Franke's Realization Functor and Monoidal Products

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Abstract

In 1996, Jens Franke in an unpublished paper states that the homotopy category of $E(1)$-local spectra is equivalent as a triangulated category to $\mathcal{D}^1(\mathcal{A})$, the derived category of quasi-periodic cochain complexes of period 1 for primes $p \geq 3$. This is Franke's realization functor $\mathcal{R}\colon \mathcal{D}^1(\mathcal{A}) \to \mathrm{Ho}(L_1\mathrm{Sp})$. However, Irakli Patchkoria spotted gaps in the proof of J.Franke that were filled in a series of papers and put in a firm ground that for primes $p \geq 5$ Franke's realization functor is a triangulated equivalence. The categories $\mathcal{D}^1(\mathcal{A})$ and v are in fact tensor-triangulated, that is, both categories posses a monoidal structure that are compatible with the triangulated structure. In this thesis we prove that Franke's realization functor commutes with the monoidal products up to a natural isomorphism, that is, $\mathcal{R}$ i is tensor-triangulated functor.

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Chapter 1

Introduction

1.1 Overview

One can say that the main object of study in stable homotopy theory is the *stable homotopy category*, a category which contains an immense amount of geometrical and topological information. This geometric and topological information comes up as homotopy groups of spheres, invariants of vector bundles of manifolds such as characteristic classes and K-theory, and homology and cohomology groups of CW-complexes or simplicial sets. This information is encoded in the notion of a (sequential) *spectrum*, a sequence of spaces

$$X_0, X_1, X_2, \ldots$$

together with specified structure maps $\sigma_n \colon \Sigma X_n \to X_{n+1}$. A map $f \colon X \to Y$ between spectra is a collection of maps $\{f_n \colon X_n \to Y_n\}_{n \in \mathbb{N}}$ that are compatible with the structure maps of the spectra X and Y. The various flavours of geometrical and topological information appear now as the *stable homotopy groups* of a spectrum X, which is the $\mathbb{Z}$-graded abelian group given by the colimit of homotopy groups of the component spaces (or of their geometric realization if they are given as simplicial sets),

$$\pi_*(X) := \operatorname*{colim}_k \pi_{*+k}(X_k).$$

In order to study the stable homotopy category one uses various *models* that represent the same avatar of stable homotopy category. These models use the theory of *model categories*, a machinery introduced by Quillen in [37] and risen to prominence in the subsequent years. A *model category* $\mathcal{C}$ is a category equipped with three classes of morphisms, the *weak equivalences*, the *cofibrations* and *fibrations* satisfying certain axioms that are reminiscent of weak homotopy equivalences and (co)fibrations of topological spaces. These axioms ensure that it is possible to *localize* or *invert* with respect to the class of weak equivalences bypassing set-theoretic issues. Thereby, one obtains a new category $\mathrm{Ho}(\mathcal{C})$ (in a universal way) which is called the *homotopy category* of $\mathcal{C}$. This category contains the vital information about the homotopy theory in $\mathcal{C}$. One way to compare the underlying homotopy theories of two model categories is by a relation known as *Quillen equivalence*. A Quillen equivalence between model categories $\mathcal{C}$ and $\mathcal{N}$ is a functor $F\colon \mathcal{C} \to \mathcal{N}$ with certain conditions that ensure the induced functor on homotopy categories $\mathbb{L}F\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{N})$ is an equivalence of categories but also preserves much more structure.

Nowadays there are many *models* that all have the stable homotopy category SH as their homotopy category. In the terminology of model categories, there are many model categories, which are all Quillen equivalent to each other and have as homotopy category the stable homotopy category SH. One of the oldest is the category of Bousfield-Friedlander spectra or sequential spectra [13] $\mathrm{Sp}^{\mathbb{N}}$ which we alluded to in the first paragraph. With these definitions in mind, we can define the stable homotopy category as $\mathrm{SH} := \mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}})$. These model categories are *stable* in the sense that their homotopy categories have the structure of a *triangulated category*. Now suppose we have a stable model category $\mathcal{C}$ with homotopy category $\mathrm{Ho}(\mathcal{C})$ and assume that $\mathrm{Ho}(\mathcal{C})$ is equivalent as triangulated category to SH. This gives rise to the following natural question. Does this equivalence come from a Quillen equivalence? Schwede shows in [45] that the stable homotopy category SH is *rigid*. This means that every stable model category $\mathcal{C}$ such that

$$\Phi\colon \mathrm{SH} \xrightarrow{\cong} \mathrm{Ho}(\mathcal{C}),$$

that is, the homotopy category $\mathrm{Ho}(\mathcal{C})$ is triangulated equivalent to the homotopy category $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}})$ is Quillen equivalent to $\mathrm{Sp}^{\mathbb{N}}$. The stable homotopy category has much more structure than just a triangulated category. In particular interest to us, it has a compatible monoidal structure, i.e., it is a tensor-triangulated category in the sense of Balmer [4]. The rigidity property of the stable homotopy category can be extended to take into account monoidal product on spectra [46]. One further structural property of the stable homotopy category is that it is not algebraic. This means that the triangulated category SH is not triangulated equivalent to a triangulated category that arises from chain complexes in an additive or abelian category.

One of the most useful tools of modern stable homotopy theory is *Bousfield localization*. It is a formal method to "invert" a class of maps in a model category $\mathcal{C}$, in way that one can consider these maps to be weak equivalences in $\mathcal{C}$. Since the stable homotopy category is so complicated, mathematicians have used the tool of Bousfield localization to simplify SH and break apart into smaller pieces that are easier to understand. These localizations involve *localizations at a prime p*, that is, one inverts those maps that induce isomorphism after tensoring with $\mathbb{Z}_{(p)}$. Even though the p-local stable homotopy category $\mathrm{SH}_{(p)}$ is much easier to handle, still has the structural properties of SH. In particular, like SH, the p-local version is also rigid and also it cannot be algebraic. In sharp contrast, if one inverts all primes at once, that is, if one inverts those maps that induce isomorphisms after tensoring with $\mathbb{Q}$, then one obtains the stable homotopy category of rational spectra denoted by $\mathrm{SH}_{\mathbb{Q}}$. A key result in rational stable homotopy theory essentially states that the stable homotopy theory of rational spectra is Quillen equivalent to that of chain complexes of rational vector spaces $\mathrm{Ch}(\mathbb{Q})$, which is an *algebraic model category*. So, there is a huge structural difference between $\mathrm{SH}_{\mathbb{Q}}$ and $\mathrm{SH}_{(p)}$.

In the previous paragraph we mentioned that one can break apart the stable homotopy category into building blocks, one for each prime. By the work of Devinatz, Hopkins, Smith, Ravenel, Morava, [22] one can further break apart the p-local parts into smaller parts, one for every natural number

n that fit into a sequence. Thus, for every spectrum X there exists a tower

$$\ldots \to L_2X \to L_1X \to L_0X \cong X_{\mathbb{Q}},$$

satisfying various properties which we will not discuss here. The *chromatic convergence theorem* states mild conditions under which the homotopy limit over this tower is the p-localization $X \to X_{(p)}$. In this sense, there is for each prime number a countable family of stable homotopy categories between $\mathrm{SH}_{\mathbb{Q}}$ and $\mathrm{SH}_{(p)}$, which we denote for now $\{\mathrm{SH}_n\}_{n\in\mathbb{N}}$. Like rationalization and p-localization, these stable homotopy categories come also from Bousfield localization with respect a homology theory. In this case, it is represented by the Johnson-Wilson spectrum $E(n)$. The localized categories $\mathrm{SH}_{E(n)} = \mathrm{SH}_n$ promise to be simpler and indeed there are *algebraic descriptions* of these categories for some n and p. The algebraic description of these categories started with the paper [11]. Bousfield provides a purely algebraic description of the objects of the $E(1)$-local stable homotopy category at an odd prime. However, there is not a full description for the homotopy classes of maps. Franke in the unpublished paper [16] uses Bousfield's work to prove that the homotopy category of $E(1)$-local spectra is equivalent, through a functor which he calls *realization* to the derived category of a specific type of cochain complexes in an abelian category. This derived category is denoted by $\mathcal{D}^1(\mathrm{Comod}_{E(1)_*E(1)})$ and we call this *Franke's algebraic model.* Furthermore, this equivalence of triangulated categories cannot come from a Quillen equivalence and in this sense, one calls these type of equivalences *exotic.* Franke continues to generalize the result to ''higher chromatic primes'', i.e., the case for $E(n)$-local spectra depending on p. However there were some gaps in the proof that were filled in the paper by Patchkoria [34], where the author also provides many more cases of exotic equivalences. Also in the paper [43] a more streamlined proof can be found for the case $E(1)$.

Finally, we come to the subject of this thesis. The homotopy category of $E(n)$-local spectra and Franke's algebraic model are actually tensor-triangulated categories. The main result of Ganter in [18] is that that in the case $n = 1$, Franke's realization functor maps the derived tensor product to the smash product of $E(1)$-local spectra in a natural way. That is, there is a functor isomorphism

$$\mathcal{R}(- \otimes^{L}_{E(1)} -) \cong \mathcal{R}(-) \wedge^{\mathbb{L}} \mathcal{R}(-). \tag{1.1}$$

However, this equivalence *cannot be monoidal* for the case $p = 3$ since it cannot be an associative functor. To illustrate this, consider $p = 3$ and denote the mod 3 Moore spectrum by $M(3)$. This Moore spectrum has a unique multiplication which is not associative. But

$$\mathcal{R}(M(3)) \cong \ldots \to 0 \to E(1)_* \xrightarrow{\cdot p} E(1)_* \to 0 \to \ldots$$

has an associative multiplication. This means that we cannot hope to find an associative functor isomorphism between $-\wedge^{\mathbb{L}}$ and the derived tensor product of twisted complexes. However, for sufficiently large prime number the question of associativity of the functor $\mathcal{R}$ remains open. Retrospectively, one could argue this is good enough, since in the realm of tensor triangulated categories, functors between tensor-triangulated categories are only required to preserve the monoidal products up to natural isomorphism, see [4, Definition 3] and [3, Remark 2.4].

We now come to the more technical aspects of the thesis. The category $\mathcal{C}^1(\mathrm{Comod}_{E(1)_*E(1)})$, which is a version of Franke's algebraic model, *does not have enough projectives*, so the derived tensor product $- \otimes^L_{E(1)} -$ in (1.1) is not trivial. Ganter defines the monoidal product on $\mathcal{D}^1(\mathrm{Comod}_{E(1)_*E(1)})$ as the tensor product of underlying degree-wise *flat replacements*, i.e., resolutions that are flat as $E(1)_*$-modules which does not use model category theory techniques. This technique is very similar to how derived tensor products are defined in algebraic geometry, where usually the abelian category of quasi-coherent sheaves lacks enough projectives, but does have enough flat objects. We use the theory from the paper [6] by Barnes and Roitzheim which constructs a *monoidal model category* on $\mathcal{C}^1(\mathrm{Comod}_{E(1)_*E(1)})$, with operation denoted again $\otimes$ which is equivalent in a sense to Franke's algebraic model. This means, the homotopy category of the aforementioned model structure together with the derived tensor product, i.e., the pair $(\mathcal{D}^1(\mathrm{Comod}_{E(1)_*E(1)}), \otimes^{\mathbb{L}})$ is naturally a tensor-triangulated category. The derived tensor product $\otimes^{\mathbb{L}}$, like in all monoidal model categories, is defined by cofibrant replacements on both arguments and then tensoring on point-set level. Another aspect in that we diverge from Ganter's paper, is that, like [16] which is one of the founding papers of derivators, Nora's paper is in the language of *derivators*. We chose to work in model category theory instead, which we believe makes our exposition more explicit and more straightforward. Another novelty is the use of homology of categories with coefficients in the proof of Theorem 4.3.1. This tool simplifies and streamlines the computation of homology groups that goes into a spectral sequence that computes homology of homotopy colimits. A theme that is common throughout the thesis is the use of techniques that come from simplicial homotopy theory. Simplicial objects play a very important auxiliary role, both in the theory of homotopy colimits and in calculating homology of categories with coefficients. Lastly, the author was unable to follow the argument in the proof of [18, Proposition 7.2.5] that a certain crowned diagram realizes the tensor product of disks of twisted-periodic complexes. In particular, the author could not find a relative discussion about the obligatory sign that appears in the definition of the tensor product of complexes. We hope our proof elucidates the introduction of the signs in the tensor product.

Overall, we have relied in more modern methods and we refer to modern literature. Our techniques put Ganter's theorem in firm rigorous footing and in better context with other existing literature. We hope it will be possible to extend the result to other settings as well.

1.2 Main Result & Strategy

In this section we will explain in more technical detail the main result and our strategy. We have hinted that the triangulated categories $\mathcal{D}^1(\mathcal{A})$ and $\mathrm{Ho}(L_1\mathrm{Sp})$ admit a triangulated structure and a monoidal structure compatible with the triangulation. The overall plan of this thesis is to rigorously prove that the Franke's realization functor

$$\mathcal{R}\colon \mathcal{D}^1(\mathcal{A}) \to \mathrm{Ho}(L_1\mathrm{Sp})$$

is compatible with the respective monoidal structures. That is, we want to show that

$$\mathcal{R}\colon (\mathcal{D}^1(\mathcal{A}), \otimes^{\mathbb{L}}) \to (\mathrm{Ho}(L_1\mathrm{Sp}), \wedge^{\mathbb{L}})$$

is a $\otimes$-functor. We now state it formally.

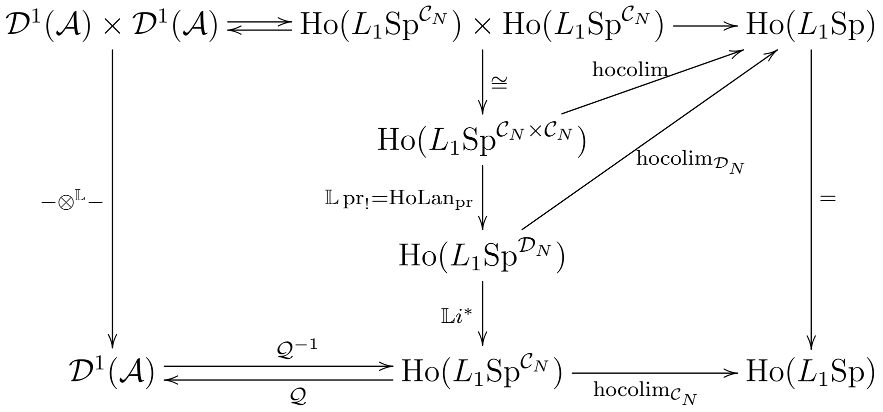
Theorem 1.2.1. *Franke's realization functor* $\mathcal{R}\colon (\mathcal{D}^1(\mathcal{A}), \otimes^{\mathbb{L}}) \to (\mathrm{Ho}(L_1\mathrm{Sp}), \wedge^{\mathbb{L}})$ *commutes with respect the monoidal products up to a natural isomorphism, that is,*

$$\mathcal{R}(C_*^\bullet \otimes^{\mathbb{L}} D_*^\bullet) \cong \mathcal{R}(C_*^\bullet) \wedge^{\mathbb{L}} \mathcal{R}(D_*^\bullet).$$

In other words, we have to show that the following diagram commutes, up to a natural isomorphism.

$$\begin{array}{ccc} \mathcal{D}^1(\mathcal{A}) \times \mathcal{D}^1(\mathcal{A}) & \xrightarrow{\mathcal{R}\times\mathcal{R}} & \mathrm{Ho}(L_1\mathrm{Sp}) \times \mathrm{Ho}(L_1\mathrm{Sp}) \\ {\scriptstyle\otimes^{\mathbb{L}}}\downarrow & & \downarrow{\scriptstyle\wedge^{\mathbb{L}}} \\ \mathcal{D}^1(\mathcal{A}) & \xrightarrow{\quad\mathcal{R}\quad} & \mathrm{Ho}(L_1\mathrm{Sp}). \end{array} \tag{1.2}$$

We stress yet again that for the case $p = 3$ the functor $\mathcal{R}$*cannot be monoidal.* So, there will be no discussion on associativity of $\mathcal{R}$. In order to prove that the above diagram commutes we will break it down two big parts, namely the left half and the right half side of the following diagram.



(1.3)

The left hand side and right half side are further broken down into smaller pieces. Our task at hand now is to prove that the left half is a commutative diagram and all the all the smaller triangles on the right hand side are also commutative. Since all the functors are natural, once we show all the smaller triangles commute it will follow that (1.2) also commutes. This makes the exposition easier since the proofs are broken apart into smaller propositions and easier-to-prove statements. We will properly define all the functors in (1.3) in Chapter 4.

1.3 Organization of the Chapters

In this section we will outline how the thesis is organized. The structure is as follows:

(i) Chapter 2 contains the necessary background in homotopy theory. We will recall relevant theory from simplicial homotopy theory, model category theory, homotopy colimits and homotopy Kan extensions. Lastly we will recall some notation and some theory from stable homotopy theory, spectra, and Bousfield localizations.

(ii) Chapter 3 contains the necessary background in homological algebra. We will recall relevant theory from homological algebra and Franke's algebraic model. Tangentially to that we will introduce various model structures on Franke's algebraic model which are essential for our exposition. We will also recall an another tool that we will use in this thesis, homology of a category with coefficients in a functor. This algebraic gadget will be essential in our computation of a spectral sequence that computes the homology of a homotopy colimit of spectra in the following chapter.

(iii) Chapter 4 is the heart of the thesis. It deals with the left half side of (1.3). The main result is Theorem 4.3.1. In summary, there exists a bifunctor $i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)$ such that under certain conditions it preserves objects in $\mathcal{L}$, a full subcategory of $\mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N})$ and

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes^{\mathbb{L}} \mathcal{Q}(Y).$$

In order to prove the theorem, we will break apart the proof into three parts. These three parts cover Section 4.5, Section 4.6 and Section 4.7. In Section 4.5 we will employ homology of categories with coefficients in order to show that the bifunctor $i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)$ preserves objects in $\mathcal{L}$. Then, in Section 4.6 we show that $\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y))$ is a good candidate for the tensor product The proof relies on the theory of homotopy Kan extensions and the structural properties of crowned diagrams. Lastly, in Section 4.7 we prove, after a reduction to a simpler case, that indeed $\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y))$ is naturally isomorphic to the tensor product $\mathcal{Q}(X) \otimes^{\mathbb{L}} \mathcal{Q}(Y)$. The proof relies on Section 4.6 and is the most technically involved in the thesis.

(iv) Chapter 5 deals with the right hand side of the above diagram. In summary, we show that for the bifunctor $i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)$ we have a canonical isomorphism

$$\operatorname*{hocolim}_{\mathcal{C}_N} i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{\mathcal{C}_N} X \wedge^{\mathbb{L}} \operatorname*{hocolim}_{\mathcal{C}_N} Y.$$

The main result is Theorem 5.1.1 which also is broken down to smaller lemmas. The proofs rely on standard methods of homotopy theory, i.e., smash products for diagram model categories, homotopy Kan extensions and a homotopy finality argument.

(v) Lastly, in Chapter 6 we will conclude the proof of our main result, Theorem 1.2.1. Since we have done most of the work in the previous chapters, our last proof is a short one.

More details of contents of each chapter can be found the introduction of each chapter.

Chapter 2

Background in Homotopy Theory

In this chapter we recall the elements of homotopy theory that are necessary for the proof of our result. The chapter is structured as follows. In Section 2.1 we will recall some relevant facts from simplicial homotopy theory. Simplicial objects play an auxiliary role in our exposition but nevertheless quite an important one. In Section 2.2 we will recall some relevant definitions from model category theory which we heavily rely on throughout. In Section 2.3 we will recall the concepts of monoidal model categories and enriched model category. We also discuss some relevant material on monoidal model categories on the arrow category and how this interacts with the monoidal product. In Section 2.4 we will discuss the Bousfield-Kan definition of homotopy colimits in a simplicial model category. Inherently the Bousfield-Kan construction is a statement about the interaction of model category theory and simplicial homotopy theory. This means we will return to simplicial objects and we will use the material discussed in the first section. In Section 2.5 we recall some definitions of homotopy Kan extensions which are central in our exposition. In Section 2.6 we will recall some definitions from spectra, *i.e.*, stable homotopy theory which we mostly treat as a black box. Throughout the chapter we do not try to be complete or prove every single proposition and we assume the reader is familiar with the notions defined. We will provide all the necessary references.

2.1 Simplicial Objects

In this section we will recall some relevant facts about simplicial objects that will be useful later on. For reference see [19, Chapter I]. A more classical reference is [29].

2.1.1 Main Definitions

The main reference for this subsection is [19, Chapter I]

Definition 2.1.2. Let $\mathbf{\Delta}$ denote the following category. The objects of $\mathbf{\Delta}$ are the totally ordered sets $[n] = \{0, 1, 2, \ldots n\}$ for $n \geq 0$. A morphism $\sigma\colon [n] \to [k]$ is an order-preserving function, *i.e.*, a function $\sigma\colon [n] \to [k]$ such that $\sigma(i) \leq \sigma(j)$ for $0 \leq i \leq j \leq n$.

Definition 2.1.3. A *simplicial set* is a functor $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$, where Set is the category of sets. The functor category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set})$ is called the *category of simplicial sets*. We will usually write the value of the functor X at an object $[n]$, as X_n

As usual, the standard n-simplex Δ^n is the simplicial set

$$\Delta^n \colon \Delta^{\text{op}} \to \text{Set}$$
$$[m] \mapsto \text{Hom}_{\mathbf{\Delta}}([m],[n]) = \mathbf{\Delta}([m],[n]),$$

and with notation from Definition above, the value of the n-simplex at $[k]$ is given by $\Delta^n_k := \mathbf{\Delta}([k],[n])$.

There is a more traditional way to define a simplicial set than the one given in Definition 2.1.3, which we describe below. Among all of the functors (order preserving maps) $[m] \to [n]$ appearing in $\mathbf{\Delta}$ there are special ones, namely

$$d_i \colon [n-1] \to [n] \quad 0 \leq i \leq n \quad \text{(cofaces)}$$
$$s_j \colon [n+1] \to [n] \quad 0 \leq j \leq n \quad \text{(codegeneracies)}$$

where, by definition,

$$d_i(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases}$$

and

$$s_j(k) = \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i. \end{cases}$$

These functors satisfy a list of identities called the cosimplicial identities which we do not reproduce here, see [19, Figure 1.2, p4]. The maps d_j, s_i

and these relations can be viewed as a set of generators and relations for Δ. Thus, in order to define a simplicial set Y, it suffices to write down sets $Y_n,\ \ n \geq 0$ (sets of n-simplices) together with maps

$$d_i \colon Y_n \to Y_{n-1}, \quad 0 \leq i \leq n \quad \text{(faces)}$$
$$s_j \colon Y_n \to Y_{n+1}, \quad 0 \leq j \leq n \quad \text{(degeneracies)}$$

satisfying the simplicial identities, which are dual to the *cosimplicial identities*, see [19, p5]. We can depict a simplicial set $Y \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ as follows

$$Y_0 \leftleftarrows Y_1 \;\substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\; Y_2 \cdots$$

where we usually omit the degeneracy maps for typographical reasons.

Notation 2.1.4. The letter I will stand for the simplicial set Δ^1, and $(I, 0)$ for the pointed at 0. S^0 denotes the pointed simplicial 0-sphere, i.e., th union of the standard 0-simplex with a disjoint base point. S^1 denotes the simplicial circle $I/(0 \sim 1)$, i.e., $\Delta^1/\partial\Delta^1$.

Simplicial sets are ubiquitous since every category $\mathcal{C}$ gives rise to a simplicial set as the next example shows.

Example 2.1.5. Let $\mathcal{C}$ be a small category and let $N(\mathcal{C})$ be the following simplicial set

$$N(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C}),$$

where $\mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$ denotes the set of functors from $[n]$ to $\mathcal{C}$. In other words a n-simplex is a string

$$c_0 \to c_1 \to \ldots \to c_n$$

of composable morphisms in the category $\mathcal{C}$.

We will encounter nerves of certain categories later on. Simplicial sets are very closely related to topological spaces as our next example will show.

Example 2.1.6. There is a standard functor

$$\begin{gathered} \mathbf{\Delta}^{\text{op}} \to \text{Top} \\ [n] \mapsto |\Delta^n| \end{gathered}$$

The topological standard n-simplex $|\Delta^n| \subset \mathbb{R}^{n+1}$ is the space

$$|\Delta^n| = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \colon \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\},$$

with the subspace topology. The map $\theta_* \colon |\Delta^n| \to |\Delta^m|$ induced by $\theta \colon [n] \to [m]$ is defined by

$$\theta_*(t_0, \ldots, t_m) = (s_0, \ldots, s_n),$$

where

$$s_i = \begin{cases} 0 & \theta^{-1}\{i\} = \emptyset \\ \Sigma_{j \in \theta^{-1}(i)} t_j & \theta^{-1} i \neq \emptyset. \end{cases}$$

Conversely, every topological space gives rise to a simplicial set. Let T be a topological space and consider the simplicial set

$$\text{Sing}(T) \colon \mathbf{\Delta}^{\text{op}} \to \text{Set} \quad [n] \mapsto \text{Hom}_{\text{Top}}(|\Delta^n|, T).$$

This is the object that leads to the singular homology of the space T.

In fact, the homotopy theory of simplicial sets and topological spaces are "equivalent" in a specific sense. We will recall this form of equivalence of homotopy theories in the next section. In the example above we mention briefly the construction of the topological standard n-simplex $|\Delta^n|$ from the simplicial set Δ^n. This construction can be thought of as " gluing" and can be made formal as a certain form of colimit. We will return to that later on when we recall the geometric realization functor.

2.1.7 Decomposition & Skeleta

In this subsection we will discuss the canonical and natural filtration of a simplicial object by its skeleton. The skeleton of a simplicial object is constructed using lower dimension simplices, and it turns out any simplicial object can be written as a colimit of its skeleta. We will return to the skeletal filtration of a simplicial object later in Subsection 2.4.20 and Subsection 3.5.9, and it will play a fundamental role in our exposition. For reference for this subsection, we refer to [19, Chapter VII].

Following Definition 2.1.3, we can replace the category Set with any other category $\mathcal{C}$ in which all small limits and all small colimits exist, that is to say $\mathcal{C}$ is complete and cocomplete. Then we consider the category $s\mathcal{C}$, *i.e.*, the category of simplicial objects in $\mathcal{C}$. Let $\Delta_{\leq n}$ be the of the full subcategory of Δ with objects $[k]$ for $k \leq n$ and denote by $i_n \colon \Delta_{\leq n} \hookrightarrow \Delta$ the inclusion.

Following the notation $s\mathcal{C}$ for the category of simplicial objects in $\mathcal{C}$ we denote

$$s_n\mathcal{C} = \mathrm{Fun}(\Delta^{\mathrm{op}}_{\leq n}, \mathcal{C}) = \mathcal{C}^{\Delta^{\mathrm{op}}_{\leq n}}.$$

There is a restriction functor,

$$i_n^*\colon s\mathcal{C} \to s_n\mathcal{C},$$

which simply forgets the k-simplices, $k > n$, that is, given $X \in s\mathcal{C}$ we have the diagram i_n^*X

$$\Delta_{\leq n} \overset{i_n}{\hookrightarrow} \Delta \overset{X}{\to} \mathcal{C}.$$

Since we assume $\mathcal{C}$ to be cocomplete, the restriction functor i_n^* has a left adjoint denoted by Lan_{i_n} which is given by

$$(\mathrm{Lan}_{i_n} X)_m = \operatorname*{colim}_{[m]\to[k]} X_k = \operatorname*{colim}_{[k]\in[m]/\Delta_n} X_k.$$

The colimit is over morphisms $[m] \to [k]$ in Δ with $k \leq n$. This is an example of a left Kan extension, a concept which we will make formal in Section 2.5. If $X \in s\mathcal{C}$, we define the nth *skeleton* of X by the formula

$$\mathrm{sk}_n X := \mathrm{Lan}_{i_n} i_n^* X. \tag{2.1}$$

There are natural maps $\mathrm{sk}_n X \to \mathrm{sk}_m X$ for $n \leq m$ and $\mathrm{sk}_n X \to X$. Since $(\mathrm{sk}_n X)_m = X_m$ for $m \leq n$, by construction, there is a natural isomorphism

$$\operatorname*{colim}_n \mathrm{sk}_n X \xrightarrow{\cong} X. \tag{2.2}$$

Dual to the notion of skeleta are the coskeleta. The restriction functor i_n^* has also a right adjoint defined by

$$(\mathrm{Ran}_{i_n} X)_m = \lim_{k \to n} X_k,$$

with the limit over all morphism $k \to m$ in Δ with $k \leq n$. This can be used to construct a coskeleton such that X

$$X \xrightarrow{\cong} \lim_n \mathrm{cosk}_n X.$$

Because $\mathcal{C}$ has limits and colimits, $s\mathcal{C}$ has a canonical structure as a simplicial category, i.e., the hom-objects are not just sets but actually simplicial sets. Thus, we have a bifunctor

$$\underline{\mathrm{Hom}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to s\mathrm{Set},$$

and further there is a functor, the so-called *tensor*

$$- \wedge - \colon s\mathrm{Set} \times s\mathcal{C} \to s\mathcal{C}.$$

which is left adjoint to $\underline{\mathrm{Hom}}$, see [42, Definition 3.3.1.] for details. We will make the concept formal later on Section 2.3. In particular, if $X \in s\mathcal{C}$ and $K \in s\mathrm{Set}$, then a natural choice of tensor action of simplicial sets is the following.

$$(K \odot X)_n = \coprod_{K_n} X_n. \tag{2.3}$$

We denote this particular tensor with $\odot$ to make it distinct with another form of natural tensor action of simplicial sets on an arbitrary diagram category given in Example 2.3.15. It is now a straightforward to prove that if $X \in s\mathcal{C}$ is a constant simplicial object, i.e., all the sets X_n for $n \geq 0$ are equal and all the face and degeneracy maps are the identity map and $K \in s\mathrm{Set}$, then there is a natural isomorphism

$$\operatorname{sk}_n X \odot \operatorname{sk}_n K \cong \operatorname{sk}_n(X \odot K). \tag{2.4}$$

There is more constructive way to obtain the skeleton $\operatorname{sk}_n X$ of a simplicial object $X \in s\mathcal{C}$ which we now briefly explain. First we have the definition of the *latching space* of a simplicial object. We also give the definition of *matching space* since we are at it.

Definition 2.1.8. Let $X \in \mathcal{C}^{\Delta^{\mathrm{op}}}$. The nth *latching object* of X is

$$L_nX := (\operatorname{sk}_{n-1} X)_n \cong \operatorname*{colim}_{[k]\in[n]/\Delta_{n-1}} X_k.$$

Dual to the latching space we have the nth *matching space*

$$M_nX := (\operatorname{cosk}_{n-1} X)_n = \lim_{k\to n} X_k = \lim_{[k]\in\Delta_{n-1}/[n]} X_k,$$

in other words where $k \to n$ runs over all morphisms (or all monomorphisms) in Δ with $k < n$.

If $Z \in \mathcal{C}$, we may regard Z as a constant object in $s\mathcal{C}$. Then, there is an adjunction isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(Z, X_n) \cong \operatorname{Hom}_{s\mathcal{C}}(\Delta^n \odot Z, X)$$

for all $n \geq 0$. By choosing $X = \operatorname{sk}_n X$ and $Z = X_n$ we obtain a natural map in $s\mathcal{C}$

$$\Delta^n \odot X_n \to \operatorname{sk}_n X.$$

Setting $X = \operatorname{sk}_{n-1} X$ and $Z = L_n X$ we obtain a natural map

$$\Delta^n \odot L_n X \to \operatorname{sk}_{n-1} X.$$

Furthermore, by (2.4),

$$\operatorname{sk}_{n-1}(\Delta^n \odot X_n) = \operatorname{sk}_{n-1} \Delta^n \odot X_n = \partial\Delta^n \odot X_n$$

and we obtain a commutative diagram

$$\begin{array}{ccc} \partial\Delta^n \odot L_n X & \longrightarrow & \Delta^n \odot L_n X \\ \downarrow & & \downarrow \\ \partial\Delta^n \odot X_n & \longrightarrow & \operatorname{sk}_{n-1} X. \end{array}$$

We have the following proposition. For the proof we refer to [19, Proposition 1.7, Chapter VII].

Proposition 2.1.9. *For all $X \in s\mathcal{C}$ and $n \geq 0$ there is a natural pushout square.*

$$\begin{array}{ccc} \partial\Delta^n \odot X_n \coprod_{\partial\Delta^n \odot L_n X} \Delta^n \odot L_n X & \longrightarrow & \operatorname{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \Delta^n \odot X_n & \longrightarrow & \operatorname{sk}_n X \end{array}$$

There is a dual version for coskeleta that is given by certain pullbacks, but we do not need this in our exposition.

2.2 Model Categories

In this section we will introduce some relevant facts from model category theory. Model categories were introduced by Quillen in 1967 [37]. We will introduce the basic definitions of a model category and of derived functors. We will also discuss various model structures on diagram (functor) categories, homotopy colimits and stable model structures. We assume the reader is familiar with the basics of the theory and most of the proofs are skipped and refer to the large literature of model categories.

2.2.1 A Short Reminder of Model categories

A *model category* $\mathcal{C}$ is a complete and cocomplete category equipped with three classes of morphisms called weak equivalences, fibrations and cofibrations, satisfying certain axioms. A map that is both a weak equivalence and a cofibration resp. fibration, is called acyclic cofibration resp. acyclic fibration. We will not list the axioms here and reader is advised to look up the references, for example see [24, Definition 1.1.4]. The main point of this structure is that it allows one to do "homotopy theory" in the category $\mathcal{C}$. The most prominent examples of model categories are the categories $s\mathrm{Set}$ of simplicial sets, Top of topological spaces and $\mathrm{Ch}(k)$, the category of chain complexes of modules over a ring k.

For every model category $\mathcal{C}$, one has the associated *homotopy category* $\mathrm{Ho}(\mathcal{C})$ which is defined as the localization of $\mathcal{C}$ with respect to the class of weak equivalences. The model structure ensures that we do not face any set theoretic problems when passing to localization, *i.e.* $\mathrm{Ho}(\mathcal{C})$ has sets of morphisms. Note however that $\mathrm{Ho}(\mathcal{C})$ admits other equivalent descriptions as well. For example $\mathrm{Ho}(\mathcal{C})$ is equivalent to the homotopy category of cofibrant objects in $\mathcal{C}$. Then $\mathrm{Ho}(\mathcal{C})$ is equivalent to the localization $\mathcal{C}_{\mathrm{cof}}[\mathcal{W}^{-1}]$ where $\mathcal{W}$ is the class of weak equivalences in $\mathcal{C}_{\mathrm{cof}}$ see for example [24, Proposition 1.2.3]. Given objects X and Y of $\mathcal{C}$, the notation $[X, Y]$ will stand for the set of morphisms in $\mathrm{Ho}(\mathcal{C})$ between X and Y.

We will assume that in a model category $\mathcal{C}$ there are *functorial* fibrant and cofibrant replacements. We write any cofibrant replacement functor $Q\colon \mathcal{C} \to \mathcal{C}$ that comes with weak equivalence $q\colon Q \to 1_{\mathcal{C}}$

Convention 2.2.2. In what follows, we let $\mathrm{Ho}(\mathcal{C})$ denote the category $\mathcal{C}_{\mathrm{cof}}[\mathcal{W}^{-1}]$.

Next, we recall the definition of a Quillen adjunction, which is the primary way to relate different model categories.

Definition 2.2.3. A *Quillen adjunction* between two model categories $\mathcal{C}$ and $\mathcal{N}$ is a pair of adjoint functors

$$F\colon \mathcal{C} \rightleftarrows \mathcal{N}\colon G$$

where the left adjoint F preserves cofibrations and acyclic cofibrations (or, equivalently G preserves fibrations and acyclic fibrations).

We refer to F as a *left Quillen* functor and to G as a *right Quillen* functor. The usefulness of the above definition is that any such pair of adjoint functors, induces an adjunction on the level of homotopy categories, that is,

$$\mathbb{L}F\colon\ \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{N})\colon \mathbb{R}G.$$

We refer $\mathbb{L}F$ as the *left derived functor* of F and $\mathbb{R}G$ as the *right derived functor* of G. If $\mathbb{L}F$, or equivalently $\mathbb{R}G$ is an equivalence, then the Quillen adjunction is called a *Quillen equivalence*.

Convention 2.2.2 allows us to provide a very simple description of the functor $\mathbb{L}F$. Indeed, the functor

$$F_{\mathcal{C}_{\mathrm{cof}}} \colon \mathcal{C}_{\mathrm{cof}} \to \mathcal{N}_{\mathrm{cof}}$$

preserves weak equivalences and, therefore, it induces a functor between the localizations. This functor is precisely $\mathbb{L}F$ in terms of Convention 2.2.2.

So, any Quillen equivalence

$$F \colon \mathcal{C} \to \mathcal{N}$$

induces an equivalence on homotopy categories

$$\mathbb{L}F \colon \ \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{N}).$$

However, there are examples of functors $\Phi \colon \ \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{N})$ that are equivalences of categories, but nevertheless, they are not induced by any zig-zag of Quillen equivalences. Such equivalences are called *exotic equivalences*.

Finally, an important class of model categories is the class of *simplicial model categories*. These are model categories which are enriched, tensored and cotensored over $s\mathrm{Set}$ and which satisfy the pushout-product axiom. If a simplicial model category is pointed, i.e., the terminal object is isomorphic to the initial one, then $\mathcal{C}$ is enriched over the category $s\mathrm{Set}_*$ of pointed simplicial sets. In particular, we have functors

$$- \wedge -\colon s\mathrm{Set}_* \times \mathcal{C} \to \mathcal{C}, \quad \mathrm{Map}_{\mathcal{C}}(-.-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to s\mathrm{Set}_*,$$

and the adjunction

$$\mathrm{Hom}_{\mathcal{C}}(K \wedge X, Y) \cong \mathrm{Hom}_{s\mathrm{Set}}(K, \mathrm{Map}_{\mathcal{C}}(X, Y)),$$

see [5, Definition 6.1.28]. We will define simplicial model categories properly in Section 2.3.

2.2.4 Diagram Model Categories

For $\mathcal{C}$ a model category and J any small category there are two natural ways to put a model category structure on the functor category $\mathrm{Fun}(J, \mathcal{C}) = \mathcal{C}^J$. These are the *projective* and the *injective* model structures. In the greatest generality, neither one need exist, but there are rather general conditions that ensure their existence. One way is to restrict the properties of the model category $\mathcal{C}$ in order to ensure a model structure on the category of diagrams $\mathcal{C}^J$. The other way is to restrict the properties of the indexing category. We recall the basic definitions.

Definition 2.2.5. We define the following classes of maps in $\mathcal{C}^J$.

1. The projective weak equivalences and projective fibrations are the natural transformations that are objectwise such morphisms in $\mathcal{C}$.

2. The injective weak equivalences and injective cofibrations are the nat-

ural transformations that are objectwise such morphisms in $\mathcal{C}$.

If either of these choices defines a model structure on $\mathcal{C}^J$, we call it the *projective model structure* $\mathcal{C}^J_{\text{proj}}$ or *injective model structure* $\mathcal{C}^J_{\text{inj}}$ respectively. Of course, the projective cofibrations and injective fibrations can then be characterized by the appropriate lifting properties.

Because the projective and injective model structures on $\mathcal{C}^J$ have the same weak equivalences, the identity functor Id is a Quillen equivalence between them if they both exist. The projective model structure exists as long as $\mathcal{C}$ is cofibrantly generated, see [21, Definition 11.1.2], while both model structures exist if the underlying category$\mathcal{C}$ is a locally presentable (a set-theoretic condition) and cofibrantly generated, see [26, Proposition A.2.8.2]. On the other hand if the indexing category is restricted enough say, a finite poset, then both model structures exist without any restriction on the model category $\mathcal{C}$.

Below we introduce the definition of a *direct category* which is a generalization of the concept of a poset. The structure of a direct category allows us to be more explicit with the cofibrations in the projective model structure.

Definition 2.2.6. Let ω denote the poset category of the ordered set $\{0, 1, 2, \ldots\}$. A small category J is called *direct* if there is a functor $f\colon J \to \omega$ that sends non-isomorphims to non-isomorphisms. We refer to $f(j)$ as the *degree* of the object j. Dually, J is an *inverse* category if there is a functor $J^{\text{op}} \to \omega$ that sends non-isomorphisms to non-isomorphisms.

For further details, see [24, Definition 5.1.1.]. Any finite poset J is a direct category, and dually J^{op} is an inverse category. We provide some examples that will be useful later on.

Example 2.2.7. Let $[1]$ denote the poset $0 \leq 1$.

Example 2.2.8. Consider the poset

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \\ (0,1) & & \end{array}$$

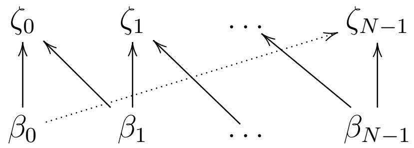
denoted by $\ulcorner$. Let $\iota\colon [1] \to \ulcorner$ be the map of posets which sends 0 to $(0,0)$ and 1 to $(1,0)$. In other words, ι includes the interval $[1]$ to the top horizontal line. Furthermore, consider the product of the interval posets $[1] \times [1]$. It is the following poset

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \downarrow \\ (0,1) & \longrightarrow & (1,1) \end{array}$$

and we let $i_{\ulcorner}\colon \ulcorner \to [1] \times [1]$ be the inclusion. The product $[1] \times [1]$ is usually denoted by $\square$.

