Models of Imperfect Repair

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Abstract Repair is a type of maintenance carried out on an item after it fails. A failure may occur any time, hence the times to repair cannot be pre-specified. Methods used to model times to failures are normally stochastic processes such as the renewal process and the homogeneous Poisson process, depending on the effectiveness of a repair. Apparently, the effectiveness of repair will in turn affect the probability of failures. As such, there have been developed many stochastic processes to model the failure processes in the literature. This paper reviews existing failure process models and discusses future development that is needed.

Key words: imperfect repair, renewal process, non-homogeneous process, geometric process

1 Introduction

It is accepted that no technical systems can last for ever without any failures. As such, repair is needed in order to restore a failed item to a working state. For an asset management firm, it is vital to estimate the number of failures of a typical technical system and then to estimate the capital expenditure spending on repair and maintenance. For example, a water company may wish to estimate how many failures of each asset such as a water pumper or a mixer will have in the next five years, so it can plan their budget accordingly.

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To ensure a system to operate and to reduce the probability of failures, three types of maintenance may be adopted: corrective maintenance, preventive maintenance and predictive maintenance. Corrective maintenance is a synonym of the term *repair*; preventive maintenance is carried out at pre-specified time points in order to reduce the probability of failure; and predictive maintenance is condition-based maintenance, with which maintenance is performed once the condition of the maintained system indicates the need for maintenance.

Once a failure occurs, repair upon the failure may end up with the following five situations:

Better-than-perfect repair. In the case that the failed item is replaced with a new item, which is not identical to the failed one and is more reliable than the failed one, we say the repair is *better than perfect*. Due to technological advance, such a situation may happen when a more advanced item is used to replace the failed item.

Perfect repair. If the failed item is replaced with a new identical item, we say the repair is perfect, or a perfect repair. That is, the item used to replace the failed item has the same reliability as the failed one. In the reliability literature, perfect repair is also called as *good-as-new repair*.

Minimal repair. The minimal repair can restore the failed item to the status just as before it failed. In this case, the effectiveness of the repair is minimal as it simply brings the item back to an operating status but it does not improve the reliability of the repaired item. Hence, if the effectiveness of a preventive maintenance is minimal, then the maintenance is not needed as the purpose of a preventive maintenance is to improve the reliability of the maintained item.

Worse-than-minimal repair. If a repair unfortunately brings the maintained item to a worse status than the status just before its failure, then the repair is a worse-than-minimal repair. Such a repair may largely be caused by unskilled repairmen.

Imperfect repair. If the effectiveness of a repair is between that of the perfect repair and that of the minimal repair, the repair is said *imperfect repair*. Imperfect repair may occur more often than the above four scenarios. This is especially true for a complex system that is composed of many components. If a component fails and is then replaced, the reliability of the system is improved. That is, the repair effectiveness is better than that of the minimal repair. However, since the entire system is not replaced, the repair effectiveness is worse than the perfect repair.

Modelling the effectiveness of imperfect repair is an essential requirement in various scenarios, for example, when people plan maintenance strategies, or estimate the residual lifetime for some important systems, like nuclear power plants, aeroplanes, trains. Sometimes, these systems seem to be still in normal working conditions, when they come to the end of their planned life. To extend their functioning life, one must justify some reliability requirements. One way to do so is to take into account the effectiveness of repair actions or corrective maintenance. Repair is carried out after a failure and intends to put the system into a state in which it can perform its function again. Modelling the effect of these repair actions is of great practical interest and is the first step in order to be able to assess maintenance efficiency [12].

