le b \le c$$

$$(iii) \qquad A = \begin{pmatrix} 1 & 1 & -a & -b & -c & -1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad 0 < a \le b \le c \quad ,$$

where a, b and c are non-negative integers and the variables are u, v, x, y, z, t. The conditions on the order of a, b, c imply that in all cases above the variables x, y, z are ordered with $x \preceq y \preceq z$. And if $\mathcal{F}$ is of type (ii) or (iii), then $t \preceq x$.

Table 4.2 below gathers some computations of the anti-canonical class of $\mathcal{F}$ and X, which we use later.

In the next two subsection, we explicitly analyse cases which do not occur in Table 4.1 and give arguments why each of them fails. Our arguments are based on the materials provided in this part, namely Lemma 4.4.1, Theorem 4.4.2, Lemma 4.4.4 and Proposition 4.4.7.

	Type (i)	Type (ii)	Type (iii)
$-K_{\mathcal{F}}$	$(2-a-b-c)L+4M$	$(2-a-b-c)L+4M$	$(1-a-b-c)L+4M$
$-K_X$	$(2+e-a-b-c)L+M$	$(2+e-a-b-c)L+M$	$(1+e-a-b-c)L+M$

Table 4.2: Anticanonical classes of $\mathcal{F}$ and X

4.4.2 Hypersurfaces in scrolls of Type (ii) or (iii)

Proposition 4.4.8. *If $\mathcal{F}$ is of type (iii), then X does not have a link to any other Mori fibre space except for $e=2, a=b=c=1$.*

Proof. If $e=2$, then Lemma 4.4.4 implies $a+c<3$, and that means $a=b=c=1$. Under these numerical conditions a general X passes the first step of the 2-ray game isomorphically and then maps to $\mathbb{P}^2$ with conic fibres. This forms Family 6 in Table 4.1. The case $e>2$ is not concerned, due to Lemma 4.4.1. For $e<2$, Lemma 4.4.4 does the elimination. □

Proposition 4.4.9. *Suppose $\mathcal{F}$ is of type (ii), and consider its 2-ray game of Type* III *or* IV. *Exactly one of the following cases occurs:*

1. *X does not have an $\mathcal{F}$-link, or*
2. *X does have an $\mathcal{F}$-link but it does not lead to an $\mathcal{F}$-Sarkisov link on X, or*
3. *X follows the 2-ray game of $\mathcal{F}$ to a Sarkisov link, and we are in one of the cases*
   (A) $e=a=0$, $b=c=1$,
   (B) $e=a=b=0$, $c=1$,
   (C) $e=-1$, $a=b=c=0$.

Proof. Suppose the given 2-ray game on $\mathcal{F}$ does restrict to a Sarkisov link on X. In particular, X has terminal singularities, so $e\leq 0$ by Lemma 4.4.1. If $e<0$, Lemma 4.4.1 requires $S_k(u,v)t^2\in f$, where S_k is a general polynomial with variables u,v of degree $-e=k>0$. The numerology presented in Table 4.2, shows that $-K_X\sim(2-k-a-b-c)L+M$.

This, together with Lemma 4.4.4, gives the inequality $k + a + c < 2$. But this can be satisfied only if $k = 1$ and $a = b = c = 0$, which is the case (3C).
In the case $e = 0$, a similar argument using the result of Lemma 4.4.4 forces $a + c < 2$, and this leads immediately to cases (3A,3B) or $e = a = b = c = 0$. but this case gets eliminated by Theorem 4.4.2. □

In fact, all solutions (3A–3C) provide Sarkisov links when X is general; these are respectively families No. 5, 2 and 1 in Table 4.1.

4.4.3 Families embedded in Type (i) scrolls

Let us recall that the variable with ratio weight zero is fixed to be x throughout this part. The following lemma forces strong restrictions on f, the defining polynomial of X. It uses the condition on the singularities of X.

Lemma 4.4.10. *Let $X \subset \mathcal{F}$ be a hypersurface of $\mathcal{F}$ of a Type (i), defined by the polynomial f as*

$$\begin{pmatrix} -e \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -a & -b & -c \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix},$$

where $a, b, c > 0$. If there is no term of the form $S_d(u,v)x^k l(y,z,t)$ in the equation of f, then X is not terminal, where l is either a linear form on y, z, t or is a constant.

Proof. By Theorem 4.4.2, f must include at least a monomial with no u or v in it. This already means $e \geq 0$. Let Γ be the curve defined by $(y = z = t = 0)$. If $e = 0$, then $x^4 \in f$ and there is nothing to prove. If $e > 0$, then $\Gamma \subset X$ and in fact by easy computations one could see that $\Gamma \subset \operatorname{Bs}|D|$. If there is no term of the form $S_d(u,v)x^k l(y,z,t)$ in f, then X is singular along Γ. In particular, the singular locus of X is not isolated and hence X cannot be terminal. □

If a, b, c are all nonzero, then by Theorem 4.4.2 f must include at least one pure monomial in the x, y, z, t variables. But this monomial cannot be x^4, as if otherwise holds, then Lemma 4.4.4 implies $a + c < 2$ which cannot be satisfied for any pair of positive integers a and c. Hence $abc \neq 0$ implies $e \neq 0$.

On the other hand, if one of a, b, c is zero, then Proposition 4.4.7 implies $e = 0$. If only two of a, b, c is zero, then irreducibility of X forces $e = 0$. The case $a = b = c = 0$ has been considered in Theorem 4.2.3.

The following families have already been studied in Theorem 4.2.3.

$$X \in \begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}$$

$$X \in \begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{pmatrix}$$

$$X \in \begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{pmatrix}$$

Now we consider the families with $e > 0$. We will specify each family by a sequence of positive integers correspond to $(a, b, c; e)$ which represent the following:

$$X \in \begin{pmatrix} -e \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -a & -b & -c \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Note that the columns of the action matrix of $\mathcal{F}$ are not necessarily in order. But the 2-ray game is played each time after considering the appropriate order.
We also introduce two numbers n and κ, which will simplify our notation, by

$$n = a + b + c, \qquad \kappa = 2 + e - a - b - c \quad .$$

Note that the number κ is associated to the degree of the anticanonical class of X and determines it uniquely as $-K_X \sim \kappa L + M$. Let us recall that L is the divisor linearly

equivalent to $(u = 0)$ and M is the one equivalent to $(x = 0)$.
We will be considering every X defined by $(a, b, c; e)$ by varying $n \in \mathbb{N}$ and spot families which link to a different Mori fibre space. The cases $n = 0, 1, 2$ have already been analysed.

- $n = 3$

The only option for $n = 3$ is when $a = b = c = 1$. By Lemma 4.4.1 $e \leq c$, which can only be satisfied by $e = 1, 2$. The analysis of the case $(1, 1, 1; 1)$ is the Family 7 in Table 4.1.

A general X defined by $(1, 1, 1; 2)$ is not terminal as it does not not satisfy conditions of Lemma 4.4.10.

- $n = 4$

This case has only two possibilities: $(1, 1, 2; e)$ and $(1, 2, 1; e)$. By Lemma 4.4.10 we must have $e \leq 2$. If $e < 2$, for both cases X fails to satisfy Lemma 4.4.4. Remaining cases provide $\mathcal{F}$-Sarkisov links to other Mori fibre spaces. These are Families 8 and 9 in Tables 4.1.

- $n = 5$

Different partitions of 5 allow us to have $(1, 1, 3; e)$, $(1, 3, 1; e)$, $(1, 2, 2; e)$ or $(2, 2, 1; e)$. For the first two cases, e cannot be less than 3 as otherwise it fails to fulfil the criteria of Lemma 4.4.4. It also cannot be more than 3 because of the condition imposed by Lemma 4.4.10. A similar argument for the other two cases bounds e to be equal to 2. However, $(1, 3, 1; 3)$ does not have Picard number two by Proposition 4.4.7. $(1, 2, 2; 2)$ also fails to satisfy Lemma 4.4.4 condition. The only remaining cases win to provide $\mathcal{F}$-Sarkisov links form Families 10 and 11 in Table 4.1.

- $n = 6$

Possible partitions of 6 give three candidates $(1, 1, 4; e)$, $(1, 2, 3; e)$, $(2, 2, 2; e)$. Applying numerical conditions imposed by Lemma 4.4.4, Lemma 4.4.10 and Proposition 4.4.7, and running the elimination process, we are left with the $(1, 1, 4; 4)$ and $(1, 2, 3; 3)$. In

Lemma 4.4.11, a reason for failure of $(1, 1, 4; 4)$ is given. The case $(1, 2, 3; 3)$ is precisely the Family 12 in Table 4.1.

Lemma 4.4.11. *Let $X \subset \mathcal{F}$ be defined by*

$$\begin{pmatrix} -4 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -4 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \quad ,$$

with variables u, v, x, y, z, t and equation f. Then a general X has Picard number strictly bigger than 2.

Proof. The proof here is the standard method used in Proposition 4.4.7. The only difference here is that instead of working with X we consider X_1, obtained by flopping a curve in X. Considering the 2-ray game of X restricted from that of $\mathcal{F}$, there is an Atiyah flop on X because we have a term $x^2t \in f$, which allows one to eliminate t in a neighbourhood of Γ_x. As X_1 is obtained by flopping a curve in X, they have isomorphic Picard groups. Hence $\rho_{X_1} > 2$ implies $\rho_X > 2$.

