A QUANTUM DEFORMATION OF THE SECOND WEYL ALGEBRA: ITS DERIVATIONS AND POISSON DERIVATIONS

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Abstract

Since the introduction of quantum algebras in the 1980's, many have introduced quantum deformations of the Weyl algebras. Two such examples are the quantum Weyl algebras and the Generalized Weyl algebras. In this thesis, we use a different approach to find deformations of (a quadratic extension of) the second Weyl algebra $A_2(\mathbb{C})$, and compare some properties of these deformations to those of $A_2(\mathbb{C})$.

Let $\mathfrak{n}$ be a nilpotent Lie algebra, $U(\mathfrak{n})$ the enveloping algebra of $\mathfrak{n}$ and Q a primitive ideal of $U(\mathfrak{n})$. Dixmier [12] proved that the factor algebra $U(\mathfrak{n})/Q$ is isomorphic to an n^{th} Weyl algebra $A_n(\mathbb{C})$, where $n \in \mathbb{N}_{\geq 1}$. This isomorphism gives a route to construct potential deformations for any Weyl algebra. Let $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$ represent a simple Lie algebra. Now, Dixmier's result holds for $\mathfrak{n} = \mathfrak{g}^+$. Since $U_q^+(\mathfrak{g})$ is a q-deformation of $U(\mathfrak{g}^+)$, it is natural to consider $U_q^+(\mathfrak{g})/P$, where P is a primitive ideal of $U_q^+(\mathfrak{g})$, as a potential deformation of the Weyl algebras.

This thesis focuses on the case where $\mathfrak{g} = G_2$. We find a family of primitive ideals $(P_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{C}^2\setminus(0,0)}$ of $U_q^+(G_2)$ whose corresponding quotients $A_{\alpha,\beta} := U_q^+(G_2)/P_{\alpha,\beta}$ are simple noetherian domains of Gelfand-Kirillov dimension 4. In view of Dixmier's result, we consider $A_{\alpha,\beta}$ as a q-deformation of (a quadratic extension of) $A_2(\mathbb{C})$. The derivations of the Weyl algebras are all known to be inner derivations [5]. Motivated by this, we also study the derivations of $A_{\alpha,\beta}$ and compare them to those of the Weyl algebras. The final part of the thesis studies a Poisson derivation of a semiclassical limit $\mathcal{A}_{\alpha,\beta}$ of $A_{\alpha,\beta}$. Interestingly, the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$ and the derivations of $A_{\alpha,\beta}$ are congruent.

Contents

Chapter 0

Introduction

One particular class of algebras in non-commutative algebras that has been widely studied since its inception is the class of Weyl algebras. Dirac is well noted to be the one who introduced the Weyl algebras in the field of quantum mechanics. Since then, the notion and concept of the Weyl algebras has diffused into other areas of non-commutative algebras from different perspectives. For example, one can understand the Weyl algebras from the perspective of differential operators, quotient of free algebras, and of course quantum mechanics. In line with this, we will provide the definition of Weyl algebras from three different perspectives. Before we do that, it is imperative to also add that the Weyl algebras come in series. That is, for every $n \in \mathbb{N}_{>0}$, one can associate a Weyl algebra denoted by $A_n(\mathbb{C})$. Now, $A_n(\mathbb{C})$ is naturally called the n^{th} Weyl algebra. The smallest of this series of algebras is called the first Weyl algebra $A_1(\mathbb{C})$, and it is the building block for any n^{th} Weyl algebra $A_n(\mathbb{C})$. That is, given n copies of $A_1(\mathbb{C})$, one can define $A_n(\mathbb{C})$ as follows:

$$A_n(\mathbb{C}) = A_1(\mathbb{C}) \otimes A_1(\mathbb{C}) \otimes \cdots \otimes A_1(\mathbb{C}) \quad (n \text{ copies}).$$

As a result of this, it is sufficient to define the first Weyl algebra $A_1(\mathbb{C})$. We are now ready to understand $A_1(\mathbb{C})$ from the following three points of view.

In theoretical physics (particularly, quantum mechanics), one can describe $A_1(\mathbb{C})$ as follows. Take the momentum operator $\hat{p}$ and position operator $\hat{q}$, it is well known that $[\hat{p}, \hat{q}] := \hat{p}\hat{q} - \hat{q}\hat{p} = i\hbar$, where $\hbar$ is called Plank's constant. One can therefore define $A_1(\mathbb{C})$ as an algebra of quantum mechanics generated by the operators $\hat{g}$ and $\hat{q}$ subject to the relation $\hat{g}\hat{q} - \hat{q}\hat{g} = 1$, where $\hat{g} = -\frac{i}{\hbar}\hat{p}$.

Secondly, in the ring of differential operators, $A_1(\mathbb{C})$ can be described in the following context. Let $\mathbb{C}[x]$ be a polynomial ring and, $\hat{x}$ and $\hat{y} := \dfrac{d}{dx}$ be operators of $\mathbb{C}[x]$ with $\hat{x}$ defined as $\hat{x}(f) = xf,\ \forall f \in \mathbb{C}[x]$. Observe that $\hat{y}\hat{x}(f) = \dfrac{d}{dx}(xf) = x\dfrac{df}{dx} + f = \hat{x}\hat{y}(f) + f$. This implies that $\hat{y}\hat{x} - \hat{x}\hat{y} = 1$. As a result, $A_1(\mathbb{C})$ can be defined as a $\mathbb{C}-$subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$ generated by the operators $\hat{x}$ and $\hat{y}$ (subject to the relation $\hat{y}\hat{x} - \hat{x}\hat{y} = 1$).

Algebraically, one can also define $A_1(\mathbb{C})$ as follows. Let $R := \mathbb{C}\langle X, Y\rangle$ be a free algebra generated by X and Y. Now, $I := \langle XY - YX - 1\rangle$ is a two-sided ideal of R. The factor algebra R/I satisfies the relation $xy - yx - 1 = 0$, where $x := X + I$ and $y := Y + I$. Therefore, $A_1(\mathbb{C})$ can be defined as a $\mathbb{C}-$algebra generated by x and y subject to the relation $xy - yx = 1$.