In the reliability literature, widely used methods of estimating the number of failures are stochastic processes. There are many models that have been proposed to model the effectiveness of imperfect repair, for example, the Brown-Proschan models [6], the virtual age models [18], and the geometric process models [39]. It is noted that models for preventive maintenance and corrective maintenance are essentially different in the sense that preventive maintenance is pre-scheduled and hence the methods to model the effectiveness of a series of preventive maintenance on a maintained item are deterministic models; corrective maintenance cannot be pre-scheduled and hence the methods to model the effectiveness of a series of corrective maintenance on a maintained item are stochastic processes [12]. Nevertheless, the ways to depict the effectiveness of a maintenance action, no matter whether it is preventive or corrective maintenance, are similar. For example, age-reduction models are used in both preventive maintenance modelling [38] and corrective maintenance modelling [12].

In this article, the term *item* and the term *system* are exchangeable.

There has been a lot of research on modelling the failure process of a repairable system, which mainly concentrates on modelling the repair effect of a repairable system through considering: (1) the working time probability functions after repairs (for example, the geometric process [19]); (2) the effective age of the maintained item (for example, the virtual age models [18]); (3) the failure intensity of the maintained item (for example, the intensity modification model [12]), and (4) the virtual component methods (for example, [36]). Those models can be categorised as the following.

Basic models. This category includes the renewal process (RP) and the nonhomogeneous Poisson process (NHPP). The RP is used in modelling perfect repair and the NHPP is used in modelling minimal repair. They are the bases of many further developments. That is, to a certain degree, many failure process models can be regarded as the extensions of those two models. The extensions of the RP include: the geometric process introduced by Lam [19] and its many versions of extensions [5, 33, 7, 4, 33]. The extensions of the NHPP include, for example, [15] introduce an intensity function that can depict a failure process exhibiting the bath-tub curve pattern; [27, 16] introduce segmented failure intensity functions; [22] introduce the time-transformed renewal process (or the trend renewal process) that have both the ordinary renewal process and the NHPP as special cases. [21] introduce a new model that incorporates both time trends and renewal-type behaviour.

Age reduction models. This class may have an intensity function (precisely, hazard function) $\lambda_0(a_1 t + a_2)$, where a_1, a_2 are estimable parameters, respectively. The virtual age models [18, 35]. It also includes the two virtual age models [18], and the ARA models [12]. Work that extends this subclass also includes the model discussed in [11].

Intensity modification models. In this class, its intensity function $b_1\lambda_0(t)+b_2$. This class mainly modifies the intensity function after repair. It includes the arithmetic reduction of intensity (ARI) models.

Hybrid intensity models. In this class, an intensity function is obtained by combining different intensity functions [6] or the same intensity with different arguments [40, 24]. There is a widely studied type of models, i.e., the $(p,1-p)$ type, is originated from [6] who assume that at the time of each failure a perfect maintenance/repair occurs with probability p and a minimal repair occurs with probability $1-p$, independently of the previous history of repair and maintenance. [3] generalise the Brown-Proschan model by allowing the probability of a perfect repair to depend on the age of the failed item: assuming that at the time of each failure a perfect maintenance/repair occurs with probability $p(t)$ and a minimal repair occurs with probability $1-p(t)$. Other extensions of the Brown-Proschan model have been made, see [40, 24] for examples.

Virtual component models. [36] proposed two models to model the failure process of a repairable series system composed of multiple components. Both models assume a real-world system can be analogised to virtual systems composed of multiple virtual components. Correspondingly, the failure intensity of each model is a mixture of two different failure intensities, which does not follow the $(p,1-p)$ rule.

Other types. There are other types of failure process models that may not be categorised into the above classes, for example, the superimposed renewal process [17], the branching Poisson process [1], the Markovian models [2], etc.

2 Existing models of imperfect repair models

Table 1 Notations

Symbol	Description
T_k	The time of kth failure of a system.
$N(t)$	The number of failures of the system up to time t.
X_k	The time between $(k-1)$th and kth failures.
$\lambda(t)$	The failure intensity function.
$\lambda_I(t)$	The initial failure intensity function before the first failure.
$F(.)$	Cumulative Distribution Function of a random variable.
$f(.)$	Probability Distribution Function of a random variable.
ρ	The effectiveness of repair on failure intensity of a system in ARI/ARA models.
S_k	The effectiveness of the kth repair on failure intensity of a system in GRI/GRA models.

