UNRESTRICTED FREE ALGEBRAS

BY

A. L. ALLEN

A THESIS SUBMITTED FOR THE DEGREE DOCTOR OF PHILOSOPHY

AT

THE UNIVERSITY OF KENT AT CANTERBURY

ON

11th JULY 1968

ACKNOWLEDGEMENTS

First I would like to express my thanks to my supervisor, Dr. S. Moran, who suggested the problem and advised me on numerous occasions. Next I would like to thank the Science Research Council for generous financial support while I was researching at The University of Kent at Canterbury from October 1965 until September 1968.

CONTENTS

UNRESTRICTED FREE ALGEBRAS Ph.D A.L.Allen 1968.

In section 1 we construct the unrestricted free nonassociative algebra $A$ ,we show that $A$ is afree nonassociative algebra.

In sections 2 and 3 we consider the problem of showing that the unrestricted free associative algebra $L$ is a free associative algebra. We make use of results due to S.Moran(23), P.Cohn(5) and the Poincaré-Birkhoff-Witt Theorem.

In section 4 we show that the unrestricted free commutative algebra $C$ is a free commutative algebra, using a corollary of the Poincaré-Birkhoff-Witt Theorem.

In the fifth and final section we establish some results on the completion of $\Omega$-groups following M.Hall (11),and then establish via a subalgebra theorem of S.Feigelstock(6) that the projective limit of free anarchic algebras is a free anarchic algebra. We conclude with this last result.

# REVIEW OF THE LITERATURE ON UNRESTRICTED PRODUCTS OF GROUPS, AND ALGEBRAS.

## PRELIMINARIES

For an understanding of this review it is necessary to have some knowledge of verbal products and nilpotent products . We discuss these under the general heading ' products of groups'. The list of papers quoted in this discussion can be found in Combinatorial Group Theory, Magnus, Karrass and Solitar, Inter--science, Wiley, 1966. We will therefore give only the year and author, not the full description of the publication. Unrestricted products will be dealt with under a separate heading ,and then descriptions of publications will be given in full. Finally, we will outline the relevant details of the research contained in this thesis. Indicating ,in particular , how it follows on from the work on unrestricted products of groups.

## PRODUCTS OF GROUPS.

Golovin in 1950, investigated the question of whether the concepts of the direct product and free product of groups are special cases of a wider class of products . To present his results we use the following notation . Given any two groups, $A$ and $B$ ,we denote by $A \times B$ their direct product and $A * B$ their free product. By $A \circ B$ ,we denote the direct or free product or any multiplicative operation, still to be defined . Also , we shall use the notation $A^G$ to denote the normal closure of $A$ (the smallest normal subgroup containing $A$ )in the group $G$ ,where $A$ will usually be a subgroup of $G$ ,but may be any set of elements of $G$ .

Now we list six properties of multiplication of groups, all of which are satisfied by the direct and free product:

I. Given any two groups $\tilde{A}$ and $\tilde{B}$ ,there exist a group $G$ denoted by $A \circ B$ and called the product of $\tilde{A}$ and $\tilde{B}$ ,such that $G$ contains an isomorphic copy $A$ of $\tilde{A}$ and an isomorphic copy $B$ of $\tilde{B}$ and is generated by $A$ and $B$ .

II. $A^G$ intersects $B$ in the identity, and $B^G$ intersects $A$ in the identity .

III. $A \circ B \simeq B \circ A$ under the isomorphism which maps the subgroups $A$ , $B$ of the first product into the subgroups $A$ , $B$ respectively ,of the second product. (We call this isomorphism the natural isomorphism)

IV. If $\tilde{A}$, $\tilde{B}$ , $\tilde{C}$ are any three groups (with $C \simeq \tilde{C}$ ), then

$$(A \circ B) \circ C \simeq A \circ (B \circ C)$$

again under the natural isomorphism.

V. Let $M$ be any normal divisor of $A$ and let $N$ be any normal divisor of $B$ .Then, if $G = A \circ B$

$$(A/M) \circ (B/N) \simeq G/(M^G \cdot N^G)$$

under the natural isomorphism , mapping $A/M$ and $B/N$ in $(A/M) \circ (B/N)$ onto the subgroups $(A \cdot M^G \cdot N^G)/(M^G \cdot N^G)$, $(B \cdot M^G \cdot N^G)/(M^G \cdot N^G)$ of $G/(M^G \cdot N^G)$ respectively. The dot denotes the usual group operation.

VI. Let $H \subset A$ and $K \subset B$ be any subgroups of $A$ and $B$ respectively . Then the subgroup $S$ of $G = A \circ B$ generated by $H$ and $K$ is isomorphic to $T = H \circ K$ under the isomorphism which maps $H \subset S$ onto $H \subset T$ and $K \subset S$ onto $K \subset T$.

(We call this isomorphism the natural isomorphism). Golovin in 1950, called a product satisfying I & II a <u>regular product</u>, and a product that also satisfies III and IV a <u>fully regular product</u> If we use the notation $(A,B)$ for the subgroup generated by the commutators $(a,b)$ with $a$ in $A$, $b$ in $B$, we see that $(A,B)$ is a normal divisor of $A*B$ and under the natural isomorphism, $A\times B \simeq (A*B)/(A,B)$. Golovin showed that any regular product $A\circ B$ has the property that, under the natural isomor--phism, $A\circ B \simeq (A*B)/N$ where $N\subset(A,B)$ and $N$ is a normal subgroup of $A*B$ .Ruth Struik in 1956, showed that there exist products for which I and II but not III, or I,II andIII but not IV, or I,II,III andIV but not V is satisfied. In addition, she gave an example of a product satisfying I,II,III and V but not IV. Regular products satisfying III but not IV were also constructed by S.Moran 1956, and by Benado 1956, 1957. Golvin, 1950, constructed an infinite sequence of fully regular products all of which also satisfy condition V. His construction was given in different forms by S.Moran 1956, and R.Struik 1956. Following Golovin, we shall define, for $k=1,2,3,\ldots$, a product $A \underset{k}{\circ} B$ which will be called the <u>nilpotent product</u>(more properly, <u>$k$th nilpotent product</u>) of $A$ and $B$ .To do so, we need the following notation:

Let $G$ be any group and let $H$ be any subgroup of $G$ .We define for $k=0,1,2,\ldots$ : ${}_0H_G = H^G$ , ${}_kH_G = ({}_{k-1}H_G, G)$ .Then the $k$th nilpotent product is defined by

$A \underset{k}{\circ} B = (A * B) / {}_k(A, B)_G$ (Golovin)

$A \underset{k}{\circ} B = (A * B) / ({}_0A_G, {}_kB_G) \cdot ({}_kA_G, {}_0B_G)$ (Struik)

$A \underset{k}{\circ} B = (A * B) / (A, B) \cap {}_k(A * B)_G$ (Moran)

where throughout $G$ is used as an abbreviation for $A * B$. The equivalence of the three definitions can be derived from an identity proved independently by Struik, 1956, and Moran, 1956:

$${}_k(A, B)_G = \prod_{n+m=k} ({}_mA_G, {}_nB_G) = ({}_0A_G, {}_kB_G) \cdot ({}_0B_G, {}_kA_G)$$

where $G = A * B$ .Golovin, Moran , and Struik proved that ,in general, $A \underset{k}{\circ} B$ and $A \underset{\ell}{\circ} B$ are not isomorphic under the natural isomorphism if $k \neq \ell$ .

Moran, 1956, showed that Golovin's nilpotent products are special cases of a much more general class of products which he called <u>verbal products</u> .To construct them ,we define first a fixed (but otherwise arbitrary) verbal subgroup $V(G)$ for every group $G$ . Then we define the $V$-product $A \underset{V}{\circ} B$ by

$$A \underset{V}{\circ} B = (A * B) / (A, B) \cap V(A * B)$$

For all possible types of verbal subgroups the $V$ -product of groups is fully regular and satisfies postulate V. In addition, it is possible to write down explicitly the $V$-product of any set of groups $\tilde{A}_\alpha$ , where $\alpha$ runs through an arbitary index set $L$ . (Since we shall use the free product of the $A_\alpha$ we may identify them with their replicas in their free product, omitting the use of the $\tilde{A}_\alpha$). For this purpose, let $G$ be the free product of all the $A_\alpha$, and let $C(G)$ be the product of the normal closures of all $(A_\alpha, A_\beta)$ in $G$ , where $\alpha \neq \beta$ , $\alpha, \beta \in L$.
We call $C(G)$ the <u>Cartesian subgroup</u> of the free product of the $A_\alpha$ .

Obviously, $G/C(G)$ is the (restricted) direct product of the $A_\alpha$. Then Moran showed that the $V$-product of the $A_\alpha$ is given by $G/C(G)\cap V(G)$. Also S. Moran, 1956, proved that $C(G)$ is always a free group.

Except for verbal products, no other products satisfying I to V are known. On the other hand, the free and direct products are the only verbal products known to satisfy postulate VI. In fact Wiegold has shown (unpublished) that free and direct products are the only verbal products which satisfy postulate VI. However, Moran in two papers published in 1959 constructed larger classes of regular products of groups satisfying postulates III and IV which are not verbal products in general.

Golovin, 1950, had shown that each decomposition of a given group into a regular product corresponds to a set of orthogonal idempotent endomorphisms. Benado, 1956,1957, used this result as a starting point for an investigation of associative products, and for constructing examples of nonassociative products.

In general, it is a difficult task to prove that two verbal pro-ducts are different if the verbal subgroups used for their definition are different in a free group on sufficiently many generators. If we define the $l$-th soluble product as the verbal product arising from the case when $V(G)$ is the $l$-th derived group of $G$, then it can be shown (Moran 1958) that for $l \geqslant 2$ the soluble product of Abelian groups contains a locally infinite sub-group. Since Golovin, 1950, 1951, had proved that the nilpotent products of a finite number of finite groups are themselves, finite it follows that the soluble products are, for $l \geqslant 2$, not nilpotent products. R.Struik, 1959, proved that a large class of verbal products (defined by using 'complex' commutators for the definition of the underlying verbal subgroups) are different from each other and from Golovin's nilpotent products.

All of the problems that have been studied in connection with the definition of the direct and of the free product also can be investigated for verbal and other fully regular products of groups. These investigations have been carried out in part for nilpotent, especially for 2-nilpotent, products by Golovin 1951 (two papers). Maximality conditions have been found by S.Moran 1958, who also showed that a group isomorphic to a verbal product $A \overset{\circ}{\vee} B$ where $\overset{\circ}{\vee}$ does not denote the free product, cannot be decomposed into a free product (of non trivial groups) except when $A$ and $B$ are of order two.

There are many other results of Golovin, Moran, Benado and Struik, but for our purposes we do not need to give a too detailed discussion.

<u>UNRESTRICTED PRODUCTS</u>

We begin our survey of the literature on unrestricted products by first giving some of the results of R. Baer ( 1 ) on torsion free abelian groups. We will only give those results which relate to the unrestricted sums of infinite cyclic groups; usually called the complete direct sum of infinite cyclic groups see Fuchs (27).

First we have some definitions. A torsion free abelian group is <u>completely decomposable</u> (irreducible) if it is a complete direct sum of groups of rank 1. A homogenous abelian group is an abelian torsion free group all of whose elements $(\neq 0)$ have the same type (for a definition of type see Fuchs (27)). A torsion free abelian group $G$ is separable if every finite subset of the group $G$ can be embedded in a completely decomposable direct sum of $G$ .

Now it is shown by R. Baer ( 1 ) that the complete direct sum of an infinite set of infinite cyclic groups is torsion free. More particularly, the essential part of the following result was proved by R. Baer ( 1 ) "A complete direct sum of an infinite set of infinite cyclic groups is $\aleph_1$-free, but is not free".

Also in the same paper R. Baer showed that if $G$ is a homogenous group of finite rank $r$, $G$ is completely decomposable if and only if $G/B$ is finite whenever the subgroup $B$ of $G$ is the direct sum of $r$ pure subgroups of rank 1. Note a pure subgroup $S$ of $G$ is a subgroup $S$ containing the solution to $nx=a$ if the equation also has a solution in $G$ ; $a \in S$, $\forall n$ an integer. It can be deduced that a particular case of an homogenous group is a complete direct sum of an infinite number of infinite cyclic groups. Conditions for an homogenous group to be completely decomposable are given in ( 1 ) and it is remarked that a complete direct sum of infinite cyclic groups does not satisfy one of these conditions.

An important example of a separable not completely decomposable group is a complete direct sum of an infinite number of infinite cyclic groups. As it turns out, separable subgroups are completely decomposable.

There are many more results contained in this most important paper of R. Baer ( 1 ). However, we have tried to consider only those results which relate directly to the complete direct sum of an infinite number of infinite cyclic groups. For more information consult the original paper or L. Fuchs ( 27 ).

E. Specker in (24) considered formal sequences of integers as a particular case of the complete direct sum of an infinite number of infinite cyclic groups. His results are as follows: Let $F$ be the additive group of sequences $\{a_n\}$ of integers. A growth-type is any subset $\phi$ of the totality $X$ of increasing sequences $\{p_n\}$ of natural numbers such that

$$\text{(i)}\ \{p_n\} \in \phi\ ,\ \{q_n\} \in X\ \&\ q_n \leq p_n \Rightarrow \{q_n\} \in \phi$$

$$\text{(ii)}\ \{p_n\}, \{q_n\} \in \phi \Rightarrow \{p_n + q_n\} \in \phi.$$

To each growth type $\phi$ is associated a subgroup $F_\phi \subseteq F$ $\{a_n\} \in F_\phi$ if $\{\max_{i \leq n}(1, |a_i|)\} \in \phi$ . The growth type consisting of all bounded sequences in $X$ is denoted by $\eta$ . Then the following results are deducted.

(i) Totality of growth types has cardinal number $2^{\aleph}$,

(ii) $F_\phi \cong F_\psi$ only if $\phi = \psi$

(iii) All subgroups of $F$ of cardinal number $\aleph_0$ and all subgroups of $F_\eta$ of cardinal number $\aleph_1$, are free abelian.

(iv) If $\phi \neq \eta$ , $F_\phi$ has a non-free subgroup of cardinal number $\aleph_1$ .

In 1952 G. Higman ( 13 ) published a paper in which he constructed the unrestricted free product of a family of groups. We now give this construction and discuss the salient features of that paper.

Let $G_\alpha$ , $\alpha \in A$ , be a family of groups indexed by a set $A$ . The finite subsets of $A$ form a directed set if ordered by inclusion. For each finite subset $r$ of $A$ , construct $D_r$ , the free product of the groups $G_\alpha$ , $\alpha \in r$ . If $r \subseteq S$ are finite subsets of $A$ , then a natural homomorphism of $D_S$ onto $D_r$ is given by mapping onto the identity those $G_\alpha$'s with $\alpha \in S$ , $\alpha \notin r$ . These homomorphisms and the directed sets determine the inverse (or projective) limit of the groups $D_r$ and this is taken as the definition of the unrestricted free product $F$ of the groups, $G_\alpha$ . This is in analogy to the unrestricted direct product, which could be constructed in a similar way using direct products.

The unrestricted free product $F$ contains the ordinary free product $F^{(\omega)}$ as a subgroup; $F$ and $F^{(\omega)}$ coincide only if $A$ is finite. If $A$ is infinite $F$ can be regarded as the completion of $F^{(\omega)}$ in terms of the subgroup topology of M.Hall ( 11 ) where we take as neighbourhoods of the identity the kernels of the natural homomorphisms mapping $F$ onto $D_r$ .

A number of results are found for the case in which $F$ is the unrestricted product of infinite cyclic groups . $F$ is not a free group. The derived group is not closed. $F$ contains a subgroup $P$ which is not free but is such that the only freely irreducible( that is, cannot be written as the free product of proper subgroups) subgroups of $P$ ~~are~~ are infinite cyclic groups. In 1953 G. Higman ( 14 ) put these results to good use by disproving a conjecture of Takahasi. Takahasi asked. 'Is every countable group $G$ which satisfies (i) $G$ is locally free, (ii) $G$ possesses no infinite properly ascending sequence of $m$ -generator subgroups, for any fixed integer $m$ ,a free group'. G.Higman showed that the subgroup $P$ of the unrestricted free product of a countable number of infinite cyclic groups satisfies (i) and (ii) ,but by his previous result given above this subgroup is not free.

As recently as 1964 M. Burrow ( 3 ) showed that if $F$ is the unrestricted product of countably many free cyclic groups and $G$ is a homomorphic image of $F$ with a chain :

$$G = N_0 \supset N_1 \supset N_2 \; . \; . \; . \; . \; . \; N_n = 1$$

of subgroups $N_i$ , such that $N_{i+1} \triangleleft N_i$ and $N_i/N_{i+1}$ is free abelian, then $G$ is finitely generated.

A.Hulanicki, and M.F. Newman ( 16 ) 1963, obtained some specialised information on an unrestricted direct product with one amalgamated subgroup.

**They amalgamated a central subgroup $H$ and showed that the unrestricted direct product exists, if and only if, $H$ is algebraically compact.**

By an example the authors showed that when the product exist it need not be unique. An error which occured in this paper was corrected by A.Hulanicki in ( 17 ). We mention in passing that A.Hulanicki and K.Golema ( 15 ), obtained some results on the structure of the factor group of the unrestricted sum by the restricted sum of Abelian groups.

Following on the work of finding more general products of groups S.Moran in (22), 1961 constructed the unrestricted regular product of an arbitary family of groups, using the method developed by G.Higman ( 13 ) when he constructed the unrestricted free product. The only change made by S.Moran was that the free product was replaced by an associative regular multiplication. The results obtained were the following.

(i) The regular product can be embedded as a subgroup in the unrestricted regular product.

(ii) The unrestricted direct product, of the family of groups, is a factor group of the unrestricted regular product.

Among the classes of associative regular multiplications introduced by S.Moran are the verbal multiplications, and a definition of unrestricted verbal product is given in terms of these. Several other properties of **The unrestricted free products were obtained**. For example: a group cannot be decomposed into a restricted **and unrestricted free** product of proper subgroups.

In 1962 S.Moran (23), turned his attention to unrestricted nilpotent products. He defined the unrestricted verbal product, but mainly concerned himself with the case when the verbal subgroup function determined the $(n+1)$-th term of the lower central series, each $G_\alpha$ was infinite cyclic and $A$ was countable. Under these conditions the corresponding $V$-product is a free nilpotent group of class $n$ and countable rank. Now let us denote by $G$ the corresponding unrestricted $V$-product. The following results were obtained.

(i) If $H$ is a countable subgroup of $G$ such that $H$ modulo its centre is finitely generated, then $H$ is isomorphic to a subgroup of a free nilpotent group of class $n$.

(ii) The Mal'cev completion of $G$ is isomorphic to the Mal'cev completion of a subgroup of a free nilpotent group of class $n$.

The <u>Mal'cev completion</u> of a torsion-free nilpotent group $A$ is a divisible and torsion-free group $B$, which contains $A$, is nilpotent of the same class as $A$ and is such that some positive power of every element of $B$ lies in $A$.

(iii) If $\psi$ is a homomorphism of $G$ into a free nilpotent group then $\psi$ maps the unrestricted product of all but a finite number of the factors onto the unit element.

The proof of (ii) relies on the following fact : the unrestricted Lie algebra over a field $\Omega$ is a free Lie algebra over $\Omega$ . This as far as we are concerned is of great importance and is taken as the starting point of this thesis . However ,this will be discussed further on.

In the last two sections of this paper the author investigates similar problems for the unrestricted soluble products of infinite cycles (i.e., where the verbal subgroup function $V$ determines some term of the derived series) and the unrestricted 3rd. Burnside products of (a) infinite cycles and (b) cycles of order three (verbal subgroup function determined by the word $x^3$ ).

H.B. Griffith in ( 7 ) discussed the unrestricted free product of a sequence of groups $\{G_n\}$ . This is the inverse limit $K$ of the free product $K_n$ of $G_1, G_2, \dots G_n$ with the homomorphisms $K_{n+1} \to K_n$ determined by $G_{n+1} \to 1$ . It is shown that in the natural topology for $K$ the derived group $[K, K]$ is not closed.

This result rests on showing that in an ordinary free product the product of more than $12n-2$ elements from distinct $G's$ is not a product of $n$ commutators.

