Geometrical Bodies as Material Forms
Luca Pacioli's *Summa*, *Divina proportione, Euclid's Elements,* and the architecture of the late fifteenth-early sixteenth century

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Contents

Notes

All translations from Pacioli's texts are my own. Other translations from Greek, Latin, and Italian are as indicated in the footnotes and bibliography. Except when indicating scribal abbreviations, parentheses are retained when found in the original; square brackets enclose additional clarifications; square brackets with italicised text enclose text as in the original. Transcriptions from the texts of Pacioli in footnotes follow conservative criteria; I distinguished u from v in the single grapheme u/v. I have retained the form ç.

Abstract

This thesis examines Luca Pacioli's arguments on geometrical bodies and their application to the architecture of Donato Bramante, in their philosophical and theoretical contexts. Pacioli's works stand at the intersection of the theoretical and practical knowledge of the late fifteenth and early sixteenth centuries. Pacioli was a lecturer in mathematics, was exceptionally well-placed in some of the most learned courtly environments and was very receptive to philosophical and mathematical trends. He was also very able when these had to be summarised or developed in his works. These are therefore extraordinary documents with which to trace the interrelation of philosophical, mathematical, and architectural arguments and consider the work of the practitioners with whom Pacioli interacted.

There are two main ways to approach geometrical bodies and their application to architecture: either for their conceptual and symbolic meanings or for their formal and visual properties. Whilst the first has been the focus of much attention in architectural scholarship, the second has been almost completely neglected. However, as this study shows, Pacioli was primarily motivated by the distinctive formal and visual properties of regular and other geometrical bodies derived from these. This thesis considers how and why geometrical bodies were regarded as exemplary forms in Pacioli's works and by which means these were applied to architecture, and compares Pacioli's position with those of other architectural theorists. To accomplish this, it relies on a hierarchical understanding of human visuality coherent with the latest researches in the psychology and philosophy of perception.

This thesis sheds light on Pacioli's unprecedented presentation of a set of material models of geometrical bodies and their perspectival representations in drawings drafted by Leonardo da

Vinci, and argues that the theory and practice of perspective were central to Pacioli's arguments that related geometrical bodies with architecture. The growing interest in methods of mathematical quantification which Pacioli discussed, and the development of perspectival techniques of representation which Bramante mastered, opened novel possibilities in architectural design and visualisation. Ultimately, the contextual analysis of Pacioli's arguments and Bramante's innovative design process clarifies not only their inherent correlations, but also their relevance to present-day approaches to the formal, visual, and spatial properties of architecture.

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Finally, I am thankful for the support of my extended family and friends all along the way. I should like to thank in particular my mother Margherita and my sister Giulia. A special note is for my partner Gimin, without whom this project would not be the same. Her support and advice was and still is immense. This thesis is dedicated to my late grandmothers, Maria and Bibi.

0.1—Jacopo de Barbari (?), c. 1495, Portrait of Luca Pacioli and Gentleman. Museo di Capodimonte, Naples.

Seek no secret initiation
beneath the veil; leave alone what is fixed.
If you want to live, poor fool,
Look only behind you, toward empty space.[1]

Introduction

Aims and research background

The sixteenth-century mathematician and biographer Bernardino Baldi included Luca Pacioli (c. 1447-1517) among the most excellent mathematicians since "[he] was at his time among the most diligent populariser [*divulgatore*] of these [mathematical] disciplines, and for this reason held in high esteem by many", concluding that "there was no painter, sculptor, or architect of his times that would not be a close friend of him, [and] among these was Piero della Francesca, his compatriot and most excellent in painting and perspective [*pittore eccellentissimo e perspettivo*]".[2] Pacioli's correlation with Piero della Francesca and his interactions with scholars and practitioners in the courtly and learned environments of his time was as well-known to sixteenth-century scholars as it is today. Even by the standards of modern scholarship, Pacioli's works stand out for their significance beyond the originality of some of their contents.

Pacioli published in print the first comprehensive volume (*Summa de arithmetica geometria, proportioni, et proportionalita*) that covered the state of the art in commercial arithmetics and practical geometry. The *Compendium de divina proportione*, prepared at the Sforza court of Milan,

[1] Johann Wolfgang von Goethe, "Genius die Büste der Natur enthullend," cited in Pierre Hadot, *The Veil of Isis: An Essay on the History of the Idea of Nature*, trans. Michael Chase (London: Harvard University Press, 2006), 249.

[2] "È degno frate Luca d'essere connumerato fra quei matematici più eccellenti de' quali da noi si scrivono le vite, e ciò per essere stato ne' suoi tempi diligentissimo illustratore di queste discipline, e da tutti per questa cagione molto stimato […] Non vi fu pittore, scultore, o architetto de' suoi tempi che seco non contrahesse strettissima amicitia, fra' quali vi fu Pietro de Franceschi suo compatriota, pittore eccellentissimo e perspettivo". Bernardino Baldi, *Le vite de' matematici.* ed. Elio Nenci (Milano: Franco Angeli, 1998), 331, 343-344.

was the first to include perspectival drawings (drafted by Leonardo da Vinci) of the five regular (so-called Platonic) and other bodies derived from these. The mathematical background of this work was derived, above all, from the last books of Euclid's *Elements.* Pacioli published a Latin edition of these (based on Campanus of Novara's thirteenth-century edition)[3] and stated that he had prepared a (now lost) translation in vernacular Italian, excerpts of which are found in others of his works.[4] This would have been the first of its kind to be published. The *Compendium* was later printed and collated with a short treatise on architecture (*Tractato de l'architectura*), a section on regular bodies based on the translation of a work by Piero della Francesca, and a set of letters drawn with ruler and compass in the *Divina proportione*.[5]

Moreover, in other works, Pacioli collected mathematical puzzles and dealt with games such as chess, to entertain his readers and particularly, his learned patrons.[6] All these works confirm that Pacioli was well informed and often at the forefront of the main trends of his time both in mathematics and in the arts. Few scholars in the late fifteenth century were more connected than Pacioli with theorists and practitioners in courtly and learned environments. Among these, he claimed to have had close contacts with Leon Battista Alberti in Rome, Leonardo da Vinci collaborated with him in Milan, and he referred in his *Compendium* to architectural works now associated with Donato Bramante.

It is therefore unsurprising that scholars have looked at his works. However, together with being of great interest, Pacioli's works also pose serious challenges to the scholars that

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[3] Luca Pacioli, *Euclid magarensis philosophi acutissimi mathematicorumque omnium sine controversia principis opera a Campano interprete fidissimo tralata …* (Venice: A. Paganius Paganinus, 1509)

[4] Luca Pacioli, *Summa de arithmetica geometria, proportioni et proportionalita* (Venice: A. Paganius Paganinus, 1494).

[5] Luca Pacioli, *Divina proportione* (Venice: A. Paganius Paganinus, 1509).

[6] Luca Pacioli, *De viribus quantitatis.* ed. Augusto Marinoni (Milano: Ente Raccolta Vinciana, 1997); Luca Pacioli, *De ludo scachorum* (Sansepolcro: Aboca Museum, 2007).

approach them. Pacioli was exceptionally receptive to the trends that circulated in the environments in which he operated, and very able when these had to be reproposed or summarised in his works. On the one hand, this has historically led to claims of plagiarism or the dismissal of his works as unoriginal, particularly when almost literal appropriations of other authors' writings have been traced.[7] On the other hand, this also means that a great deal of arguments can be found in Pacioli's works, and that at times, these may even appear contradictory. Consequently, readings of his works have mostly been fragmentary and (particularly among art and architecture historians) limited to a few well-known passages.[8]

Outside disciplines such as the history of accounting or mathematics, scholarly studies have rarely extensively dealt with Pacioli's works; and even less considered their relationships with architectural theory and practice. Beginning with Rudolf Wittkower's very influential *Architectural Principles in the Age of Humanism*,[9] scholars have usually relied on a limited and established number of passages (mostly from Pacioli's *Compendium* and sometimes from the *Summa*) linked to the Renaissance theory of proportions and its role in theories of Universal Harmony. With good reasons, these passages have been quoted to show the belief, widely held among scholars in the Renaissance, that proportional relationships among quantities found in nature were part of the inherent structure of the world, and that these linked the world and the celestial spheres as creations of the Highest Maker. Needless to say, this process was considered to be the blueprint for every act of human making, and Pacioli made no exception to this.

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[7] Particularly in relation to his undeclared borrowings from the mathematical writings of Piero della Francesca. As sometimes noted, in his works Pacioli seems to mention authoritative past authors in mathematical subjects but not authors that are closer to him chronologically, although borrowing materials from both. Argante Ciocci, *Luca Pacioli: La Vita e le Opere* (Umbertide: Digital Editor, 2017), 203.

[8] It should be noted that none of these works has been fully translated in English.

[9] Rudolph Wittkower, *Architectural Principles in the Age of Humanism* (London: Warburg Institute, 1949). Wittkower only quoted from Pacioli's *Tractato de l'architectura*, a relatively short (12 folios) treatise dedicated to stonecutters from Borgo San Sepolcro, adding in a footnote that Pacioli was 'a friend of Alberti's'. ibid., 16n.

Nonetheless, although this framework describes some general underlying principles, it does not explain Pacioli's arguments concerning three-dimensional geometrical bodies and their application to architecture. Pacioli's works, especially the *Compendium*, devote a great deal of attention to the formal and visual properties of regular and other bodies derived from these. Relying mostly on writings by Piero della Francesca, Pacioli had already introduced the mathematical properties of these bodies in the last section of the *Summa*. In the *Compendium*, the perspectival drawings drafted by Leonardo da Vinci played a crucial role in the process of correlating formal and visual properties. These were most probably based on a set of three-dimensional physical models that Pacioli claims to have provided with the text. In this text, Pacioli argues for the application of geometrical bodies to architecture, and expands his argument in the later published *Tractato de l'architectura*.

Pacioli's arguments, and the philosophical and theoretical context in which they are framed, are much more concerned with visual and material properties than modern scholars have been willing or able to acknowledge. For instance, if compared with Wittkower's claims about the commensurability of ratios and the employment of the so-called harmonic proportions derived from musical theory, Pacioli shows very little, if any, interest in proportions in music in his works, even omitting them from the section in his *Summa* dealing with the theory of proportions and proportionality. At the same time, he is not particularly concerned with the incommensurability of ratios (one of which—the golden section—is dedicated in the title of one of his works),[10] when these can be visually quantified, dealt with by geometrical means based on aesthetic judgements and, if needed, by practical approximations.

[10] As it is well-known, two among the five regular bodies (the icosahedron and dodecahedron) have their sides in incommensurable ratio (involving the so-called golden section) to the diameter of the circumscribed sphere.

Notwithstanding, the point made by this study is not (or not only) about whether Pacioli was arguing for incommensurable ratios among linear measurements in architecture (as opposed to their commensurability associated—as argued by Vitruvius—with the principle of symmetry). Rather, it concerns Pacioli's argument that three-dimensional regular and other bodies derived from these are exemplary for their formal, visual and ultimately spatial properties and are determined by proportions acknowledged both mathematically and visually. In other words, and this is the crucial thesis of this study, Pacioli argues for the correlation of the visual, formal, and ultimately spatial properties of three-dimensional material geometrical bodies within a theoretical framework that relies on the universality of proportions. Pacioli therefore considers proportional relationships as belonging to a three-dimensional body with precise (and pre-established) formal and visual properties and as such, applied to architecture.

When approaching Pacioli's works, it is clear that scholars have seldom, if ever, positioned the concept of the geometrical body in the context of Pacioli's wider theoretical and philosophical framework. Following Wittkower, scholarly approaches in recent decades have often been marred both by a lack of attention to the philosophical and theoretical framework of Pacioli's arguments and to the correlations between theory and practice that Pacioli aimed to demonstrate in most cases. Consequently, the visual, formal, and aesthetic properties of these bodies have rarely, if ever, been considered in their contexts. The crucial questions that this thesis addresses concern the *how* and the *why* geometrical bodies are considered exemplary forms in Pacioli's works, and by which means these are applied to architecture.

Overall, the study of these aspects is particularly important because the scholarship on Pacioli's works has instead almost exclusively privileged meanings, often examining them in isolation and detached from their theoretical and practical contexts. This has led, in many cases, to the generic association of Pacioli's works with Platonism,[11] or even (in one case) with alchemy, although Pacioli shows a clearly circumscribed interest in arguments linked with the former, and no sign of interest in the latter.[12] As a result, the basic visual perceptual principles on which conceptual (verbal) meanings rely, and their correlations with the mathematical principles employed by Pacioli have been almost completely overlooked.

Pacioli's works are better understood in the context of the Aristotelian and Scholastic framework that was prevalent in the environments where he was educated and in those in which he operated throughout his life. Ultimately, when considered in this philosophical context, even passages in Pacioli's texts that are usually interpreted as mystical or symbolic can be successfully located in a coherent system that has clear theoretical and practical implications. As I intend to show, Pacioli's broad reliance on this underlying philosophical framework is arguably explicit or at least traceable in many aspects of his theory of geometrical bodies. This involves, for instance, the Aristotelian principle of causality and the universal role assigned to proportions in spatial and visual relationships.

Moreover, the arguments developed by Pacioli are closely correlated to whether or not the transparent medium through which the material world is visually perceived (and materially composed of) is itself considered material. The air is a very special material that is not

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[11] Even a perceptive reader as Paul Lawrence Rose, in his influential introduction to Renaissance mathematics, called Pacioli's *Divina proportione* "platonistic". Nonetheless, he also added that Pacioli "was also closely connected with the Aristotelians of the city [of Venice], to judge from his warm praise of Domenico Bragadino, head of the Aristotelian Scuola di Rialto who had taught Pacioli Mathematics at Venice in the 1460's." Paul Lawrence Rose, *The Italian Renaissance of mathematics. Studies on humanists and mathematicians from Petrach to Galileo* (Geneva: Librairie Droz, 1975), 144.

[12] Alberto Pérez-Gómez has argued that Pacioli traced an "alchemical path". However, these assumptions are largely based in the text on collectivist concepts such as "Renaissance culture". Pérez-Gómez also mentions a "Platonic influence" for Pacioli's discussion of the proportions of the head in profile. Alberto Pérez-Gómez, "The Glass Architecture of Fra Luca Pacioli," in *Chora 4: Intervals in the Philosophy of Architecture*, ed. Alberto Perez-Gomez and Stephen Parcell (Montreal: McGill-Queen's University Press, 2004), 251, 273.

ordinarily visually perceived, except in some occurrences. Aristotle's discussion of this subject led to his rejection of the void (or vacuum) in the *Physics*.[13] At the same time, if correlated with the geometrical properties of three-dimensional bodies (and considered independently of Aristotle's assumptions on the nature of light), this is an aspect of central importance in the context of the principles of linear perspective that Pacioli endorses and employs in his works. In this case, air is doubtless considered as a homogenous material in which proportional relationships can take place in three-dimensions. It is in this theoretical and philosophical framework that Pacioli's extensive reliance on the visual properties of three-dimensional geometrical bodies, and his views on the universality of proportions, should be understood.

All these aspects are fundamental in considering Pacioli's references to the application of geometrical bodies to architecture. Since Pacioli believes that mathematical principles should be at the foundation of every practical enterprise, as they are the basis of the phenomenal world, these arguments have inevitable implications for the visual, formal and spatial properties of architecture. In many ways, Pacioli follows Alberti's lead in the belief that mathematical principles (combined with the linear geometry of light) determine visual perception and are essential in the definition of the formal properties of architecture. Nonetheless, Alberti's process was geometrically reliant and was resolved with immaterial lines and angles, whereas Pacioli crucially relied instead on the three-dimensional geometrical body. Such bodies are applied to architecture in Pacioli's works by means of their three-dimensional formal, spatial and visual properties in the context of the principles of perspective. The links established in Pacioli's texts with some of the works of Donato

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[13] Aristotle, *Physics,* IV.7-8, 213a12-218a20 (esp. 216a28-30).

Bramante in the application of geometrical bodies to architecture should thus be considered in this theoretical framework.

Bramante's mastery in the practice of linear perspective was already widely recognised in the sixteenth century by theorists such as Sebastiano Serlio and Cesare Cesariano. Since then, scholars have frequently noted the importance of visual and spatial properties in his architecture in Milan and Rome. Nonetheless, to justify Bramante's visual and formal solutions, architectural historians in recent decades have mostly looked at functional aspects and symbolic meanings, at Bramante's possible sources in the works of coeval architects, or in local and antique buildings. However, such analyses have provided only a very limited understanding of the design process employed by Bramante to define the visual, formal, and spatial properties of architecture. They have not examined, for instance, the process of precise quantification and visualisation that these solutions necessarily required prior to their material application. At the same time, Bramante's interactions with other architects and mathematicians in the environments in which he operated and the observations made in Pacioli's works, have received remarkably little attention in the literature.

Given the scarcity of theoretical writings by Bramante, the correlation of theory and practice expressed in his works has received very limited attention from architectural historians. At the same time, the arguments concerning geometrical bodies in Pacioli's works and Bramante's design process as reconstructed from his architectural works have never been compared. One of the main questions addressed by this study therefore is whether the arguments found in Pacioli's texts help to explain at least some of the formal solutions found in Bramante's works. The results are promising in this respect. Ultimately, as this study begins to demonstrate, clarifying aspects of Pacioli's arguments and Bramante's design process in their relationship with architecture's formal, visual and spatial properties, holds valuable and timely lessons for today's theorists and practitioners.

Methodology

To account for the process of visual perception and the definition of the formal properties of architecture in the works by Pacioli and Bramante, this study endorses a hierarchically structured understanding of human visuality. This methodological approach reflects the latest studies in the psychology and philosophy of perception and follows in principle the structure outlined by the cognitive scientist and philosopher Zenon Pylyshyn, and more recently confirmed by Chaz Firestone and Brian Scholl.[14] When dealing with the spatial properties of architecture, a consideration of both the process of visual perception and visual imagination is fundamental. The relationships of this process to architecture has been widely considered by Branko Mitrović, on whose studies this study extensively relies.[15]

Essentially, this framework rests on the assumption that there is a basic stage in visual perception that is independent of any assigned verbal (conceptual) category. This relies on the visual perception (firstly taking place in the retina of the observer) of lines, angles, two-dimensional shapes and colours as received by the brain. After this stage, the brain relies on a series of what psychologist call 'constraints' to determine the possible or probable three-

[14] Zenon Pylyshyn, "Is vision continuous with cognition? The case for cognitive impenetrability of visual perception," *Behavioural and Brain Sciences* 22 (1999): 341-423; Zenon Pylyshyn, *Seeing and Visualizing. It's Not What You Think* (Cambridge, MA: MIT Press, 2003), 49-157. Chaz Firestone and Brian Scholl, "Cognition does not affect perception: Evaluating the evidence for 'top-down' effects," *Behavioural and Brain Sciences* 39, e229 (2016): 1-77.

[15] Many aspects of this relationship have been summarised in: Branko Mitrović, *Visuality for Architects*, Architectural Creativity and Modern Theories of Perception and Imagination (Charlottesville: University of Virginia Press, 2013). More recently in Branko Mitrović, "Visuality and Aesthetic Formalism," *British Journal of Aesthetics,* vol. 58, n. 2, (April 2018): 147-163.

dimensional shape of objects from two-dimensional retinal images.[16] Although this is not a basic perceptual feature, it should be considered as closely correlated to the basic stage of perception, since this is a fundamental background ability in human visuality.[17] For geometrical bodies, this means that the three-dimensional spatial arrangement of a body has already been perceived at this stage, independent of any verbal (conceptual) meaning that can be linked to it at the higher levels of cognition.[18] In other words, this study supports a formalist assumption that human visuality is multi-layered. This means that at least some aesthetic attributes can be assigned to geometrical bodies or architecture purely according to visual properties derived from two-dimensional images and a three-dimensional mental model, and independent of conceptual or symbolic meanings.

A certain degree of ambiguity in the literature concerning human visuality is determined by the fact that all these stages can be referred to under generic terms such 'visual perception' or 'visual experience'. However, for the purpose of this study, a fundamental distinction should be drawn between the basic (lower) stages (including the two-dimensional basic perceptual features and the background mastery of the principles of perspective) and the higher stages of visual cognition or experience involving verbal (conceptual) or symbolic meanings. It is significant that Pacioli made a similar distinction in his writings and referred to an

[16] Branko Mitrović, "Visuality After Gombrich: the Innocence of the Eye and Modern Research in the Philosophy and Psychology of Perception" *Zeitschrift für Kunstgeschichte,* 76. Bd., H. 1 (2013): 78. As discussed by Donald Hoffman, there are specific rules or constraints that are involved when the brain interprets a two-dimensional representation three-dimensionally. Hoffman listed 35 such 'so-called' constraints. Donald D. Hoffman, *Visual Intelligence: How we create what we see* (New York: Norton, 1998). The philosopher John Searle called this stage the "background mastery of the principles of perspective". John R. Searle, *Seeing things as they are*, (Oxford: Oxford University press, 2015), 138-140.

[17] An overview of the neural basis of visual shape and object perception in primates is given in: Anitha Pasupathy, Yasmine El-Shamayleh, Dina V. Popovkina, "Visual Shape and Object Perception," in *The Oxford Research Encyclopaedia of Neuroscience* (Oxford: Oxford University Press, 2019).

[18] Arguably, to the basic stages of visual perception should also be correlated the phenomenon (experienced by most viewers) that psychologists usually called 'constancy'. In brief, 'constancies' determine that an object is perceived with the same metrical properties despite any changes that may have occurred in the retinal stimulus (e.g., its relative size or perception on a slant). There is, however, a further distinction to be made between the perception of 'constancies' and the inference that an object has retained certain metrical properties despite changes in the proximal stimulus at the level of the retina. Mitrović, *Visuality*, 44-47.

analogously arranged hierarchical structure in the visual and aesthetic appreciation of architecture. Moreover, Pacioli is also clear that the theory and practice of perspective would certainly be of no value if unable to deliver these same stages of visual experience.[19]

A consideration of the basic stages of human visuality so understood (including basic perceptual features and the background mastery of the principles of perspective) is thus unavoidable in approaching Pacioli's arguments on the visual and formal properties of geometrical bodies and their application to architecture. Additionally, since they are independent from conceptual meanings and categorisation, the basic principles underpinning these stages cannot have changed from Pacioli's times to the present (and indeed for a considerably longer time). This is demonstrated by the fact that perspectival drawings made in the Renaissance still determine the intended three-dimensional visual experience relying on the geometry of lines, angles and two-dimensional figures, independent of symbolic or conceptual meanings that were or are associated with them. In other words, if it can be assumed that the logical structure of the basic stages of human visual perception has remained the same, then the arguments of a theorist like Pacioli and their relationship with works by Leonardo da Vinci or Donato Bramante can be approached purely in terms of visual and formal properties. This can be done based on our present-day knowledge of how the process of human visuality works, and independent from conceptual (verbal) and symbolic meanings.

With a hierarchically structured understanding of human visuality in mind, the aim is therefore first to consider Pacioli's arguments on geometrical bodies in their own right. This means that Pacioli's works should be allowed to speak for themselves, without being seen as

[19] This is discussed in the last section of Chapter 1 and in Chapter 3.

the product of collective historical forces, as was often done in the past. This will provide the basis for a comparison between Pacioli's arguments on geometrical bodies and perspective and aspects of Bramante's design approach to the visual, formal, and spatial properties of architecture. Ultimately, this thesis argues that this can only be achieved through an analysis of the arguments in the texts by Pacioli and Bramante, and a consideration of empirical data (surveys, drawings, direct and photographic visual experience and so on) gathered from the latter's architectural works.

Moreover, as noted, if the relationship between the arguments on architectural theory and practice found in Pacioli's texts and the work of coeval architects such as Bramante is to be clarified at all, this can only take place in the theoretical-mathematical and philosophical contexts in which these arguments were formulated. It is thus necessary to consider first the environments in which Pacioli was educated and the individuals with whom he was able to interact throughout his life. This will then support the analysis of Pacioli's arguments and their mutual correlations found in his works. This should help clarify Pacioli's original responses to the particular questions and concerns he was facing.

Finally, Pacioli's arguments dealing with geometrical bodies and their application to architecture should also be compared with the reasoning and design process found in (or deducted from) the works of contemporary or near-contemporary architects and theorists such as Leon Battista Alberti, Francesco di Giorgio Martini, and Leonardo da Vinci. In essence, this study will consider, based on available writings and data, Pacioli and Bramante's answers to contingent philosophical and practical problems, and compare them with the arguments of other theorists and practitioners. Overall, this means that cultural-based historical generalisations will be of very limited help and that the thesis will instead rely on the study of individual arguments in their contexts and on empirical data.

The structure of the study

The relationship between theory and practice, central to Pacioli's arguments concerning geometrical bodies, has determined the arrangement of the study in three main sections. The first section (Chapter 1) should help to introduce the contexts in which Pacioli was educated and establish how he built up his connections with scholars and architectural practitioners. Chapter 1 begins with a short biography that mostly relies on documental evidence gathered in recent decades. However, particular attention is given to the contexts with which the philosophical and theoretical framework found in Pacioli's works can be meaningfully related. This analysis provides the background for the introduction (in the last part of the chapter) to the main theoretical themes found in Pacioli's works. These are further examined in the ensuing chapters.

The second section (Chapters 2 and 3) concerns Pacioli's arguments dealing with geometrical bodies in their philosophical and theoretical framework. Chapter 2 examines aspects of the philosophical and mathematical context that Pacioli had to address and in which he developed his arguments. Properties such as quantities and proportions, their relationship with philosophical arguments and their correlation with the visual, formal and spatial properties of geometrical bodies are particularly considered. Chapter 3 considers the process of the quantification and visualisation of geometrical bodies and the correlation with the theory and practice of linear perspective. Pacioli's employment of theoretical and practical arguments found in Piero della Francesca's writings on the subject, and the perspectival representations of regular and other bodies derived from these drafted by Leonardo da Vinci for Pacioli's *Compendium,* are key passages in this respect.

Finally, the third section (Chapters 4 and 5) examines the consequences of the process of the visualisation and quantification of geometrical bodies when these are applied to architecture. The theoretical framework and design process of sacred spaces (temples) found in fifteenth-century architectural treatises is introduced in Chapter 4. These are then compared with Pacioli's arguments dealing with the visual and formal properties of architecture in the *Compendium* and *Tractato de l'architectura,* and Bramante's recorded *Opinio* on the *tiburio* of the Duomo of Milan. Chapter 5 considers the visual, formal and spatial properties of some of the buildings associated with Bramante's design in Milan and Rome. Although Leonardo da Vinci's drawings of central-plan churches are also examined, these are considered in the context of Leonardo's interactions with Bramante and Pacioli in their Milanese period. The analysis then focusses on aspects of an empirical theory of Bramante's design approach to the formal properties of architecture and their relationships with Pacioli's arguments. The upshot of this section is that the concept of the geometrical body as understood in the context of mutually correlated arguments in Pacioli's and Leonardo's works is helpful in clarifying Bramante's design process.

An appendix follows at the end of the thesis. The Appendix presents a survey of relevant technical terms employed in Pacioli's *Summa*, *Divina proportione,* and *Euclid's Elements*. These terms are: *dispositione*, *forma* and *figura*, and *ingegno*. The survey focuses on the texts authored by Pacioli, listing the contexts in which these terms are found. The Appendix provides the textual background to the interpretative problems and readings found in the main body of the thesis.

1. The life and works of Luca Pacioli - an introduction

This chapter is divided into two main interrelated parts. In the first, biographical information is considered and the works of Pacioli are introduced. This includes the environments and the individuals with whom Pacioli interacted throughout his life, the contents of his works and passages that are important when his theoretical and philosophical frameworks are examined. The second part looks at themes developed across Pacioli's works in the context of one of the foremost stated aims of his writings—the correlation of theory and practice.

The first part largely relies on studies completed in the last few decades. These are particularly linked to a number of conferences held in Pacioli's native Sansepolcro and to biographical studies that were published separately. In recent decades, archival studies have clarified a number of points in Pacioli's biography. At the same time, there are passages that have been discussed less often, but are nonetheless important in approaching the theoretical and philosophical framework of his works. A number of environments and individuals with whom Pacioli interacted throughout his life will be discussed in this context. Following this, some of the arguments established across Pacioli's works will be introduced in the second part of the chapter. These will mainly concern the theoretical and practical foundations of the concept of the geometrical body and the process of the aesthetic judgement of architecture.

Overall, the two sections are intrinsically mutually related. The examination of the historical-social context in which Pacioli operated, and his actions and interactions with other individuals will support the analysis of the theoretical foundation of his works. Conversely, an understanding of the theoretical and philosophical framework of Pacioli's works is needed if the nature of the relationships with other individuals in the specific context of the environments in which he operated is to be clarified at all. How did an initially non-university, provincial individual like Pacioli establish his authority in some of the most learned courtly environments of the fifteenth century? Ultimately, this first chapter should help to establish how Pacioli built up his multiple connections, and provide the foundations for an analysis of some of the philosophical and geometrical arguments presented in his works.

1.1 Luca Pacioli, lecturer of theoretical and practical mathematics and magister theologiae: an essential biography

1.1.1 The early years of Pacioli in the trade-oriented environment of Sansepolcro and the relationship with Piero della Francesca

This section will consider Luca Pacioli's early years and aspects of the cultural, social and economic environment of his native Sansepolcro in the mid fifteenth century (at the time Borgo San Sepolcro or simply Borgo), based on the available documentation. Luca Pacioli was born in the town between October 1446 and October 1448.[20]

Located on a plain in the upper Tiber valley, Sansepolcro had economic and strategic relevance in the fifteenth century. The town lies midway between Florence and the Adriatic coast, at the crossing of trading routes that linked the city of Florence with Rimini and Ancona, Venice and Rome via Cesena and Perugia.[21] Following the secular dominion of the Malatesta family of Rimini dating from 1371, it was eventually sold to Florence by the papacy in 1441.[22] The town had a socially active population and a thriving trading community, as testified by the numerous local markets and the documented trading links with

[20] Elisabetta Ulivi, "Nuovi documenti su Luca Pacioli," in *Pacioli 500 anni dopo,* ed. Enrico Giusti and Matteo Martelli (Sansepolcro: Centro Studi "Mario Pancrazi", 2010), 32.

[21] Sansepolcro was also a location of regional prominence in the exchange of agricultural and artisanal products in the upper Tiber Valley, in an area reaching the towns of Arezzo, Perugia, Cesena and Ancona. James R. Banker, *Piero della Francesca Artist & Man* (Oxford: Oxford University Press, 2014), 2.

[22] As a counterpart to the help received the year before in the Battle of Anghiari. Gian Paolo Giuseppe Scharf, *Borgo san Sepolcro a metà del Quattrocento*, *Istituzioni e società* (Firenze: Olschki, 2003) 29-32.

cities in the centre and the north of the Italian peninsula. The town also had a sizeable population. Around the year 1450, Sansepolcro had approximately 4,400 inhabitants, a number comparable with the (now considerably larger) towns of Arezzo or Pistoia in the same period.[23]

Among the trades, the marketing of walnut wood, wool, leather and indigo (the blue dye extracted from the woad plant cultivated around the town), were particularly relevant. In the fifteenth century, merchants from Sansepolcro were recorded as travelling to Florence, Pisa and Venice,[24] and Luca Pacioli most likely traveled to Venice with a merchant.[25] Merchants imported coarse and fine cloth, sugar, iron, and salt into Sansepolcro. [26] The town produced raw materials such as wood and also specialised in fine products from its workshops. Piero della Francesca's father was among the merchants associated with the production and craftsmanship of leather in fifteenth-century Sansepolcro.[27]

Pacioli's father, Bartolomeo, was not an artisan or trader but probably owned some agricultural land. Pacioli's mother, who most likely died in his childhood, was Maddalena di Francesco di Matteo Nuti.[28] They married in 1427 and had four children: Antonio (who died young), Ginepro, Ambrogio and Luca.[29] After the death of their father in 1459, both Ginepro

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[23] Angelo Tafi, *Immagine di Borgo San Sepolcro: Guida Storico-Artistica della Città di Piero* (Cortona: Calosci, 1994) 80. Tafi refers the number to the year 1450. The comparison with Pistoia and Arezzo in numbers of 'mouths' [*bocche*] is made in a document of dating to the 1440s, discussed in Scharf, *Borgo san Sepolcro,* 1.

[24] James R. Banker, *Piero della Francesca*, 2.

[25] James R. Banker, "La formazione di Luca Pacioli," in *Luca Pacioli - Maestro di contabilità - Matematico - Filosofo della natura*, ed. Esteban Hernández-Esteve and Matteo Martelli (Umbertide: University Book, 2018), 61.

[26] Banker, *Piero della Francesca*, 2.

[27] The workshops [*botteghe*] of Della Francesca are a case in point. As noted by Banker, Renaissance *botteghe* often combined craftwork and sales, however probably the Della Francesca or their employees performed only the ultimate decorative leather work in these central shops. Banker, *Piero della Francesca*, 98-100. Benedetto di Pietro della Francesca (1375-1464) started out as a leather worked but through business and marriages developed a life as a relatively wealthy and respected merchant of indigo. Brüggen Israëls, *Piero della Francesca*, 12. Piero della Francesca was born around 1412. Banker, *Piero della Francesca*, 1.

[28] Ulivi, "Nuovi documenti," 22-23.

[29] Francesco Paolo Di Teodoro, "Pacioli, Luca," Dizionario Biografico degli Italiani, Volume 80, 2014, https://www.treccani.it/enciclopedia/luca-pacioli_(Dizionario-Biografico)/.

and Ambrogio took Franciscan vows (as did Luca later) and entered the local convent of San Francesco in the 1460s.[30] The younger Luca instead joined the house of Conte dei Bofolci, a wealthy local land-owning family who lived nearby, most likely as a servant and companion to their teenage son, Folco.[31] He remained with the Bofolci until at least the mid-1460s and established a long-term connection with Folco.[32] At this point Pacioli left his hometown for Venice, probably in 1464, or in 1466 at the latest. In Venice, Pacioli entered the service of a local merchant and, as he later noted, developed his knowledge of mathematics. However, Pacioli was again recorded in Sansepolcro in October 1466.[33] Although constant travels and teaching appointments brought him away from his hometown, Sansepolcro remained a periodic point of return throughout his life.[34]

Considering the example of cities such as Florence, with established schools of commercial arithmetic and practical geometry (the so-called schools of abacus), it is legitimate to ask which links, if any, existed between the trading community and the school system in Sansepolcro.[35] As far as we know, Sansepolcro did not have an abacus school and the local institutions did not support any formal teaching of commercial arithmetic or geometry in the fifteenth century. As noted by James Banker, given the large number of masters involved in

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[30] Banker, "Piero della Francesca and Luca Pacioli," 44.

[31] Ulivi, "Nuovi documenti," 23.

[32] ibid., 46. Pacioli mentions the Bofolci in all of his three testaments, adding in 1508 that Conte dei Bofolci "in pueritia me nutrì e alevò". Venezia, Archivio di Stato, *Testamenti Pedretti*, busta 786, ins. 201, 9 novembre 1508 and Vianello, *Luca Pacioli nella storia della ragioneria con documenti Inediti* (Messina,1896), 171. Borgo Sansepolcro (2 febbraio 1510, Archivio di Stato di Firenze, *Notarile antecosimiano*, 17712, ins. 231; 21 novembre 1511, ibid., 6938, 130r-131r.

[33] When he joined a deed with his brothers to sell their family house. Pacioli is also documented in Ferrara the 23 April 1466, during the celebrations of St George's day. This may be taken to point that Pacioli was already in Venice and returned to Sansepolcro. Ulivi, "Nuovi documenti," 33-34.

[34] Both of his brothers remained in the convent of San Francesco in Sansepolcro, of which his brother Ginepro, master of theology, was also the general guardian in (at least) 1472-1473. Ulivi, "Nuovi documenti," 24.

[35] The schooling system of Sansepolcro did not change substantially in the time intervening between Luca Pacioli's and Piero della Francesca's youth, recent studies that focused on the context of the youth of Piero will be useful to outline the early years of Pacioli in Sansepolcro. The studies of James R. Banker have been fundamental in this regard. Piero della Francesca was born around 1412 in Sansepolcro. Banker, *Piero della Francesca,* 1; James R. Banker, *The culture of San Sepolcro during the youth of Piero della Francesca* (Ann Arbor: University of Michigan Press, 2003).

abacus schools in fifteenth-century northern and western Tuscany, it is certainly noteworthy that the communal government of Sansepolcro did not establish this kind of school.[36] Surviving documents suggest that the sale and taxation of land presumed knowledge of the land's exact size, which customarily required the services of *agrimensores*. Since the town councillors failed to provide training on surveying in the fifteenth century, the people of Sansepolcro either relied on men with little formal training or on others from nearby areas.[37]

Nonetheless, when the services of these surveyors are discussed in the documents from Sansepolcro in the early fifteenth century, the primary activity mentioned is not measuring land, but gauging the volume of three-dimensional objects such as casks of wine.[38] This may be because, as noted by Banker, locals could measure land in two dimensions, but estimates of three-dimensional objects required an *agrimensores*'s specific knowledge of three-dimensional geometry.[39] Banker concluded that the absence of a formal mathematical education, the monopoly of the *agrimensores* on three-dimensional objects, and an eventual appeal from an unknown merchant to Piero della Francesca to prepare a treatise on abacus for Borgo San Sepolcro in the mid-late 1460s, all seem to suggest that Piero and later Pacioli had first accessed mathematical knowledge through informal networks.[40] This means that they would have learned directly from the *agrimensores* or from merchants with mathematical knowledge, and that they looked for additional sources and furthered their mathematical knowledge outside Sansepolcro.[41]

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[36] Consistent instruction in *abaco* is not mentioned in the documents of the southeastern portion of Tuscany and upper Tiber Valley until the second half of the fifteenth century. Furthermore, as pointed by Banker, the documents of Borgo San Sepolcro do not reveal a single abacus master active in the town in the fifteenth century. Banker, *The culture of San Sepolcro,* 85-86.
[37] ibid.
[38] As noted by Banker, in 1378 two agrimensores set up a partnership (*societas*) for measuring three-dimensional volumes. These concerned primarily casks of wine. ibid., 86.
[39] ibid.
[40] ibid., 89.
[41] ibid.

What clearly emerges from the documents of this period is the presence in Sansepolcro of a grammar school, established at least from the late fourteenth century. Students could learn reading, writing in vernacular Italian and Latin, and familiarise themselves with texts on rhetorical and moral subjects. A humanistic curriculum was probably introduced in the grammar school in Sansepolcro from the middle of the fifteenth century.[42] Although from a family of modest means, given Pacioli's later knowledge of Latin and his intellectual aspirations, it is probable that he attended the public grammar school in the mid-1450s to learn letters and eventually Latin.[43] If he had attended, Pacioli would have studied under Maestro Matteo di ser Paolo.[44] Indeed, in the *Summa*, Pacioli may well have described his former grammar-school teacher while introducing the translator of Piero della Francesca's treatise *On Perspective in Painting* [*De prospectiva pingendi*]:

> In that [book] highly of painting [Piero] speaks, engaging always his words with the way and figure of making. And all of that book we have read and discussed, it was done [by Piero] in vernacular, and then the famous orator, poet, and rhetorician, in Greek and Latin (his regular associate, and fellow townsman) *maestro Matteo* translated it in Latin in the most ornate way, *de verbo ad verbum*, with exquisite words […].[45]

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[42] ibid., 62.

[43] After his father death in 1459, whether either the Pacioli or Bofolci family chose to pay the small amount for him to continue to attend grammar school, we do not know. However, we do know that Pacioli did not attend a school of *abaco* in Borgo San Sepolcro because the town did not have such a school at the time. Paul Grendler noted that primary education remained unchanged across the fifteenth century, but secondary education underwent a 'revolution' in which medieval methods and materials were replaced by a humanistic curriculum. The new curriculum, especially the adoption of the Letters of Cicero as prose models, began at different times and places throughout the century. *Studia humanitatis* were introduced by Guarino, Barzizza and Vittorino da Feltre to secondary education in the Veneto and Lombardy area, but their teaching was primarily in independent schools (where students' families paid for the education) or schools in courts of princes. However, as noted by Gendler, several cities in northern Italy hired students of these great masters for their communal schools; town governments of northern and central Italy increasingly sought out masters trained in the humanities for schools paid primarily through civic taxation. Paul E. Grendler, *Schooling in Renaissance Italy: Literacy and Learning, 1300-1600* (Baltimore: The Johns Hopkins University Studies in Historical and Political Science) 133-141; Banker, *The culture of San Sepolcro,* 62.

[44] James Banker, "Piero della Francesca and Luca Pacioli: Master and Student?*,"* in *Luca Pacioli e i grandi artisti del Rinascimento italiano*, ed. Matteo Martelli (Umbertide: Digital Editor, 2016), 46.

[45] "Nel qual altamente de la picture parla, ponendo sempre el suo dir ancona el modo e la figura del fare. El quale tutto habiamo lecto e discorso, el quale lui feci vulgare, e poi el famoso oratore, poeta, e rhetorico, greco e latino (suo assiduo co(n)sotio, e similmente co(n)terraneo) maestro Matteo lo reccò a le(n)gua latina ornatissimamente, de verbo ad verbum, con exquisiti vocabuli" Pacioli, *Summa*, 68v (Pars prima).

Following this passage, the connection between Luca Pacioli and Piero della Francesca and the nature of the interactions between them should also be considered. Piero della Francesca was around 35 years older than Pacioli and it has often been assumed that Pacioli was a direct disciple of Piero.[46] However, although Pacioli made extensive use of materials from Piero's works in his writings, and both were born and sometimes resided at the same period in Sansepolcro,[47] their relationship is better understood in the context of their network of mutual connections in the town, according to the present state of documented knowledge.

The example of Maestro Matteo di ser Paolo, mentioned by Pacioli in the passage, is a case in point. The texts by Pacioli do not provide any clear account on personal interactions, beyond the praises of Piero as a painter and the writer of a work on perspective.[48] Simultaneously, they show that Pacioli was able to access works by Piero that were only available in manuscript form in Sansepolcro as he had them at his disposal for some time. Overall, for the purposes of this study, we should consider the relationship between Piero della Francesca and Luca Pacioli as defined by Pacioli's multiple acknowledgments of the relevance of Piero's works and theories, particularly in relation to perspective and architecture, and by the connections to individuals–especially patrons, architects, and painters–that they had in common.[49]

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[46] The borrowings of Pacioli were presented in these terms by Vasari. The pupil-master relationship has been recently claimed, without additional specifications, by Alberto Pérez-Gómez. *Attunement* (Cambridge MA: The MIT Press, 2016), 52.

[47] For example, James Banker noted that in light of the documents that record the whereabouts of Piero della Francesca and Luca Pacioli, in the years 1459 to 1463 both were in Sansepolcro. Barker, "Piero della Francesca e Luca Pacioli," 52.

[48] Although there is a notary deed that mention both of them in the convent of San Francesco in Sansepolcro, where Pacioli resided, among other five witnesses in 1484, it is important to emphasise that there is no evidence of a teacher-pupil relationship or any other type of direct personal transmission or exchange of knowledge between the two. Ulivi, "Nuovi documenti," 45.

[49] James R. Banker has taken great care in censing the time times they both spent in Sansepolcro. As Banker concluded, "they may have conversed in passing or even in Piero's house or bottega, but we see no evidence that Piero influenced Luca except possibly in the most general way of inspiring Luca to study mathematics and to emphasise the role of proportion in geometry and painting". Banker, *Piero della Francesca e Luca Pacioli,* 54. The correlation of Piero della Francesca's work on perspective and Pacioli's text will be examined in Chapter 3.

1.1.2 The first period in Venice: the abacus mathematics of merchants and the natural philosophy of the School of the Rialto

In his request for the publication rights for his works issued to the Venetian senate in 1508, Pacioli stated that his studies in mathematics had begun around 44 years before [*p(er) anni circa 44*],[50] which was therefore around 1464. Whether in this year Pacioli was already in Venice or still in Sansepolcro, he had most likely started to learn the rules of commercial arithmetic and practical geometry as the apprentice of a merchant.[51] Frequently this departure is described as if Pacioli had chosen to attend a school of abacus in Venice.[52] However, as pointed out by Banker, if he was simply selecting a destination on this basis, he would have chosen Florence, the main centre of abacus schools in the Italian peninsula and a city with political and commercial links with Sansepolcro. As an orphan and probably with no established contacts in Venice, Pacioli's departure could only have been as an apprentice to a merchant travelling to Venice.[53] This first period in Venice is considered by Pacioli in a passage in his *Summa.* While discussing what is "called by the vulgar the 'rule of the thing', that is *algebra* and *amucabala",* Pacioli added that he "will use the shortened versions, or characters, in the same way as done in the four volumes concerning similar subjects previously prepared", giving a summary of three treatises prepared before the *Summa*:

> […] the one that I addressed to the young [students] of Perugia in 1476, in which not at length we discussed [of this], and also in the one that was composed in Zadar in 1481 [where we] dealt with more subtle and difficult cases; and in the one that in 1470 was dedicated to the education of our disciples [*a lo nostri relevati discipluli*] *ser*

[50] Baldassarre Boncompagni, "Intorno alle vite inedite di tre matematici (Giovanni Danck di Sassonia, Giovanni de Lineriis e Fra' Luca Pacioli da Borgo San Sepolcro)," *Bullettino di bibliografia e di storia delle scienze matematiche e fisiche*, XII, (1879): 431.

[51] From years 1464 to 1471 there is only one time when Luca Pacioli is documented in Sansepolcro (October 1466), on the April of the same year is documented in Ferrara. Ulivi, "Nuovi documenti," 33-34

[52] Banker, *Piero della Francesca and Luca Pacioli,* 46-48.

[53] Indeed, there are a number of contracts of merchants from Sansepolcro with Venetian cloth manufacturers dating from the 1450s and 1460s. As noted by Banker, one of these merchants active in Venice in at least 1454 and 1459 was Antonio, the brother of Piero della Francesca. In late 1463 a merchant of Borgo San Sepolcro established a *societas* and went to Venice, bringing with him the dye guado that he sold in Venice. ibid, 48.

> Bartolomeo, Francesco and Paolo, brothers Rompiasi of the Giudecca, respectable merchants in Venice, sons of *ser* Antonio, under which paternal and fraternal shadow, in their own house, I was educated [*me relevai*]. And at similar sciences [were dedicated the studies] under the supervision of *Domeneco Bragadino,* excellent lecturer in all sciences, as appointed by the public authorities of Venice. He was the immediate successor of his preceptor, the most perspicacious and revered doctor, and *canonico* of St Mark, master *Paolo da la Pergula.* And now to [*Bragadino*] [has succeeded] the magnificent and eminent doctor *Antonio Cornaro*, a fellow student of us under the teachings of the aforementioned *Bragadino*.[54]

Pacioli employed similar terms (*relevati* and *relevai)* when discussing the teaching of mathematics, the treatise prepared for the sons of Antonio Rompiasi (a wealthy merchant of the Giudecca), and his own education in these subjects. Once he entered the service of the Rompiasi, Pacioli was able to develop his knowledge of commercial mathematics and the practical needs linked to trading. As confirmation of this, Pacioli informed us in a later work that he traveled many times [*asai volte*] with goods sent by ship from Venice for the Rompiasi in this period, although with no mention of the ships' destinations.[55] Towards the end of this period (in 1470), he prepared a (now lost) treatise on commercial mathematics to be used by the Rompiasi.

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[54] "Per l(')operare de l(')arte magiore: ditta dal volgo la regola de la cosa over algebra e amucabala servaremo noi in questo le qui da lato abreviature over caratteri: si commo ancora nell(')altri nostri quatro volumi de simili discipline per noi co(m)pilati havemo usati: cioe in quello che a li gioveni de peroscia in titulai nel 1476. Nel quale non co tanta copiosita se tratto. E anche in quello che a çara nel 1481 de casi piu sutili e forti compone(m)mo. E anche in quello che nel 1470 derica(m)mo a lo nostri relevati discipluli ser bart(olome)o e Francesco e Paulo fratelli de ro(m)piasi da la çudeca, degni mercatanti in vinegia, figliuoli gia de ser Antonio. Sotto la cui ombra paterna e fraterna i(n) lor propria casa me relevai. E a simili scientie sotto la disciplina de miser Domeneco bragadino li in vinegia da la excelsa signoria lectore de ogni scie(n)tia publico deputato. Qual fo I(m)mediate successore, al perspicacissimo e r(everen)do doctore, e di San Marco canonico meastro paulo da la pergola suo preceptore. E ora a lui, al presente el Magnifico et eximio doctore miser Antonio cornaro nostro condiscipulo, sotto la doctrina del ditto bragadino." Pacioli, *Summa*, 67v (Pars prima).

[55] Pacioli, *De Viribus Quantitatis*, LVI (Prima parte).

In the same passage, Pacioli stated that he had attended the lessons of Domenico Bragadino. Bragadino was then a lecturer in Venice at the School of the Rialto,[56] where he had succeeded (as noted by Pacioli), his preceptor Paolo della Pergola (in 1454). Bragadino was in turn succeeded by his pupil and Pacioli's fellow student Antonio Cornaro in 1479.[57] This is a fundamental passage in the biography of Pacioli, although it has been little considered thus far.[58] It is often argued that Pacioli studied mathematics or even abacus at the School of the Rialto.[59] However, the school was established after a bequest by a wealthy merchant in 1408 to provide the teaching of logic and philosophy, and at least from the times of Paolo della Pergola, had followed the programmes *in artibus* taught at the University of Padua, where Paolo had been a disciple of Paul of Venice.[60] The lectures held at the school were also widely considered as propaedeutic to the studies at the *studium* of Padua.[61] During the tenure of Paolo della Pergola, theology was added to the original studies of logic and natural philosophy. These subjects continued to be taught, with the addition of dialectics and geometry under the tenure of his successor, Domenico Bragadino.[62]

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[56] On the School of Rialto are still fundamental two essays by Bruno Nardi: "Letteratura e cultura veneziana del Quattrocento" in *La civiltà veneziana del Quattrocento* (Firenze: Sansoni, 1957), 99-145, and "La scuola di Rialto e l'Umanesimo veneziano" in *Umanesimo europeo e umanesimo veneziano*, (Firenze: Sansoni, 1964) 93-139; both essays were reprinted in Bruno Nardi, *Saggi sulla Cultura Veneta del Quattro e Cinquecento*, ed. Paolo Mazzantini (Padova: Editrice Antenore, 1971) 3-98. These were partly based on the information collected by Segarizzi in Arnaldo Segarizzi, *Cenni sulle scuole pubbliche a Venezia nel secolo XV e sul primo maestro d'esse* (Venezia: Venezia, Premiate Officine Grafiche di Carlo Ferrari, 1916). A later essay by Fernando Lepori expanded on the materials presented by Nardi. Fernardo Lepori, "La scuola di Rialto dalla fondazione alla metà del cinquecento" in *Storia della Cultura Veneta: Dal Primo Quattrocento al Concilio di Trento*, vol. 3/II, ed. Girolamo Arnaldi and Manlio Pastore Stocchi (Vicenza: Neri Pozza, 1980), 539-605.

[57] Paolo della Pergola was the appointed lecturer at the School of Rialto between 1421 and 1454. Nardi, *Saggi*, 17-18. After his studies with Bragadino and before succeeding him at the School of Rialto, Antonio Cornaro completed his studies *in artibus* at the University of Padua, where in 1478 he was lecturing in metaphysics. Lepori, "La scuola di Rialto," 577.

[58] The attendance of the lectures of Domenico Bragadino has so far received no specific attention beyond being briefly noted in the biographies of Pacioli. However, Pacioli will return at the School of Rialto in 1508 to deliver an introductory lecture to the fifth book of Euclid's *Elements*, before the start of the annual cycle of lessons in the school. The lecture took place, as reported by Pacioli in the edition of Euclid's *Elements* printed in Venice in 1509, on the 11 August 1508.

[59] Banker, *Piero della Francesca and Luca Pacioli,* 48-50.

[60] Paul of Venice (Paulus Nicolettus Venetus) (c. 1369–1429). Padua was the only *studium* in the territory under the control of Venice.

[61] Lepori, *La scuola di Rialto*, 549-551

[62] "Pauli discipulus and commensalis, qui non modo in dialecticis, philosophia ac theologia, sed geometria quoque doctissimus extitit". Segarizzi, *Cenni sulle scuole*, 67; Lepori, *La scuola di Rialto*, 571

An important account of the school's curriculum under the tenure of Paolo della Pergola is given in the dialogue *De mortalium felicitate* written by Nicolò da Cattaro, bishop of Modrussa and student at the school with Bragadino.[63] The dialogue was completed in 1461-62 and is set at the school. It presents Paolo della Pergola and two fellow-students of the author, Domenico Bragadino and Giovanni Cesariense, and discusses the role endowed to natural laws in the context of the doctrine of divine revelation. The writings of Aristotle, "the only one to whom Nature disclosed his secrets",[64] together with the commentaries of Avicenna and Averroes are considered in the dialogue through the lens of the causal and mathematical reasoning of John Duns Scotus, the primal authority of reference throughout the whole text.

Although in the fictionalised context of a literary dialogue, the contents discussed are both reliable and consistent with other sources that place natural philosophy, the texts of Aristotle, the commentary of Averroes, and the commentaries on Aristotle's texts by Duns Scotus, at the core of the school's curriculum, both under Paolo della Pergola and his successor Bragadino.[65] Bruno Nardi famously summarised the approach of the School of the Rialto as "philosophical Aristotelian-Averroist, naturalistic and scientific", and contrasted it with the two other contemporary public schools in San Marco that privileged the humanistic studies of rhetoric, classical literature and history.[66] Ultimately, as Pacioli attended the lectures "in all

[63] The work was completed in 1461-62 and dedicated to Pope Paul II.

[64] "Aristoteles autem, quem solum natura ipsa singula sua secreta perscrutari passa est, beatitudinem nostram esse dixit in operatione speculativa sitam, quae est secundum optimam speculandi virtutem" (Rome, Biblioteca Casanatense, ms. 276, 5r). Lepori, "La scuola di Rialto," 561.

[65] Lepori, "La scuola di Rialto," 566-570.

[66] Two other public schools were established in Venice in the central decades of the fifteenth century, both supported by the senate and located in the area of San Marco. These schools, of which the first was established in 1447, had a humanistic vocation and were propaedeutic to employment in the Venetian chancellery. The subject lectured where grammar, rhetoric, oratory and history. The second school in particular, established in 1460, had a high profile in the philological and historical analysis of ancient texts and had as lector at his opening Giovanni Mario Fidelfo, to him succeeded George of Trebizond (1460-1462), Giorgio Merula (1462-1484) and Giorgio Valla. Lepori, "La scuola di Rialto," 601-602. Bruno Nardi summarised the approach of the School of Rialto as "philosophical Aristotelian-Averroist, naturalistic and scientific" and the School of San Marco as "Platonic, humanistic-philosophical and moral-religious", adding that historian of literature, especially in recent decades, had almost exclusively focused on the latter and ignored the former. Although the neat and generic division proposed by Nardi has been proved to be not completely true since there were occasional exchanges among the two schools, it still holds if their general theoretical frameworks are considered. Nardi, *La Scuola di Rialto*, 51; for a somewhat different view: Lepori, "La scuola di Rialto," 603-605.

sciences" by Domenico Bragadino,[67] the School of the Rialto is not a secondary detail in his biography. Indeed, Duns Scotus's *Ordinatio* (a commentary of Peter Lombard's *Sententiae*) and in particular Duns Scotus's use of mathematical reasoning in theological arguments is referenced by Pacioli in three of his texts.[68] Moreover, the interest in the writings of Duns Scotus in the decades before Pacioli's period in Venice led to the preparation of two critical editions of his *Ordinatio* in the Franciscan convent of Saint Anthony in Padua, the first in 1472 and second (the first complete edition) in 1477. Additional late fifteenth-century editions of his works were produced in Padua (1490) and in the Franciscan Frari convent in Venice (1497), both belonging to the same province of the Franciscan order.[69] At the same time, Pacioli's older bother Ginepro graduated in theology as Franciscan in Padua in 1469.[70]

When considering this phase of Pacioli's life, it is significant that he could develop his knowledge in commercial, practical, philosophical, and theoretical applications of mathematics at the same time. As Pacioli recalled, he was apprentice to the merchant Antonio Rompiasi and taught commercial mathematics to his three sons (writing a treatise on arithmetic and algebra for them), but he also followed Domenico Bragadino's lectures on mathematics concerning Aristotelian logic and natural philosophy. In the fifteenth century, these two schools of mathematics were notoriously separate in their aims, contents and audiences. As we shall see, in keeping with his experience during his formative years in Venice in 1464/66-1470, Pacioli strove to unify them in all of his later texts, and particularly in the *Compendium de divina proportione*.

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[67] Pacioli, *Summa*, 67v (Pars prima).

[68] Pacioli, *Summa,* 2v-3r (Dedicatory letter)*;* Pacioli, *Divina proportione (1509)*, 2v (Compendium)*;* Pacioli, *De Viribus Quantitatis,* LII (Parte seconda).

[69] As noted by Antonino Poppi, all printed editions of Duns Scotus's *Ordinatio* in the second half of the fifteenth century, apart from one now lost and edited in Aurl, were critically edited by the Minor Friars of Padua or the Minor Friars of Santa Maria dei Frari in Venice. Antonino Poppi, "Il contributo dei formalisti *padovani,"* in *Problemi e figure della scuola scotista del santo*, (Padova: Edizioni Messaggero-Basilica del Santo, 1966), 663-664. Lepori, "La scuola di Rialto," 566-570.

[70] Andrea Czortek, *Studiare, Predicare, Leggere* (Città di Castello: Edizioni Pliniane, 2016), 152.

1.1.3 The meeting with Alberti and an itinerant life of lectures in schools and courtly environments

At the end of his first period in Venice, Pacioli met, by his own account, Leon Battista Alberti "with whom for many months in the *alma Roma* at the time of the pontifex Paulo Barbo from Venice in his own house with him [and] at his expenses [I was] always well treated. [He is] certainly a man of great acumen, doctrine, humanity and rhetoric as manifest in the elevated language of his work on architecture."[71] Given that Pietro Barbo (Pope Paul II) died in July 1471, most scholars have thus located the meeting between 1470 and early 1471, when Pacioli was around 23 years of age. It has often been questioned as to which connections helped the young, provincial, and non-university-educated Pacioli to gain access to the house of the elderly Alberti. Considering the available information, Pacioli's links with learned environments in Venice and in Padua, the School of the Rialto and its Aristotelian philosophical framework, should undoubtedly all be considered important in this respect.

Pacioli shared Alberti's mathematical interests found in works such as the *Ex ludis rerum mathematicarum*, *Elementa picturae* and *Descriptio urbis Romae.* Additionally, Pacioli continued and to some extent completed the programme of mathematical quantification of human visual and spatial experience discussed by Alberti in works such as *De statua, De pictura* and *De re aedificatoria.*[72] Moreover, his acquaintance with Alberti certainly helped Pacioli's subsequent connections with learned patrons and scholars in courtly environments. At the end of this period in Rome, Pacioli returned to Sansepolcro and entered the local Franciscan convent.[73]

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[71] "Non so pensare carissimi miei p(er)che el nostro co(m)patriota Leo(n) Batista de li Alberti Fiore(n)tino, con lo quale piu e piu mesi ne l'alma Roma al te(m)po del pontifice Paulo Barbo de Vinegia in p(ro)prio domicilio con lui a sue spesi sempre ben tractato, home certamente de grandissima perspicacita e doctrina i humanita e rethorica, comme apare nel suo alto dire nela sua op(er)a de architectura." Pacioli, *Divina proportione (1509)*, 29v (Tractato de l'architectura).

[72] Alberti's programme of the quantification of visual experience has been described by Branko Mitrović. Branko Mitrović, *Serene Greed of the Eye: Leon Battista Alberti and the Philosophical Foundations of Renaissance Architectural Theory* (München: Deutscher Kunstverlag, 2005), 73-76.

[73] Pacioli is firstly recorded there as friar in February 1471. Ulivi, "Nuovi documenti," 35.

It was suggested that Alberti may have influenced him to choose a clerical life as a means to further his intellectual interests despite his limited resources.[74] Although this is possible, if considered in the context of the lectures attended at the School of the Rialto recalled in the *Summa*, and the positions of his two brothers at this point in time, it is less surprising that Luca Pacioli choose to become a Franciscan friar after this period and eventually became a *magister* of theology in the early 1480s.[75]

Towards the end of the 1470s, Pacioli was documented as lecturer in abacus in a public school in Perugia where he wrote the *Tractatus mathematicus ad discipulos perusinos* for his students.[76] This is a manuscript treatise surviving in one copy, it is largely in Pacioli's own handwriting and presented in vernacular Italian. The presentation, contents and style of writing are coherent with coeval manuals of abacus and practical geometry.[77] The treatise was composed of 396 folios (of which 367 survive) subdivided into 17 parts, with a total of about 800 cases, plus a "description of many and different currencies" and a review of commercial tariffs.[78] Blank folios are left between the different sections, since Pacioli had expressively argued that "on these subjects it is not possible to compose a finished and limited

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[74] Banker, "Piero della Francesca e Luca Pacioli," 52.

[75] As customary, and noted by Pacioli in the *Summa*, he undertook at this point a series of travels to complete his noviciate and it is not recorded in Sansepolcro in the ensuing years. "Ma da poi che l(')abito indegnamante del seraphyco san francesco ex voto piglia(m)mo, p(er) diversi paesi ce convenuto andare peregrinando". Pacioli, *Summa*, 67v (Pars prima). The first document bearing the name of Pacioli with the new title of *magister* is dated June 1484; Pacioli is called here *Magister sacre pagine professor*. Banker, "La formazione*,"* 63; Ulivi, "Nuovi documenti," 35.

[76] Now codex Vat. lat. 3129. In the *Summa*, 67r-v (Pars prima) Pacioli states to have completed the treatise in 1476. However, in the manuscripts is stated that the text was begun on the 12th December (most probably of 1477) and finished the 29th April 1478 (2r). As noted by Derenzini, is likely that work on the text continued until 1480. Giovanna Derenzini, "Il Codice Vaticano Latino 3129 di Luca Pacioli," in *Luca Pacioli e la Matematica del Rinascimento*, ed. Enrico Giusti (Perugia: Petruzzi Editore, 1998), 187-191. Pacioli states to have written the text between the 12 December 1477 and the 29 April 1478, but probably completed around 1480. Di Teodoro, "Pacioli, Luca."

[77] The treatise also includes a note on an exchange of mathematical problems with a master of abacus in Florence (Giovanni Sodi or del Sodo). Pacioli, *Tractatus mathematicus ad discipulos perusinos*, 259v.

[78] "molte e diverse monete" Pacioli, *Tractatus*, 2r-3v; Calzoni, *Il trattato inedito di Luca Pacioli*, 192

work [and], as attested by Aristotle, the number of these [cases] progress towards infinity […]".[79]

The cases mainly concern problems of commercial arithmetic, but also included a section on algebra (now lost), and a section on geometry occupying 48 folios.[80] As in coeval manuals of practical geometry, here the problems ask for the height of towers and the surface of a conical pavilion with a cylindrical base, the distances between and around these, the measures of pyramidal and cubic bodies, the quantities of goods contained in sacks and boxes and so on. The practical relevance of the problems is always made clear. Euclid is mentioned four times in the geometrical section, including in the introduction to Pythagoras' theorem. When Pacioli states that the surface of the sphere is four time the surface of its major circle and references Book XII of Euclid's *Elements*, the name of Euclid was later rightly corrected (probably by Pacioli himself) with Archimedes.[81] In the text overall, brief theoretical arguments are always presented as instrumental to the resolution of practical problems of quantification.

Pacioli makes plain the practical and material importance of a knowledge of mathematical principles that regulate commercial practices, and importantly adds that these procedures

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[79] Therefore, Pacioli seems to argue, the work can always be revised and extended. "[…] e così a la giornata volendoci agiognere una cosa più che l(')altra potrai in fine in alcune carti vachue, qual ò posto per respeto, e così fra le compagnie e baratti e anche in principio, però che in questa facultà non è possibile a far opera finita e limitata, teste Aristotele qui ait quod si quid infinitum est, numerus est ille ergo etc…" Pacioli, *Tractatus mathematicus ad discipulos perusinos*, 3v; Giovanna Derenzini, "Il Codice Vaticano Latino 3129 di Luca Pacioli," in *Luca Pacioli e la Matematica del Rinascimento*, ed. Enrico Giusti (Perugia: Petruzzi Editore, 1998), 182.
[80] Banker, "La formazione di Luca Pacioli," 62.
[81] Whether the later correction, made in a different ink from the main text, is by the hand of Pacioli is not clear. Pacioli, *Tractatus mathematicus ad discipulos perusinos*, 289v.

should be judged with knowledge of their necessity and without theological prejudices.[82] A final annotation of the last folio of the treatise (added by the guardian of the Franciscan convent in Perugia), tells that Pacioli returned a copy of Euclid's *Elements* in December 1480.[83] Pacioli was doubtless deepening his knowledge of the *Elements* in the edition of Campanus during this period, and this was to be the edition referenced in all his subsequent works.

An episode also most likely took place in 1480, as recalled by Pacioli in the *Compendium de divina proportione*. It took place in Rome at the palace of Girolamo Riario, nephew and associate of Sixtus IV (Pope between 1471-1484); and involved the count, Pacioli, the painter Melozzo da Forlì, and a unnamed stonecutter working at the palace.[84] To the detriment of this unwary stonecutter, Pacioli, in partnership with Melozzo, showed his acquaintance with the geometrical properties of the five regular bodies presented in Book XIII of the *Elements*, and the geometrical reasons why only five bodies with all equal bases can be inscribed in a sphere with all their vertices touching the sphere's surface.[85]

Chronologically, this is the first attestation of Pacioli's interest in the five regular bodies. As we shall see, a discussion of the geometrical properties of these bodies are, from this point on, recurrent in the works of Pacioli. Indeed, when compared with the geometrical contents of

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[82] In the treatise, a disciple of Pacioli is made to speak in defence of commercial practices and profits derived from monetary transaction, and dismiss the critiques of such practices brought forward by modern and ancient theologians. Clearly, if Pacioli did not share this opinion it would not have transcribed it in his treatise. "[…] molto me stupischo de multi moderni et antiqui theologi li quali dan questo cambio como inclito essendo in lui certo lucro […] Io dubito che lo caso non fo' inteso da colui che dettono questo iuditio. Io sono mercatante et intendo l'arte, e doy anni ho fatto lo exsercitio nante che lo habbia possuto intendere, et ho hauto non mediocre ingegno, et ho voluto et desiderato de intenderlo, siché non si meraviglieno li religiosi si tanto audacemente dico che lo quesito dammodo impossibile a uno religioso intenderlo per informatione, et per consequens non po' giudicare tam quam cecus de coloribus." Pacioli, *Tractatus mathematicus ad discipulos perusinos*, …; Giuseppe Calzoni, "Il trattato inedito di Luca Pacioli e l'insegnamento della matematica applicata agli affari nel secolo XV a Perugia'" in *Luca Pacioli e la Matematica del Rinascimento*, ed. Enrico Giusti (Perugia: Petruzzi Editore, 1998), 197.

[83] The note is dated 11 December 1480. Derenzini, *Il Codice Vaticano Latino 3129*, 191.

[84] Based on the documented whereabouts of Luca Pacioli, Melozzo da Forlì and Girolamo Riario, the episode has been convincingly dated to the second half 1480, when the palace was in construction. Banker, "La formazione di Luca Pacioli," 79.

[85] Luca Pacioli, Pacioli, *Compendium*, 67r-v. The episode will be further examined in Chapter 3.

the *Tractatus* compiled in Perugia, the *Summa* (which Pacioli published in 1494), is not only a more extended collection of cases found the in treatises of practical geometry, but also includes a specific section dealing with all five regular and two semi-regular bodies.

1.1.4 The contents and context of the *Summa de arithmetica geometria, proportioni, et proportionalita*

Passages in the *Summa* attest that in the 1480s, Pacioli had other occasions to teach geometry and display his knowledge of the five regular bodies to eminent clerics or to Duke Guidobaldo of Urbino in Rome. Pacioli claims to have shown a set of models of the regular and other bodies derived from these to Guidobaldo in the palace of Giuliano della Rovere (the future Pope Julius II and nephew of Sixtus IV) in 1489, and to have initially made the set for the bishop of Carpentras, Monsignor *Pietro de Valetarii* of Genoa, who was also a nephew of Pope Sixtus IV.[86] Like Pacioli and his uncle Sixtus IV, Giuliano had joined the Conventual Franciscans, and in 1474 was made Cardinal protector of the order.[87] This network of connections which Pacioli had already encountered in the early 1480s, explains the patronage received, while also outlining a common Franciscan theological and philosophical framework in which he operated. By all accounts, Pacioli presented himself as an authority in the geometry of the five regular and other bodies derived from these and often linked this to his acquaintance of the last books of Euclid's *Elements*.[88]

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[86] Banker, "La formazione di Luca Pacioli," 68. The name is mistakingly spelled "Pietro de Valerari". Notably, the duke Guidobaldo of Urbino was a relative by marriage to Pope Sixtus IV, since in 1478 his sister Giovanna Montefeltro had married Giovanni della Rovere, a nephew of Sixtus IV and brother of Giuliano della Rovere. After the death of the childless Guidobaldo in 1508, the duchy passed under the control of Giovanna and Giovanni's son, Francesco Maria I della Rovere.

[87] The multiple connections of the Della Rovere with the Franciscans are traced in Ian F. Vestegen's introduction to *Patronage and Dynasty: The Rise of della Rovere in Renaissance Italy;* Ian F. Vestegen, *Patronage and Dynasty: The Rise of della Rovere in Renaissance Italy,* ed. Ian F. Vestegen (Kirksville: Truman State University Press, 2007), xiii-xvii.

[88] Indeed, Euclid is the first to be mentioned among the sources acknowledge by Pacioli in the introductory part of the text: "And all these things will be according to the ancients and also the modern mathematicians, especially from the most perspicacious philosopher *Euclide* of Megara, and of Severin Boetio, and [among] the modern Leonardo of Pisa, Giordano [*Nemorario*], Biagio [*Pelacani*] of Parma, Giova(nni) Sacrobusco (sic) and Prodocimo of Padua, from whom I took the most of this volume." Pacioli, *Summa,* (no number) (Summario)

In the *Summa*, Pacioli also claims that he was giving lectures in Rome in the same period, even if their place and subjects are not specified.[89] This employment followed a second appointment in Perugia as lecturer in commercial arithmetics (abacus) and practical geometry between 1487 and 1488.[90] Additionally, Pacioli states in two occasions that he had taught geometry in Naples for three years in the period before the publication of the *Summa*, although without specifying in which years. He added that he had taught and discussed the contents of Euclid's *Elements* and other volumes on *de re militari* with the "noble and excellent man of arms" Camillo Vitelli of Città di Castello, Pietro Vittori, ambassador of the Florentines with the King of Naples, Gian Giacomo Trivulzio, Milanese *condottiero* and commander of the Neapolitan forces, and Giovanni Pontano, the humanist *segretario maggiore* at the court of the King of Naples.[91] This testifies to the level of authority in lecturing about Euclid's *Elements* that Pacioli had acquired by the late 1480s.

On approaching the contents of Pacioli's *Summa* overall, it is important to highlight his multiple and concurrent engagements in theoretical and practical discussions of geometry. After his first formative period in Venice and his meeting with Alberti in Rome, Pacioli was engaged in teaching commercial mathematics and practical geometry in public schools, while also establishing a series of affiliations with learned and politically influential circles such as

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[89] Banker, "La formazione di Luca Pacioli," 69.

[90] Pacioli is documented another three times in his hometown in the year 1485, before, as noted by Ulivi, being in Florence and then again in Perugia to teach mathematics in the period between May 1487 and April 1488. A document attests that he is again in Sansepolcro the 16 February 1488. ASF, Notarile Antecosimiano 12223, 25v. Ulivi, "Nuovi documenti," 35.

[91] "E lor proportioni li troveranno fabricati e formati, dela qual cosa piu volte col nobil homo excellente armigero, Camillo Vitelli de Castello sopra di questo conferendo apertamente trovato habiamo. Nel tempo che optimamente el suo perspicacissimo ingegno co(m)prese el sublime volume de Euclide p(er) piu mesi da me expostili, e in nel degno gimnasio de Napoli legendo. El simile con lo Magni(fic)o oratore delo Illustrissisi(m)o D(omino) fiorenti(n)o al hora Piero Victori, e con la S. de miser Giovan Jacomo Trauççi. De parte in parte scorrendo per li antichi volumi Quinto Curtio, Frontino, Vegetio. E gli altri cha de re militari hanno scritto a deto si tocava." Pacioli, *Summa*, 2v. (Epistola); "E non manco con la signoria miser Giovan Iacomo Traulzi [Trivulzio], con lo degno oratore del dominio fiorentino alora Pier Vetori, con presenza del Pontano nel palazzo del conte de Sarno in Napoli; e non manco con lo magnifico e degno condottiere signore Camillo Vitelli de la Cita de Castello, legendoli io per tre anni el sublime volume del nostro euclide, e in Milano con lo mio a quell tempo peculiar patrone meser Galeazo Sanseverino e piu volte con lo eccellentissimo duca Ludovico Maria Sforza …".. Pacioli, *Divina proportione (1509)*, 24v (Tractato de architectura).

the family network of the Della Rovere. When both the documental evidence and the information provided by Pacioli throughout his texts are considered, it becomes clear that he had experienced four main types of environments. Firstly, Pacioli had multiple engagements as a teacher of commercial arithmetics and practical geometry in public schools. Secondly, there were appointments as lecturer of arithmetics and algebra in both public schools and universities. Thirdly, there were interactions with circles of patrons, architects, and artists in learned and courtly environments. Lastly, there were studies and activities undertaken from the early 1470s as a member, and by 1484 as *magister* of theology, of the Conventual Franciscans.[92]

It is clear that all these engagements were instrumental in Pacioli establishing connections and joining learned and courtly circles. Crucially, the notes included by Pacioli in the *Summa* and in later writings show that his knowledge of the theory of the regular bodies in Euclid's *Elements,* and the presentations of models of regular and other bodies derived from these, were always the subjects discussed by Pacioli on these occasions. Indeed, it is significant that in the *Summa*, Pacioli directly addresses Duke Guidobaldo of Urbino in two instances, in the opening dedication letter and before the section on regular and other bodies derived from these in the second (geometrical) part of the text. As Pacioli claimed, it was under the patronage of the Cardinals Della Rovere and of Guidobaldo, that he presented a set of regular and other geometrical bodies derived from these. Significantly, a portrait of Pacioli and a student (probably Duke Guidobaldo of Urbino) has been dated to the years immediately following the publication of the *Summa* (illus. 0.1).[93] This well-known painting

[92] When theological studies are considered, Pacioli received his degree in theology before June 1484. However, where Pacioli received the degree has not been conclusively established. As noted by Banker, it was not unusual for promising friars to be sent outside their province of origin and in a centre of university education to complete their degrees. If this is the case, Pacioli could have attended his studies in the convent of Santa Croce in Florence. Indeed, there are a number of passages in later writings of Pacioli that attest his familiarity with environments in Florence. Banker, "La formazione," 63.

[93] The painting (now in Naples, Museo nazionale di Capodimonte, but originally in Urbino) has been tentatively attributed to Jacopo de' Barbari.

shows Pacioli in a Franciscan habit lecturing on Book XIII of Euclid's *Elements*. The *editio princeps* of Campanus's edition lies in front of Pacioli. Above a copy of the *Summa* stands a wooden model of a regular body (dodecahedron), while on the opposite side a hanging semi-regular body (rhombicuboctahedron) is displayed. Although not mentioned in the *Summa*, this would be discussed in the *Compendium de divina proportione*. This body is shown as made of glass and half-filled with water. There are refractions and three reflections of the ducal palace of Urbino. The level of skill on perspective required for such representations and the mathematical and material properties of the geometrical bodies in the painting have often been highlighted.[94]

The *Summa* was printed in Venice in November 1494. Overall, the text is divided in two main parts. The first is dedicated to arithmetic and the second to geometry. The work opens with a dedication to Marco Sanuto,[95] and an epigram addressed to the reader, followed by dedicatory letters in Italian and Latin addressed to Guidobaldo da Montefeltro, Duke of Urbino. A summary of the subjects discussed in the text is followed by a table of the contents of the first part on arithmetic. This is composed of 9 distinctions [*distinctiones*], each divided into treatises [*tractati*], the number of which varies for each distinction. Among the most relevant, the first distinction discusses number theory, the fifth "the rule of three", the sixth concerns proportions and proportionality [*De proportionibus et proportionalitatibus*], while the seventh discusses "the rule *helcataym,* called by the vulgar of the double false position".[96] The

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[94] Renzo Baldasso, John Logan, "Between the Golden Ratio and a Semiperfect Solid: Fra Luca Pacioli and the Portrayal of mathematical Humanism,"in *Visual Culture and Mathematics in the Early Modern Period,* ed. Ingrid Alexander-Skipnes (London: Taylor and Francis, 2017), 130-149; Enrico Gamba, "Proviamo a rileggere il 'Doppio ritratto' di Luca Pacioli," in *Le tre facce del poliedrico Luca Pacioli*, ed. Francesca Maria Cesaroni, Massimo Ciambotti, Enrico Gamba and Vico Montebelli (Urbino: Quaderni del Centro Internazionale di Studi Urbino e la Prospettiva, 2010), 81-97. Argante Ciocci, "Luca Pacioli, Leonardo da Vinci e il disegno dei poliedri," in *Arte e Matematica in Luca Pacioli e Leonardo da Vinci*, ed. Matteo Martelli (Città di Castello: Nuova Prhomos, 2020), 58.

[95] Marco Sanudo, a Venetian patrician "co(n)sumatissimus astrologus, in Arithmetica emine(n)tissiums, in Geometria excelle(n)tissimus" as noted by Pacioli, covered the costs of printing of the *Summa*. Pacioli, *Summa*, (not numbered opening letter).

[96] "De regulis helcataym que vulgo due false positiones nuncupantur". Pacioli, *Summa*, 98v (Pars prima).

ninth distinction deals with commercial arithmetics. Here the first treatise deals with the arrangement of partnerships, the calculation of interests and dividends, etc. Of particular importance is Treatise XI on bookkeeping [*De scripturis*], where for the first time in a printed manual, Pacioli described the double-entry system used by Venetian merchants.[97] The concluding treatise (XII) describes the weights and measures of various European cities.[98]

The second part (geometry) has a separate numeration of folios and is formed of eight distinctions, all composed of chapters [*capitula*] except for the eighth. Apart from the eighth, the contents of all the other distinctions do not substantially differ from what could be found, although usually not in the same volume, in coeval manuals of practical geometry.[99] As Pacioli notes, the eighth distinction is not divided in chapters but is organised into 100 "subtle and practical" problems.[100] These discuss volumes, areas, heights and distances measured through proportions. Additionally, it includes problems of quantification concerning the five regular bodies and two semi-regular bodies derived from these by truncation, spheres and conical bodies. These are discussed in a "Particular treatise on regular and ordinary bodies".[101] These problems were largely derived from Piero della Francesca's *Trattato d'abaco,* while they were not included in other manuals of practical geometry. A section discussing a

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[97] Pacioli, *Summa*, 197v-210r.

[98] Pacioli, *Summa*, 211r-224v; As noted by Francesco P. Di Teodoro, this is a copy of the *Libro che tracta de mercatantie et usanze de' paesi*, a work published anonymously in Florence in 1481. Di Teodoro, "Pacioli, Luca."

[99] The contents of the eight distinctions of the second (geometrical) part of the *Summa* can be summarised as follows: 1. Excerpts from Euclid *Elements* book I, II, and VI, problems concerning the areas of plane figures; 2. (Problems concerning) triangles; 3. Rhombic figures, trapezoids, polygons 4. Excerpts from *Elements* III, circles and curved surfaces; 5. Division of triangles and other polygons and circles; 6. Excerpts from *Elements* XI, measuring of volumes and spherical surfaces; 7. Measurement of distances, heights, etc. employing similar triangles and proportion (towers, walls, battlements, etc.); 8. 100 "subtle and practical" problems on volumes, areas, heights and distances, problems on perspective, measure of volumes, surfaces and sides of the five regular bodies and two other (semi-regular) bodies derived from these, conical bodies and spheres.

[100] Picutti has shown that a large part of these problems (but not the ones dealing with regular and other bodies derived from these) were borrowed from a treatise on practical geometry produced in the 1460s, or one closely related to this: *Tractato di praticha di geometria* (*Palatino* 577, Biblioteca Nazionale Centrale, Firenze) attributed to Maestro Benedetto (Ettore Picutti, "Sui plagi matematici di Frate Luca Pacioli," *Le Scienze*, 246, (1989): 72-79. The contents of this part of the *Summa* will be examined in more detail in Chapter 3 of the thesis.

[101] *Particularis tractatus circa corpora regularia et ordinaria.* This treatise is at folios 68v-74r of the *Summa* (Pars secunda).

method to measure barrels and a summary of the whole second part where "is discussed of geometry in all manners theoretical and practical" conclude the volume.[102]

In the text, Pacioli states that he came to Perugia from Florence in 1487 for a teaching appointment.[103] He states in another section, to have consulted books on mathematics in the library of the Dominican convent of St Mark in Florence, and to have seen a copy of Witelo's *perspectiva,* from which he learnt about natural optics.[104] A number of other sources are mentioned in the summary at the beginning of the *Summa*. These include "especially [...] the most perspicacious philosopher *Euclide* of Megara, and of Severin Boetio, and [among] the modern Leonardo of Pisa, Giordano [*Nemorario*], Biagio [*Pelacani*] of Parma, Giova(nni) Sacrobusco (sic) and Prodocimo of Padua, from whom I took the most of this volume."[105] Nevertheless, the list is certainly not exhaustive, as it does not include Piero della Francesca, from whom Pacioli took large parts of the materials on regular and other bodies derived from these, although Piero is mentioned in two other passages in the text.[106] Moreover, it does not include the authorities that Pacioli mentions in other parts of the text: Aristotle, Plato, Albert

[102] "del modo a far li stagiuoli e tavola de scemi in ogni luogo"; "[...] dove se tratta de geometria i(n) tutti li modi theorica e pratica". Pacioli, *Summa.*

[103] "Qui in questa alma e augusta cità de Peroscia, dove a loro communa satisfatione partendomi del fior del mondo, ciò de Fiorenza harivai. E tal peso presi, per la perpetua obligatione o con tutti di questa cita. Nel 1487 a di primo magio". Pacioli *Summa*, 98v, (Pars prima).

[104] "La qual regola io la cavo dal speculative auctore de perspectiva per nome ditto Vitelone; qual me ricordo haver lecto in la bibliotecha de'frati de San Marco in Firenza, de l'ordine di San Domenico ditti della observantia. La qual libraria feci e ordinò el Magnifico homo Cosmo de Medici; in la quale de ciascuna facultà in greco e latino copiossisimamente feci mettere libri, boni e belli, e maxime in tutte le arti mathematiche asai vine, feci porre. Li quali in parte in quel luogo (per l'umanità de quelli padri) legendo, trascorsi, secondo quel poco senso che Idio per sua gratia ci a dato; sempre qualche utilità reportandone etc." Pacioli, *Summa,* 79r (Pars prima). Banker, "La formazione*,"* 64.

[105] "E queste cose tutte con le sequenti siranno secondo li antichi e ancora moderni mathematici. Maxime del perspicacissimo phylosopho megarense, Euclide, e del Severin Boetio e de' nostri moderni Leonardo pisano, Giordano, Biagio da Parma, Giova(n) Sacrobusco e Prodocimo padoano, da' quali in magior p(ar)te cavo el presente volume." Pacioli, *Summa*, Summario della prima parte principale, c.n.n.

[106] The failure to name Piero della Francesca and his *Trattato d'abaco*, for which Pacioli substantially borrowed for his short treatise of regular and other bodies derived from this in the *Summa*, has lead to the well known accusations of plagiarism of Vasari. Di Teodoro, "Pacioli, Luca."

of Saxony and John Duns Scotus, among others. Among these authors, it is undoubtedly not accidental that the last two were both Franciscans, as noted by Pacioli.[107]

As often noted, the *Summa* is by its own name, a collection of the available knowledge on commercial mathematics and practical geometry consolidated in the centuries before its publication, with particular prominence given to the theory of proportions and its practical applications. Nonetheless, to approach the *Summa* only as a manual of abacus and practical geometry would be misleading. Rather, Pacioli strove firstly in the *Summa,* and later more explicitly in the *Compendium de divina proportione*, to justify the certainty and practical usefulness of mathematical principles by presenting them in the context of a precise theoretical and philosophical framework. Ultimately, as noted by Pacioli in the dedicatory *Epistola*, the foundation of every operative action is explained in the *Summa* with "clear rules and most perfect canons" and always derived from the writings of the "ancient and modern philosophers".[108]

1.1.5 The interactions in the environment of the Sforza court of Milan (1496-1499), and the presentation of the *Compendium de Divina proportione*

Among the environments that Pacioli visited in the years following the publication of the *Summa*, the court of Ludovico Sforza in Milan is particularly important. It was in this period that one of the best known works by Pacioli dealing with regular and other bodies derived

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[107] Pacioli calls Albert of Saxony "profound philosopher and doctor or our seraphic order", "p(ro)fo(ndo) ph(ilosoph)o e sacro doctore de l(')ordine nostro seraphyco". Pacioli, *Summa*, 68r (Pars prima).

[108] "In which (as shown at lenght in the ensuing summary table) I have collected many and different sections about arithmetic, geometry, proportions and proportionality, with all that is very necessary to the practice, (and expressed it) with clear rules and most perfect canons. And (I have taken) the foundation of every operative action from the ancient and modern philosophers" "In lo quale (co(m)e diffusame(n)te in la seque(n)te tavola appare) molte varie e diverse p(ar)ti necessarissime de Arithmetica, Geo(metri)a, Proportioni e p(ro)portionalita, sinsiemi ho recolto con q(ue)lle tutte che a lor pratica se ricerca, co(n) regole ferme e canoni p(er)fectissimi. E de ciascun atto op(er)ativo suoi fondame(n)ti, s(econd)o li antichi e ancor moderni ph(ilosoph)y." Pacioli, *Summa*, (Epistola)

from these, the *Compendium de divina proportione*, was completed and dedicated to the duke.[109] If the presence of Pacioli in the environment of the court of Urbino is often evoked by scholars but scarcely documented,[110] the period at the Sforza court allows for more consideration of his position, his interactions with other individuals and the contents of his works. As noted by Pacioli, the contents of the *Compendium* originated as the outcome of discussions held in the environment of the court.

In 1496 Pacioli was appointed as the chair of arithmetics and geometry in a public school in Milan, also called *Achademia*.[111] Although the Duchy's only *studium* was located in Pavia, the duke maintained a separate school in Milan, that if not awarding degrees, was hosting public lectures.[112] As noted by Pacioli, the school was under the direct jurisdiction and financial responsibility of Duke Ludovico Sforza, and thus had direct links with the environment of the court. Pacioli added that his lectures on "arithmetics, geometry, proportions and proportionality" had covered the first ten books of Euclid's *Elements*, while dealing "with the

[109] The work was prepared in three manuscript copies, one offered to the duke Ludovico Sforza (now codex Langues Etrang res n. 210 at the Bibliothèque de Genève), one to the "personal patron" [*peculiar patrone*] of Pacioli at the court, Galeazzo Sanseverino (now inv. S.P.6 ex Cod. F 170 sup. at the Veneranda Biblioteca Ambrosiana in Milan) and one (now lost) to Pier Soderini, *gonfaloniere* of the Florentine Republic.

[110] Since a large part of archival documentation was lost after its transfer to Florence, it has become extremely difficult to trace the whereabout of the individuals involved in the environment of the court between the 1460s and 1480s.

[111] In a letter of 1490, Ludovico Sforza referred to it as the "Achademia, dove se leze publicamente". Luigi Banfi, "Scuola ed Educazione nella Milano dell'ultimo Quattrocento," in *Milano nell'Età di Ludovico il Moro*, vol. II (Milano: Archivio Storico e Biblioteca Trivulziana, 1983): 383-395. Other sources are discussed in Agostino Sottili, "L'Università di Pavia nella Politica Culturale Sforzesca," in *Gli Sforza a Milano e in Lombardia e i loro rapporti con gli stati Italiani ed Europei (1450-1535)* (Milano: Cisalpino-Goliardica, 1982), 519-580. Ulivi, "Nuovi documenti," 38.

[112] In the years when Pacioli lectured, the chair of mathematics was joined by appointments on rhetoric, poetry and Greek. As shown by Sottili, the salaries of lecturers at the school of Milan where generally higher than the salaries of lecturers in Pavia. Sottili, "L'Università di Pavia nella Politica Culturale Sforzesca."

theory and the practice", adding that his time in Milan was divided between lecturing and working on the *Compendium* presented to the duke.[113]

Indeed, since the school was under the direct jurisdiction and remuneration of the duke, Pacioli was necessarily involved in the dynamics of patronage and exchanges linked to the environment of the court.[114] This becomes clear as soon as the dedicatory letter and preface (Chapters 1 and 2) of Pacioli's *Compendium* are examined. Pacioli opens the dedicatory letter with a reference to a laudable and scientific disputation [*laudabile e scientifico duello*] that took place in the residence of Ludovico Sforza in February 1498, in which many learned scholars, both religious and secular, took part.[115] Among these, there were a number ecclesiastics and theologians, duly acknowledged by Pacioli, and Galeazzo Sanseverino, the *condottiero* of Ludovico's army, personal patron of Pacioli, and "keen imitator of our [mathematical] disciplines".[116] Pacioli also recorded the participation of "eminent orators, expert in the noble arts of medicine and astronomy", briefly described by name and qualification. As shown by Monica Azzolini, all these figures were important representatives of Ludovico's courtly

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[113] "E questo al presente de le mathematici a lor com(m)mendazione. De le quali gia el numero in questa vostra inclita cita a la giornata come(n)ça per gratia de v(ostra) D(ucale) celsi(tudine) non poco a crescere per l(')assidua publica de lor lectura novellamente per lei introducta col proficere de li egregi audenti seco(n)do la gratia in quelle a me da l(')altissimo concessa chiaramente e con tutta diligentia (a lor iudicio) el sublime volume del prefato Euclide in the scientie de Arithmetica e Geometria, proportioni e p(ro)portionalita expone(n)dola. E gia a li suoi x libri dignissimo fine imposto interponen(n)do sempre a sua theorica ancora la pratica nostra a piu utilita e amplia intellige(n)tia de q(ue)lli, e a(l)la p(rese)nte expedition de questo el residuo del te(m)po deputando." Pacioli, *Compendium*, 9r.

[114] Monica Azzolini, "Anatomy of a Dispute: Leonardo, Pacioli and Scientific Courtly Entertainment in Renaissance Milan," *Early Science and Medicine*, vol. 9, no. 2, (2004): 117.

[115] "Correndo gli anni de nostra salute excelso D(uca) 1498 a di 9 de Febrario. Essendo nell(')inspugnabil arce del(l')inclita vostra cita de Milano dignissimo luogo de sua solita residentia a la presentia di q(ue)lla constituto in lo laudabile e scientifico duello da molti de ogni grado celeberrimi sapentissimi acompagnata si reliosi co(m)mo seculari de li quali assidue la sua magnifica corte abunda." Pacioli, *Divina proportione (1509)*, 1r (Compendium).

[116] As noted by Monica Azzolini, when the role of mathematics for practice, as clearly advocated in the *Compendium* is considered, it should be added that the "personal patron" of Pacioli of the court was the military captain Galeazzo Sanseverino. Azzolini, *Anatomia*, 118. Additionally, Pacioli had lectured on Euclid's *Elements* to a group including *condottieri* in Naples a few years before. Additionally, as made clear in the text, the mathematical and philosophical arguments and the background of Pacioli were closely correlated with the stance of artists and architects at the court.

entourage. Not only were they learned physicians at the court but, in at least two cases, were mathematicians practicing astronomy and astrology.[117]

Continuing his account, Pacioli added that there were other doctors in law, advisers, secretaries and chancellors, and very shrewd architects and engineers, "regular inventors of new things", such as Leonardo da Vinci "our Florentine compatriot, that in sculpture and painting each one knows by name".[118] To support his claim, Pacioli mentioned the approximate volume and weight of the equestrian statue of Francesco Sforza, and the fresco at Santa Maria delle Grazie, both commissioned to Leonardo by the court. In both cases, Pacioli compared Leonardo to the ancient artists and highlighted the superiority of his skills. Moreover, Pacioli added that Leonardo had started to write an invaluable work on local motion and accidental weights, having completed books on painting and human motion.[119] It is easy to note that the section dedicated to Leonardo is much richer in details and praise when compared to the preceding text.

Considering the rest of the dedicatory letter and preface and given the insistence of Pacioli on the necessity and certainty of mathematical disciplines, it is clear that the disputation in some way involved the standing and purposes of mathematical knowledge. Pacioli began his argument by referring to Aristotle and Averroes, claiming that the mathematical disciplines are true in the first degree and that every other practical and mechanical operation follows on from them, and correlating to this the preeminent role played by sight in acquiring

[117] Azzolini added that these university educated scholars likely considered themselves superior to scholars like Pacioli. Azzolini, *Anatomy*, 119-122.

[118] "Leonardo da venci compatriota nostro fiorentino, qual de sculptura getto e pictura con ciascuno el cog(n)ome verifica." Pacioli, *Compendium*, 2r.

[119] "[…] e non de queste satio a l(')opera inextimabile del moto locale e del(l)e percussioni e pesi e de(l)le forze tutte, cioe pesi accidentali. Havendo gia con tutta diligentia al degno libro de pictura e movimenti humani posto fine […]". Pacioli continued with a praise of "his like brother Iacomo andrea de Fer(r)ara, very knowledgable disciple of Vitruvius and not less (knowledgable) in military things." Information on Giacomo Andrea da Ferrara is very scarce and no buildings have been related to his input. ibid., 2r-v.

knowledge.[120] In what is accounted as a critique of at least some of the other scholars involved, Pacioli also lamented the paucity of good astronomers due to "the lack of knowledge of arithmetic, geometry, proportions and proportionality", adding that nine out of ten of their judgements were based on tables and measurements derived from ancient authors which were often mistakenly transcribed, leading to errors and prejudices. Pacioli concluded by claiming that mathematics was indeed a true science, and in this different from poetry since the latter "falsely deceive the ears as smoke".[121]

At this point, the purpose and claimed character of the *Compendium* become clearer if we consider the context of the courtly environment in which it took place. As Pacioli indicated to the duke, following on from the *Summa*, which was dedicated to Duke Guidobaldo of Urbino, a "close associate of yours", the *Compendium* was configured as the "complement and culmination of all other works on these mathematical disciplines ".[122] Additionally, Pacioli argued that this is a "most useful treatise", that "with all the material forms of the bodies that are included in it, will give not less admiration to whom will visit your library than the other volumes dealing with the most worthy subjects". This is because, Pacioli concluded, "these

[120] These passages also highlight that courtly patrons were receptive to experimentations and to challenges to the traditional hierarchy of disciplines. As noted by Azzolini, while the universities of the Renaissance tended to promote conservative views, it was at court where unconventional ideas and new theories could circulate more freely. Azzolini, *Anatomy*, 132

[121] "Conciosia che no(n) sieno fau(o)le arinimi e altre ridiculose e false Facetie ne anoco(ra) mendaci e i(n)credibili poetic(h)e inventioni. Le q(ua)li per un fume le orechie pascano." Pacioli, *Divina proportione 1498*, 3v. However, Pacioli adds that according to the Philosopher, the false things can be useful to the acknowledgement of the true; even if the true will be more useful and fruitful.

[122] "El che tutto a chi con diligentia la grandopera nostra di simil discipline e faculta compilata, e al Magnanimo d(i) v(ostra) Cel(situdi)ne affine Duca de Urbino Guido Ubaldo dicata […] Ma da quelle grandemente excitato represi lena a la piagia diserta e per condimento de ogni altra opera nostra di simil faculta composta. E a summo e delectevol gusto de tutte le prefate scientie e mathematici discipline a V(ostra) Cel(situdi)ne e a utilita de li reverenti subditi di quella […]" Pacioli, *Compendium*, 2r-3r.

forms have so far remained concealed to our contemporaries."[123] Here, Pacioli was referring to regular and other geometrical bodies derived from these by truncation.

The whole dedicatory letter and the reference to the *scientifico duello* can thus be interpreted as an affirmation of Pacioli's authority in the environment of the court, and the choice to focus on regular and other bodies derived from this should be considered in this context. Moreover, as Pacioli explained (again quoting Aristotle), all knowledge derives from seeing and "sight is considered, among the senses, the most noble by sages. Hence, it is not without reason that the eye is said by the vulgar [to be] the first door from which the intellect understands and savours".[124] This is the reason why, Pacioli argued, "I have decided to compose a *Compendium* of the mathematical disciplines and together with it, with my own hand and for the common good, materially give form in due manner to [geometrical] bodies in their proper form [*forma p(ro)pria*]. And these, with the present *compendium*, [I decided to] offer to your ducal celsitude".[125] As Pacioli concluded, the duke would take great pleasure [*grandissimo piacere*] in the rare appearance [*inusitato aspecto*] of the bodies,[126] especially when he will be able to find

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[123] "A decore ancora e p(er)fecto ornamento de la sua dignissima bibliotheca de inumerabile multitudine de volumi in ogni faculta e doctrina adorna a disponere q(ue)sto breve co(m)pendio e utilissimo tractato detto de divina proportione. El q(ua)le co(n) tutte sue forme materiali de li corpi che in ditto se co(n)tengono non menore admiratione a chi q(ue)lla visitara darano che tutti gli altri volumi co(n) l(')altre sue dignissime cose in q(ue)lla reposte si facino. Per esser dicte forme a li vive(n)ti finora state ascoste. Nel quale diremo de cose alte e sublimi quali verame(n)te sonno el cimento e copella de tutte le prelibate scientie e discipline e da quello ogni altra speculativa op(er)atione scientifica pratica e mecanica deriva." ibid., 3r-v.

[124] "Propter admirari ceperu(n)t ph(ilosoph)ari. Vole Excelso D(uca) la p(ro)posta auctorita del maestro di color ch(e) sa(n)no che dal vedere avesse initio el sape(re). Si co(m)mo el medesimo i(n) un altro luogo afferma dicendo Quod nihil est in intellectu q(uo)n (sic) prius sit in sensu: cioe che niuna fia nel(l')intellecto che quella priam non se sia per alchun modo al senso offerta. E de li nostri sensi per li savii el vedere piu nobile se conclude. Onde non immeritamente anchor de vulgari fia detto l(')ochio esser la prima porta per la qual lo intellecto intende e gusta." ibid., 4r.

[125] ""[…] la qual cosa fra me stesso pensando a questo utilissimo compendio de le scientie mathematici electo, la penna prender deliberai & insiemi con quello de mia propria mano materialmente per la comune utilita in forma propria li lor corpi debitamente formare. E quelli con lo presente compendio a V. D. Cel. offerirlo." ibid., 4v.

[126] "Pel cui inusitato aspecto, commo cosa a nostri tempi dal ciel venuta, non dubito el suo ligiadro e perspicaci intellecto prenderne grandissimo a piacere." ibid.

"the causes and most sweet harmony [of these bodies], with the help and intercession of the present treatise".[127]

After this introductory section, the *Compendium* discusses the *divina proportione* (golden ratio) (Chapters 6-7) and its effects (Chapters 11-22) linking these to the five regular and other bodies derived from these. The most extended section of the *Compendium* (Chapters 23-69) describes regular and other (semi-regular) bodies derived from these. It is in the ratios between the sides and the diameter of the circumscribing sphere of two of the regular bodies, as Pacioli noted, that the golden section is found.[128] Finally, Chapter 70 refers to the perspectival illustrations of the bodies and to a set of physical models,[129] while Chapter 71 gives a glossary of mathematical terms that precedes sixty coloured representations of regular and other geometrical bodies derived from these. Overall, it will become clear when other sections of the work are examined that the greatest achievement Pacioli assigned to the work resides in the visual and formal properties of the geometrical bodies presented in the treatise.

Pacioli clearly believed his courtly patrons to be an attentive and supportive audience and that Leonardo had expressed correlated arguments. The interactions of Pacioli and Leonardo extended well beyond the praise included in the *Compendium*. Leonardo annotated that he had bought a copy of the *Summa* and an extended group of his mathematical notes, particularly referring to the contents of the first ten books of Euclid's *Elements*, can be dated to the period

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[127] "[…] di tal forme sue cause, e dolcissima armonia, con l(')aiuto e suffragio del presente tractato retrovara." ibid.

[128] These passages will be examined in Chapter 2.

[129] In the *Tractato de l'architectura*, published in 1509, Pacioli confirmed to have made three set of models of regular and other bodies derived from these to accompany the three manuscript copies. Pacioli, *Divina proportione (1509),* 29v (Tractato de l'architectura).

in which the two collaborated at the Sforza court of Milan.[130] Additionally, as noted, Leonardo contributed to Pacioli's *Compendium* with the drafts of the perspectival representations of regular and other geometrical bodies derived from these that complement the text.[131]

Correspondences between Pacioli's texts and Leonardo's notes have attracted considerable attention from scholars in the last few decades.[132] The privileged position assigned to sight, and the link between sensory experience and knowledge are found both in Pacioli's texts and in a number of notes left by Leonardo.[133] In a famous passage of the *Paragone* (part of the treatise on painting collected after his death), Leonardo argued that painting is a *scientia*, being based on the principles of mathematics and geometry and that "no human investigation may claim to be a true science if it does not pass through mathematical demonstrations"; adding shortly after that "nothing renders certainty of itself without experience".[134] In another passage, Leonardo contended that painting, based on the noblest sense of vision, is superior to all the other arts, especially poetry, since the latter relied on the inferior sense of hearing.[135] In the *Compendium*, Pacioli similarly argued that as perspective relies on sight, it should be considered a liberal art as much as music. The same passage significantly ends with a

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[130] Notably, Leonardo transcribed passages from a version of Euclid's *Elements* in vernacular Italian and from Pacioli's *Summa*.As suggested by Di Teodoro, this could have been a copy of the translation of Campanus's edition of Euclid's *Elements* that Pacioli claimed to have made in a document of 1508. The interaction between Leonardo and Pacioli, and the consequent improvements in Leonardo's mathematical knowledge have been often noted, whereas Leonardo drafted a set of perspectival representation of regular and other bodies derived from these for Pacioli's *Compendium de divina proportione*. In Codex Atlanticus (120 rd) Leonardo recorded: "Impara da maestro Luca la multiplicazione delle radici".

[131] These will be discussed in more detail in Chapter 3 of the thesis.

[132]A good overview is given by Augusto Marinoni in the introduction to Pacioli's *De divina proportione*: Marinoni, *Luca Pacioli e il "De divina proportione"*, 16-18; and Monica Azzolini: Azzolini, *Anatomy of a dispute,* 124-128.

[133] "Nissuna humana investigatione si pò dimandare vera scientia se essa non passa per le Mathematiche dimostrationi. E se tu dirai che le scientie che principiano e finischano nella mente habbiano verita, questo non si concede, ma si niega per molte raggioni. E prima che in tali discorsi mentali non accade esperientia, sanza la quale nulla dà di sè certezza." Claire J. Farago, *Leonardo da Vinci's Paragone* (Leiden: Brill, 1992), 178-179.

[134] The Italian reads: "Nissuna humana investigatione si pò dimandare vera scientia se essa non passa per le Mathematiche dimostrationi. E se tu dirai che le scientie che principiano e finischano nella mente habbiano verita, questo non si concede, ma si niega per molte raggioni. E prima che in tali discorsi mentali non accade esperientia, sanza la quale nulla dà di sè certezza." ibid.

[135] ibid. The passages have been discussed by Monica Azzolini. Azzolini, Anatomy, 124-125

reference to the naturalistic visual qualities of Leonardo's painting of the Last Supper at the convent of Santa Maria delle Grazie.[136]

At this point it is useful to note that among the other individuals associated with Leonardo that Pacioli most likely met at the Sforza court, or met again after Urbino, is Donato Bramante.[137] Possible interactions between Pacioli and Bramante have received very little attention; however, two architectural works in Milan associated with Bramante are described in Pacioli's *Compendium* as examples of the employment of a 72-bases body in architecture.[138] Bramante was involved in the environment of the court of Milan from at least the early 1480s and there are no doubts as to his expert standing at the court when Pacioli was involved. Bramante's geometrical skills and knowledge of perspective are well testified by two of his earliest works in Milan, the Prevedari Print and the choir of Santa Maria presso San Satiro.[139] As often noted, Bramante most likely acquired his knowledge of geometry and theory of perspective whilst practicing in the environment of the court of Urbino. Giorgio Vasari wrote that Bramante had practiced the abacus at length in his childhood, had studied the works of Fra' Carnevale of Urbino, and since he always delighted [*si dilettò*] in practicing architecture

[136] Pacioli, *Compendium*, 10r-v.

[137] Although there are no known works of Bramante in Urbino, it is generally accepted that his training in painting, perspective and architecture took place in or around the environment of the court of Urbino. Bramante was born in the proximities of Urbino (while Vasari indicated Castel Durante as the place of birth other sources point to Fermignano) around the year 1444. A general overview of the formative period of Bramante in Urbino was given by Arnaldo Bruschi, "La formazione e gli esordi di Bramante," in *Bramante Milanese e l'Architettura del Rinascimento Lombardo,* ed. Christoph L. Frommel, Luisa Giordano, and Richard Schofield (Venice: Marsilio, 2002), 33-66. The analysis of Bruschi also focussed on possible contributions of Bramante to paintings produced in Urbino in the period and to the design of spaces inside Palazzo Ducale, like the alcove in Federico's bedroom (decorated by Fra' Carnevale) or the chapels of the 'Spirito Santo' and 'Delle Muse', attributed to Bramante by Pasquale Rotondi; Pasquale Rotondi, *Il Palazzo Ducale di Urbino* (Urbino: Istituto statale d'arte per il libro, 1950). However, there is no strong evidence to support these claims.

[138] Pacioli, *Compendium*, 60v.. These will be examined in Chapter 4 of the thesis.

[139] After a passage in Bergamo in 1477, where he painted the facade of the Palazzo del Podestà, a work today unfortunately almost completely lost, Bramante arrived in Milan in the late 1470s. The first surviving work recorded in Milan is an extremely important document. This is mentioned in a contract between the painter Matteo Fedeli and the engraver Bernardo Prevedari, dated 24 October 1481.

and perspective, he had moved to Lombardy.[140] The Milan-born humanist Sabba da Castiglione, wrote instead in a work published in 1554, that Bramante was a "disciple of Mantegna and great in the use of perspective [*gran prospettivo*] as a pupil of Piero della Francesca".[141] Personal interactions between Leonardo da Vinci and Bramante are recorded among the notes by the former.[142] Although meetings between Bramante and Pacioli in Urbino or Milan are not documented, it is clear that they shared multiple connections and mathematical interests in these environments, while being about of the same age.[143] It would be surprising if Bramante did not know Pacioli's works. In any case, what we know for certain is that a lecturer of arithmetics and geometry who was involved at the Sforza court such as Pacioli gave a geometrical interpretation of elements of Bramante's architectural works in his *Compendium*. A comparison between the arguments found in Pacioli's text and the design process deduced from the analysis of Bramante's works will be made in the last part of this thesis.

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[140] "[Bramante] nacque in Castello Durante nello Stato di Urbino, d'una povera persona, ma di buone qualità. E nella sua fanciullezza oltra il leggere e lo scrivere, si esercitò grandemente nello abbaco. Ma il padre che aveva bisogno che e' guadagnasse, vedendo che egli si dilettava molto de 'l disegno, lo indirizzò ancora fanciulletto a l'arte della pittura, nella quale studiò egli molto le cose di fra' Bartolomeo, altrimenti fra' Carnovale da Urbino, che fece la tavola di Santa Maria della Bella in Urbino. Ma perché egli sempre si dilettò de la architettura e de la prospettiva, si partí da Castel Durante e, condottosi in Lombardia, andava ora in questa ora in quella città, lavorando il meglio che e' poteva […]' Giorgio Vasari, *Le vite de' più eccellenti archittetti, pittori, et scultori italiani, da Cimabue insino a' tempi nostri …* (Florence: Lorenzo Torrentino, 1550).

[141] "discepolo del Mantegna e gran prospettivo, come creato di Piero del Borgo" Sabba Da Castiglione, *Ricordi ovvero ammaestramenti* (Venice: Paulo Gherardo, 1554), 139.

[142] Leonardo mentions Bramante in at least three occasions in the surviving corpus of his notes, notably once referring to some unspecified buildings of Bramante"'Edifiti di Bramante" in a list of notes related to the fall of Ludovico Sforza, and another time with the familiar name "Donnino". "Manner of the drawbridge that Donnino showed me"', "Modo del ponte levatoio che mi mostrò Donnino" in the original. Notably "Donnino" is the same name that Bramante's parents used in their will. Leonardo da Vinci, Codex M, folio 53v, datable c. 1499. The third instance refers to the "gruppi del Bramante" that is the knots, the drawings of twines as conceived by Leonardo for the Sala delle Asse in Castello Sforzesco or for the emblem of the "Achademia Leonardi Vinci", Codex Atlanticus, folio 225rb, datable c. 1490. Donato Bramante, *Sonetti e altri scritti*, ed. Carlo Vecce (Rome: Salerno Editrice, 1995), 10-12.

[143] Vasari says that Bramante died in 1514 at 70 years of age "visse Bramante anni LXX". There are no reason to doubt this information. Vasari, *Le vite.*

1.1.6 The return to Venice's School of the Rialto and the publication of Euclid's *Elements* and *Divina proportione*

At the fall of Ludovico Sforza in September 1499, Pacioli, Leonardo da Vinci, and Bramante left Milan. After a passage of time at the court of Mantua (where Pacioli dedicated a short treatise on chess to Isabella d'Este),[144] Pacioli and Leonardo were both documented as being in Florence. Pacioli continued lecturing on arithmetics and geometry with a probable appointment at Perugia in 1500 and a number of recorded appointments in the *studium* of Pisa in Florence from 1500 to 1506.[145] In this period Pacioli completed another text, *De viribus quantitatis.* This is a vast collection of mathematical puzzles. It is composed of a dedicatory letter to an anonymous dedicatee, a prologue and a text divided into three main parts. Although not original in its contents, the text is a comprehensive survey of the literature in mathematical puzzles up to the late fifteenth century. The first part deals with "numeral forces, that is arithmetic" and concerns problems employing whole numbers and progressions. The second part deals with "of the virtues and geometrical force" and with problems of a physical-mechanical character and is followed by other secondary subjects such as proverbs. The third part is "of the force and natural virtue in writing", concerning games, riddles, recipes, etc.[146]

In May 1504, Pacioli was elected provincial minister of the Conventual Franciscans and in 1508 a papal bull of Julius II dispensed him of the vow of poverty, an additional sign of his

[144] This is *De ludo scachorum dicto Schifanoia*, a short treatise (48 folios) with game strategies on chess. Probably begun in Milan, the treatise is mentioned by Pacioli in *De viribus quantitatis*, 2r and among the list of works included by Pacioli in the request of copyrights issued to the doge Leonardo Loredan and dated 19 December 1508. Archivio di Stato di Venezia, *Collegio*, *Notatorio*, reg. 16, cc. 34v-35r; Boncompagni, "Intorno alle vite," 431; Di Teodoro, "Pacioli, Luca."

[145] Di Teodoro, "Pacioli, Luca."; Ulivi, "Nuovi documenti," 39. As noted by Ulivi, Pacioli is documented as lecturer in the *Studium* of Pisa in Florence from November 1500 to October 1506. Pacioli is also recorded in the register of lecturers at the *Studium* Bologna, in the academic year 1501-1502. However, he may have turned down this appointment. In the same period (1500-1506) Pacioli is documented in several occasions in his native Sansepolcro.

[146] Di Teodoro, "Pacioli, Luca."; Pacioli, *De Viribus*; Boncompagni, "Intorno alle vite."

established connections with the Della Rovere.[147] In the same year, he returned to the School of the Rialto in Venice, where he gave an introductory lecture to the fifth book of Euclid's *Elements,* and perhaps delivered other lectures on the subject. By inviting Pacioli some forty years after his time as student, Sebastiano Foscarin (the newly appointed head of the school) and his collaborators clearly believed that there was a correlation between Pacioli's interests, his theoretical framework and the subjects taught at the school. This is important because all the contemporary information on Foscarin's teachings and philosophical frame of reference point to the works of Aristotle and the Great Commentator Averroes.[148]

The lecture introduced the theory of proportions expounded in book five of Euclid's *Elements.* Pacioli thus dealt with one of the subjects that was more familiar to him, the theory of proportions in Euclid's *Elements*. This is at the core of both the *Summa* and the *Compendium de divina proportione*. The contents of the lecture and a list of selected attendees were included before the fifth book of Campanus's Euclid's *Elements,* an edition that was amended, commented on and printed by Pacioli in Venice in 1509.[149] The lecture was mostly an overview of the philosophical framework of the theory of proportions found in book V of the *Elements*.[150] Pacioli started by presenting theological and metaphysical arguments that demonstrated the universal status endowed to proportions. Pacioli noted that this was the first

147 Ulivi, "Nuovi documenti," 56. As noted, Pacioli claimed in the Summa to have visited the house of the then cardinal Giuliano Della Rovere in 1489 and shown here a set of models of regular and other bodies to the duke of Urbino.

148 As noted by Fernando Lepori, the historian Marco Guazzo, a contemporary of Foscarin, wrote that Foscarin "expounded the difficult theories of *Aristotile* and of *Averois* the great commentator" [*espose le cose difficili di Aristotile e di Averois il gran commentatore*]; Panfilo Monti dedicated to Foscarin his edition of the *Opera omnia* of the averroist Alessandro Achillini; Marcantonio Venier offered to Foscarin his *Phisiologia*, in which dealt with "many opinions of Aristotle and Averroes" [*molte opinioni di Aristotele e di Averroè*], among others. Lepori, "La scuola di Rialto," 596-597.

149 As Pacioli noted, the lecture took place in the church of San Bartolomeo on the 11th August 1508. As noted by Bruno Nardi, recognised authorities in the subjects taught at the school were commonly asked to deliver such prolusions. Nardi, *La Scuola di Rialto,* 67.

150 Bruno Nardi has transcribed the whole prolusion included by Pacioli in his amended edition of Campanus's Euclid *Elements.* ibid., 67-72.

principle used to create and arrange every visible thing in the celestial and terrestrial realm.[151] Pacioli also noted that Aristotle in the *Physics* studied the proportions involved in motion with great acumen, while in the *Categories* proportions were indicated as the special means to investigate the relationships occurring among natural things.[152]

From this, Pacioli followed with a list of practical disciplines that rely on proportions and proportionality: medicine, astronomy, chorography and cosmography, architecture, painting and perspective, sculpture, music, rhetoric, grammar, laws, and mechanical applications, as proved by experience. The lecture ended with a discussion of the arithmetical and geometrical properties of proportions and proportionality. As noted, a list of the selected attendees was attached to the text. Senatorial and political figures are followed by a group of professors in theology [*Reverendi Sacre Theologie Professores*]. Among these are Fra Giovanni Giocondo da Verona [*Frater Iocundus Veronensis Antiquarius. Omnes prelibati Eiusdem Minoritariae Familiae*] who would publish an illustrated copy of Vitruvius's *De architectura* in 1511, Bernardo Bembo, and Sebastiano Foscarin, *philsophie p(ro)fessor clarissimus*. A group of *Medici Illustres* then follows, among them *Aldus Manutius Romanus,* the three brothers Rompiasi taught by Pacioli during his stay in Venice in the late 1460s, the sculptor and architect Pietro Lombardo, Francesco Rosselli *Florentinus Cosmographus*, etc.[153] The composition of the audience attests the level of authority reached by Pacioli at this point in his life.

The remaining contents of the edition of Euclid's *Elements* published by Pacioli in 1509 [*Euclidis megarensis philosophi acutissimi mathematicorumq(ue) omnium sine controversia principis Op(er)a a*

[151] "Hanc prae oculis summus opifex in caelestium terrestiumque rerum dispositione semper habuit dum orbium motus cursusque syderum et planetarum omnium ordinatissime disponeret […]" Pacioli, *Euclidis*, 30r

[152] "Naturales aut(em) et ipsi ut paulo ante diximus psedulo rerum naturalium proportiones quaesivere, prout in eoru(m) codicibus passim habetur, pr(a)esertim Aristotelis, cuius opera prae aliis assidue prae manibus habe(n)tur. Nam in de physico auditu proportionem motuum inter se subtilissime perscrutatur. Et ex decem predicamentis, quo numero debario o(mn)es ph(ilosoph)i contenti extitere, unu(m) relatio(n)is seu ad aliq(ui)d hiuc ta(m) sublimi indagatrici p(ro)portioni, s(cilicet) sp(eci)aliter addicavit." Pacioli, *Euclidis*, 30r.