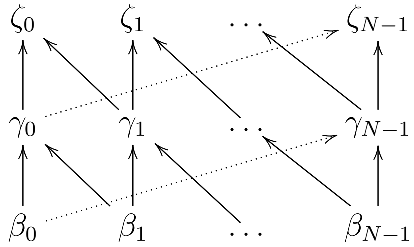
Next we are introducing two posets that we will work with

Example 2.2.9. Let $\mathcal{C}_N$ be the poset consisting of elements $\{\beta_i, \zeta_i \mid i \in \mathbb{Z}/N\mathbb{Z}\}$ such that $\beta_i < \zeta_i$ and $\beta_i < \zeta_{i-1}$ for $i \in \mathbb{Z}/N\mathbb{Z}$. As a diagram it looks as follows:



The other poset that will be of interest is the following.

Example 2.2.10. Let $\mathcal{D}_N$ be the poset consisting of elements $\{\beta_n, \gamma_n, \zeta_n \colon n \in \mathbb{Z}/N\mathbb{Z}\}$ such that $\beta_n \leq \gamma_n \leq \zeta_n$ and $\beta_{n+1} \leq \gamma_n$ and $\gamma_{n+1} \leq \zeta_n$. As a diagram it looks as follows



Definition 2.2.11. Suppose $\mathcal{C}$ is a small category with small colimits, J a small direct category, z an object in J and J_z the category of all non-identity morphisms with codomain z. The *latching space functor* $L_z \colon \mathcal{C}^J \to \mathcal{C}$ is the composition

$$\mathcal{C}^J \to \mathcal{C}^{J_z} \xrightarrow{\text{colim}} \mathcal{C},$$

where the first arrow is the restriction functor. Note that we have a natural transformation

$$L_z X \to X_z$$

for any fixed object $z \in \mathcal{C}$. Dually the *matching space functor* $M_z \colon \mathcal{C}^J \to \mathcal{C}$ is the composition

$$\mathcal{C}^J \to \mathcal{C}^{J^z} \xrightarrow{\lim} \mathcal{C},$$

where J^z is the category of all non-identity morphisms with domain z.

Notice that, equivalently the latching space of a diagram X is given by

$$L_z X = \text{colim}\left(J_z \hookrightarrow J \xrightarrow{X} \mathcal{C}\right),$$

where $J_z \hookrightarrow J$ is the inclusion. Dually, the matching space of a diagram X is given by

$$M_z X = \lim \left(J^z \hookrightarrow J \xrightarrow{X} \mathcal{C} \right).$$

The following proposition is proved in [24, Theorem 5.1.3].

Proposition 2.2.12. *Given a model category $\mathcal{C}$ and a direct category J, there is a model structure on $\mathcal{C}^J$ in which a morphism $f\colon X \to Y$ is a weak equivalence (resp. fibration) if and only if the map $f_z\colon X_z \to Y_z$ is a weak equivalence (resp. fibration) for all $z \in J$. Furthermore, $f\colon X \to Y$ is an (acyclic) cofibration if and only if the induced map*

$$X_z \coprod_{L_z X} L_z Y \to Y_z$$

is an (acyclic) cofibration for all $z \in J$.

To clarify the above proposition proves that the projective pre-model structure is a model structure; it does not introduce a new and different model structure

Example 2.2.13. Let I be a finite poset and any model category $\mathcal{C}$. We have the model structure 2.2.12 on $\mathcal{C}^I$.

Remark 2.2.14. In what follows, when we have a direct category I and a model category $\mathcal{C}$, the category of diagrams $\mathcal{C}^I$ will always have the model structure defined in Proposition 2.2.12 without further mention. If not, we will explicitly say so.

2.2.15 Reedy Model Structure on Simplicial Objects

In this subsection we will review the Reedy model structure on simplicial objects. For a reference see [5, Definition 6.9.4], also [24, Section 5.2]. We start with the definition of a Reedy category.

Definition 2.2.16. A *Reedy category* is a triple (J, J^+, J^-) consisting of a small category J and two wide subcategories (that is, they contain all the objects) J^+, and J^-, such that there exists a functor $d\colon J \to \omega$, called a *degree function*, such that every nonidentity map in J^+ raises the degree, every nonidentity map in J^- lowers the degree, and every map $f \in J$ can be factored uniquely as $f = gh$, where $h \in J^-$and $g \in J^+$. In particular, J^+ is a direct category and J^- is an inverse category, see Definition 2.2.6.

Example 2.2.17. Every direct category J, see Definition 2.2.6 is a Reedy category, by considering $J^+ = J$ and the discrete $J^- = \mathrm{Ob}(J)$ (with degree on J^- given by identity). Dually, any inverse category is a Reedy category.

Example 2.2.18. The prototype example of a Reedy category is the simplex category $\mathbf{\Delta}$. The Reedy category structure on Δ is defined by as follows.

(i) The degree function $d\colon \mathrm{Ob}(\mathbf{\Delta}) \to \mathbb{N}$ is defined by $[k] \mapsto k$,

(ii) A map $[k] \to [n]$ is in Δ^+ precisely if it is injective.

(iii) A map $[n] \to [k]$ is in Δ^- precisely if it is surjective.

The Reedy category structure on $\mathbf{\Delta}^{\mathrm{op}}$ is defined by switching $\mathbf{\Delta}^+$ and $\mathbf{\Delta}^-$.

Suppose $\mathcal{C}$ is a category with all small colimits and limits, and J is a Reedy category. For each object z of J, we define the latching space functor

L_z as the composite

$$\mathcal{C}^J \to \mathcal{C}^{J^+} \xrightarrow{L_z} \mathcal{C},$$

where the latter functor is the latching space functor defined for direct categories in Definition 2.2.11 . Similarly, we define the matching space functor M_z as the composite

$$\mathcal{C}^J \to \mathcal{C}^{J^-} \xrightarrow{L_z} \mathcal{C},$$

where the latter functor is the matching space functor defined for inverse categories in Definition 2.2.11. Note that we have natural transformations

$$L_z X \to X_i \to M_i X$$

defined for $X \in \mathcal{C}^J$.

Proposition 2.2.19. *Let $\mathcal{C}$ be a model category and consider the category $\mathcal{C}^{\Delta^{\mathrm{op}}}$, i.e., the category of simplicial objects in $\mathcal{C}$. A morphism $f\colon X \to Y$ in $s\mathcal{C}$ is*

(i) a Reedy weak equivalence if $f_n\colon X_n \to Y_n$ is a weak equivalence in $\mathcal{C}$,

(ii) a Reedy (trivial) cofibration if for every $n \geq 0$ the canonical map,

$$X_n \coprod_{L_n X} L_n Y \to Y_n$$

is a (trivial) cofibration in $\mathcal{C}$

(iii) a Reedy (trivial) fibration if for every $n \geq 0$ the canonical map

$$Y_n \to Y_n \times_{M_n Y} M_n X$$

is a (trivial) fibration in $\mathcal{C}$.

Remark 2.2.20. More generally, in the above proposition the Reedy category Δ^{op} can be interchanged with any other Reedy category J, but for the course of our exposition we only make use of simplicial objects. The Reedy model structure for arbitrary Reedy category J enjoys the following property. Let $\mathcal{C}$ be a combinatorial model category and recall the projective and injective model struture on $\mathcal{C}^J$, Definition 2.2.5. The identity functors provide left Quillen equivalences

$$\mathcal{C}^j_{\text{proj}} \xrightarrow{\text{Id}} \mathcal{C}^J_{\text{Reedy}} \xrightarrow{\text{Id}} \mathcal{C}^J_{\text{inj}}$$

from the projective model structure on functors to the injective one.

It follows from the definition that a Reedy cofibrant object has a simple description. For future reference we state it as a definition.

Definition 2.2.21. A simplicial object $X \in s\mathcal{C}$ is *Reedy cofibrant* if the maps $L_n X \to X_n$ are cofibrations in $\mathcal{C}$ for all n. A sequence of objects $\{X_i\}_{i \in \mathbb{Z}_{\geq 0}}$ as follows

$$* \to X_0 \to X_1 \to X_2 \to \dots$$

is Reedy cofibrant if all the maps in the sequence are cofibrations.

We will return to Reedy cofibrant objects and Reedy cofibrant sequences in Section 2.4 and in Section 3.5.

2.2.22 Homotopy Colimits in Model Categories

In this subsection we will recall some definitions of homotopy colimits in general model categories. Recall that the functor $\operatorname{colim}\colon \mathcal{C}^J \to \mathcal{C}$ has a right adjoint which is given by the constant functor

$$\operatorname{const}\colon \mathcal{C} \to \mathcal{C}^J.$$

It follows from Proposition 2.2.12 that for any model category $\mathcal{C}$ and direct category J, there is a Quillen adjunction

$$\operatorname*{colim}_J\colon \mathcal{C}^J \rightleftarrows \mathcal{C}\colon \ \operatorname{const}.$$

One can see this by noticing that the constant functor

$$\operatorname{const}\colon \mathcal{C} \to \mathcal{C}^J$$

sends weak equivalences and fibrations in $\mathcal{C}$ to projective weak equivalences and projective fibrations in $\mathcal{C}^J$, that is, it is a right Quillen functor.

Definition 2.2.23 (Homotopy Colimit)**.** Let $\mathcal{C}$ be a model category and let J be a poset. Consider the projective model structure $\mathcal{C}^J$. The left derived functor of $\operatorname{colim}\colon \mathcal{C}^J \to \mathcal{C}$ is called *homotopy colimit* and denoted by

$$\operatorname*{hocolim}_J\colon \ \operatorname{Ho}(\mathcal{C}^J) \to \operatorname{Ho}(\mathcal{C}), \quad X \mapsto \operatorname*{colim}_J QX,$$

where QX is some cofibrant replacement of X in $\mathcal{C}^J$. We will explain the concept of homotopy colimit more in the next section. If J is not a poset we have to require the existence of the projective model structure on $\mathcal{C}^J$ (in other words, $\mathcal{C}$ should be cofibrantly generated) to consider the homotopy colimit.

In the case the poset is $\ulcorner$ defined in Example 2.2.8, the homotopy colimit of a diagram $F \in \mathrm{Ho}(\mathcal{C}^{\ulcorner})$, *i.e.*, $\mathrm{hocolim}_{\ulcorner} F$ is called *homotopy pushout*. A particular form of homotopy pushout is the so-called homotopy cofiber or cone, a homotopical version of ordinary cofiber (or cokernel) of a map. Recall that the cofiber (cokernel) of a map is given by the pushout

$$\mathrm{colim}\left(\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \\ * & & \end{array}\right).$$

Our next example recalls the homotopy cofiber.

Example 2.2.24 (Homotopy Cofiber)**.** Let $f\colon X \to Y$ be a morphism in a pointed model category $\mathcal{C}$. The *homotopy cofiber* or *cone* of f is defined as the homotopy colimit (homotopy pushout)

$$\mathrm{hocolim}\left(\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \\ * & & \end{array}\right).$$

We can define the homotopy cofiber with the help of homotopy Kan extensions which we define later. We will denote the homotopy cofiber by $\mathrm{cone}(f)$ or with $\mathrm{hocofib}(f)$. Another way to write the above colimit is $\mathrm{cone}(f) = \mathrm{hocolim}(* \leftarrow X \to Y)$

We barely touched the subject of homotopy colimits in model categories but this will suffice for our purposes. We will return to the subject of homotopy colimits in Section 2.4

2.2.25 Stable Model Categories

Recall that the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a pointed model category $\mathcal{C}$ supports a *suspension* functor

$$\Sigma\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$$

given by

$$\Sigma X := \mathrm{hocolim}(* \leftarrow X \rightarrow *),$$

with a right adjoint functor

$$\Omega\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$$

given by

$$\Omega X = \mathrm{holim}(* \rightarrow X \leftarrow *).$$

In the more familiar examples of model categories like spaces and chain complexes, suspension and loop functors usually admit a simple description.

Example 2.2.26. Consider Top_* the model category of pointed topological spaces and let $X \in \mathrm{Top}_*$. The suspension ΣX is defined as the smash product $S^1 \wedge X$ and the loop space ΩX is defined by the mapping space of pointed maps $\mathrm{Map}(S^1, X)$. It follows we have the adjoint pair (Σ, Ω), i.e.,

$$\mathrm{Map}(\Sigma X, Y) \cong \mathrm{Map}(X, \Omega Y).$$

Example 2.2.27. Let $\mathcal{C}(K)$ be the category of cochain complexes of K-modules for a commutative ring K. In this case the suspension of $C^\bullet$ is simply the shift, i.e., $\Sigma C^\bullet = C^{\bullet+1}$. Notice in this case the suspension has an inverse Σ^{-1} which shifts in the opposite direction.

A *stable* model category is an axiomatization of the above fact, that is, if the adjoint pair (Σ, Ω) is an adjoint equivalence.

Definition 2.2.28. A *stable model category* is a pointed model category for which the unit and counit functors of the adjoint pair (Σ, Ω) are equivalences.

As we will see later, the category Sp of spectra is also a stable model category. Leaving examples for now, we continue our discussion for an arbitrary stable model category $\mathcal{C}$.

Remark 2.2.29. If $\mathcal{C}$ is a pointed simplicial model category, then the suspension functor

$$\Sigma\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$$

admits a simple description. Indeed, by the simplicial model category axioms the functor

$$S^1 \wedge -\colon \mathcal{C} \to \mathcal{C}$$

is a left Quillen functor, where S^1 stands for the simplicial circe, i.e., $\Delta[1]/\partial\Delta[1]$.

Then, Σ can be defined as the left derived functor of $S^1 \wedge -$, i.e.,

$$\Sigma X := S^1 \wedge^{\mathbb{L}} X = S^1 \wedge QX,$$

see, [24, p. 6.1.1]

Note that if $\mathcal{C}$ is stable, then the homotopy category $\mathrm{Ho}(\mathcal{C})$ is a triangulated category with Σ a shift functor, see [5, Theorem 4.2.1] and [24, p. 7.1.6]. We will not recall here the definition of a triangulated category, we refer to [5, Definition 4.1.2]. In a simplicial model category $\mathcal{C}$ we can choose a particular model for the homotopy cofiber or cone, see Example 2.2.24 of a morphism that helps with computations. It is called the *mapping cone* construction.

Definition 2.2.30. Suppose $\mathcal{C}$ is a simplicial stable model category and $f\colon X \to Y$ a morphism in $\mathcal{C}_{\mathrm{cof}}$. Let $\mathrm{cone}(f)$ be the pushout of f along the morphism canonical morphism

$$\mathrm{incl} \wedge 1\colon S^0 \wedge X \to (I, 0) \wedge X = CX,$$

that is, $\mathrm{cone}(f)$ comes with the pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{incl}\wedge 1}\downarrow & & \downarrow \\ CX & \longrightarrow & \mathrm{cone}(f). \end{array}$$

Here CX denotes the *cone* of X, i.e.the smash product $(I, 0) \wedge X$, where

$(I, 0)$ is the simplicial set Δ^1 pointed at 0. The natural map

$$\pi\colon (I, 0) \wedge X \to S^1 \wedge X$$

and the trivial map

$$*\colon Y \to S^1 \wedge X$$

induce, using the universal property of pushput, a map

$$\partial\colon \operatorname{cone}(f) \to S^1 \wedge X$$

The fact that the mapping cone construction $\operatorname{cone}(f)$ represents the homotopy cofiber and further details can be found in the proof of [24, Proposition 6.3.4].

Definition 2.2.31. Let $\mathcal{C}$ be a simplicial stable model category and $f\colon X \to Y$ a morphism in $\mathcal{C}_{\mathrm{cof}}$. The *elementary triangle* associated to f is the triangle

$$X \xrightarrow{f} Y \xrightarrow{\iota} \operatorname{cone}(f) \xrightarrow{\partial} S^1 \wedge X.$$

A triangle (f, g, h)

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

in $\mathrm{Ho}(\mathcal{C})$ is called *distinguished* if it is isomorphic to an elementary one.

Note, that a morphism (α, β, γ) is an isomorphism of triangles if all components are isomorphisms.

2.3 Monoidal & Enriched Model Categories

In this section we will review some relevant facts about monoidal model categories, that is to say, model categories that support a compatible monoidal structure. In order to do so we will make a small detour and introduce the notion of a *Quillen bifunctor*. The advantage of introducing this more abstract formulation first is that it axiomatizes various interesting structures in homotopy theory that we will be using. This includes the structure of a monoidal model category and an enriched model category. For references for Quillen bifunctors we refer to [24, Chapter 4], and for a contemporary reference on monoidal model categories see [5, Chapter 6].

2.3.1 Quillen Bifunctors

In this subsection we introduce Quillen bifunctors. We do not recall here the definition of a closed symmetric monoidal category. See, [5, Definition 6.1.1]. It is usually denoted by $(\mathcal{C}, \otimes, I, \mathrm{Hom})$, where I is the unit of the monoidal product, and Hom is the inner hom. We will abuse notation and write it as $(\mathcal{C}, \otimes)$. We start with the definition of a two-variable adjunction.

Definition 2.3.2. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be categories. An adjunction of two variables or two-variable adjunction

$$(\otimes, \mathrm{Hom}_l, \mathrm{Hom}_r)\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$$

consists of bifunctors

$$\otimes\colon \mathcal{C}\times\mathcal{D}\to\mathcal{E}$$
$$\mathrm{Hom}_r\colon \mathcal{D}^{\mathrm{op}}\times\mathcal{E}\to\mathcal{C}$$
$$\mathrm{Hom}_l\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{E}\to\mathcal{D}$$

together with natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(c,\mathrm{Hom}_r(d,e))\cong\mathrm{Hom}_{\mathcal{E}}(c\otimes d,e)\cong\mathrm{Hom}_{\mathcal{D}}(d,\mathrm{Hom}_l(c,e)).$$

We often abuse notation by referring to $(\otimes,\mathrm{Hom}_r,\mathrm{Hom}_l)$,or even $\otimes$ alone as an adjunction of two variables, leaving the adjointness isomorphisms implicit.

Example 2.3.3. [42, Definition 3.7.2] If $\mathcal{V}$ is a closed symmetric monoidal category and let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a tensored $\mathcal{V}$-category, if for each $v\in\mathcal{V}$ and $c\in\mathcal{C}$ there is an object $v\otimes c\in\mathcal{C}$ together with natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(v\otimes c,c')\cong\mathrm{Hom}_{\mathcal{V}}(v,\mathrm{Hom}(c,c'))\quad\forall\ v\in\mathcal{V},c,c'\in\mathcal{C}.$$

Similarly we say that $\mathcal{C}$ is cotensored if for each $v\in\mathcal{V}$ and $c\in\mathcal{C}$ there is an object $c^v\in\mathcal{C}$ together with natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(c',c^v)\cong\mathrm{Hom}_{\mathcal{V}}(v,\mathrm{Hom}_{\mathcal{C}}(c',c))\quad\forall\ v\in\mathcal{V},c,c'\in\mathcal{C}.$$

Then, the tensor cotensor, and hom-objects define a two-variable adjunction

$$\mathcal{V} \times \mathcal{C} \to \mathcal{C}.$$

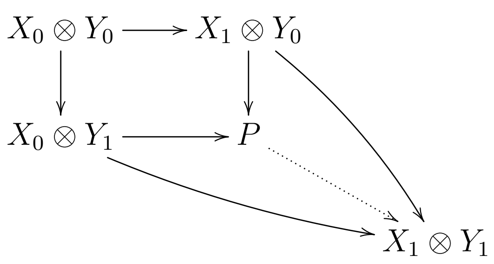
Example 2.3.4. Any closed symmetric monoidal category is tensored, cotensored and enriched over itself. These functors define a two-variable adjunction.

The above are the examples of two-variable adjunctions that most frequently appear "in nature".

Definition 2.3.5. Let $(\mathcal{C}, \otimes)$ be a closed symmetric monoidal monoidal category and let $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$ be maps in $\mathcal{C}$. The pushout-product map is the universal arrow

$$f \,\square\, g\colon X_0 \otimes Y_1 \coprod_{X_0 \otimes Y_0} X_1 \otimes Y_0 \to X_1 \otimes Y_1.$$

In other words, it is the universal map out of the following pushout as shown below



where we have denoted the pushout $P = (X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0)$.

Remark 2.3.6. The pushout-product map $f \square g$ can be defined alternatively as a left Kan extension. We will see it in Example 2.5.5.

Definition 2.3.7. Given model categories $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$, an adjunction of two variables

$$(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l) \colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$$

is called a *Quillen adjunction of two variables* or *a left Quillen bifunctor* if, given a cofibration $f \colon U \to V$ in $\mathcal{C}$ and a cofibration $g \colon X \to Y$ in $\mathcal{N}$, the universal map $f \square g$ defined in Definition 2.3.5 is a cofibration in $\mathcal{E}$ which is acyclic if either f or g is. We refer to the left adjoint $\otimes$ of a Quillen adjunction of two variables as a *Quillen bifunctor*, and often abuse notation by using the term "Quillen bifunctor $\otimes$".

An important point of left Quillen bifunctors is that they preserve weak equivalences on the subcategories of cofibrant objects and furthermore preserve cofibrant objects, see [42, Lemma 11.4.2] for a proof of this fact and further motivation on left Quillen bifunctors.

Our first example of Quillen bifunctor, which also can be taken as a definition is that of a monoidal model category see [24, p. 4.2.6] and [42, Definition 11.4.6]

Definition 2.3.8. A *symmetric monoidal modal category* is a closed symmetric monoidal category $(\mathcal{C}, \otimes, I)$ with a model structure such that the following hold:

(i) The monoidal structure $- \otimes - \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a Quillen bifunctor, and

(ii) the maps

$$QI \otimes v \to I \otimes v \cong v, \quad \text{and} \quad v \otimes QI \to v \otimes I \cong v$$

are weak equivalences if v is cofibrant.

The second condition is redundant if the unit I is cofibrant.

Example 2.3.9. Consider the monoidal categories $(s\mathrm{Set}, \times)$ and $(\mathrm{Top}, \times)$. These monoidal categories equipped with the standard model structures (Serre or Hurewicz) are both (symmetric) monoidal model categories. The same is true for the pointed versions, i.e., $(s\mathrm{Set}_*, \times)$ and $(\mathrm{Top}_*, \times)$

Example 2.3.10. Let K be a commutative ring and consider the monoidal category $(\mathrm{Ch}(K), \otimes)$. This monoidal category equipped with the *projective model structure* is monoidal model category. However, $(\mathrm{Ch}(K), \otimes)$ equipped with the *injective model structure* is *not* a monoidal model structure. To recall these two model structures on chain complexes, see [24, Theorem 2.3.11] and [24, Theorem 2.3.13] respectively.

A monoidal category $\mathcal{C}$ gives rise to more monoidal categories by considering diagrams from small categories into $\mathcal{C}$. In our next example we discuss how this is related to model category theory.

Example 2.3.11. Let $(\mathcal{C}, \otimes)$ be a monoidal model category and let J be a direct category. Consider the diagram category $\mathcal{C}^J$ with the projective model structure, Definition 2.2.12. The category $\mathcal{C}^J$ inherits a monoidal structure

$$\mathcal{C}^J \times \mathcal{C}^J \to \mathcal{C}^J, \quad (X, Y) \mapsto X \otimes Y,$$

where $X \otimes Y$ is the diagram $j \mapsto X_j \otimes Y_j$. By [8, Proposition 4.15] $(\mathcal{C}^J, \otimes)$ is a monoidal model category.

As we mentioned in the introduction of this section, Quillen bifunctors also capture the structure of an enriched model category. We now state the long-waited definition of a simplicial model category.

Example 2.3.12 (Simplicial Model Category)**.** A simplicial model category is a model category $\mathcal{C}$ that is tensored, cotensored, and simplicially enriched and such that

$$s\text{Set} \times \mathcal{C} \to \mathcal{C}$$

is a left Quillen bifunctor.

We can generalize the above and consider enriched in monoidal model categories other than $(s\text{Set}, \times)$.

Example 2.3.13 (Enriched Model Category)**.** Let $\mathcal{E}$ be a monoidal model category. An $\mathcal{E}$*-model categor*y is a category $\mathcal{C}$ enriched, tensored and cotensored over $\mathcal{E}$ together with a model structure such that

$$- \otimes - \colon \mathcal{E} \times \mathcal{C} \to \mathcal{C}$$

is a left Quillen bifunctor.

Enriched model categories, i.e., $\mathcal{E}$ model categories are also known as $\mathcal{E}$-modules.

Example 2.3.14. As a particular example of the above, an *algebraic* model category is a $\text{Ch}(\mathbb{Z})$-model category.

Our next example of simplicial structure plays will play a fundamental role in the thesis. Given a simplicial model category $\mathcal{C}$, there is a natural choice of simplicial model structure for the category of diagrams $\mathcal{C}^J$.

Example 2.3.15. Let J be a direct category. If $\mathcal{C}$ is a simplicial model category, then so is $\mathcal{C}^J$. Indeed, we define tensors and cotensors levelwise, that is, for $T \in s\mathrm{Set}$ and $X \in \mathcal{C}^J$

$$(T \wedge X)_z := T \wedge X_z, \quad (X^T)_z := (X_z)^T,$$

and the mapping spaces for $\mathcal{C}^J$ are given by the end construction

$$\mathrm{Map}_{\mathcal{C}^J}(X, Y) = \int_{z \in J} \mathrm{Map}_J(X_z, Y_z).$$

Now that we have a simplicial structure on the functor category $\mathcal{C}^J$, it will be useful to make the mapping cone construction on this category explicit, see Definition 2.2.30. Let J be a direct category and $\mathcal{C}$ be a simplicial model category. We equip $\mathcal{C}^J$ with the simplicial structure given in Example 2.3.15. If $f\colon X \to Y$ is a natural transformation of diagrams in $\mathcal{C}^J$ we can define the mapping cone $\mathrm{cone}(f) \in \mathcal{C}^J$ of the natural transformation f, see Definition 2.2.30. Since colimits and tensors in $\mathcal{C}^J$ are computed objectwise, we can evaluate $\mathrm{cone}(f)$ at $z \in J$ by the pushout

$$\begin{array}{ccc} X_z & \longrightarrow & Y_z \\ \downarrow & & \downarrow \\ CX_z & \longrightarrow & \mathrm{cone}(f)_z, \end{array}$$

that is,

$$\mathrm{cone}(f)_z = \mathrm{cone}(f_z\colon X_z \to Y_z). \tag{2.5}$$

This remark will be important in Section 4.6.

Next, we discuss that the homotopy category of a monoidal category is a closed monoidal category. In order to do so, we first present a result that a Quillen bifunctor of model categories induces an adjunction of two variables on the homotopy categories.

Proposition 2.3.16. *Suppose $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ are model categories and*

$$(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l)\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$$

is Quillen bifunctor. Then the total derived functors define an adjunction of two variables

$$(\otimes^{\mathbb{L}}, \mathbb{R}\mathrm{Hom}_r, \mathbb{R}\mathrm{Hom}_l)\colon\ \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{D}) \to \mathrm{Ho}(\mathcal{E})$$

For a proof see [24, Proposition 4.3.1.]. For objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the derived functor $- \otimes^{\mathbb{L}} -$ is defined as follows

$$C \otimes^{\mathbb{L}} D := QC \otimes QD \tag{2.6}$$

where Q is a cofibrant replacement, in the model structures $\mathcal{C}$ and $\mathcal{D}$ respectively. The above proposition specializes to the case of a monoidal model category $(\mathcal{C}, \otimes)$, see [5, Theorem 6.1.11] and [24, Theorem 4.3.2].

Corollary 2.3.17. *Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal model category. Then $(\mathrm{Ho}(\mathcal{C}), \otimes^{\mathbb{L}})$ is a closed symmetric monoidal category.*

2.3.18 Monoidal Model Structures on the Arrow Category

In this subsection we will explore the model structures on the arrow category $\mathrm{Arr}(\mathcal{C}) = \mathcal{C}^{[1]}$ and the interaction with the cofiber (cokernel) functor. Recall the interval poset $[1] = \{0 \leq 1\}$ from Example 2.2.7. The functor category $\mathcal{C}^{[1]}$ is also known as the category of arrows $\mathrm{Arr}(\mathcal{C})$. This is because, an object in $\mathcal{C}^{[1]}$ is a morphism $f\colon X \to Y$ and a morphism in $\mathcal{C}^{[1]}$ is a natural transformation of diagrams $[1] \to \mathcal{C}$ given by $\alpha\colon f \to g$ which is the following commutative square

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ Y_0 & \longrightarrow & Y_1. \end{array}$$

The arrow category $\mathcal{C}^{[1]}$ has two natural functors $\mathrm{Ev}_0\colon \mathcal{C}^{[1]} \to \mathcal{C}$ and $\mathrm{Ev}_1\colon \mathcal{C}^{[1]} \to \mathcal{C}$ which just evaluates a diagram at 0 and 1 respectively. This is the same as the functor that gives the source and target of an arrow f in $\mathcal{C}$.

Lemma 2.3.19. *[25, Lemma 1.1] Suppose $\mathcal{C}$ is a closed symmetric monoidal category. The evaluation functors $\mathrm{Ev}_0, \mathrm{Ev}_1\colon \mathcal{C}^{[1]} \to \mathcal{C}$ have left adjoints L_0, L_1 and right adjoints U_0, U_1*

Theorem 2.3.20. *[25, Theorem 1.2.] Let $(\mathcal{C}, \otimes)$ be a closed symmetric monoidal category. The category $\mathcal{C}^{[1]}$ has two different closed symmetric monoidal structures. In the* tensor product *monoidal structure, the monoidal product of $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$ is given by*

$$f \otimes g\colon X_0 \otimes Y_0 \to X_1 \otimes Y_1.$$

The unit is $L_0 e$*, and the closed structure is given by the projection map*

$$\underline{\mathrm{Hom}}_{\otimes}(f,g) = \mathrm{Hom}(X_0,Y_0) \times_{\mathrm{Hom}(X_0,Y_1)} \mathrm{Hom}(X_1,Y_1) \to \mathrm{Hom}(X_1,Y_1).$$

In the pushout-product *monoidal structure, the monoidal product of* f *and* g *is the pushout product*

$$f \,\square\, g \colon (X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \to X_1 \otimes Y_1.$$

The unit is $L_1 e$*, and the closed structure* $\mathrm{Hom}_{\square}(f,g)$ *is given by*

$$\mathrm{Hom}_{\square}(f,g) = \mathrm{Hom}(X_1,Y_0) \to \mathrm{Hom}(X_0,Y_0) \times_{\mathrm{Hom}(X_0,Y_1)} \mathrm{Hom}(X_1,Y_1).$$

Consider now the monoidal category $(\mathcal{C}^{[1]}, \square)$ equipped with the projective model structure, see Proposition 2.2.12. By [25, Theorem 3.1] this is a monoidal model structure. By Corollary 2.3.17, we have the monoidal category $(\mathrm{Ho}(\mathcal{C}^{[1]}), \square^{\mathbb{L}})$. For the next lemma, recall from Example 2.2.24 the cone or homotopy cofiber $\mathrm{cone}(f)$ or $\mathrm{hocofib}(f)$ of a map in a model category $\mathcal{C}$. It is the homotopical invariant version of the cokernel or cofiber of a morphism, that is, the pushout

$$\mathrm{coker}(f) = \mathrm{colim} \begin{pmatrix} X & \xrightarrow{f} & Y \\ \downarrow & & \\ * & & \end{pmatrix}$$

We can regard the cokernel as a functor

$$\operatorname{coker}\colon \mathcal{C}^{[1]} \to \mathcal{C}^{[1]}, \quad (X \xrightarrow{f} Y) \mapsto (Y \to \operatorname{coker}(f)).$$

Dually we have the kernel of a morphism, given by

$$\ker(f) := \lim \left(\begin{array}{ccc} & & X \\ & & \big\downarrow{\scriptstyle f} \\ * & \longrightarrow & Y \end{array} \right)$$

The following [25, Theorem 1.4.]

Theorem 2.3.21. *Suppose $(\mathcal{C}, \otimes)$ is a pointed closed symmetric monoidal category. The functor*

$$\operatorname{coker}\colon (\mathcal{C}^{[1]}, \Box) \to (\mathcal{C}^{[1]}, \otimes)$$

is a (strongly) symmetric monoidal functor. Its right adjoint is the kernel.

The isomorphism $\operatorname{coker}(f \,\Box\, g) \cong \operatorname{coker}(f) \otimes \operatorname{coker}(g)$ comes from commutation of pushouts which we briefly explain. Consider the following diagram.

$$\begin{array}{ccccc} X_1 \otimes Y_1 & \longleftarrow & X_0 \otimes Y_1 & \longrightarrow & * \\ \big\uparrow & & \big\uparrow & & \big\downarrow{\scriptstyle =} \\ X_1 \otimes Y_0 & \longleftarrow & X_0 \otimes Y_0 & \longrightarrow & * \\ {\scriptstyle =}\big\uparrow & & \big\downarrow & & \big\downarrow{\scriptstyle =} \\ X_1 \otimes Y_0 & \underset{=}{\longleftarrow} & X_1 \otimes Y_0 & \longrightarrow & * \end{array}$$

If we take vertical pushouts in the above diagram we get the diagram

$$X_1 \otimes Y_1 \leftarrow X_0 \otimes Y_1 \coprod_{X_0 \otimes Y_0} X_1 \otimes Y_0 \to *,$$

whose pushout is $\operatorname{coker}(f \,\square\, g)$. On the other hand, if we take horizontal pushouts , we get the diagram

$$\operatorname{coker}(f) \otimes Y_1 \leftarrow \operatorname{coker}(f) \otimes Y_0 \to *,$$

whose pushout is $\operatorname{coker}(f) \otimes \operatorname{coker}(g)$. Since pushouts commute with each other, we see that

$$\operatorname{coker}(f \,\square\, g) \cong \operatorname{coker}(f) \otimes \operatorname{coker}(g),$$

both as objects in $\mathcal{C}$ and as maps in $\mathcal{C}$.

Next, consider the monoidal category $(\mathcal{C}^{[1]}, \otimes)$ with the injective model structure, $(\mathcal{C}^{[1]}_{\text{inj}}, \otimes)$, see Definition 2.2.5. By [25, Theorem 2.1] it is a monoidal model category and the evaluation functor $\operatorname{Ev}_1 \mathcal{C}^{[1]} \to \mathcal{C}$ is left Quillen. By [25, Proposition 4.1.] the cokernel functor is a left Quillen and hence so is the composition

$$(\mathcal{C}^{[1]}_{\text{proj}}, \square) \xrightarrow{\text{coker}} (\mathcal{C}^{[1]}_{\text{inj}}, \otimes) \xrightarrow{\operatorname{Ev}_1} (\mathcal{C}, \otimes).$$

So, the fact that both functors are monoidal and left Quillen imply the following corollary.

Corollary 2.3.22. *Let $(\mathcal{C}, \otimes)$ be a pointed symmetric monoidal model cate-*

gory, and let f and g be morphisms in $\mathcal{C}$. There is a canonical isomorphism

$$\mathrm{cone}(f) \wedge^{\mathbb{L}} \mathrm{cone}(g) \cong \mathrm{cone}(f \,\square^{\mathbb{L}}\, g),$$

which comes from the commutation of homotopy colimits and evaluating at the target, that is,

$$(\mathrm{Ho}(\mathcal{C}^{[1]}), \square^{\mathbb{L}}) \xrightarrow{\mathrm{cone}} (\mathrm{Ho}(\mathcal{C}^{[1]}), \otimes^{\mathbb{L}}) \xrightarrow{\mathrm{Ev}_1} (\mathrm{Ho}(\mathcal{C}^{[1]}), \otimes^{\mathbb{L}}).$$

This corollary will be fundamental later on in Chapter 4. We will also see a different proof of the above fact using homotopy left Kan extensions, see Proposition 2.5.11.

2.3.23 Smash Products for Diagram Model Categories

In the following, given posets I and J, the product $I \times J$ will always have the product order. If I and J are more generally direct categories then the degree of an element $(i, j) \in I \times J$ is defined to be the sum of degrees of i and j.

Definition 2.3.24. Let $(\mathcal{C}, \otimes)$ be a monoidal category and let I and J be direct categories. We define the *external product*, or *objectwise product*, which is the bifunctor

$$- \,\overline{\otimes}\, - \colon \mathcal{C}^I \times \mathcal{C}^J \to \mathcal{C}^{I \times J}$$

sending (X, Y) to the diagram $X \,\overline{\otimes}\, Y \colon (i, j) \mapsto X_i \otimes Y_j$.

The external product is part of a two-variable adjunction. Since we do not use the extra structure we will not define the other two functors in the two-variable adjunction.

We have the following proposition.

Proposition 2.3.25. *Let $(\mathcal{C}, \otimes)$ be a monoidal model category and let I and J be direct categories . Then, the bifunctor*

$$- \overline{\otimes} - \colon \mathcal{C}^I \times \mathcal{C}^J \to \mathcal{C}^{I \times J}$$

is a Quillen bifunctor, that is to say, it has a total left derived functor

$$- \overline{\otimes}^{\mathbb{L}} - \colon \operatorname{Ho}(\mathcal{C}^I) \times \operatorname{Ho}(\mathcal{C}^J) \to \operatorname{Ho}(\mathcal{C}^{I \times J}).$$

In fact, the above assertion follows from the more general statement that I and J are Reedy categories, see [8, Proposition 4.15]. Alternatively, suppose that $\mathcal{C}^I_{\text{inj}}, \mathcal{C}^J_{\text{inj}}$ and $\mathcal{C}^{I \times J}_{\text{inj}}$ exist, e.g., if $\mathcal{C}$ is a combinatorial model category. Since in the injective model structures the cofibrations are the objectwise cofibrations objectwise in $\mathcal{C}$, the above proposition follows directly. The universal property of $- \overline{\wedge}^{\mathbb{L}} -$ implies that up to canonical isomorphism both constructions give the same result.

Corollary 2.3.26. *There is a functor isomorphism*

$$\operatorname*{hocolim}_{I \times J}(X \overline{\otimes}^{\mathbb{L}} Y) \cong (\operatorname*{hocolim}_{I} X) \otimes^{\mathbb{L}} (\operatorname*{hocolim}_{J} Y)$$

Proof. From Proposition 2.3.25, it follows that the external product preserves cofibrant objects and preserves trivial cofibrations between diagram cofibrant objects. The result now follows from the strict formula

$$\operatorname*{colim}_{I\times J}(X \barwedge Y) \cong (\operatorname*{colim}_{I} X) \wedge (\operatorname*{colim}_{J} Y).$$

□

2.4 Homotopy Colimits in Simplicial Model Categories

In Definition 2.2.23 we saw a general definition of a homotopy colimit of a diagram in a model category $\mathcal{C}$ as a left derived functor. This is called the “global” definition. In the case that $\mathcal{C}$ is a simplicial model category there is another equivalent formulation, called the “local” definition. There is both a theory for homotopy limits and colimits but since we only use homotopy colimits we will only refer to that and leave homotopy limits out. In Subsection 2.4.8 we review the “local” definition of homotopy colimit and in the next subsection 2.4.11 we will review a closely related way to construct homotopy colimits via simplicial objects.

2.4.1 Nerves of Overcategories and Undercategories

In this subsection we will recall overcategories and undercategories. They are also known as *comma* or *slice* category. They will be useful later on.

Definition 2.4.2. If $\mathcal{C}$ and $\mathcal{D}$ are categories, $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, and d is an object of $\mathcal{D}$, then the category of objects of $\mathcal{C}$ over d denoted by F/d

or $(F \downarrow d)$ is the category in which an object is a pair (c, σ) where c is an object of $\mathcal{C}$ and σ is a map $F_c \to d$ in $\mathcal{D}$, and a morphism from the object (c, σ) to the object (c', σ') is a map $\tau\colon c \to c'$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} F_c \arrow[rr] \arrow[dr, "\sigma"'] & & F_{c'} \arrow[dl, "\sigma'"] \\ & d & \end{tikzcd}$$

Notice that for every all the categories F/d there exists a natural forgetful functor

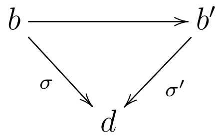
$$\pi\colon F/d \to \mathcal{C}, \quad (c, \sigma) \mapsto c. \tag{2.7}$$

Let $\mathrm{const}_d\colon F/d \to \mathcal{D}$ be the constant functor sending every (c, σ) to $d \in \mathcal{D}$ and any morphism $h \in \mathrm{mor}(F/d)$ to 1_d. Then, there is a natural transformation $\eta\colon F \circ \pi \Rightarrow \mathrm{const}_d$ given by $\eta = \left\{ \eta_{(c,\sigma)}\colon F_c \xrightarrow{\sigma} d \right\}$. A convenient way to organize this data is the following diagram

$$\begin{tikzcd} F/d \arrow[r, "\pi"] \arrow[d] \arrow[dr] & \mathcal{C} \arrow[d, "F"] \\ * \arrow[r] & \mathcal{D}. \end{tikzcd} \tag{2.8}$$

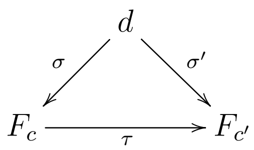
Here $*$ denotes the category with one object $*$ and one morphism. Our first example of an overcategory is the following

Example 2.4.3. Let $\mathcal{C} = \mathcal{D}$ and $F = 1_{\mathcal{C}}$. Then, the category F/d is usually denoted by $\mathcal{C}/d$. That is, an object in $\mathcal{C}/d$ an object is a pair (b, σ) where b is an object of $\mathcal{C}$ and σ is a map $b \to a$ in $\mathcal{C}$, and a morphism from the object (b, σ) to the object (b', σ') is a map $\tau\colon b \to b'$ that makes the following triangle commutative



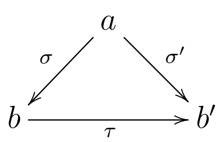
Dually to Definition 2.4.2 we can define the following category.