In a later paper ( 8 ) the author improves this latter result. [illegible] shows that if $h$ is the free product of $m$ elements each from $G_1, \dots G_m$ [illegible] $h$ is not the product of fewer than [illegible] commutators by his [illegible]

conclusion holds if $m > n$ . The proof is complicated.

It may be suspected that G. Higman's results given in ( 13 ) can be applied to semi-groups with an identity. This was done by H.B. Griffiths in ( 9 ). Some of the results obtained are as follows. Let a semigroup mean a semigroup with identity. Then there is a natural retraction of the free product of two semigroups on either factor and the unrestricted free product $A_H$ of the sequence $\{A_i\}$ of semigroups can be defined in the same way as G.Higman did in ( 13 ). If $\{B_i\}$ is a second sequence of semigroups, and $\gamma_i$ is a homomorphism of $B_i$ into $A_i$ $(i = 1, 2, \ldots)$ there is a naturally defined homomorphism $\gamma : B_H \rightarrow A_H$ . Even if each $\gamma_i$ is onto $A_i$ $\gamma$ is not in general onto $A_H$ . In particular, if each $B_i$ is a free semigroup $\gamma$ maps $B_H$ onto a subsemigroup $A_T \subseteq A_H$ which the author calls the "Topologists' Product". If $S_n$ denotes the subsemigroup of $A_H$ generated by $A_1, A_2 \ldots A_n$, and the unrestricted free product of the sequence $\{A_{n+i}\}$, then $A_T$ can also be characterised as the inter--section of all $S_n$ . Lastly, if all the $A_i$ are all groups, Let $X_i$ be a metric space having $A_i$ as fundamental group, let the diameter of $X_i$ tend to zero as $i$ increases. Let the $X_i$ have a single point in common, at which each is in a suitable topology locally connected ( 9 ). Then $A_T$ is the fundamental group of their union. There are some applications to the theory of local connectivity.

This review brings us almost up to date ,in respect of the work done on unrestricted products of groups. In 1966 O.N. (Macedonskaja (21) introduced some results on polyverbal operations. In fact, those polyverbal operations of Macedonskaja were nonassociative. Various other writers in particular P.W. Stroud (25) 1965 discussed verbal and marginal subgroups. It seems likely that some of these generalisations may be applicable to the unrestricted product.

UNRESTRICTED FREE ALGEBRAS

We conclude this review of the literature with a summary of results contained in this thesis.

In section 1 we construct the unrestricted free nonassociative algebra A. We show that the subalgebra A of all elements of finite degree in A is a free non-associative algebra

In sections 2 and 3 we show the subalgebra L of all elements of finite degree in the unrestricted free associative algebra L is a free associative algebra. We make extensive use of results due to S.Moran ( 23 ), P.Cohn ( 5 ) and the Poincaré-Birkhoff-Witt Theorem.

In section 4 we show the subalgebra C of all elements of finite degree in C is a free commutative algebra, using a corollary of the Poincaré-Birkhoff-Witt Theorem (P.B.W. Theorem).

In the fifth and final section we establish some results on the completion of $\Omega$ -group following M. Hall ( 11 ) and then establish a subalgebra theorem of S. Feigelstock*( 6 ) We conclude with this last result.

* The reference to Feigelstock is in fact, a well known result, see for example P.Cohn Universal Algebras, Harper Row, pp 125 Ex.5. (x3)

## INTRODUCTORY REMARKS

All the unrestricted algebras considered in this thesis are formed from a countable number of factor algebras of increasing rank. It is clear, however, that the results we obtain can be extended to an arbitary family of factor algebras. This is not done explicitly in the text since the notation is likely to become somewhat cumbersome.

The two main tools used throughout are the inverse (projective) limit and the Poincaré-Birkhoff-Witt Theorem: Theorem (2·9) in the text. It was felt, therefore, that a proof of the Poincaré-Birkhoff-Witt Theorem, should be included. This was done, the proof being that as given in N. Jacobson, Lie Algebras, Interscience, 1962.

Excluding the survey of the literature on unrestricted products, the work is divided into five main sections.

### Notation

The standard notation of decimal point system of numbering of equations is used throughout the text.

<u>DEFINITION 1.1</u> An inverse system of sets

An inverse system of sets $\{X, \pi\}$ over a directed set $M$ is a function which assigns to each $\alpha \in M$, a set $X_\alpha$, and to each pair $\alpha$, $\beta$ such that $\alpha < \beta$ a mapping

$$\pi_{\alpha\beta} : X_\beta \longrightarrow X_\alpha$$

such that

$$\pi_{\alpha\alpha} = 1_{X_\alpha} \quad \text{the identity on } X_\alpha,$$

for $(\alpha < \beta < \gamma)$ $\pi_{\alpha\beta}\pi_{\beta\gamma} = \pi_{\alpha\gamma}$. The mappings $\pi_{\alpha\beta}$ are called the <u>projections</u> of the system.

<u>DEFINITION 1.2</u> The projective or inverse limit

Let $\{X, \pi\}$ be an inverse system of sets over a directed set $M$. The <u>projective or inverse limit</u> of the $\{X, \pi\}$ is the subset of the cartesian product

$$\prod_{\alpha \in M} X_\alpha$$

consisting of those functions $x = (x^{(\alpha)})$ such that, for $\alpha < \beta$ in $M$

$$\pi_{\alpha\beta} x^{(\beta)} = x^{(\alpha)} .$$

We denote the projective limit by

$$P.L.(X_\alpha)$$

. Given a topology for each $X_\alpha$ we can assign a topology to $P.L.(X_\alpha)$, namely the Tychonoff topology induced by

$$\prod_{\alpha \in M} X_\alpha .$$

THE PROJECTIVE LIMIT OF FREE NONASSOCIATIVE ALGEBRAS

Let $A_K$ denote the free nonassociative algebra of rank $K$ having the elements $x_1, x_2, \ldots x_K$ as its free generators over a field $\Omega$ . If $m > K$ , there exist a natural homomorphism $\pi_{Km}: A_m \to A_K$ which maps $x_{K+1}, x_{K+2}, \ldots x_m$ into the zero element of $A_K$. For any given $K$ , a basis for $A_K$ is given by the fundamental monomials.

DEFINITION 1.3 (Fundamental monomials and degree)

A fundamental monomial of $A_K$ is a suitably bracketed nonassociative product in the free generators $x_1, x_2, \ldots, x_K$ . We assign an integer to each fundamental monomial called the degree. Each of $x_1, x_2, \ldots x_K$ has degree one and the degree of any other fundamental monomial is obtained by adding up the degrees of the free generators which occur in that fundamental monomial.

Under the homomorphisms $\pi_{Km}: A_m \to A_K$ we form the projective limit of the free nonassociative algebras $\{A_K\}_{K=1,2,\ldots}$ and denote it by $A = \varprojlim (A_K)$

The fact that the subalgebra $\underline{A}$ of all elements of finite degreeis a free nonassociative algebra is demonstrated below

First we have some preliminary lemmas, a construction and some notation.

NOTATION

In what appears below $\sum$ and $\sum^*$ will denote the restricted and unrestricted sum respectively. $\phi^{(K)}$ will denote the natural projection homomorphism of $\underline{A}$ onto $A_K$ obtained by mapping $x_{K+1}, x_{K+2}, \ldots$ onto the zero element of $A_K$ . The image of $a \in \underline{A}$ under $\phi^{(K)}$ will be denoted by $a^{(K)}$ . Every element $a$ of $\underline{A}$ can be written uniquely in the form

$$a = \sum_{l=1}^{\infty} \sum_{i=1}^{\infty}{}^{*} \alpha_{il}\, b_i(l)$$

where $\alpha_{il} \in \Omega$ and the unrestricted summation runs over all the fundamental monomials of fixed degree $l$ in the free generators $x_1, x_2, \ldots$ . These monomials are denoted by $b_i(l)$ in the above summation. For fixed $l$ a fundamental monomial $b_i(l)$ occurs before $b_j(l)$ in the unrestricted sum if for some positive integer $K$ $\phi^{(k)}(b_j(l)) = 0$ while $\phi^{(k)}(b_i(l)) \neq 0$ , An element of the form $a = \sum_{i=1}^{\infty}{}^{*} \alpha_i\, b_i(l)$ is said to have <u>degree</u> $l$ in $\underline{A}$. ${}_l\underline{A}$ will denote all those elements of $\underline{A}$ which have degree not less than $l$ together with the zero element. ${}_l\underline{A}$ is an ideal.

<u>CONSTRUCTION 1.4</u>

First we notice that since ${}_l\underline{A}/{}_{l+1}\underline{A}$ is a vector space over a field $\Omega$, it must therefore have a basis. Let $B_1 = C_1$ be a set of elements of $\underline{A}$ that is linearly independent modulo ${}_2\underline{A}$ . Suppose that the sets $B_\nu, C_\nu$ have already been defined for all $\nu < n$ where $n > 1$ and the elements of the sets $B_\nu$ $(\nu = 1, 2, \ldots)$ have been so ordered that an element of $B_\nu$ is greater than element of $B_{\nu'}$ if $\nu > \nu'$ . We define $C_n$ to be the set of all fund--amental monomials on the elements of the sets $B_1, B_2, \ldots B_{n-1}$ which belong to ${}_n\underline{A}$ but do not belong to ${}_{n+1}\underline{A}$ . Finally, $B_n$ is a set of elements of ${}_n\underline{A}$ which is linearly independent modulo the subalgebra generated by ${}_{n+1}\underline{A}$ and the set $C_n$.

Before we can apply this construction we prove the following.

<u>LEMMA 1.5</u>

Let $a_1, a_2, \ldots a_r$ be elements of the unrestricted sum $\underline{A}$ of a countably infinite number of one dimensional vector spaces over a field $\Omega$ . Then $a_1, a_2, \ldots a_r$ may be embedded in a direct summand of $\underline{A}$.

That is, $\underline{A} = \sum_{\lambda=1}^{\infty}{}^{*} \Omega_\lambda$ is the direct sum of a subspace containing $a_1, a_2, \ldots a_r$ and the subspace $\sum_{\lambda \geq s}^{*} \Omega_\lambda$, for some positive integer $s$.

Proof: Let $\Lambda$ be the set of positive integers. Suppose for each value of $\lambda$ we have a fixed field $\Omega$ ( recall $\underline{A}$ is defined over $\Omega$ ). If we form the unrestricted sum of $\Lambda$ copies of this field $\Omega$ then every element $a$ of $\underline{A}$ has a unique representation of the form

(1.5.1) $$a = \sum_{\lambda \in \Lambda}{}^{*} \alpha_\lambda \qquad (\alpha_\lambda \in \Omega)$$

The proof proceeds by induction on $r$. If $r = 1$, then we must embed $a_1$ in $\underline{A}$. Let $\alpha_{\lambda_1}$ be the first non-zero coefficient of $a_1$ in its unrestricted representation corresponding to (1.5.1).

We write

$$\underline{A} = \{a_1\} + \underline{A}'$$

where $a$ belongs to $\underline{A}'$ if and only if in the representation of $a$ $\alpha_\lambda = 0$ for $\lambda = \lambda_1$

The summation is obviously direct. Suppose now that we have two elements $a_1, a_2$, let $\alpha_{\lambda_1}, \alpha_{\lambda_2}$ be their corresponding non-zero coefficients in their corresponding representations. If $\lambda_1 \neq \lambda_2$ then write

$$\underline{A} = \{a_1, a_2\} + \underline{A}''$$

where $a$ belongs to $\underline{A}''$ if and only if in the representation of $a$ $\alpha_\lambda = 0$ for $\lambda = \lambda_1$ and $\lambda = \lambda_2$. If $\lambda_1 = \lambda_2$ let $\alpha_{\lambda_2'}$ be the next nonvanishing coefficient in the representation of $a_2$, then write

$$\underline{A} = \{a_1, a_2\} + \underline{A}''$$

where $a$ belongs to $A''$ if and only if in the representation of $a$ $\alpha_\lambda = 0$ for $\lambda = \lambda_1$ & $\lambda = \lambda_2'$.

Finally if we have elements $a_1, a_2, \ldots a_r$ after $r$ steps we obtain

$$\underline{A} = \{a_1, a_2, \ldots a_r\} + \underline{A}^r$$

where $a$ belongs to $A^r$ if and only if in the representation of $a$, $\alpha_\lambda = 0$ for a finite sequence of values of $\lambda$; $\lambda_{i_1}, \lambda_{i_2} \ldots \lambda_{i_r}$.
This completes the proof of lemma 1.5.

LEMMA 1.6

If $B_1, B_2, \ldots B_{n-1}$ are finite sets, then $C_n$ is a set of linearly independent elements of ${}_n\underline{A}$ modulo ${}_{n+1}\underline{A}$, for $n = 1, 2, \ldots$

Proof: We proceed by induction on $n$. The result is obviously true by construction 1.4 when $n = 1$. Suppose that the result is true for $C_1, C_2, \ldots, C_{n-1}$. Now as these sets are finite for every $m$ $(1 \leq m \leq n-1)$, it is possible by the lemma 1.5 to use a direct decomposition of the space $\{{}_m\underline{A}/{}_{m+1}\underline{A}\}_{m=1,2,\ldots}$ and bring all the elements of the sets $B_1, B_2 \ldots B_{n-1}$ together with the fundamental monomials on the elements of these sets through to a direct summand of ${}_m\underline{A}/{}_{m+1}\underline{A}$ $(m = 1, 2, \ldots)$
We now do this.

There exist elements $d_i(m)$ of ${}_m\underline{A}$ and positive integers $q(m), N(m)$ such that

$$(1.6.1) \qquad {}_m\underline{A}/{}_{m+1}\underline{A} = \left(\sum_{i=1}^{q(m)} \{d_i(m) + {}_{m+1}\underline{A}\}\right) + \left(\sum_{i>N(m)}^{*} \{b_i(m) + {}_{m+1}\underline{A}\}\right)$$

and $$(C_m \cup B_m) + {}_{m+1}\underline{A} \subseteq \left(\sum_{i=1}^{q(m)} \{d_i(m) + {}_{m+1}\underline{A}\}\right)$$

In (1.6.1) $\sum, \sum^*$ denote the restricted and unrestricted direct sums respectively while $\sum\limits_{i>N(m)}^{*}$ is to mean those and only those fundamental monomials of degree $m$ on $x_1, x_2, \ldots$ occur in the unrestricted direct sum which satisfy the condition

(1.6.2) $$\phi^{(N(m)+1)}(b_i(m)) = 0$$

Suppose the contrary to our lemma, the elements of $C_n$ are linearly dependent modulo ${}_{n+1}\underline{A}$ . This implies there exist scalars (not all zero) such that

(1.6.3) $$c = \sum_i \gamma_{n_i} c_{n_i} \quad \text{belongs to } {}_{n+1}\underline{A}$$

Now let $N = \max\{N(1), N(2), \ldots N(n-1)\}$.

consider the image under $\phi^{(N)}$ of (1.6.3). This yields

(1.6.4) $$c^{(N)} = \sum_i \gamma_{n_i} c_{n_i}^{(N)} \quad \text{belongs to } {}_{n+1}A^{(N)}$$

This implies via the decomposition (1.6.1) that for some $\ell (\leq n-1)$ the set $C_\ell^{(N)} \cup B_\ell^{(N)}$ is a set of linearly dependent elements of ${}_\ell\underline{A}$ modulo ${}_{\ell+1}\underline{A}$.

For if not, $C_\ell^{(N)} \cup B_\ell^{(N)}$ is a linearly independent **module** ${}_{\ell+1}A^{(N)}$ for every $\ell$ . Further to this every subalgebra of a free nonassociative algebra is then free vide E. Witt ( 26) and the elements of the set

$$B_1^{(N)} \cup B_2^{(N)} \ldots \cup B_{n-1}^{(N)}$$

freely generate a subalgebra of $A^{(N)}$ also by E. Witt ( 26).

No nontrivial relation of the form (1.6.4) exists between the elements of the set

$$B_1^{(N)} \cup B_2^{(N)} \ldots B_{n-1}^{(N)}$$

. Hence the set $C_\ell^{(N)} \cup B_\ell^{(N)}$ is a set of linearly dependent elements of ${}_\ell\underline{A}$ modulo ${}_{\ell+1}\underline{A}$.

This implies that there exist scalars (not all zero) such that:

(1.6.5) $$a_\ell^{(N)} = \sum_{i=1}^{k} \varepsilon_{\ell_i} b_{\ell_i}^{(N)} + \sum_{j=1}^{k'} \varepsilon'_{\ell_j} c_{\ell_j}^{(N)} \in {}_{\ell+1}\underline{A}$$

where $b_{\ell_i}^{(N)} \in B_\ell^{(N)}$, $c_{\ell_j}^{(N)} \in C_\ell^{(N)}$. $(i=1,2,\ldots k); (j=1,2,\ldots k')$

Thus the element

(1.6.6) $$a_\ell = \sum_{i=1}^{k} \varepsilon_{\ell_i} b_{\ell_i} + \sum_{j=1} \varepsilon'_{\ell_j} c_{\ell_j}$$

has the following properties

(I) $a_\ell$ does not belong ${}_{\ell+1}\underline{A}$ by the induction hypothesis in the construction 1.4 and since $B_\ell$ is linearly independent modulo the subalgebra generated by ${}_{\ell+1}\underline{A}$ & $C_\ell$. $(\ell = 1, 2, \ldots n)$

(II) $a_\ell$ belongs to $\left(\sum_{i=1}^{q(\ell)} \{d_i(\ell) + {}_{\ell+1}\underline{A}\}\right)$ modulo ${}_{\ell+1}\underline{A}$ by the decomposition (1.6.1)

(III) $a_\ell$ belongs to $\left(\sum_{i>N(\ell)}^{*} \{b_i(\ell) + {}_{\ell+1}\underline{A}\}\right)$ modulo ${}_{\ell+1}\underline{A}$ by (1.6.6) and (1.6.5).

But these properties of $a_\ell + {}_{\ell+1}\underline{A}$ contradict the direct decomposition given (1.6.1) of the vector space ${}_\ell\underline{A}/{}_{\ell+1}\underline{A}$.

This concludes the proof of lemma 1.6.

THEOREM 1.7 The subalgebra of all elements of finite degree in the unrestricted nonassociative algebra is a free nonassociative algebra.

Proof: Now choose $B_1, B_2, \ldots B_n, \ldots$ to be maximal sets satisfying the above construction. The elements of the set $C_n$ are linearly independent modulo $\{{}_{n+1}\underline{A}\}_{n=0,1,2\ldots}$ Hence the elements of the set $\bigcup_{i=1}^{\infty} B_i$ are free generators for $\underline{A}$. This proves theorem 1.7.

SECTION 2. Poincaré-Birkhoff-Witt Theorem, Free Associative Algebras.

INTRODUCTION

In the previous section we showed that the projective limit of free nonassociative algebras contains a free subalgebra. We now try to determine whether the projective limit of free associative algebras contains a free finite degree subalgebra.. Our approach is indirect. It is well known that we can associate with each Lie algebra a corresponding associative algebra called the universal enveloping algebra. And then, if we have a basis for the Lie algebra the Poincaré-Birkhoff-Witt Theorem(P.B.W.Theorem) states that a basis for the universal enveloping algebra is given by the unit element and the ordered products of basis elements of the corresponding Lie algebra. Thus in what follows with each free Lie algebra of rank $K$, denoted by $L_K$, we associate the corresponding free associative algebra denoted by $L^e_K$, $(k=1,2,\ldots)$

Now it was shown by S. Moran in (23)*, that the projective limit of free Lie algebras of increasing rank is a free Lie algebra called the Unrestricted Free Lie Algebra. We denote this by $\mathcal{L}$. Plainly, $\underline{\mathcal{L}}$ will have a corresponding universal enveloping algebra, we denote this by $\underline{\mathcal{L}}^e$. As it turns out, $\underline{\mathcal{L}}^e$ is not large enough to contain all the elements arising from the completion process inherent in the taking of the projective limit of the Lie algebras $L_K$, of increasing rank ($K=1,2,\ldots$). We do show, however, that $\underline{\mathcal{L}}^e$ is contained in a larger algebra $\underline{L}$ contained in (the projective limit of the $L^e_K$) which contains the completion elements and the embedding of $\underline{\mathcal{L}}^e$ in $\underline{L}$ is injective. In fact, the closure of $\underline{\mathcal{L}}^e$ is the associative algebra $\underline{L}$.