In order to finish the proof, we need to show that there are at least three divisors on X_1, which are linearly independent. We specify three divisors below and then conclude by proving they have non-linearly dependent intersections with three specific curves inside X_1. After a suitable change of coordinates we can assume $f = yz(y-z)(y-\lambda z)+t(x^2+g)$ (for some fixed cross ratio λ), where g is a polynomial of bi-degree $(0, 2)$. Setting $t = 0$ in X_1 leaves 4 divisors above the four roots $0, 1, \lambda, \infty$ of the quartic in y, z, each of them a divisor in X_1 isomorphic to $\mathbb{P}^2_{u:v:x}$. Let D be the divisor defined by $(y = 1, z = t = 0)$ and suppose $L \sim (u = 0)$ and $M \sim (x = 0)$ are two other divisors of X_1. We show that these divisors are linearly independent.

Define three curves on X_1 by

$$C_1 = (v = x = f = 0), \quad C_2 = (v = z = f = 0), \quad C_3 = (x = y = t = 0) \quad .$$

Computing the intersections leads to

$$\begin{pmatrix} C_1.L & C_1.M & C_1.D \\ C_2.L & C_2.M & C_2.D \\ C_3.L & C_3.M & C_3.D \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} .$$

This matrix has full rank and this completes the proof. □

- $n = 7$

Considering different partitions of 7 and applying the numerical elimination process as before, it turns out that there is only one family of three-folds for which a general member is not birationally rigid, which is $(1, 2, 4; 4)$. This forms Family 13 in Table 4.1.

The following lemma shows that we only need to consider cases where $n \leq 7$.

Lemma 4.4.12. *Any X with $n > 7$ does not link to any other Mori fibre space by an $\mathcal{F}$-link.*

Proof. Let X be defined by

$$\begin{pmatrix} -e \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 0 & 0 & 1 & \beta_1 & \beta_2 & \beta_3 \end{pmatrix} ,$$

where $\{\beta_1, \beta_2, \beta_3\} = \{1, 1, 2\}$ and variables are in order $u = v \prec x \preceq x_1 \preceq x_2 \preceq x_3$. Lemma 4.4.10 implies $e \in \{\alpha_i - m \,|\, 1 \leq i \leq 3\,, m = 0, 1\}$. By the adjunction formula $-K_X \sim (2 - m + \alpha_i - \Sigma\alpha_j)L + M$. To fulfil $-K_X \in \mathrm{Int}(\mathrm{Mob}(X))$, the requirement of Lemma 4.4.4, we must have

$$m + \alpha_1 + \alpha_2 + \alpha_3 - \alpha_i < 2 + \frac{\alpha_2}{\beta_2} .$$

Proposition 4.4.7, together with Lemma 4.4.4 and Theorem 4.4.2, shows that this inequality has no solution for any choice of m and i. □

Chapter 5

Stability and rank 3 toric varieties

We begin this chapter by discussing previous works regarding the behaviour of some **bad** models of low degree del Pezzo fibrations. In the examples we know, this occurs when the 3-fold X has a critical locus in some sense; for example an unexpected singular point. The main idea concerns a stability condition for degree 3 del Pezzo fibrations due to Corti and Kollár. We summarise these notions in Section 5.1. In Section 5.2, we show how our methods of working with rank 3 toric varieties, explained in Chapter 3, allow one to analyse Kollár's maps explicitly. The remainder of this chapter is a first attempt to generalise the notion of stability to degree 2 del Pezzo fibrations which coincides with Corti's notion of standard model [Cor96].

5.1 Historical notes

Many examples of nonrigid dP_2 fibrations were produced in Chapter 4. It would be interesting to see under what conditions a dP_2 fibration is birationally rigid. Section 5.1.1 presents the work of Grinenko, which seems to provide the most suitable conditions for a low degree del Pezzo fibration to be birationally rigid.

5.1.1 Grinenko's results

Construction of dP_2 fibrations.

Grinenko in [Gri00a] §3.1, [Gria] §3.3 proposes the following construction for degree 2 del Pezzo fibrations over $\mathbb{P}^1$.

Consider a rank 3 vector bundle $\mathcal{E}$ over $\mathbb{P}^1$ defined as

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1}(n_2) \qquad ,$$

where $0 \leq n_1 \leq n_2$, and let $V = \mathrm{Proj}_{\mathbb{P}^1} \mathcal{E}$ that is, the fibrewise projectivisation of $\mathcal{E}$. There is, of course, a natural projection $\pi \colon V \to \mathbb{P}^1$. Let M denote the class of the tautological bundle which satisfies $\pi_* \mathcal{O}(M) = \mathcal{E}$, and L the class of a fibre of π. Let X be a 2-to-1 cover of V branched over the divisor

$$R = 4M + 2aL \qquad ,$$

for some integer a.

The 3-fold X constructed this way, defines a smooth dP_2 fibration for suitable (n_1, n_2, a). After this construction, Grinenko considers the 2-ray game of these varieties as a way to study their rigidity. In contrast with our construction in Chapter 4, Grinenko's varieties form the smooth cases of our studies, i.e. when X is defined by

$$\begin{pmatrix} -2a \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -n_1 & -n_2 & -a \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad .$$

Rigidity

After studying birational geometry of these varieties, Grinenko proposes the following statement.

Conjecture 5.1.1. [[Gric], Conjecture 1.5] [[Gri00a], Conjecture 1.6] [[Gria], Conjecture 2.5] *Let X be a (smooth) del Pezzo fibration of degree 1,2 or 3. Then X is birationally* *rigid if and only if* $-K_X \notin \mathrm{Int}(\mathrm{Mob}(X))$.

In fact, this conjecture was proposed for smooth cases only in [Gric] and [Gri00a], while it is stated without the smooth condition in [Gria].

Remark 5.1.2. This conjecture is proved for smooth fibrations with degree 1 in [Gri00a] Theorem 2.6, and it is proved for smooth degree 2 del Pezzo fibrations which satisfy some extra conditions in [Gri00a] Theorem 3.1.

An interesting question, arising naturally, is the following:

Question 5.1.3. Up to what extent of generality does the conjecture above make sense?

Example 4.4.4 in [BCZ04] shows that this conjecture does not hold in general for degree 3 del Pezzo fibrations. In that example, they consider a 3-fold X, which is a degree 3 del Pezzo fibration over $\mathbb{P}^1$ with an isolated cD_4 singularity. They show that $-K_X \notin \mathrm{Int}(\mathrm{Mob}(X))$, but X is birational to another dP_3 fibration X' with $-K_{X'} \in \mathrm{Int}(\mathrm{Mob}(X'))$ and X' is birational to a conic bundle over $\mathbb{P}^2$. In other words, X is not birationally rigid. This example is explained in Example 5.1.7 below. This example suggests that the notion of stability (for cubic surface fibrations) may be the solution to Question 5.1.3.

We construct an analogous example of $dP_2/\mathbb{P}^1$ in Example 5.3.2 which demonstrates that the conjecture does not hold with weakened conditions. In fact it shows that the conjecture is not true even if we consider Gorenstein varieties.

5.1.2 Kollár's stability and rigidity of cubic fibrations

In [Kol97], Kollár introduced the notion of stability for hypersurfaces inside projective space defined over a principal ideal domain. After a brief review of his work, we discuss the relation of stability to rigidity of cubic fibrations. Everything in this section is taken from [Cor96], [Kol97] and [BCZ04].

Definition 5.1.4. ([Kol97] 2.1) Let $\mathcal{O}$ be a principal ideal domain with field of fractions K. A *weight system* (x, ω) on $\mathcal{O}[y_0, \ldots, y_n]$ is a choice of coordinates

$$(x_0, \ldots, x_n)^t = M(y_0, \ldots, y_n)^t, \quad \text{where} \quad M \in \mathrm{SL}(n+1, \mathcal{O})$$

and weight of each x_i is $\omega_i \in \mathbb{R}$. The weight system (x, ω) is called integral (respectively rational), if $\omega_i \in \mathbb{Z}$ (respectively $\omega_i \in \mathbb{Q}$). The weight system is called *trivial* if ω_i are equal.

Definition 5.1.5. ([Kol97] 3.2) Let $f_K \in K[y_0, \ldots, y_n]$ be a polynomial and $p \in \mathcal{O}$ be a prime element. There exist $s \in \mathbb{Z}$ and $p' \in \mathcal{O}$ prime to p such that $f = p^{-s}.p'.f_K \in \mathcal{O}[y_0, \ldots, y_n]$. Such f is called the $\mathcal{O}$-model of f_K and the largest s with this property is called the *multiplicity* of f_K at p; we denote this by $\mathrm{mult}_p f_K$.

Definition 5.1.6. ([Kol97] 3.3) Let the notation be the same as in Definition 5.1.5. Suppose $f \in \mathcal{O}[y_0, \ldots, y_n]$ is a homogeneous polynomial and $X_{\mathcal{O}} \subset \mathbb{P}^n_{\mathcal{O}}$ the hypersurface defined by the equation $(f = 0)$.

