Consensus Control for a Class of Linear Multiagent Systems using a Distributed Integral Sliding Mode Strategy

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Abstract

In this paper, a consensus framework is proposed for a class of linear multiagent systems subject to matched and unmatched uncertainties in an undirected topology. A linear coordinate transformation is derived so that the consensus protocol design can be conveniently performed. The distributed consensus protocol is developed by using an integral sliding mode strategy. Consensus is achieved asymptotically and all subsystem states are bounded. By using an integral sliding mode control, the subsystems lie on the sliding surface from the initial time, which avoids any sensitivity to uncertainties during the reaching phase. By use of an appropriate projection matrix, the size of the equivalent control required to maintain sliding is reduced which reduces the conservatism of the design. MATLAB simulations validate the effectiveness and superiority of the proposed method.

Key Words: Multiagent system, Linear system, Consensus control, Integral sliding mode control, Regular form

I. Introduction

Cooperative control of multiagent systems has received considerable attention in recent years due to its relevance in fields including microgrids, spacecraft formation and industrial cooperative robotics [1]. The behaviour is characterised by cooperation between subsystems via a communication network whereby each subsystem shares information with its neighbours to ensure that all agents reach an agreed goal. Consensus control is a typical and fundamental collective behavior of cooperative control. In a distributed system, consensus control generally focuses on how the agents come to agreement on certain quantities by using their own information together with information received from their neighbours [2]. Consensus control can be widely applied in practice. For instance, in order to increase production, multiple reactors are used to simultaneously perform a chemical reaction where controllers communicate with each other and maintain the temperature, pressure and flow across the reactors in order to maintain consistency of the product.

In process control, uncertainties or modeling errors can seriously affect the behaviour of subsystems. Within a multiagent network this behaviour can spread across the systems because of the interactions between the agents. The performance can decrease in terms of control accuracy if such uncertainties are not considered[3][4][5]. Robust control is an effective approach to cope with such uncertainty. H_∞ control is a typical robust control strategy which has been widely applied in consensus theory [6][7]. The adaptive control paradigm is also commonly used to deal with uncertainties in multiagent systems [8][9]. However, in much of this research, a high control gain is required to suppress uncertainties which may be undesirable in practice. In some cases, a disturbance observer can be systematically designed to observe and then compensate for disturbances and uncertainties [10][11][12]. However, well parameterised models are typically required to define the disturbance observer. Sliding mode control possesses useful characteristics such as total invariance to matched uncertainties, straightforward implementation and fast global convergence [13][14]. There are several contributions which consider distributed control using sliding mode approaches. Consensus is achieved using a decoupled distributed sliding mode control for second-order multiagent systems in [2]. Leader-following containment control is investigated for linear systems in [15]. Scaled consensus is studied for linear systems by means of an H_∞ sliding mode control in [16]. It should be noted that during the reaching phase in classical sliding mode control, the system behaviour is still affected by matched uncertainties [17][18]. Integral sliding mode control serves as a solution to this problem as it eliminates the reaching phase. Finite-time consensus is achieved for second-order multiagent systems with disturbances using an integral sliding mode approach in [19]. Fixed-time consensus tracking is studied for second-order nonlinear systems in [20]. The consensus protocols in [19] and [20] are not applicable for more general classes of linear system. A nearly optimal integral sliding-mode consensus protocol is designed for multiagent systems in the presence of matched disturbances in [21]. Note that the unmatched uncertainties have not been considered in this work. Consequently, it is valuable to develop a method to cope with matched and unmatched uncertainties for linear multiagent systems.

Much of the existing research in distributed control considers consensus for multiagent systems, but does not consider the stability of the subsystems. For example, in [2][19] and [22], second order systems are usually considered as position-velocity systems, in which position increases over time, i.e., the subsystems are unstable after achieving consensus. Theoretically, this is due to the existence of multiple zero eigenvalues and their linearly dependent eigenvectors in the system matrix. However, in physics, the second order system can also act as a mathematical model of a sensor system [23] or a motor system [24]. In these application scenarios, divergence of the states to infinity over time is undesirable. For other known research, though the states reach the equilibrium point ultimately, there is no direct proof of the stability of the subsystems. In [1][8][9] and [25], a robust adaptive strategy is utilized to achieve consensus, but it is difficult to synthesize this method to prove stability of the subsystems. As a consequence, it is challenging to develop a consensus protocol which will stabilize the subsystems and where proof of stability can be achieved constructively.

Motivated by the above discussion, in this paper a consensus framework is proposed for linear multiagent systems which are subject to uncertainties by using an integral sliding mode strategy. Firstly, the distributed linear system is transformed into a novel regular form by a linear coordinate transformation, which facilitates designing the distributed consensus protocol. In comparison with the traditional regular form [26], the novel regular form inherits the property that matched and unmatched uncertainties can be separated. Further, the transformed representation facilitates analysis of the consensus error. Secondly, despite the presence of model uncertainties, an integral sliding mode strategy is employed so that the states start on the sliding surface from the initial time, which achieves better robustness characteristics than classical sliding mode control [27][28]. Meanwhile, by using the integral sliding mode control, the matched uncertainties are eliminated, and the effect of unmatched uncertainties is minimized by the projection matrix. Even if the states deviate from the sliding surface, the integral sliding mode control can drive them back to it in finite time. In addition, when the system states move along the surface, a nominal dynamics is exhibited. In consequence only a nominal protocol needs to be designed to guarantee consensus. In this way, couplings are in the nominal control protocol, while not in the integral sliding control protocol, which simplifies control protocol design in contrast to other robust methods [29][30]. Thirdly, in light of the novel regular form and integral sliding mode strategy, a consensus control protocol is proposed for a distributed linear system, and it renders all the subsystem states bounded. Considering subsystem stability is desirable, but it is usually ignored or not directly proved in other papers [31][32][33][34][35]. Moreover, the proposed protocol is fully distributed without requiring global information when compared to [7][36] and [37].

In this paper, the main contributions are twofold. On the one hand, an integral sliding mode based consensus protocol is proposed so that matched uncertainties are eliminated while the effect of unmatched uncertainties is minimized. On the other hand, in light of the consensus control framework, consensus for the multiagent system can be achieved asymptotically, while the subsystem states are rendered bounded.

The rest of this paper is organized as follows. In Section II, some basic concepts are stated, a linear coordinate transformation is given and the problem to be solved is formulated. In Section III, the integral sliding mode control is designed and sliding motion stability is analyzed. In Section IV, consensus and subsystems' stability are analyzed. In Section V, simulation results are analysed and finally in Section VI, conclusions are drawn.

II. Preliminaries and problem formulation

Graph theory is used to illustrate the communication among subsystems [38]. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ denote an undirected graph consisting of N nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, a set of undirected edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = (a_{ij})_{N\times N}$. An undirected edge $\mathcal{E}_{ij}$ in the undirected graph $\mathcal{G}$ is denoted by a pair of unordered nodes (v_i, v_j), which indicates v_i and v_j are neighbours and can communicate with each other. The set of neighbours for node v_i is denoted by $\mathcal{N}_{v_i} = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}, i \neq j\}$. The weights $a_{ij} = a_{ji}$= 1 in the weighted adjacency matrix $\mathcal{A}$ if and only if the edge (v_i, v_j) exists, and $a_{ij} = a_{ji}$=0 otherwise. Define $a_{ij} = 0$ when $i = j$. The Laplacian matrix $\mathcal{L} = (\mathcal{L}_{ij})_{N\times N}$ is defined by $\mathcal{L}_{ij} = -a_{ij}$ for $i \neq j$, and $\mathcal{L}_{ii} = \sum\limits_{j=1,j\neq i}^{N} a_{ij}$. A path is a sequence of connected edges in a graph, and a graph is connected if there is a path between every pair of vertices.

$0_{n\times m}$ denotes an $n-$row and $m-$column matrix with all the entries being 0. 0_n denotes an $n-$row vector with all the entries being 0. I_m denotes an $m \times m$ identity matrix. Let $\|M\|_F = \sqrt{\sum\limits_{i=1}^{p}\sum\limits_{j=1}^{q} |m_{ij}|^2}$ be the Frobenius norm of $M = (m_{ij})_{p\times q}$. $\|\varpi\|_\infty$= $\max\limits_{1\leq i\leq n} |\varpi_i|$ denotes an infinite norm of $\varpi \in R^n$. $\|\centerdot\|$ denotes the Euclidean norm in this paper unless additionally stated. $\lambda_i(P)$ denotes an eigenvalue of $P \in R^{n\times n}$, where $i = 1, 2, \ldots, n$, $\lambda_{\max}(P)$ denotes the maximum eigenvalue of P. $h(o)$ is a function with o being the argument, and then $o^* = \arg\min\limits_{o} h(o)$ represents the argument o^* which minimizes $h(o)$.

Consider a distributed multiagent system with N subsystems where the communication among subsystems is denoted by an undirected topology graph $\mathcal{G}$. Each subsystem has the following identical nominal linear dynamics which is subject to model uncertainties

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + \phi_i(t, x_i), i = 1, 2, \ldots, N \tag{1}$$

where $x_i(t) \in R^n$, $u_i(t) \in R^m$, $A \in R^{n\times n}$, $B \in R^{n\times m}$ are the state, control protocol, system matrix and input matrix of the ith subsystem respectively. The uncertainties are lumped together and denoted as $\phi_i(t, x_i) \in R^n$.

The following assumptions will be imposed on system (1).

Assumption 1: The pair (A, B) is controllable.

Assumption 2: B has full column rank, i.e., $rank(B) = m$.

Assumption 3: The continuous uncertainty $\phi_i(t, x_i) \in R^n$ is unknown but bounded, i.e., $\|\phi_i(t, x_i)\| \leq \beta$, where $\beta \in R$ is a known constant.

Assumption 4: The undirected graph $\mathcal{G}$ is connected.

Under Assumption 2, it follows from Lemma 5.3 in [26] that there exists a linear coordinate transformation $\tilde{z}_i \triangleq \begin{bmatrix}\tilde{z}_{i1}^T & \tilde{z}_{i2}^T\end{bmatrix}^T = T_1 x_i$ such that (1) can be described as

$$\begin{aligned}\dot{\tilde{z}}_{i1}(t) &= \tilde{A}_{11}\tilde{z}_{i1}(t) + \tilde{A}_{12}\tilde{z}_{i2}(t) + \tilde{\phi}_{i1}(t, \tilde{z}_i)\\ \dot{\tilde{z}}_{i2}(t) &= \tilde{A}_{21}\tilde{z}_{i1}(t) + \tilde{A}_{22}\tilde{z}_{i2}(t) + B_2 u_i(t) + \tilde{\phi}_{i2}(t, \tilde{z}_i)\end{aligned} \tag{2}$$

where T_1 is an invertible matrix, $\tilde{z}_{i1}(t) \in R^{n-m}$, $\tilde{z}_{i2}(t) \in R^m$, $\tilde{A}_{11} \in R^{(n-m)\times(n-m)}$, $\tilde{A}_{22} \in R^{m\times m}$, $rank(B_2) = m$, $\tilde{\phi}_{i1}(t, \tilde{z}_i) \in R^{n-m}$ and $\tilde{\phi}_{i2}(t, \tilde{z}_i) \in R^m$ are unmatched and matched uncertainties respectively.

Perform a coordinate transformation $z_i \triangleq \begin{bmatrix}z_{i1}^T & z_{i2}^T\end{bmatrix}^T = T_2\begin{bmatrix}\tilde{z}_{i1}^T & \tilde{z}_{i2}^T\end{bmatrix}^T$, where $T_2 = \begin{bmatrix}K_1 & 0_{(n-m)\times m}\\ K_2 & I_m\end{bmatrix}$, such that in the new coordinates $\begin{bmatrix}z_{i1}^T & z_{i2}^T\end{bmatrix}^T$, system (1) can be described by

$$\begin{aligned}\dot{z}_{i1}(t) &= A_{11}z_{i1}(t) + A_{12}z_{i2}(t) + \phi_{i1}(t, z_i)\\ \dot{z}_{i2}(t) &= A_{21}z_{i1}(t) + A_{22}z_{i2}(t) + B_2 u_i(t) + \phi_{i2}(t, z_i)\end{aligned} \tag{3}$$

where T_2 is an invertible matrix, $z_{i1}(t) \in R^{n-m}$, $z_{i2}(t) \in R^m$, $A_{11} = K_1\left(\tilde{A}_{11} - \tilde{A}_{12}K_2\right)K_1^{-1} \in R^{(n-m)\times(n-m)}$ is a real negative symmetric definite, $A_{22} \in R^{m\times m}$, $\phi_{i1}(t, z_i) = K_1\tilde{\phi}_{i1}(t, \tilde{z}_i)$, $\phi_{i2}(t, z_i) = K_2\tilde{\phi}_{i1}(t, \tilde{z}_i) + \tilde{\phi}_{i2}(t, \tilde{z}_i)$ are unmatched and matched uncertainties respectively.