Above are three different contexts in which $A_1(\mathbb{C})$ can be understood. Generally, every n^{th} Weyl algebra $A_n(\mathbb{C})$ can precisely be described as a $\mathbb{C}-$algebra generated by $x_1, \cdots, x_n, y_1, \cdots, y_n$ subject to the following defining relations: $[x_i, x_j] = [y_i, y_j] = 0$ and $[x_i, y_j] = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. The Weyl algebras $A_n(\mathbb{C})$ are simple, noetherian domains, and have a Gelfand-Kirillov dimension of $2n$ [30, Chapter 8]. Dixmier has also studied the automorphism group of $A_1(\mathbb{C})$, and concluded in [11, Theorem 8.10] that the automorphism group is generated by two families of automorphisms $\phi_{s,\mu}$ and $\phi'_{s,\mu}$ of $A_1(\mathbb{C})$ defined as follows: $\phi_{s,\mu}(x) = x$, $\phi_{s,\mu}(y) = y + \mu x^s$ and $\phi'_{s,\mu}(x) = x + \mu y^s$, $\phi'_{s,\mu}(y) = y$, where $\mu \in \mathbb{C}$ and $s \in \mathbb{N}$. For $n > 1$, to the best knowledge of the author, the automorphism groups of $A_n(\mathbb{C})$ are not known. The center of the Weyl algebras is reduced to scalars. Another property of the Weyl algebras that is worth mentioning is that all its derivations are inner [5, Lemma 1], and the first Hochschild cohomology group is of dimension zero. What has even made the Weyl algebras very famous and instigated a lot of research in non-commutative algebras are the Jacobian

Conjecture [28], and the Dixmier Conjecture [11]. These two Conjectures are stated below.

Conjecture 1 (Jacobian, 1939). *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If the Jacobian determinant of F is a non-zero constant, then F must have an inverse polynomial map function.*

The case where $n = 1$ is considered trivial since one can easily show that the associated polynomial map is a linear map, hence, has an inverse map which is also linear. To achieve this, let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial map defined by $x \mapsto f(x)$. Suppose that df/dx is a non-zero constant, then clearly $f(x)$ is of the form $f(x) = a_0 + a_1x$, where $a_1 \in \mathbb{C}^*$ and $a_0 \in \mathbb{C}$. Since $a_1 \neq 0$, one can easily confirm that the inverse polynomial map of f is also of the form $f^{-1}(x) = -a_0a_1^{-1} + a_1^{-1}x$ as desired. For $n > 1$, to the author's best knowledge, the Jacobian Conjecture remains open.

Conjecture 2 (Dixmier, 1968). *Every algebra endomorphism of Weyl algebras $A_n(\mathbb{C})$ is an automorphism.*

Dixmier's Conjecture remains open even for the case where $n = 1$. Tsuchimoto [43], and Belov-Kanel and Kontsevich [6] independently proved that Conjecture 1 and Conjecture 2 are stably equivalent. That is, the Jacobian Conjecture implies the Dixmier Conjecture and vice versa.

With the appearance of quantum groups in the 1980's, people have introduced various quantum deformations (or analogues) of $A_n(\mathbb{C})$. The notable deformations are the quantum Weyl algebras. For example, the first quantum Weyl algebra is defined as $A_1^{(q)}(\mathbb{C}) := \mathbb{C}\langle x, y \mid xy - qyx = 1\rangle$. Clearly, $A_1^{(1)}(\mathbb{C}) = A_1(\mathbb{C})$. As a result of this, we say that $A_1^{(q)}(\mathbb{C})$ is a *quantum deformation* (or q-deformation for short) of $A_1(\mathbb{C})$. By extension, the quantum Weyl algebras $A_n^{(q)}(\mathbb{C})$ are q-deformations of the Weyl algebras $A_n(\mathbb{C})$ for each $n \in \mathbb{N}_{>0}$. Note, the properties of the quantum Weyl algebras do not always reflect the properties of the Weyl algebras. For example, the first quantum Weyl algebra is not simple when q is not a root of unity, however, the first Weyl algebra is simple. Are there any other deformations of $A_n(\mathbb{C})$? Yes, Bavula [3] has introduced a generalization of the Weyl algebras called the *generalized Weyl algebras (GWAs)*, but they are also not simple in general. As a result, their properties too do not always reflect the properties of the Weyl algebras. Let D be a $\mathbb{C}-$algebra with a central element a, and σ denote an automorphism of D. The *first generalized Weyl algebra* $D(\sigma, a)$ is simply defined as an algebra generated over D by the indeterminates x and y subject to the following relations:

$$yx = a, \quad xy = \sigma(a), \quad xd = \sigma(d)x, \quad yd = \sigma^{-1}(d)y, \quad \text{for all} \quad d \in D.$$

Let $q \in \mathbb{C}^*$ and $\lambda' \in \mathbb{C}$. From [41, Proposition 2.1.1], we have that the GWA $\mathbb{C}[H'](\sigma', a')$, where $\sigma(H') = qH' - \lambda'$, is isomorphic to one of the following:

i. $\mathbb{C}[H](\text{id}, a)$,

ii. $\mathbb{C}[H](\sigma, a)$ with $\sigma(H) = H - 1$,

iii. $\mathbb{C}[H](\sigma, a)$ with $\sigma(H) = qH$ and $q \in \mathbb{C} \setminus \{0, 1\}$.

The third family (iii) is usually called the *quantum GWA* provided q is not a root of unity, and the second family (ii) is well known to be isomorphic to the first Weyl algebra $A_1(\mathbb{C})$ (for example, see [2, §4]). Other specific examples of GWAs are the quantum plane and quantum Weyl algebra $A_1^{(q)}(\mathbb{C})$ (see [2, §4] for further details).