[153] Pacioli, Euclidis, 31r-v. An epistle of Isidoro Bagnoli concludes this section.

Campano interprete fidissimo tra[ns]lata] are now considered. The texts of the 15 books of Euclid were taken from the *editio princeps* based on the edition of Campanus, and separate comments were added by Pacioli.[154] The volume opens with a dedicatory epistle addressed to Francesco Soderini, the Cardinal of Volterra and brother of Pier Soderini, perpetual *gonfaloniere* of the Florentine Republic, where, as we have seen, Pacioli took refuge after leaving Milan. In the opening epistle, Pacioli recalls the texts dedicated to the Duke of Urbino and the Duke of Milan, and his collaboration and closeness with Leonardo da Vinci [*carissimo Leonardo Vincio*].[155]

Conjointly with Euclid's *Elements* Pacioli printed the *Divina proportione*. This includes five epigrams (including "a sonnet of the author", a dedication letter to Pier Soderini, an epistle of Daniele Gaetani ad Andrea Mocenigo, a glossary related to geometrical bodies, and a summary of the contents of the work). These are followed by the *Compendium* completed in Milan in 1498 (*Pars Prima*), a treatise on architecture [*Tractato de l'architectura*] (*Pars Secunda*) dedicated to local sculptors and stonecutters from Sansepocro, and a treatise on regular and other bodies derived from these, the *Libellus in tres partiales divisus q*(*ui*)*nq*(*ue*) *corpor*(*um regularium et depe*(*n*)*dentiu*(*m*) again dedicated to Pier Soderini (mostly a translation in vernacular Italian of the *Libellus de quinque corporibus regularibus* of Piero della Francesca). To these were added a full-page drawing of a head in profile linked to the treatise on architecture, a series of full-page capital letters drawn with a ruler and a compass, a drawing of a temple's door [*porta speciosa*] (attached to the treatise on architecture), 59 full-page xylographies of regular and other

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154 Menso Folkerts compared Pacioli's edition of the text of Campanus with the *editio princeps* of Euclid's *Elements* based on Campanus's edition printed by Erhard Ratdolt in Venice in 1482, and showed that the amendments made by Pacioli to the text of Campanus were minor. The main contributions of Pacioli are undoubtedly in the added separate comments (introduced by Pacioli as *Castigator* and thus intended to be corrections or clarifications). Overall, Folkerts has counted 136 comments of this type, 42 of which are more than 10 lines in lenght. Menso Folkerts, *The Development of Mathematics in Medieval Europe: The Arabs, Euclid, Regiomontanus* (Aldershot: Ashgate, 2006), 228-230.

155 Pacioli, *Euclidis*, 2r. The letter is followed by an epigram of Daniele Caetani, and an epistle of Daniele Caetani to Rainerio *patricio veneto,* and another epistle of Francesco Massari to Giacomo Cocco.

bodies derived from these numbered from I to LIX, interrupted by two full-page drawings of architectural details between the bodies numbered XIII and XIV. Finally, a tree diagram displaying the relationship among different types of proportions [*Arbor proportio et proportionalitas*] is found. Essentially, the work is a collection of a wide range of materials linked by the central role of proportions, and developed by Pacioli in a period of more than ten years.

From the title page of the *Divina proportione*, Pacioli makes it clear that this is a "work necessary to all the perceptive and curious intellects, where every scholar of Philosophy, Perspective, Painting, Sculpture, Architecture, Music and other mathematical subjects, suave, subtle, admirable knowledge will accomplish and will find delight with various matters of the most secret science".[156] Indeed, the relationship between theory and practice, articulated within Pacioli's theoretical framework, is an ever-recurring argument throughout the *Summa*, *Compendium de divina proportione* and *Divina proportione*.

When the *Divina proportione* and *Euclid's Elements* were printed, Pacioli was more than sixty years old. A further appointment as lecturer of arithmetic and geometry at the *studium* of Perugia in 1510-11 followed. One of Pacioli's last documented appointments was as lecturer in arithmetic and geometry in Rome in 1514, which was the *studium* under the direct control of the papacy.[157] Pacioli died in 1517.[158]

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[156] "Opera a tutti glingegni perspicaci e curiosi necessaria Oue ciascun studioso di Philosophia: Prospettiua Pictura Sculptura: Architectura: Musica: e altre Mathematice: suavissima: sottile: e admirabile doctrina consequira: e delecterassi: co[n] varie questione de secretissima scientia." Pacioli, *Divina proportione (1509)*, Title page.

[157] Ulivi, "Nuovi documenti," 41-43.

[158] As noted by Ulivi, after the 15 April and before the 6 July 1517. Ulivi, "Nuovi documenti," 57.

1.2 The theoretical and practical foundation of Pacioli's *Summa, Divina proportione* and *Euclid's Elements*

1.2.1 The format and readership of the works of Pacioli

One of the elements that emerges throughout the biography and the works of Pacioli alike is the consistent and interrelated engagement with both practical-commercial and theoretical-philosophical mathematical knowledge. This is reflected in Pacioli's appointments as lecturer of commercial mathematics and practical geometry in public schools and as lecturer of arithmetics and algebra in universities. Additionally, by 1484, Pacioli was a *magister theologiae* and showed throughout his work to be particularly aware that the knowledge of mathematical principles and their deductive structures were instrumental in theological investigations.[159] The correlation of theory and practice that Pacioli argued for and displayed in the *Summa* and *Divina proportione* emerged from the network of contacts and interests that he was able to organise throughout his scholarship. In Milan, Pacioli wrote that he was lecturing on "the theory and the practice", adding that his time was divided between lecturing and working on the *Compendium* presented to the duke.[160] The novelty and clarity of Pacioli's approach in connecting theory and practice, and in their dissemination, has been often noted. However, what has been missing is an examination of how this was materially linked with the work of practitioners. In other words, what was the relationship between the work of practitioners who interacted with Pacioli and the theoretical-practical framework found in his works?

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[159] For example, Pacioli mentions John Duns Scotus's commentary on the *Sententiae* (sic) of Peter Lombard as exemplar in this regard. Pacioli, *Compendium*, 7r.

[160] "E questo al presente de le mathematici a lor com(m)mendazione. De le quali gia el numero in questa vostra inclita cita a la giornata come(n)ça per gratia de v(ostra) D(ucale) celsi(tudine) non poco a crescere per l(')assidua publica de lor lectura novellamente per lei introducta col proficere de li egregi audenti seco(n)do la gratia in quelle a me da l(')altissimo concessa chiaramente e con tutta diligentia (a lor iudicio) el sublime volume del prefato Euclide in the scientie de Arithmetica e Geometria, proportioni e p(ro)portionalita expone(n)dola. E gia a li suoi x libri dignissimo fine imposto interponen(n)do sempre a sua theorica ancora la pratica nostra a piu utilita e amplia intellige(n)tia de q(ue)lli, e a(l)la p(rese)nte expedition de questo el residuo del te(m)po deputando." Pacioli, *Compendium*, 9r.

Pacioli had studied and lectured in commercial arithmetic and practical geometry. These subjects, together with algebra, had already been discussed in his first known treatise, the *Tractatus mathematicus ad discipulos perusinos,* and in the now lost manuscripts prepared in Venice and Zadar.[161] As remarked in the *Summa* and in his introductory lecture at the School of the Rialto, Pacioli was well aware of the increasing importance that mathematical knowledge was acquiring in accounting, surveying and in its practical applications in crafts and trading in the late fifteenth century. Although some of these works are lost, from what can be assumed from Pacioli's remarks, these treatises were all redacted in vernacular Italian (*volgare*) as this was the language mostly known by the cultural strata of the practitioners, including merchants, artisans, artists and architects, and was the language used in schools of commercial arithmetics and practical geometry.

Pacioli was also appointed as lecturer in arithmetics and algebra in universities and public schools on a number of occasions. Euclid's *Elements* had been the core of Medieval universities' curricula and continued in the fifteenth and sixteenth centuries. However, out of the fifteen books of the *Elements* (of which the last two are now considered spurious), only the first six were usually required in a course of study.[162] The contents of these books were important for astronomical calculations used by clerics for ecclesiastical recurrences and by medics for their use of astrology in diagnostics and predictions. However, Pacioli noted that

[161] Pacioli mentions both texts in the *Summa,* the manuscript in Venice was redacted in the year 1470, the one in Zadar in 1481. Pacioli, *Summa*, 67v (Pars prima). Elisabetta Ulivi, "Giovanni del Sodo, un maestro d'abaco fiorentino*,"* in *Luca Pacioli - Maestro di contabilità - Matematico - Filosofo della natura*, ed. Esteban Hernández-Esteve and Matteo Martelli (Umberyide: University Book, 2018), 99.

[162] Edward Grant, "*Science and The Medieval University*," in Rebirth, Reform and Resilience, Universities in Transition 1300-1700, ed. James. E. Kittelson and Pamela J. Transue (Columbus: Ohio State University Press, 1984), 71. Notably, these are the books dealing with the construction of planar geometrical figures and with the theory of proportions. Books I-IV of Euclid's *Elements* in fact include a set of definitions related to planar geometrical figures, they deal with two-dimensional problems and with the construction of figures such as triangles, squares, regular pentagons, hexagons, etc. Problems involving magnitude and proportions are introduced from Book V, while three-dimensional objects are only discussed starting from a set of definitions in Book XI.

he had covered the first ten books of the *Elements* while lecturing in Milan.[163] Since Pacioli was lecturing not in a *studium* but in a school under the direct control of the duke, the curriculum was undoubtedly different and also covered practical applications, as noted by Pacioli. While Pacioli's university lectures and his introductory lecture at the School of the Rialto were delivered in Latin, his lectures in commercial mathematics and practical geometry were most likely delivered in *volgare*.

Unlike most practitioners, Pacioli was thus able to read and lecture on theoretical and practical mathematics both in Latin and vernacular Italian, while being well aware that to reach a wider public of practitioners he needed to publish his works in the latter. It is commonly opined that the meeting with Alberti in the early 1470s contributed to Pacioli's decision to publish most of his works in *volgare.* In the proemium to *Della Famiglia*, Alberti famously justified the use of volgare as linked to the need to reach a wider public beyond the rarity of *letterati*.[164] The *volgare* was used by Pacioli both in the *Summa* and *Divina proportione* (both in the *Compendium,* completed in 1498, and in all the sections added in 1509).[165] Additionally, although Pacioli published a commented edition of Euclid's *Elements* in Latin starting from Campanus's, he also listed a translation of Euclid's *Elements* in *volgare* in his

[163] "E questo al presente de le mathematici a lor com(m)mendazione. De le quali gia el numero in questa vostra inclita cita a la giornata come(n)ça per gratia de v(ostra) D(ucale) celsi(tudine) non poco a crescere per l(')assidua publica de lor lectura novellamente per lei introducta col proficere de li egregi audenti seco(n)do la gratia in quelle a me da l(')altissimo concessa chiaramente e con tutta diligentia (a lor iudicio) el sublime volume del prefato Euclide in the scientie de Arithmetica e Geometria, proportioni e p(ro)portionalita expone(n)dola. E gia a li suoi x libri dignissimo fine imposto interponen(n)do sempre a sua theorica ancora la pratica nostra a piu utilita e amplia intellige(n)tia de q(ue)lli, e a(l)la p(rese)nte expedition de questo el residuo del te(m)po deputando." Pacioli, *Compendium*, 9r .

[164] Leon Battista Alberti, *Opere volgari*, ed. Cecil Grayson, vol. I (Bari: Laterza, 1960), 155. In the dedication of the *Teogenio* to Lionello d'Este, a work written in *volgare*, Alberti adds "in modo ch'io fussi inteso da' miei non litteratissimi cittadini". ibid., vol. II (Bari: Laterza, 1966), 55. Enzo Mattesini, "Luca Pacioli e l'uso del volgare," in *Luca Pacioli e la Matematica del Rinascimento,* in *Luca Pacioli e la Matematica del Rinascimento*, ed. Enrico Giusti (Perugia: Petruzzi Editore, 1998), 259-260.

[165] However, excerpts in Latin, taken from other works and particularly from Euclid's *Elements* in the edition of Campanus and Vitruvius's *De Architectura* in the Florentine edition of 1496 are found both in Pacioli's *Summa* and *Divina proportione (Tractato de l'architectura)*. However, although only fragmentary and often paraphrased, large parts of Pacioli's *Tractato de l'architectura* are an attempted transposition in vernacular Italian of Vitruvius's *De architectura,* predating the first complete Italian edition of Cesariano (1521).

request for publication rights issued to the Venetian senate in 1508.[166] Although a copy of this translation has still to be found, passages from it occur both in the *Summa* and in the *Compendium de divina proportione.* It should be noted that these particularly concern book V and books XII-XV of the *Elements,* discussing the theory of proportions and three-dimensional geometry.

Although it was common for a treatise on commercial arithmetics and practical geometry to be written in *volgare*, the *Summa,* as noted, also included additional and systematic references to theoretical knowledge commonly found only in texts redacted in Latin. A passage from the dedicatory letter that opens the work is particularly eloquent in this regard:

> […] our primary aim has been to give with due diligence norms that give a good knowledge on how to work in these fields, since this can be well understood through an ordered process. And because in our times a clear knowledge of these terms in Latin is almost lost for the scarcity of good preceptors that can explain them; [and] in order to show respect to all the subjects of V. D. S.[167], you should not expect a style higher than that of Cicero, as if [these] writings were a fount of eloquence. [This is] because considering that this [style] is not for everyone, I have arranged [this work] in maternal and vernacular language, in a way that both the literate and the vulgar, beyond its usefulness, will have a great pleasure in practicing it. And [this work] may be given to those of all arts, crafts and faculties, since the ample [number of] cases that it covers can find application in all things.[168]

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[166] Boncompagni, "Intorno alle vite inedite di tre matematici," 431.

[167] This refers to the duke Guidobaldo of Urbino, dedicatee of the work.

[168] "[…] el nostro principale inte(n)to è stato de dar norma con summa diligentia da ben sap(er)e che in ditte facultà op(er)are: si co(m)e p(er) suo ordinato p(ro)cesso manifestamente si po co(m)prehe(n)dere. E p(er)che ali t(em)pi n(ost)ri la chiara notitia de lor scabrosi termini fra li latini quasi e de p(er)dita per la rarita de buoni p(re)ceptori che la demostrino. Habiando sempre respetto ala commune utilità de tutti li revere(n)ti subditi de V. D. S. a q(ue)lla un più alto chel Ciceroniano stilo non saspetaste, co(m)e a fonte de eloque(n)tia scrive(n)do sap(er)tene. Ma ateso che a ognuno cio non sia capaci. Pero in materna e vernacula lengua mi so messo a disponerla. In modo che litterati e vulgari Oltra l'utile ne haranno grandissimo piacere in essa exercitandose. E sieno dati a che arti, mestieri, e faculta si voglia. Per l'ampla generalita che in lei si contene: da poterse a tutte cose applicare." Pacioli, *Summa*, 2r (Pars prima). Notably, Pacioli published the dedicatory letter that opens the Summa both in Italian and Latin, a rare case of auto-translation that, as noted by Di Teodoro, is found in the *Decennale* (1504) of Niccolò Machiavelli, in the *Universal vocabulario* (1490) di Alonso de Palencia or in the two lexicographic works (1492 e 1495) of Antonio Elio de Nebrija. Di Teodoro, "Pacioli, Luca."

The reference to the act of practicing [*exercitandose*] is particularly relevant in this passage, as is the reference to a broad range of practical uses that the text aims to address. In another passage of the *Summa*, Pacioli states that the *pratici vulgari* (practitioners who can read only in vernacular Italian) are the primary addressees of his work.[169] This clearly highlights the type of audience Pacioli had in mind and marks a clear difference with all his declared sources.[170] Finally, the emphasis given by Pacioli to theoretical principles that "find application in all things" [*da poterse a tutte cose applicare*] is particularly noteworthy. Pacioli claims in the *Summa* a direct relationship between the theory and the practice, between the mathematical and material realm.[171]

Pacioli declares in the *Summa* to be willing to write in his *materna e vernacula lengua.* As noted by Enzo Mattesini, the actual language employed is a mix of the local *volgare* of Sansepolcro and its adjacent areas, the language used in the courtly environments which Pacioli visited, and other inflections found in the north of the Italian peninsula, particularly in the area of Venice.[172] It is therefore unsurprising that in the sixteenth century, the mathematician and biographer Bernardino Baldi criticised the poor language of the *Summa*, after praising the mathematical skills and close relationships of Pacioli with painters, sculptors and architects,

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[169] Discussing perfect numbers and the previous authors that discussed this subject, Euclid and Boethius, Pacioli notes that nevertheless this is "unknown to many, and especially to the practitioners that speak in vernacular [*practici vulgari*], to whom this work is primarily addressed" "[…] a la cui instantia principalemte questopera intendo" Pacioli, *Summa*, 8v (Pars prima, distinctio prima, tractatus primo).

[170] Indeed, the works of authors such as Boethius, Euclid or Jordanus to mention some of the acknowledged sources of Pacioli, circulated only in Latin in the fifteenth century. (*Summa*, Summario della prima parte principale, c.n.n.).

[171] This certainly recalls the Aristotelian discussion of mathematics. The relevance of Aristotelian writings in the texts of Pacioli is discussed in the ensuing sections of the thesis.

[172] Enzo Mattesini, "Luca Pacioli e l'uso del volgare," in *Luca Pacioli e la Matematica del Rinascimento,* in *Luca Pacioli e la Matematica del Rinascimento*, ed. Enrico Giusti (Perugia: Petruzzi Editore, 1998), 249-282.

and particularly with Piero della Francesca. He noted nonetheless that "in this way his work could be studied and understood by merchants and practitioners".[173]

Overall, three qualities related to the format and circulation of the three considered works by Pacioli are particularly noteworthy. Firstly, as noted, Pacioli's texts testify a two-way exchange of knowledge, a reception and transmission of theoretical knowledge on the one hand, and multiple correlations of theoretical and practical knowledge on the other. The *Summa* and *Divina proportione* can be read in this way as acting as the meeting point between two distinct groups and two languages.[174] Secondly, Pacioli was able to exploit the opportunities for marketing and dissemination offered by both the employment of vernacular Italian and the introduction of the printing press in the Italian peninsula in the second half of the fifteenth century.[175] His previous interactions in Venice had most probably provided the connections to see his *Summa*, *Divina proportione* and Euclid's *Elements* printed in the city. The circulation through printing and the use of *volgare* can both be read as precise editorial choices by Pacioli

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[173] Baldi, a pupil of Federico Commandino, includes Pacioli among "the most excellent mathematicians" and praises him as "extremely diligent illustrator of the mathematical disciplines and as such kept on high esteem by many". Bardi did no mention the accusation of plagiarism made by Vasari and included Piero della Francesca among Pacioli's compatriots and friends. He also gave a detailed account of the structure of the *Summa* and added that Commandino had planned a review of the work that couldn't be completed due to his death. He continued arguing that "there was no painter, sculptor or architect of his times that was not a close friend" with Pacioli. Baldi's impression follows the opinion of other mathematicians of the sixteenth century. Their views on the works of Pacioli have been mainly concerned his *Summa*. "È degno frate Luca d'essere connumerato fra quei matematici più eccellenti de' quali da noi si scrivono le vite, e ciò per essere stato ne' suoi tempi diligentissimo illustratore di queste discipline, e da tutti per questa cagione molto stimato […] Non vi fu pittore, scultore, o architetto de' suoi tempi che seco non contrahesse strettissima amicitia, fra' quali vi fu Pietro de Franceschi suo compatriota, pittore eccellentissimo e perspettivo […] Scrisse frate Luca ne la sua lingua materna, come si disse, accioché da' mercatanti et artefici l'opera sua potesse essere studiata et intesa, nondimeno poco felicemente gli successe poiché il suo dire è di maniera barbaro, irregolato, rozo ed infelice che rende nausea a quelli che leggono le cose sue, e certo che se sotto cotanta sordidezza di parole non vi fossero considerationi così belle et utili, non sarebbe quell'opera degna della luce". Bernardino Baldi, *Le vite de' matematici*, 331, 338, 343-344.

[174] This can be referred to what Pamela O. Long calls 'trading zones', emerging in the fifteenth and sixteenth century Europe. These are period that precedes the development of professionalism and that are characterised by "a kind of fluidity and openness to discussion concerning issues of design and problems in engineering in which a variety of people from diverse background offered opinions, suggested alternatives, conversed with one another, and produced relevant writings and drawings" The centres where this exchange of knowledge take place are the arsenals, the mines, the metal processing sites, but also the workshops, the courts, the printing presses. Pamela O. Long, *Artisan/Practitioners and the Rise of the New Sciences, 1400-1600* (Corvallis: Oregon State University Press, 2001), 94.

[175] As a sign of the popularity and circulation of the *Summa*, a second edition was published in 1523. Luca Pacioli, *Summa de arithmetica geometria, proportioni et proportionalita:* Novamente impressa. Et per esso Paganino di novo impressa. In Tuscolano, sula riva dil laco Benacense, 1523.

that responded to the evolving dynamics of knowledge exchange of his times. As is often remarked, Pacioli was the first to present a comprehensive discussion of problems of commercial arithmetic and practical geometry in printed form, with the publication of the *Summa* in 1494.[176] The *Compendium de divina proportione* instead, was first presented in three manuscript copies, and was later printed together with additional materials in 1509 to allow its wider circulation.

Lastly, as exemplified by Vasari's claim of Pacioli's appropriation of Piero della Francesca's works,[177] Pacioli was very able in collecting and organising the knowledge that was circulating in the environments he experienced, even if often without acknowledging all his sources. However, if not exclusively for their originality, his works should be examined for the way in which they correlated not only theory and practice, but also the works of different authors, and different subjects, in an unprecedented manner. In other words, the works of Pacioli should be investigated not only for their sources but also for their internal logic and interrelation of arguments in their theoretical framework of reference. Pacioli's texts not only present arguments linked to an established theoretical context, but also display original ways of correlating them. This can be demonstrated, as we shall see, in relation to arguments involving geometrical bodies, as considered across the texts by Pacioli examined in this thesis.

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[176] As noted in the previous section, Leonardo da Vinci annotated to have bought a copy of the *Summa* of Pacioli. It can be presumed that other artists or architects may have done the same or have consulted the text.

[177] Famously Vasari pointed to Pacioli's borrowings from the writings of Piero della Francesca. While stating that Pacioli was a pupil of Piero, Vasari added that being Piero old and unwell, Pacioli appropriated Piero's works and published them as his own. "Perché Maestro Luca dal Borgo frate di San Francesco che sopra i corpi regolari della geometria scrisse, fu un suo discepolo: et venendo in vecchiezza Pietro, che aveva composto di molti libri, Maestro Luca facendoli stampare tutti gli usurpò per se stesso come già s'è detto di sopra". Giorgio Vasari, *Le vite de' più eccellenti archittetti, pittori, et scultori italiani, da Cimabue insino a' tempi nostri* (Firenze: Lorenzo Torrentino, 1550), 366-367.

1.2.2 The concept of the geometrical body

Before examining single arguments in Chapters 2 and 3 related to the concept of geometrical body in Pacioli's writings, it is important to give an overview of the correlation among the arguments in the three texts and introduce some of the sources that Pacioli is shown to have known and employed. The correspondences established by Pacioli between theory and practice are again crucial in this respect, and these are undoubtedly linked to the individuals and sources with whom he interacted and knew about throughout his life. An analysis of the correlation of theory and practice is particularly important when considering the concept of the geometrical body as presented in Pacioli's works. As noted, geometrical bodies (and particularly regular and other bodies derived from these), were a privileged subject for Pacioli in his interactions with courtly and learned environments: with the Della Rovere in Rome, with Duke Guidobaldo of Urbino and at the Sforza court of Milan. Coherently, sections dealing with the theoretical and practical properties of regular bodies are the culmination of the contents of Pacioli's *Summa* and *Divina proportione*. As Pacioli noted, these were similarly the subjects of the last books of Euclid's *Elements* and the culmination of its geometrical logic-deductive system.[178]

At this point it is important to clarify that there are two main types of geometrical bodies, both referred to as *corpi* by Pacioli. They are discussed in the *Summa*, *Divina proportione (Compendium, Tractato de l'architectura* and *Libellus)* and Pacioli's Euclid's *Elements,* with the latter based on Campanus's edition. The first acceptation refers to three-dimensional material bodies for everyday use such as boxes, casks, barrels, vats, etc. Stemming from Leonardo

[178] Indeed, books XI-XV of Euclid's *Elements* in the edition of Campanus, together with books V and VI, are the most quoted source throughout Pacioli's *Summa* and *Divina proportione*. In book V the concept of magnitude (without defining it) and proportionality among magnitudes are introduced, and the subject is further developed in book VI. The propositions of books V and VI are instrumental for developing the three-dimensional geometry of books XI-XV, particularly in book XIII, where regular bodies are introduced.

Pisano's (Fibonacci) treatise on practical geometry (*practica geometriae*), problems on dimensions, with some dimensions known, and the capacities of these bodies were also included (although not always), in manuals of practical geometry circulating in the fifteenth century.[179] Crucially, to undertake such measurements, material bodies had to be geometrically considered as abstract geometrical shapes such as rectangular prisms, truncated cones, cylinders, etc. The knowledge of a few basic geometrical rules to deal with three-dimensional bodies was therefore essential for their resolution. The second acceptation refers to the five regular and other bodies derived from these by the truncation of their vertices. Each of these bodies can be inscribed in a sphere with all the vertices touching its surface. Pacioli's *Summa* and *Compendium* were not the first texts to discuss practical geometrical problems involving three-dimensional items of common use such as casks, barrels, vats, etc.; they were, however, the first to collate these with a philosophical and theoretical discussion of the geometrical properties of the five regular bodies and, additionally, to do so in a widely circulated printed text. The five regular bodies were famously constructed and inscribed in a sphere in book XIII of Euclid's *Elements*, while a 72-bases body was employed in the 14° proposition of book XII in the edition by Campanus.[180] There is no mention of materiality in the *Elements.* Although Campanus called them *corporum* following earlier Latin translations from the

[179] *Tractato di praticha di geometria* (*Palatino* 577, Biblioteca Nazionale Centrale, Firenze) attributed to Maestro Benedetto (Picutti, "Sui plagi matematici di Frate Luca Pacioli," 72-79.

[180] Additional semi-regular bodies were geometrically described in the fifteenth century in the writings of Piero della Francesca and Luca Pacioli. Johannes Campanus or Campanus of Novara (1220–1296) was a mathematician and translator active at the papal court of Viterbo around the third quarter of the thirteenth century.

Arabic,[181] it is hard to believe, if this section is considered in isolation, that any of its readers would have described these bodies as in any way material. Nevertheless, Pacioli described their geometrical properties but also associated these with material instantiations (models) in the *Summa, Compendium* and *Tractato de l'architectura*. In Pacioli's texts, not only are everyday material bodies discussed as abstract geometrical shapes, but regular and other bodies derived from these are discussed as material bodies and considered in the *Compendium* for their applications in architecture.[182]

Nonetheless, while in Pacioli's *Summa* the discussion of geometrical bodies, and among them regular bodies, takes place in the context of an extensive treatise dealing with many aspects of commercial arithmetic and practical geometry, the *Compendium* and the later published *Libellus in tres partiales tractatus divisus q(ui)nque corpor(um) regularium et depe(n)de(n)tiu(m)* […] are mostly concerned with the geometrical analysis, quantification, perspectival representation and material instantiation of regular and other bodies derived from these. Only the examination of the context in which these claims were made and the wider theoretical framework of the works can help analyse the correlation of arguments established by Pacioli.

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181 As shown by Menso Folkerts, Adelard of Bath translated from Arabic and integrated his translation with excerpts of the (now mostly lost) version of Boethius. Menso Folkerts, *Euclid in Medieval Europe* (Munich: Benjamin catalogue, 1989). An abridged version was derived from this, a version probably accomplished by Robert of Chester (Ketton) around 1440/1441 and now usually referred to as the 'Adelard II'. Hubert L. L. Busard, Menso Folkerts, *Robert of Chester's (?) Redaction of Euclid's Elements, the so-called Adelard II Version,* 2 vols (Basel: Birkhäuser 1992). This was the most circulated among the translations of the *Elements* accomplished in Toledo around the middle of the twelfth century, and was later the basis for the edition prepared by Campanus of Novara in the thirteenth century. In Arabic versions of Euclid's *Elements* dating from before the twelfth century, the Greek term 'στερεόν' in Book XI had already moved from the original acceptation of something solid, firm, and established a relationship to the 'body'. An example of this can be found in the translation made by Ibn al-Muqaffa. As noted by Sonja Brentjes, his technical language is characterised by terms which resurface in Arabic extracts of the *Elements* linked to the Hajjaj tradition of the eighth-ninth century AD. In this case the term use for geometrical body was the Arabic 'juththa' in Book XI of the *Elements*. The term relates to the sphere of corporeality and translates as 'body' or 'corpse.' Ahmad Al-Karabisi, Sonia Brentjes, "Commentary on Euclid's 'Elements'," in Sic itur ad astra: Studien zur Geschichte der Mathematik und Naturwissenschaften, Festschrift für den Arabisten Paul Kunitzsch zum 70, ed. Menso Folkerts and Richard P. Lorch (Geburtstag, Wiesbaden: Harrassowitz, 2000), 31-75.

182 A third acceptation, considered in Chapter 4, is found in Pacioli's *Tractato de l'architectura*, added in the printed *Divina proportione* to the *Compendium* and refers to the human body. Following ancient authors as Vitruvius, the body or a part of the body is here first geometrised and then related to the design of architectural elements.

Leonardo Pisano (Fibonacci) had discussed the geometrical properties and quantified elements of three among the five regular bodies.[183] However, this section was usually omitted from the manuscripts in *volgare* that had stemmed from the Latin text of Fibonacci and were circulated in the fifteenth century.[184] The texts that future artists, architects and other practitioners usually studied in this period commonly presented an exemplified structure, with a reduced number of geometrical cases that mostly dealt with the resolution of problems concerning linear and planar quantities.

One only needs to look at the writings of Leon Battista Alberti and Francesco di Giorgio Martini that dealt with problems of practical geometry and measuring by sight to see that the three-dimensional geometrical properties of regular, or other types of bodies, were not a subject that was commonly discussed. Both Alberti and Francesco di Giorgio did not include any problems dealing with the gauging or proportional arrangement of bodies of everyday use, or the regular bodies discussed by Euclid. Rather, they mostly employed geometrical procedures involving similar triangles and calculations with ratios to measure the depth of

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[183] Fibonacci completed his manuals of commercial arithmetics, algebra, and practical geometry in the first decades of the thirtieth century. Fibonacci had collected a large amount of geometrical and arithmetical knowledge in methods of measuring from the manuals of the ancient *agrimensores*, Arabic and Greek authors such as Euclid, Archimedes and the Banū Mūsā. Marshall Clagett, "The Impact of Archimedes on Medieval Science," *Isis*, vol. 50, no. 4 (December, 1959): 426. For what concerns practical geometry, Fibonacci gave a summary of the subjects dealt with at the opening of book VI of his *Practica geometrae*, completed in 1220: "Corporum quidem genera sunt plura, ex quibus sunt hec: Solidi, Seratilia, Piramides, Columne, Spere, et eorum partes, nec non et corpora, que multorum basium nuncupatur, et describuntur circa speras: proprie quidem solidum est, quod latum longum et altum habet; et constat ex sex superficiebus: ut sunt tassilla: scrinea,: cisterne et similia. […] Solida multarum basium sunt multimoda, ex quibus sunt solida viii basium, et xii basium, et xx basium equalium, que Euclides in xiiii libro infra speram construere docet. Sunt etiam alie infinite corporum speties, quorum mensure habebuntur per ea que super prescripta corpora dicere procurabimus: et ut modus mensurandi corpora perfecte habeatur. […] Et notandum cum dicimus aliquod corpus continere tot ulnas, vel palmos, vel digitos, seu uncias, vel aliquas alias mensuras, tunc intelligimus, illas mensuras esse corporales, seu cubicas; que tribus equalibus dimensionibus, latitudine scilicet, et longitudine, atque altitudine constant." Baldassarre Boncompagni, *La Pratica Geometrie di Leonardo Pisano secondo la lezione del Codice Urbinate n. 292 della Biblioteca Vaticana* in Scritti di Leonardo Pisano matematico del secolo decimoterzo, vol. II, (Rome: Tipografia delle Scienze Matematiche e Fisiche, 1862), 158-159; Fibonacci, *Fibonacci's De Practica Geometrie*, ed. Barnabas Hughes (New York: Springer, 2008).

[184] Often the section of *abacus,* devoted to arithmetical and algebraic calculations, and the section of *practica geometriae* were condensed in one volume. There are around three-hundred surviving codices of *abacus* or/and *practica geometriae* dating from the thirteenth to the fifteenth century; although they are found in many Italian cities, they are mostly located in Florence. Out of these, about one-hundred and eighty of these were produced in the fifteenth century. Elisabetta Ulivi, "Luca Pacioli una biografia scientifica," in *Luca Pacioli e la matematica del Rinascimento*, ed. Enrico Giusti and Carlo Maccagni, (Firenze: Giunti, 1994), 37.

wells, the height of towers, and the area and perimeter of fields considered as polygons, etc. Alberti mentioned geometrical bodies [*corpi*] in the final section of his *Ex ludis rerum mathematicarum*:

> If you want to ask more of me, I would be happy to take it further. The measures of bodies, such as square, round or pointed columns, with several faces, spherical, and the capacity of vases and the like, are topics more difficult to deal with. If it would be to your pleasure, however, I could go over them. I doubt that I can say anything about them any differently than the ancients did, and they said it in such a way that even with effort and a knowledge of mathematics they are barely understandable.[185]

This is undoubtedly because only linear and planar dimensions were required in the resolution of all the practical problems and mathematical puzzles that were usually proposed in this kind of manuals. For instance, when towers were discussed in these texts, properties such as the volume, or the exact three-dimensional shape of the tower were not considered. In other words, for the purpose of the problems presented it was always sufficient to geometrically consider a tower abstracted as a linear dimension. It should be added that Alberti dealt with geometrical shapes more extensively in other works, such as the *De pictura, Elementa picturae, De statua, Descriptio urbis Romae,* and famously argued in the preface of *De re aedificatoria* that "a building is a kind of body that consists of lineaments and matter".[186] Alberti's position will be considered in Chapter 4, but it is worth adding here that lineaments

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185 "Se altro mi chiederete, lo farò volentieri. Le misure de' corpi, come sono colonne quadre, rotunde e aguzze, di più faccie, sperice e simili, sono materie più aspre a trattare. Pur quando a voi dilettasse, potrò ricorvele. Dubito non poterle dire se non come le dissono gli antichi, e loro le dissono in modo che con fatica e cognizione di matematica e appena si comprendano." Leon Battista Alberti, *Opere Volgari*, vol. III, ed. Cecil Grayson (Bari: Laterza, 1973), 172-173. A slightly different version, based on another manuscript, is given by Williams, Marc and Wessell: 'Se altro mi chiederete lo faro volentieri le misure di capi (corpi come sono colonne tonde, quadre et aguze di piu faccie sperice et simili capi et tenute di vasi et simile sono materie piu aspre a trattar pur quando a voi dilectassi potro ricorvele dubito no[n] poterle dir se no[n] come dixono gli antiqui et loro le dixono in modo ch[e] co[n] faticha et cognitione di mate maticha et a pena si comprehendono dicovi ch[e] molte cose lasciovi et no[n] dissi bench[é] fussino dilectevoli solo p[er]ch[é] no[n] vedevo modo di poterle dir chiar et ap[er]te come cercavo dirle et in queste durai faticha et no[n] pocha ad exprimerle et farvile intender[e]. Leon Battista Alberti, *The Mathematical Works of Leon Battista Alberti*, ed. Kim Williams, Lionel March, Stephen R. Wessell, (Basel: Springer, 2010), 68-69.

186 "Nam aedificium quidem corpus quoddam esse animavedvertimus, quod lineamentis veluti alia corpora constaret et materia". Alberti added that the one is the product of thought, the other of Nature. Leon Battista Alberti, *L'Architettura* (Milano: Il Polifilo, 1966), Prologue, 15.

are determined by lines and angles; and that Alberti considered three-dimensional shapes as geometrically defined by lineaments. As we shall see, Pacioli's arguments are certainly correlated with those of Alberti, and although centred on a somewhat different geometrical concept, can be interpreted as a further development in Alberti's programme of the quantification of spatial relationships and visual experience.[187]

In relation to this, a crucial role is undoubtedly played by the works of Piero della Francesca among Pacioli's sources. The substantial and undeclared borrowings by Pacioli from Piero della Francesca's study of geometrical bodies in the *Trattato d'abaco* and *Libellus de quinque corporibus regularibus* in the *Summa* and *Divina proportione (Libellus)* respectively, have been examined by many scholars.[188] It should be noted that all Piero's texts remained in manuscript form until the twentieth century (with Piero's *Trattato* and *Libellus* now known in a single copy).[189] However, as we will see, the context and theoretical frameworks in which these arguments take place should also be examined, while there are correlations at a conceptual

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[187] The latter has been described by Branko Mitrović, "Leon Battista Alberti, Mental Rotation, and the Origins of Three-Dimensional Computer Modeling," *Journal of the Society of Architectural Historians*, vol. 74, no. 3 (September 2015): 318, 312-322.

[188] As demonstrated by Margaret Daly Davis, the main source of the mathematical problems involving regular and semi-regular solids presented in the *Paticularis tractatus circa corpora regularia et ordinaria* of Pacioli's second part of the *Summa* is Piero della Francesca's *Trattato d'Abaco,* a work probably completed by the end of the 1460s. More than half of the problems presented in the *Trattato* (27) are reproposed almost verbatim in Pacioli's *Summa*, 10 were slightly shortened, while 6 were shortened consistently, 3 were extended, 4 partially re-elaborated and only 5 were completely reviewed. Pacioli's *Libellus in tres partiales divisus […],* included in the *Divina proportione*, is mostly as translation in *volgare* of Piero's *Libellus de quinque corporibus regularibus*. Margaret Daly Davis, *Piero della Francesca's Mathematical Treatises* (Ravenna: Longo Editore, 1977), 98-106; Di Teodoro, "Pacioli, Luca."

[189] Piero della Francesca, *Trattato d'abaco* (Florence, Biblioteca Medicea Laurenziana, MS. Ashb. 280/359-291) (1460-1470 c.); Piero della Francesca, *Libellus de quinque corporibus regularibus,* ms Urb. Lat 632, Rome, Biblioteca Apostolica Vaticana (1480-1485 c.); The treatise on perspective in painting (*De prospectiva pingendi*), first redacted in 1470-1480, survives in seven manuscript versions, three in Italian and four in Latin: Parma, Biblioteca Palatina, MS. 1576 (with autograph drawings and notes) in Italian; Milano, Biblioteca Ambrosiana, Cod. D 200 inf., (XVI cent., without drawings) in Italian; Reggio Emilia, Biblioteca Municipale A. Panizzi, Cod. Reggiano A 41/2 (before A 44), (XV cent., with autograph drawings and notes) in Italian; Milan, Biblioteca Ambrosiana, Cod. Ambr. C. 307 inf., (XV cent., with autograph drawings and notes) in Latin; Bordeaux, Cod. 616, (XV cent., with autograph drawings and notes) in Latin; London, British Library, Cod. Add. 10366, (XV cent.) in Latin, Parigi, Bibliothèque Nationale, Cod. Lat. 9337, (XVI cent.) in Latin. For the time of writing of the three works of Piero della Francesca I refer to the studies of James R. Banker, summarised in: Banker, *Piero della Francesca*. Moreover, while Piero's *Libellus,* dedicated to the Duke Guidobaldo of Urbino was redacted in Latin, a version in vernacular Italian was included in Pacioli's *Divina proportione*. Pacioli, *Libellus in tres partiales divisus […], Divina proportione (1509).*

level between Piero della Francesca and Luca Pacioli that still need to be considered.[190] Piero geometrically and numerically investigated the properties of all regular and of six semi-regular bodies as purely abstract shapes in his mathematical works, the *Trattato d'abaco* and *Libellus*. However, he had prepared his treatise *On Perspective in Painting* (*De prospectiva pingendi*) between these two works. Here, he dealt with the same underlying geometrical principles based on quantities and proportionality, and applied them to the perspectival representation of regular, everyday, and architectural bodies such as a capital and base of a column, a coffered half dome, an octagonal temple, a cubic house, etc.

Although correlations between the works of Alberti, the three texts by Piero della Francesca and the works of Pacioli are further examined in Chapters 3 and 4, is important to introduce here their common theoretical foundation in Campanus's edition of Euclid's *Elements*. As noted, a commented edition based on Campanus's text was published by Pacioli. Among the many references to Euclid's *Elements*, Pacioli notes in the *Summa* that the five regular bodies were "scientifically discussed by Euclid in the last books [of the *Elements*]" and adds that is not useless to present some cases to the vulgar practitioners, and let them "hear about the sweetness of their dimensions".[191] Interestingly, a connection between the geometrically abstract and arithmetically quantifiable also emerges if the philosophical framework of

[190] Differently from Pacioli, Piero included no references to ancient authors or to their theories, apart from mathematical authorities such as Euclid and Archimedes from which he take great care in noting where he found the relevant propositions.

[191] These, Pacioli continues, are also discussed by philosophers, and "by Plato in his *Timaeus* (as noted by Saint Augustine) that assigns the bodies to the natural elements." However, Pacioli concludes, from the assignment of the tetrahedron to fire, Aristotle in his *De cælo et mundo* started his discussion." At this point Pacioli refers for the first time in his writings to a set of physical models of the regular and other bodies derived from these that he presented to the duke Guidobaldo of Urbino, dedicatee of the *Summa,* in Rome in 1489. "E Bènche di sopra i q(ue)sto nella distition 6ª al capitolo 4º de la mesura de la sp(her)a succinteme(n)te fosse ditto abasta(n)ca: no(n) dimeno me par q(ue) excelso Duca p(ar)ticularme(n)te dir(e) de alqua(n)ti corpi esse(n)tiali i ditta spera locabili de li q(ua)li un angolo tocca(n)do subito tutti toccano: e p(ri)ncipalmente lo fo p(er) la notitia de li 5 regulari di q(ua)li Euclide a pieno nelli ultimi soi libri scie(n)tificame(n)te tratta. Di che me pare no(n) i(n)utile a ponere de loro certi casi acio li pratici vulgari ancora essi qualche dolceçça di loro dime(n)sio(n)i sentino. Di q(ue)sti fra phy(losofi) si fa gra(n) discussione. E maxime se be(n) el thymeo del divin ph(ilosof)o Platone (seco(n)do lo Aurelio doctor sa(n)cto Augustin) con diligentia satende. Dove de universi natura diffusamente parla(n)do spesso a suo p(ro)posito li i(n)duci. Attribue(n)do lor separatamente a li 5 corpi semplici: cioe Terra Aqua Aeri Fuoco E Cielo. Si co(m)mo apieno di sopra in q(ue)sto nella parte de Arithmetica in la distinction prima. Nel seco(n)do suo trattato al terço articolo di loro parl(i)amo." Pacioli, *Summa*, 68v (Pars secunda).

Campanus's edition of Euclid's *Elements* is considered. Paul L. Rose described the *Elements* edited by Campanus as "scholasticised" since they privileged the *Elements*' axiomatic structure, adding, however, that "Campanus moved away from geometrical proofs to an emphasis on arithmetical proofs", but did not elaborate further on how this was achieved.[192]

This framework becomes apparent as soon as one examines Book V of the *Elements* in Campanus' edition, commented and published by Pacioli. Here, the concept of quantity (*quantitas* in the editions of Campanus and Pacioli, but "magnitude" [*μέγεθος*] in Euclid's Greek) is introduced as the foundation of proportionality, but—as in Euclid's version—without being defined.[193] Commenting on the first five definitions, Campanus and later Pacioli, largely relied on the theory of quantity expounded in Medieval translations of Aristotle's *Categories,* and interpolated the philosophical conception of Aristotle with the principles of proportionality discussed by Euclid. The implications of this connection in the context of Pacioli's writings are substantial and will be examined in Chapter 2, however, it is worth outlining that in the *Metaphysics,* Aristotle defined magnitude as a continuous entity: "that which is divisible into continuous parts"[194] and in *De cælo* argued that "the continuous may be defined as that which is divisible into parts which are themselves divisible in ever-divisible parts, body [*soma*] as that which is divisible in all ways. Magnitude divisible in one direction is a line, in two directions a surface, in three directions a body [*soma*]", concluding

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[192] Rose, *The Italian Renaissance*, 81.
[193] Ian Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements* (Cambridge MA: MIT Press, 1981), 121.
[194] Aristotle, *Metaphysics*, V, 1020b12. The examples given were geometrical: the one-, two-, and three-dimensional magnitudes. ibid., 1020a8–12.

that "beyond these there are no other magnitudes, because the three dimensions are all that there are."[195]

Aristotle largely developed the theory of *De cælo* as a counterargument to the geometrical construction of regular bodies and their assignment to the four basic Empedoclean elements in Plato's *Timaeus*.[196] It is to this context and correlation of arguments that Pacioli refers both in the *Summa* and *Divina proportione*.[197] Crucially, although Aristotle's three-dimensionality is not the essence of a physical substance, this should nonetheless be understood as tied to its essence. This is because, as noted by Christian Pfeiffer, a three-dimensional extension is always a concomitant of a physical substance,[198] as it is always a concomitant to any body.[199]

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[195] "The continuous may be defined as that which is divisible into parts which are themselves divisible in ever-divisible parts, body [*soma*] as that which is divisible in all ways. Magnitude divisible in one direction is a line, in two directions a surface, in three directions a body [*soma*] […] body [*soma*] is the only complete magnitude, since it is the only one which is defined by extension in three directions, that is, which is an 'all'. Being divisible in three directions, it is divisible in all." *De cælo (On the Heavens)*, 268a9-13, 23-25. In the Metaphysics Aristotle added that surfaces, lines and points are always derived from a body, and never apprehended in themselves. This has brought some commentators to speak of an ontological parasitising of surfaces, lines and points upon bodies, as these cannot be separated from them and never stand in actuality on their own. "[…] if that which is posterior in generation is prior in substantiality, body [*soma*] will be prior to plane and line, and in this sense it will also be more truly complete and whole, because it can become animate; whereas how could a line or plane be animate? The supposition is beyond our powers of apprehension. Further, body [*soma*] is a kind of substance, since it already in some sense possesses completeness." *Metaphysics* XI, 1060b12–16, 185.

[196] Plato geometrically describes the regular bodies in the *Timaeus* as an ordered assemblage of elementary triangles. Plato, *Timaeus,* 53c6-8.

[197] Pacioli, *Divina proportione (1509)*, opening sonnet.

[198] An updated overview of the scholarly debate on body and magnitude in the writings of Aristotle, and their ontological role in physical substances, is given by Christian Pfeffeir. As noted by Pfeiffer, Aristotle explains this via the principle that what is prior in generation is later in substance and relying on geometrical arguments to argue that a body is prior in generation and later in substance. This is famously discussed in *Metaphysics,* XIII, 2, 1077a24–31. Christian Pfeiffer, *Aristotle's Theory of Bodies* (Oxford: Oxford University Press, 2018), 84.

[199] However this does not imply, both for Aristotle and Pacioli, that is not legitimate for a mathematician to consider the geometrical properties of a body as separate from matter. The classic reference for this is to Aristotle's argument in *Physics* II: "Now the mathematician, though he too treats of these things, nevertheless does not treat of them as the limits of a natural body; nor does he consider the attributes indicated as the attributes of such bodies. That is why he separates them; for in thought they are separable from motion, and it makes no difference, nor does any falsity result, if they are separated. The holders of the theory of Forms do the same, though they are not aware of it; for they separate the objects of natural science, which are less separable than those of mathematics. This becomes plain if one tries to state in each of the two cases the definitions of the things and of their attributes" 193b31–194a4. Pacioli argues in the *Compendium de divina proportione* "that the good ingeniousness of mathematical disciplines is very useful in any other science, because these are of great abstraction and subtlety and always considered separate from sensible matter";"[…] chel buono ingegno a le mathematici [discipline] fia aptissimo a cadauna scientia. Conciosia che le sieno de grandissima abstarctione e subtigliezza perche sempre fuora de la materia sensibile se hano a considerare" Pacioli, *Compendium*, 8v. Indeed, the concept of three-dimensionally extended body, combined with the rejection of an empty place (or vacuum), supports both Aristotle and Pacioli's arguments against Plato's geometrical construction of regular bodies by means of triangles in the *Timaeus*.

Indeed, as shown by David Summers, Medieval commentators of Aristotle's *De anima* did not fail to note and expand on this assumption.[200] As we shall see, by relying on these premises Pacioli argued in the *Summa* that quantity had been considered by many philosophers, and could thus be considered equal to substance.[201]

Ultimately, an analysis of these arguments, through the translations and commentaries of the Aristotelian sources with which Pacioli operated, is unavoidable when considering the concept of the geometrical body expressed in the *Summa* and *Divina proportione*. Only the examination of the multiple arguments that Pacioli correlates to the concept of the geometrical body, and their wider theoretical framework, can account for the claims that Pacioli made in the application of regular and other bodies derived from this in architecture, in the *Compendium de divina proportione* and *Tractato de l'architectura*. At the same time, an analysis of the relationship between theory and practice is crucial not only in making sense of Pacioli's arguments, but also for the inescapable implications that these have in arguments dealing with the process of human visual experience.

1.2.3 Aesthetic theory and judgement in Pacioli's works

It is now useful to introduce how Pacioli coordinated arguments on nature as divine creation, perspective and visual experience of architecture in his writings. Leaving a more detailed analysis of each argument for the next chapters, it is important to anticipate the overall order in which Pacioli considered their manifestation in works of architecture. The role assigned by Pacioli to visual perception in the process of aesthetic judgement is crucial in this regard, the theory of quantity and proportionality developed in his works cannot be fully considered

[200] David Summers, *The Judgement of Sense* (Cambridge: Cambridge University Press, 1987), 151-153.
[201] Pacioli, *Summa*, 1r (Epistle).

without accounting for its relationship with vision and perspective. Indeed, the position assigned by Pacioli to the concept of the geometrical body both in the *Summa* and *Divina proportione* (in the *Compendium, Tractato de l'architectura* and *Libellus),* as discussed in the previous section, already suggests that for Pacioli aesthetic judgements are intrinsically linked to geometrical and formal properties.

This is not to say that Pacioli's texts are devoid of additional associated meanings. As we have already begun to see, geometrical, theological and philosophical concepts provided Pacioli with the theoretical framework in which to develop his own propositions. However, as noted, the scholarship on Pacioli's works has almost obsessively privileged meanings, while often examining them in isolation and detached from their mathematical-theoretical framework. As a result, the basic formal and perceptual principles discussed in Pacioli's works have been almost completely overlooked. Although Pacioli did not include a definition of beauty of the kind given by Alberti in *De re aedificatoria*,[202] in a series of passages distributed among his *Summa*, *Compendium* and *Tractato de l'architectura,* he clearly argued for a hierarchical structure in human perception and mental cognition. As we will see, in this process, Pacioli crucially described the aesthetic judgement of formal properties of architecture as the first-immediate stage of visual experience.

The relationship between theory and practice, that Pacioli was so keen to stress in all of his works, is again crucial in this respect. If the contents of the *Summa* and *Divina proportione* are in fact clearly addressing practitioners, it is essential for Pacioli to be able to link practice and visual experience within a wider theoretical framework. Pacioli famously overcame this

[202] Alberti, *De re aedificatoria*, Prologue, 15; VI. 2, 445-451. The aesthetic formalism of Alberti, and particularly the implications of his definition of beauty in *De re aedificatoria* have been examined by Branko Mitrović. Branko Mitrović, *Serene Greed of the Eye: Leon Battista Alberti and the Philosophical Foundations of Renaissance Architectural Theory* (München: Deutscher Kunstverlag, 2005), 101-125.

difficulty by likening crafts and artistic creation to the imitation of nature's works and referring to their common dependance on proportions. This is certainly not surprising, as the claim that artistic creation should imitate nature and that proportions were crucial in this process was widely held in the context in which Pacioli operated. As we shall see, Pacioli argued that proportions are among the universal principles by which the Highest Maker's Divine Wisdom has arranged the world. However, Pacioli's insistence throughout his works on a passage from the *Book of Wisdom*, where it is said that "god created everything according to number, weight and measure" also highlights a clear link to a series of quantitatively accounted properties. Quantity is central both to the theory of proportions in Books V-VI of Euclid's *Elements* in Campanus's edition and, as we have begun to see, to the medieval Aristotelian concept of the body. As posed by Pacioli:

> […] Because it is not possible in nature for anything to persist if it is not rightly proportionate to its necessity. Whence the Divine Wisdom (as Augustine says with praise and commendation) *Omnia fecit deus in numero, pondere et mensura,* that is to every thing was given what was necessary, pondered according to weight, measure and number. In all these three things you always find proportion, according to it [proportion] (the Highest Maker) *cuncta bene disposit* […] If you consider well, you will find proportion as mother and queen of all arts, and without it none of them could be practiced.[203]

[203] "Peroché impossibile e alcuna cosa in natura persistere: se la no(n) e debitamente p(ro)portionata a sua necessita. E pero la divina sapientia (commo dici Augustino in sua laude e com(m)endazione) O(mn)ia fecit deus in numero: pondere: et mensura: cioe che a ogni cosa dette la sua debita exige(n)tia: considerata secondo el peso: el numero: e la mesura. In le quali tre cose: sempre se a retrovare la proportione: secondo la quale (commo summo opefici) cuncta bene disponit. […] Se tu ben discorri, in tutte le arti tu troverai la proportione de tutte esser madre e regina e senza lei niuna poterse exercitare" Pacioli, *Summa*, folio 68v (Pars prima). In the *Compendium de divina proportione* Pacioli further expanded on this argument: "Therefore the said mathematical disciplines are the foundation and the ladder to reach knowledge of any other science, because they are in the first degree of certainty, as was said by the Philosopher, the mathematical sciences are in the first degree of certitude and all the naturals follow from them. […] And without their knowledge is not possible to know well any of the others, and in the [Book of] wisdom is written that *omnia consistunt in numero pondere et mensura,* that is everything in the superior and inferior universe is subject to number, weight and measure." "Conciosia che dicte mathematici sieno fondamento e scala da pervenire a la notitia de ciascun altra scientia per eser loro nel primo grado de la certeça affermandolo el philosopho cosi dicendo mathematice enim scientie sunt in primo gradu certitudinis & naturales sequuntur eas.[…] e senza lor notitia fia impossibile alcun(')altra bene intendere. E nella sapientia ancora e scripto quod omnia consistunt in numero pondere & mensura cioe che tutto cio che per l(')universo inferiore e superiore si squaterna quello de necessita al numero peso e misura fia sottoposto" Pacioli, *Compendium*, 2r (Proemium).

Nevertheless, although proportion was widely held to be the mathematical principle underlying divine creation and to be followed in any artistic endeavour, the question of *how* this was to be pursued was undoubtedly answered in different ways.[204] Famously, the argument that the architect should imitate nature by means of proportions and symmetry among parts was made by Vitruvius in *De architectura.* Since the human body was understood to be the most perfect creation in nature, it was considered by Vitruvius and Renaissance architectural theorists as the object for imitation. Pacioli was no exception, as he made reference to the proportioning of the human body both in the *Summa* and the *Tractato de l'architectura*.[205] At the same time, Pacioli clearly articulated his position with respect to the relationship of proportions and vision already in the dedicatory letter that opens the *Summa*:

> And also in architecture, as demonstrated by Vitruvius in his volume and Leon Battista Alberti of Florence in his perfect work on architecture, have arranged with proportion great and sublime buildings. [And] as in our own times, the new light of Italy, the admirable *fabrica* of the respectable palace of V. D. S.[206], accomplished by and standing in memory of your father. Certainly, nature displayed in it its power and art more than in any other visible [building]. Which language could express the beautiful order [*bell'ordine*] of its disposition [*dispositione*]. Surely no one can suffice to this purpose, if not its own, very eloquent language; [because] it not only immediately pleases the sight, but even more one is stupefied when with the intellect examines with how much artifice and ornament it was composed. Perspective, if you pay attention, undoubtedly would be [worth] nothing if not able to accommodate both [of these], as

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[204] One only has to look at the writings of Leon Battista Alberti, whom Pacioli said to have met, Francesco di Giorgio Martini, that was active in Urbino in the second half of 1470s, and Leonardo da Vinci to find different approaches to this process. An overview of the process of imitation in the sixteenth century has been given by Battisti: Eugenio Battisti, "Il concetto di imitazione nel Cinquecento italiano," in *Rinascimento e Barocco* (Turin: Einaudi, 1960), 175-215; and James S. Ackerman, "Art and Science in the drawings of Leonardo da Vinci," in *Origins, Imitations, Conventions* (Cambridge, MA: MIT Press, 2002).

[205] Notably, Pacioli here refers to *De re aedificatoria* of Leon Battista Alberti, "where he demonstrate that every building should always be done with proportion, in a way that the lenght should correspond to the width, and also to the height". The reference to the text of Alberti "man of great ingenuity, which work was recently printed in Florence, although without figures", followed the remark that Augustine described the biblical Ark based on the length, width and height of the body of Christ. Pacioli, *Summa*, 68v (Pars prima). Famously, the Ark is mentioned for the same reason by Alberti in Book IX, 7 of *De re aedificatoria.*

[206] Here Pacioli is referring to Duke Guidobaldo of Urbino, the dedicatee of the *Summa.*

> fully demonstrated by the monarch of painting, *Pietro de franceschi* (sic), our fellow countryman […][207]

This process is described by Pacioli while introducing the certainty and necessity of proportions in every created thing, both divine and human. The clear connection established between proportions and visual perception is particularly significant in the passage, as it is the parallel to perspective at the end. Notably, Pacioli claims that no language is able to fully express the aesthetic (formal) properties of the building and match its visual experience. The Aristotelian theory of mental cognition based on the power of apprehension of the senses should be considered in this context. Pacioli assigns here a first ability to aesthetically judge architecture to the material, sensorial level of the soul, and acknowledges two distinct and connected levels of visual appreciation: the immediate [*non solo alla vista subito veduto piaci*], and the intellectual [*ma ancor piu riman stupefatto chi con intelletto va discorrendo con quanto artifitio e ornamento è stato composto*]. As Pacioli concludes, perspective would be worth nothing if not able to deliver both these stages, as demonstrated by Piero della Francesca.

Moreover, Pacioli employs the term *dispositione* to refer to the beautiful order of the architecture of the palace. Later in the sixth distinction of the *Summa* that deals with the theory of proportion, Pacioli clarifies that *dispositione* is also called *proportione* among the parts of a whole.[208] As often noted, the Latin *dispositio* was used in rhetoric to refer to the apt

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[207] "A l'architettura ancora. Vitruvio in suo volume. E Leon Battista degli Alberti fiorentino in sua p(er)fetta op(er)a de architectura molto d(e)mostrano esserli accomodata p(ro)portiona(n)do suoi magni et excelsi hedifitii. Si co(m)e al di n(ost)ri la nova luce de ytalia. Admiranda fabrica del degno pallazzo de V. D. S. p(er) la felicissima sua paterna memoria initiata e co(n)sumata manifesta. Nel qual certamente ben mostro natura sua forza e arte fra quanti p(er) altri sienno stati visti. Qual lengue el bell'ordine de tutta sua degna dispositione poria exprimere certo Niuna seria bastante. Se non la sua eloquentissima, el qual non solo alla vista subito veduto piaci, ma ancor piu riman stupefatto chi con intelletto va discorrendo con quanto artifitio e ornamento è stato composto. La perspectiva se be(n) si guarda senza dubio nulla serebbe se queste non li se accomodasse. Co(m)e a pieno demostra el monarcha d(e) la pictura maestro Pietro di franceschi n(ost)ro co(n)terraneo […]" Pacioli, *Summa*, 2r (Epistola).

[208] "Del fabro legnaro no(n) e dubio cio che si facia (o cassi, o tavole, o banche, o letieri, o usci, o tine, o botti, o mesure da pa(n)ni, o colme co(m)mo si voglia) sempre secondo certa dispositio(n)e (la quale si chiama proportione) asetta." Pacioli, *Summa*, 69r (Pars prima).

arrangement of the parts of a discourse.[209] However, *dispositio* had also been used by Vitruvius in two different contexts, firstly among the six qualities that architecture should possess, and secondly to indicate the modes of architectural representation.[210] As we shall see in Chapter 4, the same term (*dispositione*) is employed by Pacioli in the *Compendium* to describe the three-dimensional disposition of the parts of a geometrical body, but also to indicate the disposition of the parts of a building in a geometrical whole.

Successive stages in the process of visual perception and experience of architecture are confirmed in another passage of the same sixth distinction of the *Summa* that discusses the theory of proportions and proportionality. Here, Pacioli argues that "architecture also is not of value (as Vitruvius, Dinocrates, Frontinus and Pliny prove), if it is not properly proportionated". This is because, Pacioli continues, "it does not please or delight the eye, neither [is it pleasing or delightful] to inhabit it. And it is also said to be unhealthy". Significantly, once again Pacioli expresses here a sequence of aesthetic values that starts from the visual, and then extends to unspecified functional and wholesome qualities.

Overall, Pacioli implies in these passages a kind of universalism of visual aesthetic judgement akin to the one argued by Alberti in *De re aedificatoria*.[211] Moreover, as in the case of Alberti, Pacioli argues in at least one instance in his *Tractato de l'architectura*, that the ability to judge also

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[209] *Dispositio* can be read in conjunction with its role in the arrangement of argument in Classical rhetoric, a point particularly relevant in reading Alberti's use of the term. As noted by Lee in his classic study "dispositio for the rhetoricians means a preliminary blocking out of the oratorical discourse, so as to give a clear indication of the structural outlines of its final form with the relation of parts to the whole" Rensselaer W. Lee, "Ut Pictura Poesis: The Humanistic Theory of Painting," *The Art Bulletin*, vol. 22, no. 4 (December 1940), 264.

[210] "Dispositio autem est rerum apta conlocatio elegantesque conpositionibus effectus operis cum qualitate"; "Species dispositionis, quae graecae dicuntur ideae, sunt hae: ichnographia, orthographia, sacenographia.". Vitruvius, *De Architectura*, 1, 2, 2. Moreover, *dispositio* was also used, with *habitus,* in Medieval translations of Aristotle *Categories* as a *genus* of quality.

[211] Mitrović, *Serene*, 115-117.

determines the ability to make, correct or decide, based on the judgement of the eye.[212] As for Alberti, this is undoubtedly linked to Pacioli's belief in the certainty and universality of geometrical knowledge, and to the belief that there are common geometrical principles underlying the physical world, visual perception and perspective. Nonetheless, as we shall see in Chapter 3, in the case of Pacioli the arguments of Piero della Francesca, as developed in his writings and outlined at the outset of the third book of his *De prospectiva pingendi,* are also important in this regard.[213]

The same hierarchical structure that places the perception of formal properties at the basic level of visual experience can be found in the *Compendium de divina proportione.* Here, however, the objects of perception are regular and other bodies derived from these. When perspectival representations of regular and other bodies are considered, Pacioli argues that the vision of the perspectival representation of each body allows the experience of parts of the body that

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[212] In a passage of the *Tractato de l'architectura,* Pacioli refers to the visual judgement of someone skilled in perspective: […] And all the parts described so far in the profile [of the head] are rational and known. But when the irrationality of proportions occur, when they cannot in any way be named with numbers, these are left to the good judgement [*degno arbitrio*] of the *p(er)spectivo* that with grace [*gratia*] will have to complete them. This is because art imitates nature as much as possible. And if the maker made what nature made you would not call it art but another nature, that is in all similar to the first, and therefore the same. "E queste p(ar)ti narrate finora al suo p(ro)filo tutte vengano a essere rationali e a noi note. Ma dove intervene la irrationalita de le proportioni cioe che p(er) alcu(n) mo(do) non se possono nominare per numero restano al degno arbitrio del p(er)spectivo qual con sua gratia le ha a terminare. Perochè l'arte imita la natura quanto li sia possibile. E se ap(p)onto l(')arteficio facesse q(ue)llo che la natura ha facto non se chiamaria arte ma un(')altra natura totalitier a la prima simile che verebe a essere la medesima." Pacioli, *Divina proportione* (1509), 25v-26r (Tractato de l'architectura). Indeed, this is not the only passage in the *Tractato* where Pacioli refers to the 'judgement' of the maker; another passage mentions 'ad libitum of the eye' [*a libito de l(')ochio*] with regard to the ornamentation of the column's shaft. ibid., 30r.

[213] "I say that perspective by its own name expresses how to deal with objects seen from a distance represented within certain given limits [*termini*], proportionally according without which it is impossible to degrade accurately. Because painting is nothing other than demonstrations of surfaces and of bodies [*de superficie et de corpi*] degraded or enlarged on the picture plane [*termine*] placed so that the true things [*le cose vere*] seen by the eye under diverse angles are represented on the said picture plane […]" "Dico che la prospectiva sona nel nome suo commo dire 'cose vedute da lungi, rapresentate socto certi dati termini con proportione, secondo la quantità de le distantie loro', sença de la quale non se pò alcuna cosa degradare giustamente. Et perché la pictura non è, se non dimostrationi de superficie et de corpi degradati o acresciuti nel termine, posti secondo che le cose vere vedute da l'occhio socto diversi angoli s'apresentano nel dicto termine […]" Della Francesca, *De prospectiva*, 190; I have amended the translation of James R. Banker in Banker, *Piero della Francesca*, 176. The passage will be further discussed in Chapter 3. Notably, *De prospectiva pingendi* is the only work of Piero della Francesca expressively mentioned by Pacioli. Pacioli states to have made a *compendium* of this work. Pacioli, *Divina proportione (1509)*, 23r (Tractato de l'architectura).

are not shown in the drawing and must be imagined at the level of the intellect.[214] In the *Compendium,* some of these bodies (or parts of them), are described as employed in architecture to make domes, vaults or choir apses.[215] The process is repeated in the *Tractato de l'architectura*. Here, Pacioli suggests that capitals or bases of columns can also be produced in the form or regular bodies, and "they will be a reason of high praise for your building, because not only will they make it adorned, but they will also stimulate speculations by the learned and the wise." Pacioli implies that the first effects produced by the employment of regular bodies as architectural elements are formal and aesthetic. Only after this, as separate stage, it follows that the bodies will also make the learned and wise speculate about them.[216] The passage continues with a story that Pacioli claims to have read in some annals when in Rome, concerning the employment of an icosahedron by Phidias in the temple of Cerere. Beyond the probably fictitious character of the story, the passage demonstrates once again Pacioli's belief in the independent-from-meaning aesthetic visual appreciation of geometrical regular bodies. Overall, both in the *Compendium de divina proportione* and in the later *Tractato de l'architectura*, regular and other bodies derived from these are for Pacioli among the most perfect geometrical and material entities, and thus, as we shall see, the highest visual manifestation of formal and aesthetic properties.

Ultimately, it is important to note that in the *Summa* and the *Divina proportione*, Pacioli consistently describes a process of visual and aesthetic appreciation of architecture that relies on a multi-layered arrangement of visual experience. Additionally, as noted, in the *Summa*

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[214] "The solid or void elevated tetrahedron, or we can say pointed [tetrahedron], similarly has 18 lines of which 6 are common and has 36 superficial angles e 8 solid [angles] of which 4 belong to the superficial pyramids and 4 are common to the 5 pyramids, this [fifth pyramid] is the interior [pyramid] that the eye cannot see, but only the intellect can comprehend" Pacioli, *Compendium*, 12v.

[215] Pacioli, *Compendium*, 60r.

[216] "Ançe saran(n)o de dignissima co(m)mendatio(n)e del v(ostt)ro opifitio p(er)che no(n) solo lo re(n)deran(n)o adorno ma ancora a li docti e sapie(n)ti dara(n)no da speculare conciosia che sempre siano fabricati co(n) quella sc(r)a e divina p(ro)portione h(ave)nte medium duoq(ue) extrema etc." Pacioli, *Divina proportione (1509)*, 34v (Tractato de l'architectura).

Pacioli clarifies that perspective would indeed be worth nothing if not able to accommodate all the visual stages discussed. Since for Pacioli perspectival representations have a role akin to the visual experience of the actual architectural objects themselves, the visual experience of formal properties on such representations must be a universal element in aesthetic judgement. This is undoubtedly related to the role that Pacioli assigns to mathematical properties, and to the aesthetic value that he correlates to formal properties that are geometrically determined and visually experienced.

The last point has implications for the application of proportional relationships among the components of three-dimensional geometrical bodies in architecture. These could be 'objectively' employed without optical distortions, or as to be 'subjectively' perceived as such from a given viewing angle.[217] However, this issue is also related to where the perception of three-dimensionality (and the related mastery of the principles of perspective) in perspectival representations is located in the hierarchy outlined by Pacioli. The theory of perception summarised by Pylyshyn, which like Pacioli's, relies on a hierarchically structured organisation of human visuality and considers this as an implicit ability of any competent visual perceiver, it is useful in considering this issue.[218] If three-dimensional geometrical bodies are judged in the intellect of the observer and are based on universal mathematical properties, their proper shape can be mentally (objectively) constituted. Ultimately, the stance already articulated in

[217] Since architectural elements are never perceived in their entirety in a single glance, the issue is important at both the psychological and philosophical level. An overview of the debate is given by Branko Mitrović, "Objectively speaking," *Journal of the Society of Architectural Historians*, vol. 52, no. 1 (March 1993): 59-67. Indeed, as noted by Mitrović, this is a question that can be traced at least back to Plato, *Sophist*, 236a, where the works of sculptors are discussed. Famously Vitruvius discusses both the use of a module in architecture and the necessity of optical corrections in elevation design. Likewise and mostly relying on Vitruvius's text, Pacioli discussed both approaches in the *Tractato de l'architectura.* However, it should be noted that Pacioli never discusses optical corrections in the visual appreciation of regular or other bodies derived from these when they are employed in architecture.

[218] Pylyshyn, *Seeing and Visualizing.* 49-157. Firestone and Scholl, "Cognition does not affect perception: Evaluating the evidence for 'top-down' effects," 4. As discussed by Donald Hoffman, there are specific rules or constraints that are involved when the brain interprets a two-dimensional representation three-dimensionally. Donald Hoffman, *Visual Intelligence: How we create what we see* (New York: Norton, 1998). This approach is further considered in Chapter 3 and 5.

Pacioli's *Summa*, and expanded to correlate regular and other bodies derived from these and architecture in the *Compendium* and *Tractato d'architectura*, has repercussions not only for Pacioli's theory of visual perception, but also for the theory and practice of architecture.

Conclusion

The analysis of Pacioli's *Summa*, *Divina proportione* and *Euclid's Elements,* introduced in this chapter, should be correlated to an assessment of the specific environments and the sources that Pacioli experienced and could access throughout his life. Most importantly, this involves the individuals with whom Pacioli interacted, the texts he was able to read and the other works that he was able to see. As noted, Pacioli was first educated in commercial mathematics and practical geometry. He later became a sought-after lecturer on these subjects, appointed in public and private schools throughout his life. He was primarily a teacher, and this is certainly noticeable in the didactic character of his works. On this account, his aims were eminently pragmatic and addressed the needs of contemporary practitioners.

Nevertheless, at the time of his first period in Venice, Pacioli had access to philosophical and theoretical mathematical learning. This first occurred, as far as we know, at the School of the Rialto in the late 1460s. Although an often neglected passage in his biography, its consideration is essential if Pacioli's works are to be understood in their theoretical framework. Pacioli recalled this passage while introducing geometry and algebra in the *Summa*, and undoubtedly laid the foundations of his theoretical and philosophical framework in Venice. As we have begun to see, this framework is (not uncommonly), largely Aristotelian. Moreover, having joined the Conventual Franciscans, Pacioli later completed his theological studies to become *Magister theologiae.* It is thus not surprising that he interacted with a number of Franciscans, and that Franciscan authors, such as John Duns Scotus or Albert of Saxony, are praised and referenced in his writings.

As often noted, Pacioli also claimed multiple connections with artists, architects and scholars. Among these, the importance of his meeting with Alberti in the early 1470s should not be underestimated. Indeed, although developed somewhat differently, Pacioli showed in his works a persistent commitment to the programme of the quantification of the material world that links back to Alberti. As noted and shall see in later chapters, Pacioli's different kinds of links with Piero della Francesca, Leonardo da Vinci and Donato Bramante were also important in this regard. Overall, one of the foremost stated aims of Pacioli was to clarify the correlations between theory and practice. Moreover, Pacioli claimed connections with the circle of Della Rovere in Rome, with individuals at the court of Guidobaldo da Montefeltro in Urbino, and with other individuals at the Sforza court of Milan. Crucially, his works show that his knowledge of the geometrical properties of the five regular and other bodies derived from these was instrumental to access and establish his authority in these circles. In his writings, Pacioli described their formal and aesthetic properties while claiming, unprecedentedly, their practical usefulness, material instantiation and application in architecture. Pacioli claimed to have discussed and presented set of models of these bodies in all of these environments.

Where formal and aesthetic properties are concerned, the discussion of regular and other bodies derived from these is consistent with Pacioli's interpretation of the process of visual judgement of works of architecture. Crucially, this is hierarchically organised for Pacioli, with visual aesthetic judgments based on visual and formal properties located at the basic levels of perception and experience; while verbal properties are considered as separate. Ultimately, both the concept of the geometrical body as described in the works of Pacioli, and the outlined hierarchical process of visualisation, have implications that reach back to the instantiation of visual and formal properties in architecture. In order to clarify these connections, and their role in coeval practice, it is essential to firstly examine the theoretical and philosophical framework of Pacioli's works.

2. Pacioli's arguments concerning regular and other bodies derived from these in their philosophical and mathematical framework

This chapter examines Pacioli's theoretical framework with particular reference to the correlation of arguments involving regular geometrical bodies and their formal properties. It will consider how Pacioli was able to reconcile contemporary approaches to Aristotelian, Medieval Scholastic and Platonic texts in the context of the underlying Aristotelian framework outlined in his works. Regular and other bodies derived from these are a geometrical subject discussed in all texts of Pacioli under examination. As noted, these bodies are considered in a separate section in the *Summa*, whilst regular bodies are both constructed and geometrically examined in Books XI-XV of Pacioli's edition of Euclid's *Elements*, based on the Campanus edition. Finally, Pacioli's *Compendium de divina proportione,* prepared in the environment of the Sforza court of Milan, mainly concerns a description of the formal properties of these bodies and presents them in the well-known set of perspectival representations drafted by Leonardo da Vinci.

Since Pacioli always presented regular and other bodies derived from these in learned and courtly environments, we must assume that he believed that these corresponded to the interests of his patrons, other scholars and practitioners who interacted in the same environments. On all these occasions Pacioli emphasised both the theoretical and practical relevance of the geometrical knowledge of these bodies. It is, however, necessary—before considering their practical applications—to introduce Pacioli's theoretical and philosophical framework. A number of aspects of what can be approached as a theory of geometrical bodies can be identified in Pacioli's texts. Simultaneously, some of the section of the *Summa* dealing with regular bodies are referenced in the *Compendium* and both rely on Euclid's *Elements* (in Campanus's edition). Overall, the *Elements* are Pacioli's most frequently quoted source, referred to in both the *Summa* and *Compendium* to support theoretical and mathematical arguments. At the same time, the philosophical framework in which Pacioli, as Campanus, operated is predominantly Aristotelian. Although this may be seen an inherent fissure given these two different systems of reference, we shall see how Pacioli was able to develop a set of coherent arguments concerning regular and other geometrical bodies derived from these in this philosophical and mathematical context. As is well known, the works of Aristotle had been passed down through translations and commentaries which although highly regarded, were subject to corrections and interpretations either through precedents, or necessity.[219] Ultimately, when approaching Pacioli's works, it is clear that overall scholars have seldom positioned the concept of regular or other bodies derived from these in his wider mathematical-theoretical and philosophical framework. As a result, the formal and aesthetic properties of these bodies as expressed in Pacioli's texts have rarely, if ever, been considered and clarified in their context.

2.1 Quantity and proportion

In the late fifteenth century and in the environments where Pacioli operated, theoretical mathematical knowledge and methods of precise quantification were actively pursued and highly praised. Consequently, the services of mathematicians were in this respect, increasingly sough after. This is evident even at a first reading of Pacioli's texts. For instance, at the beginning of his *Compendium*, Pacioli claims to have measured and quantified the weight of

[219] An example of this approach is in two annotations attributed to Pietro Pomponazzi: "Et si dicatur: Aristoteles non fuit istius opinionis (quod tamen est dubium), respondetur quod Aristoteles non fuit deus et ipse non omnia novit". And with regard to the interpretation of Averroes: "Commentator erravit, neque ipse est deus". Pietro Pomponazzi, *In 1 Physicorum* (ms. 390, Biblioteca della Fraternita de' Laici, Arezzo) 41v, 47v; cited in Nardi, *La Scuola di Rialto*, 52. Indeed, as noted by Nardi, Pomponazzi expressed this position towards the works of Aristotle in a number of occasions. Bruno Nardi, *Saggi sull'Aristotelismo Padovano dal secolo XIV al XVI* (Firenze: Sansoni, 1958), 263.

the equestrian monument of Francesco Sforza designed by Leonardo.[220] At the same time, it was increasingly important (at least for a scholar such as Pacioli), to be able to philosophically frame, and thus justify the prominent role held by quantity and the process of quantification in many practical applications.[221]

The importance of these two concurrent enterprises in the late fifteenth century is demonstrated not only by the ostentatiously practical and theoretical-philosophical character of Pacioli's works, but also by his multiple engagements in schools, universities and courtly environments. It is in this wider context that Pacioli's discussion of the concept of quantity in the opening to the *Summa* should be understood. Pacioli argues this at the very beginning of the text:

> Quantity, Magnanimous Duke, is such a noble and excellent thing, that many philosophers have judged it equal to substance [*substantia*], and together with it eternal. This is because they have not known of anything in nature that can exist without [quantity]; therefore I aim to discuss it (with the help of him who governs our senses); and [this is] not because the ancients have not discussed it extensively already, both in theory and practice, but [because] what they said is nowadays very obscure, and by many poorly learnt, and [it is] poorly applied by the vulgar practitioners; they thus [now] employ [quantity] in many different ways, and with great and laborious difficulties in their works, both with numbers or with measurements. Therefore, I do not aim to discuss it if not [for what concerns] the practical works and the works of crafts. [I will be] mixing with it, in the appropriate way, the theory […] both in numbers and in geometry. […] And so you will be able to follow what Aristotle says

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220 "Commo l(')admiranda e stupenda equestre statua. La cui altezza da la cervice a piana terra sonno braccia 12 cioe 36 tanti de la qui presente linea ab. E tutta sua ennea massa a libre circa 200000 ascende che di ciascuna l(')oncia comuna fia el duodecimo a la Sanctissima invicta vostra paterna memo(r)ia dicata […]." Pacioli, *Compendium*, 2r.

221 Indeed, this is not surprising but in keeping with what John E. Murdoch defined as a consistent tendency in medieval theories of proportions: to bring arithmetical conceptions in the framework of geometrical and in theories of general magnitudes. However, as we shall see, Pacioli placed particular emphasis on geometrical measurements, and therefore both in rational and irrational quantities. John E. Murdoch, "The Medieval Language of Proportions: Elements of the Interaction with Greek Foundations and the Development of New Mathematical Techniques," in *Scientific Change*, ed. A. C. Crombie (London: Heinemann, 1963), 270.

> in the second [book] of the Posterior [Analytics]: Thus in fact someone is the most expert when possessing [knowledge of] their own [work] etc.[222]

It is unsurprising that quantity should hold a fundamental position in a text dealing with practical and theoretical mathematics. The correlated concept of magnitude is at the core of both Euclid's *Elements* and Aristotle's *Physics*. Mathematical texts from late Antiquity had followed the relationship established by Aristotle, in which the concept of quantity is clearly linked to the essence of physical bodies. This is noticeable, for instance, in Boethius's *De institutione arithmetica,* which in turn was based on a late Antique text on arithmetic by Nichomacus of Gerasa.[223] Drawing on these sources, and on the medieval authors that further elaborated on them such as Campanus of Novara or Albert of Saxony,[224] Pacioli was therefore able to argue, as we have seen, that quantity could be considered equal to substance, and then correlate this to the importance of quantity in practical applications.

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[222] "La quantita Magnanimo Duca e si nobile et eccellente cosa, che molti philosophi per questo giudicata l'ànno alla substantia para, e co(n)essa coeterna. Peroche hanno cognosciuto per vero modo alchuna cosa in rerum natura senza lei non potere existere. Per laqual cosa de lei intendo (co(n) l'adiuto de Colui che li nostri sensi regge) tractarne: non che per altri prischi e antichi philosophi non ne sia copiosamentetractato: e in Theorica e practica. Ma perche lor dicti gia alli tempi nostri sonno molto obscuri, e da molti male apresi, e alle pratiche vulgari male applicati, di che in loro operationi molto variano, e con grandi e laboriosi affanni mettano in opera, s. de numeri commo de misure; unde di lei parlando non intendo se non quanto che alla practica e operare sia mestiero, mescolandoci secondo i luoghi oportuni anchora la theorica: e causa de tale operare, si de numeri co(m)mo de geometria. Ma prima accio meglio quello che sequita se habia apprendere, essa quantita dividiremo seco(n)do el nostro proposito; e dividendola a ciaschuno suo membro assegneremo sua propria e vera diffinitione e descriptione. E allora poi seguir. quello che Aristotile dice in secondo posteriorum: Tum enim maxime scitur aliquid cum habetur suum quid est etc.' Pacioli, *Summa*, 1r (Epistle).

[223] Two elements taken from the texts of Nichomacus and Boethius are relevant for the role they have in the texts of Luca Pacioli. Firstly, magnitude is defined as an 'essence' of all physical bodies. In *De Institutione Aritmetica* Boethius states: "Essentia autem gemiuae partes sunt, una continua et suis partibus juncta nec ullis finibus distributa ut est arbor lapis, et omnia mundi hujus corpora, quae proprie magnitudine appellantur. Alia vero disjuncta a se, et determinata partibus, et quasi acervatim in unum redacta concilium […]" Boethius, *Boetii, Ennodii Felicis …* Opera Omnia, Boetii, Tomus Prior, ed. Jacques-Paul Migne (Paris: Garnier Fratres, 1882), 1081. Moreover, in the section concerning quantity constant of itself, which is considered in geometric figures and in which is a common element of all magnitudes [*De per se constante quantitate, quae in figuris geometricis consideratur; communis ratio omnium magnitudinum*], Boethius considers the three-dimensionality of bodies: "Haec enim tria circa omne corpus inseparabili conjunctione versantur, et in natura corporum constituta sunt. Quare quidquid uno intervallo caret, illud corpus solidum not est" Boethius, *De Institutione Aritmetica*, 1119-1120.

[224] Albert of Saxony (c. 1316-1390) wrote a *Tractatus proportionum* in turn derived from the *Tractatus de proportionibus velocitatum in motibus* (1328) proportionibus" written by Thomas Bradwardine. As noted by Ciocci, after Euclid's *Elements* in the edition of Campanus, the treatise of Albert of Saxony is the most important source of Pacioli for the sixth distinction of the *Summa* (Pars prima), dealing with the theory of proportion. The text of Albert of Saxony was printed in Venice in 1480 and 1496, in Bologna in 1506. Argante Ciocci, "Luca Pacioli e l'albero delle proporzioni," in in *Pacioli 500 anni dopo,* ed. Enrico Giusti and Matteo Martelli (Sansepolcro: Centro Studi "Mario Pancrazi", 2010), 64.

The argument that quantity is equal to substance is intrinsically linked by Pacioli to the universal role held by proportions in every act of creation in such a way, as noted by Pacioli, "that all our studies in any faculty are aimed at understanding the relationship [*convene(n)tia*] between the one and the other".[225] If proportions are fundamental in every act of creation, it is quantity, and first of all the concept of the *part of* a quantity, Pacioli argues, that allows proportional ratios to be established.[226] Thus, Pacioli opens the sixth distinction of the *Summa* dealing with the theory of proportions by emphasising the connection between numbers, measurements and proportions: "everyone that has written anything on numbers", Pacioli notes, "has also dealt with measurements, and together with them, with their common envelope [*co(m)mun lor vestime(n)to*] called proportion".[227] A list of ancient and medieval mathematicians and philosophers who have dealt with proportion follows, with a central role granted to Euclid. The *Elements*, Pacioli argues, were studied in all philosophical schools after their publication.[228] Pacioli then lists the books of the *Elements* that concern geometry and those that deal with arithmetic. He leaves aside, however, Book V because it is "common to

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225 In the sixth distinction of the *Summa*, where the theory of proportions is expounded, Pacioli argues in passages often quoted that "nothing in nature can be known without having understood its proportion", and as Pacioli adds, "nothing in nature can survive if not rightly proportioned to its necessity".

226 As noted by Pacioli: "Euclides before defining what proportion and also proportionality is, defined the *part*, because since [the part] is a property of quantity, having defined the part, immediately you have defined the whole, by calling it a multiple in respect to the part, being it an addition or multiplication, based on the way in which [the whole] contains the parts". "Euclides prima che diffinisse la proportione e anche proportionalita nel qui(n)to diffini essa parte. […] E diffinita la parte, im(m)ediate diffinisci el tutto chiama(n)dolo multiplici respetto a la sua parte, o sia agregativa overo mutiplicativa, commo li se contiene". Pacioli, *Summa*, 69r-v.

227 "Ognuno che di numeri alcuna cosa ha scripto sempre ancora insiem co(n) q(ue)lli de mesure al qua(n)to ha tractato, e i(n) co(m)pagnia d(')esse del co(m)mun lor vestime(n)to ditto p(ro)portione." Pacioli, *Summa*, 67v (Pars prima).

228 Pacioli lists Ameto son of Joseph, Jordanus (de Nemore), Thomas Bradwardine, Biagio Pelacani of Parma, Albert of Saxony, member of the Franciscans. Pacioli also adds that Plato dealt with proportions in his *Timaeus*, in the *Laws* and *Republic*, and Aristotle shows to have greatly understood and dealt with proportions in *De Cielo et mundo:* "De Ameto figliuolo de Joseph (del qual el campano expone(n)do el quoi(n)to de Euclide fa me(n)tione) la sua epistola p(ri)ncipaliter, de la p(ro)portione, p(ro)portionalita co(m)pose. Giodano ancora i(n) la sua arithmetica pure de la p(ro)portio(n)e e p(ro)portionalita parlo. Tomas beduardin similiter particulare trattato de le p(ro)portio(n)i. Blasius de parma tarcta(n)do de l(')una e l(')altra q(ua)ntita i(n) certo suo co(m)pe(n)dio, de la p(ro)portio(n)e similme(n)te tratto. Albertutius a(n)cora de Saxonia p(ro)fondo ph(ilosoph)o e sacro dottore de l(')ordine nostro seraphyco, trattato p(ar)ticulare co(m)pose de le p(ro)portio(n)i, el q(ua)l molto p(er) le scole a la te(m)pesta nostra se rivolta." Pacioli, *Summa*, 67v-68r (Pars prima).

all the others",[229] later adding that this is the "living soul of all the others".[230] Pacioli here also points out the accomplishments of Archimedes, since he was able to apply combined geometrical and arithmetical means, "with numbers and measurements", to approximate the ratio of the diameter to the circumference of a circle.[231] To this follows a section titled "On the necessity of knowledge of proportions", that lists practical applications to which the employment of proportion is fundamental. Pacioli here mentions jurisprudence, medicine, perspective in painting and architecture.[232] To these are added stonemasonry, shoemaking, wood-making and weaving.[233] As in other sections of the text, Pacioli first emphasises the relationship between theory and practice in order to frame the rest of the discourse in these terms. At this point, employing a passage found in Campanus' commentary on the *Elements,* Pacioli refers to the discussion of quantity in Aristotle's *Categories,* and clarifies the correlation of quantity and proportions:

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[229] "E q(ue)sto fia manifesto se be(n) si leggi di molti ph(ilosoph)y lor libri, de mathematiche foculta co(m)pilati. Co(m)mo de Euclide megarense: La cui op(er)a sempre tutti li antichi ginnasii, cie de stoyci, academci, peripateticim, platonici etc.ane di degna disciplina mathematica illustrati. Nella q(ua)le lui de arithmetica, cioe de numeri trattando, ancora de geometria, cioe de misure largame(n)te disse, e con quelle agio(n)se (co(m)me e ditto) el lor co(m)mun velo ditto p(ro)portio(n)e. Unde divise ditta op(er)a i(n) 15 libri partiali. De li q(ua)li 10 so(n)no de geometria, cioe p(rim)o, 2˚, 3˚, 4˚, 6˚, 11˚, 12˚, 13˚, 14˚, 15˚. E quatro sonno de arithmetica p(ri)ncipalme(n)te, cioe 7˚, 8˚, 9˚, 10˚ e uno (a tutti q(ue)sti co(mun)e, cioe el q(ui)nto fo de la p(ro)portio(n)e. La q(ua)l (co(m)mo se dira) cosi se aspecta al nu(mer)o, co(m)mo a la mesura. Del Severino Boetio a(n)cora se ben si guarda i(n) la sua arithmetica, trovasse le força de geometria, e anche particulare me(n)tio(n)e de le p(ro)portio(n)i. De thebit ancora degno ph(ilosoph)o (del q(ua)l molto Boetio expone(n)do Euclide fa mee(n)tione, maxime nel qui(n)to) p(er) sue op(er)e diffusamente d(e) l(')una e di l(')altra i(n)siemi co(n) le p(ro)portioni tratto. De Ameto figliuolo de Joseph (del qual el campano expone(n)do el quoi(n)to de Euclide fa me(n)tione) la sua epistola p(ri)ncipaliter, de la p(ro)portione, p(ro)portionalita co(m)pose. Giodano ancora i(n) la sua arithmetica pure de la p(ro)portio(n)e e p(ro)portionalita parlo. Tomas beduardin similiter particulare trattato de le p(ro)portio(n)i. Blasius de parma tarcta(n)do de l(')una e l(')altra q(ua)ntita i(n) certo suo co(m)pe(n)dio, de la p(ro)portio(n)e similme(n)te tratto. Albertutius a(n)cora de Saxonia p(ro)fondo ph(ilosoph)o e sacro dottore de l(')ordine nostro seraphyco, trattato p(ar)ticulare co(m)pose de le p(ro)portio(n)i, el q(ua)l molto p(er) le scole a la te(m)pesta nostra de rivolta. Pacioli, *Summa*, 67v-68r (Pars prima).

[230] Pacioli, *Summa*, 77r (Pars prima).

[231] "De l(')archima(n)dritta de li phylosopha(n)ti platone, ancora evide(n)teme(n)te apare, i(n) suo thimeo,, et de republica et de legibus p(er) tutto aco(m)modatame(n)te de le p(ro)portioni haver trattato. De AR(istotle), li libri che de celo e mo(n)do p(ro)pose, manifestame(n)te mostrano lui altame(n)te de le p(ro)portio(n)i haver i(n)teso e trattato. De archimede siracusano, l(')opera che feci de quadratura circuli, e de ce(n)tro gravitatis, et de figura ovali e de harene ne(mer)o ap(un)to mostra co(n) dilige(n)tia haverne ditto. Maxime qua(n)do co(n) sua sutilita de i(n)gegno, trovo la p(ro)pinq(ui)ta de la co(n)venentia del diametro del cerchio a la sua circu(m)fere(n)tia, per vie geometriche e arithmetich(e) p(ro)cede(n)do […]" Pacioli, *Summa*, 68r (Pars prima).

[232] Pacioli, *Summa*, 69v (Pars prima).

[233] ibid., 69v-70r (Pars prima).

> […] not only in number and measure is found proportion, but also in sounds, in places, in times, in weights, and in powers as Plato wants and affirms in his Timaeus, [proportion] will not be found among things unless one is greater than the other or one is equal to the other. And this is a property of quantity, as Aristotle says in the *predicamenti*: *Quantitatis aute p(ro)pius est secundu ip(er)as equale vel inequale dici*. (sic) For this reason it is clear that proportion first and foremost is found in quantity, be it discrete or continuous […] and never in any place you will find proportions or anything similar if not in quantity […] Based on this, both quantities have to be of the same kind. Such as two walls, or two lines, or two surfaces, or two places, or two times, or two bodies, or two sounds […][234]

Although the passage closely follows the comment added by Campanus to the third proposition of Book V of the *Elements,* and later included by Pacioli in his own revised edition, neither 'walls' nor 'sounds' were included by Campanus, and are therefore Pacioli's own additions.[235] Moreover, unlike Campanus' Latin, Pacioli presents the *Summa* in Italian. This is undoubtedly among the first known instances of the translation and circulation in print of a meaningful passage from Aristotle's *Categories.*[236] Theoretical discussions on quantity and commented translations of passages from Aristotle's *Categories* (although most probably sourced from Campanus' edition of Euclid's *Elements*), were not usually included in manuals of commercial mathematics or practical geometry before Pacioli's *Summa*. Since these manuals were mostly collections of practical problems, the whole theory of proportions was, but for a few exceptions, usually completely omitted and only the practical applications were

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[234] '[…] non solamente in lo numero e misura se ritrova la proportione ma etiam i(n) li suoni, e in li luoghi, e in li tempi, e in li pesi, e in le potentie si commo plato vole e afferma in suo thi(me)o, la quale mai se ritrova i(n) niuna cosa se non in qua(n)to l'una e magio(r)e de l'altra, o veramente una e equale a l'altra. E questa debita p(ro)prieta de la quantita si co(m)mo dice AR ne li predicamenti. Quantitatis aute p(ro)pius est secundu ip(er)as equale vel inequale dici Per laqual cosa se ma(n)ifesta la p(ro)portione prima e principalmente retrovarse in le q(uan)tita o sie(n)no co(n)tinue o sie(n)no discrete […] Per la qual cosa se manifesta tutte doi ditte q(ua)ntita dover esser de un medesimo genere. Co(m)mo doi muri, o doi li(n)ee: o doi sup(er)ficie, o doi luoghi, o doi tempi, o doi corpi, o doi suoni […]' Pacioli, *Summa*, 69r (Pars prima)

[235] Pacioli, *Euclidis,* 32r-v

[236] Eugenio Refini noted that "no Aristotelian works on logic or metaphysics were translated into romance languages before the sixteenth century"; Eugenio Refini, "'Aristotile in parlare materno': Vernacular Readings of the Ethics in the Quattrocento," *I Tatti Studies in the Italian Renaissance*, vol. 16, no. 1/2 (September 2013): 314-315, 315n. Although Refini refers here to the translation of a whole work. I am also not aware of the passage as being circulated in vernacular in print or manuscript form before the year of publication of Pacioli's *Summa* (1494).

presented.[237] Pacioli continues here instead by clarifying the mathematical consequences of this assumption:

> Because you cannot say that a line is greater or smaller than a surface, or indeed equal to it, or a body, and the same for a time of a place. But a line is greater or smaller or equal to another line, and the same a body to another body, and a surface to another surface, as Euclid demonstrate by induction [*i(n)duc*i] in the 6° [book] for surfaces, in the 7°, 8°, 9°, 10° for numbers, in the 11°, 12°, 13°, 14°, 15° for bodies. In short, to conclude, proportion is born out of quantity, as we said above.[238]

Pacioli then adds that there are two possible ways to express proportion. The first is called *communiter*, which is defined as the *habitudine* of two things compared in a term that is univocal between them. However, Pacioli notes that this is due to a misuse allowed by language when you compare, for instance, "the sharpness of a stylus and the sharpness of a voice".[239] The other, *proprie dicta*, "establishes a proportion, as noted by Euclid, between two quantities that are of the same species or kind".[240] The proportion "*proprie dicta*", Pacioli notes is divided into the arithmetical, geometrical and harmonic. The first is found "in continuous quantities", the second "in discrete quantities", the third only in sounds and chants. However, Pacioli argues that it also relies on the other two by taking place in a continuous quantity (time) and is articulated in a sequence of discrete notes. Nonetheless, Pacioli concludes that he would deal

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[237] Other instances in which some notions of the theory of proportions are considered are: Leonardo Pisano's (Fibonacci) *Liber abaci* in Latin, the Praticha d'arismetrica of Maestro Benedetto da Firenze (ms. L.IV.21, Biblioteca Comunale di Siena), e la Pratica d'arismetricha (ms. Pal. 573, Biblioteca Nazionale di Firenze). Argante Ciocci, "Luca Pacioli e l'albero delle proporzioni," 71n.

[238] "P(er)che no(n) se po co(m)modamente dire la linea esser magiore over minore de la sup(erfici)e, over a lei eq(ua)le e anche del corpo, e cosi el tempo del luogo. Ma si bene una linea magiore o menore o equale de un(')altra linea. E cosi un corpo d(e) un(')altro corpo, e una superficie de un(')altra superficie si como de le sup(erfici)e nel 6° e de li numeri nel 7°, 8°, 9°, 10°, e di corpi in 11°, 12°, 13°, 14°, 15°, induci Euclide. Si che breviter co(n)clude(n)do la p(ro)port(i)o(n)e nasci da la q(uan)tita co(m)mo sopra e ditto. Pacioli, *Summa*, 69r (Pars prima).

[239] "Proportione communiter dicta e habitudine de doi cose as(s)siemi comparate una a l(')altra e l(')altra a l(')una, in alcun termino a loro univoco. E per questo se exclude una certa abusione de vulgar parlar che dici la voci e stilo ognuna esser acuto e l(')un piu de l(')altro." Pacioli, *Summa*, 69v (Pars prima). Indeed, Pacioli refers in his commentary to the third definition of Euclid's *Elements* to the conventional nature of naming. The passage is derived from the commentary to the *Categories* ascribed to Boethius but it refers back to the conventionalist theory argued by Aristotle in *De Interpretatione*. Pacioli, *Euclidis*, 32v; Pacioli, *Summa*, 69v (Pars prima).

[240] This is a paraphrase of definition 3, Book V of Euclid's *Elements*, already commented by Pacioli in the text. In the Elements of Campanus and Pacioli the definition states: "Proportio est duarum quantecumq(ue) sint eiusdem generis quantitatum certa idest determinata alterius ad alteram habitudo." Pacioli, *Euclidis*, 32r.

only with the first two, since the latter "is not [relevant] for the purpose of this work".[241] This statement is unexpected in an extensive treatise such as the *Summa*, particularly if compared with other Renaissance scholars who placed particular importance on the theory of the so-called harmonic proportions. However, it is less surprising when both Pacioli's interests and the intended readers of his works are considered. As we have already begun to see, Pacioli argues throughout his works for formal and aesthetic properties to be based on quantities purely derived from visual properties. At the same time, he never considers that visual properties can be derived from harmonic ratios linked to intervals in music.

Following the *Elements*, Pacioli adds in this section that proportion exists between two quantities of the same kind, in a known and determined *habitudine*. Nonetheless, as Pacioli clarifies following the comment by Campanus,[242] *certa e determinata habitudine* does not mean that the proportional ratio between quantities can always be known either by ourselves or by nature, as this can be irrational and not knowable by numbers. Rather, it means that quantities can be determined by their (geometrical) relation with other quantities, in a way, that "they cannot be anything else [other] than what they are".[243] This clarification has evident repercussions for the visual experience of quantities that are in proportional ratios and crucially depends on the universal and necessary nature of mathematical principles. Although a quantity may not be known numerically, Campanus and Pacioli argue that nevertheless, it can be inferred through geometrical (visual) properties. Of course, as noted in two examples discussed later by Pacioli, we do not need to know the actual measure of the

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[241] "E questa p(er)he non fia cosi al proposito a lo intento nostro in quest(')opera da canto la lasciere(m)o e autendaremo a la discreta e co(n)tinua." Pacioli, *Summa*, 70r (Pars prima).

[242] The commentary of Campanus, included by Pacioli in his edition of the Elements states: "Quod aut(em) dicit certa habitudo. No(n) sic intelligas q(ua)si nota vel scita, sed quasi determinata: ut sit sensus. Proportio e(st) determinata h(ab)itudo duar(um) quantitatum ita inqua(m) determinata q(uod) hec et non alia." Pacioli, *Euclidis*, 32r-v.

[243] "E pero se asegna un(')altra diffinitio(n)e conforme a quella de Euclide dece(n)do proportione e de doi quantita d(')un medesimo genere de l(')una a l(')altra et e converso certa e determinata habitudine. Cera non dico perch(e) sia sempre a noi nota ne a la natura ma perche cosi e quella che non po esser altra. E a questo si conforma Euclide. Pacioli, *Summa*, 69v (Pars prima).

diagonal and the sides of a square to visually experience that they are in a specific ratio and proportion. Accordingly, we can visually (mentally) measure the ratio between the volume of a cube and a dodecahedron inscribed in the same sphere, even if we do not know or cannot know the precise numerical value of the proportional ratios among their sides (as determined by the golden ratio). Moreover, if it cannot be anything other than what they are, as argued by Campanus and Pacioli, it follows that knowing the denomination of such proportions would not have any effect on their visual experience.

Crucially, these proportional relations can spatially involve linear distances and take place in three-dimensions, as in the case of regular bodies. Consequently, if such bodies are defined through the proportional relationship among their parts, the proportional ratios among their quantities can be understood in the same terms. If we follow this argument through, as Pacioli notes in the *Compendium*, even their internal quantities (i.e. intervening in the volume between their superficial boundaries), can be visually measured. As Campanus and Pacioli summarise, the application of continuous (geometrical) quantities is certainly more extensive than the application of discrete (arithmetic) ones.[244] Referring to Aristotle's *Categories,* Pacioli concludes by reinstating that "a geometric proportion occurs when a continuous [quantity] is compared with another, such as a line with another line, a surface with another surface, a body with another body, a time and a place with another time and another place".[245] Ultimately, in keeping with the Aristotelian conception of magnitude and its relationship with bodies, here, bodies are again considered as the highest expression of possible proportional relationships among geometrical entities. Since bodies are divisible in all ways, they can be

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[244] "E p(er) q(ue)sto se ma(n)ifesta ch(e) la p(ro)port(ion)e geometrica e de magior abstractio(n)e e co(n)sideratio(n) che no(n) e quella arithmetica e piu largamente se ritrova la p(ro)portione in le q(uan)tita continue che in le quantita discrete. Pero chel geometra de la rationale e inrationale indifferentemente considera e lo arithmetico solamente de la proportione rationale che per qualche numero si possa nominare." ibid., 70v-71r (Pars Prima).

[245] ibid., 70r (Pars Prima).

compared with other bodies, their surfaces can also be compared with other surfaces in the same body, or with the surfaces of another body, and the same can be said of lines.

2.2 The role of the *divina proportione* in Pacioli's *Compendium de divina proportione*

While examining the theoretical framework of Pacioli's works regarding the concept of the geometrical body, it is essential to assess the role of the proportion associated with the title of one of Pacioli's most renowned works. This is the *divina proportion*e, or as it is now commonly known, the golden section.[246] However, the name of this proportion is the "extreme and mean ratio" in Book VI of Euclid's *Elements*,[247] and is employed in Book XIII to construct two regular geometrical bodies, the icosahedron and dodecahedron. The proportion is in fact found in the ratio between their sides and the diameter of the circumscribing sphere.[248] The mathematical properties of the proportion where well known since Antiquity; in the thirteenth century, a "method for dividing a line into mean and extreme proportion", with the use of numbers, was already included at the end of the sixth chapter of Fibonacci's *Practica geometriae.*[249]

Nonetheless, Pacioli's addition of the adjective *divina* (divine) and the justification of the title adduced in Chapter 5 of his *Compendium*, has led scholars to a number of metaphysical speculations on the assumed symbolic and analogical role of that proportion in the text. The

[246] It is worth noting that the 'divine proportion', golden section or mean is a continuous proportion determined by three terms in constant (continuous) proportion among them. This could be expressed by three values a, b and a+b, where the ratio of a/b is equal to (a+b)/a.
[247] "ἄκρον καὶ μέσον λόγον" Euclid, *Elements*, 188.
[248] The definition given in Book VI of the *Elements* states, in modern English translation: "A straight-line is said to have been cut in extreme and mean ratio when as the whole is to the greater segment so the greater (segment is) to the lesser". Euclid, *Elements*, 156. A construction to cut a line by means of this ratio is first encountered in Book II, before ratios are defined. The construction in Book II involves the use of rectangles constructed on the two segments and of a square constructed on the greater segment. This construction is later used in Book IV to contract regular pentagons and 15-sided polygons.
[249] Fibonacci, *De Pratica Geometrie*, 335.

divine character of this proportion has become a recurrent reference over time for historians of art and architecture, often to support metaphysical analyses of its role in informing the two-dimensional mathematical order of architecture and its symbolic meanings.[250]

Nevertheless, as I will argue in this section, if the subject is analysed in its wider context, Pacioli's theoretical system becomes clearer. Although Pacioli describes a number of mathematical properties that in his view justify the adjective *divina*, he never establishes a link, either in the *Summa* or the *Compendium de divina proportione*, between any symbolic-religious meaning and the role of the proportion in art or architecture, or indeed any other craft. In other words, he never argues that it should be used in producing material artefacts to convey symbolic meanings. The only instance when possible metaphysical speculations are associated with the employment of regular bodies in architecture is in Pacioli's later *Tractato de l'architectura,* but these are kept distinct from, and acknowledged only after, aesthetic and formal properties.[251] Additionally, rather than insisting on its linear or two-dimensional applications, Pacioli sees the proportion as fundamental to the geometrical formation of two among the five three-dimensional regular bodies discussed in Euclid's *Elements*.

It is important to examine the contents of Pacioli's text in order to validate these claims. The *Compendium* can be divided into four major sections characterised by different contents,

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[250] A well-known example is in Matila Ghyka's works, see Matila Ghyka, *The Geometry of Art and Life* (New York: Sheed and Ward, 1946) and Matila Ghyka, *Le Nombre d'Or, Rites and Rythmes Pythagoriciens dans le dévelopment de la civilisation occidentale* (Paris: Gallimard, 1959). As noted by Alina Payne, there is a tradition of associating the golden section with classical and Renaissance architecture. Burckhardt devotes a chapter to it; the argument was further amplified by Wölfflin. From then on the discussion has become common place. Jakob Burckhardt, *Architecture of the Italian Renaissance* (Chicago, 1987), 70-76 [1st ed. Stuttgart, 1867]; Heinrich Wölfflin. *Renaissance and Baroque*, trans. by K. Simon (London: The Fontana Library, 1964) 48-51 [1st ed. Munich, 1888]. However, Wittkower dismissed the argument because as leading to irrational, hence incommensurable numbers, in contrast to an "organic, metrical and rational" Renaissance *weltanschauung*. Aline Payne, "Rudolf Wittkower and Architectural Principles in the Age of Modernism," *Journal of the Society of Architectural Historians,* vol. 53, no. 3 (September 1994): 327, 327n.

[251] "Ançe saran(n)o de dignissima co(m)mendatio(n)e del v(ostt)ro opifitio p(er)che no(n) solo lo re(n)deran(n)o adorno ma ancora a li docti e sapie(n)ti dara(n)no da speculare conciosia che sempre siano fabricati co(n) quella sc(r)a e divina p(ro)portione h(ave)nte medium duoq(ue) extrema etc." Pacioli, *Divina proportione (1509)*, 34v (Tractato de l'architectura).

mathematical approaches and related illustrations. The divine proportion is introduced in the first section of the text. After the dedication and anticipation of the subject in Chapter 1, Pacioli begins the preface (Chapter 2) by acknowledging the role of vision in acquiring knowledge, citing its role as assigned by Aristotle. He praises perspective in Chapter 3 for uniting the certainty of mathematics—among the natural sciences the foremost in certitude[252]—with the judgement of sight. The following chapter is of a technical character, and here Pacioli explains some of the recurrent abbreviated terms that he will use in the text.

At this point, Pacioli introduces reasons to support the use of the adjective *divina* for this proportion. Chapter 5 in fact is significantly entitled "on the title of the present treatise".[253] Here Pacioli discusses five 'conveniences' [*convenientie*] that "belongs to God" [*a ipso dio spectanti*] and are related to the adjective *divina*. The descriptions always start from mathematical properties that are then related to divinity through formal causes. The first property discussed is the unity of the divine proportion with other proportions. Pacioli affirms that the former cannot be assigned to another species by any difference. This, Pacioli argues, is regarded as the unity commonly assigned to God by theological and philosophical schools.[254] The second property points to the three mathematical terms of the divine proportion, leading to the sacred trinity. The third mentions the irrational and arithmetically unknowable nature of the proportion, while the fourth mentions its unchanging character independent from an assigned quantity. Finally, the fifth and last refers to the fifth essence, or celestial matter related to the dodecahedron in Plato's *Timaeus*. Pacioli states that "as it will later be shown, this body cannot be formed without this proportion",[255] and through it, the

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[252] It should be note that this passage was also usually ascribed to Aristotle in the fifteenth century and Pacioli relates it to Averroes. Pacioli, *Divina proportione (1509)*,1v (Compendium).
[253] "Del condecente titulo del presente tractato" Pacioli, *Compendium*, 12v; *Divina proportione (1509)*, 3v (Compendium).
[254] "la quale unita Sia il el supremo epiteto de ipos idio" Pacioli, *Compendium*, 12v; *Divina proportione (1509)*, 4r.
[255] "El quale commo de sotto se mostrera sença la nostra proportione non e possibile poterse formare". Pacioli, *Divina proportione (1509)*, 4r (Compendium); Pacioli, *Compendium,* 13r.

other four regular bodies corresponding (as argued in the *Timaeus*) to fire, water, air and earth, can be proportioned and inscribed in a sphere.[256]

For Pacioli, the description of the proportion's mathematical properties and the reasons for its divine character culminate in the formation of the regular three-dimensional bodies. Pacioli's programme becomes clear by the end of the section. The Platonic linkage of the five regular bodies with the four basic elements and the universe is brought, by means of the mathematical and divine properties of the proportion, under the framework of Christian theology. Pacioli concludes the chapter with a meaningful passage:

> […] these forms and figures [*forme e figure*] are called by the sages [*sapienti*] regular bodies [*corpi regulari*], and [I] will discuss separately about each of them. And from these, [we will have] an infinite number of other bodies called derived [*dependenti*] [from these]. The five regular [bodies] cannot be in reciprocal proportion, nor they can be inscribed in a sphere, without [employing] our proportion, as it will later become clear. [The discussion of] these properties [*convenientie*], even though many others could also be brought forward, are sufficient to describe the name of this compendium [*compendio*].[257]

As made explicit in the passage, and visible from reading the text, the mathematical properties of the divine proportion are the crucial element that leads to the formal properties of the five regular bodies. Pacioli begins the chapter that follows in the text with a significant passage worth quoting in full:

> This proportion excellent duke is of so much privilege and excellence deserving as much [praise] as you can say [of it]; and with regard to its infinite potency, without its knowledge many things worthy of the highest admiration, both in philosophy or in

[256] ibid.

[257] "E queste tal forme e figure da li sapienti tutti corpi regulari sonno nuncupate. Commo separatamente di sotto de cadauno se dira. E poi medianti questi a infiniti altri corpi detti dependenti. Li quali 5 regulari no(n) e possibile fra loro poterse proportionare, ne da la sphera poterse intendere circumscriptibili senza la nostra detta proportione. El che de sotto aparera. Le quali convenentie, ben che altre assai se ne potesse adure, queste a la condecente a le nominatione del presente compendio scienno per sufficientia assegnate." Pacioli, *Compendium*, 14v and Pacioli, *Divina proportione (1509)*, 4r (Compendium).

> any science, never to light would have come. Certainly this [proportion] has been given by the invariable nature of the superior causes [*invariabile natura de li superiori principii*], as is said by the great philosopher Campanus, our very famous mathematician on the tenth of the 14th book. The most important thing considering it [this proportion] is that a great diversity of solids, both in size and in multitude of bases, and also of figures and forms [*figure et forme*] with a certain irrational symphony among their accords [can be made], as it will be seen in our process, placing the stupendous effects that (from a line divided according to this proportion) not natural but divine should actually be called.[258]

Here, Pacioli closely follows the commentary that Campanus added to proposition 10 of Book 14 of the *Elements*. However, two differences between Campanus and Pacioli should be noted. Firstly, Pacioli dropped Campanus' explicit reference only to lines in conjunction with the proportion. This is an important element because it is consistent with Pacioli's application of proportions not only to lines, but to all types of quantities. Secondly, where Campanus mentioned only the *figura* of solids, Pacioli also added the *forma.*[259] By using only *figura,* Campanus made it clear that he was referring to a shape, therefore avoiding the ambiguities of the term *forma* that were carried from the Latin word, when used in Medieval translations to refer to Aristotle's form not only as essence [Aristotles's *eidos* or *morphe*], but also as shape [Aristotle's *schema*].[260] However, *figura* was also commonly used to signify a two-dimensional shape in Latin translations of Euclid's *schema* and Campanus' edition made no exception.

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[258] "De la sua degna commendazione Cap. VI. Questa nostra proportione excelso D(uca) e da tanta prerogativa e de excellentia degna quanto dir mai se potesse e per respecto de la sua infinita potentia, conciosia che sença sua notitia moltissime cose de admiratione dignissime ne in philosophia ne in alcuna scientia mai a luce poterieno pervenire. El qual dono certame(n)te de la invariabile natura de li superiori principii, commo dici el gran philosopho Campanno (no)stro famosissimo mathematico sopra la decima del 14 glie co(n)cesso. Maxime vedendo lei esser quella che tante diversita de solidi si de grandeççi si de moltitudine de basii si ancora de figure et forme con certa irrationale simphonia fra loro acordi, commo nel nostro processo se intendera ponendo li stupendi effecti quali (de una linea secondo lei divisa) non naturali ma divini verament(n)te sonno d'appellare" *Compendium*, 13v-14r; *Divina proportione (1509)*, 4r (Compendium).

[259] The commentary of Campanus in the edition of Euclid's *Elements* published by Pacioli is as follows: "Mirabilis itaq(ue) est potentia linee pm p(ro)portionem habentem medium duoq(ue) extrema divise: cui cum plurima pholosophantium admiratione digna conveniant hoc principium vel precipuum ex superiorum principiorum invariabili procedit natura ut tam diversa solida tu(m) magnitudine tu(m) basium numero tu(m) etiam figura irrationali quadam simphonia rationabiliter conciliet. Quippe demonstratum est q(uod) proportio duodecedri corporis ad ycocedron corpus que ambo spera una coambit est quasi proportio linee potentis super quam libet lineam pm prefatam proportionem divisam et super eius maiorem partem ad quamlibet lineam potemtem super eandem et eius minorem partem, quoniam vero de tribus ceteris corporibus regularibus non habemus aliquid dictum studeamus de ipsis aliquid dicere." Pacioli, *Euclidis*, 137v.

[260] An overview of Medieval translations of Aristotle's *eidos*, *morphe* and *schema* is given by Branko Mitrović, Mitrović, *Serene*, 55.

Pacioli's addition of *forma* may thus have served to clarify that three-dimensionality was implied here. The process of the formation [*formatione*] of the regular bodies was expressly linked to the proportion "with the middle and the two extremes" in the incipit to Book XIII added by Pacioli in his commentary on Campanus' edition of the *Elements*.[261] As this will be discussed in the last section of this chapter, *forma* should be understood in this context as referring to the shape-defining properties of the body rather than to its essence.

Continuing a close reading of the *Compendium*, Chapters 7-23 further discuss the mathematical properties of the divine proportion, largely relying on Euclid's *Elements*. The nature of the proportion is further clarified at the beginning of this section:

> But the middle and the extremes of this [proportion] cannot be interchanged, as we will see. This is the same that the four *conventia* with the highest maker [*sommo opefice*] and [this proportion] should be placed as among the other proportions without a specific species or any difference under the conditions of their definitions; in this it resemble our saviour that came not to change the laws but rather to abide them and in joining mankind became subject and obedient to Mary and Joseph. Similarly our proportion arrived from the heavens can join all the others both in definition and conditions, it does not degrade them, on the contrary it expands them by equally upholding the principle of unity among all quantities and never changing, the same [properties] that were assigned to the great god by our Saint Severino [Boethius][262]

The reference to the Aristotelian theory of universals is rather explicit in this passage. The proportion is positioned in the same species as other proportions, being subject to the same definition and conditions and therefore expressing the same essence. Consequently, the

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[261] "Liber tertiusdecimus Euclidis de admiranda vi linee secundum proportionem haventem medium duoque extrema divise et quinque corporum regularium formatione" Pacioli, *Euclidis*, 122r.