The steps required to render A_{11} real negative symmetric definite are presented as follows:

(a) Apply pole assignment to $\tilde{A}_{11} - \tilde{A}_{12}K_2$. Under Assumption 1, the pair $\left(\tilde{A}_{11}, \tilde{A}_{12}\right)$ is controllable according to Proposition 3.3 in [26], so there exists $K_2 \in R^{m\times(n-m)}$ such that $\tilde{A}_{11} - \tilde{A}_{12}K_2$ has $n-m$ real distinct negative eigenvalues $\lambda_i\left(\tilde{A}_{11} - \tilde{A}_{12}K_2\right)$, $i = 1, \ldots, n-m$. In this case, $\tilde{A}_{11} - \tilde{A}_{12}K_2$ is Hurwitz stable and $rank\left(\tilde{A}_{11} - \tilde{A}_{12}K_2\right) = n-m$.

(b) Since $\tilde{A}_{11} - \tilde{A}_{12}K_2$ has $n-m$ real distinct negative eigenvalues, it follows from Theorem 1.3.9 in [39] that $\tilde{A}_{11} - \tilde{A}_{12}K_2$ can be transformed into the corresponding real diagonal matrix $\Lambda \triangleq diag(\lambda_1, \ldots, \lambda_{n-m})$ by using the nonsingular matrix

$K_1 \in R^{(n-m)\times(n-m)}$.

(c) Let $A_{11} \triangleq \left(a_{ij}^{11}\right)_{(n-m)\times(n-m)}$ and $A_{12} \triangleq \left(a_{ij}^{12}\right)_{(n-m)\times m}$. Select K_1 such that $a_{ii}^{11} \le -\sum\limits_{j=1}^{m}\left|a_{ij}^{12}\right|$, then $A_{11} = K_1\left(\tilde{A}_{11} - \tilde{A}_{12}K_2\right)K_1^{-1}$, and $A_{11} = \Lambda$.

Remark 1: The invertible matrix T_2 renders A_{11} to be a real negative symmetric definite in (3), which is helpful for consensus protocol design and synthesis. This will play an important role in achieving consensus and ensuring the subsystems' stability.

Remark 2: The condition that $a_{ii}^{11} \le -\sum\limits_{j=1}^{m}\left|a_{ij}^{12}\right|$ is given to make the subsystems stable. In step (c), K_1 is straightforward to determine. First, consider that $a_{ii}^{11} = \lambda_i\left(A_{11}\right) = \lambda_i\left(\tilde{A}_{11} - \tilde{A}_{12}K_2\right)$, a_{ii}^{11} is only determined by K_2. Second, because $A_{12} = K_1\tilde{A}_{12}$, in order to satisfy $a_{ii}^{11} \le -\sum\limits_{j=1}^{m}\left|a_{ij}^{12}\right|$, the closer to 0 for the elements of K_1, the easier the condition is to satisfy. For $A_{11} = K_1\left(\tilde{A}_{11} - \tilde{A}_{12}K_2\right)K_1^{-1}$, as both sides are multiplied by K_1 and K_1^{-1} respectively, the elements of K_1 can be very close to 0.

From [40], $\phi_i\left(t, x_i\right)$, $\phi_{i1}\left(t, z_i\right)$ and $\phi_{i2}\left(t, z_i\right)$ may be expressed in the following form:

$$\left[0_{n-m}^T, \phi_{i2}^T\left(t, z_i\right)\right]^T = T_2T_1BB^{+}\phi_i\left(t, x_i\right) \tag{4}$$

$$\left[\phi_{i1}^T\left(t, z_i\right), 0_m^T\right]^T = T_2T_1B^{\perp}B^{\perp+}\phi_i\left(t, x_i\right) \tag{5}$$

where $B^{+} \triangleq \left(B^TB\right)^{-1}B^T \in R^{m\times n}$ is the left inverse of B, and the columns of $B^{\perp} \in R^{n\times(n-m)}$ span the null space of B^T, i.e., $B^TB^{\perp} = 0_{m\times(n-m)}$. Moreover, the following identity holds

$$BB^{+} + B^{\perp}B^{\perp+} = I_n \tag{6}$$

Definition 1: Consensus is said to be achieved for the distributed multiagent system (1) if for any initial conditions, $\lim\limits_{t\to\infty}\left\|x_i\left(t\right) - x_j\left(t\right)\right\| = 0$, $\forall i, j = 1, 2, \cdots, N$.

This paper concentrates on utilizing local information to develop a control protocol such that consensus can be achieved when each subsystem (3) is affected by bounded uncertainties. In this case, the consensus problem for (1) can also be solved correspondingly.

Before presenting the main results, some lemmas and definitions are given as follows.

Lemma 1: [41] (Global Invariant Set Theorem) Consider the autonomous system $\dot{x} = f\left(x\right)$ with f continuous, and let $V\left(x\right)$ be a scalar function with the continuous first order partial derivatives. Assume that $V\left(x\right) \to \infty$ as $\left\|x\right\| \to \infty$, and $\dot{V}\left(x\right) \le 0$ over the whole state space. Let $\mathcal{R}$ be the set of all points where $\dot{V}\left(x\right) = 0$, and $\mathcal{M}$ be the largest invariant set in $\mathcal{R}$. Then all solutions globally asymptotically converge to $\mathcal{M}$ as $t \to \infty$.

Lemma 2: [42] Consider the non-homogeneous system of differential equation

$$\dot{x} = \mathbf{A}\left(t\right)x + f\left(t, x\right), t \in [\xi, \infty) \tag{7}$$

where $\mathbf{A} : [\xi, \infty) \to \mathbf{B}\left(R^n\right)$, $f : [\xi, \infty) \times R^n \to R^n$ are continuous. Assume that the Cauchy problem

$$\begin{cases} \dot{x} = \mathbf{A}\left(t\right)x + f\left(t, x\right), t \in [\xi, \infty) \\ x\left(t_0\right) = x_0, x_0 \in R^n \end{cases} \tag{8}$$

has a unique solution defined in $[\xi, \infty)$

$$x\left(t, t_0, x_0\right) = C\left(t, t_0\right)x_0 + C\left(t, t_0\right)\int_{t_0}^{t} C\left(t_0, s\right)f\left(s, x\left(s, t_0, x_0\right)\right)ds \tag{9}$$

Suppose that the mapping f satisfies

$$\left\|f\left(t, x\right)\right\| \le L\left(t, \left\|x\right\|\right) \tag{10}$$

$$0 \le L\left(t, u\right) - L\left(t, v\right) \le M\left(t, v\right)\left(u - v\right), u \ge v \ge 0 \tag{11}$$

where M is nonnegative continuous. If the trivial solution $x \equiv 0$ of the corresponding homogeneous system is stable, i.e.,

$\|C(t,t_0)\| \le \mu(t_0)$ for all $t \ge t_0$, and the following conditions

$$\int_{t_0}^{\infty} \|C(t_0,s)\| L(s,\mu(t_0)\|x_0\|)ds \le M_1(t_0,x_0) < \infty \tag{12}$$

$$\int_{t_0}^{\infty} \|C(t_0,s)\| M(s,\|C(s,t_0)x_0\|)ds \le M_2(t_0,x_0) < \infty \tag{13}$$

hold, then there exists an $\tilde{M}(t_0,x_0) > 0$ such that

$$\|x(t,t_0,x_0) - C(t,t_0)x_0\| \le \tilde{M}(t_0,x_0) \tag{14}$$

for all $t \ge t_0$.

Definition 2: [43] Consider the system

$$\dot{x} = f(x,u) \tag{15}$$

Assume that $\dot{x} = f(x,0)$ has a uniformly asymptotically stable equilibrium point at the origin. The system (15) is said to be globally input-to-state stability (ISS) if there exist a $\mathcal{KL}$ function η, a class $\mathcal{K}$ function ϑ such that

$$\|x\| \le \eta(\|x_0\|,t) + \vartheta(\|u\|_\infty), \forall t \ge 0 \tag{16}$$

for any initial state $x_0 \in R^n$ and any bounded input $u \in R^m$.

Definition 3: [43] A continuously differentiable function $V: R^n \to R$ is said to be an ISS global Lyapunov function on R^n for the system (15) if there exist class $\mathcal{K}_\infty$ functions $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\mathcal{X}$ such that:

$$\varepsilon_1(\|x\|) \le V(x(t)) \le \varepsilon_2(\|x\|), \forall x \in R^n, t > 0 \tag{17}$$

$$\frac{\partial V(x)}{\partial x} f(x,u) \le -\varepsilon_3(\|x\|), \forall u \in R^m : \|x\| \ge \mathcal{X}(\|u\|) \tag{18}$$

Lemma 3: [43] (Globally ISS Theorem) Consider the system (15) and let $V: R^n \to R$ be an ISS global Lyapunov function for this system. Then (15) is globally ISS according to Definition 2 with

$$\vartheta = \varepsilon_1^{-1} \cdot \varepsilon_2 \cdot \mathcal{X} \tag{19}$$

Remark 3: According to Definition 2, the response of $\dot{x} = f(x,0)$ with initial state x_0 satisfies

$$\|x\| \le \eta(\|x_0\|,t), \forall t \ge 0 \tag{20}$$

As t increases to infinity, $\eta(\|x_0\|,t) \to 0$, then

$$\|x\| \le \vartheta(\|u\|_\infty) \tag{21}$$

Remark 4: Note that there is no contradiction between $\|x\| \ge \mathcal{X}(\|u\|)$ in (18) and $\|x\| \le \vartheta(\|u\|_\infty)$ in (21). Definition 3 indicates that the derivative of the Lyapunov function V is negative definite whenever the trajectories of x are outside of $H_{\mathcal{X}}$ which represents a hypersphere centred at the origin given by $H_{\mathcal{X}} = \{x \,|\, \|x\| \ge \mathcal{X}(\|u\|)\}$; Remark 3 indicates that the trajectories of x will remain ultimately bounded by the hypersphere H_ϑ which represents another hypersphere centred at the origin given by $H_\vartheta = \{x \,|\, \|x\| \le \vartheta(\|u\|_\infty)\}$.

Lemma 4: [44] If $\mu_1, \mu_2, \ldots, \mu_n \ge 0$ and $0 < p < q$, then

$$\left(\sum_{i=1}^{n} \mu_i^q\right)^{1/q} \le \left(\sum_{i=1}^{n} \mu_i^p\right)^{1/p} \tag{22}$$

III. Integral sliding mode control protocol design and stability analysis of the sliding motion

This section aims to design an integral sliding mode control protocol and analyze the stability of the sliding motion for the multiagent system (3). To simplify notation, some of the function arguments will be omitted.

The sliding function is presented as follows

$$s_i(t) = \alpha_i G\Bigg(\begin{bmatrix} z_{i1}^T(t) & z_{i2}^T(t) \end{bmatrix}^T - \begin{bmatrix} z_{i1}^T(t_0) & z_{i2}^T(t_0) \end{bmatrix}^T \\ - \int_{t_0}^{t} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_{i1}^T(\tau) & z_{i2}^T(\tau) \end{bmatrix}^T + \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} u_i^{con}(\tau)\, d\tau \Bigg) \tag{23}$$

where $G \in R^{m\times n}$ is a projection matrix that will be designed later according to the projection theorem and satisfies $rank\left(G\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\right) = m$, $\alpha_i \in R$ is a small positive parameter which can be chosen by the designer, $z_{i1}(t_0)$ and $z_{i2}(t_0)$ are the initial values and $u_i^{con}(t)$ is a consensus control protocol that is defined by

$$u_i^{con}(t) = B_2^{-1}\left(\sum_{j=1}^{N} a_{ij}\left(z_{j2}(t) - z_{i2}(t)\right) + A_{12}^T \sum_{j=1}^{N} a_{ij}\left(z_{j1}(t) - z_{i1}(t)\right)\right) - A_{21}z_{i1}(t) - A_{22}z_{i2}(t) \tag{24}$$

The corresponding sliding surface is

$$\left\{ \left(z_{11}^T, \cdots, z_{N1}^T, z_{12}^T, \cdots, z_{N2}^T\right)^T \middle| \, s_i(t) = 0_m, \forall i = 1, 2, \cdots, N \right\} \tag{25}$$

where $s_i(t)$ is defined in (23).

The control protocol for the multiagent system (3) is given by

$$u_i(t) = u_i^{dis}(t) + u_i^{con}(t) \tag{26}$$

where $u_i^{dis}(t)$ is a discontinuous control protocol and selected as

$$u_i^{dis}(t) = -\rho \frac{\left(G\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\right)^T s_i(t)}{\left\| \left(G\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\right)^T s_i(t) \right\|} \tag{27}$$

where $\rho > \beta\|B^+\|_F$ is a control gain.