In this thesis, we used a different approach to find a deformation of (a quadratic extension of) $A_2(\mathbb{C})$. Let $\mathfrak{n}$ be a finite dimensional nilpotent Lie algebra over $\mathbb{C}$, $U(\mathfrak{n})$ be the enveloping algebra of $\mathfrak{n}$, and Q a primitive ideal of $U(\mathfrak{n})$. Dixmier has shown that the factor algebra $U(\mathfrak{n})/Q$ is isomorphic to an n^{th} Weyl algebra $A_n(\mathbb{C})$ [12, Theorem 4.7.9].

Let $\mathfrak{g}$ be a simple Lie algebra. Then, the enveloping algebra $U(\mathfrak{g})$ has a quantum deformation, introduced independently by Drinfeld [13] and Jimbo [27], called *the quantized enveloping algebra* $U_q(\mathfrak{g})$ of $\mathfrak{g}$. There is a triangular decomposition for $\mathfrak{g}$ as $\mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$; where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g}^-$ and $\mathfrak{g}^+$ are the negative and positive nilpotent subalgebras of $\mathfrak{g}$ respectively. As a result of this triangular decomposition, there are corresponding decompositions for $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ as $U(\mathfrak{g}^+) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}^-)$

and $U_q^+(\mathfrak{g}) \otimes U^0(\mathfrak{h}) \otimes U_q^-(\mathfrak{g})$ respectively. It is well known that $U_q^+(\mathfrak{g})$ is isomorphic to $U_q^-(\mathfrak{g})$ (for example, see [45, §2]). Now, Dixmier's theorem applies to $\mathfrak{n} = \mathfrak{g}^+$. That is, $U(\mathfrak{g}^+)/Q \cong A_n(\mathbb{C})$. Since $U_q^+(\mathfrak{g})$ is a q-deformation of $U(\mathfrak{g}^+)$, it is natural to ask whether there exists a primitive ideal P of $U_q^+(\mathfrak{g})$ such that $U_q^+(\mathfrak{g})/P$ is also a q-deformation of $U(\mathfrak{g}^+)/Q \cong A_n(\mathbb{C})$? If yes, then this gives another deformation of $A_n(\mathbb{C})$. In [33], Launois described a family of simple quotients of $U_q^+(B_2)$, and under certain conditions, retrieved the first Weyl algebra from these simple quotients when $q = 1$. In line with Dixmier's result (i.e. $U(\mathfrak{g}^+)/Q \cong A_n(\mathbb{C})$) and motivated by Launois' example, we also aimed to find a family of simple quotients of a quantized enveloping algebra whose deformation gives (a quadratic extension of) $A_2(\mathbb{C})$ and then compare some properties of these simple quotients to those of $A_2(\mathbb{C})$. This is the main aim of the thesis which is organized as follows.

Chapter 1 focuses on studying preliminary materials, and Chapter 2 studies some properties of the positive part of the quantized enveloping algebra of type G_2, $U_q^+(G_2)$. The algebra $U_q^+(G_2)$ is a $\mathbb{C}-$algebra generated by the indeterminates E_1 and E_2 subject to the quantum Serre relations (omitted for now). It is a noetherian domain and can be written as an iterated Ore extension over $\mathbb{C}$. The center of $U_q^+(G_2)$ is the polynomial ring $\mathbb{C}[\Omega_1, \Omega_2]$, where Ω_1 and Ω_2 are central elements of $U_q^+(G_2)$. The torus $\mathcal{H} = (\mathbb{C}^*)^2$ acts by automorphism on $U_q^+(G_2)$ via $(\alpha_1, \alpha_2) \cdot E_i = \alpha_i E_i$; $i = 1, 2$; for all $(\alpha_1, \alpha_2) \in \mathcal{H}$. As a result, one can use the $\mathcal{H}-$*stratification* of Goodearl and Letzter [21] to study the prime spectrum of this algebra, and identify primitive ideals. The $\mathcal{H}-$*stratification* partitions the prime spectrum of $U_q^+(G_2)$ into disjoint strata and these partitions are indexed by the $\mathcal{H}-$invariant prime ideals. Now, the ideals $\langle 0 \rangle$, $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$ are all $\mathcal{H}-$invariant prime ideals of $U_q^+(G_2)$ of at most height one, with the following strata:

- $\langle 0 \rangle-$stratum of $U_q^+(G_2) = \{\langle 0 \rangle\} \cup \{P(\Omega_1, \Omega_2) \mid P(\Omega_1, \Omega_2) \in \mathcal{P},\ P(\Omega_1, \Omega_2) \neq \Omega_1, \Omega_2\} \cup \{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \mid \alpha, \beta \in \mathbb{C}^*\}$. Note, $\mathcal{P}$ is the set of all unitary irreducible polynomials of $\mathbb{C}[\Omega_1, \Omega_2]$.

- $\langle \Omega_1 \rangle-$stratum of $U_q^+(G_2) = \{\langle \Omega_1 \rangle\} \cup \{\langle \Omega_1, \Omega_2 - \beta \rangle \mid \beta \in \mathbb{C}^*\}$.

- $\langle\Omega_2\rangle-$stratum of $U_q^+(G_2) = \{\langle\Omega_2\rangle\} \cup \{\langle\Omega_1 - \alpha, \Omega_2\rangle \mid \alpha \in \mathbb{C}^*\}$.