* This result is incorrect. S.Moran showed that the subalgebra $\underline{\mathcal{L}}$ of all elements of finite degree is a free Lie algebra.

## UNIVERSAL ENVELOPING ALGEBRA, POINCARÉ-BIRKHOFF-WITT THEOREM.

CONVENTION. Throughout this section 'algebra' will be taken to mean associative algebra with unity element $1$ ,'subalgebra' will mean a subalgebra of the associative algebra containing the unit element $1$ ,and 'homomorphism' will be taken in the usual sense for algebras, Further it will be understood that the homomorphisms map $1$ into $1$ .

Notation $A_L$ denotes the Lie algebra of the algebra $A$ obtained by defining the product in $A$ as the Lie product ,or(additive) commutator product $[x\,y] = xy - yx$ for $x$, $y$ in $A$ .

DEFINITION 2.1 UNIVERSAL ENVELOPING ALGEBRA.

Let $L$ be a Lie algebra (arbitary dimension and characteristic). A pair $(E, i)$ where $E$ is an algebra and $i$ is a homomorphism of $L$ into $E_L$ is called a universal enveloping algebra (U.E.A.) of $L$ if the following holds. If $A$ is any algebra and $\theta$ is a homomorphism of $L$ into $A_L$,then there exist a unique homomor--phism $\theta'$ of $E$ into $A$ such that $\theta' i = \theta$ .Diagramatically, we are given

(2.1.1)
$$\begin{tikzcd} E = E_L & \\ L \arrow[u, "i"] \arrow[r, "\theta"] & A = A_L \end{tikzcd}$$

where $i$ and $\theta$ are homomorphisms of $L$ and we can complete this diagram to the commutative diagram

(2.1.2)
$$\begin{tikzcd} E = E_L \arrow[dr, dashed, "\theta'"] & \\ L \arrow[u, "i"] \arrow[r, "\theta"] & A = A_L \end{tikzcd}$$

where $\theta'$ is a homomorphism of $E$ into $A$ . As a consequence of the definition 2.1 we have the following results.

THEOREM 2.2

1.
Let $(E, i)$, $(E', i')$ be universal enveloping algebras for $L$ .Then there is a unique isomorphism $j$ of $E$ onto $E'$ such that $i' = ji$.

2. $E$ is generated by the image $iL$ .

3. Let $L_1$, $L_2$ be Lie algebras with $(E_1, i_1)$ ,$(E_2, i_2)$ respectively universal enveloping algebras and let $\alpha$ be a homomorphism of $L_1$ into $L_2$ .Then there is a unique homomorphism $\alpha' : E_1 \to E_2$ such that $i_2\alpha = \alpha' i_1$ that is ,we have a commutative diagram:

$$\begin{tikzcd} L_1 \arrow[r, "\alpha"] \arrow[d, "i_1"'] & L_2 \arrow[d, "i_2"] \\ E_1 \arrow[r, "\alpha'"] & E_2 \end{tikzcd}$$

4. Let $B$ be an ideal in $L$ and let $K$ be the ideal in $E$ generated by $iB$ .If $l \in L$ then $j: l+B \to il+K$ is a homomorphism of $L/B$ into $E'_L$ $(E' = E/K)$ and $(E', j)$ is the U.E.A. for $L/B$ .

Proof.

1. If we use the defining property of $(E, i)$ and the homomorphism $\theta = i'$ of $L$ into $E'$ we obtain a unique homomorphism $j$ of $E$ into $E'$ such that $i' = ji$ .Similarly, we have a homomorphism $j'$ of $E'$ into $E$ such that $i = j'i'$. Hence $i = j'ji$ and $i' = jj'i'$. But $i' = 1_{E'} i'$ and by uniqueness of the defining property of $(E', i')$ applied to $\theta = i'$ we see that $jj' = 1_{E'}$ .Similarly, $j'j = 1_E$: thus $j$ is an isomorphism of $E$ onto $E'$ .

2. Let $E'$ be the subalgebra of $E$ generated by $iL$. The mapping $i$ can be considered as a mapping of $L$ into $E'_L$. Hence there is a unique homomorphism $i': E \to E'$ such that $i = i'i$. Since $i = 1_E i$ and $i'$ can be considered as a mapping of $E$ into $E$, the uniqueness condition gives $i' = 1_E$. Hence $E = 1_E E = i'E \subseteq E'$ and $E = E'$.

3. If $\alpha$ is a homomorphism of $L_1$ into $L_2$, then $i_2\alpha$ is a homomorphism of $L_1$ into $E_{2L}$. Hence there is a unique homomorphism $\alpha'$ of $E_1$ into $E_2$ such that $\alpha' i_1 = i_2\alpha$.

4. We first note that the mapping $l \to il + K$ of $L$ into $E'(= E/K)$ is a homomorphism of $L$ into $E'_L$. Since $iB \subseteq K$, $B$ is mapped into zero by this homomorphism. Hence we have an induced homomorphism $j: l + B \to il + K$. Now let $\theta: L/B \to A_L$, $A$ an algebra. Then $\eta: l \to \theta(l+B)$ is a homomorphism of $L$ into $A_L$. Hence there is a homomorphism $\eta': E \to A$ such that $\eta = \eta' i$.

$$\begin{tikzcd}
(E = E_L) \arrow[r, "\eta'"] & (A = A_L) & (E'_L = E') = E/K \arrow[l, "\theta'"'] \\
L \arrow[u, "i"] \arrow[ur, "\eta"] \arrow[rr] & & L/B \arrow[u, "j"'] \arrow[ul, "\theta"']
\end{tikzcd}$$

From the definition of $\eta$ if $b \in B$ $\eta b = 0$ in $A$. Thus $ib \in \mathrm{Ker}(\eta')$ and this implies that $K \subseteq \mathrm{Ker}(\eta')$. Hence we have an induced homomorphism $\theta': u + K \to \eta' u$ of $E'$ into $A$. Now $\eta' i(l) = \eta(l) = \theta(l + B)$ and $\theta' j(l+B) = \theta'(il+K) = \eta' i l$. Hence $\theta = \theta' j$ as required. It remains to show that $\theta'$ is unique. This will follow by showing that $j(L/B)$ generates $E'$. Now by 2, $E$ is generated by $iL$, which implies that $E'$ is generated by the cosets $il + K$. Since $j(l+B) = il + K$ we have $E'$ is generated by the set of elements $j(l+B)$, that is, by $j(L/B)$. This proves the theorem.

We now give a construction of the U.E.A. But first we must define the <u>tensor algebra</u> based on the vector space $L$ .

**DEFINITION 2.3 Tensor product of two $R$-modules.**

The <u>tensor product</u> $A\otimes B$ of two left $R$-modules($R$ commutative ring with unity) $A$ and $B$ is the $R$-module generated by the set of all pairs $(a,b)$ $a\varepsilon A, b\varepsilon B$ with relations

$$
\begin{aligned}
&(a_1+a_2, b)-(a_1,b)-(a_2,b)=0\\
&(a, b_1+b_2)-(a,b_1)-(a,b_2)=0 \qquad (2.3.1)\\
&(ra,b)-r(a,b)=r(a,b)-(a,rb)=0.
\end{aligned}
$$

Now $A\otimes B$ is obtained as follows . Let $R(A,B)$ be the free $R$-module generated by the set of pairs $(a,b)$ and let $Y(A,B)$ be the least subgroup of $R(A,B)$ consisting of all the elements of the form

(2.3.2) $(a_1+a_2,b)-(a_1,b)-(a_2,b)$ ; $(a,b_1+b_2)-(a,b_1)-(a,b_2)$
$(ra,b)-r(a,b)$; $(a,rb)-r(a,b)$

then $A\otimes B = R(A,B)/Y(A,B)$. The element of $A\otimes B$ which is the image of the generators $(a,b)$ of $R(A,B)$ will be denoted by $a\otimes b$ . These elements generate the group $A\otimes B$ and the relations are:

$$
\begin{aligned}
&(a_1+a_2)\otimes b = a_1\otimes b + a_2\otimes b\\
&a\otimes(b_1+b_2) = a\otimes b_1 + a\otimes b_2 \qquad (2.3.3)\\
&(ra)\otimes b = r(a\otimes b) = a\otimes(rb)
\end{aligned}
$$

With the obvious specialisation we can consider the tensor product of vector spaces , over a field $R$ $(=\Omega)$. Further it is easily seen that the product of any finite number of such spaces may be defined, mutatis mutandis.

We may regard a Lie algebra $L$ over a field $\Omega$ , as a vector space. For the purpose of our next definition we do this.

DEFINITION 2.4 TENSOR ALGEBRA ON VECTOR SPACE.

The tensor algebra on a vector space $L$ is

(2.4.1) $\mathcal{T} = \Omega 1 \oplus L_1 \oplus L_2 \oplus \dots \oplus L_i \oplus \dots$

where $L_1 = L$, $L_i = L \otimes L \otimes \dots \otimes L$, $i$ times, and $\Omega$ is the field.

The vector space operations in $\mathcal{T}$ are as usual and multiplication in $\mathcal{T}$ is indicated by $\otimes$ and is characterised by

(2.4.2) $(x_1 \otimes x_2 \dots \otimes x_i) \otimes (y_1 \otimes \dots \otimes y_j) = x_1 \otimes \dots \otimes x_i \otimes y_1 \dots \otimes y_j$.

Let $K$ be an ideal in $\mathcal{T}$ which is generated by all the elements of the form

(2.4.3) $[ab] - a \otimes b + b \otimes a$, $a, b$ belong to $L_1$

and let $E = \mathcal{T}/K$. Let $i$ denote the restriction to $L = L_1$ of the canonical homomorphism of $\mathcal{T}$ onto $E$. We have

$[ab]i - ai \otimes bi + bi \otimes ai = ([ab] - a \otimes b + b \otimes a) + K = K = 0$ in $E$.

Hence $i$ is a homomorphism of $L$ into $E_L$.

THEOREM 2.5 $(E, i)$ is a universal enveloping algebra for the Lie algebra $L$.

Proof:. We recall first the basic property of the tensor algebra, that any linear mapping $\theta: L \to A$ where $A$ is an algebra can be extended to a homomorphism of $\mathcal{T}$ into $A$. Thus let $\{u_j \mid j \in J\}$ be a basis for $L$, then it is well known that the distinct 'monomials' $u_{j_1} \otimes u_{j_2} \dots \otimes u_{j_n}$ of degree $n$ form a basis for $L_n$. Here $u_{j_1} \otimes u_{j_2} \dots \otimes u_{j_n} = u_{k_1} \otimes \dots \otimes u_{k_n}$ if and only if $j_r = k_r$, $r = 1, 2, \dots n$. The elements $1$ and the different monomials of degrees $1, 2, 3, \dots$ form a basis of $\mathcal{T}$. And it is easily seen that a linear mapping $\theta'': \mathcal{T} \to A$ such that $\theta'' 1 = 1$, $(u_{j_1} \otimes \dots \otimes u_{j_n})\theta'' = (u_{j_1}\theta)(u_{j_2}\theta) \dots (u_{j_n}\theta)$ is a homomorphism of $\mathcal{T}$ into $A$ such that $\theta'' a = \theta a$ for all $a \in L$ $(= L_1 \subset \mathcal{T})$.

Now let $\theta$ be a homomorphism of $L$ into $A_L$ and let $\theta''$ be its extension to a

homomorphism of $T$ into $A$. If $a$, $b$ belong to $L$,
$[ab]\theta'' - (a\theta'')(b\theta'') + (b\theta'')(a\theta'') = [ab]\theta - (a\theta)(b\theta) + (b\theta)(a\theta) = [a\theta, b\theta] - (a\theta)(b\theta) + (b\theta)(a\theta) = 0.$
Hence the generators (2.4.3) of $K$ belong to the kernel of $\theta''$.
We therefore have an induced homomorphism $\theta'$ of $E$ into $A$ such that $\theta' i(a) = \theta'(a+K) = \theta'' a = \theta a$. Thus $\theta = \theta' i$ as required. The tensor algebra $T$ is generated by $L$ and this implies that $E$ is generated by $iL$. Since two homomorphisms which coincide on the generators are necessarily identical we have that the homomorphism $\theta'$ such that $i\theta' = \theta$ is unique.

THE POINCARÉ-BIRKHOFF-WITT THEOREM

NOTATION AND REMARKS

We have noted that if $\{u_j \mid j \in J\}$, where $J$ is a set, is a basis for (the Lie algebra) $L$, then the monomials $u_{j_1} \otimes u_{j_2} \cdots \otimes u_{j_n}$ of degree $n$ form a basis for $L_n$. We suppose now that the set $J$ of indices is ordered and we proceed to use this ordering to introduce a partial order into the set of monomials of any given degree $n \geqslant 1$. We define the index of a monomial.

DEFINITION 2.6 Index of a monomial.

The index of a monomial $u_{j_1} \otimes u_{j_2} \cdots \otimes u_{j_n}$ is defined thus. For $i, k$, $i < k$

set

$$\eta_{ik} = \begin{cases} 0 & \text{if } j_i \leq j_k \\ 1 & \phantom{\text{if }} j_i > j_k \end{cases}$$

and define index

(2.6.1) $\operatorname{ind}(u_{j_1} \otimes u_{j_2} \cdots \otimes u_{j_n}) = \sum_{i<k} \eta_{ik}$.

Note that the $\operatorname{ind} = 0$ if and only if $j_1 \leq j_2 \leq \cdots \leq j_n$.

Monomials having this property will be called standard monomials.

We now suppose that $j_k > j_{k+1}$ and we wish to compare

$ind(u_{j_1} \otimes u_{j_2} \dots \otimes u_{j_n})$ and

$ind(u_{j_1} \otimes u_{j_2} \dots u_{j_{k+1}} \otimes u_{j_k} \dots \otimes u_{j_n})$ , where the second

monomial is obtained by interchanging $u_{j_k}$ , $u_{j_{k+1}}$. Let $\eta'_{ik}$ denote the $\eta$'s for the second monomial . Then we have $\eta'_{ij} = \eta_{ij}$ if, $i$, $j \neq k, k+1$ ; $\eta'_{ik} = \eta_{i\,k+1}$, $\eta'_{i\,k+1} = \eta_{ik}$ $(i<k)$; $\eta'_{kj} = \eta_{k+1\,j}$, $\eta'_{k+1\,j} = \eta_{k,j}$ $(j>k+1)$ and $\eta'_{k\,k+1} = 0$ , $\eta_{k,k+1} = 1$ , Hence

$$ind(u_{j_1} \otimes u_{j_2} \dots \otimes u_{j_n}) = 1 + ind(u_{j_1} \otimes \dots \otimes u_{j_{k+1}} \otimes u_{j_k} \dots \otimes u_{j_n})$$

We apply these remarks to the study of the algebra $E = T/K$ for which we prove first the following.

LEMMA 2.7

Every element of $T$ is congruent modulo $K$ to a $\Omega$-linear combination of $1$ and the standard monomials .

Proof:. It suffices to prove the statement for monomials . We order these by degree and for given degree by index. To prove the assertion for a monomial $u_{j_1} \otimes u_{j_2} \dots \otimes u_{j_n}$ it suffices to assume it for monomials of lower degree and for those of the same degree $n$ which are of lower index than the given monomial. Assume the monomial is not standard and suppose that $j_k > j_{k+1}$. We have

$$u_{j_1} \otimes u_{j_2} \dots \otimes u_{j_n} = u_{j_1} \otimes \dots u_{j_{k+1}} \otimes u_{j_k} \dots \otimes u_{j_n} + u_{j_1} \otimes \dots [u_{j_k} \otimes u_{j_{k+1}} - u_{j_{k+1}} \otimes u_{j_k}] \dots \otimes u_{j_n}$$

Since $u_{j_k} \otimes u_{j_{k+1}} - u_{j_{k+1}} \otimes u_{j_k} = [u_{j_k} u_{j_{k+1}}] \pmod K$

$$u_{j_1} \otimes \dots \otimes u_{j_n} = u_{j_1} \otimes \dots \otimes u_{j_{k+1}} \otimes u_{j_k} \dots \otimes u_{j_n} + u_{j_1} \otimes u_{j_2} \dots \otimes [u_{j_k} u_{j_{k+1}}] \dots \otimes u_{j_n} \bmod (K).$$

The first term on the right-hand side is of lower index than the given monomial while the second is a linear combination of monomials of lower degrees .The result follows from the induct--ion hypothesis.

We wish to show that the cosets of $1$ and the standard monomials are linearly independent and so form a base for $E$ .For this we introduce the vector space $B_n$ with basis $u_{i_1} u_{i_2} \dots u_{i_n}$, $(i_j \in J)$ $i_1 \leq i_2 \leq \dots \leq i_n$, and the vector space $B = \Omega 1 \oplus B_1 \oplus B_2 \dots \oplus B_n \oplus \dots$
The required independence will follow easily from the lemma 2.8

LEMMA 2.8

There exist a linear mapping $\sigma$ of $T$ into $B$ such that, $\sigma(1) = 1$

(2.8.1) $(u_{i_1} \otimes u_{i_2} \dots \otimes u_{i_n})\sigma = u_{i_1} u_{i_2} \dots u_{i_n}$

(2.8.2) $(u_{j_1} \otimes u_{j_2} \dots \otimes u_{j_n} - u_{j_1} \otimes u_{j_2} \dots \otimes u_{j_{k+1}} \otimes u_{j_k} \dots \otimes u_{j_n})\sigma =$
$= (u_{j_1} \otimes \dots \otimes [u_{j_k} u_{j_{k+1}}] \otimes \dots \otimes u_{j_n})\sigma$

Proof:. Set $\sigma(1) = 1$ and let $L_{n,j}$ be the subspace of $L_n$ spanned by the monomials of degree $n$ and index $\leq j$ .Suppose a linear mapping $\sigma$ has already been defined for $\Omega 1 \oplus L_1 \oplus L_2 \dots \oplus L_{n-1}$ satisfying (2.8.1), (2.8.2) for the monomials of this space. We extend $\sigma$ linearly to $\Omega 1 \oplus L_1 \oplus \dots \oplus L_{n-1} \oplus L_{n,0}$ by requir--ing that $(u_{i_1} \otimes u_{i_2} \dots \otimes u_{i_n})\sigma = u_{i_1} u_{i_2} \dots u_{i_n}$ for the standard mono--mials of degree $n$ .Next assume that $\sigma$ has already been defined for $\Omega 1 \oplus L_1 \oplus L_2 \dots L_n \oplus L_{n,i-1}$ ,satisfying (2.8.1), (2.8.2) for the monomials belonging to this space and let $u_{j_1} \otimes u_{j_2} \dots \otimes u_{j_n}$ be of index $i \geq 1$ . Suppose $j_k > j_{k+1}$ .Then we set

(2.8.3) $(u_{j_1} \otimes u_{j_2} \dots \otimes u_{j_n})\sigma = (u_{j_1} \otimes \dots u_{j_{k+1}} \otimes u_{j_k} \dots \otimes u_{j_n})\sigma \dots$
$\dots + (u_{j_1} \otimes \dots \otimes [u_{j_k} u_{j_{k+1}}] \dots \otimes u_{j_n})\sigma.$

This makes sense since the two terms on the right-hand side are in $\Omega 1 \oplus L_1 \oplus L_2 \dots L_n \oplus L_{n,i-1}$. We first show that (2.8.3) is independent of the choice of the pair $(j_k, j_{k+1})$, $j_k > j_{k+1}$. Let $(j_e, j_{e+1})$ be a second pair with $j_e > j_{e+1}$. There are essentially two cases I. $l > k+1$, II $l = k+1$.