(5.1.6.1) An integral weight system (x, ω) over $\mathcal{O}$ is called

$$\left\{\begin{array}{l} \textit{properly stable} \\ \textit{semi-stable} \\ \textit{unstable} \end{array}\right. \quad \textit{on } f \textit{ at } p \textit{ if } \ \mathrm{mult}_p f(p^{\omega} x) \left\{\begin{array}{l} < \\ \leq \\ > \end{array}\right. \quad \frac{\deg f}{n+1} \sum_i \omega_i \,.$$

(5.1.6.2) f (or $X_{\mathcal{O}}$) is called *properly stable* (resp. *semistable*) at p over $\mathcal{O}$ if every weight system is properly stable (resp. semistable) on f at p.

(5.1.6.3) f (or $X_{\mathcal{O}}$) is called *unstable* at p over $\mathcal{O}$ if there is an unstable weight system on f at p.

(5.1.6.4) f (or $X_{\mathcal{O}}$) is called *properly stable* (resp. *semistable*) over $\mathcal{O}$ if it is properly stable (resp. semistable) at p over $\mathcal{O}$ for every prime $p \in \mathcal{O}$. f (or $X_{\mathcal{O}}$) is called *unstable* over $\mathcal{O}$ if it is unstable at p over $\mathcal{O}$ for some p.

With $(p^\omega x)$ we mean the obvious set of coordinates $(p^{\omega_0}x_0, \dots, p^{\omega_n}x_n)$.

A procedure to find semistable models.([Kol97] 4.3) We start with a homogeneous polynomial $f_K \in K[y_0, \dots, y_n]$.

Step 1: Find any $\mathcal{O}$-model f_1 of f_K.

Step 2: Assume that we already have f_j. If f_j is semi-stable at every prime p, then we are done.

Step 3: Otherwise there is a prime p and an integral weight system (x, ω) which is unstable on f_j. Set

$$f_{j+1} = p^{-s}.f_j(p^{\omega_0}x_0, \dots, p^{\omega_n}x_n), \quad \text{where } s = \text{mult}_p\, f_i(p^\omega x) \quad ,$$

and go back to Step 2.

Example 5.1.7. [[Cor96] Example 5.8., [Kol97] Example 6.4.3., [BCZ04] 4.4.4.] Let $\mathcal{F}$ be a rank 2 toric 4-fold defined by

(1) $\text{Cox}(\mathcal{F}) = \mathbb{C}[u, v, x, y, z, t]$,

(2) the irrelevant ideal $I = (u, v) \cap (x, y, z, t)$ and

(3) the action of $(\mathbb{C}^*)^2$ by

$$\begin{pmatrix} 1 & 1 & 0 & -2 & -2 & -4 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} .$$

Consider the hypersurface $X \subset \mathcal{F}$ defined by polynomial f of bi-degree $(-4, 3)$. For a general X, this polynomial is a combination of polynomials from the Newton polygon of

X:

deg of u, v coefficient	
0	$xy^2 \quad xyz \quad xz^2 \quad x^2t$
2	$y^3 \quad y^2z \quad yz^2 \quad z^3 \quad xyt \quad xzt$
4	$y^2t \quad yzt \quad z^2t \quad xt^2$
6	$yt^2 \quad zt^2$
8	t^3

Let X be defined by a special f with the property that u^i divide the coefficient polynomials according to the table

i	monomial
1	$xyt \quad xzt$
2	$y^2t \quad yzt \quad z^2t$
3	xt^2
4	$yt^2 \quad zt^2$
6	t^3

and general coefficients otherwise. The base locus of this system is the point $p_{v,t} = (0 : 1; 0 : 0 : 0 : 1) \in X$. By Bertini's theorem, X is nonsingular away from this point. The germ $(p_{vt} \in X) \cong (0 \in (C^4, x^2+y^3+z^3+u^6)$, in particular this point has a cD_4 singularity. The fact that $x^2t \in f$ implies $-K_X \notin \mathrm{Int}(\mathrm{Mob}(X))$.

Let us consider X as a cubic surface in $\mathbb{P}^3_K$, where $K = \mathbb{C}(u, v)$. By Definition 5.1.6, X is unstable with respect to the weight system $(3, 2, 2, 0)$ as

$$\mathrm{mult}_u f_K(u^3x, u^2y, u^2z, t) = 6 < \frac{3}{4}(3+2+2) \quad .$$

Running the stabilisation process on X, explained below, gives a square birational 3-fold X' to X. This X is defined by a polynomial g in the ambient variety of the weight system

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad ,$$

where the variables are u, v, x', y', z', t' for

$$x' = u^3 x \quad y' = u^2 y \quad z' = u^2 z \quad t' = t \quad .$$

The polynomial g is obtained by replacing x, y, z, t in f with x', y', z', t' and cancelling the factor u^6. The variety X' is smooth with an Eckardt point. It is easy to see that $-K_{X'} \in \mathrm{Int}(\mathrm{Mob}(X'))$ and the $\mathcal{F}'$-link of X' shows that X' (and hence X) is birational to a conic bundle. In particular, X is not birationally rigid even though $-K_X \notin \mathrm{Int}(\mathrm{Mob}(X))$.

Proposition 5.1.8 ([Kol97] Proposition 6.4.2). *In order to check semi-stability of a family of cubic surfaces over a one dimensional regular scheme, it is sufficient to use weight systems with the following five weight systems:*

$$(0,0,0,1), \quad (0,0,1,1), \quad (0,1,1,1), \quad (0,1,2,2), \quad (0,2,2,3) \quad .$$

5.2 Factorisation of Kollár maps by rank 3 Cox rings

In this section we show how the stabilising maps of Kollár for cubic surface fibrations, with $\mathbb{Q}$-factorial terminal singularity and $\rho_X = 2$, factor through Sarkisov links. In order to give our explicit links, we use our results on rank 3 toric varieties from Chapter 3.

Lemma 5.2.1. *Let X be a dP_3 fibration, in particular X is $\mathbb{Q}$-factorial with terminal singularity and* $\mathrm{Pic}(X) \cong \mathbb{Z}^2$. *If X is unstable, then it is unstable with respect to one of the weight systems*

$$(2,2,1,0) \quad or \quad (3,2,2,0) \quad .$$

Proof. Without loss of generality we can assume that X is unstable at $(u = 0) \in \mathbb{P}^1_{u:v}$. Let the variables of $\mathbb{P}^3$ be x, y, z, t. If X is unstable with respect to $(1,0,0,0)$, then the defining polynomial of X is of the form $uf = xg$ and hence the central fibre $(u = 0)$ splits into a conic $(g = 0)$ and a plane $(x = 0) \cong \mathbb{P}^2$. Therefore $\rho_X \geq 3$.

If X is unstable with respect to $(1,1,0,0)$, then collecting terms by increasing powers of x, y up to quadratic gives the defining polynomial of X as

$$F = u^2 f + uxg_1 + uyg_2 + x^2 h_1 + xyh_2 + y^2 h_3 \qquad ,$$

for some polynomials $f, g_1, g_2, h_1, h_2, h_3$ such that F has the right degree. It is clear that X is singular along the line $(u = x = y = 0)$ as each point of this line has multiplicity at least 2. Therefore X is not terminal.

If X is unstable with respect to $(1, 1, 1, 0)$, a similar argument shows that the point p_{vt} has a canonical singularity $x^3 + y^3 + z^3 + v^k + \dots$. □

Remark 5.2.2. Lemma 5.2.1 can be compared to [Cor96] §3, where Corti defined standard models of cubic fibrations and proves any non-standard model can be mapped to a standard one using one of the 3 weights $(1, 1, 1, 0)$, $(1, 1, 0, 0)$ or $(1, 0, 0, 0)$.

Theorem 5.2.3. *Let X be a dP_3 fibration, in particular X is $\mathbb{Q}$-factorial with terminal singularities and $\mathrm{Pic}(X) \cong \mathbb{Z}^2$. If $X/\mathbb{P}^1$ is unstable then the stabilising map has a factorisation through a Sarkisov link in one of the following ways:*

(i) If it is unstable with respect to $(3, 2, 2, 0)$, then it can be stabilised by a weighted blow up at a point, followed by 3 Francia flips and a divisorial contraction to a point.

(ii) If it is unstable with respect to $(2, 1, 1, 0)$, then it can be stabilised by a weighted blow up at a point, followed by 3 Atiyah flops and a divisorial contraction to a point.

Proof. Let $X \subset \mathcal{F}$ be a dP_3 fibration, where $\mathcal{F}$ is defined by

(a) $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[u_0, v_0, x_0, y_0, z_0, t_0]$,

(b) the irrelevant ideal is $I_{\mathcal{F}} = (u_0, v_0) \cap (x_0, y_0, z_0, t_0)$ and

(c) the action of $(\mathbb{C}^*)^2$ is

$$\begin{pmatrix} 1 & 1 & -\alpha & -\beta & -\gamma & -\delta \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

for $\alpha, \beta, \gamma, \delta \geq 0$ integers. Suppose X is defined by the polynomial f of bi-degree $(-e, 3)$.