For each $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$, the prime ideal $\langle\Omega_1 - \alpha, \Omega_2 - \beta\rangle$ is maximal in its respective stratum, as a result, it is a primitive ideal of $U_q^+(G_2)$ [7, Theorem II.8.4]. Moreover, $\langle\Omega_1 - \alpha, \Omega_2 - \beta\rangle$ is a height 2 maximal ideal of $U_q^+(G_2)$. Given this family of height 2 maximal ideals of $U_q^+(G_2)$, Chapter 3 focuses on studying the simple quotients of $U_q^+(G_2)$, which are the main algebras of interest in this thesis. The simple quotient $A_{\alpha,\beta} := U_q^+(G_2)/\langle\Omega_1 - \alpha, \Omega_2 - \beta\rangle$ is a noetherian domain of GKdim 4, with a trivial center (i.e. central elements are all scalars). One could observe that $A_{\alpha,\beta}$ shares some common properties with $A_2(\mathbb{C})$. That is, they are both simple, noetherian domains, have GKdim of 4, and have trivial central elements. Could it be that $A_{\alpha,\beta}$ can be deformed to obtain $A_2(\mathbb{C})$? Generally, this is not guaranteed. However, in this particular case, at appropriate choices of α and β, when $q = 1$, we have that $A_{\alpha,\beta}$ is isomorphic to (a quadratic extension of) $A_2(\mathbb{C})$.

In Chapter 4, we explore the derivations of $A_{\alpha,\beta}$. Note, to the best of the author's knowledge, there is no known general results for the derivations of the simple quotients of $U_q^+(\mathfrak{g})$, with the exception of the case where $\mathfrak{g} = \mathfrak{sl}_3$ (also known as the quantum Heisenberg algebra (see for instance [1, §2.2])). In this case, the simple quotients are known to be isomorphic to the GWAs [2, §4] whose derivations have been studied by Kitchin [29, Chapter 7], and are known to be the sum of inner and scalar derivations (note, scalar derivations act on the set of generators of an algebra by multiplication by scalars). More precisely, the first Hochschild cohomology group of the GWAs is of dimension one. We therefore take interest in knowing the derivations of $A_{\alpha,\beta}$, and comparing them to those of $A_2(\mathbb{C})$. Recall that the derivations of $A_2(\mathbb{C})$ are known to be inner. Since $A_{\alpha,\beta}$ and $A_2(\mathbb{C})$ share some common properties, could it be that the derivations of $A_2(\mathbb{C})$ reflect those of $A_{\alpha,\beta}$? In fact, when α and β are non-zero, then similar to $A_2(\mathbb{C})$, the derivations of $A_{\alpha,\beta}$ are all inner. However, if either α or β is zero, then the derivations of $A_{\alpha,\beta}$ are the sum of inner and scalar derivations. Precisely, the first Hochschild cohomology group of $A_{\alpha,\beta}$ has dimension zero when α and β are both non-zero, and dimension one when either α or β is zero. To achieve these results, we use Cauchon's theory of *deleting derivations algorithm* [8] to successively embed $A_{\alpha,\beta}$ into a suitable quantum torus. Now, every derivation of the quantum torus, through the work of Osborn and Passman [40], is known to be the sum of an inner and a scalar/central derivation. Since $A_{\alpha,\beta}$ can be embedded into a quantum torus, we extend every derivation of $A_{\alpha,\beta}$ to a derivation of the quantum torus via localization, and then "pull back" the derivations of the quantum torus to $A_{\alpha,\beta}$, a process Launois and Lopes [36] called *restoring derivations algorithm.* We conclude that every derivation of $A_{\alpha,\beta}$ is inner when α and β are non-zero, and the sum of an inner and a scalar derivation when either α or β is zero.

The final part of the thesis (Chapters 5 and 6) study the Poisson derivations of a semiclassical limit $\mathcal{A}_{\alpha,\beta}$ of $A_{\alpha,\beta}$ by following procedures similar to that of $A_{\alpha,\beta}$. In fact, the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$ are similar to their non-commutative counterparts. That is, every Poisson derivation of $\mathcal{A}_{\alpha,\beta}$ is inner (or hamiltonian) when α and β are non-zero, and the sum of an inner and a scalar Poisson derivation when either α or β is zero.

Notations and conventions

- $\mathbb{K}$ is a field with characteristic zero and $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$.
- $q \in \mathbb{K}^*$ is not a root of unity.
- Let $a, b \in \mathbb{N} := \{0, 1, 2, \cdots\}$. Then, $a \leq i \leq b$ means the set of natural numbers from a to b. We will used " $<$ " instead of " $\leq$ " to indicate the exclusiveness of either a or b.
- $\mathbb{N}_{>a}$ denotes the set of all natural numbers greater than a.
- $q^{\bullet}$ means an arbitrary integer power of q. This symbol will often be used whenever the power of q is of no interest.
- $|\triangle|$ denotes the cardinality of the set $\triangle$.

- $\langle\Theta\rangle_R$ means an ideal generated by the element Θ in the algebra R. Where no doubt arises, we will simply write $\langle\Theta\rangle$.

- $Z(R)$ denotes the center of the algebra R. If R is a Poisson algebra, then we will denote its Poisson center by $Z_P(R)$.

- Spec(R) represents the set of all prime ideals of the algebra R. If R is a Poisson algebra, then we will denote its set of Poisson prime ideals by P.Spec(R).

- Fract(R) denotes the right ring of fractions of the ring R.

- DDA and PDDA represent *deleting derivations algorithm* and *Poisson deleting derivations algorithm* respectively.

Chapter 1

Preliminaries

In this chapter, we present the background materials needed for the subsequent chapters. The subtopics to be considered include root systems, quantized enveloping algebras, Ore and iterated Ore extensions, localization and ring of fractions, quantum affine space and quantum torus, and Cauchon's theory of deleting derivations algorithm.

1.1 Root systems

Most of the materials presented in this section can be found in [25] and [15].

Let a be a non-zero element in the euclidean vector space $\mathbf{E}$, and s_a be a reflection associated to a in $\mathbf{E}$. Recall that $s_a : \mathbf{E} \longrightarrow \mathbf{E}$ is defined by:

$$s_a(b) := b - 2\frac{(a,b)}{(a,a)}a,$$

for all $b \in \mathbf{E}$. Note, $(a,b) = ||a||||b||\cos(\theta)$ and $s_a s_a = s_a^2 = \text{id}$.