[262] "Ma fral meçço e gli extremi de questa nostra non e possibile poterse variare commo se dira. Di che meritamente fo la quarta convenentia col summo opefici e che la sia co(n)numerata fra l(')altre proportioni sença specie o altra differentia serva(n)do le conditioni de loro deffinitioni in questo la possiamo asemigliare al nostro salvatore qual venne non per solvere la legi ançi per adempirla e con gli omini converso facendose subdito e obediente a Maria e Ioseph. Cosi questa nostra proportione dal ciel mandata con l(')altre s(')acompagna i(n) diffinitione e condictioni e non le degrada ançi le magnifica piu amplame(n)te tenendo el principato de l(')unita fra tutte le quantita indifferenteme(n)te e mai mutando se commo del grande idio dici el nostro Sancto Severino. Videlicet stabilisq(ue) manens dat cuncta moveri." Compendium, 16r; *Divina proportione (1509)*, 4v (Compendium).

'divine' status of the proportion should not be interpreted as a difference in essence, rather according to Pacioli, it confirms and expands the principle of unity among all material quantities.

In this section (Chapters 7-24), Pacioli continues by adding that quantities divided according to the proportion can be dealt with by "operating according to the documents given by us on the speculative practice called *algebra* and *almucabala*, also called the rule of the *cosa*, that you can find solved [in] our other work".[263] A list of the 13 'admirable effects' [*mirabili effetti*] related to the proportion ensues, these concern its arithmetical and geometrical mathematical properties. Here, Pacioli closely follows the Campanus edition of Euclid's *Elements* and quotes the relevant propositions and definitions in each chapter.[264]

As is made clear at the end of the section, the effects associated with the divine proportion are fundamental to the formation of three-dimensional bodies. Moreover, from this point in the text, the proportion is mentioned either without further specifications, or by its Euclidian name of "proportion with the middle and the two extremes". The section significantly ends with a chapter discussing "how the discussed effects concur in the composition of all regular bodies and the ones derived from these", with Pacioli summarising that "the virtue and potency of the aforementioned proportion with its own effects it is expressed maximally, as we

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[263] This is discussed under the chapter "Comme se intendi la quantita divisa secondo la proportione h(abente) el m(eçço) e doi estremi"; Pacioli continues "Dobiamo sapere che queste cose be(n) notate a dividere una quantita facendo la p(ro)portione havente el meçço e doi extremi, vol dir di quella far doi tal parti inequali che producto de la menore in tutta dicta qua(n)tita indivisa sia qua(n)to el quadrato de la magior parte, co(m)me p(er) la 3 diffinitio(n)e del 6 dechiara el nostro ph(ilosoph)o. […] Questo caso e altri simili operando secondo li docum(n)ti da noi dati nella pratica speculativa detta algebra et almucabala p(er) altro nome la regola de la cosa posta in la p(re)alegata op(er)a nostra se trovava soluto." Pacioli is here referring to the *Summa*. Pacioli, *Divina proportione (1509)*, 5r (Compendium).

[264] "[…] commo aperto demostra la prima del 13 de nostra guida" ibid., 5v (Compendium). Pacioli refers in the discussion of the 13 effects twice to Euclid's Book 2, twice to Book 4, three times to Book 6, once to Book 10, eight times Book 13, three times to Book 14. Eleven out of nineteen references are to books dealing with three-dimensional geometry (Euclid's Books 13-15 in Medieval editions). Although the effects could be infinite, Pacioli explains to have decided to enumerate 13 of them in reference to the devotion of the Duke to Christ and the twelve apostle, as demonstrated by the work of Leonardo da Vinci in Santa Maria delle Grazie.

said above, in the formation and composition [*formatione e co(m)positione*] of the regular and [from these] derived bodies."[265]

What follows (Chapter 26-31) is a description of the construction of the five regular bodies derived from Book XIII of Euclid's *Elements*. Here, the proportion is described by Pacioli as determining the ratio between the sides of two regular bodies, the icosahedron and dodecahedron, and the diameter of the sphere circumscribing them. Pacioli highlights the proportion as found applied both in the two (at the level of the pentagonal surface) and the three dimensions (at the level of the ratio between the diameter and the sides of the body). At the same time, as Pacioli adds at the end of this section, the proportion can be applied to other quantities beyond linear dimensions:

> The surfaces [of the regular bodies], excellent Duke, we can say to be in proportional [ratio] as it was said for their corporeal mass, and therefore irrational by means of the malice of the pentagonal figure of the dodecahedron. However, others can at times have the ratio [between their surfaces] rational, as between the tetrahedron, the cube,

[265] The text continues "But first we should clarify why these bodies are called 'regular', secondly is to be proved how in nature is not possible to form a sixth [body]. Since these bodies are called regular because they have their sides, angles and bases equal and one is related to the other and they correspond to the five elementary bodies [*corpi semplici*] in nature that are earth, water, air, fire, and the fifth essence otherwise celestial virtue that substantiate all the others in their being. And these five are [to be understood as] sufficient in nature, otherwise you would argue that god [created something] superfluous or [something is] lacking to what is needed in nature. This would be absurd as affirms the Philosopher that god and nature don't operate in vain, that is [they] don't fail [to provide] what is needed and don't exceed it, the same [should be said] on the forms of these five bodies […] And therefore not without reason the ancient Plato in his Timaeus the figures [*figure*] of the said regular [bodies] assigned to the elementary bodies and at the fifth *convenentia* linked to the divine name of our [divine] proportion as we mentioned above." "Co(m)mo li dicti effecti co(n)corino a la compositione de tutti li corpi regulari e lor dependenti. Cap XXIIII. Hora excelso D(uca) la virtu e pote(n)tia de l(')antedicta nostra proportione co(n) suoi singulari effecti maxime co(m)mo de sopra dice(m)mo se manifesta in la formatione e co(m)positione de li corpi si regulari co(m)mo dependenti. De li q(ua)li a cio meglio se prenda qui seque(n)te ordinatament(n)te ne diremo. E prima de li 5 essentiali quali p(er) altro nome sono chiamati regulari. E poi successivamente de alqua(n)ti abastanza loro egregii dependenti. Ma prima e da chiarire p(er)che sieno dicti corpi regulari. Seco(n)damente e da p(ro)vare co(m)mo in natura non sia possibile formare un 6. Onde li dicti sonno chiamati regulari p(er)che sonno de lati e anguli e basi equali e lu(n)o de l(')altro a po(n)cto de contiene co(m)mo se mostrara e co(r)respondeno a li 5 corpi semplici in natura cioe terra, aqua, airi, fuco e q(ui)nta essentia cioe virtu celeste che tutti gli altri sustantia in suo essere. E si co(m)mo questi 5 semplici sonno bastanti e sufficienti in natura altrame(n)te seria arguire I dio superfluo overo diminuito al bisogno naturale. La q(ua)l cosa e absurda co(m)mo afferma el ph(ilosoph)o che I dio e la natura non op(er)ano in vano cioe non ma(n)cano al bisogno e non excedono quello cosi a simili le forme de questi 5 corpi de li q(ua)lli sa adire a po(n)cto sonno 5 ad decorem universi e no(n) possano esser piu per quel che sequira. E p(er)o non imeritatamente co(m)mo se dira di socto l(')antico Platone nel suo thymeo le figure de dicti regulari atribui a li 5 corpi semplici co(m)mo in la q(ui)nta co(n)venentia del divin nome a la nostra p(ro)portione atribuita de sopra fu decto e questo a la loro denominatione." Pacioli, *Divina proportione (1509)*, 8r (Compendium); *Compendium*, 17v.

> [and] the octahedron, for being triangular and square; and [also] knowable in their proportional ratio with the diameter of the sphere circumscribing them [and] in which they are formed [*si formano*].[266]

After this section, no other mention of the proportion is made.[267] Although prisms and other bodies derived from the regular by truncation are still discussed in the remaining parts of the *Compendium*, Pacioli never argues that these express, or should be materially employed to express, a symbolic or divine meaning.

The two bodies that Pacioli explicitly addresses as relevant for their practical use in architecture are those with twenty-six and seventy-two bases. Despite them sharing a number of geometrical properties with regular bodies and being described by Pacioli and illustrated by Leonardo da Vinci, they do not depend on the divine proportion for their geometrical construction, nor does Pacioli mention any metaphysical associations to justify their use in architecture. What is coherently demonstrated throughout the *Compendium* is the mathematical role of proportions, and particularly the divine proportion, in the formation of regular and other bodies derived from these. Overall, Pacioli's positioning of proportions in the problem of universals is correlated with the Aristotelian process of formal causation.

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[266] "Le loro superficie ex(cellent) D(uca) fra loro similmente possiamo dire al medesimo modo e(ss)er p(ro)portio(n)ali co(m)mo de lor massa corporea se dicto cioe irr(ati)onali per la malitia de la figura pe(n)tagona che i(n) lo duodecedro(n) se i(n)terpone. Ma del(l')altre possa(n)o a le volte e(ss)ere r(ati)onali como q(ue)lle del tetracedron cubo octocedron per e(ss)ere tria(n)gule e q(ua)drate e note i(n) p(ro)portione co(n) lo diemetro de la loro sp(her)a i(n) la q(ua)le si forma(n)o […]" Pacioli, *Divina proportione (1509)*, 12v (Compendium).

[267] Throughout the section that ensues in the text, in most cases only enunciations are included, with a few demonstrations of theorems taken from the *Elements.* Geometrical drawings are taken from the edition of Campanus. The text continues with a discussion of the geometrical relationships established by inscriptions of a body inside another body, with the vertices of the former touching the surfaces of the latter (Chapter 34-47). Finally, in what can be considered the third section of the *Compendium* (Chapters 48-62) the mathematical approach changes dramatically if compared from the first two sections. Here Pacioli gives only a description of the semi-regular and irregular geometrical bodies discussed, enumerating the lines, superficial and solid angles and polygonal bases of each. Mathematical demonstrations are not included and the correlation with the text of the *Elements* is lost as these bodies were non discussed in Euclid's text. Often practical examples are made and Pacioli frequently refers the reader to the sixty illustrations drafted by Leonardo da Vinci. It can certainly be argued that in this section the clarity and visual character of the illustrations acquires prominence over the text. The *Compendium* ends with a section on pyramidal and oblong bodies (Chapters 58-68), with a final glossary of geometrical terms (Chapter 71).

However, not only proportional ratios, but other properties linked to the concept of the whole body should remain central to this analysis. The examination of two relevant concepts linked with regular and other bodies derived from these in Pacioli's text will complement the theoretical survey of his writings. These, as we will see, should also help to complete an assessment of Pacioli's arguments in the framework outlined above.

2.3 Pacioli's references to Plato's *Timaeus* in an Aristotelian framework

After considering the role of proportions and quantity in Pacioli's writings, it is now useful to see how these concepts were involved in the definition of geometrical bodies, and particularly of the five regular bodies, famously discussed in Plato's *Timaeus*. Neither the environments in which Pacioli is documented throughout his life, nor any individual with whom Pacioli interacted can be meaningfully linked with any profession of Platonism. Moreover, explicit references to his contemporary Marsilio Ficino, who published the first printed edition of Plato's *Opera Omnia* in 1484,[268] are notably absent from Pacioli's works. Pacioli generically refers in some instances to the prominent role of geometry (and particularly of proportions) in the works of Plato. However, all other references are limited to the association of regular geometrical bodies and Empedoclean basic elements found in the *Timaeus*. These passages have been the grounds for much speculation on Pacioli and Platonism, even if such claims are not usually based on an analysis of the wider philosophical framework of Pacioli's texts.[269] For instance, it is usually not noted that such a correlation had already been established in

[268] A second printed edition not directly supervised by Ficino, was published in Venice in 1491. Paul Oskar Kristeller, "Marsilio Ficino as a beginning student of Plato," *Scriptorium*, Tome 20, n°1 (1966): 43.

[269] The Platonism of Pacioli's *Divina proportione* is argued in Alberto Perez-Gomez, "The Glass Architecture of Fra Luca Pacioli," 245-286. However, to make this claim Perez-Gomez relies on collectivist assumptions of what took place "in Renaissance culture"(ibid., 254), or an unspecified "Platonic influence" for Pacioli's discussion of the proportions of the head in profile (ibid., 273). On the other hand, the texts of Aristotle, extensively referenced by Pacioli, are completely overlooked in Perez-Gomez's analysis. Another study that emphasised elements of Neoplatonism in the texts of Pacioli is Edoardo Mirri, "Elementi di filosofia platonica in Luca Pacioli," in *Filosofia e Cultura in Umbria tra Medioevo e Rinascimento*, (Gubbio-Perugia: Centro di Studi umbri-Università degli studi di Perugia, 1967), 377-399.

Campanus' edition of Euclid's *Elements,* where a link to Plato's theories is explicitly made in Book XV, and each regular body is associated with a basic element.[270]

At the same time, as Pacioli notes on many occasions in his *Summa, Compendium* and *Tractato de l'architectura*, his purpose is to clarify and strengthen the existing links between theory and practice. This is notably different to Plato's arguments in works such as the *Philebus* or *The Republic*.[271] Another difference is found in Pacioli's theory of proportions. If proportions are linked to the formal and aesthetic properties of geometrical bodies, as in Pacioli, proportions necessarily involve geometrical components of such bodies. However, in the late fifteenth century, some among the Neoplatonists implied that beauty could only be an affection of unity without parts and was thus incompatible with this type of reasoning.[272] Finally, by his own repeated admission, rather than Plato Pacioli was following Augustine (as did other Renaissance theorists), arguing that eternal forms derived from the act of creation by the Highest Maker.[273]

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[270] The passage is found in the comment to proposition 12, Book XV of Pacioli's edition of the *Elements*, based on Campanus's. Pacioli, *Euclidis*, 144r-v.

[271] In the *Philebus* Plato contrasted the art of calculating and measuring, used by craftsmen, with the geometry and measurements accomplished by philosophers. It is suggested that those "undertaken by genuine philosophers are infinitely more precise and true in the use of measurements and numbers". Plato, *Philebus*, 57c15-d2. Additionally, in the so called 'theory of art' expressed in Book X of Plato's *Republic,* Plato establishes a clear hierarchy of levels of possible knowledge and perfection that propagates from the creator of all things. The first level underneath the creator is occupied by nature, the creator most direct creation. To this follows, at the next degree of removal, the works of craftsmen and artisans that manufacture their products. This is the level of the *techné*, of the technical and artistic knowledge embodied by the works of craftsmen. The last level under this, that is three times removed form the creator, is the level of the mere representations, of appearances fashioned by painters, by the poets and dramaturges. The unifying role that Pacioli assigns to proportions is undoubtedly incompatible with this system. Plato, *Republic*, 592b7-605b12.

[272] As noted by Mitrović, the argument goes back to Plotinus. This argument has indeed consequences in considering all kinds of immaterial entities, and if beauty depended on proportional relationships between parts that are material, then neither God or the Angels, being immaterial, could be said to be beautiful. The view was held in the late fifteenth century by Ficino and later subscribed by a number of authors in the sixteenth century. Branko Mitrović, "Paduan Aristotelianism and Daniele Barbaro's Commentary on Vitruvius' *De Architectura*," *The Sixteenth Century Journal*, vol. 29, no. 3 (Autumn, 1998): 673.

[273] Pacioli uses the terms "sommo opefice" in the *Divina proportione,* while discussing *De civitate Dei* of St. Augustine. Pacioli, *Compendium*, 5r.

Pacioli concluded the first passage in the *Summa* that introduces the Platonic association of regular bodies and basic elements in the *Timaeus* by indirectly endorsing Aristotle's criticism of the process of assemblage of their parts and the related rejection of the vacuum.[274] Essential to Pacioli's argument is the application of the Aristotelian theory of causality to the eternal abstract forms of geometrical bodies which is made clear particularly in the C*ompendium*. Here Pacioli argued that proportions are the universal principle that allow this process to take place. At the same time, although Pacioli argues that mathematical entities can be mentally considered abstracted from matter, it is also clear in his texts that he sees these as always correlated to a material instantiation through formal causes. It has been noted that such a process of causation, linking material instantiations and eternal abstract forms through formal causes, was not precluded in the sixteenth-century Aristotelianism that developed in Padua.[275] As we shall see, the texts of Pacioli show that this approach was already (at least in principle) established in the fifteenth century and most probably already when Pacioli attended lectures at the School of the Rialto in Venice under Domenico Bragadino.[276]

The problem of universals, is often noted, is fundamental to the history of philosophy.[277] Those like Plato, who believed there were some actual abstract entities existing outside the material world, were known as realists; whereas those who tried to avoid extra-mental universals were called nominalists. However, although critical of Plato's position, it is well known that Aristotle's view on universals is far from being settled;[278] he maintained a critical position towards the Platonic theory of forms in *De Caelo* (which is often referenced by Pacioli)

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[274] Pacioli, *Summa*, 4r (Pars prima).

[275] Mitrović, "Paduan Aristotelianism," 668-670.

[276] Indeed, as noted by Mitrović, this approach was already to be found in Porphyry's Isagoge, a text that had a good circulation throughout the Middle Ages and fifteenth century.

[277] Paul Vincent Spade gives an overview of the problem from Aristotle focusing on five among the commentators of his texts: Porphyry, Boethius, Abelard, Duns Scotus, Ockham. Paul Vincent Spade, *Five Texts on the Medieval Problem of Universals* (Indianapolis: Hackett, 1994).

[278] An overview is given in Cohen, S. Marc and C. D. C. Reeve, "Aristotle's Metaphysics", *The Stanford Encyclopaedia of Philosophy* (Winter 2020), ed. Edward N. Zalta, https://plato.stanford.edu/archives/win2020/entries/aristotle-metaphysics.

and other writings, relying, however, on abstract concepts in his theory of causes and categories.[279] Following Porphyry's attempt to present (without answering) the problem and consider the two views in the Isagoge, arguments were developed during the Middle Ages by authors such as John Duns Scotus. Duns Scotus linked the two positions recurring to a form of so-called 'moderate realism' that relied both on formal and final causes.[280] As discussed in Chapter 1, the writings of Duns Scotus were most probably well known in the environment of the School of the Rialto in Venice when Pacioli attended, as they were at the s*tudium* at Padua, and are indeed noted in the works of Pacioli.

Moreover, Pacioli's argument specifically involves regular geometrical bodies and their association with the basic natural elements. The five regular bodies are introduced by Pacioli at the beginning of the *Summa,* in the section discussing the properties of whole numbers, where Pacioli mostly refers to the contents of Boethius's *De Institutione Arithmetica.*[281] Discussing the properties of the number five, Pacioli adds an introduction to the five regular bodies. The passage is important because it contains crucial themes and fundamental authorities in Pacioli's theoretical framework:

> […] the fifth [number], that this is present in nature can be perceived by means of our experience. Especially if we consider the first causes that concur to our own composition, such as the simple [regular] bodies: these are earth, water, air and fire and the fifth essence, that is the celestial virtue. Without doubt these [elements] are

[279] Indeed, as noted by Mitrović, undoubtedly a radical nominalist stance was not possible during the Renaissance since mathematical objects (numbers and geometrical forms) were commonly assumed to have a separate existence. Mitrović, "Paduan Aristotelianism," 669n. Pacioli refers explicitly to this in the *Compendium de divina proportione.* Pacioli, *Compendium*, 8v.

[280] The approach found in the texts of John Duns Scotus, that as we have seen circulated widely in Padua and Venice in the 1460s, is particularly important in this regard. The theory of Duns Scotus is sometimes called 'moderate realism'. Timothy B. Noone, "Universals and Individuation," in *The Cambridge Companion to Duns Scotus*, ed. Thomas Williams (Cambridge: Cambridge University Press, 2003), 100-128. Duns Scotus argued that universals have extra-mental existence only in correlation to the material of the individual in which they exist in actuality; the sum of the properties expressed to each individual instantiation later famously came to be known as haecceity (*haecceitas*). Spade, *Five Texts,* xiii. In short, if mathematical properties are considered, the moderate realism of Scotus thus allows to postulate extra-material concepts only insofar they are linked to individual material instantiations.

[281] Boethius, *De Institutione Arithmetica,* I, xix.

> preordained in the number of five. And to these (as Plato in his Timaeus recommends and the mathematicians demonstrate) correspond five other bodies in nature called regular. Euclid discusses these at length in his 13º, 14º, 15º [book].[282]

From this point, a long extract is taken verbatim from proposition 17 of Book XIII of Campanus' edition of Euclid's *Elements*. The passage describes the number and types of bases for each of the five geometrical bodies, and refers to the two propositions in other books that demonstrate why there are only five geometrical bodies with equiangular and equilateral bases inscribed in a sphere.[283] Pacioli concludes as follows:

> […] for the reasons discussed, only 5 and no more could be the aforementioned regular bodies [*corpi regulari*]. These have been discussed here outside of the [section] of practical geometry, as I thought that was useful to introduce them. But much more extensively I will discuss them when we will speak about geometry. Giving to them forms [*forme*], measurements [*misure*] and [a way of] fabrication [*fabriche*] clear and open, with beautiful and subtle rules for their quantification. And therefore here I will not discuss them any longer. And even if beyond these [five regular] are the bodies [*corpi*] in infinite number (as we will later see), nevertheless none could be ever found similar in form and figure [*forma et figura*] to the ones just discussed.[284]

The passage ends with a description of the reasons why each body has been assigned to a specific basic natural element. The description follows the contents of Campanus' edition of the *Elements*, although the passage included by Pacioli is not in Latin, but in Italian.

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[282] "[…] el quinario: quale quanto ancora lui se ampli in natura: per experientia palpabile se manifesta. Maxime se quardamo ale prime cause a nostra compositione concurrenti: commo so(n)no li corpi simplici: cioe. Terra. Aqua. Aere. Fuoco. E Quinta essentias: cioe. Virtu celestiale. Senza dubio nel numero quinario sonno ordinati. E a q(ue)sti (si commo Plato in suo thymeo aprova e li mathemathici i(n)ducano) so(n)no conresponde(n)ti .5. altri corpi in natura dicti regolari. Di quali difusamente tracta Euclide nel. 13º. 14º. 15º." Pacioli, *Summa*, 4r (Pars prima, De septe(r)narij generalitate. Articulus tertius)

[283] "Hec hautem q(uin)q(u)e solida regularia dicuntur, quoniam ipsa equiangula sunt, atq(qu)e equilatera, et a sphera atq(u)e ad in vicem circunscriptibilia" Pacioli, *Summa*, 4r (Pars prima). A references to the 32˚ proposition of the first book and the 21˚ of the eleventh Book follows to clarify the geometrical reasons why a sixth body with the same geometrical properties cannot be formed. These propositions demonstrate the maximum sum of angles around a point in two and three-dimensions.

[284] "Concludese brevemente per le ragioni prossime assignate che solo. 5. E no(n) piu possono essere li ditti corpi regulari. Li quali qui qua(n)tunche incidenter fuor dela pratica geometrica mesiano occorsi: me parso non inutile haverli indutti. Ma assai piu ampliamente: quando de geometria parlaremo: ne tocaro. Assegnando lor forme e misure: e fabriche commodi chiari e aperti: e regole aloro quadrature bellissime e sotile. Epero qui non ne toco piu. Equantunche oltra questi infiniti siano li corpi (commo li se dira) non dimeno nium si porra mai trovare: simili in forma e figura: ad alcuno deli predetti." Pacioli, *Summa*, 4v (Pars prima, De septe(r)narij generalitate. Articulus tertius).

Nonetheless, at the end of this section, Pacioli introduces the counter arguments found in *De Cælo,* in which Aristotle confuted the correspondence of basic elements and regular bodies on the basis of his rejection of the vacuum. "[Plato's] opinions", Pacioli argues "would bring many and great issues in nature. To the greater extent in relation to the error of the vacuum, that in nature is rejected by everyone and in no way can be conceived".[285] A similar argument is repeated by Pacioli in the second (geometrical) part of the *Summa.*[286]

At this point, it is worth considering how regular bodies were described in Plato's *Timaeus*. In introducing the construction of regular bodies, Plato argued that "it is absolutely necessary that depth should be bounded by a surface and this is composed of triangles."[287] Regular bodies were then described by the ordered assemblage of the two basic triangles: "[…] and when four equilateral triangles are combined so that three plane angles meet in a point, they form one solid angle, which comes next in order to the most obtuse of the plane angles. And when four such angles are produced, the first solid form (εἶδος στερεόν-*eîdos stereón*) is constructed", while "the second solid form is formed from the same triangles, but constructed out of eight equilateral triangles, which produce one solid angle out of four planes; and when six such solid angles have been formed, the second body in turn is completed […]".[288] In the

[285] "Le quali opinioni quantumche a lui così paresse, non di manco Aristotele suo discipulo contra lui, de simel cosa parlando, in quel *De celo et Mundo*, reprova tutte le pretatte opinioni per molti e grandi inconvenienti che in natura sequitarebbero quando così fosse come Plato dice. Maxime fra gli altri sequirebbe lo errore del vacuo, che in natura da tutti se niegase per nullo modo potersi dare" Pacioli, *Summa*, 4v (Pars prima).

[286] Pacioli, *Summa*, 68v (Pars secunda).

[287] Plato, *Timaeus, Critias, Cleitophon, Menexenus, Epistles,* ed. by Jeffrey Henderson, trans. R. G. Bury, (London: Harward University Press, 1929), 53c6-8.

[288] ibid., 54e2-12. As noted by Charles Mugler, the use of the term στερεόν ('solid', 'firm') to indicate a geometrical body was a recent innovation at the time of Plato. Mugler considered the use of the term in Platonic dialogues as the first 'authentic example' of this use in the history of the Greek science. Charles Mugler, *Platon et la recherche mathématique de son époque* (Strasbourg: Éditions P. H. Heitz, 1948), 40. The reason seems to be clear, what is firm or solid has depth, and follows therefore the definition previously given by Plato. However, one cannot help but find a certain uneasiness in Plato's use of στερεόν/α for something immaterial; indeed, differently from other dialogues, the term is used in the *Timaeus* only as adjective, 'the solid form', 'the solid surface'. However, *στερεόν* used in other Platonic dialogues such as the *Meno* as a noun. "*Socrates:* Well then, you speak of surface [*ἐπίπεδον*], and also of solid [*στερεόν*]—the terms employed in geometrical problems? *Meno.* I do. *Socrates:* So now you are able to comprehend from all this what I mean by figure [*σχῆμα*]. In every instance of figure I call that figure [*σχήματος*] in which the solid ends [*στερεὸν περαίνει*]; and I may put that more succinctly by saying that figure is 'limit of [what is] solid' [*στερεοῦ πέρας σχῆμα εἶναι*]." Plato, *Meno,* 76A.

early Middle Ages, a similar principle of subdivision in triangles was repeated by Boethius in *De Instititione Arithmetica* for every type of polygon; even though three-dimensional bodies were described by Boethius, and later by Pacioli, as composed of pyramids.[289]

Pacioli gives proof in a number of passages to be aware that regular bodies were constructed from triangles in the *Timaeus*, or at least that their construction involved only the assemblage of the surfaces at their boundaries. However, he made no direct reference to the process of the construction of regular solids as devised by Plato. It should be added that this part of the *Timaeus* was not included in the late antique or medieval commentaries that Pacioli mentions in the *Compendium.*[290] Here the association of each regular body to the four basic elements and fifth essence is repeated by Pacioli in two passages. In one instance, Pacioli meaningfully clarifies the reasons why some of Plato's arguments have been included in the text:

> […] because if in nature there would be a sixth regular body to be assigned, the Highest Maker would have a lack of something, and we could judge that everything that was needed was not provided at the beginning. It is certainly for this reason, and not for any other, that I understand Plato's arguments in attributing each of these [regular] bodies to the basic elements, [these arguments are] made as [someone] very skilled in geometry and most profoundly [knowledgeable] in mathematics […][291]

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[289] "[…] just as for plane figures the triangle is the first number, so in solids the figure called pyramid is the principle of depth". "quemadmodum in planis figuris triangulus numerus primum est, sic in solidis qui vocatur pyramis profunditatis esse principium" Boethius, *De institutione Arithmetica,* II, xxi; Boethius, ed. By Jacques-Paul Migne, *Boetii, Ennodii Felicis …* Opera Omnia, Boetii Tomus Prior, (Paris: Garnier Fratres, 1882), 1119; translated in Michael Masi, *Boethian Number Theory: A Translation of the De Institutione Arithmetica* (Amsterdam: Rodopi, 1983), 142 (translation amended).

[290] "E queste tali forme da Calcidio celeberrimo philosopho expone(n)do el dicto Timeo molto sonno co(m)mendate. E cosi da Macrobio Apuleio e moltissimi altri perche in vero sonno de ogni commendatione degni, per le ragioni che in loro fabriche se aducano mostrando la sufficientia de ditte 5 forme si commo quella de li 5 corpi semplici non potere per alcun modo esser piu […]" Pacioli, *Divina proportione (1509)*, 17r (Compendium).

[291] "Perche se in natura se potesse un sexto corpo semplici asegnare, el summo opefici verrebe a esser stato i(n) sue cose diminuito, e senza prudenza da giudicarlo, non havendo a pri(n)cipio tutto el bisogno oportuno a lei cognosciuto. E per questo certamente, e non per altro mosso comprehendo Platone queste tali commo e dicto a ciascuno de li dicti semplici atribuisse cosi argumentando cioe Commo buonissimo geometra e p(ro)fondissimo mathematico […]" Pacioli, *Compendium*, 63v-64r. Famously the commentary of Calcidius, largely circulated in the Middle Ages, arrived only to 53c, thus before the assemblage of four regular bodies by means of two elementary triangles.

Pacioli adds that the geometrical reasons why there cannot be a sixth body are demonstrated in the penultimate (17°) proposition of Book XIII of Euclid's *Elements*. At the end of this passage, rather than returning to Aristotle's objections to Plato's geometrical construction of regular bodies, Pacioli adds the geometrical construction of the sphere and argues that: "these five [bodies] are called regular, however they should also include the sphere, the most regular of all [bodies], [and] from it all others derive, as the supreme cause of all causes". However, unlike Euclid, Pacioli notes here that he was describing the sphere not as a surface created by the revolution of an arc of circumference around its diameter "as done in the 11° book of the *Elements*",[292] but as the revolution of a semicircle cutting through a material. This is because, as Pacioli concluded, "if only the [semicircular] arc is taken, this does not leave a trace [*non fa vestigio*] [of the body], since it is a line without width and depth" concluding that "as much is said for [the sphere] knowledge and causation."[293] The reference to the process of causation is made explicit in this passage. This is because, Pacioli concludes, the sphere is the geometrical body from which all the others can be inscribed and derived.

2.4 Quantity and matter: the problem of the *vacuum*

The process of the natural causation of regular bodies is correlated to Aristotle's well-known discussion of the 'error of the *vacuum*', a problem mentioned by Pacioli in both the *Summa* and the opening of the *Divina proportione*. It is worth considering first the composition included by Pacioli at the opening of the *Divina proportione*:

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[292] The definition of sphere is at the beginning of Book XI of the *Elements* (definition 14 in modern editions of the *Elements*). Campanus's edition, commented and published Pacioli, states: "Sphera est transitus arcus circu(m)ferentie dimidii circuli quotiens sumpto vel supremo semicirculo lineaq(ue) diametri fixa donec ad locum suum redeat arcum ipse circumducitur". Pacioli, *Euclidis*, 100v.

[293] "E avenga che questi cinque sono chiamati regulari non pero se exclude la spera che non sia sopra tutti regularissima, e ogni altro da quella derivarse comme da la causa delle cause piu sublime […] e spera ymaginando co(m)mo se deve che dicto semicirculo gratia exampli sia un meçço taglieri materiale che aliter non formaria corpo, peroche solo l(')arco circu(n)ducto non fa vestigio siando linea sença ampieça e p(ro)fondita e questo a sua notitia e causatione sia detto." Pacioli, *Divina proportione (1509)*, 17v (Compendium).

> Five bodies in nature are produced / By natural [philosophers] 'the simple ones' called / Because each is assigned to a element / by order they are all arranged / unmixed: clear and neat they were constructed / so called 'four elements and the ether' / Plato as such wants them to be figured / they generate infinite fruits / But because the void nature abhors / Aristotle in 'De cielo et mundo' / did not understood them as figures / However, the profound geometric ingeniousness / of Plato and Euclid [I] liked to present / Five circumscribed by a sphere / Regular, of delightful appearance / As you can see, with equal bases and equal sides / And a sixth cannot be made.[294]

In its summarising character, the text underlines important elements for Pacioli's theoretical framework. It mentions that although "Plato as such wants [the bodies] to be figured" [*Quali Platone vol che figurati*] Aristotle "did not deal with them as such figures" [*per se non figurati volse porre*]. Pacioli explains the reason for this disagreement between Plato and Aristotle in the ensuing lines, because as Aristotle argued, "the void nature abhors". Two references to Aristotle's discussion of the impossibility of the vacuum in regular bodies are also found in Pacioli's *Summa.*

As noted, a fundamental aspect of the geometrical construction of the five regular bodies in both Plato's and Euclid's texts is that this happens through the construction of their bases or their boundaries. This is because the process of construction relies on solid planar triangular surfaces in Plato's *Timaeus*, or of the sides of their perimetric polygonal bases in the case of Euclid's *Elements*. It is from the knowledge of this theoretical framework that Pacioli expresses the position of Aristotle and the 'error of the void'. This would in fact occur if the solids-bodies were to be considered as only materially composed of their perimetrical surfaces or

[294] "Cinque corpi in natura son producti / Da' naturali semplici chiamati./ Perché a ciascun composito adunati / per ordine concorran fra lor tutti / Immixti: netti e puri fur constructi / Quattro elementi e ciel così nomati / Quali Platone vol che figurati / L'esser dien a infiniti fructi. / Ma perché el vacuo la natura abhorre / Aristotil in quel de cielo et mundo / per se non figurati volse porre / Però l'ingegno geometra profondo / di Plato e d'Euclide piacque exporre. / Cinqualtri che in spera volgan tundo / Regolari; d'aspeto iocundo / Come vedi de lati e basi pare / E un altro sexto mai se pò formare" Pacioli, *Divina proportione (1509)*, opening sonnet.

sides. It is therefore relevant to look at a passage in the *Summa* where the problem is introduced:

> Those opinions, even if they reflected what [Plato] believed, nonetheless his disciple Aristotle spoke against them discussing similar things in *De cælo et Mundo*. [Aristotle] gives evidence that these opinions would bring many and great issues in nature. To a greater extent in relation to the error of the vacuum [*errore del vacuo*], that in nature is rejected by everyone and in no way can be conceived: *quia vacuum nihil est*. And this [void] is what would follow if the aforementioned bodies were to be outlined [*lineati*] only through their sides, their surfaces and their angles; [this is the void] which nature rejects.[295]

Argante Ciocci, an historian of science that has worked extensively on Pacioli's texts, argued that Pacioli is here referring to the problem known in the Middle Ages as *impletio loci,* that is, the void that would occur in nature if regular bodies were to be lined up together around a point.[296] Although the problem of which geometrical bodies could entirely fill the space around a point was debated in the Middle Ages, Pacioli does not seem to follow this argument. Rather, he seems to refer to the inner space within the external boundary of each body. It should be noted that Pacioli never explicitly mentions in his texts the joining of more

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[295] "Le quali opinioni quantumche a lui così paresse, non di manco Aristotele suo discipulo contra lui, de simel cosa parlando, in quel *De celo et Mundo*, reprova tutte le pretatte opinioni per molti e grandi inconvenienti che in natura sequitarebbero quando così fosse come Plato dice. Maxime fra gli altri sequirebbe lo errore del vacuo, che in natura da tutti se niegase per nullo modo potersi dare: quia vacuum nihil est. E ciò sequiri, quando ditti corpi simplici fossero lineati per lati e superficie e angoli; il che la natura abhorre" Pacioli, *Summa*, 4v (Pars prima).

[296] Ciocci discussed the problem of the vacuum only as concerning the void that will be created if regular bodies were to be lined up together around a point. Aristotle in *De caelo* affirmed that "This attempt to assign geometrical figures to the simple bodies is on all counts irrational. In the first place, the whole will not be filled up. Among surfaces it is agreed that there are three figures which fill the place that contains them—the triangle, the square, and the hexagon: among solids only two, the pyramid and the cube". (Translation amended) Aristotle*, On the Heavens,* 306b3-8. Although Aristotle does not specify that the pyramids should be regular, the context suggests he was indicating regular pyramids. However, regular pyramids cannot fill the space around a point and this lead to various interpretations of the passage from Antiquity through the Middle Ages. As noted by Ciocci, Averroes and numerous others medieval commentators of Aristotle, among them Albertus Magnus, Roger Bacon, Nicole Oresme, and especially Thomas Bradwardine, discussed the problem of which and how many solids could fill the vacuum around a point. The solution of Bradwardine (presented in his *Geometria Speculativa,* written in the fourteenth century) is particularly significant and cannot be considered only as related to the problem of the 'impletio loci'. Bradwardine in fact argues that the 20 pyramids that form the icosahedron would fill the space around a point. His argument focuses therefore only on one solid-body and not on the problems of space filling of multiple solids-bodies. The argument of Bradwardine, Ciocci noted, was later perfection by Paul of Middleburg in his predications of 1480-1481, that argued that 20 pyramids would fill an icosahedron but that this pyramids are not regular tetrahedrons. Argante Ciocci, "Il filosofo della natura: i poliedri regolari e l'immagine geometrica del mondo," in *Luca Pacioli - Maestro di contabilità - Matematico - Filosofo della natura*, ed. Esteban Hernández-Esteve and Matteo Martelli (Umbertide: University Book, 2018), 271-275.

bodies and the philosophical and physical problem of the 'vacuum' that would ensue from this, but always focuses instead on singular geometrical bodies. This interpretation is further supported by the other two instances in Pacioli's writings, where the terms *lineati/e* are found.[297] In both cases, they refer to the action of marking with lines.[298] Pacioli in fact never argues in his writings against Plato's assignment of geometrical bodies to the basic natural elements, but objects to their geometrical and material composition as described in the *Timaeus*. If these bodies were to be materially instantiated only by means of their external surfaces and angles (i.e. their boundaries) Pacioli argues following Aristotle, this would imply a vacuum in the body. It is worth looking at the theory as expressed in Aristotle's *De cælo.* In this text, Aristotle confutes different aspects of the theory of regular bodies expressed in Plato's *Timaeus*. In the first part of Book III, Aristotle criticises the process of the construction of regular bodies:

> Finally there are those according to whom all bodies [*soma*] are generated, being constructed out of planes and resolved into planes again [*epípeda kaì ex epipédōn*]. […] as for this last theory, which constructs all bodies [*somata*] out of planes, a glance will reveal many points in which it is in contradiction to the findings of mathematics […] the composition of solids from planes clearly involves, by the same reasoning, the composition of planes from lines and lines from points (a view according to which a part of a line need not be a line); and this is something which we have already considered in the work on motion, where we concluded that there are no indivisible lines.[299]

If the regular bodies are composed only of surfaces, argues Aristotle, they would be resolved in them and they would therefore have a different mathematical and physical nature. Additionally, this would determine the paradox of reducing three-dimensionality to two-

[297] The interpretation of Ciocci seems to rely on a translation of Pacioli's *lineati* with the present-day Italian term 'allineati'. However, an analysis of all occurrences of the term in the writings of Pacioli does not support this reading.
[298] The term indicates the lines of a square table 'in quadrilatero lineato' in Pacioli, *Summa*, 28r (Pars prima); and refer to lines made on a hourglass in Pacioli's *De Viribus Quantitatis* "le ditte vocce [de vetro] vogliano essere lineate quanto più speso si pò, proportionando ditte linee…]" Pacioli, *De Viribus*, Capitulo LXL, 267.
[299] Aristotle, *On the Heavens*, 298a34-299a15.

dimensionality and the whole body to one of its parts. Consequently, if the five cosmic figures discussed by Plato are in the realm of the intelligible, they cannot have essential properties such as weight. Although three-dimensionality is not the essence of a physical substance for Aristotle, this should nonetheless be understood as tied to its essence, and as such it is always concomitant to any kind of physical body. Otherwise, the body would not exist in actuality, quoting again from the *De Cælo*:

> In sum, either there is no magnitude at all in their arguments, or magnitude can be annihilated, once granted that as the point is to the line, so the line is to the surface and the surface to body [*soma*]; for all can be resolved into one another, and hence can be resolved into the one which is primary, so that it would be possible for there to exist nothing but points, and no body [*soma*] at all.[300]

It is clear that magnitude is central both to Aristotle's conception of the body and his rejection of the vacuum in geometrical regular bodies.[301] Aristotle famously returned to arguments that rejected the void in multiple passages in his writings. In *Physics IV*, Aristotle again considered the void while dealing with motion:

> It is evident from what has been said, then, that, if there is a void, a result follows which is the very opposite of the reason for which those who believe in a void set it up. They think that if movement in respect of place is to exist, the void must exist, separated by itself; but this is the same as to say that place is separate; and this has already been stated to be impossible.[…] For as, if one puts a cube in water, an amount of water equal to the cube will be displaced, so too in air (but the effect is imperceptible to sense).[302]

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[300] Aristotle, *On the Heavens*, 300a7-13.

[301] In the *Physics IV* Aristotle considers the correlation between the (outer) boundary and form of a body considered as magnitude in relationship to the concept of place: "Now if place is what primarily contains each body, it would be a limit, so that the place would be the form or shape of each body which the magnitude or the matter of the magnitude is defined; for this is the limit of each body. If, then, we look at the question in this way the place of a thing is its form. But, if we regard the place as the extension of the magnitude, it is the matter". Aristotle, *Physics,* IV.2, 209b1–11

[302] Aristotle, *Physics* IV.8, 216a22-26, 216a28-30. It is important to note that Aristotle's *τόπον* ('place') was customarily rendered with 'locus' in medieval translations: "Opinantur quidem igitur vacuum esse, si vere erit secundum locum motus, discretum secundum se; hoc autem idem est quod est locum dicere esse aliquid separatum; hoc autem quod sit inpossibile esse, ptrius dictum est. […] Sicut enim in aqua si ponat aliquis cubum, distabit tanta aqua quanta est cubus, sic et in aere, sed sensui inmanifestum est." Aristoteles Latinus, *Physica,* vol. 1.2, ed. Fernand Bossier and Josef Brams (Leiden: Brill, 1990), 164.

The passage is relevant because it is correlated with Pacioli's arguments, both directly and through the writings of Duns Scotus. In this context, it is important to compare Pacioli's concept of quantity with Duns Scotus' arguments on whether angels occupy a place (locus) discussed in his commentary to the *Sententiae* by Peter Lombard. Although scholars have paid little attention to Pacioli's referencing of this passage by Duns Scotus, the argument has clear consequences when the concept of quantity as applied to geometrical bodies is considered.

Pacioli mentions the geometrical arguments brought forward by Duns Scotus in three of his texts.[303] Moreover, Duns Scotus' mastery of Euclid's *Elements* and its argument as to whether angels occupy a place are explicitly linked by Pacioli in the *Compendium*.[304] In this case, Pacioli argues that Duns Scotus was able to develop his theological argument by mathematical, geometrical and proportional means. Furthermore, this argument is framed in the context of Aristotle's concept of place developed in *Physics* IV, and famously defined as "the first (i.e., innermost) motionless boundary of the thing that contains" or in other words, "the limiting surface of the body continent".[305] Although it is beyond the scope of this study to conclusively define Aristotle's concept of place, it is important to keep in mind the content-container

[303] Pacioli, *Summa,* 2v-3r (Dedicatory letter)*;* Pacioli, *Divina proportione (1509)*, 2v (Compendium)*;* Pacioli, *De Viribus Quantitatis,* LII (Parte seconda).

[304] As Pacioli argues: "By no means [rather than mathematical, geometrical and proportional] were accomplished the great speculations in theology of our *subtilissimo* (sic) Scotus if not through the knowledge of the mathematical disciplines, as it is visible in all his sacred writings. Especially, if one looks carefully, in the theory discussed in the second book of his *Sententiae* (sic) where he enquires if angels inhabit a proper and determined place through their existence, [and] where [Scotus] demonstrates to have understood all of the volume of our most shrewd philosopher Euclid of Megara. In the same way, if all the texts of the prince of the sages: Physics, Metaphysics, Posterior [Analytics] and others seem difficult, it is because of the scarce knowledge of the aforementioned disciplines."; "Non per altri mezzi anchora al(l)e grande speculationi de sacra theologia el nostro subtilissimo Scotto pervenne se non per la noticia de le mathematici discipline comme per tutte sue sacre opere apare. Maxime se ben si guarda la questione del suo 2° libro de le sententie quando inquirendo domanda se l(')angelo habia sup proprio et determinato luoco a sua existentia in la quale ben dimostra haver inteso tutto el sublime volume del nostro perspicacissimo Megarense philosopho Euclide. Non per altro similmente li texti tutti del principe di color che sanno physica: metaphysica Posteriora e gli altri se mostran difficili se no(n) per la ignorantia de le gia ditte discipline." Pacioli, *Compendium*, 7r. Notably, Pacioli stressed the importance of arithmetics, geometry, proportions and proportionality in reading the second, third and fourth book of Scotus's commentary to Peter Lombard's *Sententiae* in the *Summa*. Pacioli, *Summa*, 2v-3r, 4r (Epistola)

[305] Aristotle, *Physics* IV 4, 212a20-21, 212a7-8.

relationship established by Aristotle. In this case, a body's quantity (understood as extension) is defined in relationship to its place and its boundary.

It is by referring to this kind of relationship that Duns Scotus was able to reject the hypothesis that angels, as immaterial entities, occupy place. It is important to review the argument as developed by Duns Scotus. Firstly he argued that any body occupying a place possess five features: "to be in an actual place, to be in a determinate because equal place, to be in a place commensurately, to be determinately in this place or in that, and to be naturally or violently in a place."[306] Duns Scotus continued by correlating physical bodies as mathematically extended bodies and the role of quantity in bodies. It is worth looking at the passage from the text:

> The first four belong to body [*corpus*] insofar as it is extended, a quantum, or a body; the last one belongs to it insofar as it is a natural body. For although no extended thing exists unless it also has qualities, yet it is naturally an existent with an extension before it is an existent with qualities - and in this regard it is an object of mathematics before it possesses quality, that is, it is first such as is considered *per se* and first by a mathematician. This is what the Philosopher means in *Physics* [IV, 8, 216a27-b8] 'On the Vacuum', because he maintains that 'if a cube is put in air or water, even if it has no natural qualities, yet it causes as much displacement as an inserted body,' so that it causes a distance as great as the body; and this does not belong to it insofar as it is natural only, but insofar as it is precisely an extension, a quantum, and so a mathematical object.[307]

[306] "Ad quaestionem igitur istam solvendam, primo videndum est de loco corporis. Corpori enim cuilibet, praeter 'ultimum' (cuius non est aliud extra continens), quinque conveniunt: esse in loco actuali, esse in loco determinato quia aequali, esse in loco commensurative, esse in loco hoc determinate vel in alio, et esse in loco naturaliter vel violenter.' Duns Scotus, *Ordinatio*, II, D2, P2, Q2, The Logic Museum, ed. C. Balic, C. Barbaric, S. Buselic, B. Hechich, L. Modric, S. Nanni, R. Rosini, S. Ruiz de Loizaga, and C. Saco Alarcón. Vatican City: Typis Polyglottis Vaticanis, 1973. Accessed on 12 January 2021. http://www.logicmuseum.com/wiki/Authors/Duns_Scotus/Ordinatio/Ordinatio_II/D2/P2Q2.

[307] "Prima quattuor conveniunt corpori in quantum 'quantum' vel corpus, ultimum convenit sibi in quantum est corpus naturale. Licet enim nullum quantum exsistat nisi ipsum sit quale, tamen 'exsistens' prius naturaliter est quantum quam quale, - et secundum hoc, prius naturaliter est mathematicum quam quale, hoc est, tale quale primo consideratur a mathematico, per se et primo. [218] Hoc intendit Philosophus IV Physicorum 'De vacuo', quia vult quod 'si corpus cubicum ponatur in aerem vel in aquam, - licet nullam habeat passionem naturalem, tantum tamen facit distare quantum est ipsum corpus impositum', ita quod quantum est ipsum corpus, tantum facit distare; et hoc non convenit sibi in quantum scilicet est tantum naturale, sed in quantum in se est 'quantum' praecise, et ita mathematicum." ibid.

With regard to being in place commensurately, Duns Scotus adds the following:

> […] an extended body is actually in a place, because it is in what actually precisely contains it; for it cannot be in place without the ultimate limit (which is what proximately contains it) making it actual, because it makes the sides of the containing body to be [materially] distant (*IV Physicorum*).[308] I say—because of sameness of quantity—that a body necessarily requires a place equal to it. And for this reason a body is in place commensurately, such that a part of the contained surface corresponds to a part of the containing surface, and the whole of it to the whole.[309]

Since the quantity of the containing place and the contained body are always the same, they should have the same extension. Since angels cannot be in place commensurately, Duns Scotus concluded that they cannot occupy a place. Duns Scotus' argument has important consequences if compared to the application of geometrical bodies in architecture, as argued by Pacioli. Since a body is commensurately in place, its material extension (including voids whose matter is the air) and shape are retained when the body is applied to architecture. Although this passage will be further considered in Chapter 4, it is important to note that Pacioli seems to refer to a similar process in his description of the application of a body with seventy-two bases to a wall of the church of Santa Maria San Satiro in Milan:

> Due to [this 72-bases body] usefulness [*convenientia*] many [building] in different places you can find arranged and built […] And here as well, in your Milan, in the respectable *Sacello de San Scetro* in the ornate chapel there is a part of this [body],

[308] "De secundo articulo dico quod - supposito primo - corpus 'quantum' est in loco in actu, quia in praecise continente actualiter; non enim potest esse in loco, quin illud ultimum (quod est proximum continens) faciat illud actu, quia facit latera corporis continentis distare. Secus autem est de parte in toto, quae non facit superficiem in potentia continentem, ipsam in actu; et ideo non est pars in toto sicut locatum in loco (IV Physicorum)." ibid., par. 232.

[309] "De tertio dico quod - propter eandem quantitatem - necessario coexigit corpus locum sibi aequalem. Et propter illud est in loco commensurative, ita quod pars superficiei contentae correspondet parti superficiei continentis, et totum toti." ibid., par. 233. Also quoted in Helen S. Lang, "Bodies and Angels: The Occupants of Place for Aristotle and Duns Scotus," *Viator*, vol. 14 (1983): 262.

> broken and with this convex [part] applied to the wall [*con reservatione de alquanto convexo al muro aplicata*] […][310]

Both in the *Summa* and *Divina proportione*, quantity is consistently presented by Pacioli for its relevance to works for the *pratici vulgari*, that is practitioners who could only access sources in Italian.[311] Ultimately, when considered in its Aristotelian context, it is important to remember that quantity is presented by Pacioli as acting at the three-dimensional level of the body. As will become clear, this was not only a theoretical problem for Pacioli; rather, it concerned the practical and the visual, the formal and aesthetic properties of artefacts and architecture. This link is expressed both in the *Summa* and *Divina proportione*. Ultimately, when geometrical bodies are considered as material instantiations, and are causally related to eternal abstract forms by means of their correspondence to the basic natural elements, this argument has consequences for any practical application. If regular bodies are visually perceived, the employment of regular bodies can justify the use of proportional relationships established among their dimensions, as they are in their corresponding eternal forms. At the same time, as we shall see in Chapter 3, if regular and other bodies derived from these are a phenomenal component of the material world, it is possible to visually experience their form by means of perspectival representations. Since geometrical bodies were ultimately understood as instantiated in matter, their relationship with perspective and visual perception should be carefully considered.

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310 "Anchora qui nel suo Milano nel degno sacello de san scetro, l(')ornata Capella fia una parte de questo spaccata, e con reservatione de alquanto convexo al muro aplicata, e in ciascuna sua basa hiontovi un rosone che adorna la rende. E in lo devoto e sacratissimo vostro templo de le gratie la sua tribuna al primo altare e laterali, gia non e se non una parte a simil de questo, pur in suoi basi a piu vaghezza giontovi quelli." Pacioli, *Compendium*, 60r-v.

311 Among them particularly to craftsmen, artisans and architects. What Pacioli's texts reveal is that quantity is at the core of the proportional nature of bodies, in crafts and in architecture that goes beyond a simply linear or two-dimensional geometrical interpretation. Indeed, a different nature of proportional relationships is embedded in the notion of a three-dimensional body as mathematical, and this involves both the relationship established among its constitutive elements and the order of their apprehension.

2.5 The form of regular and other bodies derived from these

Whilst analysing the theoretical framework expressed in Pacioli's texts, it is important to examine the concepts of form [*forma*], that Pacioli employs in conjunction with regular and other bodies derived from these. Although, as noted, the philosophical framework in which he operated was predominantly Aristotelian;[312] this section will highlight how Pacioli established correlations among arguments developed in different sources to support his use of the concept of *forma* as applied to regular and other bodies derived from these.

Unlike Campanus' edition of Euclid's *Elements* or the mathematical writings of Piero della Francesca, Pacioli used the concept of form extensively and expressly to refer to regular and other three-dimensional bodies.[313] The analysis of significant passages from Pacioli's texts should be contextualised within the Aristotelian interpretations of the concept. As is well known, the concept of form as essence is central to the Aristotelian understanding of bodies, and it is clear that for Aristotle form and shape are not identical for animated bodies. The properties that determine the shape of a body are mere accidents, while a form (essence) can only be a universal shared by all bodies of a certain species.

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312 Although, as discussed, necessarily filtered through the extended Medieval tradition of translations and commentaries on the works of the Philosopher.

313 Piero della Francesca did not use the concept of form when discussing regular bodies in his mathematical writings (*Trattato d'abaco* and *Libellus de quinque corporibus regularibus*). However, the case of Campanus is more articulated and Campanus's edition of Euclid's *Elements* is a major source for Pacioli. Campanus uses the concept only in his comments to proposition 12 of Book XV of Euclid's *Elements*, where he discusses the Platonic theory that assigns the basic natural elements to the five regular bodies. The passage was reproposed by Pacioli in his edition of Euclid's *Elements*, based on Campanus's one: "Duodecedron autem nulli cetero(que) sue ambitionis denegavit hospicium immo cunctoru(m) receptator existit. Unde non inco(n)venienter duodecedri figura antiqui Platonis discipuli vel ascripsere celo quemadmodum piramidis formam igni eo q(ue) sursum sub piramidali figura evolet ac octocedri aeri. Quippe sicut aer ignem motus parvitate sequitur sic octocedri forma piramidis formam ad motu(m) habilitate comittat. Viginti vero basium figuram aque dictaverunt nacum ipsa basium pluralitate plus ceteris circuletur in speram fluentis rei motus magis q(ue) scandentis co(n)venire vis est. Cubon vere figuram q(ue) dadedere terre. Quid n in figuris maiori ad motum viole(n)tia indiger qua thessera. At in elementis quid fixius constantiusq(ue) repetitur terra. Si igitur ex 20 inscriptionibus 3 quas piramis non substinet binasq(ue) a quibus naturam cubi et octocedri aliena est. Rursusq(ue) unam cui repugnat ycocedri figura reieceris erunt relique tantum, 12, inscriptiones. Piramidis quidem sola cubi vero octocedriq(ue) bine. Ycocedri autem tres Duoddecedri autem quator. Duquibus omnibus ut arbitror suffienter alias disputandum est." Pacioli, *Euclidis*, 144r-v.

However, following a discussion of the properties of perfect numbers in the *Summa*,[314] Pacioli extends the concept of perfection to natural bodies [*corpi creati e da natura p(ro)ducti*] and links this with aesthetic considerations. According to Pacioli, monstrous bodies are lacking or excessive in their members. Conversely;

> Others are bodies proportioned and formed [*p(ro)portionati e formati*] according to their necessity: and these are the ones in which nothing is lacking, and they have neither less nor more of what is due, like every beautiful man or other body that has five fingers on each hand or foot, two eyes and ultimately every other member well placed. And these are deservedly called perfect [*perfecti*] because, as said, they are not lacking or have anything in excess of the formal corporeality [*formale corporatio(n)e*] sought. And therefore the association is not illogical of the said [perfect] numbers with the things produced and created in nature.[315]

Although Pacioli does not discuss the process by which proportions are formally instantiated in the body, the link established between mathematical properties and a formal and aesthetic understanding of the body is crucial. Since a correspondence between perfect numbers and perfect bodies is established in the passage, Pacioli makes no mention of the essence of the body, but merely highlights mathematical properties as determining beauty in a body.[316] Nevertheless, proportional arrangements in animated bodies can still be understood as an expression of their form (essence), and therefore coherently fit with Aristotle's theory of

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[314] Pacioli gives the example of number 6. This is a perfect number because the sum of the integers derived through its division by 1/2, 1/3, 1/6 gives as total the initial 6. The next perfect number is 26, followed by 496 and so on. Pacioli, *Summa*, 3r (Pars prima).

[315] "Altri son(n)o corpi seco(n)do sua exige(n)tia p(ro)portionati e formati: e so(n)no quelli a cui alcuna cosa no(n) ma(n)ca, e no(n) hano ne piu ne ma(n)co a lor dovere, si co(m)mo ciascun bello homo o altro corpo che a 5 deta p(er) mano e pede, doi ochi e finaliter ogni altro membro ben situato. E tali son(n)o meritame(n)te chiamati perfecti perche a loro no(n) ma(n)ca co(m)mo e ditto ne ava(n)ça alcuna cosa a la formale corporatio(n)e recercata. " Pacioli, *Summa*, 3r (Pars prima). The passage follows in principle the link established by Boethius between perfect numbers and the human body in I, xix of *De Institutione Arithmetica*. However, some differences should be noted. Boethius makes no mention of the form or proportions of the body, nor Boethius explicitly made any aesthetics consideration.

[316] The relationship established between necessity and the form of the bodies by means of proportions is certainly noteworthy. Towards the end of the section Pacioli further underlines that beauty is the result of the form instantiated in the body. Pacioli shows here to be aware of the Aristotelian theory of form as essence for natural bodies. According to Aristotle, a form of a thing is that-which-is-to-be-that-thing, its *essence.* However, as noted by Mitrović, while natural bodies have natural essences, it is much more difficult to say what constitutes the essence of an artificial body. Although Aristotle argued in the Physics that art cannot change the essence of an object, these does not mean that they are devoid of essence.

essential form. More problems arise, however, when artificial or geometrically defined bodies are concerned. For Aristotle, every body is an individual instantiation of a universal, that is, of a property shared by many bodies. This would also apply to mathematical entities instantiated in matter. However, although mathematical entities can be thought of as separate, they are always instantiated in matter as individual bodies.

Aristotle argued in the *Physics* that art (*techne*) cannot alter the essence of a body, since "of things that exist, some exist by nature, some from other causes. By nature the animals and their parts exist, and the plants and the simple bodies (*apla ton somaton*) (earth, fire, air, water) […]".[317] The classic example that follows from this is that if you were to bury a bed, and the bed were to rot and start sprouting, you would end up with more wood, not more beds.[318] But still this does not mean that artificial bodies are completely devoid of essence for Aristotle. As noted by Mitrović, Aristotle considered the form (essence) of a house in more than one occasion in the *Metaphysics*. In the third book he listed the causes of a house, while in the eighth book considered three ways of stating its definition;[319] however, in all the proposed arguments, Aristotle did not consider that the essence of a house could be described by its shape.[320] However, if bodies are considered *qua* magnitudes their identity can also be defined by their boundaries. In the fifth book of the *Metaphysics,* Aristotle argued that "[the boundary is] that which is the form [*εἶδος*] of a magnitude or of something having a magnitude."[321] Indeed, if the boundary (i.e. the shape) of a cube is altered by removing a portion of its

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[317] *Physics*, II.1, 192b9-192b11.
[318] ibid., 193a13-193a16.
[319] Mitrović, *Serene Greed of the Eye*, 57.
[320] As noted by Mitrović, the causes discussed by Aristotle in Book three are as follow: the material cause is earth and stone, the final cause is the function, the efficient cause are the architecture and the builder and the formal cause is the definition, a verbal account of the house, its logos. ibid.
[321] "καὶ ὃ ἂν ᾖ εἶδος μεγέθους ἢ ἔχοντος μέγεθος". *Metaphysics*, V.17, 1022a5–6. Interestingly, however, Medieval and fifteenth-century translators, including Bessarion, commonly translated εἶδος is this passage as species. One cannot but notice a certain uneasiness to refer the form of a body to its magnitude, they thus interpreted it as referred to its species, in keeping with the assumption that its form is a body's essence. Aristoteles Latine, *Interpretibus Variis*, ed. Academia Regia Borussica (Berlin: Wilhem Fink Verlag, 1831), 501.

magnitude, it ceases to be a cube and becomes something else. The connection between the (outer) boundary and form of a body *qua* magnitude was further taken up by Aristotle in his discussion of place in *Physics* IV subject nonetheless to historically problematic interpretations.[322] The form of artificial bodies was later reconsidered by Thomas Aquinas in the thirteenth century, a discussion of which Pacioli was most probably aware.[323] However, by introducing the concept of accidental or artificial form [*forma accidentalis* or *artificialis*] which is separated from the substantial form [*forma substiantialis*], Aquinas sidestepped the problem of accounting for the form of artificial bodies in Aristotle's theoretical framework.[324] In his commentary to Peter Lombard's *Sententiae*, Duns Scotus criticised Aquinas' position and argued instead for a plurality of forms.[325] This is unsurprising because Duns Scotus considered in his commentary a three-dimensional body as defined mathematically by

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[322] Here, Aristotle considers a second kind of form that applies when its boundary defines the identity of the body in question. This can be considered as a special case in the form-matter distinction that applies to geometrical bodies, since, as noted by Christian Pfeiffer, it is their topological form (shape) that defines the identity of these bodies. Overall, as concluded by Pfeiffer, magnitudes are defined in by their shapes and limits in *Physics,* IV.2, 209b1–11. Pfeiffer, *Aristotle's Theory of Bodies,* 97. Moreover, in all the Latin editions of the Physica surveyed by Fernand Bossier and Josef Brams in the Aristoteles Latinus employ 'forma' in this passage. It is worth considering the whole passage: "Su igitur est locus primum continens unumquodque corporum, terminusquidam utique erit; quoare videtur species et forma uniuscuiusque locus esse, quo determinatur magnitudo et materia magnitudinis; hoc enim est uniuscuiusque terminus. Sic quidem igitur considerantibus locus uniuscuiusque species est; secundum autem quod videtur locus Esse distantia magnitudinis, materia; hec namque altera est a magnitudine, hec autem est contenta sub specie et definita, sicut sub plano et termino, est autem huiusmodi materia et infinitum; cum autem removeantur terminus et passiones spere, relinquitur nichil preter materiam." Aristoteles Latinus, *Physica*, 140. Another fundamental although problematic passage in this respect is Aristotle's discussion of place in Physics IV.4.

[323] That Pacioli was aware of Aquinas solution can be inferred from a passage in the *Summa*. In a section discussing the properties of the number three, Pacioli listed the three "natural principles [*principii naturali*] that concur to every composite: matter, form and privation. Although these were derived from Aristotle's theory of substantial change, Pacioli repeats the formulation found in Chapter three of Aquinas's *De Principiis naturae.*

[324] While discussing these principles, Aquinas brought an instructive example concerning artificial bodies: "When an idol is made out of copper, the copper, which is in potency to the form of the idol, is the matter. And this matter, which is without shape or order, is lacking in form. The idol is so called because of its form [*forma*]; but this is not substantial form, for before the copper acquires this form it already has actual being, and its being is not dependent upon the shape [*figura*] that it will acquire. Rather, it is accidental form [*forma accidentalis*]. For all artificial forms are accidental forms [*formae artificiales sunt accidentales*]; art can operate only upon something which is already naturally in being"; "Sicut quando ex cupro fit idolum, cuprum quod est potentia ad formam idoli est materia; hoc autem quod est infiguratum sive indispositium est privato formae. Illud a qua dicitur idolum est forma, non autem substantialis, quiacuprum ante adventum illius formae habet esse in actu, et eius esse non dependent ab illa figura, sed est forma accidentalis. Omnes enim formae artificiales sunt accidentalis; ars enim non operatur nisi supra id quod iam constitutum est in esse natura." Umberto Eco, *The Aesthetics of Thomas Aquinas* (London: Radius, 1988), 175. In a noteworthy example involving architecture, Aquinas argued that: "the form of a house, which is the form of the whole and not of each part, is an accidental form [*forma accidentalis*]"; "Sicut forma domus, quae est forma totius et non singularium partium, est forma accidentalis" Aquinas, *Summa contra Gentiles*, II, 72, 3 quoted in Eco, *The Aesthetics,* 178.

[325] Scotus, *Ordinatio* 4, D11. An overview is given in Williams, Thomas. "John Duns Scotus", *The Stanford Encyclopedia of Philosophy,* Winter 2019, https://plato.stanford.edu/archives/win2019/entries/duns-scotus.

quantity.[326] As is becoming clear, by concentrating solely on the natural and mathematical properties of the five regular bodies, Pacioli was able to develop a similar argument and consider the form (shape) of geometrical bodies as determined by their quantitative properties.

Lastly, in the passage of the *Categories* describing the genus of quality, Aristotle argued that both form [*morphe*] and shape [*schema*] are one of the four qualities of things. Importantly, Aristotle discussed the fourth quality by means of geometrical examples and added that although visible qualities such as "rare and dense, rough and smooth" might be "appearing at first sight to indicate quality" they "are foreign, in fact, from that class." These qualities "will rather be found to denote a particular position of the parts".[327] Although Medieval translations of Aristotle's *Categories*, as noted by Mitrović, failed to convey the differentiation between shape and form (understood as essence),[328] the translations of Boethius, William of Moerbeke and Aquinas, all conveyed here the combination of the two distinct concepts of form [*forma*] and figure [*figura*].[329] That Pacioli was aware that both concepts were associated with the fourth quality is proven by his deliberate addition of *forma* to *figura* in the passage of

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326 See the passages from Duns Scotus' commentary of Peter Lombard's *Sententiae* discussed in section 2.4.

327 "Of quality the fourth kind consists of the figure [*schema*] and form [*morphe*] of things; […] And things have a definite nature by being 'triangular,' 'quadrangular,' by being 'straight,' 'crooked' and so on. In virtue, indeed, of its form [*morphe*] is each thing qualified. Rare and dense, rough and smooth, while appearing at first sight to indicate quality, are foreign, in fact, from that class. They will rather be found to denote a particular position of the parts." Aristotle, *Categories,* 10a13-21.

328 Mitrović, *Serene*, 55.

329 I translated μορφὴν as 'form' to follow the Latin *formam* used by William of Moerbeke in the thirteenth century. However, previous Latin translations as the one usually attributed to Boethius used 'figuram' for *morphe* and *formam* for Aristotle's *schema*. Aristoteles Latinus, *Categoriae Vel Praedicamenta*, ed. Laurentius Minio-Paluello (Bruges: Desclée de Brouwer, 1961), 27, 104. The divergence between Aristotles's text and Boethius's translation was also highlighted by Mitrović; Mitrović, *Serene*, 55. It is useful to list the four types of qualities as described in Aquinas' Medieval translation of Aristotle's *Categories*: (1) a habit or disposition [*habitus vel dispositio*], (2) a natural potency or lack of it [*potentia vel impotentia naturalis*] (3) an affection or the quality of being affected [*passio vel passibili qualitas*], and (4) the form and shape surrounding something [*forma et circa aliquid constans figura*]. I am using here the version of Aquinas's *Summa Theologiae*, I-II 3 ob 3 employed in Eco, *The Aesthetics,* 176. However, the Latin passages are consistent with versions based on the translations usually attributed to Boethius and the translation of William of Moerbeke. Aristoteles Latino, *Categoriae*, 27, 104. *Dispositio* was also used, with *habitus,* in Medieval translations of Aristotle *Categories* as a *genus* of quality.

his commentary on Campanus' edition of Euclid's *Elements* that discussed regular bodies.[330] At the same time, as previously argued, Pacioli described proportions in Aristotelian terms as among the universals that underline the form of each of the regular bodies. This is confirmed by a passage that introduces the five regular bodies in the *Summa*, where Pacioli made use of the concepts of *forma* and *figura* of bodies:

> But more at length I will touch on them when we will discuss geometry, assigning them forms and measures [*forme e misure*], and [a way of] making [them] with clear and open instructions, and beautiful and subtle rules for their squaring […] and although beyond these there are infinite bodies (as we will discuss), nevertheless no one can be found similar to these in form and figure [*forma e figura*] […] And if we proceed further, we would find these five bodies by Plato, where we mentioned before (according to his evident reasons) being rightly linked to the unmixed bodies, also said elementary. And his opinion of each of their forms and figures was that [it should be linked to] each element. And as a consequence [that] all the elementary bodies should

[330] "This proportion excellent duke is of so much privilege and excellence deserving as much [praise] as you can say [of it]; and with regard to its infinite potency, without its knowledge many things worthy of the highest admiration, both in philosophy or in any science, never to light would have come. Certainly this [proportion] has been given by the invariable nature of the superior causes [*invariabile natura de li superiori principii*], as is said by the great philosopher Campanus, our very famous mathematician on the tenth of the 14th Book. The most important thing considering it [this proportion] is that a great diversity of solids, both in size and in multitude of bases, and also of figures and forms [*figure et forme*] with a certain irrational symphony among their accords [can be made], as it will be seen in our process, placing the stupendous effects that (from a line divided according to this proportion) not natural but divine should actually be called. In the original: "De la sua degna commendazione Cap. VI. Questa nostra proportione excelso D(uca) e da tanta prerogativa e de excellentia degna quanto dir mai se potesse e per respecto de la sua infinita potentia, conciosia che sença sua notitia moltissime cose de admiratione dignissime ne in philosophia ne in alcuna scientia mai a luce poterieno pervenire. El qual dono certame(n)te de la invariabile natura de li superiori principii, commo dici el gran philosopho Campanno (no)stro famosissimo mathematico sopra la decima del 14 glie co(n)cesso. Maxime vedendo lei esser quella che tante diversita de solidi si de grandeççi si de moltitudine de basii si ancora de figure et forme con certa irrationale simphonia fra loro acordi, commo nel nostro processo se intendera ponendo li stupendi effecti quali (de una linea secondo lei divisa) non naturali ma divini verament(n)te sonno d'appellare." Pacioli, *Divina proportione (1509)*, 4r (Compendium). The commentary of Campanus, as published by Pacioli has: "Mirabilis itaq(ue) est potentia linee pm p(ro)portionem habentem medium duoq(ue) extrema divise: cui cum plurima pholosophantium admiratione digna conveniant hoc principium vel precipuum ex superiorum principiorum invariabili procedit natura ut tam diversa solida tu(m) magnitudine tu(m) basium numero tu(m) etiam figura irrationali quadam simphonia rationabiliter conciliet. Quippe demonstratum est q(uod) proportio duodecedri corporis ad ycocedron corpus que ambo spera una coambit est quasi proportio linee potentis super quam libet lineam pm prefatam proportionem divisam et super eius maiorem partem ad quamlibet lineam potemtem super eandem et eius minorem partem, quoniam vero de tribus ceteris corporibus regularibus non habemus aliquid dictum studeamus de ipsis aliquid dicere." Pacioli, *Euclidis*,137v.

> have a figure [*fossero figurati*] and each its own form and figure [*forma e figura*], like me and you, the third and the fourth, oxen, donkeys, etc.[331]

The reference to the Platonic theory of natural elements is therefore of the essence here and provides Pacioli with an important argument in establishing a link between mathematical and natural causes. In another passage in the *Summa,* Pacioli clarifies that it "is impossible in nature for anything to be if not rightly proportioned to its necessity".[332] As we have seen, this connection was further developed by Pacioli in the *Compendium*. The last section of the passage is also particularly significant. Following the distinction between the form *qua* substance and the form *qua* extended body found in Duns Scotus' commentary, Pacioli and his readers would share the same essential form, but not the same topological form nor the same figure. The same would apply to the third and the fourth (we can assume that Pacioli here is speaking about regular bodies, but the same would also be true for numbers), oxen, donkeys, etc. In this regard, Pacioli emphasises that he is referring to each individual (in a species') own (not essential) form and figure.

Pacioli's interpretation of the concept of form in relation to regular bodies was not further discussed in the *Summa*. Although the concept was first introduced in the *Summa* and had a precedent in Campanus' commentary on proposition 12 of Book XV of *Euclid's Elements,* ultimately it is in the *Compendium* that it finds its most coherent expression. A survey of all the instances where *forma* or the verb *formare* are used in the *Compendium* shows that these can always be related to an arrangement of elements in a certain disposition, being lines, angles

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[331] "Ma assai piu ampliamente, quando de geometria parlaremo, ne tocaro. Assegnado lor forme e misure, e fabriche com modi chiari e aperti, e regole e loro quadrature bellissime e sotile. E pero qui non ne toco piu. E quantunque oltra questi infiniti sieno li corpi (comme li se dira) non dimeno niun si porra mai trovare, simili in forma e figura, ad alcuno de li predetti. […] E se piu oltra andremo, trovaremo ditto quinario de corpi regolari per Plato(n)e nel luogo preallegato essere (secondo sue evidenti ragioni) meritamente attribuito a li corpi i(n)misti. Cioe semplici. E de ciascuna di loro forme e figure fo opinione d(')esso che li elementi. E per conseque(n)te tutti li corpi semplici fossero figurati e per ognuno havesse la propria sua forma e figura, si comme, Io e tu, el terzo el quarto, bovi, asini e(t)c." Pacioli, *Summa*, 4v (Pars prima).

[332] "Peroche impossibile e alcuna cosa in natura persistere: se la no(n) e debitamente p(ro)portionata a sua necessita." ibid., 68v (Pars prima).

or polygons that are joined in a proportional relationship with their quantities in the form (shape) of a regular or other body derived from these. *Figura* refers to two-dimensional representations of regular or other bodies, in accordance with its use in Latin translations of Aristotle's works and Campanus' edition of Euclid's *Elements*.[333] Notably, the explanation adduced by Pacioli in the *Compendium* for the form (shape) of the five regular bodies follows the clarification that "our proportion, as mentioned before, is always involved in the compositions and formations [*co(m)positioni e formationi*] [of the five regular bodies and other derived from these]"[334]. Pacioli continues as follows:

> But first we should clarify why these bodies are called 'regular', secondly it is to be proved how in nature it is not possible to form a sixth [body]. Since these bodies are called regular because they have their sides, angles and bases equal and one is related to the other and they correspond to the five elementary bodies [*corpi semplici*] in nature that are earth, water, air, fire, and the fifth essence, otherwise celestial virtue, that substantiate all the others in their being. And these five are [to be understood as] sufficient in nature, otherwise you would argue that God [created something] superfluous or [that something is] lacking to what is needed in nature. This would be absurd, as affirms the Philosopher: God and nature don't operate in vain, that is [they] don't fail what is needed and don't exceed it; and similarly the forms [*forme*] of these five bodies that are indeed five *ad decorem universi* and cannot be more […] And therefore not without reason the ancient Plato in his Timaeus assigned the figures

[333] This is also consistent with the use of the term by Aquinas. As noted by Eco, *figura* signifies in the writings of Aquinas the boundary of any continuous quantity, its external shape; in the case of the human body, its substantial (or essential) form disposes its matter in accordance with an inner necessity and confers upon the matter a boundary or limit, manifested externally in the accidental quality of *figura.* Eco, *The Aesthetics,* 177.

[334] "Le p(ro)portioni de l(')uno a l(')altro sempre sira(n)no irrationali per respecto de la n(ost)ra p(ro)portione sopra aducta la q(ua)le i(n) loro co(m)positioni e formationi se interpone co(m)mo se detto excepto del tetracedron e lo cubo e l(')octocedron […]" Pacioli, *Divina proportione (1509)*, 12v (Compendium).

> [*figure*] of the said regular [bodies] to the elementary bodies and at the fifth *convenentia* linked to the divine name of our proportion, as we mentioned above.[335]

The reference to both Aristotle and Plato is particularly significant if considered in the context in which is made. It is by means of the mathematical properties of the five regular bodies that Pacioli correlates Aristotle's theory of natural causation with Plato's theory of creation. Additionally, in a section significantly entitled: "On how to form [*formare*] other [bodies] beyond the said [regular], and how their forms [*forme*] continue ad infinitum"[336], Pacioli makes no distinction between natural and artificial bodies, since both are predicated on the universal of proportions:

> […] from the 5 regular [bodies] the virtue always is instilled, similarly to the five simple [elements] that concur to the formation of every created composite. For this reason, as it was mentioned above, Plato was compelled to assign the delectable 5 regular forms to the 5 elementary bodies, that are earth, air, fire and sky, as at length is discussed in his Timaeus, where he dealt with the nature of the universe.[337] […] And these forms as the very famous philosopher Calcidius discussing the said Timaeus, are much praised. And similarly Macrobius, Apuleius and many others because in truth they are worth much praise, for the reasons that are brought in their making,

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[335] "Co(m)mo li dicti effecti co(n)corino a la compositione de tutti li corpi regulari e lor dependenti. Cap XXIIII. Hora excelso D(uca) la virtu e pote(n)tia de l(')antedicta nostra proportione co(n) suoi singulari effecti maxime co(m)mo de sopra dice(m)mo se manifesta in la formatione e co(m)positione de li corpi si regulari co(m)mo dependenti. De li q(ua)li a cio meglio se prenda qui seque(n)te ordinatament(n)te ne diremo. E prima de li 5 essentiali quali p(er) altro nome sono chiamati regulari. E poi successivamente de alqua(n)ti abastanza loro egregii dependenti. Ma prima e da chiarire p(er)che sieno dicti corpi regulari. Seco(n)damente e da p(ro)vare co(m)mo in natura non sia possibile formare un 6. Onde li dicti sonno chiamati regulari p(er)che sonno de lati e anguli e basi equali e lu(n)o de l(')altro a po(n)cto de contiene co(m)mo se mostrara e co(r)respondeno a li 5 corpi semplici in natura cioe terra, aqua, airi, fuco e q(ui)nta essentia cioe virtu celeste che tutti gli altri sustantia in suo essere. E si co(m)mo questi 5 semplici sonno bastanti e sufficienti in natura altrame(n)te seria arguire I dio superfluo overo diminuito al bisogno naturale. La q(ua)l cosa e absurda co(m)mo afferma el ph(ilosoph)o che I dio e la natura non op(er)ano in vano cioe non ma(n)cano al bisogno e non excedono quello cosi a simili le forme de questi 5 corpi de li q(ua)lli sa adire a po(n)cto sonno 5 ad decorem universi e no(n) possano esser piu per quel che sequira. E p(er)o non imeritatamente co(m)mo se dira di socto l(')antico Platone nel suo thymeo le figure de dicti regulari atribui a li 5 corpi semplici co(m)mo in la q(ui)nta co(n)venentia del divin nome a la nostra p(ro)portione atribuita de sopra fu decto e questo a la loro denominatione." Pacioli, *Compendium*, 8r; Pacioli, *Compendium*, 17v.

[336] "Del modo a saperne oltra li dicti piu formare e commo loro forme in infinito procedano" Pacioli, *Divina proportione (1509)*, 16v (Compendium).

[337] "Non me pare Excelso Duca in dicti corpi piu externe con cio sia chel lor processo tenda in infinito per la continua e sucessiva abscisione de mano in mano de li suoi angoli solidi e secondo quella lor varie forme si vengono a multiplicare […] E questo solo habiamo finor sequito per monstrare co(m)mo da quelli 5 regulari la virtu sempre negli altri dependenti se distilla a similitudine de li 5 semplici che a la formatione de ogni creato composto concorrano. Per la qual cosa (co(m)mo de sopra fo acenato) Platone fo constretto le prelibate 5 forme regulari a li 5 corpi semplici atribuire, cie a la terra aiere aqua fuoco e cielo co(m)mo difusamente apare nel suo Thimeo dove de la natura de l(')universo tratto." ibid.

> demonstrating the sufficiency of the said 5 forms [that] like the 5 elementary bodies cannot be more […] And from these [5 bodies] any other form [*forma*] depends.[338]

Pacioli's insistence of on divisions *ad infinitum* of bodies should be paralleled with Aristotle's argument in *De Cælo,* where he explained that the"continuous [magnitude] may be defined as that which is divisible into parts which are themselves divisible to infinity, body [*soma*] as that which is divisible in all ways.".[339] In short, Pacioli somewhat followed Aristotle arguing that the form of regular bodies rests on mathematical properties, but diverged by exclusively pointing to this, and not to the form (essence) of the material in which these were instantiated. As argued by Pacioli, mathematical properties, and specifically the proportional arrangements of their quantitative components, determine the form of the regular and other bodies derived from these.[340] Consequently, it is perfectly possible for regular geometrical bodies to be instantiated in wood or indeed any material without a change of form. Following the instantiation of the form (shape) of regular bodies in a set of physical models, Pacioli meaningfully refers to these in the *Summa*, *Compendium* and *Tractato de l'architectura* as the "material forms" [*forme materiali*] of geometrical bodies.

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[338] "E queste tali forme da Calcidio celeberrimo philosopho expone(n)do el dicto Timeo molto sonno co(m)mendate. E cosi da Macrobio Apuleio e moltissimi altri perche in vero sonno de ogni commendatione degni, per le ragioni che in loro fabriche se aducano mostrando la sufficientia de ditte 5 forme si commo quella de li 5 corpi semplici non potere per alcun modo esser piu, e si commo el numero de dicti semplici non si po in natura accrescere, cosi queste 5 regulari non e possible asegnaene piu che de basi e de lati e de anguli sinno e quali, che in spera collocati toccando un angolo tutti tocchino. Perche se in natura si potesse un sexto corpo semplici asegnare el summo opefici verebbe a esser stato i(n) le sue cose diminuito e sença prudença da giudicarlo, non havendo a principio tutto el bisogno oportuno a lei cognosciuto. E per questo certame(n)te e no(n) per altro mosso comprendo Platone queste tali commo e dicto a ciascuno de li dicti semplici attriuisse e cosi argumenta(n)do ; cioe commo buonissimo geometra e p(ro)fondiss(i)mo mathematico, vedendo le 5 varie forme de questi non poter per alcun modo alcun(')altra che al sperico tenda de lati basi e angoli commo e dicto e quali ymaginarse formare commo in la penultima del 13 se mostra e per noi a lo portuno saduci non immeritamente argui le ditte advenire al 5 semplici. E da quelle ogni altra forma dependere." Pacioli, Divina proportione (1509), 17r.

[339] Aristotle concluded that "magnitude divisible in one direction is a line, in two directions a surface, in three directions a body. There is no magnitude not included in these; for three are all, and "in three ways" is the same as "in all ways." Aristotle, *On the Heavens*, 268a10-13.

[340] Here the difference with Aquinas's concept of artificial form [*formae artificialis*] becomes clear. Nevertheless, by the addition of the concept of artificial form Aquinas also moved towards understanding the form of artificial bodies as based on their visual and shape-defining properties.

Overall, although regular bodies correspond to natural elements in Pacioli's theory, their definition is essentially mathematical and based on quantity. This does not mean that when instantiated in matter, such bodies cannot have a form (essence) resting on this and other verbally describable properties. However, their identity rests on the three-dimensional quantitative proportional arrangement of their components, which determines their form (shape). As we have seen, Aristotle had first considered this possibility and (most importantly in approaching Pacioli's works) this was further considered in the Middle Ages by Duns Scotus. This, and its correlation with the theory of the elementary natural bodies found in Plato's *Timaeus* and its medieval commentaries, was instrumental in developing Pacioli's argument on regular geometrical bodies in the *Summa* and *Divina proportione*. The relationship between form and geometrically defined bodies is therefore regulated, for Pacioli, by quantities and proportions. Consequently, each body has its own proportional arrangement expressed by its *forma* and visible in its *figura*. Ultimately, the regular and other bodies derived from these discussed and represented in perspective in Pacioli's works, express in themselves the relationship of form (shape) and matter (extension) in three-dimensions.

Conclusion

This chapter outlined elements of Pacioli's philosophical framework by examining some of the arguments presented in his texts and their mutual correlations. It was argued that the philosophical background of Pacioli's texts is predominantly Aristotelian. Indeed, Aristotle was the most recognised and established authority in the late fifteenth-century philosophical and mathematical contexts in which Pacioli operated. However, acknowledging their philosophical framework does not mean that Pacioli's arguments are fully justifiable by relating them only to Aristoteles's works. Rather, these were Pacioli's responses to specific philosophical and mathematical issues, such as the central role assumed by processes of quantification of three-dimensional geometrical bodies in the context in which he operated. The analysis of Pacioli's texts has shown that he was able, in the context of the late Medieval Aristotelian framework, to develop the implications of these theoretical arguments through the involvement of multiple philosophical sources. In particular, the analysis has highlighted the development of a series of correlated arguments linked to regular and other bodies derived from these.

The relationship between the natural order of creation and mathematical properties has been indicated as central in all texts by Pacioli examined. By resorting to Aristotles's arguments and Campanus' commentary on Euclid's *Elements,* Pacioli was already able to set out a theory of quantity with proportions at its core in the *Summa.* Consistent with his Aristotelian context, quantity was also presented as acting at the level of the body, and the argument was mathematically and theologically framed by placing the unifying character of proportions as the universal which underlies the form of each regular body. To reach this conclusion, the analysis of the mathematical role and the properties of the divine proportion in Pacioli's *Compendium* have been instrumental in addressing the correlations among arguments. Critically, Pacioli employs the same theoretical tools in the formal and aesthetic judgment of all creations, including architecture, and regardless of what can be verbally conveyed.[341]

Finally, it was emphasised that Pacioli was able to overcome the difficulties of the Aristotelian and Scholastic distinction between substantial (essential) and artificial forms by referring to the linking of natural and regular geometrical bodies to their form *qua* extended magnitude

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[341] As noted in Chapter 1, Pacioli significantly discusses the aesthetic and formal value of architecture in the dedicatory letter that opens the *Summa*: "Which language could express the beautiful order of its disposition [*dispositione*]. Surely no one can suffice to this purpose." Pacioli, *Summa*, 2r..

and therefore quantity.[342] Consequently, it was possible for Pacioli to argue that the forms of regular bodies rest completely on their proportional mathematical order rather than the matter in which they are instantiated.[343] *Forma* was thus conceived as the three-dimensional arrangement of a body to which a visible two-dimensional *figura* corresponds. It is important to underline that when Pacioli discusses the *forma* of regular bodies, this comprises all the elements that concur with the shape of a three-dimensional body, and not only those that may be visible from a single viewpoint. Nonetheless, as Pacioli makes clear, visual, formal and shape-defining properties are necessarily interrelated. Ultimately, the arguments developed by Pacioli present inevitable consequences for the aesthetic and formal evaluation of all works designed and made by individuals.

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342 I am here referring to Pacioli's introduction of the regular bodies as the *forme* of the five basic elements of being and the universe.

343 As will be noted in the next section, the reading of *forma* when referred to bodies as their individual shape is further reinforced in Pacioli's *Compendium* by the addition of the term *propria.* As we will see, *propria forma* had been used before Pacioli to refer to the shape-defining qualities of bodies in Piero della Francesca's *De prospectiva pingendi.* Additionally, Pacioli also uses *forma materiale* to refer to regular bodies (and bodies derived from these) to specify the instantiation in matter of their form.

3. The quantification and visualisation of regular and other bodies derived from these in Pacioli's *Summa* and *Divina proportione*

As the previous chapter examined mathematical properties such as quantity and proportion in a theoretical framework in the context of Pacioli's works, it is now important to examine the perceptual and practical implications of these properties of regular and other bodies derived from these. It will be shown that Pacioli's arguments have inevitable consequences when the formal, visual and spatial properties of geometrical bodies are considered. The purpose of this chapter is to continue the examination of Pacioli's works by tracing how the forms of regular and other bodies derived from these were visually and spatially approached in the context of the processes of quantification and visualisation argued by Pacioli.

Following studies in the psychology and philosophy of perception from the last few decades, the analysis developed in this section will rest on the assumption that human visual perception is hierarchically structured and that there is a background level of perception independent of conceptual thinking.[344] This means that there is a stage in the visual perception of shapes and colours independent of any assigned category. The crucial role played by geometry is clear in this regard. Although the visual stimulus is two-dimensional, the principles of perspective, and specifically the experience of three-dimensionality, should be considered as a further phase closely correlated to the background capacities of visual perception.[345] Ultimately, where

[344] As noted, this is in essence the arrangement proposed by the cognitive scientists and philosopher Zenon Pylyshyn and more recently confirmed by Chaz Firestone and Brian Scholl. The latter have published an empirically anchored presentation of the six general pitfalls affecting all hundreds of studies that claimed so-called 'top-down' effects on visual perception, or "cognitive penetrability". Pylyshyn, "Is vision continuous with cognition? The case for cognitive impenetrability of visual perception"; Firestone and Scholl, "Cognition does not affect perception: Evaluating the evidence for 'top-down' effects". The separation of what Pylyshin calls "early vision" from "higher" cognitive aspects in the human brain had been already outlined in the groundbreaking researches of the neuroscientist and psychologist David Marr in the 1980s. David Marr, *Vision: Computational Investigation into the Human Representation and Processing of Visual Information*, Cambridge, MA: MIT Press, 2010. Important aspects of the relationship between this hierarchical approach to human visuality and architecture have been summarised by Branko Mitrović. Branko Mitrović, *Visuality for Architects*, 75-90.

[345] Pylyshyn, "Is vision continuous with cognition? The case for cognitive impenetrability of visual perception," 354-355; Firestone and Scholl, "Cognition does not affect perception: Evaluating the evidence for 'top-down' effects," 4.

geometrical bodies are concerned, this means that the background knowledge of the three-dimensional spatial arrangement of shapes determines the experience of a whole (composed of parts), that is geometrically coordinated in three-dimensions. This process is related to a set of so-called natural constraints that are employed in the mental visualisation of a three-dimensional object based on two-dimensional images.[346] Only at an additional (and separate) stage can the verbal (conceptual) categorisation of 'body', and its definition as a particular body (for instance "a dodecahedron" or "a twelve-pentagonal-bases", "a cylindrical barrel" and so on) take place.

Considering the senses involved in the analysis, this chapter will thus mostly concern sight. Where architecture is concerned, sight is by and large the primary sense involved and the same can be argued when geometry and spatial properties are considered. In the context in which Pacioli operated, the theory and practice of linear perspective played a crucial role in this respect, and he was able to base his arguments on the visual and geometrical principles established in his contemporary theory and practice of perspective. It will be shown that crucially, Pacioli argued for the similarity between the visual perception of real objects and the perception of their perspectival representation. This was possible, according to Pacioli, because both perceptions followed a set of mathematically determined principles.[347] The same mathematical principles, Pacioli noted, are central to the problems of quantification that take up large parts of the *Summa* and *Divina proportione*.

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[346] Branko Mitrović, "Visuality and Aesthetic Formalism," *British Journal of Aesthetics,* vol. 58, n. 2, (April 2018): 161. As discussed by Donald D. Hoffman, there are specific rules or constraints that are involved when the brain interprets a two-dimensional representation three-dimensionally. Donald D. Hoffman, *Visual Intelligence: How we create what we see* (New York: Norton, 1998).

[347] Linear Perspective is important because by considering the geometry of light rays it aims to graphically and two-dimensionally reproduce the visual stimulus received. In this chapter it will be argued that in Pacioli's *Summa* and *Compendium de divina proportione* the arguments concerning the theory and practice of perspective are essentially correlated with arguments concerning the mathematical quantification and visualisation of regular and other bodies derived from these.

Although this chapter is thematically organised into two sections concerning the mathematical quantification and the visualisation of regular and other bodies derived from this, the two themes are inherently interrelated. While in the first section, the analysis of the process of mathematical quantification predominates, the second focuses on the correlated process of visualisation. However, although organised in distinct sections here, as noted the mathematical, formal and visual properties of geometrical bodies are all substantially interrelated in Pacioli's arguments. Geometrical bodies need to be mentally visualised (and rotated) if the quantitative relationships among their components are to be spatially acknowledged. It is not possible, for instance, to visualise the proportional (quantitative) relationship between the diameter of the sphere and the side of a cube inscribed in it if both elements have not been mentally visualised and rotated. In other words, what can be (geometrically) quantified is implicitly related with what can be visualised.

When a hierarchical structure of human visual perception is assumed, it provides a framework to explain the relationship between the perception, visualisation, quantification and conceptualisation of geometrical bodies. Following on from the contextualisation of the concept of the geometrical body in Pacioli's texts in Chapter 2, the analysis in Chapter 3 will begin by considering the arguments on the quantification of geometrical bodies in the *Summa* and progress towards their aesthetic, formal and visual character as presented in the *Compendium*. Ultimately, it will be argued that it is in the progression and correlation of arguments, and in the formal, visual and spatial linkage established between geometrical bodies and architecture, where one of Pacioli's most original contribution is found.

3.1 Quantification

3.1.1 Problems of the quantification of regular bodies and their context in Pacioli's *Summa*

Manuals of commercial mathematics (abacus) and practical geometry demonstrated how to determine unknown quantities by solving exemplar problems of a practical character. Consequently, the passage from the *Book of Wisdom,*[348] often quoted by Pacioli both in the *Summa* and *Divina proportione,* that everything is arranged according to number, weight and measure, expressed fundamental properties investigated through the mathematical procedures included in these manuals.[349] This also clarifies the important role acquired by algebra when dealing with unknown quantities in mathematically determined relationships.[350] Following a section on elementary geometry derived from the first books of Euclid's *Elements*, a large part of practical geometry manuals concerned what would now be called surveying and gauging. The objects of enquiry were therefore presented at the beginning of the problem, while the quantity to be found was among the mathematical properties of the objects considered.

Stemming from these established manuals, the most important historical aspect of Pacioli's *Summa* as is often noted, lay in its role in the systemisation and circulation of knowledge rather than its originality. It was the first comprehensive and printed volume to have collated problems in arithmetic and geometry derived from the manuals of abacus and practical geometry that preceded it. The sources employed by Pacioli for the second part of the *Summa* that deals with geometry [*Tractatus Geometrie*] include not only Fibonacci's treatises in their fifteenth-century manuscript copies, but also manuals that collated problems developed from

[348] Also called *Wisdom of Solomon*, XI, 20.
[349] Pacioli, *Summa*, 68v (Pars prima); Pacioli, *Divina proportione (1509),* 2r, 16r (Compendium).
[350] Algebraic methods of resolution were already included treatise of abaco and practical geometry edited by Leonardo Pisano (Fibonacci) in the early thirteenth century, and although not always part of the manuals of practical geometry that followed, were included both in Piero della Francesca's *Trattato d'abaco* and Luca Pacioli's *Divina proportione*.

Fibonacci's text. These include the manuscript Palatino 577 (or a manual very close in its contents to this)[351] and the *Trattato d'abaco* of Piero della Francesca, a manuscript which remained in Piero's possession until around 1480, and is known in a single autograph copy.[352] When considering the section on regular bodies, in most cases the problems taken from Piero's treatise are included by Pacioli without relevant alterations.[353] However, the context in which these problems occur in the two texts is markedly different. Piero's treatise, apart for a single instance, never referred to physical objects, but rather discussed geometrical bodies as abstract shapes. Moreover, the textual approach was matched by the character of the illustrations in its margins, where three-dimensional bodies were shown in orthogonal or parallel projection and with uniform line thickness.[354] Pacioli's *Summa* on the contrary, included a multitude of theoretical and practical subjects, with extended sections dealing with three-dimensional objects of daily use. The original character of Pacioli's work therefore lies in his attempt to make a series of different geometrical subjects cohere by highlighting their mutual correspondences.

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[351] Ms. Palatino 577, Biblioteca Nazionale di Firenze. The borrowings of Pacioli from this, or another manuscript very close to this, where first highlighted by Ettore Picutti. Ettore Picutti, "Sui plagi matematici di Frate Luca Pacioli," Le Scienze, 246, (1989): 72-79. 119 folios of the *Summa* can be referred to this or a similar codex, while Pacioli copied or re-elaborated problems from 15 folios from the *Trattato d'abaco* of Piero della Francesca. Daly Davis, *Piero della Francesca's Mathematical Treatises*.

[352] This is the codex MS. Ashb. 280/359-291, in Biblioteca Medicea Laurenziana, Florence. The dating and possession of the manuscript are described in: Machtelt Brüggen Israëls, *Piero della Francesca and the Invention of the Artist* (London: Reaktion Books, 2020), 236-237.

[353] The *Trattato d'abaco* of Piero della Francesca is undoubtedly the primary source in Pacioli's *Summa* when regular or other bodies derived from these are concerned. As it is well known, Pacioli included, without acknowledgements, geometrical problems derived from Piero's treatise in the eighth distinction of this part of the *Summa.* Although already acknowledged in Vasari's *Vite* in 1550, the appropriations of Pacioli from Piero's *Trattato d'abaco* and *Libellus* have been traced with precision only in the twentieth century, and there is now no doubt that Pacioli consciously avoided to mention his source in these cases. These problems examined the five regular and other two semi-regular bodies derived from these, and were included under a separate treatise (*Paticularis tractatus circa corpora regularia et ordinaria*). As demonstrated by Margaret Daly Davis, the main source of the mathematical problems involving regular and semi-regular solids presented in the *Particularis tractatus circa corpora regularia et ordinaria* of Pacioli's second part of the *Summa* (1494) is Piero della Francesca's *Trattato d'Abaco,* a work probably completed by the end of the 1460s. Daly Davis, *Piero della Francesca*, 98-106.

[354] There is a geometrical problem enquiring the volume of a barrel. However, after the initial identification as a barrel, Piero swiftly states that this should be considered as "two truncated pyramids". The problem is reproposed by Pacioli in the *Summa*. Pacioli, *Summa*, 70r.

Although the three-dimensional objects investigated in manuals of practical geometry were not usually abstract, but were physical objects of the material world, Pacioli argued in the *Summa* that the practical processes of quantification could be extended to regular and other bodies if their visual, formal and spatial properties were perceived as such. In other words, regular and other bodies derived from these were understood as sharing geometrical and physical properties with objects of daily use, while retaining at the same time their formal and aesthetic qualities.

To understand the practical context of Pacioli's arguments concerning geometrical bodies it is therefore crucial to consider the contents of the second (geometrical) part of Pacioli's *Summa*, particularly the arguments discussed in its second half (distinctions six-eight). This section collates a range of materials, but mainly deals with three-dimensional geometry. The first (sixth distinction) begins with the definitions given in Fibonacci's *practica geometriae* and introduces the types of geometrical bodies that will be considered. These include rectangular and cubic bodies, round and multisided columns, spherical bodies, etc. In this context Pacioli also briefly refers to Book XI of Euclid's *Elements*, where three-dimensional bodies are introduced. However, differently from Euclid, here Pacioli (like Piero della Francesca and all their contemporary writers in practical geometry) referred to the volume of a body as its squaring [*quadratura*] or its corporeal area [*area corporale*]. Problems dealing with the measurement of the surface of the sphere and its volume are also found in this distinction.

Nonetheless, the ensuing (seventh) distinction lists problems that deal with measurement by sight. These discuss, in Pacioli's own summary: "the instruments and different ways by which it is customary to measure with the appearance, that is by sight every distance of length, height, and width without changing place." Notably, Pacioli adds that the distinction also concerns "the way to measure the ones that for any obstruction cannot be seen, [but] that can be measured by means of proportions from the ones that are visible."[355] The examples listed by Pacioli include the measurements of towers, wells, rivers, moats among others. These methods were well known in the Middle Ages and were found in manuals of practical geometry.

Finally, the distinction that follows this (the eighth), is the most extensive in the second part of the *Summa* and collates a wide range of subjects. However, it is not divided into chapters but into enumerated cases, all deemed by Pacioli as very useful to the practice of geometry.[356] Among the quantities enquired about in this section are lengths, areas, and volumes, the latter often described as the material content or holding capacity of objects of common use such as barrels, casks, boxes, conical accumulations of goods, etc, some of which are illustrated on the margins of the text (illus. 3.1). A survey of the terms employed in connection to the enquired quantities shows indications as "it holds" [*tiene*], "the inside" [*dentro*], "when empty" [*voita*], etc.[357] The use of these terms highlights not only a clear distinction between inside and outside, but that the mathematical process of quantification involves knowledge of the formal and spatial properties of each body. It is useful to look at an example:

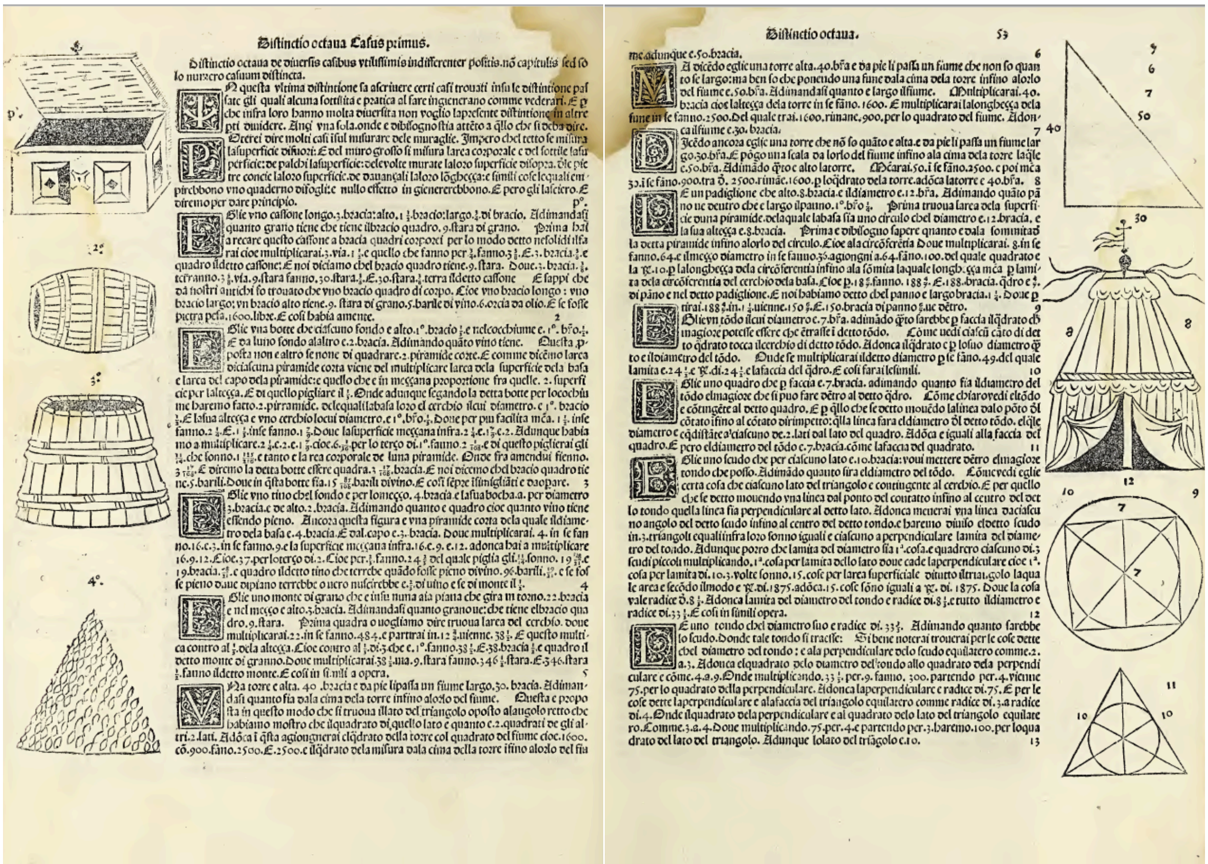
> There is a box that is 3 *braccia* long, 1 1/2 *bracio* high, 3/4 of a *bracio* wide. I ask how much wheat it can hold given that the square *bracio* holds [*tiene*] 9 *stara* of wheat.[358] Firstly you should find the corporeal square *bracia* [*bracia quadri corporei*] of the box, in the way said before for solids [*solidi*] and you will find it by multiplying 3 by 1 1/2 and the result by 3/4 […] And you should remember that the Ancients found that one

[355] "De li strumenti e mo(d)i diversi co(n) li q(ua)li se costu(m)a mesurare solo co(n) l(')aspecto: che col veder(e) ogni dista(n)tia d(e) longhe(çç)a, alte(çç)a, largh(eçç)a, e p(eo)fo(n)dita per ch(e) l(')ochio la possa scorgere se(n)za moverse de luogo. E anche del modo a mesurare q(ue)lle ch(e) p(er) alcu(n) i(m)pedime(n)to no(n) se potessero vedere p(er) la notitia de le visibili lor certa mesura p(ro)portionalmente arguire." Pacioli, *Summa*, 75v (Pars secunda).

[356] The distinction is significantly called: "Distinctio octava de diversis casibus utilissimis indifferenter positis" Pacioli, *Summa*, 52v (Pars secunda).

[357] Pacioli, *Summa*, 52v (Pars secunda).

[358] The *stara* was an old Italian unit of measure for capacity used for wheat.


Distinctio octava Casus primus.

Distinctio octava.

3.1—Luca Pacioli, *Summa de arithmetica proportioni et proportionalita*, *Tractatus geometriae, Distinctio octava,* 52v-53r (Pars secunda).

> square braccio of a body [*bracio quadro di corpo*], that is a braccio long, a braccio high and a braccio wide, holds 9 *stara* of wheat, 5 barrels [*barile*] of wine, 6 jars of oil, and if it was stone would weight 1600 pounds [*libbre*] and so keep these in mind.[359]

Other problems ask for the quantity held by different bodies: “There is a vat, its bottom through the middle is 4 *braccia*, its mouth has a diameter of 3 *braccia* and its 2 *braccia* high. I ask what is squared [*quadro*] that is how much wine it holds when it is full […]”[360] or “A cask holds 10 barrels and I want to take away every day a tenth part of what I can find inside it; I ask in how many days will I have taken away 5 barrels […]”.[361] In all these cases, the dimensions involved are comparable to those that the real object would likely have.

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[359] “Eglie uno cassone longo 3 bracia: alto 1 1/2 bracio: largo 3/4 di bracio. Adimandasi quanto grano tiene che tiene il bracio quadro 9 stara di grano. Prima hai a recare questo cassone a bracia quadri corporei per lo modo detto nei solidi il farai cioe multiplicherai 3 via 1 1/2 e quello che fanno per 3/4 […] E sappi che da nostri antichi fo trovato che uno bracio quadro di corpo. Cioe uno bracio longo: uno bracio largo: uno bracio alto tiene 9 stara di grano, 5 barile di vino, 6 orcia da olio. E se fosse pietra pesa 1600 libbe. E cosi habia a mente.” Pacioli, *Summa*, 52v, (Pars secunda).

[360] ibid.

[361] ibid., 66r-v (Pars secunda).

At this point, at the beginning of the section devoted to the measurement of three-dimensional bodies, Pacioli introduces how to calculate the volume of a cube: "There is a cube, or we can say a dice, that has every face [side] equal to 10 *braccia*. I ask you the measurement of the corporeal area [*area corporale*] of the aforementioned cube […]"[362] In quantitative terms, even if variations in the local *braccio* are considered, the cube would have a side measuring between 5 and 6 metres.[363] Similar numerical values are employed for all regular bodies later discussed in the text, even if the unit of measurement is often omitted. Overall, when dimensions are given in problems involving regular geometrical bodies in the *Summa*, these are usually larger than the dimensions given for object such as vats, barrels, or boxed and more akin to the dimensions of a space such as a room. As we shall see, similar dimensions are also found in the treatises of Piero della Francesca.

A number of planar geometrical problems involving polygons follows from this,[364] and then a section with six practical cases concerning perspective and involving the measuring of distances and heights by means of similar triangles.[365] The ensuing section considers how to operate a weighing instrument similar to a steelyard balance and gain the measuring of distances on a plane before returning to consider three-dimensional bodies. Here, a problem asks which is the biggest geometrical body that can be inserted in another and implies the use of algebraic methods of resolution. This is the last problem before the opening of the

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362 Pacioli, *Summa*, 44r (Pars secunda). Interestingly, while the text of Fibonacci used palms as the unit of measurement in this problem, the problem as discussed by Pacioli used *braccia* instead. The change happened sometime before the compilation of the *Summa,* as *braccia* were already employed when the same problem was discussed in the codex Palatino 577 (c. 1460) now at the Biblioteca Centrale di Firenze.

363 As noted by Banker, in fifteenth-century Sansepolcro the majority of measurements of length and width were expressed in *braccia* (arm-lengths); one *braccio* of Sansepolcro equals 56 centimetres. Banker, *Piero della Francesca*, xvii. The lengths of the *braccio* used in other areas have been summarised by Zupko. Ronald E. Zupko, *Italian Weights and Measures from the Middle Ages to the Nineteenth Century* (Philadelphia, PA: American Philosophical Society, 1981).

364 These are problems of measurement in two-dimensions, trigonometry and the circumscription of planar shapes. In last group of cases, the dimensions of the biggest given number of shapes (circles for example) that can be inscribed in a singular shape is asked. For example, "in a circle that has a diameter of 40, I want to place the largest four circles that I can; I ask the diameter of each of them". Pacioli, *Summa*, 57v (Pars secunda).

365 These are discussed in the ensuing section 3.2.1 of the thesis.

particular treatise [*particularis tractatus*] that investigates numerical quantities of regular and other bodies derived from these.[366] As meaningfully summarised by Pacioli later in the text, this treatise concerns all five regular bodies and their measures, which "are of great artifice, and their science is the most subtle that there can be in all mathematical disciplines. And [this science] is presupposed by every philosopher".[367] It is important to give an overview of three-dimensional bodies involved. The treatise is composed of fifty-six numbered cases.[368] Five concern the measurement of the body of four triangles (tetrahedron), five the cube, three the sphere or ball [*palla*], two a (semi-regular) body of eight bases (four hexagons and four triangles) inscribed in the sphere, two a (semi-regular) body of fourteen bases (eight square and six triangles), one a barrel (considered as two truncated cones), one 'irregular bodies' such as statues (to be measured by immersion in water), four the body of twelve pentagonal bases (dodecahedron), seven the body of twenty triangular bases (icosahedron), three the body of eight triangular bases (octahedron); from these follow nine additional cases discussing pyramids with four triangular bases, four involving the cube inscribed in the sphere, two on pyramids inscribed in the sphere, and seven dealing with three-dimensional sections cut out of the sphere. All through this section, the folios are densely furnished with geometrical figures that illustrate the written formulations. Although the contents of this short treatise are essentially mostly borrowed from Piero della Francesca's *Trattato d'abaco*,[369] Pacioli's introduction to this section is particularly significant.

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[366] Pacioli, *Summa*, 68v-73v (Pars secunda). The discussion of methods to precisely calculate and optimise the volume of barrels will become in the decades following the publication of the *Summa* a favoured subject among mathematicians, at least until Kepler's book on the volume of wine barrels published in 1615. Kepler's *Nova stereometria doliorum vinariorum* (New solid geometry of wine barrels).

[367] "Tractato p(ar)ticulare d(e) tutti li 5 corpi regolari e sue mesure q(ua)li so(n)no d(e) gra(n)dissimo artificio e a la lor sci(enz)a e la subtilissima che esser possi i(n) tutte le discipline mathematici. E da ciascu(n) Ph(ilosoph)o prosuposta." Pacioli, *Summa*, 75v (Pars secunda).

[368] The first numerated case is the introduction.

[369] As noted by Margaret Daly Davis, of these 56 cases, Pacioli copied about 27 of them from Piero's *Trattato* almost verbatim, he slightly shortened another 10, shortened consistently 6, and extended 3. Another 4 were partially re-elaborated and only 5 completely reworked by Pacioli. Daly Davis, *Piero della Francesca*, 98-106.

Pacioli argues that although the five regular bodies were briefly mentioned in the first (arithmetical) part of the *Summa*, nevertheless they deserve to be discussed again in more detail. These are the "essential bodies", Pacioli remarks, which are each circumscribed by a sphere, with all their vertices touching the surface of the sphere, and they are "scientifically discussed by Euclid in the last books [of the *Elements*]". Importantly, Pacioli highlights here that he thinks it would be useful for the "vulgar practitioners" to hear about the sweetness of the dimensions of these bodies. Pacioli adds that these are also discussed by philosophers, and "by Plato in his *Timaeus* (as noted by Saint Augustine) that assigns the bodies to the natural elements."[370] At this point Pacioli refers for the first time in his writings to a set of physical models of the regular and other bodies derived from these, that he presented to Duke Guidobaldo of Urbino (dedicatee of the *Summa*) in Rome in 1489:

> These are the ones, magnanimous duke, the material forms [*forme materiali*] of which, greatly adorned, [were] in your hands in the sublime palace of the most revered cardinal and protector *Mo(n)segnor de san piero in vincula*, when visiting his highness, the Pontifex Innocent VIII, in the year 1489. […] and together with them there were many others derived from the regular [bodies] and called dependants [*dependenti*]. I made [*fabricai*] these for the most revered *mo(n)segnor meser Pietro de valetarii de Genoa,* bishop of *Carpentras* […] Hence of these, the ensuing propositions will enquire their manner of operation [and] we will teach [them] in [a way that is] sufficiently practical, as it will later become clear. And first [we will discuss] about the first regular [body], called tetrahedron [*tetracedron*]: that is the [body with] 4 triangular bases of

[370] "E Bènche di sopra i q(ue)sto nella distition 6ª al capitolo 4º de la mesura de la sp(her)a succinteme(n)te fosse ditto abasta(n)ca: no(n) dimeno me par q(ue) excelso Duca p(ar)ticularme(n)te dir(e) de alqua(n)ti corpi esse(n)tiali i ditta spera locabili de li q(ua)li un angolo tocca(n)do subito tutti toccano: e p(ri)ncipalmente lo fo p(er) la notitia de li 5 regulari di q(ua)li Euclide a pieno nelli ultimi soi libri scie(n)tificame(n)te tratta. Di che me pare no(n) i(n)utile a ponere de loro certi casi acio li pratici vulgari ancora essi qualche dolceçça di loro dime(n)sio(n)i sentino. Di q(ue)sti fra phy(losofi) si fa gra(n) discussione. E maxime se be(n) el thymeo del divin ph(ilosof)o Platone (seco(n)do lo Aurelio doctor sa(n)cto Augustin) con diligentia satende Dove de universi natura diffusamente parla(n)do spesso a suo p(ro)posito li i(n)duci. Attribue(n)do lor separatamente a li 5 corpi semplici: cioe Terra Aqua Aeri Fuoco E Cielo. Si co(m)mo apieno di sopra in q(ue)sto nella parte de Arithmetica in la distinction prima. Nel seco(n)do suo trattato al terço articolo di loro parl(i)amo." Pacioli, *Summa*, 68v (Pars secunda).

> which the figure was assigned to fire by Plato, [although] Aristotle argued against his [Plato's] opinion in *De cælo et mundo*.[371]

If considered in their contexts, the reference to a set of physical models of the regular and other bodies derived from these, and the decision of Pacioli to build and present such a set of geometrical bodies, become clearer. The formal properties of each body can be better understood and mentally visualised if seen from different viewpoints and, perhaps simultaneously, manipulated. What follows is a list of cases in which the processes to find numerical quantities of volumes, surfaces, sides, radii or diameters of the five regular, and two semi-regular bodies are described. The semi-regular bodies are derived by truncation from the tetrahedron and the cube, while the sphere is presented as circumscribing or inscribed in some of the bodies.