Let $\mathcal{O}$ be the coordinate ring of $\mathbb{P}^1_{u_0:v_0}$ with field of fractions K and consider the cubic surface $X_{\mathcal{O}} : (f = 0) \subset \mathbb{P}^3_K$. Assume $X_{\mathcal{O}}$ is unstable with respect to $(3, 2, 2, 0)$. In other words $\mathrm{mult}_{u_0} f(x_0 u_0^3, y_0 u_0^2, z_0 u_0^2, t_0) \geq 6$. We do the stabilising map for $X_{\mathcal{O}}$ in a slightly different way. This is done in 4 steps.

Step 1. Toric decomposition

Consider the rank 3 toric variety $\tilde{\mathcal{F}}$ defined by

(a) $\mathrm{Cox}(\tilde{\mathcal{F}}) = \mathbb{C}[u, v, x, y, z, t, w]$,

(b) the irrelevant ideal is $I_{\tilde{\mathcal{F}}} = (u, v) \cap (x, y, z, t) \cap (v, w) \cap (t, w) \cap (u, x, y, z)$ and

(c) the action of $(\mathbb{C}^*)^3$ is

$$\begin{pmatrix} 1 & 1 & -\alpha & -\beta & -\gamma & -\delta & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 3 & 2 & 2 & 0 & -1 \end{pmatrix} .$$

By Proposition 3.3.2, $\tilde{\mathcal{F}}$ is a $(1, 3, 2, 2)$ blow up of $\mathcal{F}$ at the point $p = (0 : 1; 0 : 0 : 0 : 1)$. The blow up map is

$$\varphi \colon \mathrm{Cox}(\mathcal{F}) \to \mathrm{Cox}(\tilde{\mathcal{F}})$$

$$(u_0, v_0, x_0, y_0, z_0, t_0) \mapsto (uw, v, xw^3, yw^2, zw^2, t) \quad .$$

Under this map the divisor $\tilde{E} : (w = 0) \subset \tilde{\mathcal{F}}$ gets contracted to the point $p \in \mathcal{F}$. Using the two components (w, v) and (w, t) of the irrelevant ideal of $\tilde{\mathcal{F}}$ one can stabilise the actions of $(\mathbb{C}^*)^3$ on $\tilde{E}$ by fixing v and t to be nonzero. This shows that $\tilde{E} \cong \mathbb{P}(1, 2, 2, 3)$.

Now consider the toric variety $\tilde{\mathcal{F}}'$, where $\mathrm{Cox}(\tilde{\mathcal{F}}')$ is the same as that of $\tilde{\mathcal{F}}$ but the irrelevant ideal is

$$I_{\tilde{\mathcal{F}}'} = (u, v) \cap (u, x) \cap (w, y, z, t) \cap (v, w) \cap (x, y, z, t)$$

and the action

$$\begin{pmatrix} 1 & 1 & -\alpha & 1-\beta & 1-\gamma & 3-\delta & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 & -1 & -3 & -1 \end{pmatrix},$$

which is obtained by an action of $\mathrm{SL}(3, \mathbb{Z})$ on the action matrix of $\tilde{\mathcal{F}}$.

Let $\mathcal{F}'$ be the rank two toric variety with

(a) $\mathrm{Cox}(\mathcal{F}') = \mathbb{C}[w_1, v_1, x_1, y_1, z_1, t_1]$,

(b) the irrelevant ideal is $I_{\mathcal{F}} = (w_1, v_1) \cap (x_1, y_1, z_1, t_1)$ and

(c) the action of $(\mathbb{C}^*)^2$ is

$$\begin{pmatrix} 1 & 1 & -\alpha & 1-\beta & 1-\gamma & 3-\delta \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Similar to $\tilde{\mathcal{F}}$, the variety $\tilde{\mathcal{F}}'$ is the blow up of $\mathcal{F}'$ at the point $q = (0:1:1:0:0:0)$, with the map

$$\psi \colon \mathrm{Cox}(\mathcal{F}') \to \mathrm{Cox}(\tilde{\mathcal{F}}')$$
$$(w_1, v_1, x_1, y_1, z_1, t_1) \mapsto (uw, v, x, uy, uz, u^3 t) \quad .$$

Similarly, the exceptional locus is the divisor $\tilde{E}' \cong \mathbb{P}(1, 1, 1, 3)$.

Step 2. Flipping map $\tau \colon \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}'$.

Let $\bar{\tau} \colon \mathrm{Cox}(\tilde{\mathcal{F}}') \to \mathrm{Cox}(\tilde{\mathcal{F}})$ be the identity map. We show that the induced map $\tau \colon \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}'$ is an anti-flip.

Let us recall the irrelevant ideals of these varieties:

$$I_{\tilde{\mathcal{F}}} = (u, v) \cap (x, y, z, t) \cap (v, w) \cap (t, w) \cap (u, x, y, z)$$

$$I_{\tilde{\mathcal{F}}'} = (u, v) \cap (u, x) \cap (w, y, z, t) \cap (v, w) \cap (x, y, z, t) \quad .$$

Our proof here is set theoretic. Consider the set $(v = 0)$, we show that τ restricted to this set is an isomorphism. It is easy to see from the ideals above that $(v = 0)$ implies $uw \neq 0$ on both $\tilde{\mathcal{F}}'$ and $\tilde{\mathcal{F}}$. Applying these conditions to the two ideals makes them identical; hence restriction of τ to $(v = 0)$ is the identity.

Now let $(v \neq 0)$. Hence we can use the first component of the $(\mathbb{C}^*)^3$ action to stabilise v; we are left with

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -1 & -3 & -1 \end{pmatrix}$$

and ideals

$$I_1 = (x, y, z, t) \cap (t, w) \cap (u, x, y, z), \qquad I_2 = (x, y, z, t) \cap (t, w, y, z) \cap (u, x) \quad .$$

An elementary set-theoretic argument shows that τ is an isomorphism away from $(u = x = 0) \subset \tilde{\mathcal{F}}$. But the locus $(u = x = 0) \subset \tilde{\mathcal{F}}$ is isomorphic to the surface

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

with irrelevant ideal $(y, z) \cap (t, w)$. Similarly, the inverse of τ is an isomorphism away from $(t = w = 0) \subset \tilde{\mathcal{F}}'$, which is the Hirzebruch surface $\mathbb{F}(1)$. One can check this by restricting the ideal I_1 and the action of $(\mathbb{C}^*)^2$ to this set. In fact τ can be factorised as

$$\begin{tikzcd} \tilde{\mathcal{F}} \arrow[dr, "\tau^-"'] & & \tilde{\mathcal{F}}' \arrow[dl, "\tau^+"] \\ & \mathcal{G} & \end{tikzcd}$$

where τ^- contracts the surface $(u = x = 0)$ to the line $\mathbb{P}^1_{y:z}$ and τ^+ contracts the surface $(t = w = 0)$ to the same line. In other words, τ is a transverse flip $(2, 1, -1, -1)$ along the line $\mathbb{P}^1_{y:z}$.

Step 3. Everything restricts to X.

The restriction of φ to X is the blow up of the point $p \in X$. Of course $p \in X$, as otherwise we must have $v^k t^3 \in f$; but this contradicts the assumption of X being unstable.

We claim that f has at least one monomial containing only the variables y, z. If, on the contrary, this condition does not hold, it is easy to see that X is unstable with respect to one of the weight systems $(1, 0, 0, 0)$, $(1, 1, 0, 0)$ or $(1, 1, 1, 0)$. This can be checked by drawing the Newton polygon of a cubic in $\mathbb{P}^3$ and arguing on the multiplicity. Therefore the exceptional locus of the blow up of $p \in X$ is the weak Fano surface $S_3 \subset \mathbb{P}(1, 2, 2, 3)$. The multiplicity is 6 by the assumption on the stability of X. It has become obvious that the restriction of the flip from the toric variety consists of 3 Francia flips.

Now consider the restriction of ψ to the image of this flip. We claim that the exceptional divisor in this case is isomorphic to $\mathbb{P}^2$. Note that f must include the monomial $x^2 t$, as otherwise the multiplicity condition on X implies that X is unstable with respect to $(1, 1, 1, 0)$ and hence not terminal. On the other hand, $\tilde{E}'$ is the set $(u = 0)$. Using this and the irrelevant ideal of $\tilde{\mathcal{F}}'$, we can eliminate the variable t in a neighbourhood of this set using $x^2 t \in f$. Hence the exceptional divisor is isomorphic to $\mathbb{P}^2$.

Step 4. 2-ray game on X.

So far we have constructed a decomposition of maps from X to X', another

$dP_3/\mathbb{P}^1$. In fact the two copies of $\mathbb{P}^1$ that are considered in these two fibrations have coordinates (u_0, v_0) and (w_1, v_1). But the construction of our maps show that $(u_0, v_0) = (w_1, v_1) = (uw, v)$; hence this $\mathbb{P}^1$ is preserved by this decomposition. The following picture show this sequence of maps, which is nothing but a type I Sarkisov link.

$$\begin{tikzcd}
 & \tilde{X} \arrow[r, dashed, "\text{flip}"] \arrow[dl, "\text{blow up}"'] & \tilde{X}' \arrow[dr, "\text{blow up}"] & \\
X \arrow[dr, "dP_3/\mathbb{P}^1"'] & & & X' \arrow[dl, "dP_3/\mathbb{P}^1"] \\
 & \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 &
\end{tikzcd}$$

The proof for the case where X is a $dP_3/\mathbb{P}^1$ fibration and is unstable with respect to $(2, 2, 1, 0)$ is similar to this case. $\square$

5.3 Stability of hypersurfaces in WPS

In this section, we give a stability condition for hypersurfaces of weighted projective space. This is the simplest generalisation of Kollár's stability. A procedure to find semistable models of an unstable model could be lifted from Kollár's procedure to this case, although termination is a different issue. We rewrite this in our case and then compare the differences in examples in the next section.