I. Set $u_{j_k} = u$, $u_{j_{k+1}} = v$, $u_{j_e} = \omega$, $u_{j_{e+1}} = x$,. Then the induction hypoth-esis permits us to write for the right-hand side (2.8.3)

$$(\dots v \otimes u \dots \otimes x \otimes \omega \otimes \dots)\sigma \dots$$
$$\dots + (\dots v \otimes u \dots \otimes [\omega x] \otimes \dots)\sigma \dots$$
$$\dots + (\dots \otimes [uv] \dots \otimes x \otimes \omega \dots)\sigma \dots$$
$$\dots + (\dots \otimes [uv] \dots \otimes [\omega x] \otimes \dots)\sigma.$$

If we start with $(j_e, j_{e+1})$ we obtain

$$(\dots u \otimes v \dots x \otimes \omega \dots)\sigma + (\dots u \otimes v \dots \otimes [\omega x] \otimes \dots)\sigma =$$
$$= (\dots v \otimes u \dots x \otimes \omega \dots)\sigma \dots$$
$$\dots + (\dots \otimes [uv] \otimes \dots x \otimes \omega \dots)\sigma \dots$$
$$\dots + (\dots \otimes v \otimes u \otimes \dots \otimes [\omega x] \otimes \dots)\sigma \dots$$
$$\dots + (\dots \otimes [uv] \otimes \dots \otimes [\omega x] \otimes \dots)\sigma$$

This is the same as the value obtained before.

II. Set $u_{j_k} = u$, $u_{j_{k+1}} = v = u_{j_e}$, $u_{j_{e+1}} = \omega$,. If we start by using the induction hypothesis we can change the right-hand side of (2.8.3) to

$$(2.8.4) \quad (\dots \omega \otimes v \otimes u \dots)\sigma + (\dots [v\omega] \otimes u \dots)\sigma + (\dots v \otimes [u\omega] \dots)\sigma \dots$$
$$+ (\dots [uv] \otimes \omega \dots)\sigma$$

Similarly if we start with

$$(\dots u \otimes w \otimes v \dots)\sigma + (\dots u \otimes [vw] \dots)\sigma$$

we can end with

(2.8.5) $(\dots w \otimes v \otimes u \dots)\sigma + (\dots w \otimes [uv] \dots)\sigma \dots$

$$\dots + (\dots [uw] \otimes v \dots)\sigma + (\dots u \otimes [vw] \dots)\sigma.$$

Hence we have to show that $\sigma$ annihilates the following elements of $\Omega 1 \oplus L_1 \oplus L_2 \dots \oplus L_n$ :

$$(\dots [vw] \otimes u \dots) - (\dots u \otimes [vw] \dots) \dots$$

(2.8.6) $\dots + (\dots v \otimes [uw] \dots) - (\dots [uw] \otimes v \dots) \dots$

$$\dots + (\dots [uv] \otimes w \dots) - (\dots w \otimes [uv] \dots)$$

Now it follows easily from the properties of $\sigma$ in $\Omega 1 \oplus L_1 \dots \oplus L_n$ that if $(\dots a \otimes b \dots) \varepsilon L_{n-1}$, where $a$, $b$ belong to $L_1$, then $(\dots a \otimes b \dots)\sigma - (\dots b \otimes a \dots)\sigma - (\dots [ab] \dots)\sigma = 0$

(2.8.7)

Hence $\sigma$ applied to (2.8.6) gives

(2.8.8) $(\dots [[vw]u] \dots)\sigma + (\dots [v[uw]] \dots)\sigma \dots$

$$\dots + (\dots [[uv]w] \dots)\sigma$$

Since $[[vw]u] + [v[uw]] + [[uv]w] = [[vw]u] + [[wu]v] + [[uv]w] = 0$

(2.8.8) has the value zero. Hence in this case, too, the right-hand side of(2.8.3) is uniquely determined. We now apply (2.8.3) to define $\sigma$ for the monomials of degree $n$ and index $i$ . The linear extension of this mapping to the space $L_{n,i}$ gives a map--ping on $\Omega 1 \oplus L_1 \oplus L_2 \dots L_{n-1} \oplus L_{n\,i}$ satisfying our conditions. This completes the proof of the lemma.

We can now prove the following.

THEOREM 2.9 (Poincaré-Birkhoff-Witt)

The cosets of $1$ and the standard monomials form a basis for $E = T/K$

Proof:. Lemma2.7 shows that every coset is a linear combination of $1 + K$ and the cosets of the standard monomials . Lemma 2.8 gives a linear mapping $\sigma$ of $T$ into $B$ satisfying (2.8.1) and (2.8.2). It is easy to see that every element of the ideal $K$ is a linear combination of elements of the form

$$(u_{j_1} \otimes u_{j_2} \ldots \otimes u_{j_n}) - (u_{j_1} \otimes u_{j_2} \ldots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes u_{j_n}) \ldots$$
$$\ldots - (u_j \otimes \ldots \otimes [u_{j_k} u_{j_{k+1}}] \otimes \ldots \otimes u_{j_n})$$

.Since $\sigma$ maps these elements into zero we have ,$\sigma(K) = 0$ and so $\sigma$ induces a linear mapping of $E = T/K$ into $B$ . Since (2.8.1) holds,the induced mapping sends the cosets of $1$ and the standard mono--mials $u_{i_1} \otimes u_{i_2} \ldots \otimes u_{i_n}$ into $1$ and $u_{i_1} u_{i_2} \ldots u_{i_n}$ respectively. Since the images are linearly independent in $B$ , we have the linear independence in $E$ of the cosets of $1$ and the standard monomials . This completes the proof.

COROLLARY 2.10

The mapping $i$ of $L$ into $E$ is 1-1 and $\Omega 1 \cap iL = 0$

Proof:. If $(u_j)$ is a basis for $L$ over $\Omega$ ,then $1 = 1 + K$ and the cosets $i(u_j) = u_j + K$ are linearly independent . This implies both statements.

REMARKS

We shall now simplify our notation in the following way . We write the product in $E$ in the usual way for associative algebras: $xy$ . We write $1$ for the identity in $E$ and we identify $L$ with its image $iL$ in $E$ . This is a subalgebra of $E_L$ since the identity mapping $i$ is an isomorphism of $L$ into $E_L$ .

Also $L$ generates $E$ and the P.B.W. Theorem states that if $\{u_j \mid j \in J\}$, $J$ ordered , is a basis for $L$ ,then the elements $1$ , $u_{i_1} u_{i_2} \dots u_{i_n}$ , $i_1 \leq i_2 \dots \leq i_n$ form a basis for $E$ .In the light of these remarks the defining pro--perty of $E$ can be restated in the following way. If $\theta$ is a homomorphism of $L$ into $A_L$, $A$ an algebra , then $\theta$ can be extended to a <u>unique</u> homomorphism $\theta$ (formerly $\theta'$, see defn.2.1) of $E$ into $A$ .

## FREE LIE ALGEBRAS

In order to construct a free Lie algebra . We must first define and construct a **free associative algebra.**

<u>DEFINITION 2.11</u> <u>Free Algebra (or Free Lie Algebra)</u>.

The notion of a free algebra (or free Lie algebra ) generated by a set $X = \{x_j \mid j \in J\}$ can be formulated in a manner similar to that of defn. 2.1 of a universal enveloping algebra of a Lie algebra. We define this to consist of the pair $(F, i)$ (or $(FL, i)$) consisting of an algebra $F$ (or Lie algebra $FL$ )and a mapping $i$ of $X$ into $F$ (or $FL$) such that if $\theta$ is any mapping of $X$ into an algebra $A$ (or Lie algebra $L$ ),then there exist a unique homomorphism $\theta'$ of $F$ or ($FL$) into $A$ (or $L$ )such that $\theta = \theta' i$ . Diagramatically,

$$\begin{tikzcd} F \arrow[dr, "\theta'"] & \\ X \arrow[u, "i"] \arrow[r, "\theta"'] & A \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} FL \arrow[dr, "\theta'"] & \\ X \arrow[u, "i"] \arrow[r, "\theta"'] & L \end{tikzcd}$$

where both diagrams are,of course ,commutative.

<u>CONSTRUCTION 2.12</u> Free Algebra generated by $X$ .

To construct a free algebra generated by a set $X$ .We form the vector space $M$ over a field $\Omega$, with basis $X$ and then we

form the tensor algebra $T(=F)=\Omega 1 \oplus M \oplus (M\otimes M)\ldots$ based on $M$.
The mapping $i$ is taken to be the injection of $X$ into $F$. Now let $\theta$ be a mapping of $X$ into an algebra $A$. Since $X$ is a basis for $M$, $\theta$ can be extended to a unique linear mapping of $M$ into $A$ and this can be extended to a unique homomorphism $\theta$ of $F$ into $A$. Hence $F$ and the injection mapping of $X$ into $F$ is a free algebra generated by $X$.

<u>CONSTRUCTION 2.13</u> Free Lie algebra generated by a set $X$.

The construction is indirect and uses the free algebra $F$ generated by $X$. Let $FL$ denote the subalgebra of the Lie algebra $F_L$ generated by the set $X$. Let $\theta$ be a mapping of $X$ into a Lie algebra $L$ and let $E$ be the U.E.A. of $L$, which by the P.B.W. Theorem we suppose contains $L$. Then $\theta$ can be considered as a mapping of $X$ into $E$, so this can be extended to a homomorphism $\theta$ of $F$ into $E$. Moreover, $\theta$ is a homomorphism of $F_L$ into $E_L$ and since $\theta$ maps $X$ into a subset $L(\subseteq E)$, the restriction of $\theta$ to the subalgebra $FL$ of $F_L$ generated by $X$ is a homomorphism of $FL$ into $L$. We have therefore shown that $\theta$ can be extended to a homomorphism of $FL$ into $L$. Since $X$ generates $FL$, $\theta$ is unique. Hence $FL$ and the injection mapping of $X$ into $FL$ is a free Lie algebra generated by $X$.

<u>THEOREM 2.14 (Witt)</u>

Let $X$ be an arbitary set and let $F$ denote the free algebra (freely) generated by $X$. Let $FL$ denote the subalgebra of $F_L$, generated by the elements of $X$. Then $FL$ is a free Lie algebra generated by $X$ and $F$ is the U.E.A. of $FL$.

Proof: Let $\theta$ be a homomorphism $FL$ into a Lie algebra $A_L$, $A$ an algebra. Then there exist a homomorphism $\theta$ of $F$ into $A$ which coincides with the restriction of $\theta$ to $X$. Then $\theta$ is a homomorphism of $F_L$ into $A_L$ so the restriction $\theta'$ of $\theta$ to $FL$ is a homomorphism of $F_L$ into $A_L$. Since $\theta'(x)=\theta(x)$ for $x$ in $X$ and $X$ generates $FL$, it is clear that $\theta'$ coincides with the given homomorphism $\theta$ of $FL$ into $A_L$. Thus we have extended the homomorphism $\theta$ to a homomorphism of $F$ into $A$. Since $FL$ generates $F$ it is clear that the extension is unique. Hence $F$ is the U.E.A. of $FL$.

DEFINITION 2.15 Lie element*

An element of $F$ is called a Lie element if the element belongs to $FL$.

REMARKS

We quote the Theorem 2.14 of E.Witt for sake of completeness. For our purposes it is necessary to have a basis for a free Lie algebra constructed from the free generators $x_1, x_2, \dots$. We therefore construct the standard or basic monomials of M.Hall, see P.Serre (24) and M.Hall (10).

DEFINITION 2.16 (BASIC MONOMIALS)

Let $X=\{x_j \mid j\in J\}$, $J$ ordered, be free generators for a free Lie algebra $FL$ over a field $\Omega$. Then the free generators are taken as the basic monomials of degree $1$. If we have defined the basic monomials of degree $1, 2, \dots (n-1)$, and they are simply ordered in some way so that $u<v$ if $d(u)<d(v)$, where $d$ is the degree function mapping the elements $u, v$ into the positive integers. If $d(u)=r$ $d(v)=s$ and $r+s=n$ then $[u\,v]$ is a basic monomial

*Note: Lie elements are defined in terms of a fixed set of free generators; in our case the set of generators is the set $X$ as given above.

of degree $n$ if both the following conditions hold

B(i) $u$ & $v$ are basic monomials and $u > v$

B(ii) If $u = [xy]$ is the form of a basic monomial, then $v \geqslant y$. Thus we have defined the degree of all basic monomials. If we now order the basic monomials of fixed degree lexicographically, then the basic monomials are well ordered.

Now let $L_k$ denote the free lie algebra having the elements $x_1, x_2, \dots x_k$ as its free generators ,over the field $\Omega$ $(k = 1, 2, \dots n, \dots)$ .And let $(L^e_k, i_k)$ denote the corresponding universal enveloping algebra of $L_k$. We construct the basic monomials for $L_k$ and order them as indicated in the above definition. Then if the basic monomials for $L_k$ are denoted by $b_j(x_1, x_2, \dots x_k)$, $j \varepsilon J'$ more briefly by $b_j$, $j \varepsilon J'$, $J'$ ordered. A basis for $L^e_k$ is given by the ordered products

$$b_J = b_{j_1} b_{j_2} \dots b_{j_s}, \ (j_1 \leq j_2 \leq \dots \leq j_s).$$

where capital $J$ subscript always denotes the ordered set $j_1 \leq j_2 \dots \leq j_s$ for some $s$. Such a product $b_J$, will be called a <u>standard or basic</u> product . The <u>degree</u> of a standard product is defined as the sum of the degrees of the separate **basic** monomials occuring in the said product. If $m > k$ ,then there exist a natural homomorphism of $L_m$ onto $L_k$. Hence it is possible to form the projective limit of the $L_k$. We denote this by $\mathcal{L}_0$ .

Consider the following diagram

$$
\begin{tikzcd}
\cdots \arrow[r, "\pi_{k+1}"] & L_k \arrow[r, "\pi_k"] \arrow[d, "i_k"] & L_{k-1} \arrow[r, "\pi_{k-1}"] \arrow[d, "i_{k-1}"] & \cdots\ (\mathcal{L}) \\
\cdots \arrow[r, "\pi'_{k+1}"] & L^e_k \arrow[r, "\pi'_k"] & L^e_{k-1} \arrow[r, "\pi'_{k-1}"] & \cdots\ (L)
\end{tikzcd}
\tag{2.17}
$$

It follows directly from theorem 2.2 part 3 that there exist a unique homomorphism $\pi'_k$ which makes each of the above squares commutative, in that $\pi'_k i_k = i_{k-1}\pi_k$. Since the mapping of $L_m$ into $L_k$ for $m>k$ is given by $\pi_{k+1}\circ\cdots\circ\pi_m$ using Theorem 2.2 part 3 again we can see that the whole diagram is commutative. The uniqueness of the corresponding natural mapping $\pi'_{k+1}\circ\pi'_{k+2}\cdots\circ\pi'_{m-1}\circ\pi'_m$ of $L^e_m$ into $L^e_k$, $m>k$, enables us to define the projective limit of $L^e_k$. We shall denote this by $L$ . Our next result shows that $\mathcal{L}$ can be injectively embedded in $L$ .

But first we have a definition.

<u>DEFINITION</u> 2.17

We define $i = P.L.(i_k)$ where $i: \mathcal{L} \to L$

by $i(x) = (i_k(x^{(k)}))$ for all $x \in \mathcal{L}$, $x^{(k)} \in L_k$

Now let us suppose we have the discrete topology on each factor $L_k$ (or $L^e_k$), $(k=1,2\ldots)$ of $\mathcal{L}$ (or $L$) and $\mathcal{L}$ (& $L$) have the corresponding induced Tychonoff topology. Then we can prove the following

<u>THEOREM 2.18</u>

If $i = P.L.(i_k)$ , then $i: \mathcal{L} \to L$ and $i$ is a continuous injective homomorphism*

Proof:

(i) $i$ is injective. We first show that $\ker i = P.L.(\ker i_k)$. Let

*It is easily seen that $i$ is a homomorphism

$x \in \ker i$ then $i(x) = 0 = (i_k(x^{(k)})) = (0^{(k)})$ for all $k$, hence $x^{(k)}$ belongs to $\ker i_k$ for all $k$. But for $l \geq k$ $\pi_{k+1} \circ \pi_{k+2} \circ \cdots \circ \pi_l(\ker i_l) \subseteq \ker i_k$. Hence $x$ belongs to $P.L.(\ker i_k)$. Similarly, if $x$ belongs to $P.L.(\ker i_k)$ then $i_k(x^{(k)}) = 0$ for all $k$. Thus $i(x) = (i_k(x^{(k)})) = (0^{(k)})$ and $x \in \ker i$. Thus $\ker i = P.L.(\ker i_k)$. $i$ is injective since $i_k$ is for each $k$.

(ii) $i$ is continuous.

We show that the inverse image of a member of a basis of neighbourhoods of zero in $L$ is a member of a basis of neighbourhoods of zero $\mathcal{L}$. If $N$ is a neighbourhood of zero in $L$, then $N$ is a union of sets of the form

$$B = \{y : y \in L \ \& \ y^{(n)} = 0 \text{ for all } n \leq m\}$$

where $y^{(n)}$ is the image of $y \in L$ under the projection $\phi^{(n)} : L \to L^e_n$

Now

$$i^{-1}(B) = \{x : x \in \mathcal{L} \ \& \ i(x) \in B\}$$

and this set is a neighbourhood basis of zero in $\mathcal{L}$ if for all $x \in i^{-1}(B)$, $x^{(M)} = 0$ for some $M$, depending on the element $x$. Since $i(x) \in B$ $0 = (i(x))^{(M)}$ for some $M$, i.e.

$$\phi^{(M)} \circ i(x) = 0 = y^{(M)} \in L^e_M$$

This implies $i(x) \in \ker \phi^{(M)}$ and from the definition of $i$ we have $0 = y^{(M)} = i_M x^{(M)}$, $i_M$ is injective. Hence $x^{(M)} = 0$. Thus $i^{-1}(B)$ is a member of a neighbourhood basis of zero in $\mathcal{L}$.

$$\begin{tikzcd} A_k \arrow[r, "\pi_{kl}"] \arrow[d, "\phi_k"'] & A_l \arrow[d] \\ B_k \arrow[r, "\pi'_{kl}"] & B_l \end{tikzcd} \quad (k \geq l)$$

COROLLARY 2.18

From part (i) of the proof we deduce: For any homomorphism $\phi_k : A_k \to B_k$

$\mathrm{Ker}\,\phi = P.L.(\mathrm{Ker}\,\phi)_k$ where $\phi : P.L.(A_k) \to P.L.(B_k)$ and the $\{A_k, \pi_{ke}\}$ $\{B_k, \pi'_{ke}\}$ are inverse systems of suitable algebraic structures in the same category.

To summarise our results so far we have the following commutative diagram

$$\begin{tikzcd} \underline{\mathcal{L}}^e \arrow[drr, "\phi"] & & \\ \underline{\mathcal{L}} \arrow[u, "j"] \arrow[rr, "i"'] & & \underline{\mathrm{L}} \end{tikzcd}$$

where $i$ and $j$ are injective mappings, $\underline{\mathcal{L}}$ is a free Lie algebra,[23] $\underline{\mathcal{L}}^e$ is a free associative universal enveloping algebra of $\underline{\mathcal{L}}$. $\underline{\mathrm{L}}$ is an associative algebra.*

LEMMA (2.19)

If $\{z_\gamma : \gamma \in \Gamma\}$ is a set of free generators for $\underline{\mathcal{L}}$, then we can arrange for $\phi(z_\gamma) = z_\gamma$ for all $\gamma \in \Gamma$ $\phi$ is as given in the above diagram.

Proof:

From the definition of U.E.A. $\underline{\mathcal{L}}^e$ of $\underline{\mathcal{L}}$ we know that $\underline{\mathcal{L}}^e$ is generated by $j(\underline{\mathcal{L}})$ and $\phi$ is a unique homomorphism.

But $\phi|_{j(\underline{\mathcal{L}})} = i \circ j^{-1}$, where $j^{-1}$ is defined on $j(\underline{\mathcal{L}})$. Hence from the injectiveness of $i$ and $j^{-1}$ the uniqueness of $\phi$, the lemma follows.

Our main aim is to show that $\phi$ is an injective homomorphism of $\underline{\mathcal{L}}^e$ into $\underline{\mathrm{L}}$.

Before we proceed we give some information about $\underline{\mathrm{L}}$, the unrestricted associative algebra formed from the projective limit of the free universal enveloping algebras $\mathrm{L}^e_k$.

From the definition of $\underline{\mathrm{L}}$ we know that every element $a$ of $\underline{\mathrm{L}}$

* $\underline{\mathrm{L}}$ is the subalgebra of $\mathrm{L}$ consisting of all elements of finite degree.

can be written in the following form

$$a = \sum_{\ell=1}^{\infty} \sum_{I}{}^{*} \alpha_{\ell I}\, b_I(\ell)$$

where $\alpha_{\ell I} \in \Omega$ , for all values of $I$ & $\ell$, $b_I(\ell)$ runs through all the basic products on the free generators $x_1, x_2, \ldots$ (for fixed $\ell$ ).