Definition 2.4.4. If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between small categories and d is an object of $\mathcal{D}$, then the category d/F or $d \downarrow F$ of *objects of* $\mathcal{C}$ *under* d is the category in which an object is a pair (c, σ) where c is an object of $\mathcal{C}$ and σ is a map $\sigma\colon d \to F_c$ in $\mathcal{D}$, and a morphism from the object (c, σ) to the object (c', σ') is a map $\tau\colon c \to c'$ in $\mathcal{C}$ that makes the triangle



Similarly, we have the dual to Example 2.4.3.

Example 2.4.5. If $\mathcal{C}$ is a small category and a is an object of $\mathcal{C}$, then the category of *objects of* $\mathcal{C}$ *under* a denoted by $a/\mathcal{C}$ or $a \downarrow \mathcal{C}$ is the category in which an object is a pair (b, σ) where b is an object of $\mathcal{C}$ and σ is a map $a \to b$ in $\mathcal{C}$, and a morphism from the object (b, σ) to the object (b', σ') is a map $\tau\colon b \to b'$ that makes the triangle



commute.

In fact, the categories $c/\mathcal{C}$ for various c can be assembled into a single functor which we define below

Definition 2.4.6. Let $\mathcal{C}$ be a small category. We define the following two functors.

- $-/\mathcal{C}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$, is the diagram that on an object c of $\mathcal{C}$ takes the value $c/\mathcal{C}$. A morphism $c \to c'$ induces a functor $c'/\mathcal{C} \to c/\mathcal{C}$ by pulling back,

- $N(-/\mathcal{C})\colon \mathcal{C}^{\mathrm{op}} \to s\mathrm{Set}$, is the diagram that an object c of $\mathcal{C}$ takes the value $N(c/\mathcal{C})$, and that takes the map $\sigma\colon c \to c'$ to the map of simplicial sets $\sigma^*\colon N(c'/\mathcal{C}) \to N(c/\mathcal{C})$.

Remark 2.4.7. In all of our cases of interest we only consider slice categories of posets. So it will be useful to have a more explicit form of the slice categories for morphisms of posets. Given a morphism of posets $f\colon \mathcal{C} \to \mathcal{D}$, the slice categories f/d and d/f are given by

$$f/d = \{c \in \mathcal{C}\colon f(c) \leq d\}$$

$$d/f = \{c \in \mathcal{C}\colon f(c) \geq d\}$$

A similar statement holds for the slice categories $\mathcal{C}/c$ and $c/\mathcal{C}$. For details see [36, Section 1].

2.4.8 Local Definition and Functor Tensor Products

As we said above, in this section we will review the local definition of homotopy colimits. The main reference for this subsection is [21, Chapter

18]. More contemporary references are [47, Section 6], while we follow the overview given at [15, Chapter 5].

Since we would like to formulate a homotopy invariant version of functor colim: $\mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ it will be convenient to reformulate it in a way that is easier to construct a homotopical invariant version of colim. First we recall the *functor tensor product.*

Definition 2.4.9 (Functor tensor product)**.** [21, Definition 18.3.2] Let $\mathcal{C}$ be a simplicial model category and let I be a small category. If F is a I-diagram in $\mathcal{C}$ and G is a I^{op}-diagram of simplicial sets ($G\colon I^{\mathrm{op}} \to s\mathrm{Set}$), then the functor tensor product $G \otimes_I F$ is defined as follows. The object $G \otimes_I F$ of $\mathcal{C}$ is the coequilizer of the maps

$$\mathrm{coeq}\left(\coprod_{\sigma\colon i \to i'} G_{i'} \otimes F_i \rightrightarrows \coprod_{i \in \mathrm{ob}(\mathcal{C})} G_i \otimes F_i\right)$$

where the top map on the summand $\sigma\colon i \to i'$ is the composition of the map

$$\sigma_* \otimes 1\colon X_i \otimes K_j \to X_i \otimes K_i$$

with the natural injection into the coproduct where $\sigma_* = X(\sigma)\colon X_i \to X_{i'}$. Similarly, the lower map is the composition of the map

$$1 \otimes \sigma^*\colon G_i \otimes F_j \to G_i \otimes F_i$$

with the natural injection into the coproduct (where $\sigma^*\colon K_j \to K_i$).

The functor tensor product $G \otimes_I F$ from the functor $G \otimes F\colon I^{\mathrm{op}} \times I \to \mathcal{C}$ is an example of the general construction knows as *coend*. Usually it is denoted as

$$\int^{i \in I} G_i \otimes F_i.$$

Here $\otimes$ is the tensor action of simplicial sets on the category $\mathcal{C}$. Note that if we take K to be the constant functor at the terminal simplicial set, which has a unique simplex in each dimension, then we obtain

$$*_I \otimes_I X \cong \operatorname*{colim}_I X.$$

Thus, the tensor product $G \otimes_I F$ can be thought of as the colimit of F "fattened up" by G.

The point of the functor tensor product is that we can replace the constant diagram $*_I$ at the terminal simplicial set in the construction $* \otimes_I X$ by a diagram of larger contractible spaces. This replacement gives us the homotopical "wiggle" room we were looking for and produces the homotopy colimit. The minimal natural choices for these contractible spaces come from the nerves of over-categories and under-categories in the diagram. Given a small category $\mathcal{C}$, recall the diagram of simplicial sets $N(-/\mathcal{C})^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \to s\mathrm{Set}$, see Definition 2.4.6.

Definition 2.4.10 (Uncorrected Homotopy Colimit)**.** Let X be a I-diagram in a simplicial model category $\mathcal{C}$. Then the *uncorrected homotopy colimit* is

$$\mathrm{uhocolim}_I X := N(-/I) \otimes_I X$$

Spelled out from Definition 2.4.9 the object uhocolim X is given by

$$\operatorname{uhocolim} X = \operatorname{coeq}\left(\coprod_{i\to j} N(j/I)\otimes X_i \rightrightarrows \coprod_{i\in I} N(i/I)\otimes X_i\right).$$

This is [21, Definition 18.1.1] where the author does not use the adjective "uncorrected". The reason that we call it uncorrected is that in full generality, in an arbitrary simplicial model category $\mathcal{C}$, the above construction is not homotopy invariant, so it does not induce a functor on the level of homotopy categories. Problems tend to arise when the objects of our diagram are not cofibrant. We will come back to that later one.

To briefly explain how we reached Definition 2.4.10 as a first approximation, lets look back at how we first defined the derived colimit, Definition 2.2.23. We defined it as first taking a cofibrant replacement and then calculating the ordinary colimit. With the formulation of a colimit as a functor tensor product, Definition 2.4.9 this is the following.

$$\begin{aligned} \operatorname{hocolim}\colon \operatorname{Ho}(\mathcal{C}^I) &\to \operatorname{Ho}(\mathcal{C}) \\ X &\mapsto *_I \otimes_I X_{\mathrm{cof}}. \end{aligned}$$

Here X_{cof} is a cofibrant replacement in the projective model structure $\mathcal{C}^I_{\mathrm{proj}}$. This cofibrant replacement can be very hard to calculate, so instead of replacing X we replace the constant functor $*_I$. If X is objectwise cofibrant, then $*_I \otimes_I X_{\mathrm{cof}}$ and $(*_I)_{\mathrm{cof}} \otimes_I X$ are weakly equivalent. Here X_{cof} is, as before, the cofibrant replacement in the projective model structure on $\mathcal{C}^I$, while $(*_{\mathcal{I}})_{\mathrm{cof}}$ is the cofibrant replacement in the projective model structure on $s\mathrm{Set}^{I^{\mathrm{op}}} = \mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, s\mathrm{Set})$. The nerve $N(-/I)\colon I^{\mathrm{op}} \to s\mathrm{Set}$ is cofibrant in the projective model structure on $s\mathrm{Set}^{\mathcal{I}^{\mathrm{op}}}$ and so provides a particular choice of a cofibrant replacement for the constant functor, and thereby we reached the definition of uncorrected homotopy colimit Definition 2.4.10, i.e.,

$$\mathrm{uhocolim}_I\, G := N(-/I) \otimes_I G.$$

We can reestablish homotopy invariance by precomposing this functor with an objectwise cofibrant replacement which we assume that exists in any model category, see Subsection 2.2.1. We denote by $C_{\mathrm{obj}}G$ the objectwise functorial cofibrant replacement. The result is the *corrected* homotopy colimit

$$\operatorname*{hocolim}_I\colon \mathrm{Ho}(\mathcal{C}^I) \to \mathrm{Ho}(\mathcal{C}) \quad G \mapsto N(-/I) \otimes_I C_{\mathrm{obj}}G \tag{2.9}$$

The (corrected) homotopy colimit represents the derived functor, as desired: the functor tensor product $*_I \otimes_I G_{\mathrm{cof}}$ (which represents the derived colimit) is weakly equivalent to the corrected homotopy colimit $N(-/I) \otimes_I C_{\mathrm{obj}}G$. Here G_{cof} is the cofibrant replacement in the projective model structure on $\mathcal{C}^{\mathcal{I}}$.

2.4.11 Bousfield-Kan Construction aka Simplicial Replacement

In this subsection we will provide one more way to construct homotopy colimits. This is done via simplicial techniques. After introducing some definitions we explain briefly how this method constructs a good theory of homotopy colimits. The main references for this subsection are [42, Chapters 4, 5] and [47, Section 7].

Let $\mathcal{C}$ be a model category and consider the category of simplicial objects $s\mathcal{C} = \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$. We consider $s\mathcal{C}$ as a simplicial category with tensors defined objectwise, as in Example 2.3.15. This means that if $K \in s\mathrm{Set}$ and $X \in s\mathcal{C}$, then $K \otimes X$ is the simplicial object defined by

$$(K \otimes X)_n = K \otimes X_n.$$

Note the difference from the tensor action that we used in (2.3) which was denoted by

$$(K \odot X)_n = \coprod_{K_n} X_n.$$

Now, let $\mathcal{C}$ be a simplicial model category. Given a simplicial object $X \in \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ we can construct an object in $\mathcal{C}$ by "gluing" the simplices. This is called *geometric realization*, see [21, Definition 18.6.2].

Definition 2.4.12 (Geometric Realization)**.** Let $X \in \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$. *The geometric realization* of X, denoted as $|X|$ is defined as the coequalizer

$$\mathrm{coeq}\left(\coprod_{\sigma\colon [n]\to[k]\in\mathbf{\Delta}} \Delta^k \otimes X_n \rightrightarrows \coprod_{[n]\in\mathbf{\Delta}} \Delta^n \otimes X_n \right).$$

This is an example of a functor tensor product Definition 2.4.9 (coend). In this case, the geometric realization is the functor tensor product of $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ and the functor $\Delta^\bullet\colon \mathbf{\Delta} \to s\mathrm{Set}, [n] \mapsto \Delta^n$. In other words, the realization

$|X|$ is the object

$$\Delta^\bullet \otimes_{\mathbf{\Delta}^{\mathrm{op}}} X = \int^n \Delta^n \otimes X_n.$$

The following theorem is the cornerstone of our exposition of homotopy colimits using geometric realizations. Like all theorems of importance, it has many proofs which some can be found in [19, p. VII 3.6], [21, p. 18.4.11] and [42, Corollary 14.3.10]. Recall the Reedy model structure on $s\mathcal{C}$, Proposition 2.2.19.

Theorem 2.4.13. *If $\mathcal{C}$ is a simplicial model category, then*

$$|-| : s\mathcal{C} \to \mathcal{C}$$

is a left Quillen functor with respect to the Reedy model structure. In particular, $|-|$ sends Reedy cofibrant simplicial objects to cofibrant objects and preserves objectwise weak equivalences between them.

At this level of generality, this is the strongest result possible. It is not true that geometric realization preserves all objectwise weak equivalences. However this will suffice for our purposes. With the power of above theorem, we start to work our way towards how this is related to the homotopy colimit of a diagram $X \in \mathcal{C}^I$ in a simplicial model category $\mathcal{C}$.

Our first definition towards this goal is the *simplicial replacement functor.* That is to say, given any diagram $F\colon I \to \mathcal{C}$ we can replace it with simplicial object in $\mathcal{C}$ with good properties.

Definition 2.4.14 (Simplicial replacement)**.** Let I be a small category and consider a diagram $X\colon I \to \mathcal{C}$. The *simplicial replacement* of X is the simplicial object in $\mathcal{C}$, denoted as $\operatorname{srep} X$ or as X^{Δ} given in simplicial degree $[n]$

$$(\operatorname{srep} X)_n = \coprod_{(i_0 \to i_1 \to \ldots \to i_n) \in N(I)_n} X_{i_0}.$$

The coproduct is indexed over the set of n-chains

$$\sigma = [i_0 \to i_1 \to \ldots \to i_n]$$

over the nerve of I. If $0 \leq k < n$, then

$$d_k \colon (\operatorname{srep} X)_n \to (\operatorname{srep} X)_{n-1}$$

maps the term X_{i_n} indexed on σ to the term X_{i_n} indexed on

$$\sigma(k) = [i_0 \to i_1 \to i_{k-1} \to i_{k+1} \to \ldots \to i_n]$$

via the identity, while for $k = n$, the map d_n sends the term X_{i_n} to $X_{i_{n-1}}$ indexed on

$$\sigma(n) = [i_0 \to i_1 \to \ldots \to i_{n-1}]$$

via the induced map $X(i_n \to i_{n-1})$. The degeneracy maps

$$s_j \colon (\operatorname{srep} X)_n \to (\operatorname{srep} X)_{n+1}, 0 \leq j \leq n$$

are easier to define. Each s_j sends the summand X_{i_n} corresponding the summand

$$[i_0 \to i_1 \to \ldots \to i_n]$$

to the identical summand X_{i_n} corresponding to the chain in which one has inserted the identity map $i_j \to i_j$.

In other words, the simplicial replacement is the following simplicial object

$$\coprod_{i_0} X_{i_0} \leftleftarrows \coprod_{i_0 \to i_1} X_{i_0} \; \substack{\leftarrow\\[-1ex]\leftarrow\\[-1ex]\leftarrow} \coprod_{i_0 \to i_1 \to i_2} X_{i_0} \cdots$$

where degeneracy maps are omitted. Usually this is called the *simplicial bar construction* or *Bousfield-Kan construction* denoted by $B(*, I, X)$. We will stick to $\operatorname{srep}(X)$ or X^{Δ} for now. The reader can find more details in the references mentioned in the beginning of the subsection.

Remark 2.4.15. The colimit of a diagram $X \in \mathcal{C}^I$ if it exists, agrees with the colimit of $\operatorname{srep}(X) \in s\mathcal{C}$. Indeed, consider the colimit of the diagram $\operatorname{srep}(X)$, that is $\operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \operatorname{srep}(X)$. Since we assume that $\mathcal{C}$ is cocomplete we can writh this colimit as a coequilizer of coproducts, that is to say

$$\coprod_{i} X_i \leftleftarrows \coprod_{j \leftarrow i} X_i,$$

where the arrows are "source" and "target". But this is precisely the colimit of X. Therefore in this case $\operatorname{srep}(X)$ has the augmentation

$$\operatorname{srep}(F) \to \operatorname*{colim}_{I} F,$$

where here we regard the object $\operatorname{colim}_I F$ as a constant simplicial object.

Formally we have presented the following result.

Lemma 2.4.16. *Given a diagram $X \in \mathcal{C}^I$ and its simplicial replacement* srep$(X) \in s\mathcal{C}$, *there is a canonical isomorphism*

$$\operatorname*{colim}_I X \cong \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} (\mathrm{srep}(X))$$

The proof can be found in [42, Lemma 4.4.2]. The following lemma will be very important to us [42, Lemma 5.1.2], [47, Lemma 8.7]. Since it is so important in our exposition we include a proof for completeness.

Lemma 2.4.17. *Let I be a small category and let $\mathcal{C}$ be a simplicial model category. If $F \in \mathcal{C}^I$ is objectwise cofibrant, then* srep$(F) \in s\mathcal{C}$ *is Reedy cofibrant.*

Proof. Recall $B_n(*, I, F)$ is the coproduct over NI_n of the image under F of the first object in the sequence of composable arrows. Hence, the nth latching object sits inside $B_n(*, I, F)$ as the coproduct indexed by degenerate simplices in NI_n. But cofibrations, and hence cofibrant objects, are closed under coproducts. Similarly cofibrations are closed under pushout and hence coproduct inclusions of cofibrant objects are also cofibrations. By our hypothesis, every object in these coproducts is cofibrant, so the above observations imply that the nth latching map is a cofibration. □

The following theorem relates what we have discussed so far, that is, the uncorrected and corrected homotopy colimit constructed in Definition 2.4.10 and (2.9) and the geometric realization of simplicial replacements. For a proof see [42, Theorem 6.6.1].

Theorem 2.4.18. *Let $F \in \mathcal{C}^I$ be any diagram in a complete and cocomplete,* *simplicial model category $\mathcal{C}$. There are natural isomorphisms*

$$|\operatorname{srep}(F)| \cong N(-/I) \otimes_I F.$$

In particular, the homotopy colimit of a objectwise cofibrant diagram F can be computed by the functor tensor product with $N(-/I)$.

So, based on the Theorem above, if we have a diagram $F\colon I \to \mathcal{C}$ that is objectwise cofibrant, then the geometric realization of the simplicial replacement captures the correct notion for derived colimit and can be used as a model for it. One of the reasons that this is happening is that simplicial replacements of objectwise cofibrant diagrams are Reedy cofibrant, see Lemma 2.4.17, and that geometric realization preserves weak equivalences between them.

Remark 2.4.19. In model categories such as topological spaces, simplicial sets and spectra, the geometric realization of the simplicial replacement captures the correct homotopy type. No objectwise cofibrant replacement is needed. The reader can find more details in [28], [30, Appendix C] and a more thorough treatment in [21, Chapter 18].

2.4.20 Skeletal Filtration of the Geometric Realization

In this subsection we return to our discussion of the skeleton filtration defined in (2.2) and how it relates with model category structure. Again we let $\mathcal{C}$ be a simplicial model category and let J be a direct category. We start with a remark.

Remark 2.4.21. On the category of simplicial objects in $\mathcal{C}$, that is, $s\mathcal{C}$ we have defined two different simplicial tensor actions, one given in Example 2.3.15, and the other (2.3). We recall them here briefly for clarification. For $K \in s\mathrm{Set}$ and $X \in s\mathcal{C}$ we have defined the simplicial actions, that is, the simplicial objects in $\mathcal{C}$ $K \otimes X$ and $K \odot X$ as follows

$$(K \otimes X)_n = K \otimes X_n$$
$$(K \odot X)_n = \coprod_{K_n} X_n$$

We need to know how they interact. We give a result that we won't prove. Consider the category $s\mathcal{C}$ with simplicial tensor action on $s\mathcal{C}$ given by (2.3) , i.e.,

$$s\mathrm{Set} \times s\mathcal{C} \to s\mathcal{C} \quad (K, X) \mapsto K \odot X,$$

with $(K \odot X)_n = \coprod_{K_n} X_n$. Let $c \in \mathcal{C}$ and consider it as a constant simplicial object, i.e., $\mathrm{const}(c) \in s\mathcal{C}$. Then there is a natural isomorphism

$$|K \odot \mathrm{const}(c)| \cong K \otimes \mathrm{const}(c)$$

where the simplicial tensor action on the right hand side is as in Example 2.3.15, see [19, Lemma 3.4] for a proof.

Recall from Proposition 2.1.9 that for a simplicial object X and for every

$n \geq 0$ there is a pushout square

$$\begin{array}{ccc} \partial\Delta^n \odot X_n \coprod_{\partial\Delta^n \odot L_n X} \Delta^n \odot L_n X & \longrightarrow & \mathrm{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \Delta^n \odot X_n & \longrightarrow & \mathrm{sk}_n X. \end{array}$$

Therefore, for $X \in s\mathcal{C}$ Reedy cofibrant, the geometric realization comes with a natural skeletal filtration. We define

$$\mathrm{sk}_n |X| = |\mathrm{sk}_n X| .$$

Then, by the above and the natural isomorphism $|K \odot A| \cong K \otimes A$, Remark 2.4.21 yields that there are natural pushout squares

$$\begin{array}{ccc} \partial\Delta^n \otimes X_n \coprod_{\partial\Delta^n \otimes L_n X} \Delta^n \otimes L_n X & \longrightarrow & \mathrm{sk}_{n-1} |X| \\ \downarrow & & \downarrow \\ \Delta^n \otimes X_n & \longrightarrow & \mathrm{sk}_n |X| . \end{array} \tag{2.10}$$

This is because $|-|$ as a left adjoint commutes with all colimits. If X is Reedy cofibrant, then Theorem 2.4.13 implies that each of the maps

$$\mathrm{sk}_{n-1} |X| \to \mathrm{sk}_n |X|$$

is a cofibration. Furthermore, again since $|-|$ commutes with colimits

$$\operatorname*{colim}_n \mathrm{sk}_n |X| \cong |X| . \tag{2.11}$$

We end this subsection by discribing how equation (2.11) can be equivalently stated as a homotopy colimit. This is will come up in Subsection 3.5.9 where we will discuss a spectral sequence of a filtration of spectra. Recall from Definition 2.2.21 that a sequence of objects $\{C_i\}_{i\in\mathbb{Z}_{\geq 0}}$ in $\mathcal{C}$ as follows

$$* \to C_0 \to C_1 \to C_2 \to \dots$$

is Reedy cofibrant if all the objects are cofibrant and all the maps are cofibrations. This means that the colimit of the above sequence coincides with the homotopy colimit of the sequence. Now, if $X \in s\mathcal{C}$ is Reedy cofibrant, then the morphisms $|X|_{n-1} \to |X|_n$ are cofibrations which of course implies that the sequence

$$* \to |X|_0 \to |X|_1 \to |X|_2 \to \dots$$

is Reedy cofibrant. Altogether we have $\mathrm{hocolim}_i\, |X|_i \cong_{\mathrm{RC}} \mathrm{colim}_i\, |X|_i = |X|$ where the subscript RC serves as a reminder that an equivalence depends upon Reedy cofibrancy of X. Lastly, there is another homotopy colimit, that is, the homotopy colimit of the Δ^{op}-diagram itself. For completeness we state the following proposition.

Proposition 2.4.22. *[21, p. 18.7.4] If the simplicial object X is Reedy cofibrant, then there is a natural weak equivalence (The Bousfield-Kan map)*

$$\underset{\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\, X \xrightarrow{\cong} |X|.$$

We do not use the above proposition in our exposition.

2.4.23 Changing the Indexing Category & Homotopy Final Functors

Let $\alpha\colon \mathcal{C} \to \mathcal{D}$ a functor between small categories. Note that given any diagram $X\colon \mathcal{D} \to \mathcal{C}$ we get a natural morphism of simplicial objects

$$\phi_\alpha\colon (\operatorname{srep} F^*X) \to (\operatorname{srep} X).$$

Taking geometric realizations gives a natural map

$$\phi_\alpha\colon \operatorname*{hocolim}_{\mathcal{C}} \alpha^* X \to \operatorname*{hocolim}_{\mathcal{D}} X. \tag{2.12}$$

Suppose now that $\mathcal{C}$ is a simplicial model category. It is a natural question to ask the following. What are sufficient conditions for a functor $K\colon \mathcal{C} \to \mathcal{D}$ such that the map (2.12) is a weak equivalence in $\mathcal{C}$? Turns out that a sufficient conditions is the homotopical analogue of a *final functor* which we define below. Recall the category d/K, the category of objects of $\mathcal{C}$ under an object $d \in \mathcal{D}$, Definition 2.4.4. Also, recall that a simplicial set is *contractible* if the unique map to the terminal object is a weak homotopy equivalence.

Definition 2.4.24 (Homotopy Finality)**.** [42, Definition 8.5.1] A functor between small categories $K\colon \mathcal{C} \to \mathcal{D}$ is *homotopy final* (or *homotopy terminal*) if for every object $d \in \mathcal{D}$, the simplicial set $N(d/K)$ is contractible and *homotopy initial* if each $N(K/d)$ is contractible.

Recall that a functor $K\colon I \to J$ is final, if we can restrict diagrams on J

to diagrams on I along K without changing their colimit, that is,

$$\operatorname*{colim}_{I} K^* F \cong \operatorname*{colim}_{J} F.$$

The definition of homotopy final functor is justified by the following theorem

Theorem 2.4.25. *[42, Theorem8.5.6] Let $F\colon \mathcal{D} \to \mathcal{C}$ be a diagram in a simplicial model category. If $K\colon \mathcal{C} \to \mathcal{D}$ is homotopy final then*

$$\operatorname*{hocolim}_{\mathcal{C}} K^* F \to \operatorname*{hocolim}_{\mathcal{D}} F$$

is a weak equivalence.

Dually, a homotopy initial functor induces a weak equivalence on homotopy limits. Since we are only interested in diagrams of posets, by Remark 2.4.7 we can be more specific when a morphism of posets is homotopy final.

Corollary 2.4.26. *A map $f\colon \mathcal{C} \to \mathcal{D}$ of posets is homotopy terminal if for for every $d \in \mathcal{D}$*

$$d/f = \{c \in \mathcal{C}\colon f(c) \geq d\}$$

is contractible.

Remark 2.4.27. In Chapter 5 in the proof of Proposition 5.2.4 we will need a convenient way to check whether a poset is contractible. Recall from [36, Section 1.5] that a poset $\mathcal{C}$ is *conically contractible* if there is an object $c_0 \in \mathcal{C}$ and a map of posets $f\colon \mathcal{C} \to \mathcal{C}$ such that $c \leq f(c) \geq c_0$ for every $c \in \mathcal{C}$. In this case one can show that the identity $1_{\mathcal{C}}$, the map f, and the constant map with value c_0 from $\mathcal{C}$ to itself are homotopic (that is to say, their realizations are homotopic), and hence $\mathcal{C}$ is contractible.

Another useful fact that will come handy later is the following.

Lemma 2.4.28. *If $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ is an adjoint pair, then F is homotopy initial and G is homotopy terminal.*

Proof. For any object $d \in \mathcal{D}$, the category F/d of objects of $\mathcal{C}$ over d is isomorphic to $\mathcal{C}/G_d$. However, the category $\mathcal{C}/G_d$ has $1\colon G_d \to G_d$ as a terminal object. This implies that $N(F/d)$ is contractible, so F is homotopy initial. A dual argument shows that the right adjoint G is homotopy terminal. □

2.5 Homotopy Kan Extensions

In this section we will recall some basic definitions and facts about homotopy Kan extensions which we will use throughout. To help motivate the definitions we will briefly recall how Kan extensions work and then we will recall homotopy Kan extensions.

2.5.1 Kan Extensions

In this subsection we will recall some relevant facts for ordinary Kan extensions. For our exposition, [21, Section 11.9] is enough. For more details see [27, Chapter X] and [42, Chapter 1].

Let I and J be a small categories, and let $f\colon I \to J$ be a functor. If $\mathcal{C}$ is a cocomplete category and $X : I \to \mathcal{C}$ is a diagram, we can ask whether there is any natural extension of X to a functor $J \to \mathcal{C}$, denoted by $\operatorname{Lan}_f X \in \mathcal{C}^J$. That is, we wish to find a functor filling in the dotted arrow in the diagram

$$\begin{array}{ccc} I & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ J & & \end{array}$$

in a universal way. That means that if $Y\colon J \to \mathcal{C}$ is another diagram and f^*Y is the restriction to I we have a natural isomorphism

$$\operatorname{Hom}_{\mathcal{C}^J}\left(\operatorname{Lan}_f X, Y\right) \cong \operatorname{Hom}_{\mathcal{C}^I}\left(X, f^*Y\right).$$

If j is an object of J, then we must define $(\operatorname{Lan}_f X)_j$ so that for every object $i \in I$ and every map $\sigma\colon f(i) \to j$ we have a map

$$(\operatorname{Lan}_f X)_\sigma\colon X_i \to (\operatorname{Lan}_f X)_j$$

. If $\tau\colon i_1 \to i_2$ is a map in I then we must ensure the triangle

$$\begin{array}{ccc} X_{i_1} & \xrightarrow{X_\tau} & X_{i_2} \\ & \searrow \quad \swarrow & \\ & (\operatorname{Lan}_f X)_j & \end{array}$$

commutes. This suggests that we define $(\operatorname{Lan}_f X)_j$ to be the colimit indexed by f/j, the category of objects of I over j, see Definition 2.4.2.

Definition 2.5.2. Let $f\colon I \to J$ a functor between small categories, let $\mathcal{C}$ be a cocomplete category and let $X\colon I \to \mathcal{C}$ be a functor. The *extension* $\operatorname{Lan}_f X$ of X to J is the functor $\operatorname{Lan}_f X$ which on the object $j \in J$ is defined by

$$(\operatorname{Lan}_f X)_j = \operatorname*{colim}_{i \in f/j} X_i, \tag{2.13}$$

On maps $\tau \colon j_1 \to j_2$ in J, it is the natural map of colimits induced by $\tau_* \colon f/j_1 \to f/j_2$.

Theorem 2.5.3. *Let $f \colon I \to J$ a functor between small categories, let $\mathcal{C}$ be a cocomplete category and let $X \colon I \to \mathcal{C}$ be a functor. If $\operatorname{Lan}_f X$ is the extension of X to J, then for every diagram $Y \colon J \to \mathcal{C}$ there is a natural isomorphism (in X and in Y)*

$$\operatorname{Hom}_{\mathcal{C}^J}(\operatorname{Lan}_f X, Y) \cong \operatorname{Hom}_{\mathcal{C}^I}(X, f^* Y).$$

For a proof see [21, p. 11.9.3]. By the above theorem, the restriction functor (pullback)

$$f^* \colon \mathcal{C}^J \to \mathcal{C}^I,$$

has a left adjoint Lan_f, constructed in Definition 2.5.2

$$\operatorname{Lan}_f = f_! \colon \mathcal{C}^I \rightleftarrows \mathcal{C}^J \colon f^*. \tag{2.14}$$

Expanding the definition of the left Kan extension we have that the value $(f_! X)_j$ is given by the colimit

$$(f_! X)_j = \operatorname{colim}\left(f/j \xrightarrow{\pi} I \xrightarrow{X} \mathcal{C}\right) = \operatorname*{colim}_{f/j} \pi^* F$$

where π is the forgetful functor (2.7). For a morphism $j_1 \to j_2$ in J the induced map

$$(f_!X)_{j_1} \to (f_!X)_{j_2} \tag{2.15}$$

is the map on colimits induced by the functor $f/j_1 \to f/j_2$

$$\begin{array}{ccccccc}
(f/j_1 & \xrightarrow{\pi} & I & \xrightarrow{X} & \mathcal{C} \,) & \longrightarrow & \operatorname{colim}_{f/j_1} \pi^* X \\
\downarrow & & \| & & \| & & \vdots \\
(f/j_2 & \xrightarrow[\pi]{} & I & \xrightarrow[F]{} & \mathcal{C} \,) & \longrightarrow & \operatorname{colim}_{f/j_2} \pi^* X
\end{array}$$

Example 2.5.4. Consider the case where J is the terminal category $*$, so for any small category I there is a unique functor $t\colon I \to *$. Then, in this case, the left Kan extension along this functor

$$\mathrm{Lan}_t \colon \mathcal{C}^I \to \mathcal{C} \colon t^*$$

is just the functor colim_I.

A more interesting example is given by the pushout-product map, Definition 2.3.5. It can be defined as a left Kan extension.

Example 2.5.5. Consider a cocomplete, (closed) monoidal category $(\mathcal{C}, \otimes)$. Let $[1] = \{0 \le 1\}$ and the product $[1] \times [1]$. Furthermore the following map of posets

$$\begin{aligned}
\mathrm{pr}\colon [1] \times [1] &\to [1] \\
(0,0), (1,0), (0,1) &\mapsto 0 \\
(1,1) &\mapsto 1
\end{aligned}$$

Now let f and g be morphisms in $\mathcal{C}$. We can consider them as as objects in the arrow category $f, g \in \mathcal{C}^{[1]}$. The functors $f\colon [1] \to \mathcal{C}$ and $g\colon [1] \to \mathcal{C}$ give rise to their objectwise tensor product, $f \mathbin{\overline{\otimes}} g$, see Definition 2.3.24. That is, the functor

$$f \mathbin{\overline{\otimes}} g\colon [1] \times [1] \to \mathcal{C}$$

which is the following commutative diagram.

$$\begin{array}{ccc} X_0 \otimes Y_0 & \longrightarrow & X_1 \otimes Y_0 \\ \downarrow & & \downarrow \\ X_0 \otimes Y_1 & \longrightarrow & X_1 \otimes Y_0. \end{array}$$

Notice that the slice category $\mathrm{pr}/0$ is the poset $\ulcorner$ and the slice $\mathrm{pr}/1$ is the whole $[1] \times [1]$. It follows from (2.15) that the map

$$\operatorname*{colim}_{\ulcorner} (f \mathbin{\overline{\otimes}} g) \to \operatorname*{colim}_{[1]\times[1]} (f \mathbin{\overline{\otimes}} g)\,,$$

induced by the inlcusion $\ulcorner \hookrightarrow [1] \times [1]$ is exactly the map

$$f \mathbin{\square} g\colon X_0 \otimes Y_1 \coprod_{X_0 \otimes Y_0} X_1 \otimes Y_1 \to X_1 \otimes Y_1.$$

So, indeed $(\mathrm{Lan}_{\mathrm{pr}}(f \mathbin{\overline{\otimes}} g)) = \mathrm{pr}_!(f \mathbin{\overline{\otimes}} g) = f \mathbin{\square} g$.

2.5.6 Homotopy Kan Extensions

In this subsection we briefly introduce *homotopy Kan extensions*. They are the homotopy invariant version of ordinary Kan extensions. Nowdays the theory of homotopy Kan extensions is subsumed in the theory of *derivators*.

For our purposes the discussion in [21, Section 11.9] suffices. See [20], for a detailed exposition.

Now, let $\mathcal{C}$ be a model category. Furthermore, let I, J be finite posets and $f\colon I \to J$ a map of posets. The pullback functor

$$f^*\colon \mathcal{C}^J \to \mathcal{C}^I$$

preserves weak equivalences, so it defines a functor between homotopy categories, which we denote by the same letter. Recall the functor $\mathrm{Lan}_f = f_!$, left adjoint to f^*, see Definition 2.5.2. We have the following proposition.

Proposition 2.5.7. *Let $\mathcal{C}$ be a model category and let $f\colon I \to J$ be a map of finite posets. Then the adjunction*

$$f_!\colon \mathcal{C}^I \rightleftarrows \mathcal{C}^J\colon f^*$$

is a Quillen adjunction.

Proof. This follows from the definition of the projective model structure, see Proposition 2.2.12. The functor f^*, by construction is a right adjoint, it preserves weak equivalences and projective fibrations which means that f^* is right Quillen. □

This means that the derived functors of the adjoint pair $(f_!, f^*)$ define an adjoint pair on the level of homotopy categories

$$\mathbb{L}\mathrm{Lan}_f := \mathbb{L}f_!\colon \mathrm{Ho}(\mathcal{C}^I) \rightleftarrows \mathrm{Ho}(\mathcal{C}^J)\colon f^*.$$

A useful fact about homotopy Kan extension is that it does not change the homotopy colimit of a diagram which is similar to ordinary Kan extensions.

Corollary 2.5.8. *Let $\mathcal{C}$ be a model category, $f\colon I \to J$ be map of direct categories and let $X \in \mathcal{C}^I$. Then there is a canonical isomorphism in* $\mathrm{Ho}(\mathcal{C})$

$$\operatorname*{hocolim}_J \mathbb{L}f_! X \cong \operatorname*{hocolim}_I X.$$

Proof. This follows from the fact that for every pair of left Quillen functors F and G there is a natural isomorphism

$$\mathbb{L}F \circ \mathbb{L}G \to \mathbb{L}(F \circ G),$$

see [24, Theorem 1.37], together with the natural isomorphism

$$\operatorname*{colim}_J \mathrm{Lan}_f X \cong \operatorname*{colim}_I X.$$

□

To conclude this section, we will shortly discuss how one calculates the values and edges of a homotopy Kan extension. It is a fact that left homotopy Kan extensions can be calculated pointwise, like in ordinary category theory.

Proposition 2.5.9. *Let $f\colon I \to J$ be a map of posets and let X be any functor $I \to \mathcal{C}$. For any object $j \in J$ there is a canonical isomorphism in* $\mathrm{Ho}(\mathcal{C})$*,*

$$(\mathbb{L}f_! F)_j \cong \operatorname{hocolim}\left(f/j \xrightarrow{\pi} I \xrightarrow{X} \mathcal{C}\right).$$

In other words we can calculate the values of a homotopy left Kan extension much like as in (2.13) but taking homotopy colimits instead.

Remark 2.5.10. Recall that in the case of the diagram $F \in \mathcal{C}^I$ being objectwise cofibrant, the homotopy colimit of F may be calculated as the realization of its simplicial replacement, i.e.

$$B(*, I, F) = |\mathrm{srep}(F)| \cong \operatorname*{hocolim}_I X.$$

There is a similar construction that uses the two-sided bar construction, [42, Defintion 4.2.1]. With this, we could define the homotopy Kan extension of F along f as the functor

$$j \mapsto B(\mathrm{Hom}_J(f(-), d), I, F).$$

With the above technology we present another proof of Proposition 2.3.22. Recall from Example 2.2.24 that

$$\mathrm{cone}(f) = \mathrm{hocolim}\left(\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \\ * & & \end{array}\right)$$

where the underlying poset is the poset $\ulcorner$ from Example 2.2.8, i.e.,

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \\ (0,1) & & \end{array}$$

For easier notation we will write it as $\mathrm{cone}(f) = \mathrm{hocolim}(* \leftarrow X \rightarrow Y)$.

With the theory of homotopy left Kan extensions we now give a different proof of Proposition 2.3.21.

Proposition 2.5.11. *Let $(\mathcal{C}, \otimes)$ be a pointed symmetric monoidal model category, and let $f\colon X \to Y$ and $g\colon U \to V$ be morphisms in $\mathcal{C}$. There is a canonical isomorphism*

$$\operatorname{cone}(f) \wedge^{\mathbb{L}} \operatorname{cone}(g) \cong \operatorname{cone}(f \,\square^{\mathbb{L}}\, g).$$

Proof. We may assume that X, Y, U, V are cofibrant in $\mathcal{C}$. By definition,

$$\operatorname{cone}(f) \wedge^{\mathbb{L}} \operatorname{cone}(g) = \operatorname{hocolim}(* \leftarrow X \xrightarrow{f} Y) \wedge^{\mathbb{L}} \operatorname{hocolim}(* \leftarrow U \xrightarrow{g} V).$$

By Corollary 2.3.26, this is isomorphic to

$$\operatorname{hocolim}\left(\begin{array}{ccccc} * & \longleftarrow & X \wedge V & \longrightarrow & Y \wedge V \\ \uparrow & & \uparrow & & \uparrow \\ * & \longleftarrow & X \wedge U & \longrightarrow & Y \wedge U \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & * & \longrightarrow & * \end{array}\right). \tag{2.16}$$

We denote the above underlying $\ulcorner \times \ulcorner$-diagram by $\mathcal{Z}$. We define the following map of posets

$$\begin{aligned} \mathrm{pr}\colon \ulcorner \times \ulcorner &\to \ulcorner \\ ((1,0),(1,0)) &\mapsto (1,0) \\ ((0,0),(0,0)), ((0,0),(1,0)), ((1,0),(0,0)) &\mapsto (0,0) \\ \text{else} &\mapsto (0,1), \end{aligned}$$

and consider the homotopy left Kan extension

$$\mathbb{L}\mathrm{pr}_! \colon \mathrm{Ho}(\mathcal{C}^{\ulcorner \times \ulcorner}) \to \mathrm{Ho}(\mathcal{C}^{\ulcorner}).$$

First, applying the formula from Proposition 2.5.9 to the diagram $\mathcal{Z}$ firstly we have $(\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(1,0)} = Y \wedge V$. Next, for the object $(0,0)$ the slice category $\mathrm{pr}/(0,0)$ is just the poset $\ulcorner$ and we have

$$(\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(0,0)} = \mathrm{hocolim}\begin{pmatrix} X \wedge U & \xrightarrow{f \wedge 1} & Y \wedge U \\ \big\downarrow{\scriptstyle 1 \wedge g} & & \\ X \wedge V & & \end{pmatrix}$$

and finally $(\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(0,1)} \cong *$. Notice

$$(\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(0,0)} \to (\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(1,0)} = f \,\square^{\mathbb{L}}\, g.$$

Hence, the homotopy left Kan extension of the underlying diagram (2.16) is the following $\ulcorner$-diagram

$$\begin{array}{ccc} X \wedge V \coprod^h_{X \wedge U} Y \wedge U & \longrightarrow & Y \wedge V \\ \big\downarrow & & \\ * & & \end{array}$$

whose homotopy colimit is $\mathrm{cone}(f \,\square^{\mathbb{L}}\, g)$. □

2.6 Spectra

Many phenomena and invariants such as homology and cohomology of simplicial sets or CW-complexes are "stable" in the sense that suspending acts as a degree shift that can be reversed. Spectra are meant to capture the essential properties of homology and cohomology theories in a manner that is easier to handle. For a general introduction we refer to [9, Section 2] , [7], and the seminal [1].