Unlike the propositions dealing with abstract magnitudes of three-dimensional bodies in Euclid's *Elements,* that Pacioli mentions at the beginning of the passage, here, the processes of resolution follow the quantitative concerns typical of manuals on abacus or practical geometry. One case, for instance, asks about the surface of a dodecahedron with the diameter of a circumscribed sphere known: "This is a spherical body and its diameter is the root of 48. It contains a body with 12 pentagonal equilateral faces and I ask [the numerical value of] its sides."[372] Although the same problem is found in Piero's *Trattato d'abaco,* the visual experience

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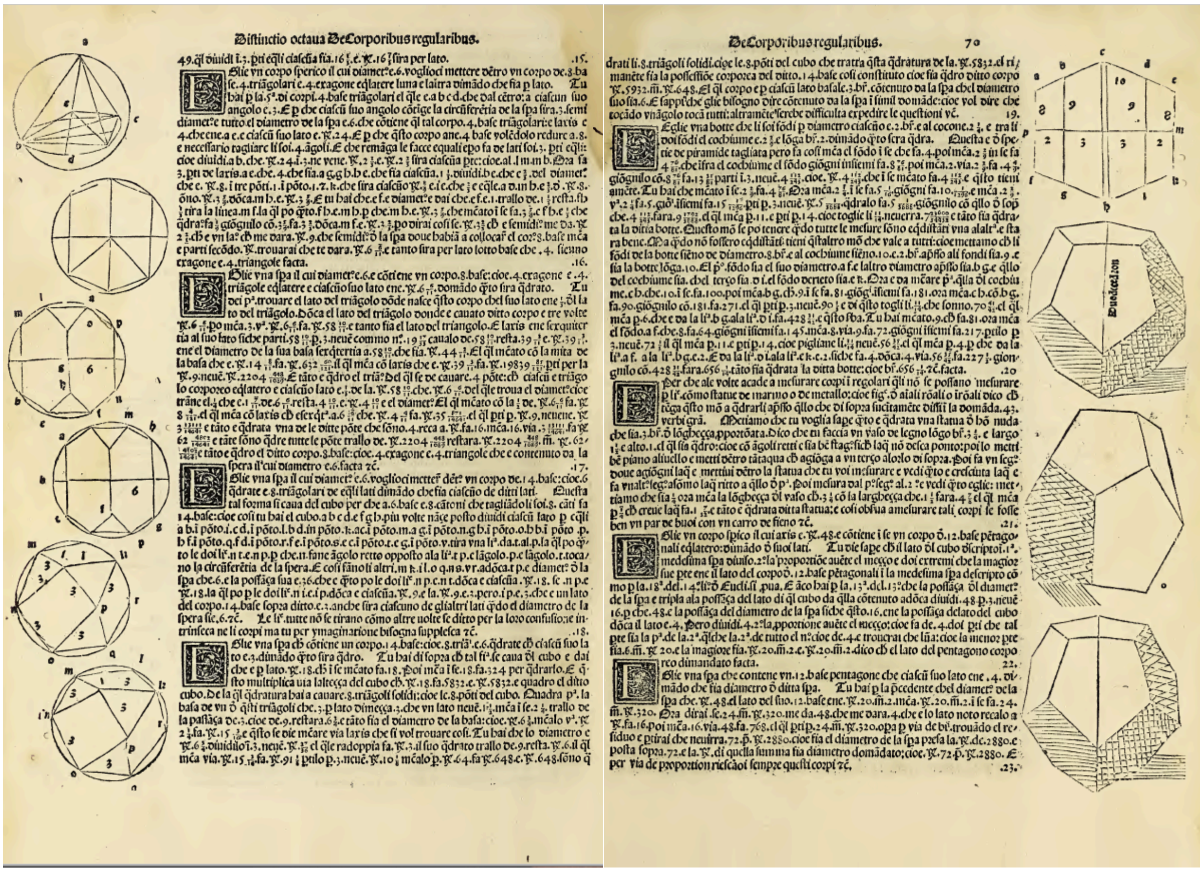
[371] "Questi son q(ue)lli Magnanimo Duca di quali le forme materiali (co(n) assai adorneçςe nelle proprie mani di U. D. S. Nel sublime palaçço Del Revere(n)dissimo cardinale n(ost)ro p(ro)tectore Mo(n)segnor de san piero in vincula: qua(n)do q(ue)lla venne ala visitatio(n)e Del su(m)mo po(n)tefice Innoce(n)tio octavo: negli an(n)i de la salute n(ost)ra 1489. Del mese de aprile che gia sonno 5 anni elapsi. E insiemi co(n) quelli vi foron molti altri da ditti regulari depe(n)denti. Quali fabricai per lo Revere(n)do mo(n)segnor meser Pietro de valetarii de Genoa dignissimo vescovo de carpe(n)tras: al cui obsequio alora foi deputato i(n) casa de la felicissima momoria del R[mo] Cardinale de Fois nel palaçço ursino i(n) ca(m)po de fiore. Siche di questi le seguenti petizioni inq(ue)riranno el lor modo opere(n)tivo abastança pratica insegnaremo commo apresso se intendera E pri(m)a del pri(m)o regulare ditto tetracedon: cioe 4 base tria(n)golari la cui fi(gur)a Platone al fuoco attribui: co(n)tra la cui opinio(n)e A Ri(stotele) q(ue)l de celo e(t) mu(n)do prese ardire e(t) c(on) E diro cosi." Pacioli, *Summa*, 68v (Pars secunda)

[372] "Eglie un corpo sp(her)ico il cui axis e R48 e co(n)tiene i(n) se un corpo d(e) 12 base pe(n)tagonali eq(ui)latero: dima(n) d(ie) suoi lat" Pacioli, *Summa*, 70r-70v (Pars secunda) It can be compared with the problem found in Piero della Francesca's Trattato d'abaco: "E gle uno co(r)po sperico ch(e) il sou assis e R de 48. Il q(ua)le co(n)tene u(n)o co(r)po de 12 base pe(n)tagonale eq(ui)latero doma(n)do de suoi lati: […]" Della Francesca, *Trattato d'Abaco*, 110r.


Distinctio octaua De Corporibus regularibus.

De Corporibus regularibus. 70

3.2— Luca Pacioli, *Summa de arithmetica proportioni et proportionalita*, *Tractatus geometriae, Distinctio octava*, 69v-70r (Pars secunda).

of the body as a physical object determined by its different context in Pacioli's text is further emphasised in the *Summa* by its illustration in perspective and with shadows in the margin of the text (illus. 3.2).[373]

Finally, at the end of this section and the whole of the *Summa*, Pacioli considers a mathematical-geometrical and another empirical method to calculate the capacity of barrels. The former involves the calculation of the volume of half of the barrel considered as a truncated cone and the doubling of the result, a procedure customary in manuals of practical geometry. The latter method instead involves the creation of equally distanced holes in a barrel to be taken as a benchmark. Starting with a barrel full of liquid, holes are to be made at regular distances and a table should be drawn to annotate the quantity of liquid (how

373 Indeed, the illustrations in turn resemble the dodecahedron shown in the painting portrain Luca Pacioli and a Gentleman, and currently in the Museum of Capodimonte in Naples. As a copy of the *Summa* is shown in the painting, and the painting has been dated the year after the *Summa's* publication (1495).

many mugs [*boccali*]) released after each hole is made. Pacioli admits that the procedure may need to be repeated and other tables made, as the shapes of barrels and their related quantities differ in other places. Nevertheless, at the conclusion to the whole case, Pacioli adds the following: "However, all the measurements of the same quality and nature are in proportion between them, in a way that from the small you can find the large, and from the large the small when these are similar [in shape] […] And [it is] in this way that you can deal with bodies."[374] Overall, what makes Pacioli's work different from Piero della Francesca's *Trattato d'abaco* (from which the former's eighth distinction largely depend) is the insistence on the usefulness of geometrical regular and other bodies derived from these for practitioners. Their location and contextualisation at the end of the second part of the *Summa* points to the exemplar formal and proportional quantitative properties of the three-dimensional geometrical bodies considered.[375]

3.1.2 Quantification and materialisation of regular and other bodies derived from these in Pacioli's *Compendium de divina proportione*

On first examination, the contents of Pacioli's *Compendium* are different in many respects to the *Summa*. While the latter is usually considered the first comprehensive manual in the tradition of books of commercial arithmetic and practical geometry to be printed, the former was presented in its first redaction of 1498, as a manuscript text edited in only three copies and addressed to learned patrons.[376] However, when the arguments of the *Compendium* are examined, it becomes clear that they are also in many respects a coherent development of the

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374 "Pero tutte le misure de una q(ua)lita e natura sonno tra loro proportionate i(n) modo che per la piccola se trova la grande. E per la grande si trova la piccola quando sien simili: cioe tutti cerchi o tutti quadri o tutti tria(n)goli. Et e cosi riesci ne li corpi aponto." Pacioli, *Summa*, 75r (Pars secunda).

375 Although this could have been also implied by Piero della Francesca, the context and abstract mathematical character of the cases presented in the *Trattato d'abaco* did not suggest any explicitly practical value.

376 As noted by Pacioli, the three copies were dedicated to the duke of Milan Ludovico Sforza (Bibliothèque Publique et Universitaire de Genève. (ms. Langues Etrangères n. 210), Galeazzo Sanseverino (Biblioteca Ambrosiana di Milano (inv. S.P.6 ex Cod. F 170 sup.) and Pier Soderini, with the latter now lost. The treatise, called *Compendium de divina proportione*, will become the first section in the printed edition of 1509.

arguments presented in the *Summa* on regular and other bodies derived from these. Indeed, throughout the *Compendium*, Pacioli often refers the reader to the section of the *Summa* that deals with regular and other bodies derived from these to avoid repeating arguments that could easily be found there. Nevertheless, the discussion of these bodies in the *Summa* took up a relatively short section of the text compared to the multiple arguments of practical geometry it presented, whereas this becomes the main argument in the *Compendium*.

As noted in Chapter 2, the proportional properties of regular bodies are Pacioli's starting point. After a discussion of the title of the work and the thirteen effects of the divine proportion (Chapters 4-14), there follows an Italian translation of large part of Book XIII of Euclid's *Elements* from the Campanus edition (Chapters 15-25). The mathematical relationships among the geometrical components of each body are then clarified (Chapters 26-33). Throughout the ensuing section, only enunciations are included in most cases, with a few demonstrations of theorems taken from the *Elements,* while the geometrical figures are still taken from the Campanus edition. The text continues with a discussion of the geometrical relationships established by inscriptions of a body inside another body, with the vertices of the former touching the surfaces of the latter (Chapters 34-47).

Finally, in what can be considered the third section of the *Compendium* (Chapters 48-62), the mathematical approach substantially changes compared to the first two sections. Here, Pacioli gives only a description of the regular, semi-regular and irregular geometrical bodies discussed, enumerating the lines, superficial and solid angles and polygonal bases of each body. Mathematical demonstrations are not included and the relationship with the *Elements* is less direct, as many of these bodies are not discussed in Euclid's text. Practical examples are given instead and Pacioli constantly refers the reader to the sixty illustrations drafted by

Leonardo da Vinci. As is often noted, the visual clarity of the illustrations acquires prominence over the text in this section.[377]

Following this, Pacioli devotes Chapter 54 to the formation of the spherical body.[378] The description starts with the construction found in Book XI of the *Elements* in the Campanus edition and continues by setting a semicircle drawn from a line (segment) taken as the diameter. Following the procedure found in the *Elements*,[379] the sphere is constructed by the revolution of the semicircle around the diameter. However, Pacioli adds that the spherical body or sphere should be imagined by considering the semicircle as a section cut through a material [*meçço taglieri materiale*], since otherwise this would not form a body [*non formaria corpo*].[380] In other words, Pacioli makes it clear in this passage that the sphere should be understood as composed of its perimetric surface and its internal space. This remark should be linked to the problems in the *Summa* dealing with sections cut out of the sphere in the treatise investigating numerical quantities of regular and other bodies derived from these. One of these problems asks how much would be taken away from the squaring of the body (the volume of the sphere) if its diameter of 14 *braccia* is cut on one side with a plane by 5 *braccia* and the resulting section of sphere is removed; while another asks what would be the squaring of the solid body [*corpo solido*] remaining after a sphere (again with a diameter of 14 *braccia*) is cut by two parallel planes and only the three-dimensional section of sphere between

377 This section will be examined in the last section of this chapter.

378 "Del corpo sperico la sua formatione" Pacioli, *Divina proportione (1509)*, 17r (Compendium).

379 The definition of sphere is at the beginning of Book XI of the *Elements* (definition 14 in modern editions of the *Elements*). Campanus's edition, commented and published Pacioli, states: "Sphera est transitus arcus circu(m)ferentie dimidii circuli quotiens sumpto vel supremo semicirculo lineaq(ue) diametri fixa donec ad locum suum redeat arcum ipse circumducitur". Pacioli, *Euclidis*, 100v.

380 "[…] e spera ymaginando co(m)mo se deve che dicto semicirculo gratia exampli sia un meçço taglieri materiale che aliter non formaria corpo, peroche solo l(')arco circu(n)ducto non fa vestigio siando linea sença ampieça e p(ro)fondita e questo a sua notitia e causatione sia detto." Pacioli, *Divina proportione (1509)*, 17v (Compendium).

them is considered.[381] Moreover, it should be noted that 14 *braccia* correspond to the diameter of a small dome. Significantly, the next chapter (Chapter 57) examines how the five regular bodies can materially be placed inside the sphere:

> And inside this sphere, excellent Duke, we can imagine all the five regular bodies in the following way: first is the tetrahedron; if over the [sphere's] surface, that is [over] its robe or dress [*cioe la sua spoglia over veste*] we trace or imagine [to trace] four equidistant points in every direction, and through these we join six rectilinear lines, these will necessarily pass through the inside of the sphere, and we will have the aforementioned body inscribed in it. And if you strike a cut with the imagination, with a planar surface [*e chi tirasse el taglio per yimaginatione con una superficie piana*] through every side of the body, you would have then the bare tetrahedron. If (so that it can be more easily understood by others) the aforementioned sphere was a bombard stone with the four equidistant points marked on it, and a stonecutter was to break it and trim it with a scalpel, leaving the four mentioned points, from that stone we would have a tetrahedron […] hence applying a similar imagination, all bodies will be placed in the sphere, in a way that their angular points, with one touching the surface of the sphere all touch, and it is not possible for one to touch and not the others, once placed in the sphere.[382]

Pacioli continues arguing that the geometrical principles underlining the construction of the five regular bodies, and their consequent maximum number of five, are often not well known by stonecutters and that this can be proved by asking them to make a base, a capital and a cymatium of a column, with a form having its lines, bases and angles all equal and which is not one of the five regular bodies. As an example, Pacioli mentions a body with four bases

381 "Egli e(') una sphera il cui diame(tr)o e 14 taglione co(n) una linea piana 5 br(accia), domando che levara de la q(ua)dratura del corpo."; "Egli e una sphera il cui diametro e 14 a.d. meno doi line(e) piane quidistante l(')una dall(')altra. L'una sega el diamtero in ponto e l(')altra in ponto h, a.e e(') 3 e a.h e(') 6 e la superficie che tra l(')una e l(')altra e(') 132. Domando che si levara da quadratura del corpo solido fra una linea e l(')altra." Pacioli, *Summa*, 73v (Pars secunda).

382 "E in questa spera excelso D[uca] se ymaginano tutti li 5 corpi regulari in questo modo. Prima del tetracedron se sopra la sua superficie. cioè la sua spoglia over veste se segnino over yimaginano 4 ponti equidistanti per ogni verso l'uno da l'altro. e quelli per 6 linee recte se conghionghino le quali de necessità passaranno dentro dala spera sirà formato aponto el corpo predetto in essa. E chi tirasse el taglio per yimaginatione con una superficie piana per ogni verso secondo linee recte protracte remarebe nudo aponto dicto tetracedron. Commo (ació per questo gli altri meglio se aprendino) se la dicta spera fosse una pietra da bombarda e sopra lei fossero dicti 4 ponti con equidistantia segnati se uno lapicida over scarpelino con suoi ferri la stempiasse over sfaciasse lasciando li dicti 4 ponti aponto de tutta dicta Petra arebe facto el tetracedron […] Onde co[n] simili ymaginationi tutti seranno in la spera situati…" Pacioli, *Divina proportione (1509)*, 17v (Compendium).

that are not triangular, or one of six bases that are not square, etc.[383] The text continues with the recounting of an episode in Rome at the time of the construction of the palace now known as the Palazzo Altemps.[384] On this occasion, Pacioli was with the painter Melozzo da Forlì who was at the time involved in the "fabrica" of the palace,[385] and suggested that Count Girolamo Riario ask one of the stonecutters working on his palace to make a capital with such formal properties. Although the stonecutter reportedly stated that the task would be easy, Pacioli emphasises how the lack of geometrical knowledge prevented him from recognising that this was impossible in reality. The chapter ends with the stonecutter asking to be taught on the subject by Pacioli.[386]

The whole episode clearly aims to support Pacioli's argument for the necessity of the formal and spatial geometrical principles that are the basis of the material world. It is therefore particularly significant that these principles were discussed, and their knowledge tested on a building site, and that by the end of the story, the stonecutter was willing to learn these principles from Pacioli. When compared with the argument in the *Summa,* where the mathematical skills of Melozzo, among those of other painters, sculptors and architects are praised, what emerges is the difference in knowledge of the rules of quantification in three-

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[383] "E p(er) q(ue)sta sci(enti)a i(n)fallibile porra V. cel(situdine) a le volte (co(m)mo noi habiamo fato) con dicti lapicidi havere solaçço in questo modo argua(n)do loro ignora(n)ça . Ordina(n)doli che queste simil pietre ne facino qualche forma de lati facie e anguli equali e che nun sia simile a le 5 de li regulari, verbi gratia obligandoli a fare un capitello o basa o cimasa a qualche colonna che sia de quatro o de sei facce e quali a modo dicto e che quella de le 4 non sie(n)no triangule overo quelle de le 6 non sienno quadrate. E cosi de 8 e 20 facce e Niuna sia triangula over de 12 e niuna sia pentagona, le quali cose tutte sonno impossibile." Pacioli, *Divina proportione (1509)*, 17v (Compendium).

[384] Based on the documented whereabouts of Luca Pacioli, Melozzo da Forlì and Girolamo Riario, the episode has been convincingly dated to the second half of 1480, when the palace was in construction. Banker, *La formazione di Luca Pacioli*, 79.

[385] Melozzo da Forlì (1438-1494) has been often considered by scholars a disciple of Piero della Francesca. Although the type of relationship with Piero della Francesca is still debated, Melozzo, and his disciple Marco Palmezzano (1460-1539), display a rigorous knowledge of perspective in their works. Indeed, they were both mentioned, among other painters, for their skills in perspective in Pacioli's *Summa*. Pacioli, *Summa,* 2r (Epistola).

[386] Pacioli, *Divina proportione (1509)*, 18r (Compendium).

dimensions between this latter group and those who merely execute others' instructions.[387] Notably, this is fully in keeping with the role of the architect as expressed in Alberti's *De re aedificatoria*.[388] Nonetheless, rather than only focusing on the role of a specific category of professionals, Pacioli is keen to underline that everyone is subject to the geometrical properties that rule the relationships among all quantities and thus all three-dimensional forms in the material world.

The *Compendium* ends with a section on pyramidal and oblong bodies (Chapters 58-68),[389] that precedes a chapter discussing the measurement of regular and other bodies derived from these (Chapter 69). Here, Pacioli closely follows the geometrical procedures of quantification commonly found in manuals of practical geometry, adding (following Book XII of Euclid's *Elements*) the relationships of magnitudes among geometrical bodies having the same base and height. The chapter following this section (Chapter 69) concerns the measurement of regular and other bodies derived from these [*depe(n)de(n)ti*].[390] For the measurement of regular bodies, here called "the most excellent and perfect" among all bodies, Pacioli refers back to the section of the *Summa* where "sufficient about these was said", adding that many copies of the *Summa* could be found in Milan at the time he was writing.[391]

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[387] In the same section, discussing the works of Melozzo da Forlì and Marco Palmezzano, Pacioli added that "they always with the level and compass proportionate their works with admirable perfection. In this way, they are represented to our eyes not as human but as divine, and to all their figures only life seems to lack". "Quali sempre con libella e circino lor op(er)e p(ro)potiona(n)do a p(er)fection mirabile conducano. In modo che non humane ma divine negli occhi n(ostr)ri saprese(n)tano. E a tute lor figure solo el spirito par che ma(n)chi."Pacioli, *Summa,* 2r (Epistola).

[388] Alberti, *De re aedificatoria*, II.1, IX.11. The "authorial" status bestowed upon the architect by Alberti in *De re aedificatoria* juxtaposed to the expertise of the executors of the building has been examined by Mario Carpo. Carpo, *The alphabet*, 22-23.

[389] These are, as explained by Pacioli, bodies that are longer, or taller, then large such as various types of columns.

[390] "De la mesura de tutti li altri corpi regulari e depe(n)de(n)ti" Pacioli, *Divina proportione (1509)*, 21v (Compendium).

[391] "Onde de dicti regulari non mi curo altrame(n)te q(ui) extenderme p(er) haverne gia co(m)posto p(ar)ticular tractato a lo illustis(simo) affine de v. D. Celsitudine Guido ubaldo Duca de Urbino nella n(ost)ra op(er)a a S. S. Dicata e al lectore facile a q(ue)lla fia el recorso p(er) essercala co(m)e utilita pervenuta co(m)mo dena(n)ce fo detto. E i(n) q(ue)sta vostra inclita cita asai se ne trovano. La cui mesura ta(n)to e piu speculativa qua(n)to piu degli altri corpi sonno q(ue)lli piu excelle(n)ti e p(er)fecti. Materia certama(n)te da coturno e no(n) da sciocco. E in q(ue)l luogo a sufficientia ne fo detto." ibid.

Concerning the quantities of other bodies derived from the regular (Pacioli later makes it clear that he is referring to truncated or elevated (stellated) bodies), he refers to the measurement of pyramids to be added or deducted from the "perfect" (i.e. regular) version of the body, the quantity [*q(uan)tita*] (volume) of which was measured in the *Summa*.[392] The number of pyramids to be deducted or added is linked to the number of bases for each body, while all the pyramids would be of the same quantity (volume).[393] This precedes Pacioli's reference to the set of perspectival representations and physical models of regular and other bodies derived from these found in the last chapter of the *Compendium.*[394] Overall, in this last section Pacioli argues that the quantitative properties of these bodies, discussed in both the *Compendium* and in the *Summa,* are the same as those visible in the set of physical models and perspectival representations presented to Duke Ludovico Sforza.[395]

3.1.3 Piero della Francesca and Luca Pacioli's quantification of bodies in the *Libellus in tres partiales tractatus divisus q(ui)nque corpor(um) regularium et depe(n)de(n)tiu(m) […]*[396] of the *Divina proportione (1509)*

The treatise examined in this section was collated by Pacioli with the *Compendium de divina proportione* in the printed edition of 1509. It is well-known that the treatise is a translation in Italian of the *Libellus* on the five regular and other "irregular" bodies, completed by Piero

[392] "Ma al mo(do) de li altri da q(ue)lli depe(n)de(n)ti sia simile a q(ue)llo che de le pyramide corte fo dato, cioe che bisogna redurli a li suoi totali p(er)fecti e q(ue)lli p(er) le regole n(ost)re date al luogo detto co(n) diligenza mesurarli, e q(ue)lla q(uan)tita serbare e poi al supleme(n)to facto al suo i(n)tero da parte p(er) le regole de le piramidi ancora mesurare. E q(ue)l che fa cavare de la q(uan)tita de tutto el suo regulare el rimnae(n)te fia ap(p)onto la q(uan)tita de dicto depe(n)de(n)te." ibid.

[393] Pacioli adds here that is important to be able to acknowledge the whole process of construction that determines the form and of each body, since all measurements of quantities "always follows from the practice." "E p(er)o una mesurata subito p(er)q(ue)lla l(')ltre tutte sie(n) note secondo el num(e)ro che a lor lati over basi o altri se posto p(er)o el quale bisogna i(n) la pratica sempre regerse." ibid.

[394] Chapter 70 is significantly entitled: "How you will find all the aforementioned *bodies* in order as they are arranged and made in perspective and in their material forms according to their particular table placed in public display". "Commo se habino retrovare tutti li dicti corpi ordinatamente comme sono posti in questo facti in prospectiva: e anchora le loro forme materiali […]". Pacioli, *Compendium*, 83r.

[395] Pacioli, *Summa*, 68v (Pars secunda). The construction of a set of models is further confirmed by a document dated 30 August 1502 where Pacioli received a payment of 52 lire and 9 soldi for a set of wooden models sold to the city of Florence. Archivio di Stato di Firenze, Operai di Palazzo, 10, 67r. Ulivi, "Luca Pacioli: una biografia scientifica," 26.

[396] "Libellus in tres partiales tractatus divisus q(ui)nque corpor(um) regularium et dependentiu(m) active perscrutationis, D. Petro Soderino principi perpetuo Populi Florentini a M. Luca Paciolo Burgense Minoritano particolariter dicatus" The treatise has a separate numeration in the *Divina proportione (1509).*

della Francesca in the 1480s. Piero's treatise survives in a single manuscript copy.[397] Since the treatise is often discussed as a work by Piero, it is seldom analysed for its correlations to the arguments that Pacioli was developing in his writings. However, it will be argued here that the treatise should also be considered as a further development in the process of the quantification of regular and other bodies derived from those initiated by Pacioli in the *Summa* and continued in the *Compendium*.[398]

Although named *Libellus in tres partiales tractatus divisus* by Pacioli, the treatise is divided into four parts with separately numerated cases, most of which were already presented in Piero's *Trattato d'abaco*.[399] Again, following traditional treatises on commercial mathematics and practical geometry, the *Libellus* is organised into problems to be resolved through numerical quantities. The contents of the treatise can be read as a logical progression in the study of regular bodies. As such, the *Libellus* is rightly regarded as Piero's highest mathematical achievement, and the culmination of the process of the quantification of regular and other three-dimensional bodies started in the *Trattato d'abaco*.

[397] Codex Urb. lat. 632, Biblioteca Apostolica Vaticana, Rome. Brüggen Israëls recently argued that the analysis of the drawings suggests that Piero made all the illustrations in the volume while the text was transcribed by unknown scribes. Brüggen Israëls, *Piero della Francesca*, 243. The *Libellus* was probably completed by Piero della Francesca in the early 1480s and presented to Duke Guidobaldo of Urbino after the death of his father in September 1482. The text was probably written firstly in Italian, and was then translated in Latin; only one copy survives. Piero's Latin manuscript laid unpublished until 1916, however it was included by Luca Pacioli in a vernacular Italian version in the printed edition of the *Divina proportione* (1509), with no mention of Piero's authorship. Although this would later bring accusations of plagiarism to Pacioli, this was certainly instrumental to the circulation and epistemological role of the text. Pacioli undoubtedly recognised its value and importance and had probably access to Piero's original manuscript in vernacular Italian. As noted by Banker, here Piero demonstrates his understanding of being in the process of creating a 'novel mathematical method', one that combined the prose propositions and procedures of the Latin elite culture of the university with the tradition of the schools of *abacus*. Banker, *Piero della Francesca*, 199-205; Brüggen Israëls, *Piero della Francesca*, 241-249.

[398] Indeed, if the treatise was prepared by Piero della Francesca as a presentation copy in vellum, written in Latin and dedicated to a learned patron, it was collated by Pacioli after the *Compendium de divina proportione,* and a short treatise on architecture [*Tractato de l'architectura*], in the printed edition of the *Divina proportione*. Together with the *Libellus*, Pacioli added a short treatise on architecture (*Tractato de l'architectura*-discussed in Chapter 4 of the thesis) and an illustrated set of capital letters drawn with ruler and compass. Pacioli, *Divina proportione (1509).*

[399] As noted by Banker, the *Libellus* presents 140 problems and 174 geometrical diagrams. Of the problems, 88 were already present in Piero della Francesca's *Trattato*; most of these were placed in the first two parts of the *Libellus*. Banker, *Piero della Francesca*, 202.

The first section of the *Libellus* deals with planar geometry and considers the types of polygons involved in the formation of regular bodies, which are equilateral triangles, squares and pentagons. In the second section the five regular bodies are introduced, and each is presented as inscribed in the sphere. The numerical quantities of their sides, surfaces and volumes are here considered in their quantitative relationship to the internal diagonal of the bodies that is also the diameter of their circumscribed sphere. The third section investigates the quantitative relationship of sides, surfaces and volumes resulting from the inscription of a regular body inside another regular body, with both sharing a common side, or with a vertex of one body dividing the side of another.[400] The fourth and final section considers other bodies beyond the regular bodies and derived from these by truncation. Five of the bodies discussed in this section are now known as semi-regular or Archimedean bodies. As is often noted, no mathematical text had dealt with the geometrical properties of these bodies since Late Antiquity, and these sources were not available in the environments in which either Piero or Pacioli operated.[401] Additionally, two cases proposed in this section examine bodies that are not strictly semi-regular, but that can still be studied as inscribed on the sphere, the seventy-two-bases body and the body derived from the intersection of two cylinders.

Piero della Francesca was certainly familiar with these last two bodies since he had applied similar geometries to construct architectural elements in perspective in his treatise *On*

[400] These problems were not included in the *Trattato* and roughly follow the order that could be found in Book XV of Euclid's *Elements,* cited in twelve occasions. Banker, *Piero della Francesca*, 202.

[401] In each semi-regular body, the surfaces are arranged identically around each vertex and these bodies can still be inscribed in the sphere with all their vertices touching the sphere's surface. In the *Trattato d'Abaco* Piero delle Francesca had already discussed two semi-regular bodies, and Pacioli had also reproposed them in his *Summa*. However, four more semi-regular bodies were added in the *Libellus* and derived by truncation of the vertices of the five regular bodies; in this way, the *Libellus* presented a semi-regular body derived from the truncation of each of the regular bodies. As noted by scholars, there were only few accounts on the description and even less on the construction of semi-regular bodies in Antiquity and Late Antiquity. These were related to the work of Archimedes and the Greek mathematician Pappus. Pappus of Alexandria lived between the third and fourth century AD. However, his writings were not available during the Middle Ages in the West and will not be until at least the sixteenth century. In the same section of Piero's *Libellus* are also discussed a series of miscellaneous cases concerning pyramids, spheres and polygons.

Perspective in Painting [*De Prospectiva Pingendi*], completed before the *Libellus*.[402] It is often noted that Piero, in his dedicatory letter addressed to the duke of Urbino, was willing to have the *Libellus* placed beside his treatise on perspective in the duke's library, and this may be taken as an indication of the relationship that Piero envisaged between the two works.[403] However, perspectival representations of regular or other bodies are not provided in the *Libellus*. Independent of Pacioli's direct knowledge of Piero's *Libellus* before its publication in the *Divina proportione* in 1509, it is only in Pacioli's *Compendium* that all regular bodies were represented in perspective.

As Pacioli clearly believed the *Libellus* to be a fitting addition to his *Compendium* and *Trattato de l'architectura*, an overview of two cases presented in the *Libellus* is important in examining the arguments in other writings by Pacioli. The first case presented in the fourth section concerns a body with seventy-two bases with the longest of its sides known. The problem asks "to find the diameter of the sphere that circumscribes the body and the superficial area [of the body]."[404] As noted at the outset of the problem, "the construction of this body was demonstrated by Campanus in the 14° of [book] 12 of Euclid [*Elements*],[405] but [Campanus] did not demonstrate the quantity of its sides if not with lines and did not demonstrate the

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[402] Banker proposed with good arguments for the four manuscripts of *De prospectiva pingendi* where the hand of Piero is visible, and for Piero's copy of the works of Archimedes (now codex Lat. 106 in the Riccardiana Library in Florence) the period 1477-1481. This chronology is further supported by the analysis of watermarks in the paper used for these copies. In this period, Banker noted, Piero is documented in Borgo San Sepolcro and here he could have established a *scriptorium*. Banker, *Piero della Francesca*, 185-188. James R. Banker, "A manuscript of the works of Archimedes in the hand of Piero della Francesca," *The Burlington Magazine*, vol. 147, no. 1224 (Mar, 2005): 165-169 .

[403] "Nec dedignabitur celsitudo tua ex hoc iam emerito et fere vetustate consumpto agello, unde et illustrissimus genitor tuus, uberiores percepit, hos exiles et inanes fructus suscipere, et libellum ipsum inter innumera amplissimae tuae paternaeque bibliothecae volumina penes aliud nostrum de prospectiva opusculum, quod superioribus annis edidimus, pro pedissequo et aliorum servulo, vel in angulo collocare." Della Francesca, *Libellus*, 1v. Additionally, the two works shared a passage taken from Vitruvius's *De architectura.*

[404] "Eglie uno corpo de 72 base, 24 triangulare et 48 q(ua)trangule non d(')anguli ne de lati equali che illato loro magiore cioe doi lati de ciaschuna basa e 2, domandase il diametro de la sphera che lo circumscrive et de la superficie" Pacioli, *Divina proportione (1509)*, 20r (Libellus).

[405] Proposition 17 in modern editions of the *Elements*.

surface [of the body], which is asked by the problem."[406] Considering the body as composed of seventy-two pyramids with their apex at the centre of the circumscribed sphere is fundamental in the process of resolution followed by Piero and Pacioli. The second step is to consider that right-angled triangles can be derived from each pyramid if this is cut by a plane passing through the centre of its base and apex. To illustrate this process, the body is shown in both Piero and Pacioli's version in a double orthogonal projection of plan and elevation, with a third drawing showing half of the body partially in section and partially in elevation (illus. 3.3). While aiding the understanding of the resolution process to find the quantity requested, the three illustrations also make the formal and spatial properties of the body intelligible. However, such visual aids, such as the view of the same body in perspective drafted by Leonardo da Vinci for Pacioli's *Compendium* were all unprecedented in manuals of mathematics or practical geometry.[407]

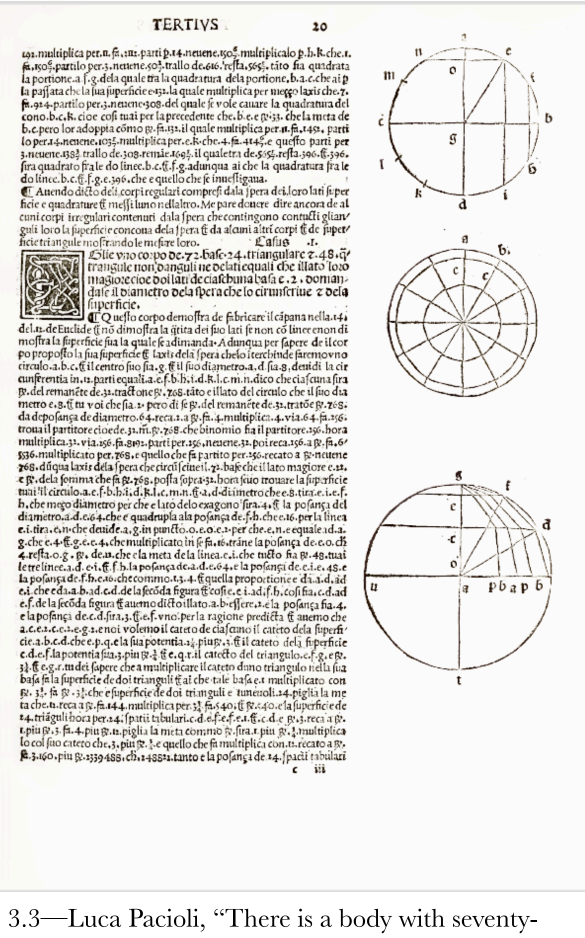
Case 10 asks instead what would be the shared volume of two columns (geometrically considered as two cylinders) crossing at a right angle (illus. 3.4). This is double the volume of a pavilion vault (also called cloister vault), and is instrumental in finding the surface of a crossing vault, enquired in Case 11.[408] The complexity and exceptional character of this problem of quantification has often been emphasised and therefore needs careful

[406] "Questo corpo demostra de fabricare il ca(m)pana nella 14 del 12 de Euclide et no(n) dimostra la q(uan)tita dei suo lati se non co(n) linee e non dimostra la superficie sua la quale se dimanda" Pacioli, *Divina proportione (1509)*, 20r (Libellus).

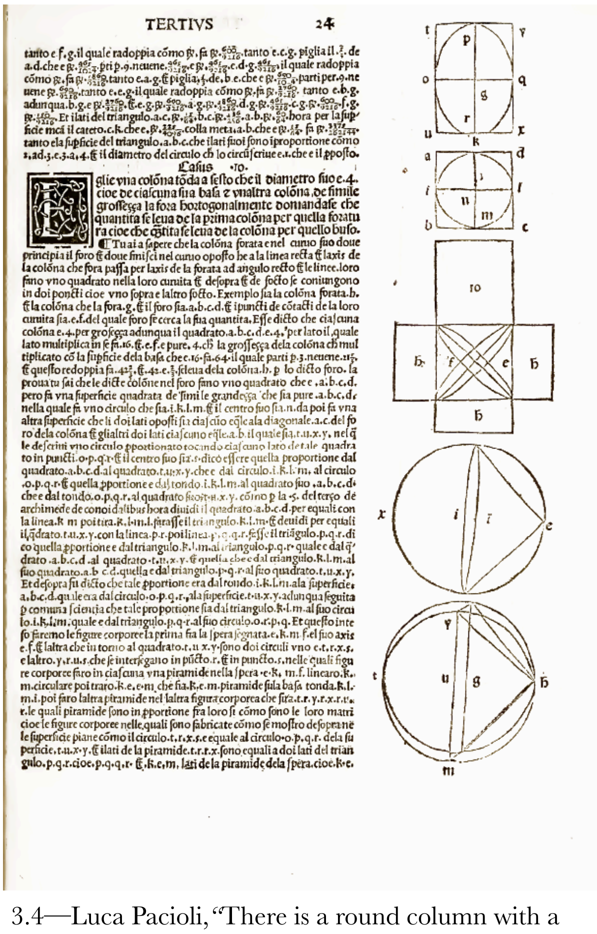
[407] Additionally, in Pacioli's *Compendium de divina proportione* this body was explicitly linked with architecture for its usefulness in vaults and domes. The passage is discussed in Chapter 4 of the thesis.

[408] Enrico Gamba, Vico Montebelli, Pierluigi Piccinetti, "*La matematica di Piero della Francesca*," *Lettera matematica PRISTEM* (2006)*:* 49-60.

consideration.[409] The result provided can be summarised (in present-day notation) in the following formula: volume shared between the two cylinders = *2/3d*^{3}, where *d* is the diameter common to the two cylinders and the side of the cube that circumscribes their shared volume. The procedure of resolution starts from the circle and the ellipse that derive from cutting the cylinders with two vertical planes rotated at 45 degrees. Triangles are then inscribed in each section. Two pyramids are then derived from the triangles (illus. 3.4-bottom), and the proportional relationships of their volumes is considered to find the volume of the double



3.3—Luca Pacioli, "There is a body with seventy-two bases […] with the longest of its sides known […]". *Libellus in tres partiales…*, *Divina proportione (1509)*, 20r.



3.4—Luca Pacioli,"There is a round column with a diameter of 4 […] and another column of similar size intersects it orthogonally; it is asked which quantity is taken away by that intersection […]". *Libellus in tres partiales…, Divina proportione (1509)*, 24r.

[409] Independently from the ancient mathematician, Piero and later Pacioli revived a problem previously discussed by Archimedes. Banker, *Piero della Francesca*, 204-205. Archimedes stated on *The Method* to have sent the same problem to Eratosthenes, however the treatise had disappeared in the Middle Ages to resurface only in 1906. Notably, whether Piero had access to commentaries that mentioned the problem is not know and he does not mention Archimedes in the passage. Marshall Clagett in his comprehensive work on Archimedes in the Middle Ages admitted to be 'profoundly puzzled' be the achievements of Piero in the construction of the Archimedean irregular polyhedra and development of solutions to Archimedean problems. Marshall Clagett, *Archimedes in the Middle Ages*, The Medieval Archimedes in the Renaissance, 1450-1565, vol. III, (Philadelphia PA: American Philosophical Society, 1978), 405-407.

vault. This can be summarised as follows: volume (square pyramid) : volume (double vault) = volume (cone) : volume (sphere).[410] As noted in the text, for the "XXXIII primi Sperae et Coni Archimedis" the volume of the sphere is four times the volume of the inscribed cone that has the biggest circle (in the sphere) as its base and half the diameter of the sphere as its height; therefore, Piero concluded, the volume of the double pavilion vault is four time the volume of the square pyramid.[411] The process as stated by Piero and Pacioli clearly requires the prior mental visualisation of the three-dimensional shapes derived from planar sections cut through the intersecting cylinders before the geometrical solution can be devised. In addition, as in the previous case, it is noteworthy the role played by triangular sections in the process of quantification. Basically, both resolution processes develop from the proportional relationship of two-dimensional polygons understood as triangular sections of pyramids or cones, as discussed in Books VI and XII of Euclid's *Elements.*

Overall, problems of quantification of three-dimensional bodies are often resolved in the *Libellus* by first reducing them to a series of pyramids studied through their planar sections. The parallel with the theory of perspective described by Piero is fitting here, since both procedures examine the three-dimensional material extension of space by means of sections cut through pyramids. However, Piero did not explicitly establish this connection in his writings, but rather discussed the two approaches in separate works; whereas Pacioli clearly emphasised the link between the formal, spatial and visual properties of regular and semi-regular bodies, demonstrated by their geometrical analysis and quantification. As noted, these

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410 Gamba, Montebelli, Piccinetti, *La matematica di Piero della Francesca*, 49-52.

411 Della Francesca, P. (1480-1485 c.), *Libellus de quinque corporibus regularibus*, ms Urb. Lat 632, Rome: Biblioteca Apostolica Vaticana, folio 60v; Pacioli, (1509) *Divina proportione*, *Libellus in tres partiales*…,, 24v. This gives the volume of the double vault $V = 4(1/3d^2 \times d/2)=4/6d^3=2/3d^3$. The resolution process is discussed in Gamba, Montebelli, Piccinetti, *La matematica di Piero della Francesca*, 49-52. Although the text seems to refer to a proposition in Archimedes's *Conoids and Spheroids* (Sperae et Coni), the description points instead to the first book of Archimedes's *On the Sphere and the Cylinder,* proposition 34 in modern editions. Reviel Netz, *The works of Archimedes*, Translation and Commentary, vol. 1 (Cambridge: Cambridge University Press, 2004), 148-153.

correlations were already clearly established in Pacioli's writings, regardless of his knowledge of Piero's *Libellus* before 1509.

In the context of Pacioli's texts, a relationship between a quantitative analysis of regular and other bodies derived from these and their visual perception was already introduced in the second part of the *Summa* and finds its clearest expression in the *Compendium*. The formal properties of regular and other bodies derived from these that can be inscribed in a sphere with all their vertices touching the sphere, hold spatial properties that Pacioli aimed to demonstrate in visual terms. This analysis will now focus on the visual properties of regular and other bodies derived from these in Pacioli's works, firstly by examining arguments involving the theory and practice of perspective.

3.2 Visualisation

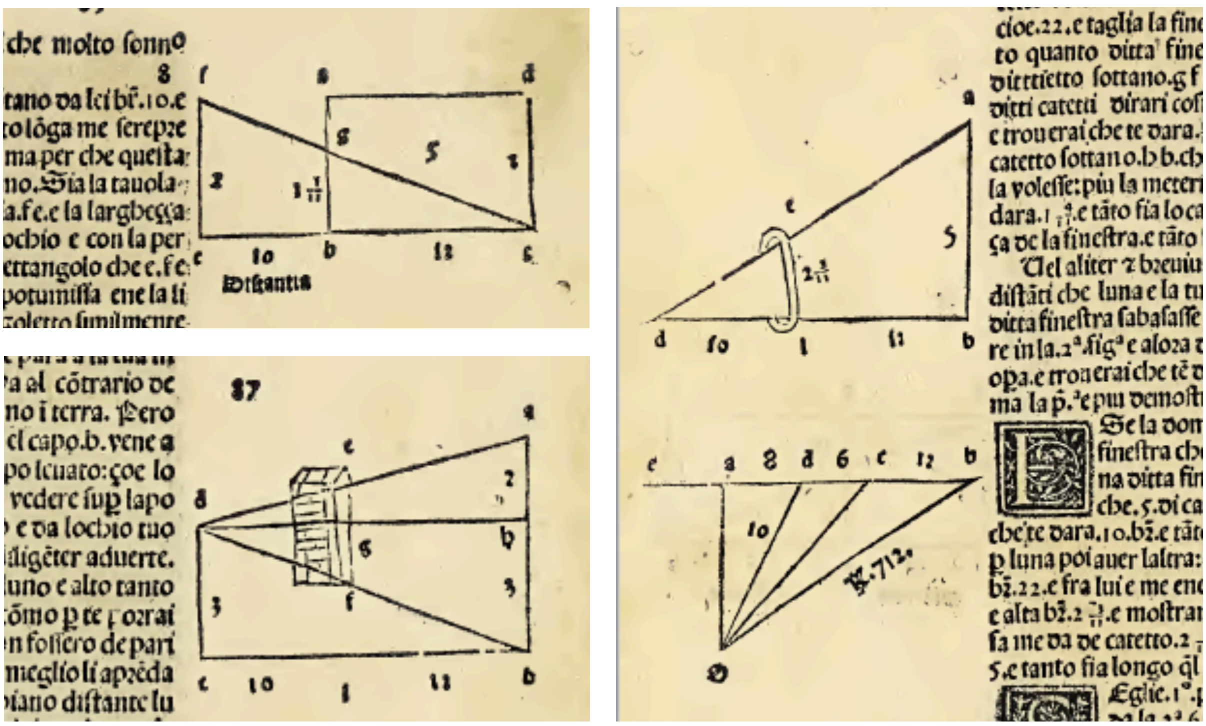
3.2.1 The mathematical and phenomenal nature of perspective in Pacioli's *Summa* and *Divina proportione*

In the eighth distinction of the second part of the *Summa*, which is devoted to geometry, Pacioli introduces perspective in problems concerning measuring by sight. Following the tradition of manuals of *practica geometiae,* here, perspective is used for problems involving the measuring of distances and heights. The first problem begins thus:

> There is a table [that is] 12 *braccia* long and 2 *braccia* wide [which lays] plane on the ground, I am [distant] from it 10 *braccia*, and [the distance] from my eye to the ground, that is my height, is 2 *braccia*. I ask how long the aforementioned table will be when represented to me. You should know that this a question of perspective

> [*perspectiva*], but because this science is subordinate to geometry and arithmetic we will solve it […][412]

Pacioli's solution employs the definition and proportional relationships of similar triangles found in Euclid's *Elements,*[413] which is also used in the other problems of measuring by sight that follow in the text (illus. 3.5). An additional problem asks for the height and distance of a window that acts as an intersection of the visual pyramid, "knowing that in painting the height of a man is usually taken as 3 *braccia*". Clearly, the relationship can be reversed, and the real dimensions of the object seen can be found by knowing its length at the intersection of the visual triangle. Following traditional manuals of practical geometry, these problems highlight two fundamental elements of the theory of perspective known to Pacioli: the eye is considered as the apex of a bundle of straight rays that touches the object seen and forms a



3.5—Luca Pacioli, Problems concerning measuring by sight (perspective). Tractatus geometriae, Distinctio octava, *Summa de arithmetica proportioni et proportionalita,* 65r-66v (Pars secunda) [details].

[412] "Egli è una tavola longa braccia .12. larga braccia .2. piana in terra e io sto da lei braccia .10. e da l'occhio mio a terra cioè la mia statura è braccia .2. Dimando quanto longa me se representa dita tavola. Sapi che questa domanda è de perspectiva ma perché questa scientia è subalternata a geometria e aritmetica si la solveremo…" Pacioli, *Summa*, 65r (Pars secunda).

[413] The theory of proportional similarity is discussed in Book VI of Euclid's *Elements.*

pyramid; and the intersection of the visual pyramid as the plane of representation, where the proportional diminution of the object's real dimensions is measured.[414] This is not surprising, as the methods of measuring by sight had already been included in Euclid's *Optics*,[415] and circulated in other treatises on practical geometry. As seen, they were also discussed in another distinction of the *Summa.*[416] However, as in other treatises on practical geometry,[417] these methods were here integrated with the theory of proportional relationships of similar triangles found in Book VI of Euclid's *Elements.* Crucially, this could account for quantities in proportional ratios in the plane of perspectival representation.

Following these six practical cases in the *Summa*, Pacioli theoretically elaborates on perspective in Chapter 3 of the *Compendium.* Here, perspective is listed among other "mathematical sciences and disciplines": "Arithmetics, Geometry, Astrology, Music, Perspective, Architecture, Cosmography and other that are derived from these".[418] The text continues thus:

> Nonetheless commonly for the sages the first four are taken, that is Arithmetic, Geometry, Astronomy and Music, and the others are said to be subordinated, that is dependent from these. This is what is mentioned by Plato and Aristotle, Isidore in his

[414] Although the connection of Alberti's perspectival construction with surveying has been highlighted by Martin J. Kemp and William M. Ivins among others, it should be noted here that in the case of Pacioli this is explicitly stated. Martin J. Kemp, *The Science of Art: Optical Themes in Western Art from Brunelleschi to Seurat* (New Haven: Yale University Press, 1990); William M. Ivins, . Ivins, *On the Rationalisation of Sight, with an Examination of Three Renaissance Texts on Perspective…* (New York, 1973), reprinted from Papers of the Metropolitan Museum of Art, 8 (New York: The Metropolitan Museum of Art, 1938).

[415] Euclid's *Optics* includes four propositions (18-21) that discuss surveying by sight: "to know how great is a given elevation when the sun is shining", "to know how great is a given elevation when the sun is not shining", "to know how great is a given depth", and "to known how great is a given lenght". Euclid, "The Optics of Euclid," translated by Harry Edwin Burton, *Journal of the Optical Society of America* vol. 35, no. 5 (May 1943): 360-361. For the compatibility between Euclid's *Optics* and Linear Perspective: C. D. Brownson, "Euclid's *Optics* and Its Compatibility with Linear Perspective," *Archive for History of Exact Sciences* vol. 23, no. 3 (1981): 181-193.

[416] Pacioli, *Summa*, Distinctio septima (Pars secunda)

[417] The method of measuring by sight with a line cutting the visual triangle was described in a fourteenth-century treatise called *De visu* prepared by Grazia de' Castellani. Notably this has been transmitted, although in shortened form, in a treatise compiled by a master of abacus (codex Ottoboniano Latino 3307, *Libro di praticha d'arismetrica*, Biblioteca Apostolica Vaticana, 172r). Additionally, as noted by Filippo Camerota, in a treatise on arithmetics completed c. 1410, the author defines practical geometry as the discipline by which "we will measure through perspective [*p(erspecti)va*], with the eye, the unknown quantity of a thing" [*noi mesureremo per p(erspecti)va a occhio la quantitade dubia duna cossa*] *Aritmetica* (ca. 1410), Biblioteca Nazionale Marciana di Venezia, It. IV. 497 (5263); 16r-19v. Camerota interpreted *pva* in the passage above with *perspectiva.* Camerota, *La Prospettiva,* 32, 54n, 65; Gino Arrighi, *Un estratto del "De visu" di M. Grazia de' Castellani* (Firenze: Tip. Baccini e Chiappi, 1967), 54.

[418] Pacioli, *Compendium*, 9v.

> 'Etymologie' and Severinus Boethius in his 'Arithmetica'. But in our opinion, even if it is shallow and imbecile, three or five should be included, that is Arithmetics, Geometry and Astronomy, excluding Music for the same reasons as Perspective, or [alternatively] Perspective should be added to the first three for the same reasons as Music. If they say that Music pleases hearing, one of the natural senses, [Perspective] [pleases] sight that is even more worthy for being the first door of the intellect; if [Music] follows audible numbers and measures applied to the time of its prolations, [perspective] [follows] the natural number according to every definition and measure of the line of vision. If that [Music] pleases for the soul through the use of harmony, this [Perspective] pleases very much with [seeing things] at the right distance and with a variety of colour; if [Music] considers the harmonic proportions, [Perspective] the arithmetic and geometric ones.[419]

The primacy accorded to sight and the arguments Pacioli brought to support the comparison, and the precedence of perspective over music, are essential here. In writings that follow this passage, Pacioli highlights how much "painting resembles nature",[420] and mentions examples of paintings that resembled reality so closely as to deceive their viewers. Pacioli mentions here the last supper of Leonardo "where with actions and gestures the one to the other and the other to the one, with lively [*viva*] and troubled admiration [they] seem to speak".[421] Another example describes the basket of grapes drawn by Zeuxis that deceived the birds as real, and the veil painted by Parrhasius believed real by Zeuxis, both described by Pliny the Elder.[422] Nonetheless, Pacioli adds that the representation painted by Parrhasius is certainly the most noteworthy, since it was able to deceive not animals but a rational human, and a master of

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[419] "Non di meno comunamente per li savii, le quatro prime se prendano, cioe Arithmetica, Geometria, astronomia e musica e l'altre sieno dette subalternate cioe da queste quatro dependenti. Cosi vol Platone e Aristotile e Isidoro in le sue ethimologie el Severin Boetio in sua Arithmetica. Ma il nostro iudicio benche imbecille et basso sia o tre o cinque ne costringne: cioe Arithmetica, Geometria e astronomia, excludendo la musica da dicte per tante ragioni quante loro de le 5 La prospectiva e per tante ragione quella agiongnendo a le dicte quatro per quante quelli a le dicte nostre 3 la musica. Se questi dicano la musica contentare l'udito uno dei sensi naturali e quella el vedere quale tanto e piu degno quanto egli e la prima porta al(l)'intelletto se dicano quella satende il numero sonoro e a la misura importata nel tempo de sue prolationi. E quella al numero naturale secondo ogni sua diffinitione e a(l)la misura de la linea visuale. E questa per debita distantia e varieta di colori molto delecta. Se q(ue)lle sue armoniche proportioni considera e questa le arithmetrici e geometrici" Pacioli, *Compendium*, 9v-10r.

[420] Pacioli, *Compendium*, 10r.

[421] "Dove, con acti e gesti l'uno a l'altro e l'altro a l'uno con viva e afflicta admiratione par che parlino" Pacioli, *Compendium*, 10r-10v.

[422] This is the well known painterly contest between Parrhasius and Zeuxis told by Pliny the Elder's in his *Naturalis Historia*e. Pacioli mentions Pliny's *De Picturis* as his source. Pacioli, *Compendium*, 10v.

that technique.[423] Overall, however, the argument Pacioli develops in this section is not only that perspectival representations are based, like music, on mathematical principles, but that perspective produces an effect on the viewer similar to those of the actual objects, since vision itself is organised according to mathematical principles.

If the remaining sections of the *Compendium* are considered, while pointing to the perspectival representations of regular and other bodies drafted by Leonardo da Vinci, Pacioli often emphases the visual clarity that comes from these. When perspectival representations based on physical models are provided, Pacioli limits the description of regular and other bodies derived from these to the type and number of surfaces, lines and angles that concur with the form, adding that this is "made clear by its material form"[424] or that "the experience of its material (form) will satisfy the eye".[425] Additionally, a comment added to Pacioli's translation of Piero della Francesca's *Libellus*, laments the lack of someone as "skilled in perspective" as Leonardo. This is the only reason, Pacioli argues, why he did not provide a visual representation of all the geometrical bodies described in the text.[426]

Continuing the survey of explicit references to the theory and practice of perspective in the *Summa* and *Divina proportione*, it should be noted that Pacioli mentions Piero's treatise on perspective in painting [*De prospectiva pingendi*] (completed in the 1470s and circulated both in

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[423] Pacioli, *Compendium*, 10r.

[424] "Co(m)mo la fo(r)ma sua materiale a noi fa manifesto." Pacioli, *Divina proportione (1509)*, 15r (Compendium) or "Che tutto ancora la p(ro)pria forma sua ma(teria)le fa ap(er)to" ibid.

[425] "E in q(ua)lu(n)che modo q(ue)sto se getti in spatio pia(n)o semp(re) in su(e) 3 po(n)te o co(n)i pyramidali si ferma che la experie(n)tia del suo ma(teria)le ancora a l(')ochio satisfara." Pacioli, *Divina proportione (1509)*, 16r.

[426] "Reader you should not be surprised if not always is presented in the margin the figure of similar bodies composed of various bases because these are difficultly rendered by a drawing, and [therefore] should be made by the hand of someone skilled in perspective and these are not always available, as was done for his humanity by Lionardo da Vinci (sic), being [both] in Milan and both employed by the most excellent lord Duke Ludovico Maria Sforza […]" "Lectore non te maraviliare se de simili corpi composti di diverse e varie base non te se mette sempre in margine loro figure conciosia che le sieno difficilime farle in desegno, per che bisogna che sieno facte per mano de bono perspectivo quali non si possano sempre havere a sua posta, s. comme per sua humanit. feci el nostro Lionardo da Vinci, siando a Milano, ali medesimi stipendii delo excellentissimo signor Duca di quello Ludovico Maria Sforza etc." ibid., 22r (Libellus).

Latin and Italian) in four instances.[427] Although rarely considered by scholars, these instances are important and should be examined in their contexts and for their purposes in Pacioli's arguments. If the wider context in which the remarks were made is considered, it can be noted that Pacioli establishes a link between perspective and architecture in all four cases, and that the link is sustained by mathematical arguments.

Pacioli mentions Piero twice in both the *Summa* and the *Tractato de l'architectura* included in the printed edition of the *Divina proportione*. However, the four passages have seldom been considered together. It has been noted that in all instances, Pacioli is adamant in praising Piero's skills in perspective and painting, but apart from a brief and passing occurrence,[428] never mentions either his mathematical writings or his achievements.[429] In the dedicatory letter of the *Summa*, after acknowledging the crucial role that proportions hold in the treatises on architecture by Alberti and Vitruvius and the "beautiful order of the arrangement [*dispositione*]" of the ducal palace of Urbino "that no language can express if not its own", Pacioli significantly adds that "perspective, if you pay attention, would be [worth] nothing if not able to accommodate these, as fully demonstrated by the monarch of painting, *Pietro de*

[427] While Piero della Francesca's *Trattato d'abaco* and *Libellus* are known in a single manuscript copy, the treatise on perspective (*De prospectiva pingendi*) survived in seven manuscript versions, four in Latin and three in Italian. *Trattato d'abaco* (Florence, Biblioteca Medicea Laurenziana, MS. Ashb. 280/359-291), the *Libellus de quinque corporibus regularibus* (Rome, Biblioteca Vaticana, MS. Vat.Urb.lat. 632), and the *De prospectiva pingendi:* Parma, Biblioteca Palatina, Ms. 1576 (Piero della Francesca 1899, 1942), Milano, Biblioteca Ambrosiana, Cod. D 200 inf., (XVI cent., without drawings, Reggio Emilia, Biblioteca Municipale A. Panizzi, Cod. Reggiano A 41/2 (before A 44), (XV cent.), (with autograph drawings and notes) in Italian; Milan, Biblioteca Ambrosiana, Cod. Ambr. C. 307 inf., (XV cent.) (with autograph drawings and notes), Bordeaux, Cod. 616, (XV cent.) (with autograph drawings and notes), London, British Library, Cod. Add. 10366, (XV cent.), Parigi, Bibliothèque Nationale, Cod. Lat. 9337, (XVI cent.) in Latin.

[428] "Che dele mathematici lo rende chiaro el monarca ali dì nostri della pictura e architectura […]" Pacioli, *Divina proportione (1509)*, 23r (Tractato de l'architectura), 35r. The passage is further discussed later in this section.

[429] James R. Banker, "Piero della Francesca e Luca Pacioli: Maestro e Alunno?", 45-70. Given that Pacioli clearly borrowed from Piero's mathematical writings, this certainly appears as a deliberate choice on his part to bolster his authority in mathematics. As well highlighted by James Banker, although Pacioli extensively borrowed from his mathematical texts, he never mentions Piero as a geometrician.

franceschi, our fellow countryman […]".[430] This is the first instance where Pacioli links architecture and perspective. It is significant that it occurs in the dedicatory letter that opens the *Summa,* which introduces the role of proportions in mathematics. Later in the text, Pacioli argues that proportion is the "mother and queen of all arts and none could be practiced without it", in a passage often quoted by scholars. However, he continues that this is proved by perspective in painting and gives the example of a human figure that in painting should be given the right size "in the eyes of the one that looks at it".[431] In other words, the size of the figure should be proportionate, as the rest of the painting, to the distance of the observer. Pacioli continues by underlining the importance of good proportional dispositions of colours according to their power [*potentia*], reiterating that shapes and their colours should be proportioned according to the distance from where they are seen, and adding that this also applies to the "lineaments and dispositions [*liniamenti e dispositioni*] of any other shape". Importantly, while *lineamenti* refers here to outlines in paintings, Pacioli also adds *dispositioni* to denote the proportionate arrangement of lines and colours that contribute to the whole

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[430] "A l'architettura ancora. Vitruvio in suo volume. E Leon Battista degli Alberti fiorentino in sua p(er)fetta op(er)a de architectura molto d(e)mostrano esserli accomodata p(ro)portiona(n)do suoi magni et excelsi hedifitii. Si co(m)e al di n(ost)ri la nova luce de ytalia. Admiranda fabrica del degno pallazzo de V. D. S. p(er) la felicissima sua paterna memoria initiata e co(n)sumata manifesta. Nel qual certamente ben mostro natura sua forza e arte fra quanti p(er) altri sienno stati visti. Qual lengue el bell'ordine de tutta sua degna dispositione poria exprimere certo Niuna seria bastante. Se non la sua eloquentissima, el qual non solo alla vista subito veduto piaci, ma ancor piu riman stupefatto chi con intelletto va discorrendo con quanto artifitio e ornamento è stato composto. La perspectiva se be(n) si guarda senza dubio nulla serebbe se queste non li se accomodasse. Co(m)e a pieno demostra el monarcha d(e) la pictura maestro Pietro di franceschi n(ost)ro co(n)terraneo […]" Pacioli, *Summa*, 2r (Epistola).

[431] "Se tu ben discorri in tutte l(')arti: tu troverai la proportione d(e) tutte esser madre e regina: e sença lei Niuna poterse exercitare. Questo el prova prospectiva in sue picture. Le quali se ala statura de una figura humana: non li da la sua debita grosseçça ne gli ochi de chi la guarda, mai ben responde. E ancora el pictore mai ben dispone suoi colori se non attende a la potentia de l(')uno e de l(')altro, cioe che ta(n)to de bianco (verbi gratia p(er) incarnare) over negro o giallo etc. voltando de rosso etc. E cosi nelli piani dove hano a postare tal figura, molto li co(n)vene haver cura, de farla stare co(n) debita proportione de dista(n)tia. E cosi li panni che li fanno, venghino con debito modo a parere. E cosi facendo una figura a sedere sotto qualche fornice, bisogna che la proportionino in tal modo, che levandose in piedi, ella non desse del capo fore del coperchio. E cosi in altri liniamenti e dispositioni de qualunche altra figura si fosse. Del qual documento, a cio ben s(')abino a disponere. El sublime pictore (a li di nostri ancor vivente) maestro Pietro de li Franceschi nostro conterraneo del borgo sa(ncto) sepolchro, ha ne i(n) questi di co(m)posto degno libro de ditta prospectiva. Nel qual altamente de la pictura parla, ponendo sempre al suo dir ancora el modo e la figura del fare. El quale tutto habiamo lecto e discorso, el qual lui fece vulgare, e poi el famoso oratore, poeta, e rhetorico, geco e latino (suo assiduo co(n)sotio, e similmente co(n)terraneo) maestro Matteo lo recco a l(e)ngua latina ornatissimamente de verbo ad verbum, con exquisiti vocabuli. Ne la quale opera de le 10 parole le 9 recercano la p(ro)portione. E cosi co(n) instrume(n)ti li insegna proportionare piani e figure, con quanta facilita mai si possa, e vie apertissime etc." Pacioli, *Summa*, 68v (Pars prima).

figure. Interestingly, an analysis of the use of the terms in Piero's *De prospectiva pingendi* shows that *lineamento/i* is used four times and always refers to an outline traced by the means of lines; however, there are no occurrences of *dispositione/i,* nor was the term used in the other two mathematical works by Piero.

Following his remark that Piero had composed the most respectable book on perspective; Pacioli further argues that Piero speaks thoroughly about painting, always presenting the theory by way and through examples of the practice [*el modo e la figura del fare*].[432] Pacioli meaningfully concludes, referring again to the argument of the centrality of proportion to both perspective and painting, that "out of ten words" in Piero's treatise, "nine refer to proportion".[433] At this point, Pacioli establishes a link between the role of proportions in painting and architecture. The passage opens with a general reference to the works of Vitruvius, Dinocrates, Frontinus and Pliny, after which Pacioli adds that if architecture "is not properly proportioned, it does not please or delight the eye, neither [is it pleasing or delightful] to inhabit it. And it is also said to be unhealthy […]"[434] The last sentence is particularly important as Pacioli establishes a clear connection between the instantiation of proportions and the aesthetic, spatial and wholesome properties of architecture. Significantly,

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[432] Pacioli adds here that the treatise was composed in *volgare* by Piero and was later translated in Latin "da verbo ad verbum" and with excellent words by Maestro Matteo. "El quale tutto habiamo lecto e discorso, el qual lui fece vulgare, e poi el famoso oratore, poeta, e rhetorico, geco e latino (suo assiduo co(n)sotio, e similmente co(n)terraneo) maestro Matteo lo recco a l(e)ngua latina ornatissimamente de verbo ad verbum, con exquisiti vocabuli." Pacioli, *Summa*, 68v (Pars prima).

[433] Pacioli adds that Piero della Francesca teaches in the treatise "to proportionate planes and figures, with as much ease as much possible, and with accessible methods". It is interesting that Pacioli does not mention bodies in the passage. As we will see, these are a central geometrical theme in Piero's treatise. The omission of Pacioli may be fortuitous, or it can be read as another instance of Pacioli avoiding to openly acknowledge the achievements of Piero in geometry. Pacioli, *Summa*, 68v (Pars prima).

[434] "L(')architettura ancora nulla vale (si commo vitruvio, Dinocrate, Frontino e Plinio approbano) se debitamente non e proportionata ne a l(')ochio, ne a l(')abitare mai piaci ne diletta. E ancor dici non esser sana. E si prova anche di templi in loro formationi e constructioni, e di loro cori, l(')armonia de li divini officii poco valere, se con debita proportione non sonno disposti. E dimostra commo la co(m)muna longheçça loro, debia a la loro largheçça correspondere inducendo la forma del corpo humano a la cui similitudine vole le chiesi essere fabricate, commo al corpo del nostro redemptore Jesu Christo volse la divina bonta che Noe l(')arca fabricasse tanti co(m)biti longa larga e alta commo expone Augustino in de civitate dei. E cosi anche de palaççi e altre case da habitare, in uno e piu solari, commo se co(n)tiene ne la sua grande e difficile opera de architettura." Pacioli, *Summa*, 68v (Pars prima).

Pacioli continues that this is proved by the formations and constructions [*formationi e constructioni*] of temples [*templi*] and of their choirs [*cori*], and that their harmony cannot be achieved if not through the right proportions. While the discussion of temples is not surprising and is found both in Vitruvius's *De architectura* and Alberti's *De re aedificatoria*, the reference to choirs is Pacioli's addition. As will be seen, this is particularly relevant when considered with the other instances where choirs are discussed as instantiations of geometrical bodies in Pacioli's *Compendium*.

The last two references to Piero's *De prospectiva pingendi* occur in Pacioli's *Tractato de l'architectura*, where a correlation between perspective and architecture is again established. In his introduction to the *Tractato,* Pacioli claims to have made a *compendium* of Piero's treatise and promises to give an account of this in future publications on architecture.[435] The final mention of Piero's treatise is in Chapter 29 of Pacioli's *Tractato*. This is the first instance where Piero is mentioned for his achievements in mathematics, however Pacioli does not follow with any details about mathematical works and continues by calling Piero "monarch of painting and architecture". What ensues from this is a list of the works and places where Piero has painted. A reference to Piero's treatise on perspective and its place in the library of the ducal palace of Urbino concludes the chapter.[436] No specific architectural work of Piero is mentioned. However, Pacioli may well refer here to Piero's painted architecture and to the

[435] However, no records of this work, if ever completed, are extant. "Sì che vi prego che interim, con questo operando, non ve sia tedio l'aspectare, del qual (se pregio non adviene) sperio in breve sirete a pieno da me satisfacti. E anco con quella prometto darve piena notitia de prospectiva medianti li documenti del nostro conterraneo e contemporale, di tal facultà a li tempi nostri Monarcha Maestro Petro de Franceschi, de la qual già feci dignissimo compendio, e per noi ben apreso. E del suo caro quanto fratello Maestro Lorenzo Canozo da Lendenara qual medesimamente in dicta faculta fo ali tempi suoi supremo che'l dimstrano per tutto le sue famose opere si intarsia nel degno coro del Sancta a Padua e sua sacrestia e in Vinegia ala Ca Grande […]". Pacioli, *Divina proportione (1509)*, 23r (Tractato de l'architectura).

[436] "Che dele mathematici lo rende chiaro el monarca ali dì nostri della pictura e architectura, Maestro Piero de li Franceschi con suo pennello mentre pote comme apare in Urbino, Bologna, Ferrara, Arimono, Ancona e in the terra nostra in muro e tavola a oglio e guazzo, maxine in la cità di Arezzo la magna capella della tribuna del'altar grande una della degnissime opere d'Italia a da tutte commendate. E per lo libro de prospectiva compose qual si trova in la dignissime bibliotheca de lo illustrissimo Duca de Urbino nostro. Sìche ancora voi ingegnative el simile fare". ibid., 35r (Tractato de l'architectura).

perspectival construction of several architectural elements in *De prospectiva pingendi*, rather than to any specific building.

If all these passages are considered, Pacioli is keen to establish a linkage between painting and architecture through their common bearing on mathematical principles and specifically on proportions, expressed in both visual and spatial terms. Since proportions are the universal principle that links perspective, painting and architecture, this fully justify, as Pacioli argues, the interrelation of all the properties that can be visually perceived. As will become clear in the remaining sections of this chapter, this is instrumental in supporting Pacioli's argument for the importance of visual, formal and ultimately aesthetic properties in regular and other bodies derived from these. Piero della Francesca's approach to the theory and practice of perspective is a crucial source in this respect and should now be examined.

3.2.2 Geometrical bodies in Piero della Francesca's *De prospectiva pingendi* and their role in understanding Luca Pacioli's approach to perspective

Considering the role played by Piero's treatises throughout Pacioli's texts where regular and other bodies derived from these are concerned, an analysis of the concept of body in Piero's *De prospectiva pingendi* is essential to understand the context in which Pacioli operated. Despite its circulation and its importance as the first detailed practical manual on perspective,[437] and although it is mentioned by Pacioli in four instances, scholars have never closely examined Piero's treatise on painting in comparison with Pacioli's arguments, but have rather focused

[437] As far as we know, and as noted by Judith V. Field as it would seem as far as Piero knew, *De prospectiva pingendi* was the first treatise to give a detailed practical account of perspective. The work is clearly addressed to practitioners and follows the customs of manuals of abacus or practical geometry that circulated at the time. The geometrical procedures are described step by step and diagrams are always provided in support of the text. Cases are organised as problems to be solved, the reader is addressed with the 'tu' and drawing instructions mostly delivered in the imperative. Passages or explanatory notes of discursive character are rare. Judith V. Field, *Piero della Francesca: A mathematician's Art* (New Haven: Yale University Press, 2005), 130.

on the extent of the latter's borrowings from Piero's *Trattato d'abaco* and *Libellus*.[438] In other words, it is important to acknowledge the similarities and differences between their works on a formal, visual, spatial, and conceptual level, rather than focusing only on the extent of Pacioli's borrowings.[439]

This section will demonstrate that an analysis of the concept of geometrical bodies discussed in Piero's treatise on perspective is needed when approaching Pacioli's works. A comparison of Piero's *De prospectiva pingendi* and Pacioli's *Compendium de divina proportione* is particularly fruitful when the visual properties of regular and other bodies derived from these are examined. After an introduction dealing briefly with natural optics and the conformation of the eye, Piero considers progressively more complex geometrical constructions, as in Euclid's *Elements*. From an empirical definition of point and line in Book I, the treatise culminates with a discussion of complex three-dimensional bodies such as a capital or a human head in Book III. Piero's treatise deals exclusively with geometrical and visual properties and never indulges in symbolic, compositional or any considerations of meanings that can be related to painting. In other words, in *De prospectiva pingendi* (as in his other treatises of a mathematical character), Piero never considers what Alberti called *compositio* and *historia* in *De pictura*.[440]

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[438] I refer here to Piero della Francesca's *Trattato d'abac*o (completed in the 1450s) and *Libellus de quinque corporibus regularibus* (probably completed in the early 1480s after August 1482), both treatise survived in their original form in a single manuscript copy.

[439] Rather than only on the basis of the correspondence with their sources, as scholars have often done, the appropriations of Pacioli from Piero's *Trattato* and *Libellus* should be considered at the level of the arguments proposed. The final purpose will be to examine in which measure Piero's writings were instrumental in the development of the concepts presented by Pacioli.

[440] Composition, Alberti notes, "is that procedure in painting whereby the parts are composed together in a picture". Alberti, *On Painting*, 67 (II, 33). To this follows a reference to the 'historia', that is the narrative, the story that the picture is aimed to convey. This because, Alberti remarks, "the great work of the painter is not a colossus but 'historia'". Importantly, to this follow the explanation that the bodies [*corpi*] are part of the 'historia' as the member is a part of the body and the surface part of the member. Finally, circumscription "is the procedure in painting whereby the outlines of the surfaces are drown". Alberti, *On Painting*, 67-68 (II, 33). However, the role assigned to the 'historia' should be correlated in this context, despite Alberti's familiarity with mathematical principles and the artists' workshop, with the fact that he was writing for a learned audience receptive of meanings. Alberti, *De pictura*, II, 35. *On painting*, 71, 99n; Banker, *Piero della Francesca*, 170.

After introductory propositions that refer to Euclid's geometry and optics, the treatise presents (in the manner of treatises on abacus and practical geometry), a series of cases complemented by drawings. The work opens with an introduction to the three parts that paintings consist of, and the reasons for discussing only proportioning [*commensuratione/commensuratio*], in addition to the "outlines and contours contained in things" from drawing. This is because without these, Piero adds, "perspective cannot be shown in action". Piero concludes that colouring will not be discussed, and that he will only deal with what "can be demonstrated by means of lines, angles and proportions, speaking of points, lines, surfaces and bodies."[441] Piero continues by clarifying the subdivision of perspective which incidentally provides a definition of the terms employed:

> [Perspective] contains in itself five parts: the first is seeing, that is the eye; the second is the form of the thing seen [*forma de la cosa veduta/rei forma*]; the third is the distance from the eye to the thing seen; the fourth is the lines which go from the edge [*estremita/extremitas*] of the things and come to the eye; the fifth is the limit [*termine/terminus*] that is between the eye and the thing seen, which is where it is intended to put the things.[442]

[441] "La pictura contiene in sé tre parti principali, quali diciamo essere disegno, commensuratio et colorare. Desegno intendiamo essere profili et contorni che nella cosa se contene. Commensuratio diciamo essere essi profili et contorni proportionalmente posti nei lughi loro. Colorare intendiamo dare i colori commo nelle cose se dimostrano, chiari et uscuri secondo che i lumi li devariano. De le quali tre parti intendo tracta[re] solo de la commensuratione, quale diciamo prospectiva, mescolandoci qualche parte de desegno, perciò che sença non se pò dimostrare in opera essa prospectiva; il colorare lasciaremo stare e tractaremo de quella parte che con line, angoli et proportioni se pò dimostrare dicendo de puncti, linee, superficie et de corpi." Piero Della Francesca, *De prospectiva pingendi*, ed. Chiara Gizzi, (Venezia: Edizioni Ca' Foscari, 2016), 81, Book I, Proemio.

[442] "La qual parte contiene in sé cinque parti: la prima è il vedere, cioè l'ochio, seconda è la forma de la cosa veduta, la terça è la distantia da l'ochio a la cosa veduta, la quarta è le linee che se partano da l'estremità de la cosa e vanno a l'ochio, la quinta è il termine che è intra l'ochio e la cosa veduta dove se intende ponere le cose." ibid.

What follows is a discussion of each of these five parts of perspective. Piero first discusses the eye, then introduces the *forma*.[443] The distance between the eye and the thing seen, the lines that move from the extremities of things and reach the eye, and the *termine,* the surface where the eye describes through its rays the things seen proportionally and can judge their measures. The *forma* is crucial, Piero notes, because "without it the intellect cannot judge and the eye cannot understand that thing."[444] Piero further adds that "it is necessary to know how to draw with lines [*lineare*] all the things that one intends to deal with, in their proper form [*propria forma*] on the plane."[445] This is the geometrical form (shape) of the object considered, before any distortion due to the distance or position of the viewpoint.

Propria forma is a combination of terms used ninety-seven times throughout the treatise, referring to the proper form of the objects to be represented and the related set of orthogonal projections (plans, elevations or sections) that represent their lineaments and spatial arrangement. Consequently, the proper form and the perspectival representations of a body are linked by mathematically ruled transformations based on proportions, as in the methods of measuring by sight found both in treatises on practical geometry and Pacioli's *Summa*. At this point, the form of the objects discussed by Piero can be compared with the form of geometrical bodies, as argued by Pacioli in the *Summa* and *Compendium de divina proportione.*

[443] "La prima dissi essere l'ochio, del quale non intendo tractare se non quanto fie necessario a la pictura; dunqua dico l'ochio essere la prima parte, perché gl'è quello in cui s'apresentano tucte le cose vedute socto diversi angoli: cioè quando le cose vedute sono equalmente distante da l'ochio, la cosa magiore s'apresenta socto magiore angolo che la minore, et similmente, quando le cose sono equali et non sono a l'ochio equalmente distante, la più propinqua s'apresenta socto magiore angolo che non fa la più remota, per le quali deversità se intende il degradare d'esse cose. La seconda è la forma de la cosa, perhò che sença quella l'intellecto non poria giudicare né l'ochio comprendare essa cosa. La terça è la distantia da l'ochio a la cosa, perché, se no ci fusse la distantia, seria la cosa con l'ochio contingente overo contigui, e quando la cosa fusse magiore de l'ochio, non seria capaci a receverla. La quarta sono le linee, le quali s'apresentano da l'estremità de la cosa e terminano nell'ochio, infra le quali l'ochio le receve e discerne. La quinta è uno termine nel quale l'ochio descrive co' suoi raggi le cose proportionalmente et posse in quello giudicare la lor mesura: se non ci fusse termine non se poria intendere quanto le cose degra[1v]dassaro, sì che non se porieno dimostrare." ibid., 81-82, Proemio

[444] "La seconda è la forma de la cosa, perhò che sença quella l'intellecto non poria giudicare né l'ochio comprendare essa cosa." ibid., 82, Book I, Proemio

[445] "Oltra di questo è necesario sapere lineare im propia forma sopra il piano tucte le cose che l'omo intende fare." ibid.

Notably, *forma* is mentioned only once by Piero without the adjective *propria*, in association with the three-dimensional body [*corpo*], when the latter is introduced by Piero at the outset of Book II:

> [The] body has in itself three dimensions: longitude, latitude and altitude; its boundaries [*termini*] are the surfaces. Such bodies [*corpi*] are of different forms [*forme*]: like a cubic body, like a tetragon with different sides, like a [body] round, like a [body] with [multiple] sides, like pyramids with [multiple] sides, and other [bodies] with many and different sides, as you can see in the natural and man-made things.[446]

In a survey of Piero's three treatises, it becomes clear that he never used the term *forma* in his *Libellus* or *Trattato d'abaco*, but only in his treatise on perspective. The *forma* therefore concerns for Piero only the visual properties of bodies. However, as noted in Chapter 2, *forma* was consistently employed by Pacioli instead to refer to three-dimensional mathematical and material bodies throughout the *Summa* and *Compendium*. A reason for this usage can be found in the properties conveyed by the form. The form is in fact understood by Pacioli as indicating the shape-defining properties of regular and other bodies derived from these.[447]

After introducing the five components of perspective, Piero presents the three books into which the work is organised: the first book will concern "points, lines and plane surfaces"; the second is of "cubical bodies, squared pilasters, round columns and columns with more faces"; finally, the third will concern "heads, capitals and bases of columns, toruses of more faces and

[446] "Corpo ha in sé tre demensioni: longitudine, latitudine et altitudine; li termini suoi sono le superficie. I quali corpi sono de diverse forme: quale è corpo chubo, quale tetragono che non sono de equali lati, quale è tondo, quale laterato, quali piramide laterate, et quale di molti et diversi lati, sì commo ne le cose naturali et ancidentali se vede." ibid., 139, Book II, Proemio.

[447] The similarity with the definition of forma provided by Alberti in *De pictura* should be noted here. As noted by Mitrović, in the Italian version Alberti writes that "the rays […] bring the *forma* of the things seen to the sense", however in the Latin version the term was replaced with *rerum simulacra*. This is certainly noteworthy, however as pointed by Mitrović, the great variety of manuscripts of *De pictura* prevents conclusions based on the terminology employed. Mitrović, *Serene*, 56-57. Nevertheless, a difference with Alberti's text is Piero's reference to the mental process associated to the visual perception of the *forma*. A survey of the occurrences of the concept shows that the term *formam* is found in treatises dealing with natural optics. Indeed, it has been noted that Piero collated elements of natural optics, what medieval theorists called *perspectiva communis*, that is the analysis of the physics of sight, with the mathematical principles of *pespectiva artificialis*, the practical perspective of painters. Banker, *Piero della Francesca*, 173.

other bodies [*corpi*] arranged in different ways."[448] The method of perspectival construction in Books I and II starts from the plan in proper form of the object to be drawn in perspective and employs a diagonal to transfer the dimensions to the diminished plane. If needed, verticals are then elevated starting from the diminished plan of the object. Although differently expressed, the procedure is essentially the same as the method presented in Alberti's *De pictura/Della pittura*.[449] The method is applied in Book II to a number of isolated architectural bodies including a cubic building, an octagonal temple, an hexagonal well, a cube with a base and cymatium, and a crossing vault.

Following this, Book III of *De prospectiva pingendi* introduces the so called "Other Method" [*altro modo*]. This method, as Piero explains, deals with bodies that would be difficult to foreshorten with the method presented in Books I and II.[450] The "Other Method" requires as a minimum, a double orthogonal projection at the same scale of the body to be represented in perspective. As is often noted, a method of representation with orthogonal projections was already being used by architects, although it did not necessarily imply the same scale when an object was shown either in plan or in elevation. Differently, the method devised by Piero

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[448] "Intese le sopradecte cose, seguitaremo l'opera, facendo di questa parte dicta pro[s]pectiva tre libri. Nel primo diremo de puncti, de linee et superficie piane. Nel secondo diremo de corpi chubi, de pilastri quadri, de colonne tonde et de più facce. Nel terço diremo de le teste et capitelli, base, torchi de più base et altri corpi diversamente posti." Della Francesca, *De prospectiva pingendi*, 83 (Proemio).

[449] Although, differently from Piero, Alberti did not prove the mathematical correctness of the perspectival construction he described and did not use drawings in his text. As noted by Field, the first proof of the correctness of Alberti's construction was given by Giovanni Battista Benedetti (1530-90), a professional mathematician, in a work published in 1585. In 1625, a misreading of Benedetti, taking him to have proved that it was only Alberti's construction that was correct, led to the construction being given the title 'legitimate construction' [*costruzione legittima*] by which it is sometimes still known. Judith V. Field, "Alberti, the Abacus and Piero della Francesca's proof of perspective," *Renaissance Studies*, Vol. 11, No. 2 (June 1997): 69.

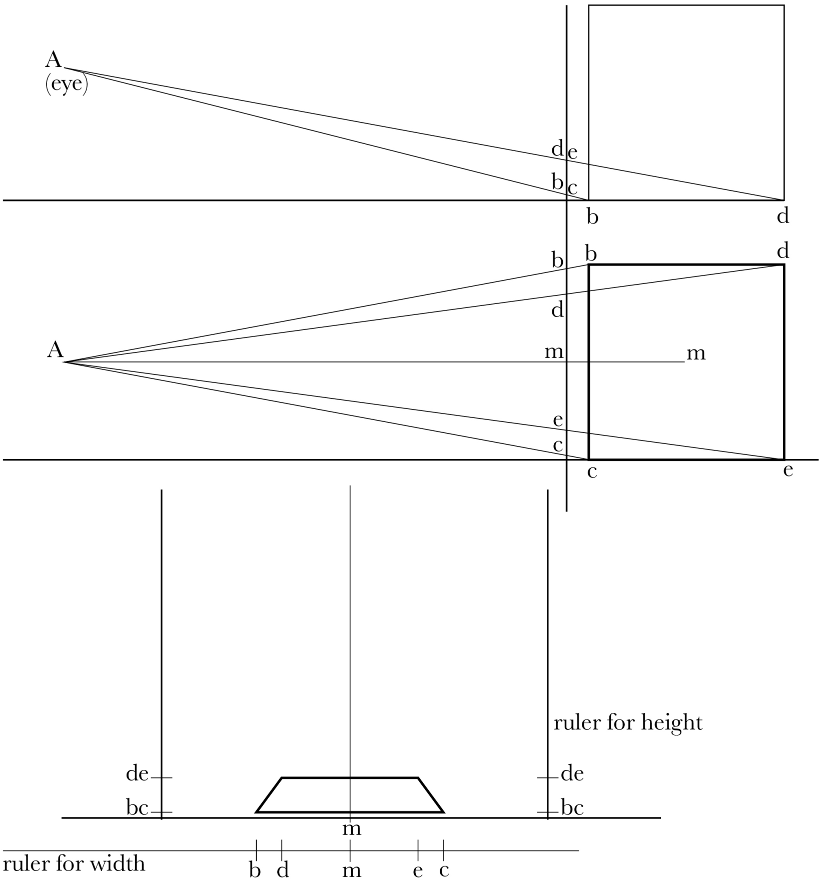
[450] It has been shown that many elements in the paintings of Piero may well have been devised through the Other Method, although probably obtained with that idea and practice in mind rather than rigorously by projection. This has been noted by Robin Evans, for example in relation to a group of heads in *The Proving of the True Cross* in the fresco cycle of the Golden Legend in the Arezzo, in the coffered barrel vault of the Brera Altarpiece, 'in which the Other Method was almost certainly used to draw the details', or in the London *Nativity* among others. Robin Evans, *The Projective Cast: Architecture and Its Three Geometries* (London: MIT Press, 1995), 159-167. Martin Kemp argued instead that in the painted works of Piero "there are occasions, however, where the control is so apparent that the use of calculations is strongly implied", and pointed to the heads of the sleeping soldiers in Piero's *Resurrection* in Sansepolcro. Kemp added that "a tendency to generalise form into semi-geometrical units is inherent in Piero's procedure, and such form as the hemispherical Adam's apple and lidded eyes clearly show the signs of his perspectival attention." Martin Kemp, *The Science of Art*, 34.

required the same dimensional scale to be applied to both projections in a way that the body should have, as Piero's clarifies "the same size and all the correspondent parts".[451]

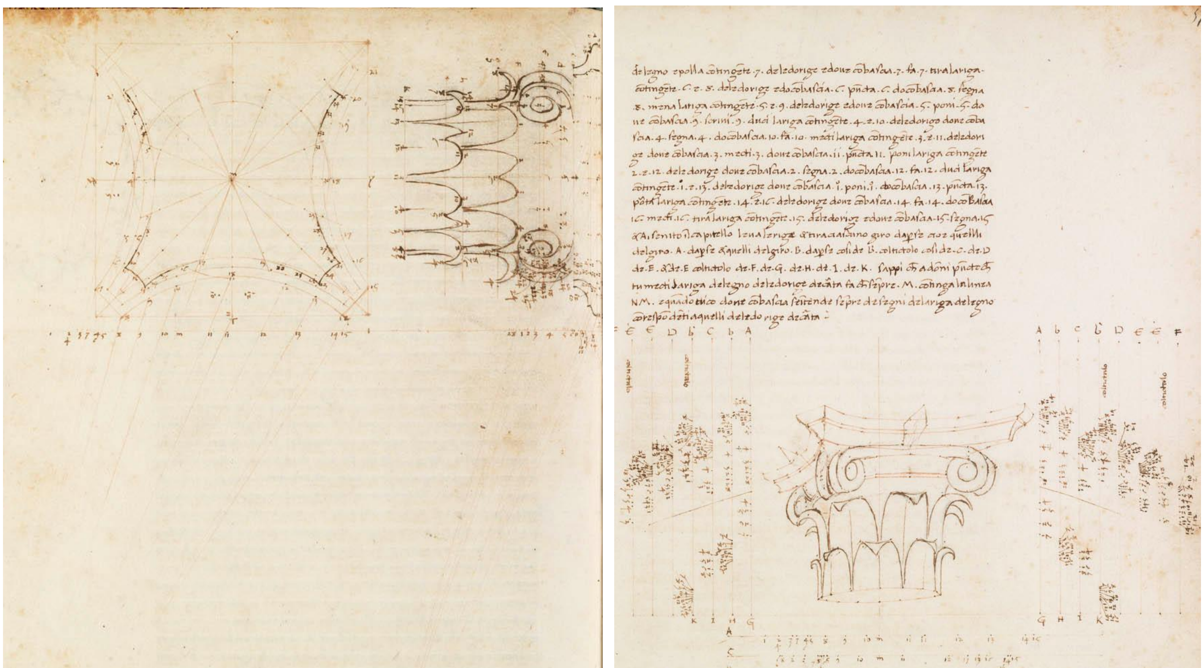
Crucially, a body here should be pre-formed (i.e. its outer boundary and spatial extension should be fully known in three-dimensions) before the procedure begins and orthogonal projections of its shape in plan (or section/s) and elevation are drawn. This method then allows the mapping of every point of a body in the plane of representation by the projection of the visual line concurrently in plan and elevation (illus. 3.6). The result is a foreshortened shape of the body from whichever visual angle has been spatially defined in three-dimensions. This method is used by Piero to construct perspectival representations of a square and octagonal surface, a *torculo* or *mazzocchio*, a cube, a human head seen from two different positions, and architectural bodies such as the base and the capital of a column (illus. 3.7), and a coffered half-dome with twenty-eight bases. The introduction to Book III is particularly relevant as it involves a discussion of the concept of three-dimensional body in its relation to perspective:

> Many painters blame perspective because they do not understand the force of lines and of angles, which are produced from it, [and] by which proportionally every contour and line is described. Hence, it seems I must show how this science is necessary for painting. I say that perspective by its own name expresses how to deal with objects seen from a distance represented within certain given limits [*termini*], proportionally according without which it is impossible to diminish [*degradare*] accurately. Because painting is nothing other than demonstrations of surfaces and of bodies [*de superficie et de corpi*] diminished or enlarged [*degradati o acresciuti*] on the picture plane [*termine*] placed so that the true things [*le cose vere*] seen by the eye under diverse angles are represented on the said picture plane. And the intellect is not able to judge by itself their measure, that is the [exact] quantity of what is near and what is in the distance. And moreover, one part of every object is always nearer than the other to the

[451] "[…] de quela medessima grandeça et tucte le parti conrespondenti." Della Francesca, *De Prospectiva Pingendi,* III, VIII, 279. The process is also described by Filippo Camerota. Camerota, *La Prospettiva*, 93-94. As already noted by Robin Evans, it is seldom observed that there is no perspective projection in Piero's Other Method of perspective; there is a perspective result, which is achieved entirely by orthographic means, just like architecture. Robin Evans, "When the Vanishing-Point Disappears" *AA Files*, No. 23 (Summer 1992): 6.



3.6—Piero della Francesca (redrawn after), 1470-1480, "Above the plane, the square surface rationally [*con ragione*] reduce" *De prospectiva pingendi*, ms Palatino, Book III,1, 33r. Biblioteca Palatina, Parma.



3.7—Piero della Francesca, 1470-1480, "From the given point, in the placed plane [*termine*], to [proportionally] reduce the capital described." *De prospectiva pingendi*, ms Palatino, Book III, 8, 54v, 57r. Biblioteca Palatina, Parma.

> eye and the nearer represents itself always under a greater angle than the more distanced one on the picture plane. Therefore, I say perspective is necessary, which discerns all the objects proportionately as a true science, diminishing and increasing every object through the force of lines.[452]

The passage confirms once more that Piero was perfectly aware that although the form (shape) was altered when viewed in perspective, both the quantities that determined it and their proportional relationships always remained the same throughout the process.[453] Particular emphasis in the passage is placed on the "true things seen by the eye" that are thus represented in the picture plane and to the fact that the intellect, without the mathematical certainty of the theory of perspective, cannot judge accurately their measures. This is a clear and important statement that needs careful consideration. What the intellect alone cannot measure, Piero argues, can be mathematically controlled by the theory and practice of linear perspective. Hence, conversely, it can be assumed that through the knowledge and application of these measures (or their alteration) painters can determine (or at least affect) what will be experienced at the level of the intellect by the viewers of their works.

Moreover, in Problem IX of Book III, the "Other Method" is used to construct a perspectival representation of a coffered half dome (illus. 3.8). Significantly, when the distance of the eye

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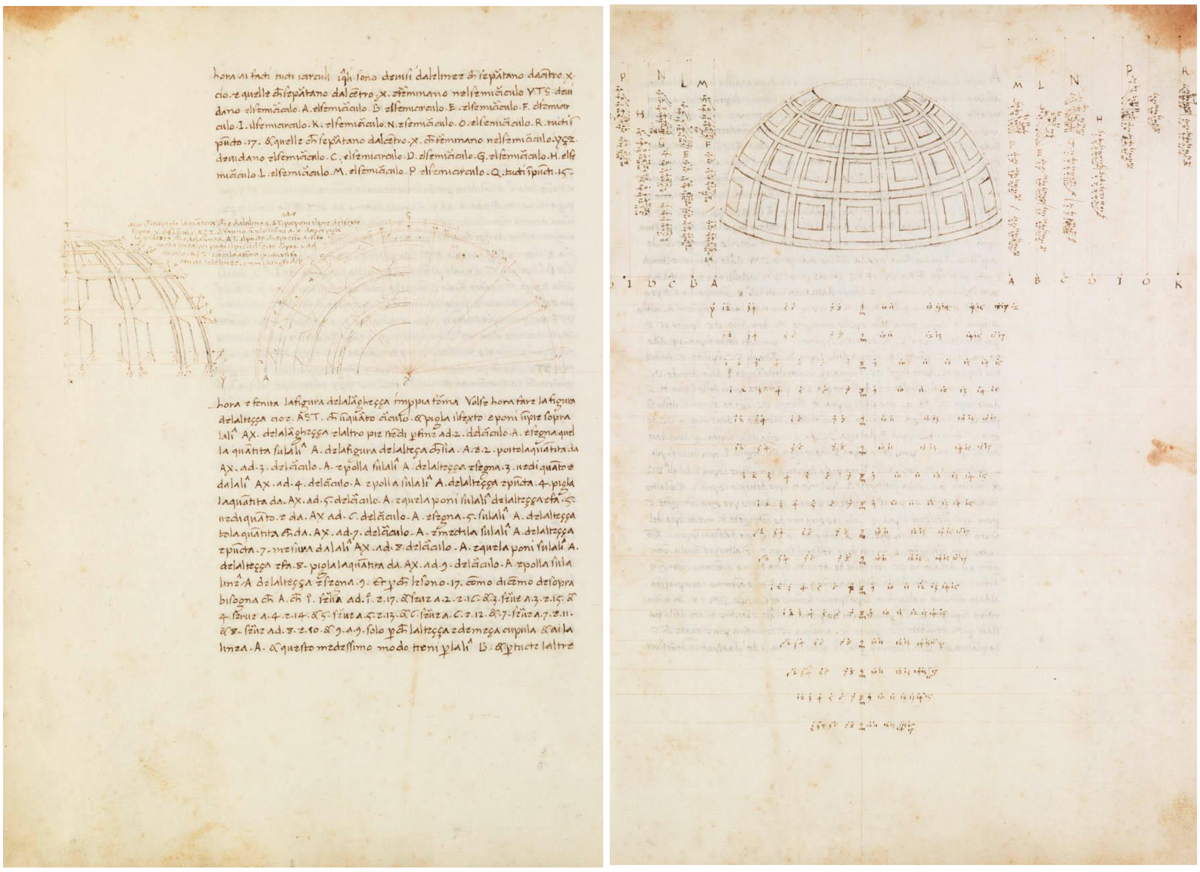
[452] "Molti dipintori biasimano la prospectiva perché non intendano la força de le linee et degl'angoli che da essa se producano: con li quali commensuratamente onni contorno e lineamento se descrive. Perhò me pare de dovere mostrare quanto questa scientia sia necesaria alla pictura. Dico che la prospectiva sona nel nome suo commo dire 'cose vedute da lungi, rapresentate socto certi dati termini con proportione, secondo la quantità de le distantie loro', sença de la quale non se pò alcuna cosa degradare giustamente. Et perché la pictura non è, se non dimostrationi de superficie et de corpi degradati o acresciuti nel termine, posti secondo che le cose vere vedute da l'occhio socto diversi angoli s'apresentano nel dicto termine, et però che d'onni quantità una parte è sempre a l'ochio più propinqua che l'altra, et la più propinqua s'apresenta sempre socto magiore angolo che la più remota nei termini assegnati, et non posendo giudicare da sé lo intellecto la loro mesura, cioè quanto sia la più propinqua et quanto sia la più remota, però dico essere necessaria la prospectiva, la quale discerne tucte le quantità proportionalmente commo vera scientia, dimostrando il degradare et acrescere de onni quantità per força de linee. La quale seguitando molti antichi dipinctori aquistaro perpetua laude; commo Aristomenes Thasius, Polides, Apello, Andramides, Nitheo, Çeusis, et molti altri. Et benché a molti sença prospectiva sia dato laude, è data da quelli che non àno notitia de la virtù de l'arte con falso giuditio." Della Francesca, *De prospectiva*, 190; I have employed and amended the translation of James R. Banker in Banker, *Piero della Francesca*, 176.
[453] A point also emphasised in Camerota, *La Prospettiva*, 91.



3.8—Piero della Francesca, 1470-1480, "a dome […] that is like a quarter of a ball on its concave side, divided in square in which are roses." *De prospectiva pingendi*, ms Palatino, Book III, 9, 78v, 82r. Biblioteca Palatina, Parma. (autograph text and drawings).

is indicated, this is placed by Piero at 10 *braccia* from the object represented.[454] From the Sansepolcro version of *braccio*, this gives a distance of approximately 5.6 metres.[455] Based on this distance, the diameter of the half dome would be about 4.5 *braccia* (approximately 2.52 metres), while it would be placed with its base sitting at approximately 2.8 metres above the level where the observer would be standing. This demonstrates that Piero was thinking about the visual experience of this architectural element as if one was standing and visually

[454] The autograph at the Biblioteca Palatina of Parma (ms. 1576) has: "Tira una linea equidistante ad SA de la largheçça, […] che sirà termine dove se dei mectere le rige, remosso da SA quanto te piaci; dapoi discosta da la linea […] quanto te piaci, mectamo che sia dieci bracci, et in quello luogo fa' puncto O, che sia l'occhio, nel quale ficcha l'acho col filo sutilissimo." Della Francesca, *De Prospectiva Pingendi,* III, VX, 79r, 343. Notably, the manuscripts in Latin that depend from the ms. Palatina (Bordeaux and London) have "intervalu(m) ulnaru(m) decem", The Reggio Emilia manuscript, in *volgare* and prepared by a copyist, and the Latin manuscripts depending on this, have "uno/unius" *braccio* instead of "dieci/decem"—this is understandable, since the text refers to a procedure executed on a drawing. Nevertheless, Piero wrote "dieci" in letters and not in numbers, so he could not have been possibly mistaken here. It is thus clear that he was indicating a specific scale and a correlated perception of this body. This has never, as far as I am aware, being pointed out.

[455] In fifteenth-century Sansepolcro one *braccio* equaled approx. 56 centimetres. Banker, *Piero della Francesca*, xvii.

experiencing it in a building.[456] In other words, he was referring to a scale compatible with the visual experience of its material architectural instantiation.

At the same time, it should be noted that Piero's *De prospectiva pingendi*, his other mathematical treatises, and Pacioli's works always deal with and construct a singular geometrical body in perspective at a time.[457] In other words, the subject of mathematical and perspectival construction is always a singular, isolated body. However, although Piero also considered geometrical bodies in his two other mathematical works (with a 72-bases body among them), he did not establish any more explicit correlation between regular or other bodies and the material world, nor he did represent them in perspective in these works. Overall, if this is further considered in comparison with the arguments in Pacioli's *Summa* and *Compendium*, clearly Pacioli believed that perspective was the necessary representational means to express the formal, visual and spatial qualities of geometrical bodies in his texts. Ultimately, although one the most significant contributions of Piero's treatise rests on the assumption of the mathematical relationship linking visual experience and three-dimensional geometrical bodies, the inherent correlation between these and the visual experience of the material world was made explicit only in Pacioli's *Compendium*.

[456] Moreover, as noted, Piero constructed in perspective in the same work a capital and a base of a column, an octagonal temple and a cubic house, even if with no indication of scale.

[457] It should be added that when discussing the optimal angles of vision determined in proposition 30 of Book I: Piero argued for a maximum right angle of vision at the eye, and prescribed that it should make an angle of 60˚or less. The proof given by Piero has been examined by Judith V. Field. As noted by Field, the proof is not correct if understood in three-dimensions. However, by prescribing an angle of 60º or less Piero seems to avoid the unacceptable excessive lengths. Field, *A Mathematician's Art,* 155.

3.2.3 The perspectival representations of geometrical bodies drafted by Leonardo da Vinci and their role in the context of Pacioli's *Compendium de divina proportione*

This section stands at the culmination of the chapter's analysis of the processes of the quantification and visualisation of geometrical bodies in Pacioli's *Summa* and *Compendium de divina proportione*. The set of physical models of regular and other bodies derived from these, and their perspectival representations in the *Compendium,* are undoubtedly among the most original and immediate expressions of Pacioli's arguments.

By concentrating on the formal properties and visual perception of geometrical bodies, Pacioli's text underlines a hierarchical structure in human visual perception. As will be seen, the bodies are described by Pacioli in terms of their shape-defining properties (lines, angles and shapes), but their full understanding is deferred to the visual perception of their material instantiations. Once these basic features have been perceived, depth can then be added by relying on a background knowledge of perspective, thus determining the three-dimensional experience of each geometrical body.[458] From the level of sense perception we can now move, as noted by Pacioli, to the level of the intellect.

However, rather than considering the perspectival representations drafted by Leonardo *per se*, this section will examine the relationship established with the set of physical models from which these were derived, and their role in Pacioli's *Compendium*. The process described by Pacioli, the set of models and the perspectival representations added to the text are both necessary and unprecedented in their mode and immediacy in conveying the formal and aesthetic properties of the geometrical bodies considered. By introducing the work as a "most

[458] It is useful to correlate here the description of Pacioli and the account of perspective given by Searle. Searle, *Seeing*, 150-151.

useful treatise" to the duke of Milan in the first chapter of the *Compendium*, Pacioli added that this "with all the material forms of the bodies that are included in it, will give not less admiration to whoever will visit your library than the other volumes dealing with the most worthy subjects. For these forms have so far remained concealed to our contemporaries."[459] Ultimately, by reversing the logical progression of the whole chapter, it will become clear that the process of quantification is inseparable from the mental visualisation of the three-dimensional geometrical spatial arrangements of the bodies examined.

Before examining passages from Pacioli's text, it is worth considering that the perspectival representations of regular and other bodies derived from these resulted from the collaboration of Luca Pacioli and Leonardo da Vinci in Milan, between the late 1496 and 1498.[460] The representations survive in two manuscript copies completed in 1498. These are copies of the originals prepared by Leonardo da Vinci that were also the blueprints for the woodcuts that appeared in the printed edition of 1509.[461] These were most likely based on a set of models that Pacioli claims on two occasions to have made and presented to his

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459 "A decore ancora e p(er)fecto ornamento de la sua dignissima bibliotheca de inumerabile multitudine de volumi in ogni faculta e doctrina adorna a disponere q(ue)sto breve co(m)pendio e utilissimo tractato detto de divina proportione. El q(ua)le co(n) tutte sue forme materiali de li corpi che in ditto se co(n)tengono non menore admiratione a chi q(ue)lla visitara darano che tutti gli altri volumi co(n) l(')altre sue dignissime cose in q(ue)lla reposte si facino. Per esser dicte forme a li vive(n)ti finora state ascoste. Nel quale diremo de cose alte e sublimi quali verame(n)te sonno el cimento e copella de tutte le prelibate scientie e discipline e da quello ogni altra speculativa op(er)atione scientifica pratica e mecanica deriva." Pacioli, *Divina proportione (1509)*, 1v (Compendium).

460 Pacioli was called as lector of mathematics in the Studio of Milan by Ludovico Sforza in October 1496. Interestingly, sometime before Leonardo annotated to have bought a copy of Pacioli's *Summa* for "119 soldi." Codex Atlantico, Milan, Biblioteca Ambrosiana, folio 288r.

461 The two surviving manuscripts with copies of the drawings of Leonardo are the codex. inv. S.P.6 ex Cod. F 170 sup. at the Veneranda Biblioteca Ambrosiana in Milan, dedicated to Galeazzo Sanseverino, and the Codex Langues Etrangers n. 210 at the Bibliothèque de Genève, dedicated to Ludovico Sforza. The originals are now lost. Pacioli mentions of being still in possession of the originals after the completion of the *Compendium de divina proportione* in chapter CXVI of *De viribus quantitatis*. When speaking of perspective Pacioli adds: "Commo di lei in quello facemmo Della Divina Proportione alla excellentia del Duca di Milano, Ludovico Maria Sforza, apieno provano e suo effecto largamente manifesta l'opera del nostro Leonardo Venci, compatriota fiorentino, quando con tutta forza feci in ditto libro de sua gloriosa mano li corpi mathematici, qual ancora apresso di noi tenemo maravigliosi a ognuno che li mirano". Pacioli, *De Viribus Quantitatis*, 305; Marinoni, *Luca Pacioli e il "De divina proportione"*, 18.

patrons.[462] As noted by Judith Field and Martin Kemp, the two surviving copies of drawings show no lines of construction, while the position of the viewpoint is slightly shifted in each representation. All these elements attest to the hypothesis that these representations were made using some kind of sighting device after physical models.[463] This device could have been similar to the instrument drawn by Leonardo in his notebooks, or indeed the net [*rete*] placed between the painter and the object to be represented as described by Pacioli in the second chapter of the *Tractato de l'architectura*.[464]

If the models were created to convey the formal, visual and aesthetic properties of regular and other bodies derived from these, the perspectival representations drafted by Leonardo da Vinci aim, as Pacioli argues, to convey the same properties. Both means were referred to by Pacioli towards the end of the *Compendium,* in a chapter named "How you will find all the aforementioned bodies in order as they are arranged in this [*Compendium*] made in perspective and also their material forms according to their specific table [*tau(o)la particulare*][465] placed publicly visible".[466] Pacioli continues by describing the set of models and their perspectival drawings:

[462] The claim is made by Pacioli once in the *Summa,* once in the *Compendium de divina proportione* and once in the *Tractato de l'architectura.* Pacioli, *Summa*, 68v (Pars secunda); Pacioli, *Divina proportione (1509*), 22r (Compendium); Pacioli, *Divina proportione (1509*), 28v (Tractato de l'architectura). That a set of models was still among the possessions of Pacioli after the completion of the *Compendium de divina proportione* 1498 is confirmed by a document dated 30 August 1502. The document states that Pacioli received a payment of 52 lire and 9 soldi for a set of wooden models sold to the city of Florence. Archivio di Stato di Firenze, Operai di Palazzo, 10, c. 67r; Ulivi, "Luca Pacioli una biografia scientifica," 26; Ciocci, "Luca Pacioli, Leonardo da Vinci e il disegno dei poliedri," 57n.

[463] Judith V. Field, *Rediscovering the Archimedean Polyhedra: Piero della Francesca, Luca Pacioli, Leonardo da Vinci, Albrecht Dürer, Daniele Barbaro, and Johannes Kepler* (Basel: Springer, 1997), 262; Kemp, *The Science of Art,* 171.

[464] Pacioli, *Divina proportione (1509)*, 26r. For an English translation of Pacioli's *Tractato de l'architectura* see the thesis's Appendix 1. With regard to Leonardo's notebooks, I am referring to Codex Atlanticus 1ra, c. 1510, Milan, Biblioteca Ambrosiana. The drawing shows a draughtsman using a transparent plane to draw an armillary sphere. Also discussed in Kemp, *The Science of Art,* 171.

[465] The 1498 edition has *tauola* instead of the *taula* found in the 1509 edition.

[466] "Co(m)mo se habino retrovare tutti li dicti corpi ordinatame(n)te comme sono posti in questo facti in p(ro)ospectiva e ancora le lor forme materiali seco(n)do la loro taula particulate posta patente in publico" Pacioli, *Divina proportione (1509)*, 22r (Compendium).

> […] to be able to find all the proper figures in perspectival appearance in this [*Compendium*] placed and also their material [forms] according to their public arrangement your celsitude should do as follow: when you read in the previous chapters of the creation and formation [*creatione e formatione*] [of the bodies], you should look at the number marked in the ancient abacus in that place […] And that precise figure of that body will be shown planar in all perfection in perspective as was done by our Leonardo Vinci. And the same numbers among the material forms [*forme materiali*] of these bodies, hung with their names in Greek and Latin placed in a document above each of them and joined to its knotted cords, will refer to the number placed in the margin of [the passage] whereabouts this body is discussed. And your celsitude will have [them] in their arrangements [*hara lor disposizio(n)i*] in the one and the other way, made not of cheap material [*vil materia*] (as for dearth was inevitable for me), but with precious metal and fine gems, [as these bodies] deserve to be adorned.[467]

The correspondence established by Pacioli between the textual descriptions of regular and other bodies derived from these, their physical models and their perspectival drawings is central here. As will be seen, Pacioli highlights this correspondence after the description of each body in what can be considered the third section of the *Compendium* (Chapters 48-54).[468] Rather than abstract mathematical bodies, Pacioli is referring throughout this section to a set of physical bodies belonging to the material world. At the same time, Pacioli does not consider symbolic or analogical meanings associated with the geometrical bodies presented in this section.

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467 "Perche dove n(on) e ordi(n)e semp(re)fia co(n)fusio(n)e p(er)o a piu piena i(n)tellige(n)tia de q(ue)sto n(ost)ro co(m)pe(n)dio p(er) spaer retrovare tutte le p(ro)prie figure i(n) p(ro)spectivo aspecto i(n) q(ue)sto p(ro)poste a anco le materiali s(econd)o lor publica taula la v. cel. Observera q(ue)sto mo(do) cioe q(ua)n(do) legiarete di sopra i(n) lor capitoli de lor creatio(n)i e formationi guarderete i(n) q(ue)l luogo del libro el nu(mer)o segnato p(er) abaco antico, cope cosi come(n)çando dal I al 48(°) cap(itolo) dice(n)do I,II,III, IIII, V. E seq(ue)ndo fine a lor termine. E q(ue)l medesimo nu(mer)o apo(n)to farete de trovare dena(n)çe dove i(n) q(ue)sto dicti corpi so(n)no p(er) ordi(n)e tutti figurati. El q(ua)l nu(mer)o similme(n)te i(n) q(ue)l luogo sira posto, refere(n)do I a I e II a II e III a III e cosi i(n) tutti. E q(ue)lla tal figura sira del d(i)c(t)o corpo f(a)c(t)o i(n) piano co(n) tutta p(er)fect(i)o(n)e de p(ro)spectiva co(m)mo fa el n(ost)ro Lio(n)ardo vi(n)ci. E q(ue)sti medesimi nu(mer)i a(n)cora recercarete fra le for(m)e ma(teria)li de dicti corpi pe(n)de(n)ti co(n) lor nome i(n) greco e i(n) lati(n)o posti i(n) un breve sopra ciascu(n)o afixo nel suo cordiglio fra doi a(m)bre negre, pur refere(n)do ognu(n)o co(m)mo e dicto al nu(mer)o posto in margine dove di q(ue)l tal se tracta. e. V. Cel. al(l')u(n)o e al(l')altro mo(do) hara loro dispositio(n)i, le q(ua)li n(on) de vil materia (co(m)mo p(er) i(n)opia a me e stato força), ma de p(re)tioso metallo e fine gemme meritarieno essere ornati" Pacioli, *Divina proportione (1509)*, 22r (Compendium).

468 This section follows in the *Compendium de divina proportione* the introduction of the "divina proportione" (Chapters 5-25), the proportional relationships among the five regular bodies and their circumscribed sphere (Chapters 26-31), and the proportional relationship among regular bodies and bodies derived from them (Chapters 32-47).

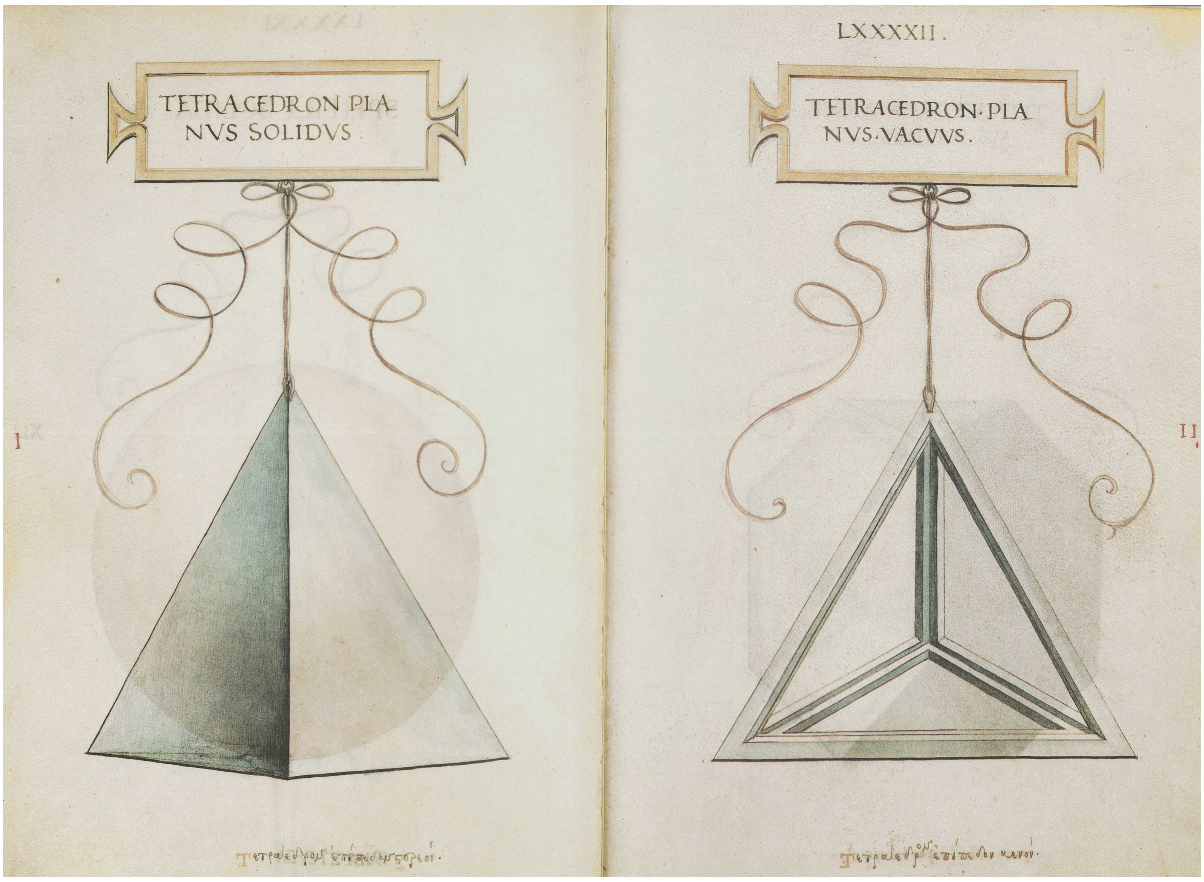
Each regular body is described in its "planar" [*planus*] (its original or perfect form), "truncated" [*abscisum*] (corresponding to semi-regular or Archimedean bodies), and "elevated" forms, when pyramids are added to its triangular, square or pentagonal surfaces in planar or truncated forms. Additionally, all these bodies are presented both in their "solid" and "void" or skeletal versions. A description of the tetrahedron and other bodies derived from this by truncation and elevation opens Chapter 48: "On the form and arrangement [*forma e dispositione*] of the tetrahedron flat solid or void, truncated solid or void, and elevated solid or void."[469] Firstly, Pacioli introduces the "planar" tetrahedron: "The tetrahedron planar solid or void is formed by 6 equal lines which contain 12 superficial and 4 solid angles and make among them 4 triangular bases equilateral and equiangular".[470] After referring to the numbers of the "ancient abacus" (i.e., Roman numerals) paired with the perspectival representation of each body described (illus. 3.9-3.12), Pacioli continues by describing the truncated version:

> The snipped or truncated [*scapezzo o absciso*] planar solid or void [*solido piano over vacuo*] tetrahedron is contained by 18 lines that cause 56 superficial and 12 solid angles and it is surrounded by 8 bases of which 4 are hexagonal of six equal sides and 4 triangular, similarly equilateral and equiangular. But of the said 18 lines, 12 are common to the triangular and hexagonal bases, nonetheless these all belong to the hexagons, because necessarily 4 (sic) hexagons joined together at some of their sides cause [the formation of] triangles; as the experience of its proper material form [*propria forma materiale*] clearly shows to the eye, and [this body] derives from the previous one by uniformly cutting off a third of its sides.[471]

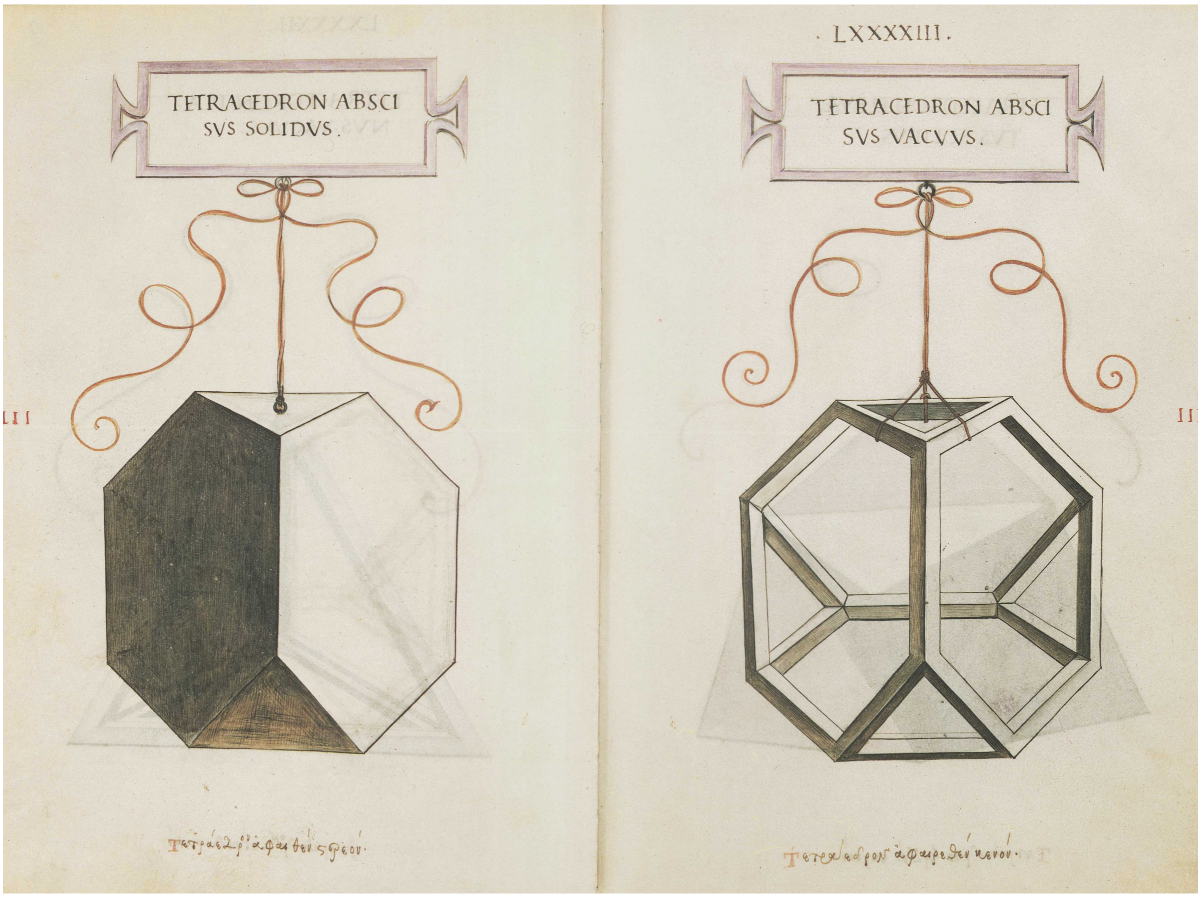
[469] "De la forma e dispositione del tetracedro(n) pia(n)o solido o vacuo e de l(')abciso solido piano over vacuo e de lo elevato soldio over vacuo" Pacioli, *Divina proportione (1509)*, 14r (Compendium).