What is missing? We do not claim that this definition of stability is the universal one as it is not constructed by means of any geometric invariant theory. However, it should become clear at the end of this section that if there is such a universal stability, then it should include ours as a part of it. This is because

(1) It does coincide with the notion of *standard* models of Corti for dP_2 surfaces and his method of finding the standard models when restricted to $\mathbb{P}(1, 1, 1, 2)$ in all cases except for the special case [Cor96] 4.10.3.

(2) It works well in the cases we study.

In fact, there is no GIT theory behind our definition as the automorphism group of weighted projective space is not reductive. A better attempt could perhaps be a similar consideration to the work of Ross and Thomas in [RT] where they study similar stability conditions of orbifolds.

On the other hand, a disadvantage of this definition is that we do not have a Sarkisov decomposition as in Theorem 5.2.3. This is shown in Example 5.3.2. However, we show in Proposition 5.3.5 that there is a way of stabilising that variety for which the stabilisation process is a Sarkisov link.

Stability. Let $\mathcal{O}$ be a principal ideal domain with field of fractions K as before. An integral weight system (x, ω) is called trivial if $\omega_i = ka_i$ for all i and some integer k.

Definition 5.3.1. Let $f \in \mathcal{O}[y_0, \dots, y_n]$ be a homogeneous polynomial with respect to the weights of the variables in the weighted projective space $\mathbb{P} = \mathbb{P}_{\mathcal{O}}(a_0, \dots, a_n)$ and let $X_{\mathcal{O}} \subset \mathbb{P}$ be the hypersurface defined by the equation $(f = 0)$.

(5.3.1.1) An integral weight system (x, ω) over $\mathcal{O}$ is called

$$\left\{\begin{array}{l} \textit{properly stable} \\ \textit{semi-stable} \\ \textit{unstable} \end{array}\right. \quad \textit{on } f \textit{ at } p \textit{ if } \operatorname{mult}_p f(p^{\omega}x) \left\{\begin{array}{l} < \\ \leq \\ > \end{array}\right. \quad \frac{\deg f}{\sum_j a_j} \sum_i \omega_i \; .$$

(5.3.1.2) f (or $X_{\mathcal{O}}$) is called *properly stable* (resp. *semistable*) at p over $\mathcal{O}$ if every weight system is properly stable (resp. semistable) on f at p.

(5.3.1.3) f (or $X_{\mathcal{O}}$) is called *unstable* at p over $\mathcal{O}$ if there is an unstable weight system on f at p.

(5.3.1.4) f (or $X_{\mathcal{O}}$) is called *properly stable* (resp. *semistable*) over $\mathcal{O}$ if it is properly stable (resp. semistable) at p over $\mathcal{O}$ for every prime $p \in \mathcal{O}$. f (or $X_{\mathcal{O}}$) is called *unstable* over $\mathcal{O}$ if it is unstable at p over $\mathcal{O}$ for some p.

A procedure to find semistable models. We start with a polynomial $f_K \in K[y_0, \dots, y_n]$, which is homogeneous with respect to $\mathbb{P}$.

Step 1: Find any $\mathcal{O}$-model f_1 of f_K.

Step 2: Assume that we already have f_j. If f_j is semi-stable at every prime p, then we are done.

Step 3: Otherwise there is a prime p and an integral weight system (x, ω) which is unstable on f_j. Set

$$f_{j+1} = p^{-s}.f_j(p^{\omega_0}x_0, \dots, p^{\omega_n}x_n), \quad \text{where } s = \text{mult}_p\, f_i(p^{\omega}x) \quad ,$$

and go back to Step 2.

5.3.1 Unstable models of dP_2 fibrations

Definition 5.3.1 gives a stability condition for $dP_2/\mathbb{P}^1$. The following example shows that Conjecture 5.1.1 does not hold in general. In fact, it shows that this conjecture is not even true for Gorenstein case. As a matter of fact, if our definition of stability proves to be sensible then it might be reasonable to ask for the conjecture to be true for (semi-)stable $dP_2/\mathbb{P}^1$. Otherwise, there must be a better stability condition which suits the following example.

Example 5.3.2. Let $\mathcal{F}$ be a rank 2 toric variety defined by

(i) $\text{Cox}(\mathcal{F}) = \mathbb{C}[u, v, x, t, y, z]$,

(ii) irrelevant ideal $I = (u, v) \cap (x, t, y, z)$ and

(iii) the action of $(\mathbb{C}^*)^2$ by

$$\begin{pmatrix} 1 & 1 & 0 & -2 & -2 & -4 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix} .$$

Let $X \subset \mathcal{F}$ be a hypersurface defined by $(f = 0)$, where f is homogeneous of bi-degree $(-4, 4)$. A general such polynomial has the following Newton polygon:

deg of u, v coefficient	
0	$x^3z \quad t^2$
2	$xy^3 \quad ty^2 \quad x^2yz \quad xzt$
4	$yzt \quad xy^2z \quad y^4 \quad x^2z^2$
6	$xyz^2 \quad tz^2 \quad y^3z$
8	$xz^3 \quad y^2z^2$
10	yz^3
12	z^4

Let X be defined by a special f with the property that u^i should divide the coefficient polynomials according to the table

i	monomial
1	x^2yz
2	$xzt \quad xy^2z$
3	$yzt \quad y^3z$
4	x^2z^2
5	xyz^2
6	$tz^2 \quad y^2z^2$
8	xz^3
9	yz^3
12	z^4

and general coefficients otherwise. A simple computation using the Bertini theorem shows that X is smooth away from the point $p = (0 : 1 : 0 : 0 : 0 : 1)$. The germ at this point is isomorphic to $0 \in (\mathbb{C}^4, t^2 + x^3 + y^4 + u^{12})$; in particular p is a cE_6 singularity. Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1, p_u}$. Following our notation, $X_{\mathcal{O}}$ is unstable at u with respect to $(2, 3, 2, 0)$. Let

$X^1_{\mathcal{O}}$ be the model obtained after one loop through the algorithm above that finds the semistable model. One can see that $X^1_{\mathcal{O}}$ is unstable with respect to $(1,1,0,0)$. One could compare this with [Cor96] 4.7.1 to see that X^1 is singular along a line and is not standard. Let us denote by $X^2_{\mathcal{O}}$ the model obtained by running the procedure one round for $X^1_{\mathcal{O}}$. It is again easy to check that $X^2_{\mathcal{O}}$ is unstable with respect to $(1,2,1,0)$. The model obtained by stabilising $X^2_{\mathcal{O}}$ is stable; we denote it by X'. This 3-fold is defined by

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{pmatrix}$$

and is general; in particular X' is smooth and has an $\mathcal{F}$-link by Table 4.1 to a Fano 3-fold.

Remark 5.3.3. Example 5.3.2 shows that Conjecture 5.1.1 does not necessarily hold for Gorenstein $dP_2/\mathbb{P}^1$ with terminal singularities.

Remark 5.3.4. In comparison with stability of $dP_3/\mathbb{P}^1$, the example above shows that the stabilisation with respect to a weight system for $dP_2/\mathbb{P}^1$ does not necessarily factor through Sarkisov links. However, in the next proposition we show that there is another weight system such that X is unstable with respect to it and its stabilising process is a Sarkisov link.

Proposition 5.3.5. *Let $X \subset \mathcal{F}$ be the $dP_2/\mathbb{P}^1$ in Example 5.3.2 and X' be its stable model obtained by running the procedure as in the example. X is also unstable with respect to the weight system $(4,6,3,0)$, and the procedure to stabilise X with respect to this weight system is a Sarkisov link.*

Proof. One checks easily that $X_{\mathcal{O}}$ is also unstable with respect to $(4,6,3,0)$ with multiplicity 12. Running the procedure for this weight system can be factorised in a blow up of the point $p \in X$, followed by 2 flips $(3,1,-1,-1)$ and a divisorial contraction to a point with exceptional divisor isomorphic to $\mathbb{P}(1,1,2)$. These can all be done in exactly the same way as in the proof of Theorem 5.2.3 using the rank 3 toric variety T defined by

(i) $\mathrm{Cox}(T) = \mathbb{C}[u,v,x,t,y,z,w]$,

(ii) irrelevant ideal $I = (u, v) \cap (x, y, z, t) \cap (u, x, t, y) \cap (w, v) \cap (w, z)$ and

(iii) the action of $(\mathbb{C}^*)^3$ on $\mathrm{Cox}(T)$ by

$$\begin{pmatrix} 1 & 1 & 0 & -2 & -2 & -4 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 6 & 3 & 0 & -1 \end{pmatrix}.$$

Note that the essential monomials for this computation are t^2, ty^2 and x^3z, which allow one to restrict all maps of the 2-ray game of T of Type I to X.