2.6.1 Modern Foundations of Spectra

The first popular model category of spectra was due to Bousfield-Friedlander [13], and for many years it was the only one in common use. By a spectrum we mean the following: a spectrum X is a collection $X_n \in \mathcal{S}_*$ for $n \geq 0$ together with morphisms $\sigma_n \colon \Sigma X_n \to X_{n+1}$. A morphism $f \colon X \to Y$ of spectra is a collection of morphisms $f_n \colon X_n \to Y_n$ that commute with the structure maps, i.e., the following diagram commutes.

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\sigma_n} & X_{n+1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma Y_{n+1} & \xrightarrow{\sigma_n} & Y_{n+1}. \end{array}$$

Here by $\mathcal{S}_*$ we mean either the category of Top_* (some convenient category for pointed topological spaces) or $s\mathrm{Set}_*$, the category of pointed simplicial sets. We denote the above category of spectra by $\mathrm{Sp}^{\mathbb{N}}$. We give two elementary examples.

Example 2.6.2. Every based space X gives rise to a spectrum Σ^∞, the *suspension spectrum*. Its nth space is given by $\Sigma^n X$ and the structure maps are the canonical homeoomorphisms.

$$\Sigma\Sigma^n X \cong \Sigma^{n+1} X \to \Sigma^{n+1} X.$$

In fact, this extends to a functor

$$\Sigma^\infty \colon \mathcal{S}_* \to \mathrm{Sp}^{\mathbb{N}},$$

which sends a space A to $\Sigma^\infty A$. The functor Σ^∞ is left adjoint to the functor $\Omega^\infty \colon \mathrm{Sp}^{\mathbb{N}} \to \mathcal{S}_*$ which sends a spectrum to its zeroth space.

Example 2.6.3. The Eilenberg-Mac Lane prespectrum HA, where A is an abelian group, has nth space $K(A, n)$, i.e., the Eilenberg-Mac lane space associated to A. That is, it is the sequence of spaces

$$K(A, 0), K(A, 1), K(A, 2), \dots .$$

The structure maps of HA are the adjoints to the weak homotopy equivalences $K(A, n) \to \Omega K(A, n{+}1)$. A homomorphism of abelian groups $A_1 \to A_2$ gives rise to a map of spectra $HA_1 \to HA_2$, hence it defines a functor

$$H \colon \mathrm{Ab} \to \mathrm{Sp}^{\mathbb{N}}.$$

However, the category $\mathrm{Sp}^{\mathbb{N}}$ has the disadvantage that does not admit a suitable smash product on the model category level. This means that it is ill-equipped to deal with multiplicative properties of homology and cohomology theories,i.e., ring spectra. These days, there are many monoidal model categories that have the same underlying stable homotopy theory. Each model has their various advantages and disadvantages. Here is a list of the main players:

(i) Bousfield-Friedlander spectra, $\mathrm{Sp}^{\mathbb{N}}$

(ii) Symmetric spectra Sp^{Σ}

(iii) Orthogonal spectra Sp^{O},

(iv) S-modules or EKMM spectra,

(v) functors with smash product.

Symmetric and orthogonal spectra can be considered as objects in $\mathrm{Sp}^{\mathbb{N}}$ with extra structure. The other models are more abstract and harder to define. However, for the purposes of this thesis, it is irrelevant in which model we chose to work with. All that it matters to us is that we work on stable, monoidal model category. Henceforth we will assume the existence of a pointed category Sp equipped with a closed symmetric monoidal smash product $\wedge$, with unit $\mathbb{S}$ (the sphere spectrum) and inner hom $F(-,-)$. Additionally, we will assume we are given adjoint functors

$$\Sigma^{\infty}\colon \mathcal{S}_{*} \rightleftarrows \mathrm{Sp}\colon \Omega^{\infty},$$

that generalize the adjoint pair given in Example 2.6.2 as well as a simplicial, stable model structure on Sp, see Example 2.3.12 and Definition 2.2.28. The following is a non-exhaustive list some of the following properties that we will assume.

(i) The adjoint pair $(\Sigma^\infty, \Omega^\infty)$ is a Quillen pair.

(ii) There is a smash product $\wedge$ which makes $(\mathrm{Sp}, \wedge)$ into a monoidal model category see, Definition 2.3.8. This means that that stable homotopy category $(\mathrm{Ho}(\mathrm{Sp}), \wedge^{\mathbb{L}})$ is a tensor-triangulated category, i.e., a triangulated category with a compatible monoidal product.

(iii) There exists a a weak equivalence $\epsilon\colon \Sigma^\infty S^0 \to \mathbb{S}$.

(iv) There is a Quillen equivalence $\Phi\colon \mathrm{Sp} \to \mathrm{Sp}^{\mathbb{N}}$.

For more properties of the tensor-triangulated category $(\mathrm{Ho}(\mathrm{Sp}), \wedge^{\mathbb{L}})$ see [5, Theorem 6.1.14]. The objects of Ho(Sp) are simply called spectra. As we noted above the category Ho(Sp) is a triangulated category. Since by assumption Sp is a simplicial model category, the shift operator is given by the suspension $\Sigma(-)$ constructed in simplicial model categories, see Remark 2.2.29. Elementary triangles are as in Definition 2.2.31. Since Sp is also stable, in Ho(Sp) becomes inverse to $\Sigma^{-1}(-) = \Omega(-)$.

We briefly explain the importance of the monoidal structure on the category Sp. One of the main reasons for introducing a symmetric monoidal products on the category of spectra Sp or on its homotopy category Ho(Sp) is to take care of multiplicative properties on homology or cohomology theories. For example the cohomology theory that one first encounters, singular cohomology, has the structure of a graded ring. By the Brown representability theorem, this gives rise to maps

$$H\mathbb{Z} \wedge H\mathbb{Z} \to H\mathbb{Z}$$

for the Eilenberg–MacLane spectrum $H\mathbb{Z}$. Many (co)homology theories come equipped with such structure. A *ring spectrum* is a spectrum $R \in (\mathrm{Sp}, \wedge)$ together with a multiplication map $\mu\colon R \wedge R \to R$ and a unit map $\eta\colon \mathbb{S} \to R$ such that the expected associativity and unitality diagrams commute. In other words, (R, μ, η) is a monoid in $(\mathrm{Sp}, \wedge, \mathbb{S})$. In the same vein one can define commutative rings and modules over rings, i.e., one can do "algebra" in the symmetric monoidal category $(\mathrm{Sp}, \wedge)$

2.6.4 Homotopy Groups of Spectra

Recall that in stable a model category $\mathcal{C}$ we write $[-,-]$ for the set of morphisms in $\mathrm{Ho}(\mathcal{C})$. For spectra $X, Y \in \mathrm{Sp}$ we define

$$[X, Y]_r := [\Sigma^r X, Y] \tag{2.17}$$

and we let $[X, Y] = [X, Y]_0$. These are abelian groups for all X, Y and $r \in \mathbb{Z}$. Notice that $[X, Y]_*$ is a $\mathbb{Z}$-graded abelian group. Next, we define stable homotopy groups of spectra. For the sphere spectrum $\mathbb{S}$, we can define $\mathbb{S}^n$ recursively as $\mathbb{S}^n = \Sigma\mathbb{S}^{n-1}$

Definition 2.6.5. For an integer n, we define the nth stable homotopy group $\pi_n(X)$ of a spectrum X as the group $[\mathbb{S}^n, X]$ of morphisms from $\mathbb{S}^n$ to X in the stable homotopy category $\mathrm{Ho}(\mathrm{Sp})$. However, this should not be confused with homotopy classes of maps, unless X is fibrant in Sp.

If X is a ring spectrum and $f\colon \mathbb{S}^p \to X$ and $g\colon \mathbb{S}^q \to X$, i.e., $f \in \pi_p(X)$ and $g \in \pi_q(X)$ we may form the composite

$$\mathbb{S}^{p+q} \xrightarrow{\gamma} \mathbb{S}^p \wedge \mathbb{S}^q \xrightarrow{f \wedge g} X \wedge X \xrightarrow{\mu} X$$

which defines a pairing

$$\cdot\colon \pi_p(X) \times \pi_q(X) \to \pi_{p+q}(X). \tag{2.18}$$

More generally we have the following proposition.

Proposition 2.6.6. *There is a natural pairing*

$$\pi_*(X) \otimes \pi_*(Y) \to \pi_*(X \wedge Y),$$

and a homeomorphism

$$\mathbb{Z} \to \pi_*(\mathbb{S}),$$

that makes the functor

$$\pi_*\colon (\mathrm{Sp}, \wedge) \to (\mathrm{Ab}_*, \otimes)$$

lax symmetric monoidal.

For a concrete construction of the above map using the model of symmetric spectra, see [44, Chapter I, Subsection 6.2] [44, Theorem 6.16].

Lastly, let us mention that any spectrum defines a generalized homology theory as follows. We will use this concept in the following subsection where we will discuss Bousfield localizations.

Definition 2.6.7. Let E be a spectrum. For any other spectrum X and an integer k, we define the kth E-homology group of X as

$$E_k(X) = \pi_k(E \wedge^{\mathbb{L}} X) = [\mathbb{S}^k, E \wedge^{\mathbb{L}} X].$$

Collecting together all the $E_k(X)$ we have a $\mathbb{Z}$-graded module

$$E_*(X) := [\mathbb{S}, Z \wedge^{\mathbb{L}} X]_* = \pi_*(E \wedge^{\mathbb{L}} X).$$

We will expand on the homological properties of the above functor in Chapter 3. We will see there that the functor

$$E_*(-)\colon \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ab}_* = \mathrm{Ab}^{\mathbb{Z}},$$

is *homological* where $\mathrm{Ab}_* = \mathrm{Ab}^{\mathbb{Z}}$ is the category of $\mathbb{Z}$-graded abelian groups.

2.6.8 Bousfield Localization at a Spectrum

Bousfield localization, is an established tool in model category theory. It is a method to formally add more weak equivalences to a model category $\mathcal{C}$. One chooses a set S of maps in $\mathcal{C}$ that one wants to add to the class of weak equivalences. That is to say, one can construct a new model structure $L_S\mathcal{C}$ such that now that the set of maps S are weak equivalences. As a consequence these maps become isomorphisms in $\mathrm{Ho}(L_S\mathcal{C})$. Therefore, we can think of

Bousfield localization as a formal framework for inverting maps in the homotopy category. For our purposes, the most common setting is localization of spaces or spectra with respect to a homology theory $E_*(-)$. Specializing to this case, rather than the weak equivalences being weak equivalence of spaces, or stable equivalences of spectra, one constructs a model structure with the E_*-equivalences as the weak equivalences in the localized model structure. Informally speaking, the general idea of localization at a spectrum E is to associate to any spectrum X the "part of X that E can see", denoted by $L_E X$. In particular, it is desirable that L_E is a functor with the following equivalent properties:

$$E \wedge X \simeq * \text{ implies } L_E X \simeq *.$$

For reference for the general construction and existence of the left Bousfield localization in model categories at a set of maps see, [21, Chapter 3]. For localizations with respect to homology theories see [40, Chapter 7] and [5, Chapter 7] for a more contemporary exposition.

To make things formal, we start with the following definition that gathers all the necessary terminology.

Definition 2.6.9. Let E be a spectrum.

(i) A spectrum X is *E-acyclic* if the smash product with E is zero, i.e., $E \wedge X \simeq *$.

(ii) A map $f \colon X \to Y$ is an *E-equivalence* if $1_E \wedge f \colon E \wedge X \to E \wedge Y$ is a weak equivalence, hence, if $E_*(f) \colon E_*(X) \to E_*(Y)$ is an isomorphism in Ho(Sp).

(iii) A spectrum X is *E-local* if the following equivalent conditions hold.

- For every E-equivalence $f\colon A \to B$ the map $f^*\colon [B, X]_* \to [A, X]_*$ is an isomorphism;
- Every morphism $Y \to X$ out of an E-acyclic spectrum Y is zero in Ho(Sp), i.e., $[Y, X]_* = 0$

(iv) A spectrum Y with a map $X \to Y$ is called an *E-localization* of X if Y is E-local and $X \to Y$ is an E-equivalence.

(v) If a localization of X exists, then it is unique up to homotopy and will be denoted by $X \xrightarrow{\eta_X} L_E X$.

Localizations of this kind were first studied by Adams [1], but set-theoretic difficulties prevented him from actually constructing them. In the paper [12], Bousfield showed that localization functors exist for arbitrary spectra E. Next we discuss how localizations can be put in terms of model category theory. The E-local equivalences, that we defined above are the weak equivalences of a model structure on Sp.

Proposition 2.6.10. *Let* Sp *be either the category of Bousfield-Kan, symmetric or orthogonal spectra and let E be an object in* Sp*. There is a model structure on* Sp *such that a map $f\colon X \to Y$ is an*

(i) weak equivalence if f is an E_*-equivalence

(ii) cofibration if each $X_n \coprod_{\Sigma X_{n-1}} \Sigma Y_{n-1} \to Y_n$ is a cofibration in $\mathcal{S}$, and

(iii) fibration if f has the right lifting property with respect to acyclic cofibrations.

We will use the notation $L_E\mathrm{Sp}$ for the model E-local model structure on Sp. The fibrant objects are the E-local Ω-spectra. The existence of the above model structure is given by a left Bousfield localization at a set of maps. For details see [5, Chapter 7.2]. Let's see some examples

Example 2.6.11. Let $E = H\mathbb{Q}$. Then $L_E X = E \wedge L_{\mathbb{Q}}\mathbb{S} = E \wedge H\mathbb{Q}$, it is the rationalization of X

Example 2.6.12. Let $E = M\mathbb{Z}_{(p)}$ be the Moore spectrum of the abelian group $\mathbb{Z}_{(p)}$. In this case $L_E X \cong X_{(p)}$ is the Bousfield p-localization.

Much of contemporary homotopy theory relies on the machinery of Bousfield localizations. Particularly, the main object of study of the stable homotopy theory, the tensor-triangulated category Ho(Sp) is an extremely complicated category. It is very beneficial to break apart this category into smaller pieces, so to speak, that are easier to understand and to work with. These smaller pieces are all Bousfield localizations with respect to various spectra. This is the starting point of *chromatic* homotopy theory. One starts by breaking apart Ho(Sp) each prime at a time as in Example 2.6.12, and then further breaking it apart using the so called Johnson-Wilson theories $E(n)$ and the closely related Morava K-theories $K(n)$. For a reference see [5, Subsection 7.4.3] and the seminal [39].

For the purposes of this thesis we will working with the following.

Example 2.6.13. Let p be an odd prime and let $n \in \mathbb{Z}$. There exists a ring spectrum $E(1)$ also known as the *Adams summand* of the p-local complex K-theory spectrum and one has

$$KU_{(p)} = \bigvee_{i=0}^{p-2} \Sigma^{2i} E(1).$$

Therefore, the localizations at $E(1)$ and at $KU_{(p)}$ are the same. The construction of the spectrum is highly non-trivial and will not concern us here. What does concern us that it defines a multiplicative homology theory and that it has coefficients

$$E(1)_* = \mathbb{Z}_{(p)}[v_1, v_1^{-1}] \ \ |v_1| = 2p - 2.$$

In this case, it is customary to write the Bousfield localization $L_{E(1)}\mathrm{Sp}$ as $L_1\mathrm{Sp}$.

Lastly, we would like the Bousfield localization $L_E\mathrm{Sp}$ also to be a monoidal model structure. We have the following proposition.

Proposition 2.6.14. *Let* $(\mathrm{Sp}, \wedge)$ *a model of spectra and let* E *be an object of* Sp. *Then* $(L_E\mathrm{Sp}, \wedge)$ *is also a monoidal model category.*

For a proof we refer to [5, Theorem 7.3.11].

Chapter 3

Background in Homological Algebra

In this chapter we recall the elements of homological algebra that are necessary for the proof of our result. Also one of its goals is to set up the necessary language to state Franke's theorem and all the necessary components. The chapter is structured as follows. In Section 3.1 we will set up some notation regarding chain and cochain complexes since we are going to use both in our exposition. We will review some relevant definitions from generalized homology theory defined by a spectrum and the tensor product of graded objects (abelian groups). In Section 3.2 we will discuss how homology of a spectrum has actually much more structure than a graded abelian group. In other words it has the structure of a *comodule over a Hopf algebroid*. We will also introduce our main example, comodules over the Johnson -Wilson spectrum at height 1. After this, in Section 3.3 we will discuss Franke's algebraic model, that is to say, twisted complexes over the abelian category of comodules. It has at least two descriptions because our abelian category $\mathcal{A} = \text{Comod}_{E(1)_*E(1)}$ has a splitting of period $2p-2$ and the homology theory $E(1)_*(-)$ is periodic. We will also discuss another simpler algebraic model for twisted complexes, that is, differential graded comodules. After this, we will discuss the monoidal structure of these algebraic models. In Section 3.4 we will discuss again some model category theory applied to the algebraic models of twisted complexes. In order to do so, we will make a small tour of relative homological algebra. In Section 3.5 we will introduce another key ingredient, homology with coefficients in a functor. This is a generalization of the homology of a group or a category and it also plays a very important role in this thesis. Lastly we will see how homology with coefficients is used in the spectral sequence that computes the homology of a homotopy colimit of a diagram of spectra.

3.1 Homology

In this section we will discuss how a spectrum gives rise to a homology theory. Before that we establish some notation about chain and cochain complexes in additive categories.

3.1.1 Chain and Cochain Complexes

Let $\mathcal{A}$ be an abelian category. A *cochain complex* in $\mathcal{A}$ is a sequence of maps

$$\dots \to X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \dots$$

such that

$$d^n \circ d^{n-1} = 0$$

for all $n \in \mathbb{Z}$, so differentials raise the degree. We denote the category of complexes in $\mathcal{A}$ where a morphism $\phi\colon X \to Y$ between complexes consists of maps $\phi^n\colon X^n \to Y^n$ compatible with the differential by $\mathrm{Ch}(A)$ or by $\mathcal{C}(\mathcal{A})$. Now suppose that $\mathcal{A}$ is abelian. For a complex $X^\bullet$ and each $n \in \mathbb{Z}$ one defines the nth-*cohomology* as

$$H^n X = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}.$$

Setting $X_{-i} = X^i$, we get a chain complex, i.e., an object

$$\ldots \to X_2 \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \ldots$$

and we get the *homology* of a chain complex. We will denote chain complexes with lower indices.

Remark 3.1.2. In our exposition we will use both cochain and chain complexes and this is not done by accident. In particular, due to the shape of the poset $\mathcal{C}_N$ and the functor $\mathcal{Q}\colon \mathcal{L} \to \mathcal{C}^{2p-2}(\mathcal{B})$ that we will construct in Section 4.2, see (4.2), will produce cochain complexes, that is, the differentials raise the degree. On the other hand, the spectral sequence that computes the homology of a homotopy colimit that we will see later, Proposition 3.5.11, has as input *homology* of certain chain complexes (differential lowers the degree), that originate from simplicial objects via the Dold-Kan correspondence.

We will denote the category of cochain complexes by $\mathcal{C}(\mathcal{A})$ and the derived category by $\mathcal{D}(\mathcal{A})$.

3.1.3 Generalized Homology Defined by a Spectrum

In this subsection we discuss how a spectrum E determines a "generalized homology theory" both for spaces (or simplicial sets) and spectra. It also determines a "generalized cohomology theory" but since we are only interested in homology theory we skip the definitions for cohomology. Recall that from Definition 2.6.7, a spectrum $E \in \mathrm{Ho}(\mathrm{Sp})$ determines a functor

$$E_*(-)\colon\ \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ab}_*, \quad X \mapsto E_*(X) = \pi_*(E \wedge^{\mathbb{L}} X).$$

As the terminology suggests, E-homology is a homological functor. Recall that if $\mathcal{T}$ is a triangulated category, a covariant functor $E\colon \mathcal{T} \to \mathrm{Ab}$ is called homological if it takes coproducts in $\mathcal{T}$ to coproducts of abelian groups, and if for every distinguished triangle (f, g, h) in $\mathcal{T}$ the sequence of abelian groups

$$E(A) \xrightarrow{E(f)} E(B) \xrightarrow{E(g)} E(C) \xrightarrow{E(h)} E(\Sigma A)$$

is exact.

Proposition 3.1.4. *Let E be a spectrum and k any integer.*

(i) For every distinguished triangle (f, g, h) in the stable homotopy category, the sequence

$$E_k(A) \xrightarrow{E_k(f)} E_k(B) \xrightarrow{E_k(g)} E_k(C) \xrightarrow{E_k(h)} E_k(\Sigma A),$$

is exact.

(ii) For every family $\{X_i\}_{i\in I}$ of spectra the natural map

$$\bigoplus_{i\in I} E_k(X_i) \to E_k(\bigvee_{i\in I} X_i)$$

is an isomorphism.

If E is a ring spectrum, one can define various algebraic operations, such as products on homology groups, that arise from the operations on E. We will only be interested in the following.

Construction 3.1.5 (External Product)**.** Let E be a ring spectrum and let X and Y be spectra and k, l integers. Then we define the external products

$$\times\colon E_k(X) \otimes E_l(Y) \to E_{k+l}(X \wedge^L Y) \tag{3.1}$$

as follows

$$E_k(X) \otimes E_l(Y) = \pi_k(E \wedge^{\mathbb{L}} X) \otimes \pi_l(E \wedge^{\mathbb{L}} Y) \dot{\to} \pi_{k+l}(E \wedge^{\mathbb{L}} X \wedge^{\mathbb{L}} E \wedge^L Y)$$
$$\xrightarrow{(1_E \wedge \tau_{X,E} \wedge^{\mathbb{L}} 1_Y)_*} \pi_{k+l}(E\wedge E\wedge X\wedge Y) \xrightarrow{(\mu\wedge X\wedge Y)_*} \pi_{k+l}(E\wedge^{\mathbb{L}} X\wedge^{\mathbb{L}} Y) = E_{k+l}(X\wedge^{\mathbb{L}} Y).$$

The first map is the pairing (2.18).

For more details on pairings and products see [1, Part III, Section 9]. Before the next proposition, we recall the following.

Definition 3.1.6 (Tensor product)**.** Let R_* be a graded ring and let M_* and M_* be graded R_*-modules. The tensor product of $M_* \otimes_{R_*} N_*$ is defined as having n-th component

$$(M_* \otimes_{R_*} N_*)_n := \left(\bigoplus_{p+q=n} M_p \otimes_{\mathbb{Z}} N_q \right) / (mr \otimes n - m \otimes rn | a+b+c = n, m \in M_a, r \in R_b, n \in N_c) \tag{3.2}$$

The following proposition is the cornerstone [44, Proposition 6.20, Chapter II] and will be used implicitly throughout.

Proposition 3.1.7. *Let E, X and Y be as in 3.1.5. If $E_*(X)$ is flat as a (right) graded E_*-module or if $E_*(Y)$ is a flat (left) graded E_*-module then map*

$$E_*(X) \otimes_{E_*} E_*(Y) \to E_*(X \wedge^{\mathbb{L}} Y)$$

induced by the exterior product is an isomorphism.

For a proof see, [48, Theorem 13.75].

3.2 Comodules over a Hopf Algebroid

In this section we will review some relevant facts about the abelian category of comodules over a Hopf algebroid. None of the results in this section are new, and we refer the reader to [23, Section 1], [38, App. A1] and [38, Section 2] for further details.

We saw briefly in Definition 2.6.7 that any spectrum $E \in \mathrm{Sp}$ defines a homological functor to abelian groups, the E-homology, i.e.

$$E_*(-) \colon \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ab}_*, \quad X \mapsto E_*(X) = \pi_*(E \wedge^{\mathbb{L}} X).$$

In fact the value $E_*(X)$ has a lot more structure than just a plain graded abelian group, i.e., it is a $E_*(E)$-*comodule*, a notion which we will introduce below. The pair $(\pi_*(E), E_*(E))$ has a structure of a *Hopf algebroid* which we will define shortly. Let K be a commutative ring. A *Hopf algebroid* is a pair (A, Ψ) of commutative graded K-algebras such that it is a cogroupoid object in the category of commutative K-algebras. We will commit a small abuse of notation by omitting. Unwinding the above, that means in particular that it consists of a pair (A, Ψ) of graded commutative K-algebras along with K-algebra morphisms

$$
\begin{aligned}
\eta_L\colon A &\to \Psi \quad \text{(left unit, inducing source)}\\
\eta_R\colon A &\to \Psi \quad \text{(right unit, inducing source)}\\
\Delta\colon \Psi &\to \Psi \otimes_A \Psi \quad \text{(coproduct, inducing composition of morphisms)}\\
\epsilon\colon \Psi &\to A \quad \text{(augmentation, inducing idenity morphisms)}\\
c\colon \Psi &\to \Psi \quad \text{(conjugationm inducing inverses),}
\end{aligned}
$$

satisfying certain identities. These ensure that, for any commutative ring B, the sets $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(\Psi, B)$ are the objects and morphisms of a groupoid, respectively. Here, η_L gives Ψ the structure of a left A-module, η_R gives Ψ a right A-module structure, and $\Psi \otimes_A \Psi$ refers to the bimodule tensor product. We now come to the definition of a comodule over a Hopf algebroid.

Definition 3.2.1 (Comodule)**.** A (A, Ψ)-*comodule* is a left A-module M

together with a map

$$\psi_M : M \to \Psi \otimes_A M$$

satisfying a coassociativity and a counit condition.

We refer to [38, Proposition 2.2.8.] for a detailed account on how $(\pi_*(E), E_*(E))$ is Hopf algebroid and $E_*(X)$ is a $E_*(E)$-comodule. We write Comod_Ψ for the category of Ψ-comodules; we will always assume that Ψ is a flat A-module, which ensures that Comod_Ψ is a cocomplete abelian subcategory of Mod_A [23, Lemma 1.1.1]. The forgetful functor $\mathrm{Comod}_\Psi \to \mathrm{Mod}_A$ has a right adjoint that sends an A-module M to the Ψ-comodule $\Psi \otimes_A M$, with structure map $\Delta \otimes \mathrm{id}$ [38, A1.2.1]. Such a comodule is called an *extended comodule*. It follows that the colimit in Comod_Ψ is just the colimit in A-modules equipped with its canonical comodule structure. The forgetful functor does not preserve limits, or even infinite products, and so in general it is difficult to construct right adjoints in Comod_Ψ. Nonetheless, the category Comod_Ψ is also complete, see [23, Proposition 1.2.2].

The category of Ψ-comodules is symmetric monoidal. Let M and N be Ψ-comodules and define their comodule tensor product $M \otimes N$ to be the module $M \otimes_A N$, with comodule structure given by the composite

$$M \otimes N \xrightarrow{\psi_N \otimes \psi_N} (\Psi \otimes_A M) \otimes (\Psi \otimes_A N) \to \Psi \otimes_A (M \otimes N) \tag{3.3}$$

where the last map sends $g_1 \otimes m \otimes g_2 \otimes n \to g_1 g_2 \otimes m \otimes n$. The tensor product has a right adjoint $\underline{\mathrm{Hom}}_\Psi(-,-)$ (the underline is to remind us that they are internal Hom) making Comod_Ψ a closed symmetric monoidal category [23,

p. 1.3.1], although it is harder to explicitly give a formula for $\underline{\mathrm{Hom}}_\Psi(M,N)$ and we will do not use it. With this closed monoidal structure, we can define the *dual* of a Ψ-comodule M to be $DM = \underline{\mathrm{Hom}}_\Psi(M,A)$.

The following proposition which characterizes dualizable comodules will be fundamental to us in the subsequent chapters.

Proposition 3.2.2. *[23, Prop. 1.3.4] A comodule M is dualizable in* Comod_Ψ *if and only if M is dualizable as an A-module,* i.e., *M is finitely generated and projective as an A-module.*

A Hopf algebroid (A,Ψ) will be called an *Adams Hopf algebroid* if it satisfies Adams' condition, *i.e.*, if Ψ is a filtered colimit of dualizable Ψ-comodules. We also note if one has a flat Adams Hopf algebroid then Comod_Ψ is a Grothendieck abelian category. This implies that Comod_Ψ is locally presentable, see [10, Proposition 3.10]. We have the category $\mathcal{C}(\mathrm{Comod}_\Psi)$, the category of cochain complexes in the abelian category Comod_Ψ.

3.2.3 Comodules over the Johnson-Wilson spectrum

In this subsection we will recall some properties of a particular Hopf algebroid $(E(1)_*, E(1)_*E(1))$. There is a very concrete albeit complicated description of this category which goes back to Bousfield, see [11]. Since we do not use this description we will not recall it here. Let X be a spectrum. Then the $E(1)_*E(1)(X)$-comodule $E(1)_*(X)$ is an object of $\mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)}$ by taking

$$M_n := E(1)_n(X), \quad n \in \mathbb{Z}.$$

Since the homology theory $E(1)_*(-)$ is $2p-2$-periodic we will also use subcategory $\mathcal{B} = \mathrm{Comod}^0_{E(1)_*E(1)}$ of $\mathrm{Comod}_{E(1)_*E(1)}$ consisting of those objects $(M_n)_{n\in\mathbb{Z}}$ such that

$$M_n = \begin{cases} M & : n \equiv 0 \mod 2p-2 \\ 0 & : \text{ else} \end{cases} \tag{3.4}$$

This describes a so-called *split* of period $2p-2$ of $\mathcal{A}$: $\mathcal{B} \subset \mathcal{A}$ is a Serre class such that

$$\bigoplus_{0\le i<2p-2} \mathcal{B} \to \mathcal{A}$$
$$(B_i) \to \bigoplus_{0\le i<2p-2} B_i[i]$$

is an equivalence of categories.

3.3 Twisted Cochain Complexes

In this section we will review some essential material regarding twisted complexes.

3.3.1 Main Definitions

Let $\mathcal{A}$ be an arbitrary abelian category (we will assume that it is also $\mathbb{Z}$-graded and we suppress the grading), $T\colon \mathcal{A} \to \mathcal{A}$ a self-equivalence and N a natural number.

Definition 3.3.2. The category $\mathcal{C}^{(T,N)}(\mathcal{A})$ of (T,N)-*twisted cochain complexes* or *quasi-periodic complexes* with values in $\mathcal{A}$ is defined as follows. The objects are cochain complexes $C^\bullet_*$ with $C^i_* \in \mathcal{A}$ for all i together with specified isomorphism of cochain complexes

$$\phi^\bullet_C \colon T(C^\bullet_*) \to C^\bullet_*[N] = C^{\bullet+N}.$$

The morphisms are morphisms of cochain complexes $f\colon C^\bullet_* \to D^\bullet_*$ that are compatible with those isomorphisms, that is, the following diagram commutes.

$$\begin{array}{ccc} T(C^\bullet) & \longrightarrow & C^\bullet[N] \\ \downarrow & & \downarrow \\ T(D^\bullet) & \longrightarrow & D^\bullet[N] \end{array}$$

Note that the shifted complex $C^\bullet_*[N] = C^{\bullet+N}$ has differentials $(-1)^N d$, where d is the differential of the cochain complex $C^\bullet_*$. In our particular case where $\mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)}$, the self-equivalence is the last section's $T^{p-1}\colon \mathcal{A} \to \mathcal{A}$. We denote the category $\mathcal{C}^{(T^{p-1},1)}(\mathcal{A})$ by $\mathcal{C}^1(\mathcal{A})$. Secondly we are interested in the category $\mathcal{C}^{T^{((2p-2)(p-1),2p-2)}}(\mathcal{B})$ where $\mathcal{B}$ is the split of $\mathcal{A}$ introduced above. This category of twisted cochain complexes will be denoted by $\mathcal{C}^{2p-2}(\mathcal{B})$. Actually, these two categories are equivalent, that is,

$$\mathcal{C}^1(\mathcal{A}) \cong \mathcal{C}^{2p-2}(\mathcal{B})$$

and we explain why below. Consider

$$C^\bullet_* = (\ldots \to C^0_* \to C^1_* \to C^2_* \to \ldots)$$

an object of $\mathcal{C}^1(\mathcal{A})$, that is, $C^i_* \in \mathcal{A}$ and $T(C^i_*) \cong C^{i+1}_*$ via α_C. Since $\mathcal{A}$ splits into $2p-2$ copies of $\mathcal{B}$, each C^i_* splits into

$$C^i_* = C^i_{(0)} \oplus C^i_{(1)} \oplus \ldots \oplus C^i_{(2p-1)}.$$

with $C^i_{(j)} \in \mathcal{B}[j]$. So $C^\bullet_*$ gives us a complex taking values in $\mathcal{B}$ by setting

$$C^*_{(0)} := (\ldots \to C^0_{(0)} \to C^1_{(0)} \to C^2_{(0)} \to \ldots).$$

The self-equivalence T act on each C^i by cyclically permuting the summands:

$$T(C^i_{(j)}) \cong T(C^i)_{(j+1)} \cong C^{i+1}_{(j+1)},\ j \in \mathbb{Z}/(2p-2)\mathbb{Z}.$$

As a consequence we have

$$T^{(2p-2)(p-1)}(C^i_{(0)}) \cong T^{(2p-3)(p-1)}(C^{i+1}_{(1)}) \cong \ldots \cong C^{i+2p-2}_{(0)},$$

and thus $C^\bullet_{(0)}$ is a $2p-2$-periodic, that is, $C^\bullet_{(0)} \in \mathcal{C}^{2p-2}(\mathcal{B})$. On the other hand, an object of $\mathcal{C}^{2p-2}_*(\mathcal{B})$ carries the same information as an object of $\mathcal{C}^1_*(\mathcal{A})$: given

$$D^\bullet_* = (\ldots \to D^0_* \to D^1_* \to D^2_* \to \ldots) \in \mathcal{C}^{2p-2}_*(\mathcal{B})$$

one obtains a corresponding object $\overline{D}^\bullet_* \in \mathcal{C}^1(\mathcal{A})$ by setting $\overline{D}^i_{(j)} := T^j(D^{i-j}_*)$.

The category $\mathcal{C}^1(\mathcal{A})$ of twisted periodic complexes of period 1 can be given equivalently as the category whose objects are cochain complexes in $\mathcal{A}$

together with an isomorphism between internal and external shifts, *i.e.*,

$$C^{\bullet}_{*-1} \cong C^{\bullet+1}_{*}.$$

Similarly, for $\mathcal{C}^{2p-2}(\mathcal{B})$, can be given as the category whose objects are cochain complexes in $\mathcal{B}$ together with an isomorphism

$$C^{\bullet}_{*-2p+2} \cong C^{\bullet+2p+2}_{*}.$$

Notice that the definition of the category $\mathcal{C}^1(\mathcal{A})$ is redundant since it consists of a complex

$$\dots \xrightarrow{d^{-1}} C^0_* \xrightarrow{d^0} C^1_* \xrightarrow{d^1} C^2_* \xrightarrow{d^2} \dots,$$

in which $\phi^n \colon T(C^n_*) \cong C^{n+1}_*$, i.e., all the objects are isomorphic. There is a simpler algebraic structure that captures the same information and it's a lot more simple.

Definition 3.3.3. A *differential comodule* M_* is a pair (M_*, d) where

$$M_* \in \mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)} \quad \text{and} \quad d \colon M_* \to M_*,$$

where d is an endomorphism of degree -1 satisfying $d^2 = 0$. We denote the abelian category of differential comodules as $d\mathcal{A} = d\mathrm{Comod}_{E(1)_*E(1)}$

To ease notation for the following proposition we will write $\psi^n \colon C^{n+1}_* \to T(C^n_*)$, i.e., the inverse of ϕ^n. See also [35, Proposition 3.3].

Proposition 3.3.4. *The functor*

$$F : \mathcal{C}^1(\mathcal{A}) \to d\mathcal{A}, \quad (C^\bullet_*, d) \mapsto (C^0_*, \psi^0 \circ d^0)$$

is an equivalence of categories.

Proof. The functor F picks only the entry at $n = 0$, i.e., C^0_*, with differential $d\colon C^0_* \to C^0_*[1] = T(C^0_*)$ defined by the composition

$$C^0_* \xrightarrow{d^0} C^1_* \xrightarrow{\psi^0} T(C^0_*).$$

The functor $G\colon d\mathcal{A} \to \mathcal{C}^1(\mathcal{A})$ defined by $G(M_*, d)_k = T^k(M_*)$ with differential $d^k = (-1)^k T^k(d)$ in an inverse to F. □

3.3.5 The Tensor Product of Twisted Complexes

In this subsection we will review the tensor product on $\mathcal{C}^1(\mathcal{A})$. By definition, an object $C^\bullet_* \in \mathcal{C}^1(\mathcal{A})$ is a cochain complex of objects in $\mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)}$, i.e., it is a *bigraded* object.

Definition 3.3.6. Let $X^\bullet_*, Y^\bullet_* \in \mathcal{C}^1(\mathcal{A})$. The tensor product is defined by

$$(X \otimes Y)^p_q = \bigoplus_{\substack{m+s=p \\ n+t=q}} X^m_n \otimes Y^s_t$$

with differential

$$d(x \otimes y) = d(x) \otimes y + (-1)^{|x|} x \otimes d(y).$$

Note that the formula uses the total degree in the sign convention, that is, we are using the notation $|x|$ for the sum of horizontal and vertical degree.

The category $d\mathcal{A}$ of differential objects is also symmetric monoidal. Let $X_*, Y_* \in d\mathcal{A}$. The tensor product $X \otimes Y$ is defined

$$(X \otimes Y)_n = \bigoplus_{p+q=n} X_q \otimes Y_p$$

with differential defined by

$$d(x \otimes y) = d(x) \otimes y + (-1)^{\deg(x)} x d(y).$$

Recall from Proposition 3.3.4 the functor $F\colon \mathcal{C}^1(\mathcal{A}) \to d\mathcal{A}$, which picks the object C^0_*

Proposition 3.3.7. *The equivalence*

$$F\colon (\mathcal{C}^1(\mathcal{A}), \otimes) \to (d\mathcal{A}, \otimes)$$

is strong symmetric monoidal.

Proof. The assertion follows directly from the definitions. □

3.4 Model Structures for Twisted Complexes

In this section we will make a small thematical detour and recall various model structures on twisted cochain complexes in the category $\mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)}$. The main result of this section which is also fundamental to our thesis is that there is a monoidal model structure on $\mathcal{C}^1(\mathcal{A})$ which is Quillen equivalent to the *injective* model structure $\mathcal{C}^1(\mathcal{A})_{\mathrm{inj}}$. Our first definition is to recall injective model structure.

Definition 3.4.1 (Injective Model Structure)**.** In this model structure, a map $f\colon X \to Y$ in $\mathcal{C}^1(\mathcal{A})$ is

(i) a weak equivalence if it is a quasi-isomorphism,

(ii) a cofibration if it is monomorphism, and

(iii) a fibration if it is a degreewise split epimorphisms with strictly injective kernel

We denote this model structure by $\mathcal{C}^1(\mathcal{A})_{\mathrm{inj}}$.

This model structure was the one considered initially by J.Franke. The homotopy category of the above model structure is called the Franke's model *i.e.*, $\mathcal{D}^1(\mathcal{A}) = \mathrm{Ho}(\mathcal{C}^1(\mathcal{A})_{\mathrm{inj}})$. Note that since the abelian category $\mathcal{A}$ does not have enough projectives, there is no projective-type model structure on $\mathcal{C}^1(\mathcal{A})$. Recall from Section 3.2 that the category $\mathcal{A}$ is a symmetric monoidal abelian category with the tensor product as operation. However, the symmetric monoidal category $(\mathcal{C}^1(\mathcal{A}), \otimes)$ equipped with the injective model structure is *not* a symmetric monoidal model category. This is because, in the injective model structure, the monoidal product $\otimes$ is not a Quillen bifunctor, *i.e.*, the pushout product axiom fails, see Definition 2.3.7. An example that illustrates it is the following. Consider $U = \mathbb{P}E(1)_*, V = \mathbb{P}(E(1)_* \otimes \mathbb{Q}), W = 0$ and $X = E(1)_* \otimes \mathbb{Z}/p$ and let $f\colon U \to V$ and $g\colon W \to X$. Note that

$$\mathbb{P}(C) \otimes \mathbb{P}(D) \cong \mathbb{P}(C \otimes D),$$

we see that $f \square g$ is tha map $X \to 0$ which is not a monomorphism and hence not a cofibration in $\mathcal{C}^1(\mathcal{A})_{\text{inj}}$.