An element of the form

$$a = \sum_{I}{}^{*} \alpha_I\, b_I(\ell) \qquad (\alpha_I \in \Omega)$$

is taken as having <u>degree</u> $\ell$ . In the unrestricted sum the basic products are so ordered that if $\phi^{(k)}(b_I(\ell)) = b_I(\ell)$ while $\phi^{(k)}(b_J(\ell)) = 0$ for some positive integer $k$ , then the basic product $b_I(\ell)$ appear before the basic product $b_J(\ell)$ in the above unrestricted sum $\sum^{*}$ .

<u>NOTATION</u>

All the elements of $\underline{L}$ which involve only basic products of degree not less than $\ell$ (together with zero) in the above representation form an ideal in $\underline{L}$ which we denote by ${}_{\ell}\underline{L}$ .

Throughout the following lemma (2.20) we let $\{z_\gamma : \gamma \in \Gamma\}$ denote the free generators of $\underline{\mathcal{L}}$ as constructed by S.Moran in (23).

<u>LEMMA (2.20)</u>

If $z_1, z_2, \ldots z_r$ is a finite subset of free generators $\{z_\gamma : \gamma \in \Gamma\}$ for $\underline{\mathcal{L}}$ then for all sufficiently large $n$, $z_1^{(n)}, z_2^{(n)}, \ldots z_r^{(n)}$ generate a free Lie subalgebra of $\underline{L}_n$ .

Proof: Since we have a finite subset of the free generators $\{z_\gamma : \gamma \in \Gamma\}$, $\Gamma$ is a well ordered set. We can apply the usual decomposition to write ${}_{\ell}\underline{\mathcal{L}}/{}_{\ell+1}\underline{\mathcal{L}}$ $(\ell = 1, 2, \ldots m)$ as a direct sum of two subspaces one of which is spanned by $z_1, z_2, \ldots z_r$ of degree $\ell$. Let $B_1 = C_1$ be the $z_1, z_2, \ldots z_r$ which have degree one. The degree of an element $z$ of $\underline{\mathcal{L}}$ being defined in terms of the free generators $x_1, x_2, \ldots$ which are all taken to have degree one. We let $B_n$ be those $z$'s belonging to $z_1, z_2, \ldots z_r$ which have degree $n$. We order the elements of $B_1, B_2, \ldots B_n$, by agreeing that $z > z'$ if $z \in B_i$ & $z' \in B_j$ & $i > j$.

We define $C_n$ to be the set of all basic monomials of degree $n$ on the elements of the sets $B_1, B_2, \ldots B_{n-1}$.

Now as the sets $C_1, C_2, \ldots C_n$ are finite we have the following decomposition. For every integer $m\ (1 \leq m \leq n)$, there exist elements $d_i(m) \in \underline{\mathcal{L}}$ of degree $m$ and positive integers $N(m)$ & $q(m)$ such that

$$_m\underline{\mathcal{L}} / {}_{m+1}\underline{\mathcal{L}} = \left(\sum_{i=1}^{q(m)} \{d_i(m) + {}_{m+1}\underline{\mathcal{L}}\}\right) + \left(\sum_{i>N(m)}^{*} \{b_i(m) + {}_{m+1}\underline{\mathcal{L}}\}\right)$$

$$\& \quad (C_m \cup B_m) + {}_{m+1}\underline{\mathcal{L}} \subseteq \left(\sum_{i=1}^{q(m)} \{d_i(m) + {}_{m+1}\underline{\mathcal{L}}\}\right)$$

where $\sum_{i>N(m)}^{*}$ means those and only those basic monomials of degree $m$ on $x_1, x_2, \ldots$ occur in the unrestricted direct sum which satisfy

$$\phi^{(N(m)+1)}(b_i(m)) = 0$$

Now choose $N = \max\{N(1), N(2) \ldots N(n)\}$ and suppose the contrary to our lemma.

Then

$$\sum_i \alpha_{iN} c_i^{(N)} + \sum_j \beta_{jN} b_j^{(N)} = 0 \tag{2.20.1}$$

where $c_i^{(N)}$ belong to $C_1^{(N)} \cup C_2^{(N)} \ldots \cup C_n^{(N)}$

where $b_i^{(N)}$ belong to $B_1^{(N)} \cup B_2^{(N)} \ldots \cup B_n^{(N)}$.

and the $\alpha_{iN}, \beta_{jN}$ are not all zero.

Let

$$\sum_i{}' \alpha_{iN} c_i^{(N)} + \sum_j{}' \beta_{jN} b_j^{(N)} = 0 \tag{2.20.2}$$

be the subsum of (2.20.1) containing the homogeneous terms of least degree which genuinely occur in (2.20.1) Note the $z_i$ are homogeneous, see page 52.
By the decomposition we can write (2.20.2) as

$$\sum_i{}' \alpha_{iN} c_i + \sum_j{}' \beta_{jN} b_j = 0$$

However the $z_1, z_2, \ldots z_r$ freely generate a free Lie subalgebra of $\underline{\mathcal{L}}$ ~~by E.Witt ( 26~~ )

Thus all the $\alpha_{iN} \,/\, \beta_{jN}$ occurring in (2.20.2) are zero. It follows immediately that all the $\alpha_{iN}/\beta_{jN}$ occurring in (2.20.1) are zero. This contradiction proves our lemma 2.20.

<u>THEOREM</u> (2.21)

The homomorphism $\phi : \underline{\mathcal{L}}^e \to \underline{L}$ as given in the above commutative diagram, is injective.

Proof:

By the P.B.W. Theorem if $\{z_\gamma : \gamma \varepsilon \Gamma\}$ is a set of free generators of $\underline{\mathcal{L}}$, $\Gamma$ well ordered, then the basic products

$$b_K = b_K(z) = b_{k_1}(z)\, b_{k_2}(z) \ldots\ldots b_{k_n}(z) \quad (k_1 \leq k_2 \leq \ldots \leq k_n)$$

form a basis for $\mathcal{L}^e$, where the $b_{k_i}(z)$ are the usual basic monomials.
Now if $g \; (g \neq 0)$ belongs to $\underline{\mathcal{L}}^e$, we have

$$(2.21.1) \qquad g = \sum_{\underline{K}} \alpha_{\underline{K}} b_{\underline{K}} \quad , \; (\alpha_{\underline{K}} \varepsilon \Omega)$$

and only a finite number of the $\alpha_{\underline{K}}$ are different from zero.
Suppose that

$$(2.21.2) \qquad \phi(g) = 0 = \sum_{\underline{K}} \alpha_{\underline{K}} b_{\underline{K}}(\phi(z)) = \sum_{\underline{K}} \alpha_{\underline{K}} b_{\underline{K}}(z) \text{ in } \underline{L}$$

By lemma (2.19). Now only a finite number of the free generators say $z_1, z_2 \ldots z_r$ occur in (2.21.2). Take the projection of (2.21.2) under

$\Phi^{(n)}$ for large $n \geq N$. We have then

(2.21.3) $$0 = \sum \alpha_K b_K(\mathfrak{z}^{(N)}) \quad \text{in } L^e_N$$

Now by lemma (2.20), the $\mathfrak{z}_1^{(N)}, \mathfrak{z}_2^{(N)} \ldots \mathfrak{z}_r^{(N)}$ are free generators for a free Lie subalgebra of $L_N$, for large $N$. Thus the $b_{K_i}(\mathfrak{z}^{(N)})$ form a basis for a subalgebra and the basic products $b_K(\mathfrak{z}^{(N)})$ forma basis for the universal enveloping algebra of this free Lie subalgebra. Hence all the $\alpha_K$ are zero. This contradiction proves our theorem (2.21).

We have just shown that $\underline{\mathcal{L}}^e$ can be injectively embedded in $\underline{L}$. We now prove that there exist elements which belong to $\underline{L}$ but do not belong to $\underline{\mathcal{L}}^e$.

THEOREM (2.22)

If $W^{(2n)} = x_1x_2 + x_2x_3 \ldots + x_{2n-1}x_{2n}$, then $W = (W^{(2n)})$ is an element of $\underline{L}$ which does not belong to $\underline{\mathcal{L}}^e$

Proof: Let $\{x_i, y_\alpha\}$ be an ordered base for the vector space of elements of degree 1 in $\underline{\mathcal{L}}$.

A typical $y_\alpha = \sum\limits_{i>N}^{*} \alpha_i x_i$. Then

$$\{\mathfrak{z}_\gamma, [x_{i_k}, x_{i_{k+1}}], [x_{i_k}, y_\alpha], [y_\alpha, y_\beta]\}, (i_k < i_{k+1} < \alpha < \beta)$$

is a base for the vector space of the elements of degree two, the $\mathfrak{z}_\gamma$ are Lie elements of $\underline{\mathcal{L}}$ of degree two. By the P.B.W. Theorem a basis for the vector space of elements of degree two in the enveloping algebra $\underline{\mathcal{L}}^e$ of $\underline{\mathcal{L}}$ is given by:

$$\{\mathfrak{z}_\gamma, [x_{i_k}, x_{i_{k+1}}], [x_{i_k}, y_\alpha], [y_\alpha, y_\beta], x_{i_k}^2, y_\alpha^2, (x_{i_k} x_{i_{k+1}}) \ldots$$
$$\ldots (x_{i_k} y_\alpha), (y_\alpha y_\beta)\}; (i_k < i_{k+1} < \alpha < \beta)$$

Now let us assume that $\omega$ belongs to $\underline{\mathcal{L}}^e$, then

$$w = \sum' \lambda_\gamma z_\gamma + \sum' \alpha_k [x_{i_k}, x_{i_{k+1}}] + \sum' \beta_k [x_{i_k}, y_{\alpha_k}] \ldots$$
$$\ldots + \sum' \gamma_{\alpha\beta} [y_\alpha, y_\beta] + \sum' \alpha'_k x_{i_k}^2 + \sum' \beta'_k y_{\alpha_k}^2 \ldots$$
$$\ldots + \sum \gamma'_k (x_{i_k} x_{i_{k+1}}) + \sum \delta'_k (x_{i_k} y_\alpha) + \sum \varepsilon'_k (y_{\alpha_k} y_{\beta_k}).$$

Now $w^{(2n)}$ is given by

$$w^{(2n)} = \sum' \lambda_\gamma z_\gamma^{(2n)} + \sum' \beta_k [x_{i_k}, y_\alpha^{(2n)}] + \sum' \alpha_k [x_{i_k}, x_{i_{k+1}}] \ldots$$
$$\ldots + \sum' \gamma_{\alpha\beta} [y_\alpha^{(2n)}, y_\beta^{(2n)}] + \sum' \alpha'_k x_{i_k}^2 \ldots$$
$$\ldots + \sum' \beta'_k (y_{\alpha_k}^{(2n)})^2 + \sum' \gamma'_k (x_{i_k} x_{i_{k+1}}) + \sum' \delta'_k (x_{i_k} y_{\alpha_k}^{(2n)}) + \sum' \varepsilon'_k (y_{\alpha_k}^{(2n)} y_{\beta_k}^{(2n)})$$

where the dashes over the summation signs indicates that some of the summands may be zero.

If we now factor by the ideal generated in $L^e_{2n}$ by the elements

$$\{ z_\gamma^{(2n)}, [x_{i_k}, x_{i_{k+1}}], [x_{i_k}, y_\alpha^{(2n)}], [y_\alpha^{(2n)}, y_\beta^{(2n)}] \}$$

then the image of $w^{(2n)}$, denoted by $w'^{(2n)}$ has the form.

(2.22.1) $$w'^{(2n)} = \sum' \alpha'_k x_{i_k}^2 + \sum \beta'_k (y_{\alpha_k}^{(2n)})^2 + \sum' \gamma'_k (x_{i_k} x_{i_{k+1}}) \ldots$$
$$\ldots + \sum \delta'_k (x_{i_k} y_\alpha^{(2n)}) + \sum' \varepsilon'_k (y_{\alpha_k}^{(2n)} y_{\beta_k}^{(2n)})$$

Now for large $n$ (2.22.1) will have constant rank as given by examining the matrix of the quadratic form.

But

(2.22.2) $$w'^{(2n)} = w^{(2n)} = x_1 x_2 + x_2 x_3 \ldots + x_{2n-1} x_{2n}$$

is easily seen to have rank $n$ . Hence by increasing $n$ we may make the rank arbitarily large in (2.22.2) whilst the rank of $\omega^{i(2n)}$ in (2.22.1) is constant for large $n$ . This contradiction proves the result. Although in Theorem 2.22 we have only found an element of degree $2$ which is $\underline{L}$ but not in $\underline{\mathcal{L}}^{e}$. It is not difficult to see that there exist elements of arbitary degree having the property.

Notation

Let $\{w_\beta : \beta \in B(l)\}$ be a set of homogeneous elements of degree $l$ which form a basis for $_l\underline{L}$ modulo the subalgebra generated by $_{l+1}\underline{L}$ and the associative monomials formed by taking products of elements from the set $\bigcup_{j=2}^{l-1}\{w_\beta : \beta \in B(j)\} \cup \underline{\mathcal{L}}^{e}$ which have degree $l$.

section 3 we introduce some of the work of P.Cohn (5) which is useful for a further velopment of the ideas contained in this thesis.

## RING WITH A DEGREE FUNCTION. FREE SUBALGEBRAS OF FREE ASSOCIATIVE ALGEBRAS.

We now set out to prove that the projective limit of the universal enveloping algebras of increasing rank $L^e_K$ contains a free associative subalgebra $\underline{L}$. We were able to show that the projective limit of nonassociative algebras $A_K$, of increasing rank $K$, (the $A_K$ are, of course, free) contains a free nonassociative subalgebra $\underline{A}$. And in so doing we made use of the result due to E.Witt( 26 ) : 'every subalgebra of a free nonassociative algebra is free'. Put another way, free nonassociative algebras over a field form a Schreier variety. However, the free associative algebras over a field do not form a Schreier variety. For example, let $\Omega[x]$ denote the free associative algebra in a single variable $x$ over a field $\Omega$, then $\Omega[x^2, x^3]$ is a subalgebra of $\Omega[x]$ but $\Omega[x^2, x^3]$ is not free. Hence any attempt to show that the associative algebra $\underline{L}$ is free depends on characterising those subalgebras of free associative algebras which are free.

This problem was discussed in a paper of P.Cohn( 5 ) and many of the results given there depend on a generalisation of Euclids' algorithm to a ring with a degree function. See a previous paper of P.Cohn( 4 ). For our purposes '<u>ring</u>' will mean associative ring different from zero, '<u>Field</u>' will mean commutative field. Since we wish to apply the results given by P.Cohn( 5 ) to the associative algebra we must amend the definition of $L$ to make it an algebra with unit element $1$. Henceforth when we write $L$ we will understand $(\Omega 1)\oplus L$. All subalgebras will include the unit element.

<u>DEFINITION 3.1</u>(Ring with a degree function)

A ring with a degree function is a ring $R$, together with a degree function $d$, which satisfies the following:

(i) For all $x$ belonging to $R\,(x \neq 0)$, $d(x)$ is a non-negative integer, $d(0) = -\infty$.

(ii) $d(x-y) \leq \max(d(x), d(y)) \quad (x, y \in R)$

(iii) $d(xy) = d(x) + d(y)$

Consider now a free associative algebra $A$ on an arbitary generating set $X$, over a field $\Omega$. In P.Cohn( 4 )an abstract characterisation of free associative algebras is given which can be used to obtain a criterion for subalgebras to be free. Unfortunately, this criterion is not easily applicable since it depends on the extension of a degree function defined on the subalgebra which may not be related to a degree function for the whole algebra. The next definition gives some indication of how this difficulty is overcome: we regard the algebra as a module over the subalgebra.

<u>DEFINITION 3.2</u>( Right $R$ -module with a degree function)

Let $R$ be any algebra over a field $\Omega$ with degree function $d$. A right $R$ -module $M$ $(M \times R \to M)$is said to possess a degree function if a non-negative integer $d(x)$ is associated with each $x$ belonging to $M$, $(x \neq 0)$, $d(0) = -\infty$, such that:

(i) $d(x-y) \leq \max(d(x), d(y))$, (ii) $d(xa) = d(x) + d(a)$, $a \in R, y \in M$

Thus ,for example, $R$ considered as a module over itself has a degree function , namely $d$ ,so that the apparent ambiguity in the terminology is resolved . More generally, let $S$ be a sub--ring then $S$ inherits the degree function from $R$ by restrict--ion. Then the original degree function on $R$ may still be used when $R$ is considered as an $S$- module.

From (i) and(ii) of Definition 3.2 ,it follows that for any $u_i$ in $M$ , $a_i$ in $R$ , $d(\sum u_i a_i) \leq \max_i (d(u_i a_i))$ . This introduces our next definition.

<u>DEFINITION 3.3</u>( $R$- independence in an $R$- module $M$ with a degree function).
A family $U = \{u_i : i \in I\}$ of elements of $M$ is said to be $R$- independent, if for any family $\{a_i : i \in I\}$ of elements of $R$ almost all zero , $d(\sum u_i a_i) = \max_i \{d(u_i a_i)\}$ .

We next introduce the concept of 'elementary transformations'. Suppose that $X$ is a finite subset of a free associative algebra $A$ over a field $\Omega$ .

<u>DEFINITION 3.4</u> ( Elementary transformations)
An elementary transformation is understood to mean one of the following applied to $X$ :

(i) A non-singular linear transformation applied to $X$ with coefficients from the field $\Omega$ .

(ii) An element $x$ belonging to $X$ is replaced by

$$x + p(x_1, x_2, \ldots x_k)$$

where $p$ is a non-commutating polynomial function in the elements $x_1, x_2, \ldots x_k$ of $X$ and $x$ is distinct from the $x_1, x_2 \ldots x_k$ .

DEFINITION 3.5 (An irreducible set)

If $U = \{u_i \mid 1 \leq i \leq p\}$ is a finite subset contained in $X$ and $d(U) = \sum_{i=1}^{p} d(u_i)$, we shall say that $U$ is irreducible if:

(i) $0$ does not belong to $U$.

(ii) We cannot reduce $d(U)$ by elementary transformations.

Our next result is the main theorem contained in P.Cohn( 5 ). It enables us to characterise those subalgebras of free associative algebras that are free.

THEOREM 'Let $A$ be a free associative algebra and let $U$ be a finite irreducible subset of homogeneous elements . Then $B$, the subalgebra generated by $U$ is free if and only if $U$ is right $B$- independent.'

We aim to show that $\underline{L}$, the projective limit of the free associative enveloping algebras $L^e_K$, of rank $(K=1,2,\ldots)$, is free. In order to demonstrate this we must decide what set of elements to take as a possible free generating set; write down an arbitary polynomial relation in an arbitary finite subset of these free generators and show that all the coefficients in this polynomial vanish. We now do precisely this. Let $W$ be the set of generators of $\underline{L}$ $\{z_\gamma, \omega_\beta\}$ (as defined on page 48 ). The degree of these generators has already been defined and it is easily seen that we can take the $\{z_\gamma, \omega_\beta\}$ to be homogeneous.

We now show that the generating set $W$ of $\underline{L}$ is a free generating set. That is, $W$ freely generates $\underline{L}$ as a free associative algebra over the field $\Omega$. However, we must verify the conditions of irreducibility and right-independence.

We first show irreducibility of this arbitary finite subset $U$ of $W$ .Recall that the homomorphic projection $\phi^{(n)}$ maps elements of $\underline{L}$ onto $L^{e}_{n}$ .Denote by $U^{(n)}$ the homomorphic image of the subset $U$ .

THEOREM 3.6

If $U^{(n)}$ is the homomorphic image under $\phi^{(n)}$ of an arbitary finite subset $U$ contained in $W$, then there exist $n \geqslant N$ for some large $N$ (defined in the decomposition given below) such that $U^{(n)}$ is irreducible for all $n \geqslant N$ .