Remark 5: It should be noted that full knowledge of the initial conditions is assumed in selecting the sliding function (23) as part of the integral sliding mode strategy. In more classical sliding mode control, the system is affected by matched uncertainties during the reaching phase. Integral sliding mode control can ensure the system states slide on the sliding surface from the very beginning at the cost of assuming the initial conditions are known. It follows that integral sliding mode control has no reaching phase and has strong robustness to matched uncertainties throughout the evolution of the system state. Note that discontinuous feedback control is used in integral sliding mode control, which can stabilize the states to reach sliding mode in finite time when they leave the sliding mode. Accordingly, the full knowledge of initial conditions is not very restrictive in practice but desirable from the point of view of robustness.

Remark 6: For the integral sliding manifold (23), the system states slide on it from the initial time, and will not escape from it under application of the integral sliding control. It is a suitable choice for distributed control of multiagent systems. A nominal dynamics occurs when the system states move along the integral sliding manifold, while only the nominal protocol (24) is needed to guarantee consensus. In this way, couplings are in the nominal control protocol, while not in the integral sliding control protocol, which simplifies control protocol design.

Next, the behaviour when each subsystem is subjected to uncertainty effects will be analyzed when the system is controlled by the discontinuous control protocol (27). Closing the loop in (3) with (26), the derivative of $s_i(t)$ with respect to time is given by

$$\dot{s}_i(t) = \alpha_i G\left(\begin{bmatrix} \dot{z}_{i1}^T & \dot{z}_{i2}^T \end{bmatrix}^T - \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} z_{i1}^T & z_{i2}^T \end{bmatrix}^T + \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} u_i^{con}\right)\right) \tag{28}$$

$$
\begin{aligned}
&= \alpha_i G\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} z_{i1}^T & z_{i2}^T \end{bmatrix}^T + \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\left(u_i^{dis} + u_i^{con}\right) + \begin{bmatrix} \phi_{i1}^T & \phi_{i2}^T \end{bmatrix}^T \right. \\
&\left. - \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} z_{i1}^T & z_{i2}^T \end{bmatrix}^T + \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} u_i^{con}\right)\right) \\
&= \alpha_i G\left(\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} u_i^{dis} + \begin{bmatrix} \phi_{i1}^T & \phi_{i2}^T \end{bmatrix}^T\right)
\end{aligned}
$$

The equivalent discontinuous control u_{ieq}^{dis} is obtained from this as

$$
u_{ieq}^{dis}(t) = -\left(G\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\right)^{-1} G\begin{bmatrix} \phi_{i1}^T & \phi_{i2}^T \end{bmatrix}^T \tag{29}
$$

By substituting (29) as $u_i^{dis}(t)$ in (3), the sliding dynamics can be obtained as

$$
\begin{aligned}
&\dot{z}_{i1}(t) = A_{11} z_{i1}(t) + A_{12} z_{i2}(t) + \phi_{i1}(t, z_i) \\
&\dot{z}_{i2}(t) = A_{21} z_{i1}(t) + A_{22} z_{i2}(t) + B_2 u_i^{con}(t) - B_2\left(G\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\right)^{-1} G\begin{bmatrix} \phi_{i1}^T & 0_m^T \end{bmatrix}^T
\end{aligned} \tag{30}
$$

As can be seen, the action of the integral sliding mode control strategy has transformed the original uncertainties $\begin{bmatrix} \phi_{i1}^T & \phi_{i2}^T \end{bmatrix}^T$ into the following equivalent uncertainties [17][40]

$$
\phi_{ieq}(t, z_i) \triangleq \begin{bmatrix} \phi_{i1} \\ -B_2\left(G\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\right)^{-1} G\begin{bmatrix} \phi_{i1}^T & 0_m^T \end{bmatrix}^T \end{bmatrix} = \left(I_n - \begin{bmatrix} 0_{(n-m)\times n} \\ B_2\left(G\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\right)^{-1} G \end{bmatrix}\right)\begin{bmatrix} \phi_{i1}^T & 0_m^T \end{bmatrix}^T \tag{31}
$$

Theorem 1: Since $G\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}$ has full rank, $B^+(T_2T_1)^{-1}$ is a matrix which minimizes the norm of $\phi_{ieq}(t, z_i)$, i.e.,

$$
G^* = B^+(T_2T_1)^{-1} = \arg\min_{G\in R^{m\times n}} \left\|\left(I_n - \begin{bmatrix} 0_{(n-m)\times n} \\ B_2\left(G\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\right)^{-1} G \end{bmatrix}\right)\begin{bmatrix} \phi_{i1}^T & 0_m^T \end{bmatrix}^T\right\| \tag{32}
$$

***Proof*:** Notice that

$$
\left\|\left(I_n - \begin{bmatrix} 0_{(n-m)\times n} \\ B_2\left(G\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\right)^{-1} G \end{bmatrix}\right)\begin{bmatrix} \phi_{i1}^T & 0_m^T \end{bmatrix}^T\right\| = \left\|\begin{bmatrix} \phi_{i1}^T & 0_m^T \end{bmatrix}^T - \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\varphi_i\right\| \tag{33}
$$

where $\varphi_i = \left(G\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\right)^{-1} G\begin{bmatrix} \phi_{i1}^T & 0_m^T \end{bmatrix}^T$. Thus (32) can be transformed into

$$
\varphi_i^* = \arg\min_{\varphi_i\in R^m} \left\|\begin{bmatrix} \phi_{i1}^T & 0_m^T \end{bmatrix}^T - \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}\varphi_i\right\| \tag{34}
$$

which has $\varphi_i^* = B^+(T_2T_1)^{-1}\begin{bmatrix} \phi_{i1}^T & 0_m^T \end{bmatrix}^T$ as a solution according to the classical projection theorem in page 51 of [45]. Making $G = B^+(T_2T_1)^{-1}$, it can be obtained that $\varphi_i = B^+(T_2T_1)^{-1}\begin{bmatrix} \phi_{i1}^T & 0_m^T \end{bmatrix}^T = \varphi_i^*$, which implies that (32) is true.

Remark 7: By substituting $\varphi_i^* = B^+(T_2T_1)^{-1}\begin{bmatrix} \phi_{i1}^T & 0_m^T \end{bmatrix}^T$ into (33) and combining (6), it follows that $\left\|\phi_{ieq}^*\right\| = \left\|\begin{bmatrix} \phi_{i1}^T & 0_m^T \end{bmatrix}^T\right\|$, i.e., the norm of the equivalent uncertainties is driven by the unmatched uncertainties and the effects of the uncertainties are minimized by designing the projection matrix G optimally.

Theorem 2: Assume Assumptions 1-3 hold. Then the control from (27) can keep the subsystem (3) on the sliding surface

(25) from the initial time with $G = B^{+}(T_2T_1)^{-1}$.

The proof of Theorem 2 is provided in Appendix A.

It follows that the subsystem (3) will slide on the surface (25) despite the presence of the uncertainties [26]. Even if the states deviate from the sliding surface, the discontinuous control can drive them back to it in finite time. Because the subsystem starts on the sliding surface at the initial time, it will remain on the sliding surface thereafter, i.e., $s = \dot{s} = 0$ when $t \geq 0$.

IV. Consensus and stability analysis of subsystems

In this section, consensus will be analyzed for the distributed system in the presence of the control protocol. The stability of each subsystem is then considered.

When the subsystem is restricted on the sliding surface (25), substitute $G = B^{+}(T_2T_1)^{-1}$ and the consensus control protocol (24) into (30). The sliding dynamics can then be described as

$$\begin{aligned} \dot{z}_{i1}(t) &= A_{11}z_{i1}(t) + A_{12}z_{i2}(t) + \phi_{i1}(t, z_i) \\ \dot{z}_{i2}(t) &= \zeta_i(t, z_i, z_j) \end{aligned} \tag{35}$$

where $\zeta_i(t, z_i, z_j) = \sum\limits_{j=1}^{N} a_{ij}(z_{j2}(t) - z_{i2}(t)) + A_{12}^T \sum\limits_{j=1}^{N} a_{ij}(z_{j1}(t) - z_{i1}(t))$.

The lumped form of the closed-loop system (35) is shown as

$$\begin{aligned} \dot{z}_{\mathrm{I}}(t) &= (I_N \otimes A_{11})\, z_{\mathrm{I}}(t) + (I_N \otimes A_{12})\, z_{\mathrm{II}}(t) + \phi_{\mathrm{I}}(t, z) \\ \dot{z}_{\mathrm{II}}(t) &= -\left(\mathcal{L} \otimes A_{12}^T\right) z_{\mathrm{I}}(t) - (\mathcal{L} \otimes I_m)\, z_{\mathrm{II}}(t) \end{aligned} \tag{36}$$

where $z_{\mathrm{I}}(t) = \left[z_{11}^T, z_{21}^T, \cdots, z_{N1}^T\right]^T \in R^{N(n-m)}$, $z_{\mathrm{II}}(t) = \left[z_{12}^T, z_{22}^T, \cdots, z_{N2}^T\right]^T \in R^{Nm}$, $z(t) = \left[z_{\mathrm{I}}^T, z_{\mathrm{II}}^T\right]^T \in R^{Nn}$, $\phi_{\mathrm{I}}(t, z) = \left[\phi_{11}^T, \phi_{21}^T, \cdots, \phi_{N1}^T\right]^T \in R^{N(n-m)}$.

Let $\bar{A} \triangleq \begin{bmatrix} I_N \otimes A_{11} & I_N \otimes A_{12} \\ -\mathcal{L} \otimes A_{12}^T & -\mathcal{L} \otimes I_m \end{bmatrix}$. In Appendix B, it is proved that the corresponding homogeneous system of (36) is marginally stable, i.e., $\left\| e^{\bar{A}(t-t_0)} \right\| \leq \bar{\mu}(t_0)$. The following assumption will be imposed on the system (35) and (36).

Assumption 5: For the closed loop system (35) and (36), $\phi_{i1}(t, z_i)$ satisfies $\|\phi_{i1}(t, z_i)\| \leq \gamma(t) \|z_i(t)\|$, where $\gamma(t) \leq -\lambda_{\max}(A_{11})$, $\int_{t_0}^{\infty} \left\| e^{\bar{A}(t_0 - s)} \right\| \gamma(s)\, \mu(t_0) \|z(t_0)\|\, ds \leq \iota$, and ι is a known constant.

Remark 8: Assumption 5 is reasonable. First, $\phi_{i1}(t, z_i)$ donotes the uncertainty and it is a function of z_i, so it is reasonable to assume $\|\phi_{i1}(t, z_i)\| \leq \gamma_i(t) \|z_i\|$. Second, recall that in Remark 1, $\lambda_i(A_{11})$ is only determined by K_2, and the elements of K_1 can be very close to 0. In this way, though $\phi_{i1}(t, z_i) = K_1 \tilde{\phi}_{i1}(t, \tilde{z}_i)$, the norm of K_1 can be very small, and the value of $-\lambda_{\max}(A_{11})$ can be large enough to make Assumption 5 hold. Assumption 5 will play its role in the proof of consensus and stability.

Theorem 3: Suppose Assumptions 1-5 hold. The distributed system (35) can achieve consensus asymptotically.

***Proof*:** The consensus problem can be transformed into the following stabilisation problem:

$$\begin{aligned} \dot{e}_i^a(t) &= A_{11}e_i^a(t) + A_{12}e_i^b(t) + e_i^{\phi_1}(t, z_i, z_j) \\ \dot{e}_i^b(t) &= \zeta_i(t, z_i, z_j) - \bar{\zeta}(t, z_i, z_j) \end{aligned} \tag{37}$$

where $e_i^a(t) \triangleq \left(e_{i1}^a, \ldots, e_{i,n-m}^a\right)^T = z_{i1} - \frac{1}{N}\sum\limits_{j=1}^{N} z_{j1}$, $e_i^b(t) \triangleq \left(e_{i1}^b, \ldots, e_{i,m}^b\right)^T = z_{i2} - \frac{1}{N}\sum\limits_{j=1}^{N} z_{j2}$, $e_i^{\phi_1}(t, z_i, z_j) \triangleq \phi_{i1} - \frac{1}{N}\sum\limits_{j=1}^{N} \phi_{j1}$, $\bar{\zeta}(t, z_i, z_j) \triangleq \frac{1}{N}\sum\limits_{j=1}^{N} \zeta_j$.