In order to remedy the fact that $(\mathcal{C}^1(\mathcal{A})_{\text{inj}}, \otimes)$ is not a monoidal model category we will construct a monoidal model category which is called *quasi-projective model structure*. This model structure together with the tensor product of complexes turns out to be a symmetric monoidal model category and moreover, it is Quillen equivalent to the injective model structure. We will not recall all of the proofs and definitions for the construction of this model structure. Instead we will define only the necessary ingredients and we will refer mainly to the paper [6] for a detailed exposition.

Recall from Definition 3.3.2 that an object $C_*^\bullet \in \mathcal{C}^1(\mathcal{A})$ is a cochain complex in the abelian category $\mathcal{A}$ together with an isomorphism $T(C_*^\bullet) \cong C_*^{\bullet+1}$. Equivalently, a quasi-periodic cochain complex of period 1 is a cochain complex $C_*^\bullet$, together with an isomorphism between internal and external shifts *i.e.*, $C_{*-1}^\bullet \cong C_*^{\bullet+1}$. By "forgetting" this isomorphism between internal and external shifts we have the forgetful functor

$$U\colon \mathcal{C}^1(\mathcal{A}) \to \mathcal{C}(\mathcal{A}).$$

The forgetful functor has a left and a right adjoint. We will be interested only in the left adjoint to the forgetful functor. Define a functor $\mathbb{P}$ as follows

$$\mathbb{P}\colon \mathcal{C}(\mathcal{A}) \to \mathcal{C}^1(\mathcal{A}), \quad M_*^\bullet \mapsto \bigoplus_{k\in\mathbb{Z}} T^k M_*^{\bullet+k}. \tag{3.5}$$

In particular, this means that

$$\mathbb{P}M_*^n = \bigoplus_{k\in\mathbb{Z}} T^k M_*^{n+k}.$$

The proof that indeed the functor $\mathbb{P}$ is left adjoint to the forgetful functor can be found in [6, Lemma 1.2]. With the help of the adjunction

$$\mathbb{P}\colon \mathcal{C}(\mathcal{A}) \rightleftarrows \mathcal{C}^1(\mathcal{A})\colon U$$

we are going to lift various model structure from $\mathcal{C}(\mathcal{A})$ to $\mathcal{C}^1(\mathcal{A})$, see [21, Theorem 11.3.2].

The first step in order to construct the quasi-projective model structure is the *relative projective model structure*. We are going to summarize the relative projective model structure on $\mathcal{C}(\mathcal{G})$ for an arbitrary Grothendieck abelian category $\mathcal{G}$. It is a generalization of the projective model structure on $\mathcal{C}(\mathrm{Mod}_R)$ where R is a commutative ring. It was introduced by Christensen and Hovey in the paper [14]. Recall that if P is an object in $\mathcal{G}$ and X is a object in $\mathcal{C}(\mathcal{G})$ we write $\mathrm{Hom}_{\mathcal{G}}(P, X)$ for the cochain complex that has the abelian group $\mathrm{Hom}_{\mathcal{G}}(P, X_n)$ in degree n with differentnial

$$(d_X)_*\colon \ \mathrm{Hom}_{\mathcal{G}}(P, X^n) \to \mathrm{Hom}_{\mathcal{G}}(P, X^{n+1}).$$

To construct the relative projective model structure, one begins by choosing a projective class in $\mathcal{G}$, that is, a collection $\mathcal{P}$ of objects in $\mathcal{G}$ and a collection $\mathcal{E}$ of maps in $\mathcal{G}$ such that the pair satisfy certain conditions which we define below. Recall that given a collection of objects $\mathcal{P}$ in $\mathcal{G}$, a morphism $f\colon A \to B$

in $\mathcal{A}$ is called $\mathcal{P}$-*epimorphism* if it induces an epimorphism $\mathrm{Hom}_{\mathcal{G}}(P, A) \to \mathrm{Hom}_{\mathcal{G}}(P, B)$ for all P in $\mathcal{P}$.

Definition 3.4.2 (Projective class)**.** Let $\mathcal{G}$ be an abelian category. *A projective class* in $\mathcal{G}$ is a collection $\mathcal{P}$ of objects of $\mathcal{G}$ and a collection $\mathcal{E}$ of maps in $\mathcal{G}$ such that

(i) An object U is in $\mathcal{P}$ if and only if $\mathrm{Hom}_{\mathcal{G}}(U, X) \to \mathrm{Hom}_{\mathcal{G}}(U, Y)$ is surjective for every $X \to Y$ in $\mathcal{E}$,

(ii) A map $X \to Y$ lies in $\mathcal{E}$ if and only if $\mathrm{Hom}_{\mathcal{G}}(U, X) \to \mathrm{Hom}_{\mathcal{G}}(U, Y)$ is surjective for every U in $\mathcal{P}$

(iii) For every X in $\mathcal{G}$, there is a morphism $P \to X$ in $\mathcal{E}$ such that P is in $\mathcal{P}$.

When a collection $\mathcal{P}$ is part of a projective class $(\mathcal{P}, \mathcal{E})$, the projective class is unique, and so we say that P determines a projective class or even that $\mathcal{P}$ is a projective class. An object of P is called a $\mathcal{P}$-*projective*, or, if the context is clear, a *relative projective*. Objects of $\mathcal{P}$ are called $\mathcal{P}$-projectives or relative projectives. Elements of $\mathcal{E}$ are called $\mathcal{P}$-epimorphisms or relative epimorphisms.

See [14, Definition 1.1] for further details and examples of projective classes.

We are ready now to define the relative projective model structure on $\mathcal{C}(\mathcal{G})$. We have the following definition.

Definition 3.4.3. Suppose that $\mathcal{P}$ is a projective class on the abelian category $\mathcal{G}$. We say that a map $f\colon X \to Y$ in $\mathcal{C}(\mathcal{G})$ is:

(i) a $\mathcal{P}$-equivalence if $f_*\colon \operatorname{Hom}_{\mathcal{G}}(P, X) \to \operatorname{Hom}_{\mathcal{G}}(P, Y)$ is a quasi-isomorphism in $\mathcal{C}(\mathbb{Z})$ for all $P \in \mathcal{P}$,

(ii) a $\mathcal{P}$-fibration if $\operatorname{Hom}_{\mathcal{A}}(P, f)$ is a degreewise surjection for all $P \in \mathcal{P}$,

(iii) a $\mathcal{P}$-cofibration if it has the left lifting property with respect to all $\mathcal{P}$-fibrations that are also $\mathcal{P}$-equivalences.

When this model structure exists we denote it by $\mathcal{C}(\mathcal{G})_{\text{rel.proj}}$.

By [14, Lemma 1.5], assuming that $\mathcal{G}$ is cocomplete, one way to obtain a projective class is to take any set S and define $\mathcal{P}$ to be the collection of retracts of coproducts of objects in S and $\mathcal{E}$ to be the collection of S-epimorhisms. Recall from Section 3.2 that a comodule $M \in \operatorname{Comod}_{E(1)_*E(1)}$ is dualizable if and only if M is finitely generated and projective as an $E(1)_*$-module.

Definition 3.4.4. Let S be the set of isomorphism classes of dualizable comodules in $\mathcal{A} = \operatorname{Comod}_{E(1)_*E(1)}$.

As above, the set of isomorphism classes of dualizable comodules defines a projective class on $\operatorname{Comod}_{E(1)_*E(1)}$. In what follows we will always use this projective class on the abelian category $\mathcal{A} = \operatorname{Comod}_{E(1)_*E(1)}$. To provide motivation for the choice of dualizable comodules, see [23, Section 1.4]. By [23, Theorem 2.1.1] the relative projective model structure $\mathcal{C}(\mathcal{A})_{\text{rel.proj}}$ exists and by [6, Proposition 1.3] the forgetful functor $U\colon \mathcal{C}^1(\mathcal{A}) \to \mathcal{C}(\mathcal{A})_{\text{rel.proj}}$ creates a model structure on $\mathcal{C}^1(\mathcal{A})$ and we denote it by $\mathcal{C}^1(\mathcal{A})_{\text{rel.proj}}$. Next, we would like to have a more explicit version of the cofibrant objects in $\mathcal{C}^1(\mathcal{A})_{\text{rel.proj}}$.

Lemma 3.4.5. *A quasi-periodic cochain complex $C_*^\bullet$ is cofibrant in $\mathcal{C}^1(\mathcal{A})_{rel.proj}$ if and only if it is degreewise relative projective and every map from $C_*^\bullet$ to a weakly $\mathcal{P}$-contractible quasi-periodic chain complex $K_*^\bullet$ is nullhomotopic with quasi-periodic homotopy.*

So, in particular, a cofibrant cochain complex in $\mathcal{C}^1(\mathcal{A})_{\text{rel.proj}}$ is degreewise dualizable. For a proof of the above Lemma, see [6, Lemma 5.6].

Finally, we are ready to define the quasi-projective model structure $\mathcal{C}(\mathcal{A})_{\text{q.proj}}$. It can be defined as the left Bousfield localization of $\mathcal{C}(\mathcal{A})_{\text{rel.proj}}$ with respect to the class of quasi- isomorphisms, *i.e.*, maps that induce isomorphism in cohomology. To be specific we have the following definition.

Definition 3.4.6 (Quasi-projective model structure)**.** Let $\mathcal{A} = \text{Comod}_{E(1)_*E(1)}$ together with the projective class $\mathcal{P}$ generated by the set of dualizable comodules. We call a map $f\colon X \to Y$ in $\mathcal{C}(\mathcal{A})$

(i) a weak equivalence if f is a quasi-isomorphism,

(ii) a cofibration if it is a $\mathcal{P}$-cofibration, (see Definition 3.4.3) and

(iii) a fibration it if has the right lifting property with respect to the acyclic cofibrations.

The existence of this model structure is technical and we refer to [6, Proposition 6.2] for the relevant details. As a left Bousfield localization, the cofibrations of $\mathcal{C}(\mathcal{A})_{\text{q.proj}}$ are the same as that of $\mathcal{C}(\mathcal{A})_{\text{rel.proj}}$. In particular, the cofibrant objects are the same as in Lemma 3.4.5. Like above, the adjoint pair $(\mathbb{P}, U)$ creates a model structure on quasi-periodic complexes which we denote by $\mathcal{C}^1(\mathcal{A})_{\text{q.proj}}$.

To conclude this section, so far we have three model structure on $\mathcal{C}^1(\mathcal{A})$ and it will be useful to describe how are these three model structures are related to each other. The following adjunction

$$\mathrm{Id}\colon \mathcal{C}^1(\mathcal{A})_{\mathrm{rel.proj}} \rightleftarrows \mathcal{C}^1(\mathcal{A})_{\mathrm{q.proj}} \colon \mathrm{Id}$$

is a Quillen adjunction, and the adjunction

$$\mathrm{Id}\colon \mathcal{C}^1(\mathcal{A})_{\mathrm{q.proj}} \rightleftarrows \mathcal{C}^1(\mathcal{A})_{\mathrm{inj}} \colon \mathrm{Id}$$

is a Quillen equivalence, see [6, Theorem 6.5] for a proof. Furthermore, by [6, Theorem 6.9], the model category $\mathcal{C}^1(\mathcal{A})_{\mathrm{q.proj}}$ together with the tensor product of complexes $\otimes$ is a symmetric monoidal model category. As a corollary [6, Corollary 6.10], Franke's model $\mathcal{D}^1(\mathcal{A})$ is a symmetric monoidal category with the derived tensor product $\otimes^{\mathbb{L}}$, see Corollary 2.3.17.

3.5 Homology of a Category with Coefficients in a Functor

In this section we will introduce one our main tools, namely homology of a category with coefficients in a functor. It is a particular case of functor homology that assigns the groups $\mathrm{Tor}_*^I(F, G)$, to functors $F\colon I \to \mathcal{A}$ and $G\colon I^{\mathrm{op}} \to \mathcal{A}$ with $\mathcal{A}$ an abelian category. Since we do not need such generality we will introduce it in a more down-to-earth way that goes back to Quillen and uses simplicial techniques. Traditional references include [31] and [32].

More contemporary references include [17] and [41, Chapters 15, 16].

3.5.1 Definition of Homology of a Category with Coefficients

For the following definition, we let $\mathcal{A}$ be an arbitrary abelian category and $s\mathcal{A}$ be the category of simplicial objects in $\mathcal{A}$, i.e., $s\mathcal{A} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{A})$. Recall from Section 2.1 that $D \in s\mathcal{A}$ can be given equivalently as a collection D_n of objects of $\mathcal{A}$ and a collection of maps

$$d_i \colon D_n \to D_{n-1}, \quad 0 \leq i \leq n \quad \text{(faces)}$$
$$s_j \colon D_n \to D_{n+1}, \quad 0 \leq j \leq n \quad \text{(degeneracies)}$$

for all $n \in \mathbb{Z}_{\geq 0}$ satisfying the simplicial identities.

Before we define the homology of a category with coefficients in a functor we will first define the associated complex of a simplicial object in an abelian category.

Definition 3.5.2. Let $D \in s\mathcal{A}$ be a simplicial object in $\mathcal{A}$. We define the *associated complex* $(C_\bullet(U), \partial) \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ by

$$C_n(D) = D_n \qquad \partial_n = \sum_{i=0}^{n} (-1)^n d_i \colon C_n(D) \to C_{n-1}(D).$$

Note that the simplicial identities imply $\partial^2 = 0$, so $C_\bullet(D)$ is indeed a chain complex. Moreover, this evidently defines a functor $C \colon s\mathcal{A} \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$

In other words, the associated complex to a simplicial object $D \in s\mathcal{A}$ is the following chain complex.

$$D_0 \xleftarrow{d_0 - d_1} D_1 \xleftarrow{d_0 - d_1 + d_2} D_2 \leftarrow \dots . \tag{3.6}$$

We could just as well consider cochain complexes and cosimplicial objects, that is, diagrams $D\colon \Delta \to \mathcal{A}$. We chose to work with simplicial objects since it fits better thematically. Now, let $D\colon I \to \mathcal{A}$ be a diagram in an abelian category. We will define the objects $H_p(I; D)$ in $\mathcal{A}$ for $p \geq 0$.

Definition 3.5.3. Let I be a small category and consider a diagram $D\colon I \to \mathcal{A}$. The *homology of the category I with coefficients in the functor D*, is defined as the homology of the complex $C_\bullet(D)$, i.e., the homology of the associated complex of the simplicial replacement $\operatorname{srep}(D) \in s\mathcal{A}$.

So, unwinding the definition, we start by first taking the simplicial replacement of D, see Definition 2.4.14, that is, the diagram $\operatorname{srep}(D)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{A}$

$$\bigoplus_{i_0} D_{i_0} \leftleftarrows \bigoplus_{i_0 \to i_1} D_{i_0} \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\; \bigoplus_{i_0 \to i_1 \to i_2} D_{i_0} \cdots$$

Then we consider the associated chain complex, (3.6), $C_\bullet(D)$. Then we defined $H_p(I; D)$ to be the pth homology group of the chain complex $C_*(D)$.

Lemma 3.5.4. *Let $\mathcal{A}$ be a cocomplete abelian category, and let I be a small category. For any diagram $D\colon I \to \mathcal{A}$ there is a canonical isomorphism*

$$H_0(I; D) = \operatorname*{colim}_I D$$

Proof. By construction, the zeroth homology $H_0(C_\bullet(D))$ is $C_1(D)/\operatorname{im}(d_0 -$

d_1), i.e., $\mathrm{coker}(d_0 - d_1)$ which is

$$d_0 - d_1 \colon \bigoplus_{i_0 \to i_1} D_{i_0} \to \bigoplus_{i_0} D_{i_0}.$$

But this solves the universal problem defining the colimit, so $H_0(\mathcal{C}; D) = \mathrm{colim}_I\, D$. □

Although homology of a category with coefficients is easy to define, in practice it can be very difficult to calculate.

Definition 3.5.5. Let $A \in s\mathcal{A}$. We define the *degenerate subcomplex* $D_\bullet(A)$ of $C_\bullet(A)$ by

$$D_0(A) := 0 \quad \text{and} \quad D_n(A) := \sum_{i=0}^{n-1} \mathrm{im}(s_i) \quad \text{for} \quad n \geq 1.$$

That is, $D_\bullet(A)$ is generated by the images of the degeneracy maps. Note that by the simplicial identities,

$$d(s_j) = \sum_{i=0}^{n} (-1)^i d_i s_j = \sum_{i=0}^{j-1} (-1)^i s_{j-1} d_i + \sum_{i=j+2}^{n} (-1)^i s_j d_i.$$

Definition 3.5.6. Again let $A \in s\mathcal{A}$ be simplicial object in $\mathcal{A}$. We define the *normalised chain complex* $N(A) \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ which in dimension n, $N_n(A)$ consists of the subobject of A_n that is killed by the face maps $d_i, i < n$. That is,

$$N_0(A) := A_0 \text{ and } N_n(A) := \bigcap_{i=0}^{n-1} \ker(d_i) \subseteq A_n, \;\; \partial_n := (-1)^n d_n \colon N_n(A) \to N_{n-1}(A) \text{ for } n \geq 1.$$

The simplicial identities imply both that $d_n(N_n(A)) \subseteq N_{n-1}(A)$, assumed in the above definition of d_n, and that $\partial^2 = 0$. Again this gives a functor $N\colon s\mathcal{A} \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$.

Proposition 3.5.7. *[19, Theorem 2.1] Let $A \in s\mathcal{A}$. For all $n \geq 0$ the natural map*

$$\phi\colon N_n(A) \oplus D_n(A) \to A_n = C_n(A).$$

induced by the inclusions is an isomorphism. Therefore we have a natural isomorphism

$$N_\bullet(A) \cong C_\bullet(A)/D_\bullet(A).$$

Furthermore, the inclusion $N(A) \hookrightarrow C(A)$ is a natural chain homotopy equivalence.

Consider now a diagram $D \in \mathcal{A}^I$ and its simplicial replacement $\mathrm{srep}(D) \in s\mathcal{A}$, see Definition 2.4.14. By construction, an n-simplex in $\mathrm{srep}(D)$ is degenerate if and only if there is an identity map in the n-chain. This means that by in order to compute the homology objects $H_*(I; D)$ it suffices to consider the normalized complex which by Proposition 3.5.7 means we can can focus only on the non-degenerate simplices.

The following example will appear frequently in Section 4.5.

Example 3.5.8. Consider the pre-pushout diagram $\ulcorner$ given in Example 2.2.8 and let $D\colon \ulcorner \to \mathcal{A}$ be a diagram as follows.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array}$$

Because of the shape of the poset, the simplicial replacement $\operatorname{srep}(D) \in \mathcal{A}^{\Delta^{\mathrm{op}}}$ of the diagram X is 2-skeletal. That is to say, for $n \geq 3$ all of the elements in $\operatorname{srep}(X)_n$ are degenerate. This implies that the normalized complex associated to the simplicial object $\operatorname{srep}(D)$ is the following chain complex.

$$0 \to A \oplus A \xrightarrow{\partial} A \oplus B \oplus C.$$

Here the differential ∂ is the following map

$$\begin{aligned} \partial \colon A \oplus A &\to A \oplus B \oplus C \\ (x, y) &\longmapsto (x + y, -f(x), -g(y)). \end{aligned}$$

From the above, we have directly that

$$\begin{aligned} H_0(\ulcorner, D) &= \operatorname*{colim}_{\ulcorner} D \\ H_1(\ulcorner, D) &= \ker \partial = \ker f \cap \ker g. \end{aligned}$$

3.5.9 A Spectral Sequence for Homotopy Colimits of Diagrams of Spectra

In this subsection we introduce a very important tool that we will use in Chapter 4. This tool is a spectral sequence such that given a spectrum E and a diagram of spectra $X \in \mathrm{Sp}^I$ one can compute the E_*-homology of the homotopy colimit $\operatorname{hocolim}_I X$, that is,

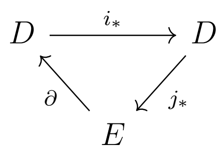
$$E_*(\operatorname*{hocolim}_I X).$$

In order to introduce this spectral sequence, we will venture once again to the garden of simplicial objects. References for this subsection include [15, Section 3.2.1, Chapter 5].

Suppose we have a sequence $\{X_i\}_{i\in\mathbb{Z}_{\geq 0}}$ in the stable model category of spectra

$$* = X_0 \to X_1 \to X_2 \to \dots \tag{3.7}$$

We set $C_i = \operatorname{hocofib}(X_i \to X_{i+1})$ and $X = \operatorname{hocolim}_i X_i$. Here by a sequence we mean nothing more than just a sequence of maps, we do not impose any kind of conditions. Usually sequences like the above are considered as "filtrations" of some object and we think of the sequence of spectra X_i as a filtration of $\operatorname{hocolim}_i X_i$. Taking homotopy groups, see Definition 2.6.5 of the spectra X_i, and of the homotopy cofibers C_i, we wrap up the resulting triangles into an exact couple with



in which

$$D = \bigoplus_{p,n} \pi_n X_p$$

and

$$E = \bigoplus_{p,n} \pi_n C_p.$$

The maps i_* and j_* are the direct sums of the maps on homotopy induced by $i_p\colon X_p \to X_{p+1}$ and $j_p\colon X_p \to C_p$ respectively, and ∂ is the direct sum of the boundary maps $\partial\colon \pi_n C_p \to \pi_{n-1} X_{p-1}$. This produces a spectral sequence with $E^1_{pq} = \pi_{p+q}C_p$ and converges strongly to $\pi_{p+q}X = \pi_{p+q}\operatorname{hocolim}_i X_i$:

$$E^1_{pq} = \pi_{p+q}C_p \Longrightarrow \pi_{p+q}X. \tag{3.8}$$

Now that we have a spectral sequence of a sequence of spectra we generalize the above spectral sequence for an arbitrary simplicial object in spectra.

Proposition 3.5.10. *Let $X \in \mathrm{Sp}^{\Delta^{\mathrm{op}}}$ be a simplicial spectrum which is Reedy cofibrant. Then there is a spectral sequence*

$$E^1_{pq} = \pi_q NX_p \Rightarrow \pi_{p+q}(|X|),$$

where $NX_p := X_p/L_pX = \operatorname{cofib}(L_pX \to X_p)$.

Proof. Consider the skeletal filtration 2.4.20 of A namely,

$$* \to |X|_0 \to |X|_1 \to |A|_2 \to \dots.$$

Since we assume that X is Reedy cofibrant 2.2.21 all of the maps above are cofibrations, hence the homotopy cofiber $|X|_{n-1} \to |X|_n$ is simply the ordinary cofiber. Moreover,

$$\operatorname*{hocolim}_i |X|_i \cong \operatorname*{colim}_i |X|_i = |A|.$$

Since the ordinary cofiber is just the quotient of the target by the image of the map, by Proposition 2.1.9 we can identify the cofiber of $|X|_{n-1} \to |X|_n$ with $\Sigma^n NX_n$. Therefore the filtration (3.7) together with the cofibers is the following.

$$\begin{array}{ccccccccc}
* & \longrightarrow & |X|_0 & \xrightarrow{\phi_0} & |X|_1 & \xrightarrow{\phi_1} & |X|_2 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & X_0 & & \Sigma N X_1 & & \Sigma^2 N X_2 & & \dots
\end{array}$$

where each sequence

$$|X|_{n-1} \to |X|_n \to \Sigma^n N X_n$$

is a cofibration sequence. Hence the E^1_{pq} term of the spectral sequence (3.8) is the following.

$$\pi_{p+q}(\Sigma^p N X_p) = \left[\mathbb{S}^{p+q}, \Sigma^p N X_p\right] \cong [\mathbb{S}^q, N X_p] = \pi_q N X_p$$

□

Next we generalize from a simplicial spectrum to an arbitrary diagram of spectra. Let now I be a direct category. We equip the category Sp^I with the model structure 2.2.12 and consider a diagram $X \in \mathrm{Sp}^I$. We may assume that X is objectwise cofibrant. If not, we replace it cofibrantly. In any case, by Lemma 2.4.17, its simplicial replacement $\mathrm{srep}(X) \in s\mathrm{Sp}$ is Reedy cofibrant. Recall from Definition 2.6.7 that given spectra E and X, the E_*-homology of X is given by $E_*(X) = \pi_*(E \wedge X)$.

Proposition 3.5.11. *Let E be a spectrum and let $X \in \mathrm{Sp}^I$ be a diagram of spectra that is objectwise cofibrant. There is a spectral sequence*

$$E^2_{pq} = H_p(I; E_q X) \Rightarrow E_{p+q}(\operatorname{hocolim}_I X).$$

Proof. Consider the simplicial replacement of X, namely, $\operatorname{srep}(X)$ which by Lemma 2.4.17 is Reedy cofibrant. So we have the skeletal filtration of $\operatorname{srep}(X)$, i.e.the following sequence of spectra.

$$* \to |\operatorname{srep}(X)|_0 \to |\operatorname{srep}(X)|_1 \to |\operatorname{srep}(X)|_2 \to \dots.$$

Notice that again by Proposition 2.1.9 we can identify

$$\operatorname{cofib}(|\operatorname{srep}(X)|_{n-1} \to |\operatorname{srep}(X)|_n) = \Sigma^n \coprod_{i_0 \to \dots i_n} X_{i_0}.$$

Here in the coproduct we consider only the non-degenereate simplices. Next, we objectwise smash the simplicial object $\operatorname{srep}(X)$ with E to get the diagram

$$E \wedge \operatorname{srep}(X) \colon \Delta^{\mathrm{op}} \to \mathrm{Sp}, \quad [n] \mapsto E \wedge^{\mathbb{L}} \operatorname{srep}(X)_n.$$

Smashing preserves cofiber sequences and the geometric realization of the simplicial spectrum $E \wedge^{\mathbb{L}} \operatorname{srep}(X)$ is naturally isomorphic to $\operatorname{hocolim}_I E \wedge^{\mathbb{L}} X$. Since homotopy colimits commute with $- \wedge^{\mathbb{L}} -$ this is naturally isomorphic to $E \wedge^{\mathbb{L}} \operatorname{hocolim}_I X$. So the terms of the spectral sequence

$$\pi_{p+q}\left(\Sigma^p(E \wedge^{\mathbb{L}} \coprod_{i_0 \to \dots \to i_p} X_{i_0})\right) \cong \pi_q(E \wedge^{\mathbb{L}} \coprod_{i_0 \to \dots \to i_p} X_{i_0}) = E_q(\coprod_{i_0 \to \dots \to i_p} X_{i_0}) = H_p(I; E_q X)$$

and we are done. □

Notice that given $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{X} \mathrm{Sp}$ by (2.4.23) we have an induced natural map of simplicial objects

$$\mathrm{srep}(F^*X) \to \mathrm{srep}(X),$$

hence the above proposition gives a morphism of spectral sequences $f\colon \left\{E^2_{pq}\right\} \to \left\{E'^2_{pq}\right\}$. More generally we have the following

Proposition 3.5.12. *There is a functorial assignment of a spectral sequence to each simplicial object of* Sp *converging (in good cases) to its homotopy colimit (geometric realization) with a canonical identification of the E_2 terms given in the previous proposition.*

Chapter 4

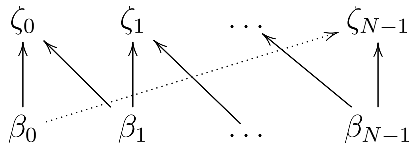
Monoidal Properties of the Functor $\mathcal{Q}$

4.1 Introduction to the Chapter

This chapter constitutes the heart of the thesis and it is the longest one. It is structured as follows. In the second section, with all the tools at our disposal from Chapters 2 and 3 we can finally recall Franke's realization functor $\mathcal{R}$ and it's construction. We will recall all the necessary definitions and concepts to make the construction precise. We will also establish some notation that we will use throughout. In the next section we state the main theorem and result of the chapter. We will explain what the theorem entails how the theorem will be broken down into smaller propositions. In the fourth section we start our way to set up the necessary definitions and notation. We will be interested in a homotopy left Kan extension of a particular functor, so we will discuss the slice categories of that functor in detail. This is because the values of a homotopy left Kan extension are computed using homotopy colimits of slice categories, as explained in Proposition 2.5.9. In the fifth section, after we have set up the necessary notation, we will explain in more detail our strategy and various technicalities and we will start the proof of the first part of Theorem 4.3.1. That is, we are going to construct a crowned diagram that computes the tensor product of twisted periodic complexes. In the sixth section we will discuss the cones of the new diagram that we will construct. In the last section we will discuss the differentials of the new twisted-periodic complex.

4.2 Construction of Franke's Functor

In this section we are ready to recall the constructions of Franke's functor, our main character for this thesis. For more detailed exposition, see [43]. Recall the poset $\mathcal{C}_N$ defined in Example 2.2.9 for $N := 2p - 2$ for p an odd prime. It is the following poset



Also, recall Proposition 2.6.10 for the $E(1)$-local model structure, $L_1\mathrm{Sp}$. We equip the category $L_1\mathrm{Sp}^{\mathcal{C}_N}$ with the projective model structure 2.2.12. For a diagram $X \in L_1\mathrm{Sp}^{\mathcal{C}_N}$ we denote the structure morphisms as follows

$$l_n \colon X_{\beta_n} \to X_{\zeta_n}, \quad k_n \colon X_{\beta_{n+1}} \to X_{\zeta_n}.$$

Consider now the full subcategory of objects X of $\mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N})$ satisfying the following conditions.

(i) The objects $Z_*^n := E(1)_{*-n}(X_{\zeta_n})$ and $B_*^n := E(1)_{*-n}(X_{\beta_n})$ are concentrated in degrees congruent to 0 modulo N, i.e., Z_*^n and B_*^n are objects of $\mathcal{B} = \mathrm{Comod}^0_{E(1)_*E(1)}$.

(ii) The maps $\lambda_n := E(1)_{*-n}(l_n)\colon B_*^n \to Z_*^n$ are injective.

Definition 4.2.1. Denote the subcategory of $\mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N})$ defined above by $\mathcal{L}$. Furthermore, we let $\mathcal{L}' \subset \mathcal{L}$ be the full subcategory of $\mathcal{L}$ such that Z_*^n and B_*^n are projective $E(1)_*$-modules.

Now, let X be an object of $\mathcal{L}$. We define

$$C_*^n(X) := E(1)_{*-n}(\mathrm{cone}(X_{\beta_{n+1}} \to X_{\zeta_n})) = E(1)_{*-n}(\mathrm{cone}(k_n)) \qquad (4.1)$$

Since we are working in $\mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N})$ and not $\mathrm{Ho}(L_1\mathrm{Sp})^{\mathcal{C}_N}$, all of the constructions above are properly functorial.

If we apply $E(1)_{*-n}(-)$ to the exact triangle

$$X_{\beta_{n+1}} \xrightarrow{k_n} X_{\zeta_n} \to \mathrm{cone}(k_n) \to \Sigma X_{\beta_{n+1}}$$

we obtain a short exact sequence

$$B_{*+1}^{n+1} \to Z_*^n \to C_*^n \to B_*^{n+1} \xrightarrow{0} Z_{*-1}^n.$$

It follows that C_*^n is also an object of $\mathcal{B} = \mathrm{Comod}^0_{E(1)_*E(1)}$. In order to define the differential we apply $E(1)_{*-n}(-)$ to

$$\mathrm{cone}(X_{\beta_{n+1}} \to X_{\zeta_n}) \to \Sigma X_{\beta_{n+1}} \xrightarrow{\Sigma l_{n+1}} \Sigma X_{\zeta_{n+1}} \xrightarrow{\Sigma\iota} \Sigma\mathrm{cone}(X_{\beta_{n+2}} \to X_{\zeta_{n+1}})$$

and we have

$$d^n \colon C^n_*(X) \to B^{n+1}_*(X) \to Z^{n+1}_*(X) \to C^{n+1}_*(X).$$

So, we have a well-defined functor

$$\mathcal{Q} \colon \mathcal{L} \to \mathcal{C}^{2p-2}_*(\mathcal{B}) \quad X \mapsto \mathcal{Q}(X) = (C^\bullet_*(X), d). \tag{4.2}$$

Franke proves that this functor is an equivalence. Choose an inverse $\mathcal{Q}^{-1}$ of $\mathcal{Q}$ once and for all. Franke further shows that

$$\mathcal{R} := \operatorname*{hocolim}_{\mathcal{C}_N} \circ \mathcal{Q}^{-1} \colon \mathcal{C}^{2p-2}_*(\mathcal{B}) \to \mathrm{Ho}(L_1\mathrm{Sp}). \tag{4.3}$$

factors over the derived category of $\mathcal{C}^{2p-2}_*(\mathcal{B})$ and induces an equivalence of categories

$$\mathcal{R} \colon \mathcal{D}^{2p-2}(\mathcal{B}) \to \mathrm{Ho}(L_1\mathrm{Sp})$$

Here $\mathcal{D}^{2p-2}(\mathcal{B}) = \mathrm{Ho}(\mathcal{C}^{2p-2}_*(\mathcal{B})_{\mathrm{inj}})$, the homotopy category of the *injective* model structure on $\mathcal{C} *^{2p-2} (\mathcal{B})$, see Definition 3.4.1.

Remark 4.2.2. There is another category which is implicit in the above discussion. Let ce be the class of morphisms in $\mathrm{Ho}(\mathcal{L})$ that are sent to equivalences by the homotopy colimit functor. The above claim is that these are the same as the morphisms sent by $\mathcal{Q}$ to quasi=equivalences, so we have

$$\mathrm{Ho}(L_1\mathrm{Sp}) \cong \mathrm{Ho}(\mathcal{L})[\mathrm{ce}^{-1}] \cong \mathcal{C}_*^{2p-2}(\mathcal{B})[\mathrm{qi}^{-1}] = \mathcal{D}^{2p-2}(\mathcal{B}),$$

where qi is the class of weak equivalences in $\mathcal{C}_*^{2p-2}(\mathcal{B})$, that is to say the quasi-isomorphisms.

Finally, for all $C, D \in \mathcal{C}_*^{2p-2}(\mathcal{B})$, the mapping space $\mathrm{Map}(C, D)$ is weakly equivalent to a product of Eilenberg-MacLane spaces. However, the mapping space $\mathrm{Map}_{L_1\mathrm{Sp}}(\mathbb{S}, \mathbb{S})$ is not a product of Eilenberg-MacLane spaces. It follows that $\mathcal{C}_*^{2p-2}(\mathcal{B})$ and $L_1\mathrm{Sp}$ cannot be Quillen equivalent. For details of that fact see [43]. So, the functor $\mathcal{R}$ is an equivalence of homotopy categories which cannot rise from a Quillen equivalence. Moreover, this equivalence can be made stronger in certain cases. Both model structures $\mathcal{C}_*^{2p-2}(\mathcal{B})$ and $L_1\mathrm{Sp}$ are *stable* model categories, see Definition 2.2.28, therefore the homotopy categories $\mathcal{D}^{2p-2}(\mathcal{B})$ and $\mathrm{Ho}(L_1\mathrm{Sp})$ are triangulated categories. In the paper [33, Section 4.2], it is proven that if $p \geq 5$ the functor $\mathcal{R}$ is a *triangulated equivalence*. However, the question, whether the equivalence is triangulated in the case $p = 3$, remains open.

4.3 Statement of the Main Result

In this section we will state the main result of this chapter and we will provide motivation.

Theorem 4.3.1. *There exists a bifunctor*

$$i^*\mathbb{L}\mathrm{pr}_!(-\,\overline{\wedge}^{\mathbb{L}}\,-)\colon\ \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N})\times\mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N})\to\mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N})$$

such that the following holds. Let $X, Y \in \mathcal{L}$ such that for any $n \in \mathbb{Z}/(2p-2)\mathbb{Z}$ and any $\alpha \in \{\beta, \zeta\}$, the underlying $E(1)_*$-modules of the $E(1)_*E(1)_*$-comodules

$$E(1)_*(X_{\alpha_n}) \quad \text{and} \quad E(1)_*(Y_{\alpha_n}),$$

are projective. Then,

$$i^*\mathbb{L}\mathrm{pr}_!(X\,\overline{\wedge}^{\mathbb{L}}\,Y)\in\mathcal{L}$$

and there is a natural isomorphism

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X\,\overline{\wedge}^{\mathbb{L}}\,Y))\cong\mathcal{Q}(X)\otimes\mathcal{Q}(Y).$$

The theorem has two parts. For brevity we will use the subcategory $\mathcal{L}'$ introduced in Definition 4.2.1. The first part is the existence of the bifunctor $i^*\mathbb{L}\mathrm{pr}_!(-\overline{\wedge}^{\mathbb{L}}-)\colon \mathcal{L}'\times\mathcal{L}'\to\mathcal{L}'$ and the second part is the existence of a natural isomorphism

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X\,\overline{\wedge}^{\mathbb{L}}\,Y))\cong\mathcal{Q}(X)\otimes\mathcal{Q}(Y),$$

given that X and Y satisfy certain hypotheses. The theorem above (the two parts combined) tells us that the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{C}^1(\mathcal{A}) \times \mathcal{C}^1(\mathcal{A}) & \longleftarrow & \mathcal{L}' \times \mathcal{L}' \\ {\scriptstyle \otimes}\downarrow & & \downarrow \\ \mathcal{C}^1(\mathcal{A}) & \longleftarrow & \mathcal{L}' \end{array}$$

where

(i) The top horizontal arrow is the functor

$$\mathcal{Q} \times \mathcal{Q} \colon \mathcal{L}' \times \mathcal{L}' \to \mathcal{C}^1(\mathcal{A}) \times \mathcal{C}^1(\mathcal{A}).$$

(ii) The left vertical arrow is the tensor product of twisted periodic complexes of period 1 in the abelian category $\mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)}$.

(iii) The right vertical arrow is our, yet to be constructed, bifunctor

$$i^*\mathbb{L}\mathrm{pr}_!(- \,\overline{\wedge}^{\mathbb{L}}\, -) \colon \mathcal{L}' \times \mathcal{L}' \to \mathcal{L}'.$$

(iv) The bottom horizontal arrow is the functor

$$\mathcal{Q} \colon \mathcal{L}' \to \mathcal{C}^1(\mathcal{A}).$$

The first part of Theorem 4.3.1, i.e., the existence of the bifunctor

$$i^*\mathbb{L}\mathrm{pr}_!(- \,\overline{\wedge}^{\mathbb{L}}\, -) \colon \mathcal{L}' \times \mathcal{L}' \to \mathcal{L}'$$

is the content of Section 4.5. The second part of Theorem 4.3.1, i.e., the natural isomorphism

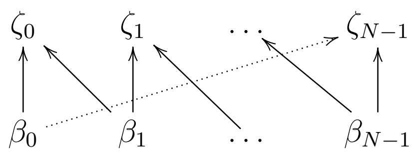
$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y)$$

is the content of Sections 4.6 and 4.7.

4.4 Crowned Diagrams

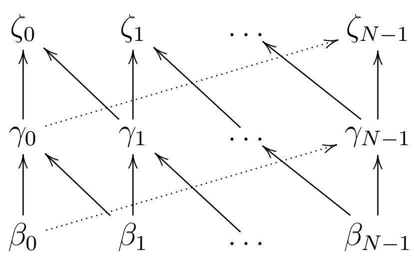
4.4.1 Main Definitions of Crowned Diagrams

Recall the poset $\mathcal{C}_N$ that was defined in Example 2.2.9 as follows. It is the poset consisting of elements $\{\beta_i, \zeta_i \mid i \in \mathbb{Z}/N\mathbb{Z}\}$ such that $\beta_i < \zeta_i$ and $\beta_i < \zeta_{i-1}$ for $i \in \mathbb{Z}/N\mathbb{Z}$. As a diagram it looks as follows.



We will set $N = 2p - 2$ where p is an odd prime.

Also recall the poset $\mathcal{D}_N$ in Example 2.2.10 consisting of elements $\{\beta_n, \gamma_n, \zeta_n \colon n \in \mathbb{Z}/N\mathbb{Z}\}$ such that $\beta_n \leq \gamma_n \leq \zeta_n$, $\beta_{n+1} \leq \gamma_n$ and $\gamma_{n+1} \leq \zeta_n$. As a diagram it looks as follows.