1.1.1 Definition. A subset $\Phi \not\ni 0$ of $\mathbf{E}$ is a *root system* in $\mathbf{E}$ if the following axioms are satisfied:

(A1) Φ is finite and spans $\mathbf{E}$.

(A2) If $\alpha \in \Phi$, then the only scalar multiples of α in Φ are α and $-\alpha$.

(A3) If $\alpha \in \Phi$, then the reflection s_α permutes the elements of Φ.

(A4) If $\alpha, \beta \in \Phi$; then $\dfrac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$.

Note, the elements of Φ are called *roots.* A subset Π of Φ is called a *base* for Φ if Π is a basis of $\mathbf{E}$, and for all $\theta \in \Phi$, θ belongs to the span of Π over all non-positive or non-negative integers. When θ belongs to the span of Π over non-negative integers (resp. non-positive integers), we say that θ is a positive root (resp. a negative root). Let us denote the collection of all positive roots (resp. negative roots) by Φ^+ (resp. Φ^-). Then, $\Phi^+ = -\Phi^-$. The roots in Π are called *simple roots.* Again, Φ is *irreducible* if it cannot be written as a disjoint union of two proper subsets Φ_1 and Φ_2 such that $(\alpha_1, \alpha_2) = 0$ for all $\alpha_1 \in \Phi_1$ and $\alpha_2 \in \Phi_2$. Let Π be a base of Φ, it is well known that Φ is irreducible if and only if Π is irreducible (for example, see [25, Section 10.4]). Every root system Φ can be written as a disjoint union of some irreducible root systems. As a result, we will only focus on the irreducible root systems since they are the building blocks for root systems. If Φ is irreducible, then every root of Φ is either of the same length (usually referred to as simply laced) or of two lengths: short and long (usually referred to as non-simply laced). The short root is normally normalized to have a length of $\sqrt{2}$. The dimension of the euclidean vector space $\mathbf{E}$ (denoted by dim$(\mathbf{E})$) is called the *rank* of Φ (denoted by rk(Φ)). Since Π is a basis of $\mathbf{E}$, the cardinality of Π (denoted by $|\Pi|$) coincides with the dimension of $\mathbf{E}$. As a result, we have the following equality: dim$(\mathbf{E}) = |\Pi| =$ rk$(\Phi) = n$.

One important property of irreducible root systems is that, to every irreducible root system, one can associate one of the following complex simple Lie algebras: A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, E_n $(n = 6, 7, 8)$, F_4 and G_2 [25, Section 11.4]. Moreover, given any of these simple Lie algebras, one can also construct an irreducible root system associated to the simple Lie algebra [25, Section 12.1]. Note, each of A_n, B_n, C_n, D_n and E_n is the notation for a collection of a series of simple Lie algebras, and n is the rank of the associated Lie algebra or the root system. Of course, F_4 and G_2 are of rank 4 and 2 respectively. The roots of A_n, D_n, E_6, E_7 and E_8 are all simply laced, and the rest are non-simply laced. For the purpose of this thesis, we will only discuss the root system for G_2. We do this in line with the presentation in [25, Section 12.1]. Let **E** be a 2-dimensional subspace of $\mathbb{R}^3$, and $\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3$ denote the standard unit vectors of $\mathbb{R}^3$. Set $\alpha_1 := \hat{\varepsilon}_2 - \hat{\varepsilon}_1$ and $\alpha_2 = 2\hat{\varepsilon}_1 - \hat{\varepsilon}_2 - \hat{\varepsilon}_3$. One can verify that the set $\Phi = \pm\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$ is a root system for G_2 in **E**. Moreover, one can also observe that the set $\{\alpha_1, \alpha_2\}$ is a base for Φ. Hence, α_1 and α_2 are the simple roots of G_2. The root system of G_2 is non-simply laced since the simple roots α_1 and α_2 are of different lengths: short root (α_1) and long root (α_2), which are of lengths $\sqrt{2}$ and $\sqrt{6}$ respectively. That is, $(\alpha_1, \alpha_1) = ||\alpha_1||^2 = 2$ and $(\alpha_2, \alpha_2) = ||\alpha_2||^2 = 6$. In total, G_2 has 12 roots of which 6 are short and the other 6 are long. Furthermore, it has 6 positive roots and 6 negative roots. The obtuse angle between the two simple roots α_1 and α_2 is $5\pi/6$. The root diagram of G_2 with only the positive roots is shown in the figure below.

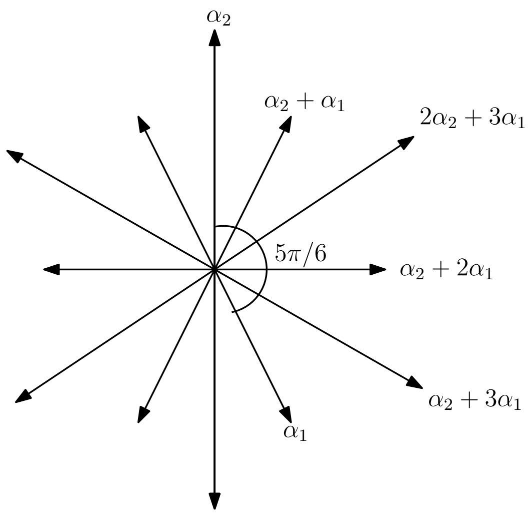


Figure 1.1: G_2 root diagram.