□

5.3.2 Other stability conditions

It has become clear that our definition of stability for dP_2 surfaces over $\mathcal{O}$ is sensible but not quite universal. There are few options that one could think of as the *better* stability set up. But studying each of these in a appropriate way requires much more work; perhaps as a project. We just outline the ideas of each.

(1) Take $S : (f_4 = 0) \subset \mathbb{P}(1, 1, 1, 2)$ and consider the embedding of $\mathbb{P}(1, 1, 1, 2)$ in $\mathbb{P}^6$ by $\mathcal{O}(2)$. Then consider the hypersurface defined by the image of $(f = 0)$ in $\mathbb{P}^6$; define some stability on it in the sense of Corti–Kollár which restricts suitably to the image of S. This is inspired by [Cor96] 4.11.

(2) Consider the Kodaira embedding of $\mathbb{P}(1, 1, 1, 2)$ in $\mathbb{P} = \mathbb{P}(1, 1, 1, 2, 2, 2, 2, 2, 2, 2)$. Consider the reductive part of the automorphism group of $\mathbb{P}$, that is $\mathrm{GL}(3, \mathbb{Z}) \times \mathrm{GL}(7, \mathbb{Z})$, and look at the image of $(f = 0)$ in $\mathbb{P}$. Again, define a stability which is compatible in components and restrict it to the image of S. The advantage of this construction is that the restriction of this reductive part is exactly the automorphism group of $\mathbb{P}(1, 1, 1, 2)$. This idea is inspired by the work of Ross and Thomas on stability of orbifolds [RT].

(3) Instead of working with integral weight systems, consider rational weights.

In fact, we do not know if these lead to different theories or not, or which of them is the most suitable and the *correct* one! However, we know that the good candidate must match with the set up that we worked with throughout this section. On the other hand the following theorem shows that at each step of the semi-stabilisaion process, introduced in this chapter, the singularities of the variety improve.

Theorem 5.3.6. *Let $X \subset \mathcal{F}$ be a $dP_2/\mathbb{P}^1$ fibration as before. Suppose X is unstable with respect to $(\omega_0, \ldots, \omega_3)$ at $p = (u = 0)$ and let $D = (u = 0) \subset \mathcal{F}$ be the fibre above p, considered as a divisor in $\mathcal{F}$. Let $X' \in \mathcal{F}'$ be the model obtained after running the process of stabilisation one time. Then*

$$a(\nu, K_{\mathcal{F}'} + X') \geq a(\nu, K_{\mathcal{F}} + X + (\sum \omega_i - m)D) \quad ,$$

where $a(\nu, A)$ denotes the discrepancy of the divisor A with valuation ν.

Proof. By Lemma 2.21 in [Cor96], it is sufficient to show that $a(\nu_E, K_{\mathcal{F}} + X + (\sum \omega_i - m)D) \leq 0$, where ν_E is the valuation correspond to the ϕ^{-1}-exceptional divisor E for $\phi\colon \mathcal{F} \dashrightarrow \mathcal{F}'$. A simple calculation (c.f. [Cor96] Lemma 2.21) together with the conditions on unstability of X at p implies that $a = \sum \omega_i - m \leq 0$. $\square$

5.4 A nonrational variety with big pliability

In this section, we give an example of a nonrational Mori fibre space with pliability strictly bigger than 2. This example is interesting only because of its exciting nature that allows us to run many explicit calculations on it.

Let $\mathcal{F}$ be a rank 2 toric variety with

(i) $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[u, v, x, t, s, y, z]$,

(ii) irrelevant ideal $I = (u, v) \cap (x, y, z, t)$,

(iii) and the action of $(\mathbb{C}^*)^2$ given by

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -1 & -1 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 \end{pmatrix}.$$

Let $D_1 \sim 3(x = 0) - (u = 0)$ and $D_2 \sim 4(x = 0) - 2(u = 0)$ be two divisors on $\mathcal{F}$ and let f and g be general polynomials with $f \in \mathrm{H}^0(\mathcal{F}, D_1)$ and $g \in \mathrm{H}^0(\mathcal{F}, D_2)$. Suppose X is the 3-fold complete intersection defined by $(f = g = 0) \subset \mathcal{F}$. We have

$$f = us + x^y + xt + v^2y^y + \dots \quad \text{and} \quad g = xs + t^2 + x^2y^2 + \dots \quad .$$

It is easy to check, using the methods in Chapter 4, that X has at worst terminal singularities, $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ and generic fibre of $X \to \mathbb{P}^1$ is a dP_2 surface; Hence X is a $dP_2/\mathbb{P}^1$ model.

One can run the 2-ray game of $\mathcal{F}$ and restrict it to X as in Chapter 4 to check that X has an $\mathcal{F}$-link starting with a flop, followed by a flip $(3, 1, -1, -1)$ to a dP_4 fibration over $\mathbb{P}^1$. Note that X is singular away from the point $p = (0 : 1; 0 : 0 : 1 : 0 : 0)$, which is terminal quotient of type $\frac{1}{3}(1, 1, 2)$ and the flip in the 2-ray game of X gets rid of this singularity. The $dP_4/\mathbb{P}^1$ at the end of the link is smooth and general in the family. Following [Shr06], by [Ful98] Example 3.2.11 one can see that the Euler characteristic of this $dP_4/\mathbb{P}^1$ is $\chi = -28$. Using the following result of Alexeev we conclude that X is not rational.

Theorem 5.4.1 ([Ale87] Theorem 2). *Let V be a standard fibration by del Pezzo surfaces of degree 4 over $\mathbb{P}^1$. If the topological Euler characteristic $\chi(V) \neq 0, -4, -8$, then V is not rational.*

So far we have shown that X is not rational but it is nonrigid with a $dP_4/\mathbb{P}^1$ model. The aim is to find another model for X. We do this by blowing up the singular point on X.

Let T be a rank 3 toric variety defined by

(i) $\mathrm{Cox}(T) = \mathbb{C}[u', v', x', t', s', y', z', w]$,

(ii) irrelevant ideal $I = (u', v') \cap (x', y', z', t') \cap (u', x', t', y', z') \cap (v', w) \cap (s', w)$,

(iii) and the action of $(\mathbb{C}^*)^3$ given by

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 & 0 \\ 3 & 0 & 4 & 2 & 0 & 1 & 1 & -3 \end{pmatrix}.$$

By Proposition 3.3.2, T is the blow up of $\mathcal{F}$ at the point p. The blow up map $T \to \mathcal{F}$ in coordinates is

$$(u', v', x', t', s', y', z', w) \mapsto (u'w, v', x'w^{\frac{4}{3}}, t'w^{\frac{2}{3}}, s', y'w^{\frac{1}{3}}, z'w^{\frac{1}{3}}) \quad .$$

One can see that this map restricts to X as the Kawamata blow up of the point $\frac{1}{3}(2, 1, 1)$. It is also easy to see that T is not well formed. Using our methods in Chapter 3, one can get the well formed matrix

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now define T' to be a rank 3 toric variety with

(i) $\mathrm{Cox}(T') = \mathrm{Cox}(T)$,

(ii) irrelevant ideal $I = (u', v') \cap (x', y', z', t') \cap (u', x') \cap (t', y', z', s', w)) \cap (v', w)$,

(iii) and the action of $(\mathbb{C}^*)^3$ given by

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 & -4 & -1 & -1 & -1 \end{pmatrix}.$$

A similar computation to the one in the proof of Theorem 5.2.3 shows that there is a flop $T \dashrightarrow T'$, which restricts to the blow up of X as the flop of 4 disjoint lines. In particular T' has a divisorial contraction to to the rank 2 variety $\mathcal{F}'$ with

(i) $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[w, v, x, t, s, y, z]$,

(ii) irrelevant ideal $I = (w, v) \cap (x, y, z, t)$,

(iii) and the action of $(\mathbb{C}^*)^2$ given by

$$\begin{pmatrix} 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 \end{pmatrix}.$$

Note that we abuse the notation for coordinate systems on $\mathcal{F}$ and $\mathcal{F}'$; But this must not cause any problem! This map restricts to the birational transform of X as a divisorial contraction to X', contracting a copy of $\mathbb{P}(1,1,2)$ to a nonsingular point. Considering all these maps, the defining polynomials of X' are

$$f' = s + w^2 + x^2y + wxt + v^2y^3 + \dots \quad \text{and} \quad g' = xs + t^2 + w^2x^2y^2 + \dots \quad .$$

The key point is that we are left with a linear term in s. This allows us to eliminate the variable s globally on X. In other words, X defined by

$$\begin{pmatrix} -2 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

It was shown in Theorem 4.2.3 that X' is birational to a conic bundle.

Chapter 6

Cubic surface fibrations over $\mathbb{P}^2$

Cubic surface fibrations over the projective 2-space form a class of Mori fibre spaces in dimension four. In this chapter we construct some families of varieties in this class, constructed as hypersurfaces inside rank two toric varieties, by methods similar to the dP_2 fibrations in Chapter 4. The aim is to study their birational geometry and compare with the situation in 3 dimensions.