Proof:. Suppose that $U^{(n)} = \{u_i^{(n)}\}_{1 \le i \le p}$. An elementary transformation which is non-singular , takes

(3.6.1) $\quad u_i^{(n)} \longrightarrow u_i'^{(n)} = \sum_{j=1}^{p} \alpha_{ij}\, u_j^{(n)}, \ (1 \le i \le p)$

where $|(\alpha_{ij})| \neq 0$ and $\alpha_{ij} \in \Omega$ . If $U^{(n)}$ is not irreducible under such a trans-formation , then for some $i$ , $(1 \le i \le p)$

(3.6.2) $\quad d(u_i^{(n)}) > d(u_i'^{(n)}) = d\left(\sum_{j=1}^{p} \alpha_{ij}\, u_j^{(n)}\right), \ (1 \le i \le p)$

From the condition that $|(\alpha_{ij})| \neq 0$, we see that not all $\alpha_{ij}$ are zero. The $\alpha_{ij}$ are functions of $n$ . Now the inequality of (3.6.2)is satisfied if,(i) cancellation occurs , (ii) a nonsingular transformation exist which will reduce the degree. If cancellation occurs we may equate terms of highest degree in the right hand side of( 3.6.2) to obtain the homogenous expression

(3.6.3) $$\sum_{k} \alpha_{ij_k}\, u_{j_k}^{(n)} = 0$$

In a similar manner , we consider an elementary transformation of the form

(3.6.4) $\quad u_i^{(n)} \rightarrow u_i'^{(n)} = u_i^{(n)} + p(u_1^{(n)}, u_2^{(n)} \ldots \hat{u}_i^{(n)}, \ldots u_p^{(n)})$.

where $p$ is a noncommutative polynomial in $u_1^{(n)}, u_2^{(n)}, \ldots u_{i-1}^{(n)}, u_{i+1}^{(n)} \ldots u_p^{(n)}$, the circumflex indicating that the element $u_i^{(n)}$ is omitted . If $U^{(n)}$ is not irreducible under such a transformation, then for some $i \ (1 \leq i \leq p)$ we have

(3.6.5) $$d(u_i^{(n)}) > d(u_i'^{(n)}) = d(u_i^{(n)} + p(u_1^{(n)} \ldots \hat{u}_i^{(n)} \ldots u_p^{(n)}))$$

This inequality implies that cancellation of the highest degree terms occurs in the right hand side of (3.6.5). Equating the terms of highest degree we obtain an homogenous expression

(3.6.6) $$u_i^{(n)} = p'(u_1^{(n)} \ldots \hat{u}_i^{(n)} \ldots u_p^{(n)})$$

where $p'$ is a subpolynomial of $p$. ~~, such that the degree of~~ $\cancel{p'}$ ~~is equal to the degree of~~ $\cancel{u_i^{(n)}}$ ~~.~~

The proof of the theorem now proceeds in two parts : we use a construction and decomposition to show that with a suitable choice of $h$ cancellation cannot take place in either of the relations (3.6.2) or(3.6.5). Secondly, we show an elementary transformation which is nonsingular cannot reduce the degree .

DECOMPOSITION

et $A_\ell$ denote the set of elements of degree $\ell$ in an arbitrary finite subset $U$ of the generators of $W$ as defined on page 52) and let $C_\ell$ denote the associative monomials of degree $\ell$ formed from the elements of the sets $A_1, A_2, \ldots \ldots A_{\ell-1}$. The sets $A_\ell$ are finite in number since $U$ has an element of greatest degree. It is, therefore, possible to use a direct decomposition of the space $\underline{\underline{L}}$ and bring all the elements of $U(\ell)$ through to a direct summand of $\underline{\underline{L}}$. We now do this. There exist elements $d_I(\ell)$ of ${}_\ell\underline{\underline{L}}$ and positive integers $q(\ell)$ and $N(\ell)$ such that:

(3.6.7)

$$ {}_\ell\underline{\underline{L}} / {}_{\ell+1}\underline{\underline{L}} = \Big(\sum_{I=1}^{q(\ell)} \{d_I(\ell) + {}_{\ell+1}\underline{\underline{L}}\}\Big) + \Big(\sum_{I>N(\ell)}^{*} \{b_I(\ell) + {}_{\ell+1}\underline{\underline{L}}\}\Big) $$

$$ (C_\ell \cup A_\ell) + {}_{\ell+1}\underline{\underline{L}} \subseteq \Big(\sum_{I=1}^{q(\ell)} \{d_I(\ell) + {}_{\ell+1}\underline{\underline{L}}\}\Big) \quad (1 \le \ell \le m) $$

where $m$ is the degree of the highest degree element of the set $U$, $\sum$ and $\sum^*$ denote the restricted and unrestricted sums, respectively while $\sum^*_{I>N(\ell)}$ is to mean those and only those basic products of degree $\ell$ on $x_1, x_2, \ldots$ which satisfy

(3.6.8) $\quad \phi^{(N(\ell)+1)}(b_I(\ell)) = 0 \quad$ occur in the unrestricted sum. Let $N = \max\{N(1), N(2) \ldots N(m)\}$.

We now apply these results to the relations (3.6.3),(3.6.6) If we replace $n$, in these relations, by $N = \max\{N(1) \ldots N(m)\}$ Then (3.6.3) in particular can be written $\sum_k \alpha_{ijk} u^{(N)}_{jk} = 0$

Using the above decomposition this can be written $\sum_k \alpha_{i j_k} u_{j_k} = 0$. That is, we have a linear relationship between the elements of $A_\ell$ for some $\ell (1 \leq \ell \leq m)$ .Contradicting the definition of $A_\ell$ given in the above decomposition In a similar manner relation(3.6.6)

can be written in one of the forms

(3.6.9) $w_\beta^{(n)} = \sum_i \alpha_i a_{\ell i}^{(n)} + \sum_j \beta_j c_{\ell j}^{(n)}$
(3.6.10) $z_\gamma^{(n)} = \sum_i \alpha_i' a_{\ell i}^{(n)} + \sum_j \beta_j' c_{\ell j}^{(n)}$
for some $\ell$ and not all $\alpha_i, \beta_j$ or $\alpha_i', \beta_j'$ are zero

By the decomposition (3.6.9) contradicts the definition of the generators $W$ (see p.p 52).

Similarly if we write (3.6.10) in the form $z_\gamma^{(N)} = f(W_\beta^{(N)}, z_\alpha^{(N)})$ with all $\alpha \neq \gamma$ and consider this relation for large $N$, we see by lemma (2.20) that for large the $z_\alpha^{(N)}$'s form a set of free generators for $L_N$

Now substitute for $W_\beta^{(N)}$ in terms of these $z_\alpha^{(N)}$'s where no $z_\gamma^{(N)}$ will appear, then we obtain a relation of the form $z_\gamma^{(N)} = g(z_\alpha^{(N)})$ with $\gamma \neq \alpha$. This contradicts lemma (2.20)

Finally, it remains to consider the possibility that there exist a nonsingular transformation which reduces the degree of the elements of $U^{(n)}$ even when cancellation does not take place in(3.6.2). If $d(u_i^{(n)}) > d(u_i'^{(n)}) = d(\sum_{j=1}^{p} \alpha_{ij} u_j^{(n)})$ where $|(\alpha_{ij})| \neq 0$ . Consider the system of equations:

(3.6.10) $\sum_{j=1}^{p} \alpha_{ij} u_j^{(n)} = u_i'^{(n)}, (1 \leq i \leq p)$ ,which can be written $A\underline{u}^{(n)} = \underline{u}'^{(n)}$ . Now recall that any non-singular matrix can be written as a finite product of elementary matrices. The elementary matrices are defined as follows :

(i) An elementary matrix is the identity matrix with two rows interchanged , or(ii) an elementary matrix is the identity matrix with one of its rows multiplied by a nonzero element of $\Omega$ . or (iii) an elementary matrix is the identity matrix with one row added to another.. The elementary matrices are nonsingular

A matrix of type (i) merely reorders the elements of $U^{(n)}$, so this cannot reduce the degree. A matrix of type (ii) multiplies an element of $U^{(n)}$ by a nonzero scalar, so this cannot reduce the degree. Finally, a matrix of type (iii) adds two elements of $U^{(n)}$ together and since cancellation does not take this type of elementary matrix cannot reduce the degree. Thus a finite sequence of such elementary matrices applied to $U^{(n)}$ cannot reduce the degree of $U^{(n)}$ and we have proved our theorem.

We now show that the set $U^{(n)}$ is also right $B$-independent where $B$ is the subalgebra of the free associative enveloping algebra $L^e_n$, generated by $U^{(n)}$. We shall make implicit use of the construction and decomposition of Theorem 3.6.

THEOREM (3.7)

The set $U^{(n)}$ is right $B$-independent for sufficiently large $n$ where $B$ is the subalgebra of $L^e_n$ generated by the set $U^{(n)}$

Proof:

Suppose that the set $U^{(n)}$ is not right $B$-independent. Then there exist elements $b_q^{(n)}$ belonging to $B$, such that

(3.7.1) $$a^{(n)} = \sum_{q=1}^{p} u_q^{(n)} b_q^{(n)}$$

where $d(a^{(n)}) < \max_q \{ d(u_q^{(n)} b_q^{(n)}) \}$

By equating terms of the highest degree we obtain an homogenous equation of the form

(3.7.2) $$\sum_{k=1}^{p'} u_{q_k}^{(n)} b_{q_k}^{(n)} = 0$$

$b_{q_k}^{(n)}$ belong to $B$. We recall that the elements $\{ u_q^{(n)} : 1 \le q \le p \}$ are ordered according to increasing degree. Hence in relation (3.7.2) will be of least degree. The proof of our theorem now proceeds by induction on the degree of $b^{(n)}_{q_{p'}}$.

If $d(b^{(n)}_{q_{p'}})=0$, then $b^{(n)}_{q_{p'}}$ is a field constant and (3.7.2) implies that the set $U^{(n)}$ is not irreducible. However, for $n \geq N = \max\{N(1),..,N(m)\}$ this contradicts theorem (3.6). Hence no such relation as (3.7.2) exist for $d(b^{(n)}_{q_{p'}})=0$.

Now let us assume that (3.7.2) does not exist when $n \geq N = \max\{N(1)..N(m)\}$ and the degree of $b^{(n)}_{q_{p'}}$ is less than $m$.

Consider

(3.7.3) $$F = \sum_{k=1}^{p'} u^{(n)}_{q_k} b^{(n)}_{q_k} = 0$$

where $n \geq N$ and $d(b^{(n)}_{q_{p'}}) = m$. Since each of the $b^{(n)}_{q_k} \in B$ they will be generated by the elements of the set $U^{(n)}$, thus we may write

(3.7.4) $$b^{(n)}_{q_k} = \sum_{i=1}^{n} f_{q_k i}(u^{(n)})\, x_i$$

where the $x_i$ are free generators for $L^e_n$ and the $f_{q_k i}(u^{(n)})$ are non commutative polynomial functions on the elements of some finite subset of $W^{(n)}$ which contains the set $U^{(n)}$. We shall denote this finite subset of $W^{(n)}$ also by $U^{(n)}$. Now substitute for the $b^{(n)}_{q_k}$ in (3.7.3) rearrange summations and we have

(3.7.5) $$F = \sum_{i=1}^{n} \Big( \sum_{k=1}^{p'} u^{(n)}_{q_k} f_{q_k i}(u^{(n)}) \Big) x_i = 0$$

Hence $\sum_{k=1}^{p'} u^{(n)}_{q_k} f_{q_k i}(U^{(n)}) = 0$ for $i = 1, 2, \dots n$

By induction each $f_{q_k i}(U^{(n)}) = 0$ for all $i$ & $k$

Hence $F$ is identically zero and this proves our Theorem.

The last two theorems enable us to deduce that $U^{(n)}$ freely generates a subalgebra of $L^e_n$ for sufficiently large $n$ .

We now show that $\underline{L}$ is a free associative algebra over a field $\Omega$ . Freely generated by the set $W$ .

THEOREM 3.8 The subalgebra of all elements of finite degree in the unrestricted free associative algebra is a free associative subalgebra.

Proof:

Suppose the contrary. Then there exist a finite subset of the generators $W$ , which we denote by $U$, such that we have a non-commutative polynomial relation of the form.

$$\sum_i \alpha_i u_{i_1} u_{i_2} \ldots u_{i_k} = 0 \qquad (u_i \in U)$$

$(\alpha_i \in \Omega)$ and not all the $\alpha_i$ are zero.

By the **decomposition and construction** this is equivalent to

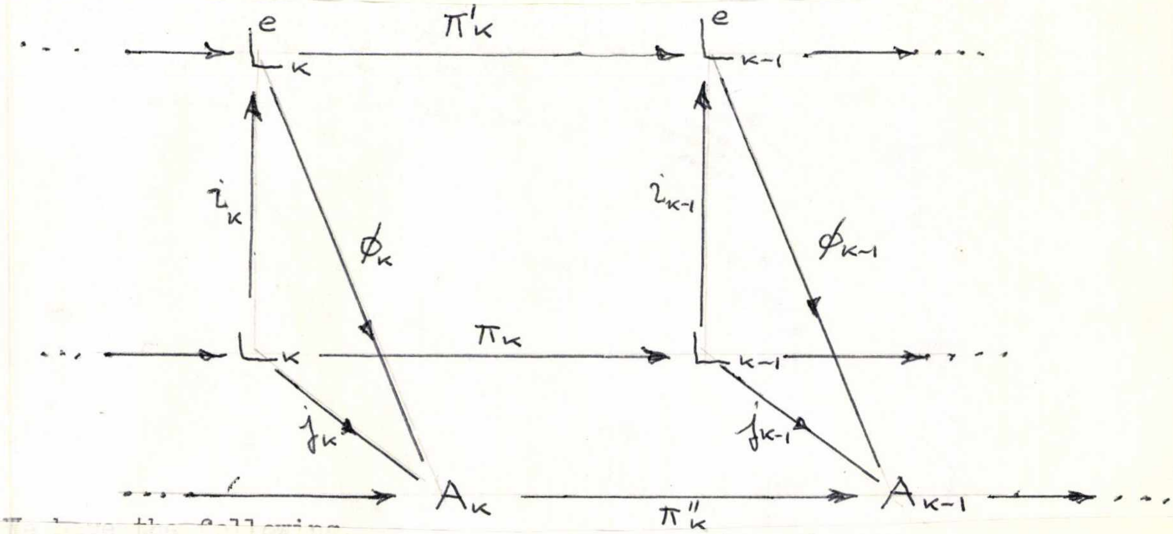
for large $n$ . $\sum_i \alpha_i u^{(n)}_{i_1} u^{(n)}_{i_2} \ldots u^{(n)}_{i_k} = 0$

But this contradicts our result at the end of the previous theorem. Namely, $U^{(n)}$ freely generates a subalgebra of $L^e_n$ for large $n$ .

Hence we have proved theorem (3.8).

We now define and study an unrestricted associative algebra which we denote by $L$, Note that from now on our notation is not the same as was used above.

Consider the following three dimensional diagram.




Where it is supposed we are given lie algebras $L_k$ $(k=1,2,\dots)$ and homomorphisms. $\pi_k: L_k \to L_{k-1}$ $(k=1,2,\dots)$ which enable us to define the inverse limit, $\varprojlim(L_k)$ Similarly it is supposed we are given associative algebras $A_k$ $(k=1,2,\dots)$ which form an inverse limit under homomorphisms $\pi''_k$ $(k=1,2,\dots)$ and lie homomorphisms $j_k$ $j_k: L_k \to A_k$ $(k=1,2,\dots)$ which make the base of the above diagram commutative.

~~We now deduce that~~

We now deduce that by the universal property there exist homomorphisms $\phi_k$ $(k=1,2\dots)$ which make the whole diagram commutative. Where $\phi_k: L^e_k \to A_k$ and the $L^e_k$ are U.E.A. for $(k=1,2,\dots)$

If we now denote the $\varprojlim(A_k)$ by $A$, $\varprojlim(L^e_k)$ by $L$ and $\varprojlim(L_k)$ by $\mathcal{L}$ also $\varprojlim(i_k)=i$, $\varprojlim(j_k)=j$; $\phi = \varprojlim(\phi_k)$ we then have the commutative diagram.

$$\begin{tikzcd} L \arrow[dr, "\phi"] & \\ \mathcal{L} \arrow[u, "i"] \arrow[r, "j"'] & A \end{tikzcd}$$

where $\mathcal{L}$ is a Lie algebra, $L$ is associative algebra and $A$ is an associative algebra.

By taking the discrete topology on each factor of the projective limits $L, \mathcal{L}$ and $A$ and endowing each of $L$, $\mathcal{L}$ and $A$ with the induced Tychonoff topology we see that:

(i) $\phi$ is continuous and unique since each $\phi_k$ which makes up $\phi$ has these properties $(k=1,2,\dots)$

(ii) $i$ is injective and continuous since each $i_k$ is injective and continuous $(k=1,2,\dots)$

(iii) $j$ is an arbitary homomorphism since each $j_k$ is such $(k=1,2\dots)$.

We call $(L, i)$ a <u>topological associative enveloping algebra</u> of the free Lie algebra $\mathcal{L}$, or more briefly $(L, i)$ is a T.E.A. of $\mathcal{L}$.

<u>Proposition</u> (3.9) The T.E.A. $(L, i)$ of $\mathcal{L}$ is <u>unique</u>.

Proof: Suppose $(L', i')$ is another T.E.A. for $\mathcal{L}$.

Put $(L', i') = (A, j)$ in the above diagram. This gives rise to a unique $\phi'$ such that

$$\underline{\phi' \circ \phi = \text{identity on } L}$$
$$\underline{\phi \circ \phi' = \text{identity on } L'}$$

Hence $L' \simeq L$.

<u>Proposition</u> (3.10) $i(\mathcal{L})$ generates topologically the T.E.A. $L$

Proof:

Let $\overline{B}$ be the proper subalgebra of $L$ generated by $\overline{i(\mathcal{L})}$.

The continuous mapping $i$ <u>has</u> the property that $i(\mathcal{L}) \subseteq i(\overline{\mathcal{L}}) \subset \overline{i(\mathcal{L})}$ by continuity <u>and</u> $i : \mathcal{L} \longrightarrow B_L$.

Now $\overline{B_L} = \overline{i(\mathcal{L})}$ and hence there is a unique continuous homomorphism $i' : L \longrightarrow \overline{B_L}$ such that $i'i = i$.

$$\begin{tikzcd}
L_L & \\
\mathcal{L} \arrow[u, "i"] \arrow[r, "i"'] & \bar{B}_L \arrow[ul, "i'"']
\end{tikzcd}$$

Since $i = 1_L i$,

$i'$ can be considered as a continuous homomorphism of $L$ into $L$, the uniqueness condition gives $i' = 1_L$.

Hence $L = 1_L(L) = i'(L) \subseteq \bar{B}_L$

$\quad$ i.e, $\bar{B}_L = L$ $\quad$ Thus $i\mathcal{L}$ is dense in $L$.

<u>Proposition</u> (3.11) $L$ has no zero divisors.

Proof: Suppose $x, y \in L$ $(x \neq 0), (y \neq 0)$ but $xy = 0$. Then by assumption there exist positive integers $n$ and $n'$ such that $x^{(m)} \neq 0$ for all $m \geqslant n$ and $y^{(m)} \neq 0$ for all $m \geqslant n'$.

Let $N = \max\{n, n'\}$ then $(xy)^{(m)} = x^{(m)} y^{(m)} \neq 0$ for $m \geqslant N$ since each $x^{(m)}, y^{(m)} \in L^{e\,d}_{m}$ has no divisors of zero - N. Jacobson ( 18 ) page 166. Hence $xy \neq 0$. This contradiction proves proposition.

This concludes the discussion of the topological enveloping algebra $L$.

In the next section we consider the unrestricted commutative algebra.

COMMUTATIVE CASE

We now consider the projective limit $\mathcal{C}$ ,of free associative commutative algebras of increasing rank , denoted by $\{C_k\}_{k=1,2,\ldots}$. We show that the subalgebra $\underline{C}$ is a free commutative associative algebra. First, we develop some results we will require . In particular, a corollary of the P.B.W. Theorem (2.9). We shall use the usual convention given at the beginning of Section 2 regarding the words algebra and subalgebra.