In the rest of this section and beyond, $\mathfrak{g}$ will denote a complex simple Lie algebra, and Φ will represent the root system of $\mathfrak{g}$. Fix $\mathfrak{g}$ with rank n. We are going to discuss another concept called the *Weyl group* of $\mathfrak{g}$. Let Φ denote a root system of $\mathfrak{g}$, and Π denotes the set of simple roots of Φ. For every root $\alpha_i \in \Phi$, one can associate a reflection s_{α_i}. The collection of all these reflections s_{α_i} forms a group called the *Weyl group* of $\mathfrak{g}$. We denote this group by $\mathscr{W}$. Moreover, if $\alpha_i \in \Pi$, then the associated reflection s_{α_i} is called *a simple reflection*. The set of all the simple reflections generate $\mathscr{W}$. That is,

$\mathscr{W} := \langle s_{\alpha_i} \mid \alpha_i \in \Pi \rangle$.

Let $\alpha_i, \alpha_j \in \Pi$ (note, i and j are non-zero natural numbers); and set

$$\eta_{ij} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}. \tag{1.1.1}$$

It is well known that $\eta_{ij}\eta_{ji} \in \{0, 1, 2, 3\}$ (see [15, Lemma 11.4] for further details). Let $s_{\alpha_i}, s_{\alpha_j} \in \mathscr{W}$; where $\alpha_i, \alpha_j \in \Pi,\ i \neq j$. There exists $m_{ij} \in \mathbb{Z}$ such that $(s_{\alpha_i} s_{\alpha_j})^{m_{ij}} = s_{\alpha_i}^2 = 1$, where m_{ij} is the order of $s_{\alpha_i} s_{\alpha_j}$ in $\mathscr{W}$, and 1 is the identity element of $\mathscr{W}$. Again, for $\eta_{ij}\eta_{ji} = 0, 1, 2, 3$, we have that $m_{ij} = 2, 3, 4, 6$ respectively [7, Page 41].

Set $s_i := s_{\alpha_i}$. If $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ is the set of simple roots of Φ, then

$$\mathscr{W} = \langle s_1, \cdots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1, \quad i \neq j \rangle.$$

The relations $(s_i s_j)^{m_{ij}} = 1$ and $s_i^2 = 1$ imply that

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots .$$

Each side of the equality has exactly m_{ij} factors. The Weyl group $\mathscr{W}$ acts on Φ as follows:

$$s_i(\alpha_j) = \alpha_j - \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\alpha_i = \alpha_j - \eta_{ij}\alpha_i,$$

where $s_i \in \mathscr{W}$ and $\alpha_j \in \Phi$. This action just permutes the elements of Φ as stated in axiom A3.

Denote the cardinality of the set of positive roots Φ^+ of Φ by N, and let $\omega \in \mathscr{W}$. Since ω is generated by s_i, one can write ω as a product of simple reflections. Let $r \in \mathbb{N}$ be minimum such that ω can be written as $\omega = s_{i_1} s_{i_2} \cdots s_{i_r}$. We call r the *length* of ω, and the notation $r = l(\omega)$ will often be used. It is well known that $l(\omega) \leq N$. Furthermore, there exists a unique element $\omega_0 \in \mathscr{W}$ such that $l(w_0) = N$. The element ω_0 is called the *longest element* of $\mathscr{W}$. One can recover the positive roots of Φ from any reduced decomposition of ω_0. That is, if $\omega_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$, then the positive roots of Φ

are exactly $\beta_1, \cdots, \beta_N$ defined by

$$\beta_r := s_{i_1} s_{i_2} \cdots s_{i_{r-1}}(\alpha_{i_r}) \quad (1 \leq r \leq N).$$

For example, given the set of simple roots $\{\alpha_1, \alpha_2\}$ of G_2 above, the associated set of simple reflections is $\{s_1, s_2\}$. One can easily confirm that $\eta_{12}\eta_{21} = 3$, hence the order of $s_1 s_2$ in the Weyl group $\mathscr{W}$ of G_2 is 6. Therefore, $\mathscr{W} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^6 = 1 \rangle$. Moreover, $|\mathscr{W}| = |\Phi| = 12$. Given that $(s_1 s_2)^6 = 1$ and $s_1^2 = s_2^2 = 1$, it follows that $s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 = 1$. This implies that $s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1$. This expression gives the longest and unique element ω_0 of $\mathscr{W}$. Thus, $\omega_0 = s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1$. Note, $l(\omega_0) = N = 6$ as expected. Choose $\omega_0 = s_1 s_2 s_1 s_2 s_1 s_2$. Then, given the relation $s_i(\alpha_j) = \alpha_j - \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\alpha_i$, one can equally confirm that the positive roots of G_2 (see Figure 1.1) are as follows:

$$\begin{aligned} &\beta_1 = \alpha_1 && \beta_4 = s_1 s_2 s_1(\alpha_2) = 2\alpha_2 + 3\alpha_1 \\ &\beta_2 = s_1(\alpha_2) = \alpha_2 + 3\alpha_1 && \beta_5 = s_1 s_2 s_1 s_2(\alpha_1) = \alpha_2 + \alpha_1 \\ &\beta_3 = s_1 s_2(\alpha_1) = \alpha_2 + 2\alpha_1 && \beta_6 = s_1 s_2 s_1 s_2 s_1(\alpha_2) = \alpha_2. \end{aligned}$$

1.2 Quantized enveloping algebras

In this section, we present the defining relations and a class of automorphisms (called the *braid group* introduced by Lusztig [38]) of the quantized enveloping algebra $U_q(\mathfrak{g})$ of a finite dimensional complex simple Lie algebra $\mathfrak{g}$. Unless otherwise stated, we follow the presentations and conventions in [7, Chapter I.6]. For further details of the material presented here, [7, Chapter I.6] and [26, Chapter 8] will be helpful. Throughout this section, we assume that q is not a root of unity.

1.2.1 Definition. Let v be an indeterminate and $m,\ t \in \mathbb{N}$; we have the following definitions:

(i). $[t]_v := \dfrac{v^t - v^{-t}}{v - v^{-1}} = v^{t-1} + v^{t-3} + \cdots + v^{1-t}$.

(ii). $[m]_v! := [m]_v[m-1]_v \cdots [1]_v$. By convention, $[0]_v! := 1$.