In this section, we borrow the definitions of the symbols from [36].

Denote the successive failure times of a repairable system by $\{T_k\}_{k\geq 1}$, from $T_0 = 0$. Denote the times between failures by $\{X_k\}_{k\geq 1}$ and $\{X_k = T_k - T_{k-1}\}$. Assume a repair task is performed after each failure and the repair times are negligible. Let $N(t)$ denote the number of failures of the system up to time t. The failure process of the system can be defined equivalently by the random processes $\{X_k\}_{k\geq 1}$ or $\{N(t)\}_{t\geq 0}$ and is characterised by the intensity function,

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P\{N(t+\Delta t) - N(t) \geq 1 | \mathscr{H}(t)\}}{\Delta t}, \tag{1}$$

where $P\{N(t+\Delta t) - N(t) \geq 1 | \mathscr{H}(t)\}$ is the probability that the system fails within the interval $(t, t+\Delta t)$, given the history of failures up to time t, $\mathscr{H}(t)$ [9].

Another basic assumption is that the initial intensity, i.e. the failure intensity before the first failure, is a deterministic and continuous function of time, $\lambda_I(t)$, and the system is wear-out continuously, i.e. the initial intensity is strictly increasing.

2.1 Geometric process and its extensions

The GP and discusses its limitations in detail. We begin with an important definition on stochastic order.

Assume that X and Y are two random variables. If for every real number r, the inequality $P(X \geq r) \geq P(Y \geq r)$ holds, then X is stochastically greater than or equal to Y, or $X \geq_{st} Y$. Equivalently, Y is stochastically less than or equal to X, or $Y \leq_{st} X$ (p. 404 in [25]).

Given a sequence of non-negative random variables $\{X_k, k = 1, 2, \dots\}$, if they are independent and the cdf of X_k is given by $F(a^{k-1}t)$ for $k = 1, 2, \dots$, where a is a positive constant, then $\{X_k, k = 1, 2, \cdots\}$ is called a geometric process (GP) [19].

The above definition is given by Lam [19], although it is likely that this definition was around earlier. For example, in [26], it reads "*we consider the situation in which failing components are replaced by new ones with better statistical properties. Specifically, it is assumed that the nth replacement has a lifetime distribution $F(a^k t)$*" and also gives the GP-version renewal function. Nevertheless, most publications typically credit the geometric process to Lam [19].

$\{X_k, k = 1, 2, \cdots\}$ in the GP may be stochastically increasing (decreasing) if $a < 1$ ($a > 1$). If $a = 1$, then $\{X_k, k = 1, 2, \cdots\}$ reduces to a renewal process. That is, when $a \neq 1$, the GP offers an alternative that can model the effectiveness of imperfect maintenance.

Some authors either proposed similar definitions to that of the GP [14, 32] or made an attempt to extend the GP [5, 34, 20]. Those different versions can be unified: they replace a^{k-1} with $g(k)$, where $g(k)$ is a function of k and is defined differently by different authors, as discussed below.

For a sequence of non-negative random variables $\{X_k, k = 1, 2, \dots\}$, different consideration has been laid on the distribution of X_k, as illustrated in the following (in chronological order).

(a). [14] proposes a process, named the *general deteriorating renewal process*, in which the distribution of X_k is $F_k(t)$, where $F_{k+1}(t) \leq F_k(t)$. A more specific model is defined such that $F_k(t) = F(a_k t)$ where $1 = a_1 \leq a_2 \leq a_3 \leq \dots$ and a_k are parameters. In this model, $g(k) = a_k$.

(b). [32] defines a quasi-renewal process, which assumes $X_1 = W_1, X_2 = aW_2, X_3 = a^2 W_3, \dots$, and the W_k are independently and identically distributed and $a > 0$ is constant. Here, $g(k) = a^{1-k}$.