Based on the errors defined above, $\zeta_i(t, z_i, z_j)$ can be rewritten as

$$\zeta_i(t, z_i, z_j) = \sum_{j=1}^{N} a_{ij}\left(e_j^b - e_i^b\right) + A_{12}^T \sum_{j=1}^{N} a_{ij}\left(e_j^a - e_i^a\right) \tag{38}$$

Because $a_{ij}=a_{ji}$, for $\bar{\zeta}(t,z_i,z_j)$ it can be obtained that

$$\begin{aligned}\bar{\zeta}(t,z_i,z_j) &= \frac{1}{N}\sum_{j=1}^{N}\zeta_j \\ &= \frac{1}{N}\sum_{j=1}^{N}\left(\sum_{k=1}^{N}a_{jk}\left(e_k^b-e_j^b\right)+A_{12}^T\sum_{k=1}^{N}a_{jk}\left(e_k^a-e_j^a\right)\right) \\ &= \frac{1}{2N}\sum_{j=1}^{N}\sum_{k=1}^{N}a_{jk}\left[\left(e_k^b-e_j^b\right)+\left(e_j^b-e_k^b\right)+A_{12}^T\left(e_k^a-e_j^a\right)+A_{12}^T\left(e_j^a-e_k^a\right)\right] \\ &= 0_m\end{aligned} \tag{39}$$

A Lyapunov candidate function is constructed as

$$V_2(t)=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n-m}\int_0^{e_{ik}^a-e_{jk}^a}a_{ij}y\,dy+\frac{1}{2}\sum_{i=1}^{N}\left(e_i^b\right)^Te_i^b \tag{40}$$

The derivative of V_2 along the errors e_i^a and e_i^b is given by

$$\begin{aligned}\dot{V}_2(t) &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n-m}a_{ij}\left(e_{ik}^a-e_{jk}^a\right)\left(\dot{e}_{ik}^a-\dot{e}_{jk}^a\right)+\sum_{i=1}^{N}\left(e_i^b\right)^T\dot{e}_i^b \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n-m}a_{ij}\left(e_{ik}^a-e_{jk}^a\right)\dot{e}_{ik}^a+\sum_{i=1}^{N}\left(e_i^b\right)^T\dot{e}_i^b \\ &= \sum_{i=1}^{N}\left(\dot{e}_i^a\right)^T\sum_{j=1}^{N}a_{ij}\left(e_i^a-e_j^a\right)+\sum_{i=1}^{N}\left(e_i^b\right)^T\dot{e}_i^b\end{aligned} \tag{41}$$

Combined with (35), it can be obtained that

$$\begin{aligned}\dot{V}_2(t) &= \sum_{i=1}^{N}\left(A_{11}e_i^a+A_{12}e_i^b+e_i^{\phi_1}\right)^T\sum_{j=1}^{N}a_{ij}\left(e_i^a-e_j^a\right)+\sum_{i=1}^{N}\left(e_i^b\right)^T\left(\sum_{j=1}^{N}a_{ij}\left(e_j^b-e_i^b\right)+A_{12}^T\sum_{j=1}^{N}a_{ij}\left(e_j^a-e_i^a\right)\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}(e_i^a)^TA_{11}^T\left(e_i^a-e_j^a\right)+\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(e_i^b\right)^T\left(e_j^b-e_i^b\right)+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\left(e_i^{\phi_1}\right)^T-\left(e_j^{\phi_1}\right)^T\right)\left(e_i^a-e_j^a\right) \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(e_i^a-e_j^a\right)^TA_{11}^T\left(e_i^a-e_j^a\right)-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(e_i^b-e_j^b\right)^T\left(e_i^b-e_j^b\right)+\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\phi_{i1}^T\left(e_i^a-e_j^a\right) \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}(z_{i1}-z_{j1})^TA_{11}^T\left(z_{i1}-z_{j1}\right)-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}(z_{i2}-z_{j2})^T\left(z_{i2}-z_{j2}\right)+\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\phi_{i1}^T\left(z_{i1}-z_{j1}\right)\end{aligned} \tag{42}$$

Further, note that A_{11} is negative real definite, so A_{11}^T is negative real definite. Combined with Assumption 5, the following inequalities can be obtained

$$\begin{aligned}\dot{V}_2(t) &\le \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\lambda_{\max}\left(A_{11}^T\right)\|z_{i1}-z_{j1}\|^2-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\|z_{i2}-z_{j2}\|^2+\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\|\phi_{i1}\|\|z_{i1}-z_{j1}\| \\ &\le \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\lambda_{\max}\left(A_{11}^T\right)\|z_{i1}-z_{j1}\|^2-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\|z_{i2}-z_{j2}\|^2+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\gamma\|z_i\|+\gamma\|z_j\|\right)\|z_{i1}-z_{j1}\| \\ &\le \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\lambda_{\max}\left(A_{11}^T\right)+\gamma\right)\left(\|z_i\|+\|z_j\|\right)\|z_{i1}-z_{j1}\|-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\|z_{i2}-z_{j2}\|^2\end{aligned} \tag{43}$$

The analysis of (43) is presented as follows: $\left(\lambda_{\max}\left(A_{11}^T\right)+\gamma\right)\left(\|z_i\|+\|z_j\|\right)\|z_{i1}-z_{j1}\|\le 0$, equality holds if and only if $z_{i1}-z_{j1}=0_{n-m}$ ($z_{i1}=z_{j1}=0_{n-m}$ included); $\|z_{i2}-z_{j2}\|^2\ge 0$, equality holds if and only if $z_{i2}-z_{j2}=0_m$. Hence, $\dot{V}_2\le 0$.

Referring to Lemma 1, it can be obtained that (a) $V_2(t)$ is radially unbounded over e_i^a and e_i^b; (b) Since the undirected graph is connected, if $\dot{V}_2 \equiv 0$, then $z_{i1} \equiv z_{j1}$, $z_{i2} \equiv z_{j2}$, $\forall i, j = 1, 2, \ldots, N$. That is, $\lim_{t\to\infty} \|z_{i1} - z_{j1}\| = 0$ and $\lim_{t\to\infty} \|z_{i2} - z_{j2}\| = 0$, $\forall i, j = 1, 2, \ldots, N$, i.e., $\lim_{t\to\infty} \|x_i - x_j\| = 0$, $\forall i, j = 1, 2, \cdots, N$. Based on the above analysis, system (35) can be driven to consensus asymptotically.

Due to the presence of the unmatched uncertainties $\phi_{i1}(t, z_i)$, the evolution of $z_{i1}(t)$ and $z_{i2}(t)$ should be discussed.

Theorem 4: Suppose Assumptions 1-5 hold. The states of the system (36) are bounded.

***Proof*:** (a) According to Assumption 3, $\phi_{\mathrm{I}}(t, z)$ is continuous. (b) Because $\|\phi_{i1}(t, z_i)\| \le \gamma(t) \|z_i(t)\|$, then $\sum_{i=1}^{N} \|\phi_{i1}(t, z_i)\|^2 \le \gamma^2(t) \sum_{i=1}^{N} \|z_i(t)\|^2$, i.e., $\|\phi_{\mathrm{I}}(t, z)\|^2 \le \gamma^2(t) \|z\|^2$. As a result, $\|\phi_{\mathrm{I}}(t, z)\| \le \gamma(t) \|z\|$, which satisfies (10) and (11), and $L = M = \gamma(t) \|z\|$. (c) Recall that $\int_{t_0}^{\infty} \left\| e^{\bar{A}(t_0 - s)} \right\| \gamma(s) \mu(t_0) \|z(t_0)\| ds \le \iota$, which satisfies (12). Due to $\left\| e^{\bar{A}(t-t_0)} \right\| \le \mu(t_0)$, $\int_{t_0}^{\infty} \left\| e^{\bar{A}(t_0 - s)} \right\| \gamma(s) \left\| e^{\bar{A}(s-t_0)} z(t_0) \right\| ds \le \int_{t_0}^{\infty} \left\| e^{\bar{A}(t_0 - s)} \right\| \gamma(s) \left\| e^{\bar{A}(s-t_0)} \right\| \|z(t_0)\| ds \le \int_{t_0}^{\infty} \left\| e^{\bar{A}(t_0 - s)} \right\| \gamma(s) \mu(t_0) \|z(t_0)\| ds \le \iota$, which satisfies (13). (d) Note that the corresponding homogeneous system of (36) is marginally stable. In the light of Lemma 2, the states of the system (36) are bounded.

The functional relationship between the states and uncertainties is analyzed as follows.

In Assumption 3 $\|\phi_i\| \le \beta$, thus $\|\phi_{i1}\| \le \beta$. Because the states are bounded, z_{i2} is also bounded in (35).

A Lyapunov candidate function is constructed as

$$V_3(t) = \frac{1}{2} z_{i1}^T z_{i1} \tag{44}$$

Let $-1 < \theta < 0$, then the derivative of $V_3(t)$ is given by

$$\begin{aligned} \dot{V}_3(t) &= z_{i1}^T \dot{z}_{i1} \\ &= z_{i1}^T (A_{11} z_{i1} + A_{12} z_{i2} + \phi_{i1}) \\ &= (1+\theta) z_{i1}^T A_{11} z_{i1} + z_{i1}^T A_{12} z_{i2} + z_{i1}^T \phi_{i1} - \theta z_{i1}^T A_{11} z_{i1} \\ &\le (1+\theta) \lambda_{\max}(A_{11}) \|z_{i1}\|^2 \end{aligned} \tag{45}$$

provided that $z_{i1}^T A_{12} z_{i2} + z_{i1}^T \phi_{i1} - \theta z_{i1}^T A_{11} z_{i1} \le 0$.

Assume that $\left\| z_{i1}^T A_{12} z_{i2} + z_{i1}^T \phi_{i1} \right\| \le \left\| \theta z_{i1}^T A_{11} z_{i1} \right\|$. In the left-hand side of this inequality, $\left\| z_{i1}^T A_{12} z_{i2} + z_{i1}^T \phi_{i1} \right\| \le \|z_{i1}\| (\|A_{12} z_{i2}\| + \|\phi_{i1}\|) \le \|z_{i1}\| (\|A_{12}\|_F \|z_{i2}\| + \|\phi_{i1}\|)$; in the right-hand side, $\left\| \theta z_{i1}^T A_{11} z_{i1} \right\| \ge \lambda_{\max}(A_{11}) \theta \|z_{i1}\|^2$. Suppose that $\|z_{i1}\| (\|A_{12}\|_F \|z_{i2}\| + \|\phi_{i1}\|) \le \lambda_{\max}(A_{11}) \theta \|z_{i1}\|^2$, then equivalently

$$\|z_{i1}\| \ge \frac{\|A_{12}\|_F \|z_{i2}\| + \|\phi_{i1}\|}{\lambda_{\max}(A_{11}) \theta} \tag{46}$$

According to Definition 3, it can be shown that $\varepsilon_1(\|z_{i1}\|) = \varepsilon_2(\|z_{i1}\|) = \frac{1}{2}\|z_{i1}\|^2$, $-\varepsilon_3(\|z_{i1}\|) = (1+\theta) \lambda_{\max}(A_{11}) \|z_{i1}\|^2$, $\mathcal{X} = \frac{\|A_{12}\|_F \|z_{i2}\| + \|\phi_{i1}\|}{\lambda_{\max}(A_{11}) \theta}$, where z_{i1} is taken as the state and z_{i2} and ϕ_{i1} as the inputs in Definition 3. It follows from Lemma 3 that the subsystem is globally ISS with

$$\vartheta(\|u_\infty\|) = \frac{\|A_{12}\|_F \|z_{i2}\| + \|\phi_{i1}\|}{\lambda_{\max}(A_{11}) \theta} \tag{47}$$

Therefore, appealing to Remark 3, z_{i1} is bounded with

$$\|z_{i1}\| \le \frac{\|A_{12}\|_F \|z_{i2}\| + \|\phi_{i1}\|}{\lambda_{\max}(A_{11}) \theta} \tag{48}$$

Remark 9: The above results indicate that the Lyapunov function V_3 is negative definite along the trajectories of z_{i1} whenever the trajectories are outside of the hypersphere defined by $\left\{ z_{i1} \middle| \|z_{i1}\| \ge \frac{\|A_{12}\|_F \|z_{i2}\| + \|\phi_{i1}\|}{\lambda_{\max}(A_{11}) \theta} \right\}$, and the trajectories will remain ultimately bounded by the hypersphere of radius $\frac{\|A_{12}\|_F \|z_{i2}\| + \|\phi_{i1}\|}{\lambda_{\max}(A_{11}) \theta}$.

Remark 10: This section considers stability of the subsystems. Note that the stability of the subsystems is not considered in [1]. When the subsystem dynamics (1) is a class of second-order systems, the states may diverge due to the existence of uncertainties. See Appendix C for a detailed analysis.

V. SIMULATIONS AND ANALYSIS

In this section, two simulation examples are presented to demonstrate the validity of the proposed method.

Example 1. This example aims to demonstrate the effectiveness of the theoretical results in the presence of matched and unmatched uncertainties. Consider a multiagent system with four subsystems, whose topology connection is shown in Fig.1.