[470] "El tetracedron, piano solido over vacuo fia formato da 6 linee equali, quali contengano 12 angoli superficiali e 4 solidi e fanno fra loro 4 basi triangulari equilatere & equiangule." ibid.; Pacioli, *Compendium*, 12r.

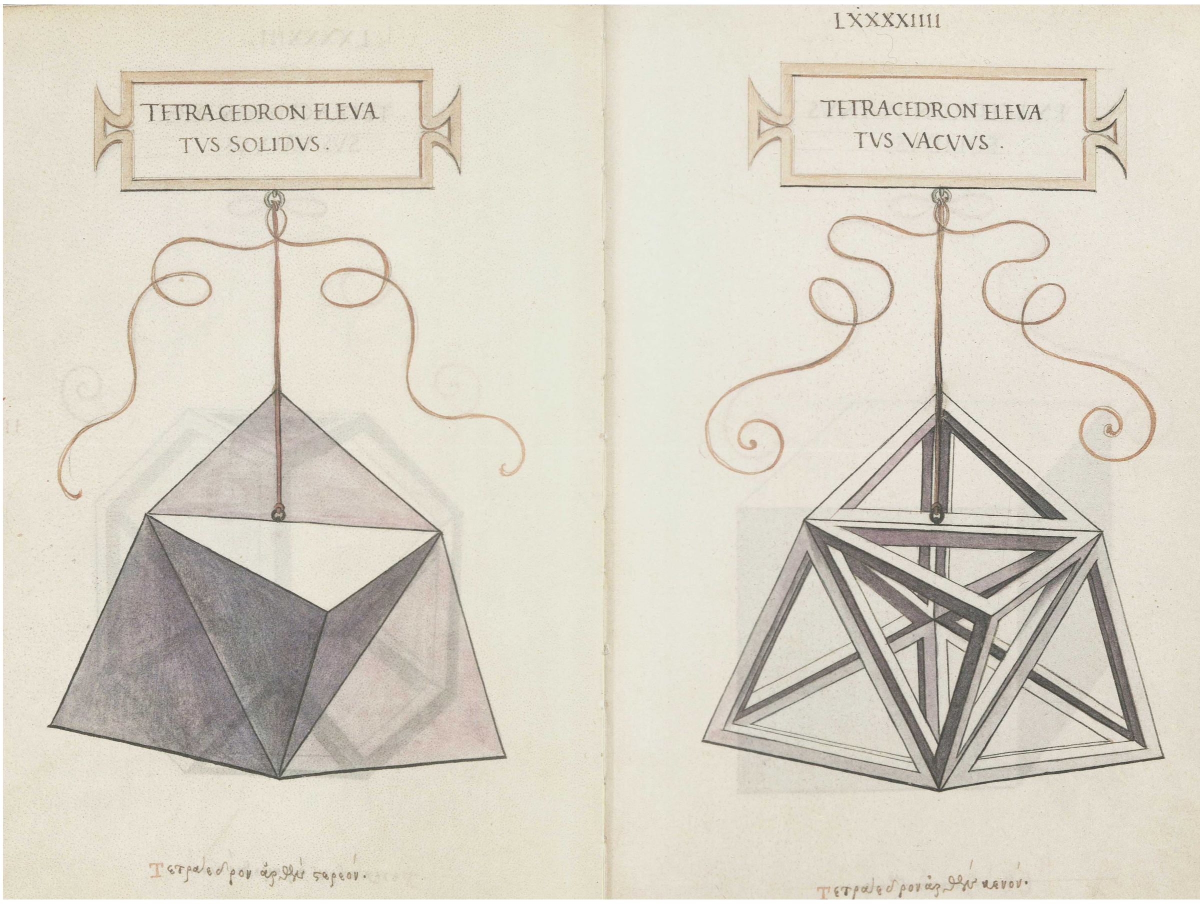
[471] "El tetracedron scapezzo o vogliam dire absciso solido piano over vacuo fia contenuto da 18 linee quali causano 36 anguli superficiali e 12 solidi e 8 basi lo circundano. De le quali 4 sonno exagone cioe de 6 basi equali e le altre 4 sonno triangole, similmente equilatere, e anco equiangole. Ma de le dicte 18 linee le 12 sonno commune a le basi triangule e a le exagone. Le quali non di meno sonno tutte proprie de quelli exagoni, perche de necessita quelli 4 exagoni gionti asciemi con alchuni soi lati causano quelli 4 triangoli. Si co(m)mo la experientia nella sua propria forma materiale a l(')ochio nostro rende chiaro e nasci dal precedente neli suoi lati per terzo uniformi tagliati." Pacioli, *Compendium*, 12r-v. Part of this passage, from "similmente equilatere e anco equiangule" until "sua propria forma" was (apparently mistakenly) omitted in the printed version edited in 1509. Pacioli, *Divina proportione (1509)*, 13r-v (Compendium).



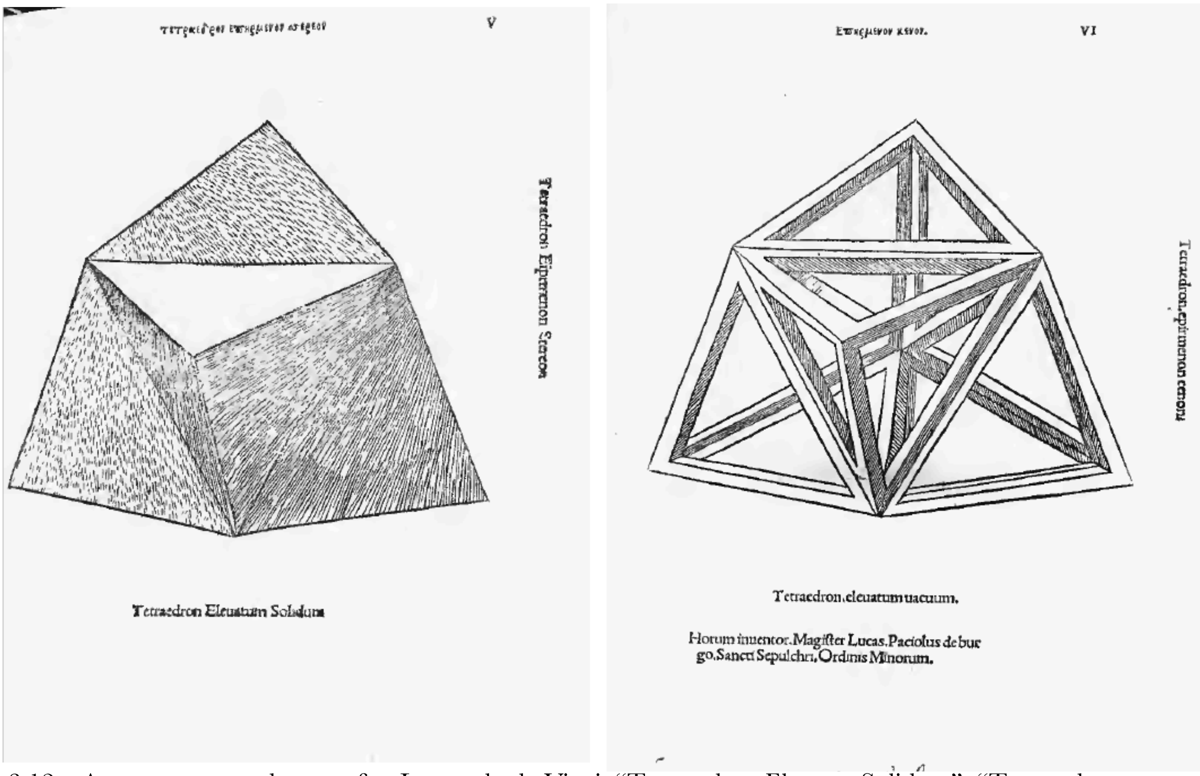
3.9—Leonardo da Vinci, "Tetracedron Planus Solidus"and "Tetracedron Planus Vacuus," in Luca Pacioli, *Compendium de divina proportione.* Veneranda Biblioteca Ambrosiana, Milan.



3.10—Leonardo da Vinci, "Tetracedron Abscisus Solidus"and "Tetracedron Abscisus Vacuus". In Luca Pacioli, *Compendium de divina proportione.* Veneranda Biblioteca Ambrosiana, Milan.



3.11—Leonardo da Vinci, "Tetracedron Elevatus Solidus"and "Tetracedron Elevatus Vacuus," in Luca Pacioli, *Compendium de divina proportione.* Veneranda Biblioteca Ambrosiana, Milan.



3.12—Anonymous woodcutter after Leonardo da Vinci, "Tetracedron Elevatus Solidum", "Tetracedron Elevatus Vacuum," in Luca Pacioli, *Divina proportione (1509).*

As noted, Pacioli limits the description of the bodies to the number of lines, solid angles and types of surfaces at their boundaries, pointing the reader to the corresponding representations to visualise the bodies and clarify their formal properties. By directing the reader to the "proper material form" of each body, Pacioli can refer both to the physical models and their perspectival representations. The purpose of the models and of the perspectival representation attached to the work is to clarify the formal properties of the bodies described by the number of surfaces and perimetric angles that concur with their formation. A survey of the description of the five regular and other truncated or elevated bodies derived from these in Chapters 48-52 of the *Compendium* shows that the form or formation of these bodies is mentioned eight times, and the "material form" [*forma materiale*] of the bodies is referred to in eight additional instances. Moreover, the visual perception of the form of each body allows, according to Pacioli, the apprehension of parts of the body that cannot be visually perceived but must be comprehended at the level of the intellect:

> The solid or void elevated tetrahedron, or we can say pointed [tetrahedron], similarly has 18 lines of which 6 are common and has 36 superficial angles and 8 solid [angles] of which 4 belong to the superficial pyramids and 4 are common to the 5 pyramids; this [fifth pyramid] is the interior [pyramid] that the eye cannot see, but only the intellect can comprehend; and [as] the other 4 exterior [pyramids] of the 5 which this body is composed of are between them [made of] equilateral triangles and equiangular as demonstrated by its proper material form. And the surfaces that dress it, which inappropriately are called bases, are a total number of 12 and are all triangular.[472]

Similarly, the truncated octahedron is "composed by 8 triangular pyramids with equal sides and angles, all of the same height as it appears from the outside, and in addition by the

[472] "El tetracedron elevato o vogliam dire pontuto solido over vacuo ha similmente 18 linee de le quali 6 sonno comune e ha 36 anguli superficiali e 8 solidi de le quali 4 sonno co(mu)ni de le pyramidi superficiali e 4 sonno comuni a le 5 pyramidi cioe a quella interiore che l(')occhio non po veder, ma solo l(')intellecto l(')aprende , e le altre 4 exteriori de le quali 5 pyramidi dicto corpo fia composto quando le sienno fra loro equilatere triangule & equiangule commo la sua propria forma materiale a noi dimostra. E le sue superficie che lo vestano quali non propriamente sonno dette basi in tutto sono 12 per numero tutte triangule. E de questo non se po per alchun modo asegnare lo elevato absciso per defecto de li exagoni che non fanno anguli solidi." Pacioli, *Compendium*, 12v.

intrinsic octahedron that can only be perceived by the imagination in the intellect, of which the bases are the bases of the aforementioned [8] pyramids, as it is made manifest by its material form."[473]

The hexahedron or cube, and the bodies derived from its truncation or elevation by pyramids added to its surfaces, follow in Chapter 49. After a description of the number of its lines, angles and surfaces, Pacioli argues that the surfaces that "contain [this body]", have "a form similar to the diabolic instrument otherwise called *dado* or *taxillo"*[474], linking a geometrical body to a physical object in the material world by means of its form. Additionally, the truncated form of the body is generated by cutting each side of the cube, "as demonstrated to the eye by its proper material form",[475] while the elevated version "[…] is composed by 6 extrinsic quadrilateral pyramids, that can all be presented to the eye according to the position of the body. And in addition, by an intrinsic cube above which these pyramids rest, [which] can only be imagined by the intellect, since it is completely hidden to the eye by the superimposition of the said pyramids."[476] Similarly, the truncated elevated hexahedron has 48 triangular faces or surfaces that surround it and "it is composed by a solid intrinsic hexahedron that can be perceived only by the intellect, and by 14 pyramids. And when thrown over a plane space, this [body] always rests on three pyramidal cones or points, as demonstrated by its form".[477]

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[473] "E questo tal corpo e co(m)posto de 8 pyramidi laterate tria(n)gule eq(ui)latere et eq(ui)angule de medesima alteçça q(ua)li tutte de fore ap(ai)ano, e ancora del octocedron i(n)trinseco p(er) sola ymaginatione da l(')intelletto p(er)ceptibile del q(ua)le octocedron le basi sonno le basi de le dicte 8 pyra(m)idi. Vo(m)mo la fo(r)ma sua materiale a noi fa manifesto." Pacioli, *Divina proportione (1509)*, 15r (Compendium).

[474] "Simile a la forma del diabolico instr(ument)to altrame(n)te detto dado o taxillo." Pacioli, *Divina proportione (1509)*, 14v (Compendium).

[475] "[…] co(m)mo demostra a l(')ochio la sua p(ro)p(ri)a forma ma(teria)le." ibid.

[476] "[…] e fia co(m)posto dicto corpo de 6 pyramidi laterate q(ua)drilatere extri(n)seci q(ua)li a (')ochio tutte sa(re)sentano seco(n)do la situazione del corpo. E ancora del cubo i(n)tri(n)seco sopra el q(ua)le dicte pyramidi se posano e solo l(')intellecto lo ymagi(n)a p(er)che a l(')ochio tutto s(')asco(n)de p(er) la suppositione a lui de dicte pyramidi" ibid.

[477] "E ha 48 faç(ç)e o sup(er)ficie che lo circu(n)da(n)o tutte triangulari e q(ue)sto si facto corpo se co(m)po(n)e de lo exacedro(n) taglato solido i(n)tri(n)seco p(er) i(n)tellecto solo p(er)ceptibile w da 14 pyramidi co(m)mo e dicto e gettato i(n) pia(n)o spacio se(m)p(re) sefer(ma) sopra 3 coni pyramidali o po(n)ti co(m)mo la for(m)a demonstra." ibid.

As in the problems of measuring by sight and perspective discussed in Pacioli's *Summa*, what can be perceived by sight can more easily be quantified, and what is not visible can be measured by means of what is visible. Simultaneously, as argued by Pacioli in the *Compendium*, to quantify what is visible aids the visualisation and quantification of what it is not visible at the level of the intellect. This involves both quantities and formal properties of the geometrical bodies considered, since for Pacioli, the two are necessarily interrelated. The six square bases of the truncated octahedron, Pacioli notes, are "formed by the 8 hexagons when these touch each other, and all the eye makes this truth clear to the intellect [when] the material form [is seen]".[478] Clearly, in Pacioli's view, the verbal description of each geometrical body is not sufficient to understand or imagine the form of the body. This can more easily be understood by the visual perception of its material form, be it a physical model or its perspectival representation. A reference to the former is found in the description of the truncated elevated dodecahedron in Chapter 52. As argued by Pacioli, the body is composed by a truncated dodecahedron that can only be perceived through the imagination at the level of the intellect, and by thirty-two pyramids, twelve with pentagonal and twenty with triangular bases and all with the same height. Pacioli continues that if thrown over a plane, the body will rest on the top of six pyramids, one with pentagonal and five with triangular bases. However, Pacioli recognises that "the same thing seems absurd to the eye when [the body] is [seen] suspended in the air".[479] This is the same manner in which the body was drawn by Leonardo and Pacioli could refer both to the visual experience of its perspectival drawings or to the three-dimensional physical model.

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[478] "E q(ue)lli tali q(ua)drati se forma(n)o da li exagoni q(ua)n(do) uniformi tutti 8 se contangino che di tutto l(')ochio ne la forma sua materiale chiaro all(')intellecto la verita fa nota." ibid.

[479] "E cascando in piano questo sempre si ferma in 6 ponte over coni pyramidali. De li quali coni uno fia de pyramide pentagona. E li altri 5 sonno de le pyramidi triangule. La qual cosa in aiere suspeso pare alocchio absurda, che simil ponte sienno a un paro. E questo tale Ex. D. E de grandissima abstarctione e de profonda scientia che chi intende so non me lassara mentire. E a la sua dimensione se perviene con subtilissima pratica maxime de algebra & almucabala a rari nota. E da noi nella nostra opera ben demonstra le vie facilmente a poterla aprehendere."

The unprecedented way of representing geometrical bodies endowed with material properties in Pacioli's *Summa* has been noted by scholars.[480] This was achieved by the additional use of colours, shadows and chiaroscuro. The difference is particularly striking if compared with the diagrams that furnished the manuscript editions of Euclid's *Elements* in the fifteenth century and before, as in the Campanus' edition .[481] However, the point is less about the way the geometrical bodies were represented, but rather the visual and material linkage, first established by Pacioli, between physical models and perspectival drawings that were able to show the bodies "with all perfection"[482] and thus provide visual information that could not be conveyed by verbal descriptions. Ultimately, when the whole *Compendium* is considered, Pacioli's main original achievement was the formal, spatial and aesthetic properties he was able to convey through the unmediated visual perception of physical models and (with the aid of Leonardo da Vinci) their perspectival drawn representations.[483]

At this point, it is essential to consider the process of visual imagination as described by Pacioli in the context of the functioning of the human soul and its cognitive processes, as

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[480] This was firstly pointed out by Marisa Dalai Emiliani. Marisa Dalai Emiliani, "Figure rinascimentali dei poliedri platonici qualche problema di storia e autografia," in *Fra Rinascimento, Manierismo e Realtà. Scritti di storia dell'arte in memoria di Anna Maria Brizio*, ed. Pietro C. Marani (Firenze: Giunti Barbera, 1984), 9.

[481] It was also underlined that such a way of presenting geometrical bodies had no precedents in previous literature. The difference is particularly striking when considered in comparison with another text edited by Pacioli, Campanus's edition of Euclid's *Elements*. In the latter, diagrams were used to visually represent the process of geometrical constructions described in the text and aiding the imagination of the resulting shape. As a consequence, they included only the lines considered necessary to this task. This meant avoiding all additional lines that, although described in the proposition, could as Campanus argue rather "confound the intellect". "Has autem diametros in hac plana figura protrahere contempsi ne multitudo linearum confunderet intellectum. Si igitur figuram hanc ut oportet actu vel animo compleveris videbis ex sex diagonalibus lineis sex superficies ipsus cubi dividentibus piramidem quatuor basium triangularium esse perfectam qua(m) cubo p(ro)posito ex diffinitione constat esse inscriptam." Pacioli, *Euclidis megarensis*, 141r; Ciocci, "Luca Pacioli, Leonardo da Vinci e il disegno dei poliedri," 58.

[482] E q(ue)lla tal figura sira del d(i)c(t)o corpo f(a)c(t)o i(n) piano co(n) tutta p(er)fect(i)o(n)e de p(ro)spectiva […]" Pacioli, *Divina proportione (1509)*, 22r (Compendium).

[483] The way in which the concepts of *solidus* and *vacuus* were applied to two perspectival representations of the same geometrical body also had consequences for the perception of its formal and spatial properties. *Solidus* is used here as an adjective and refers to the quality of the surfaces that bound the geometrical body. The Greek terms used in the caption are in fact *epipedon stereon*. In the "void" [*vacuus*] version, *epipedon kenon* again refers to the quality of the surface. The *vacuus* is not a geometrically correct representation of the body concerned (its surfaces are certainly not the polygons verbally described and found in the "solid" version, as the number of lines and angles increases). However, this is secondary to the visual and three-dimensional clarity that the "void" [*vacuus*] version allows over the "solid" [*solidus*].

described by Aristotle in *De anima*. For Aristotle, the higher level of the soul, the intellect, contemplates representations or *phantasms* (created by the imagination and based on the information gathered in the common sense), and extracts their form (essence).[484] However, as noted by Mitrović, in Aristotle's cognitive scheme, the intellect extracts the essence of the thing from the phantasm, but there is no account of any (mathematical) feedback that this may provide to the imagination. It is consequently not clear in Aristotle's account, how the imagination can solve the geometrical problems that arise from imagining a three-dimensional object from different sides in a series of two-dimensional phantasms.[485] Nonetheless, although Pacioli considered the main stages of human visuality and reflected the terminology to be found in translations of Aristotle's *De Anima*, he did not follow Aristotle's account of human vision on the whole. As done by Alberti before him, he avoided Aristotle's account of light as the actualisation of the medium of vision and consistently referred instead to the principles of linear perspective.[486] Additionally, as noted above, Pacioli argued that the form of regular bodies visually perceived rest completely on their mathematical-quantitative properties. *Forma* was thus conceived by Pacioli as a three-dimensional geometrical body visualised at the level of the intellect, to which images or figures [*figure*] correspond. Additionally, although Pacioli did not draw a distinction between the agent and the potential division of the intellect described by Aristotle,[487] nor considered the permanence of the soul, it is clear that this stage include the principles of perspective and that these are based on mathematical properties and are considered universal by Pacioli.[488]

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[484] Aristotle, *On the Soul* (*De anima)*, Book III, Chapters 4-8, 429a10-432a14. As noted by Mitrović, the higher strata of the soul comprising the intellect have the task of recognising the universal in the phantasm. Branko Mitrović, "Aesthetic Formalism in Renaissance Architectural Theory," *Zeitschrift für Kunstgeschichte,* 66. Bd. (2003): 326.

[485] Mitrović, "Leon Battista Alberti, Mental Rotation, and the Origins of Three-Dimensional Computer Modeling," 319.

[486] ibid., 317-318.

[487] Aristotle, *On the Soul* (*De Anima)*, 430a10–14.

[488] Coherently, in the later *Tractato de l'architettura* he will describe the application of geometrical bodies to architecture as linked to the architect's *ingegno*. Pacioli, *Divina proportione (1509),* 31r.

The correlation between the formal properties of the geometrical bodies examined by Pacioli and their visual, spatial and physical instantiations in the material world is strengthened in Chapters 53 and 54, in the latter section of the *Compendium*. Here, Pacioli introduces both the body of twenty-six bases and the body of seventy-two bases. The usual description of the number of lines, angles and superficial surfaces that form each body, and the reference to its perspectival representation, are followed in both cases by the claim that both bodies find common applications in architecture. The body of twenty-six bases is "derived from the hexahedron cut uniformly in all its corners, as demonstrated to the eye by its material form"[489] and "its science is very useful in many respects, particularly to those that know how to use it in architecture".[490] The body of seventy-two bases instead "is frequently used by architects for being a very useful form [*forma assai acomodata*] in the arrangement of buildings [*in loro dispositioni di edifici*], especially when it happens to make domes or other vaults or what we can call ceilings [*cieli*]", although, as Pacioli adds, sometimes only a part of the whole body is used.[491] In these chapters in the *Compendium,* Pacioli argues for a direct correlation between the formal properties of geometrical bodies, the visual experience of physical models and their perspectival drawings, and their material instantiation in architecture. Ultimately, Pacioli's argument holds a series of inevitable consequences in its approach to the formal, visual and spatial properties of architecture, particularly when the relationship between geometry, perspective, proportions and the aesthetic properties of architecture are concerned.

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489 However, this is not geometrically correct or Pacioli should have specified that to have the body described and represented in perspective the truncation should not be uniform to all corners. The uniform truncation of a hexahedron (cube) does not result in the body described by Pacioli, but in a similar body that has rectangular polyhedra instead of square. This was noted by Judith V. Field. Field, *Rediscovering the Archimedean Polyhedra*, 253-266.

490 "E l(')origine de q(ue)sto fia da lo exacedro(n) uniforme seco(n)do ogni suoi p(ar)ti tagliato come a l(')ochio la sua material forma ci demostra. E fia la sua sci(enti)a i(n) molte considerationi utilissima a chi b(e)n(e) la (sa) acomodare maxime in architectura […]" Pacioli, *Divina proportione (1509)*, 15v (Compendium).

491 "[…] E q(ues)to 72 basi molto da li architetti fia freque(n)tato i(n) loro dispositio(n)i de hedificii, p(er) e(ss)er forma assai acomodata maxi(m)e dove occurrese fare tribu(n)e o altre volte, o volia(m)o dire cieli. E ave(n)ga che non semp(re) apo(n)to, se pre(n)dino in detti edifici ta(n)te facce pure a q(ue)lla similitudine se regano squata(n)dolo sterça(n)dolo in tutti modi seco(n)do el luogo e sito dove tal edificio inte(n)dan porre. A la cui co(n)venie(n)tia asaissimi in diversi p(ar)ti se trova(n)o disposto e fabricati." Pacioli, *Divina proportione (1509)*, 16r (Compendium). These passages are further discussed in Chapter 4 of the thesis.

These arguments establish that the formal and visual properties of a geometrical body can be applied to architecture.

Conclusion

During the fifteenth century (and the contexts in which Luca Pacioli operated), there was an increasing interest and accuracy in mathematical processes of quantification. It is often noted that this followed a more general trend linked to the teachings of schools of commercial arithmetics and practical geometry, and the manuals that originated from these and were then circulated. Needless to say, the purposes of the procedures presented in these manuals were eminently practical and aimed at determining the formal properties of material objects in order to deduct the quantities sought. They were addressing the needs of traders, artisans and technicians. However, the knowledge of mathematical procedures of quantification in the *Summa, Compendium* and *Libellus* by Pacioli demonstrate that more advanced mathematical skills could be employed to examine bodies with a higher level of geometrical complexity.

Most of these materials in Pacioli's texts were derived from Piero's *Trattato d'abaco* and *Libellus de quinque corporibus regularibu*s, and above all, referred to the last books of Euclid's *Elements* in the medieval edition by Campanus. However, Piero's texts survived in a single manuscript copy, while both texts by Pacioli were printed and widely circulated. Moreover, Pacioli was the first to explicitly argue that the same mathematical processes of quantification could be applied both to three-dimensional bodies in common use and to regular and other bodies derived from these. This was, as shown above, no minor achievement. Even more so, because the same mathematical principles based on proportions were systematically applied to the perspectival representation of bodies in Piero's *De prospectiva pingendi.* Again, while Piero did not explicitly correlate formal, visual and spatial arguments and discussed the arithmetical quantification and the perspectival visualisation of bodies in separate works, Pacioli did not fail to recognise and argue for their unification. Overall, Pacioli was able both to exploit and make visible the nature of the inherent correlations among these subjects in his arguments.

Pacioli linked the visual perception of apparent quantities to the measurement of real quantities, arguing for their mathematical correlation based on proportions.[492] In other words, the process of mathematical quantification of regular and other bodies derived from these becomes at once mathematical, visual and experiential, in Pacioli's *Compendium*. Additionally (and crucially), the visual perception of geometrical bodies supports, according to Pacioli, the process of mental visualisation of the formal properties that are not materially represented in perspectival drawings. As shown, Pacioli consistently (and coherently with the Aristotelian system of cognitive processes) describes this additional stage as taking place at the level of the intellect. Pacioli clearly refers, however, this stage to the visual experience of the form (shape) of geometrical bodies as mathematically defined.

Additionally, when dealing with regular and other bodies derived from these in Chapters 48-54 of the *Compendium*, Pacioli makes clear that these should be considered for the aesthetic value of their visual and formal properties and does not refer to any symbolic or allegorical meaning. As Pacioli argues, the formal properties of geometrical bodies can best be conveyed visually by physical or perspectival representations, or by a combination of these with a verbal description, but never by the latter alone.

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[492] In essence, as Pacioli implies, from the knowledge of the one, the others can be mathematically determined.

Ultimately, these assumptions clarify Pacioli's presentation of a set of physical models to accompany the *Compendium* and the perspectival representations based on the models, derived from their direct visual experience. In Pacioli's view, all geometrical bodies partake of the same mathematical, visual and spatial properties shared by all physical objects in the material world. As such, they were instantiated in a set of models, while objects of common use or architectural elements could have the same formal, visual and spatial properties of regular or other bodies derived from these. This follows, in Pacioli's argument, from the process of quantification of visual perception and the relationship of form (shape) and matter in regular and other bodies derived from these, as defined by mathematical properties (their extension). The physical world can thus be understood as a homogenous compound of form (shape) and matter which can be manipulated through three-dimensional bodies. Although this is in principle consistent with the Aristotelian framework found in Pacioli's works, it is in the interconnection of visual (perceptual and experiential) and formal arguments by means of quantifiable mathematical properties in this philosophical context that lay one of the most original outcomes of Pacioli's arguments. The important and inevitable consequences that this assumption has for architecture will be examined in the forthcoming chapters. Chapter 4 considers and compares the theoretical framework in the context of fifteenth-century treatises on architecture with Luca Pacioli's writings on the subject. Chapter 5 examines correlations with some among the architectural works of Donato Bramante in Milan and Rome.

4. The visualisation and application of quantities to architecture in fifteenth-century treatises, in Pacioli's *Compendium* and *Tractato de l'architectura,* and in Bramante's *Opinio*

This chapter examines the implications of the concept of regular and other geometrical bodies derived from these, as discussed in Pacioli's works, for the theory of architecture. Given the correlations established among architecture, perspective, visual perception and mathematical principles in Pacioli's works and the environments in which he operated, his claims concerning architecture need careful examination and a comprehensive introduction.

Although Pacioli consistently advocates a closer relationship between theory and practice in his writings, he was not a practicing architect nor was he (at least not primarily), an architectural theorist. As we have seen, he was primarily a scholar and lecturer in practical and theoretical mathematics and a very receptive compiler of the theoretical, philosophical, and practical mathematical arguments that were circulating in his milieu. Since Pacioli demonstrates the belief that mathematical principles are the foundations of every practical enterprise, as they are the basis of the phenomenal world, works of architecture would necessarily conform to the same principles. Pacioli as is often noted, was not original in his stance, as the belief that mathematical principles explain visual perception and are essential to architecture was previously argued by other authors in the fifteenth century, most notably by Leon Battista Alberti.

Nevertheless, this point is critical when Renaissance authors and their writings on architecture are examined, because these are the same geometrical principles that allow perspectival representations (based on the geometry of light rays) to produce a visual experience similar to the experience derived by the perception of the object itself. When the application of linear quantities to architecture is considered, it is crucial to examine if and when the mastery of the principles of perspective was employed in devising architecture's formal properties. While the theories of perspective and architecture were discussed in separate works and thus kept separate by Alberti, the theories and the geometrical principles underlining them are integral to the concept of the regular and other bodies derived from these as presented by Pacioli in his *Compendium de divina proportione*. This, as noted, should be linked with other elements of the theoretical framework expressed in his writings. As discussed above, the physical world is considered by Pacioli as a homogenous composite of form (shape) and matter (spatial extension), and is defined by three-dimensional bodies, among which regular and other bodies derived from these are the most perfect visual expression.

Regular or other bodies derived from these are geometrical entities in which proportional relationships among components in three-dimensions are fixed. In the terms discussed in the previous chapters, this means that quantitative relationships between a body's form (its shape, or outer boundary) and matter (understood as three-dimensional spatial extension, or its volume) cannot change without changing the body's identity. If a sphere is compressed and the proportional relationships among the quantities of its components change, it ceases to be a sphere. To better understand this concept, the applications of these bodies to architecture in the late fifteenth century as described by Pacioli, should be compared with the process of applying geometrical entities to architecture, as discussed in coeval treatises. Like Vitruvius's *De architectura*, these treatises discussed linear quantities in proportional relationships, and systematically applied these to the width, length and height of architectural elements. Since these quantities were based on a linear dimension measured in the plan or elevation of the building,[493] these quantities could be proportionally arranged through an itemised process to determine the dimensions of the architectural elements. Accordingly, as we shall see, to achieve the desired three-dimensional composition, the process always began with the discussion and arrangement of linear quantities and two-dimensional figures. This means that unlike Pacioli's *Compendium,* a three-dimensional shape (or geometrical body) was not discussed and applied to architecture in the terms outlined above—that is, as a body composed conjointly of form (shape) and matter (spatial extension)—but rather that three-dimensional shapes were firstly quantified and visualised by means of linear quantities to be spatially arranged.

At this point, it is important to clarify that this and the next chapter will not focus on symbolic or any other conceptual meanings associated with architecture, but only on geometrical properties, that is, lines, surfaces, and three-dimensional bodies understood as geometrical and material entities. At the same time, they will not primarily concern the quantitative values of the proportional ratios employed (although they will be noted), but rather *how* geometrically and spatially these quantities were visualised and applied to architecture. In other words, it will consider the process by which quantities were employed in architecture as geometrical entities, and the implications that this process had for architecture's formal properties.

This is not to say that symbolic meanings are completely missing in the writings of Pacioli that deal with architecture, as these have been grounds for speculation in the scholarship

[493] Although, differently from Vitruvius, Renaissance theorists such Alberti did not derive this module from an element of the building (e.g. the diameter of a column, a triglyph, etc.) but understood it as an abstract linear module by which define the width or diameter of the plan and then the lenght and height of the building. In practice, this can taken to correspond to a unit of measurement, e.g. a *braccio,* a *piede,* a *palmo*, etc.

(although, as noted, the whole theoretical context of his works was often not considered).[494] However, Pacioli never argues that these contribute to the formal or aesthetic properties of architecture. As noted in the previous chapters, if the process of human visual perception is hierarchically organised, conceptual meanings should be located at a higher levels of cognition.[495] At the same time, if the visual imagination of three-dimensional architectural shapes can be considered as a basic stage of human visuality, this chapter also concerns what this process determines at the geometrical level of lines, angles, and two-dimensional shapes.[496] In other words, this is the central question of this and the following chapter. If a building (or a part of it) is mentally visualised (by means of two-dimensional images) and quantified as a three-dimensional geometrical body composite of form (shape) and spatial extension, how does this affect the design process and ultimately the formal properties of the building? Clearly, this question involves the whole process of definition of the visual, formal, and spatial properties of architecture.

Overall, the writings of Pacioli that deal with architecture can be divided into two main groups. On the one hand, as he was not primarily an architectural theorist, Pacioli had to consistently rely not only on Vitruvius's *De architectura*, but also on the interpretations of Vitruvius's text and on treatises by other architectural theorists. This is particularly noticeable in a work in which a systematic discussion of architecture is attempted, as in Pacioli's short treatise *Tractato de l'architectura*. His references here are Vitruvius's *De architectura*, (from which

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[494] One has only to look at the analysis of Byrna Rackusin to see how far these interpretations could go. Byrna Rackusin, "The Architectural Theory of Luca Pacioli: De Divina Proportione, Chapter 54," 479-502.

[495] Lines, angles, two-dimensional shapes, and colours are at the basic level of human visual perception. However, after this initial stage of perception depth intervenes as a non-basic feature because of, as summarised by Searle, our "background mastery of the principles of perspective". This is an ability possessed by any competent perceiver and allows the visual perception of depth and other spatial relations. Searle, *Seeing*, 139-140.

[496] I should here refer to the discussion of visual imagination and mental rotation of architecture by Branko Mitrović, summarised in: Branko Mitrović, "Leon Battista Alberti, Mental Rotation, and the Origins of Three-Dimensional Computer Modeling," *Journal of the Society of Architectural Historians*, vol. 74, no. 3 (September 2015): 312-315. The term 'mental rotation' derives from the experiments in visual imagination of Roger Shepard in the 1970s. Roger Shepard and Jacqueline Metzler, "Mental Rotation of Three-Dimensional Objects," *Science*, vol. 171, no. 3972 (1971): 701-703.

entire sections are repurposed verbatim), Alberti's *De re aedificatoria*, Francesco di Giorgio Martini's *Trattati,* and even Fra Giovanni Giocondo's edition of Vitruvius's text, seen by Pacioli before its publication.[497] However, among all these sources, Pacioli only explicitly acknowledged his borrowings from Vitruvius. Moreover, more general references to architecture that mention the works of both fifteenth-century (such as Alberti) and ancient authors (such as Vitruvius) were already included in Pacioli's *Summa*. On the other hand, although Pacioli discusses only a limited group of architectural elements in the *Tractato de l'architectura* (mainly the classical orders and the proportional arrangement of the parts of a building) it is clear that the concept of geometrical body as discussed in other works is presented and related to this theoretical context. These are a group of writings that will be discussed in the second part of this chapter and clearly refer to the employment of regular and other bodies derived from these to architecture. They are mainly found in Pacioli's *Compendium,* but also, as noted, in the context of the later *Tractato de l'architectura*.

As we shall see, this is certainly the most productive group of writings for the purposes of this study; however, they have received much less scholarly attention than the first group. The reasons for this are multiple. Firstly, it is difficult to relate and compare these writings with the work of other architectural theorists, such as Alberti or Francesco di Giorgio Martini. Secondly, scholars have often failed to consider the relationship with architecture at both a formal and an aesthetic level, privileging instead symbolic and analogical meanings or the problem of sources.[498] However, since Pacioli is referring in the *Compendium* to existing buildings, its indications should have been considered both empirical and theoretical.

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[497] Pacioli, *Divina proportione (1509)*, 27r (Tractato de l'architectura). Francesco P. Di Teodoro, "Due Quaestiones Vitruviane riconosciute: la base attica e il capitello composito nel terzo libro del de prospectiva pingendi di Piero della Francesca e un plagio conclamato di Luca Pacioli," in *L'Apelle Vitruviano: Riflessioni sulla cultura architettonica dei pittori nella prima età moderna*, ed. Filippo Camerota (Madrid: Universidad Nacional de Educación a Distancia, 2019), 58-61.

[498] As noted, examples of this approach are found in the writings of Alberto Perez-Gomez and Byrna Rackusin: Alberto Perez-Gomez, "The Glass Architecture of Fra Luca Pacioli,"and Byrna Rackusin, "The Architectural Theory of Luca Pacioli: De Divina Proportione."

Ultimately, these arguments have not been considered in the wider theoretical and philosophical context of Pacioli's works or compared with the arguments and works of other individuals in the environments in which Pacioli operated. If Pacioli was primarily an educator and a compiler, these are all reasons to believe that these writings are interrelated to the theoretical and practical knowledge that circulated in these environments, and that Pacioli believed that these principles were expressed in the buildings referred to in his writings. It will be argued that Pacioli's *Compendium* was produced in response to its theoretical and practical context, and as such was also the outcome of discussions held in the environment of the Sforza court of Milan.

4.1 The application of linear quantities and planar figures to the three-dimensional shapes of architecture

4.1.1 The human body *qua* quantitative body

As discussed in Chapter 1, since the human body was understood to be the most perfect creation in nature and the repository of proportional relationships, it was considered by Vitruvius and Renaissance architectural theorists as the privileged object for imitation. The metaphor of the human body was also employed as an organisational or functional principle for architectural works and in this sense should be considered as metaphorical, with no bearing on the actual shape that architects were supposed to give to buildings.

However, some linear quantities (namely the width, length, and height of a body's members or of the whole body) were also considered for their quantitative values. This section will concern only the geometrical properties of the human body and how these were first abstracted and then considered applicable to architecture. Although the process by which geometrical properties were abstracted from the human body and applied to architecture is often considered together with other conceptual meanings that are derived from it, an analysis of the formal and spatial dynamics associated with its geometrical properties is crucial, if the process of visualisation and the application of quantities to architecture is to be clarified.

At the opening of the first book of *De Architectura,* Vitruvius defines proportions as determined by quantities, and quantity as depending on the modules (linear units of measurement) of the parts of a work of architecture.[499] While considering the properties associated with symmetry, a quantitative description of the human body follows through a series of linear measurements (with commensurable ratios among them) that determine the symmetry of the whole:

> Symmetry also is the appropriate harmony arising out of the details of the work itself; the correspondence of each given detail among the separate details to the form of the design as a whole. As in the human body, from cubit, foot, palm, inch and other small parts comes the symmetric quality of eurythmy; so is it in the completed building. [500]

Vitruvius continues with a description of how the linear module that is the basis of this proportional system should be applied when employed in architecture.[501] Later in the work, a similar argument is developed in the opening chapter of Book III. Here, Vitruvius considers the design of temples dedicated to the sacred gods and discusses the role of proportions following the principles introduced in the first book:

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[499] "Ordinatio est modica membrorum operis commoditas separatim universeque proportionis ad symmetriam comparatio. Haec conponitur ex quantitate, quae graece posotes dicitur. Quantitas autem est modulorum ex ipsius operis sumptio e singulisque membrorum partibus universi operis conveniens effectus." Vitruvius, *De Architectura*, I.2.2.

[500] "Item symmetria est ex ipsius operis membris conveniens consensus, ex partibusq(ue) separatis ad universae figurae speciem, ratae partis responsus ut in homini corpore, e cubito, pede, palmo, digitis, caeterisq(ue) particulis." Vitruvius, *De Architectura,* I. 2. 4.

[501] Vitruvius lists among the suitable linear lenght of reference the diameter of a column, a triglyph, or a module for a sacred building, a perforation for a balista, the space between the rowlocks in a ship, etc. ibid.

> Proportion consists in taking a fixed module, in each case, both for the parts of a building and for the whole, by which the method of symmetry is put into practice. For without symmetry and proportion no temple can have a regular plan; that is, it must have an exact proportion worked out after the fashion of the members of a finely-shaped human body.[502]

As a result, a list of the linear quantities associated with the members of the human body and their proportional relationship as planned by Nature is given: "the face from the chin to the top of the forehead and the roots of the hair is a tenth part (of the whole height of the body)"; "the palm of the hand from the wrist to the top of the middle finger is as much […] the length of the foot is one sixth of the height of the body", etc.[503] Similarly, Vitruvius argues that the members of temples are to be proportioned having their (linear) quantities arranged commensurably the one with the other.[504] His well-known description of the human body lying on its back and circumscribed in a square and a circle follows this passage.[505] The use of terms such as *schema, figurationem* and *figurae* in this section clearly indicate that these linear quantities in proportional relationships are to be understood in two-dimensional planes. The principles of proportionality among linear quantities thus arranged represents the basis for the discussion of the process of the design of temples and of their elements such as their orders, undertaken in Books III and IV. In these passages, Vitruvius geometrically describes

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[502] ibid., III. 1. 1-2.

[503] ibid., III. 1. 2.

[504] It should be noted that although Vitruvius speaks in these passages only of commensurable modules and thus quantities, incommensurable ratios among quantities in other passages of *De architectura* are also used. ibid., IV. 1. 8; VI. 3. 3. The first involves the ratio among the width of the abacus and the lower column diameter in the corinthian order, the second one of the three preferred width/lenght ratios for a room. Both instances involve thus the side/diagonal ratio of a square. At the same time, there are no doubts that Vitruvius was aware of the incommensurability of this ratio, since he discusses the doubling of the area of a square in the preface of Book IX, ascribing its discovery to Plato, and adding that this problem has to be solved by means of geometry. Indeed, incommensurable ratios among linear quantities are occasionally found also in the treatises of Renaissance authors. However, this does not affect the fact that all the quantities are considered as linear.

[505] Again, it is worth looking at the Latin of the second edition of *De Architectura* to appreciate the terminology used: "Namq(ue) si homo conlocatus fuerit supinus ma(n)ibus ac pedibus pa(n)sis circum(que) collocatum centrum in umbilico eius […] Non minus quemadmodum *schema* rotundationis in corpore efficitur. Item quadrata *designatio* in eo invenietur. […] Ego si ita natura composuit corpus hois uti proportio(n)ibus membra ad summa(m) *figurationem* eius respo(n)deant; Cum eam co(n)struisse videantur antiqui: ut etiam in oper(um) perfectionibus singolor(um) membror(um) ad universa(m) *figurae* spe(cie)m habeant co(m)mensus exactione(m)" ibid., III. 1. 4.

the three-dimensional extension of the human body only through linear quantities (the width, length and height of its members or of the whole body) or two-dimensional figures. Vitruvius then proceeds to associate these geometrical properties with the shape that temples should have in plan, or to the diameter and height of the elements composing the orders.

Following *De architectura*, and sometimes referring to later Christian authors such as Augustine, the human body was discussed by Renaissance theorists as the repository of proportional relationships among commensurable linear quantities or corresponding to the two types of lines, the straight and the curved.[506] As in Vitruvius, the body was therefore geometrically considered and abstracted in a series of linear quantities or two-dimensional figures. As we have seen, Pacioli acknowledges in the *Summa* that the proportioning of churches should be undertaken through the employment of linear measurements derived from the human body. The passage opens with a general reference to the works of Vitruvius, Dinocrates, Frontinus and Pliny. As noted, Pacioli argues here that if architecture "is not properly proportioned, it does not please or delight the eye, neither [is it pleasing or delightful] to inhabit it. And it is also said to be unhealthy […]". Pacioli continues;

> And this is proved for the temples in their formations and constructions, and for their choirs, the harmony of the sacred buildings is worth not much if they are not disposed with due proportion. And [Vitruvius] demonstrates how their length should commensurately correspond to their width according to the human body, similarly to which the churches require to be built, in the same way that the divine Goodness wanted the ark of Noah being built how many cubits long, wide and high according to the body of our saviour Jesus Christ, as written by Augustine in *De Civitate Dei.* And the

[506] However, as noted by Mario Carpo, while Vitruvius discussed sequences of geometrical operations that could be carried out on site and at full scale, with rulers and compasses, Renaissance theorists gradually replaced this approach with number-based operations that could take place in a scaled and measured drawings. Mario Carpo, *The alphabet and the algorithm* (Cambridge MA: The MIT Press, 2011), 28-29.

> same [is said] of palaces or other residential houses, with one or more floors, as it is discussed in his great and difficult work on architecture.[507]

The biblical Ark was commonly referenced by Renaissance theorists in relation to the proportions of the human body. However, no author tried to show the Ark's actual shape. Rather, this reference was made only to support the argument that assigned to the human body a central and symbolic role in the divine process of creation, and that linear quantities in proportional and commensurable ratios should be applied to sacred buildings. Notably, Augustine himself made reference to the human body while referring to a number of symbolic associations between the Ark and the body of Christ.[508] This was perfectly understood by fifteenth-century architectural theorists. No one, including Pacioli, argued that a church should have been built exactly following the quantities given by Augustine, or their arrangements in space. Ultimately, the spatial and proportional relationships among these quantities were not fixed but could be adjusted, provided that their commensurability was retained.

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[507] "L(')architettura ancora nulla vale (si commo vitruvio, Dinocrate, Frontino e Plinio approbano) se debitamente non e proportionata ne a l(')ochio, ne a l(')abitare mai piaci ne diletta. E ancor dici non esser sana. E si prova anche di templi in loro formationi e constructioni, e di loro cori, l(')armonia de li divini officii poco valere, se con debita proportione non sonno disposti. E dimostra commo la co(m)muna longheçça loro, debia a la loro largheçça correspondere inducendo la forma del corpo humano a la cui similitudine vole le chiesi essere fabricate, commo al corpo del nostro redemptore Jesu Christo volse la divina bonta che Noe l(')arca fabricasse tanti co(m)biti longa larga e alta commo expone Augustino in de civitate dei. E cosi anche de palaçça e altre case da habitare, in uno e piu solari, commo se co(n)tiene ne la sua grande e difficile opera de architettura." Pacioli, *Summa*, 68v (Pars prima)

[508] A references to ark as based on the dimensions of the human figure is also found at the outset of Chapter 7 of the ninth book of Alberti's *De re aedificatoria*, when the analogy between the human body and columns is introduced. As noted, since the human body was understood to be the most perfect creation in nature, it was considered by Vitruvius and Renaissance architectural theorists as the object for imitation. In *De Civitate Dei* Augustine describes the Ark as three-hundred cubit long, fifty wide, and thirty high. Augustine adds that the door was placed on the side since it was there that the crucified body of Christ had been perforated, and that the ark was made of squared timber to signify the saints's stable life. "Nam et mensurae ipsae longitudinis et altitudinis et latitudinis eius significant corpus humanum, in cuius veritate ad homines praenuntiarus est venturus et venit. Humani quippe corporis longitudo a vertice usque ad vestigia sexiens tantum habet quam latitudo, quae est ab uno latere ad alterum latus, et deciens tantum quam altitudo, cuius altitudinis mensura est in latere a dorso ad ventrem; velut si iacentem hominem metiaris supinum seu pronum, sexiens tantum longus est a capite ad pedes, quam latus a dextra in sinistram vel a sinistra in dextram, ed deciens, quam altus a terra. Unde facta et arca trecentorum in longitudine cubitorum et quinquaginta in latitudine et triginta in altitudine.Et quod ostium in later accepit, profecto illud est vulnus, quando latus crucifixi lancea perforatum est; hac quippe ad illum venientes ingrediuntur, quia inde sacramenta manarunt, quibus credentes initiantur. Et quod de lignis quadratis fieri iubentur, undique stabilem vitam sanctorum significat; quacumque enim verteris quadratum, stabit; et cetera, quae in eiusdem arcae constructione dicintur, ecclesisticarum signa sunt rerum." Augustinus, *De Civitate Dei* (Turnholti: Brepols, 1955), XV, 26.

Given the authority enjoyed by *De Architectura* and patristic writings in the geometrical interpretation and abstraction of the human body in the fifteenth century, it is understandable that Renaissance theorists did not consider taking the width, length, and height of the human body conjointly and base their designs on this shape.[509] The only geometrical property to be taken from the human body and applied to architecture in fifteenth-century architectural treatises was the commensurability among its linear quantities of depth, width, and height. This is further demonstrated by the use of the same analogy for elements at markedly different scales and with different formal properties, such as a whole building and a column of a building.

At the same time, however, a tendency towards a closer quantitative correspondence between the geometrical extension of the human body and architectural elements is noticeable in architectural writings of the late fifteenth century. It is found, for instance, in Francesco di Giorgio Martini's *Trattati*. Although the process still essentially relies on linear quantities applied to a two-dimensional plane, Francesco di Giorgio attempts to more accurately employ quantities derived from the human body to the plan and elevation of churches. The height of the human body is divided into nine or seven parts, and from this division, Francesco di Giorgio derives the position of the columns and the walls of a church in plan.[510] A similar process is applied to the facade of the church, and the height of the human body is divided into seven parts.[511]

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[509] Although its should be noted that the column since the column had an already a pre-established circular (and in Renaissance times also squared) base shape. Indeed, the reliance on a procedure that involved only linear quantities in proportional relationship understood in a plan and elevation was possible because some of these elements, such as the column, had an already established (standard) shape.

[510] Francesco di Giorgio Martini, *Trattati di architettura ingegneria e arte militare,* ed. Corrado Maltese, 2 vols. (Milan: Il Polifilo, 1967), 403-405.

[511] ibid., 394.

Similarly, in his *Tractato de l'architectura*, Pacioli accurately describes the quantities associated with the profile of the head and their proportional relationships. This follows an introduction that considered human proportions derived from *De architectura*.[512] The discussion begins by relating the head's profile with an equilateral triangle: "[…][the body's] most noble exterior member, that is the head, if you look well, you will see that it is formed on the first figure with straight lines, namely the equilateral triangle called *ysopleuros* standing as foundation and origin of all the ensuing books of our Euclid in the first part of his first book".[513] He continues with a lengthy explanation of the proportioning of the head's profile. However, in a meaningful passage that concludes the description of the lines of the profile that can be expressed by ratios of whole numbers, Pacioli explicitly addresses a *p(er)spectivo,* (i.e. someone skilled in practicing perspective), while discussing the nature of abstract mathematical knowledge and its material-physical instantiations:

> And all the parts described so far in the profile [of the head] are rational and known. But when the irrationality of proportions occurs, when they cannot in any way be mentioned by numbers, these are left to the good judgement [*degno arbitrio*] of someone skilled in perspective [*p(er)spectivo*] that with grace will complete them. This is because art imitates nature as much as it can. And therefore, if the maker made what nature made you would not call it art but another nature, that is in all similar to nature, and therefore this would be the same nature [again].[514]

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512 "Cioe p(ri)ma diremo de la Humana p(ro)portione respecto al suo corpo e membri, pero che del corpo humano ogni mesura con sue denominationi deriva e in epso tutte sorti de proportioni e p(ro)portionalita se ritrova con lo det(t)o de l'altissimo mediante li intrinseci secreti de la natura. E per q(ue)sto tutte nostre mesure e instrumenti a dimensioni deputati per li publici e privati comme se dicto sonno denominate dal corpo humano l'una detta bracio l'altra passo, l'altra pede, palmo, cubito, digito, testa, etc. E cosi comme dici el nostro V(itruvio) a sua similitudine dobiam proportionare ogni hedificio con tutto el corpo ben a suoi membri proportionato. E per questo prima diremo de epsa mesura Humana con suoi proportioni a suoi membri secondo la quale ve avete a regere in vostre opere lapicide maxime de frontespicii e altre degne faciate de templi porti e palaççi quali sempre se costumo adornarli de colonne cornici e architravi comme a pieno ne dici el nostro V(itruvius)." Pacioli, *Divina proportione (1509)*, 24v-25r (Tractato de l'architectura) (folio 25 is marked as 17).

513 Pacioli writes "nel primo luogo del suo p(ri)mo libro". I translated 'luogo' with 'part', but this could also be translated in a more literal way with 'place'.

514 "E queste p(ar)ti narrate finora al suo p(ro)filo tutte vengano a essere rationali e a noi note. Ma dove intervene la irrationalita de le proportioni cioe che p(er) alcu(n) mo(do) non se possono nominare per numero restano al degno arbitrio del p(er)spectivo qual con sua gratia le ha a terminare. Perochè l'arte imita la natura quanto li sia possibile. E se ap(p)onto l(')arteficio facesse q(ue)llo che la natura ha facto non se chiamaria arte ma un(')altra natura totalitier a la prima simile che verebe a essere la medesima. Questo dico a cio non vi dobiate maravegliare se tutte cose aponto non r(e)ndano a le mani de l(')orefice peroche non e possibile. E di qua nasci che li savii dicano le sci(nti)e e discipline mathematici essere abstracte e mai actualiter no(n) e possibile ponerle in esservibili" " Pacioli, *Divina proportione (1509)*, 25v (Tractato de l'architectura).

Here, Pacioli almost literally repeats Aquinas's dictum that "ars imitatur naturam in quantum potest".[515] However, Pacioli highlights the role of the maker's eye in judging what mathematical irrationality leaves undefinable with numbers and further clarifies this point: "And the more parts are known, the more it is useful to the *p(er)spectivo,* since then more easily the eye can apprehend the quantity of the thing that needs to be placed [in the work], be they animals, trees or buildings".[516] The text continues with a description of a net [rete] "with many fine threads of cittern or silk or lashes small or large as they prefer [and] pulled tightly" placed by painters "between [them] and the thing they wants to portray in its real shape [propria forma]" to draw the head's profile.[517] An illustration of the net superimposed to the head's profile accompanies the text (illus. 4.1). The whole passage is most probably referring to the same process employed by Leonardo da Vinci to portray regular and other bodies derived from these for Pacioli's *Compendium de divina proportione*. Although the *Tractato* is dedicated to "respectable stonecutters that practice sculpture and architecture",[518] here, Pacioli clearly praises the skills of painters and the relevance of their knowledge of perspective while dealing with architecture. A higher level of accuracy in the quantification of the human body corresponds, in Pacioli's' argument, with a greater ability of judgment in a maker skilled in perspective.

In relation to this, it is worth adding a final and frequently noted point when the works of late fifteenth-century architectural theorists and practitioners are considered in their relationship with *De architectura*. Vitruvius dealt primarily with Greek and Hellenistic architecture, while

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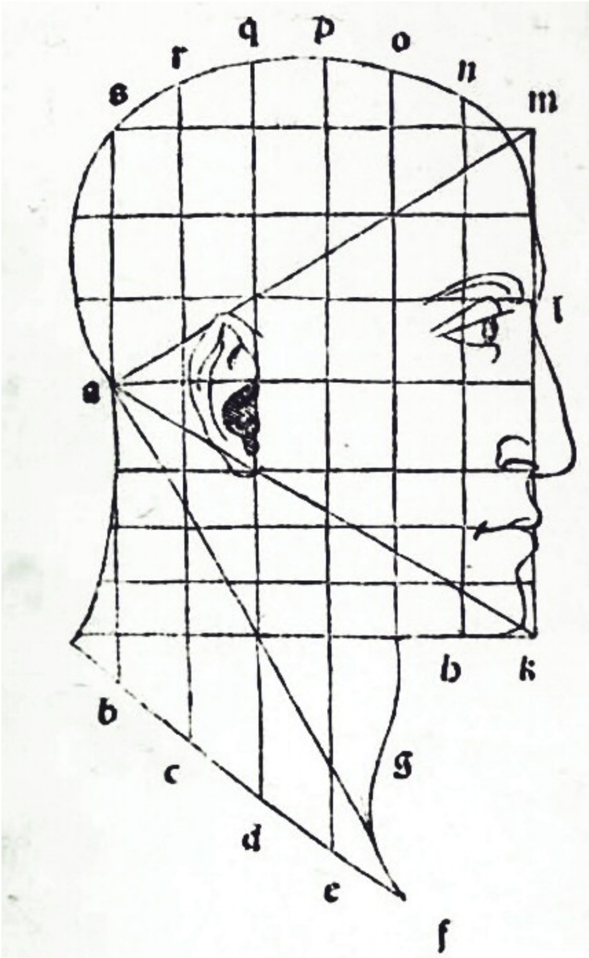
[515] "Art imitates nature as much as it can", as noted by Pacioli. Aquinas, *Commentarium in Libros Posteriorum Analyticorum*, I, i, 5; cited in Eco, *The Aesthetics of Thomas Aquinas*, 173.

[516] "E quante piu parti se fa note tanto fia piu comodo al prospectiuo peroche meglio vene con lochio aprenhendere la quantita dela cosa che vol porre o sia testa o sia che altra cosa sevolia comme animali albori hedifitii" Pacioli, *Divina proportione (1509)*, 25v.

[517] ibid.

[518] The dedication reads: "A li suoi caris discipuli e alievi Cesaro del Saxo, Cera del Cera, Rainer Fra(n)cesco de Pippo, Bernard(n)o e Marsilio da Mo(n)te e Hieronymo del Sacciarino e co(m)pagni del Borgo San Sepulchro, degni lapicidi de scultura e architectonica faculta solertissimi sectatori". ibid., 23r.



4.1—The proportions of the head's profile highlighted by means of a superimposed net. Note the location of lines marking the base of the chin, the mouth, the base of the nose and forehead, and the location of the ear. At the same time, as noted by Pacioli, the outlines of other parts of the head do not follow the net. Luca Pacioli, *Tractato de l'architectura*, *Divina proportione*, 25v [detail].

the Roman buildings and ruins that could be seen in Rome and across the Italian peninsula dated mostly to the Imperial age and showed formal and spatial solutions that differed from those discussed in the text. As Pacioli proves in his *Tractato* while discussing the proportional relationship of the head to the whole body, there was a growing tendency in fifteenth and early sixteenth-century theorists and practitioners towards measuring ancient works and comparing their direct experiences of them with the proportional relationships found in the available copies of *De architectura*.[519]

519 Discussing the lenght of the head in relation to the lenght of the whole body Pacioli adds: "E questa altezza comunamente da li pittori e statuarii antichi se prende per una testa in loro opere, comme per statue e altre figure in Roma la esperienza sempre ci ha dimostro e de continuo li nostri con tutta diligenza el medesimo demostrano." Pacioli, *Divina proportione (1509)*, 26v (Tractato de l'architectura).

However, where the human body was concerned, its three-dimensional extension (volume) was not applied to architecture unless through two-dimensional or linear quantities. In other words, when dealing with (and in some cases defining) the proportional relationships of the human body, only linear and two-dimensional quantities derived from its formal properties were found to be sufficient.[520] The forthcoming sections of this chapter concern the quantitative relationships found in nature and their visual perception, that as we have seen, are recurrent throughout Pacioli's works, and are inevitably correlated to his discussion of architecture.

4.1.2 The process of visualisation and the design of sacred architecture in fifteenth-century treatises

It is should now be somewhat easier to consider the same process when linear quantities are applied in proportional relationships to design sacred architecture in fifteenth-century treatises. As noted, this analysis will mainly concentrate on the process by which these are applied to the definition of the formal properties of architecture, rather than on the specific ratios employed.

It should be clarified as to why sacred architecture is considered in this and the next sections of the thesis. Scholars have sometimes considered the formal properties of sacred architecture conjointly with (or even depending on) their function and their symbolic meaning. But these are clearly separate. Formal properties have historically evolved, even when the functions and meanings attached to them have remained essentially the same. Moreover, sacred buildings usually have a large covered central space, with spaces of minor dimensions (chapels or apses)

[520] Notably, except for Filarete's *Trattato* (and it should be noted that Filarete was a sculptor-architect), the discussion of all-round anthropomorphic column (with obvious analogical meanings) is completely absent in fifteenth-century and early sixteenth-century treatises of architecture, although discussed by Vitruvius. *De architectura*, I, 1, 5-6.

arranged around it. As such, their proportions are more easily controlled and discussed. Simultaneously, these buildings are considered and understood in fifteenth-century treatises as isolated from their immediate surroundings and therefore with a clearly bounded three-dimensional extension. As argued by Alberti, it is preferable for temples to be perfectly visible from every direction.[521] In addition, their parts could also be read in isolation; indeed, quarter or domes in the apses of two churches are explicitly referenced when the employment of geometrical bodies is discussed in Pacioli's *Compendium.* It is important to compare Pacioli's arguments (examined in the second part of this chapter) with treatises of other coeval authors. However, before turning to Pacioli's writings in the second part of this chapter, this section will concentrate on treatises by other authors.

The overall process is developed in the treatises of Alberti and Francesco di Giorgio Martini through the employment of two-dimensional figures at the level of the plan, and linear quantities for the height of the building. The latter is based on the diameter of the plan if circular or inscribed in a circle, or on its width if based on a square. Although this is not the same rigid proportioning based on a square grid proposed by Filarete to regulate the plan for the Duomo of Sforzinda,[522] it still indicates that in the process of designing sacred architecture, the three-dimensional shape of the building was derived by the elevation of the plan.

It is however important to consider in detail the steps undertaken in the processes of setting-up and proportional arrangement. The design of temples is notably considered in Book VII

[521] Alberti, *De re aedificatoria*, VII, 3, 549. "Ea re pro fronte habebit ampla et se dignam plateam, circuetur stratis laxioribus, vel potius plateis dignissimis, quod undevis praeclare conspicuu sit".
[522] Filarete starts from the subdivision [*scompartitio*] of the plan based on a square grid and follows this to establish the location of the elements of the building. Overall, the square grid is composed by 15 modules on each side. Antonio Averlino detto il Filarete, *Trattato*, ed. Anna Maria Finoli e Liliana Grassi, 2 vols. (Milan: Il Polifilo, 1972),183 ff., esp. 191-192.

of Alberti's *De re aedificatoria*. After an initial introduction, Alberti notes at the beginning of Chapter 4 that the temple is made up of two parts, the portico and the interior cella, and that each varies from temple to temple. A discussion of the various types of two-dimensional shapes that the plan of a temple can take then follows. The first and most perfect, Alberti argues, is the circle. He adds that this is manifest by nature as nature privileges the circle, but also delights in hexagonal shapes.[523] Alberti continues by describing the types of plans derived from quadrilateral shapes by the ancients, namely the square and one-half, the square and one-third and the square doubled. Polygonal shapes derived from the circle are then described. Overall, Alberti lists nine basic planar shapes that can in turn be encircled by chapels or apses. These can be rectangular or semi-circular in plan, depending on their position and number.[524] The width of the entrances to chapels within apses, Alberti continues, should be commensurate with the width and length of the temple. Having completed the discussion of a temple's internal spaces, Alberti moves to consider the portico in front of or around it. As in the previous case, the discussion in this section concerns only the proportioning and arrangement of elements in the plan of the building.[525]

After describing the shapes that a temple may take in plan, and the proportioning of the portico in front of or around its cella in the same plane, Alberti begins in Chapter V an extended section dedicated to the proportioning of the orders.[526] As a result, the height of the cella is discussed only in Chapter X. Here, Alberti argues that in temples with a circular plan,

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[523] Alberti, *De re aedificatoria*, VII, 4, 549, 551.
[524] To summarise, these two-dimensional shapes are: the circle, the square, the hexagon, the octagon, the decagon, the dodecagon, and three shapes derived from the square. ibid.
[525] ibid.
[526] In the chapter that follows from this (Chapter V) Alberti considers the vertical proportioning of the orders. This section occupies all chapters until Chapter X. As noted by Carpo, Alberti here does not spell out the measurements of the parts, but gives instead a method that itemising the steps of a process generates the right proportional relationships among the parts. The proportioning of the elements of the capitals is followed by the proportioning of the bases and the proportioning of entablatures. Mario Carpo, "Drawing with Numbers: Geometry and Numeracy in Early Modern Architectural Design," *Journal of the Society of Architectural Historians*, vol. 62, no. 4 (Dec., 2003): 448-469.

the wall is never less than half the diameter, in many cases two/thirds of this, and in others three/fourths. However, Alberti notes, the most skilled divided the circumference of the plan into four parts and after straightening one of these, applied this quantity to the internal wall with a ratio (to the circumference) of about eleven to four.[527] Moreover, if chapels have to be built on both sides of the plan, sometimes the walls are made to a height that is equal to the width of the plan.[528]

The same process highlighted in the case of temples is repeated when basilicas are considered in Chapters XIV and XV of the same book. In summary, linear quantities are first applied to the proportioning of the plan (in the case of a basilica, the length should be double the width). These are then subdivided to provide the width and length to be associated with each architectural element in the plan. Secondly, a discussion of the linear quantities from which the elevation of the building is dictated follows. Regarding the orders, Alberti notes that their diameters should be reduced if compared with the ones given for temples, while all other elements should conform to the proportional relationships previously determined.[529] Overall, the sequence of operations by which the three-dimensional shapes of temples are geometrically constructed is clear. In order to achieve the desired three-dimensional composition, the process described by Alberti deals with the progressive arrangement of

[527] Alberti, *De re aedificatoria*, VII, 10, 607. The last ratio, as noted by Wittkower and Rykwert, Tavernor, and Leach is foreign to Alberti's theory and of difficult comprehension. Wittkower, *Principles,* 7-8; Alberti, *On the Art,* 219, 394. An explanation was given by Vasily P. Zubov. Since Alberti expressively says this is a fourth of the circumference that has been straightened, he most likely referred to 11/14, that is 22/7 (the common approximate ratio of circumference and diameter in the fifteenth century) divided by 4. Vasily P. Zubov, "Quelques aspects de la théorie des proportions esthétiques de L.-B. Alberti," *Bibliothèque d'Humanisme et Renaissance*, t. 22, no. 1 (1960), 56.

[528] Alberti, *De re aedificatoria,* ibid.

[529] "Columniationes etiam nequicquam habebunt gravitatem, quam habeant quae templis apponuntur. Qua da re, si praesertim trabeata utemur columnatione, sic ratiocinabimus. Nam, si erunt illae quidem corinthiae, adimetur earum ex crassitudine pars duodecima; si autem ionicae, adimetur decima; si vero doricae, nona. In caeteris deinceps capitolorum et trabis et fasciae et coronae et eiusmodi cioaptationibus templa imitabuntur." Alberti, *De re aedificatoria*, VII, 14, 639, 641. Two chapters, devoted to the discussion of commemorative monuments and the use of statues complete Book VII.

linear quantities and two-dimensional figures.[530] Geometrical bodies of the kind discussed by Pacioli are not taken into consideration in determining the formal properties of temples.

A similar itemised process is described in Francesco di Giorgio Martini's *Trattati*. As in the case of Alberti, both geometrical proportional relationships and arithmetical quantities are employed. However, it should be noted that Francesco di Giorgio also adds examples with dimensions given in *piedi*. The section of the *Trattati* dealing with the design of temples begins, like Alberti, with an introduction to this typology of buildings. However, in this section, Francesco di Giorgio mainly concentrates on the role of man as microcosmos, or small material world, linked to the eternal perfection of immaterial substances.[531] The argument is instrumental in introducing the role of linear quantities derived from the human body, and thus in describing the proportioning of the orders that follows the initial prologue. As is well-known, Francesco di Giorgio places particular emphasis on the proportioning of the human body. This subject, he argues, should be considered first because the proportional quantities of the column derive from the proportioning of the human body, and from the former, many proportional relationships applied to the various parts of the temple are derived, as columns are located in various parts of a temple.[532]

Francesco di Giorgio had previously introduced the three main types of temples in respect to their form and figure, from which an infinite number of other figures can be derived. The

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[530] From this follows indications on which ornaments should be applied to the walls of the temple, while the vaults and the roof to cover the building, the proportioning of openings and altars are taken up next. ibid., VII, 16-17, 648-663.

[531] "Cosí per opposito considerando quale sia el principio della vita sua, argumento, stato e decremento e finalmente corruzione o morte, cose tutte comune eziando alli vilissimi animali, dall'altra parte la perfezione delle sustanzie immateriali e la altezza delle opere incomprensibili [perfettamente] del primo fattore […]" Francesco di Giorgio, *Trattati*, 370.

[532] ibid., 373. The passage is obviously dependent on Vitruvius's *De architectura*. In this section Francesco di Giorgio also correlates the proportioning of the elements that composed a man's head in front elevation and profile as divided in three parts with the symmetry of the orders, an argument that was reproposed by Pacioli in the *Tractato de l'architectura.*

first and most perfect is round,[533] the second is angular and has many sides, and the third is a combination of the first two.[534] He adds that there are three external parts of a temple: the portico, the vestibule and the ante, noting that all are composed by an arrangement of columns. The shape of the interior part of the temple—the cella— is considered by Francesco di Giorgio only after the proportioning of the orders. Significantly, he first clarifies the shape that the cella should have in plan. Like Alberti, this can be a circle, a polygon inscribed in a circle or an oblong figure that is squared but with different sides, or a combination of the previous two types.[535] After stating these principles, he continues that if the temple is circular or similar, its height can be determined in two ways. To find the height, the first method takes the diameter of the plan and adds two/thirds of its length, the second method takes instead the diameter and adds seven/elevenths of its lenght. Similarly, Francesco di Giorgio adds that the chapels around the perimeter of the cella should follow the latter's proportional ratio in their width/height, even if their width is left to the choice of the architect.[536] Two methods to find the linear quantities that determine the height of the internal cornices [*ricinti*] in the walls of the temple follow,[537] with the second acting as the base of the *cupola* or *tolo*.[538] The *cupola*, Francesco di Giorgio argues, should always be—without exception— half the diameter of its base.[539] This is the same proportional relationship indicated by Vitruvius for the roof of the *tholus,* although Vitruvius refers to the diameter of the whole work and this can also be understood as including the peristasis around

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[533] A similar incipit to Alberti's. *De re aedificatoria*, VII, 4, 549.
[534] Francesco di Giorgio, *Trattati*, 372.
[535] "Dico adunque che due sono le figure le quali ricercano diverse proporzioni. La prima è la rotunda con tutte le figure composte di linee rette che al tondo traeno, come la esagona e la pentagona, ortogon<i>a e così in infinito multiplicando li anguli e non diminuendo. La siconda è le figura oblonga, cioè quadra inequilatera, con tutte le altre figure che a questa se si assimigliano […] dichiarate le condizioni e proprietà delle prime due, la terza per sé rimane chiara e manifesta […]". ibid. 395.
[536] ibid., 395-396.
[537] The first recinto should be, according to Francesco di Giorgio at the five/ninth or the six/eleventh of the total height. ibid., 396.
[538] In another instance, few lines after, Francesco di Giorgio Martini class this as a *tiburio,* following the name given to this part of the building in Lombardy. ibid.
[539] "et ancora del tolo overo cupola, però dico che sempre senza eccezione alcuna l'altezza della cupola o tolo debba essere la metà del diametro della larghezza da piedi, cioè avere proporzione subdupla a quello." ibid.

the cella.[540] While Alberti does not deal with domes in the seventh book of *De re aedificatoria,*[541] this is the only instance where Francesco di Giorgio discusses this type of element. It is geometrically described first by means of its circular base, and then by the proportional relationship between its height and diameter, with no other geometrical indications or indeed variations of its shape. Examples of the employment of numerical values follow this description, which are illustrated by line drawings on the margins of the treatise. Francesco di Giorgio takes a diameter of 55 and 120 *piedi* to illustrate the described proportional relationships among linear quantities. The quantities annotated in the drawings provide a clear visualisation of the process as verbally described, although their values sometimes do not follow what is stated in the text. It should be noted that no attempt is made to show any thickness of the walls or any materiality, as the construction is purely abstract and illustrated only by means of lines and arithmetical quantities.

After this section, temples with an oblong of cross-shaped plans are considered. Although the geometry varies, the steps undertaken in proportioning the elements and their sequencing is essentially the same. As noted by Francesco di Giorgio, "the most important principle, although not much followed by our contemporary architects […], is that every maker should assign the due proportion to the dimensions among them— that is, to a diameter of whatever quantity should follow a proportionate height".[542] The treatise concludes by considering the proportioning of the entrances, and the position of the simulacrum, chapels and vaults among others. The quantities used are all commensurable, however, in the section discussing the height/width ratio of residential rooms, he also describes two practical methods to

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540 "In medio tecti ratio ita habeatur, uti, quanta diametros totius operis erit futura, dimidia altitudo fiat tholi praeter florem". Vitruvius, *De architectura*, IV. 8. 3.
541 Alberti considers the process of construction of vaults and domes in Book III.
542 Francesco di Giorgio, *Trattati*, 405.

approximate incommensurable ratios.[543] Nonetheless, as noted by Francesco P. Fiore, surveys of buildings designed by Francesco di Giorgio have shown that commensurable linear quantities account for most of what was actually built, and that these were mostly accurately applied to the internal surfaces of buildings, both in plan and elevation.[544]

Overall, it should be noted that the sequence of steps to be taken in the design and quantification of sacred architecture, as described by Alberti and Francesco di Giorgio, closely corresponds to the sequence of phases to be undertaken for the actual construction of the building, and that both largely rely on the employment of two-dimensional figures and linear quantities in the main planes of the building.[545] On the one hand it is not unexpected that architects and architectural theorists would describe design stages that reflected stages in the construction process. However, this also has consequences for the process of the definition of its formal properties. On the other hand, although this is in principle also true for Francesco di Giorgio, Alberti in *De re aedificatoria* clearly emphasised a separation between the creative process in the mind of the author/architect and the manual work of the artisan who would give form to the material of the building according to its lineaments.[546] It is well known that Alberti argued that it was quite possible to design forms in the mind without having recourse to any material and employing only immaterial lines and angles.[547] Thus, the relationship between the material and immaterial properties of architecture as discussed in

543 As Martini argues, although these ratios are incommensurable [*non possano avere se non sorda radice*], all can be approximated and composed of lines in proportional relationship "e perché tutte le altezze (di) detti sale e ticlini le proporzioni lo(ro) si truova di numari semplici essere tratti, et essi numari non possano avere se non sorda radice e così restano mobili, ma tutti hanno modi e regole composte di più varie e proporzionate linee […]" ibid., 349.

544 Francesco Paolo Fiore, "Principi architettonici di Francesco di Giorgio," in *Francesco di Giorgio alla corte di Federico da Montefeltro,* ed. Francesco P. Fiore (Firenze: Olschki, 2001), 396, 398.

545 Firstly the plan is traced and the elevation of the building is begun from the foundations, the building's height should then be determined; lastly, the roof is built and its shape determined.

546 "periti artificis manus, quae lineamentis materiam conformaret."Alberti, *De re aedificatoria*, Prologue, 15.

547 "Erit ergo lineamentum certa constansque per- scriptio concepta animo facta lineis et angulis perfectaque animo et ingenio erudito."Alberti, *De re aedificatoria*, I, 2, 21.

the *De re aedificatoria* will next be considered. This will later support the comparison between Alberti's and Pacioli's arguments.

4.1.3 The relationship between *concinnitas* and linear quantities in Book IX of Alberti's *De re aedificatoria*

At this point, it is essential to compare Pacioli's concept of the geometrical body in the light of the formal and aesthetic properties considered in the previous chapters, with Alberti's notion of beauty in creation and in architecture according to *De re aedificatoria.* In Book IX of *De re aedificatoria,* Alberti makes clear the three principal theoretical components from which a work's *concinnitas* (which results in beauty) is achieved. These are number (*numerus*), *finitio*, and position (*collocatio*). As Alberti concludes, "beauty is a certain harmony (consensus) and agreement (*conspiratio*) of parts of that to which they belong, according to a definite number, *finitio* and placement, that *concinnitas*—the absolute and primary reason (proportion, *ratio*) of nature, requires."[548]

Numerus refers to a quantity, and *collocatio*, as Alberti argues, relates to the arrangement of parts. Although the exact meaning of *finitio* has been the subject of debate among scholars, Alberti clearly states later in this same chapter that *finitio* "is the correspondence between lines that define dimensions such as the width, length, and height".[549] As summarised by Mitrović, the explanation proposed by Orlandi, that *finitio* is a "determined proportion between lines" can be taken as a fair account of its use in the context of Book IX and initial paragraphs of Book X of *De re aedificatoria*, while the translation as 'outline', proposed by Rykwert, Tavernor,

[548] I have employed here the translation by Branko Mitrović; Mitrović, *Serene*, 111. Alberti says: "Quae si satis constant, statuisse sic posumus: pulchritudinem Esse quendam consensum et conspirationem partium in eo, cuius sunt, ad certum numerum finitionem collocationemque habitam, ita uti concinnitas, hoc est absoluta primariaqueratio naturae, postularit." Alberti, *De re aedificatoria*, IX, 5, 817.

[549] "Finitio […] est correspondentia quaedam linearum inter se, quibus quantitates dimentiatur." ibid., IX, 5, 821.

and Leach, works better in the remaining sections of the treatise.[550] Overall, as Mitrović concluded, it is important to highlight that in both contexts, *finitio* pertains to shape-defining properties,[551] and, I would add that in its relationship with *concinnitas, finitio* is always linked to lines.

In Alberti's text, the definition of *concinnitas* precedes a section often-quoted by scholars, which aims to clarify the nature of *concinnitas* and the application of its three components. Alberti relies here on the theory of the so-called harmonic ratios derived from musical intervals. These, Alberti argues, are then employed by architects in the most convenient way. They can be used in pairs to define the width and length of architectural elements proportionately to the *area* in which they are placed, or they can be determined in groups of three (i.e. width, length, and height), according to the measurements that are most convenient for the work.[552] As Vasily Pavlovich Zubov summarised, commenting on Alberti's design process, Alberti begins with lines (*lineae*, *lineamenta*) to then verify his geometric constructions by arithmetic calculations, and finish the whole thing by examining three-dimensional models.[553]

Unlike Pacioli, the relationship between the form (shape) and matter (spatial extension) of a body is not considered by Alberti where *concinnitas* is concerned. Conversely, Pacioli shows

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[550] The term has been considered as a synonym of 'proportion' by Orlandi,, since this is the way in which it was used in *De statua*. Rykwert, tavernor and Leach have always translated it instead with 'outline', since this term is better suited to translate *finitio* in parts of the text other than Book IX and the initial paragraphs of Book X. An overview of the debate has been given by Mitrović. Mitrović, *Serene*, 113-114.
[551] ibi., 114.
[552] "Itaque diffiniet diametros ternatim numeris, quot recensuimus, uti accommodatiores eos venire suum ad opus intelligat." Alberti, *De re aedificatoria,* IX, 6, 829.
[553] "Ainsi donc, comme l'avoue Alberti lui-même, il commençait par des lignes (lineae, linementa) pour vérifier ensuite ses constructions géométriques par des calculs arithmétiques et achever le tout en examinant des modèles tridimensionaux." This followed Alberti's remarks at the end of Chapter 10 of Book IX: "De me hoc profiteor: multas incidisse persaepius in mentem coniectationes operum, quae tum quidem maiorem in modum probarim; eas cum ad lineas redegissem, errores inveni in ea parte ipsa, quae potissimum delectassent, et valde castigandos; rursus cum perscripta pensitavi et numero metiri adorsus sum, indigentiam cognovi meam atque redargui; postremo, eadem cum modulis exemplaribusque mandassem, nonnunquam singula repetenti evenit, ut me etiam numerum fefellisse deprehenderim." Alberti, IX, 10, 861,863. Vasily P. Zubov, "Quelques aspects de la théorie des proportions esthétiques de L.-B. Alberti," 61.

little interest in harmonic ratios derived from music. As noted, he omits their full explanation even from the distinction of the *Summa* dealing with the theory of proportions. However, when considering the formal properties of a regular body, it is still legitimate to ask how they would fit into Alberti's theoretical framework. In other words, on which elements of a geometrical body would *concinnitas*, and therefore beauty, depend. Alberti indirectly provides the answer to this question in Chapter 6 of Book IX. Although Alberti's proportioning theory is largely based on commensurable quantities derived from musical intervals, he argues that some relationships in nature cannot be defined by means of numbers but only through their roots. Incommensurable ratios are thus mentioned among the linear quantities employed to find the width, length, and height of rooms.[554]

The properties of the cube are here introduced, followed by a discussion of its linear dimensions. It is worth considering the whole process as described by Alberti. The cube, Alberti notes, is consecrated to the divinity, because originating from the unit, it is the unit in its entirety. Moreover, it is the most stable among all shapes, as it rests equally on all its bases. However, Alberti adds, if the unit is the source of number and not the number itself, the first number is 2, of which the square is 4, and the cube is 8. "This is the cube", Alberti notes, "from which we will determine our proportions".[555] At this point Alberti takes the side of the cube and the diameter of its squared base, adding that the latter is not known in numbers, although it is known that it is the square root of 8 [or $2\sqrt{2}$]. Then, Alberti adds, "there is the diagonal of the cube, and we know that this is the square root of 12 [or $2\sqrt{3}$]. Lastly, there is the hypotenuse of the triangle of which one of the cathetus is the root of 4 and the other the root of 12". These, Alberti concludes, "are the three relationships of numbers and quantities to be used in defining the diameters (sic) of the cube, from which it is customary to assign the

554 "Diamteris etiam finiundis innatae sunt quaedam correspondentiae, quae numeris nequicquam terminari possunt, sed captantur radicibus et potentiis." ibid.
555 ibid., IX, 6, 831.

shortest to the width of the area, the longest to the length, and the middle to the height"; even though" Alberti adds “they can also be interchanged to suit the building”.[556] Overall, Alberti clearly did not consider that the relationship between the shape and three-dimensional (spatial) extension of the cube had to be retained when this was applied to architecture; rather, he only considered the linear quantities derived from the side and diagonals of the cube and applied them to the width, length, and height of a room.

This does not mean that Alberti could not conceive of a space in a cubical shape, as the ratio 1:1:1 is among the width/length/height ratios described for a temple or residential room in Book VII and Book IX; however, this ratio was defined (as are all other ratios), by means of linear quantities and not as a composite of shape and spatial extension as derived from a cubic geometrical body. Indeed, when Alberti introduces, arithmetic, geometric, and harmonic means in the passage that follows, he argues that they are methods to determine the three dimensions of architectural objects “without recurring to bodies or harmonies”.[557]

That Alberti considered lines the geometrical means to apply quantities to architecture is fully in keeping with the role assigned to the concept of lineaments throughout the whole of *De re aedificatoria*. At the opening of the first book of the treatise, Alberti explains that lineaments are immaterial and that the same lineaments are perceived in different buildings that correspond in all angles and lines. Moreover, in considering a building as a kind of body composed of lineaments and matter, with architects dealing only with the former; he clearly wanted, as noted by Mitrović, to highlight a building’s shape as composed of immaterial lines

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[556] “Tales igitur, quales recensuimus, diametris finiendis et numerorum et quantitatum corrispondentiae innatae sunt. Istorum omnium usus est, ut minima linea detur areae latitudini, maxima vero dentur altitudini. Sed interdum pro aedificiorum commoditate commutabuntur.” ibid.

[557] As Alberti concludes, using these three types of proportional means, “architects have found a great number of excellent solutions—both for the whole building and for single parts of it— which would take too long to discuss. “Huismodi mediocritatibus architecti et totum circa aedificium et circa partes operis perquam plurima dignissima adinvenere, quae longum esset prosequi. Atqui mediocritatibus quidem istiusmodi ad altitudinis diametrum extollendam apprime usi sunt.” ibid., IX, 6, 835.

and angles.[558] Ultimately, lines and angles are thus for Alberti the fundamental geometrical entities that define architecture at the level of human cognitive capacities.[559]

As we have seen, Pacioli concentrates instead on the relationship between form (shape) and matter (understood as three-dimensional extension) that define geometrical bodies. As noted in Chapter 3, this directly follows from Pacioli's belief in the correspondence of the visual perception of real objects and the perception of their perspectival representations, both based on a set of mathematically determined spatial principles ruled by proportions. Ultimately, for Pacioli, the concept of the geometrical body is central to this correlation. This argument should also be compared with Alberti's separation of the purposes of perspectival representations and architecture. For Pacioli however, these were essentially interrelated, since they took place in the same extended material medium (i.e. air) and were subject to the same mathematical and spatial properties. However, as Pacioli makes clear, this does not mean that immaterial lines cannot be abstracted from geometrical bodies; but rather, that the essence of geometrical bodies rests on the correlation between their form (shape) and matter (spatial extension). This is therefore a different issue from the one posed by the representation of a building in two-dimensional drawings or in a model; rather, it is related to the definition of architecture's formal properties in the design process.

Ultimately, the concept of shape in Alberti's texts is to be understood as defined by lineaments, whether they are the three-dimensional shapes of architectural objects conveyed by two-dimensional drawings, three-dimensional models or two-dimensional paintings. As

[558] "Neque habet lineamentum in se, ut materiam sequatur, sed est huiusmodi, ut eadem plurimis in aedificiis esse lineamenta sentiamus, ubi una atque eadem in illis spectetur forma, hoc est, ubi eorum partes et partium singularum situs atque ordines inter se conveniant totis angulis totisque lineis." Alberti, *De re aedificatoria*, I, 1, 19, 21. This process has been explained in detail by Branko Mitrović; Branko Mitrović, "Leon Battista Alberti, Mental Rotation, and the Origins of Three-Dimensional Computer Modeling," 315-316.
[559] Mitrović, *Serene Greed of the Eye*, 38.

lineaments are fully geometrically describable, Alberti argues that this can be done by means of lines and angles. However, if a building's shape is geometrically defined by immaterial lineaments, it cannot be simultaneously defined as a geometrical body composite of form (shape) and matter (three-dimensional spatial extension).

4.2 The application of geometrical bodies as composite of form (shape) and matter (three-dimensional spatial extension) to architecture

4.2.1 The discussion of regular and other geometrical bodies derived from these in the context of Pacioli's *Tractato de l'architectura*[560]

Procedures in the design of temples described by Leon Battista Alberti and Francesco di Giorgio Martini should first be compared with the contents of Pacioli's late *Tractato de l'architectura*. As noted, this is a short treatise dedicated to "disciples and students of Borgo San Sepolcro, respectable stonecutters that practice sculpture and architecture". [561] It was added by Pacioli to the printed edition of 1509, between the *Compendium* and the *Libellus in tres partiales divisus*, both of which deal mostly with regular and other bodies derived from these.

Although the *Tractato* is often remembered in architectural historiography only for its original association of psychological properties with Ionic and Corinthian capitals,[562] there are a number of references to regular and other bodies derived from these that should be carefully

[560] For my English working translation of extended passages of Pacioli's *Tractato de l'architettura* see Appendix 2.

[561] Pacioli, *Divina proportione (1509)*, 23r-33r (Tractato de l'architectura). As noted, it is a short text (about ten folios) dedicated to "disciples and students of Borgo San Sepolcro, respectable stonecutters that practices sculpture and architecture". In the opening section, Pacioli warns that he will satisfy his students only in part, deferring a more comprehensive undertaking of architectural theory at future times: "a piu commodi tempi e Ocio che a tali discipline s'aspectano". ibid., 23r. However, no additional theoretical writings on architecture of Pacioli have survived and probably pacioli never fulfilled his promise.

[562] "Ma li loro capitelli sonno diversi. Quello de la Ionica o voi dire pilvinata fia malenconico, pero che non leva in su ardito che representa cosa melanconica e flebile e vedovile […] Ma solo ha li voluti circu(m)circa revolti in giu verso la lo(n)gheçça de la colonna a similitudine de le donne afflicte e scapegliate. Ma la corinta ha el suo capitello elevato e adorno de fogliami e voluti co(n) suo abaco e cimasa co(m)me se dicto a similitudine de le donne de le giovine polite alegre e adorne con loro balçi." Pacioli, *Divina proportione (1509)*, 29r (Tractato de l'architectura). Hanno-Walter Kruft, *A History of Architectural Theory* (London: Zwemmer, 1994), 63; Vaughan Hart "Paper Palaces, from Alberti to Scamozzi," in *Paper Palaces, the rise of the Renaissance Architectural Treatises*, ed. Vaughan Hart and Peter Hicks (London: Yale University Press, 1998), 23-24.

examined and compared with the theoretical framework outlined in other works by Pacioli, and discussed in previous chapters of this thesis. Although somewhat scattered and arguably unaccomplished, the *Tractato* can be read, similar to Pacioli's other texts, as a collection of theoretical knowledge linked to the environments in which Pacioli operated and the architectural theorists and practitioners he would have met. In this regard, it is significant that in the last chapter, Pacioli recalls a discussion concerning the Duomo of Milan that took place in the residence of Ludovico Sforza in 1498. The duke and "many other renowned" authorities, Pacioli notes, were in attendance and the event included an exposition of the books of Vitruvius's *De architectura*.[563]

Pacioli mentions at the beginning of the *Tractato* that the following three main subjects should be discussed: "the first is public spaces, such as temples, the second buildings useful to the well-being of the city and for the defence of small and large republics or private and particular places, the third is residential architecture".[564] However, as already anticipated in the dedication, Pacioli adds that since dealing with all these subjects would take too long, their full undertaking should be postponed. Consequently, "of the temples we will not say much although they would deserve much [more attention] for [being linked to] to sacred functions. On the other hand, not less should be said of buildings designated to defend, since there are

[563] "Questa dignissima auctorita dilectissimi miei a certi p(ro)positi del domo de Milano nel 1498 sia(n)do nella sua inexpugnabile arce nella camera detta de Moroni all p(rese)ntia de lo exel. D. de q(ue)llo L. M. SF. Con lo Revere(n)dissimo Car. Hipolyto da este suo cognato lo Illustre S. Galeaçço San. Se. Mio peculiar patro(n)e e molti altri famosissimi co(m)me acade in co(s)pecto de simili. Fra glia Ltri lo eximio V. I. Doctore e co(n) e cavalie(r)e Meser Onofrio de Paganini da Brescia detto Ceveli. Il qual ibi coram egregiame(n)te exponendo a tutti astanti a gra(n)dissima affectione del nostro V(itruvius) indusse nelle cui opere parea che acunabilis fosse instructo". Pacioli, *Divina proportione (1509)*, 35r.

[564] "Prima divideremo l(')architectura i(n) tre parti p(ri)ncipali de li luoghi publici che l(')una fia de li templi sacri, l'altra de quelli deputati a la salute e defensione de le piccole e gra(n)di republiche e de li luoghi ancora privati e particulari la terça de quelli a la p(ro)pria oportunita necessari de li p(ro)prii domicilii quali ci hano da le cose contrarie a li corpi n(ost)ri nocive sempre a defendere. Pero che in queste e circa queste dicta faculta sue forçe extende, etc." ibid., 23r. This subdivision does no mention of notions related to materials and construction method and skills discussed in other architectural treatises, and indeed Pacioli does not discuss these in the treatise. This circumscribes the subjects anticipated by Pacioli to the topics discussed in Book III, IV, V, VI of Vitruvius *De Architectura* and Book V, VII, VIII, IX of Alberti's *De re aedificatoria*.

infinite machines and dispositions for military reasons".[565] After this, Pacioli begins a long digression concerning men-at-arms from Sansepolcro, only to conclude that this type of architecture will be left to future undertakings. Finally, Pacioli returns to the third part, namely buildings "for opportunity and necessity", arguing that these should be [designed] "with due symmetry of proportions and proportionality to the whole building and its parts, interior and exterior members, which our V[itruvio] fully discusses […]"[566]

In this respect, the *Tractato* follows some of the arguments that Pacioli developed in the *Compendium*. He refers to these on several occasions, mainly dealing with the application of proportions to architecture. Pacioli concludes the introductory section that addresses the dedicatees of the *Tractato* by noting that he will mainly deal with the "ornaments of architecture such as columns, cornices and frontispieces that are applied to every type of building".[567] These, Pacioli argues, will be organised in three parts. Firstly, "the proportions of the human body and its members will be discussed", then "we will discuss the round columns and how these should be duly arranged with your scalpels, both for the strength and support of buildings and for their ornament", and lastly, "we will deal with the *epistilio* or architrave and its composition".[568]

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565 "Conciosia che de li templi non se ne potria dir tanto che piu non meritassero per loro sacratissimo culto. Comme a pieno el nostro V(itruvio) ne parla. De l(')altra parte a la defensione deputata non menore sarebe al dire; conciosia che infinite quodammodo sieno le machine e dispositioni militari." ibid., 23r-v.

566 "Questa terça parte de dicta Architectura a la oportunita e necessita comme de palaççi e altri casamenti dentro e de fora con tutti suoi membri […] con loro debite symmetrie de p(ro)portioni e p(ro)portionalita al corpo tutto de lo hedificio e suoi parti e membri interiori et exteriori di quali a pieno parla el nostro V[itruvio]. ibid., 24v.

567 "Peroche nulla parte de dicta Architectura non e possibile al tutto bene essere adorna se de conçi ligiadri marmorei Porfirii sepe(n)tini o altre forti differenti prete (sic) [pietre] non sieno adorni comme de colonne, cornici e frontespicii a altri ornamenti a la parte refensiva e publica oportunita comme a la parte de le sacre." ibid.

568 " E del nostro discorso faremo tre succi(n)te parti secondo el numero de li tre exe(m)pli posti in principio de questa opera detta de la divina p(ro)portione. Cioe p(ri)ma diremo de la Humana p(ro)portione respecto al corpo e membri […] diremo de le colonne tonde co(m)me in li edifitii se habiate con vostri scalpelli debitame(n)te disponere per la forteça e substentatione de lo hedifitio co(m)me per loro ornamento. E poi diremo de lo epistilio o vero architrave e sua compositione." ibid., 24v-25r.

However, despite Pacioli's stated intentions, the *Tractato* presents a heterogeneous collection of materials and sources. It is safe to argue, as others have done, that it is a hastily arranged collection. It is, at the same time, an attempt to translate, and for the first time publish in print, sections of Vitruvius's *De Architectura* that were not available to practitioners in the early sixteenth-century.[569] However, as briefly mentioned, there are a number of references to the employment of regular and other bodies derived from these in architecture included among the heterogeneous materials, and these should be considered in their context. Having examined the introductory section of the treatise, it is now important to consider a more detailed overview of its contents.

As anticipated by Pacioli, the first part of the treatise (Chapters I-III) concerns the proportional relationships that can be derived from the human body.[570] As we have seen, the first member discussed is the head (Chapters I-II), followed by the whole body (Chapter III). Chapters IV-VII describe different types of columns and the proportional relationships among their parts. As noted, the misunderstandings of *De architectura* are for the most part the same as those that can be found in Francesco di Giorgio's *Trattati*.[571] Moreover, Pacioli clearly draws his terminology from both Alberti's *De re aedificatoria* (*latastro, orbicolo, gola* …) and Francesco di Giorgio (*stilobate, cardinale* …).[572] After a description in Chapter VIII of where duly proportioned columns can be found in Italy, Pacioli briefly considers multi-sided [*laterate*]

[569] As stated in the dedication, one of Pacioli's aims was to give proportioning rules and architectural precepts useful to his disciples and therefore to practitioners that could not have direct access to the text in the early sixteenth century.
[570]"del corpo humano ogni mesura con sue denominationi deriva". Pacioli states that all kind of measures in fact derive their name from the human body. Pacioli, *Divina proportione (1509),* 24v-25r (Tractato de l'architectura), (folio 25 is marked as 17).
[571] Bruschi, "Luca Pacioli De divina proportione," in *Scritti Rinascimentali d'architettura*, ed. Arnaldo Bruschi, Corrado Maltese, Manfredo Tafuri, Renato Bonelli (Milan: Edizioni il Polifilo, 1978), 107-121.
[572] Some of these have been noted by Bruschi and Di Teodoro. Bruschi, "Luca Pacioli De divina proportione," 114-115; As noted by Francesco P. Di Teodoro, Pacioli *Trattato* is one of the few cases, together with Leonardo notebooks, that show to have adopted the terminology of Alberti. Francesco P. Di Teodoro, "Vitruvius in the Trattato dell'Architettura by Luca Pacioli," in *Illuminating Leonardo,* A Festschrift for Carlo Pedretti Celebrating His 70 Years of Scholarship (1944–2014), ed. Constance Moffatt and Sara Taglialagamba (Leiden: Brill, 2016), 115.

columns and pyramids with round and multi-sided bases [*tonde e laterate*] in Chapters XI and X, respectively. Regular and other bodies are briefly mentioned in these chapters, with references to both the contents of the *Compendium* and their perspectival representations drafted by Leonardo da Vinci. When discussing pyramids, Pacioli notes that their "weight and measure" is always a third of the corresponding cylinder of the same height, as proved by Euclid.[573] It should be added that weight and its relationship with the strength of architectural elements is consistently mentioned throughout the *Tractato*. For instance, while discussing the stylobate or *pilastrello* of the column, Pacioli considers its proportioning appropriate in two ways: "The height of the stylobate should be twice the width of the plinth of the column, and this is because it should be proportioned on the one and the other way, that is [according] to the strength to its weight and beauty to the eye as responding to the other parts [of the work]".[574]

In Chapter XI, Pacioli introduces the set of capital letters drawn with a compass and ruler to be applied to buildings. Notably, Pacioli makes clear here that the geometry of letters, as the geometrical entities discussed in Euclid's *Elements*, are independent from the meanings to which these are associated in different idioms.[575] In other words, although letters can also convey meanings at a separate level, Pacioli is here considering only their geometrical and visual properties. After this, Pacioli mentions that he will deal with the arrangement of

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[573] As discussed in Book XII of Euclid's *Elements*.

[574] "alto due larghezze o volete dire quanto due larghezze del plinto columnare e cio debitamente sia proportionata all'uno e all'altro modo, cioe a forteçça del peso e venusta de l'occhio come respondente alle alre parti". Pacioli, *De divina proportione (1509)*, 28v, Chapter VI (Tractato de l'architectura).

[575] Pacioli gives the example of the contents of Euclid's *Elements*, that "although [it may be] written in Greek, the geometrical figures still do not change", and recounts an episode in the San Marco Square of Venice, where he explained the geometrical contents of Euclid's *Elements* ("at the presence of perhaps 50 other gentlemen") to a *barbaresco* speaking an idiom unknown to him. "In questo a suo principio me parso ponere l(')alphabeto antico. Solo p(er) demostrare a cadauno che, sença altri instrumenti, co(n) la linea retta e curva no(n) che quello ma tutto apresso cadauna natione sia ebrea greca caldea o latina, co(m)me piu volte me so retrovato a dire con effecto a p(ro)varne, be(n)che a me loro idiomi non sieno noti. […] comme qui in Vinegia, a certo barbaresco un dì i(n) su la piaça de San Marco, presenti forsi 50 gentilomini. Ma non mutando el greco le figure geometriche, cioe che no(n) facesse el quadro con 5 cantoni me oferei in tutto e per tutto li lor passi in Euclide nostro; chiaritomi da loro quid nominis, el quid rei promisi darlile io." ibid., 30v, Chapter XI.

columns in buildings in Chapter XII. However, after initially considering the intercolumniation (arguing that this should not exceed two and a half diameters for the related weakness of the architrave), Pacioli extends his argument to include the spaces [*spatii*] and intervals that concur to the whole building, and how these should be proportioned. As Pacioli argues: "as noted by V[itruvio], the architect should be very skilled in considering the place, the distances, and the weights of buildings since not in every place can these be formed with symmetries and proportions, due to the limited extension [*angustia*] of the places and other obstacles".[576] It is therefore of great skill, Pacioli concludes, "to be able to follow the squared [right-angled shapes] and round [shapes] or their parts, if possible with numbers, otherwise always with lines".[577] This, Pacioli notes, "is stated by the authority of V[itruvio] clearly in Book V".[578] Although Vitruvius was explicitly referring to theatres, Pacioli deliberately omitted this reference from his introduction and summary of the passage in *volgare* (and thus extended its application to all kind of buildings). Following this well-known passage, Pacioli significantly concludes with the suggestion that regular and other bodies derived from these could be employed in these instances:

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[576] "Peroche come dici V[itruvio] bisogna molto a l(')architecto esser sveghiato in sul facto in co(n)siderare luoghi, distantii e pesi de li edifitie, co(n)ciosia che no(n) i(n) ogni luogo sempre se po servare le symmetrie e p(ro)portioni p(er) l(')a(n)gustia de li luoghi e altri impedime(n)ti. Onde molti so(n)no co(n)strecti formarli altrame(n)te che suo volere." ibid., 31r.

[577] "E p(er) questo fia misteri q(an)to piu si po, tenerse al q(ua)dro o(ver) to(n)do e lor p(ar)ti p(er) (qua)lche mo(do) note se possibil fia per nu(m)ero, al ma(n)co per linea no(n) ma(n)chi. Il che tutto lui [Vitruvio] el co(n)chiude in questa aurea auctorita nel q(ui)nto libro posta formaliter, v(idelicet): […]" ibid.

[578] The passage from Book V (6.7) of *De architectura* as included by Pacioli reads: Nec t(ame)n in O(mn)ibus theatris symmetriae ad o(mn)es r(ati)ones et effectus possunt, sed oportet architectum a(n)i(m)advertere q(ui)bus rationibus necesse sit sequi symmetria(m), et q(ui)bus p(ro)portionibus ad loci naturam aut magnitudinem operis te(m)perari sunt (e)n(im) res quas et in pusillo et in magno theatro necesse e(st) eade(m) magnitudine fieri propter usum uti gradus diaçeumata, pluteos, itiner, asces(s)us, pulpita, tribunalia et si qua alia intercuru(n)t, ex quibus necessitas cogit diccendere a symmetria ne impediatur usus. Non minus si qua exiguitas copiarum Id est marmoris materie relinquarumq(ue) rerum que parantur in opere defuerint Paulum demere, aut adicere, dum id ne nenimium improbe fiat. Sed consensu non erit alienum. Hoc autem erit si architectus erit usus peritus preterea ingenio nobili solertiaq(ue) non fuerit visuatus etc." ibid. The passages from *De architectura* included by Pacioli are, as noted by Francesco P. Di Teodoro (apart few limited variations) consistent with the Florentine printed edition of 1496, in turn related (although with a number of changes) to the *editio princeps* published in Rome (c. 1486-1492). This is therefore the *terminus post quem* for the preparation of this parts of Pacioli *Tractato*. Francesco P. Di Teodoro, "Vitruvius in the Trattato dell'Architettura by Luca Pacioli, 114-119. For the changes between the editio princeps and 1496 edition: Lucia A. Ciapponi,"Fra Giocondo da Verona and his edition of Vitruvius," *Journal of the Warburg and Courtauld Institutes*, vol. 47 (1984): 73, appendix IA.

> I briefly conclude that in addition to *arte* the good architect should also have *ingegno* in compensating for what is missing and reducing what is superfluous according to the circumstances and disposition [*dispositione*] of places so that their buildings will not appear monstrous. And for this reason, I studied with many difficulties and long vigils for you and anyone else the forms [*forme*] of all 5 regular bodies, with the others dependant from these; and all these [were] included in this work with their canon of due proportions, and [if you] look at them, I am sure [you] will be able to employ [them] for your needs. And the others [dealing with] mechanics and science will [also] benefit from their usefulness, and [these bodies] should be given to whatever art, crafts or sciences requires them, as the divine Plato makes clear in his Timaeus.[579]

Pacioli argues here that regular and other bodies derived from these should be recommended for their employment in devising the formal properties of buildings when there would be difficulty in achieving proportional relationships for the particular arrangement of places. The passage should be compared to the discussion of quantities in proportional relationships and their application to architecture at the end of Chapter 5 in Book IX of Alberti's *De re aedificatoria,* discussed in the previous section. Alberti was referring to harmonic proportions expressed by linear quantities, to be employed in the most convenient way: in pairs to define the width and length of architectural elements proportionately to the *area* in which they are placed; or in groups of three (i.e. width, length, and height), "according to the measurements that are more convenient to the work".[580] Here, Pacioli somewhat expands on the mathematical premises established by Alberti pointing to the application of regular and other bodies derived from these, where proportional relationships among components in three-dimensions are pre-established. Pacioli significantly relates the employment of these geometrical bodies with the architect's *ingegno,* and therefore to the process of design and the

[579] "Co(n)chiude breviter che oltra l(')arte el buono architeto bisogna habia ingegno a suplire el dimenuto e smenuire el superfluo secondo la oportunita e dispositione de li lochi a cio non parino loro edifitii monstruosi. E a q(ue)sto effecto a voi a qualumch(')altro mi son messo a trovare co(n) grandissimi afanni e lo(n)ghe vigilie le forme de tutti li 5 corpi regulari co(n) altri loro dependenti e quelli posti in questa nostra opera con suoi canoni a farne piu con debita lor proportione acio in epsi spechiandove mi rendo certo ch(e) voi a li vostri p(ro)positi li saprete acomodare. E li altri mecanici e scientifici ne conseguiranno utilita non poca e siano dati a che arte mistieri e scientie si vogliano co(m)me nel suo Thymeo ed divin Plato(n)e el re(n)de ma(n)ifesto." Pacioli, *Divina proportione (1509),* 31r (Tractato de l'architectura).

[580] "Itaque diffiniet diametros ternatim numeris, quot recensuimus, uti accommodatiores eos venire suum ad opus intelligat." Alberti, *De re aedificatoria,* IX, 6, 829.

capacity for invention. At the same time, he links this to the disposition (i.e. the geometrical arrangement), of the places where these will be employed.

Overall, it is important to note here that Pacioli believes that the geometrical properties of regular and other bodies derived from these are perfectly suited to provide formal and aesthetic solutions ("so that their buildings will not appear monstrous")[581] in architecture. For Pacioli, their inclusion in the context of a treatise dealing with Vitruvius's *De architectura* is justified by the proportional disposition (even if not always following the Vitruvian symmetry) of their parts that correlates their spatial and visual properties.

The *Tractato* continues in Chapters XIII-XVII by returning to one of the three elements mentioned at the beginning, the *epistilio* (or architrave) and its composition. Here, Pacioli offers an attempted interpretation of the some of the difficult passages of *De architectura*,[582] but briefly concludes with the disclaimer that although there was much more to say, this was all that could be put together in the limited time at his disposal before printing, "in the manner of suggestions to what we hope to discuss more extensively in the future".[583] Nonetheless, in the ensuing chapter, Pacioli returns "despite there being no mention of them by Vitruvius", to the employment of regular bodies and specifically to how these can be used by sculptors.[584] Here, he claims to have shown the bodies "in their material and proper form [*ma(teria)li in*

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581 "[…] oltra l(')arte el buono architeto bisogna habia ingegno a suplire el dimenuto e smenuire el superfluo secondo la oportunita e dispositione de li lochi a cio non parino loro edifitii monstruosi." Pacioli, *Divina proportione (1509)*, 31r (Tractato de l'architectura).

582 As noted, this has been identified with the Florentine printed edition of 1496. Francesco P. Di Teodoro, "Vitruvius in the Trattato dell'Architettura by Luca Pacioli, 114-119. For instance, Pacioli completely misunderstood the position and conformation of the triglyphs in the doric order, arguing in Chapter XVII that these are shaped as small pilasters placed above the architrave and cornice. "Poi sopra tutta questa compositione d(')epistilio e cornice i(n) ultimo a presso el tutto se pongano li tigraphi cioe certi pilastrelli con tre coste facti e doi canellati co(m)me certe colonnette quadre distanti uno da l(')altro doi loro largeçce e a e volte 3 et."ibid., 32r.

583 "Seria asai da dire circa questo ma el tempo non me per ora concesso. Peroche de continuuo di e nocte me conviene in su li torcoli e lor calographi agovernar l(')operenostre con tutta diligentia co(m)me se richiede. Ma questo poco a vostra compiacença ho voluto ponere qui co(m)me per cenno a quello che speramo conpiu dilatatione de dicta architectura tractare." ibid.

584 "Q(ua)li piu volte ma(teria)li in p(ro)pria forma ve ho mostrati avenga che di loro p(ar)ticularmente no(n) ne facia me(n)tione alcu(n)a el nostro Victruvio" ibid., 34v.

p(ropr)ia forma] several times" to his disciples.[585] Pacioli concludes that "they will be a reason of high praise for your building, because not only will they make it adorned, but they will also stimulate speculations by the learned and the wise if they will be built with that sacred and divine proportion that has the middle [term] and the two extremes etc.";[586] concluding that "the forms [*forme*] of these bodies," prepared by him, could be found "in the palace of the *Cancelleria* in Rome, in Florence and in Venice"[587] The aesthetic value of the forms of regular and other bodies derived from these and their visual experience is central to this passage.

Finally, in Chapter XIX, Pacioli further considers how architects should rely on the disposition of the elements of a building in places which are constricted [*angustia*], as discussed in Chapter XII.[588] Here, Pacioli repeats his previously mentioned arguments about the proportionality of parts, again adding that, if possible, these should be in arithmetically known ratios, or alternatively geometrically. As Pacioli concludes, this is because proportion is much more extended in continuous quantities than among the discrete, and the geometer, unlike the arithmetician, considers both.[589] This last passage once more confirms Pacioli's commitment to the visualisation of quantities by geometrical means, which he also discusses in relation to the quantification of parts of the human body. This also explains the reference to Piero della Francesca that ends the chapter, *monarcha,* according to Pacioli, both of painting and architecture.[590]

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[585] ibid.

[586] As noted, here Pacioli argues that regular bodies should be employed in bases or capitals for two reasons: firstly because they will make the building aesthetically "adorned" and pleasant, and secondly for how they will be interpreted by the learned.

[587] "Ançe saran(n)o de dignissima co(m)mendatio(n)e del v(ostt)ro opifitio p(er)che no(n) solo lo re(n)deran(n)o adorno ma ancora a li docti e sapie(n)ti dara(n)no da speculare conciosia che sempre siano fabricati co(n) quella sc(r)a e divina p(ro)portione h(ave)nte medium duoq(ue) extrema etc." ibid., 34v.

[588] "Co(m)me nelli loghi angusti lo architectto se habia a regere in sua dispositione." ibid.

[589] "Co(n)ciosia che la p(ro)portione sia molto piu ampla in q(uan)tita continua che in la discreta." ibid. These , pacioli adds, are discussed by Euclid in the fifth book of the *Elements* and, following Euclid, in the *Summa* both in theory and practice. Pacioli brings here the examples of the capitals letters drawn by ruler and compass, added to show "how from the mathematical [disciplines] everything derives".

[590] Nevertheless, it is important to correlate this with the overall mathematically controlled arrangement (what Pacioli calls *dispositione*) of the spatial and formal properties of the building. ibid., 35r. The passage is discussed in section 3.2.1 of the thesis.

Ultimately, as argued above, although published in 1509, the *Tractato* is a repository of a broad number of sources and architectural principles, and it is therefore legitimate to believe that at least some of these refer to Pacioli's presence in the Sforza court of Milan. It is thus important to consider the passages in the *Compendium* where Pacioli first applied regular and other bodies derived from these to architecture in his writings.

4.2.2 The application of the 26 and 72-bases body to architecture in Pacioli's *Compendium de divina proportione*[591]

Having outlined Pacioli's philosophical and theoretical framework, it should now be easier to consider the instances when regular and other bodies derived from these are applied to architecture in his writings. The first discussion of this takes places in Chapter 53 of the *Compendium de divina proportione*.[592]

These passages can only be understood both in their wider theoretical framework and in the context of the *Compendium*. Thus, the prominence given by Pacioli to visual perception and perspective in the first part of the text should be kept in mind. Moreover, in the sections leading up to Chapters 53 and 54, Pacioli describes regular and other bodies derived from these in their material forms (as composite of form (shape) and matter (three-dimensional spatial extension), and presented both in a set of models and through the perspectival representations drafted by Leonardo da Vinci. Pacioli consistently refers throughout the *Compendium* to such material instantiations in order to clarify the formal properties of the bodies.

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[591] For my English working translation of *Capitulo LIII and LIIII* of the *Compendium de divina proportione* see Appendix 1.

[592] As noted, this text was produced in the environment of the Sforza court of Milan and in the text Pacioli refers to two buildings of the city as examples of the employment of a 72-bases body in architecture. Both building can be correlated to the work of Donato Bramante, at the time among the most established authorities in architecture at the Sforza court. These passages deserve therefore careful consideration.