We will be interested in two functors between these two categories. The first functor is the *projection functor*

$$\begin{aligned} \mathrm{pr}\colon \mathcal{C}_N \times \mathcal{C}_N &\to \mathcal{D}_N \\ (\beta_i, \beta_j) &\mapsto \beta_{i+j} \\ (\zeta_i, \zeta_j) &\mapsto \zeta_{i+j} \\ (\zeta_i, \beta_j) &\mapsto \gamma_{i+j} \\ (\beta_i, \zeta_j) &\mapsto \gamma_{i+j}. \end{aligned} \tag{4.4}$$

Note, that we really should be writing $\beta_{i(\mathrm{mod} N)}$ and $\gamma_{i+j(\mathrm{mod} N)}$ etc. but we do a small abuse of notation and avoid this. The other functor that we will deal with is the functor

$$i\colon \mathcal{C}_N \to \mathcal{D}_N, \quad \zeta_n \mapsto \zeta_n, \quad \beta_n \mapsto \gamma_n. \tag{4.5}$$

As we can see from the definition, the functor i embeds $\mathcal{C}_N$ into $\mathcal{D}_N$ in the two top "floors". Notice that the functor $i^*\colon L_1\mathrm{Sp}^{\mathcal{D}_N} \to L_1\mathrm{Sp}^{\mathcal{C}_N}$ preserves weak equivalences, hence it defines a functor on the homotopy categories, which we denote by the same letter

$$i^*\colon \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N}) \to \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N}).$$

Next, recall from Definition 2.3.24, the objectwise (exterior) smash product for diagrams $X \in \mathcal{C}^I$ and $Y \in \mathcal{C}^J$, for I and J finite posets. It follows formally by choosing $I = J = \mathcal{C}_N$, we have a bifunctor

$$- \,\overline{\wedge}\, -\colon L_1\mathrm{Sp}^{\mathcal{C}_N} \times L_1\mathrm{Sp}^{\mathcal{C}_N} \to L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N}. \tag{4.6}$$

By Proposition 2.3.25, this external product has a total left derived functor

$$- \,\overline{\wedge}^{\mathbb{L}}\, -\colon \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N}) \times \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N}) \to \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N}). \tag{4.7}$$

Given diagrams $X, Y \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N})$, we can define the homotopy left Kan extension of the objective (exterior) smash product diagram

$$X \,\overline{\wedge}^{\mathbb{L}}\, Y \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N})$$

along the functor $\mathrm{pr}\colon \mathcal{C}_N \times \mathcal{C}_N \to \mathcal{D}_N$, that is,

$$\mathbb{L}\mathrm{Lan}_{\mathrm{pr}}(X \,\overline{\wedge}^{\mathbb{L}}\, Y) = \mathbb{L}\mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N}).$$

Now that we have all the necessary ingredients, we can finally define the bifunctor that we need for Theorem 4.3.1.

Definition 4.4.2. The bifunctor $i^*\mathbb{L}\mathrm{pr}_!(-\overline{\wedge}^{\mathbb{L}}-)$ is defined as the composition

$$\mathcal{L} \times \mathcal{L} \xrightarrow{\overline{\wedge}^{\mathbb{L}}} \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N}) \xrightarrow{\mathbb{L}\mathrm{pr}_!} \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N}) \xrightarrow{i^*} \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N}).$$

To ease notation we will set

$$E = \mathbb{L}\mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N})$$

and we will use both interchangeably throughout.

The values of this functor are given by the formula in Subsection 2.5.6. That is, the values of E at the objects are given by the following.

$$E_{\gamma_n} = \underset{\mathrm{pr}/\gamma_n}{\mathrm{hocolim}}(X \wedge^{\mathbb{L}} Y) \tag{4.8}$$

$$E_{\zeta_n} = \underset{\mathrm{pr}/\zeta_n}{\mathrm{hocolim}}(X \wedge^{\mathbb{L}} Y) \tag{4.9}$$

$$E_{\beta_n} = \underset{\mathrm{pr}/\beta_n}{\mathrm{hocolim}}(X \wedge^{\mathbb{L}} Y) \tag{4.10}$$

The edges of the homotopy Kan extension, $\widehat{l}_n \colon E_{\gamma_n} \to E_{\zeta_n}$ and $\widehat{k}_n \colon E_{\gamma_{n+1}} \to E_{\zeta_n}$ are given by the natural maps

$$E_{\gamma_n} = \underset{\mathrm{pr}\,/\gamma_n}{\mathrm{hocolim}}(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \to \underset{\mathrm{pr}\,/\zeta_n}{\mathrm{hocolim}}(X \,\overline{\wedge}^{\mathbb{L}}\, Y) = E_{\zeta_n} \tag{4.11}$$

$$E_{\gamma_{n+1}} = \underset{\mathrm{pr}\,/\gamma_{n+1}}{\mathrm{hocolim}}(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \to \underset{\mathrm{pr}\,/\zeta_n}{\mathrm{hocolim}}(X \,\overline{\wedge}^{\mathbb{L}}\, Y) = E_{\zeta_n} \tag{4.12}$$

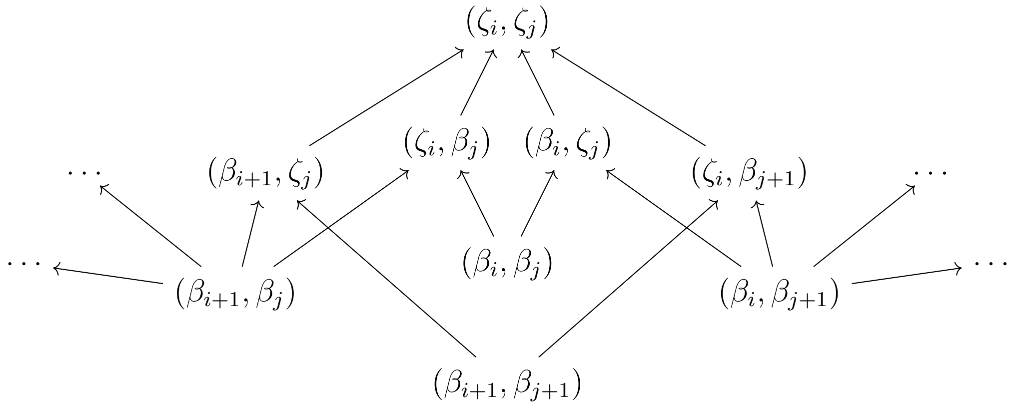
induced by functors ϕ and ψ, respectively, see Subsection 2.12.

Since we are interested in the homotopy Kan extension of the functor $\mathrm{pr} \colon \mathcal{C}_N \times \mathcal{C}_N \to \mathcal{D}_N$, we need to have an explicit discription of all the slice categories $\mathrm{pr}\,/\zeta_n, \mathrm{pr}\,/\gamma_n$ and $\mathrm{pr}\,/\beta_n$. Recall from Remark 2.4.7 that given $\mathcal{C}$ and $\mathcal{D}$ posets and a functor $f \colon \mathcal{C} \to \mathcal{D}$, the slice category is defined as

$$f/d = \{c \in \mathcal{C} \colon f_c \leq d\}$$

for $d \in \mathcal{D}$.

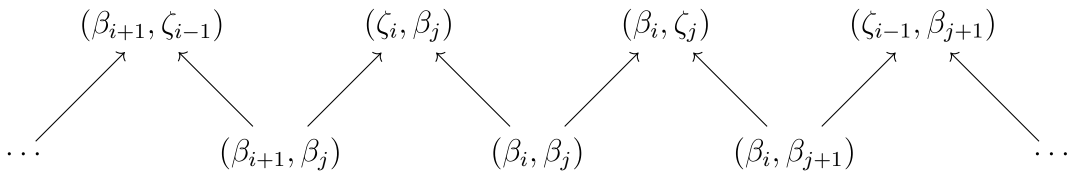
Example 4.4.3. For $n \in \mathbb{Z}/(2p-2)\mathbb{Z}$ and the object ζ_n we have the slice category pr/ζ_n as below.



Here, $i+j \equiv n \pmod{2p-2}$. Notice that all the morphisms (non-idenity) are of the form (id, l_i) or (l_i, id) and similarly (id, k_i) or (k_i, id) for $i \in \mathbb{Z}/(2p-2)\mathbb{Z}$. The poset pr/ζ_n follows the same pattern to the left and to the right.

Our next example of interest is the following.

Example 4.4.4. Let n again as above but now consider the slice category pr/γ_n which looks as follows.



Again $i + j \equiv n \pmod{2p-2}$. Analogously to the above example, all the non-identity morphisms are of the form (id, l_i) or (l_i, id) and similarly (id, k_i) or (k_i, id) for $i \in \mathbb{Z}/(2p-2)\mathbb{Z}$.

Example 4.4.5. Let again $n \in \mathbb{Z}/(2p-2)\mathbb{Z}$ but now we consider the slice category pr/β_n. Notice that it is

$$\cdots \qquad (\beta_{i+1}, \beta_{j-1}) \qquad (\beta_i, \beta_j) \qquad (\beta_{i-1}, \beta_{j+1}) \qquad \cdots$$

in which $i + j \equiv n \pmod{2p-2}$. In other words, it is a discrete category. This means that

$$E_{\beta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\beta_n}(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \cong \bigoplus_{i+j=n} X_{\beta_i} \wedge^{\mathbb{L}} Y_{\beta_j}.$$

This is the only case that we can be explicit about the values of the homotopy left Kan extension

$$E = \mathbb{L}\mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y).$$

As a last example that will be useful for us is the following.

Example 4.4.6. [18, pp 35] For the poset pr/ζ_n, consider the following subposet $J_n \subseteq \mathrm{pr}/\zeta_n$ defined as follows:

$$\begin{array}{ccccccccc}
 & & (\zeta_{i+1}, \zeta_{i-1}) & & & (\zeta_i, \zeta_j) & & (\zeta_{i-1}, \zeta_{i+1}) & \\
 & \nearrow & & \nwarrow & \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \\
\cdots & & & (\beta_{i+1}, \beta_j) & & & (\beta_i, \beta_{j+1}) & & \cdots
\end{array}$$

where $i + j \equiv n \pmod{2p-2}$. Notice that in this poset the non-identity morphisms are of the form (k_i, l_i) or (l_i, k_i), unlike the examples above where one arrow was always the identity arrow.

Suppose now that we have a pair of adjoint functors

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G.$$

If $\mathcal{C}$ and $\mathcal{D}$ are posets (and F and G are map of posets) then the condition that the pair (F, G) defines an adjoint pair becomes

$$F_c \leq d \text{ if and only if } c \leq G_d. \tag{4.13}$$

For more details on posets adoints, see [2, Section 9.4].

Remark 4.4.7. Now let $\theta_n\colon J_n \to \mathrm{pr}/\zeta_n$ denote the inclusion of the subposet defined in Example 4.4.6. Below, we will define a map of posets

$$L\colon \mathrm{pr}/\zeta_n \to J_n, \tag{4.14}$$

which it suffices to define for the part of the poset visible in Example 4.4.3, the rest can be defined analogously. The map L is defined as follows.

$$\begin{aligned} L\colon \mathrm{pr}/\zeta_n &\to J_n \\ (\beta_{i+1}, \beta_j) &\mapsto (\beta_{i+1}, \beta_j) \\ (\beta_i, \beta_{j+1}) &\mapsto (\beta_i, \beta_{j+1}) \\ \text{else} &\mapsto (\zeta_i, \zeta_j) \end{aligned}$$

Notice that the condition (4.13) holds for L as the left adjoint and θ_n as the right adjoint. In other words, the functor L as constructed above, is a left adjoint to the inclusion $\theta_n\colon J_n \to \mathrm{pr}/\zeta_n$. As a consequence, since the inclusion map $\theta_n\colon J_n \to \mathrm{pr}/\zeta_n$ is a right adjoint, by Lemma 2.4.28, for any $F \in \mathrm{Ho}(\mathcal{C}^{\mathrm{pr}/\zeta_n})$

$$\operatorname*{hocolim}_{J_n} \theta_n^*(F) \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} F.$$

In other words, the value E_{ζ_n} in (4.8) can be calculated

$$E_{\zeta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \cong \operatorname*{hocolim}_{J_n} \theta_n^*(X \,\overline{\wedge}^{\mathbb{L}}\, Y). \tag{4.15}$$

Given any of subposet of $\mathcal{C}_N \times \mathcal{C}_N$, *e.g.*, pr/γ_n from Example 4.4.4 we can define the restriction of the objectwise smash product $X \,\overline{\wedge}\, Y \in L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N}$ to pr/γ_n by taking the pullback along the inclusion $\nu\colon \mathrm{pr}/\gamma_n \to \mathcal{C}_N \times \mathcal{C}_N$, that is,

$$\nu^*\colon L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N} \to L_1\mathrm{Sp}^{\mathrm{pr}/\gamma_n}.$$

Notice that ν^* preserves weak equivalences so it induces a functor on homotopy categories

$$\nu^*\colon\ \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N}) \to \mathrm{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\gamma_n}).$$

The diagram

$$X \,\overline{\wedge}^{\mathbb{L}}\, Y\colon\ \mathrm{pr}/\gamma_n \xrightarrow{\nu} \mathcal{C}_N \times \mathcal{C}_N \to L_1\mathrm{Sp}$$

which looks as follows:

$$\begin{array}{ccccccccc}
 & X_{\beta_{i+1}} \wedge^L Y_{\zeta_{i-1}} & & X_{\zeta_i} \wedge^L Y_{\beta_j} & & X_{\beta_i} \wedge^L Y_{\zeta_j} & & X_{\zeta_{i-1}} \wedge^L Y_{\beta_{j+1}} & \\
\nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \\
\cdots & & X_{\beta_{i+1}} \wedge^L Y_{\beta_j} & & X_{\beta_i} \wedge^L Y_{\beta_j} & & X_{\beta_i} \wedge^L Y_{\beta_{j+1}} & & \cdots
\end{array}$$

Moreover we have maps between the subposets of $\mathcal{C}_N \times \mathcal{C}_N$. The morphisms $\gamma_n \to \zeta_n$ and $\gamma_{n+1} \to \zeta_n$ induce maps of posets

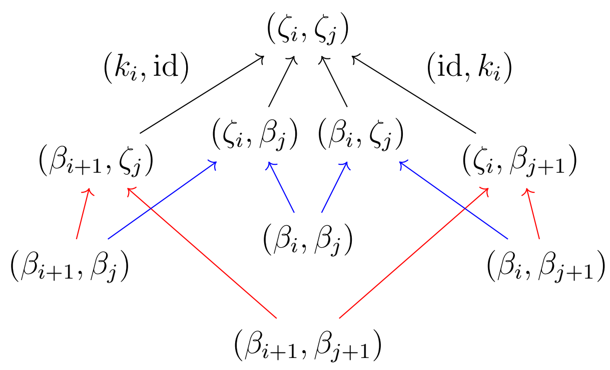
$$\psi\colon \operatorname{pr}/\gamma_n \to \operatorname{pr}/\zeta_n$$
$$\phi\colon \operatorname{pr}/\gamma_{n+1} \to \operatorname{pr}/\zeta_n$$

which in turn also induce pullback functors on the homotopy cateogies, that is,

$$\phi^*\colon \operatorname{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\zeta_n}) \to \operatorname{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\gamma_n}), \text{ and } \psi^*\colon \operatorname{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\zeta_n}) \to \operatorname{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\gamma_n}).$$

We conclude this section with a convention.

Convention 4.4.8. To ease notation we will make the following convention. Consider the poset pr/ζ_n.



The red color shows the image of the morphism of posets

$$\phi\colon \operatorname{pr}/\gamma_{n+1} \to \mathrm{pr}/\zeta_n,$$

and the blue color shows the image of the morphism

$$\psi\colon\ \mathrm{pr}\,/\gamma_n \to \mathrm{pr}/\zeta_n.$$

We regard both of them as subposets of pr/ζ_n. Because of this, we will commit an abuse of notation, so instead of writing, for example,

$$\phi^*(X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y) \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\gamma_n})$$

we will simply write

$$X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\gamma_n}),$$

with the understanding that this diagram was given by the pullback functor

$$\phi^*\colon\ \mathrm{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\zeta_n}) \to \mathrm{Ho}(\mathrm{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\gamma_n}))$$

unless we need the extra notation for clarification.

4.5 The Spectral Sequence

In this section we will prove the first part of Theorem 4.3.1, that is, the existence of the bifunctor

$$i^*\mathbb{L}\mathrm{pr}_!(- \wedge^L -)\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}.$$

After we review some relevant notions, we will state formally in Proposition 4.5.2 our main result of this section. Before we start with the proof we will explain some relevant technical details and the overall strategy for the proof of Proposition 4.5.2.

4.5.1 Preliminaries

Recall Definition 4.4.2. The goal of this subsection is to show that for the $\mathcal{D}_N$-diagram

$$E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N})$$

given by the homotopy left Kan extension along the projection functor $\mathrm{pr}\colon \mathcal{C}_N \times \mathcal{C}_N \to \mathcal{D}_N$, is what we need, namely that the $\mathcal{C}_N$-diagram i^*E is in the subcategory $\mathcal{L}$. This means we have to show that

$$E(1)_*(i^*E_{\beta_n})[n] = E(1)_*(E_{\gamma_n})[n] = E(1)_{*-n}(E_{\gamma_n})$$

and

$$E(1)_*(i^*E_{\zeta_n})[n] = E(1)_*(E_{\zeta_n})[n] = E(1)_{*-n}(E_{\zeta_n})$$

are not just objects of $\mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)}$, but objects of the splitting $\mathcal{B} = \mathrm{Comod}^0_{E(1)_*E(1)}$ and that the induced morphism

$$E(1)_*(E_{\gamma_n})[n] \to E(1)_*(E_{\zeta_n})[n]$$

is a monomorphism for all n.

We now formally state the proposition that we are going to prove.

Proposition 4.5.2. *Let $X, Y \in \mathcal{L}$ such that for every $n \in \mathbb{Z}/(2p-2)\mathbb{Z}$ and*

$\alpha \in \{\beta, \zeta\}$ the underlying $E(1)_*$-modules of

$$E(1)_*(X_{\alpha_n}) \quad \textit{and} \quad E(1)_*(Y_{\alpha_n})$$

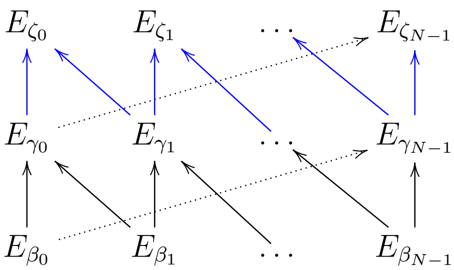
are projective. Consider the homotopy left Kan extension of

$$X \,\overline{\wedge}^{\mathbb{L}}\, Y \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N})$$

along

$$\mathrm{pr}\colon \mathcal{C}_N \times \mathcal{C}_N \to \mathcal{D}_N, \qquad E = \mathbb{L}\mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N})$$

with the values and morphisms given in (4.8)-(4.10) and (4.11), (4.12) respectively.



(4.16)

Then, the $E(1)_*E(1)$-comodules $E(1)_*(E_{\alpha_n})[n]$ with $\alpha \in \{\beta, \gamma, \zeta\}$ are not just objects of $\mathcal{A}$ but objects of the splitting $\mathcal{B}$ and the morphisms

$$E(1)_*(E_{\gamma_n}) \to E(1)_*(E_{\zeta_n})$$

induced by $E_{\gamma_n} \to E_{\zeta_n}$ are monomorphisms.

For the following corollary, recall the map of posets $i\colon \mathcal{C}_N \to \mathcal{D}_N$, from

(4.5).

Corollary 4.5.3. *Let $X, Y \in \mathcal{L}$ satisfying the hypothesis in the above Proposition. The top two rows of the diagram $E = \mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y)$ is an object in $\mathcal{L}$, that is, the diagram $i^*E \in \mathcal{L}$.*

Since the values E_{ζ_n}, E_{γ_n} and E_{β_n} are computed by homotopy colimits, we will use Proposition 3.5.11, the spectral sequences converging to the $E(1)_*$-homology of the homotopy colimit.

Lemma 4.5.4. *There are spectral sequences*

$$E^2_{pq} = H_p(\mathrm{pr}\,/\gamma_n; E(1)_q(X \overline{\wedge}^L Y)) \Rightarrow E(1)_{p+q}\left(\underset{\mathrm{pr}\,/\gamma_n}{\mathrm{hocolim}}(X \,\overline{\wedge}^L\, Y)\right) = E(1)_{p+q}(E_{\gamma_n}) \tag{4.17}$$

and

$$E'^2_{pq} = H_p(\mathrm{pr}\,/\zeta_n; E(1)_q(X \overline{\wedge}^L Y)) \Rightarrow E(1)_{p+q}\left(\underset{\mathrm{pr}\,/\zeta_n}{\mathrm{hocolim}}(X \,\overline{\wedge}^L\, Y)\right) = E(1)_{p+q}(E_{\zeta_n}) \tag{4.18}$$

and a natural morphism of spectral sequences $f\colon \left\{E^2_{pq}\right\} \to \left\{E'^2_{pq}\right\}$ induced by the map in (4.11).

Before we start with the proof we briefly explain the strategy of the proof and some technicalities which we gather below.

Recollection 4.5.5. (i) We are aiming to compute $E(1)_*(E_{\zeta_n})$ and $E(1)_*(E_{\gamma_n})$ for each $n \in \mathbb{Z}/N$ and to show that they are concentrated in the correct degrees. In order to do so, we will compute the pages of the spectral sequences above. The second page of the spectral sequences are the homologies of the posets pr/γ_n and pr/ζ_n with coefficients in the functors

$E(1)_q(X \,\overline{\wedge}^{\mathbb{L}}\, Y)$ for various $q \in \mathbb{Z}$, *i.e.*,

$$H_p(\mathrm{pr}/\gamma_n; E(1)_q(X \,\overline{\wedge}^{\mathbb{L}}\, Y))$$

and

$$H_p(\mathrm{pr}\,/\zeta_n; E(1)_q(X \,\overline{\wedge}^{\mathbb{L}}\, Y)).$$

The morphism of posets $\mathrm{pr}/\gamma_n \to \mathrm{pr}/\zeta_n$ induces a morphism on the homology of categories

$$H_*(\mathrm{pr}/\gamma_n; E(1)_q(X \,\overline{\wedge}^{\mathbb{L}}\, Y)) \to H_*(\mathrm{pr}/\zeta_n; E(1)_q(X \,\overline{\wedge}^{\mathbb{L}}\, Y))$$

which will be used to show that the morphisms $E(1)_*(E_{\gamma_n}) \to E(1)_*(E_{\zeta_n})$ are monomorphisms. These two facts together will prove that indeed $i^*E = i^*\mathbb{L}\mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \in \mathcal{L}$, and we have a well defined functor $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$.

(ii) We will calculate $H_*(\mathrm{pr}/\gamma_n; E(1)_q(X \,\overline{\wedge}^{\mathbb{L}}\, Y))$ as follows. First we will consider the simplicial replacement of the diagram

$$E(1)_q(X \,\overline{\wedge}^{\mathbb{L}}\, Y)\colon \mathrm{pr}/\gamma_n \to \mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)},$$

see Definition 2.4.14 to obtain a simplicial object in $\mathcal{A}$, that is,

$$\mathrm{srep}(E(1)_q(X \,\overline{\wedge}^{\mathbb{L}}\, Y))\colon \Delta^{\mathrm{op}} \to \mathcal{A}.$$

Then we compute the homology of the associated complex, $C_*(E(1)_q(X\overline{\wedge}^{\mathbb{L}}$

$Y)$), see Definition 3.5.2.

(iii) Consider a diagram $F \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N})$. By Convention 2.2.2, F is a projective cofibrant object, so in particular, it is objectwise cofibrant. The external smash product $X \mathbin{\overline{\wedge}}^{\mathbb{L}} Y$ as defined in (4.6)

$$- \mathbin{\overline{\wedge}} -\colon L_1\mathrm{Sp}^{\mathcal{C}_N} \times L_1\mathrm{Sp}^{\mathcal{C}_N} \to L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N},$$

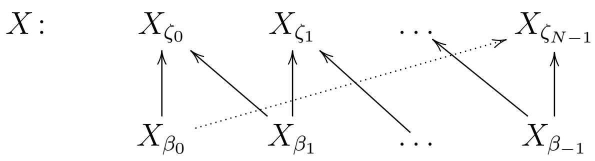
is Quillen bifunctor, so in particular it preserves cofibrant objects. This implies that $X \mathbin{\overline{\wedge}}^{\mathbb{L}} Y$ is cofibrant in $L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N}$, so in particular objectwise cofibrant. Now, for any subposet $\iota\colon \mathcal{P} \hookrightarrow \mathcal{C}_N \times \mathcal{C}_N$, e.g., any of the slice categories of the projection functor pr (4.4), we have the pullback functor

$$\iota^*\colon L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N} \to L_1\mathrm{Sp}^{\mathcal{P}}.$$

This functor is not necessarily a left Quillen functor (with respect the projective model structures, see Proposition 2.2.12). However, the diagram $\iota^*(X \mathbin{\overline{\wedge}}^{\mathbb{L}} Y)$, is objectwise cofibrant, which means that geometric realization of the simplicial replacement still models the homotopy colimit of the diagram $\iota^*(X \mathbin{\overline{\wedge}}^{\mathbb{L}} Y)$. In particular, the skeletal filtration of all the restrictions is always Reedy cofibrant, see Definition 2.2.21.

(iv) By Proposition 3.5.5, the homology of the complex $C_\bullet(E(1)_q(X \mathbin{\overline{\wedge}}^{\mathbb{L}} Y))$ is isomorphic to the homology of the normalized complex $N_\bullet(E(1)_q(X \mathbin{\overline{\wedge}}^{\mathbb{L}} Y))$. This means in practice that we will ignore degenerate simplices when we construct the associated complex.

(v) Consider a crowned diagram $X \in \mathcal{L}$ as follows.



We have that for each $i \in \mathbb{Z}/(2p-2)\mathbb{Z}$

$$Z_*^i = E(1)_*(X_{\zeta_i})[i] = E(1)_{*-i}(X_{\zeta_i})$$

and

$$B_*^i = E(1)_*(X_{\beta_i})[i] = E(1)_{*-i}(X_{\beta_i}).$$

are objects of the splitting $\mathcal{B}$, that is, the $E(1)_*E(1)$-comodules Z_*^i and B_*^i are concentrated in degrees $\equiv 0 \pmod{2p-2}$. Since *e.g.* $E(1)_{*-i}(X_{\zeta_i})$ is concentrated in degrees $\equiv 0 (\mathrm{mod}\ 2p-2)$ this implies that $E(1)_*(X_{\zeta_i})$ is concentrated in degrees $\equiv -i \pmod{2p-2}$. Similarly, given another crowned diagram Y in $\mathcal{L}$, with

$$\widetilde{Z}_*^j = E(1)_*(Y_{\zeta_j})[j] = E(1)_{*-j}(Y_{\zeta_j})$$

and

$$\widetilde{B}_*^j = E(1)_*(Y_{\beta_j})[j] = E(1)_{*-j}(Y_{\beta_j})$$

we have as above that $E(1)_*(Y_{\zeta_j})$ and $E(1)_*Y_{\beta_j}$ are concentrated in degrees $\equiv -j \pmod{2p-2}$. Now given diagrams $X, Y \in \mathcal{L}$, assume that for every $i, j \in \mathbb{Z}/N\mathbb{Z}$ and $\alpha \in \{\beta, \zeta\}$ the underlying $E(1)_*$-modules

$E(1)_*(X_{\alpha_i})$ and $E(1)_*(Y_{\alpha'_j})$ are projective as $E(1)_*$-modules. So, from Proposition 3.1.7, for any such pair we have that that the smash products $X_{\alpha_i} \wedge^{\mathbb{L}} Y_{\alpha'_j}$

$$E(1)_* \left(X_{\alpha_i} \wedge^{\mathbb{L}} Y_{\alpha'_j} \right) \cong E(1)_*(X_{\alpha_i}) \otimes_{E(1)_*} E(1)_*(Y_{\alpha'_j}).$$

Furthermore, $E(1)_*(X_{\alpha_i} \wedge^{\mathbb{L}} Y_{\alpha'_j})$ is concentrated in degrees $\equiv -i - j (\text{mod } 2p-2)$. Since objects in the split category $\mathcal{B}$, see 3.4 are concentrated in degrees congruent to 0 mod $2p-2$ and the definition of the tensor product 3.1.6 we have that for $n = i + j$

$$E(1)_{-n} \left(X_{\beta_i} \wedge^L Y_{\zeta_j} \right) \cong B^i \otimes \widetilde{Z}^j.$$

Note that we have removed the grading in B^i and $\widetilde{Z}^j$ above since we have summed them up due to the tensor product.

(vi) The above is the reason that we will assume the hypothesis that the diagrams $X, Y \in \mathcal{L}$ are objectwse projective. We will see in Chapter 6 in the proof of the main Theorem, why this is a reasonable assumption to make.

(vii) From the above discussion, it follows that if we have, for example, $X_{\beta_i} \wedge Y_{\beta_j}$ with $i + j = n$, then

$$E(1)_m(X_{\beta_i} \wedge Y_{\beta_j}) = 0$$

for $m \neq -n$ (mod $2p-2$).

(viii) The category $L_1\mathrm{Sp}$ is equipped with the $E(1)$-local model structure, a left Bousfield localization of the stable model structure on Sp, see Proposition 2.6.10. We equip $L_1\mathrm{Sp}^{\mathcal{C}_N}$ with the model structure of Proposition 2.2.12. By Example 2.3.11, $(L_1\mathrm{Sp}^{\mathcal{C}_N}, \wedge)$ is a monoidal model structure.

4.5.6 The Start of the Proof

Now we are ready to start the proof of Proposition 4.5.2.

Proof. We will start by working out the spectral sequence (4.17). Recall the poset $\mathrm{pr}\,/\gamma_n$ which looks as follows

$$
\begin{array}{ccccccccccccc}
 & & (\beta_{i+1}, \zeta_{i-1}) & & & & (\zeta_i, \beta_j) & & & & (\beta_i, \zeta_j) & & & & (\zeta_{i-1}, \beta_{j+1}) & & \\
 & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
\cdots & & & & (\beta_{i+1}, \beta_j) & & & & (\beta_i, \beta_j) & & & & (\beta_i, \beta_{j+1}) & & & & \cdots
\end{array}
$$

Here $i + j = n$, and so the functor $X \,\overline{\wedge}^{\mathbb{L}}\, Y \colon \mathrm{pr}/\gamma_n \to L_1\mathrm{Sp}$ is

$$
\begin{array}{ccccccccccccc}
 & & X_{\beta_{i+1}} \wedge^L Y_{\zeta_{i-1}} & & & & X_{\zeta_i} \wedge^L Y_{\beta_j} & & & & X_{\beta_i} \wedge^L Y_{\zeta_j} & & & & X_{\zeta_{i-1}} \wedge^L Y_{\beta_{j+1}} & & \\
 & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
\cdots & & & & X_{\beta_{i+1}} \wedge^L Y_{\beta_j} & & & & X_{\beta_i} \wedge^L Y_{\beta_j} & & & & X_{\beta_i} \wedge^L Y_{\beta_{j+1}} & & & & \cdots
\end{array}
$$

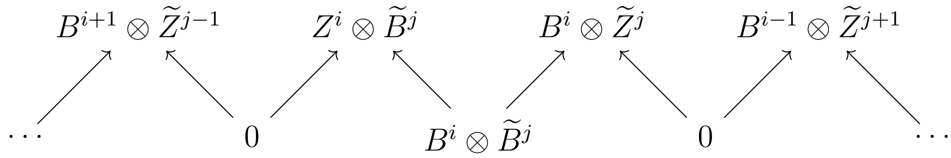
Our goal is to compute

$$
H_p(\mathrm{pr}\,/\gamma_n; E(1)_q(X \wedge^{\mathbb{L}} Y)) \qquad \text{for all } p \geq 0 \text{ and all } q \in \mathbb{Z}
$$

which form the E^2-terms of the spectral sequence (4.17). In order to do so, we apply the homological functor $E(1)_{-n}(-)$ to the above diagram to get the diagram

$$E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y)\colon \operatorname{pr}/\gamma_n \to \mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)}.$$

By Discussion (v) the diagram $E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y)$ looks as follows



(4.19)

We write the morphisms as

$$f_{ij} = l_i \otimes 1\colon B^i \otimes \widetilde{B}^j \to Z^i \otimes \widetilde{B}^j \tag{4.20}$$

$$g_{ij} = 1 \otimes \tilde{l}_j\colon B^i \otimes \widetilde{B}^j \to B^i \otimes \widetilde{Z}^j \tag{4.21}$$

to distinguish, for labeling purposes, the two different morphisms in the simplicial replacement below. Notice that since B^i and $\widetilde{B}^j$ and projective $E(1)_*$-modules, they are flat automatically, hence the morphisms (4.20) and (4.21) are monomorphisms. Next, we consider the simplicial replacement of the diagram $E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y)$. We have the diagram

$$\operatorname{srep}\big(E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y)\big) : \Delta^{\mathrm{op}} \to \mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)}.$$

Following Definition 2.4.14 we have that

$$\operatorname{srep}\big(E(1)_{-n}(X \wedge^{L} Y)\big)_0 = \bigoplus_{i+j=n} \Big((B^i \otimes \widetilde{B}^j) \oplus (Z^i \otimes \widetilde{B}^j) \oplus (B^i \otimes \widetilde{Z}^j)\Big)$$
$$\operatorname{srep}\big(E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y)\big)_1 = \bigoplus_{i+j=n} \Big((B^i \otimes \widetilde{B}^j)_{f_{i,j}} \oplus (B^i \otimes \widetilde{B}^j)_{g_{i,j}}\Big)$$

with face maps given by "source" and "target" respectively. Notice that because of the shape of the poset pr/γ_n, for all $m \geq 2$, the simplices

$$\operatorname{srep}\big(E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y)\big)_m$$

consist of solely of degenerate simplices. Now we consider the associated complex of the above simplicial object,

$$C_*(E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y),$$

see Definition 3.5.2. We shortly explain the differential of the complex $C_*(E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y))$,

$$\partial = d_0 - d_1 \colon C_1(E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y)) \to C_0(E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y)).$$

Notice from (4.19), we can consider the simpler case where the diagram 4.19 looks as follows.

$$Z^i \otimes \widetilde{B}^i \xleftarrow{\ f_{ij}\ } B^i \otimes \widetilde{B}^j \xrightarrow{\ g_{ij}\ } B^i \otimes \widetilde{Z}^j$$

Then, the differential of the associated complex of the simplicial replacement of the above diagram, in this simpler case, is as follows

$$\partial_{ij} = d_0 - d_1 \colon (B^i \otimes \widetilde{B}^j) \oplus (B^i \otimes \widetilde{B}^j) \to (B^i_* \otimes \widetilde{B}^j) \oplus (Z^i \otimes \widetilde{B}^j) \oplus (B^i_* \otimes \widetilde{Z}^j) \tag{4.22}$$

$$(x, y) \mapsto (x + y, -f_{ij}(x), -g_{ij}(y)). \tag{4.23}$$

See Example 3.5.8. By Lemma 3.5.4 the 0th homology of the above complex is just the pushout $B^i \otimes \widetilde{Z}^j \coprod_{B^i \otimes \widetilde{B}^j} Z^i \otimes \widetilde{B}^i$. The first homology is just the kernel of the differential ∂_{ij}, given in (4.22). Since the maps f_{ij} and g_{ij} are injective, this forces $d_{ij}(x, y) = 0$ if and only if $x = y = 0$, which implies the the first homology is 0. It follows from the diagram (4.19), the differential ∂ of the complex $C_*(E(1)_{-n}(X \,\overline{\wedge}^{\mathbb{L}}\, Y))$ is the direct sum of the above differentials ∂_{ij} for $i + j = n$. Now that we know the differential of the complex $C_*(E(1)_{-n}(X \,\overline{\wedge}^{\mathbb{L}}\, Y))$ we will compute its homology. The group

$$H_0(\mathrm{pr}/\gamma_n; E(1)_n(X \wedge^L Y))$$

is the colimit of the diagram $E(1)_{-n}(X \,\overline{\wedge}^{\mathbb{L}}\, Y)$. By inspecting the diagram $E(1)_{-n}(X \,\overline{\wedge}^{\mathbb{L}}\, Y)$ above we can see the colimit of the diagram is a direct sum (coproduct) of pushouts, that is,

$$\begin{aligned} H_0(\mathrm{pr}/\gamma_n; E(1)_{-n}(X \,\overline{\wedge}^{\mathbb{L}}\, Y)) &= \operatorname*{colim}_{\mathrm{pr}/\gamma_n} E(1)_{-n}(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \\ &= \bigoplus_{i+j=n} \left(Z^i \otimes \widetilde{B}^j \coprod_{B^i \otimes \widetilde{B}^j} B^i \otimes \widetilde{Z}^j \right). \end{aligned}$$

Similar to the simpler case

$$H_1(\mathrm{pr}/\gamma_n; E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y))$$

is the kernel of the differential

$$d_0 - d_1\colon \operatorname{srep}(E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y))_1 \to \operatorname{srep}(E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y))_0.$$

Since it is a direct sum of the simpler differentials ∂_{ij} in (4.22), it follows that

$$H_1(\mathrm{pr}/\gamma_n; E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y)) = 0.$$

All the higher homologies

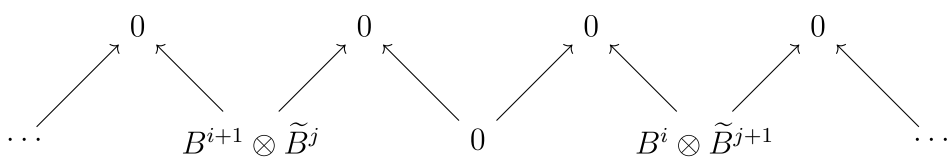
$$H_q(\mathrm{pr}/\gamma_n; E(1)_{-n}(X \overline{\wedge}^{\mathbb{L}} Y))$$

vanish for all $q \geq 2$.

Next we apply the homology functor $E(1)_{-n-1}(-)$ to the diagram $X \overline{\wedge}^{\mathbb{L}} Y\colon \mathrm{pr}/\gamma_n \to L_1\mathrm{Sp}$ and we have the diagram

$$E(1)_{-n-1}(X \overline{\wedge}^{\mathbb{L}} Y)\colon \mathrm{pr}/\gamma_n \to \mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)}$$

as follows



Clearly,

$$H_0(\mathrm{pr}/\gamma_n; E(1)_{-n-1}(X \,\overline{\wedge}^{\mathbb{L}}\, Y)) = 0,$$

and

$$H_1(\mathrm{pr}/\gamma_n; E(1)_{-n-1}(X \,\overline{\wedge}^{\mathbb{L}}\, Y)) = \bigoplus_{i+j=n+1} B^i \otimes \widetilde{B}^j.$$

By the Discussion in (vii) it follows that for all $p \geq 0$ and all $m \neq -n, -n - 1 \mod 2p-2)$, the terms

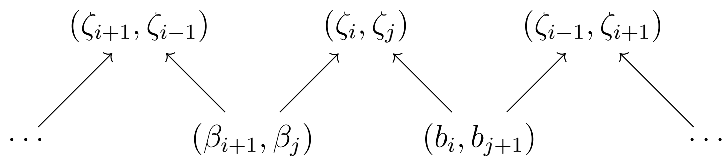
$$H_p(\mathrm{pr}/\gamma_n; E(1)_m(X \,\overline{\wedge}^{\mathbb{L}}\, Y))$$

all vanish. This completes the computation of the E^2-terms of the spectral sequence (4.17). It is concentrated in degrees $(0, m)$ and $(1, m-1)$ with $m \equiv -n \mod 2p-2$. Therefore the spectral sequence collapses and we have a short exact sequence:

$$0 \to \bigoplus_{i+j=n} \left(Z^i \otimes \widetilde{B}^j \bigoplus_{B^i \otimes \widetilde{B}^j} B^i \otimes \widetilde{Z}^j \right) \to E(1)_{-n}(E_{\gamma_n}) \to \bigoplus_{i+j=n+1} B^i \otimes \widetilde{B}^j \to 0 \tag{4.24}$$

This concludes the calculation of the spectral sequence (4.17).

We are moving on to calculate the spectral sequence (4.18). Recall the poset J_n from Example 4.4.6. It is a subposet of pr/ζ_n, which is the following.



By Remark 4.4.7 the inclusion functor $\theta_n \colon J_n \to \mathrm{pr}/\zeta_n$ has a left left adjoint L, see (4.14), and we have

$$E_{\zeta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \barwedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{J_n} \theta_n^*(X \barwedge^{\mathbb{L}} Y),$$

see (4.15). So, instead of the spectral sequence (4.18) we can compute the following spectral sequence

$$H_p(J_n; E(1)_q(\theta_n^*(X \wedge^{\mathbb{L}} Y))) \Longrightarrow E(1)_{p+q}(\operatorname*{hocolim}_{J_n} \theta_n^*(X \wedge^{\mathbb{L}} Y))$$

since both converge to the same target, namely

$$E(1)_*(\operatorname*{hocolim}_{J_n} \theta_n^*(X \wedge^{\mathbb{L}} Y)) \cong E(1)_*(\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y)) = E(1)_*(E_{\zeta_n}).$$

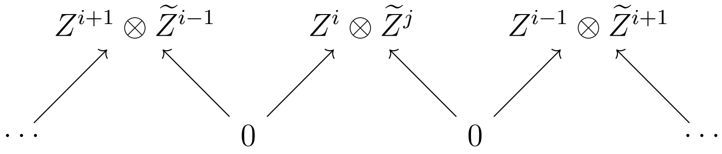
In fact this can be made stronger. The adjunction $L \colon \mathrm{pr}/\zeta_n \rightleftarrows J_n \colon \theta_n$ induces a natural isomorphism

$$H_*(\mathrm{pr}/\zeta_n; E_q(X \barwedge^{\mathbb{L}} Y)) \cong H_*(J_n, \theta_n^* E_q(X \barwedge^{\mathbb{L}} Y)).$$

By Discussion (vii), and from the diagram J_n we only need to consider again $E(1)_{-n}(-)$ and $E(1)_{-n-1}(-)$. So, firstly we apply $E(1)_{-n}(-)$ to the diagram $\theta_n^*(X \wedge^{\mathbb{L}} Y)$ and get the J_n-diagram in $\mathcal{A}$

$$E(1)_{-n}(\theta_n^*(X \wedge^{\mathbb{L}} Y)) \to \mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)}$$

which looks as follows



From the above we get that

$$H_0(J_n; E(1)_{-n}(\theta_n^*(X \overline{\wedge}^{\mathbb{L}} Y))) = \bigoplus_{i+j=n} Z^i \otimes \widetilde{Z}^j,$$

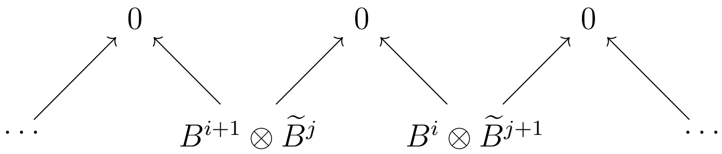
and

$$H_p(J_n; E(1)_{-n}(\theta_n^*(X \overline{\wedge}^{\mathbb{L}} Y))) = 0, \quad p \geq 1.$$

Next, we will apply the functor $E(1)_{-n-1}(-)$ and we get the diagram

$$E(1)_{-n-1}(\theta_n^*(X \overline{\wedge}^{\mathbb{L}} Y)) \to \mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)}$$

which is as follows.