(c). [5] proposes a variant, which assumes that the distribution of X_k is $F_k(t) = F(k^{-a} t)$, or $g(k) = k^{-a}$. The authors argued that the expected number of event counts before a given time, or analogously, the Mean Cumulative Function (MCF) (or, the renewal function), does not exist for the decreasing GP. As such, they propose the process as a complement.

(d). [34] set $g(k) = \alpha a^{k-1} + \beta b^{k-1}$, where α, β, a and b are parameters. Their intention is to extend the GP to model more complicated failure patterns such as the bathtub shaped failure patterns.

(e). [7] extends the GP to the threshold GP: A stochastic process $\{Z_n, n = 1, 2, ...\}$ is said to be a threshold geometric process (threshold GP), if there exists real numbers $a_i > 0, i = 1, 2, ..., k$ and integers $\{1 = M_1 < M_2 < ... < M_k < M_{k+1} = \infty\}$ such that for each $i = 1, ..., k$, $\{a_i^{n-M_i} Z_n, M_i \leq n < M_{i+1}\}$ forms a renewal process.

(f). [4] set $g(k) = a^{b_k}$ (where a and b_k are parameters) and discuss statistical properties of the process. The purpose of their extension is to overcome the limitation that the GP only allows for logarithmic or explosive growth.

(g). [37] extend the GP by relaxing the assumption that $\{X_k, k = 1, 2, \dots\}$ are independent. They introduce a definition in which a sequence of non-negative random variables $\{X_k, k = 1, 2, \dots\}$ in which $\{X_k, k = 1, 2, \dots\}$ are dependent and the cdf of X_k is given by $F(a^{k-1} t)$ for $k = 1, 2, \dots$.

(h). [33] proposes a definition, called *doubly geometric process*, in which a sequence of non-negative random variables $\{X_k, k = 1, 2, \dots\}$ in which $\{X_k, k = 1, 2, \dots\}$ are independent and the cdf of X_k is given by $F(a^{k-1} x^{h(k)})$ for $k = 1, 2, \dots$, where $h(k)$ is a function of k and the likelihood of the parameters in $h(k)$ has a known closed form.

2.2 Reduction of intensity models

The reduction of intensity models are used when the effect of repair is considered to reduce the failure intensity. The reduction methods can be categorized into different groups such as arithmetic reduction of intensity (ARI) [12], geometric reduction [13], etc.

The basic idea of ARI considers that each repair activity can reduce the failure intensity of an amount depending on the past of the failure process. In literature, the ARI models are constructed with two assumptions [12]:

1. Each maintenance action reduces the failure intensity by sub-tracking an amount possibly depending on the past of the failure process;
2. after failure, the wear-out speed is the same as before failure.

By considering different effects of the past failure process on current failure intensity, the ARI models can be classified into ARI_∞, ARI_1, and ARI_m models. The ARI_∞ means the arithmetic reduction of intensity with infinite memory, which is built with the assumption that repair reduces the failure rate of an amount proportional to the current failure rate. With consideration of Assumption 1, the ARI_∞ failure intensity is

$$\lambda(t) = \lambda_I(t) - \rho \sum_{j=0}^{N_t-1} (1-\rho)^j \lambda_I(T_{N_t-j}). \tag{2}$$

The ARI_1 means the repair activity can only reduce the relative wear since the last repair. This model is called the arithmetic reduction of intensity with memory one. With consideration of Assumption 1, the ARI_1 failure intensity is

$$\lambda(t) = \lambda_I(t) - \rho\lambda_I(T_{N_t}). \tag{3}$$

The ARI_m is called the arithmetic reduction of intensity model with memory m, it means there are m previous failures are involved in the current failure rate. With consideration of A1, the ARI_m failure intensity is

$$\lambda(t) = \lambda_I(t) - \rho \sum_{j=0}^{\mathbf{Min}(m-1,N_t-1)} (1-\rho)^j \lambda_I(T_{N_t-j}). \tag{4}$$