The dynamics of each subsystem is given by

$$\dot{x}_i = \begin{bmatrix} 1 & 5 & 8 & 3 \\ 4 & 7 & 5 & 9 \\ 11 & 5 & 4 & 3 \\ 9 & 6 & 0 & 9 \end{bmatrix} x_i + \begin{bmatrix} 5 & 7 \\ 4 & 1 \\ 0 & 5 \\ -8 & 6 \end{bmatrix} u_i + \phi_i \tag{49}$$

where the initial states are selected as follows:

$$\begin{aligned} x_1(0) &= \begin{bmatrix} -5 & 7 & 6 & 8 \end{bmatrix}^T, x_2(0) = \begin{bmatrix} 11 & 3 & -10 & -4 \end{bmatrix}^T \\ x_3(0) &= \begin{bmatrix} 8 & -3 & -1 & 0 \end{bmatrix}^T, x_4(0) = \begin{bmatrix} -4 & 6 & 0 & -2 \end{bmatrix}^T \end{aligned} \tag{50}$$

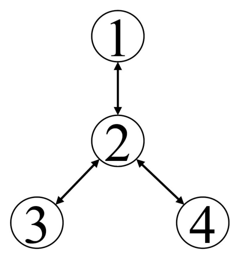


Fig. 1: Undirected graph with 4 subsystems

The uncertainties are as follows:

$$\phi_i = \begin{bmatrix} \breve{t}_{11}\gamma\sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2} + \widehat{t}_{13}x_{i3} + \widehat{t}_{14}x_{i4}\right)^2\right) + \breve{t}_{13}\left(0.1\cos\left(x_{i3}\right)\right) + \breve{t}_{14}\left(0.5\sin\left(t\right)\right) \\ \breve{t}_{21}\gamma\sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2} + \widehat{t}_{13}x_{i3} + \widehat{t}_{14}x_{i4}\right)^2\right) + \breve{t}_{23}\left(0.1\cos\left(x_{i3}\right)\right) + \breve{t}_{24}\left(0.5\sin\left(t\right)\right) \\ \breve{t}_{31}\gamma\sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2} + \widehat{t}_{13}x_{i3} + \widehat{t}_{14}x_{i4}\right)^2\right) + \breve{t}_{33}\left(0.1\cos\left(x_{i3}\right)\right) + \breve{t}_{34}\left(0.5\sin\left(t\right)\right) \\ \breve{t}_{41}\gamma\sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2} + \widehat{t}_{13}x_{i3} + \widehat{t}_{14}x_{i4}\right)^2\right) + \breve{t}_{43}\left(0.1\cos\left(x_{i3}\right)\right) + \breve{t}_{44}\left(0.5\sin\left(t\right)\right) \end{bmatrix} \tag{51}$$

where $T_2T_1 = \left[\widehat{t}_{ij}\right]_{4\times 4}$, $\left(T_2T_1\right)^{-1} = \left[\breve{t}_{ij}\right]_{4\times 4}$, $\gamma(t) = 0.01e^{-t}$, $i, j = 1, 2, 3, 4$.

The coordinate transformation matrices are

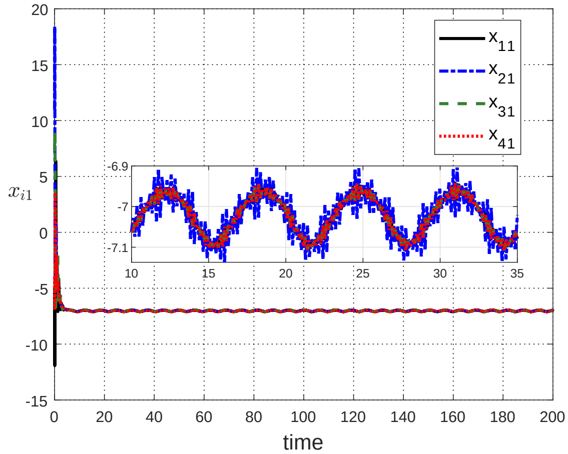
$$T_1 = \begin{bmatrix} -0.4943 & 0.3465 & 0.7856 & -0.1356 \\ -0.1300 & 0.8433 & -0.3951 & 0.3404 \\ -0.4880 & -0.3904 & 0.0000 & 0.7807 \\ 0.7076 & 0.1279 & 0.4762 & 0.5062 \end{bmatrix}, T_2 = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.3561 & 0.9858 & 1.0000 & 0.0000 \\ -0.0737 & 0.6324 & 0.0000 & 1.0000 \end{bmatrix} \tag{52}$$

and other parameters are selected as $\beta = 1.00, \alpha_i = 0.0001, \rho = 0.15, G = \begin{bmatrix} 3.6493 & 10.1017 \\ 0.4614 & -7.5056 \end{bmatrix}$.

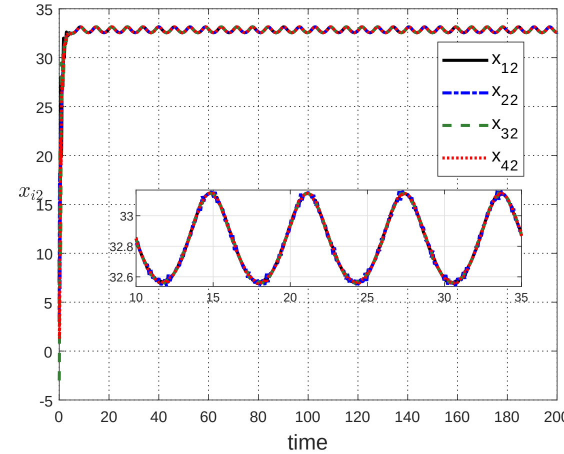
It can be verified by computations that $\|\phi_i\| \leq \beta$. In addition, by coordination transformation, it can be obtained that

$$\phi_{i1} = \gamma \begin{bmatrix} \sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2} + \widehat{t}_{13}x_{i3} + \widehat{t}_{14}x_{i4}\right)^2\right) \\ 0 \end{bmatrix} \tag{53}$$

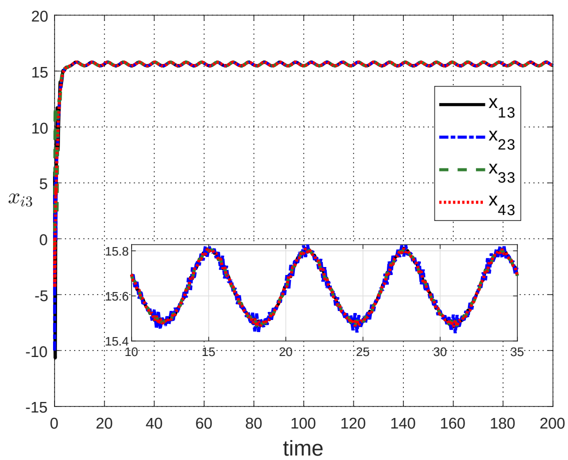
$\|\phi_{i1}\| \leq \gamma \|z_i\| = \gamma \sqrt{\sum_{j=1}^{4} \left(\widehat{t}_{j1} x_{i1} + \widehat{t}_{j2} x_{i2} + \widehat{t}_{j3} x_{i3} + \widehat{t}_{j4} x_{i4} \right)^2}$ and $\int_{t_0}^{\infty} \left\| e^{\bar{A}(t_0 - s)} \right\| \gamma(s) \mu(t_0) \|z(t_0)\| ds \leq \iota$ can be verified.



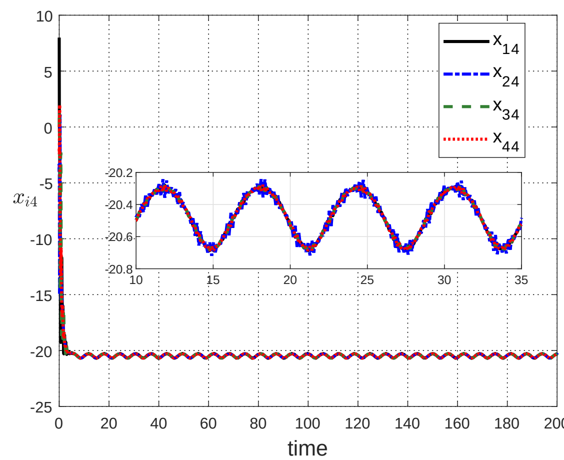
(a) x_{i1} with respect to time



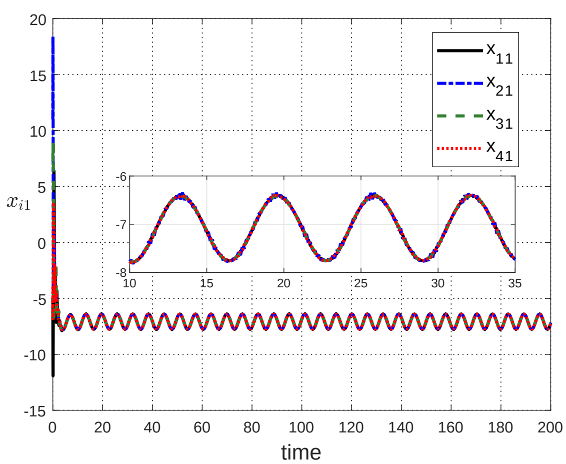
(b) x_{i2} with respect to time



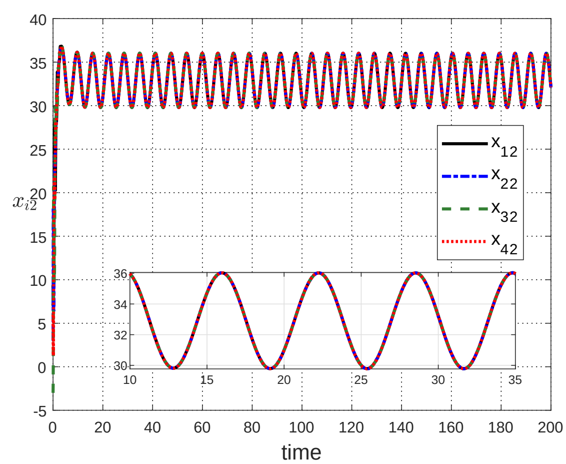
(c) x_{i3} with respect to time



(d) x_{i4} with respect to time

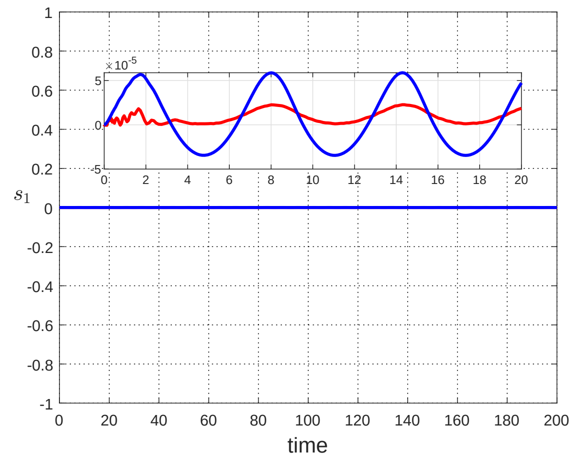


(e) x_{i1} with respect to time using boundary layer method

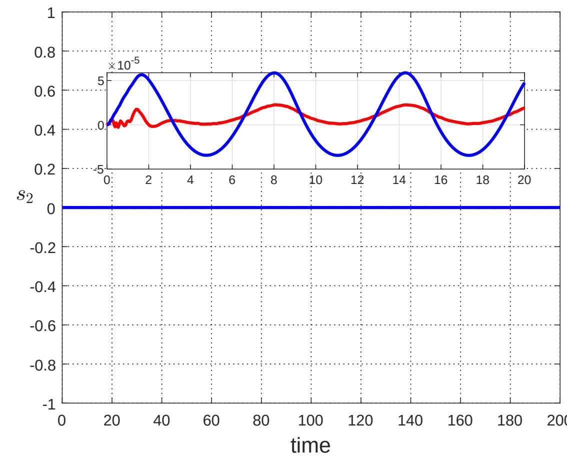


(f) x_{i2} with respect to time using boundary layer method

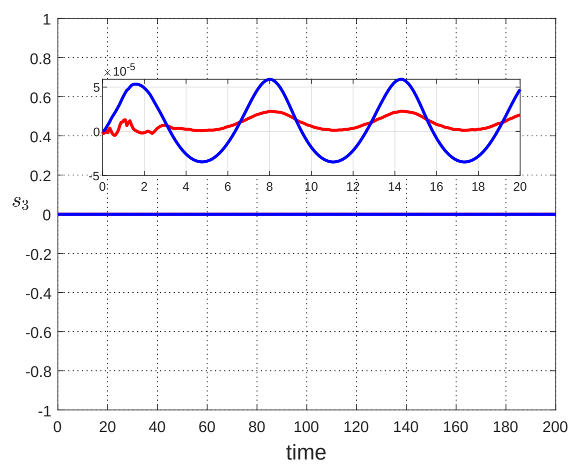
Fig. 2: Subsystems' states with respect to time