6.1 Construction

Definition 6.1.1. A 4-fold cubic fibration over $\mathbb{P}^2$ is a normal, irreducible, projective, complex variety X such that

(a) X is $\mathbb{Q}$-factorial with at worst terminal singularities,

(b) $\operatorname{Pic} X \cong \mathbb{Z}^2$,

(c) there exists an extremal morphism of fibre type $\varphi\colon X \to \mathbb{P}^2$, and

(d) the generic fibre of φ is a degree 3 del Pezzo surface.

We denote this by $dP_3/\mathbb{P}^2$.

Let $\mathcal{F}$ be a weighted bundle over $\mathbb{P}^2$ defined by

(i) $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[u, v, w, x, y, z, t]$,

(ii) $I_{\mathcal{F}} = (u, v, w) \cap (x, y, z, t)$,

(iii) $(\mathbb{C}^*)^2$ action defined by

$$\begin{pmatrix} 1 & 1 & 1 & \alpha & \beta & \gamma & \delta \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{6.1}$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$.

6.1.1 Construction as hypersurfaces

Without loss of generality we can assume that matrix above is of the form

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -a & -b & -c \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{6.2}$$

where $a \le b \le c$ are non-negative integers. In particular, the variables are in the order $u = v \prec x \preceq y \preceq z \preceq t$.

Lemma 3.3.6 in Chapter 3 shows that $\mathrm{Pic}\,\mathcal{F} \cong \mathbb{Z}^2$. We denote the basis of $\mathrm{Pic}(\mathbb{F})$ by L, M, with sections $u \in H^0(\mathcal{F}, L)$ and $x \in H^0(\mathcal{F}, M)$.

Let $D \in |4M + dL|$ be a divisor in $\mathcal{F}$ for $d \in \mathbb{Z}$ and suppose $X \subset \mathcal{F}$ is a hypersurface defined by $X = (f = 0) \subset \mathcal{F}$ for a general $f \in \mathcal{O}_{\mathcal{F}}(D)$. The aim is to study the birational geometry of those X specified by $(a, b, c; d)$, which satisfy the conditions of Definition 6.1.1.

6.2 $dP_3/\mathbb{P}^2$ models

In this section, we find those $(a, b, c; d)$ for which the 3-fold X forms a degree 3 del Pezzo surface fibration over $\mathbb{P}^2$, as in Definition 6.1.1.

Lemma 6.2.1. *Let $X \subset \mathcal{F}$ be a general hypersurface defined as in 6.1.1 by sequence of integers $(a, b, c; d)$, where $0 \le a \le b \le c$ and $d > 0$. Then a general X is a $dP_3/\mathbb{P}^2$.*

Proof. If $d > 0$, then the defining polynomial of X is of the form $f = uf_1 + vf_2 + wf_3$ for some polynomials f_i with bidegree $(d-1, 3)$. It implies that the base locus of the linear system $|3M + dL|$ is empty and hence by the Bertini theorem X is smooth. By Theorem 6.4.4 below, $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ and hence X is a $dP_3/\mathbb{P}^2$. □

Lemma 6.2.2. *Let $X \subset \mathcal{F}$ be defined by $(a, b, c; 0)$ as before. Then X forms a $dP_3/\mathbb{P}^2$ for any triple (a, b, c) except for $a = b = 0$, $c > 1$.*

Proof. It is easy to check that for any (a, b, c), the base locus of $|3M|$ is empty and therefore X is smooth. If $a = b = 0$ and $c > 1$, then a typical argument shows that the Picard number of X is at least 3. By Theorem 6.4.4 $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ for all other cases. □

Lemma 6.2.3. *Let $X \subset \mathcal{F}$ be a hypersurface defined by $(a, b, c; d)$ as in 6.1.1, where $0 \leq a \leq b \leq c$ and $d < 0$. Then X is a $dP_3/\mathbb{P}^2$ if*

(i) the defining polynomial of X includes a monomial of the form $g_k(u, v, w)x^2L(y, z, t)$, where g_k is a homogeneous polynomial in variables u, v, w of degree $k \geq 0$ and L is a linear form in y, z, t, and

(ii) one of the following holds

$$d \leq 3a \leq 3b \quad or \quad d < 3a \leq 3b \qquad .$$

Proof. If $a = b = c = 0$, then $|3M + dL|$ has no sections. If $a = b = 0$ and $c > 0$, then $f = t.g$, hence X is reducible. If only $a = 0$ and $bc \neq 0$, then a similar argument to the one in Proposition 4.4.7 shows that $\rho_X > 2$.

Let $0 < a \leq b \leq c$ and suppose one of the $d \leq 3a \leq 3b$ or $d < 3a \leq 3b$ holds. Then Theorem 6.4.4 implies that $\mathrm{Pic}(X) \cong \mathbb{Z}^2$. If $d = 3a = 3b$, then by a similar argument to Lemma 4.4.11, $\rho_X > 2$ and hence X is not a $dP_3/\mathbb{P}^2$.

Now suppose X is defined such that $0 < a \leq b \leq c$. If the polynomial f has no term of type $g_k(u, v, w)x^2L(y, z, t)$, then a generic point on the surface $S = (y = z = t = 0) \subset X$ has multiplicity at least 2. Therefore X is singular along a 2-dimensional space. Therefore

X is not terminal. If f has such a term, then it is either smooth or it is singular only at finitely many points or along a line. $\square$

Combining Lemma 6.2.1, Lemma 6.2.2 and Lemma 6.2.3 enables us to give the following characteristic theorem.

Theorem 6.2.4. *Let $X \subset \mathcal{F}$ be a general hypersurface defined by $(a, b, c; d)$. Then one of the following holds:*

(1) If $d > 0$, then X is non-singular and satisfies conditions stated in Definition 6.1.1 .

(2) If $d = 0$, then X is a dP_3 fibration by Definition 6.1.1 for any triple (a, b, c) except for $a = b = 0$, $c > 1$.

(3) $d < 0$ and

(a) $3c < -d$, $|4M + dL|$ has no sections.

(b) $3a \leq 3b < -d \leq 3c$ and X is reducible, hence not a dP_3 fibration.

(c) $3a < -d \leq 3b \leq 3c$ and X has Picard number $\rho_X > 2$, hence does not satisfy conditions of a dP_3 fibration.

(d) $-d \leq 3a$. In this case, X is a dP_3 fibration over $\mathbb{P}^2$ only if the equation of f has a term of the form $g_k(u, v, w)x^2L(y, z, t)$ in it, where g_k is a homogeneous polynomial in variables u, v, w of degree $k \geq 0$ and L is linear.

6.3 $dP_3/\mathbb{P}^2$ as Mori dream spaces

In this section we show that unlike dimension 3, all dP_3 fibrations constructed above have a 2-ray game which is the restriction of that of the ambient space we consider. The idea is based on the following lemma of Kawamata, Matsuda and Matsuki.

Lemma 6.3.1. ([KMM87] Lemma 5.1.17) *If $\psi \colon X^- \to X^+$ is a flip (flop or antiflip) with exceptional loci $E^- \subset X^-$ and $E^+ \subset X^+$, then the pair $(\dim E^-, \dim E^+)$ is exactly* *one of the pairs*

$$(2,1) \qquad (2,2) \qquad (1,2) \quad .$$

Theorem 6.3.2. *Let $X \subset \mathcal{F}$ be a cubic fibration over $\mathbb{P}^2$ obtained from one of the cases in Theorem 6.2.4. Then the Type* III *or* IV *2-ray game of $\mathcal{F}$ induces the game on X.*

Proof. We prove the theorem case by case on the sign of d and we show that in each case the conditions on the dimension of contracted loci by Lemma 6.3.1 are satisfied.

Let $d > 0$. If $a > 0$, then the 2-ray game of $\mathcal{F}$ is continued be a flip which restricts to X with dimension pair $(1,2)$. For $a = 0$ and $b > 0$, the situation is $(2,1)$ and for $a = b = 0$ the game finishes by a divisorial contraction or a fibration; Which is fine as far as the 2-ray game of X is concerned.

For $d = 0$, If $a > 0$ then the first step of the game of $\mathcal{F}$ induces an isomorphism on X and the second step is of type $(2,1)$, divisorial contraction or fibration, respectively in cases a, b, $a = b < c$ and $a = b = c$.

If $a = 0$, then the game continues with a $(2,1)$ or divisorial contraction or a fibration exactly as the previous case.

Let $d < 0$. If $a > 0$ then the 2-ray game of $\mathcal{F}$ restricts to X by a $(2,1)$ or $(2,2)$. $\square$

Corollary 6.3.3. *X is a Mori dream space with* $\mathrm{Cox}(X) = \mathrm{Cox}(\mathcal{F})/(f = 0)$. *In particular* $\mathrm{Mob}(X)$ *is generated by L and $D_z = (z = 0)$.*

6.4 Nonrigid families

In this section we give some arguments which eliminate many cases that are not going to have an $\mathcal{F}$-link to another Mori fibre space. As a result a list of nonrigid families through their Type III or IV Sarkisov links is given.