In the above models, the intensity is reduced arithmetically, they may not cop with some scenarios very well such as strong slowdown of the wear. Then, the geometric reduction of intensity is introduced by [13], recently. To build a geometric reduction of intensity (GRI) model, the subtractions and sums in ARI can be replaced by divisions and products, respectively. With consideration of Assumption 1, the GRI_∞ failure intensity is

$$\lambda(t) = \lambda_I(t) - \sum_{j=1}^{N_{t-1}} \frac{1-\frac{1}{S_j}}{\prod_{k-j+1}^{N_{t-1}} S_k} \lambda_I(T_j). \tag{5}$$

2.3 Reduction of age models

This class of models principally consider that repair can restore the system's age as repair can reduce the failure intensity of the system at time t equal to the initial intensity at time A_t, where $A_t < t$. This class of models is also called virtual age models. In this class, the real age of a system is its functioning time t; and the virtual age of a system is defined as a positive function of its real age, possibly depending on past failures: $A_t = A(t; N_t, T_1, ..., T_{N_t})$. The failure intensity is a function of its virtual age: $\lambda_t = \lambda(A)$. This idea that repair activities can reduce the virtual age of the system is mainly based on Kijima's virtual age models [30, 18], which are on the basis of Generalized Renewal Process (GRP). In Kijima's first model, the nth repair is assumed can remove the wear incurred only during the time between $(n-1)$th and nth repairs, then the virtual age is $A_n = A_{n-1} + \rho X_n$, where A_n is the virtual age after the nth repair, X_n is the time between the nth and the $(n-1)$th repairs, and ρ is the effectiveness of repair. In Kijima's second model, the nth repair is assumed can reduce all wear accumulated up to the nth repair, then the virtual age is $A_n = \rho(A_{n-1} + X_n)$. In Kijima's models, when $\rho = 0$, the repair is perfect, when $\rho = 1$, the repair is minimal.

According to [12], the reduction of age can also be arithmetic or geometric. The arithmetic reduction of age (ARA) models can be classified into, by analogy with the ARI models, arithmetic reduction of age model with infinite memory (ARA_∞), arithmetic reduction of age model with memory one (ARA_1), and arithmetic reduction of age model with memory m (ARA_m). The ARA_1 model is similar to Kijima's first model, and the ARA_∞ model is similar to Kijima's second model.

The ARA_∞ model assumes the nth repair can reduce the virtual age of the system by an proportional amount of its age before the nth repair. Then, the failure intensity of ARA_∞ model is

$$\lambda(t) = \lambda_I \left(t - \rho \sum_{j=0}^{N_t - 1} (1-\rho)^j T_{N_t - j} \right). \tag{6}$$

The ARA_1 model assumes the nth repair can reduce the virtual age of the system by an proportional amount of its age between the nth and the $(n-1)$th repair. Then, the failure intensity of ARA_1 model is

$$\lambda(t) = \lambda_I (t - \rho T_{N_t}). \tag{7}$$

The ARA_m model assumes the nth repair can reduce the virtual age of the system by an proportional amount of its age between nth and $(n-m)$ repairs. Then, the failure intensity of ARA_m model is

$$\lambda(t) = \lambda_I \left(t - \rho \sum_{j=0}^{\mathbf{Min}(m-1, N_t - 1)} (1-\rho)^j T_{N_t - j} \right). \tag{8}$$

Similar to the relationship between the ARI and GRI models, the ARA models can also be extended to geometric reduction of age (GRA) models [13]. The GRA_∞ failure intensity is [13]

$$\lambda(t) = \lambda_I \left(t - T_{n-1} + \frac{\sum_{j=1}^{n-1} \left[\prod_{k=1}^{j-1} S_k \right] X_j}{\prod_{j=1}^{n-1} S_j} \right). \tag{9}$$

After introduced by [12], the ARI and ARA models have been widely used and provided a good fit for many real maintenance data sets. [28] combine imperfect repair models and proportional intensity models to build imperfect repair proportional intensity models to cop with the field data consisting of times to failure and covariate data.