Following the description of all regular bodies and their (semi-regular) truncated versions, both in their planar and elevated forms (with a pyramid added to each base), Pacioli introduces the body with twenty-six bases, both in planar and elevated form in Chapter 53 and refers to their perspectival representations (illus. 4.2-4.3). The planar body is described as derived from a truncation of the vertices of a regular cube. It should be noted that this geometrical body was not discussed in the Campanus' edition of Euclid's *Elements* nor in the writings of Piero della Francesca. It is therefore worth considering the description of the body as given by Pacioli in full:

> Another body, excellent Duke, sufficiently dissimilar from the ones mentioned, is called the [body of] 26-bases. It is derived from the principle and origin of lightness. Of [this body] 18 [bases] are squared with equal sides and right angles, and 8 are triangular, similarly equilateral and equiangular. And this [body] has 48 sides or lines and has 96 superficial angles of which 72 are right, these are the ones of its 18 square bases, [the remaining] 24 [angles] are acute, these are the ones of its 8 equilateral triangles. And these 96 [angles] concur to the composition of 24 solid angles in it. Each of these is composed by the superficial angle of the triangle and by three right angles derived from the square. And of its 48 lines, 24 are common between the triangles and the squares in a way that if you add the 18 squares you end up with the 8 triangles formed as per the truncated [bodies] mentioned above. And the origin of this [body] is from the uniform hexahedron cut on its vertices as it is demonstrated to the eye by its material form [*material forma*].[593]

This description is followed by the remark that "the science [of this body] is in many ways very useful to whoever knows how to practice it [*accomodare*], especially in architecture."[594] No specific example is given however, and Pacioli's reference to the correspondence between the

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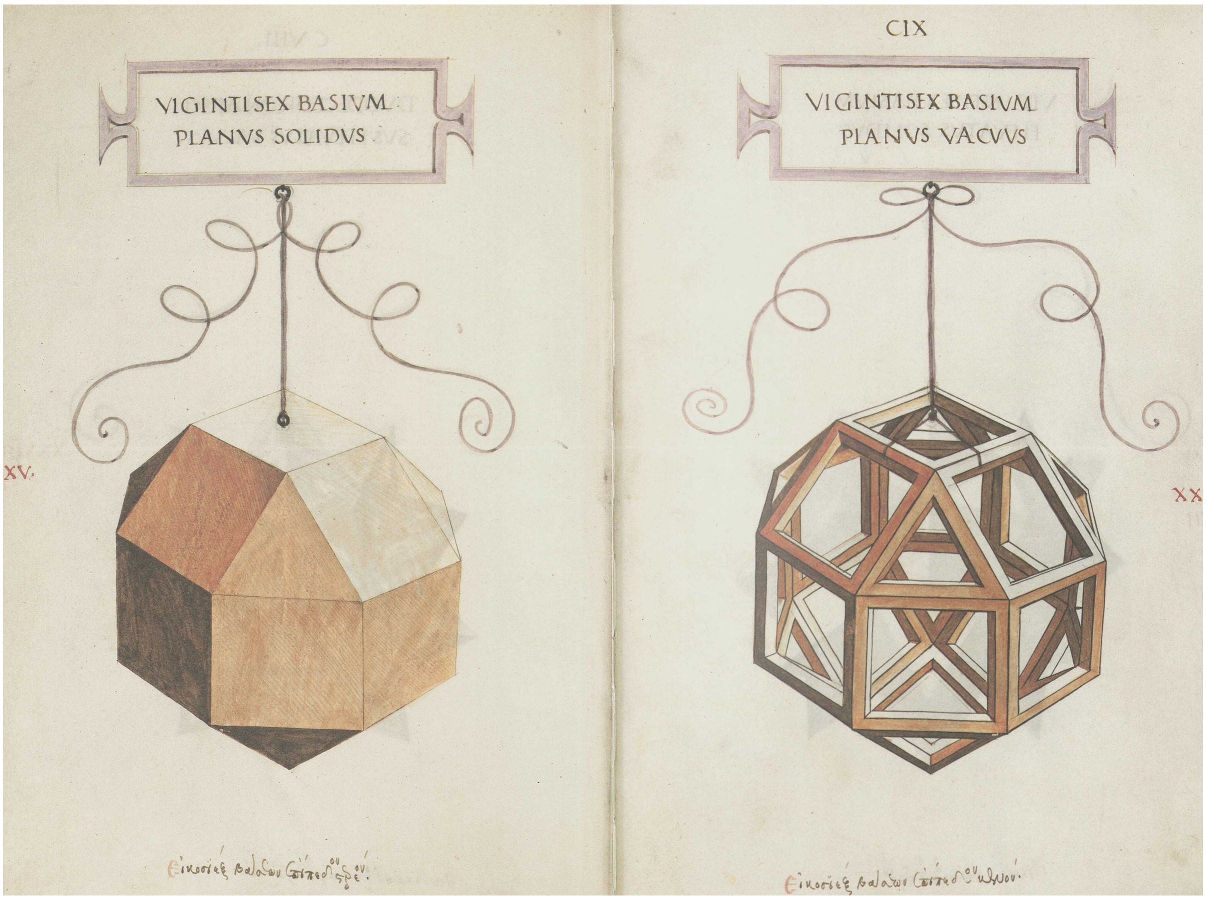
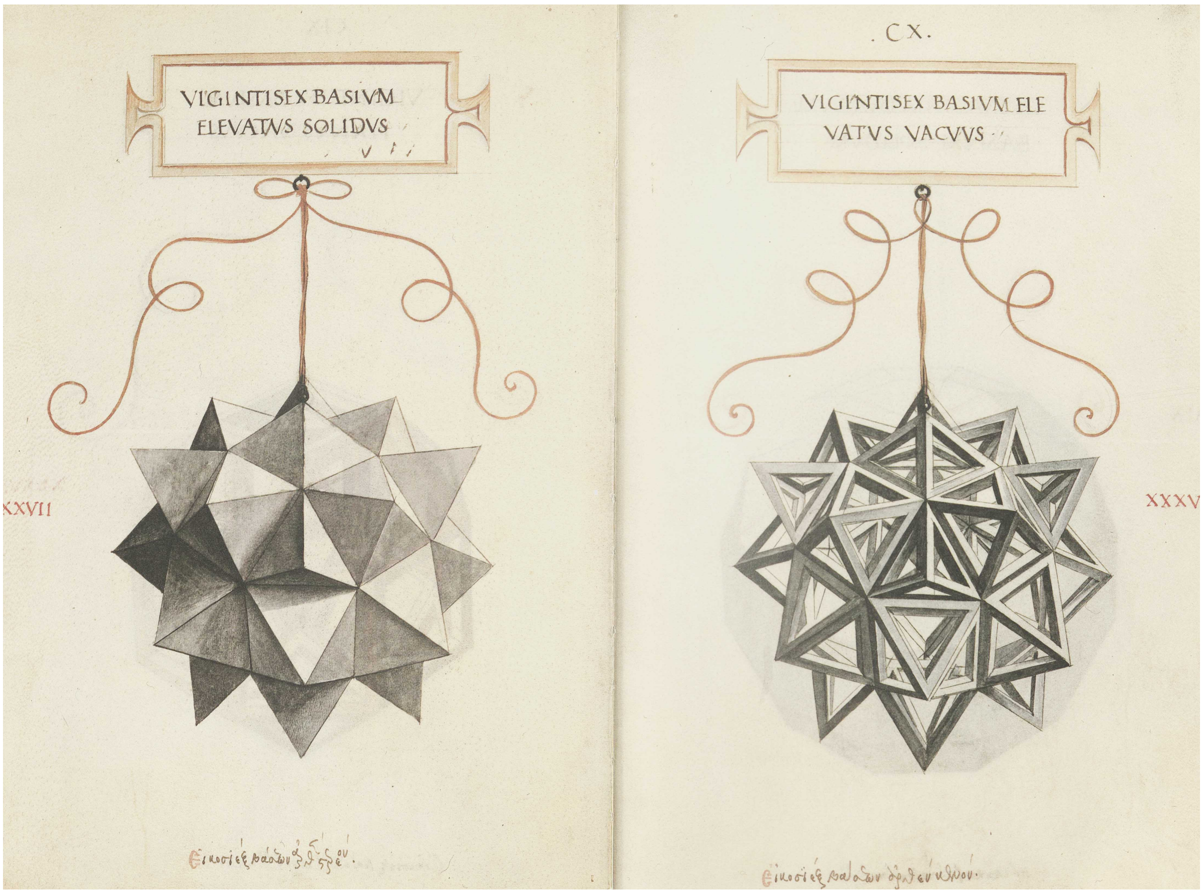
[593] "Un altro corpo Ex. D. Da li gia ditti assai dissimile se trova detto de 26 basi da principio e origine ligiadrissimo derivante. De le quali 18 sonno quadrate equilatere e rectangule e le 8 sonno triangule equilatere similmente & equiangule. E questo tale ha 48 lati overo linee, e ha 96 anguli superficiali. De li quali 72 sonno tutti recti. E sonno quelli de li suoi 8 tranguli equilateri. E questi 26 fra loro concorreno a la compositione in epso de 24 anguli solidi. De li quali ciaschuno consta de uno angulo superficiale del triangulo e de 3 anguli recti de 3 quadrati. E del 48 sue linee, 24 sonno comuni a li trigoni, e a li quadrati pero che de quelli 18 quadrati asiemi secondo la debita oportunita agionti de necessita ne resultano quelli 8 trianguli formati, si commo che degli altri abscisi de sopra se detto. E l(')origine de questo fia da lo exacedron uniforme secondo ogni sui parti tagliato, commo similmente a l(')occhio la sua material forma ci demostra" Pacioli, *Compendium*, 58r-v.

[594] "E fia la sua scientia in molte considerazioni utilissima a chi bene la sa accomodare maxime in architectura'" ibid.



4.2—Leonardo da Vinci, “Vigentisex Basium Planus Solidus”and “Vigentisex Basium Planus Vacuus,” in Luca Pacioli, *Compendium de divina proportione.* Veneranda Biblioteca Ambrosiana, Milan.



4.3—Leonardo da Vinci, “Vigentisex Basium Elevatus Solidus”and “Vigentisex Basium Elevatus Vacuus,” in Luca Pacioli, *Compendium de divina proportione.* Veneranda Biblioteca Ambrosiana, Milan.

body of twenty-six bases and architecture has as yet received no attention from scholars. However, when the body is understood as composed of spatial parts that can be isolated, as Pacioli explains in the chapter that follows from this, his claims concerning its application to architecture become clearer. The planar version of the body can also be understood as formed of two square cupolas and an octagonal prism. Clearly, numerous buildings (or parts of them) were geometrically based on an octagonal prism both before and during the fifteenth century.[595] As noted, Alberti listed the octagon among the preferred two-dimensional figures to be used in the design of temples in plan in Book VII of *De re aedificatoria*.[596] Here, however, Pacioli describes the application of a three-dimensional geometrical body by listing its components and referring to the perspectival representation of its material form.

Pacioli continues with a description of the elevated form of the body, that displays twenty-six pyramids elevated on its faces. Of these, Pacioli notes, "18 are quadrangular and 8 triangular, and all of these can be seen with the eye"; however, "the previously described body of 26-bases", Pacioli adds, "can now only be apprehended by the imagination".[597] Crucially, Pacioli is again referring to the visual experience of the material form of the body, by relying on the intellect to imagine spatial properties that cannot be represented in a drawing. Pacioli clearly believed that the same principles as to its visual perception would apply to a geometrical body employed in a building.

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[595] For example, the baptistry of Florence, the base of the dome of Florence's Cathedral and Brunelleschi's church of Santa Maria degli Angeli. The octagonal temple in the Ideal City of Baltimore should also be mentioned, this was produced in the environment of the court of Urbino in the last decades of the fifteenth century and shows two superimposed octagonal prisms.

[596] Alberti, *De re aedificatoria*, VII, 4, 551.

[597] "E fia composto dicto corpo, da 26 pyramidi laterate, de le quali 18 sonno quadrangule e 8 triaangule, quali tutte di fore intorno se possano da l(')occhio discernere, e del precedente 26 basi, solido piano intrinseco, per imaginatione solamente compreheso." Pacioli, *Compendium*, 59r.

The process is similarly repeated (although with different architectural connotations), when the seventy-two bases body (illus. 4.4) is introduced in Chapter 54.[598] Pacioli begins by referring to the proposition of Euclid's *Elements* in the Campanus edition, where the seventy-two bases body is constructed starting from a dodecagon.[599] Pacioli continues by describing the type and number of bases that bound the body, and follows with a remark on how the inside of the body should be understood: "as you can see it is made of 72 pyramids of which the bases are the same 72 [bases of the body], and these should be imagined inside [the body]" pointing therefore to the internal space of the body apprehended at the level of the imagination.[600] Furthermore, Pacioli adds here that "the form [*forma*] of the elevated [version], which I have not deducted in a material way, is again left to the reader whose *ingegno* I have no doubts about".[601] As we have seen, this is consistent with Pacioli's description of regular and other bodies derived from these. At this point the body is discussed for its employment in architecture:

> [...] And this [body] with 72 bases is frequently used by architects for being a very useful form [*forma assai accomodata*] in their dispositions of buildings [*in loro dispositioni di edifici*], especially when it happens to make domes [*tribune*] or other vaults or what we can call ceilings [*cieli*]. And it happens that not always all the faces [of this body] are taken [applied] to these buildings, nevertheless to this [body] these refer, using a fourth or third part of it cut in any way, according to the place and site where the building is intended to be.[602]

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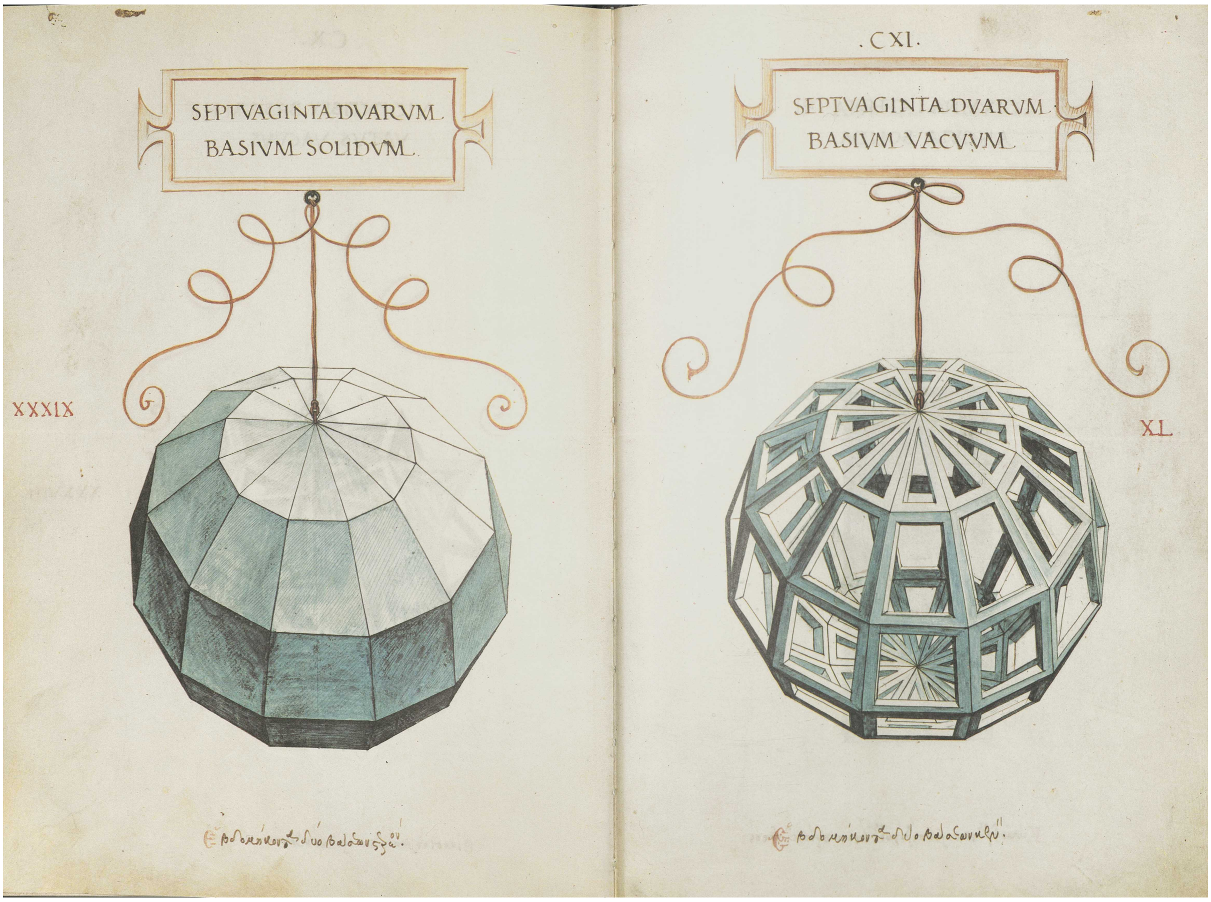
[598] "Del corpo de 72 basi piano solido e vacuo." Pacioli, *Compendium*, 59r-v. The contents of this chapter were discussed by Rackusin and commented by Bruschi in the edition of parts of the *Divina proportione* in the late 1970s. It is interesting that since that time it has received little or no scholarship attention; this is even more surprising if we consider that in the last few decades the scholarship on the writings of Pacioli has overall substantially increased. Byrna Rackusin, "The Architectural Theory of Luca Pacioli: De Divina Proportione, Chapter 54."; Bruschi, "Luca Pacioli De divina proportione.".

[599] This is, as Pacioli notes, proposition 14 of Book 12 in Campanus's editions of Euclid's *Elements*.

[600] Pacioli writes "E causariense in epso 72 pyramidi secondo il numero de(l)le suoi 72 basi de(l)le quali pyramidi le basi serienno le medesime di quelle. E lui dentro imaginato" Pacioli, *Compendium*, 59v-60r.

[601] "La forma del q(ua)le elevato non curai fra queste materialmente dedure per lassar la parte sua anchora a(l) lectore del cui ingegno non mi diffido." ibid., 60r.

[602] "[…] E questo 72 basi molto dagli architetti fia frequentato in loro dispositioni de edificii, per esser forma assai acomodata maxime dove occorresse fare tribune o altre volte, o vogliamo dire cieli. E avenga che non sempre aponto, se prendino in detti edifici tante facce, pure a quella similitudine se regano, squatandolo, sterzandolo in tutti modi secondo el luogo e sito dove tal edificio intendar porre" ibid.



4.4—Leonardo da Vinci, "Septaginta Duarum Basium Solidum"and "Septaginta Duarum Basium Vacuum," in Luca Pacioli, *Compendium de divina proportione.* Veneranda Biblioteca Ambrosiana, Milan.

Even a third or a fourth part of this body, Pacioli notes, can be applied to architecture. This, as we shall see, accounts for the references that Pacioli made to buildings in his contemporary Milan. Moreover, Pacioli argues here that the seventy-two bases body can be used by architects in their dispositions [*dispositioni*] of buildings.[603] As seen, the same term is used by Pacioli when referring to the arrangement of each of the "material shapes of the aforementioned bodies" [*forme materiali de dicti corpi*] presented to the duke of Milan,[604] while in the *Summa*, he clarified that *dispositione* is also called *proportione* among the parts of a whole.[605]. An analysis of Pacioli's texts shows that the terms *dispositione/i* or *disposto/i* occur

[603] "molto dagli architetti fia frequentato in loro dispositioni de edificii". ibid.

[604] "E V. Cel. al(l') uno e al(l')altro modo hara loro dispositioni, le quali non de vil materia, commo per inopia a me e stato forza, ma de preciso metallo e fine gemme meritarieno essere ornati" Pacioli, *Compendium,* 83v-84r.

[605] "Del fabro legnaro no(n) e dubio cio che si facia (o cassi, o tavole, o banche, o letieri, o usci, o tine, o botti, o mesure da pa(n)ni, o colme co(m)mo si voglia) sempre secondo certa dispositio(n)e (la quale si chiama proportione) asetta." Pacioli, *Summa*, 69r (Pars prima)

nine times in the *Compendium* and fourteen times in the shorter *Tractato de l'architectura.*[606] They occur fives times in Campanus edition of Euclid's *Elements* to refer to the arrangement of lines in two-dimensional figures and only once to the arrangement of lines in three-dimensional solids.[607] They are also never used in the writings of Piero della Francesca. An examination of the contexts in which they are employed in the *Compendium* reveals that they refer in all but two cases to the proportional disposition of architectural elements in a building. In the other two cases, they refer to the disposition of parts of two regular bodies, and once in combination with *forma*.[608] In Pacioli's *Tractato*, except on two occasions, the term always refers to the arrangement of architectural elements in buildings or to the arrangement of more than one three-dimensional geometrical body.[609] Notably, one of the other occurrences in Pacioli's *Tractato* is a direct quote from Vitruvius's *De Architectura* that uses the Latin term

[606] An occurrence of *despositione,* one of *despositioni* (10r, Pars prima), two of *dispositioni* (23v and 101r, Pars prima), and an occurrence of *dispositione* (101v, Pars prima) are found in Pacioli's *Summa.* All refer to the arrangement of numbers.

[607] In the edition commented by Pacioli, the terms appear in Book III (Proposition 36)"Ide(m) oste(n)sive p(ro)babit manea(n)t prior *dispositio* et ypotesis." *Euclidis*, 24r; in Book X (Proposition 18)"In hac quoq(ue) remaneat eade(m) *dispositio* eademq(ue) yphoteses que in premissa hoc solu(m) mutato q(ue) p(ro)portio numeri." ibid., 79r; in Book X (Proposition 29): "Huius quoq(ue) *dispositio* a duarum premissarum *dispositione* non sit in quoquam diversa Sint autem linee due […]" ibid., 83v; in Book XI (Proposition 8): "Fiat enim prorsus eade(m) [linee] *dispositio* que in sexta [propositio] ertiq(ue) ut ibi uterq(ue) duorum angulorum […]" ibid., 102v; in Book XI (Proposition 34)"Se(cun)dam p(ar)tem cu(m) sit co(n)versa pri(m)e co(n)verso mo(do) p(ro)basis, Sit (e)n(im) eade(m) *dispositione* mane(n)te p(ro)portio […] Dico tu(n)c *solida* […] eq(ua)lia" ibid., 109r; in Book XIII (Proposition 9): "sit (e)n(im) priori *dispositio(n)e* mane(n)te linea e c divisa in puncto b[…]" ibid., 124v. The following occurrences are addition of Pacioli to Campanus' text: "Hanc prae oculis summus opifex in caelestium terrestiumque rerum *dispositione* semper habuit dum orbium motus cursusque syderum et planetarum omnium *ordinatissime disponeret* […]" ibid., 30r; in Book IX (Proposition 39-Castigator point b)"Quia in utroq(ue) ordine sunt numeri in dupla proportione disposti et ideo p(er) equa p(ro)por(tione) sive P(er)vesrim sive directe sunt d b. Et R.e.p(ro)portio(n)ales" ibid., 74r. (Emphasis in italic is mine).

[608] "On the form and arrangement and the tetrahedron plane [and] solid or vacuous [*De la forma e dispositione de tetracedro(n) pia(n)o solido o vacuo* […]]" Pacioli, *Divina proportione (1509)*, 14r (Compendium).

[609] The other occurrences are in a passage referring to the human body but establishing a relationship with architecture: "[…] the ancient having considered the due arrangement of the human body arranged [*disponevano*] all their works and especially the sacred temples according to its proportions; […] li antichi considerata la debita dispositione del corpo humano tutte le loro opere maxime li templi sacri a sua proportione le disponevano". Pacioli, *Divina proportione (1509)*, 25r (Tractato de l'architectura) (mistakingly marked as 17r) and "I briefly conclude that in addition to *arte* the good architect(s) should also have *ingegno* in compensating for what is missing and reducing what is superfluous according to the circumstances and arrangement [*dispositione*] of places so that their buildings will not appear monstrous. And for this reason I studied with many difficulties and long vigils for you and anyone else the shapes of all 5 regular bodies, with the other dependant from these […]; Co(n)chiude breviter Che oltra l(')arte el Buono architecto bisogna habia ingeno a suplire el dimenuto e smenuire el superfluo secondo la oportunita e dispositione de li lochi a cio non parino loro hedificii monstruosi . E a q(ue)sto effecto a voi a qualumc(')altro mi son messo a trovare co(n) grandissimi afanni e lo(n)ghe vigilie le forme de tutti li 5 corpi regulari co(n) altri loro dependenti […]" Pacioli, *Divina proportione (1509)*, 31r. Interestingly, in the latter case Pacioli expresses a direct relationship between the arrangement of places, the beauty of buildings and the regular and other three-dimensional bodies derived from these.

dispositio. The term is here employed in the passage that describes the arrangements of architectural elements such as columns and architraves, with reference to the wide *intercolumnio* of the temple of Apollo and Diana.[610]

Overall, the consistent use of the term to describe the arrangement of the parts of regular and other bodies derived from these clarifies Pacioli's firm belief in the formal, visual and spatial properties of these bodies based on their determined proportional relationships, while at the same time, this is inevitably related to the concept of *dispositio* as employed in Vitruvius' *De architectura* and Pacioli's *Tractato de l'architectura*. Ultimately, this fully justifies, in Pacioli's opinion, the employment of regular and other bodies derived from these in architecture. Pacioli continues with a series of examples including the Pantheon, that confirm this principle:

> Due to [the 72-bases body's] convenience [*convenientia*] you can find many [buildings] in different places arranged and built, as in the inestimable ancient temple pantheon, today named by the Christians as *la Rotonda*, can be seen: this was organised with diligent industry and arranged in conformity to proportions [*de p(ro)portioni observantia fo disposto*], and [here] light comes from an eye left open at its highest point, which makes all the rest [of the building's interior] splendid and luminous. I will not speak of other famous and well-known cities such as Florence, Venice, Padua, Naples and Bologna, in which many buildings both sacred and secular, small or big, are built mirroring this.[611] And here as well, in your Milan, in the respectable *Sacello of San Scettro* in the ornate chapel there is a part of this [body], taken away and with a certain convex quantity applied to the wall [*con reservatione de alquanto convexo al muro aplicata*], and with a rosette in each of its bases that makes it adorned. And in your

[610] "Intaulato, Echino, Scothica e Tinia li antichi chiamano Epistilio e li nostri li dicano Architrave qual co(m)me e dicto va da l(')un capo al(l')altro incatena(n)do le colo(n)ne e questa dispositione co(m)me nel 3 libro V(itruvio) parlando de lo intervallo o thetrante del tempio de Apollo e de q(ue)ello de diana dici che p(er) troppo intervallo lo epistilio se rompea le cui parolle formali so(n)no queste videlicet: Cum trium colu(m)narum crassitudinem intercolu(m)nio interponere possumus tanq(uam) est Apollonis et Diane edes hec dispositio ha(e)c habet difficultatem q(uod) epistylia propter intervallorum magnitudinem franguntur etc." Pacioli, *Divina proportione (1509)*, 31v (Tractato de l'architectura); L. Vitruvii Polloniis, *De architectura libri decem,* Liber Tertius, Caput secundum; Vitruvius, *De Architectura*, III, 3, 4.

[611] As Pacioli explains in the preceding part of the text, this should be interpreted as the whole or a part of the body.

> devout and most sacred temple of the Grazie, the *tribuna* at the first altar and the two later ones, even if is not all similar to a part of this, vaguely resemble it in its bases.[612]

It is widely agreed that the *Sacello San Scettro* is the Sacello of San Satiro, where a *sacellum* had been located since the early Middle Ages. However, the ornate chapel [*ornata capella*] mentioned by Pacioli has not been clearly identified.[613] In the period when Pacioli was in Milan, Bramante had been commissioned to design a chapel in the church of Santa Maria presso San Satiro that was never built.[614] Pacioli could have seen drawings or a model prepared by Bramante and may have based his interpretations on these rather than on built architecture. Most probably however, as it has been suggested, Pacioli was referring to the sacristy designed by Bramante, where the termination of a niche is similar in form to the part of 72-bases body described by Pacioli.[615]

The interventions of Bramante in Santa Maria presso San Satiro will be discussed in Chapter 5. However, besides the problem of the exact location of the body mentioned by Pacioli, the

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612 "A la cui convenientia assaissimi in diversi parti se trovano disposti e fabricati. Commo de lo inextimabile anticho templo pantheon. E oggi da christiani nel capo del mondo a rotonda chiamato fia manifesto. El qual con tancta solerta industria e de proportioni observantia fo disposto, chel lume de un solo ochietto nel suo fastigio aperto relicto tutto el rende splendido e luminoso. Lascio demolte altre famose e inclite cita, commo fiorentia, vinegia, padua, napoli e bologna. In le quali assai hedeficii si sacri commo prophani, o piccoli, o grandi che sienno, al spechio de questo sonno facti. Anchora qui nel suo Milano nel degno sacello de san scetro, l(')ornata Capella fia una parte de questo spaccata, e con reservatione de alquanto convexo al muro aplicata, e in ciascuna sua basa hiontovi un rosone che adorna la rende. E in lo devoto e sacratissimo vostro templo de le gratie la sua tribuna al primo altare e laterali, gia non e se non una parte a simil de questo, pur in suoi basi a piu vaghezza giontovi quelli." Pacioli, *Compendium*, 60r-v.

613 Byrna Rackusin and Jessica Gritti tentatively identified the *capella* with the perspectival illusionistic choir of the church designed by Bramante. This is not convincing as the shape of the choir and its perspectival effect are not intended to recreate by any means a shape such a 72 bases body or part of it. As a confirmation of this, Rackusin concluded that 'there is no visual evidence that Bramante had it in mind'. The confusion of Rackusin is confirmed by his identification of the new sacristy with the original *sacellum*, while these are two distinct spaces. Rackusin, "The Architectural Theory of Luca Pacioli: De Divina Proportione," 490. ; Jessica Gritti, "'Piere cocte et intaliate'. Tramiti bramanteschi nella diffusione dei lacunari in terracotta in area cremonese," in *Porre un limite all'infinito errore*, ed. Alessandro Brodini and Giovanna Curcio (Rome: Campisano Editore, 2012), 23-36. Arnaldo Bruschi interpreted the passage as referred either to the church or to the sacristy but did not discussed further. Bruschi, "Luca Pacioli De divina proportione," 76.

614 This is the Chapel of San Teodoro. As noted by Gritti and Schofield, Bramante was given the commission by the duke, probably after the May 1496. The day of San Teodoro of the year before was in fact the day of the investiture of Ludovico as Duke. The project is documented as begun by Bramante after receiving the commission but remained unbuilt at the fall of Ludovico in 1499. Richard Schofield, "Bramante e Santa Maria presso San Satiro," in *Santa Maria presso San Satiro*, ed. Francesco Repishti (Milan: Parrocchia di San Satiro), 31, 34.

615 Bruschi, "Luca Pacioli De divina proportione," 76n.

nature of the process described in the text deserves consideration. Pacioli clarifies that "there is a part of this [body], taken away and with a certain convex quantity applied to the wall" [*sia una parte de questo, spaccata e con reservazione de alquanto convessa al muro aplicata*]. Crucially, as described by Pacioli, only if the convex extension of a geometrical body is considered material can it be applied to the wall and thus determine a concave surface on this part of the building. As we have seen, this is consistent with the theoretical framework of Pacioli's works and its correlation with the arguments of Duns Scotus. As noted by Duns Scotus, since a body is commensurately in place, its spatial extension and outer boundary should be retained when it exerts pressure and displaces any other material extension. Moreover, the same body is referred to as being applied in the church of Santa Maria delle Grazie. Here, Pacioli links the body to the half-dome of the main altar's apse and (although noting that they "vaguely resemble it in its bases"), the two half-domes on the sides of the *tiburio*.[616]

The chapter continues with emphasis on the role and relevance of proportions both in theory and practice, highlighting that artisans use proportions although they may not be aware of it: "tailors and shoemakers use geometry and don't know what [geometry] is; and stonemasons, woodcutters, blacksmiths and every maker use measurement and proportion they don't know, since, as it is said in other places, everything is formed by number, weight and measure."[617] As noted, the reference to the triad of number, weight and measure is important in this context to highlight the correlation among abstract mathematical concepts and their inevitable instantiation in every attribute of the material world. What follows from this is an

[616] As will be discussed in chapter 5, there is documental evidence for the involvement of Bramante in Santa Maria presso San Satiro from 1482, while his involvement in the church of the Grazie although not documented, has been generally accepted by scholars for what concerns the internal arrangement of the tribuna and choir. This has being argued by Arnaldo Bruschi and most convincingly in Richard Schofield, "Bramante milanese: collisioni di culture architettoniche?," *Arte Lombarda*, no. 176/177, vol. 1/2 (2016): 7-15; and Richard Schofield, "Bramante and Amadeo at Santa Maria delle Grazie in Milan," *Arte Lombarda*, no. 78 (3) (1986): 41-58.

[617] Pacioli, *Compendium*, 60v-61r.

expression of admiration for Vitruvius, called by Pacioli "a respectable architect and great mathematician".[618] By contrast, Pacioli continues by criticising some modern buildings [*moderni hedificii*], for displaying a poor understanding of structural and formal properties. "These buildings", Pacioli notes, "are doomed to ruin if the sizes of their components are based on models that due to their small dimensions cannot give a proper visual understanding" and therefore an acknowledgement of their properties.[619] Pacioli's opinion of architectural models is critical and thus undoubtedly different from Alberti's recommendations for their employment.[620] As Pacioli concludes: "And for having been wrongly understood, they ask for spending more on restoring than in building, they are called architects but they have never seen the covers of the excellent volume of our respectable architect and great mathematician Vitruvius, who composed *De Architectura* with exceptional documents discussing every type of structure".[621]

As noted, Pacioli sees a clear consonance between the formal principles found in Vitruvius's *De architectura* and the employment of regular and other bodies derived from these in the process of designing buildings. As we have seen, Pacioli also recalls a meeting in 1498 at the Sforza court of Milan, concerning the Duomo of Milan (the *tiburio* of which was still under construction at the time), where the contents of Vitruvius' text were discussed. It is thus

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618 "dignissimo architecto e gran mathematico Victruvio" Pacioli, *Compendium*, 61r.

619 "But what shall we say of modern buildings, ordered and arranged [*disposti*] according to various and different models that seems vague to the eye for their small size, and then when [the buildings are] built don't stand the weight and, instead of lasting for one-thousand years, before three years already collapse."; "Ma che diremo de li moderni hedificii in suo genere, ordinati e disposti, con varii e diversi modelli quali a'(l)ochio par che al quanto rendino vagheçça per lor esser piccoli e poi nelle fabriche non regano el peso, e non che a mill(')anni arivino nançi al terço ruinano." ibid.

620 Alberti significantly considers models while discussing the difference between the work of the architect and that of the painter. The relationship of models and architecture is based on a common reliance on the "true lines and angles". Alberti, *De re aedificatoria*, II,1, 99.

621 "E per el lor male essere intesi in refare piu che in fare fanno spendere. Chiamandose architecti, e mai non videro le coperte in cio de lo excellentissimo volume del nostro dignissimo architecto e gran mathematico Victruvio quale compose de architectura con supremi documenti a ogni structura. E chi da quel se divia sappa in aqua e fonda in rena, piu p(re)sto guasta l(')arte che architecti nominati. E no(n) sanno la differentia dal ponto a la linea commo saperanno quella degli angoli sença la quale non e possibile bene edificare chel manifesta, commo dici el prefato Victruvio […]" Pacioli, *Divina proportione (1509)*, 16r-v (Compendium); Pacioli, *Compendium,* 60v-61r.

natural to think that in the environment of the court, if not before, Pacioli was able to interact with other scholars and develop his arguments on the subject. At the conclusion of this chapter, it is therefore important to examine the only surviving theoretical and architectural text by Bramante, and to examine whether any correlation with Pacioli's arguments can be established.

4.2.3 The four bodies of the Duomo of Milan in Donato Bramante's *Opinio super domicilium seu templum magnum*[622]

The *Opinio* of Donato Bramante is a brief, consultative text (opinion) addressed to the *Magnifici Deputati* who over-saw the process of constructing the Duomo (or Cathedral) of Milan.[623] At the time of Bramante's writing (and more generally between 1486 and 1490), they were considering solutions for the still unresolved *tiburio*, or dome, positioned above the crossing of the main nave and the transept of the Duomo.[624] Besides Bramante, Leonardo da Vinci, Luca Fancelli, and Francesco di Giorgio Martini, among others, were involved in the consultation process until the solution proposed by the Lombard architects Giovanni Antonio Amadeo and Gian Giacomo Dolcebuono, with a contribution by Francesco di Giorgio

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[622] For my English working translation of Donato Bramante's *Opinio sup(er) domicilium seu templum magnum* see Appendix 2.

[623]A first critical edition of the text was published by Arnaldo Bruschi. Arnaldo Bruschi, "Pareri sul tiburio," in *Scritti Rinascimentali di Architettura* (Milan: Edizioni il Polifilo, 1978), 367-374. The text has been translated only once in English in 1989: Richard Schofield, "Amadeo, Bramante and Leonardo and the *tiburio* of Milan Cathedral," *Achademia Leonardo da Vinci* (1989), 74. Although the *Opinio* does not survive in its original copy (it was most probably destroyed in the fire of 1906), it was transcribed twice, in the 1878 and 1880. Small differences exist between the two transcriptions. For my translation I have consulted the comparison between the two in Francesco P. Di Teodoro, "Due temi bramanteschi: l'Opinio e l'incompiuta monografia di Barbot, Benois e Tierry" in *Donato Bramante, ricerche, proposte, riletture*, edited by Francesco P. Di Teodoro (Urbino: Accademia Raffaello, 2001.

[624] The *tiburio* was a longstanding issue in the construction of the Cathedral of Milan. A first octagonal *tiburio*, based on the project of Guiniforte Solari, was probably started in 1459-1460. As noted by Schofield and Repishti other projects by other Masters may have also been preexistent to this. It is not clear if a *tiburio* was completed by the death of Solari in 1481, or if it was instead concluded by September 1486 by a 'German Engineer' from the Cathedral Strasbourg and his team that were called to intervene on the issue. However throughout the 1480s the *tiburio* is recorded as having a series of unresolved structural problems. Ultimately, these brought to the demolition of the whole *tiburio* (including the supporting penditives) before 1486 or shortly after, and the evaluation of a number of proposals for its replacement. Richard V. Schofield, Francesco Repishti, "L'Intervento di Guiniforte Solari in *L'Opinio Bramanti,*" in *Ad Triangulum Il duomo di Milano e il suo tiburio: Da Stornaloco a Bramante, Leonardo e Antonio Giovanni Amedeo*, ed. Giulia Ceriani Sebregondi (Padova: Il Poligrafo, 2019), 91, 101-125, 127-130.

Martini, was ultimately chosen.[625] Although the text is not dated, as noted by Ceriani Sebregondi, the discussion of some the proposals produced by other architects, and the omission of others, demonstrate that it preceded the presentation of Bramante's model in September 1487, and was likely produced after 1486.[626] The text was thus redacted about nine years before the official arrival of Luca Pacioli at the Sforza court.[627]

The contents of the text are essentially theoretical in character, and mainly concern the formal, structural and visual properties that the new *tiburio* should have. Bramante's main point is that the *tiburio* should be made in square form. Nonetheless, as Bramante acknowledges at the end of the text, the *deputati* had already expressed their preference for an octagonal *tiburio*, which was later built. The text is therefore more theoretical than practical and aims to develop (through the analysis of the formal, structural and aesthetic properties of the *tiburio*) arguments to support Bramante's preferred solution.[628]

The text begins with Bramante's presentation of four qualities that a building should possess: "the first is strength [*forteza*]; the second conformity [*conformità*][629] with the rest of the building; the third lightness [*legiereza*]; the fourth beauty [*belleza*]". After a brief introduction claiming that only with great *ingegno* could the solutions found in the proposals and models of

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[625] Giulia Ceriani Sebregondi, Richard V. Schofield, "First Principles: Gabriele Stornaloco and Milan Cathedral," *Architectural History*, vol. 59 (2016): 63-122.
[626] Moreover, it has been argued that the text could be a transcription of an oral intervention of Bramante. However, as noted by Ceriani Sebregondi, the way in which the *Opinio* is structured seems to suggest a written intervention. It could have been requested by the *deputati* or due to an attempt of Bramante to acquire a commission. Indeed, there are documents that record in September 1487 a model based on a project of Bramante commissioned by the *deputati*. Giulia Ceriani Sebregondi, "L'Opinio Bramanti" in *Ad Triangulum Il duomo di Milano e il suo tiburio: Da Stornaloco a Bramante, Leonardo e Antonio Giovanni Amedeo*, ed. Giulia Ceriani Sebregondi (Padova: Il Poligrafo, 2019), 138.
[627] ibid., 137.
[628] As noted, the text is mainly concerning the structural and formal properties of the unresolved *tiburio* of the duomo, but it is also a theoretical description of the conformation of the whole building. As such, it needs to be examined in its wider theoretical framework.
[629] It has been translated by Schofield as 'conformity'. I have retained the same translation although it is difficult to account for the meaning of the term as intended by Bramante. Bruschi has noted that it is a concept already expressed by Leon Battista Alberti in *De re aedificatoria* . Bruschi, "Pareri sul tiburio," 349-366. However, the term 'conformatio' it is not used by Alberti in *De re aedificatoria.*

other architects be applied, Bramante considers the following: "I say that the squared is much stronger and better than the octagonal", adding that this "is more in accordance with the rest of the building." "This does not mean" Bramante adds, "that the octagon cannot stand, but with more difficulty both in itself, and for the *ingegno* of who would have to compose it".[630] Other considerations of a structural character follow this, discussing the positioning of buttresses for the *tiburio* in conformity with the rest of the building. The section that follows, the longer of the *Opinio*, is dedicated to conformity [*conformità*]. Conformity is presented as the paramount quality on which the others depend. Bramante clarifies his reading of the whole building in this section:

> Concerning the second thing, that is conformity [*conformità*], I say that this building is divided into four bodies [*corpi*] different in height and in width; where [sometimes] they are of one [width], [sometimes] of another; and in order to make you understand well, the nave in the middle is the main body that rules the whole building. To the sides there are the two wings that make the two minor bodies, which are equal in width but are different in height, as said above, in a way that the one closer to the [external] wall is lower. It follows then the first is approximately forty *braccia* [tall]; the second fifty-nine, and the middle one eighty; and since the second body is taller than the first, due to the [structural] strength of the building; from the piers of the first body spring buttresses that respond to the [thrust of the] piers of the second [body], in the same way that from the second body spring buttresses that respond to the [thrust of the] piers of the third [body], that is the middle nave.[631]

The terms *corpo/i,* have not attracted particular attention although they are used a total of eleven times in the relatively brief text of Bramante's *Opinio*. Before Bramante, the mathematician Stornaloco used these terms to describe the naves of the Duomo when

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[630] "Quanto a la prima cosa, videlicet forteza, dico che 'l quadro è molto più forte e meglio che l'octavo, perhò che più col resto de l'edificio se concorda" The transcription employed here is based on Francesco P. Di Teodoro, "Due temi bramanteschi: l'Opinio e l'incompiuta monografia di Barbot, Benois e Tierry," 87-90.

[631] "Quanto a la seconda cosa, cioè conformità, dico questo edificio essere partito in quattro corpi diversi in alteza, ma in largheza, dove sono d'una egualità, e dove d'un'altra; e perché possiati bene intendere, la nave de mezo è el corpo mastro che guida tutto l'edificio. Da parte dessendeno dove ale, che fanno i doi corpi minori, li quali sono eguali in largeza, ma differenti in alteza, como è sopradicto, perhò che quello più s'avicina a li muri, più è basso. Seguita adoncha questa a la prima, essere alta quaranta braza vel circa; la seconda cinquantadove, e quel de mezo octanta; e perchè il secondo corpo è piû alto che'l primo, como è necessità, per forteza de l'edificio, se partendo dai piloni del primo corpo contraforti che respondeno ai piloni del secondo, così dai piloni del secondo se parteno contraforti che respondono ai piloni del terzo, che è la nave de mezo." ibid., 88.

consulted in late 1391.[632] Stornaloco discussed in his letter and determined in his drawing a higher geometrical correspondency among two-dimensional figures in the vertical plane of the church. Following a well-known article by James Ackerman,[633] Stornaloco's intervention has been the subject of a lively scholarly debate, recently summarised and clarified in its principles by Ceriani Sebregondi and Schofield.[634] Stornaloco describes in his letter the naves of the building as geometrical bodies [*corporum*] most likely following the terminology used for three-dimensional geometry found in Campanus' edition of Euclid's *Elements*. At the same time, the proportioning system devised by Stornaloco was based on a horizontal grid of 16x16 braccia (taken to axes of the piers) in plan, with a regular vertical spacing of 14 braccia. In elevation, this module allowed the approximation of the height of the naves to the sides of equilateral triangles based on the plan's modules (illus. 4.5).[635]

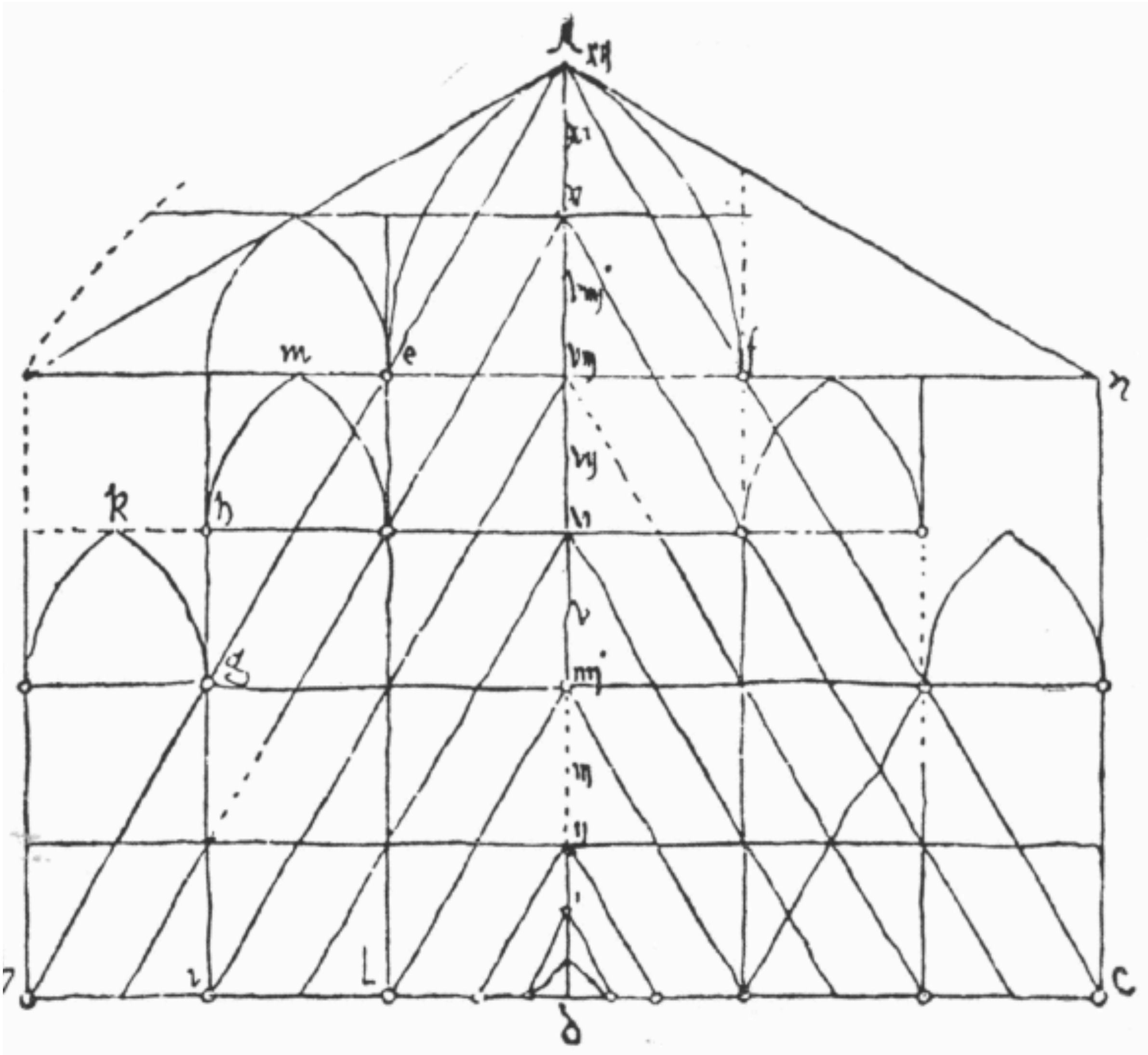
Bramante may have seen drawings showing the principles discussed by Stornaloco in the late fourteenth century (at the end of the *Opinio* he refers to "some drawings, that are in the *fabrica* and were done at the time when this duomo was built"). However, this would still be unable to justify his employment of the concept of geometrical bodies to describe the three-dimensional spatial conformation of the church. As we shall see, although Bramante mentions the height of the naves and the difference in their widths (the main nave's width—32 *braccia*—being double the others), he employs the subdivision of the building in bodies to argue that the

[632] Giulia Ceriani Sebregondi, Richard Schofield, "First Principles: Gabriele Stornaloco and Milan Cathedral," *Architectural History*, vol. 59 (2016): 115-118.

[633] James S. Ackerman, "'Ars Sine Scientia Nihil Est.' Gothic Theory of Architecture at the Cathedral of Milan," *The Art Bulletin*, vol. 31, no. 2 (Jun., 1949): 84-111.

[634] Giulia Ceriani Sebregondi, Richard Schofield, "First Principles: Gabriele Stornaloco and Milan Cathedral," 63-122.

[635] However, the height of the naves was later revised so that the imposts of the arches is lower to what initially suggested by Stornaloco. ibid. 65, 77-78.



4.5—Freehand copy (Beltrami) of Stornaloco's drawing of the elevation of the duomo based on the division of the plan.

tiburio should be considered as an additional three-dimensional body (illus. 4.6).[636] Bramante continues in the *Opinio* by considering the location of the fourth body, the *tiburio*:

> And since this middle nave has all its vaults or crossings semi-squared, there is no place for a perfect square, if not where this intersects the other similar nave [transept]. In no other place could be placed a body taller that [the nave] itself if not there. Therefore, this is the square over which we can place the fourth body, called the *tiburio*,

[636] At the same time, there is no mention of the *tiburio* in Stornaloco's letter and proportioning system, even though the height for the *tiburio* could be easily deduced and was ultimately deduced from the system of planar triangulations described by Stornaloco. Giulia Ceriani Sebregondi and Richard Schofield, "Ultime opinioni sull'*Opinio*,"in *Ad Triangulum Il duomo di Milano e il suo tiburio: Da Stornaloco a Bramante, Leonardo e Antonio Giovanni Amedeo*, ed. Giulia Ceriani Sebregondi (Padova: Il Poligrafo, 2019), 160.

> because it is square, and because all the rest of the church ends on it [and] it is developed in length.[637]

Exactly how the concept of conformity [*conformitá*] should be understood in Bramante's text is undoubtedly one of the main questions posed by the *Opinio*. Arnaldo Bruschi argued that behind Bramante's claim stands "without doubt the Albertian concept of the building conceived as a living being, an *animans* or even *animan"*.[638] But the statement by Alberti is obviously metaphorical and refers only to an organisational principle. Bramante rather is clearly referring to specific geometrically defined parts of the building. In Alberti's *De re aedificatoria* however, the body [*corpus*], acts as a metaphor for the whole building and is never presented as a part of it. This is a position that Alberti repeats in more than one passage.[639] But in the case of Bramante, there is no mention of the whole building as a body, nor it can be said that his statements refer metaphorically to any properties of living beings.

In Bramante's *Opinio*, the body [*corpo*] is a part of the duomo identified for its formal, spatial and structural properties.[640]. It can thus be argued that the conformity to which Bramante

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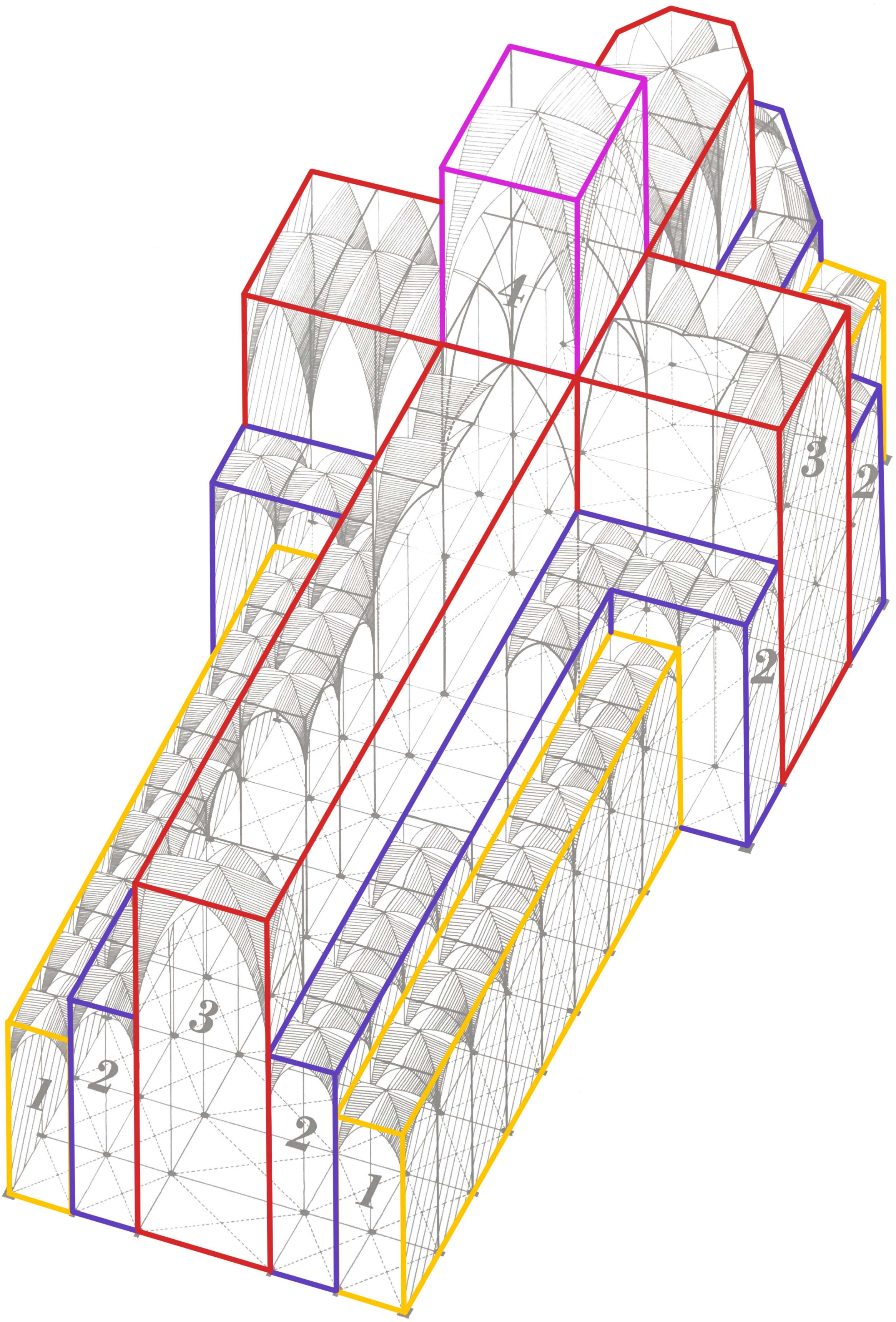
[637] "E perché questa nave de mezo fa tutte le sue volte vel crosere semiquadre, in Niuno loco non se imbatte a fare quadro perfecto, se non dove l'altra simile nave con sego se incrosa. In nesuno altro loco se può mandare corpo de magiore alteza che lei medesma, se non lì. Adoncha questo è quel quadro, sopra il quale se potrà ponere il quarto corpo, e chiamasse tiburio, perché el è quadro, e perché tutto l'altro resto de la chiesa ad quello se reduce et è longo;" Di Teodoro, "Due temi bramanteschi: *l'Opinio"*, 88-89.

[638] Bruschi quoted the following passage from *De re aedificatoria*: "[…] just as with animals members relate to members, so too in buildings part ought to relate to part; from which arose the saying, 'large buildings should have large members'";"ac veluti in animante membra membris, ita in aedificio parte partibus respondeant condecet. Ex quo illud dictum est , quod aiunt: maximorum aedificiorum maxima oportere esse membra.". Alberti, *De re aedificatoria*, I, 9, 65; Bruschi's interpretation is found in Bruschi, "Pareri sul tiburio," 359.

[639] Indeed, Bruschi also mentioned another passage from *De re aedificatoria:* "[…] just as the head, foot, and indeed any member must correspond to each other and to all the rest of the body in an animal, so in a building, and especially a temple, the parts of the whole body must be so composed that they all correspond one to another [*totum reliquum corpum referendum est*], and any one, taken individually, may provide the dimensions of all the rest."; "Quemadmodum in animante caput pes et qualecunque velis membrum ad caetera membra atque at *totum reliquum corpum referendum est*, ita et in aedificio maximeque in templo conformandae universae partes corporis sunt, ut inter se omnes corrispondeant, ut, quavis una illarum sumpta, eadem ipsa caeterae omnes partes apte dimetiantur' Alberti, *De re aedificatoria*, VII, 5, 559. English translation from Alberti, *On the Art of Building,* VII, 5, 199.

[640] Bramante continues by describing the height of the bodies and their structural arrangements: "It follows then the first being approximately forty *braccia* [tall]; the second fifty-nine, and the middle one eighty; and since the second body is taller than the first, as is due for strength of the building; from the piers of the first body spring buttresses that respond to the [thrust of the] piers of the second [body], in the same way from the second body spring buttresses that respond to the [thrust of the] piers of the third [body], that is the middle nave."



4.6—Axonometric diagram (based on Bruschi) of the four bodies of the Cathedral of Milan as described by Bramante in the *Opinio*.

refers here is the relationship between geometrical and material bodies that are considered for their uniform static behaviour in the structure of the building. In this way, the conformity discussed by Bramante is clearly related to the formal properties of the building. Formal and structural properties for Bramante are thus inherently correlated to the description of parts of the duomo as bodies. This is why, he continues, the buttresses of the *tiburio* should follow the straight walls of the body below, to conform to these and to the order of the building. Bramante further explains;

> […] it is therefore necessary, in order to conform [*conformare*] this to the other three [bodies] mentioned above, to make standing from the piers of the third body, that is the middle nave, buttresses that respond to the [thrust of the] fourth body, and in this way the building will be balanced, since otherwise it cannot be sound, and if this tiburio was made square, the buttresses would be more appropriate, because they would be placed over the straight [existing] walls. But as you break from the order of the building to make it octagonal, in the same way you should break the straight arrangement of the buttresses to make them fitting the [octagonal] tiburio, and this will move further [away] from the first order, and [even] further if [the tiburio] will be done round and without buttresses at all […][641]

Bramante developed this argument from geometrical considerations of the shape established by the supports of the *tiburio*. However, it is noteworthy that the *tiburio* of a building designed *ad triangulum* is still described by Bramante as a separate body. This has obvious repercussions for the formal properties of the *tiburio* itself. Bramante continues by briefly noting the importance of lightness [*leggerezza*] in relation to the *tiburio*'s height, and follows with considering the fourth principle mentioned at the outset of the *Opinio*, namely beauty [*belleza*]:

[641] "[…] é donca necessità, per conformare questo quarto corpo a le altre tre de soto sopradicte, far partire da' pilloni del terzo corpo, cioè da la nave de mezo, contraforti che respondeno a questo quarto corpo, e così sarà l'edificio equale, e altramente non può star bene, e se questo tiburio se havesse a fare in quadro, più verrebbeno iusti questi dicti contraforti, perhò che seguireveno sopra il drito de le muralie. Ma sì como voi romprete l'ordine del dricto de' contraforti per confarli al tiburio, e bene che questo se delongi dal primo ordine, più se delongarebe quando il tiburio si facisse tondo e sanza contraforti veruni […]" Di Teodoro, "*l'Opinio*", 89.

> Concerning the fourth thing, that is beauty [*belleza*], the more [the *tiburio*] will go [high], the more beautiful it would be, even if it should not exceed the order [of the building]; but its altitude could also be detrimental for its [related] weight. This has been seen by the Priests, *Antonio da Pandino* and *Ioanne da Molteno*, that divide the building in three. I say that if the highest point of the vaults [of the main nave] is at eighty *braccia* from the ground, they make the *tiburio* forty *braccia* from this point up, and this is a good height [*bella alteza*], with less weight, strong, with the previously mentioned buttresses outside; and who that will do the aforementioned *tiburio* in another way, will not be able to see it both from the outside and the inside, and since [the *tiburio*] should be seen above the rest of the roofs of the church, you will have to go a mile far from Milan [to see it].[642]

Why Bramante would approve a height of 40 *braccia* for the *tiburio* from the top of the crossing of the main nave (and thus a total height of 120 *braccia*) has been the subject of scholarly debate. As noted by Schofield, apart from being half of the main nave's height at this point (and the height of this point in the lowest nave as stated by Bramante), when considered with the width of the tiburio (32 *braccia*), this dimension does not follow the height/width ratio of other naves, nor any of their mutual relationships. As Schofield concluded: "for some inexplicable reason [Bramante] proceeds to ignore the relationship between the aisles that he has so carefully described and does not extrapolate from it when he fixes the height of the *tiburio*".[643] Nevertheless, it should be noted that since Bramante concentrates on the *tiburio*'s individual formal properties and visual effects, it is somewhat anticipated that the system *ad triangulum* devised by Stornaloco would not be the primary regulating device to control its height. This is a crucial passage if the hierarchy of principles employed by Bramante in the

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[642] "Quanto a la quarta cosa, che è belleza, quanto più alto se andasse, più bello sarebe, pure non se excedesse l'ordine; ma questa alteza potrebe aliquanto nocere al caricho. Questo ha visto el P(re)te, Ant(onio) de Pa(n)si(n)o e Ioanne da Molteno, che partono l'edificio per terzo. Dico se da terra a le crosere è octanta braza, loro fanno il tiburio alto quaranta de le crosere in su, et questa è bella alteza, co minema de peso, forte, havendo li dicticontraforti de fora; e chi altramente farrà dicto edificio, nè de dentro né de fora haverà vista, perhò che dovendolo vederesopra il resto del techio de la chies, converrà andare da lonze uno milio da Milano, e li contraforti che se gli faranno, saranno più tosto nocivi che utili;" ibid., 89.

[643] Schofield, "Amadeo, Bramante and Leonardo," 76.

definition of the formal properties of the *tiburio* is to be understood.[644] Besides the exact shape taken by the *tiburio* in Bramante's model, it is also crucial to note how his argument develops. Overall, despite the lack of details in Bramante's *Opinio*, the description of the building as bodies (and the discussion of their formal and structural properties) is clearly instrumental to Bramante's argument for justifying the formal and visual properties of the *tiburio* (thus understood as a separate geometrical and material body). As seen, Bramante privileges a formal, structural and above-all spatial interpretation of the building by means of geometrical bodies.

This should be correlated with Bramante's mastery of perspectival representation, already shown in works produced before the redaction of the *Opinio*. The formal properties of the *tiburio* are related by Bramante to the visual effects that these would produce from both inside and outside the building. Clearly, this can only be possible if Bramante had mentally considered the three-dimensional body of the *tiburio* and derived this judgement from the two-dimensional images that would be perceived from specific visual angles. At the same time, it is useful to correlate this, and the spatial division of the building by means of bodies, to Pacioli's concept of the geometrical body described as a composite of form (shape) and matter (three-dimensional spatial extension). As noted, the mastery of the principles of perspective applied to bodies was integral to Pacioli's description and representation of regular and other bodies derived from these as physical objects in the *Compendium*.

[644] A solution that combines a square plan *tiburio* with an height that follows the system of triangles devised by Stornaloco, is shown in the xylography included by Cesare Cesariano in his edition of Vitruvius's *De architectura* published in 1521, to illustrate Vitruvius's three modes of representation. While Cesariano's *ortographia* shows both a square and octagonal *tiburio*, the *scenographia* only shows a cubic *tiburio* in a section/elevation of the duomo, that also includes at the bottom a part of plan. It is well-known that Cesariano claims to have been a pupil of Bramante, and it has been argued that his published drawing of the duomo with a square *tiburio* (and not the octagonal *tiburio* that had been built) is a clear reference to Bramante's proposal. Christof Thoenes has argued for a blueprint for Cesariano's drawing illustrating the *scaenographia* and linked it to the period of Bramante in Milan. Christof Thoenes, "Vitruvio, Alberti, Sangallo. La teoria del disegno architettonico nel Rinascimento," in *Sostegno e adornamento. Saggi sull'architettura del Rinascimento: disegni, ordini, magnificenza* (Milan 1998), 161-175.

Similarly, Bramante considers each body as composed by spatial extension and a definite shape, described by its height, width, and length. Besides, each body is qualified by its formal and related structural properties in the system of forces that govern the building. Lastly, each body is both an independent, yet simultaneously integral part of the building. As previously discussed, this was not how the process of design and the geometrical arrangement of the architectural elements of sacred buildings was described in the treatises of near-coeval or coeval theorists, such as Leon Battista Alberti and Francesco di Giorgio Martini.
At this point, an empirical analysis of some of Bramante's architectural works including those mentioned by Pacioli in his *Compendium*, could contribute to a better understanding of how the dynamics by which the formal, spatial and visual properties of a building, understood by means of geometrical bodies, could work in the case of Bramante. Despite the limitations imposed by the incomplete quantitative data available from surveys, such an analysis could help to establish a series of principles to clarify the type of correlations between Bramante's architecture and Pacioli's arguments.

Conclusion

This chapter investigated the process by which geometrical concepts were employed in the late fifteenth century to define the formal properties of architecture. Rather than focusing on the specific quantitative values employed, this chapter concerned the procedures by which quantities were visualised as geometrical entities in architecture, namely lines, two-dimensional figures, three-dimensional shapes or bodies, and their implications for architecture's formal and visual properties.

The analysis particularly concerned the design of sacred architecture. Besides being discussed in Vitruvius's *De architectura*, this typology of building was central to fifteenth-century treatises by Leon Battista Alberti and Francesco di Giorgio Martini. At the same time, it was discussed in the writings of Luca Pacioli and Donato Bramante. However, as noted, a comparative analysis of the process by which different authors defined the formal properties of these buildings and employed geometrical concepts has not yet been undertaken, particularly in the context of Pacioli's theoretical framework.

In *De architectura*, and in the fifteenth-century treatises by Alberti and Francesco di Giorgio, linear quantities and two-dimensional geometrical figures were employed in an itemised process to control the three-dimensional formal properties of sacred architecture. Consequently, the process of abstraction employed in the spatial design of buildings was not based on the application of a (pre-formed) three-dimensionally extended body; but rather defined the formal properties of the building by the systematic employment of linear quantities or two-dimensional figures, combined in proportional relationships by arithmetical, and if needed geometrical means.

As shown, Pacioli's interest in both the process of quantification and the theory of architecture is in many ways correlated to Alberti's position. At a basic level, they both consider geometrical principles as the core of a mathematical understanding of the phenomenal world and rely on lines and angles as the basic geometrical entities. However, consistent with his mathematical approach, Alberti in *De re aedificatoria* clearly separated immaterial lines and angles from the materials in which they were instantiated in architecture, claiming that the architect should deal only with the former. Pacioli argued instead for their spatial unification in the concept of the geometrical body. In Pacioli's view, geometrical bodies (including their instantiations in architecture) partake of the same mathematical, visual, and spatial properties shared by all material objects. In essence, geometrical bodies can be applied in architecture (and this also includes architecture's spatial extension, i.e. architecture's voids, whose matter is the air) as three-dimensional entities with pre-determined formal properties.

At the same time, respecting proportional relationships among parts of a whole, Pacioli saw no contradiction between the employment of regular and other bodies derived from these in architecture in his *Tractato de l'architectura* and *Compendium,* and the theories of proportionality discussed in other architectural treatises. For Pacioli, if geometrical bodies are considered for their formal and aesthetic properties, the proportional and quantifiable relationship between their parts and their visual experience justifies their inclusion in his theory and their employment in architecture. Similar to Pacioli's formal and visual arguments are Bramante's spatial division of the Duomo of Milan by means of bodies, and the description of the formal and visual properties of the Duomo's *tiburio*, found in his *Opinio*. Bramante describes the building as being composed of four bodies. As in Pacioli's case, each is thus considered as a composite of form (shape) and spatial extension.

In the texts examined above, both Pacioli and Bramante consider the geometrical and material levels of the body to be integral to the definition and perception of the formal properties of architecture. As argued above, this is also correlated to the mastery of the principles of perspective and their application in the process of definition of the formal properties of architecture. While these were not linked to architecture in the writings of Alberti or Francesco di Giorgio, this argument was employed by Pacioli to relate the formal, visual, and spatial properties of regular and other bodies derived from these and their material instantiations. Similarly, the visual concerns expressed in the *Opinio* become clearer when Bramante's ability to employ the principles of perspective (as demonstrated in works predating his *Opinio*) is considered. Given the lack of theoretical writings by Bramante beyond the *Opinio*, only an analysis of his works in Milan and Rome can clarify the type of correlations established with Pacioli's works, and provide an empirical theory of Bramante's approach to the visual, formal, and spatial properties of architecture.

5. Donato Bramante's architectural forms in Milan and Rome, their relationships with the formal, visual and spatial properties of geometrical bodies in the works of Luca Pacioli and in the drawings of Leonardo da Vinci

This chapter will mainly concern the relationship between the theoretical framework found in Luca Pacioli's works discussed in the previous chapters, and the process of defining formal properties in the works of Donato Bramante in Milan and Rome. Although drawings of central-plan buildings and the sacred architecture of Leonardo da Vinci will also be examined, this will be considered in the context of Leonardo's interactions with Bramante in their Milanese period. The drawings by Leonardo that will be examined were in fact all drafted during or after his first period in Milan. As noted, Leonardo also drafted the perspectival representation of regular and other bodies derived from this for Pacioli's *Compendium* in Milan, and was highly praised by Pacioli for his skills in painting, perspective, and architecture among others.[645] However, there is no built architecture that can be linked to Leonardo's designs in his Milanese period. Additionally, Pacioli referred to buildings associated with Bramante when discussing the application of geometrical bodies to architecture. Ultimately, this chapter aims to determine whether Pacioli in his *Compendium* and later *Tractato de l'architectura* was theorising aspects of both Bramante's and Leonardo's design approaches to architecture and furthermore, whether the arguments that developed in his works were closely correlated with this theory.

Overall, the analysis will be developed around three closely interrelated arguments concerning the spatial, visual and formal properties of the architecture examined. Bramante's

[645] "Comme apien in le dispositioni de tutti li corpi regulari e dependenti di sopra in questo vedete quali sonno stati facti dal degnissimo pictore prospectiuo architecto musico. E de tutte virtu doctato, Lionardo da vinci fiorentino nella cita de Milano quando a li stipendii dello Excellentissimo Duca di quello Ludovico Maria Sforça Anglo ci retrouauamo nclli anni de nostra Salute 1496 fin al 99 […]" Pacioli, *Divina proportione (1509)*, 28v (Tractato de l'architectura).

works in particular, have often been noted for their innovative ways of dealing with space. Giulio Carlo Argan notably stated that Bramante's architecture is a "spettacolo della spazialità."[646] Arnaldo Bruschi moreover, argued that the concept of spatiality was central to Bramante's works in his influential biography, first published in 1969 and still widely referenced today.[647] However, in recent decades, studies on Bramante's works have mostly concerned other subjects. This has coincided with a tradition among architectural historians to privilege symbolic meanings and problems of attributions and sources, rather than the process by which formal, visual and spatial results were achieved in Bramante's architecture.

Notwithstanding, Bramante's mastery in the practice of linear perspective has been recognised since the sixteenth century.[648] Sebastiano Serlio wrote that Bramante was originally a painter and was very learned in perspective before he began to dedicate himself to architecture, meaningfully adding that whoever practices perspective cannot accomplish anything without knowing architecture, and one cannot practice architecture without perspective.[649] Similarly, in his commentary to Vitruvius's *De architectura,* Cesare Cesariano (who wrote that he had been a pupil of Bramante in his youth),[650] noted Bramante's employment of the *ratione optica* (i.e. perspective) in the conformation of the church of Santa Maria presso San Satiro in Milan, as a peripteral temple.[651]

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646 "Spectacle of spatiality." Giulio Carlo Argan, "II problema del Bramante,*" Rassegna Marchigiana* XII, (1934): 212–33.

647 Arnaldo Bruschi, *Bramante architetto* (Bari: Editori Laterza, 1969); Arnaldo Bruschi, *Bramante* (London: Thames and Hudson, 1977).

648 Bruschi, *Bramante architetto.* Filippo Camerota, "Bramante 'prospettivo'," in *Donato Bramante, ricerche, proposte, riletture*, edited by Francesco P. Di Teodoro (Urbino: Accademia Raffaello, 2001), 25.

649 "[…] il prospettivo non farà cosa alcuna senza l'architettura, né l'Architetto senza la prospettiva, e che sia il vero consideriamo uno poco gli Architetti del secolo nostro nel quale la buona Architettura ha cominciato a fiorire", and first among these "Bramante suscitatore della bene accompagnata Architettura, non fu egli prima Pittore, et molto intendente nella Prospettiva prima che si desse ad essa arte?". Sebastiano Serlio, *Tutte l'Opere d'Architettura, et Prorspetiva […]*, book II (Venice, 1569), 18v.

650 "il mio preceptore Donato, cognominato Bramante". Cesare Cesariano, *De Lucio Vitruvio Pollione De Architectura libri dece traducti de latino in vulgare, affigurati, commentati*… (Como: G. Da Ponte, 1521), 4v. Other direct references to Bramante in folios 21v, 46v, 70v, 100r, 113v. Cesariano was born in 1475.

651 "temples may have niches as chapels in a circle devised through the *ratione optica* and they seem to have a great depth, as in the aforementioned church of San Satiro was done by Bramante." ibid., 70v.

Although the theoretical principles of linear perspective in painting were well established, their systematical applications to architecture were not codified in fifteenth-century treatises. Moreover, Vitruvius in *De Architectura* either refers to corrections to rectify optical distortions in elevation and plan, or speaks about perspectival backdrops in theatres.[652] However, Vitruvius never argues that the principles of perspective should be employed simultaneously in three-dimensions to define the spatial arrangement of a building at the design stage. At the same time, as noted, Pacioli's comments on Book V of *De architectura* extended the types of corrections that could be applied to achieve the satisfactory proportional arrangement of a building,[653] and included the employment of regular and other three-dimensional bodies derived from these. The comparison of Pacioli's arguments with Bramante's architecture should be considered (based on the available data) in the context of recent theories in contemporary psychology and the philosophy of perception.

Mastery of the principles of perspective is closely related to what are commonly defined as two different kinds of visual stimuli. The first is the distal stimulus, which is the light as emitted or reflected by the object itself (corresponding to the real, actual object in its proper

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[652] Vitruvius, *De architectura*, III. 3. 11. (on increasing the diameter of angular columns), III. 3. 13. (on varying the diameter across the column's height), III. 4. 5. (on the non-planarity of the stylobate), III. 5. 9. (on increasing the height of the architrave and frieze), III. 5. 13. (on tilting elements that are located at a higher level than the point of view), IV. 4. 2-3 (on diminishing the diameter of internal columns and on determining the number of flutes on the column). These corrections were applied by Vitruvius mostly on temples. In VI. 2. 1-2 Vitruvius argued that: "Nulla architecto maior cura essa debet, nisi uti proportionibus ratae partis habeant aedificia rationum exactiones. Cum ergo constituta symmetriarum ratio fuerit et commensus ratiocinationibus explicati, tum etiam acuminis est proprium providere ad naturam loci aut usum aut speciem, adiectionibus temperaturas efficere, cum de symmetria sit detractum in aspectuque nihil desideretur. […] Non enim veros videtur habere visus effectus , sed fallitur saepius iudicio ab eo mens. Quemadmodum etiam in scenis pictis videntur columnarum proiecturae, motulorum ecphorae, signorum figurae prominentes, cum sit tabula sine dubio ad regulam plana."; and discussed the apparent breaking of oars when immersed in water, to support the argument that the eye can deceive. As translated by Cesare Cesariano: "Per che il videre non appare habere li effecti veri, ma sovente volte la mente fu fallita dal iudicio di quello". And since the "cose che sono vere paiano essere false", the architects should draw "con li acumini de li ingegnii, non solamente con le doctrine", affinché "la Eurythmia a li consideranti non sia dubio". Cesariano, *De Lucio Vitruvio Pollione De Architectura,* 96r. Camerota, *La Prospettiva*, 261, 331. Finally, in VII. Preface. 11. Vitruvius considered again the visual effects produced by perspectival backdrops in theatres.

[653] As seen, Pacioli argues here that regular and other bodies derived from these should be recommended for their employment in devising the formal properties of buildings when there would be difficulty in achieving proportional relationships for the particular arrangement of places. Pacioli, *Divina proportione (1509),* 31r (Tractato de l'architectura).

form).[654] The second is the proximal stimulus (bundles of light rays), as received by photoreceptors in the observer's retina. As noted, after this initial (two-dimensional, perspectival) visual stage the brain relies on a series of so-called 'constraints' to determine the likely three-dimensional shape of the object perceived. As summarised by Zenon Pylyshyn, there are a number of ways to describe the constraints;[655] the shared basic assumption, however, is that "the visual system follows a set of intrinsic principles that are independent of general knowledge, expectations or needs".[656] This set of principles of visual analysis, called by Donald Hoffman 'rules',[657] express as noted by Pylyshyn "the built-in constraints on how proximal information may be used in recovering a representation of the distal scene", resulting in an unambiguous three-dimensional interpretation of a two-dimensional image.[658] In other words, they apply to the way in which the proximal (perspectival) stimulus is interpreted or used to construct a three-dimensional representation of the distal perceptual world. One of the most common among such constraints involves the geometry and direction of light.[659] Another one particularly relevant to polyhedral objects is the requirement of consistency on interpreting their edges and vertices as concave or convex.[660]

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[654] An overview of the debate about the proximal and distal kind of perception is given in Mitrović, *Visuality*, 44-50.

[655] E.g., as noted by Zenon Pylyshyn, "in terms of properties of the world, in terms of the use of "general purpose models" of objects, or in terms of some world-independent mathematical principle, such as "regularisation"". Pylyshyn, *Seeing and Visualizing*, 112.

[656] ibid.

[657] Donald Hoffman listed 35 constraints (he calls them 'rules') For example, rule 1: "always interpret a straight line in an image as a straight line in 3D", rule 2; "if the tips coincide in an image, then always interpret them as coinciding in 3D", rule 3: "always interpret lines collinear in an image as collinear in 3D", etc. Among the rules particularly relevant for the visual analysis of geometrical bodies, rules 1-7, and rules 14-15, grouped by Hoffman under a section titled "spontaneous morphing". Rule 14: "rule of concave creases: Divide shapes into parts along concave creases", rule 15: "Minima rule: Divide shapes into parts at negative minima, along lines of curvature, of the principal curvatures." Hoffman, *Visual Intelligence*, 17-46, 79-90. Some of the 'rules' presented by Hoffman are summarised by Pylyshyn, *Seeing and Visualising*, 106-109.

[658] Pylyshyn, *Seeing and Visualising*, 112, 107.

[659] ibid., 107-108.

[660] As noted by Pylyshyn, "If an edge is assumed to be concave at one end, where if forms part of a vertex, then it must also be concave at its other end where it forms part of another vertex. But if such a labelling of the second vertex is physically impossible […] then that label must be discarded as inconsistent. Such a simple label-consistency constraint arises in part from the physical nature of edges and vertices and turns out to be a strong constraint that makes it possible to give a unique labelling to most images of scenes containing blocks." ibid., 97.

If considered in the three-dimensional context of architecture, while proximal stimulus is the way that architecture is visually (two-dimensionally, perspectively) perceived at a retinal level, constraints are linked to three-dimensional shapes that can be visually experienced when applied to architecture (as, for instance, regular and other bodies derived from these or their parts). Essentially, Pacioli follows the same basic visual stages in determining that some geometrical bodies had been applied to architecture. In this case, since geometrical bodies are described by Pacioli as composed of form (shape) and matter (volume or three-dimensional spatial extension), they partake to the distal perceptual world as architecture does.

Knowing the principles of the theory of perspectival representation and their applications in the fifteenth century, as described by Luca Pacioli, is essentially to master the shift between the proximal and distal kinds of stimuli and as such, necessarily involves a number of constraints. As seen, in the "Other Method" discussed in Book III of *De prospectiva pingendi*, Piero della Francesca argued that the construction of perspectival representations of three-dimensional objects begins with the precise knowledge of their (distal) geometrical properties and quantities (that Piero della Francesca calls *propria forma*), and ends with the acknolwedgment of these quantities as proportionally diminished on the plane of (perspectival) representation.[661] Piero added at the beginning of the book that what the intellect alone cannot measure, can only be (mathematically) known by means of the

[661] As noted, the formal, spatial and visual properties of geometrical bodies were discussed in the treatises of Piero della Francesca. The *Tractato d'abaco* and *Libellus de quinque corporibus regularibus* dealing with their mathematical quantification, the treatise *On Perspective in Painting* [*De prospectiva pingendi*] with the methods for their correct representation in perspective based on mathematical principles. Nonetheless, in Chapter 4 it was highlighted that if Piero discussed these properties in different works, Pacioli did not fail to recognise and argue for their correlation and unification. At the same time, when the application of these principles to architecture was considered, Pacioli pointed to two works of Bramante in Milan. Since the apprenticeship of Bramante under Piero della Francesca was acknowledged in the sixteenth century by the Milan-born humanist Sabba da Castiglione, the knowledge of the theories of Piero della Francesca is most probably a common link to which both for Luca Pacioli and Donato Bramante refer.

principles presented in the theory of perspective.[662] However, as noted, it was only in Pacioli's works that these arguments were clearly correlated with the visual perception of the three-dimensional material world, and thus with built architecture. Ultimately, in the context of human visuality, this necessarily implies that architects who employ the principles of perspective in the definition of the formal properties of their works deal with quantities in a way that will affect their observers' visual experience of shapes not only at the proximal level, but also in the associated process of construction of the distal perceptual world.

We thus arrive at a crucial question for the whole thesis: how are the visual, formal and spatial properties of regular geometrical bodies and other bodies derived from these, correlated with mastering the theory of perspective and its application in architecture? As noted, Pacioli believed that the theory and practice of perspective (based on the geometry of light rays) followed the same mathematically determined proportional relationships of vision.[663] In other words, as Pacioli maintains, if their geometrical properties are established, there would be no contradiction between the material instantiation of regular and other bodies derived from these, their simultaneous visual (proximal) perception and the experience of their (distal) three-dimensional formal properties. Similarly, as mentioned, the subdivision of a building by

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[662] Since, as noted, Pacioli was able both to exploit and make visible the inherent correlations among arguments found in the writings of Piero della Francesca, there is no reason to think that the arguments in Pacioli's *Compendium* which discuss the architecture of Bramante should not be linked to the context of the environment of the court of Milan (and probably also Urbino) were the text was produced and in which both Bramante and Pacioli operated. If a building is understood as a series of geometrical bodies composite of form (shape) and matter (spatial extension)—as argued by Pacioli—and their form (shape) is known, the process involves a certain number of constraints. As noted, Sabba da Castiglione noted in the sixteenth century that Bramante was a pupil of Piero della Francesca. Moreover, architectural elements most probably derived from Piero's paintings are found in Bramante's works. This is noticeable in particular in the works of Bramante in Milan. Richard Schofield has given the most accurate analysis of the range of possible sources for Bramante's works in Lombardy. Richard Schofield, "Bramante dopo Malaguzzi Valeri," *Arte Lombarda*, no. 167 (1) (2013): 5-51. Nonetheless, the correlations of the works of Bramante with the theories Piero della Francesca, has never been examined at the conceptual level and in the context of the theoretical framework provided in the writings of Pacioli.

[663] Pacioli both endorses the application of regular and other bodies derived from these in their proper form and proportional relationships, and their visual (perspectival) perception.

three-dimensional geometrical bodies was employed in Bramante's *Opinio* to define the structural, formal and ultimately visual properties of the *tiburio* of the Duomo of Milan.

As noted, theorists such as Alberti and Francesco di Giorgio described the process of designing temples as a sequence that reflected the stages of perception as they occurred in human visualisation: first lines, angles, and two-dimensional figures (or surfaces) are perceived, and then their orientation and rough depth subsequently lead to the formation of a three-dimensional mental representation of shapes (and their spatial organisation) that does not depend on the direction from which the object is viewed.[664] This does not mean that they could not have envisaged a three-dimensional shape before the actual design process began, but that ultimately formal properties were determined following the itemised process outlined above. Given these stages of perception, the main question of this chapter is thus whether the process of architectural design can be reversed and be led by the process of visual imagination of three-dimensional bodies. In other words, could a three-dimensional body (composed—as noted by Pacioli—of shape and matter (volume) be the initial visually defined form, and the process begin with its mental visualisation (by means of two-dimensional images)? Could the linear quantities of related architectural elements (like, for example, the orders) be defined according to this pre-formed three-dimensional body, and (if needed) then described in a series of orthogonal projections by lines, angles, and two-dimensional figures to be communicated to the builders?

Finally, a note about the case studies among Bramante's designs considered in this chapter. In recent decades scholars have made great efforts to uncover the sources of Bramante's solutions in antique Roman and Lombard architecture, and in buildings by Alberti, Francesco

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[664] This process was already outlined by David Marr in the 1980s. Marr, *Vision*, 36-37; See also Mitrović, *Visuality*, 59-60.

di Giorgio, and Brunelleschi.[665] Aspects of Bramante's relationship with Leonardo (based on the analysis of the latter's drawings), have been instead considered only recently.[666] However, beyond the singular elements discussed, such analyses have provided only limited understanding of the design process employed by Bramante. At the same time, Bramante's relationship with Pacioli's theoretical framework has been mostly neglected. While the scope of this study does not cover the whole architectural production of either Bramante or Leonardo, it will consider case studies only insofar as they are relevant to establishing the nature of the relationship between their approaches to architecture and the properties of geometrical bodies discussed by Pacioli. Ultimately, even if the study (based on the data available) would not be conclusive in all respects and further research will be required, the aim is to define aspects of an empirical theory of Bramante's approach to the visual, formal, and spatial properties of architecture.

5.1 Formal aspects of Bramante's architectural design in late fifteenth-century Milan

5.1.1 The drawing for the Prevedari Print

In 1477, Bramante painted a series of philosophers in architectural settings rendered in perspective on the facade of the Palazzo del Podestà in Bergamo (a work almost completely lost today). Following this, Bramante most probably arrived in Milan in the late 1470s.[667] Although there is little information about Bramante before this point, he doubtless practiced

[665] Most of these studies have been carried out by Richard Schofield. A summary was published in Richard Schofield, "Bramante e un rinascimento locale all'antica," in *Donato Bramante, ricerche, proposte, riletture*, ed. Francesco P. Di Teodoro (Urbino: Accademia Raffaello, 2001). Most recently in Richard Schofield, "Bramante milanese: collisioni di culture architettoniche?," *Arte Lombarda*, no. 176/177, vol. 1/2 (2016).

[666] Richard Schofield, Cristiano Tessari, "Aspects of Church Design from Brunelleschi and Alberti to Leonardo and Bramante," in *Leonardo da Vinci - Nature and Architecture*, ed. Constance Moffatt and Sara Taglialagamba (Leiden: Brill, 2019), 314.

[667] Arnaldo Bruschi, "La formazione e gli esordi di Bramante: date, ipotesi, problemi."

painting and perspective before undertaking architectural projects. The fresco in Bergamo, the later fresco of Men at Arms [*Uomini d'Arme*] in Milan and the so-called Prevedari Print, all demonstrate Bramante's interest in the perspectival representation of architectural settings. Moreover, in the first two cases at least, the painted architecture was clearly intended to establish a visual and spatial illusionistic expansion of the physical spaces in which these representations were located. The Prevedari Print is the first recorded work by Bramante in Milan and is mentioned in a contract drafted in October 1481.[668] It is worth considering it before the other works of Bramante in Milan. The contract stipulated that the engraver was to finish a large print within two months "with edifices and figures, in copper, according to the drawing on papyrus made by *magistrum Bramantem de Urbino"* (illus. 5.1).[669] This was the largest engraving from a single plate (c. 708 x 512 millimetres) produced in the Italian peninsula in the fifteenth century.[670]

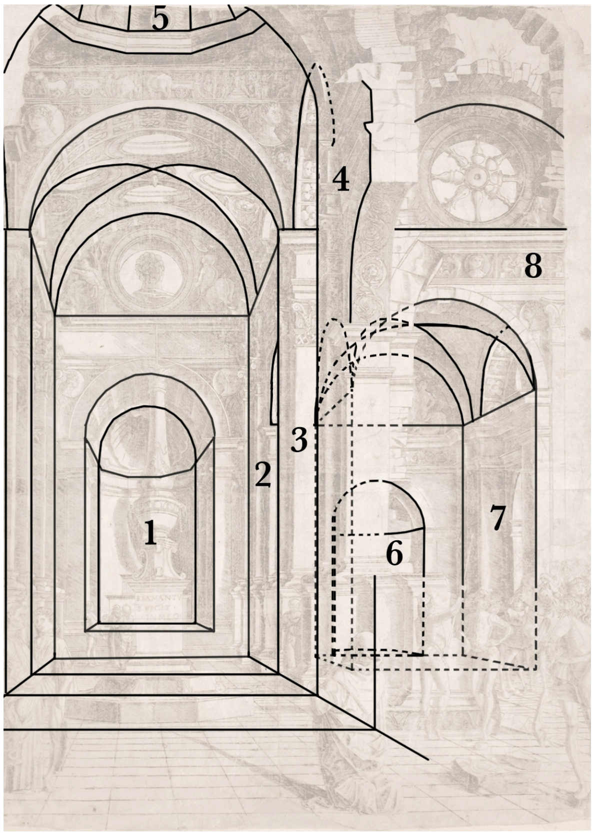
The drawing was most likely intended to display Bramante's skills and knowledge, and scholars have traced its ornamental details from many sources. Among these, the barrel vault and shell of the *Brera Altarpiece* by Piero della Francesca are almost literally referenced and are located close to the main vanishing point of the perspectival construction. Some details clearly refer to paintings by Andrea Mantegna and to the bas-reliefs by Donatello in Padua. Other architectural details have been linked to the Duomo and San Bernardino in Urbino

[668] The contract was between the painter Matteo Fedeli (also involved in the decoration of the New Church of Santa Maria of San Satiro, in which Bramante is documented from December 1482, and the engraver Bernardo Prevedari. Bruschi, *Bramante Architetto*, . Schofield, "Bramante dopo Malaguzzi Valeri,".

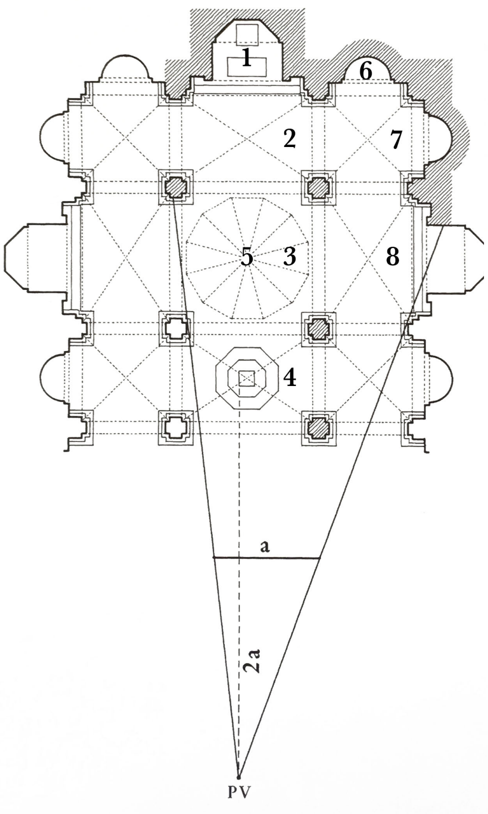
[669] '[…] cum hediffitiis et figuriis lotoni secundum designum in papiro factum per magistrum Bramantem de Urbino […]' ASMi, Notarile, 2184, Benino Cairati, in Francesco Repishti, "Bramante in Lombardia: regesto delle fonti." *Arte Lombarda*, no. 176-17, 1-2 (2016): 197.. 197. Bramante was already about 37 years of age at this point and was called *magistrum*. As the contact noted, was supposed to be ready for the 24 December of the same year.

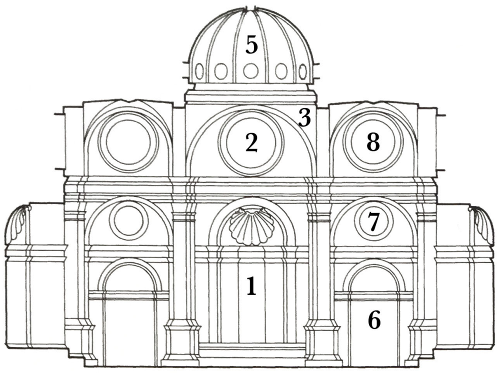
[670] Laura Aldovini, "The Prevedari Print," *Print Quarterly*, vol. 26, no. 1 (March 2009): 42.

5.1—Bernardo Prevedari (based on a drawing by Donato Bramante), Interior of a ruined church or temple with Figures [*Tempio in rovina*], 1481, 708 x 512 mm. The British Museum, London.



5.2—Main lineaments of the three-dimensional spaces (numbered) shown in the Prevedari Print.





5.3, 5.4—Hypothetical plan and section-elevation of the temple represented by Bramante (Camerota), with the position of the eye marked in plan (PV). Main spaces indicated with numbers.

designed by Francesco di Giorgio Martini, and to Alberti's buildings in Mantua.[671] At the same time, it is unsurprising that the narrative subject of the print has not yet been clearly identified. Bramante's drawing was first and foremost a display of skills in the perspectival representation of the spatial and formal properties of the architecture. The human figures deployed act mainly as a measuring tool for the architectural space.[672]

The temple's space is precisely articulated by the arrangement of a series of distinct spaces, clearly differentiated by height, geometry and ornamental details. The whole temple has often been interpreted by scholars as a part of a *quincunx* and arguably this has resulted in the simplification of its spatial articulation.[673] A *quincunx* is always defined by a central dome that is surrounded by four minor domes. However, since only a portion of the building is shown and there is a single dome suggested on the top left-hand side, there is no way of determining whether the building is a *quincunx*, thus it cannot be determined that this was Bramante's intention. It can be argued instead that one of Bramante's main purposes was to determine the mental visualisation of an articulated arrangement of three-dimensional architectural forms. Bramante therefore was simply displaying his skills in the geometrical quantification and perspectival representation of the spatial properties of architecture.

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[671] The detail of the archivolt overlapping the architrave above the barrel vault in particular can be linked with a very similar detail in the church of San Bernardino of which Bramante may have collaborated or seen drawings of Francesco di Giorgio Martini. Other details can be linked to the Colleoni Chapel in Bergamo and the Prevedari Chapel in Milan, in addition to other suggested references to details from Lombard architecture. As noted by Schofield, overall the cultural coordinates to read the Print seem clear, they are related to the North of Italy and Urbino in particular. Schofield, "Bramante dopo Malaguzzi Valeri," 13.

[672] As noted, after considering formal and aesthetic properties, Alberti's added the *historia* among the highest aims of a painting in *De pictura.* Alberti, *De pictura*, II.3. However, it can be argued that in this case the formal and spatial properties of the architecture acquire prominence over the narrative.

[673] The theme of the *quincunx* has the generative scheme for the temple in the Prevedari Print has been emphasised in particular by Arnaldo Bruschi and was also proposed in the first hypothetical reconstruction of the temple in plan and section-elevation. Bruschi, *Bramante,architetto*, 150-170. A similar criticism to the interpretation of the space of the temple as a quincunx was recently argued by Schofiled and Tessari. Richard Schofield, Cristiano Tessari, "Aspects of Church Design from Brunelleschi and Alberti to Leonardo and Bramante," in *Leonardo da Vinci - Nature and Architecture*, ed. Constance Moffatt and Sara Taglialagamba (Leiden: Brill, 2019), 322.

An analysis of the drawing shows that at least eight main spaces concur in the articulation of the visible portion of the temple. Moreover, some architectural elements are clearly arranged as to provide predetermined visual effects and to facilitate the view of the formal properties of the architecture behind. The ruined pilaster on the right hand-side, for instance, does not sit in line with the others pilasters. Simultaneously, it is broken in a way to show the arch in the background wall and thus clarify the form of the niche behind (space no. 6 in illus. 5.1-5.3).[674] Overall, although there are minor imprecisions that may have resulted from the transfer of the drawing in the print's plate, Bramante's interest in the quantification and perspectival representation of a complex architectural spatial articulation is clear. How this was instrumental in Bramante's approach to his first documented project in Milan should now be considered.

5.1.2 The design of the church of Santa Maria presso San Satiro, its choir and sacristy

The church of Santa Maria presso San Satiro in Milan is Bramante's first documented built architectural work. The new church was to stand beside the existing Sacello of San Satiro and host an image of St. Mary, that was previously located on the latter's external wall.[675] Works on the site are recorded from 1480 and Bramante's involvement is documented from 1482.[676] The initial phases of construction are uncertain,[677] but by 1482-1483 the transept of the present church was developed along the existing Via Falcone, with the main nave to be located on the opposite side and orthogonal to this (illus. 5.5).[678] The location of the transept and its adjacency to Via Falcone did not allow an arm for the church's choir, and this issue was notably addressed by Bramante with the perspectival illusion of a choir arranged in the

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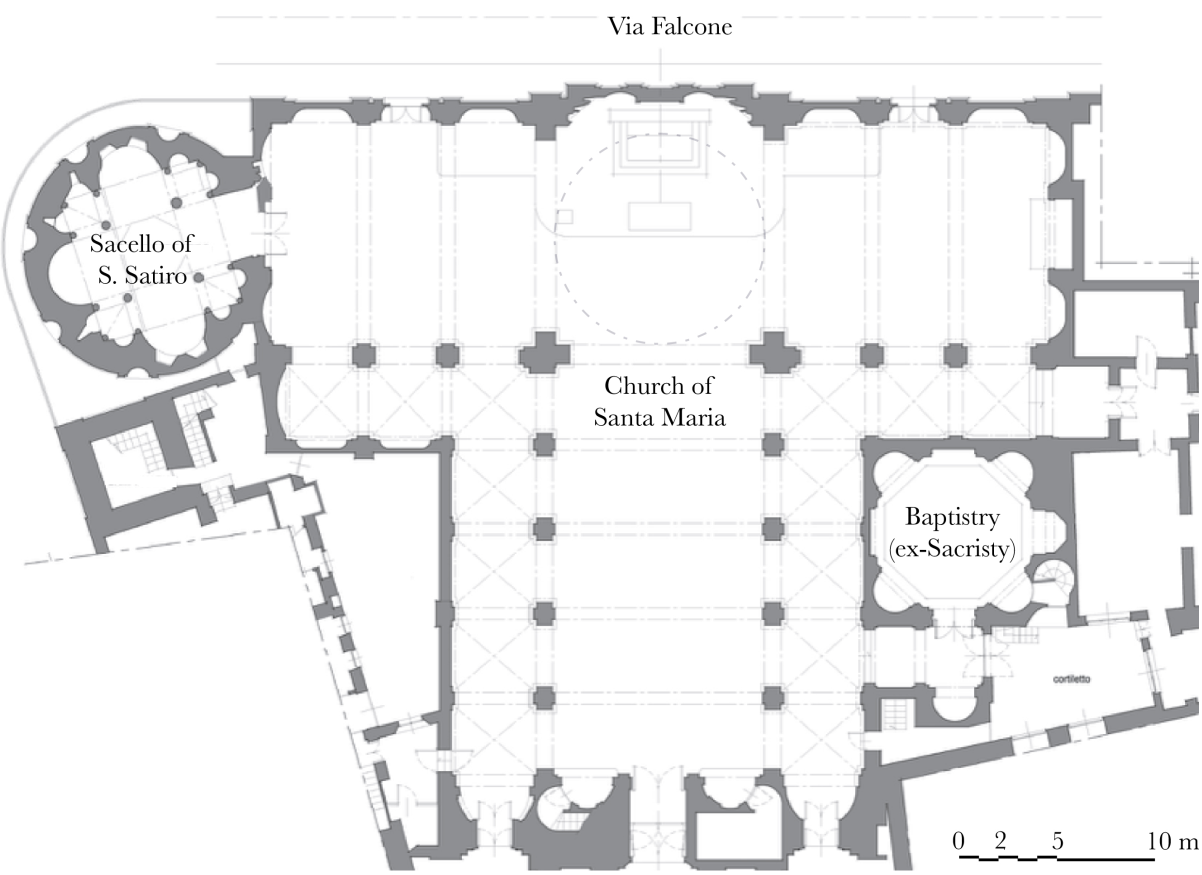
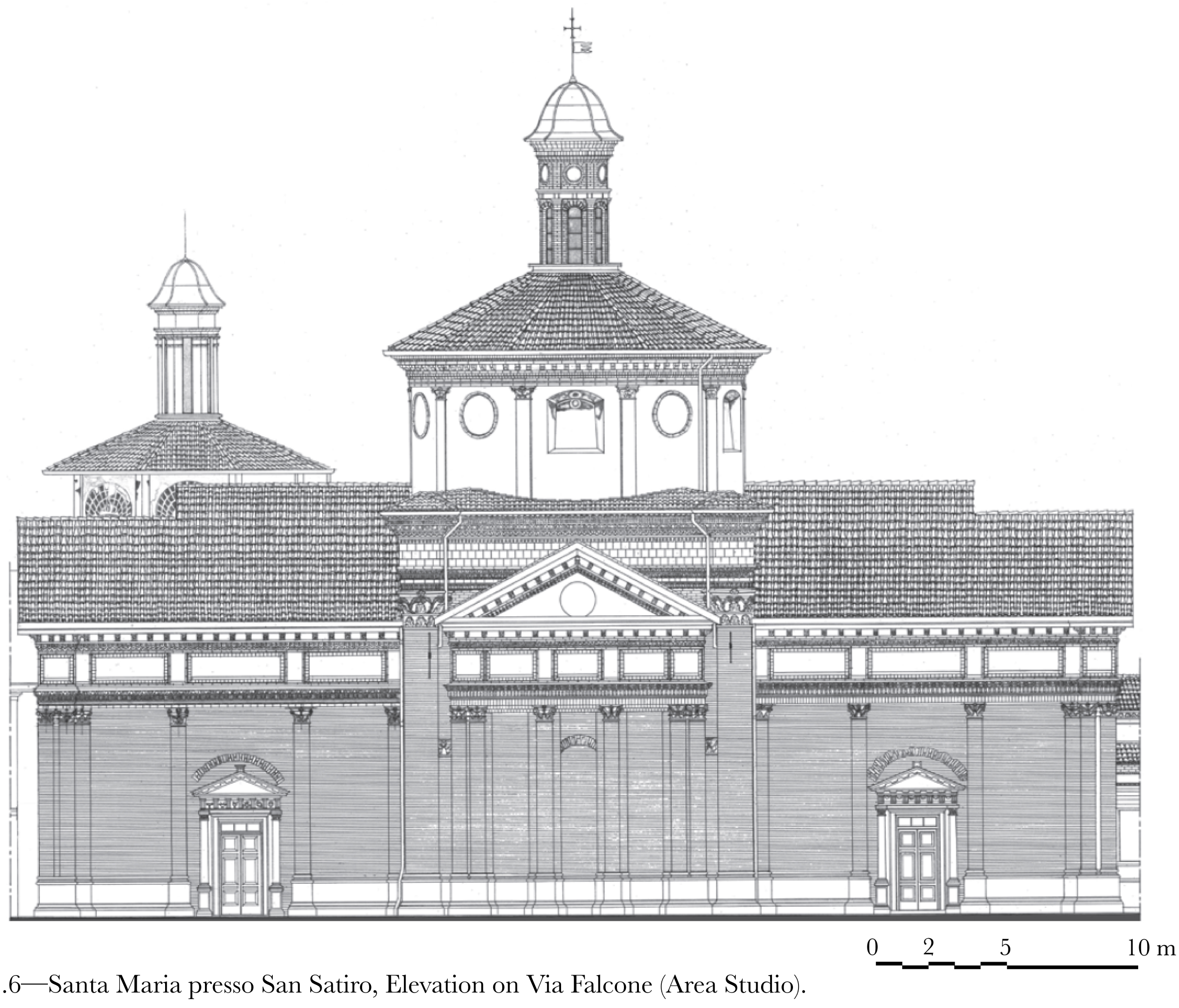
[674] Similarly, the eight-spoked oculus on the right is not centred on the arch below, but shifted as to be seen in its entirety.
[675] Richard Schofield, "Bramante e Santa Maria presso San Satiro," 21-22.
[676] Repishti, "Bramante in Lombardia: regesto delle fonti," 198.
[677] Single or double phase.
[678] ibid., 22-27.



5.5—Santa Maria presso San Satiro, plan (Area Studio).



5.6—Santa Maria presso San Satiro, Elevation on Via Falcone (Area Studio).

shallow depth of the wall. In addition, the same phase involved the construction of the main and minor naves, and the church's sacristy.[679] A coffered hemispherical dome on a low drum (the first of its kind in Lombard Renaissance architecture) was placed above the crossing of the transept and the main nave.[680]

Firstly, both the articulation of the main spaces of the church and the formal choices made by Bramante at this scale should be considered. In the definition of the formal and structural properties of the naves and the transept of the church, Bramante employed elements that have been linked to a number of precedents. The barrel vaults of the central nave and transepts and the coffered hemispherical dome at the crossing reference the Sant'Andrea by Alberti in Mantua. Bramante could have seen its model and perhaps the initial phases of its construction on site in the 1470s. The barrel vaults and the pilasters without bases have been linked to the near-contemporary Duomo of Urbino by Francesco di Giorgio, with the latter solution also found in Alberti's crypt for San Sebastiano in Mantua.[681] Other elements, such as the minor naves continuing in the transept and the folded pilasters of the sacristy have been linked (although not conclusively) to Brunelleschi's architecture in Florence, either known directly or more probably mediated through Leonardo's drawings.[682] Lastly, the architecture of Piero della Francesca's *Brera Altarpiece* is noted in the shell-shaped niches in the illusionistic choir and sacristy and in similar capitals in the choir. Nonetheless, although it is clear that some of Bramante's formal solutions are linked to his experience of works by other architects, it should be clarified as to why these elements were chosen over others, and how

[679] ibid., 27.

[680] As noted by Schofield, in the dome of Santa Maria presso San Satiro Bramante used the first ribless hemispherical cupola with coffers in Lombard architecture, thus departing from all local, Quattrocento precedents. Richard Schofield, "Bramante and Amadeo at Santa Maria delle Grazie in Milan," 46.

[681] Richard Schofield, "Bramante milanese: collisioni di culture architettoniche?," 10.

[682] The plan of Santo Spirito and Santa Maria degli Angeli are found among Leonardo's drawings in Codex B, 11v, datable in the late 1480s when Leonardo was working in the environment of the Sforza court of Milan. Codex B (1486-1488 c.). Paris, Institut de France.

Bramante employed them. When these spaces are considered in more detail, it becomes clear that Bramante arranged the formal properties of some elements of the building such as the orders, to visually correspond to a spatial hierarchy defined in three-dimensions. Bramante does this by adjusting the proportions of the columns and capitals of the main nave, transepts and minor naves to reflect the dimensions of the structure to which they are applied.[683] In other words, Bramante established in this way a visual hierarchy in the design and reading of the main spaces of the church.

A similar principle is confirmed by an analysis of the arrangement and proportioning of elements in the church's elevation on Via Falcone (illus. 5.6). As noted by Schofield and Frommel, here (for the first time in a Renaissance church in Lombardy), three orders of pilasters are employed to reflect the position of the dome, the choir, and the transept's nave, thus corresponding to the hierarchy of spaces found inside the building.[684] As mentioned above, Bramante described the Duomo of Milan in 1487 by dividing it into four bodies, each considered as uniform in its structural behaviour. Hence, it was argued that Bramante considered the Duomo by means of bodies defined by structural, formal and ultimately visual properties. Bramante followed a similar principle in Santa Maria presso san Satiro by privileging the reading of the main structural and formal elements that compose the building over the proportioning of others (such as the orders or the decorative models applied to the

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[683] As noted by Schofield and Sironi, at Santa Maria presso San Satiro the proportions of pilasters and their capitals are changed according to the requirements of the structure to which they are applied. The internal minor orders have very low pilasters with wide capitals and low capitals; beside them, the pilasters of the major order have slender proportions and corresponding pilasters. In addition, the types of capitals employed do not identify an order but, in line with the Lombard tradition, different capitals are employed in the same order. The same approach is also found later in others of Bramante's works in Milan such as the Canonica of Sant'Ambrogio. Richard Schofield and Graziano Sironi, "Bramante and the problem of Santa Maria presso San Satiro," *Annali d'Architettura*, n. 12 (2000): 34.

[684] ibid., 34-35; Christoph L. Frommel, "Lombardia," in *Bramante Milanese e l'Architettura del Rinascimento Lombardo,* ed. Christoph L. Frommel, Luisa Giordano, Richard Schofield (Venice: Marsilio, 2002).Ultimately, if the design process employed by Bramante in the definition of the formal properties of Santa Maria presso San Satiro is considered, similarly to what happens in the *Opinio,* the whole church can be read as an arranged of three-dimensional bodies that define the proportioning of minor elements such as the orders, and not the other way round.

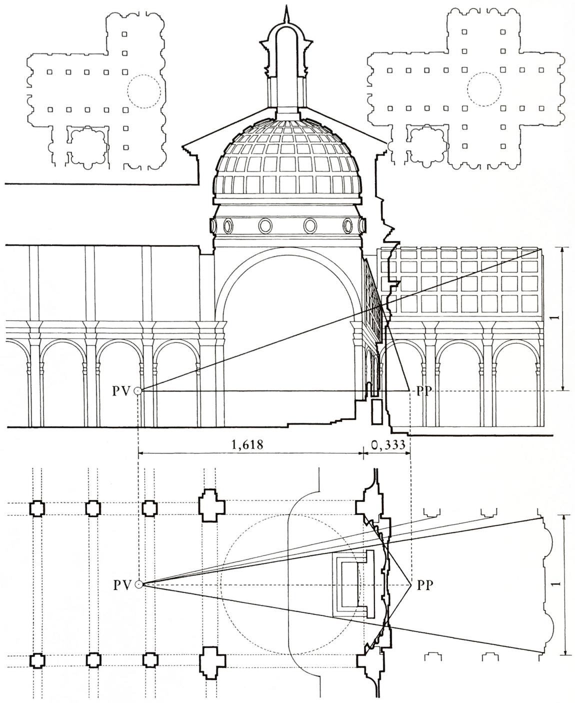
walls of the church). In other words, this is the first instance where Bramante is shown to privilege the arrangement and proportioning of the large-scale elements (such as the church's naves, transept, and dome) and define others elements applied to their surfaces accordingly.

The illusionistic perspectival choir devised by Bramante is closely correlated with this hierarchical reading of the church's architectural elements (illus. 5.7-5.8). Scholars have noted that before Bramante, no one had approached the resolution of an articulated spatial organism such as a whole church in the same visual and spatial way.[685] In the illusionistic choir, Bramante represented three bays of the main and adjoining two minor naves. To perfect the visual experience of depth, he made use of the whole depth allowed by the wall (about 120 centimetres), and employed the same materials and decorative models of other parts of the church. Since the elements are built in relief, shadows are naturally produced by the light falling from the dome. By means of its location, the choir works as a focal point for the church and is visible from the whole of the main nave and most of the transept.[686] However, the (illusionistic) visual experience of the three bays of the choir as a continuation of the three bays of the main nave after the crossing was designed to work when the observer would be positioned in the main nave. The choice of a barrel vault to cover the main nave is thus fundamental to avoid any obstruction to its view. In addition, the perspectival illusion is undoubtedly aided by the reproduction in the choir of three bays of the main nave where the observer would be located. In other words, the visual experience of depth in the choir is aided by constraints which are the same involved in the visual experience of the nave in which the

[685] The illusionistic ephemeral experiments of the Baroque period, Borromini's perspective in Palazzo Spada for example, are often mentioned in this context. However, the nature and role of the perspectival gallery of Palazzo Spada is very different from Bramante's. The gallery is visible from an internal court-garden of the Palazzo and does not contribute in the same way to the definition of an organic internal space. Filippo Camerota, "Bramante 'prospettivo'," 25.

[686] As noted by Filippo Camerota, the precursory perspectival construction of the *Trinità* of Masaccio in the church of Santa Maria Novella comes to mind. However, Masaccio's construction did not intervene in the spatial arrangement of the whole church but only on a circumscribed area. Moreover, Masaccio's construction remained a fresco applied to the wall of the church. ibid., 25-26.

5.7—Santa Maria presso San Satiro, view of the altar and illusionistic choir from the main, central nave.



5.8—Santa Maria presso San Satiro, plan and section-elevation with reconstruction of the perspectival (illusionistic) choir from the designated point of view (PV) (Camerota). The visual perception of the choir contributes to the mental spatial experience of the whole church that shifts from the real 'T shaped' conformation in plan on the top left-hand side to the more centralised scheme on the right-hand side.

observer stands.[687] Nonetheless, when the visual angle becomes appreciably narrower than the optimal angle established by Bramante (illus. 5.8), the visual experience of the depth of the choir is inevitably compromised. However, since the visual experience of the choir as an additional space in the church (extended in depth) works from the main nave where most of its viewers would usually be situated, it can be argued that this contributes to the mental spatial experience of the whole church. As such, it acts as a spatial counterweight to the main nave and establishes the centrality of the dome above the crossing.

[687] In relation to this, the use of three bays and particularly the presence of coffers in the ceiling of the main nave and choir undoubtedly contribute (similarly to the grid or tiled pavement described by Alberti) as a constraint and thus aids the three-dimensional visual experience of the space of the choir.

Although this may be among the most celebrated spatial applications of perspective to architecture, the process by which the choir was designed by Bramante and then realised on site by builders is rarely considered. Crucially, the conception of the choir required knowledge of the exact geometry of all its elements in three-dimensions before their projection and proportional measurements on the wall (and their effective communication to the builders) could take place.[688] In other words, the space of the choir had to be mentally pre-formed, that is, visualised and quantified by Bramante in its proper (distal) form, before it could be successfully designed and materially applied to the wall to create the desired perspectival visual experience.[689]

In addition, although Bramante would have dealt with other functional requirements in the design of the sacristy, his approach to visual and formal properties can nevertheless be correlated with these solutions. Scholars have usually pointed to Sant'Aquilino at San Lorenzo in Milan as Bramante's main source for the sacristy, a late antique chapel which he probably considered as Roman in origin.[690] Both spaces alternate round and rectangular niches around an octagonal nucleus. However, although both share a similar articulation of the ground level and the superimposition of a gallery and a dome on a low drum, the articulation and dimensions of the elements, and the ratios between the height and width of the two spaces are markedly different.[691] While the Chapel of Sant'Aquilino has a ratio between the height of the wall and the diameter of the octagon of about 1:1; it is about 1:1.6

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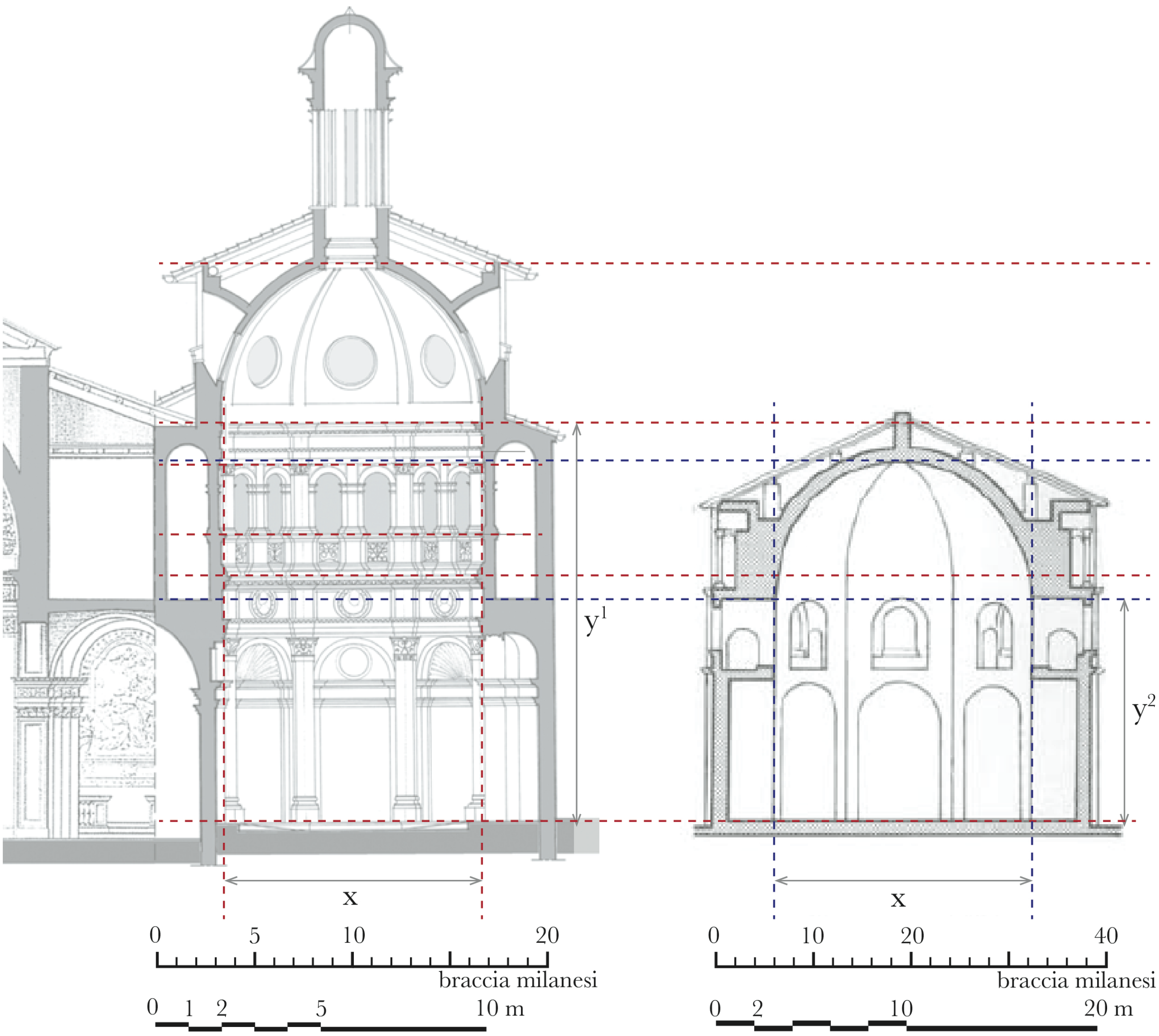
[688] As measured by Camerota, the oblique walls of the perspective define a triangle isosceles with a depth of one/third the width of the choir and the same ratio is established in the right triangle that has as hypotenuse the central axis of the vault in its proper form and minor cathetus the eight of the apex of the vault above the point of view established by the perspectival construction. ibid., 28.