From the above we get that

$$H_1(J_n; E(1)_{-n-1}(\theta_n^*(X \overline{\wedge}^{\mathbb{L}} Y))) = \bigoplus_{i+j=n+1} B^i \otimes \widetilde{B}^j$$

and

$$H_p(J_n; E(1)_{-n-1}(\theta_n^*(X \mathbin{\overline{\wedge}}^{\mathbb{L}} Y))) = 0 \quad p = 0 \text{ and } p \geq 2$$

This completes the computation of the E^2-terms of the spectral sequence. It is concentrated in degrees $(0, m)$ and $(1, m-1)$ with $m \equiv -n \bmod 2p-2$. Therefore, the spectral sequence collapses and we have a short exact sequence:

$$0 \to \bigoplus_{i+j=n} Z^i \otimes \widetilde{Z}^j \to E(1)_{-n}(E_{\zeta_n}) \to \bigoplus_{i+j=n+1} B^i \otimes \widetilde{B}^j \to 0. \tag{4.25}$$

Now that we have calculated both spectral sequences we can continue with the proof. The map of posets $\psi\colon \mathrm{pr}/\gamma_n \to \mathrm{pr}/\zeta_n$ induces morphisms on homologies of categories with coefficients $E(1)_{-n}(-)$ and $E(1)_{-n-1}(-)$ respectively, i.e.,

$$H_*(\mathrm{pr}/\gamma_n; E(1)_{-n}(X \mathbin{\overline{\wedge}}^L Y)) \to H_*(\mathrm{pr}/\zeta_n; E(1)_{-n}(X \mathbin{\overline{\wedge}}^L Y)) \cong H_*(J_n; E(1)_{-n}(\theta_n^*(X \mathbin{\overline{\wedge}}^{\mathbb{L}} Y)))$$
$$H_*(\mathrm{pr}/\gamma_n; E(1)_{-n-1}(X \mathbin{\overline{\wedge}}^L Y)) \to H_*(\mathrm{pr}/\zeta_n; E(1)_{-n-1}(X \mathbin{\overline{\wedge}}^L Y)) \cong H_*(J_n; E(1)_{-n-1}(\theta_n^*(X \mathbin{\overline{\wedge}}^{\mathbb{L}}$$

and the natural map $E(1)_*(E_{\gamma_n}) \to E(1)_*(E_{\zeta_n})$ which is induced by ψ, is compatible with the morphism of spectral sequences. Hence we have a morphism of short exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{i+j=n} \left(Z^i \otimes \widetilde{B}^j \coprod_{B^i \otimes \widetilde{B}^j} B^i \otimes \widetilde{Z}^j \right) & \longrightarrow & E(1)_{-n}(E_{\gamma_n}) & \longrightarrow & \bigoplus_{i+j=n+1} B^i \otimes \widetilde{B}^j & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & \bigoplus_{i+j=n} Z^i \otimes \widetilde{Z}^j & \longrightarrow & E(1)_{-n}(E_{\zeta_n}) & \longrightarrow & \bigoplus_{i+j=n+1} B^i \otimes \widetilde{B}^j & \longrightarrow & 0. \end{array} \tag{4.26}$$

By naturality, the left vertical map is the direct sum of the pushout-product maps

$$\lambda_i \,\square\, \widetilde{\lambda}_j \colon \left(Z^i \otimes \widetilde{B}^j \coprod_{B^i \otimes \widetilde{B}^j} B^i \otimes \widetilde{Z}^j \right) \to Z^i \otimes \widetilde{Z}^j.$$

By Lemma 4.7.6 the map $\lambda_i \square \widetilde{\lambda}_j$ is injective which means that so is the direct sum, *i.e.*, the left vertical map. The short five lemma implies now that the morphism

$$E(1)_{-n}(E_{\gamma_n}) \to E(1)_{-n}(E_{\zeta_n})$$

is an injection. In particular, $E(1)_*(E_{\gamma_n})$ and $E(1)_*(E_{\zeta_n})$ are concentrated in the correct degrees and the induced morphisms $E(1)_*(E_{\gamma_n}) \to E(1)_*(E_{\zeta_n})$ are injections. This concludes the proof of the proposition. $\square$

4.6 Cones of the New Diagram

Recall from Definition 4.4.2, and Corollary 4.5.3 we have the bifunctor

$$i^* \mathbb{L}\mathrm{pr}_!(- \,\overline{\wedge}^L\, -) \colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}.$$

Recall from (4.1), that for a diagram $X \in \mathcal{L}$ we defined

$$C^n_*(X) := E(1)_{*-n}(\text{cone}(X_{\beta_{n+1}} \to X_{\zeta_n})) = E(1)_{*-n}(\text{cone}(k_n)).$$

The goal of this section, together with the next one, Section 4.7, is to prove the following proposition.

Proposition 4.6.1. *Let $X, Y \in \mathcal{L}$ such that for every $n \in \mathbb{Z}/(2p-2)\mathbb{Z}$ and $\alpha \in \{\beta, \zeta\}$ the underlying $E(1)_*$-modules of*

$$E(1)_*(X_{\alpha_n}) \quad \textit{and} \quad E(1)_*(Y_{\alpha_n})$$

are projective. There is a natural isomorphism

$$\mathcal{Q}(i^*\mathbb{L}\text{pr}_!(X \,\overline{\wedge}^L\, Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y).$$

The above proposition will follow from Corollary 4.6.4 and Proposition 4.7.3.

4.6.2 Cones of the Diagram i^*E

In this subsection we will show that $\mathcal{Q}(i^*E)$ is a good candidate for the tensor product $\mathcal{Q}(X) \otimes \mathcal{Q}(Y)$. We start with the following proposition. We recall some notation for crowned diagrams.For $X, Y \in L_1\text{Sp}^{\mathcal{C}_N}]$ we denote the structure morphisms $k_i \colon X_{\beta_{i+1}} \to X_{\zeta_i}$ and $\widetilde{k}_i \colon Y_{\beta_{i+1}} \to Y_{\beta_i}$

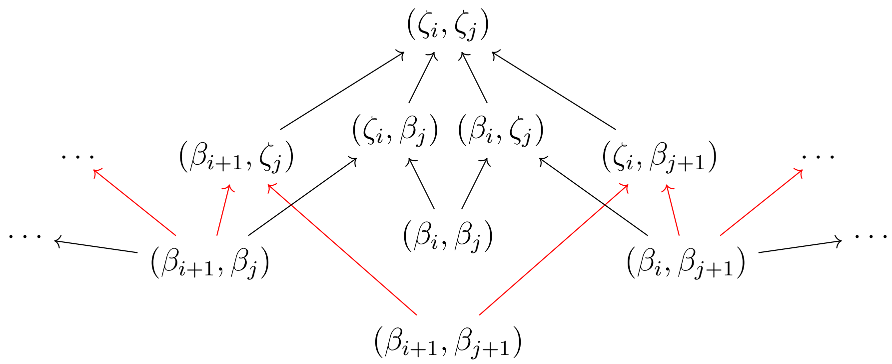
Proposition 4.6.3. *Let X, Y, E as above and consider now i^*E, the pullback of E along $i \colon \mathcal{C}_N \to \mathcal{D}_N$ as an object of $\mathcal{L}$. For every $n \in \mathbb{Z}/(2p-2)\mathbb{Z}$ there* *is a canonical isomorphism*

$$\operatorname{cone}(i^*E_{\beta_{n+1}} \to i^*E_{\zeta_n}) \cong \bigvee_{i+j=n} \operatorname{cone}(k_i) \wedge^{\mathbb{L}} \operatorname{cone}(\tilde{k}_j).$$

Proof. Recall the slice categories of the functor $\mathrm{pr}\colon \mathcal{C}_N \times \mathcal{C}_N \to \mathcal{D}_N$, namely $\mathrm{pr}\,/\zeta_n$, see Example 4.4.3 and pr/γ_n, see Example 4.4.4. Below we reproduce the functor

$$X \,\overline{\wedge}^{\mathbb{L}}\, Y\colon \mathrm{pr}/\zeta_n \to L_1\mathrm{Sp},$$

and the red color shows the image of the map of posets $\phi\colon \mathrm{pr}/\gamma_{n+1} \to \mathrm{pr}\,/\zeta_n$



(4.27)

Recall from (4.9)

$$E_{\zeta_n} = \operatorname{hocolim}(\mathrm{pr}/\zeta_n \xrightarrow{\pi} \mathcal{C}_N \times \mathcal{C}_N \xrightarrow{X \overline{\wedge}^{\mathbb{L}} Y} L_1\mathrm{Sp}),$$

and we committed an abuse of notation by writing

$$\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \,\overline{\wedge}^{\mathbb{L}}\, Y) = \operatorname*{hocolim}_{\mathrm{pr}\,/\zeta_n} \pi^*(X \,\overline{\wedge}^{\mathbb{L}}\, Y).$$

Also, recall from (4.12) that the morphism $E_{\gamma_{n+1}} \to E_{\zeta_n}$ is the canonical morphism

$$\operatorname*{hocolim}_{\mathrm{pr}/\gamma_{n+1}} \phi^*(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \,\overline{\wedge}^{\mathbb{L}}\, Y)$$

which is induced by the map of posets $\phi\colon \mathrm{pr}/\gamma_{n+1} \to \mathrm{pr}/\zeta_n$. The pullback functor

$$\phi^*\colon \mathrm{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\zeta_n}) \to \mathrm{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\gamma_{n+1}})$$

has a left adjoint defined by the homotopy left Kan extension, see Proposition 2.5.7

$$\mathbb{L}\phi_!\colon \mathrm{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\gamma_{n+1}}) \rightleftarrows \mathrm{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\zeta_n})\colon \phi^*.$$

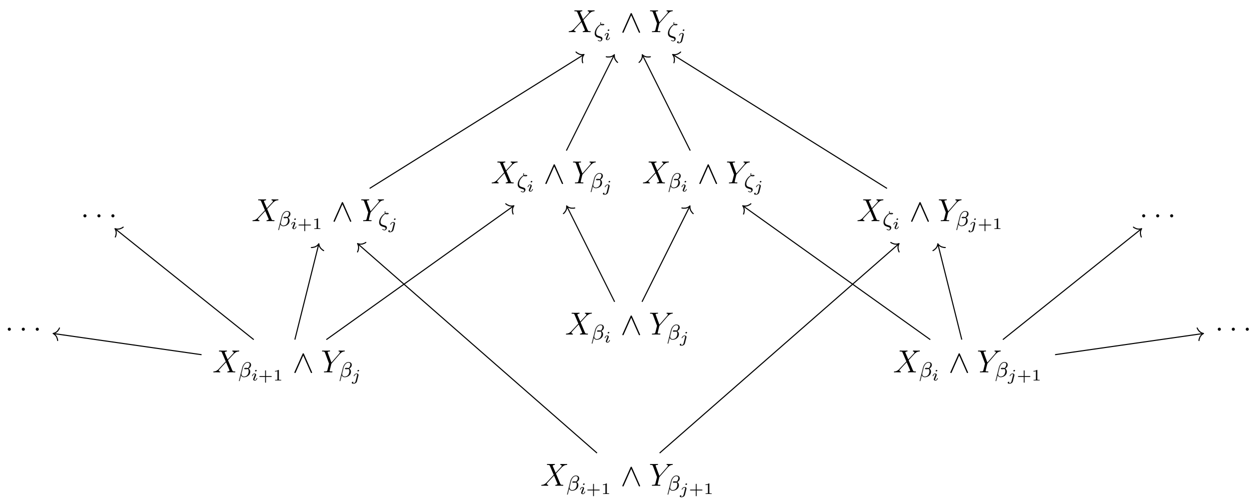
The counit of the derived adjunction $\varepsilon\colon \mathbb{L}\phi_!\phi^* \to \mathrm{Id}$ provides the canonical natural transformation

$$\varepsilon_{X\overline{\wedge}^{\mathbb{L}}Y}\colon \mathbb{L}\phi_!\phi^*(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \to X \,\overline{\wedge}^{\mathbb{L}}\, Y. \tag{4.28}$$

Lastly, since $\mathbb{L}\phi_!$ is a homotopy left Kan extension, by Corollary 2.5.8, we have the canonical isomorphism

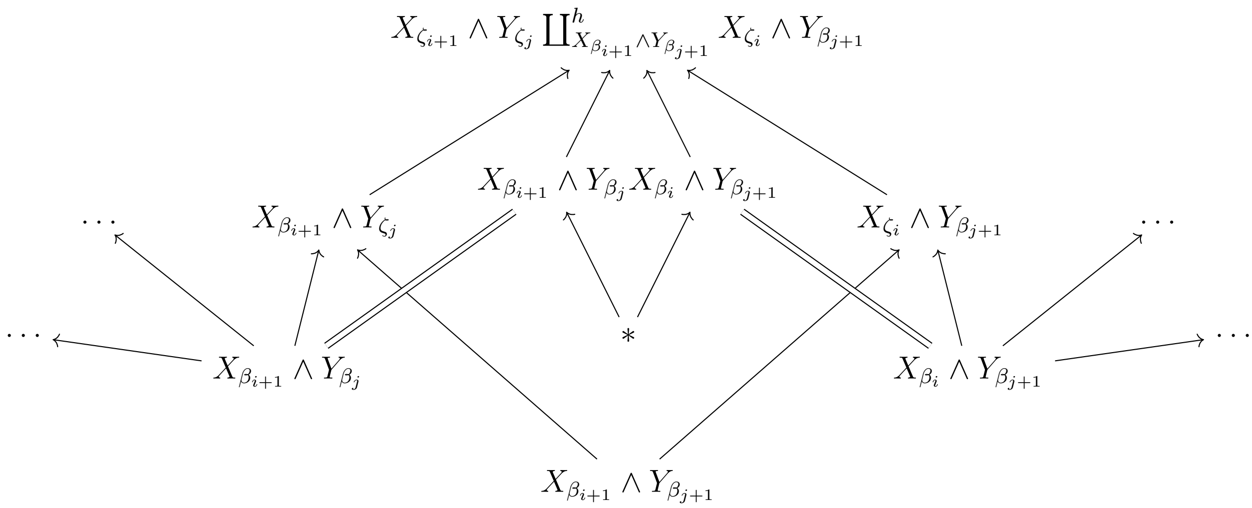
$$\operatorname*{hocolim}_{\mathrm{pr}/\gamma_{n+1}} \phi^*(X \,\overline{\wedge}^{L}\, Y) \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} \mathbb{L}\phi_!\phi^*(X \,\overline{\wedge}^{L}\, Y).$$

So, the diagram $X \,\overline{\wedge}^{\mathbb{L}}\, Y \colon \operatorname{pr}/\zeta_n \to L_1\mathrm{Sp}$ looks as follows.



(4.29)

The diagram $\mathbb{L}\phi_!\phi^*(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \in \mathrm{Ho}(L_1\mathrm{Sp}^{\operatorname{pr}/\zeta_n})$ is the following.



(4.30)

We briefly explain how we calculated the left homotopy Kan extension $\mathbb{L}\phi_!\phi^*(X\overline{\wedge}^{\mathbb{L}} Y)$. From the formula (2.5.9) of calculating homotopy Kan extensions, we can calculate the homotopy left Kan extension $\mathbb{L}\phi_!$ at an object $(\alpha_s, \alpha_t) \in \mathrm{pr}/\zeta_n$ as follows

$$(\mathbb{L}\phi_!\phi^*(X \,\overline{\wedge}^{\mathbb{L}}\, Y))_{(\alpha_s,\alpha_t)} \cong \mathrm{hocolim}(\phi/(\alpha_s, \alpha_t) \xrightarrow{\pi} \mathrm{pr}/\gamma_{n+1} \xrightarrow{\phi^*(X \overline{\wedge}^{\mathbb{L}} Y)} L_1\mathrm{Sp}).$$

Recall from Remark 2.4.7 the slice $f/d = \{c \in \mathcal{C} \mid f_c \leq d\}$ for a map of posets $f\colon C \to D$. For the object (ζ_i, β_j), the slice $\phi/((\zeta_i, \beta_j))$ consists only of the object the object (β_{j+1}, β_j), which implies

$$(\mathbb{L}\phi_!\phi^*(X \,\overline{\wedge}^{\mathbb{L}}\, Y))_{(\zeta_i,\beta_j)} = X_{\beta_{i+1}} \wedge Y_{\beta_j}.$$

For the object (β_i, ζ_j) the argument is the same as above. For (β_i, β_j) the slice category $\phi/(\beta_i, \beta_j)$ is empty which means

$$(\mathbb{L}\phi_!)_{(\beta_i,\beta_j)} \cong *.$$

For the object (ζ_i, ζ_j) the slice category $\phi/((\zeta_i, \zeta_j))$ is the poset

$$\begin{array}{ccccccccc} & & (\beta_{i+1}, \zeta_j) & & & & (\zeta_i, \beta_{j+1}) & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ (\beta_{i+1}, \beta_j) & & & & (\beta_{i+1}, \beta_{j+1}) & & & & (\beta_i, \beta_{j+1}). \end{array}$$

But the subposet

$$\begin{array}{ccc} (\beta_{i+1}, \beta_{j+1}) & \longrightarrow & (\zeta_i, \beta_{j+1}) \\ \downarrow & & \\ (\beta_{i+1}, \zeta_j) & & \end{array}$$

is homotopy final, which means

$$(\mathbb{L}\phi_!\phi^*(X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y))_{(\zeta_i,\zeta_j)} \cong \operatorname*{hocolim}_{\ulcorner}(X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y),$$

that is, the homotopy colimit (pushout)

$$\begin{array}{ccc} X_{\beta_{i+1}} \wedge Y_{\beta_{j+1}} & \xrightarrow{k_i \wedge 1} & X_{\zeta_i} \wedge Y_{\beta_{j+1}} \\ {\scriptstyle 1\wedge \widetilde{k}_j}\downarrow & & \\ X_{\beta_{i+1}} \wedge Y_{\zeta_j} & & \end{array}$$

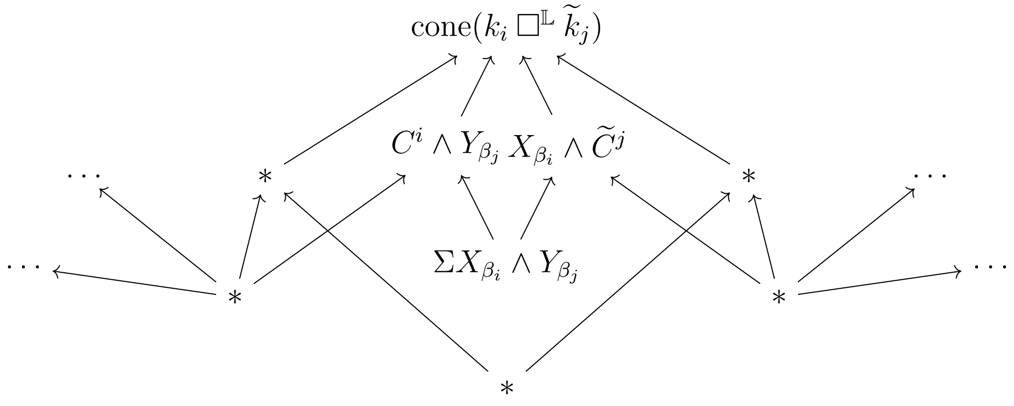
Next, we calculate the cone of the natural transformation

$$\varepsilon_{X\overline{\wedge}^{\mathbb{L}}Y} \colon \mathbb{L}\phi_!\phi^*(X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y) \to X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y$$

see (4.28), of diagrams in $\mathrm{Ho}(L_1\mathrm{Sp}^{\mathrm{pr}/\zeta_n})$. Recall the cone construction of a natural transformation in a simplicial model category of diagrams from (2.5).

$$\begin{aligned} \operatorname{cone}\left(\varepsilon_{X\overline{\wedge}^{\mathbb{L}}Y}\right) :\ & \mathrm{pr}/\zeta_n \to L_1\mathrm{Sp} \\ & (\alpha_s,\alpha_t) \mapsto \operatorname{cone}\left(\phi_!(X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y)_{(\alpha_s,\alpha_t)} \to (X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y)_{(\alpha_s,\alpha_t)}\right). \end{aligned}$$

In other words, we are taking objectwise cones of the canonical map from the diagram (4.30) to the diagram (4.29). This means that $\operatorname{cone}\left(\varepsilon_{X\overline{\wedge}^{\mathbb{L}}Y}\right)$ is the following diagram.



(4.31)

Here, we have denoted provisionally

$$
\begin{aligned}
C^i &:= \operatorname{cone}(k_i) = \operatorname{cone}(X_{i+1} \to X_{\zeta_i}) \\
\widetilde{C}^j &:= \operatorname{cone}(\widetilde{k}_j) = \operatorname{cone}(Y_{j+1} \to Y_{\zeta_i})
\end{aligned}
$$

due to spacing.

Next, we determine the homotopy colimit of the above diagram, that is, of the diagram $\operatorname{cone}\left(\varepsilon_{X\bar{\wedge}^{\mathbb{L}}Y}\right)$. One way is to observe that the homotopy colimit of the above diagram is isomorphic in $\operatorname{Ho}(L_1\mathrm{Sp})$ to the homotopy colimit of (finite) coproduct of squares

$$
\begin{array}{ccc}
\Sigma X_{\beta_i} \wedge Y_{\beta_j} & \longrightarrow & \operatorname{cone}(k_i) \wedge Y_{\beta_j} \\
\downarrow & & \downarrow \\
X_{\beta_i} \wedge \operatorname{cone}(\widetilde{k}_j) & \longrightarrow & \operatorname{cone}(k_i \,\square^{\mathbb{L}}\, \widetilde{k}_j)
\end{array}
\tag{4.32}
$$

where we can consider the above as an object in $\operatorname{Ho}(L_1\mathrm{Sp}^{[1]\times[1]})$. Formally this is by taking the visually obvious map of posets $f\colon [1]\times[1] \to \mathrm{pr}/\zeta_n$ and considering the pullback

$$f^*\colon L_1\mathrm{Sp}^{\mathrm{pr}/\zeta_n} \to L_1\mathrm{Sp}^{[1]\times[1]}.$$

The poset $[1]\times[1]$ has the bottom right corner as a final object which implies the homotopy colimit of the diagram (4.32) is naturally isomorphic to $\mathrm{cone}(k_i \,\square^{\mathbb{L}}\, \widetilde{k}_j)$. One can see this by noting that the inclusion of a terminal object in a category is a right adjoint, and the result follows from Lemma 2.4.28. Hence the homotopy colimit over pr/ζ_n is, up to natural isomorphism, the coproduct

$$\bigvee_{i+j=n} \mathrm{cone}(k_i \,\square^{\mathbb{L}}\, \mathrm{cone}(\widetilde{k}_j)).$$

The other way of seeing it is by pulling back the above diagram to $\theta_n\colon J_n \to \mathrm{pr}\,/\zeta_n$. We get the diagram

$$\begin{array}{ccccccccccc} & & \mathrm{cone}(k_{i+1}\,\square\, k_{j-1}) & & & & \mathrm{cone}(k_i\,\square\, k_j) & & & & \mathrm{cone}(k_{i-1}\,\square\, k_{j+1}) & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \cdots & & & & * & & & & * & & & & \cdots. \end{array}$$

All in all, we have that the homotopy colimit of the diagram (4.31) is

$$\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} \big(\mathrm{cone}(\varepsilon_{X \overline{\wedge}^{\mathbb{L}} Y}\big) \cong \bigvee_{i+j=n} \mathrm{cone}(k_i \,\square^{\mathbb{L}}\, \widetilde{k}_j). \tag{4.33}$$

Finally, by Corollary 2.3.22, for each pair $i, j \in \mathbb{Z}/N\mathbb{Z}$ we have the canonical isomorphism

$$\mathrm{cone}(k_i \,\square^{\mathbb{L}}\, \widetilde{k}_j) \cong \mathrm{cone}(k_i) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_j).$$

The coproduct of these isomorphisms, together with (4.33) gives us that

$$\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}\left(\mathrm{cone}(\varepsilon_{X\bar{\wedge}^{\mathbb{L}}Y})\right) \cong \bigvee_{i+j=n} \mathrm{cone}(k_i) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_j).$$

We write the above calculations formally. In order to calculate the homotopy cofiber (cone) of the morphisms $i^*E_{\beta_{n1}} \to i^*E_{\zeta_n}$ it is the same thing to calculate the homotopy cofibers $E_{\gamma_{n+1}} \to E_{\zeta_n}$.

$$\begin{aligned}
\mathrm{cone}\left(i^*E_{\beta_{n+1}} \to i^*E_{\zeta_n}\right) &= \mathrm{cone}\left(E_{\gamma_{n+1}} \to E_{\zeta_n}\right) \\
&= \mathrm{cone}\left(\operatorname*{hocolim}_{\mathrm{pr}/\gamma_{n+1}} \phi^*(X \bar{\wedge}^L Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \bar{\wedge}^L Y)\right) \\
&\cong \mathrm{cone}\left(\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} \phi_!\phi^*(X \bar{\wedge}^L Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \bar{\wedge}^L Y)\right) \\
&\cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}\left(\mathrm{cone}\left(\phi_!\phi^*(X \bar{\wedge}^L Y) \to (X \bar{\wedge}^L Y)\right)\right) \\
&\cong \bigvee_{i+j=n} \mathrm{cone}(k_i \,\square^{\mathbb{L}}\, \widetilde{k}_j) \\
&\cong \bigvee_{i+j=n} \mathrm{cone}(k_i) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_j)
\end{aligned}$$

□

We have the the following corollary.

Corollary 4.6.4. *Let* $X, Y \in \mathcal{L}$ *and* $E = \mathbb{L}\mathrm{pr}_!(X \bar{\wedge}^{\mathbb{L}} Y)$ *and assume furthermore for every* $n \in \mathbb{Z}/N$ *and* $\alpha \in \{\zeta, \beta\}$

$$E(1)_*(X_{\alpha_n}) \;\; and \;\; E(1)_*(Y_{\alpha_n}) \;\; are\ projective\ E(1)_*\text{-}modules,$$

then there is a canonical isomorphism in $\mathcal{A} = \mathrm{Comod}_{E(1)_*E(1)}$

$$C_*^n(i^*E) = E(1)_{*-n}(\mathrm{cone}(i^*E_{\beta_{n+1}} \to i^*E_{\zeta_n})) \cong \bigoplus_{i+j=n} C_*^i(X) \otimes C_*^j(Y).$$

Proof. Given a crowned diagram $X \in L_1\mathrm{Sp}^{\mathcal{C}_N}$ recall that for every $n \in \mathbb{Z}/N$ we have the following distinguished triangle

$$X_{\beta_{n+1}} \xrightarrow{k_n} X_{\zeta_n} \xrightarrow{\iota} \mathrm{cone}(k_n) \xrightarrow{\partial} \Sigma X_{\beta_{n+1}}.$$

Applying $E(1)_{*-n}(-)$ to the above distinguished triangle we have the short exact sequence

$$Z_*^n(X) \to C_*^n(X) \to B_*^{n+1}(X).$$

By our assumption, for $\alpha \in \{\zeta, \beta\}$ and every $n \in \mathbb{Z}/N$ the comodule $E(1)_*(X_{\alpha_n})$ is a projective $E(1)_*$-module and therefore also $Z_*^n(X)$ and $B_*^{n+1}(X)$ are projective $E(1)_*$-modules. By the long exact sequence of $\mathrm{Ext}(-,-)$ and the characterization of projective dimension, see [49, Lemma 4.1.6], this means that the

$$C_*^n(X) = E(1)_{*-n}(\mathrm{cone}\, k_n) \quad \text{are also projective } E(1)\text{-modules.}$$

This means that also the unshifted $E(1)_*(\mathrm{cone}\, k_n)$ are projective $E(1)_*$-modules for every $n \in \mathbb{Z}/N$. By Proposition 3.1.7,

$$E(1)_*(\mathrm{cone}\, k_s \wedge^{\mathbb{L}} \mathrm{cone}\, \widetilde{k}_t) \cong E(1)_*(\mathrm{cone}\, k_s) \otimes_{E(1)_*} E(1)_*(\mathrm{cone}\, k_t)$$

where we regard the right hand side as a comodule with the canonical comodule structure on the tensor product, (3.3). By Proposition 4.6.1 we have

$$\operatorname{cone}(i^* E_{\beta_{n+1}} \to i^* E_{\zeta_n}) \cong \bigvee_{i+j=n} \operatorname{cone}(k_i) \wedge^{\mathbb{L}} \operatorname{cone}(\tilde{k}_j),$$

and applying the functor $E(1)_*(-)$ we have

$$\begin{aligned} E(1)_*(\operatorname{cone}(i^* E_{\beta_{n+1}} \to i^* E_{\zeta_n})) &\cong E(1)_* \left(\bigvee_{i+j=n} \operatorname{cone}(k_i) \wedge^{\mathbb{L}} \operatorname{cone}(\tilde{k}_j) \right) \\ &\cong \bigoplus_{i+j=n} E(1)_*(\operatorname{cone}(k_i) \wedge^{\mathbb{L}} \operatorname{cone} \widetilde{k}_j) \\ &\cong \bigoplus_{i+j=n} E(1)_*(\operatorname{cone} k_i) \otimes E(1)_*(\operatorname{cone} \widetilde{k}_j). \end{aligned}$$

Shifting the above by $[n] = [i+j]$ we have

$$C_*^n(i^* E) \cong \bigoplus_{i+j=n} C_*^i(X) \otimes C_*^j(Y).$$

□

4.7 Differentials

4.7.1 Introduction

In the previous section, we showed in Corollary 4.6.4 that given diagrams $X, Y \in \mathcal{L}$ satisfying certain hypotheses we can construct naturally a new diagram $i^*E \in \mathcal{L}$ such that

$$C^n_*(i^*E) \cong \bigoplus_{i+j=n} C^i_*(X) \otimes C^j_*(Y)$$

as graded objects, so the diagram i^*E is a good candidate that models the tensor product $C^\bullet_*(X) \otimes C^\bullet_*(Y)$. The final step in order to show Proposition 4.6.1 is to prove that that the differentials of the complex $C^\bullet_*(i^*E)$ coincide with the differentials of the tensor product $C^\bullet_*(X) \otimes C^\bullet_*(Y)$. That is, we have to show that

$$C^\bullet_*(i^*E, d) \cong (C^\bullet_*(X) \otimes C^\bullet(Y), d_\otimes)$$

where $d_\otimes$ is the differential of the tensor product of the complexes $(C^\bullet_*(X), d)$ and $(C^\bullet_*(Y), d)$. Before we move on to the proof, let us recall from Section 4.2 how the differentials of the complex $\mathcal{Q}(X) = (C^\bullet_*(X), d)$ are defined using the structure morphisms of the crowned diagram X. So, let $X \in \mathcal{L}$ and we will construct the morphism (differential) $d^n \colon C^n_*(X) \to C^{n+1}_*(X)$. Recall Definition 2.2.31 of an elementary triangle associated to a map. We have the following sequence.

$$\text{cone}(X_{\beta_{n+1}} \to X_{\zeta_n}) \xrightarrow{\iota} \Sigma X_{\beta_{n+1}} \xrightarrow{\Sigma l_{n+1}} \Sigma X_{\zeta_{n+1}} \xrightarrow{\Sigma\iota} \Sigma\text{cone}(X_{\beta_{n+2}} \to X_{\zeta_{n+1}}) \tag{4.34}$$

We apply the functor $E(1)_{*-n}(-)$ to the above and we have.

$$d^n \colon C^n_*(X) \to B^{n+1}_*(X) \to Z^{n+1}_*(X) \to C^{n+1}_*(X). \tag{4.35}$$

4.7.2 Reduction to the Case of Quasi-Periodic Disks

In this subsection we will discuss how it suffices to actually prove a much simpler case. The following is based on Franke & Ganter, [18, Remark 7.2.4]. Let $L_*^\bullet \in \mathcal{C}^{2p-2}(\mathcal{B})$ and choose $s \in \mathbb{Z}$. Consider the following map of cochain complexes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & L^s & = & L^s & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \| & & \downarrow{\scriptstyle d^s} & & \downarrow & & \\ \cdots & \xrightarrow[d^{s-2}]{} & L^{s-1} & \xrightarrow[d^{s-1}]{} & L^s & \xrightarrow[d^s]{} & L^{s+1} & \xrightarrow[d^{s+1}]{} & L^{s+2} & \xrightarrow[d^{s+2}]{} & \cdots \end{array} \tag{4.36}$$

where the top cochain complex is still considered as an object of $\mathcal{C}^{2p-2}(\mathcal{B})$ The top cochain complex is the complex $(D^sL^s)^\bullet$, and we denote the above map as

$$f_{L,s} \colon (D^sL^s)^\bullet \to L^\bullet.$$

Under the equivalence of categories $\mathcal{Q} \colon \mathcal{L} \to \mathcal{C}_*^{2p-2}(\mathcal{B})$ there are objcts X' and X in $\mathcal{L}$ and a morphism $F \colon X \to X'$ such that the morphism $f_{L,s}$ is realized as $\mathcal{Q}(F)$. That means there are isomorphisms

$$\mathcal{Q}(X) \cong (D^sL^s)^\bullet, \qquad \mathcal{Q}(X') \cong L^\bullet$$

and the following diagram commutes.

$$\begin{array}{ccc} \mathcal{Q}(X) & \xrightarrow{\mathcal{Q}(F)} & \mathcal{Q}(X') \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ (D^sL^s)^\bullet & \xrightarrow[f_{L,s}]{} & L^\bullet \end{array}$$

Let now $M_*^\bullet$ be another cochain complex and let $t \in \mathbb{Z}$. Similarly to (4.36) we have the morphism

$$\widetilde{f}_{M,t} \colon (D^t M^t)^\bullet \to M^\bullet.$$

Again, under the equivalence $\mathcal{Q}$ there are crowned diagrams Y and Y' and a morphism $G\colon Y \to Y'$ such that

$$\mathcal{Q}(Y) \cong (D^t M^t)^\bullet, \quad \mathcal{Q}(Y') \cong M^\bullet$$

and the following diagram commutes.

$$\begin{array}{ccc} \mathcal{Q}(Y) & \xrightarrow{\mathcal{Q}(G)} & \mathcal{Q}(Y') \\ \cong\downarrow & & \downarrow\cong \\ (D^t M^t)^\bullet & \xrightarrow[f_{M,t}]{} & M^\bullet \end{array}$$

Recall that the tensor product $(D^s L^s)^\bullet \otimes (D^t M^t)^\bullet$ has only three nontrivial entries periodically, in which $n = s + t$.

$$\begin{aligned} \big((D^s L^s)^\bullet \otimes (D^t M^t)^\bullet\big)^n &= L^s \otimes M^t \\ \big((D^s L^s)^\bullet \otimes (D^t M^t)^\bullet\big)^{n+1} &= (L^s \otimes M^t) \oplus (L^s \otimes M^t) \\ \big((D^s L^s)^\bullet \otimes (D^t M^t)^\bullet\big)^{n+2} &= L^s \otimes M^t \end{aligned}$$

Tensoring the morphisms $f_{L,s}$ and $\widetilde{f}_{\widetilde{L},t}$ we have the morphism of cochain complexes

$$f_{L,s} \otimes \widetilde{f}_{M,t} \colon (D^s L^s)^\bullet \otimes (D^t M^t)^\bullet \to L^\bullet \otimes M^\bullet$$

which looks as follows.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & L^s \otimes M^t & \longrightarrow & (L^s \otimes M^t) \oplus (L^s \otimes M^t) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow {\scriptstyle (d^s \otimes \mathrm{id}, \mathrm{id} \otimes \widetilde{d}^t)} & & \\ \cdots & \longrightarrow & \bigoplus_{i+j=n} L^i \otimes M^j & \longrightarrow & \bigoplus_{i+j=n+1} L^i \otimes M^j & \longrightarrow & \cdots \end{array} \tag{4.37}$$

Here the left vertical morphism is the inclusion of the (s,t)th summand, and the right vertical map is the universal map out of the coproduct.

By Propositions 4.5.2 and 4.6.1 the diagrams $i^*\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)$ and $i^*\mathrm{pr}_!(X' \wedge^{\mathbb{L}} Y')$ are in $\mathcal{L}$ and moreover, there are natural isomorphisms.

$$C^n(i^*\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \bigoplus_{i+j=n} (D^s L^s)^i \otimes (D^t \widetilde{L}^t)^j \quad n = \{s+t, s+t+1, s+t+2\} \tag{4.38}$$

$$C^n(i^*\mathrm{pr}_!(X' \wedge^{\mathbb{L}} Y')) \cong \bigoplus_{i+j=n} L^i \otimes \widetilde{L}^j \quad n \in \mathbb{Z}. \tag{4.39}$$

By the isomorphisms (4.38) and (4.39), if we show that the differentials of the complex $C^\bullet_*(i^*\mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y))$ are the same as the differentials on the complex $(D^s L^s)^i \otimes (D^t M^t)$ we will have

$$C^\bullet_*(i^*\mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y)) \cong (D^s L^s)^i \otimes (D^t M^t).$$

If the above holds for every $s, t \in \mathbb{Z}$, by the isomorphism (4.39) and the commutativity of the squares in 4.37 we will also have

$$C^\bullet_*(i^*\mathrm{pr}_!(X' \,\overline{\wedge}^{\mathbb{L}}\, Y')) \cong L^\bullet \otimes M^\bullet$$

and the following diagram commutes:

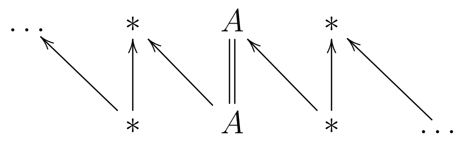
$$\begin{array}{ccc} C^\bullet(i^*\mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y)) & \longrightarrow & C^\bullet(i^*\mathrm{pr}_!(X' \,\overline{\wedge}^{\mathbb{L}}\, Y')) \\ \downarrow & & \downarrow \\ (D^sL^s)^\bullet \otimes (D^tM^t)^\bullet & \longrightarrow & L^\bullet \otimes M^\bullet. \end{array}$$

We note that the above argument holds if we consider $L^\bullet_*, M^\bullet_*$ as objects in $d\mathcal{A}$. This concludes the remark.

Following the above discussion, we move on to prove the case for disks. Recall that the differentials on the disks are the identity morphisms hence the differential for $n = s + t$ on the tensor product is

$$d^n \colon \big((D^sL^s)^* \otimes (D^tM^t)^*\big)^n \to \big((D^sL^s)^* \otimes (D^tM^t)^*\big)^{n+1} \tag{4.40}$$
$$d^n \colon L^s \otimes M^t \to (L^s \otimes M^t) \oplus (L^s \otimes M^t)$$
$$x \otimes y \longmapsto (x \otimes y, (-1)^{|x|} x \otimes y)$$

and similarly for the other nontrivial differential. The complex $(D^sL^s)^*$ is mapped via $\mathcal{Q}^{-1}$ to the following crowned diagram $X \in \mathcal{L}$

$X\colon$  (4.41)

Here the non-trivial entries are at the $(s+1)$-spot, i.e., $X_{\beta_{s+1}} = X_{\zeta_{s+1}} = A$

and $k_s\colon A \to *$. To verify, we calculate

$$C^s(X) = E(1)_{*-s}(\mathrm{cone}(k_s)) = E(1)_{*-s}(\mathrm{cone}(A \to *)) = E(1)_{*-s}(\Sigma A)$$
$$C^{s+1}(X) = E(1)_{*-s-1}(\mathrm{cone}(* \to A)) = E(1)_{*-s-1}(A).$$

Furthermore, from the crowned diagram X above we have

$$d^s = 1\colon C^s_*(X) \to C^{s+1}_*(X). \tag{4.42}$$

A similar claim holds for $(D^tM^t)^*$, which is mapped to a crowned diagram Y in which $Y_{\beta_{t+1}} = Y_{\zeta_{t+1}} = \widetilde{A}$ where the only non-trivial morphism is the identity.