(iii). for all integers $0 \leq i \leq m$, we have that:

$$\begin{bmatrix} m \\ i \end{bmatrix}_v := \frac{[m]_v!}{[i]_v![m-i]_v!} = \frac{(v^m - v^{-m})(v^{m-1} - v^{1-m}) \cdots (v - v^{-1})}{(v^i - v^{-i}) \cdots (v - v^{-1})(v^{m-i} - v^{i-m}) \cdots (v - v^{-1})}.$$

Now; $[t]_v$, $[m]_v!$ and $\begin{bmatrix} m \\ i \end{bmatrix}_v$ are all elements of the ring $\mathbb{Z}[v, v^{-1}]$. Particularly, $\begin{bmatrix} m \\ i \end{bmatrix}_v$ are called the v-*binomial coefficients*, and they are the v-analogue of the usual binomial coefficients. They can be evaluated at any given values of v. If $v = q$, with $q \in \mathbb{K}^*$, then we have the usual q-*binomial coefficients*.

Let $\mathfrak{g}$ denote a finite dimensional complex simple Lie algebra of rank n. Recall from (1.1.1) that $\eta_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. The square matrix $\mathcal{C}$ defined by $\mathcal{C} := (\eta_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{Z})$ is called the *Cartan matrix* of $\mathfrak{g}$. For example, one can verify that the Cartan matrix of G_2 has the following entries: $\eta_{11} = \eta_{22} = 2$, $\eta_{12} = -3$, and $\eta_{21} = -1$. In fact, the diagonal entries of every Cartan matrix are 2 and the off-diagonal entries are non-positive integers.

Let $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ represent the set of simple roots of $\mathfrak{g}$. For all $i \in \{1, \cdots, n\}$, set

$$q_i := q^{\frac{(\alpha_i, \alpha_i)}{2}}.$$

The quantized enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ is a $\mathbb{K}-$algebra generated by $E_1, \cdots, E_n$, $F_1, \cdots, F_n$ and $K_1^{\pm 1}, \cdots, K_n^{\pm 1}$ subject to the following relations:

$$K_i E_j = q_i^{\eta_{ij}} E_j K_i \qquad K_i F_j = q_i^{-\eta_{ij}} F_j K_i$$

$$E_i F_j = F_j E_i + \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \qquad K_i K_j = K_j K_i$$

$$\text{(QSR1)} \qquad \sum_{l=0}^{1-\eta_{ij}} (-1)^l \begin{bmatrix} 1-\eta_{ij} \\ l \end{bmatrix}_{q_i} F_i^{1-\eta_{ij}-l} F_j F_i^l = 0 \quad i \neq j$$

$$\text{(QSR2)} \qquad \sum_{l=0}^{1-\eta_{ij}} (-1)^l \begin{bmatrix} 1-\eta_{ij} \\ l \end{bmatrix}_{q_i} E_i^{1-\eta_{ij}-l} E_j E_i^l = 0 \quad i \neq j.$$

The relations QSR1 and QSR2 are called the quantum Serre relations.

Recall that given the Lie algebra $\mathfrak{g}$, there is a triangular decomposition for $\mathfrak{g}$ as $\mathfrak{g}^+ \oplus \mathfrak{h} \oplus \mathfrak{g}^-$. The associated decomposition for the enveloping algebra $U(\mathfrak{g})$ is of the form $U(\mathfrak{g}^+) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}^-)$, where $U(\mathfrak{g}^+), U(\mathfrak{g}^-)$ and $U(\mathfrak{h})$ are called the positive nilpotent, negative nilpotent and Cartan subalgebras of $U(\mathfrak{g})$ respectively. Similarly, $U_q(\mathfrak{g})$ admits a triangular decomposition of the form $U_q^+(\mathfrak{g}) \otimes U^0(\mathfrak{h}) \otimes U_q^-(\mathfrak{g})$. One can understand $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$ as the q-deformations of the subalgebras $U(\mathfrak{g}^+)$ and $U(\mathfrak{g}^-)$ respectively. The sets $\{E_1, \cdots, E_n\}$, $\{F_1, \cdots, F_n\}$ and $\{K_1^{\pm 1}, \cdots, K_n^{\pm 1}\}$ generate the subalgebras $U_q^+(\mathfrak{g}), U_q^-(\mathfrak{g})$ and $U^0(\mathfrak{h})$ respectively. Furthermore, $U_q^-(\mathfrak{g})$ and $U_q^+(\mathfrak{g})$ satisfy the relations QSR1 and QSR2 respectively. The subalgebra $U^0(\mathfrak{h})$ is commutative. There is a unique automorphism of $U_q(\mathfrak{g})$ that maps E_i to F_i, F_i to E_i and K_i to K_i^{-1} (see [26, Lemma 4.6] or [7, Lemma I.6.4]). Finally, $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$ are isomorphic (for example, see [45, §2]).

We now present a subgroup of the automorphism group of $U_q(\mathfrak{g})$ called the *braid group*. The braid group was introduced by Lusztig [38].

Braid group. Recall that $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ is the set of simple roots of $\mathfrak{g}$. Set $T_i := T_{\alpha_i}$, where $1 \leq i \leq n$. Given the Weyl group

$$\mathscr{W} = \langle s_1, \cdots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1 \Rightarrow s_i s_j s_i \cdots = s_j s_i s_j \cdots, \quad i \neq j \rangle,$$

the associated braid group is given by

$$\mathscr{B}_{\mathscr{W}} = \langle T_1, \cdots, T_n \mid T_i T_j T_i \cdots = T_j T_i T_j \cdots, \quad i \neq j \rangle.$$

Note, each side of the equality $T_i T_j T_i \cdots = T_j T_i T_j \cdots$ has exactly m_{ij} factors. The map $\varphi : \mathscr{B}_{\mathscr{W}} \longrightarrow \mathscr{W}$ defined by $\varphi(T_i) = s_i$ is an epimorphism. Note, $s_i^2 = 1$ in $\mathscr{W}$, however,