[689] Moreover, as noted, its spatial extension was reduced but not completely eliminated since the choir retained a shallow depth.

[690] Richard Schofield, Cristiano Tessari, "Aspects of Church Design from Brunelleschi and Alberti to Leonardo and Bramante," 314. Richard Schofield, "Bramante e Santa Maria presso San Satiro," 51. Federico Bellini, "Bramante milanese e il tema dell'organismo cupolato," *Arte Lombarda,* no. 176/177, vol. 1/2 (2016): 127.

[691] As noted in the previous chapter, again the analysis will not primarily concern the quantitative values employed, but rather *how* spatially these quantities were visualised and applied to architecture. In other words, it will consider the process by which quantities are employed in architecture by Bramante, and the implications that this process has for the formal and visual properties of the sacristy's space.

in the Sacristy of Santa Maria presso San Satiro (illus. 5.9). This ratio does not correspond to the width/height ratios for temples given in the treatise of Alberti; it is, nonetheless, close to the ratio established by Francesco di Giorgio for the position of the highest cornice in the interior wall. Nonetheless, if compared with the proportions given by Francesco di Giorgio, the position of the lower cornice, chromatically highlighted in the sacristy, is at a higher point (at a height with an approximate ratio of 8/13.5 to the whole height of the wall) than the 5/9 or 6/11 indicated by Francesco Di Giorgio (table 5.A). In comparison, Bramante thus altered the proportions of the first and second order of the sacristy by raising the lower cornice.



5.9—Comparison of the section-elevation of the Sacristy of Santa Maria presso San Satiro (left-hand side) and Chapel of Sant'Aquilino at San Lorenzo (right-hand side).

	wall height/ diameter ratio of central nucleus	first cornice or *recinto* height/ total wall height ratio	second order or gallery height/ total wall height ratio
Alberti, *De re aedificatoria*, Book VII (1)	≥1/2 (0.50)	-	-
Alberti, *De re aedificatoria*, Book VII (2)	2/3 (0.67)	-	-
Alberti, *De re aedificatoria*, Book VII (3)	3/4 (0.75)	-	-
Alberti, *De re aedificatoria*, Book VII (4)	11/14* (0.79)	-	-
Alberti, *De re aedificatoria*, Book VII (5)	1**	-	-
Francesco di Giorgio Martini, *Trattati* (1)	5/3 (1.67)	4/9 or 5/11 (0.44 or 0.45)	5/9 or 6/11 (0.56 or 0.55)
Francesco di Giorgio Martini, *Trattati* (2)	18/11 (1.64)	4/9 or 5/11 (0.44 or 0.45)	5/9 or 6/11 (0.56 or 0.55)
Bramante, *Sacristy of Santa Maria presso San Satiro*	~8/5 (1.60)	~16/27 (0.59)	~11/27 (0.41)

5.A—Comparison of the height/width and total wall height/intermediate height ratios of temples given in Alberti's *De re aedificatoria* (Book VII), Francesco di Giorgio Martini's *Trattati* (*Quarto Trattato*) and Bramante's Sacristy of Santa Maria presso San Satiro. All measurements of the sacristy restituted from drawings. Fractions have been rounded to two decimal places. * I employed the value proposed by Zubov, "Quelques aspects de la théorie des proportions esthétiques de L.-B. Alberti," 56. ** If chapels are arranged around the central nucleus. Alberti, *De re aedificatoria*, VII, 10, 607.

Consequently, the ratio between the width/height of the shaft of the half-pilasters in the gallery is about 1:3.6, and between the width of the pilasters and height of the entablature is about 1:4.23.[692] It should also be noted that the whole space is notably slenderer than the near-contemporary octagonal Sacristy of Santo Spirito in Florence by Giuliano da Sangallo, which has a much larger diameter and a width/height proportion close to 1 (close therefore to ratio found in the Chapel of Sant'Aquilino). Moreover, the sacristy of Bramante has a considerably narrower diameter than the Sacristy of Santo Spirito, the Chapel of Sant'Aquilino or the examples given (and measured in *braccia*) in Francesco di Giorgio's *Trattati*.

692 These ratios had also been noted by Frommel. Frommel, "Lombardia," 11. Frommel added that although the tripartition of the order is classical, the ratios use by Bramante are certainly not.

5.10—Baptistry (ex-Sacristy) of Santa Maria presso San Satiro, Milan. View of the interior.

The width/height ratio of the sacristy should be considered thus with the narrower visual angles determined by its limited diameter. In this context, if compared with the proportions of other buildings, the reduction in height of the gallery of the sacristy determines in this case a narrower visual angle when perceived from the ground level. In addition, the gallery is further subdivided by the height of the *bifore*, which is about half of the whole second order. At the same time, all these elements and subdivisions provide a series of constraints to determine the possible or probable three-dimensional (distal) shape of this level of the sacristy. Horizontally, the continuity of elements is visually enhanced by the balustrade that matches the height of the pilasters and the highly protruding cornices (illus. 5.10). Vertically, single folded pilasters are placed at each corner of the octagon tying the sacristy's first two levels and the dome as single three-dimensionally extended shapes, whilst also visually connecting the whole space by continuing from the ground to the oculus of the lantern at the top of the dome.

Since Brunelleschi's projects are often mentioned in relation to the folded pilasters in the sacristy, it is worth considering them carefully. Brunelleschi notably folded the pilasters at the interior corners of the Old Sacristy at San Lorenzo. However, there is no continuation of the pilasters beyond the first order. Only in Brunelleschi's last project, the lantern of the Duomo of Florence, are the fluted pilasters folded and shown as continuing. They are, however, still interrupted by the entablature. In other buildings with octagonal plans, such as Brunelleschi's late project of Santa Maria degli Angeli (and in the late fifteenth century Giuliano da Sangallo's Sacristy of Santo Spirito), the pilasters are always doubled at every corner of the octagon.[693] Instead, Bramante designed the folded pilaster of the sacristy to be visually continuous from the ground to the level of the oculus at the top of the octagonal dome, interrupting the ground level frieze and protruding in the central space. The dome stands on a short drum (with a height of approximately one *braccio milanese*) which is not visible from the ground, given the projection of the cornice underneath it. In addition, the continuation and flattening of the pilasters at the level of the dome conceal the corners of the octagon and make the shape of the dome to appear closer to a semi-sphere. In this way, it is more likely to be visually experienced as hemispherical. Overall, Bramante clearly privileges a logic of hierarchy and unity in the sacristy, beyond what have sometimes been recognised as typical fifteenth-century Lombard decorative elements. Although the arrangement of niches around the octagonal plan and the tripartite orders of the ground and the first level can be linked to antique and late-antique precedents, what is novel is the way in which Bramante visually and

[693] Every side of the octagon can thus be read as a separate surface.

formally achieves the hierarchical articulation and unity of the whole space.[694] The reading of the whole architecture of the sacristy becomes clearer once the hierarchy in which its elements have been organised is considered. As in the case of the church's naves, the proportioning and articulation of the architectural elements of the sacristy point to how the orders and ornamental elements are visually related to the formal and spatial organisation of the whole.

5.1.3 The choir of Santa Maria delle Grazie

Bramante's design approach in Santa Maria presso San Satiro is useful when considering his contributions in the later project of the *tiburio* and choir at Santa Maria delle Grazie. The tribuna of Santa Maria delle Grazie (composed of a *tiburio* with an internal dome standing on a square base, giant side niches and a choir with a terminal semi-circular apse) were meant to provide a new end to the church built between the 1460s and the late 1480s. The foundation stone for this part of the building was laid in March 1492, around a year after Ludovico Sforza received, according to a late sixteenth-century writer, a series of "pareri de peritissimi architetti".[695] Ludovico intended the space under the dome to be used by the Dominican friars for worship and also as a chapel, with the tombs of two of his closest collaborators under the two large niches, one on either side. The choir was to be the sepulchre for Ludovico himself and his family. By 1497, the interior of the *tiburio* and what was intended to be a sepulchral chapel were completed and the altar of the church was located at the entrance of the choir.[696]

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[694] This was also noted by Frommel. Christoph L. Frommel, "Il complesso di Santa Maria presso San Satiro e l'ordine architettonico del Bramante Lombardo, " in *La scultura decorativa del Primo Rinascimento: atti del I Convegno Internazionale di studi, Pavia 16 - 18 settembre 1980* (Rome, 1983), 156.

[695] As noted by Richard Schofield, according to Gattico, Ludovico Sforza wanted "a poco a poco demolir tutte le fabriche fatte dal conte (Gaspare Vimercate)" and began with the *capella grande* (i.e., the crossing) of the pre-existing church. Schofield, "Bramante and Amadeo at Santa Maria delle Grazie in Milan," 42.

[696] ibid.

Very few documents related to this initial phase of the work survive and none provide information on Bramante's involvement. However, the procedures followed in other similar contexts show that Bramante's suggestions could well have been incorporated without him necessarily being involved in all the phases of the project.[697] Overall, there is scholarly consensus that the scale, articulation and type of architecture employed in the interior of the building, particularly in the choir, can most probably be linked to Bramante.[698] The analysis in this section will concentrate mainly on the spatial, visual, and formal properties of the choir.

Although a range of precedents were available to Ludovico and the architects involved in the project of the tribuna, none seem to be predominant.[699] Among these, Brunelleschi's Old Sacristy at San Lorenzo in Florence, and its Milanese version—the Portinari Chapel in Sant'Eustorgio—have often been mentioned as indirect or direct sources for the spatial articulation under the dome and of the choir. In addition, the geometry of the *tiburio* itself closely resembles that of Old San Lorenzo in Milan and this was probably a major source for the architects of Delle Grazie.[700] Nonetheless, none of these sources completely explain the form given by Bramante to the choir. References to the trilobate plans of the Certosa of Pavia and San Bernardino in Urbino (a project of Francesco di Giorgio in which a contribution by Bramante is possible) are appropriate,[701] but the configuration of the Santa Maria delle

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[697] ibid. Indeed, as noted by Schofield, Bramante's role at the Grazie could certainly have been that of an authoritative expert not involved in the practicalities of the building site.

[698] The Bramantesque character of the interior articulation of the tribuna delle grazie has been highlighted by Richard Schofield and Arnaldo Bruschi among others. Schofield, "Bramante and Amadeo at Santa Maria delle Grazie in Milan," 54 and Arnaldo Bruschi, "L'Architettura," in *Santa Maria delle Grazie in Milano* (Milan: Banca Popolare di Milano, 1983), 71. More recently, it was argued in Richard Schofield, "Bramante dopo Malaguzzi Valeri," 48.

[699] Schofield, "Bramante and Amadeo at Santa Maria delle Grazie in Milan," 42

[700] ibid., 46. The correspondence of the number of sides and geometry of the external *tiburio*, as the internal dome without ribs certainly point to San Lorenzo.

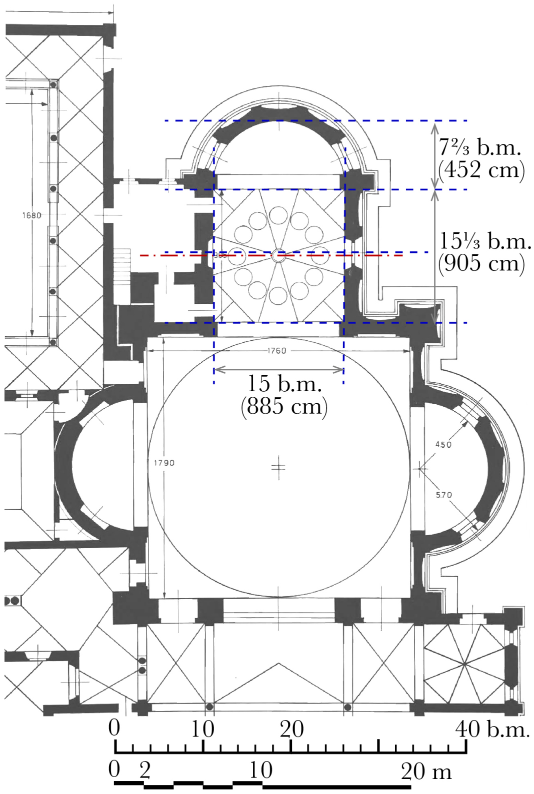
[701] The sepulchre of the Visconti (ruler of the Duchy before the Sforzas) in the arms of the transept of the Certosa of Pavia has a trilobe plan. Schofield, "Bramante and Amadeo at Santa Maria delle Grazie in Milan," 43.

Grazie differs from both precedents in the spatial articulation of the choir. This shows a level of formal, spatial, and visual articulation that cannot be found in any of the precedents mentioned above.

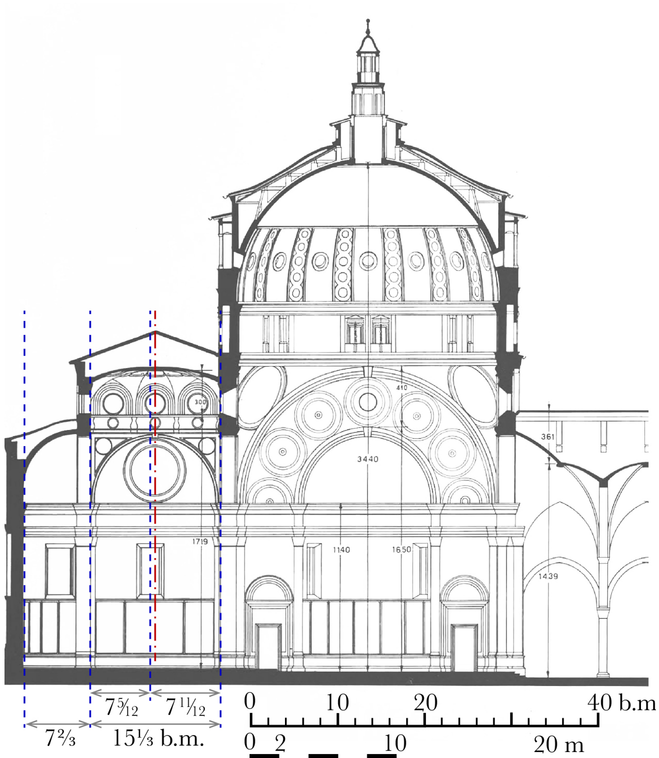
Overall, three main elements of Bramante's design approach, highlighted in different areas of Santa Maria presso San Satiro, can be found in the choir of Santa Maria delle Grazie. The first is the application of the principles of perspective to architectural elements in the design process to alter the three-dimensional visual experience of the choir. The second is the continuity of architectural elements that define the unity of the whole space. The last—as noted by Pacioli in Chapter 54 of the *Compendium de divina proportione*—is the use of parts of a geometrical body (that Pacioli identifies as a 72-bases body) in the apse that concludes the choir, as well as in the two niches flanking the space under the *tiburio*'s dome.

The choir sits at the far end of the church and as noted, was intended as a private sepulchre for Ludovico Sforza and his family. At the same time, the main altar of the church was located under the arch at the entrance of the choir on its completion. The choir's functions were therefore conceived as being restricted to a very small group, and thus the space was mainly to be visually perceived from the nave and the space under the dome. When its formal properties are considered, the geometry of the choir's plan, excluding the final apse, is often described as a square. However, an analysis of Pica's survey shows that in reality, the space is about 20 centimetres longer (about a third of a *braccio milanese*) than it is wide (905 centimetres long and 885 wide) (illus. 5.11-5.12).[702] In addition, when the side walls are considered, it can be noticed that the axes of the two tall rectangular windows (one on each side) are not located at the centre of its length, but instead are shifted about 30 centimetres (about half a *braccio)*

[702] A Milanese *braccio* is about 59 centimetres long. Agnoldomenico Pica, "Il gruppo monumentale di Santa Maria delle Grazie in Milano," in *I monumenti Italiani*, ed. Reale Accademia d'Italia (Rome: La libreria dello stato, 1937).



5.11—Santa Maria delle Grazie, Milan 1492-1497 c.. Plan of the choir-sepulchral chapel (Pica) with main dimensions in cm and *braccia milanesi*. Central axis of choir depth (905 cm) highlighted in red. All measurements restituted from drawings.



5.12—Santa Maria delle Grazie, Milan. Section-elevation of the choir-sepulchral chapel (Pica) with main dimensions in *braccia milanesi*. Central axis of choir depth (905 cm) highlighted in red. All measurements restituted from drawings.

5.13— Santa Maria delle Grazie, Milan. View of the choir-sepulchral chapel with highlighted the different three-dimensional spatial sections (A, B) determined by the position of the choir's windows and final apse. Central axis of the choir's depth (905 cm) highlighted in red. Hypothetical crossing vault dotted in black.

5.14—Santa Maria delle Grazie, Milan. View of the choir-sepulchral chapel from the central nave.

away from the nave in the depth of the choir. This shift, and the narrow angle of view from which the side walls are visually perceived from the naves and the space under the dome, mean that the difference in length between the two sections of the central space of the choir separated by the two windows cannot be easily measured by sight (illus. 5.13). Consequently, if the two sections (before and after the windows) are interpreted as equal, the space will be mentally visualised as being deeper. This was a deliberate choice by Bramante, as proven by the fact that the windows on both sides are shifted by the same amount and because there is no other plausible reason for their location in this position rather than in the centre of the walls.[703] Moreover, the pilasters at the choir's entrance have been given a wider profile (approximately double) on the depth of the choir than on the dome side, in addition to being rebated only on the side of the choir.

If all these details are taken together, Bramante could have approached the design of the choir in this way only if its different spatial sections were considered as three-dimensionally extended and, therefore, composite of form (boundary) and matter (spatial extension), as noted by Pacioli. At the same time, Bramante highlighted continuous elements with a contrasting colour to their surroundings (as in the case of the entablature). As in the Sacristy of Santa Maria presso san Satiro, continuous elements call attention to perspectival (proximal) converging lines perceived from the main points of observation. Accordingly, a series of panels runs along the three sides of the choir at the lower level. Likewise, the location of the friars' choir stalls on the three sides of the choir would have contributed to the same effect since most probably they were included in this arrangement from the beginning.

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[703] The shift of the axes of the window from the centre of the central space of the choir is also reflected in the unequal width of elements in the external elevation. The hypothesis that they could have been place off-centre to privilege the external elevation is thus not plausible. In addition, the same difference (300 mm) is found between the width and the lenght of the square space at the base of the tiburio.

Lastly, a semi-dome sits at the highest point of the final apse and above the entablature of the main order. Pacioli associated this with a quarter of the 72-bases body discussed in Book XII of Campanus' edition of Euclid's *Elements*. Since its geometry is partially covered by the arch in front, only about 20 of the body's bases are fully visible at one time from the church's nave. Nonetheless, for the arrangement of their surfaces, it can easily be visually interpreted, as Pacioli did in the *Compendium*, that this and the two niches flanking the space under the *tiburio* distally refer to a semi-dome's shape. Experiments by psychologists involving three-dimensional solid shapes as polyhedra have shown that the visual representation of a distal scene can more reliably achieved when the contours of the surfaces are planar and other possible topological information (i.e. symmetry of the object, planarity of contours, minimum variance of angles) constrain the relationships among them.[704] It should be noted that the choir's semi-dome works as the final visual element in the long axis of the church (illus. 5.14). The coffers used in the dome of Santa Maria presso San Satiro for the first time in Lombard Renaissance architecture, are here perceived from a higher distance and have thus been geometrically simplified in their decorative motifs. At the same time, their geometrical regularity, simplified arrangement, and chiaroscuro effects achieved by relief work, all contribute to the interpretation of the surfaces as distally concurring to a three-dimensional body.

This section started with the possible sources usually indicated for the architecture of Santa Maria delle Grazie, and particularly the Old Sacristy by Brunelleschi at San Lorenzo in Florence. However, as is often noted, Brunelleschi's building plan acted as the first level of

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[704] The literature in this area of research is very extended. In this instance, I refer to the summary and result of experiments on 3D shape reconstruction from a single 2D retinal image given in Zygmund Pizlo, *3D Shape: Its Unique Place in Visual Perception* (Cambridge, MA: The MIT Press, 2008), 115-144, 143-144. However, these findings are coherent with the series of 'constraints' or 'rules' highlighted by Zenon Phylyshyn and Donald D. Hoffman. Additionally, the experiments of Pizlo, Chan et all., provided additional evidence that binocular and monocular mechanisms of shape perception are "similar if not identical", and both "depend critically on shape constraints". ibid., 144.

definition for the proportional system comprised of elevated linear elements such as columns, pilasters and round arches, that were then arranged in three-dimensional units.[705] Modular units based on this system could then be combined and connected at different scales in churches, as demonstrated by an analysis of his buildings.[706] The formal properties of the Church and Old Sacristy of San Lorenzo have been coherently correlated to this system.[707] The arrangement of elements in the choir of Santa Maria delle Grazie, however, shows a coordinated geometrical and visual control of spatial properties that is not found in Brunelleschi's buildings. This is more easily explained if the space is understood in terms of three-dimensional geometrical bodies, composite of form (shape) and spatial extensions, designed as to be visually perceived from set positions. Given these aspects of Bramante's design approach, it should now be investigated as to whether any correlations can be established with the drawings of sacred architecture drafted by Leonardo da Vinci.

5.2 The definition of the formal properties of central-plan churches in the drawings of Leonardo da Vinci

Despite a large group of architectural drawings (particularly of central-plan churches) left in his manuscripts during his first Milanese period (c. 1483-1499),[708] Leonardo was not responsible for the design of any building, nor can his drawings be considered blueprints for

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[705] As noted already by Howard Saalman, once the size of the main crossings and modular subdivision in plan had been decided and the proportions to build the elevations selected from a small repertory, the builders could then proceed with oral instruction rather than detailed drawings. Saalman concluded that considering the limited schooling available to artisan and workers at the time, the advantages of this system were evident. Howard Saalman, "Early Renaissance Architectural Theory and Practice in Antonio Filarete's Trattato di Architettura," *The Art Bulletin*, vol. 41, no. 1 (March 1959): 97. Importantly, the process is also recorded by Manetti, Brunelleschi's biographer, that recalls as Brunelleschi gave showed only the plan to the deputati of Santo Spirito, agreeing to make a model after the approval of the project. ibid., 94.

[706] Arnaldo Bruschi, "Brunelleschi," in *Storia dell'Architettura Italiana: Il Quattrocento*, ed. Francesco P. Fiore (Milan: Electa, 1998), 52.

[707] ibid., 51-55.

[708] Leonardo is first documented in Milan in a contract for a painting of the *Virgin of the Rock*, dated April 1483. However, he could have well been active in Milan at least from the year before. As noted, he remained until the fall of Ludovico Sforza in September 1499. As noted, it was towards the end of this period (1496-1498) that he drafted the perspectival representations of regular and other bodies derived from these for Pacioli's *Compendium*. Richard Schofield, "Leonardo's Milanese Architecture: Career, Sources and Graphic Techniques," *Achademia Leonardi Vinci*, vol. IV (1991): 113.

anything that was supposed to be built.[709] With the exception of his design proposal for the *tiburio* of the Duomo of Milan (of which we know Leonardo presented two models),[710] the drawings of central-plan churches found in his notebooks go no further than what can be considered as the initial stages of planning.[711] Detailed drawings are largely absent and most of the proposals are variations on the same theme: a larger central nucleus (in most cases octagonal) surrounded by a series of secondary spaces. Leonardo's variations on this theme should also be related to the particular formal and visual properties of this typology of buildings. These buildings were usually visually isolated from their surrounding contexts. This was undoubtedly important for Leonardo; stating in a note placed under a perspectival bird's-eye view of a central-plan church that "a building should always be detached on all sides so that its true form may be seen".[712] Given the correlation of this subject with the works of Pacioli, the writings of fifteenth-century architectural theorists discussed in Chapter 4, and Bramante's designs for churches in Milan and later in Rome, this section will now concentrate only on the sacred architecture and central-plan buildings among the corpus of Leonardo's drawings.

Meetings between Leonardo and Bramante are recorded in documents written about the *tiburio* of the Duomo of Milan (1487-1490), while discussions between them and Francesco di Giorgio are documented in relation to Bramante's project for the Duomo of Pavia (1487-1490).[713] In addition, Bramante's name is found a few times in Leonardo's notes (c. 1490-1499) and, as discussed, parts of buildings associated with Bramante's designs are correlated in Pacioli's *Compendium* (1498) with a 72-bases body, of which Leonardo had

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709 Bellini, "Bramante milanese e il tema dell'organismo cupolato," 130.
710 He was paid for models for the *tiburio* of Milan between 1487 and 1490. Schofield, "Leonardo's Milanese Architecture: Career, Sources and Graphic Techniques," 118.
711 ibid., 132, 157.
712 "senp(r)e uno edifitio vole essere ispichato dintorno a volere dimostrare la sua vera forma" Codex B, 39v.
713 Richard Schofield, Cristiano Tessari, "Aspects of Church Design from Brunelleschi and Alberti to Leonardo and Bramante," 320.

provided perspectival representations.[714] It can be argued, following Ludwig Heydenreich, that the architectural studies of Leonardo can be best evaluated if considered and compared with the architectural themes pursued by his contemporaries.[715]

At the same time, Leonardo's drawings of central-plan buildings are mainly included in Codex B (and the related Codex Ashburnham) and have been dated c. 1486-1488.[716] Other drawings of central-plan buildings are found in the near-contemporary drawings in Codex Trivulzianus (c. 1488-1490) and in later drawings in Codex Atlanticus and Codex Forster I.[717] Writings to accompany the drawings however are notoriously rare and concise. As is well-known, drawings and short notes about architectural subjects are usually placed among miscellaneous items and mixed with drawings and written remarks that deal with the illustration of mechanical constructions and studies of light, among other subjects. However, relationships among drawings of architecture placed in successive sheets have also been noted. These were clearly intended to correlate elements in different drawings and, perhaps, to discuss these correlations with other artists and architects.

[714] Leonardo mentions Bramante in at least three occasions in the surviving corpus of his notes, notably once referring to some unspecified buildings of Bramante "Edifiti di Bramante" in a list of notes related to the fall of Ludovico Sforza. Codex L, on the verso of the cover, datable c. 1499. Another time Bramante is referred to with the familiar name 'Donnino'. 'manner of the drawbridge that Donnino showed me', 'modo del ponte levatoio che mi mostrò Donnino' in the original. Notably 'Donnino' is the same name that Bramante's parents used in their will. Leonardo da Vinci, Codex M, folio 53v, datable c. 1499. The third instance refers to the "gruppi del Bramante" that is the knots, the drawings of twines as conceived by Leonardo for the Sala delle Asse in Castello Sforzesco or for the emblem of the *Achademia Leonardi Vinci*, Codex Atlanticus, folio 225r-b, datable c. 1490. Donato Bramante, *Sonetti e Altri scritti*, 10-12.

[715] Ludwig H. Heydenreich, *Architecture in Italy 1400-1500* (London: Yale University Press, 1996), 151.

[716] Codex B is part of the manuscripts preserved at the Bibliothèque de l'Institut de France in Paris. The Codex Ashburnham 2148 should also consider part of the Codex B, its pages were cut from the Codex in the nineteenth century and later reunited. Carlo Vecce, *La biblioteca perduta: I libri di Leonardo* (Rome: Salerno Editrice, 2017), 196.

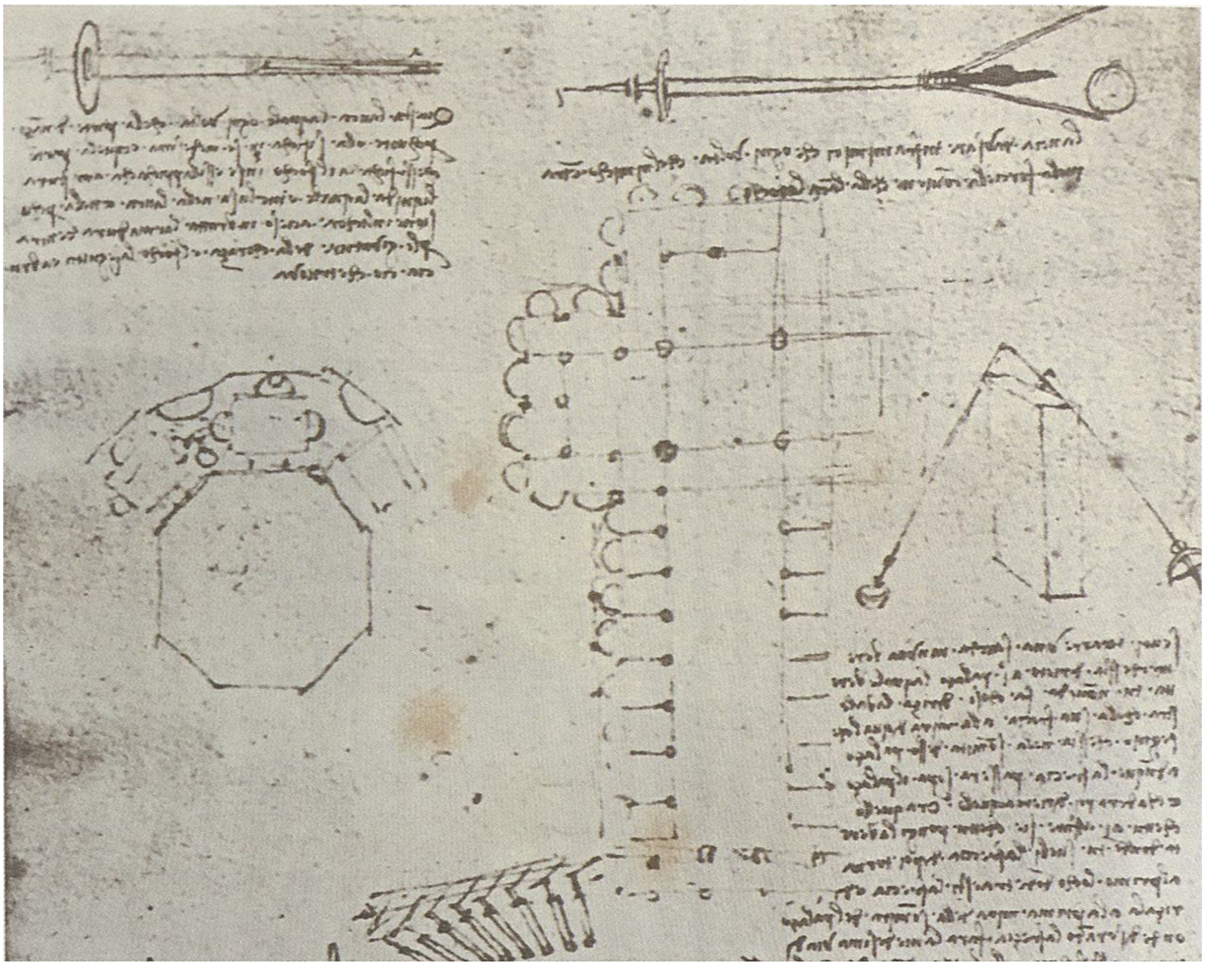
[717] Codex Atlanticus, (1478 c.-1519), Veneranda Biblioteca Ambrosiana of Milan; Codex Trivulzianus (1488-1490 c.), Biblioteca of the Castello Sforzesco of Milan; Codex Forster I (1487-1490 c. and 1505), Victoria and Albert Museum, National Library of Design. ibid., 195-196.

As in many other fields, it is the experimental and preliminary character of these investigations that allowed Leonardo to test alternative formal solutions for central-plan buildings. Consequently, proposals are rarely presented as orthogonal projections but are instead mainly developed in plan and perspective; often with a single bird's-eye view perspective that gives an overview of the three-dimensional articulation of the building in a single glance. However, drawings of near-contemporary buildings designed by Brunelleschi are also found among the range of solutions developed here. The variations applied to these buildings are particularly important in tracing both Leonardo's design process and his formal decisions.

Overall, among Leonardo's drawings of sacred architecture and central-plan buildings, two main spatial and formal approaches can be noted, which have inevitable consequences for the structural conception of a building.[718] These are exemplified by Leonardo's drawings of the plan of two buildings by Brunelleschi in Florence, namely the churches of Santo Spirito and Santa Maria degli Angeli, in Codex B, 11v (illus. 5.15). The plan of Santo Spirito shows the modular grid of pilasters along the naves and transept, and the semicircular niches all around their perimeter wherein the space of the church is organised. Essentially, the plan is arranged by a series of linear or punctiform elements representing walls and columns. A similar scheme of linear elements could then have been repeated for the elevation of the building.[719]

The plan of Santa Maria degli Angeli, drawn by Leonardo on the left-hand side, shows a further development of this approach, typical of later works by Brunelleschi such as the Tribune Morte in the Duomo of Florence. Elements such as pilasters and round arches are first organised in three-dimensional units, which are then arranged to form sides of the

[718] Bellini, "Bramante milanese e il tema dell'organismo cupolato," 131.
[719] As noted, these elements are usually arranged in three-dimensional units that are then combined to determine the whole conformation of the building.



5.15—Leonardo da Vinci, Plans of Santa Maria degli Angeli (left) and Santo Spirito (right) in Florence. Codex B, 11v [detail].

central octagonal nucleus. For Brunelleschi, every element of the system is still readable as a single linear element which is employed in the construction of a three-dimensional unit. The articulation of the pilasters at the corners of the central octagonal space are particularly important in this respect. In the case of Santa Maria degli Angeli, the corners are formed by two separate three-dimensional units. Similarly, in the case of Giuliano da Sangallo's Sacristy of Santo Spirito, the corner is therefore derived by the simple juxtaposition of two separate pilasters with a gap left between them.

As noted by Tessari and Schofield, however, in his drawing Leonardo inverts Brunelleschi's practice, showing the central octagonal space of Santa Maria degli Angeli as generated by the plan of the units placed at its perimeter.[720] In other words, the mass of the walls is defined by Leonardo by means of the geometry of the voids in the plan and not the other way round. As in the case of Bramante's Sacristy in Santa Maria presso San Satiro, the pilasters at the corners are folded here and are probably continuous above the first order. Folded pilasters at the corner of a central octagonal space are also shown by Leonardo in other drawings in Codex B. These drawings show the same design and tectonic procedure, notably an aggregation of voids in plan that is then controlled in three-dimensions by a perspectival bird's-eye view.

Interestingly, a drawing in Codex B, 21v (illus. 5.16), shows a comparison between the two corner solutions. The plan and internal perspective view on the left-hand side show Brunelleschi's double-pilaster solution for Santa Maria degli Angeli. The bottom right hand-side plan instead shows a folded pilaster flanked by columns in the central octagonal space. All drawings of central-plan buildings in Leonardo's Codex B can be assigned to either one or the other of these spatial approaches (aggregation of linear elements in plan and elevation or subtraction of voids with resulting masses), with the latter predominating. Generally, in these drawings, Leonardo tends to firstly work out the process of subtracting voids from the building's masses in plan. Moreover, in some cases, (Codex B, 17v, 21v, 22r) Leonardo covered the internal voids of the building with parallel hatchings. This is a graphic rendering of the voids of the building shown in plan which has no known precedents. James Ackerman read this as demonstrating that Leonardo was endowing the voids with a positive form.[721]

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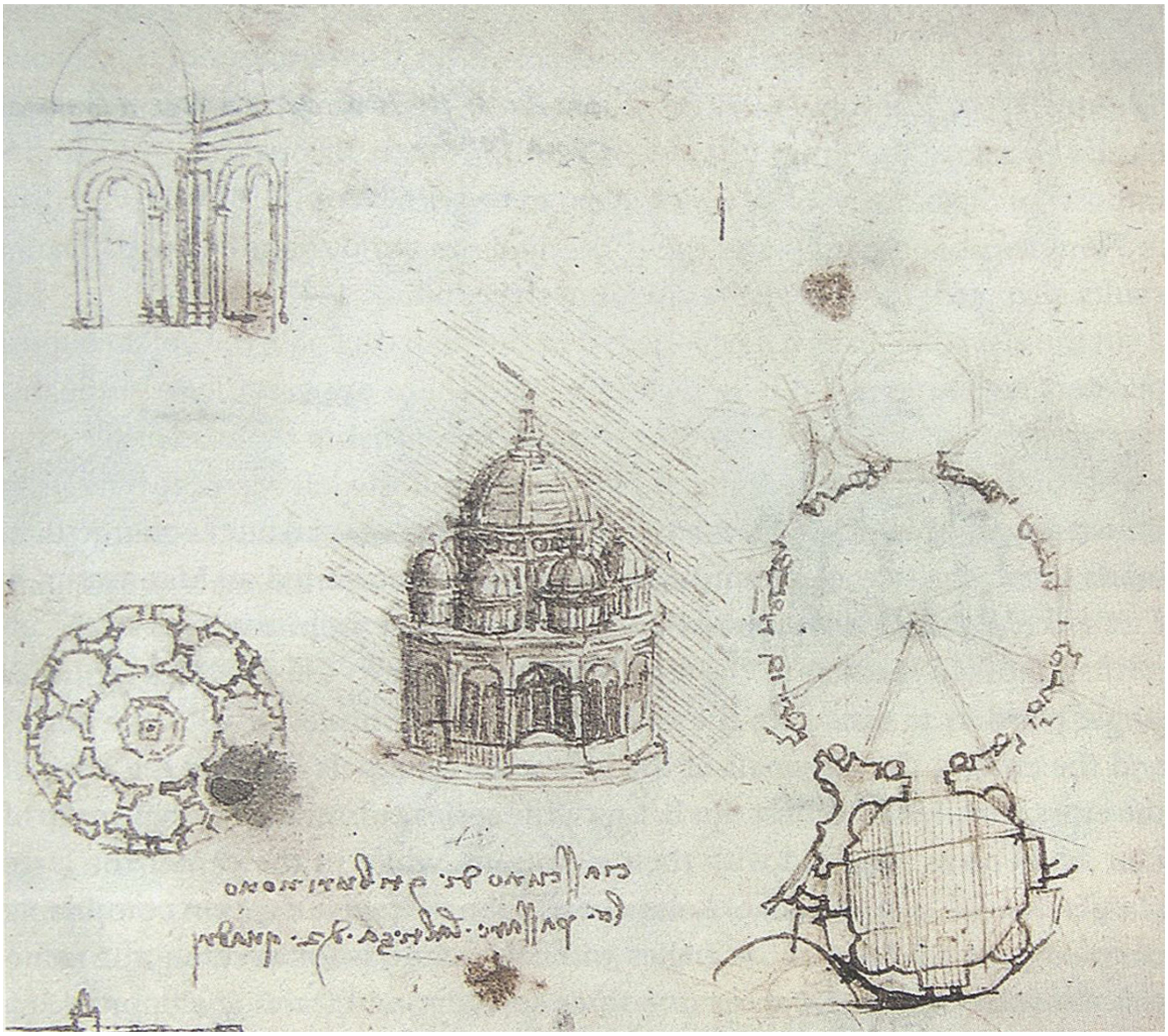
[720] Richard Schofield, Cristiano Tessari, "Aspects of Church Design from Brunelleschi and Alberti to Leonardo and Bramante," 321.

[721] James S. Ackerman, *Origins, Imitations, Conventions: representation in the visual arts* (London: The MIT Press, 2002), 70-71.



5.16—Leonardo da Vinci, Plans with exterior and interior elevation of an octagonal church. Codex B, 21v [detail].

The correlation between the formal properties of central-plan buildings found in Leonardo's drawings and Bramante's architecture in Milan has been often highlighted by the scholarship in recent years, particularly in the proposal later drafted for the New St. Peter's in Rome.[722] However, the correlation with the theoretical framework outlined in Pacioli's coeval writings, particularly in connection to the formal and visual properties of geometrical bodies understood as composite of form (shape) and matter (spatial extension), has largely been neglected. This is despite the often-discussed interactions between Pacioli and Leonardo in the environment of the Sforza Court in Milan. Although it is well established that Leonardo

[722] A recent overview is given in Richard Schofield, Cristiano Tessari, "Aspects of Church Design from Brunelleschi and Alberti to Leonardo and Bramante," 326-329.

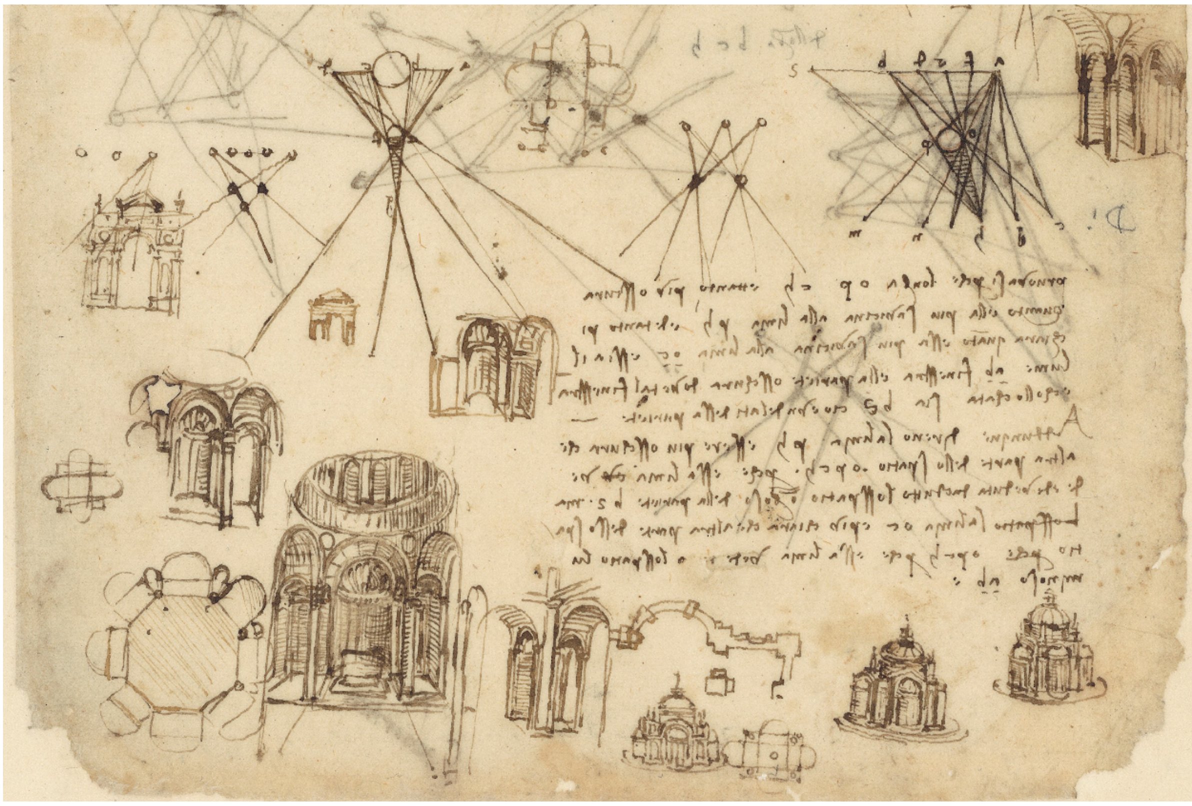
was taught by Pacioli and improved his mathematical proficiency in this period (1496-1499), providing the perspectival representations of regular and other bodies derived from these, what Pacioli could have learned and taken from Leonardo (and, indeed, Bramante) is often omitted. The recourse to multiple and alternative aggregations of two-dimensional figures in plan and their control in a bird's-eye perspectival elevation, is typical of many examples in Codex B. This indicates that in these cases drafted in c. 1486-1488, Leonardo was still beginning with the plan and first articulating figures in two-dimensions, before raising the building in elevation and controlling its form in three-dimensions. It should also be noted that Di Teodoro has shown that in Codex B, Leonardo swiftly derived the elevations from the plan, a method facilitated by the fact that the plans were drawn before the elevation, or by the way in which the two drawings were placed on the sheet.[723] Moreover, Leonardo's annotation in a perspectival view in Codex B, 18r argues that once the building was rendered in three-dimensions it could also be cut at a higher level to provide a more satisfactory arrangement.[724]

Nevertheless, some drawings in Codex B-Codex Ashburnham, and more from Codex Atlanticus that were drafted at a later stage, clearly indicate that the process must have begun from a three-dimensional shape (or geometrical body) before the definition of its plan. In general, perspectival views become more predominant in Leonardo's later architectural drawings, while the plan is sometimes omitted and replaced by perspectival-sections that instead show only part of the plan, as in the drawings in Codex Atlanticus 37r-a (c. 1508)

[723] In addition, Di Teodoro noted that in the simplest constructions the elevation is derived from an element of the plan by progression, while in the most complex the height in elevation is determined by lines passing through a particular part of the plan. Francesco P. Di Teodoro, "Leonardo da Vinci: The Proportions of the Drawings of Sacred Buildings in Ms. B, Institut de France," *Architectural Histories*, 3(1) (2015): 7, 1-10, DOI: http://dx.doi.org/10.5334/ah.cf.

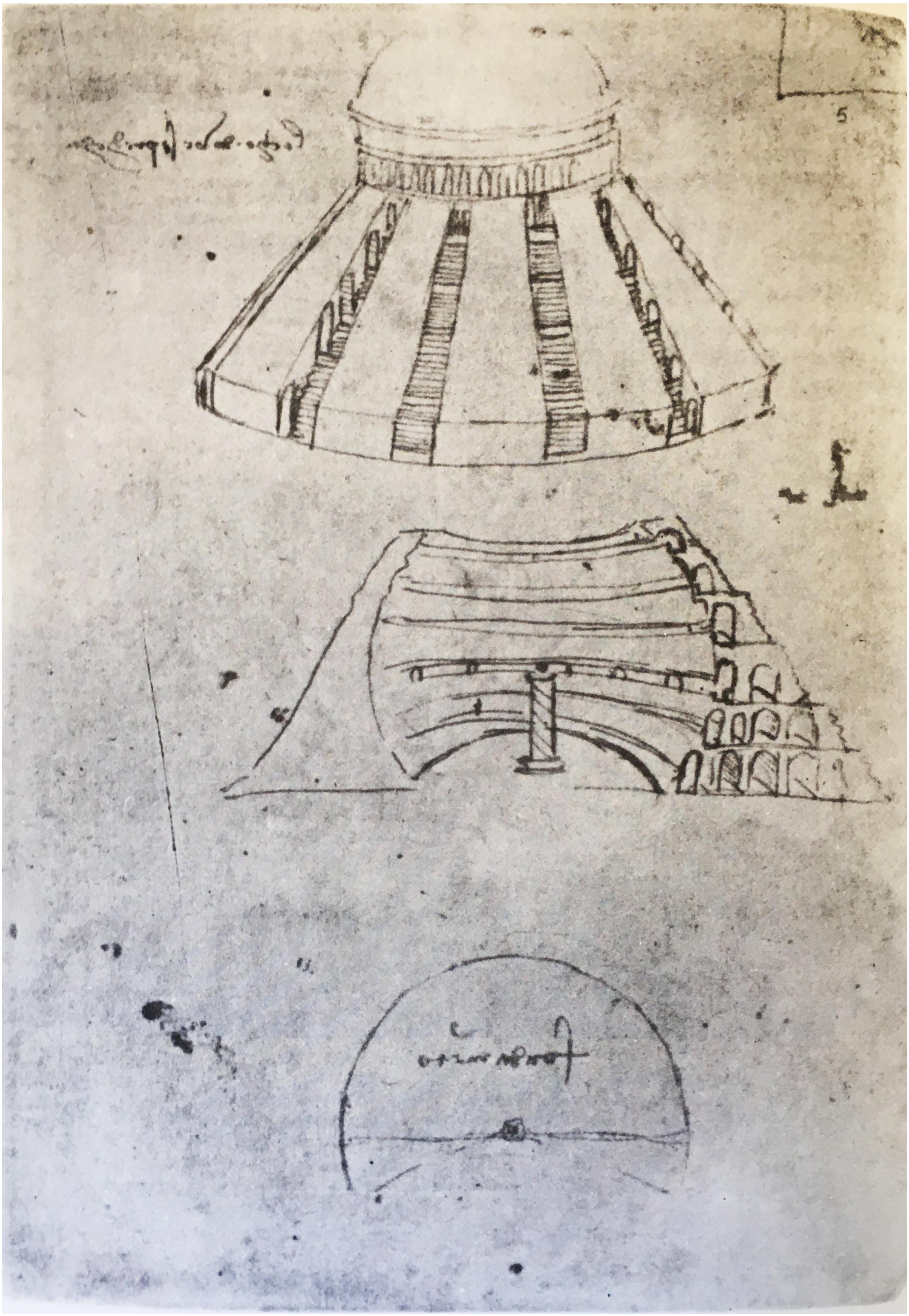
[724] "Questo edifizio ancora starebbe bene a Farlo dalla linea a b c d in su". Leonardo indicates the point where the cut can be made with a rectangle in perspective marked a-b-c-d. In addition, Leonardo employed the same octagonal dome on a drum with round oculi in all these early examples, which is clearly derived from Brunelleschi's dome in Florence's Duomo.

(illus. 5.17).[725] In brief, if the formal properties of architecture are determined in three-dimensions by the employment of geometrical bodies, it is clear that these properties can best be visually expressed in sections and in perspectival representations, or alternatively (if orthogonal projections are required) by a combination of sections, plans and elevations. A clear example of this is Leonardo's design for a "place for preaching" [*locho dove si predica*] in Codex Ashburnham, 5r (illus. 5.18). Here, the solid mass of the building is arranged around a spherical void, the centre of which is the point where the preacher would stand. This mass is articulated with galleries arranged in six levels. Leonardo provides a plan at the bottom of the sheet, but this is clearly insufficient for understanding the three-dimensional arrangement of the building. A line in the plan refers to the location where a perspectival section-elevation is



5.17—Leonardo da Vinci, Studies for central-plan churches and studies of light and shade. Codex Atlanticus, 104 r (previously 37r-a) [detail].

[725] Interestingly, the numerous drawings of architecture are here mixed with drawings and notes on studies of light and shades. The architectural drawings show a combination of five entire or partial plans, two elevations of a door on the left-hand side, five perspectival views of interiors and two perspectival bird-eye view. The numerous attempts of Leonardo do define apses around a central domed space are particularly relevant as these are tested simultaneously in plan and through internal perspectival views. Notably, the central dome is always shown on pendentives, as for the dome of Santa Maria presso San Satiro or the dome of Santa Maria delle Grazie in Milan.



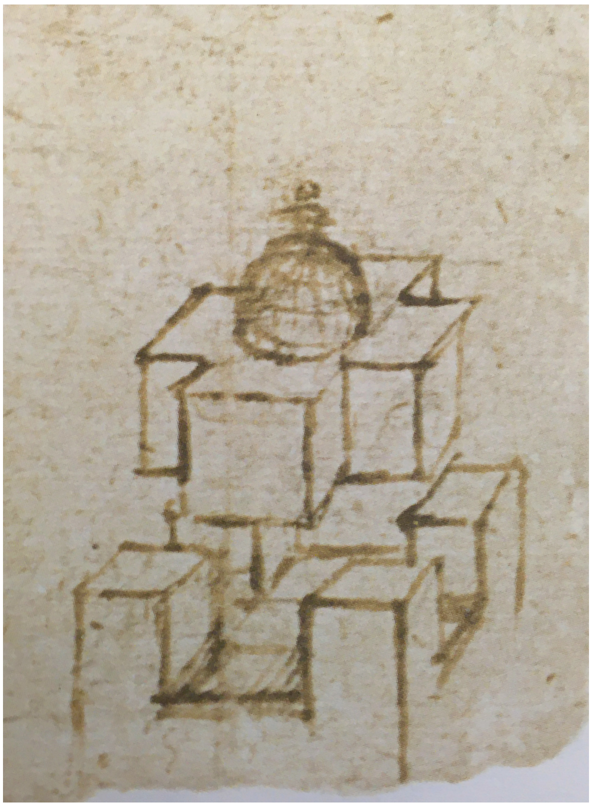
5.18—Leonardo da Vinci, “Locho dove si predica”; “Place for preaching”. Codex Ashburnham, 5 r.

cut, and this is placed at the centre of the sheet. From this, the formal and spatial properties of the building are clarified as having a spherical nucleus that takes a truncated conical shape on the outside. A perspectival elevation of the exterior is provided at the top of the sheet where the building is shown with a domed termination.

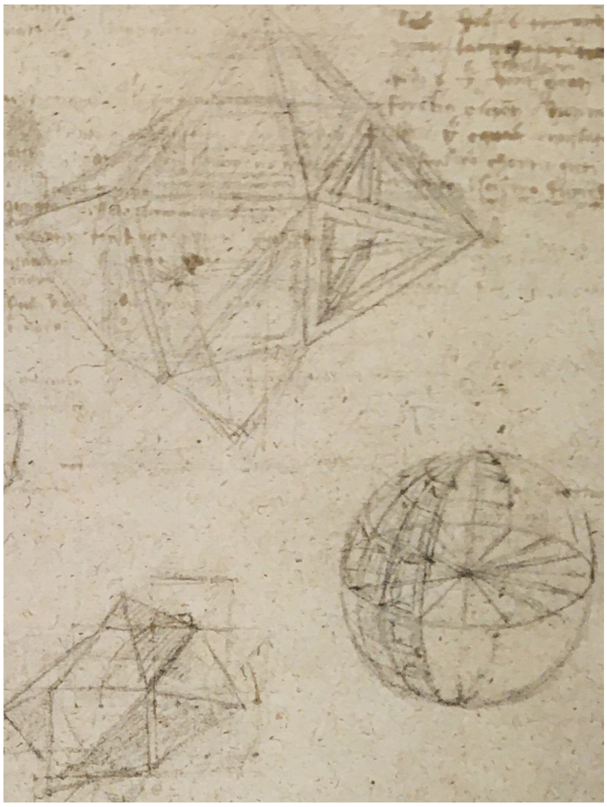
A further example is found in a later (c. 1517-1518) drawing (Codex Atlanticus, 310 v-a) (illus 5.19). The building is shown only in perspectival view and is divided into two interlocking blocks. The base block shows the outline of the four peripheral spaces and the internal nucleus of the building, above which a hemispherical dome (with bases) is placed. In the recto of the same sheet (illus 5.20), Leonardo drafted in pencil part of a geometrical body of 128-bases in skeletal form. This can clearly be linked to the perspectival representation of a 72-bases body in the same style that Leonardo had drafted for Pacioli's *Compendium* about twenty years before, and that Pacioli had associated with domed coffered ceilings. Although drawings of geometrical bodies that can be linked to the perspectival representations drafted for Pacioli's *Compendium* are frequently found among Leonardo's manuscripts,[726] a clear correlation with architecture is established in this case.

In the verso, Leonardo discusses proportional relationships of quantities and magnitudes in geometrical (two-dimensional) and arithmetical subdivisions. The process of quantifying the matter (i.e., the volume) of regular geometrical bodies by means of pyramids found in Leonardo's notes and drawings in Codex Forster I (1505), can be linked to these studies. The notebook was drafted towards the end of a period of interactions with Luca Pacioli, firstly in

[726] In Codex Atlanticus (707r, 708r) are found copies of the cartoons employed for the perspectival representations in Pacioli's *Compendium*, a icosahedron in skeletal form is found in 518r and a solid version in 920r. Other polyhedra are studied in Codex Atlanticus, 735v. Leonardo draws the five regular bodies in Codex M (Paris, Institut de France), 80v and writes down beside them the opening poem of Pacioli's *Compendium*. Argante Ciocci, "Luca Pacioli, Leonardo da Vinci e il disegno dei poliedri," 43-86. However, in his survey of the drawings of polyhedra by Leonardo, Ciocci did not mention the 128-bases body in Codex Atlanticus, 312v-a.



5.19—Leonardo da Vinci, Drawing of central-plan church divided into two interlocking blocks. The base block shows the outline of the four peripheral spaces and the internal nucleus of the building, above which a dome (with bases) is sketched. Codex Atlanticus, 849v (previously 310v-a) [detail].



5.20—Leonardo da Vinci, Drawings of geometrical bodies among which part of a geometrical body of 128-bases in skeletal form. Codex Atlanticus, 849r (previously 310r-a) [detail].

Milan and later in Florence,[727] and comprises a short treatise: "Book entitled on the transformation of a body without diminution or accretion of matter"[728] Through a number of examples in this section, Leonardo demonstrates how to transform a regular body into another without any change of matter, examining therefore the relationship of one body with

[727] As noted by Elisabetta Ulivi, a notarial deed dated 21 July 1505 records the presence of both Luca Pacioli and Leonardo da Vinci together in Florence. Archivio di Stato di Firenze, Notarile Antecosimiano 7532, c. 106r. Ulivi, "Nuovi documenti su Luca Pacioli," 50.

[728] "Libro titolato de trasformazione c[i]oe d'un corpo in un altro sanza diminuzione e accrescimento di materia". Codex Forster I, 3r, Victoria and Albert Museum, National Library of Design, London.

another when both have the same volume.[729] Significantly, these are not the only notes by Leonardo related to the quantification of three-dimensional geometrical bodies, but are indicative of his attention to the relationships between shape and matter (three-dimensional spatial extension).[730]

Overall, these and the drawings of central-plan buildings show Leonardo's interest in the relationship between spatial and formal properties in three-dimensions. As noted, the same preoccupation is expressed not only in the articulation of spaces but also in the choice of architectural elements, such as the typology of pilasters around the central nucleus of a building. As for Bramante's Sacristy of Santa Maria presso San Satiro, a similar consideration for the spatial continuity of elements can be found in at least some drawings by Leonardo. In these, a formal and spatial hierarchy is indicated by the privilege accorded to the visual arrangement in three-dimensions of architecture, considered as composite of matter (three-dimensional spatial extension) and boundary (shape). As already noted, this should also be considered together and compared with Leonardo's studies in the quantification of three-dimensional bodies in the same period. Essentially, this is consistent with the concept of the geometrical body presented in Pacioli's *Compendium* and can also be correlated with some aspects of Bramante's designs for Santa Maria presso San Satiro and Santa Maria delle Grazie, as mentioned above. With these considerations in mind, the last part of this chapter will examine aspects of Bramante's designs in Rome in the early sixteenth century.

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[729] The process involves first the division of the original body in pyramids, for example 12 pyramids with pentagonal bases are obtained from a dodecahedron. The pyramids are then subdivided in smaller pyramids, each with a triangular base, and the quantity of their matter calculated. These quantities are then transferred to a series of parallelepiped, and these in turn assembled in a cube that matches the amount of matter [volume] of the initial dodecahedron. Leonardo also proves the inverse process, from a cube to a dodecahedron. Ultimately, the pyramid can be read as the quantitative unit that both Pacioli and Leonardo interpret as basic components of the regular and other bodies derived from these. It is clear from these passages, and from the contents of Pacioli's writings discussed in previous chapters, that both Pacioli and Leonardo show to be concerned with the subdivision and quantification of bodies in three-dimensions.

[730] The quantification of three-dimensional geometrical bodies is also discussed in Codex Madrid II (1503-1504 c., with the exception of the final notebook, datable to 1493). Madrid, Biblioteca Nacional.

5.3 Formal aspects of Bramante's architectural design in early sixteenth-century Rome

5.3.1 The drawings for the New St Peter's Basilica

Bramante's proposals for the New Saint Peter's are the oldest set of drawings to have been preserved for a major building in such a number. They are moreover an exceptional group of documents for analysing Bramante's design approach to the formal properties of architecture. Since the initial studies by Franz Wolff Metternich, the amount of literature on the initial stages of the design for the New St Peter's has grown enormously.[731] Nonetheless, Bramante's drawings have been often considered in isolation and are seldom correlated with the interactions and works of his previous Milanese period. They have mainly been read as linked to Bramante's experience of Roman antique monumental architecture, or sometimes as the product of a largely unspecified process of invention.[732] Considering established data derived from drawings and surveys (particularly in the decades-long studies by Christof Thoenes), this and the following sections will instead examine aspects of Bramante's design process and compare them with his previous works and with the theoretical framework outlined in Pacioli's works. Elements will therefore be considered only insofar as they contribute to

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[731] Christoph L. Frommel, "St. Peter's: The Early History," in *The Renaissance from Brunelleschi to Michelangelo*, ed. Henry Millon and Vittorio M. Lampugnani (Milan: Bompiani, 1994), 399–423, 598–630. A summary of the phases of reconstruction of St. Peter's in Christof Thoenes, "Renaissance St. Peter's," in *St. Peter's in the Vatican*, ed. William Tronzo (Cambridge Cambridge University Press, 2005), 64–90. An overview of the historiographical process is given in Dario Donetti, "Bramante agli Uffizi. I disegni per San Pietro e la storiografia architettonica," *Annali di Architettura*, n. 26 (2014): 107-112.

[732] Cristiano Tessari, "Alle origini dell'"inventio" bramantesca per il nuovo San Pietro di Roma: incontri nella Milano del tardo Quattrocento," *Arte Lombarda*, no. 176/177, vol. 1/2 (2016): 152-153; Bruschi, *Bramante Architetto*, 555-557. As noted by Christof Thoenes, Roman antique buildings such as the imperial baths and the Basilica of Maxentius have been correlated to the projects of Bramante and considered as precedents, however in none the walls can be found conformed in a comparable way, and their structure is not geometrically pre-formed as in the case of Bramante. Thoenes, "Renaissance St. Peter's," 81. In these terms, only the dome of the Pantheon can be meaningfully compared with the coffered hemispherical dome proposed by Bramante. However, as discussed later in this section, in Bramante's process of design is found the systematic application of this approach to create a spatial hierarchy closely interrelated to the formal and visual properties of the building. In any case, precedents in themselves still do not explain how and why Bramante was willing to employ them on not other formal solutions.

clarifying the design process employed by Bramante and helping the comparison with Pacioli's theoretical framework.

Bramante's drawings for the New St Peter's were produced from 1505 to 1514, mostly under the papacy and influence of Julius II (1503-1513) and with the last drawings prepared under his successor Leo X (1513-1521). They can be grouped into three main phases of development with corresponding projects by Bramante.[733] A number of buildings already existed on site, and these had to be considered in the process. These were the old basilica of St Peter's, the two rotundas and the obelisk on the south side of the old basilica, the choir and part of transept that were begun in the 1450s under Pope Nicholas V and later interrupted, with the choir standing at a height of approximately two metres.[734]

Two sketches in GDSU 20 A verso (illus. 5.21), first isolated by Heinrich von Geymüller, show how Bramante initially intended to amend Nicholas V's project.[735] A dome on a square base is shown above the transept and two apses are added to the transept ends. The proposal is presented in plan and in a perspective view looking west. The building is here visualised by means of two main elements: a large hemispherical dome resting on a columnated drum above a square platform with smaller towers at the corners;[736] and a low and large rectangle (the transept) to which round apses are added at either end. Although their scale would change in the different projects developed by Bramante, the columnated drum and

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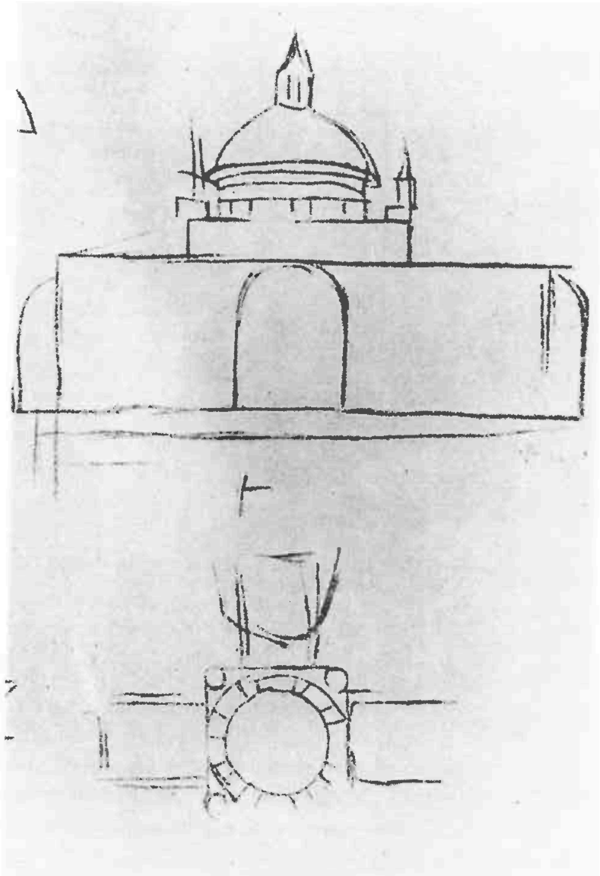
[733] Two are well documented in autograph drawings of Bramante, while the third is only known through the xylographies published by Sebastiano Serlio.

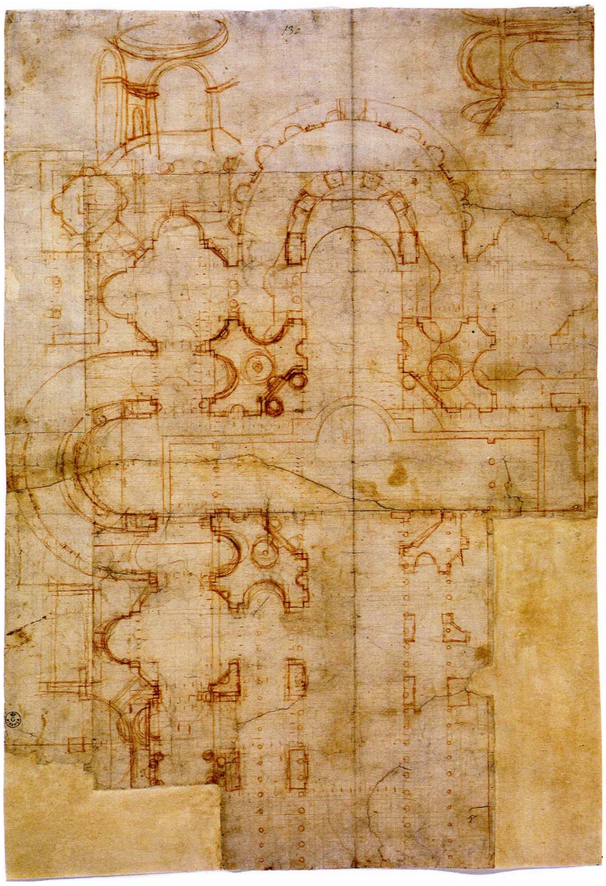
[734] Thoenes, "Renaissance St. Peter's," 65-71.

[735] It should be noted that this group of drawings of Bramante are the first instance where steps in the process of thinking and elaboration of a building are preserved. This has been linked to various factors, among these the availability and reduced cost of paper, the training of many architects in painting and sculpture, the wider need to establish a novel and distinctive role for the architect. Cammy Brothers, "What drawings did in Renaissance Italy," in *Renaissance and Baroque Architecture*, ed. Alina Payne (New York: John Wiley & Sons, 2017).

[736] Most probably this should be correlated with the intention of reusing the columns of the old basilica in the new building. Christoph L. Frommel, "Sul metodo progettuale nei disegni di Bramante, Raffaelo e Antonio da Sangallo il Giovane per San Pietro," *Annali di Architettura*, n. 30 (2018): 126, 123-136.



5.21—Donato Bramante, Sketches on the verso of GDSU 20 A showing a dome on a square base above the transept and two apses added to the transept ends, redrawn by H. von Geymüller [detail].



5.22—Donato Bramante, Plans and perspectival views of the New St Peter's overlaid on the Old St'Peter's Plan, GDSU 20 recto. Note the different dimensions and geometry of the central dome pier on the right-hand side,

hemispherical dome remained essentially the same in every phase that followed. Moreover, the perspectival representation of the basilica on a view looking west (towards the *Vaticanus Mons*) shown in the drawing reproduces the visual angle experienced by most of the pilgrims arriving in Rome and approaching the site. As in the case of the Duomo of Milan's *tiburio* discussed in the *Opinio*, clearly Bramante was here considering the dome above the transept as an isolated three-dimensional element to be visually experienced from a distance, while approaching the site.

The lower right-side of the recto of the same sheet shows a portion of a plan with the shape of the main dome's support pier. The same shape (in reduced dimensions) is also shown as the minor dome's pier (illus. 5.22). Thus, for the first time, the idea is expressed of a major and minor order of domes arranged in a *quincunx*. However, the shape of the piers shown in the drawing needs further consideration. Their shape results from the conformation of the space below the dome as an unequal-sided octagon with two different side-lengths. This arrangement allows for widening the space under the four main arches supporting the dome, even though at least part of the dome's wall would cantilever over the four chamfered piers as a consequence.[737] In other words, while the form of the piers is also obviously correlated to the geometry of the dome they support, it is firstly as a consequence of the formal properties of the central space they enclose. This shape of the piers would be retained by Bramante in all the phases of his design. Although often described as Bramante's invention, a similar chamfering of the piers can be found in coeval drawings by Leonardo, as noted.[738] Moreover, as shown in some of Leonardo's drawings and already in Bramante's Sacristy of Santa Maria presso San Satiro, the pilasters at the corners of the octagon were (as shown in Bramante's later drawings) to be folded and arranged in a single giant order. Overall, it will be shown that Bramante concentrated most of his attention on the proportioning of the dome and the piers while developing his projects for the New St Peter's. As noted, although their size would be scaled, these elements always retained the formal properties defined in the initial phase.

At this stage, the old basilica that was still operating and the transept and choir commenced under Nicholas V were also shown in the same GDSU 20 A recto. To work freehand at a correct scale, the sheet was prepared with a grid of orthogonal lines at 3.7 millimetres spacing. As noted by Thoenes, this measure corresponds to a minute (1/60 of a *palmo*). Every minute in the scale of the drawing corresponds to 5 *palmi romani* (from now *palmi*) in the real scale of the building. This means that Bramante was willing to draw at a scale of 1:300; and

[737] This would be instead much more limited if the perimeter of the central space was arranged according to the perimeter of a regular octagon.
[738] Richard Schofield, Cristiano Tessari, "Aspects of Church Design from Brunelleschi and Alberti to Leonardo and Bramante," 326.

300 *palmi* was also the total width of the naves of the Old St Peter's.[739] The grid therefore enabled Bramante to control the position of both the old and new buildings' spatial axes.[740] Consequently, in the first project, Bramante fixed the interaxial distance between the centre of the major and minor domes in plan at 150 *palmi*.[741] The major dome is here placed above the centre of the transept designed under Nicholas V, and the centre of the minor dome shown in the drawing is located on the wall of the old basilica. The diameters of the domes are in a ratio of 2:1, with the major dome's diameter at approximately 185-190 *palmi*.[742] However, although their centres are positioned to correspond with elements of both the old basilica and Nicholas V's transept, their diameters and the geometry of the supporting piers cannot be meaningfully correlated to the subdivision of the old basilica's plan. As in later cases, here, the control of the axial distance and diameters of the major and minor domes define the position of the structure below devised by Bramante.[743]

Although the scale of the drawing has been doubled to 1:150, the same quantitative relationships are shown in the well-known *piano di pergamena* (GDSU 1 A) (illus. 5.23). This is a presentation plan in parchment which most probably followed from the initial sketches by Bramante in GDSU 20 A verso and recto (lower right-hand side).[744] Only a portion is shown of what could either be a central or a longitudinal building. In addition to the portion of the previously sketched plan, the distance between the centre of the dome and the walls of the three apses is visible and can be measured at about 360 *palmi*, which is well beyond the width

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[739] A Roman *palmo* is 22.34 centimetres. Christof Thoenes, "Bramante a Roma: l'arte di progettare. La mostra del 2014-2015 al Palladio Museum di Vicenza" *Annali di Architettura*, n. 26, (2014): 49.

[740] This is in essence the same process previously employed by Bramante in the cloister of Santa Maria della Pace. Here the distance between the axes of the bays, both in length and in width, determine the spatial articulation of the whole. Moreover, the geometry of the walls is determined by subtraction of equal three-dimensional spaces arranged in a modular grid around the central nucleus. ibid., 43-44.

[741] Thoenes, "Bramante a Roma," 49.

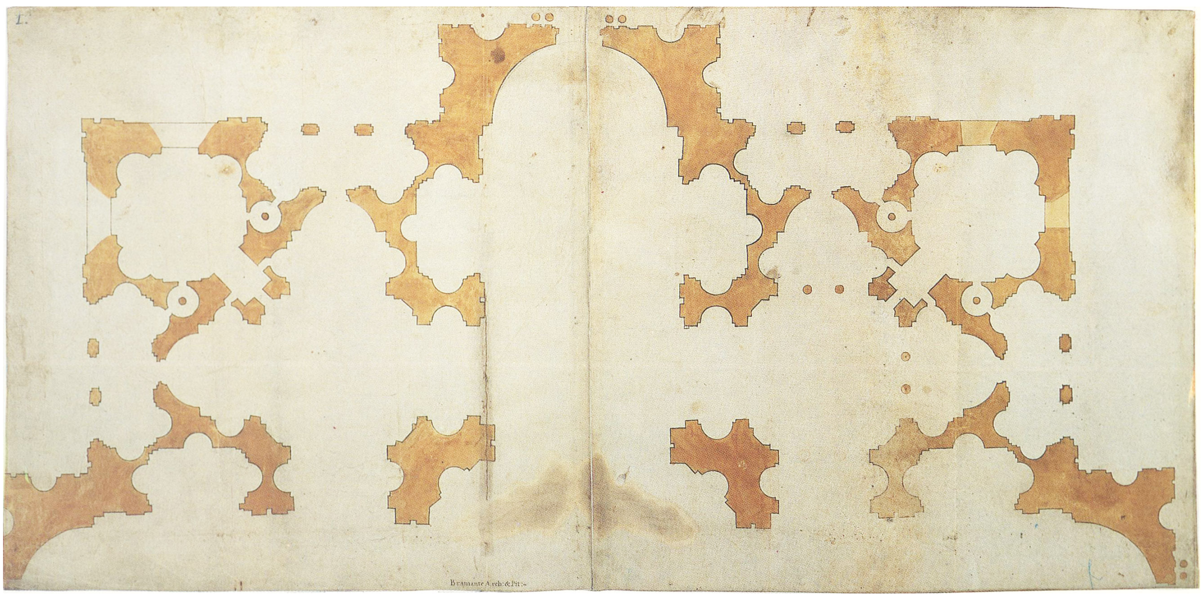
[742] ibid., 43-44. Frommel, "St. Peter's: The Early History," 604.

[743] As noted by Christof Thoenes, the interaxial distance is a crucial dimension in all of Bramante's proposals for St Peter's and from this dimensions the position of the axes of other space in plan are determined. Thoenes, "Bramante a Roma," 43-44.

[744] ibid., 49.

of Nicholas V's transept or the depth of the choir (285 *palmi* from the centre of the crossing).[745] As noted in a drawing by Antonio da Sangallo, later an assistant to Bramante, the project was "not executed"[746] for reasons that are still not completely clarified, and a revised version (namely the second project) was developed by Bramante.

The first clear development of the second project, outlined by Bramante in the remaining part of GDSU 20 A recto, should thus be dated after this point. This phase sees an increase in the interaxial distance between the main and minor domes to 190 *palmi,* and the introduction of ambulatories in the three arms around the main dome. The diameters of the main and minor domes here are 190 and 95 *palmi* respectively and thus at a ratio of 2:1, so their diameters reflect the distance of their axes in plan. Two internal perspectival views are sketched in the top margin of the sheet to visualise the formal arrangement of the space under the main dome (illus. 5.22). On the right, Bramante shows the double pilaster solution of the corner found in Brunelleschi's Santa Maria degli Angeli, with its corresponding



5.23—Donato Bramante, Parchment plan of the New St Peter's, GDSU 1 A.

[745] As pointed out by Thoenes, if the fifteenth-century (now lost) drawings of the project of Nicholas V were at the same scale these could have been overlapped to the piano di pergamena to show the transformation of the church and thus justify the scale of the latter. ibid., 50.
[746] As Antonio da Sangallo annotated in the sheet: "pianta di Sto Pietro di mano Bramante che non ebbe effetto". GDSU 1 A verso.

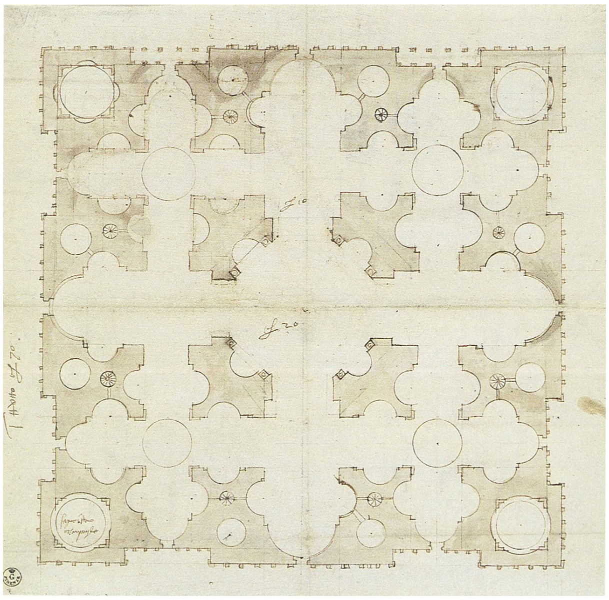
triangular-spherical pendentive. On the left, the alternative unified chamfered pilaster which is shown in Bramante's plan. The conformation of the pier in this case determines a trapezoidal-spherical pendentive above. In the second perspectival drawing, the outline of the circular dome standing above the pendentives is also visible. The visual angle is set as if the viewer were standing inside the basilica looking up. The drawing shows the base of the dome and the chamfered pilaster of the octagonal central space. A similar visual angle is employed by Bramante in a related drawing (7945 A verso, lower right-hand side), where the addition of giant columns in front of the central piers was considered to support the cantilevering portions of the dome.

As has often been noted, these drawings help clarify the design process followed by Bramante. Nevertheless, the process by which he reached these proposed solutions has yet to be fully explained. Clearly, a series of two-dimensional perspectival drawings (two of which are sketched on GDSU 20 A recto) show that Bramante had already mentally rotated and visualised the elements outlined in plan. Notably, these include not only the solutions shown in the plan, but also alternative arrangements that were not employed. As in the case of the illusionistic perspectival choir of Santa Maria presso San Satiro, these spaces had to be mentally pre-formed, that is, visualised and quantified by Bramante in three-dimensions before they could be successfully drawn. Moreover, in this case, Bramante's freehand drawings are the first documented record of the empirical employment of the principles of linear perspective at the design stage. This should not surprise, as we have seen, Bramante's initial drawings show that he had already thoroughly considered (and thus mentally visualised) the three-dimensional formal properties of at least some of the most prominent elements in the building, including the dome.

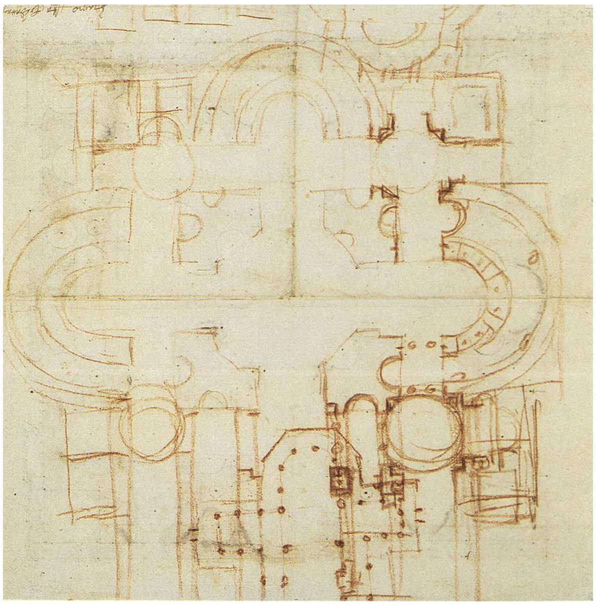
Meanwhile, Giuliano da Sangallo showed a centralised proposal in plan (GDSU 8 A), most probably based on the previous *piano di pergamena* by Bramante.[747] The spacing of the walls was regularised by Sangallo in a matrix of four-by-four elements to define the articulation of all the basilica's spaces. The three-dimensional conformation and hierarchy of spaces found in Bramante's proposals have therefore been lost. The plan simply shows a different logic in the proportioning and aggregation of the elements in plan (illus. 5.24). This is demonstrated by Bramante's corrections on the verso of the same sheet (illus. 5.25). The building is returned to a three-dimensional hierarchical order reflected in plan, the outline of the domes at the level above is added and longitudinal naves are attached to the west side of the building. As in GDSU 20 A recto, ambulatories are added in the three arms around the central dome. Two rapidly made sketches are included to further clarify the elements in the plan. The first sketch is placed beside the ambulatory of the choir and shows a plan of a solution with ambulatories as in the Old San Lorenzo in Milan. The other shows the plan of the Duomo of Milan and is placed above the central nave. As in the case of the duomo, the plan of the New St Peter's is composed of five naves with the central nave being wider that the side naves. At the same time, by means of the plan's location above the nave of the new basilica, Bramante was most probably indicating a three-dimensional hierarchy between width, length, and height, similar to the one described by means of distinct bodies in his *Opinio* on the duomo.

At this point, the building shown in the well-known foundation medal (1506) somewhat unexpectedly follows the plan of the first project in principle. However, if compared with the *piano di pergamena*, the diameter of the major dome has been increased to a ratio of about 3:1

[747] This is the drawing 8A r of the Gabinetto dei Disegni e delle Stampe delle Gallerie degli Uffizi. It has been tentatively dated to the fall of 1505, therefore just after the plan of Bramante in Christoph Frommel, "St. Peter's: The Early History,", cat. no. 287, 603. It is generally assumed that the drawing was produced a few months, if not weeks, after the 'piano di pergamena' of Bramante. Christoph Thoenes, "S. Lorenzo a Milano, S. Pietro a Roma: ipotesi sul «piano di pergamena»," *Arte Lombarda*, no. 86/87 (3-4) (1988): 94.



5.24—Giuliano da Sangallo, Plan study for the New St Peter's, GDSU 8 A recto. Note the regularised proportioning and aggregation of elements in plan.



5.25—Donato Bramante, Plan study for the New St Peter's, GDSU 8 A verso. Note how the three-dimensional hierarchical order of the building is reflected in plan.

to that of the minor domes. The building is again shown in a view looking from east to west.[748] At the time of the foundation of the main dome's first pier in April 1506, Bramante had thus already established the ratio found in the third project. Nonetheless, no drawings by Bramante survive from this phase and the project has been transmitted only in the later project by Raphael, and in the plan published by Serlio (although regularised over a 30-*palmi* grid).[749] In any case, it is most likely that at this point, the pope decided to reduce the number of naves from five to three and this determined a ratio between the major and minor domes of about 3:1. This reflected the new width given to the minor naves, while the intermediate naves were blocked by pilasters with niches. The interaxial distance between the two domes was here reduced to 165 *palmi;* consequently, the diameter of the main dome was probably slightly reduced to about 184.5 palmi (188 in the xylography published by Serlio), with the diameter of the minor domes at 65 *palmi*.[750]

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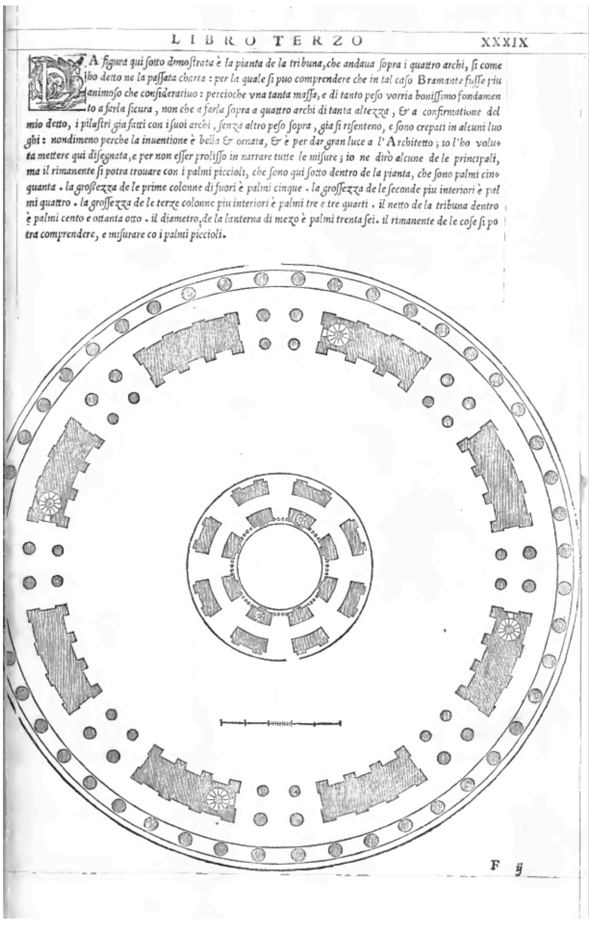
[748] As noted by Thoenes, the *Vatican Mons* determines here the building's groundline. Thoenes, "Bramante a Roma," 50.
[749] ibid., 53.
[750] ibid.

After 1506, the construction continued without major disruptions until Bramante's death in April 1514. The surviving drawings document how the design of different elements was undertaken as the construction proceeded. Among the surviving documents from this phase, a drawing by Antonio di Pellegrino (Bramante's assistant) shows in section and in elevation, the dome's pendentives as a portion of a hemisphere. He does this in plan by means of a series of lines marking the points at the same level (which are now called contour lines). This process is certain to be linked to indications received by Bramante.[751]

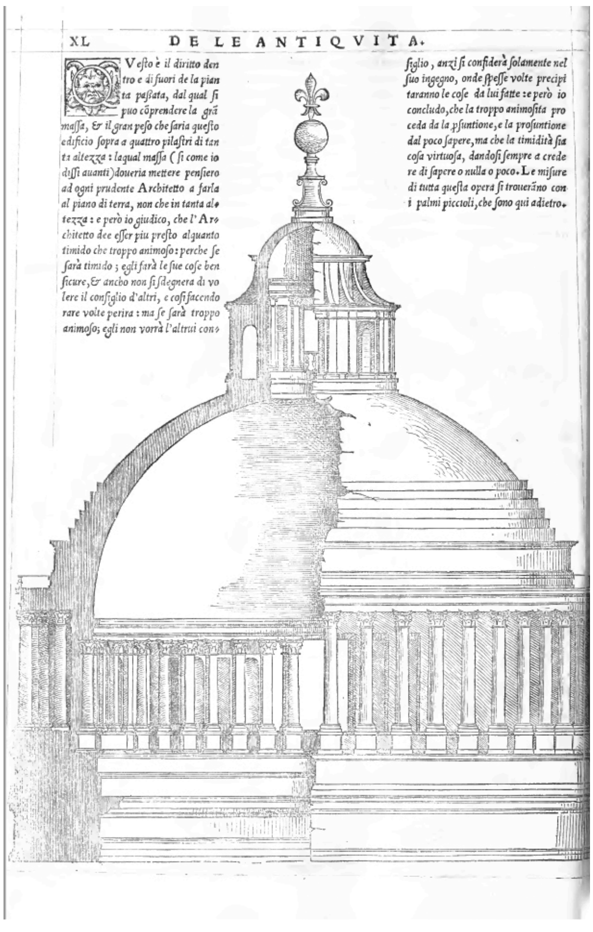
The four main piers and the four coffered arches intended to carry the dome were reaching completion just before Bramante's death. As noted by Serlio, Bramante thus further considered the design of the dome and left some drawings.[752] The original drawings have not survived, but Serlio writes to have published two xylographies derived from them (illus. 5.26-5.27).[753] Although shown in two consecutive pages by Serlio, the drawings may have originally been on a single sheet. As is often noted, Bramante showed the dome in isolation and limited himself to only two drawings, although these completely define the dome's three-dimensional formal properties.[754] If considered in the context of Pacioli's works, Bramante shows the dome as an isolated geometrical body. It is thus unsurprising that for the first time, the three main planes of orthogonal representation: plan, elevation, and section, are recorded in two drawings at the same scale to describe its formal properties. However, as noted above, these can only be employed conjointly if they are considered as sections cutting through an isolated three-dimensional body, composite of an outer boundary and matter (spatial

[751] Frommel, "St. Peter's: The Early History," 413. Thoenes, "Bramante a Roma," 53.
[752] As Serlio writes: "e questa ordinò Bramante prima ch'ei morisse. La pianta de la quale è qui nella seguente charta.". Sebastiano Serlio, *Il terzo libro di Sabastiano Serlio Bolognese* (Venice: Francesco Marcolini da Forli, 1540), 38.
[753] As noted by Thoenes, the parallel between Bramante and Michelangelo is here appropriate. While the latter bequeathed his project in the form of a wooden model, Bramante left (as far as we know) two drawings. Thoenes, "Renaissance St. Peter's," 79-80.
[754] ibid.

extension). In other words, in a similar way to how Pacioli described the sphere in Chapter 56 of the *Compendium de divina proportione*.[755] Di Teodoro noted that there is a likely precedent in Piero della Francesca's *Libellus* (translated and later published by Luca Pacioli) in the plan and section-elevation representation of the 72-bases body.[756] However, while Piero arranged each drawing on a separate sheet, it was only in the version of Pacioli published in 1509 that the section, plan, and section-elevation of the 72-bases body were displayed on the same sheet.[757] Moreover, two horizontal sections taken at different levels and cutting the columnated drum and the lantern are shown by Bramante in the same plan. As noted above, Bramante had



5.26—Sebastiano Serlio, Plan of the dome and lantern of the New St Peter's after a drawing by Bramante



5.27—Sebastiano Serlio, Section-Elevation of the dome and lantern of the New St Peter's after a drawing by Bramante

755 Additionally, as noted, in the *Summa* quantitatively dealt with the volume of sections cut from the sphere. Pacioli, *Summa*, 73v (Pars secunda). This is discussed in section 3.1.1 of the thesis.

756 The relationship between the drawings of Bramante published through the illustration in Serlio's *Trattato* and the drawings of Piero della Francesca's *Libellus* showing the 72-bases body has been highlighted by Francesco P. Di Teodoro. Di Teodoro also noted that the representation of the base and lantern of the dome in plan can be referred to the 'parallels' traced through a head to aide its construction in perspective in Book III of Piero's *De prospectiva pingendi*. Francesco P. Di Teodoro, "Vitruvio, Piero della Francesca, Raffaello," 47.

757 See illus. 3.3 at page 164.

already shown the plan of the dome superimposed on the ground plan of the building in previous drawings.

Ultimately, Bramante's conception of the dome as an isolated three-dimensional body to be visually perceived and recognisable from a distance (already found in the first surviving sketches and then employed in the hierarchical organisation of the building), finds its final definition in these drawings. As in the initial sketch, the dome is resting on a high drum that is columnated on its highest portion to let light inside the building, but also to raise the level and enhance the visibility of the hemispherical dome above. Consequently, the lantern above the latter reproduces a similar formal arrangement at a smaller scale. As we shall now see, similar formal and visual considerations to the one derived from Bramante's drawings examined above, can be applied to the choir-mausoleum designed by Bramante for Julius II.

5.3.2 The Choir of Julius II in the New St Peter's

Bramante's third project was linked to the increasing attention placed by Pope Julius II to the basilica's choir arm, where the foundations of the project by Nicholas V were still to be found.[758] During the construction of the main dome's supporting structures, the pope's attention increasingly shifted to the construction of the choir as his funerary chapel, the so-called "Cappella Iulia" where (pressed by time) the pope most probably insisted on reusing the foundations of the old choir.[759] Michelangelo was commissioned for the monument to be placed in the space, and Bramante started to concentrate on this part of the building, the only space to be completed by the time of his death in April 1514.[760] Choirs are a recurrent architectural theme among Bramante's works, both in Milan and Rome. Importantly, these

758 Thoenes, "Renaissance St. Peter's," 65-71.
759 Thoenes, "Bramante a Roma," 52.
760 Thoenes, "Renaissance St. Peter's," 78-79.

are spaces usually placed at the furthest end from the church's entrance which are not to be entered by lay people. As we have seen in Santa Maria presso San Satiro and Santa Maria delle Grazie, they are thus primarily visually experienced from vantage points located in the nave.[761] This means that although they are contiguous with the rest of the church, they can also be more easily visually and formally isolated.

Although this part of the building as designed by Bramante no longer exists, it can be reliably reconstructed in most of its formal aspects with the help of drawings. Some of these are by Bramante's assistant Antonio da Sangallo the Younger, some are copied from Bramante's drawings, and a few are attributed to Bramante himself. The drawings are important to trace some passages in Bramante's design process, although, as noted, the material constraints imposed by the structure of the previous choir should also be considered in this context. Most of the extant drawings show this portion of the building in plan, with some perspectival drawings of the space. An elevation of the whole choir by a draughtsman (GDSU 5A recto) (illus. 5.28) could also have been taken from the model based on Bramante's drawings.[762] When the plan is compared with that of the old choir, it becomes clear that Bramante was willing to significantly alter the walls of the old choir and articulate the new choir in three distinct spaces. In this way, the choir would also better conform with the other two arms of the transept. Simultaneously, to keep the diameter of the major dome as devised in the third project (and thus the same distances between the central piers), the internal width of the old choir was also reduced. Notably, this allowed possible variations in the width of section of the choir.

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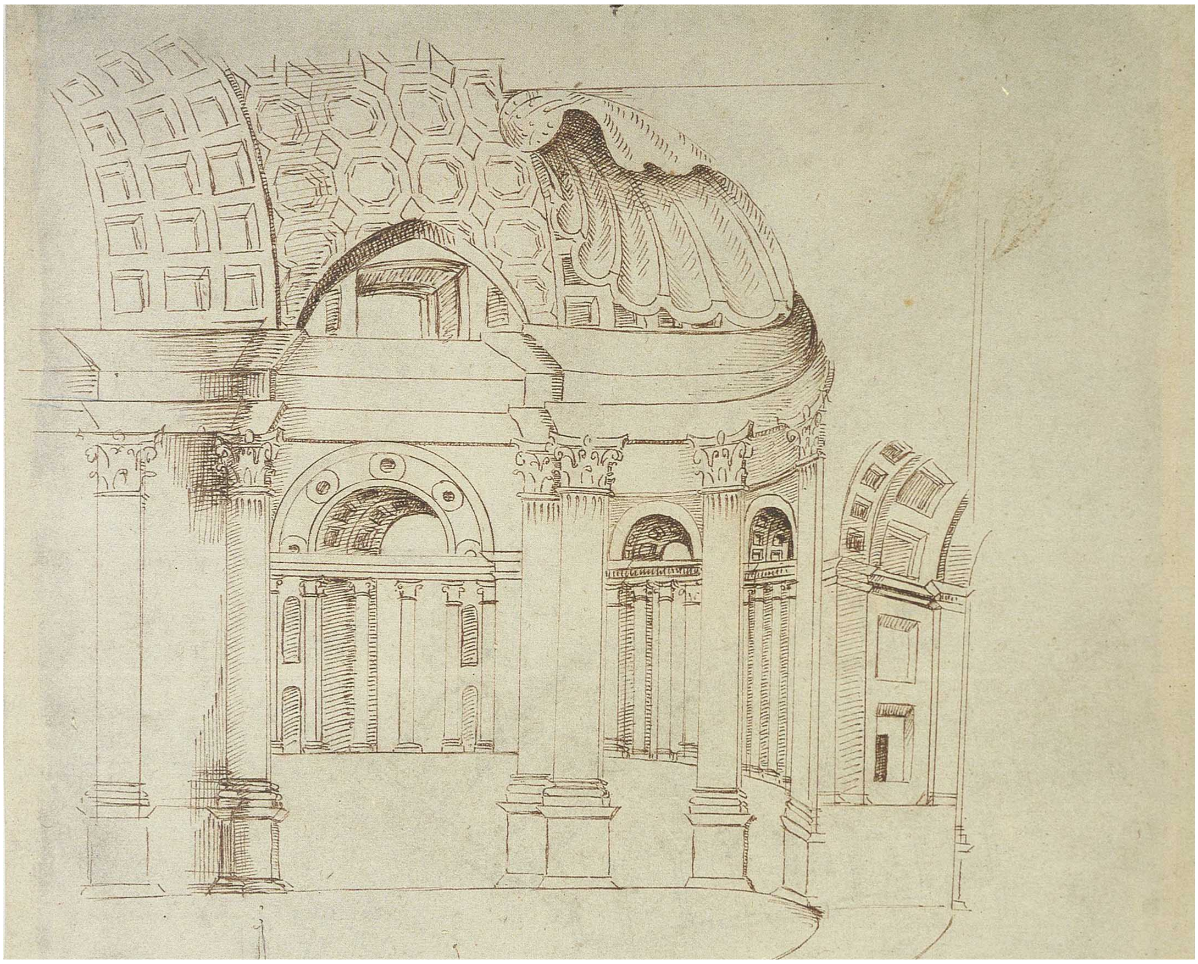
[761] This was also highlighted Wolfang Lotz in his classic study on the rendering of interior in architectural drawings of the Renaissance. Wolfgang Lotz, "Das Raumbild in der Architekturzeichnung der italienischen Renaissance," published in English as "The Rendering of the Interior in Architectural Drawings of the Renaissance," in *Studies in Italian Renaissance Architecture* (Cambridge and London: MIT Press, 1977), 15, 1–65.

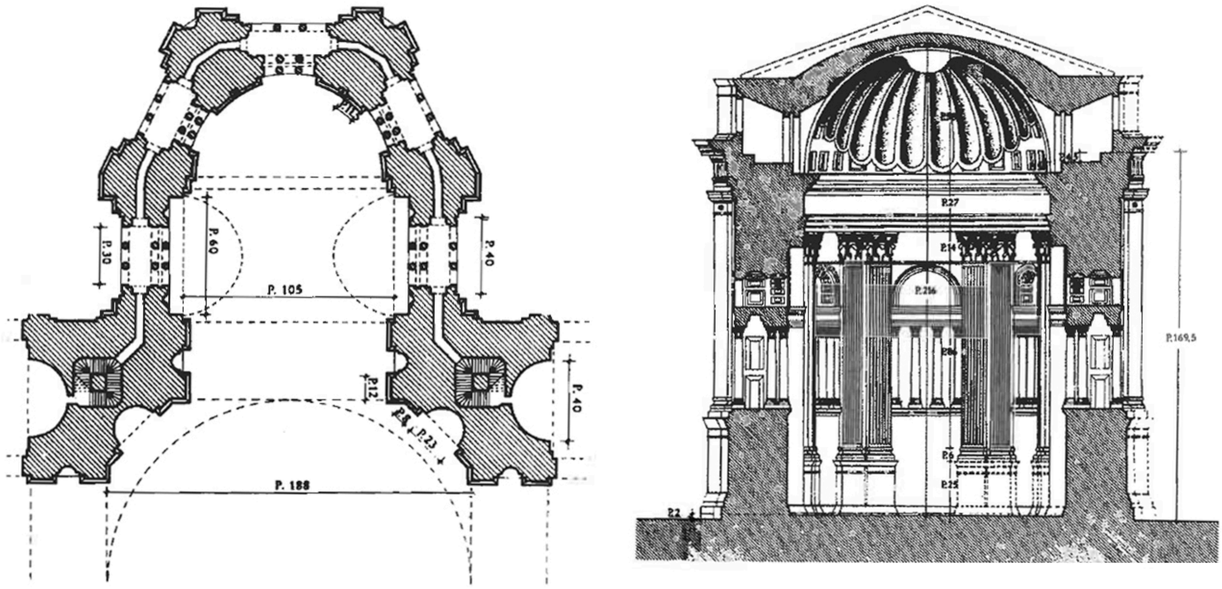
[762] Christoph L. Frommel," Sul metodo progettuale nei disegni di Bramante, Raffaello e Antonio da Sangallo il Giovane per San Pietro," 125.

Bramante continued the single order of the large pilaster of the dome's piers inside the walls of the choir. At the same time, he enlarged the central space (the mausoleum proper) by demolishing part of the old choir's walls. From a clear width of 100 *palmi* taken between the pilasters under the arches supporting the dome at the entrance of the choir, the central space was given a width of 120 *palmi* (illus. 5.29). After the central space where the tomb of Julius II was to be located, the final semicircular apse was instead given a narrower width of about 95 *palmi*, therefore just under that of the entrance arch. The main altar of the basilica was initially to be located under the arch at the entrance of the choir but was later moved under the major dome.[763] Another altar was to be in the apse under a monumental shell that Bramante applied to the quarter sphere termination at the top of the apse to act as the visual



5.28—Donato Bramante (copy from), Study for the Choir of Julius II in the New St Peter's, GDSU 5 A recto. Note the different superficial patterns and heights of the three spatial sections of the choir.

[763] Christoph L. Frommel, "St. Peter's: The Early History," 599.



5.29—Reconstruction (Bruschi) of plan and section-elevation of the Choir of Julius II in the New St Peter's according to the project of Bramante.

end to this part of the building. The shell was arranged to have the centre of radiation for its undulations at its highest point, thus having a similar formal and visual arrangement to the radial ribs of a hemispherical dome.[764]

The larger width of the central space (when compared with the pier's arch at the entrance of the choir), corresponds to an increase in length (and therefore depth) from about 45 (in the space under the piers' arch) to 60 *palmi*. Moreover, the depth increases again to about 65 *palmi* in the central axis of the narrower space in the final apse. This determined a progressive increase in depth for the three spaces, reflecting their distance from the location of most of their observers in the naves of the basilica or under the main dome. Moreover, the three spaces were developed by Bramante in height without any constraints from the old choir's wall. The height at the east side of the apse was about 215 *palmi*, which would have matched the height of the pier's arch. In the perspective-section of GDSU 5A recto (illus. 5.28), the

[764] As in previous works of Bramante such as the Prevedari Print or the Tempietto at San Pietro in Montorio.

central mausoleum space is instead about 1/26 higher (that is, to a total of about 223 *palmi*) than the space of the apse, although this is not found in other drawings.

The same perspectival drawing also shows that Bramante expressed the tripartite division of the choir found in plan and elevation with different cofferings in the barrel vault ceilings of each space (octagonal above the central space and square in the pier arch). In addition, by extending the central space both in width and height when compared to the piers' arch and the final apse, Bramante thus increased the extension and widened the corresponding visual angle of the space where the proper mausoleum was to be placed. In the central space, he also added lunettes above and coffered columnated windows below the level of the entablature, with a width of c. 30 *palmi* (and therefore half the depth of the central space). The same typology of windows (but with a reduced width of about 25 *palmi*) was placed in the last version of the semicircular final niche, interposed by double pilasters. Consequently, the central space where the tomb was to be located would have received intense light from the south from the lunette window above the entablature in the barrel vault, and from the coffered columnated arched window to the side. A lower level of illumination was instead given to the final apse, with three narrower windows arranged in its semicircular perimeter.

At the same time, although the entablature was continuous for the whole perimeter of the choir, thus tying it in with the dome's piers and the other arms of the basilica, it was folded above the central section to follow the widening of the space, with the coffered barrel vault following in the same plane. Finally, in the last phase of the design before construction, Bramante decided to widen the piers and increase the distance of the paired pilasters on its side (as shown in the probably autograph plan in GDSU 43 A recto and in the perspective-elevation GDSU 4 A verso copied from Bramante).[765] A single lower pilaster was instead to be on each side of the columnated coffered windows in the central space, and again paired high pilasters were between the windows of the apse (but with a reduced interaxial distance than in the dome's pier).[766] In other words, the three spaces into which the choir was subdivided by means of both unitary and different widths, lengths, and heights, were progressively more clearly defined in their distinct formal properties by Bramante. Ultimately (similar to the Sacristy of Santa Maria presso San Satiro), the hierarchy established in three-dimensions was reflected by Bramante at different dimensional levels in the definition of the formal and visual properties of the choir.[767]

5.3.3 The correlation of the visual, formal, and spatial properties in the Tempietto at San Pietro in Montorio

This final section considers the formal solutions employed by Bramante in the Tempietto at San Pietro in Montorio in the light of the analyses developed in previous sections of this thesis. The Tempietto was built on what was believed to be the place of the crucifixion of St Peter and is one of the most observed and studied buildings of Renaissance architecture. However, as in previous analyses in this chapter, this section will not provide a comprehensive examination of all the elements found in the building. Nor will it consider symbolic meanings or their possible sources that are often discussed in the literature and which do not provide explanations for Bramante's formal solutions. Rather, it will examine the formal properties of

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765 Frommel," Sul metodo progettuale nei disegni di Bramante, Raffaello e Antonio da Sangallo il Giovane per San Pietro, 124.

766 It should be noted that the single order of large pilaster (corinthian to the inside and doric to the outside) was repeated by Bramante to the outside wall running around the perimeter of the choir.

767 A similar principle can be found applied to the architecture of the near-contemporary choir of Santa Maria del Popolo designed by Bramante. Likewise, the space is arranged by Bramante through the longitudinal addition of spaces with different width, lenght, and height. In drawings, these can only be fully defined through the combined observation of plan and section.

the Tempietto as they are visually perceived, to consider whether correlations between visual, formal and spatial properties can be established.

As confirmed by the latest surveys, visual properties clearly had an important role in the definition of the formal properties of the Tempietto.[768] However, this relationship has never been both comprehensively and analytically examined in the context of Bramante's design process. As in Bramante's other works, so-called 'visual effects' have usually been correlated to single elements in the building rather than to the three-dimensional relationships that these establish. In other words, although the 'visual effects' employed by Bramante are often pointed out, these are not understood either in the context of the design process or the definition of the building's three-dimensional properties, including both the form (geometry of the elements employed) and the spatial extension (the space between them that determines their three-dimensional arrangement). These 'effects' are instead usually considered in isolation. Nonetheless, the amount of data available on the Tempietto allow a closer analysis of Bramante's design process than any other of his works. At the same time, the data analysis (mostly based on the latest published surveys) will inevitably be selective and aimed at clarifying correlations that have not yet been considered, particularly in the context of the theoretical framework outlined in both Bramante's and Pacioli's works.

The Tempietto was placed at a mid-point between the pre-existing wall of the Church of San Pietro in Montorio to the south, and the wall of the cloister of the Franciscan monastery to

[768] The measurements of the Tempietto referenced in this section are mostly derived by the survey coordinated by Manfred Schuller. A summary of the data resulting from this survey was published in Christoph L. Frommel,"Bramante, il Tempietto e il convento di San Pietro in Montorio," *Römisches Jahrbuch der Bibliotheca Hertziana* (München: Hirmer Verlag, 2017), 159-160. Additionally, some elements from this survey are discussed in Manfred Schuller, "Il Tempietto: analisi basata su un nuovo rilievo architettonico." in *Il Tempietto di Bramante nel Monastero di San Pietro in Montorio*, ed. Flavia Cantatore (Rome: Edizioni Quasar, 2017), 225-256.

the north.[769] For most scholars, its design took place during Bramante's early years in Rome (1500-1502). This follows from the discovery of a plate in 1628 which states that the underground walls of the structure originally on the site were excavated in 1502. However, considering the solutions employed by Bramante in the Tempietto itself, as noted by Bruschi and others, this could also be interpreted as referring only to the crypt, with the design and construction of the section above ground taking place between 1503-1504 and 1505,[770] and therefore just before Bramante's involvement in the projects for the New St Peter's Basilica.

Assuming that the circular conformation of the Tempietto was established from the initial phases of the project, when the formal properties of the Tempietto have been considered in the literature, these have been correlated with three main types of possible sources: Antique Roman peripteral temples (Bramante could have seen remains of these in and around Rome), drawings of peripteral temples in the *Trattati* of Francesco di Giorgio Martini (or other contemporary architects such as Giuliano da Sangallo),[771] and lastly the indications on the design of peripteral round temples given by Vitruvius in Book IV of *De architectura*.[772] However, only the text by Vitruvius clearly establishes correlations with some of the dimensions of the Tempietto.[773] As it is well-known, in the sixteenth century, the Tempietto had already been included by Serlio in his *Terzo Libro* (1540) as the only near-contemporary building among ancient examples. It was notably shown by Serlio in plan surrounded by a

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[769] Here a previous *spelunca* marked the spot believed to be the place of the martyrdom. Thoenes, "Bramante a Roma,", 44. Flavia Cantatore, "Bramante e il Tempietto," In *Il Tempietto di Bramante nel Monastero di San Pietro in Montorio*, ed. Flavia Cantatore, (Rome: Edizioni Quasar, 2017), 169.

[770] Most recently in Arnaldo Bruschi, "L'architettura a Roma negli ultimi anni del pontificato di Alessandro VI Borgia (1492-1503) e l'edilizia del primo Cinquecento," in *Storia dell'architettura italiana. Il primo Cinquecento*, ed. Arnaldo Bruschi (Milan: Electa, 2002), 34-75, 58.

[771] To these it can be added a (not surviving) model of an unbuilt circular temple in the Palazzo Ducale of Urbino that could have been designed by Francesco di Giorgio, and which Bramante could have seen. Francesco P. Fiore, "Fece ancora a San Pietro a Montorio di trevertino nel primo chiostro un tempio tondo," in *Il Tempietto di Bramante nel Monastero di San Pietro in Montorio*, ed. Flavia Cantatore, (Rome: Edizioni Quasar, 2017), 8-9.

[772] Vitruvius, *De architectura*, IV. 8. 2-3.

[773] Some of these dimensions have been highlighted in Hubertus Günther, "La ricezione dell'antico nel Tempietto" In *Donato Bramante, Ricerche, Proposte, Riletture*, ed. Francesco P. di Teodoro (Urbino: Accademia Raffaello, 2001), 283-295.

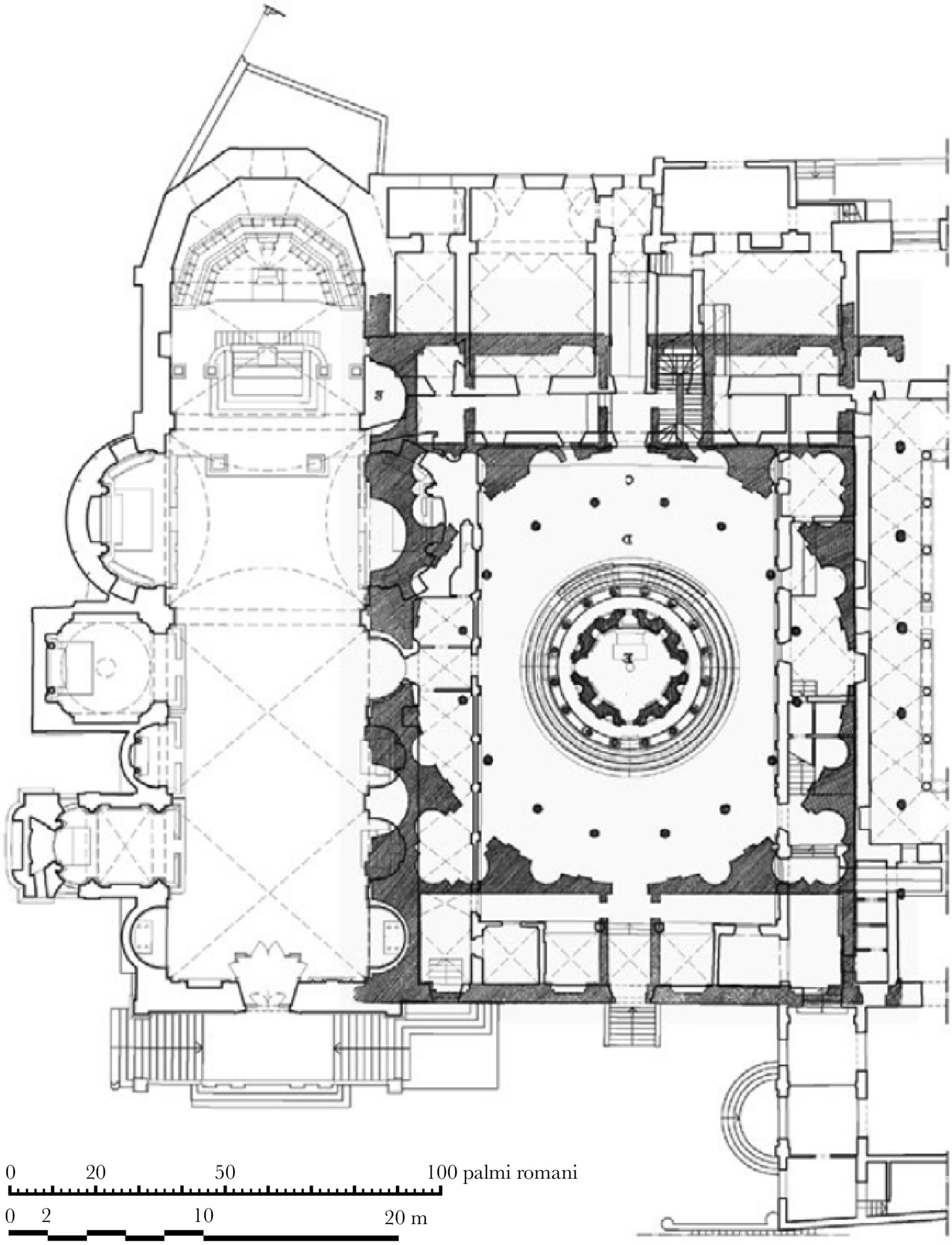
(unbuilt) circular columnated cloister, an arrangement also mentioned by Vasari.[774] Studies of the plan published by Serlio and surveys of the existing structures on site at the time of Bramante's intervention have confirmed that the circular cloister was most likely integral to the design. As first proposed by Hubertus Günther, this was most probably based on an initial square of 100 by 100 *palmi* with the Tempietto at its centre, and the existing walls to the north and south at the opposite sides (illus. 5.30).[775]

The correspondence to this scheme of the dimensions of some elements of the Tempietto in plan confirms the reliability of the plan published by Serlio. According to this reconstruction, from the initial square with a side of 100 *palmi* in plan, Bramante not only set the outer perimeter of the circular cloister, but also the internal diameter of the cella of the Tempietto. If the width and length of the square are divided by five, this in fact gives a very close approximation of the internal diameter of the cella (20.4 *palmi* and 20 *palmi* at the level of the frieze). At the same time, the diameter of other elements such as the external wall of the cella (26 *palmi*), the stylobate's diameter (38.5 *palmi*) or the doric order frieze's diameter above the stylobate (36.25 *palmi*) do not fit into a system of whole numbers subdivisions of the initial square. Only the external diameter of the ring of steps around the stylobate (50 *palmi*) can be again clearly correlated with the initial square in Serlio's plan.[776] Moreover, although alternative units of linear measurement have also been proposed, these still have not provided a coherent system to justify the dimensions and spatial arrangements of all elements of the

[774] Serlio adds that Bramante defined a plan "accordata con l'opera vecchia", and therefore following the pre-existent structure of the Church of San Pietro in Montorio on the one side and the monastery on the other. Serlio, *Terzo Libro*, 41.