PRELIMINARY RESULTS

Recall that the P.B.W. Theorem (2.9) gives a characterisation of the universal enveloping algebra in the following sense. Let $\mathcal{L}$ be a subalgebra of $A_L$ , $A$ an algebra having the property that if $\{b_j \mid j \in J\}$ is a certain ordered basis for $\mathcal{L}$ as a module, then the elements $1, b_{i_1} b_{i_2} \ldots b_{i_r}$ $(i_1 \le i_2 \ldots \le i_r)$ (where $b_{i_1} b_{i_2} \ldots b_{i_r}$ is a basic product formed from the ordered product of basic monomials) form a basis for $A$ . Under these conditions $A$ and the identity mapping form a U.E.A., $E_A$ for $\mathcal{L}$ . There is therefore a unique homomorphism $\alpha$ say, such that

$\alpha : E_A \to A$ and $\alpha|_{\mathcal{L}} = 1_{\mathcal{L}}$ this implies that $E \simeq A$. And as a consequence $A$ may be taken as the U.E.A.

Now suppose that $B$ is an ideal in $\mathcal{L}$. Let $\{b_j \mid j \in J\}$ be an ordered basis for $\mathcal{L}$, so that the index set $J$ can be partitioned into disjoint subsets $M$ and $L$, with $m < \ell$ if $m \in M$, $\ell \in L$. We arrange for $\{b_m \mid m \in M\}$ to be an ordered basis for $B$. Now by Theorem (2.2) parts 3 and 4 we know $E' = E/K$ where $K$ is an ideal in $E$ generated by $B$ and the mapping $j : a + B \to a + K$ defines a U.E.A. for $\mathcal{L}/B$. $j$ is 1-1 mapping so we may identify $\mathcal{L}/B$ with the subalgebra $(\mathcal{L}+K)/K$ of $E'_L$. This subalgebra is the set of cosets $a+K$ and it has as basis $\{b_\ell + K \mid \ell \in L\}$. Hence by the P.B.W. Theorem the cosets $1+K$, $b_{\ell_1} b_{\ell_2} \cdots b_{\ell_t} + K$ $(\ell_1 \leq \ell_2 \cdots \leq \ell_t)$ form a basis for $E'$. Thus if $D$ is the subspace spanned by the elements $1$ and the basic products $b_{\ell_1} b_{\ell_2} \cdots b_{\ell_t}$ say, then $D \cap K = 0$.

Finally, the basic products of the form

$$b_{m_1} b_{m_2} \cdots b_{m_s} b_{\ell_1} b_{\ell_2} \cdots b_{\ell_t} \quad s \geq 1, \ t \geq 0$$

or more briefly, $b_M b_L$ $(M \neq 0)$ are in $K$ and form a basis for $K$. We note that the $b_M b_L$'s and $1$ and the $b_L$'s form a basis for $E$.

Now consider the commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & L^e_K \arrow[r, "\pi'_K"] \arrow[d, "\phi_K"] & L^e_{K-1} \arrow[r] \arrow[d, "\phi_{K-1}"] & \cdots \\
\cdots \arrow[r] & L^e_K / I_K \arrow[r, "\pi'^{*}_K"] & L^e_{K-1} / I_{K-1} \arrow[r] & \cdots
\end{tikzcd}$$

We recall the projective limit of the top row is denoted by $L$.

In the diagram the $I_K$ denote the ideals generated by the Lie elements $\{[x_i x_j] \mid 1 \le i < j \le k\}$ where $x_1, x_2, \dots x_k$ are the free generators for the Lie algebra of rank $k$ , $L_k$ .Under the induced mappings $\pi_k'^*$ we denote the projective limit of the bottom row by $C$ . From the arguments given above we know that a basis for $I_k$ consists of all ascending products $b_L^{(k)}$ of basic monomials, where at least one $b_{\ell_i}^{(k)}$ in the product is not equal to any one of $x_1, x_2, \dots x_k$.

A basis for $L_k^e / I_k$ consists therefore of all distinct elements of the form

$$x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} + I_k$$

where $n_1, n_2, \dots n_k$ are non-negative integers . By Corollary 2.18 if $\phi = P.L.(\phi_k)$, then $I = \operatorname{Ker}\phi = P.L.(\operatorname{Ker}\phi_k)$.Hence we conclude from this lemma that $C \simeq L/I$ . We know that elements of $\underline{L}$ can be written as finite sums of elements of the form

$$\sum{}^{*} \alpha_L b_L + \sum{}^{*} \beta_M b_M b_L \qquad (M \neq 0)$$

where the unrestricted summation extends over basic products $b_L$ , $b_M b_L (M \neq 0)$ of fixed degree . Elements of the form

$$\sum{}^{*} \beta_M b_M b_L \qquad (M \neq 0)$$

Containing $b_M$ of degree $\geq 2$ will be a basis for $I$ . Hence the natural mapping of $\underline{L}$ into $\underline{L}/\underline{I}$ defined by

$$\sum{}^{*} \alpha_L b_L + \sum{}^{*} \beta_M b_M b_L \longrightarrow \sum{}^{*} \alpha_L b_L + \underline{I}$$

is surjective . From this result we see that it is sufficient to take $W = \{x_i, y_\alpha, w_\beta\}$ as generators for $\underline{C}$ ,as these elements do not contain Lie terms or the completion of Lie terms . We show next that the generators $\{x_i, y_\alpha, w_\beta\} = W$ are free generators for $\underline{C}$ .

To show that $\underline{C}$ is a free commutative algebra over a field $\Omega$, we apply the usual decomposition and construction to a finite subset of elements of $W$ as given below.

CONSTRUCTION (4.1)

First we notice that, with an obvious notation, ${}_{i}\underline{C}/{}_{i+1}\underline{C}$ is a vector space over $\Omega$, for all $i$, and hence it is possible to construct the following sets $A_i$ and $B_i$. Let $A_1 = B_1$ be a set of elements of $\underline{C}$ that is linearly independent modulo ${}_{2}\underline{C}$. Suppose that the sets $A_\nu, B_\nu$ have already been defined for $\nu < n\ (n > 1)$ and an element of $A_\nu$ is greater than an element of $A_{\nu'}$ if $\nu > \nu'$. We define $B_n$ to be the set of all commutative monomials on the elements of the sets $A_1, A_2, \cdots A_{n-1}$ (By a commutative monomial we understand a power product which contains a finite number of commutative elements) which belong to ${}_{n}\underline{C}$ but do not belong to ${}_{n+1}\underline{C}$. Finally, $A_n$ is a set of elements of ${}_{n}\underline{C}$ which is linearly independent modulo the subalgebra generated by ${}_{n+1}\underline{C}$ and the set $B_n$.

DECOMPOSITION (4.2)

If we have a finite set of elements in each of the sets $A_1, A_2, \cdots A_n$ then we can bring these elements and the elements of $B_1, B_2, \cdots B_n$ of degree $m$ (as defined in the construction (4.1), above) through to a direct summand of ${}_{m}\underline{C}/{}_{m+1}\underline{C}$ $(m = 1, 2, \cdots n)$

Proof:

By a slight modification of lemma (1.5) we see that it is possible to bring a finite number of elements of ${}_{m}\underline{C}/{}_{m+1}\underline{C}$ through to a direct summand. Thus there exist elements $d_i(m)$ of ${}_{m}\underline{C}$ for all $m\ (1 \leq m \leq n)$

and positive integers $N(m)$, $q(m)$ such that

$$ {}_m\underline{C}/{}_{m+1}\underline{C} = \left(\sum_{i=1}^{q(m)} \{d_i(m) + {}_{m+1}\underline{C}\}\right) + \left(\sum_{i>N(m)}^{*} \{b_i(m) + {}_{m+1}\underline{C}\}\right) $$

$$ \& \quad (B_m \cup A_m) + {}_{m+1}\underline{C} \subseteq \left(\sum_{i=1}^{q(m)} \{d_i(m) + {}_{m+1}\underline{C}\}\right) $$

As usual $\sum\limits_{i>N(m)}^{*}$ is to mean those and only those commutative monomials $b_i(m)$ of degree $m$ on the free generators $x_1, x_2, \ldots$ occur in the unrestricted direct sum which satisfy the condition

$$ \phi^{(N(m)+1)}(b_i(m)) = 0 $$

Now by choosing $n \geq N = \max\{N(1) \ldots N(n)\}$ we can insure that all the elements of the sets $A_1, A_2 \ldots A_n$; $B_1, B_2 \ldots B_n$ of degree $m$ are brought through to a direct summand of $\{{}_m\underline{C}/{}_{m+1}\underline{C}\}_{m=1,2,\ldots n}$. This completes the proof of the decomposition.

We now take an arbitary finite subset of the generators $W = \{x_i, y_\alpha, w_\beta\}$ and denote this by $U$, in conformity with our usual notation.

By the above construction and decomposition we can bring through the set $U$ to a direct summand of $\{{}_m\underline{C}/{}_{m+1}\underline{C}\}$, $(m = 1, 2, \ldots n)$.

Let $U^{(n)}$ denote the image of $U$ under $\phi^{(n)}: \underline{C} \to L^e_n/I_n$.

<u>THEOREM</u> (4.3)

If $U^{(n)}$ is a finite subset of $L^e_n/I_n$, then for **sufficiently large $n$, the subalgebra $B$* of the commutative algebra $L^e_n/I_n$ is freely generated by this set $U^{(n)}$**

Proof:

Consider the polynomial expression.

(4.3.1.) $\sum \alpha_i b_i(u_1^{(n)} \ldots u_p^{(n)}) = 0$ not all $(\alpha_i \in \Omega) = 0$

where the $b_i(u^{(n)})$ are commutative monomials on the elements $U^{(n)} = \{u_1^{(n)}, \ldots u_p^{(n)}\}$. We may assume without loss of generality, that (4.3.1.) is

* $B$ is the subalgebra generated by the set $U^{(n)}$ for some integer $n$

an homogenous equation. The proof now proceed by induction on the degree of (4.3.1). If (4.3.1) has degree $1$, then since the elements of degree $1$ in $U^{(n)}$ are $\{x_i, y_\alpha^{(n)}\}$ and these elements are linearly independent by the construction and decomposition when $n \geq N = \max\{N(1), \ldots N(n)\}$. We have a contradiction in that all the $\alpha_i$ are not zero.

**Now suppose that non trivial relation of the form**

**(4.3.2)** $$\sum \alpha \, (u_{i_1}^{(N)})^{\varepsilon_{i_1}} (u_{i_2}^{(N)})^{\varepsilon_{i_2}} \ldots (u_{i_k}^{(N)})^{\varepsilon_{i_k}} = 0$$

**exists and (4.3.2) is homogeneous.**

**If $k=1=\varepsilon_k$ for some term is the summation then we get a contradiction of the choice of the $u_i$'s using the decomposition (4.2) and the linear independence of the elements of the sets $A_1, A_2, \ldots A_n$.**

**We now apply induction to the degree of (4.3.2) and we write the above expression in the form.**

**(4.3.3)** $$\sum_{j=0}^{n} g_j (u^{(N)}) (u_\ell^{(N)})^j = 0$$

**Where $U_\ell^{(N)}$ is a $u_i^{(N)}$ of largest degree which genuinely occurs in (4.3.2). Now differentiate (4.3.3) with respect to some free generator $x_m$ giving:**

(4.3.4) $$\sum_{j=0}^{n} \left( \frac{\partial g_j}{\partial x_m} (u_\ell^{(N)})^j + j \cdot g_j \cdot \frac{\partial u_\ell^{(N)}}{\partial x_m} (u_\ell^{(N)})^{j-1} \right) = 0$$

**On equating coefficients of fixed powers of $U_\ell^{(N)}$, we have**

**(4.3.5)** $$\frac{\partial g_j}{\partial x_m} + (j+1) g_{j+1} \frac{\partial u_\ell^{(N)}}{\partial x_m} = 0 \quad \textbf{for } j = 0, 1, \ldots (n-1)$$

**and**

**(4.3.6)** $$\frac{\partial g_n}{\partial x_m} = 0.$$

**If we do this for each free generator, i.e. $m = 1, 2, \ldots N$. Then (4.3.6) implies that $g_n(U^{(N)})$ is a constant.**

Now consider

(4.3.7) $$\sum_{m=1}^{N} x_m \left( \frac{\partial g_j}{\partial x_m} + (j+1)\, g_{j+1} \frac{\partial u_l^{(N)}}{\partial x_m} \right) = 0$$

This gives using Euler's Theorem for homogeneous functions

(4.3.8) $$\lambda g_j(u^{(N)}) + \mu (j+1)\, g_{j+1}(u^{(N)})\, u_l^{(N)} = 0$$

Where $j = 0,1,\ldots (n-1)$, $\lambda$ is the degree of $g_j$, $\mu$ is the degree $u_l^{(N)}$. However, (4.3.6) and (4.3.8) together imply that $g_j(u^{(N)}), (j=1,2,\ldots n)$ is constant for (4.3.8) then implies that each $v_i^{(N)}$ can be expressed as a $g_j(u^{(N)})$ for $(j=1,2,\ldots n)$. This contradicts the choice of the $v_i^{(N)}$ for large $N$, and our Theorem is proved

Note the above proof will not work for a field of characteristic $(p > 0)$

We now deduce that the subalgebra $\underline{C}$ of all elements of finite degree in $C$ is a free commutative algebra.

THEOREM (4.4)

**$\underline{C}$ is a free commutative and associative algebra on the set $W$ as free generators over the field $\Omega$** of characteristic zero.

Suppose $\underline{C}$ is not free on the generating set $W$, then write down an arbitary polynomial relation

(4.4.1) $\sum_i \alpha_i u_{i_1} u_{i_2} \cdots u_{i_m} = 0$ and not all $\alpha_i = 0$ .

This relation defines a finite set of elements $U$ contained in $W$ and by the decomposition and construction given above (4.1) is equivalent to

(4.4.2) $\sum_i \alpha_i u^{(n)}_{i_1} \cdots u^{(n)}_{i_m} = 0$

for large $n$ .

By theorem (4.3) we have an immediate contradiction. Hence all the $\alpha_i = 0$ . This proves our theorem.

It is to be noted that we could have proved that the subalgebra $\underline{L}$ of the associative algebra $L$ was free, by the same method as used here. However, it is of interest to work within the context of the paper ( 5 ) of P.Cohn.

This concludes section 4.

## $\Omega$-GROUPS . SUBALGEBRAS OF UNIVERSAL ALGEBRAS. PROJECTIVE LIMIT OF UNIVERSAL ALGEBRAS

We now derive some results of general interest involving $\Omega$-groups. In particular we define the completion of an $\Omega$-group and show some of its properties. This is largely a generalisation of the work of M.Hall ( 11 ) on the completion of free groups. Only the more fundamental properties of $\Omega$-groups are developed ; we do indicate , however, how the more technical results required in the following may be obtained. For a comprehensive account see P.J. Higgins ( 12 ) or,less detailed,A.G. Kurosh ( 20 ).

We shall use the concept of completion of a group,considered as a uniform space, as developed originally by A.Weil. A good account of this appears in N.Bourbaki ( 2 ).

DEFINITION 5.1( $\Omega$-group ).

Let $G$ be a non-null set. An $n$-ary operation $\omega$ is defined on $G$ (where $n$ is a non-negative integer ) if to every ordered system of $n$ elements $a_1, a_2, \ldots a_n$ of $G$ there is a uniquely determined element of the same set , written: $a_1 a_2 \ldots a_n \omega$ for all $n$-ary operations $\omega$ in $\Omega$ . If $G$ also satisfies the axioms:

G.1 $G$ is a group ( not necessarily commutative ) with respect to $(+,-)$ .

G.2 $G$ admits the set $\Omega$ of finitary operations.

G.3 For all $\omega$ in $\Omega$ , $0 0 0 \ldots 0 \omega = 0$, where $0$ is the zero element of $G$ .Then $G$ is an $\Omega$-group.

DEFINITION 5.2 ( $\Omega$-subgroup)

Let $G$ be an $\Omega$ -group A non-void subset $A \subseteq G$ is called an $\Omega$-subgroup.

of $G$ if for every $n$-ary operation $\omega$ in $\Omega$ it always follows that given $a_1, a_2, \ldots a_n$ in $A$, $a_1 a_2 \ldots a_n \omega$ is in $A$.

DEFINITION 5.3 (Homomorphisms of $\Omega$-groups)

Let $G$, $G'$ be two $\Omega$-groups of the same type (i.e., both groups have the same of operations $\Omega$). If $\varphi : G \to G'$ is a mapping where for all $a_1, a_2 \ldots a_n$ and $n$-ary operations $\omega$ in $\Omega$ $(a_1 a_2 \ldots a_n \omega)\varphi = (a_1\varphi)(a_2\varphi)\ldots(a_n\varphi)\omega$. Then $\varphi$ is called a homo--morphism. The usual modifications enable one to define an iso-, endo-, auto-, morphism.

DEFINITION 5.4 ( Equivalence relation )

An equivalence relation $R$ in the set $G$ is termed a congruence relation or congruence in the $\Omega$-group $G$, if whenever $(x_i, x_i') \in R$ (for $i = 1, 2, \ldots n$) then $(x_1 x_2 \ldots x_n \omega, x_1' x_2' \ldots x_n' \omega)$ belongs to $R$. By $(x_i, x_i') \in R$ is meant that $x_i$ and $x_i'$ belong to the same $R$ equivalence class.

DEFINITION 5.5 (Quotient group)

$G/R$ the quotient group of the $\Omega$-group $G$ by a congruence $R$ denotes the $\Omega$-group $G/R$. Where the elements of the set $G/R$ are denoted by $xR$, $x \in G$; and $xR$ is the set $\{y \in G \mid (x,y) \in R\}$

**and the operations on the $xR$'s are defined as follows**

$$(x_1 R)(x_2 R)\ldots(x_n R)\omega = (x_1 x_2 \ldots x_n \omega)R$$

DEFINITION 5.6 (An ideal in an $\Omega$-group )

An ideal in an $\Omega$-group $G$ is a subset $A$ of $G$ with the follow--ing properties:

(i) $A$ is a normal subgroup of the additive group of the $\Omega$-group.

(ii) When $\omega \in \Omega$ is an arbitary $n$-ary operation, $a$ an arbitary element of $A$ ,and $g_1, g_2, \cdots g_n$ are arbitary elements of $G$, then the following inclusion relation must always hold for $i=1,2,\cdots n$

$$-(g_1 g_2 \cdots g_n)\omega + (g_1 g_2 \cdots (a+g_i) g_{i+1} \cdots g_n)\omega \in A$$

It is now easily shown that there is a 1-1 correspondence between the decomposition of an arbitary $\Omega$-group with respect to its ideals and the congruences on $G$ vide( 20 ).Thus we can speak of the $\Omega$-factor group with respect to the ideal $A$ ,namely $G/A$ as opposed to the $\Omega$-factor group with respect to the congruence $R$, $G/R$ . The ideal $A$ is ,of course, the zero element of the $\Omega$-factor group $G/A$.

**Now let $G$ be the free $\Omega$-group generated by a countable number of.** free generators $\{x_1, x_2, \cdots x_n \cdots\}$. Let $(H_m)_{m\in I}$ denote the family of kernels of the natural projections $\phi^{(m)}: G \to G_m$ ,where $G_m$ is the free $\Omega$ -group generated by $\{x_1, x_2, \cdots x_n\}$. We show that this family $(H_m)_{m\in I}$ is a family of ideals . We verify the second condition of Definition 5.6, the former is obvious.

Let $h \in H_m$ for some $m$ ,then let

$y = -(g_1 g_2 \cdots g_n)\omega + (g_1 \cdots (g_i + h) \cdots g_n)\omega$ . Map both sides under $\phi^{(m)}$ and we get

$0 = \phi^{(m)}(y) = -(\phi(g_1)\phi(g_2)\cdots\phi(g_n))\omega + (\phi(g_1)\cdots\phi(g_n))\omega$, since $\phi^{(m)}(h) = 0$

This establishes our contention.

The following properties of the family of ideals $(H_m)_{m\in I}$ are obvious from the definition of the family :

(i) $H_m \supseteq H_n$ if $n \leq m$, (ii) $\bigcap_m H_m = \{0\}$ ,(iii) the $H_m$ are normal subgroups of $(G', +')$.

The cosets $\{x+H_m\}$ of the family of ideals $\{H_m\}$ then provide us with a basis for a topology. We call this topology the <u>ideal topology</u> it is analogous to M.Hall's subgroup topology ( 11 ). Following M.Hall ( 11 ), we define a uniformity of the ideal topology in terms of the sets $H^m=\{(p,q): q \in H_m + p\}$ ,noting that if $x_1, x_2, \ldots x_n \in H^m$ we have, by definition of a congruence $x_1 x_2 \ldots x_n \omega \in H^m$.
We have therefore that the operations $\omega$ in $\Omega$ are continuous.