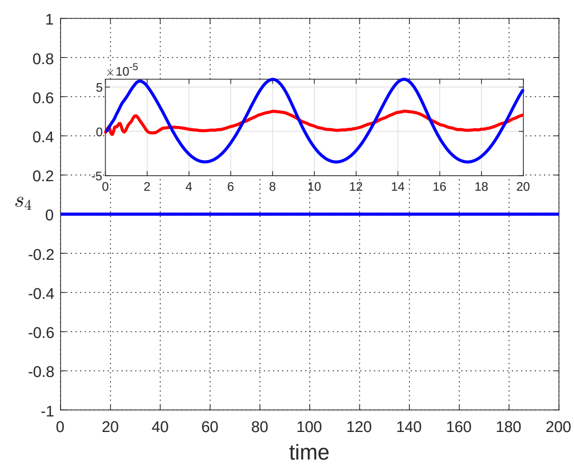
(a) s_1 with respect to time



(b) s_2 with respect to time

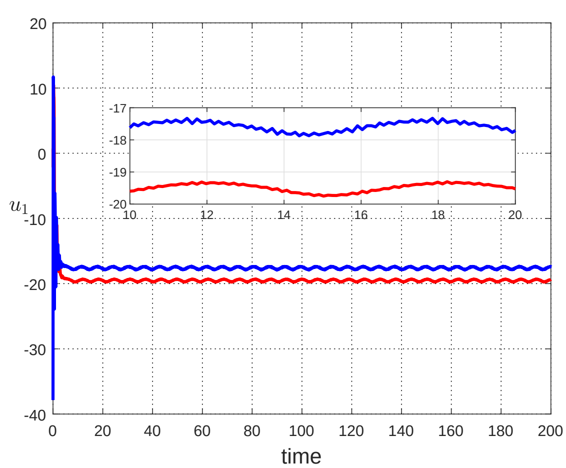


(c) s_3 with respect to time

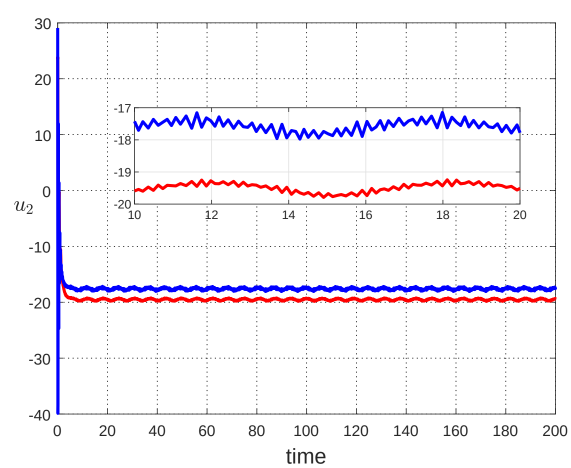


(d) s_4 with respect to time

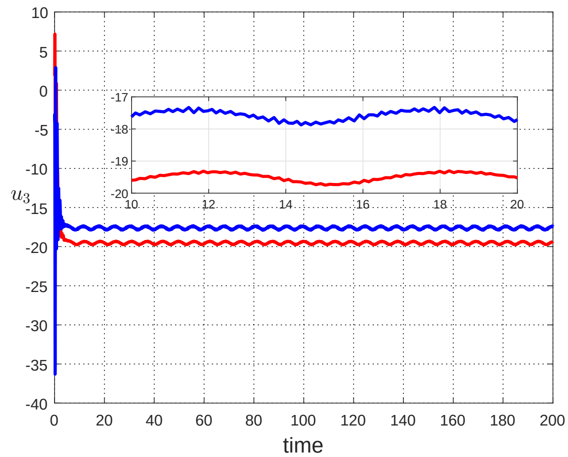
Fig. 3: Subsystems' sliding motion with respect to time



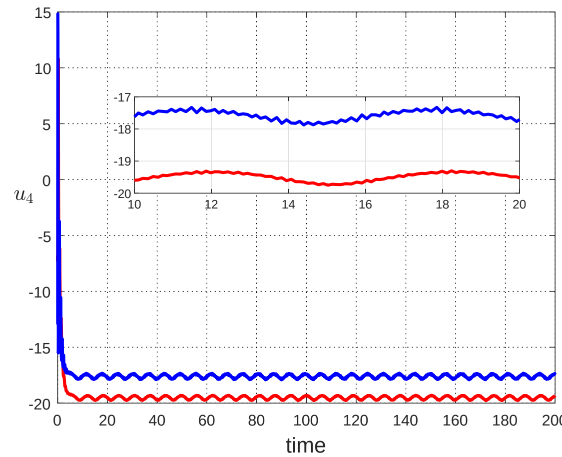
(a) u_1 with respect to time



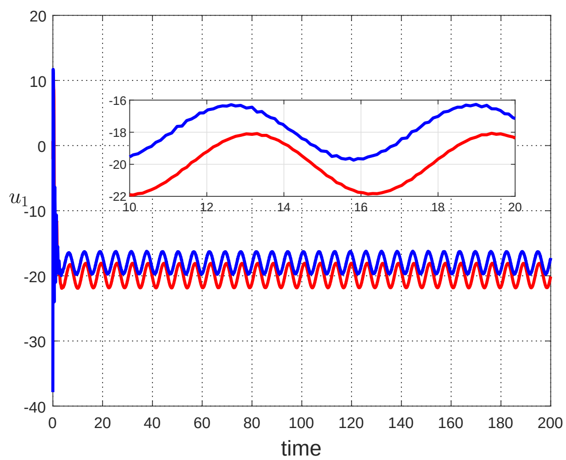
(b) u_2 with respect to time



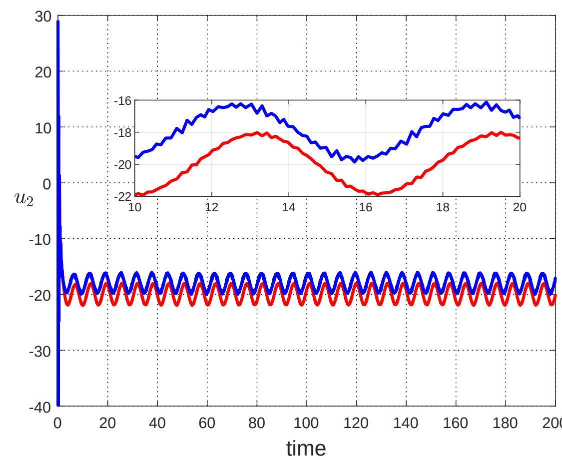
(c) u_3 with respect to time



(d) u_4 with respect to time

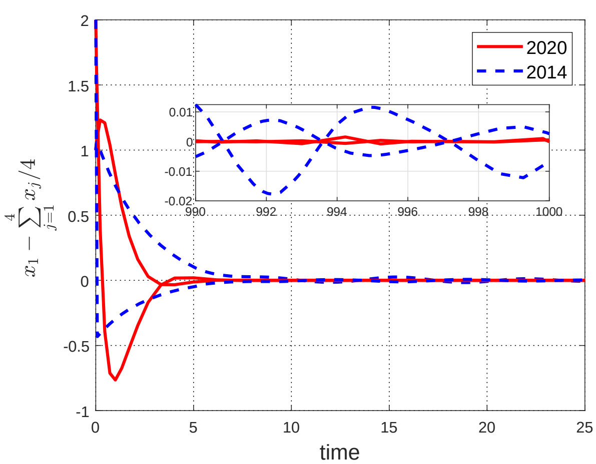


(e) u_1 with respect to time using boundary layer method

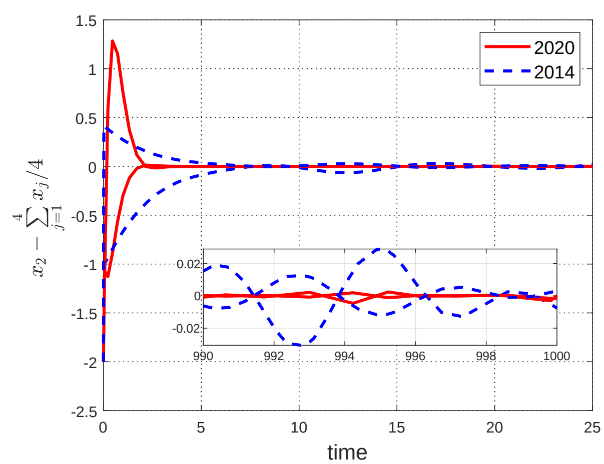


(f) u_2 with respect to time using boundary layer method

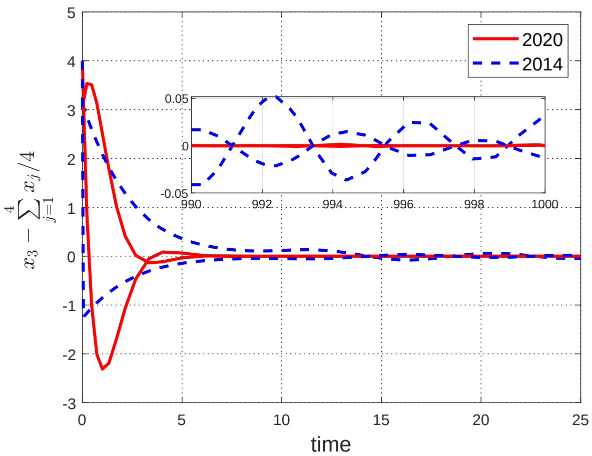
Fig. 4: Subsystems' control inputs with respect to time



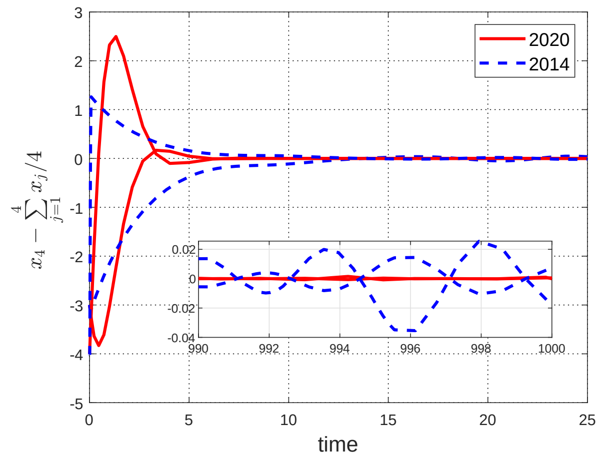
(a) Consensus errors with respect to time for the first subsystem



(b) Consensus errors with respect to time for the second subsystem



(c) Consensus errors with respect to time for the third subsystem



(d) Consensus errors with respect to time for the fourth subsystem

Fig. 5: Consensus errors with respect to time in two protocols

A boundary layer approximation is used such that $\left(G\begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}\right)^T s_i(t) / \left\| \left(G\begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}\right)^T s_i(t) + \delta \right\|$ is used to replace (27), where δ is a small positive scalar and selected as $\delta = 0.01$.

The simulation results are shown as Fig.2−4.

In Fig.2, (a)-(d) show the subsystems' states with respect to time. As can be seen, in the presence of matched and unmatched uncertainties, the system achieves consensus. Fig.3 shows the sliding variable with respect to time. It is seen that every subsystem starts on the sliding surface from the beginning which avoids sensitivity to matched uncertainties in the reaching phase.

In Fig.4, (a)-(d) show the subsystems' control inputs with respect to time. It can be seen that the control inputs remain bounded after the subsystems are stabilized. In addition, the control inputs exhibit chattering, which is caused by the discontinuous integral sliding mode control. The boundary layer method can help to reduce chattering but this is achieved at a price of the system state no longer lying on the sliding surface but remaining within a small boundary layer of the sliding surface [14][46]. In Fig.2 and Fig.4, plots (e) and (f) are added to show the subsystems' states and control inputs when a boundary layer is introduced to alleviate chattering. In this case

$$u_i^{\text{dis}}(t) = \begin{cases} -\rho\left(G\begin{bmatrix}0_{(n-m)\times n}\\ B_2\end{bmatrix}\right)^T s_i(t) \Big/ \left\|\left(G\begin{bmatrix}0_{(n-m)\times n}\\ B_2\end{bmatrix}\right)^T s_i(t)\right\|, & \left\|\left(G\begin{bmatrix}0_{(n-m)\times n}\\ B_2\end{bmatrix}\right)^T s_i(t)\right\| \geq \omega \\ -\rho\left(G\begin{bmatrix}0_{(n-m)\times n}\\ B_2\end{bmatrix}\right)^T s_i(t) \Big/ \left(\left\|\left(G\begin{bmatrix}0_{(n-m)\times n}\\ B_2\end{bmatrix}\right)^T s_i(t)\right\| + \omega\right), & \text{otherwise} \end{cases}$$

where $\omega = 0.1$. As can be seen, in comparison with (a) and (b) which apply the corresponding discontinuous control, chattering has decreased as expected.