[23] apply the ARA_∞ model on the failure dataset from a fleet of six load-haul-dump machines in a Swedish mine, as the model can help the researchers to quantify the effect of repair on each machine and to take into account the effect of the early missing data. The parameters are estimated through maximum likelihood method in this research.

[29] model the imperfect repair by ARA models with incorporating the effect of imperfect corrective and preventive maintenance. In this research, four virtual age processes are introduce to describe the different repair patterns and restoration degrees for corrective and preventive maintenance. The parameters are estimated through maximum likelihood estimation.

[10] introduce a new imperfect maintenance model based on the ARI model. The arithmetic reduction of intensity is assumed on the interarrival times of failures on a system subject to recurrent failures instead of on the failure intensity.

The parameters of ARI and ARA models can be estimated through maximum likelihood estimates [31]. [8] propose a Bayesian analysis of the ARA models and discuss the choice of prior distributions and the computation of posterior distributions. In this research, a single reliable repairable system which only has very few failures is considered. For this system, the quality of the maximum likelihood estimates is very poor because the number of observations is not enough. Then, the Bayesian analysis is employed to improve the accuracy of parameter estimations, as it can add the expert knowledge to operation feedback data. The expert knowledge on the system aging and repair efficiency can be reflected by the prior distributions.

2.4 Virtual component models

In the literature, widely used failure process models such as the generalized renewal process (GRP), geometric process (GP) and non-homogeneous Poisson process (NHPP) cannot distinguish the effect of repair upon failure of difference components in a complex system, as they consider the system as a one-component system [36]. To model the failure process of a multi-component system as a whole when the lifetime distribution of each component is unknown, [36] introduce the concept of a virtual component. The idea of [36] is: for a series system composed of multiple components, if the system fails, the failed component is replaced with an identical component and the replacement time is negligible. Assuming the times to failures of the system are known but upon each failure, which component causes the system to fail is unknown. With such data, it is not possible to estimate the failure process model for each individual component. [36] assumes that the failure process of the real system is equivalent to that of a virtual system composed of virtual components.

Whenever the real system fails, the virtual system is assumed to fail simultaneously and the failures are caused by the virtual components in turn. For example, assume a real series system composed of three components A, B, and C. We assume that the system is equivalent to a virtual series system composed of virtual components a,b and c. If we know the times of the first n failures, $T_1,...,T_{10}$, say. Then the failure process of the virtual system is assumed to be caused by virtual components a,b,c,a,b,c,a,b,c, and a, respectively. Based on such assumptions, [36] introduce two models and compares the performance of the models with several existing models on artistically simulated data. The results show that the proposed models have smaller AIC (Akaike Information Criterion).

3 Conclusions and future development

This paper reviewed some existing methods of modelling imperfect repair. The Geometric process and its extensions can be adapted to model the effectiveness of imperfect maintenance in various scenarios, as the $g(k)$ can be defined differently. However, the complexity of calculation (parameter estimation) should be considered in practice. The Reduction of Intensity and Reduction of Age models are constructed in a more intuitional way makes them more handy than the GP, but the strict assumptions should be minded and the interpretability of them in some complex scenarios also should be considered. Regarding the Virtual Component models, they provide a new gateway to model and interpret the reliability of multi-component systems, they can be developed with considering various interplays among the components.

There is much work needing further development in the future. The focus may be on the development of models for systems with different repair modes including: (1) develop a method to model the failure process of a given complex system composed of many repairable components, while the repair effectiveness of each component is assumed unknown; (2) use modern machine learning techniques to model the failure process.

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