Now with each $H_m$ there is associated the factor $\Omega$-group $G/H_m$. It is easily seen that the $H_m$ are open and closed, thus the topology induced in $G_m = G/H_m$ is the discrete topology. Also the indices $\{m\}$ are the positive integers so that we can define the projective limit of the factor groups $G_m$; this will be an $\Omega$-group $P$. The discrete topology on the $G_m$ will determine a topology on $P$, the neighbourhoods of an element $x \in P$ being all the elements of $P$ with the same $x_m$ for a finite number of $m$'s.

<u>THEOREM 5.7</u>

An $\Omega$-group $G$ with an ideal topology defined by a family $\{H_m\}$ of ideals is totally disconnected .If we take the $G_m$ factor $\Omega$-groups $G/H_m$ the indices form a directed set if whenever $H_m \subset H_n$ we write $m > n$ . A natural homomorphism is determined $\phi_{nm}: G_m \to G_n$ via the relation $G/H_n = G/H_m / H_n / H_m$. In terms of these homomorphisms and the discrete topology in the $G_m$ ,the projective limit $P$ is defined .The group $P$ is the completion of $G$ (denoted by $\hat{G}$) by Cauchy sequences. $\hat{G}$ is totally dis-connected . $\hat{G}$ will be compact if and only if the ideals regarded as normal subgroups of $G$ are of finite index.

Proof:. Since $\bigcap_{m\in I} H_m = \{0\}$ ,for any $g \neq 0$ there is an $H_m$ which does not contain $g$ .Hence as $H_m$ is open and closed , $g$ does not belong to the component of zero. Hence $G$ is totally disconnected.

Let $x \in G$ and $x \to x_m$ be the homomorphism $G \to G_m$ .Then $P$ is a subdirect product $P = \prod_m G_m$ ,where if $H_m \supset H_n$ ,then the $m$ and $n$ components $u_m$ and $u_n$ are related by the homomorphism $\phi_{mn}(u^{(n)}_n) = u^{(m)}_m$ For every $x \neq 0$ in $G$ , the element $\prod(x^{(m)})$ is an element of $P$ ,also non-zero.These elements form a subgroup of isomorphic to the $\Omega$-group $G$ and will be identified with $G$ . Since the topology for the $G_m$ is the discrete topology a neigh--bourhood in $P$ is given by all elements $u$ with a finite number of the $u^{(m)}_m$ fixed. Let $N$ be a neighbourhood determined by fixing $u^{(m)}_m$ for $m = m_1, m_2, \dots m_n$ .Then an $n$ exists following all these $m$'s .Suppose for some $u$ in $N$ the $n$ component is $u^{(n)}_n$ .Here $u^{(n)}_n$ is completely determines $u^{(m)}_m$ for $m = m_1, m_2, \dots m_n$ .Moreover ,for some $x$ in $G$ , $x \to x^{(p)}_p = u^{(p)}_p$ .Hence $x$ considered as an element of $P$ belongs to $N$ .Since every neighbourhood contains an element of $G$ , and the neighbourhoods are a basis for open sets, $G$ is everywhere dense in $P$ .Also, the topology induced in $G$ as a subgroup of $P$ is precisely the ideal topology defined by the $\{H_m\}$ with cosets of the $H_m$ as a basis of open sets . To show that $P = \hat{G}$ it is sufficient show that $P$ is complete( 2 ).In $u = \prod(u^{(m)})$ if $u^{(m)}_m = 0$ for a particular $m$ ,these $u$'s form an ideal of $P$ which includes $H_m$ .This follows from the definition of the $H_m$.The ideal so formed is $\overline{H}_m$ the closure of the ideal $H_m$ in $P$ ,since every element in $P$ is the limit of elements in $G$ and any element of $\overline{H}_m$ must be the limit of elements with $u^{(m)}=0$, we see a limit of $\overline{H}_m \cap G = H_m$. The sets $H^m$ in $P \times P$ consist of pairs $(p,q)$ where $q \varepsilon \overline{H}_m + p$. Hence $p$ and $q$ have the same $m$ component $u^{(m)}$. Since a Cauchy sequence contains 'small' sets ( 2 ); given a sequence $\{C_i\}$ in $P$ there is a set $C_i$ with $(p,q) \varepsilon H^m$ for any $p \varepsilon C_i$, $q \varepsilon C_i$ whence all elements in $C_i$ have the same $m$ component $u^{(m)}$. Since $C_i \cap C_j$ is not null every set $C_j$ of the sequence contains elements $u$ with $m$ component $u^{(m)}$. Hence a Cauchy sequence in $P$ determines a unique component $u^{(m)}$ for every $m$. If $m < n$ there will be a $u$ in some $C_k$ with components $u^{(m)}$ and $u^{(n)}$ determined by the Cauchy sequence whence $\phi_{mn}(u^{(n)}) = u^{(m)}$. Here the element $u = (u^{(m)})$ where each $u^{(m)}$ is determined by the $\{C_i\}$ is an element of $P$ since its components satis-fy the requirements of the homomorphisms. Hence $u$ is the limit of the Cauchy sequence and $P$ is the completion of $G$. The topology induced by $P$ is the ideal topology of the family $\{\overline{H}_m\}$. Hence $\hat{G}$ is totally disconnected. If any $G_m$ is infinite a sequence of elements from different cosets will have no limit point in $\hat{G}$. But if every $G_m$ is finite then the $G_m$ unrestricted direct product is compact, and $\hat{G}$ as a closed subgroup of this product is also compact.

UNIVERSAL ALGEBRAS: A SUBALGEBRA THEOREM

In reference ( 6 ) S.Feigelstock established a subalgebra theorem for an abstract class of universal algebras(the definition of abstract class is given below, defn. 5.16). We use this result to show that the projective limit of algebras belonging to this same class is free(as an algebra).

First we have some definitions from the above paper.

DEFINITION 5.8($n$-ary operation )

An $n$-ary operation $\omega$ in a set $X$ is a mapping of $X^n$ into $X$, written : $x_1 x_2 \dots x_n \omega$ ,where $(x_1, x_2, \dots x_n)$ belongs to $X^n$ and $x_1 x_2 \dots x_n \omega$ belongs to $X$ .These operations are defined for all positive integral values $n$ on the whole of $X^n$.

DEFINITION 5.9 (Algebra)

An algebra $A$ is a pair $A = (X, \Omega)$ where $X$ is non-empty and is called the carrier of $A$ ,denoted by $|A|$ ,and $\Omega$ is the set of operations defined on $X$ .

DEFINITION 5.10 (Subalgebra)

An algebra $B$ is a subalgebra of an algebra $A$ ,denoted by $B \leq A$, if $|B| \subseteq |A|$ and for all $x_1, x_2 \dots x_n$ in $B$ and all $n$-ary operations $\omega$ in $\Omega$ , $x_1 x_2 \dots x_n \omega$ is in $B$ .

DEFINITION 5.11 (Cartesian product algebra)

Let $\{A_k\} = \{(X_k, \Omega)\}_{k=1,2,\dots}$ be a family of algebras with $\Omega$ the same set of operations for each algebra $A_k$ . The cartesian product algebra is denoted by $\prod_{k=1}^{\infty} A_k$ and it consist of all 'vectors' $a = (a^{(k)})$ ,where $a^{(k)}$ belongs to $A_k$ . The operations in $\prod_{k=1}^{\infty} A_k$ are defined componentwise i.e., $a_1 a_2 \dots a_n \omega = (a_1^{(k)} \dots a_n^{(k)} \omega)$

DEFINITION 5.12 (Homomorphism of algebras)

If $A=(X,\Omega)$ and $B=(Y,\Omega)$, then the mapping $\phi : X \to Y$ is a homomorphism from $X$ to $Y$ if for every $n$-ary operation $\omega$ and for all $x_1, x_2, \dots x_n$ belonging to $X$,

$$(x_1 x_2 \dots x_n \omega)\phi = (x_1\phi)(x_2\phi)\dots(x_n\phi)\omega$$

If $\phi : X \to Y$ is a homomorphism which is also 1-1- and onto. We say that $\phi$ is an isomorphism.

We now consider the possibility of generating an algebra given a set. In particular, we note that the intersection of a family of algebras is defined if the intersection of their corresponding carriers is not void.

DEFINITION 5.13 (Generating set)

$S \subseteq |A|$ is a generator, or more precisely a generating set, of the algebra $A$ if $\bigcap\{B : B \leq A \ \& \ S \subseteq |B|\} = A$ i.e., the smallest subalgebra containing $S$ in its carrier is $A$ itself. The algebra generated by $A$ is denoted by $\langle A \rangle$.

We come now to two important concepts that of an abstract class and a variety.

DEFINITION 5.14 (Abstract class, variety)

A family $\mathcal{C}$, of algebras is called an abstract class of algebras, if $A$ belongs to $\mathcal{C}$ and $A$ is isomorphic to $B$ implies that $B$ belongs to $\mathcal{C}$. An abstract class which is closed under the formation of subalgebras, quotient algebras and cartesian products is called a variety.

DEFINITION 5.15( $\mathcal{C}$ -free algebra)

Let $\mathcal{C}$ be an abstract class of algebras. An algebra $A$ belonging to $\mathcal{C}$ is $\mathcal{C}$-free if it satisfies the following two conditions:

(i)There is a set $X \subseteq |A|$ such that $X$ generates $A$ .

(ii) For all $B$ belonging to the abstract class $\mathcal{C}$ ,and for every mapping $\theta: X \to |B|$ ;there is an extension of this mapping to a homomorphism $\theta^!: A \to B$ .The $X$ refered to in(i), (ii) above is called a $\mathcal{C}$-free generator of $A$ .

A more natural definition of what a free algebra is,will be realized in the following lemma.

LEMMA 5.16

If $\mathcal{C}$ is the variety of all $\Omega$-algebras,(i.e., the algebras having $\Omega$ as their set of operations )then given any set $X \neq \emptyset$ , $X$ generates $\mathcal{C}$ -freely an algebra $A \in \mathcal{C}$.

The algebra $A$ which will be constructed in the proof Lemma 5.16 is called the free anarchic algebra( empty set of **identical** relations)on the set $X$ .

Proof: Let $X_0 = X$ .Define $X_{i+1} = X_i \bigcup_{n=0}^{\infty} X_i^n \times \Omega_n$ where $\omega$ in $\Omega_n$ is an $n$-ary operation. Put $Y = \bigcup_{i=0}^{\infty} X_i$.If $y$ belongs to $Y$ ,then there is an $i$ such that $y$ belongs to $X_i$ . If $i > 0$ ,then $y = y_1 y_2 \cdots y_n \omega$ where $y_j$ belongs to $X_{i-1}$ ,$(j=1,2,\ldots n)$,or $y$ belongs to $X_{i-1}$.
If $y_1, y_2 \cdots y_n$ belong to $Y$ and $\omega$ belongs to $\Omega_n$ ,then $y_1 y_2 \cdots y_n \omega$ is def--ined,because there is an $m$ such that $y_j$ belongs to $X_m$ , $(j=1,2,\ldots n)$; therefore $y_1 y_2 \cdots y_n \omega$ belongs to $X_m^n \times \Omega_n$ which is contained in $Y$ and $A = (Y, \Omega)$ is an algebra.

It must now be shown that $X$ satisfies conditions (i),(ii) of Definition 5.15.

(i) It will be shown inductively that $X$ generates $A$. It is obvious that $X_0 \subseteq |\langle X \rangle|$. Suppose that $X_i \subseteq |\langle X \rangle|$. If $y_1, y_2 \cdots y_n$ belong to $X_i$ and $\omega$ belongs to $\Omega_n$, then $y_1 y_2 \cdots y_n \omega$ is in $|\langle X \rangle|$ which implies that $X_{i+1}$ is in $|\langle X \rangle|$. Therefore we have $|\langle X \rangle| = A$.

(ii) It will be shown that $X$ is a free generator of $A$. Let $B = (Z, \Omega)$ belong to the variety $\mathcal{C}$, and let $\theta : X \to Z$. Put $\theta = \theta_0$, and define $\theta_i$ as the mapping from $X_i$ into $Z$. If $x$ is in $X_{i+1} \setminus X_i$, then $x = y_1 y_2 \cdots y_n \omega$, and $\theta_{i+1} x$ is defined to be $(y_1\theta_i\, y_2\theta_i \cdots y_n\theta_i)\omega$. If $x$ is in $X_i$, then $\theta_{i+1} x = \theta_i x$. Define a mapping $\phi = \bigcup_{i=0}^{\infty} \theta_i$, such that for $y$ in $Y$, $\phi y$ is defined to be $\theta_i y$ if $y$ is in $X_i$. Let $y_1, y_2, \cdots y_n$ belong to $X_i$, and let $\omega$ be in $\Omega_n$, then:

$$[(y_1 y_2 \cdots y_n)\omega]\phi = (y_1\theta_i)(y_2\theta_i)\cdots(y_n\theta_i)\omega = (y_1\phi)\cdots(y_n\phi)\omega$$

Hence $\phi$ is a homomorphism, and the lemma is proved.

Corollary (5.16)

Each free anarchic $\Omega$-algebra is isomorphic to a free $\Omega$-algebra.

LEMMA 5.17

If $A = (X, \Omega)$ is a free anarchic algebra, then

$$x_1 x_2 \cdots x_n \omega = x_1' x_2' \cdots x_n' \omega \Rightarrow x_i = x_i' \quad (i = 1, 2, \ldots n)$$

Proof: Lemma 5.17 follows directly from the construction of the free anarchic algebra given in Lemma 5.16

THEOREM 5.18

Subalgebras of free anarchic algebras are free.

Proof. Let us take algebras $A = (E, \Omega)$, $B = (F, \Omega)$ where $B$ is a subalgebra of $A$, symbolically: $B \leq A$. Let $\mu_1, \mu_2, \ldots \mu_k$ be an arbitary finite subset of elements of $\Omega$ .with $\mu_i$ an $n_i$-ary operation .Let $u_k \varepsilon |B|$ be the element arising from the operation $\mu_k$ ,then there exist $x_1, x_2 \ldots x_{n_k}$ belonging to $|A|$ such that $x_1 x_2 \ldots x_{n_k} \mu_k = u_k$ . Define $u_k$ as irreducible in $B$ if $x_i \notin |B|$ for some $i = 1, 2, \ldots n_k$. Define the length $\ell$ of an element of $|A|$ ( or $|B|$ )as the number of ~~vector~~ symbols in the element i.e.,

$$\ell(u_k) = 1 + \sum_{i=1}^{n_k} \ell(x_i) < \infty$$

Define the set $T = \{z \mid z$ is irreducible in $B\}$ .To prove the Theorem if is sufficient to show that $T$ generates $B$ freely. We must verify conditions (i), (ii) of Definition 5.15.

Condition (i)

$T$ generates $B$ . Let $u_k$ belong to $|B|$ .If $u_k$ is irreducible then $u_k$ belongs to $T$ .If not ,then $x_i$ belongs to $|B|$ for every $i = 1, 2, \ldots n_k$ .Now $\ell(x_i) < \ell(u_k) < \infty$ ;therefore inductively there exist an $a$ belonging to $|A|$ in $u_k$ such that $a$ is irreducible in $|B|$ . $T$ therefore generates $B$ .

Condition (ii)

$T$ generates $B$ freely. Let $T^*$ be a set in 1-1 correspondence with $T$ ,such that $t^* \to t$ ,and let $B^*$ be a free anarchic algebra on $T^*$. Let $\theta$ be a map $\theta : t^* \to t$ .Extend $\theta$ to an epimorphism $\phi$ of $B^*$ onto $B$ . We show by induction that $\phi$ is a 1-1 map of $B^*$ onto $B$. Assume that $\phi$ is 1-1 map on the elements of $B^*$ of length $\leq n$ ,into $B$ . By the 1-1 corres--pondence between $T^*$ and $T$ , we may assume that $n$ is greater than $1$ . Suppose that $b^*$ belongs to $B^*$ and is of length $n+1$ . Let $b^{*i} = (x_1^{*i}, x_2^{*i} \ldots\ldots x_{n_k}^{*i})\mu_k$ where $i = 1, 2$. $\ell(x_j^{*i}) < n$ for

$(j=1,2,\ldots n_k)$ . Suppose $b^{*1}\phi = b^{*2}\phi$ , then :

$b^{*1}\phi = (x_1^1 x_2^1 \ldots x_{n_k}^1)\mu_k = (x_1^2 \ldots x_{n_k}^2)\mu_k = b^{*2}\phi \Rightarrow x_j^1 = x_j^2$ for $(j=1,2\ldots n_k)$

by Lemma 4.17 this implies that $b^{*1} = b^{*2}$ .Thus $'T$ is a free generator of $B$ , and the proof of the theorem is complete.

REFERENCES

1. R.Baer, Abelian groups without elements of finite order, Duke. Maths. Journ. 3(1937) 68-122.

2. N.Bourbaki, Elements of Mathematics, General Topology Part 1, Addison Wesley, 1966.

3. M.D. Burrows, A note on a result of G.Higman, Arch., Maths. 15(1964) pp. 258-260.

4. P.M.Cohn, A generalisation of Euclid's algorithm, Proc. Camb. Phil. Soc. 52(1961) 18-30.

5. P.M.Cohn, Subalgebras of free associative algebras, Proc. London Math. Soc., (3) 14(1964) 618-632.

6. S.Feigelstock, A universal subalgebra theorem, Amer. Math. Mon. (72) (1965) 884-888.

7. H.B. Griffiths, A note on commutators in free products (I).

8. H.B.Griffiths, A note on commutators in free products (II)

Proc. Camb. Philos. Soc. (50) 178-188 (1954) (I)

Proc. Camb. Philos. Soc. (51) 245-251 (1955) (II)

9. H.B.Griffiths, Infinite products of semigroups and local connectivity, Proc. London Math. Soc. (3) 6(1956) 455-480.

10. M.Hall Jr., A basis for free Lie rings and higher commutators in free groups, Proc. Amer. Math. Soc. 1(1950) 575-581.

11. M.Hall Jr., A topology for free groups and related groups, Annals of Maths. Vol. 52, No. 1 July 1950.

12. P.J.Higgins, Groups with multiple operators, Proc. London Math. Soc. Vol. 6, (1956) 366-416.

13. G.Higman, Unrestricted free products and varieties of topological groups, J.London Math. Soc., (1952), 73-81.

14. G.Higman, On a problem of Takahasi. J. London Math. Soc. 28(1953) 250-252.

15. A. Hulanicki, K.Golema, The structure of the factor group of the unrestricted sum by the restricted sum of Abelian groups, (II) Fund. Math. 53(1963-1964) 177-185.

16. A.Hulanicki and M.F.Newman, Existence of the unrestricted product with one amalgamated subgroup, J.London Math. Soc. 38(1963) 169-175.

17. A.Hulanicki, Correction to 12. J. London Math. Soc. 39(1964) 672.

18. N.Jacobson, Lie algebras, Interscience, Willey 1962.

19. J.Knopfmacher, On the isomorphism problem for Lie algebras, Proc. Amer. Math. Soc. Vol. 18 No. 5 October 1967.

20. A.G. Kurosh, General algebra, Chelsea Publishing Comp. 1962.

21. O.N.Macedonskaja, On neutral polyverbal operations, Math. Sb.(NS) 69(111)(1966) 286-299.

22. S.Moran, Unrestricted verbal products, J. London Math. Soc. 36(1961) 1-23.

23. S.Moran, Unrestricted nipotent products, Acta. Math. 108(1962) 61-88.

24. P.Serre, Lie algebras and Lie groups, Lecture Notes, Harvard 1964.

25. E.Specker, Additive Gruppen von Folgen ganzer Zahlen, Portugaliae Math., 9(1950) 131-140.

26. P.W.Stroud, On a property of verbal and marginal subgroups, Proc. Camb. Phil. Soc. 61(1965) 41-48.

27. E.Witt, Uber freie Ringe und ihre Unterringe, Maths. Zeitschr. Ed. 58 S113-114 (1953).

28. L.Fuchs, Abelian groups Pergamon Press 1964.