Theorem 6.4.1. *If $-K_X \notin \mathrm{Int}(\mathrm{Mob}(X))$, then the last map of the 2-ray game of X is not extremal.*

Proof. This proof is similar to that of Lemma 4.4.4. $\square$

Lemma 6.4.2. *If $d < 0$, then $a + k \leq 2$.*

Proof. Using the adjunction formula, one can compute the anticanonical divisor of X as $-K_X \sim (3 + n - a - b - c)L + M$. Theorem 6.4.1 results in $-K_X \in \mathrm{Int}(\mathrm{Mob}(X))$, which holds if and only if $a + b + c - 3 - d < b$. This implies $a + c < 3 + d$.

On the other hand, from Theorem 6.2.4 we have $d \leq c - k$. These two inequalities show that $a + k \leq 2$. □

Corollary 6.4.3. $c < 7$.

Proof. Theorem 6.4.1 implies $a + c < 3 - d$. On the other hand, Theorem 6.3.2 requires $-d < c$. One can easily check the inequality using these together with Lemma 6.4.2. □

The inequalities above provide upper limits for (a, b, c). Using these and other information provided in this chapter one can prove that Theorem 6.4.5 below has the complete list.

Theorem 6.4.4. *Let $X \subset \mathcal{F}$ be a general $dP_3/\mathbb{P}^2$ as before. If $X \in \mathrm{Int}(\mathrm{Mob}(\mathcal{F}))$, then* $\mathrm{Pic}(X) \cong \mathbb{Z}^2$.

Proof. One can apply same method as in proof of Theorem 4.3.17 to obtain this result. Note that the proof in this case is much easier as $\mathcal{F}$ and X are smooth. □

Theorem 6.4.5. *Consider a general hypersurface $X \subset \mathcal{F}$ with*

$$\begin{pmatrix} d \\ 3 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -a & -a & -c \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

where $0 \leq a \leq b \leq c$. If the the Type III *or* IV *2-ray game of X leads to another Mori fibre space, then the weights $(a, b, c; d)$ are among those appearing in the left-hand column of Table 6.1 and Table 6.2.*

The Sarkisov links generated in this way are described in the remaining columns of Tables 6.1 and 6.2.

No.	$(a,b,c;d)$	ψ_1	ψ_2	φ'	new model
1	$(0,1,1;1)$	flip	n/a	fibration	$(Y_4 \subset \mathbb{P}^4)/\mathbb{P}^1$
2	$(0,0,1;1)$	n/a	n/a	contraction	Fano $Y_4 \subset \mathbb{P}^5$
3	$(0,0,0;1)$	n/a	n/a	fibration	conic bundle over $\mathbb{P}^3$
4	$(1,1,1;0)$	$\cong$	n/a	fibration	$dP_3/\mathbb{P}^2$
5	$(0,1,1;0)$	$3 \times (1,1,1,-1,-1)$ flips	n/a	fibration	$(Y_3 \subset \mathbb{P}^4)/\mathbb{P}^1$
6	$(0,1,2;0)$	$3 \times (1,1,1,-1,-2)$ flops	n/a	contraction	$Y_6 \subset \mathbb{P}(1,1,1,1,2,2)$
7	$(0,0,1;0)$	n/a	n/a	contraction	Fano $Y_3 \subset \mathbb{P}^5$
8	$(0,2,2;0)$	$3 \times (1,1,1,-2,-2)$ antiflip	n/a	fibration	$(Y_6 \subset \mathbb{P}(1^3,2^2))/\mathbb{P}^1$
9	$(1,1,1,-1)$	$(1,1,1,-1,-1)$ flip	n/a	fibration	$dP_8/\mathbb{P}^2$
10	$(1,1,2;-1)$	$(1,1,1,-1,-2)$ flop	n/a	contraction	$Y_5 \subset \mathbb{P}(1^5,2)$
11	$(1,1,2;-2)$	$(1,1,1,-1,-1)$ flip	n/a	contraction	$Y_4 \subset \mathbb{P}(1^5,2)$
12	$(1,1,3;-2)$	$(1,1,1,-1,-1,-3;-2)$ flop	n/a	contraction	$Y_7 \subset \mathbb{P}(1^3,2^2,3)$
13	$(1,1,3;-3)$	$(1,1,1,-1,-1)$ flip	n/a	contraction	$Y_5 \subset \mathbb{P}(1^3,2^2,3)$
14	$(1,2,2;-1)$	$(1,1,1,-2,-2)$ antiflip	$(1,1,1,1,-2,-2;2)$ flop	fibration	$(Y_5 \subset \mathbb{P}(1^4,2))/\mathbb{P}^1$
15	$(1,2,2;-2)$	$(1,1,1,-2,-2)$ flop	$(1,1,1,-1,-1)$ flip	fibration	$(Y_4 \subset \mathbb{P}(1^4,2))/\mathbb{P}^1$
16	$(1,1,4;-3)$	$(1,1,1,-1,-1,-4;-3)$ flop	n/a	contraction	$Y_{10} \subset \mathbb{P}(1^3,3^2,4)$
17	$(1,2,3;-3)$	$(1,1,1,-1,-3)$ antiflip	$(1,1,1,-1,-2)$ flop	contraction	$Y_7 \subset \mathbb{P}(1^4,2,3)$
18	$(1,2,3;-3)$	$(1,1,1,-1,-2)$ flop	$\cong$	contraction	$Y_6 \subset \mathbb{P}(1^4,2,3)$

Table 6.1: Part 1 data of Type III and IV links from general degree 3 del Pezzo hypersurface fibrations over $\mathbb{P}^2$

No.	$(a,b,c;d)$	ψ_1	ψ_2	φ'	new model
19	$(2,2,2;-2)$	$(1,1,1,-2,-2)$ antiflip	n/a	fibration	$dP_2/\mathbb{P}^2$
20	$(1,2,4;-3)$	$(1,1,1,-1,-3)$ antiflip	$\cong$	contraction	$Y_9 \subset \mathbb{P}(1^3,2,3,4)$
21	$(1,3,3;-3)$	$(1,1,1,-1,-3)$ antiflip	$\cong$	fibration	$(Y_6 \subset \mathbb{P}(1^3,2,3))/\mathbb{P}^1$
22	$(2,2,3;-3)$	$(1,1,1,-2,-2)$ antiflip	n/a	contraction	$Y_6 \subset \mathbb{P}(1^5,3)$
23	$(1,3,4;-3)$	$(1,1,1,-2,-2)$ antiflip	$\cong$	contraction	$Y_9 \subset \mathbb{P}(1^4,3,4)$
24	$(2,2,4;-4)$	$(1,1,1,-2,-2)$ antiflip	n/a	contraction	$Y_8 \subset \mathbb{P}(1^3,2^2,4)$
25	$(2,3,3;-3)$	$(1,1,1,-2,-3)$ antiflip	$(1,1,1,2,-1,-1;3)$ flop	fibration	$(Y_6 \subset \mathbb{P}(1^4,3))/\mathbb{P}^1$
26	$(1,4,4;-3)$	$(1,1,1,-1,-4,-4;-3)$ antiflip	$\cong$	fibration	$(Y_9 \subset \mathbb{P}(1^3,3,4))/\mathbb{P}^1$
27	$(2,2,5;-5)$	$(1,1,1,-2,-5)$ ntiflip	n/a	contraction	$Y_{10} \subset \mathbb{P}(1^3,3^2,5)$
28	$(2,3,4;-4)$	$(1,1,1,-2,-3)$ antiflip	$(1,1,1,-1,-2)$ flop	contraction	$Y_8 \subset \mathbb{P}(1^4,2,4)$
29	$(2,3,5,-5)$	$(1,1,1,-2,-3)$ antiflip	$(1,1,2,-1,-3)$ flop	contraction	$Y_{10} \subset \mathbb{P}(1^3,2,3,5)$
30	$(2,4,4;-4)$	$(1,1,1,-2,-4)$ antiflip	$(1,1,1,1,-2,-2;2)$ antiflip	fibration	$(Y_8 \subset \mathbb{P}(1^3,2,4))/\mathbb{P}^1$
31	$(2,3,6;-6)$	$(1,1,1,-2,-3)$ antiflip	$\cong$	contraction	$Y_{12} \subset \mathbb{P}(1^3,3,4,6)$
32	$(2,4,5;-5)$	$(1,1,1,-2,-4)$ antiflip	$(1,1,2,-2,-3)$ antiflip	contraction	$Y_{10} \subset \mathbb{P}(1^4,3,5)$
33	$(2,4,6;-6)$	$(1,1,1,-2,-4)$ antiflip	$\cong$	contraction	$Y_{12} \subset \mathbb{P}(1^3,2,4,6)$
34	$(2,5,5;-5)$	$(1,1,1,-2,-5)$ antiflip	$(1,1,2,-3,-3)$ antiflip	fibration	$(Y_{10} \subset \mathbb{P}(1^3,3,5))/\mathbb{P}^1$
35	$(2,5,6;-6)$	$(1,1,1,-2,-5)$ antiflip	$\cong$	contraction	$Y_{12} \subset \mathbb{P}(1^4,4,6)$
36	$(2,6,6;-6)$	$(1,1,1,-2,-6)$ antiflip	$\cong$	fibration	$(Y_{12} \subset \mathbb{P}(1^3,4,6))/\mathbb{P}^1$

Table 6.2: Part 2 data of Type III and IV links from general degree 3 del Pezzo hypersurface fibrations over $\mathbb{P}^2$

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