Example 2. Consider the multiagent system whose topology connection is also shown as Fig.1. To further test the proposed distributed protocol, the protocol (3) developed in [1] which uses an adaptive scheme will be compared with the method proposed in this paper. The dynamics of each subsystem is given by

$$\dot{x}_i = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} x_i + \begin{bmatrix}0\\ 0.4\end{bmatrix} u_i + \phi_i \tag{54}$$

where the initial states are selected as follows:

$$x_1(0) = \begin{bmatrix}1 & 2\end{bmatrix}^T, x_2(0) = \begin{bmatrix}-1 & -2\end{bmatrix}^T, x_3(0) = \begin{bmatrix}3 & 4\end{bmatrix}^T, x_4(0) = \begin{bmatrix}-3 & -4\end{bmatrix}^T \tag{55}$$

The uncertainties are as follows:

$$\phi_i = \begin{bmatrix} \breve{t}_{11}\gamma \sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2}\right)^2\right) + 0.01\breve{t}_{12}\left(\cos\left(x_{i1}\right) + \sin\left(t\right)\right) \\ \breve{t}_{21}\gamma \sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2}\right)^2\right) + 0.01\breve{t}_{22}\left(\cos\left(x_{i1}\right) + \sin\left(t\right)\right) \end{bmatrix} \tag{56}$$

where $T_2T_1 = \left[\widehat{t}_{ij}\right]_{2\times 2}$, $(T_2T_1)^{-1} = \left[\breve{t}_{ij}\right]_{2\times 2}$, $\gamma(t) = 0.01e^{-t}$, $i, j = 1, 2$.

For the protocol proposed in this paper, the coordinate transformation matrices are

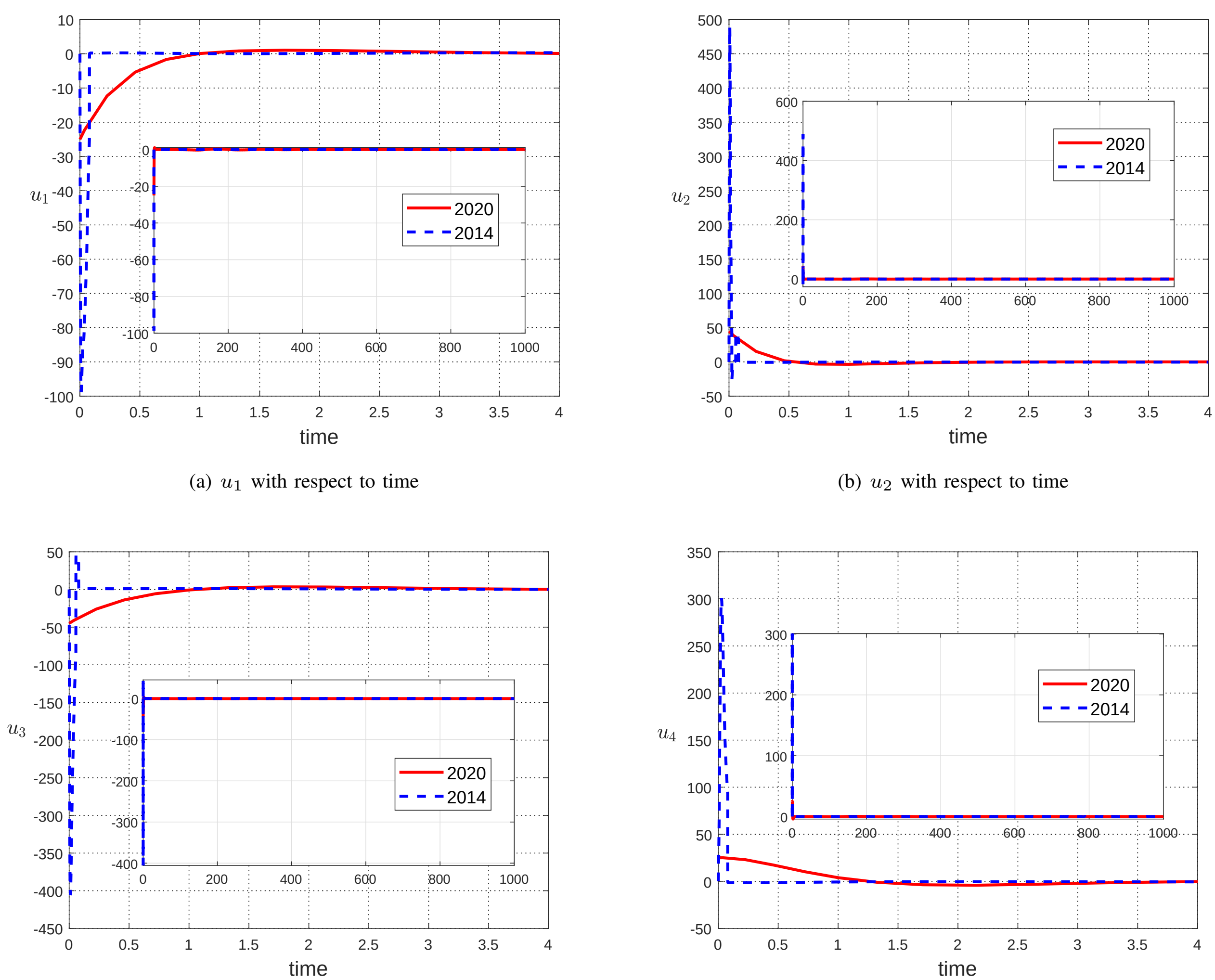
$$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{57}$$

and the other parameters are selected as $\beta = 0.15, \alpha_i = 0.0001, \rho = 0.4, G = \begin{bmatrix} 0.4 & -0.4 \end{bmatrix}, \delta = 0.0001.$

It can be verified by computations that $\|\phi_i\| \leq \beta$. In addition, by coordination transformation, it can be obtained that

$$\phi_{i1} = \gamma \sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2}\right)^2\right) \tag{58}$$

$\|\phi_{i1}\| \leq \gamma\|z_i\| = \gamma\sqrt{\sum\limits_{j=1}^{2}\left(\widehat{t}_{j1}x_{i1} + \widehat{t}_{j2}x_{i2}\right)^2}$ and $\int_{t_0}^{\infty}\left\|e^{\bar{A}(t_0-s)}\right\|\gamma(s)\,\mu(t_0)\,\|z(t_0)\|\,ds \leq \iota$ can be verified.



(a) u_1 with respect to time

(b) u_2 with respect to time

(c) u_3 with respect to time

(d) u_4 with respect to time

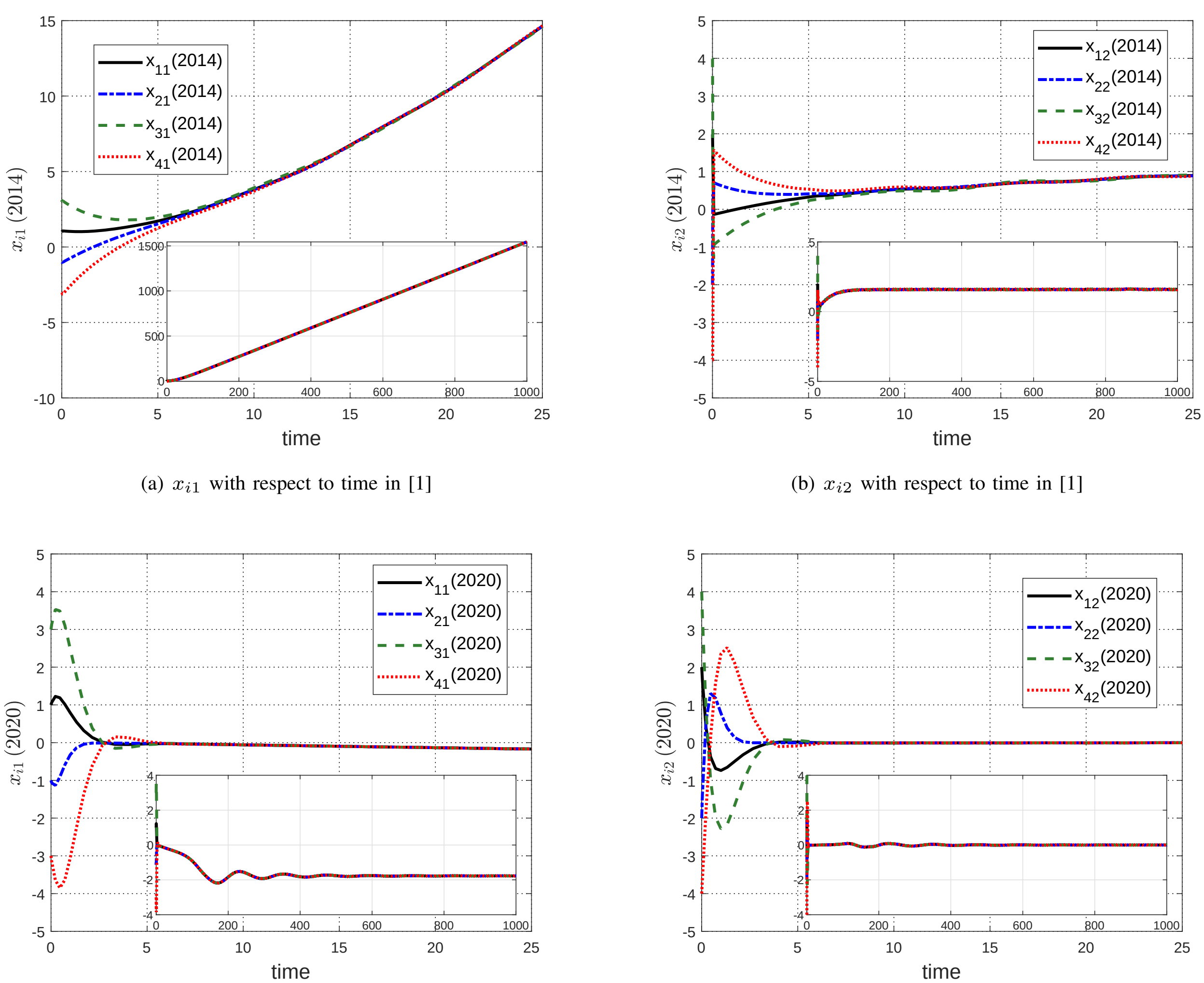
Fig. 6: Subsystems' control inputs with respect to time in two protocols

For the protocol (3) proposed in [1], the parameters are selected as $\Gamma = \begin{bmatrix} 1.0000 & 2.4495 \\ 2.4495 & 6.0000 \end{bmatrix}$, $K = \begin{bmatrix} -1.0000 & -2.4495 \end{bmatrix}$, $\bar{d}_i(0) = 0, \bar{e}_i(0) = 0, \tau_i = 10, \varepsilon_i = 10, \kappa_i = 0.5, \varphi_i = 0.05, \psi_i = 0.05$.

The simulation results are shown as Fig.5−7, where the method proposed in this paper is labeled as 2020; the method proposed in [1] is labeled as 2014. Note that although the total simulation time is 1000s, the main graphs of Fig.5−7 show the evolutions of the variables for an initial period of time and achievement of consensus, while the state evolution is shown in the embedded graph.

From Fig.5, the consensus errors with respect to time using the proposed method (2020) have a larger overshoot than those of the method in [1] (2014), but they have smaller fluctuation in the steady-state, which means higher product quality. In Fig.6, the full graphs are shown as an embedded figure, while the main graphs indicate that the control inputs of the method in [1] are several times higher than the proposed method for an initial period of time, which is energy-consuming.

Stability of the subsystems is not considered when designing the protocol (3) in [1], and the states correspondingly diverge. This can be verified by substituting numerical values into the system matrix (74) in Appendix C. No matter what values $\bar{d}_i$ take, it can be seen that there are two zero eigenvalues in the system matrix, while the corresponding eigenvectors are linearly dependent. The simulation results also illustrate this point, as shown in Fig.7 (a) and (b). With the proposed approach, the negative symmetric definiteness of A_{11} guarantees the state evolution with the protocol devised in this paper are ultimately bounded as seen in Fig.7 (c) and (d).



(a) x_{i1} with respect to time in [1]

(b) x_{i2} with respect to time in [1]

(c) x_{i1} with respect to time in this paper

(d) x_{i2} with respect to time in this paper

Fig. 7: Subsystems' states with respect to time

VI. Conclusion

A consensus framework is proposed for a class of linear multiagent systems in the presence of matched and unmatched uncertainties. An integral sliding mode strategy is utilized to ensure the subsystems lie on the sliding surface from the initial time. The impact of the uncertainties are minimized according to the projection theorem. A consensus protocol is designed and analyzed applying a linear coordinate transformation and the global invariant set theorem. The boundness of each subsystem is guaranteed by appealing to results on global ISS. Numerical simulations show the validity and superiority of the proposed method. Future work will focus on the case of a general directed switching graph and output feedback control.