[775] This was firstly argued by Günther: Hubertus Günther, "Bramantes Hofprojekt um den Tempietto und seine Darstellung in Serlios Drittem Buch," in *Studi bramanteschi* (Rome: De Luca, 1974), 483-501; and further considered, most recently, by Flavia Cantatore. Flavia Cantatore, "Bramante e il Tempietto," 171-173.

[776] Günther argued that the diameter of the external wall of the cella could be related to the diameter of the existing *spelunca* in the space of what was to be the crypt. Günther, ibid. However, the internal space of the crypt has a width of about 21 palmi (4.70 metres) and the latest investigation of the ground have confirmed a wall thickness of about 2.2 metres without any clear sign of previous structures. Cantatore, "Bramante e il Tempietto," 183; Manfred Schuller, "Il Tempietto: analisi basata su un nuovo rilievo architettonico." In *Il Tempietto di Bramante nel Monastero di San Pietro in Montorio*, ed. Flavia Cantatore, (Rome: Edizioni Quasar, 2017), 225.



5.30—Plan of cloister and Tempietto as shown by Serlio overlaid on the present-day plan of the church and Tempietto of San Pietro in Montorio (survey by F. Cantatore, graphic elaboration by V. Caniglia). The cloister around the Tempietto shown by Serlio has been given a clear diameter (to the perimetric walls) of 100 *palmi*.

Tempietto in plan and elevation.[777]

However, the two dimensions that have a 1/5 and 1/2 ratio to the initial square in plan can be meaningfully linked to dimensions in the elevation of the Tempietto. As often noted, the internal diameter of the cella (c. 20 *palmi*) corresponds to the combined height of the columns/pilasters and the entablature of the peristasis around it. The overall diameter of the ring of steps around the stylobate of the Tempietto (50 *palmi*) corresponded in elevation to the height of the building (excluding the final pinnacle-the *flos* described by Vitruvius), before the replacement of the pinnacle and the raising of the dome's external profile in the early seventeenth century.

Rather than to the Roman examples of peripteral temples that Bramante could have seen,[778] the first of these correlations between plan and elevation has been most convincingly linked to the description of the round peripteral temple given in *De architectura*, which Bramante would

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[777] Mark Wilson Jones, "The Tempietto and the Roots of Coincidence," *Architectural History*, vol. 33 (1990): 1-28. Most recently, these have been considered in Carlo Bianchini, "Quale regola per il Tempietto?" In *Il Tempietto di Bramante nel Monastero di San Pietro in Montorio*, ed. Flavia Cantatore, (Rome: Edizioni Quasar, 2017), 267-274. Bianchini proposed a system of proportioning of elements in plan based on the module (diameter) of the column taken at the at the lower part of the column shaft (*imoscapo)* and on subsequent adjustments of this. Bianchini crucially based his arguments on the fact that this would coincide with the length of a second type of Milanese *piede* which Bramante could have employed. However, at present there is no proof that this unit of measurement was employed in architecture either in Milan or in Rome.

[778] As noted by Günther, the dimensions of the Tempietto are considered, these diverge considerably from Antique Roman examples as the peripteral Temple of the Sibyl at Tivoli or the Temple of Hercules at the Foro Boario in Rome or even the Temple of Venus at Villa Adriana. If compared with the Tempietto, all these temples have a larger diameter of the cella and especially a considerably lower width/height ratio. Moreover, the stylobate has usually one or two steps instead of the three added by Bramante. Lastly, differently from the Tempietto and the solutions found in the coeval drawing of Francesco di Giorgio, none of the known Antique examples has a drum above the level of the peristasis. In the case of the drawings Francesco di Giorgio and other coeval architects, they present some formal solutions that are found in the Tempietto. However, the articulation of elements and the width/height ratio of the cella and peristyle that can be deduced from these drawings are again different from the ones found in the Tempietto. Hubertus Günther, "La ricezione dell'antico nel Tempietto," 271-283.

have been aware of.[779] Vitruvius says that the internal diameter of the cella should be equal to the height of the column in the peristasis. Although the text mentions only the column, this has always been understood as including the entablature in interpretations of the passage from the fifteenth century onwards.[780] Moreover, the passage "in the middle let the proportions of the roof be such that the height of the *tholos*, apart from the terminal, is half the diameter of the whole work"[781] was always interpreted in the Renaissance (and until the eighteenth century) as referring to the height above the peristasis.[782] Therefore, this would determine a drum above this level, and although this was not specified by Vitruvius, the roof was usually concluded by a dome in illustrations of the passage from the Fra Giocondo edition onwards.[783] Nonetheless, in all these examples, the elevation of the dome was always lower than in the Tempietto when considered in relationship to the width of the peristasis. At the same time, the internal width/height ratio that Bramante employed in the cella is instead very similar to the one found in the Sacristy of Santa Maria presso San Satiro. Notably, Vitruvius sets the external rather than the internal diameter of the cella, stating that this

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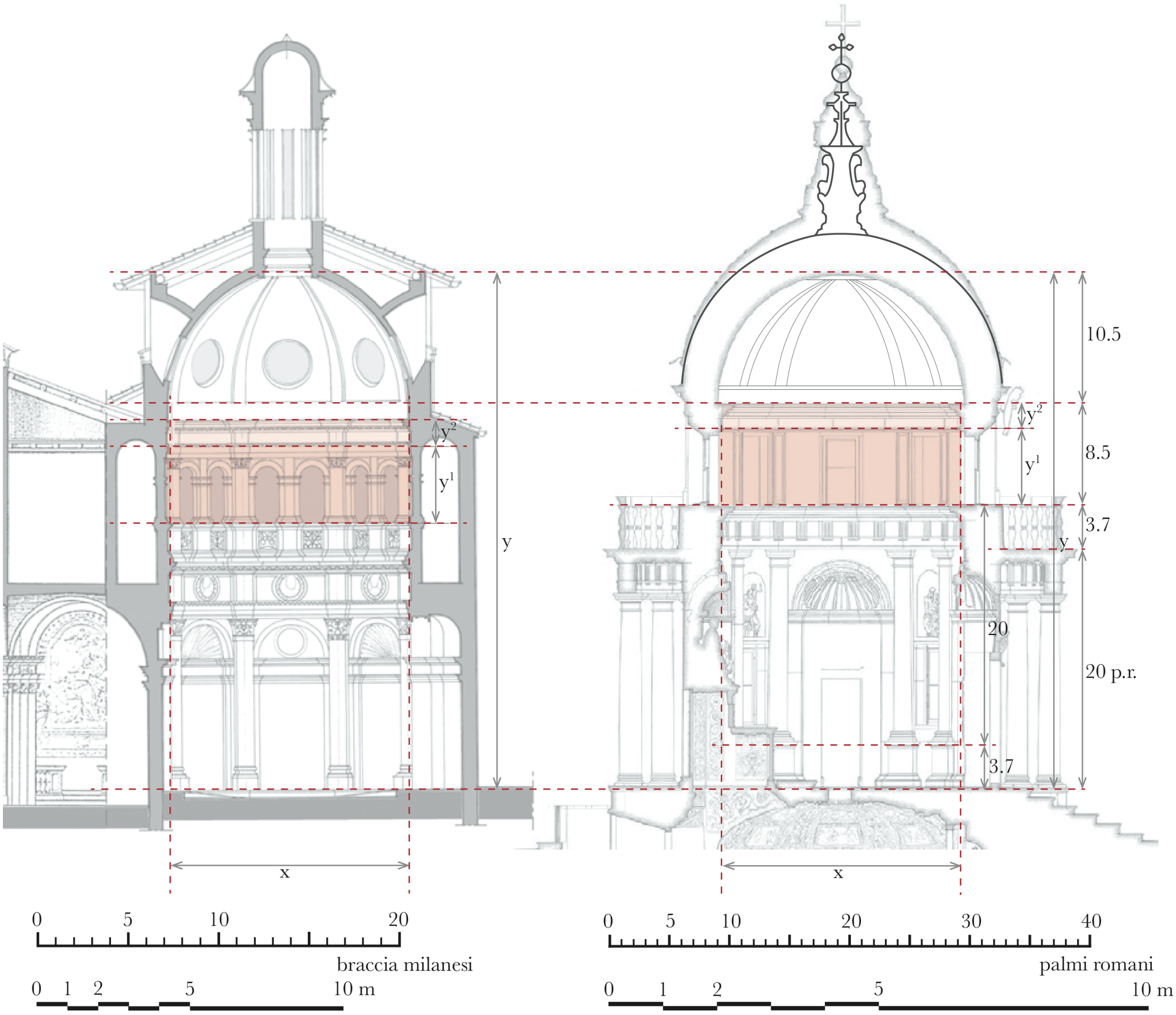
[779] "[…] if the temple is peripteral, two steps and the stylobate are to be built from the foundation; then the wall of the cella is to be built set back from the edge of the stylobate about 1/5 of the width. In the middle is to be left an opening with folding-doors for the approach. The cella within the walls and colonnade, is to have a diameter equal to the height of the column. On the stylobate, let columns be disposed round the cella and of the same proportions. In the middle let the proportions of the roof be such that the height of the dome, apart from the terminal, is half the diameter of the whole work. Let the terminal have the magnitude of the capital of the column in addition to the pyramid (on which the flower rests). The other parts are to be constructed of the proportions and symmetries as is above described." Vitruvius, *De architectura*, IV. 8. 2-3. The most recent published version of *De architectura* in the early years of the sixteenth-century was the Florentine edition (1496) based on the editio princeps published some years before (c. 1490) in Rome. This is the version that Pacioli employed in his *Tractato de l'architetura*:"Sin autem peripteros ea aedis constitietur, duo gradus et stilobatata ab imo constituantur. Deinde cellae paries collocetur cum recessu eius a stilobata circa partem latitudinis quintam medioque valvarum locus ad aditus relinquatur. Eaque cella tantam habeat diametrum praeter parietes at circuitionem quantam altitudinem columna supra stylobatam, colomne circum cellam iisdem proportionibus symmetriisque disponantur in medio tecti ratio ita habeatur, uti quanta diametros totius operis erit futura, dimidia altitudo fiat tholi praeter florem, flos autem tantam habeat magnitudinem quantam habuerit in colomne capitulum praeter piramides. Reliqua ubi supra scripta sunt ea proportionibus atque symmetriss facienda videtur." Vitruvius, *De architectura*, IV. 8. 2-3; L. Vitruvii Polloniis, *de architectura libri decem* (Florence, 1496), Liber Quartus (erroneously marked in the recto as 'Liber Quintum'), Caput Septimum.

[780] Günther, "La ricezione dell'antico nel Tempietto," 285.

[781] Vitruvius, *De architectura*, IV. 8. 3. Translation amended.

[782] Günther, "La ricezione dell'antico nel Tempietto," 291-292.

[783] Indeed, as noted by Günther, the dome understood in this way had a dimensional relationship similar to the ones found in Santa Costanza in Rome or in similar examples that in the Renaissance were considered of Roman origin. Günther, ibid.



5.31—Comparison between the section/elevation of the Sacristy of Santa Maria presso San Satiro and Tempietto at San Pietro in Montorio. When the diameter in plan is equated, the internal height is also equal. The width/height ratio of elements such as the gallery and architrave are also comparable. These elements have been shifted to a higher point in the Tempietto.

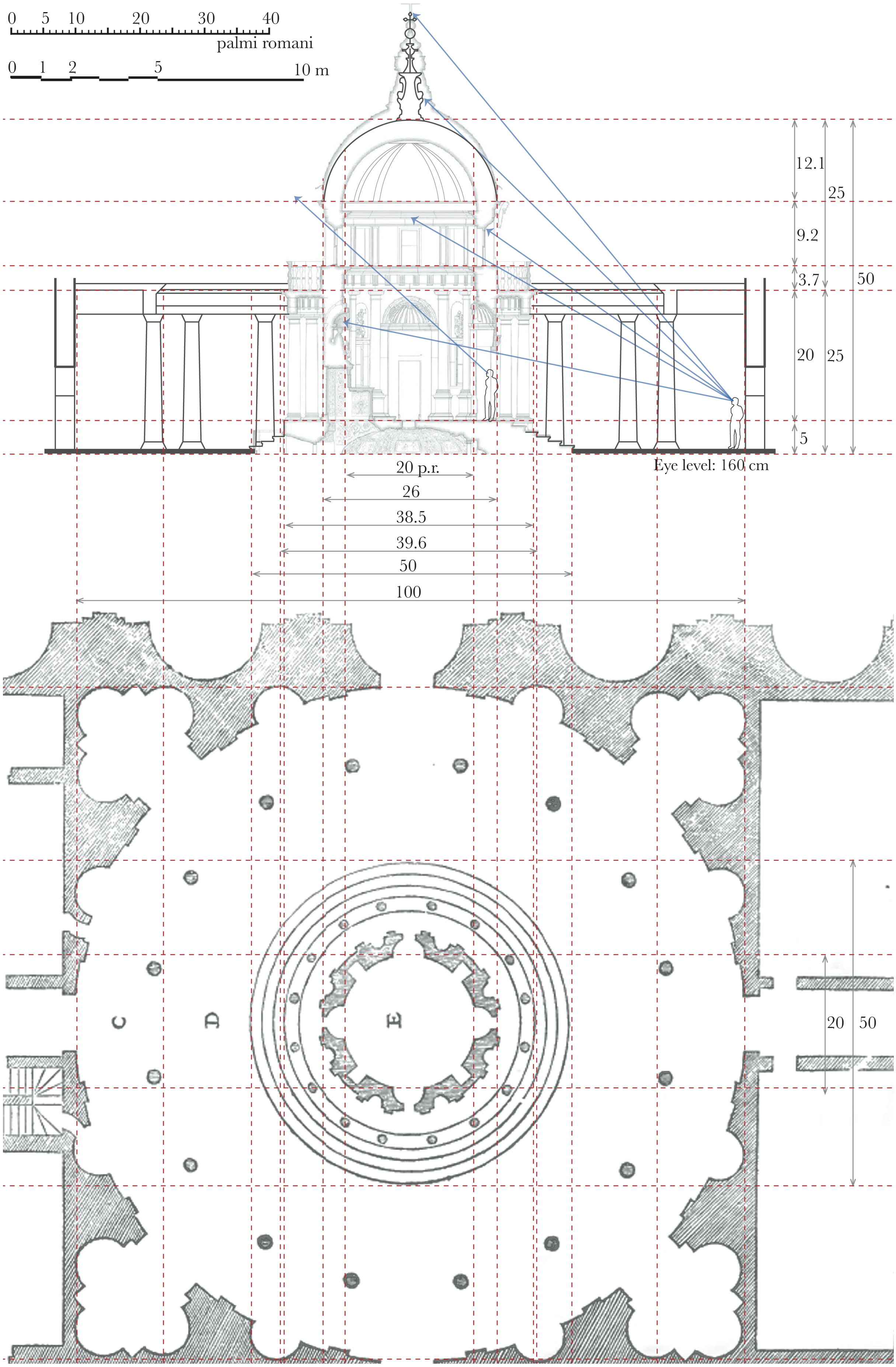
should be equal to 3/5 of the stylobate.[784] Given that this dimension is 26 *palmi* in the Tempietto, if Bramante had followed the proportions indicated by Vitruvius, the diameter of the stylobate would have been 43.33 *palmi* instead of 38.5 (that is approximately 4.83 palmi-107.9 centimetres larger). An interpretation aiming to justify the slender width/height ratio of the Tempietto argued that Bramante increased the internal height because he took perspectival effects into consideration, as he had done in the Sacristy of Santa Maria presso San Satiro.[785] However, it was not noted that the width/height ratio in the cella is about the

[784] Vitruvius, *De architectura*, IV. 8. 2.

[785] Hubertus Günther, "Gli elementi provenienti dall'Italia settentrionale nell'architettura di Bramante a Roma," *Arte Lombarda*, no. 176/177, vol. 1/2 (2016): 148.

same as the ratio of the sacristy. Moreover, a visual comparison between a section of the Tempietto and a section of the sacristy allows additional formal considerations (illus. 5.31). The width/height ratio of the area of the drum with four windows and the entablature above in the Tempietto is comparable to the area with *bifore* openings (and windows behind them) and its entablature in the sacristy. If compared with its position in the sacristy, however, this has been shifted upwards in the Tempietto to tie in with the levels to the outside of the building. Other common formal solutions are the employments of pilasters at all the levels of the building and of semicircular niches at the ground level. At the same time, these considerations indicate that the space was firstly visualised by Bramante simultaneously in plan and elevation/section (and thus in three-dimensions by means of two-dimensional images), before the dimensions of its diameter and height were established. Concurrently, Bramante most probably began by considering the relationships of elements conjointly in plan and elevation/section before establishing the position and therefore the diameter of the surrounding peristasis.

Nonetheless, the correspondences discussed so far do not help in determining the width of the peristasis around the external wall of the cella, nor the external diameter of the cella itself (a dimension also linked to the depth of the niches in the cella's wall). Both these dimensions, as noted, differ from the ratios given by Vitruvius, while at the same time, they do not correspond with the modular grid derived from the plan published by Serlio. In short, Bramante must therefore have arrived at the definition of the three-dimensional formal properties of at least this part of the building by other means and adjusted the dimensions of other spaces in correlation with this. The analysis of this logical passage may thus help to clarify the formal arrangement of other elements in the building.



5.32—Reconstruction of the section of the Tempietto showing the correlation between formal properties and visual lines. Visual point from the perimeter of circular columnated cloister as devised by Bramante according to the plan published by Sebastiano Serlio.

The diameter of the peristasis is a crucial dimension in this regard and we should first concentrate on this. The arrangement of elements in this part of the Tempietto indicates that Bramante defined this variable by means of triangulations, that is, by means of the visual angle experienced from the perimeter of the proposed cloister. In this way, he would have been able to mutually relate the visual and formal properties of the cella and the peristyle (illus. 5.32). From the perimeter of the reconstructed cloister (with an eye level at about 160 centimetres above ground), the position of the cornice of the peristasis is in fact set accurately to cover the lower section of the drum. This section therefore would not have been visible from any other part of the ground level of either the cloister or the Tempietto. Accordingly, Bramante left this part of the drum without articulation and positioned the fascia with semicircular niches and windows above it. From the same visual point, the balustrade above the peristyle would be perceived with the same height as the windows in the drum. Moreover, perceived from this point, the profile of the original hemispherical dome would have covered the unarticulated lower part of the final pinnacle and shown the tapered volutes under the final pyramid.[786] Lastly, again from the same visual point, the hemispherical niches and the statues inside the building, are visually perceived under the architrave of the three doors in the wall of the cella. The positions determined by the system of triangulation applied in section to the external elements of the building correspond to the arrangement of elements to the inside. The visual alignment of the cornice of the peristyle with the lower side of the windows of the drum corresponds to the cornice of the internal doric order. This has the same height (with an average difference of about 1 centimetre) as the orders in the peristasis.[787] The height of the pedestals on which the pilasters are raised on the inside therefore corresponds to the unarticulated lower fascia of the drum to the outside.

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[786] Hubertus Günther, "La ricezione dell'antico nel Tempietto," 269.
[787] Manfred Schuller, "Il Tempietto: analisi basata su un nuovo rilievo architettonico." 240.

As noted, although it is often generically mentioned that Bramante employed ‘visual effects’, their employment in the design process to define the formal and spatial properties of the building in three-dimension (that is, simultaneously reflected in plan and section-elevation) has not been considered. This process allows the three-dimensional relationships of each part of the building to be kept constant, thus establishing visual correlations between them. The circular conformation of the cloister (with radial vertical sections passing through the centre of the Tempietto) is particularly suitable for this process. The visual angle and its relationship with the main elements of the building is constant for all points lying on the same circumference around the building. The spatial articulation of the Tempietto and the surrounding circular cloister are thus clearly best appreciated in section.[788]

At the same time, the articulation of the orders is uniformly simplified when these are to be visually perceived from a greater distance, which facilitates the visual analysis of the three-dimensional spatial elements in the building. The pilasters have no bases or capitals at the level of the drum both to the outside and the inside, while the entablature is simplified. As in other buildings designed by Bramante, the arrangement and definition of the orders is coordinated to a hierarchy of visual, formal, and spatial properties that concur to the articulation of the building in three-dimensions. As often noted, the Tempietto was the first building displaying a doric order with a frieze of triglyphs and metopes after ancient precedents. The choice of the doric by Bramante is often explained (recurring to Vitruvius) with the *severo more* associated to the temple (as the place of St Peter’s martyrdom).[789] However, the choice of the doric with a frieze also holds inevitable visual and formal consequences. The triglyphs allow (doubtless after elaborated and precise arithmetical calculations) a uniform and continuous visual articulation of the circular friezes. Triglyphs

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788 As noted, Serlio published (and linked to original drawings of Bramante) the first set of plan, section/elevation of the dome of St Peter’s at the same scale.
789 Vitruvius, *De architectura*, I. 2. 5.

and metopes are employed four times in concentrical friezes in the building (two on both sides of the peristyle, one in the external and one in the internal wall of the cella) and their dimensions are adjusted by Bramante so that they always have a triglyph above a column or pilaster (as indicated by Vitruvius), while their spacing being visually experienced as regular.[790] Interestingly, the latest surveys have confirmed that this is one of the areas of the building constructed with the highest degree of precision. All the circular perimeters in the building approximate the circumference with a maximum deviation from the ideal radius of less than 2 centimetres.[791] Clearly Bramante made use of precise calculations of the length of the arcs of circumference covered by each stone before the design and construction of these elements. Similarly, a high level of precision has been found in the geometry of the hemispherical dome that stands on a high drum.[792] A similar precision was also probably to be found in its external hemispherical side before the raising of its profile in the early seventeenth century.

The circular arrangement and geometry of the peristyle should also be carefully considered. Vitruvius notably recommended a ratio of 1:7 for the diameter/height of a doric column.[793] The columns in the Tempietto, however, have a ratio of 1:8.56. The diameter of the column at the lower edge of the shaft in the Tempietto is about 1.8 *palmi* (averagely 39.7 centimetres) and holds no clear relationship with the diameter of the cella or the stylobate. Simultaneously, the intercolumniation is three times the diameter of the column (averagely 5.4 *palmi*—120

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[790] In the case of the frieze above the so-called triumphal rhythm of pilaster to the interior wall of the cella Bramante made use of two different width of metopes. A shorter width (29 centimetres) between each couple of pilasters and a larger (34.5 centimetres) on the other sections. However, the difference was firstly pointed out in a measured survey but never (as far as known) in the direct observation of the frieze. Schuller, "Il Tempietto," 241. Similarly, the supports of the balustrade running above the peristasis has been arranged as to have always a support above a column. At the same time, there are always three supports between each two columns.

[791] As noted by Schuller, the radius of the cornices and walls of cella taken on the internal sides do not diverge more than 2 centimetres from the ideal shape of the circle. Only at the level of the drum the cylinder has a deviation of 5 centimetres from this, however, as noted by Schuller, this is not experienced by the observer. ibid., 238.

[792] As noted by Schuller, the profile of the internal side of the dome diverges only about 3 cm from the ideal semicircle. ibid.

[793] Vitruvius, *De Architectura*, IV. 3. 3.

centimetres) as in the *dyastilos* indicated by Vitruvius,[794] and it is reduced to just above 2:1 (close to Vitruvius's *sistylos*) among the corresponding pilasters in the exterior wall of the cella. In the capitals and bases (both with a height of 1.7 *palmi* or 1.2 modules) and in the entablature (with a height of 3.6 *palmi* or 4.2 modules composed of an architrave of 1.2 modules, a frieze of 1.6 modules and a cornice of 1.5 modules), Bramante again differs from Vitruvius' indications.[795] The diameters of the columns have been explained in two main ways. The first is that the columns may have been *spolia* and therefore limited interventions were possible on their dimensions.[796] The second is that since the same material (grey granite) was used in the near-contemporary Palace of the Cancelleria (in which Bramante was probably involved), these ratios may have been a deliberate choice by Bramante to enhance the visual slenderness and transparency of the building.[797] Nonetheless, as noted above, in both cases the width/height ratio of the column and the height of the entablature could have been determined by Bramante once the position of the cornice and the diameter of the peristyle had been decided by means of triangulation from precise visual points, and then mathematically determined to give precedence to the intercolumniation.[798]

Overall, as in other buildings, some of the dimensions found correspond to whole numbers (in the unit of measurement Bramante most probably uses) and in this case, to Vitruvius' indications. Others, as we have seen, are instead linked to the consideration of visual angles in

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794 ibid., III. 3. 6.

795 For the doric order, Vitruvius gives one module (equal the radius of the column) to the height of the capital, one to the height of the architrave, one and a half to the frieze and half a module to the corona and cymatium of the cornice. To the corona and cymatium is to be added the cyma as for the ionic order. Vitruvius, *De architectura*, IV. 3. 3-6 Even if the total height is not stated by Vitruvius, with the addition of the cyma the cornice's total height would reach circa one and one/tenth modules. Moreover, it should be noted that in the editio princeps (c. 1486) and Venetian editions (1496 and 1497), the height of the frieze is indicated as half a module instead of one and a half. This dimension was rectified only in the edition of Fra Giocondo da Verona, printed in 1511.

796 Thoenes, "Bramante a Roma," 45.

797 ibid.; Frommel, "Bramante, il Tempietto e il convento di San Pietro in Montorio," 134.

798 This mathematical process would indeed be similar to what Bramante has done in other works, such as in the ramp of the Belvedere or in the cloister of Santa Maria della Pace, as reconstructed by Christof Thoenes. Thoenes, "Bramante a Roma," 43-44, 46-47.

and around the building. In all cases, however, the design process establishes spatial relationships in three-dimensions, that is, simultaneously in plan and section/elevation. In the Tempietto, the analysis of the section and plan shows that there are a number of dimensional correspondences among the elements on the outside and the inside of the building. While they have equal diameters and their position can be derived from the internal diameter of the cella, the width of the sixteen columns of the peristyle and the sixteenth half-pilaster on the exterior wall of the cella are also the same. As noted, the same height of pilasters is again found in the internal wall of the cella, while the height of the interior and exterior order of the drum also corresponds. Lastly, as noted, the internal and external profile of the dome were originally both hemispherical with semicircular radial flat ribs on their surface.

In the context of contemporary theories in the philosophy and psychology of perception, it can be argued that Bramante employs the principles of linear perspective to define the formal and spatial properties of the building in three-dimensions, while simultaneously employing established (regular) three-dimensional quantitative relationships and ‘constraints’ in some parts of the building. Similarly to what happens with regular geometrical bodies by means of the arrangement of two-dimensional bases, lines and angles (which psychologists would generically call ‘constraints’ in this case), additional three-dimensional visual information is added to the Tempietto by elements such as the semicircular ribs of the dome, the columns and pilasters arranged on the side of all concentrical spaces or the triglyphs on the doric friezes above these. In other words, Bramante employs the geometrical regularity of elements as constraints that facilitate the three-dimensional (distal) visualisation of a geometrical body (composite of boundary and volume) from perspectival (proximal) stimuli. At the same time, (with the exception of entasis to columns and pilasters) he does not apply any of the so-called ‘optical corrections’, as described by Vitruvius. It can thus be argued that Bramante believed (similar to what happens when looking at pictures painted in perspective) that these elements could be visually experienced and mentally rotated in their true (proper) shape.[799] At the same time, it should be noted here that according to Pacioli, the visual perception of regular geometrical bodies in perspectival representations supports the (distal) mental visualisation of formal and spatial properties not (proximally) shown in the drawing. As noted, Pacioli describes this judgment as based on universal mathematical properties and taking place at the level of the intellect. At the same time, he links this stage with the architect's *ingegno* and therefore the process of design and the capacity for invention.

Ultimately, this process facilitates the mental visualisation of the three-dimensional geometrical arrangement of the building based on its perspectival (proximal) perception. Indeed, already in the sixteenth century, Serlio argued that despite the building's height being more than double its width, "[…] nonetheless on site, by means of its windows and niches where the vision can expand, such a height is not displeasing; on the contrary, by means of the double cornices that encircle the interior, […] the temple proves to be much lower to its viewers".[800] As noted, recent surveys have shown a high level of precision in the quantification and construction of the circular and hemispherical elements of the Tempietto.[801] As seen in other buildings designed by Bramante in Milan and Rome, it is by mathematical considerations based on the mastery of the principles of perspective and their application to the process of design in three-dimensions that the visual, formal and spatial properties of the Tempietto are coherently coordinated.

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[799] A similar point (although not explicitly referred to Bramante) was argued by Ernst H. Gombrich when considering the principles of linear perspective. Additionally, as noted by Gombrich, so-called visual 'distortions' apply to the phenomenal world as well as to pictures painted in perspective as the latter are still part of the phenomenal world. Ernst H. Gombrich, "The 'What' and the 'How': Perspective Representation and the Phenomenal World," in *Logic and Art: Essays in Honor of Nelson Goodman*, ed. R Rudner and I Scheffler (Indianapolis: Bobbs-Merrill, 1972), 129-149.

[800] "nondimeno in opera, per le aperture delle finestre e dei nicchi che vi sono, onde la vista si viene a dilatare, tale altezza non offende, anzi per le duplicate cornici le quali girano intorno, che rubano assai all'altezza, il tempio si dimostra assai più basso ai riguardanti, ch'egli non è in effetto" Serlio, *Terzo Libro*, Book III, 44.

[801] Schuller, "Il Tempietto," 238.

Conclusion

This chapter has highlighted a number of aspects among buildings designed by Bramante and the drawings of central-plan buildings drafted by Leonardo, that can be meaningfully linked to their respective processes of design.

The first and foremost aspect (on which all the others depend to a certain degree), is the employment of the principles of linear perspective in the three-dimensional definition of the spatial properties of architecture. Rather than in the context of optical corrections applied to singular architectural elements, the principles of perspective were employed by Bramante (and in some instances Leonardo) as a design tool to define the three-dimensional articulation and visual perception of parts of the buildings, understood as a composite of architectural boundaries and matter (spatial extensions). This approach essentially coheres with the theoretical framework highlighted by Pacioli, particularly in the *Compendium* and *Tractato de l'architectura*. As noted, the process of the mathematical quantification of regular and other bodies derived from these is clearly expressed in this theoretical context, and is at once mathematical, visual, and experiential. Similarly, where reconstruction has been possible, the process of quantification and visualisation in three-dimensions, in correlation with considering the visual points from which spaces will be perceived, is a crucial element in Bramante's design process for the considered buildings.

Moreover, that Pacioli's arguments are correlated with Bramante's design approach is further supported by another aspect, namely the hierarchical correlation of formal elements established in Bramante's works. As noted, the proportioning and articulation of architectural elements such the orders and ornaments are defined according to the building's three-dimensional spatial organisation. In other words, the orders are applied to the building (or part of it) only when its main proportions have been defined in three-dimensions. As noted in the cases examined above, differentiating visually the main three-dimensional elements of a building in turn facilitates their reading as part of a defined three-dimensional hierarchical system.

The last aspect, correlated with the latter, is Bramante's (and in some cases Leonardo's) employment on several occasions of parts of geometrical bodies, in the definition of the three-dimensional formal properties of a building, without recurring to optical corrections. Among these, Pacioli identified portions of a 72-bases body in buildings in Milan and related the same body to the interior of the dome of the Pantheon in Rome, among other buildings. As noted, Bramante shows a preference in his designs for the employment of three-dimensional bodies such as hemispheres, or bodies closely geometrically related to these. Although this choice is often referred to as influenced by sources such as the dome of the Pantheon, this shows in reality only a hemispherical dome to the inside, while the dome's profile is compressed and thus visually suppressed to the outside.[802] At the same time, Bramante added formal and visual properties to these through the articulation of the orders, the chiaroscuro achieved by relief and chromatic effects that facilitate their visual analysis and three-dimensional reconstruction. These aspects can be visually correlated to the properties of the perspectival drawings of geometrical bodies, most likely based on a set of models, drafted by Leonardo da Vinci for Pacioli's *Compendium*.

Overall, all the highlighted aspects have inevitable consequences when considered in the context of the design process that Bramante could have employed in the building examined

[802] By means of the considerable reduction in thickness from the sides to the central oculus.

above. They imply that the process of three-dimensional visual imagination and quantification was central in all the phases. Moreover, they suggest that the dimensional definition and disposition of other architectural elements such as the orders were established in correlation with these, and to visually correspond with the same spatial three-dimensional hierarchy. As noted, the spatiality and visual properties of Bramante's buildings have been often pointed out by scholars. What has been missing, however, are clear correlations of these properties with the formal solutions employed in the context of his design process. The aspects and processes found in Bramante's works and highlighted above contribute to this area of the research. At the same time, although this chapter has only concerned historic examples, all these aspects are still central to the formal properties of the present theory and practice of architecture.

Ultimately, given the extent and articulation of the subject, it is inevitable that further studies are needed to comprehensively unpack and develop an empirical theory of design based on Bramante's masterpieces. However, as we have begun to see, Bramante's visual approach to architectural design undoubtedly holds many possible contributions to the present-day visuality in architecture.

Conclusion

Historically, a major limitation in studies of Pacioli's works dealing with three-dimensional geometrical bodies has been a failure to approach them as a series of coherently structured arguments in their philosophical and theoretical contexts. Although some underlying principles have been highlighted, single passages have usually been considered in isolation and often singled out in collective readings of fifteenth-century culture. In other words, Pacioli's arguments on geometrical bodies have never been approached as a coherent system and in their own right. At the same time, the correlation between theory and practice that Pacioli aimed to establish has been largely overlooked, particularly among architectural historians. However, Pacioli was primarily an educator, and the correlation of theory and practice was paramount for him. As seen, he addressed his works not only to learned patrons in the courtly environments in which he operated, but also to a wide range of practitioners who could not access scholarly works that were previously mostly written in Latin. The redressing of these gaps has required a revision of Pacioli's arguments in their philosophical, theoretical and practical contexts.

As a prominent teacher and scholar of mathematics, we have seen that Pacioli was well aware of the foundational role that mathematical knowledge held in accounting, surveying, and in practical applications in crafts and architecture in the late fifteenth century. It is often noted that this was correlated to a trend linked to the teachings of schools of commercial arithmetic and practical geometry, and the manuals that originated from these that were then circulated. Pacioli expresses these assumptions recurrently throughout his writings. It is precisely in the context of the relationship between theory and practice that Pacioli argued for the correlation of the theory of perspective and the process of visual perception and the experience of the material world, by means of the universal role of mathematical properties.

These assumptions clarify Pacioli's unprecedented presentation of a set of physical models of regular and other bodies derived from these to accompany the *Compendium de divina proportione* in the environment of the Sforza court in Milan, and their perspectival representation in a set of drawings drafted by Leonardo da Vinci and derived from their direct visual experience. In Pacioli's view, all geometrical bodies partook of the same mathematical, visual and spatial properties shared by all material objects. As such, they were instantiated in a set of models, while objects of common use or architectural elements could have the same formal, visual, and spatial properties of regular or other bodies derived from these. In Pacioli's argument, this followed from the process of the quantification of visual perception and the relationship of form (shape) and matter (spatial extension) in regular and other bodies derived from these, as defined by mathematical properties.

In order to consider Pacioli's arguments on geometrical bodies, it has required taking into account his wider philosophical context, namely the late fifteenth-century Aristotelianism he first most likely encountered in Venice. As shown, Pacioli's recourse to the texts of Plato (and among them almost exclusively to the *Timaeus*) was above all motivated by his interest in regular bodies as exemplary geometrical forms. In other words, rather than by the symbolic or mystical associations (often pointed out by scholars in recent decades) towards which he shows a correlated but subordinate interest, Pacioli was first and foremost motivated by the unique formal and geometrical properties of the regular bodies, as described in the last book of Euclid's *Elements* and in the writings of Piero della Francesca. As seen, however, Pacioli was able by means of formal causes (through the universality of mathematical properties and specifically, proportions) to correlate the five regular bodies (associated by Plato with the four basic natural elements and the fifth essence or celestial matter) as creations of the Highest Maker, to the exemplary material forms to be applied in any act of human making.

The relationship between the natural order of creation and mathematical properties is central to all the texts by Pacioli examined above. At the same time, by resorting to Aristotles's arguments and Campanus' commentary on Euclid's *Elements,* Pacioli was already able to set out a theory of quantity with proportions at its core in the *Summa.* Consistent with his Aristotelian context, quantity was also presented as acting at the level of the body, and the argument was mathematically and theologically framed by placing the unifying character of proportions as the universal which underlies the form of each regular body. It was thus possible for Pacioli to overcome the difficulties of the Aristotelian and Scholastic distinction between substantial (essential) and artificial forms by referring to the linking of natural and regular geometrical bodies to their forms *qua* extended magnitude and therefore, quantity.

Consequently, it was possible for Pacioli to argue that the forms of regular bodies rest completely on their proportional mathematical orders rather than the matter in which they are instantiated. *Forma* was thus conceived as the three-dimensional arrangement of a body to which a visible two-dimensional image or *figura* corresponds. It is important to underline that when Pacioli discussed the *forma* of regular bodies, this comprised all the elements that concur with the shape of a three-dimensional body, and not only those that may be visible from a single viewpoint. Nonetheless, as Pacioli made clear, visual, formal, and shape-defining properties are all necessarily interrelated.

By the same token, Pacioli was able to link the visual perception of apparent quantities in perspective to the measurement of real quantities, arguing for their mathematical correlation based on mathematical properties. Essentially, in Pacioli's argument, this led to the unification of the processes of quantification and the visualisation of the material world in three-dimensions. Where geometrical bodies are concerned, this means that their formal, visual, and spatial properties can be linked across the same homogenous mathematical and material system. In other words, different from Piero della Francesca'a purely mathematical works, the process of the quantification of regular and other bodies derived from these in Pacioli's *Compendium* became at once mathematical, visual, and experiential. From this point of view, one can see that starting from the *Summa* and more explicitly in the *Compendium*, Pacioli aimed to correlate and philosophically frame his theoretical and practical arguments concerning geometrical bodies, involving their visual, formal, and spatial properties.

The perspectival representation of geometrical bodies or other bodies derived from these in the *Compendium* supports, according to Pacioli, the process of the mental visualisation of their three-dimensional formal properties that are not visible in a single glance or in a single perspectival drawing. As shown, in keeping with the stages of human visuality discussed by Aristotle in *De anima*, Pacioli describes this additional stage as taking place at the level of the intellect. Nonetheless, Pacioli did not differentiate between the agent and potential intellect, neither did he follow Aristotle's theory of light. Instead, he made it clear that this stage refers to the visual cognition of the whole form (shape) of a geometrical body and that this is mathematically defined. These claims should be thus correlated to Pacioli's arguments on the universality of aesthetic judgements based on visual and formal properties, therefore involving the application of three-dimensional shapes to architecture. Pacioli clarified in the *Summa* that aesthetic judgements take place both at the level of the lower parts of the soul and at the higher level of the intellect, and that these concern visual properties independent of what can be verbally conveyed.

Overall, the process of the quantification and visualisation described by Pacioli is clearly linked to both Alberti's programme of the quantification of spatial relationships and visual experience and his claims about the universality of aesthetic judgements based on the geometry of lines and angles. Indeed, besides knowing his works well, Pacioli claimed to have spent several months in Rome in the house of the elderly Alberti. At a basic level, both Pacioli and Alberti consider geometrical principles to be at the core of visual and aesthetic experience and rely on lines and angles as the basic geometrical entities. In *De pictura,* Alberti clearly argued that beauty results purely from how surfaces are arranged in a body, and that surfaces rest on lines, as the geometry that determines human vision.[803] At the same time, however, although Alberti was particularly aware of the geometrical relationships established between the position of the observer and the lineaments of a building, he did not exploit these relationships in his accounts of architecture's design process, nor did he involve the concept of the geometrical body. In *De re aedificatoria*, Alberti referred to proportions expressed by linear quantities to be employed in the most convenient way: in pairs to define the width and length of architectural elements proportionately to the *area* in which they are placed, or in groups of three (i.e., width, length, and height), but never as a pre-established three-dimensional shape or to be perceived as such from a given visual angle.

We have seen how Pacioli, by privileging the role of three-dimensional geometrical bodies, somewhat expanded on the assumptions and the programme of the quantification of visual experience and spatial relationships developed in Alberti's works. Pacioli pointed to the

803 Alberti, *De pictura*, 2.35, 1.2. These passages have been discussed in the context of the theoretical framework of Alberti by Branko Mitrović. Mitrović, "Leon Battista Alberti, Mental Rotation, and the Origins of Three-Dimensional Computer Modeling," 312-322.

material instantiation of regular and other bodies derived from these, where proportional relationships among components are established in three-dimensions. He then correlated the employment of geometrical bodies with the architect's *ingegno,* and therefore to the process of design and the capacity for invention. At the same time, he linked these to the disposition (i.e., the geometrical arrangement) of the places where they will be employed. In brief, although the process of the abstraction and visual imagination of three-dimensional shapes (by means of two-dimensional images) is essentially coherent with Alberti's at a basic level, Pacioli differed by explicitly considering the three-dimensional visual stage as integral to the process of architectural design, that is, to the process of defining architecture's visual, formal and spatial properties through the employment of geometrical bodies.

In Pacioli's view, geometrical bodies are the most complete geometrical and material expression of the same mathematical, visual, and spatial properties shared by all objects. As such, they can be applied to architecture (including architecture's spatial extension, i.e., architecture's matter and voids, in which matter is the air) as three-dimensional entities with determined formal properties. Pacioli's belief in the universality of aesthetic judgment that has a three-dimensional geometrical body as its centre is fundamental in this regard. In essence, Pacioli understands perspective and the visual imagination of three-dimensional geometrical bodies as universal, and as such he applies them to the formal properties of every act of making, including architecture. At the same time, Pacioli assumes that the process that allows the mental understanding of two-dimensional perspectival images as representations of the shapes of three-dimensional bodies does not depend on the conceptual (verbal) or cultural meanings (for instance, in different languages) that can be associated with them at higher levels.

When compared with the process of design discussed in other contemporary or near-contemporary architectural treatises, Pacioli relied on a somewhat different geometrical concept and involved it in the process of the definition of the formal properties of architecture. In Vitruvius's *De architectura*, and in the fifteenth-century treatises by Alberti and Francesco di Giorgio, linear quantities and two-dimensional geometrical figures were employed in an itemised process to control the three-dimensional formal properties of sacred architecture. Consequently, the process of abstraction employed in the design of buildings was not based on the application of a (pre-formed) three-dimensionally extended body, but rather defined through the itemised arrangement of linear quantities or two-dimensional figures, combined in proportional relationships by arithmetical and if needed, by geometrical means. In the theory of the orders that Pacioli attempted to outline, we have seen that he had to rely to great extent not only on Vitruvius's *De architectura*, but also on the interpretations of Vitruvius's text and on treatises by other architectural theorists. Pacioli's theoretical approach to architecture could therefore be seen as ambivalent, if not outright contradictory, particularly in his short *Tractato de l'architectura*. However, when carefully considered in Pacioli's philosophical context as outlined above, one can see that such ambivalence did not exist for him. While he discussed only a limited group of architectural elements in his *Tractato* (mainly the classical orders and the proportional arrangement of the parts of a building), it is clear that he attempted to correlate and discuss the concept of the geometrical body presented in other works in this theoretical context. Respecting proportional relationships among parts, Pacioli saw no contradiction between the employment of regular and other bodies derived from these in architecture in his *Tractato* and *Compendium,* and the theories of proportionality discussed in other architectural treatises, including Vitruvius' *De architectura*. For Pacioli, if geometrical bodies are to be considered for their formal and aesthetic properties, the established proportional relationships between their parts and their visual experience justifies their application in the definition of the visual and formal properties of architecture.

Similar to Pacioli's formal and visual arguments on geometrical bodies are Donato Bramante's spatial division of the Duomo of Milan by means of bodies, and the description of the formal and visual properties of the Duomo's *tiburio*, found in his *Opinio*. Bramante describes the whole building as being composed of four bodies. As in Pacioli's case, each is considered as a composite of form (shape) and matter (volume or spatial extension). At the same time, the visual concerns expressed in the *Opinio* become clearer when Bramante's ability to employ the principles of perspective (as demonstrated in works predating his *Opinio*) is considered.

Because of the lack of theoretical writings by Bramante, Pacioli's arguments have mainly been compared in this study with Bramante's architectural works in Milan and Rome. The principles of perspective were employed by Bramante (and, as seen, in a number of instances by Leonardo da Vinci) as a design tool to define the three-dimensional articulation and visual perception of parts of buildings, understood as a composite of shape and matter (spatial extension). The process of the mathematical quantification of regular and other bodies derived from these is clearly expressed in Pacioli's theoretical framework, and is at once mathematical, visual, and experiential. Similarly, where reconstructions have been possible, the process of quantification and visualisation in three-dimensions, considering the visual points from which spaces were to be perceived, has been pointed out as a crucial element in Bramante's design process for the buildings analysed. In the *Compendium*, which was produced in the environment of the Sforza court of Milan, Pacioli linked the application of geometrical bodies to buildings now associated with Bramante. This study has shown that Pacioli's arguments concerning geometrical bodies beyond these can be coherently related with aspects of Bramante's design approach.

Although the analysis has been more fruitful in some buildings than in others, depending on the amount of available data, most cases have shown that visual, formal and spatial properties were clearly correlated by Bramante. At the same time, the reconstruction of the design process likely followed by Bramante has demonstrated his concerns for approaching a building and its parts in three-dimensions and, as described by Pacioli, as composite of shape and matter (spatial extension). This means that so-called optical corrections as indicated by Vitruvius were not employed by Bramante in one plane of a building but in three-dimensions, that is, involving the three-dimensional geometry of a body composite of shape (boundary) and spatial extension. From this point of view, one can see that the often-debated fissure in the Renaissance between optical corrections and eternal (geometrically defined) forms cannot be applied in the same terms to Bramante's works, nor to Pacioli's perspectival representations of regular and other bodies derived from these. On the one hand, if three-dimensional geometrical bodies and the buildings to which they are applied are judged by the observer's intellect and are based on universal mathematical properties, their proper form (shape) can be mentally constituted without recurring to corrections. On the other hand, the principle of perspective can be employed during the design process to determine the three-dimensional shapes that compose the building. Although this is consistent with Pacioli's arguments on regular and other bodies derived from these, being often more of an able collector and populariser rather than an original thinker on architecture means that Pacioli's position in this respect can be read to some extent as ambiguous. Interpreting Vitruvius in his *Tractato de l'architectura*, Pacioli discussed optical corrections to elevation design to be applied to parts of the orders. However, he consistently avoided mentioning optical corrections in the application of regular or other bodies derived from these to architecture.

It should be noted that this is a problem somewhat independent of the representation of a building in two-dimensional drawings. The employment of perspective in the representation of buildings was famously dismissed in Raphael and Baldassare Castiglione's letter to Leo X for distorting angles and proportions and thus providing unreliable measurements.[804] However, Raphael and Castiglione were referring to methods that would be applied to the representation of existing buildings, while in the case of Bramante, these principles were applied in the design process. Even if the principles of perspective are exploited in the design process, a building can still be represented in orthogonal projections and without distortions after its visual, formal, and spatial properties have been determined. This study has shown that Bramante's ability to quantify and two-dimensionally represent three-dimensional shapes (through mastering the principles of perspective) can be correlated to the formal solutions found in his architectural works in Milan and Rome. In short, if buildings are mentally visualised before they are represented and then built, what is built depends on our ability to quantify and two-dimensionally represent the three-dimensional shapes we have imagined.

The analysis of such a process has been most comprehensive in considering the design of Bramante's Tempietto. The Tempietto showed how the principles of perspective could be employed in the design process to define the visual, formal, and spatial properties of the building in three-dimensions. When the main dimensions of Bramante's Tempietto are considered based on available surveys and on the reconstruction of the originally planned circular cloister around the building, some of the dimensions correspond to whole numbers

[804] Raphael, Baldassare Castiglione, *Lettera a Leone X*, Mantova, Archivio di Stato, Archivio Castiglioni 2016, busta 2, carta 12 (c. 1519-1520) (autograph of Baldassarre Castiglione). Francesco P. Di Teodoro, *Lettera a Leone X di Raffaello e Baldassarre Castiglione* (Firenze: Leo S. Olschki Editore, 2020).

and to Vitruvius' indications for peripteral temples. Others, as seen, can instead be successfully correlated to established visual angles in and around the building. In all cases however, such a process establishes spatial correlations in three-dimensions, that is, simultaneously in plan and in section/elevation. For the Tempietto, as in other buildings in Milan and Rome, the analysis of surveys and surviving (or copied) drawings by Bramante has shown that there are a series of dimensional correspondences that would be explained with difficulty by means other than the primarily visual.

Additionally, the orders are applied to a building (or to part of it) only when its main proportions have been defined in three-dimensions. At the same time, the articulation of the orders and the ornaments is uniformly simplified in Bramante's Tempietto (and in other buildings, such as in the choir of Santa Maria delle Grazie) when these are to be visually perceived from a greater distance. This in turn facilitates the visual reading of the three-dimensional spatial elements in the building. The pilasters in the Tempietto have no bases or capitals at the level of the drum both to the outside and the inside, while the entablature is simplified. As in other buildings, the arrangement and definition of the orders is linked to the hierarchy of visual, formal, and spatial properties that define the articulation of the building in three-dimensions. In other words, the proportioning and articulation of architectural elements such the orders and ornaments are defined according to the building's three-dimensional spatial organisation. The visual and formal layering of architectural elements and the absence of optical corrections as discussed by Vitruvius are among the most important qualities of Bramante's Tempietto; perhaps they are among the properties to which Palladio was implicitly referring when he claimed that Bramante had been "il primo a mettere in luce la buona, e bella Architettura che da gli Antichi sin 'a quel tempo era stata nascosta".[805] Considered in the context of the contemporary psychology of perception, it can be argued that Bramante carefully modulated a series of so-called 'constraints' throughout his architecture to facilitate the three-dimensional mental visual experience of the building and its parts. This is unsurprising, since analogous visual constraints (in the form of lineaments, colours and shadows) were employed in perspectival two-dimensional representations of three-dimensional objects in the Renaissance. The drawings of regular and other bodies derived from these drafted by Leonardo da Vinci for Pacioli's *Compendium* are no exception. In this way, the visual and formal differentiation of the architectural elements of a building in turn facilitates the reading of a spatially three-dimensional hierarchical system.

This thesis has mostly dealt with the theoretical context and the architectural works of the past. However, since the logical structure of the basic stages of human visual perception has remained the same, the arguments of a theorist like Pacioli and the process of architectural design by Bramante hold particularly relevant insights for contemporary approaches to the visual and formal properties of architecture. The recent interest in the quantification and visualisation of architecture has been prompted by a transformation in the technological tools and processes dealing with architecture's visual, formal and spatial properties. Digital technologies have profoundly altered the way of dealing with the quantitative and visual properties of architecture in three-dimensions, in a way similar to what happened in the fifteenth century. In turn, this has determined a renewed and increasing interest in the formal properties of architecture. For instance, if understood in the context of the contemporary theory and practice of architecture, the visual and formal hierarchy of architectural elements

[805] "Conciosia adunque (per tornare al proposito nostro) che Bramante sia stato il primo a mettere in luce la buona, e bella Architettura, che da gli Antichi sin 'a quel tempo era stata nascosta, m'è paruto con ragione doversi dar luogo fra le antiche alle opere sue: e per. ho posto in questo libro il seguente Tempio, ordinato da lui sopra il Monte Ianiculo: e perch. fu fatto in commemoratorio di San Pietro apostolo, il quale si dice, che quivi fu crocefisso, si nomina San Pietro Montorio." Andrea Palladio, *I Quattro Libri dell'Architettura* (Venice: Dominico de Franceschi, 1570), libro IV, cap. XVII, 64.

in Bramante's works is particularly relevant. This means that instead of their suppression, as postulated by Modernist architects and scholars, the classical orders and the ornaments fundamentally partake and contribute to the three-dimensional visual, formal and ultimately spatial articulation of a building. Simultaneously, the facilitated processes of quantification, visualisation and fabrication prompted by digital technologies have determined a return of interest towards these architectural elements.[806]

Finally, further studies are needed to develop a comprehensive empirical theory of design based on Bramante's masterpieces. These will only be possible if more data is gathered and made available, particularly regarding his Milanese buildings which have received insufficient attention from this point of view. One could continue a systematic analysis of the dimensions employed in Bramante's works and compare them with coeval treatises and buildings. The same dimensions could be then methodically considered in their apparent size from a selection of established visual angles. Overall, a comprehensive visually and formally oriented study of Bramante's architectural works is needed and is long overdue, particularly in an English-speaking context. New aspects of the relationships with Pacioli's works can emerge as a consequence of this process. In this respect, this thesis has also highlighted the importance of considering empirical data in the context of the philosophical and theoretical frameworks in which these works were generated.

As we have begun to see, Pacioli's arguments and Bramante's visual approach to architectural design hold a number of valuable lessons for the present-day theoretical and practical relationship with visuality in architecture. Although not yet undertaken, a study of Bramante's

[806] As recently noted by Mario Carpo, in the current digital and 3D printing fabrication environment decoration is no longer an addition and ornament no longer a supplemental expense. This opens all sorts of theoretical issues that in turn undermine some of the core principles held by Modernist scholars and architects. Mario Carpo, *The second digital turn: design beyond intelligence* (Cambridge, MA: The MIT Press, 2016), 77-79.

design approach to the formal, visual, and spatial properties of architecture with the employment of the digital technologies currently used in architectural design will be particularly revealing. Especially, if at the same time, such a study were also coherent with the debate and the latest research developments in the psychology and philosophy of visual perception.

I believe to have shown how Pacioli's arguments on geometrical bodies contribute to an understanding of Bramante's visual and formal solutions. One thus hopes that future research will help to further clarify the inherent logic behind Bramante's *buona e bella Architettura*. Ultimately, Pacioli's theories and Bramante's design approach to the visual, formal, and spatial properties of architecture are relevant to the present-day theory and practice of architecture no less than in the sixteenth century.

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Appendix

The Appendix presents a survey of relevant technical terms employed in Pacioli's *Summa*, *Divina proportione,* and *Euclid's Elements*. These terms are: *dispositione*, *forma* and *figura*, and *ingegno*. The survey focuses on the texts authored and published by Pacioli and the contexts in which these terms are found, providing a background to the interpretative problems and readings found in the body of the thesis. Original texts are enclosed in quotation marks with key words highlighted in bold; original terms outside of quotation marks are italicised. All records are based on my own transcriptions from Pacioli's texts.

Dispositione

The survey that follows presents the contexts in which the words *dispositione/i* and closely related words are found in the texts authored by Pacioli. As it will become clear, a large part is taken by contexts in which these terms are related by Pacioli to three-dimensional geometrical bodies, architecture, and places. In a few instances in the *Summa,* the words *dispositione/*i are employed by Pacioli to refer to the arrangement of numbers as written down on the page. However, these passages are of very little help when it comes to trace Pacioli's use of these words to refer to the disposition of the parts of three-dimensional bodies, the disposition of architecture and other artificial products, or the disposition of places that the architect should consider prior to design. The contexts in which these terms appear in the *Summa* to refer to numbers are the following: "[…] el menore sempre si deve mettere sotto quello seconda la debita **dispositione**" (23v, Pars prima); "E per ventura le prenominate **dispositioni** de questa domanda qui immargine (co(m)mo se dici) no(n) havessi tu per te faciliter le farai a similitudine de quelle dinançe et drieto" (101r, Pars prima); and "Qual metti sotto la sua positione (con lo segno del me(no) di sopra) a pie la croci, commo qui la terça **dispositione** appare" (101v, Pars prima). In all other contexts, these words can be successfully considered as referring to a properly ordered disposition of parts in proportional relationship determining a three-dimensional geometrical whole.

First, in the dedicatory letter that opens the *Summa*, Pacioli uses the words *dispositione* and *disponeret* to refer to the disposition of all things as ordered by the Highest Maker: "Hanc deniq(ue) preoculis summus opifex in celestium terrestriumq(ue) rerum **dispositione** semper habuit. Dum orbium motus, corsusq(ue) syderum et planetarum omnium **ordinatissime disponeret**" (*Summa*, Epistola, 3r). This is a very important passage which is repeated almost verbatim in the text of the lecture given by Pacioli in 1508 at the School of the Rialto, and included in the commented version of Euclid's *Elements,* published by Pacioli in 1509: "Hanc prae oculis summus opifex in caelestium terrestiumque rerum **dispositione** semper habuit dum orbium motus cursusque syderum et planetarum omnium **ordinatissime disponeret** […]" (Pacioli, *Euclidis*, 30r).

The same argument is repeated at folio 68v (Pars prima) of the *Summa*. Here Pacioli links the material world and the properties of weight, number, and measure, with the way in which everything in the natural world exists and is arranged by employing the word *disponit*: "Peroché impossibile e alcuna cosa in natura persistere: se la no(n) e debitamente p(ro)portionata a sua necessita. E pero la divina sapientia (commo dici Augustino in sua laude e com(m)endazione) O(mn)ia fecit deus in numero: pondere: et mensura: cioe che a ogni cosa dette la sua debita exige(n)tia: considerata secondo el peso: el numero: e la mesura. In le quali tre cose: sempre se a retrovare la **proportione**: secondo la quale (commo summo opefici) cuncta bene **disponit**. […] Se tu ben discorri, in tutte le arti tu troverai la **proportione** de tutte esser madre e regina e senza lei niuna poterse exercitare". Importantly, in this case *disponit* is explicitly linked to *proportione* and thus the proportional relationships of parts.

In Chapter 4, it was argued that the use of the term *dispositione/i* to describe the arrangement of the parts of regular and other bodies derived from these in Pacioli's texts clarifies Pacioli's belief in the formal and visual properties of these bodies, based on their determined proportional relationships. The use of *dispositioni* to refer to the arrangement of parts of three-dimensional geometrical regular bodies (and other bodies derived from these) is found at folio 22r of the *Compendium* (1509): "e. V. Cel. al(l')u(n)o e al(l')altro mo(do) hara loro **dispositio(n)i**, le q(ua)li n(on) de vil materia (co(m)mo p(er) i(n)opia a me e stato força), ma de p(re)tioso metallo e fine gemme meritarieno essere ornati", in which the material nature of the bodies presented is also mentioned. A further occurrence is at folio 14r of the *Compendium* (1509), in the title of the section introducing the tetrahedron: "De la **forma** e **dispositione** del tetracedro(n) pia(n)o solido o vacuo e de l(')abciso solido piano over vacuo e de lo elevato soldio over vacuo". In this case the use of the word *dispositione* to refer to the arrangement of the geometrical elements that determine the tetrahedron's shape is linked by Pacioli to the tetrahedron's *forma*. A further occurrence is at folio 28v of the *Tractato de l'architectura*, here Pacioli links the act of giving a good form [*ben formare*] with the pre-defined disposition of parts [*dispositioni*] of each three-dimensional regular and other body derived from these, and represented by Leonardo da Vinci: "Le doi linee mathematici curva e recta o volino o non a perfectione le conducano comme ancora tutte le altre cose fanno co(n)ciosia che sença epse non sia possibile alcuna cosa ben formare. Comme apien in le **dispositioni** de tutti li corpi regulari e dependenti di sopra in questo vedete quali sonno stati facti dal dignissimo pictore prospectivo architecto musico, e de tutte virtu doctato, Lionardo da vinci fiorentino nella cita de Milano quando a li stipendii dello Excellentissimo Duca di quello Ludovico Maria Sforça

Anglo(?) ci retrovavamo nelli anni de nostra Salute 1496 fin al 99 donde de poi siemi per diversi sucessi in quelle parti ci partemmo e a firençe pur insiemi, trahemmo domicilio e cetera.”.

Since regular geometrical bodies and other bodies derived from these are described by Pacioli as material and are presented in a set of models, the disposition of their parts [*dispositioni*] can also be instantiated in architecture. For the same reason, the disposition of parts in proportional relationships is crucial, Pacioli argues, also in the work of smiths and woodworkers. At folio 69r of the *Summa*, we read: “Del fabro legnaro no(n) e dubio cio che si facia (o cassi, o tavole, o banche, o letieri, o usci, o tine, o botti, o mesure da pa(n)ni, o colme co(m)mo si voglia) sempre secondo certa **dispositio(n)e** (la quale si chiama **proportione**) asetta”. With regard to architecture, at folio 16r of the *Compendium* Pacioli employs the word *dispositioni* to refer to the geometrical disposition of buildings or parts of them, following the geometry of the 72-bases body: “[…] E q(ues)to 72 basi molto da li architetti fia freque(n)tato i(n) loro **dispositio(n)i** de hedificii, p(er) e(ss)er forma assai acomodata maxi(m)e dove occurrese fare tribu(n)e o altre volte, o volia(m)o dire cieli. E ave(n)ga che non semp(re) apo(n)to, se pre(n)dino in detti edifici ta(n)te facce pure a q(ue)lla similitudine se regano squata(n)dolo sterça(n)dolo in tutti modi seco(n)do el luogo e sito dove tal edificio inte(n)dan porre”. The text continues by discussing the Pantheon and the proportional disposition of its components: “A la cui convenientia assaissimi in diversi parti se trovano **disposti** e fabricati. Commo de lo inextimabile anticho templo pantheon. E oggi da christiani nel capo del mondo a rotonda chiamato fia manifesto. El qual con tancta solerta industria e de **proportioni** observantia fo **disposto**, chel lume de un solo ochietto nel suo fastigio aperto relicto tutto el rende splendido e luminoso”.

The use of the word *dispositione* to refer to the geometrical arrangement of architecture is found at folio 2v of the *Compendium.* Here the word is linked to beauty [*vagheça*] and pleasure at the level of the intellect by means of perspective [*intellectual co(n)forto prospectivo*]: "Non dico de la dolce suave armonia musicale ne de la **somma vagheça** e **intellectual co(n)forto prospectivo** e de la solertissima **dispositione** de architectura co(n) la descritione de l(')universo maritimo e terrestre e doctrina de corpi e celestiali aspecti p(er)ch(e) di lor quel che finor se detto chiaro apare".. In the dedicatory letter that opens the *Summa*, where Pacioli employs the term *dispositione* to refer to the beautiful order of the architecture of the Ducal Palace of Urbino. Pacioli asks which language could express the beautiful order [*bell'ordine*] of the palace's disposition [*dispositione*]: "Qual lengue el **bell'ordine** de tutta sua degna **dispositione"**, and answers that surely no one can suffice to this purpose, if not its own, very eloquent language; [because] it not only immediately pleases the sight, but even more one is stupefied when with the intellect examines with how much artifice and ornament it was composed: "certo Niuna seria bastante. Se non la sua eloquentissima, el qual non solo alla vista subito veduto piaci, ma ancor piu riman stupefatto chi con intelletto va discorrendo con quanto artifitio e ornamento è stato composto"; at folio 68v (Pars prima), Pacioli says that the harmony of temples and their choirs is not of value if these are not arranged [*disposti*] with proper proportions [*con debita proportione*]: "L(')architettura ancora nulla vale (si commo vitruvio, Dinocrate, Frontino e Plinio approbano) se debitamente non e proportionata ne a l(')ochio, ne a l(')abitare mai piaci ne diletta. E ancor dici non esser sana. E si prova anche di templi in loro formationi e constructioni, e di loro cori, l(')armonia de li divini officii poco valere, se **con debita proportione** non sonno **disposti**. Similarly, at folio 23r-v of the *Tractato de l'architectura*, Pacioli extends the concept of proper *dispositione* as proper arrangement of parts from temples to military architecture: "Conciosia che de li templi non se ne potria dir tanto che piu non meritassero per loro sacratissimo culto. Comme a pieno el nostro V(itruvio) ne parla. De l(')altra parte a la defensione deputata non menore sarebe al dire; conciosia che infinite quodammodo sieno le machine e **dispositioni** militari".

The same applies to architectural models, and Pacioli employs the words *disposti* and *dispose* in two passages discussing models. First at folio 16r of the *Compendium* (1509) in association with the word *ordinati*: "Ma che diremo de li moderni hedificii in suo gener(e). **Ordinati** e **disposti** co(n) varii e diversi modelli q(ua)li a l(')ochio p(ar) che al qua(n)to re(n)dino vagheçça p(er) lor e(ss)er piccoli e poi nelle fabriche no(n) rega(n)o el peso. E no(n) che a mill(')a(n)ni ariva(n)o na(n)çe al terço ruina(n)o". Next, describing the disposition of the parts of an architectural model at folio 30r of the *Tractato de l'architectura*: "In Firençe trovo dicta Architectura molto magnificata, maxime poi chel Magnifico Lore(n)ço medici se ne començo a delectare, qual de modelli molto in epsa era pro(n)tissimo cha a me fo noto per uno che con sue mani **dispose** al suo grandissimo Domenico Giuliano da magliano del degno palaçço detto doglinolo a la cita de Napoli dove in quel te(m)po me trovavo con lo nostro Catano catani dal borgo e molti altri nostri mercada(n)ti borghesi.".

Moreover, considerations about the proportional arrangements of the parts of buildings are extended by Pacioli through the employment of the word *dispositione* to the analysis of the places where architecture is built. At folio 31 of the *Tractato de l'architectura*, this consideration is also linked by Pacioli to the architect's ingegno and to the regular and other geometrical bodies derived from these: "Co(n)chiude breviter che oltra l(')arte el buono architeto bisogna habia ingegno a suplire el dimenuto e smenuire el superfluo secondo la oportunita e **dispositione de li lochi** a cio non parino loro edifitii monstruosi. E a q(ue)sto effecto a voi a qualumch(')altro mi son messo a trovare co(n) grandissimi afanni e lo(n)ghe vigilie le forme de tutti li 5 corpi regulari co(n) altri loro dependenti". Similarly, at folio 34v of the *Tractato de* *l'architectura*, Pacioli argues that the architect should rely on the disposition of parts [*dispositione*] when dealing with places that are constricted: "Co(m)me nelli loghi angusti lo architectto se habia a regere in sua **dispositione**".

In the *Tractato dell'Architectura,* the word *dispositione* is used in conjunction with the concept of proportione to argue that for the same reason and particularly in the design of temples, the ancients considered the human body as a reference. At folio 25r (mistakingly marked as 17r) of the *Tractato de l'architectura,* Pacioli writes: "[…] li antichi considerata la debita **dispositione** del corpo humano tutte le loro opere maxime li templi sacri a sua **proportione** le **disponevano**". The same principles, Pacioli argues, were also applied to the design of columns. This is discussed in the same folio (25r (mistakingly marked as 17r), and Pacioli employs both the word *dispositione* and a reference to parts in proportional relationships in architecture: "E prima diremo de le colonne tonde co(m)me in li edifitii le habiate con vostri scarpelli **debitame(n)te disponere** si per la **forteça a subtentatione de lo hedifitio co(m)me per loro ornamento**. E poi diremo de lo epistilio o vero architrave e sua compositione. De li quali habiando detto poi li situaremo i(n) l(')opera de una porta qual fia a similitudine di quella del tempio de salamone in Hierusalem prenunciata per lo propheta eçechiel con l(')altre **dispositioni**. E voi poi per vostro ingegno potresti piu e manco farne". Moreover, the proportional arrangement of columns and their relationships with the human body as discussed by the ancients, and in this case Vitruvius, is again mentioned at folio 27r of the *Tractato de l'architectura*: "E sequendo diremo a lo intento proposto cioe ala **dispositione** de le colo(n)ne rotonde e suoi pilastri, base e capitelli co(m)me vo promesso **proportionandole** a la statura humana donde prima prima derivarno co(m)me intenderete dal nostro V(itruvio), e noi in quella parte lo adurremo pone(n)do le sue parolle formaliter si che starete atenti e con diligentintia le notarete".

However, in the last instance Pacioli merely explains that he will use the text of Vitruvius to explain this passages.

This helps explaining why in all other contexts in which columns are mentioned throughout his writings, Pacioli exclusively refers to the disposition of parts [*dispositione/i*] of columns' considered as three-dimensional geometrical bodies. A first instance of this is found at folio 18v of the *Compendium* (1509): "De le [colonne] laterate la seconda sorte sonno quadrilatere e sonno quelle che hano le doi basi a modo dicto quadrangule e quatro altre superficie che la circundano sonno pur q(ua)drilatere equidista(n)ti fra loro secondo loro oppositione, e queste similme(n)te sonno a le volte equilatere a le volte i(n)equilatere secondo la **dispositione** delle loro basi". A second at folio 30v of the *Tractato de l'architectura*: "Le pyramidi ancora per le lor colonne si to(n)de comme laterate ve siranno facile imprendere, co(n)ciosia che cadauna sempre aponto sia el terço de la sua colonna, co(m)me p(ro)va el nostro Euclide, e pero di loro similme(n)te lascero loro **dispositioni** quali non e possibile a preterirle sia(n)do loro comme e dicto e al peso e a la mesura in tutti li modi sempre el terço del suo chelindro, e loro ordine e figure harete sopra i(n)questo insiemi con tutti lu altri corpi pur per mano del prelibato nostro compatriota Leonardo da vinci Fiorentino. A li cui desegni e figure mai con verita fo homo li potesse oponere ideo etc".

Lastly, Pacioli uses both the words *dispositione* and *dispositioni* at folio 27v of the *Tractato de l'architettura*. In this case the terms are employed on the one hand in connection with the proportional relationships among parts, on the other to refer to the definition of formal and visual properties to be designed and presented in models before being built. Pacioli concludes that it is necessary for the architect to achieve the proper disposition of parts [*dispositioni*] not only in the model, but also (most importantly) in the real building: "El suo tetrante se fa cavo verso el centro de dicto quadro over tondo curvandolo el nono de la costa del dicto quadro cioe curvato fin al sito de l(')ochio suo in fronte. E questo se adorna or piu or manco secondo chi fa e chi ordina la spesa con uno e doi abachi sopraposti comme meglio li agrada **a libito servando le debite proportioni** de lor gradamenti quali sempre se prosupongano servati in ogni **dispositione** degradandoli cioe a menore reducendoli e augumentandoli cioe crescendoli a magiori si co(m)me in le **dispositioni** de tutti li modelli che prima se fanno secondo li quali de necessita bisogna che l(')architetto el tutto in quelli contenuto sapia a la vera fabrica apricare et cetera".

Forma* and *figura

The purpose of this survey is to present the contexts in which the words *forma* and *figura* are used to refer to three-dimensional bodies in the texts of Pacioli and provide a more extended background to the analysis presented in the body of the thesis. This survey will therefore not cover, for example, the instances in which the word *figura* is used to refer to numbers or to two-dimensional geometrical illustrations in the margins of the texts. As standard in Campanus' edition of Euclid's *Elements* and Medieval translations, *figura* was also used by Pacioli to refer to planar two-dimensional geometrical shapes (or polygons) such as a triangle, a pentagon, etc.

As noted in Chapter 2, throughout the *Summa* and *Compendium de divina proportione* the word *forma* is consistently employed by Pacioli to refer to three-dimensional geometrical and material bodies. A reason for this can be found in the properties conveyed by the form. The form is in fact understood by Pacioli as indicating the shape-defining properties of regular and other bodies derived from these and, differently from *figura*, this includes also its material extension, or in other words, its volume. As we have seen, Pacioli was able to overcome the difficulties of the Aristotelian and Scholastic distinction between substantial (essential) and artificial forms by referring to the linking of natural and regular geometrical bodies to their form *qua* extended magnitude and therefore quantity. Consequently, it was possible for Pacioli to argue that the forms of regular bodies rest completely on their proportional mathematical and quantitative order, rather than on any other property. *Forma* was thus conceived as a three-dimensionally extended and bounded body to which a two-dimensional *figura* corresponds. Overall, it is important to underline again that when Pacioli discusses the *forma* of regular bodies, this comprises all the elements that concur with the shape of a three-dimensional body, and not only those that may be visible from a single viewpoint. Nonetheless, as Pacioli makes clear, visual, formal and shape-defining properties are all necessarily correlated. Ultimately, it was argued that the arguments developed by Pacioli and its use if the concept of *forma* present inevitable consequences for the aesthetic and formal evaluation of all works designed and made by individuals.

As we have seen, the words *forma* and *figura* were traditionally used throughout the Middle Ages to translate Aristotle's *eidos* and *morphe*. However, both words and particularly *figura* were also used sometimes as translation of Aristotle's *schema*. Simultaneously, as already noted, *figura* was also commonly used to signify a two-dimensional shape in Latin translations of Euclid's *schema* and Campanus' edition of Euclid's *Elements* made no exception. In his edition, Campanus also used *figura* at times to refer to three-dimensional bodies. Nonetheless, in Chapter 2 it was noted how Pacioli, paraphrasing Campanus' commentary to proposition 10 of Book 14 of the *Elements*, added *forma* to *figura* in the following passage: "De la sua degna commendazione Cap. VI. Questa nostra proportione excelso D(uca) e de tanta prerogativa e de excellentia degna quanto dir mai se potesse e per respecto de la sua infinita potentia, conciosia che sença sua notitia moltissime cose de admiratione dignissime ne in philosophia ne in alcuna scientia mai a luce poterieno pervenire. El qual dono certame(n)te de la invariabile natura de li superiori principii, commo dici el gran philosopho Campanno (no)stro famosissimo mathematico sopra la decima del 14 glie co(n)cesso. Maxime vedendo lei esser quella che tante diversita de solidi si de grandeççi si de moltitudine de basii si ancora de **figure et forme** con certa irrationale simphonia fra loro acordi, commo nel nostro processo se intendera ponendo li stupendi effecti quali (de una linea secondo lei divisa) non naturali ma divini verament(n)te sonno d(')appellare" (4r, *Compendium* (1509)).

In Chapter 2 it was argued that by using only *figura,* Campanus made it clear that he was referring to a shape, avoiding the ambiguities of the term *forma* that were carried from the Latin word, when used in Medieval translations to refer to Aristotle's form not only as essence [Aristotles's *eidos* or *morphe*], but also as shape [Aristotle's *schema*]. Indeed, although Campanus employed also *forma* to refer to three-dimensional geometrical bodies, he never used the combination of both *forma* and *figura* to refer to any of these bodies. Pacioli's addition of *forma,* it was concluded, served therefore to clarify that material three-dimensionality (i.e. volume) was implied here. In the contexts from Pacioli's texts listed below, *figura* is sometimes employed to refer to the (two-dimensional) external surface or boundary of a three-dimensional body. Although *forma* covers this, the word should also be understood in Pacioli's *Summa* and *Divina proportione* as referring the three-dimensional body's matter (i.e. volume or three-dimensional spatial extension). Otherwise, when Pacioli described both the *forma* and the *figura* of the same three-dimensional regular body, one word would have been redundant.

Both *forma* and *figura* are employed in the same context to refer to three-dimensional geometrical bodies in Pacioli's texts are at folio 4v of the *Summa* (Pars prima): "Concludese brevemente per le ragioni prossime assignate che solo. 5. E no(n) piu possono essere li ditti corpi regulari. Li quali qui qua(n)tunche incidenter fuor dela pratica geometrica mesiano occorsi: me parso non inutile haverli indutti. Ma assai piu ampliamente: quando de geometria parlaremo: ne tocaro. Assegnando lor **forme** e misure: e fabriche commodi chiari e aperti: e regole aloro quadrature bellissime e sotile. Epero qui non ne toco piu. Equantunche oltra questi infiniti siano li corpi (commo li se dira) non dimeno nium si porra mai trovare: simili in **forma** e **figura**: ad alcuno deli predetti"; to this follows a long passage at folio 4v of the *Summa* (Pars prima). When three-dimensional bodies are concerned, the use of *figura* can only be understood here as referring to the whole external (two-dimensional) surface of a body, while forma includes the whole body's three-dimensional matter (extension): "Ma assai piu ampliamente, quando de geometria parlaremo, ne tocaro. Assegnado lor forme e misure, e fabriche com modi chiari e aperti, e regole e loro quadrature bellissime e sotile. E pero qui non ne toco piu. E quantunque oltra questi infiniti sieno li corpi (comme li se dira) non dimeno niun si porra mai trovare, simili in **forma** e **figura**, ad alcuno de li predetti. […] E se piu oltra andremo, trovaremo ditto quinario de corpi regolari per Plato(n)e nel luogo preallegato essere (secondo sue evidenti ragioni) meritamente attribuito a li corpi i(n)misti. Cioe semplici. E de ciascuna di loro **forme** e **figure** fo opinione d(')esso che li elementi. E per conseque(n)te tutti li corpi semplici fossero **figurati** e per ognuno havesse la propria sua **forma** e **figura**, si comme, Io e tu, el terzo el quarto, bovi, asini e(t)c". As note in Chapter 2, following the distinction between the form *qua* substance and the form *qua* extended body found in Duns Scotus' commentary, Pacioli and his readers would share the same essential form, but not the same topological form nor the same figure. The same would apply to the third and the fourth (we can assume that Pacioli here is speaking about regular bodies, but the same would also be true for numbers), oxen, donkeys, etc. In this passage, Pacioli thus emphasises that he is referring to each individual (in a species') own (not essential) form and figure. The passage continues discussing the *figura* and *forma* of individual three-dimensional geometrical bodies: "Onde secondo lui conveniente a la terra atribui **la figura e forma de**

lo exacedron, cioe de le sei base quadrate equilatere et equiangole. E accio lo strinse sua argomentatione, dicendo la **figura circulare** e **spherica** maxime e atta al moto e a(l)la velocita e q(uan)to piu la cosa sacosta a la **forma e figura spherica**, tanto piu e atta al moversi. Donca per adverso, quanto piu la cosa si scostava da tal **figura** tanto piu si siontanava(?) da la velocita, e per consequente ta(n)to piu la proximava a la pigritia e tardiva. Ma infra gli altri corpi semplici, la terra e pigrissima, tarda e lenta. La qual cosa non pi havenire se non per la cagion ditta cioe che la sua **figura e forma** dev(')essere lontana e rimota da la spherica".

Both the *forma* and *figura* of three-dimensional regular bodies are discussed to in the sonnet that opens the 1509 edition of the *Compendium de divina proportione*: "Cinque corpi in natura son producti / Da' naturali semplici chiamati./ Perché a ciascun composito adunati / per ordine concorran fra lor tutti / Immixti: netti e puri fur constructi / Quattro elementi e ciel così nomati / Quali Platone vol che **figurati** / L'esser dien a infiniti fructi. / Ma perche el vacuo la natura abhorre / Aristotil in quel de cielo et mundo / per se **non figurati** volse porre / Pero l'ingegno geometra profondo / di Plato e d'Euclide piacque exporre. / Cinqualtri che in spera volgan tundo / Regolari; d(')aspeto iocundo / Come vedi de lati e basi pare / E un altro sexto mai se po **formare**"; in the text of the *Compendium*, they are found at folio 12v, with the employment of *figura* to refer to a two-dimensional polygon (the pentagon) and the reflexive *si forman* to refer to three-dimensional regular bodies: "Le loro superficie ex(cellente) D(uca) fra loro similmente possiamo dire al medesimo modo e(ss)er p(ro)portio(n)ali co(m)mo de lor massa corporea se dicto cioe irr(ati)onali per la malitia de la **figura** pe(n)tagona che i(n) lo duodecedro(n) se i(n)terpone. Ma del(l')altre possa(n)o a le volte e(ss)ere r(ati)onali como q(ue)lle del tetracedron cubo octocedron per e(ss)ere tria(n)gule e q(ua)drate e note i(n) p(ro)portione co(n) lo diemetro de la loro sp(her)a i(n) la q(ua)le si **forma(n)o** […]"; at 22r of the *Compendium* (1509), the word *figure* refers to the figures represented in perspective by Leonardo da Vinci. These are discussed together with their material (instantiations) in the models of three-dimensional bodies presented by Pacioli: "Perche dove n(on) e ordi(n)e semp(re) fia co(n)fusio(n)e p(er)o a piu piena i(n)tellige(n)tia de q(ue)sto n(ost)ro co(m)pe(n)dio p(er) spaer retrovare tutte le p(ro)prie **figure i(n) p(ro)spectivo** aspecto i(n) q(ue)sto p(ro)poste **a anco le materiali** s(econd)o lor publica taula la v. cel. Observera q(ue)sto mo(do) cioe q(ua)n(do) legiarete di sopra i(n) lor capitoli de lor creatio(n)i e **formationi** guarderete i(n) q(ue)l luogo del libro el nu(mer)o segnato p(er) abaco antico, cope cosi come(n)çando dal I al 48(˚) cap(itolo) dice(n)do I,II,III, IIII, V. E seq(ue)ndo fine a lor termine. E q(ue)l medesimo nu(mer)o apo(n)to farete de trovare dena(n)çe dove i(n) q(ue)sto dicti corpi so(n)no p(er) ordi(n)e tutti **figurati**. El q(ua)l nu(mer)o similme(n)te i(n) q(ue)l luogo sira posto, refere(n)do I a I e II a II e III a III e cosi i(n) tutti. E q(ue)lla tal **figura** sira del d(i)c(t)o corpo f(a)c(t)o i(n) piano co(n) tutta p(er)fect(i)o(n)e de p(ro)spectiva co(m)mo fa el n(ost)ro Lio(n)ardo vi(n)ci". At folio 8r of the *Compendium* (1509), the forms [*forme*] of the five regular bodies are contrasted by Pacioli with the corresponding figures [*figure*] associated to the five elementary bodies by Plato: "E si co(m)mo questi 5 semplici sonno bastanti e sufficienti in natura altrame(n)te seria arguire I dio superfluo overo diminuito al bisogno naturale. La q(ua)l cosa e absurda co(m)mo afferma el ph(ilosoph)o che I dio e la natura non op(er)ano in vano cioe non ma(n)cano al bisogno e non excedono quello cosi a simili le **forme** de questi 5 corpi de li q(ua)lli sa adire a po(n)cto sonno 5 ad decorem universi e no(n) possano esser piu per quel che sequira. E p(er)o non imeritatamente co(m)mo se dira di socto l(')antico Platone nel suo thymeo le **figure** de dicti regulari atribui a li 5 corpi semplici co(m)mo in la q(ui)nta co(n)venentia del divin nome a la nostra p(ro)portione atribuita de sopra fu decto e questo a la loro denominatione."

Other times, only the *forma* of three-dimensional geometrical bodies is considered. This happens at folio 14r of the *Compendium* (1509), where the *forma* is linked with the disposition of the parts [*dispositione*] of the tetrahedron: "De la **forma e dispositione** del tetracedro(n) pia(n)o solido o vacuo e de l(')abciso solido piano over vacuo e de lo elevato solido over vacuo"; at folio 14v, when the forma of an instrument of torture is mentioned: "Simile a la **forma** del diabolico instr(ument)to altrame(n)te detto dado o taxillo"; at 16v of the *Compendium* (1509): "Non me pare Excelso Duca in dicti corpi piu externe con cio sia chel lor processo tenda in infinito per la continua e sucessiva abscisione de mano in mano de li suoi angoli solidi e secondo quella lor varie **forme** si vengono a multiplicare [...] E questo solo habiamo finor sequito per monstrare co(m)mo da quelli 5 regulari la virtu sempre negli altri dependenti se distilla a similitudine de li 5 semplici che a la **formatione** de ogni creato composto concorrano. Per la qual cosa (co(m)mo de sopra fo acenato) Platone fo constretto le prelibate 5 **forme** regulari a li 5 corpi semplici atribuire, cie a la terra aiere aqua fuoco e cielo co(m)mo difusamente apare nel suo Thimeo dove de la natura de l(')universo tratto"; at 17r of the *Compendium* (1509): "E queste tali **forme** da Calcidio celeberrimo philosopho expone(n)do el dicto Timeo molto sonno co(m)mendate. E cosi da Macrobio Apuleio e moltissimi altri perche in vero sonno de ogni commendatione degni, per le ragioni che in loro fabriche se aducano mostrando la sufficientia de ditte 5 **forme** si commo quella de li 5 corpi semplici non potere per alcun modo esser piu, e si commo el numero de dicti semplici non si po in natura accrescere, cosi queste 5 regulari non e possible asegnaene piu che de basi e de lati e de anguli sinno e quali, che in spera collocati toccando un angolo tutti tocchino. Perche se in natura si potesse un sexto corpo semplici asegnare el summo opefici verebbe a esser stato i(n) le sue cose diminuito e sença prudença da giudicarlo, non havendo a principio tutto el bisogno oportuno a lei cognosciuto. E per questo certame(n)te e no(n) per altro mosso comprendo Platone queste tali commo e dicto a ciascuno de li dicti semplici attriuisse e cosi argumenta(n)do ; cioe commo buonissimo geometra e p(ro)fondiss(i)mo mathematico, vedendo le 5 varie **forme** de questi non poter per alcun modo alcun(')altra che al sperico tenda de lati basi e angoli commo e dicto e quali ymaginarse **formare** commo in la penultima del 13 se mostra e per noi a lo portuno saduci non immeritamente argui le ditte advenire al 5 semplici. E da quelle ogni altra **forma** dependere"; at 17v of the *Compendium* (1509): "E p(er) q(ue)sta sci(enti)a i(n)fallibile porra V. cel(situdine) a le volte (co(m)mo noi habiamo fato) con dicti lapicidi havere solaçço in questo modo argua(n)do loro ignora(n)ça . Ordina(n)doli che queste simil pietre ne facino qualche **forma** de lati facie e anguli equali e che nun sia simile a le 5 de li regulari, verbi gratia obligandoli a fare un capitello o basa o cimasa a qualche colonna che sia de quatro o de sei facce e quali a modo dicto e che quella de le 4 non sie(n)no triangule overo quelle de le 6 non sienno quadrate. E cosi de 8 e 20 facce e Niuna sia triangula over de 12 e niuna sia pentagona, le quali cose tutte sonno impossibile"; at folio 31 of the *Tractato de l'architettura*, where the forme of the five regular and other bodies derived from these are linked to architecture: "Co(n)chiude breviter che oltra l(')arte el buono architeto bisogna habia ingegno a suplire el dimenuto e smenuire el superfluo secondo la oportunita e dispositione de li lochi a cio non parino loro edifitii monstruosi. E a q(ue)sto effecto a voi a qualumch(')altro mi son messo a trovare co(n) grandissimi afanni e lo(n)ghe vigilie le **forme** de tutti li 5 corpi regulari co(n) altri loro dependenti e quelli posti in questa nostra opera con suoi canoni a farne piu con debita lor proportione acio in epsi spechiandove mi rendo certo ch(e) voi a li vostri p(ro)positi li saprete acomodare. E li altri mecanici e scientifici ne conseguiranno utilita non poca e siano dati a che arte mistieri e scientie si vogliano co(m)me nel suo Thymeo ed divin Plato(n)e el re(n)de ma(n)ifesto".

In other instances, the form [*forma*] of three-dimensional geometrical bodies is expressively linked by Pacioli to their material instantiations through the compound *forma materiale*. This is found at folio 1v of the *Compendium* (1509): "A decore ancora e p(er)fecto prnamento de la sua dignissima bibliotheca de inumerabile multitudine de volumi in ogni faculta e doctrina adorna a disponere q(ue)sto breve co(m)pendio e utilissimo tractato detto de divina proportione. El q(ua)le co(n) tutte sue **forme materiali** de li corpi che in ditto se co(n)tengono non menore admiratione a chi q(ue)lla visitara darano che tutti gli altri volumi co(n) l(')altre sue dignissime cose in q(ue)lla reposte si facino. Per esser dicte **forme** a li vive(n)ti finora state ascoste. Nel quale diremo de cose alte e sublimi quali verame(n)te sonno el cimento e copella de tutte le prelibate scientie e discipline e da quello ogni altra speculativa op(er)atione scientifica pratica e mecanica deriva"; at folio 52r-v of the *Compendium* (1498) (folio 14r-v of the *Compendium* (1509) has the central part missing in the 1509 edition): "El tetracedron scapeçço o vogliam dire absciso solido piano over vacuo fia contenuto da 18 linee quali causano 36 anguli superficiali e 12 solidi e 8 basi lo circundano. De le quali 4 sonno exagone cioe de 6 basi equali e le altre 4 sonno triangole, similmente equilatere, e anco equiangole. Ma de le dicte 18 linee le 12 sonno commune a le basi triangule e a le exagone. Le quali non di meno sonno tutte proprie de quelli exagoni, perche de necessita quelli 4 exagoni gionti asciemi con alchuni soi lati causano quelli 4 triangoli. Si co(m)mo la experientia nella sua propria **forma materiale** a l(')ochio nostro rende chiaro e nasci dal precedente neli suoi lati per terzo uniformi tagliati"; at 14v of the *Compendium* (1509): "El tetracedron elevato o vogliam dire pontuto solido over vacuo ha similmente 18 linee de le quali 6 sonno comune e ha 36 anguli superficiali e 8 solidi de le quali 4 sonno co(mu)ni de le pyramidi superficiali e 4 sonno comuni a le 5 pyramidi cioe a quella interiore che l(')occhio non po veder, ma solo l(')intellecto l(')aprende , e le altre 4 exteriori de le quali 5 pyramidi dicto corpo fia composto quando le sienno fra loro equilatere triangule & equiangule commo la sua propria **forma materiale** a noi dimostra. E le sue superficie che lo vestano quali non propriamente sonno dette basi in tutto sono 12 per numero tutte triangule. E de questo non se po per alchun modo asegnare lo elevato absciso per defecto de li exagoni che non fanno anguli solidi"; again at folio 14v of the *Compendium* (1509): "[…] co(m)mo demostra a l(')ochio la sua p(ro)p(ri)a **forma ma(teria)le**" and "[…] che di tutto l(')ochio ne la **forma sua materiale** chiaro all(')intellecto la verita fa nota". At folio 15r of the *Compendium* (1509): "Co(m)mo la **fo(r)ma** sua **materiale** a noi fa manifesto", "Che tutto ancora la p(ro)pria **forma** sua **ma(teria)le** fa ap(er)to", and "E q(ue)lli tali q(ua)drati se forma(n)o da li exagoni q(ua)n(do) uniformi tutti 8 se contangino che di tutto l(')ochio ne la **forma** sua **materiale** chiaro all(')intellecto la verita fa nota. At 15v of the *Compendium* (1509): "[…] e 20 so(n)no tria(n)gule pure eq(ui)latere tutte fra loro co(m)mo habia(m) detto reciprocame(n)te causate. Ei sua **material forma** ap(par)e", "E l(')origine de q(ue)sto fia da lo exacedro(n) uniforme seco(n)do ogni suoi p(ar)ti tagliato come a l(')ochio la sua **material forma** ci demostra. E fia la sua sci(enti)a i(n) molte considerationi utilissima a chi b(e)n(e) la (sa) acomodare maxime in architectura […]" and "E l(')origine de questo fia da lo exacedron uniforme secondo ogni sui parti tagliato, commo similmente a l(')occhio la sua **material forma** ci demostra". Moreover, at folio 22r of the *Compendium* (1509): "Co(m)mo se habino retrovare tutti li dicti corpi ordinatame(n)te comme sono posti in questo facti in p(ro)ospectiva e ancora le lor **forme materiali** seco(n)do la loro taula particulate posta patente in publico", and at folio 34v of the *Tractato de l'architectura:* "Q(ua)li piu volte **ma(teria)li in p(ro)pria forma** ve ho mostrati avenga che di loro p(ar)ticularmente no(n) ne facia me(n)tione alcu(n)a el nostro Victruvio". Part of this group should also be considered a passage at folio 16v of the *Compendium* (1509). Here Pacioli argues that the elevated form [La **forma** del q(ua)le elevato] of the 72-bases body has not been materially instantiated (a model was not made), and a perspectival representation is therefore not included. Pacioli concludes that this is left to the reader, of which mental capacity for reasoning [*ingegno*] he has no doubts about: "La **forma** del q(ua)le elevato non curai fra queste **materialmente** dedure per lassar la parte sua anchora a(l) lectore del cui ingegno non mi diffido". In an additional instance at folio 15r of the *Compendium* (1509), Pacioli seems to link the figura of a three-dimensional body shown in the perspectival representations drafted by Leonardo da Vinci, to its forma by means of their similarity: "[…] che la sua **figura similme(n)te materiale** lo dimostra xxiiii,xxiiii". At folio 30v of the *Tractato de l'architectura*, these representations are instead considered only as *figure* or *desegni,* that can be translated with drawings: "Le pyramidi ancora per le lor colonne si to(n)de comme laterate ve siranno facile imprendere, co(n)ciosia che cadauna sempre aponto sia el terço de la sua colonna, co(m)me p(ro)va el nostro Euclide, e pero di loro similme(n)te lascero loro dispositioni quali non e possibile a preterirle sia(n)do loro comme e dicto e al peso e a la mesura in tutti li modi sempre el terço del suo chelindro, e loro ordine e **figure** harete sopra i(n)questo insiemi con tutti lu altri corpi pur per mano del prelibato nostro compatriota Leonardo da vinci Fiorentino. A li cui **desegni** e **figure** mai con verita fo homo li potesse oponere ideo etc.".

According to Pacioli, the form [*forma*] of three-dimensional bodies so understood can also be applied to architecture. This is argued by Pacioli at folio 16r of the *Compendium* (1509): "[…] E q(ues)to 72 basi molto da li architetti fia freque(n)tato i(n) loro dispositio(n)i de hedificii, p(er) e(ss)er **forma assai acomodata** maxi(m)e dove occurrese fare tribu(n)e o altre volte, o volia(m)o dire cieli. E ave(n)ga che non semp(re) apo(n)to, se pre(n)dino in detti edifici ta(n)te facce pure a q(ue)lla similitudine se regano squata(n)dolo sterça(n)dolo in tutti modi seco(n)do el luogo e sito dove tal edificio inte(n)dan porre. A la cui co(n)venie(n)tia asaissimi in diversi p(ar)ti se trova(n)o disposto e fabricati". Lastly, a further occurrence of the word *forma* in a context discussed architecture is found at folio 68v of the *Summa* (Pars prima). Here Pacioli refer to the form [*forma*] of the human body. Nonetheless, rather than to the form as essence of the body, the context clarifies that Pacioli is referring, as in other cases, to the body's parts and geometrical properties (although, differently from three-dimensional geometrical bodies, not to its volume), and that these are arranged in proportional relationships: "L(')architettura ancora nulla vale (si commo vitruvio, Dinocrate, Frontino e Plinio approbano) se debitamente non e proportionata ne a l(')ochio, ne a l(')abitare mai piaci ne diletta. E ancor dici non esser sana. E si prova anche di templi in loro **formationi** e constructioni, e di loro cori, l(')armonia de li divini officii poco valere, se con debita proportione non sonno disposti. E dimostra commo la co(m)muna longheçça loro, debia a la loro largheçça correspondere inducendo la **forma** del corpo humano a la cui similitudine vole le chiesi essere fabricate, commo al corpo del nostro redemptore Jesu Christo volse la divina bonta che Noe l(')arca fabricasse tanti co(m)biti longa larga e alta commo expone Augustino in de civitate dei. E cosi anche de palaççi e altre case da habitare, in uno e piu solari, commo se co(n)tiene ne la sua grande e difficile opera de architettura".

Ingegno

The words *ingegno* is employed by Pacioli a large number of times throughout the *Summa, Compendium de divina proportione* and *Tractato de l'architectura.* In general, the survey presented here confirms that the word was used to refer to a potential capacity of reasoning and problem-solving, which Pacioli assigned to all of its readers. However, this does not mean that exemplar cases of individual *ingegno* cannot also be indicated, and Pacioli made no exception to this. At the same time, the contexts in which Pacioli employs the term *ingegno* in discussing three-dimensional geometrical bodies show that this is also related to the capacity of imagination of three-dimensional shapes. In other words, they show that according to Pacioli this ability is at least aided by *ingegno,* and that the process takes place at the level of the intellect.

In the *Summa*, the word is employed in contexts that generically refer to the *ingegno* of the reader for reasoning and problem-solving, particularly in mathematics. For this reason, the contexts listed below are only of limited help when it comes to interpreting the nature of the relationship between *ingegno* and three-dimensional bodies as understood by Pacioli. At folio 18v (Pars prima) we read: "[…] e uno solo basta che ne co(m)muti co(m)mo lo **ingegno** per te ti porra regere e ditare"; at 31v (Pars prima): "Le quali cose benche non a cadano a mercatanti in sui trafichi, non dimeno commo piu volte o ditto le pongo a magiore evidentia de **ingegno** e piu delectatione de intellecto nelle forçe e travagliamenti di dicti numeri, quali certamente so(n)no infinite e quasi inco(m)prehensibili" and "[…] el vulgar proverbio per molti sona: dura cosa la p(ar)tita. Siche bisogna ch(e) al quanto piu che a le precedenti lo **ingegno** mi presti"; at 33v (Pars prima): "Ben sai che de li picoli nelli quali lo i(n)tellecto facilme(n)te discorre l(')ho subito p(er) un poco di pratica che habia el sa dire. Ma p(er) li picoli numeri no(n) seria bisogno ta(n)ti muodi de p(ar)tire, ma p(er) li gra(n)di se fa(n)no , dove lo **ingegno** nostro p(er) la sua debilira no(n) po simul et semel ditto facto i(n) essi tra(n)correre […]"; at 34v (Pars prima), to the word *ingegno* is added the adjective *naturale,* and therefore belonging by nature to every reader: "E no(n) si po mo de simile co(n)sideratio(n) p(er) arte darne norma ne regola alcuna: se no(n)che bisogna ch(e) l(')homo da se se assetti co(n) q(ue)lla gr(azi)a ch(e) idio li ha dato d(')**ingegno** naturale. Perch(e) co(m)mo dici el ph(ilosop)ho Ars Imitatur natura(m) i(n) q(uan)tu(m) pot(est) ma no(n) totaliter […] Si che q(ue)l poco de co(n)siderare l(')**arte** lo lascia a l(')**i(n)gegni** nostri media(n)te la pratica e solicitudine del nostro negotativo op(er)are". Similarly, at folio 202r (Pars prima): "E co(n) fatiga, no(n) poca, se ritrovano lor t(i)pi(?) co(m)e sa chi p(ro)va ch(e) ogni cosa cosi apieno no(n) si puo dire. Ma biso(gn)a ch(e) a(n)cora tu alq(uan)to co(n) tuo **naturale ingegno** t(')aiuti". At 34v (Pars prima) Pacioli adds an invocation to Apollo and to the reader's *ingegno* derived by Pacioli admission from Dante: "E p(re)stame p(ar)golo **i(n)gegno** e i(n)voca apollo al(')ulti(m)o lavoro, (c)omo disse da(n)te"; at 60r (Pars prima): "De le quali monete e valute varie Asai se ne porebe dire chal presente non curo, ma solo questo notando pongo a tua memoria acio tu habi per tuo **ingegno** a investigarle, et cetera […]"; at 77v (Pars prima), where the individual quality [*peregrineçça*] of *ingegno* is also mentioned: "Peroche se bello e nelli sani rotti radici etc, simili acti, e anche in parte difficili (co(m)mo di lor ina(n)çe habiamo tractato) asai molto piu sença co(m)paratione e el travagliare delle p(ro)portioni fra loro, e co(n) piu al quanto **peregrineçça** de **ingegno** ha(n)no ad operare in la pratica vulgare."; at 81v (Pars prima): "Quanto al proposito e a lo intento principlae nostro se aspeti le gia ditte cose de le proportioni voglio sie(n)no bastanti e oltra piu di loro non inte(n)do dire. Se no(n) che qui sequente mettaremo alcune conclusioni de gran piacere a lo **ingegno** p(er)spicaci e utilita electe e aplicate secondo la força e virtu de esse proportioni. Le q(ua)li cose son certo in molto luoghi e passi in libri de philosophia (maxime in quello de celo et mundo) te serviranno"; at 86v (Pars prima): "Unde de q(ue)sto no(n) mi curo piu exte(n)derme, ma tu p(er) tuo **i(n)gegno** supplesci dove ma(n)casse parolle. E q(ui) sia fine a tal modi"; at 95v (Pars prima): […] Per che so(n)no i(n) la medesima p(ro)portio(n)ne. E cosi i(n) l(')altre p(er) tuo **i(n)gegno** sequirai. Ma per una regola assai generale tieni questo modo"; at "Siche tu p(er) tuo **i(n)gegno** bisogna che tu sia pro(n)to a sapere satisfare a li ditti pesi a chi doma(n)dasse robba, ponendo ora di qua, ora di la de la bila(n)cia […]"; at 98v (Pars prima): "Al quale intendere (sença dubio) quello che i(n) q(ue)sto habiamo detto de proportioni e proportionalita, te fara grandissima utilita in modo ch(e) per te quasi turro ditto 5° intenderai se al quanto l(')**ingegno** tuo asotigliarai."; at 105v (Pars prima): "Perche 1/4 de do(n)na semp(re) sira piu che 1/4 de fanciullo. E p(erci)o sapite reggere per tuo **ingegno** nella porre"; at 118r (Pars prima): "Per loro p(ar)tim(en)ti fra loro suci(n)tamente q(ue)sto poco exe(m)plo co(n) sua regola voglio te sia basta(n)te media(n)te tuo **i(n)gegno** etc."; at 142r (Pars prima): "Quando di tal m(an)catione sia cavato el q(ua)drato de la li(nea) etc. E tu simili che lo(n)go serebe ogni cosa extricare p(er)tuo ingegno discorri […]"; at "E cosi p(er) no(n) abundare i(n) troppo scriptura p(er) tuo **ingegno**, a moltissimi aguaglim(en)ti te regerai, ponendo semp(re) da(van)te a te un caso certo e simile app(ro)posto ch(e) cerchi dape(re)"; at 202v (Pars prima): "E cosi p(er) tuo **ingegno** a(n)darai al(l)ettando, tutte le p(ar)tite, e li no(mer)i de li debitori pers(o)ne e robee etc" and "[…] e di po(n)to a tante carti li potrai fare ritornar co(n) tuo **ingegno** etc.". At 203r (Pars prima): "E ancora nel gra(n) lib(r)o i(n) dare e havere, e di porre al alfabeto lasciaro ormai seq(ui)re al tuo **p(er)egrino ingegno** del q(ua)l molto me co(n)fido"; at 206r (Pars prima): "[…] e sire i(n) le tue p(ar)tite p(er) co(n)to de la tal cosa, o p(er) co(n)to del tal o p(er) ragio(ne) de mercantia o p(er) rafgion de contanti depositati i(n) tuo nome o d(')altri co(m)mo e d(e)tto, le quali cose so p(er) tuo **i(n)gegno** ormai reggerai etc."; at 209r-v (Pars prima), where the *ingegno* is again defined as *peregrino* (i.e. individual): "E ancora nel gra(n) lib(r)o i(n) dare e havere, e di porre al alfabeto lasciaro ormai seq(ui)re al tuo **p(er)egrino ingegno** del q(ua)l molto me co(n)fido"; at 206 (Pars prima): "[…] e sire i(n) le tue p(ar)tite p(er) co(n)to de la tal cosa, o p(er) co(n)to del tal o p(er) ragio(ne) de mercantia o p(er) rafgion de contanti depositati i(n) tuo nome o d(')altri co(m)mo e d(e)tto, le quali cose so p(er) tuo **i(n)gegno** ormai reggerai etc." A generic reference to the properties of *ingegno* in mathematical reasoning is also found in the title added to the *Divina proportione* in 1509: "Opera a tutti gli **(i)ingegni** perspicaci e curiosi necessaria […]". At the same time, at folio 2v of the *Compendium* (1509) the lack of good teachers in mathematics [*buoni preceptori*] is connected, according to Pacioli, to the lack of good mathematicians and the draining of *ingegni*: "E p(er)o non e da pre(n)dere admiratio(n)e se pochi sono a nostri te(m)pi buoni mathematici p(er)che la rarita de buoni p(re)ceptori ne fia cagio(n)e co(n) la gola, sonno, e otiose piume e i(n) p(ar)te la debilita de li recent(t)iori **i(n)gegni**".

Although *ingegno* can be understood in all contexts surveyed so far as the potential capacity of reasoning that Pacioli assigned to all of its readers, a number of individuals that have expressed *ingegno* in an exemplar way are also mentioned. Among these, first in the dedicatory letter that opens the *Summa* Pacioli modestly refer to his humble and low *ingegno:* "Deliberai il presente volume (s(econd)o el mio mio humile e basso **ingegno**)"; next, the *ingegno* of Camillo Vitelli is qualified as highly shrewd [*perspicacissimo*] in understanding the contents of Euclid's *Elements* at folio 2v of the same letter: "Nel tempo che optimamente el suo [Vitelli] **perspicacissimo ingegno** co(m)prese el sublime volume de Euclide". The *perspicacissimo ingegno* of Camillo Vitelli in approaching Euclid's *Elements* is also mentioned at folio 24v of the *Tractato de l'architettura:* "E lor proportioni li troveranno fabricati e formati, dela qual cosa piu volte col nobil homo excellente armigero, Camillo Vitelli de Castello sopra di questo conferendo apertamente trovato habiamo. Nel tempo che optimamente el suo **perspicacissimo ingegno** co(m)prese el sublime volume de Euclide p(er) piu mesi da me expostili, e in nel degno gimnasio de Napoli legendo". In the same folio, another man-at-arms from Castello is also described as someone of all *ingegno*: "S. De castello detto la masneta, ho(mo) de tutto **i(n)gegno** a(n)i(m)o(?) e gagliardia sempr(re) da n(ostr)i S. Fiore(n)tini benissimo tractato".

At folio 2v of the dedicatory letter that opens the *Summa*, Pacioli argues that John Duns Scotus demonstrated a divine *ingegno:* "Giova(n)ni Scoto (maxi(m)e p(er) 2, 3, 4, de le sententie dove su(m)mamente suo divino **ingegno** dimostro)". At 68r (Pars prima), Pacioli notes that Archimedes demonstrated *ingegno* in mathematics while studying the relationship among the diameter and circumference of the circle: "De archimede siracusano, l'opera che feci de quadratura circuli, e de ce(n)tro gravitatis, et de figura pvali e de harene nu(mer)o ap(ert)o mostra co(n) dilige(n)tia haverne ditto. Maxime qua(n)do co(n)sua sutilita de **ingegno** trovo la p(ro)pinqui)ta de la co(n)venie(n)ta del diametro del cerchio, a la sua circu(m)fere(n)tia, per vie geometriche e arithmetich(e) p(re)cede(n)do […]". Archimedes is again mentioned at folio 2r of the *Compendium* (1509) as a geometer gifted with *ingegno* and most respectable architect: "[…] ch(e) per la propria patria el **nobile ingegnoso** geometra e dignissimo architetto Archimede fesse. El qual (commo e scripto) con sue nove e rare inventioni de machine per longo t(em)po la cita Siracusana contra l(')impeto e belicoso successo de Romani […]". Among the individuals mentioned as exemplar for their *ingegno*, a preeminent position is given to Euclid. At folio 119v (Pars prima) of the *Summa,* Book X of Euclid's *Elements* is mentioned as a proof of the most shrewd ingegno [*p(er)spicacissimo ingegno*] : "E accio meglio quello che segue se habi a intendere bisogna al quanto far discorso nel ordine del 10° de Euclide. Nel quale certamente lui con suo **p(er)spicacissimo ingegno** e speculativo discorso, nelle cose matematiche, disse si altamente quanto mai alcun(l')altro i(m)maginare potesse. In modo che pochi so(n)no che ben quello aprenda. E pero quello che qui de binomij e recisi se dira, dara gran lume a la materia del ditto decimo libro". At 21r of the *Compendium* (1509), we read that Euclid's *ingegno* not human but divine should be called: "al nostro megarense Euclide […] con tante dilige(n)ti soi demostratio(n)i, commo appare in tutto suo sublime volume. El cui **ingegno** non humano ma divino se dimostra. Maxime nel suo decimo, nel quale veramente tanto lo extolse q(uan)to a lo humano fia p(er)messo". Finally, Euclid and Plato's geometrical *ingegno* is mentioned in the opening sonnet added to the *Compendium* in 1509: "[…] Quattro elementi e ciel così nomati / Quali Platone vol che figurati / L'esser dien a infiniti fructi. / Ma perché el vacuo la natura abhorre / Aristotil in quel de cielo et mundo / per se non figurati volse porre / Però l'**ingegno** geometra profondo / di Plato e d'Euclide piacque exporre. / Cinqualtri che in spera volgan tundo / Regolari; d'aspeto iocundo / Come vedi de lati e basi pare / E un altro sexto mai se pò formare". This is because, as Pacioli argues at folio 2v-3r of the *Compendium*: "Aur(um) p(ro)bat(ur) igni et **ingeno** mathematicis cioe la bonta del loro demostra el fuocho e la peregrineça de l(')ingegno le mathematici discipline, che in sente(n)tia vol dire chel buono **i(n)gegno** a le mathematici [discipline] fia aptissimo a cadau(na) [scientia] che le sie(n)no de grandissima abstarctione e subtiglieçça perche sempre fuora de la materia sensibile se hano a considerare".

Since the *ingegno* supports the capacity for mathematical reasoning, the i*ngegno* is important whenever reasoning about three-dimensional geometrical bodies. At folio 21v of the *Compendium,* where Pacioli further considers the measuring of regular and other bodies derived from these, we read that throughout the *Compendium* Pacioli always assumed the individual *ingegni* and intellects of all readers to be able to engage in geometrical thinking involving three-dimensional bodies: "De la mesura de tutti li altri corpi regulari e depe(n)de(n)ti. […] E pero di loro q(ue)sto sia el docume(n)to opportuno no(n) diffidandome de i **peregrini ingegni** e speculativi intellecti a q(ue)ste e a qualunc(')altra faculta p(ro)nti quali sempre i(n) tutto n(ost)ro p(ro)cesso habiamo p(ro)suposti, maxime per excelle(n)tia e anthonomosia fra tutti gli altri sup(re)mo de q(ue)llo de V(ostra) D(ucale) Cel(situdine)". At folio 12v of the *Compendium* (1509) we read that by means of the *peregrinezza de li ingegni*, not only usefulness, but also great pleasure can be derived from the contemplation of three-dimensional bodies: "[…] e quel ta(n)to a cio me pare dover esser basta(n)te, che in lo p(ar)ticular n(ost)ro tractato de dicti corpi, co(m)posto nell(')opera se detto, al q(ua)l per la moltitudine al(l')u(n)iverso co(mun)icata facile fia el recorso. E media(n)ti loro dime(n)sioni in quel luogo poste secu(n)do la **peregrineçça de li i(n)gegni**, sempre se ne porra co(n) l(')utilita reportarne gra(n) dilecto." At folio 16r of the same text, we read that the form [*forma*] of the elevated [version] of the 72-base body, which Pacioli has not deducted in a material way, is left to the reader's *ingegno:* "De q(ue)sto [72 basi]ancora se porra semp(re) formare el suo elevato co(m)mo negli altri se f(a)ct(o), ma p(er) la difor(m)ita de le suoi basi sera difficile sua sci(enti)a qua(n)tuncha a l(')occhio re(n)desse no(n) mediocre vagheçça. E causarie(n)se in epso 72 pyramidi seco(n)do el numero de le suoi 72 basi, de le q(ua)li pyramidi le basi sirie(n)no le medesime di q(ue)lle, e lui de(n)tro imaginato, la forma del q(ua)le elevato n(on) curai fra q(ue)ste ma(teria)lme(n)te dedure p(er) lassar la p(ar)te sua anchora al lectore del cui **ingegno** no(n) mi diffido". To the contexts in which *ingegno* is linked to the capacity of imagination of three-dimensional bodies should also be linked a passage at folio 74v (Pars secunda) of the *Summa*, dealing with the gauging of barrels. Here we read that the gauging process should be often adapted to consider the individual three-dimensional shape of each barrel. Besides *arte,* Pacioli says, one should also employ the *ingegno* and thus look at the barrel with diligence inside and outside, and carefully consider what is added or taken away from the initial benchmark [three-dimensional] shape: "Ma q(ue)sto mo(do) p(re)suppone se(m)pre nel vaso uniformita la q(ua)l cosa el piu de le volte nelle botti no(n) si trova p(er) e(ss)er mal facte e co(n) doghe no(n) eq(ua)li etc. E p(er)o bisogna a te adop(r)are oltra l(')**arte** a(n)cora el tuo **i(n)gegno** i(n) guardare la botte co(n) diligentia de(n)tro e di fore e co(n)siderare piu apo(n)tino si po q(ue)llo che si p(er)de overo ava(n)ça a la uniformita etc. E q(ue)sto non e possibile con pe(n)na i(n)segnartelo p(er)o a te lo lascio meglio saparai etc.".

Following the contemplation of three-dimensional geometrical bodies, Pacioli also employs the word *ingegno* to refer to the contemplation of three-dimensional architectural elements and the capacity for reasoning and invention involved in the process of design. As noted in Chapter 4, this is found at folio 31r of the *Tractato de l'architectura*: "Co(n)chiude breviter Che oltra l(')arte el Buono architecto bisogna habia **inge(g)no** a suplire el dimenuto e smenuire el superfluo secondo la oportunita e dispositione de li lochi a cio non parino loro hedificii monstruosi . E a q(ue)sto effecto a voi a qualumc(')altro mi son messo a trovare co(n) grandissimi afanni e lo(n)ghe vigilie le forme de tutti li 5 corpi regulari co(n) altri loro dependenti […]". At folio 25r (mistakingly marked as 17r) of the *Tractato de l'architectura*, Pacioli had introduced a number of architectural elements and their illustration in a door similar to the one of the Temple of Salamon, adding that variations to the three-dimensional dispositions of their parts [*dispositioni*] could be devised through the *ingegno*: "E prima diremo de le colonne tonde co(m)me in li edifitii le habiate con vostri scarpelli debitame(n)te disponere si per la forteça a subtentatione de lo hedifitio co(m)me per loro ornamento. E poi diremo de lo epistilio o vero architrave e sua compositione. De li quali habiando detto poi li situaremo i(n) l(')opera de una porta qual fia a similitudine di quella del tempio de salamone in Hierusalem prenunciata per lo propheta eçechiel con l(')altre dispositioni. E voi poi per vostro **ingegno** potresti piu e manco farne."; at folio 29r of the *Tractato de l'architectura,* we read that all parts of the columns can be changed *ad libitum* as different *ingegni* in different places have done: "Come se sia succede(n)do dainde in qua diversi **ingegni** e natio(n)i se costumato far a libito dicte colo(n)ne e q(ue)lle noi (f)are diversamente e lor capitelli e basi e stilobate e cosi ogni lor parte e anche in li altri hedificii". At folio 30r-v of the *Tractato de l'architettura*, Pacioli relies on the *ingegno* of his students to understand the proportioning of the parts of circular columns: "De le quali [colonne laterate] in vero non diro altro se non quello che de le tonde finora habiam detto confidandome nelli vostri **peregrini ingegni** e con quella parte, maxime a ogni operante necessaria qual da me havete con diligentia intesa, cioe de numeri e misure con la pratica de loro p(ro)portioni con le quali mi trendo certissimo che sempre saprete p(ro)prortionare co(n) li vostri acomodati strumenrti circino e libella cioe mediante la linea recta e curva, con le quali comme sopra fo detto ogni operatione a degno fine se conduce."; similarly, at folio 33r of the same *Tractato* we read that the students should have no problems in approaching contents of the text since Sansepolcro has always been a place which provided good *ingegni*, both in military arts and in other disciplines and sciences: "Ave(n)ga che lor forme sieno aplaco e qui al n(ost)ro dire porremo fine pregandove insta(n)temente che fra voi l(')uno co(n) l(')altro a uso de bon fratelli voliate co(n)ferire a piu delucidatione de tutto peroche facile fia lo arogere a le cose trovate co(m)me son certo li vostri **peregrini ingegni** fara(n)no si p(er) loro honore co(m)me de la terrea n(ost)era de la q(ua)le sempre in ogni facolta co(m)me da li vostri antenati potete avere inteso so(n)no usciti degni ho(min)i benche il luogo sia angusto pur e populoso. E buoni **inge(g)ni**. Si i(n) militaribus co(m)me di sopra sucinte score(m)mo co(m)me in altre discipline e scientie che de le mathematici lo rende chiaro el monarcha a li di nostri della pictura e architectura, maestro Pietro de li franceschi […]"; at folio 28v, we read that by means of *ingegno*, Pacioli's students can add to what in the text is missing: "E voi per vostro **ingegno** son certo che meglio aprehendarete che io no(n) dico".

Finally, the last occurrence of the word *ingegno* further clarifies the nature of the relationship established by Pacioli between the individual *ingegno* and the process of representation taking place at the level of the intellect. Here, Pacioli addresses directly the reader of his work, and argues that the eye of the reader's individual *ingegno* will represent to the intellect the [three-dimensional] architectural elements shown [by means of drawings], and which will be presented again in the last plate. The text is found in the caption to the entablature illustration appended to the *Tractato de architectura:* "[…] mi rendo certo Lectore che a l(')intellecto debitamente l(')ochio del tuo **peregrino i(n)gegno** representa co(n) li recordi che di sotto per la tavola trovarai".