Proposition 4.7.3. *Let X and Y be the objects of $\mathcal{L}$ that correspond to the quasi-periodic disks $(D^sL^s)^\bullet$ and $(D^tM^t)^\bullet$. Then*

$$(C^\bullet_*(i^*\mathbb{L}\mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y)), d) \cong (C^\bullet_*(X) \otimes C^\bullet_*(Y), d_\otimes)$$

where $(C_*(X) \otimes C_*(Y), d_\otimes)$ is the tensor product of cochain complexes.

Proof. For technical reasons we consider Franke's algebraic model as the category of differential graded objects in $\mathcal{A}$, see Definition 3.3.3. By the discussion in Subsection 3.3.5 there is no loss of information since the equivalence of categories

$$F\colon (\mathcal{C}^1(\mathcal{A}), \otimes) \to (d\mathcal{A}, \otimes)$$

is strong symmetric monoidal. By Proposition 4.5.2 and Proposition 4.6.1

we can construct a diagram

$$E = \mathbb{L}\mathrm{pr}_!(X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y) \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N}),$$

such that

$$i^*E \in \mathcal{L} \quad \text{and} \quad \mathrm{cone}(i^*E_{\beta_{n+1}} \to i^*E_{\zeta_n}) \cong \bigvee_{i+j=n} \mathrm{cone}(k_i) \wedge \mathrm{cone}(\widetilde{k}_j).$$

For convenience we will write

$$\widehat{k}_n \colon i^*E_{\beta_{n+1}} \to i^*E_{\zeta_n} \quad \text{and} \quad \widehat{l}_n \colon i^*E_{\beta_n} \to E_{\zeta_n}$$

for the structure maps of the diagram i^*E. Recall from (4.34) that the differential $C^n_*(i^*E) \to C^{n+1}_*(i^*E)$ is obtained by applying $E(1)_{*-n}(-)$ to the composition of maps

$$\mathrm{cone}(\widehat{k}_n) \to \Sigma i^*E_{\beta_{n+1}} \to \Sigma i^*E_{\zeta_{n+1}} \to \Sigma\mathrm{cone}(\widehat{k}_{n+1}). \tag{4.43}$$

This is the same as the composition of maps

$$\mathrm{cone}(\widehat{k}_n) \to \Sigma E_{\gamma_{n+1}} \to \Sigma E_{\zeta_{n+1}} \to \Sigma\mathrm{cone}(\widehat{k}_{n+1}).$$

By Proposition 4.6.1, we have:

$$\mathrm{cone}(\widehat{k}_{s+t}) \cong \mathrm{cone}(k_s) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_t)$$
$$\mathrm{cone}(\widehat{k}_{s+t+1}) \cong \Big(\mathrm{cone}(k_{s+1}) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_t)\Big) \vee \Big(\mathrm{cone}(k_s) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_{t+1})\Big),$$

and the rest are trivial maps due to the structure of the crowned diagrams X and Y. Directly from the structure morphisms of the crowned diagrams X and Y we have

$$\begin{aligned} \mathrm{cone}(\widehat{k}_{s+t}) &\cong (\Sigma A) \wedge (\Sigma \widetilde{A}) \\ \mathrm{cone}(\widehat{k}_{s+t+1}) &\cong \left(A \wedge \Sigma \widetilde{A}\right) \vee \left(\Sigma A \wedge \widetilde{A}\right). \end{aligned}$$

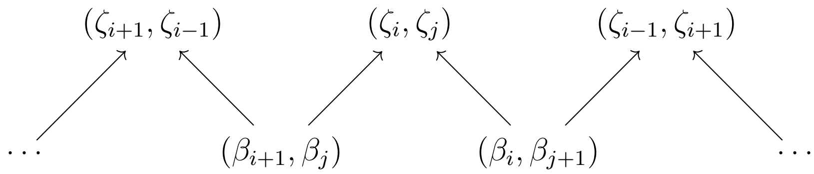
So, it remains to compute the values $E_{\gamma_{s+t+1}}, E_{\zeta_{s+t}}, E_{\zeta_{s+t+1}}$ and the map $E_{\gamma_{s+t+1}} \to E_{\zeta_{s+t+1}}$. The map

$$\mathrm{cone}(E_{\gamma_{s+t+1}} \to E_{\zeta_{s+t}}) = \mathrm{cone}(\widehat{k}_{s+t}) \to \Sigma E_{\gamma_{s+t+1}}$$

is the canonical map $\mathrm{cone}(\widehat{k}_{s+t}) \to S^1 \wedge^{\mathbb{L}} E_{\gamma_{s+t+1}}$ see Definition 2.2.31. Similarly, the map

$$\Sigma E_{\zeta_{s+t+1}} \to \Sigma \mathrm{cone}(\widehat{k}_{s+t+1})$$

is the suspension of the canonical map $E_{\zeta_{s+t+1}} \to \mathrm{cone}(\widehat{k}_{s+t+1}) = \mathrm{cone}(E_{\zeta_{s+t+2}} \to E_{\zeta_{s+t+1}})$, see again Definition 2.2.31. To compute the above, let us recall from Example 4.4.6 the poset J_n with inclusion $\theta_n \colon J_n \hookrightarrow \mathrm{pr}/\zeta_n$, which has a left adjoint $L \colon \mathrm{pr}/\zeta_n \to J_n$, and which for $i + j \equiv n$ modulo $2p - 2$ looks as follows.



Also, recall from Example 4.4.4 the poset pr/γ_n, which for $i+j \equiv n$ looks as follows.

$$\begin{array}{ccccccccccccccccc}
 & & (\beta_{i+1}, \zeta_{i-1}) & & & & (\zeta_i, \beta_j) & & & & (\beta_i, \zeta_j) & & & & (\zeta_{i-1}, \beta_{j+1}) & & \\
 & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
\cdots & & & & (\beta_{i+1}, \beta_j) & & & & (\beta_i, \beta_j) & & & & (\beta_i, \beta_{j+1}) & & & & \cdots
\end{array}$$

So, we have

$$\begin{aligned}
E_{\zeta_n} &= \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} X \,\overline{\wedge}^{\mathbb{L}}\, Y \cong \operatorname*{hocolim}_{J_n} \theta_n^*(X \,\overline{\wedge}^{\mathbb{L}}\, Y), \\
E_{\gamma_n} &= \operatorname*{hocolim}_{\mathrm{pr}/\gamma_n} (X \,\overline{\wedge}^{\mathbb{L}}\, Y).
\end{aligned}$$

Furthermore, the maps

$$E_{\gamma_{n+1}} \to E_{\zeta_n} \quad \text{and} \quad E_{\gamma_{n+1}} \to E_{\zeta_{n+1}}$$

are the maps of homotopy colimits induced by the map of posets

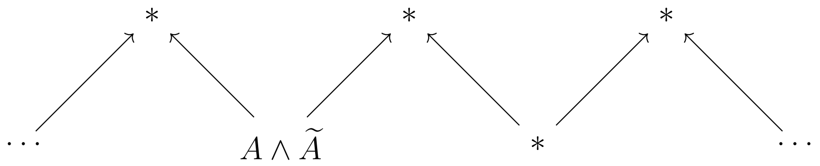
$$\psi\colon \mathrm{pr}/\gamma_{n+1} \to \mathrm{pr}/\zeta_n \quad \text{and} \quad \phi\colon \mathrm{pr}/\gamma_{n+1} \to \mathrm{pr}/\zeta_{n+1}$$

respectively. For the following we set $q = s+t$. The diagram

$$\theta_q^*(X \,\overline{\wedge}^{\mathbb{L}}\, Y)\colon J_q \to L_1\mathrm{Sp}$$

consists only of $*$ hence $E_{\zeta_q} = E_{\zeta_{s+t}} \cong *$. Continuing with calculating $E_{\zeta_{q+1}} = E_{\zeta_{s+t+1}}$, we have the poset J_{q+1}. The diagram $\theta_{q+1}^*(X \,\overline{\wedge}^{L}\, Y) \in \mathrm{Ho}(L_1\mathrm{Sp}^{J_{q+1}})$

looks as follows.

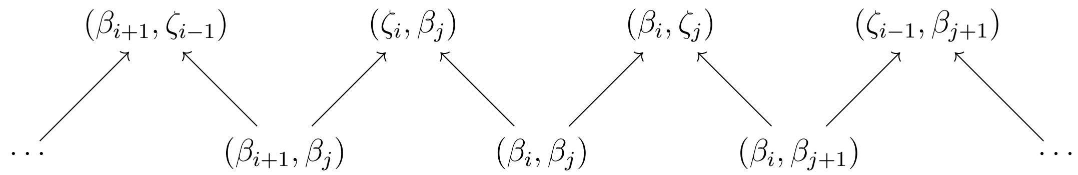


(4.44)

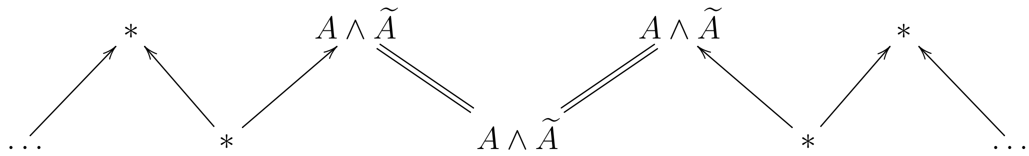
Here, there only non-trivial entry is at $(\beta_{s+1}, \beta_{t+1})$. From the diagram above we get

$$E_{\zeta_{s+t+1}} = E_{\zeta_{q+1}} = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_{q+1}} (X \wedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{J_{q+1}} \theta^*_{q+1}(X \wedge^{\mathbb{L}} Y) \cong \Sigma A \wedge \widetilde{A}.$$

We continue to do the same (briefly) for $E_{\gamma_q}, E_{\gamma_{q+1}}$ and $E_{\gamma_{q+2}}$. Recall the poset pr/γ_n from Example 4.4.4, which for $i + j \equiv n$ looks as follows.



The value $E_{\gamma_{q+2}}$ is the homotopy colimit of the diagram $X \overline{\wedge}^{\mathbb{L}} Y \in \mathrm{Ho}(L_1 \mathrm{Sp}^{\mathrm{pr}/\gamma_{q+2}})$ which is



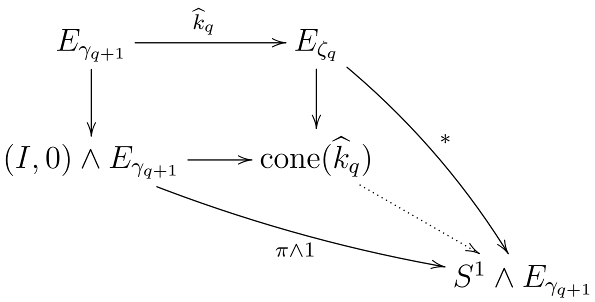
(4.45)

The non-trivial entries are at the places $(\beta_{s+1}, \beta_{t+1})$ (the bottom), $(\zeta_{s+1}, \beta_{t+1})$ (on the left) and $(\beta_{s+1}, \zeta_{t+1})$ (on the right) and the non-trivial morphisms are identities as shown. Hence $E_{\gamma_{n+2}} \cong A \wedge \widetilde{A}$. Similarly, we have

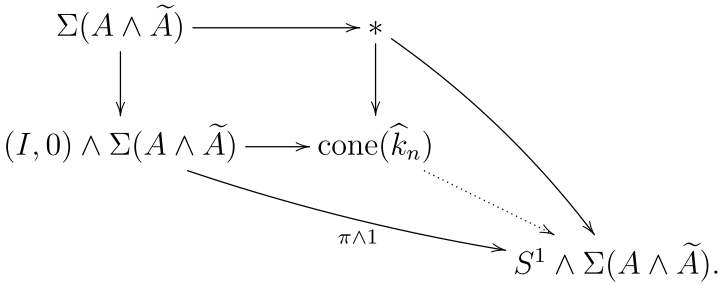
$$E_{\gamma_q} = \underset{\mathrm{pr}/\gamma_q}{\mathrm{hocolim}}(X \mathbin{\overline{\wedge}}^{\mathbb{L}} Y) \cong *$$

$$E_{\gamma_{q+1}} = \underset{\mathrm{pr}/\gamma_{q+1}}{\mathrm{hocolim}}(X \mathbin{\overline{\wedge}}^{\mathbb{L}} Y) \cong \Sigma A \wedge \widetilde{A}.$$

Let us move on to calculate the map $\mathrm{cone}(\widehat{k}_q) \to S^1 \wedge E_{\gamma_{q+1}}$. From Definition 2.2.31 we have the pushout square



which based on our computations above is the following



Recall from Proposition 4.6.3, (4.33), and Corollary 2.3.22 there is a series of canonical isomorphisms

$$\mathrm{cone}(\widehat{k}_q) = \mathrm{cone}(\widehat{k}_{s+t}) \cong \mathrm{cone}(k_s \,\square^{\mathbb{L}}\, \widetilde{k}_t) \cong \mathrm{cone}(k_s) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_t).$$

In our particular case for diagrams X and Y such that $k_s\colon A \to A$ and $\widetilde{k}\colon \widetilde{A} \to *$ this is

$$\operatorname{cone}(\widehat{k}_q) \cong \operatorname{cone}(k_s \,\square^{\mathbb{L}}\, k_t) \cong \Sigma^2 A \wedge \widetilde{A} \cong \Sigma A \wedge \Sigma\widetilde{A} \cong \operatorname{cone}(k_s) \wedge^{\mathbb{L}} \operatorname{cone}(\widetilde{k}_t).$$

This implies that the universal map out of the pushout (the dotted map) is the identity map. Thus, the map

$$\operatorname{cone}(\widehat{k}_q) \to \Sigma E_{\gamma_{q+1}}$$

is the map $\Sigma A \wedge \Sigma\widetilde{A} \to \Sigma^2(A \wedge \widetilde{A})$ which is the composition of the canonical map described above and the identity map. From the posets above we can see directly that the map

$$\widehat{l}_{q+1}\colon E_{\gamma_{q+1}} \to E_{\zeta_{q+1}}$$

is the identity map induced by

$$\psi\colon\ \mathrm{pr}\,/\gamma_{q+1} \to \mathrm{pr}/\zeta_{q+1}.$$

Therefore the map

$$\Sigma\widehat{l}_{q+1}\colon E_{\gamma_{q+1}} \to \Sigma E_{\zeta_{q+1}}$$

is the identity map

$$1\colon \Sigma^2(A \wedge \widetilde{A}) \to \Sigma^2(A \wedge \widetilde{A}).$$

Lastly it remains to figure out the map

$$E_{\zeta_{q+1}} \to \operatorname{cone}(\widehat{k}_{q+1}) = \operatorname{cone}(E_{\gamma_{q+2}} \to E_{\zeta_{q+1}}).$$

Recall from the proof of Proposition 4.6.1 that $\operatorname{cone}(\widehat{k}_{q+1})$ can be written as a homotopy colimit.

$$\operatorname{cone}(\widehat{k}_{q+1}) \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_{q+1}} \left(\operatorname{cone}(\varepsilon_{X \bar{\wedge}^{\mathbb{L}} Y})\right),$$

where $\phi\colon \mathrm{pr}/\gamma_{q+2} \to \mathrm{pr}/\zeta_{q+1}$ and where ε is the counit of the derived adjunction $(\mathbb{L}\phi_!, \phi^*)$. Pulling back the diagram $\operatorname{cone}(\varepsilon_{X \bar{\wedge}^{\mathbb{L}} Y})$ to J_{q+1} with the inclusion $\theta_{q+1}\colon J_{q+1} \to \mathrm{pr}/\zeta_{q+1}$ we get the diagram

$$\begin{array}{ccccccccccccc}
 & & A\wedge\Sigma\widetilde{A} & & & & \Sigma A\wedge\widetilde{A} & & & & * & & \\
 & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
\cdots & & & & * & & & & * & & & & \cdots
\end{array}$$

with non-trivial entries at (ζ_{s+1}, ζ_t) and (ζ_s, ζ_{t+1}) respectively. Recall the diagram $\theta^*_{q+1}(X \bar{\wedge}^{\mathbb{L}} Y)\colon J_{q+1} \to L_1\mathrm{Sp}$ from (4.44)

$$\begin{array}{ccccccccccccccccc}
 & & * & & & & * & & & & * & & & & * & & \\
 & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
\cdots & & & & A\wedge\widetilde{A} & & & & * & & & & * & & & & \cdots
\end{array}$$

The only non-trivial entry is at $(\beta_{s+1}, \beta_{t+1})$, left top being (ζ_{s+1}, ζ_t) and right top being (ζ_s, ζ_{t+1}). Because of the shape of the underlying posets and the map, we can safely ignore the trivial entries, so the map

$$E_{\gamma_{q+1}} \to \mathrm{cone}(\widehat{k}_{q+1})$$

can be taken as the map of homotopy pushouts

$$\mathrm{hocolim}(* \leftarrow A \wedge \widetilde{A} \to *) \to \mathrm{hocolim}(A \wedge \Sigma\widetilde{A} \leftarrow * \to \Sigma A \wedge \widetilde{A})$$

induced by the following map of pre-pushout posets.

$$\begin{array}{ccccc} * & \longleftarrow & A \wedge \widetilde{A} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ A \wedge \Sigma\widetilde{A} & \longleftarrow & * & \longrightarrow & \Sigma A \wedge \widetilde{A}. \end{array}$$

Now consider the above map of diagrams and the following map at the bottom.

$$\begin{array}{ccccc} * & \longleftarrow & A \wedge \widetilde{A} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ A \wedge \Sigma\widetilde{A} & \longleftarrow & * & \longrightarrow & \Sigma A \wedge \widetilde{A} \\ {\scriptstyle\tau}\downarrow & & \downarrow & & \| \\ \Sigma A \wedge \widetilde{A} & \longleftarrow & * & \longrightarrow & \Sigma A \wedge \widetilde{A} \end{array}$$

a Here τ is the map

$$A \wedge \Sigma\widetilde{A} = A \wedge (S^1 \wedge \widetilde{A}) \cong (A \wedge S^1) \wedge \widetilde{A} \xrightarrow{\tau} (S^1 \wedge A) \wedge \widetilde{A} \cong \Sigma A \wedge \widetilde{A}$$

(The first map is the associativity isomorphism.) By Lemma 4.7.5, the induced map of homotopy colimits is up to weak equivalence the diagonal map

$$\mathrm{diag}\colon \Sigma A \wedge \widetilde{A} \to (\Sigma A \wedge \widetilde{A}) \vee (\Sigma A \wedge \widetilde{A}).$$

Hence, the map (4.43) is up to weak equivalence the diagonal map but with a sign introduced by the twist map as above. This implies that that indeed the differential

$$d^{s+t}\colon C^{s+t}(i^*E) \to C^{s+t+1}(i^*E)$$

coincides with the differential of the tensor product of

$$\Big((D^sL^s) \otimes (D^t\widetilde{L}^t)\Big)^{s+t} \to \Big((D^sL^s) \otimes (D^t\widetilde{L}^t)\Big)^{s+t+1}.$$

We do not need to consider the other differential, namely to check that the differential

$$d^{s+t+1}\colon C^{s+t+1}(i^*E) \to C^{s+t+2}(i^*E)$$

coincides with the differential

$$\Big((D^sL^s) \otimes (D^t\widetilde{L}^t)\Big)^{s+t+1} \to \Big((D^sL^s) \otimes (D^t\widetilde{L}^t)\Big)^{s+t+2}$$

since by construction $(C^\bullet_*(i^*E), d)$ is a cochain complex and that means that by necessity $d^{s+t+1} \circ d^{s+t} = 0$. This concludes the proof. □

4.7.4 Technical Lemmas

In this subsection we prove two technical lemmas that are used in the the proofs. We put them here so they do not disrupt the flow of the proofs. The first lemma shows that the canonical map from a suspension of a spectrum to the wedge product of suspensions is, up to natural isomorphism, the diagonal map. The second lemma shows that the pushout-product of injective morphism is injective in the abelian category of $E(1)_*$-modules.

Lemma 4.7.5. *Let X be a spectrum and consider the following map of homotopy pushouts.*

$$\mathrm{hocolim}(* \leftarrow X \rightarrow *) \rightarrow \mathrm{hocolim}(\Sigma X \leftarrow * \rightarrow \Sigma X)$$

Then the above map is, up to isomorphism in $\mathrm{Ho}(\mathrm{Sp})$ *the diagonal map*

$$\mathrm{diag}\colon \Sigma X \rightarrow \Sigma X \vee \Sigma X$$

Proof. Let $CX = (I, 0) \wedge X$ be the cone of X and let $i\colon X \rightarrow CX$ be the canonical inclusion (this is an h-cofibration). We choose a model for ΣX as the homotopy pushout

$$\Sigma X \cong \mathrm{hocolim}(CX \leftarrow X \rightarrow CX).$$

In fact, this we can take this to be the ordinary pushout $\mathrm{colim}(CX \leftarrow X \rightarrow CX)$ since $i\colon X \rightarrow CX$ is an h-cofibration and the gluing lemma. From the model that we chose for the homotopy pushout, that is, the following pushout

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

the two maps $CX \to \Sigma X$ are the inclusions of the top and bottom "hemi-spheres''. From this model we get directly that the induced map on pushouts

$$\begin{array}{ccccc} CX & \xleftarrow{i} & X & \xrightarrow{i} & CX \\ {\scriptstyle \pi\wedge 1}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi\wedge 1} \\ \Sigma X & \longleftarrow & * & \longrightarrow & \Sigma X \end{array}$$

where $\pi\colon I \to S^1$ is the projection is indeed the diagonal map $\mathrm{diag}\colon \Sigma X \to \Sigma X \vee \Sigma X$. Hence, the induced map of homotopy pushouts is the diagonal map up to natural isomorphism. □

Lemma 4.7.6. *Let X, Y, U, V be projective $E(1)_*$-modules and let $f\colon X \to Y$ and $g\colon U \to V$ be injective $E(1)_*$-module maps. Then the pushout-product map $f \,\square\, g$ is injective.*

Proof. Since $g\colon U \to V$ is a monomorphism we have the short exact sequence

$$0 \to U \xrightarrow{g} V \xrightarrow{j} \operatorname{coker} g \to 0.$$

Notice that the dimension of the abelian category $E(1)_*$-modules is 1, which implies that $\operatorname{coker} g$ is a projective module since it is a submodule of V. Since X is flat, $X \otimes -$ is an exact functor which means the sequence

$$0 \to X \otimes U \xrightarrow{1\otimes g} X \otimes V \xrightarrow{1\otimes j} X \otimes \operatorname{coker} g \to 0$$

is short exact. Consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X \otimes U & \xrightarrow{1 \otimes g} & X \otimes V & \xrightarrow{1 \otimes j} & X \otimes \operatorname{coker} g & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f \otimes 1} & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & Y \otimes U & \longrightarrow & P & \longrightarrow & X \otimes \operatorname{coker} g & \longrightarrow & 0 \\
 & & \Big\| & & {\scriptstyle f \square g}\Big\downarrow & & \Big\downarrow{\scriptstyle f \otimes 1} & & \\
0 & \longrightarrow & Y \otimes U & \xrightarrow[1 \otimes g]{} & Y \otimes V & \xrightarrow[1 \otimes j]{} & Y \otimes \operatorname{coker} g & \longrightarrow & 0
\end{array}
$$

where P is the pushout of $1 \otimes g$ and $f \otimes 1$. Since the top left square is cocartesian, the canonical map

$$\operatorname{coker}(1 \otimes g) \to \operatorname{coker}(Y \otimes U \to P)$$

is an isomorphism so the middle row is also exact. Notice now the morphism

$$f \otimes 1 \colon X \otimes \operatorname{coker} g \to Y \otimes \operatorname{coker} g,$$

which is the right bottom vertical arrow, is injective since $\operatorname{coker} g$ is projective. Now applying the snake lemma gives us that $f \square g$ is a monomorphism.

$\square$

Chapter 5

Homotopy Colimit Calculations

5.1 Introduction to the Chapter

In this short chapter we will deal with the right half of the diagram (1.3). The goal of this chapter is to prove the following theorem.

Theorem 5.1.1. *For any pair of diagrams* $(X, Y) \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N}) \times \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N})$, *the homotopy colimit of the* $\mathcal{C}_N$*-diagram*

$$i^*\mathbb{L}\mathrm{pr}_!(X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y) \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N})$$

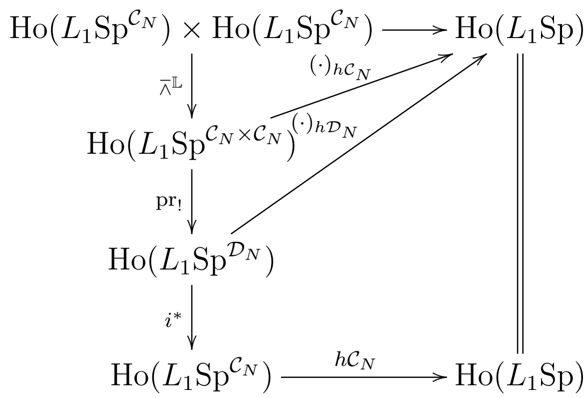
is naturally isomorphic to the smash product of the homotopy colimits of X *and* Y*, that is,*

$$\underset{\mathcal{C}_N}{\mathrm{hocolim}} \left(i^*\mathbb{L}\mathrm{pr}_!(X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y)\right) \cong \underset{\mathcal{C}_N}{\mathrm{hocolim}}\, X \wedge^{\mathbb{L}} \underset{\mathcal{C}_N}{\mathrm{hocolim}}\, Y.$$

We will prove the above theorem by breaking it apart into smaller easier to prove lemmas that we explain in the section below.

5.2 Calculations

Recall the diagram (1.3). In the previous chapter we showed that the left half side of the diagram commutes up to natural isomorphism. In this chapter we will deal with the right half side of the diagram and show that it commutes up to natural isomorphism.



Let us briefly recall all the functors involved.

(i) The top horizontal functor is the smash product of homotopy colimits of crowned diagrams, that is,

$$\operatorname*{hocolim}_{\mathcal{C}_N} X \wedge^{\mathbb{L}} \operatorname*{hocolim}_{\mathcal{C}_N} Y,$$

(ii) The first vertical functor is the derived functor of the objectwise smash products of crowned diagrams

$$- \overline{\wedge}^{\mathbb{L}} -\colon \operatorname{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N}) \times \operatorname{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N}) \to \operatorname{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N})$$

see (4.7),

(iii) The second vertical functor is the homotopy left Kan extension along $\mathrm{pr}\colon \mathcal{C}_N \times \mathcal{C}_N \to \mathcal{D}_N$,

$$\mathbb{L}\mathrm{pr}_!(- \overline{\wedge}^{\mathbb{L}} -)\colon \operatorname{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N}) \to \operatorname{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N}),$$

(iv) The third vertical functor is the pullback functor $i^*\colon \operatorname{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N}) \to \operatorname{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N})$, induced by the functor $i\colon \mathcal{C}_N \to \mathcal{D}_N$.

(v) The first diagonal functor is the homotopy colimit functor

$$\operatorname*{hocolim}_{\mathcal{C}_N \times \mathcal{C}_N}\colon \operatorname{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N}) \to \operatorname{Ho}(L_1\mathrm{Sp}).$$

(vi) The second diagonal functor is also the homotopy colimit functor

$$\operatorname*{hocolim}_{\mathcal{D}_N}\colon \operatorname{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N}) \to \operatorname{Ho}(L_1\mathrm{Sp}).$$

(vii) The bottom horizontal functor is again the homotopy colimit functor

$$\operatorname*{hocolim}_{\mathcal{C}_N}\colon \operatorname{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N}) \to \operatorname{Ho}(L_1\mathrm{Sp}).$$

(viii) finally, the right vertical functor is just the identity functor.

By Corollary 4.5.3 given crowned diagrams $X, Y \in \mathcal{L}$ satisfying certain hypotheses, we have that $i^*\mathbb{L}\mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y) \in \mathcal{L}$. So we can rewrite the above diagram as follows.

$$\begin{tikzcd}
\mathcal{L} \times \mathcal{L} \arrow[r] \arrow[d, "\overline{\wedge}^{\mathbb{L}}"'] & \mathrm{Ho}(L_1\mathrm{Sp}) \arrow[ddd, equal] \\
\mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N}) \arrow[ur] \arrow[d, "\mathbb{L}\mathrm{pr}_!"'] & \\
\mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N}) \arrow[uur] \arrow[d, "i^*"'] & \\
\mathcal{L} \arrow[r] & \mathrm{Ho}(L_1\mathrm{Sp})
\end{tikzcd} \tag{5.1}$$

Lemma 5.2.1. *The top triangle commutes, that is,*

$$\operatorname*{hocolim}_{\mathcal{C}_N} X \,\overline{\wedge}^{\mathbb{L}} \operatorname*{hocolim}_{\mathcal{C}_N} Y \cong \operatorname*{hocolim}_{\mathcal{C}_N \times \mathcal{C}_N} (X \,\overline{\wedge}^{\mathbb{L}}\, Y).$$

Proof. This follows from Corollary 2.3.26 as a direct application for $\mathcal{C} = \mathcal{D} = \mathcal{C}_N$ □

Next, we deal with the middle triangle.

Lemma 5.2.2. *The middle triangle commutes, that is,*

$$\operatorname*{hocolim}_{\mathcal{C}_N \times \mathcal{C}_N} (X \,\overline{\wedge}^{\mathbb{L}}\, Y) \cong \operatorname*{hocolim}_{\mathcal{D}_N} \mathbb{L}\,\mathrm{pr}_!(X \,\overline{\wedge}^{\mathbb{L}}\, Y).$$

Proof. This follows directly Corollary 2.5.8 as a direct application for the map of posets $\mathrm{pr}\colon \mathcal{C}_N \times \mathcal{C}_N \to \mathcal{D}_N$. □

The last lemma that we need is the following.

Lemma 5.2.3. *The bottom triangle commutes, that is,*

$$\operatorname*{hocolim}_{\mathcal{C}_N} i^*E \cong \operatorname*{hocolim}_{\mathcal{D}_N} E.$$

We will prove the above lemma by proving that the functor $i\colon \mathcal{C}_N \to \mathcal{D}_N$ is homotopy final, see Definition 2.4.24. By Remark 2.4.26, given a diagram $E \in \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N})$, to check that the canonical morphism

$$\phi_i\colon \operatorname*{hocolim}_{\mathcal{C}_N} i^*E \to \operatorname*{hocolim}_{\mathcal{D}_N} E$$

is an isomorphism it suffices to check that for any $n \in \mathbb{Z}/(2p-2)\mathbb{Z}$ and any $\alpha \in \{\zeta, \gamma, \beta\}$ the slice categories

$$\alpha_n/i = \{\alpha'_n \in \mathcal{C}_N \colon i(\alpha'_n) \geq \alpha_n\}$$

of the functor $i\colon \mathcal{C}_N \to \mathcal{D}_N$ are contractible. Recall from Remark 2.4.27 Quillen's criterion.

Proposition 5.2.4. *The functor $i\colon \mathcal{C}_N \to \mathcal{D}_N$ is homotopy final.*

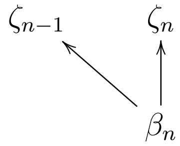
Proof. We will prove the above proposition by applying Quillen's criterion of conical contractible posets [36, Section 1.5]. First, we identity the slice categories $\zeta_n/i, \gamma_n/i$ and β_n/i and then we will check that they are indeed conically contractible. We start first with ζ_n/i. By definition

$$\zeta_n/i = \{\alpha_n \in \mathcal{C}_N \colon i(\alpha_n) \geq \zeta_n\} = \{\zeta_n\}$$

Since this poset contains only one element it is is obviously contractible. The next slice categories are of the form γ_n/i. By definition,

$$\gamma_n/i = \{\alpha_n \in \mathcal{C}_N \colon i(\alpha_n) \geq \gamma_n\}\,,$$

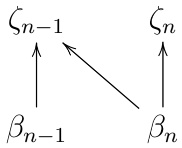
which is the poset



We choose β_n and $\mathrm{Id}\colon \gamma_n/i \to \gamma_n/i$. Directly from above we can see that γ_n/i is conically contractible. The last case are the slices β_n/i. By definition,

$$\beta_n/i = \{\alpha_n \in \mathcal{C}_N \colon i(\alpha_n) \geq \beta_n\}\,,$$

which is the poset



We choose β_n and the map of $\beta_n/i \to \beta_n/i$ as follows, $\zeta_{n-1} \mapsto \zeta_{n-1}, \zeta_n \mapsto \zeta_n, \beta_{n-1} \mapsto \zeta_{n-1}, \beta_n \mapsto \beta_n$. With these choices, we can see that the poset β_n/i is conically contractible. $\square$

Finally we end with a short proof of Theorem 5.1.1. By combining Lemma 5.2.1, Lemma 5.2.2 and Lemma 5.2.3, we get that all the smaller triangles in (5.1) commute. Combined, we have that the whole diagram (5.1), commutes and the proof is concluded.

Chapter 6

Franke's Realization functor $\mathcal{R}$ is a $\otimes$-functor

In this last chapter we will at last prove our main theorem. We will combine the results of Chapters 4 and 5.

6.1 Proof of the Main Theorem

Before we move to the proof we recall the following diagram which summarized our overall strategy.

$$
\begin{array}{ccccc}
\mathcal{D}^1(\mathcal{A}) \times \mathcal{D}^1(\mathcal{A}) & \rightleftarrows & \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N}) \times \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N}) & \longrightarrow & \mathrm{Ho}(L_1\mathrm{Sp}) \\
 & & \big\downarrow \cong & \nearrow_{\mathrm{hocolim}} & \\
 & & \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N \times \mathcal{C}_N}) & & \\
\big\downarrow -\otimes^{\mathbb{L}}- & & \big\downarrow \mathbb{L}\,\mathrm{pr}_! = \mathrm{HoLan}_{\mathrm{pr}} & \nearrow_{\mathrm{hocolim}_{\mathcal{D}_N}} & \big\downarrow = \\
 & & \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{D}_N}) & & \\
 & & \big\downarrow \mathbb{L}i^* & & \\
\mathcal{D}^1(\mathcal{A}) & \underset{\mathcal{Q}}{\overset{\mathcal{Q}^{-1}}{\rightleftarrows}} & \mathrm{Ho}(L_1\mathrm{Sp}^{\mathcal{C}_N}) & \xrightarrow[\mathrm{hocolim}_{\mathcal{C}_N}]{} & \mathrm{Ho}(L_1\mathrm{Sp}).
\end{array}
\tag{6.1}
$$

Recall from Section 3.4 that we consider Franke's algebraic model to be the model category $\mathcal{C}^1(\mathcal{A})_{\mathrm{q.proj}}$. This model category together with the tensor product $(\mathcal{C}^1(\mathcal{A})_{\mathrm{q.proj}}, \otimes)$ is a monoidal model category. Also, recall that $\mathcal{R} = \mathrm{hocolim}_{\mathcal{C}_N} \circ \mathcal{Q}^{-1}$.

Theorem 6.1.1. *The realization functor* $\mathcal{R}\colon (\mathcal{D}^1(\mathcal{A}), \otimes^{\mathbb{L}}) \to (\mathrm{Ho}(L_1\mathrm{Sp}), \wedge^{\mathbb{L}})$ *is a* $\otimes$*-functor, i.e., it preserves the monoidal products up to a natural isomorphism*

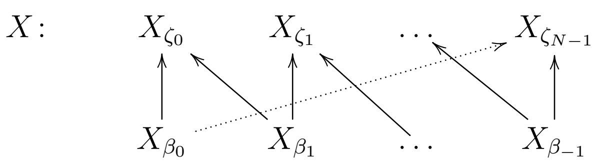
$$\mathcal{R}(C_*^\bullet \otimes D_*^\bullet) \cong \mathcal{R}(C_*^\bullet) \wedge^{\mathbb{L}} \mathcal{R}(D_*^\bullet).$$

Proof. Let $M_*^\bullet$ and $N_*^\bullet$ be objects in $\mathcal{D}^1(\mathcal{A}) = \mathrm{Ho}(\mathcal{C}^1(\mathcal{A})_{\mathrm{q.proj}})$. By our Convention 2.2.2, both objects are cofibrant. Since $M_*^\bullet$ is cofibrant, the functor

$$M_*^\bullet \otimes -\colon \mathcal{C}^1(\mathcal{A})_{\mathrm{q.proj}} \to \mathcal{C}^1(\mathcal{A})_{\mathrm{q.proj}}$$

is a left Quillen functor, see [24, Remark 4.2.3], which means it preserves cofibrant objects. Since both objects are cofibrant, the tensor product $M_*^\bullet \otimes N_*^\bullet$ represents the derived tensor product in $(\mathcal{D}^1(\mathcal{A}), \otimes^{\mathbb{L}})$ and in particular it also cofibrant. Recall from Definition 3.4.6, that the cofibrant objects in $\mathcal{C}^1(\mathcal{A})_{\text{q.proj}}$ are the same as those in $\mathcal{C}^1(\mathcal{A})_{\text{rel.proj}}$, see Lemma 3.4.5. This means that $M_*^\bullet, N_*^\bullet$ and $M_*^\bullet \otimes N_*^\bullet$ are in particular degreewise dualizable $E(1)_*E(1)$-comodules and hence by Lemma 3.2.2 they are degreewise projective as $E(1)_*$-modules.

We recall some notation from Section 4.2. Given a crowned diagram $X \in L_1\mathrm{Sp}^{\mathcal{C}_N}$



we set

$$Z_*^n(X) = E(1)_{*-n}(X_{\zeta_n}),\ B_*^n(X) = E(1)_{*-n}(X_{\beta_n}),\ C_*^n(X) = E(1)_{*-n}(\mathrm{cone}(X_{\beta_{n+1}} \to X_{\zeta_n})).$$

Next we recall Franke's functor $\mathcal{Q}$ (4.2)

$$\mathcal{Q}\colon \mathcal{L} \to \mathcal{C}^{2p-2}(\mathcal{B}) \cong \mathcal{C}^1(\mathcal{A})$$

that constructs an object $C_*^\bullet(X) \in \mathcal{C}^{2p-2}(\mathcal{B}) \cong \mathcal{C}^1(\mathcal{A})$, i.e.,

$$\dots \xrightarrow{d^{-1}} C_*^0 \xrightarrow{d^0} C_*^1 \xrightarrow{d^1} C_*^2 \xrightarrow{d^2} \dots.$$

We use the notation $Z^n_*(X) = \ker d^n$ and $B^n_* = \operatorname{im} d^n$.

By the discussion above, for the twisted complex $M^\bullet_*$ for any $n \in \mathbb{Z}$ the underlying $E(1)_*$-module M^n_* is projective. Notice that the global homological dimension of the abelian category of $\mathbb{Z}_{(p)}[v_1, v_1^{-1}]$-modules is equal to 1, i.e.,

$$\dim \mathbb{Z}_{(p)}[v_1, v_1^{-1}] = 1,$$

which implies that any submodule of a projective $\mathbb{Z}_{(p)}[v_1, v_1^{-1}]$-module is itself projective. Hence, any submodule of M^n_* is a projective $E(1)_*$-module. In particular, the submodules $\ker d^n$ and $\operatorname{im} d^{n-1}$ of M^n_* are projective $E(1)_*$-modules. Under the equivalence of the abelian categories $\mathcal{C}^1(\mathcal{A}) \cong \mathcal{C}^{2p-2}(\mathcal{B})$, the kernels and images of the differentials are also projective for $M^\bullet_* \in \mathcal{C}^{2p-2}(\mathcal{B})$. In similar fashion, the kernels and images of the differentials of the complex $N^\bullet_*$ are all projective $E(1)_*$-modules. Let now $\mathcal{Q}^{-1}$ be an inverse to Franke's $\mathcal{Q}$ functor and we let

$$X \cong \mathcal{Q}^{-1}(M^\bullet_*), \quad Y \cong \mathcal{Q}^{-1}(N^\bullet_*).$$

By our discussion above, it follows that for the crowned diagrams X and Y, for any $n \in \mathbb{Z}/(2p-2)\mathbb{Z}$ and $\alpha \in \{\beta, \zeta\}$, the underlying $E(1)$-modules of the $E(1)_*E(1)$-comodules,

$$E(1)_*(X_{\alpha_n}) \quad \text{and} \quad E(1)_*(Y_{\alpha_n})$$

are projective. Now, by Theorem 4.3.1

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y) = M_*^\bullet \otimes N_*^\bullet$$

that is, the left hand side of the diagram (6.1) commutes. By Theorem 5.1.1

$$\operatorname*{hocolim}_{\mathcal{C}_N} \left(i^*\mathbb{L}\mathrm{pr}_!(X \mathbin{\overline{\wedge}^{\mathbb{L}}} Y)\right) \cong \operatorname*{hocolim}_{\mathcal{C}_N} X \wedge^{\mathbb{L}} \operatorname*{hocolim}_{\mathcal{C}_N} Y.$$

that is, the right hand side of the diagram (6.1) commutes. Finally, Franke's realization functor (4.3) is defined by

$$\mathcal{R} = \operatorname*{hocolim}_{\mathcal{C}_N} \circ \mathcal{Q}^{-1}$$

and the proof is concluded.

□

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