Appendix A

The proof of Theorem 2 is provided as follows:

Substitute the discontinuous element from (27) with $G = B^{+}(T_2T_1)^{-1}$ into (28). Then

$$\dot{s}_i(t) = \alpha_i B^{+}(T_2T_1)^{-1}\left(-\rho T_2 T_1 B\frac{s_i}{\|s_i\|} + \begin{bmatrix}\phi_{i1}^T & \phi_{i2}^T\end{bmatrix}^T\right) \tag{59}$$

A Lyapunov candidate function is selected as

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{N} s_i^T s_i \tag{60}$$

Combining with (4), (5) and (59), the time derivative of $V_1(t)$ is given by

$$\begin{aligned}\dot{V}_1(t) &= \sum_{i=1}^{N} s_i^T \dot{s}_i \\ &= \sum_{i=1}^{N} s_i^T \alpha_i B^{+}(T_2T_1)^{-1}\left(-\rho T_2T_1 B\frac{s_i}{\|s_i\|} + \begin{bmatrix}\phi_{i1}^T & \phi_{i2}^T\end{bmatrix}^T\right) \\ &= \sum_{i=1}^{N}\alpha_i\left(-\rho\|s_i\| + s_i^T B^{+}\phi_i\right) \\ &\le \sum_{i=1}^{N}\alpha_i\left(-\rho\|s_i\| + \|s_i\|\left\|B^{+}\phi_i\right\|\right) \\ &= \sum_{i=1}^{N} -\alpha_i\left(\rho - \left\|B^{+}\phi_i\right\|\right)\|s_i\| \\ &\le \sum_{i=1}^{N} -\alpha_i\left(\rho - \left\|B^{+}\right\|_F\|\phi_i\|\right)\|s_i\| \\ &\le \sum_{i=1}^{N} -\alpha_i\left(\rho - \beta\left\|B^{+}\right\|_F\right)\|s_i\|\end{aligned} \tag{61}$$

According to Lemma 4, it follows that

$$\begin{aligned}\dot{V}_1(t) &\le \sum_{i=1}^{N} -\sigma_i\|s_i\| \\ &\le -\sigma_{\min}\sum_{i=1}^{N}\|s_i\| \\ &\le -\sigma_{\min}\sqrt{V_1}\end{aligned} \tag{62}$$

where $\sigma_{\min} = \min_i\{\sigma_i\}$, and $\sigma_i = \alpha_i\left(\rho - \beta\|B^{+}\|_F\right)$.

Appendix B

The proof that the corresponding homogeneous system of (36) is marginally stable is given as follows:

Let $\lambda_i\left(\bar{A}\right)$ be the real eigenvalues of $\bar{A}$, and $\bar{A} \triangleq \left(\bar{A}_{ij}\right)_{Nn\times Nn}$ for $i,j=1,\dots,Nn$. According to Geršgorin's theorem, $\left|\lambda_i\left(\bar{A}\right)-\bar{A}_{ii}\right| \le \sum\limits_{j=1,j\ne i}^{Nn}\left|\bar{A}_{ij}\right|$. If $\lambda_i\left(\bar{A}\right)-\bar{A}_{ii}\ge 0$, then $\bar{A}_{ii}\le\lambda_i\left(\bar{A}\right)\le\bar{A}_{ii}+\sum\limits_{j=1,j\ne i}^{Nn}\left|\bar{A}_{ij}\right|$. Because $a_{ii}^{11}\le-\sum\limits_{j=1}^{m}\left|a_{ij}^{12}\right|$ and the property of Laplacian matrix $\mathcal{L}$, $\bar{A}_{ii}+\sum\limits_{j=1,j\ne i}^{Nn}\left|\bar{A}_{ij}\right|\le 0$, else if $\lambda_i\left(\bar{A}\right)-\bar{A}_{ii}<0$, then $\bar{A}_{ii}-\sum\limits_{j=1,j\ne i}^{Nn}\left|\bar{A}_{ij}\right|\le\lambda_i\left(\bar{A}\right)<\bar{A}_{ii}<0$. Thus $\lambda_i\left(\bar{A}\right)\le 0\,(i=1,\dots,Nn)$.

For the case that $\lambda_i\left(\bar{A}\right)=0$, it can be obtained that

$$0\bar{v}=\bar{A}\bar{v} \tag{63}$$

where $\bar{v}$ is the corresponding eigenvector of 0. Further,

$$0\begin{bmatrix}\bar{v}_{\mathrm{I}}\\ \bar{v}_{\mathrm{II}}\end{bmatrix}=\begin{bmatrix}I_N\otimes A_{11} & I_N\otimes A_{12}\\ -\mathcal{L}\otimes A_{12}^T & -\mathcal{L}\otimes I_m\end{bmatrix}\begin{bmatrix}\bar{v}_{\mathrm{I}}\\ \bar{v}_{\mathrm{II}}\end{bmatrix} \tag{64}$$

where $\bar{v}=\begin{bmatrix}\bar{v}_{\mathrm{I}}^T & \bar{v}_{\mathrm{II}}^T\end{bmatrix}^T$, $\bar{v}_{\mathrm{I}}\in R^{N(n-m)}$ and $\bar{v}_{\mathrm{II}}\in R^{Nm}$, then it follows that

$$\left(I_N\otimes A_{11}\right)\bar{v}_{\mathrm{I}}+\left(I_N\otimes A_{12}\right)\bar{v}_{\mathrm{II}}=0 \tag{65}$$

$$\left(\mathcal{L}\otimes A_{12}^T\right)\bar{v}_{\mathrm{I}}+\left(\mathcal{L}\otimes I_m\right)\bar{v}_{\mathrm{II}}=0 \tag{66}$$

As $I_N\otimes A_{11}$ is invertible,

$$\bar{v}_{\mathrm{I}}=-\left(I_N\otimes A_{11}^{-1}A_{12}\right)\bar{v}_{\mathrm{II}} \tag{67}$$

Substitute (67) into (66), it follows that

$$\left(\mathcal{L}\otimes\left(A_{12}^TA_{11}^{-1}A_{12}-I_m\right)\right)\bar{v}_{\mathrm{II}}=0 \tag{68}$$

Remark 11: The two equations (68) and (63) are equivalent, the analysis of eigenvector $\bar{v}$ corresponding to eigenvalue 0 of $\bar{A}$ is equivalent to the analysis of $\bar{v}_{\mathrm{II}}$ corresponding to eigenvalue 0 of $\mathcal{L}\otimes\left(A_{12}^TA_{11}^{-1}A_{12}-I_m\right)$.

From Proposition 3.3 of [26] and Theorem 1 of [47], it follows that A_{12} has full row rank, then $A_{12}^TA_{11}^{-1}A_{12}$ is real negative symmetric definite, and $\lambda_i\left(A_{12}^TA_{11}^{-1}A_{12}\right)=\lambda_i^{-1}\left(A_{11}\right)$. Therefore, $\lambda_i\left(A_{12}^TA_{11}^{-1}A_{12}-I_m\right)<0$, and $A_{12}^TA_{11}^{-1}A_{12}-I_m$ is invertible. Note that $\mathcal{L}$ has a simple eigenvalue 0 and all the other eigenvalues are positive, then $\mathcal{L}\otimes\left(A_{12}^TA_{11}^{-1}A_{12}-I_m\right)$ has m eigenvalues 0.

Further, $\bar{v}_{\mathrm{II}}$ can be $\bar{\alpha}\left[\underbrace{1,\underbrace{0,\dots,0}_{m-1},\dots 1,\underbrace{0,\dots,0}_{m-1}}_{Nm}\right]^T$, $\bar{\alpha}\left[\underbrace{0,1,\underbrace{0,\dots,0}_{m-2},\dots 0,1,\underbrace{0,\dots,0}_{m-2}}_{Nm}\right]^T$, $\dots$, $\bar{\alpha}\left[\underbrace{\underbrace{0,\dots,0}_{m-1},1,\dots\underbrace{0,\dots,0}_{m-1},1}_{Nm}\right]^T$ $(\bar{\alpha}\in R,\bar{\alpha}\ne 0)$, and they are m linearly independent eigenvectors. The algebraic multiplicity of eigenvalue 0 is equal to the geometric multiplicity. Recall that $\lambda_i\left(\bar{A}\right)\le 0\,(i=1,\dots,Nn)$, so the corresponding homogeneous system of (36) is marginally stable.

APPENDIX C

The analysis of the case where the subsystem states are diverging in [1] is presented as follows:

(a) Substitute the consensus protocol (3, [1]) into the subsystem dynamics (2, [1]), to obtain a lumped form:

$$\dot{x}=\left[I_N\otimes A+\left(\bar{D}\mathcal{L}\right)\otimes(BK)\right]x+\left(I_N\otimes B\right)(R+F) \tag{69}$$

It should be noted that in this appendix, $(*, [1])$ refers to the corresponding equation $(*)$ in [1], and the notations also refer to the ones in [1] unless otherwise stated.

(b) Here, an eigenvalue can be acquired by the system matrix $\left[I_N \otimes A + \left(\bar{D}\mathcal{L}\right) \otimes (BK)\right]$ in (54) by which stability of the subsystem can be judged.

(b.1) When the subsystem dynamics (1, [1]) is in a linear second-order form, then $A \triangleq \begin{bmatrix} 0 & \bar{a}_{12} \\ 0 & 0 \end{bmatrix}$ and $B \triangleq \begin{bmatrix} 0 \\ \bar{b}_2 \end{bmatrix}$, where $\bar{a}_{12}, \bar{b}_2 \in R$. To guarantee the controllability of the subsystem, $\bar{a}_{12} \neq 0$ and $\bar{b}_2 \neq 0$. $K \triangleq \begin{bmatrix} \bar{k}_1 & \bar{k}_2 \end{bmatrix}$, where $\bar{k}_1, \bar{k}_2 \in R$.

(b.2) Calculate the elements item by item as follows for the system matrix $\left[I_N \otimes A + \left(\bar{D}\mathcal{L}\right) \otimes (BK)\right]$:

$$I_N \otimes A = diag\left(\underbrace{\begin{bmatrix} 0 & \bar{a}_{12} \\ 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \bar{a}_{12} \\ 0 & 0 \end{bmatrix}}_{N}\right) \tag{70}$$

$$\bar{D}\mathcal{L} = \left[\bar{d}_i \mathcal{L}_{ij}\right]_{N\times N}, i, j = 1, \ldots, N \tag{71}$$

$$BK = \begin{bmatrix} 0 & 0 \\ \bar{k}_1\bar{b}_2 & \bar{k}_2\bar{b}_2 \end{bmatrix} \tag{72}$$

$$\left(\bar{D}\mathcal{L}\right) \otimes (BK) = \left[\bar{d}_i \mathcal{L}_{ij} \begin{bmatrix} 0 & 0 \\ \bar{k}_1\bar{b}_2 & \bar{k}_2\bar{b}_2 \end{bmatrix}\right]_{N\times N} = \left[\begin{bmatrix} 0 & 0 \\ \bar{d}_i \mathcal{L}_{ij}\bar{k}_1\bar{b}_2 & \bar{d}_i \mathcal{L}_{ij}\bar{k}_2\bar{b}_2 \end{bmatrix}\right]_{N\times N} \tag{73}$$

then

$$\begin{aligned} & I_N \otimes A + \left(\bar{D}\mathcal{L}\right) \otimes (BK) \\ & = diag\left(\underbrace{\begin{bmatrix} 0 & \bar{a}_{12} \\ 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \bar{a}_{12} \\ 0 & 0 \end{bmatrix}}_{N}\right) + \left[\begin{bmatrix} 0 & 0 \\ \bar{d}_i \mathcal{L}_{ij}\bar{k}_1\bar{b}_2 & \bar{d}_i \mathcal{L}_{ij}\bar{k}_2\bar{b}_2 \end{bmatrix}\right]_{N\times N} \\ & \triangleq \left[\bar{\Lambda}_{ij}\right]_{2N\times 2N} \end{aligned} \tag{74}$$

In (74), $\bar{\Lambda}_{i1} + \bar{\Lambda}_{i3} + \bar{\Lambda}_{i5} + \ldots + \bar{\Lambda}_{i(2N-1)} = 0$, then $\bar{\alpha}\left[\underbrace{1, 0, \ldots, 1, 0}_{2N}\right]^T$ is an eigenvector of (74), where $\bar{\alpha} \in R$ and $\bar{\alpha} \neq 0$, and the corresponding eigenvalue is 0.

(c) Consider now where there is more than one 0 in the eigenvalues of $\left[I_N \otimes A + \left(\bar{D}\mathcal{L}\right) \otimes (BK)\right]$ but the corresponding eigenvectors are linearly dependent. As a consequence the states diverge. This covers the analysis of [1].

In addition to [1], there are other contributions [2][8][9][19] where the states may diverge for a second-order subsystem when subjected to uncertainties.

VII. Declaration of conflicting interests

The author(s) declare no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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