T_i^2 is not necessarily 1 in $\mathscr{B}_{\mathscr{W}}$. In line with [26, Chapter 8], Lusztig defines an action of $\mathscr{B}_{\mathscr{W}}$ by automorphism on $U_q(\mathfrak{g})$ as follows:

$$T_i(E_i) = -F_iK_i \qquad T_i(E_j) = \sum_{r=0}^{-\eta_{ij}} (-1)^r q_i^{-r} E_i^{(-\eta_{ij}-r)} E_j E_i^{(r)} \qquad i \neq j$$

$$T_i(F_i) = -K_i^{-1}E_i \qquad T_i(F_j) = \sum_{r=0}^{-\eta_{ij}} (-1)^r q_i^{r} F_i^{(r)} F_j F_i^{(-\eta_{ij}-r)} \qquad i \neq j$$

$$T_i(K_\alpha) = K_{s_i(\alpha)}, \;\; \alpha \in Q,$$

where $E_i^{(r)} := \dfrac{E_i^r}{[r]_{q_i}!}$, $F_i^{(r)} := \dfrac{F_i^r}{[r]_{q_i}!}$ and Q (the root lattice) denotes the $\mathbb{Z}-$span of Π. Hence, $\alpha = z_1\alpha_1 + \cdots + z_n\alpha_n \in Q$, with $z_1, \cdots, z_n \in \mathbb{Z}$, and $K_\alpha := K_1^{z_1} \cdots K_n^{z_n}$. Note, one can also refer to [26, Chapter 8] for the actions of the inverse automorphisms T_i^{-1} on $U_q(\mathfrak{g})$.

From any reduced decomposition of ω_0 (discussed in Section 1.1), one can construct distinguished elements $E_{\beta_1}, \cdots, E_{\beta_N}$ of $U_q(\mathfrak{g})$ as follows:

$$E_{\beta_r} = T_{i_1} \cdots T_{i_{r-1}}(E_{i_r}), \quad 1 \leq r \leq N. \tag{1.2.1}$$

The elements E_{β_r} depend on the reduced decomposition for ω_0 [32, §1.2]. Levendorskii and Soibelman [37] proved the result below (also, see [32, Theorem 1.1]).

1.2.2 Theorem. *(1) The element E_{β_r} belongs to $U_q^+(\mathfrak{g})$ for all $r \in \{1, \cdots, N\}$.*

(2) If $\beta_r = \alpha_i$, then $E_{\beta_r} = E_i$.

(3) The family of monomials $(E_{\beta_1}^{r_1} \cdots E_{\beta_N}^{r_N})_{r_1,\cdots,r_N \in \mathbb{N}}$ is a linear basis of $U_q^+(\mathfrak{g})$.

(4) For all $1 \leq i < j \leq N$, we have:

$$E_{\beta_j}E_{\beta_i} - q^{-(\beta_i,\beta_j)}E_{\beta_i}E_{\beta_j} = \sum a_{r_{i+1},\cdots,r_{j-1}} E_{\beta_{i+1}}^{r_{i+1}} \cdots E_{\beta_{j-1}}^{r_{j-1}},$$

with $a_{r_{i+1},\cdots,r_{j-1}} \in \mathbb{K}$.

Technically, the basis in point (3) is called a PBW-basis of $U_q^+(\mathfrak{g})$, where PBW stands for Poincaré-Birkhoff-Witt. Also, as a corollary to this theorem, we have that the family of monomials $(F_{\beta_1}^{s_1}\cdots F_{\beta_N}^{s_N}K_\alpha E_{\beta_1}^{r_1}\cdots E_{\beta_N}^{r_N})_{s_i,r_i\in\mathbb{N},\alpha\in Q}$ form a PBW-basis of $U_q(\mathfrak{g})$ [7, §I.6.8].

1.2.3 Remark. Recall the definitions of ω and ω_0 in Section 1.1. It follows from [7, §I.6.8] that $U_q^+(\omega) := \mathbb{K}\langle E_{\beta_1},\cdots,E_{\beta_r}\rangle$ is a subalgebra of $U_q^+(\mathfrak{g})$ for each $1\leq r\leq N$, and $U_q^+(\omega_0)=U_q^+(\mathfrak{g})$.

1.3 Ore extension

Let A be a $\mathbb{K}-$algebra, σ be an automorphism of A and δ be a $\mathbb{K}-$linear map from A to A. Then,

- δ is a derivation if $\delta(xy)=x\delta(y)+\delta(x)y,\ \forall x,y\in A$.
- δ is a $\sigma-$derivation if $\delta(xy)=\sigma(x)\delta(y)+\delta(x)y,\ \forall x,y\in A$.

Observe that when $\sigma=$ id, the $\sigma-$derivation and derivation coincide. One can easily verify that $\delta(1)=0$. From [22, Chapter 2], we have the presentations in the following definition.

1.3.1 Definition. Let σ and δ denote an automorphism and a $\sigma-$derivation of a ring A respectively. Denote $R:=A[x;\sigma,\delta]$ to mean the following:

- R is a ring containing the element x,
- A is a subring of R,
- R is a free left $A-$module with basis $(x^i)_{i\in\mathbb{N}}$,
- $xa=\sigma(a)x+\delta(a)$ for all $a\in A$.

The ring R is called an Ore extension of A or a skew polynomial ring over A. When $\sigma=$ id, then $A[x;\text{id},\delta]$ is simply written as $A[x;\delta]$ and when $\delta=0$, then $A[x;\sigma,0]$ is simply written as $A[x;\sigma]$. The Ore extension described above can easily be iterated to obtain an *iterated Ore extension* of A, written as:

$$R = A[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]\cdots[x_n;\sigma_n,\delta_n],$$

where σ_i and δ_i are the automorphisms and σ_i-derivations of
$A[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]\cdots[x_{i-1};\sigma_{i-1},\delta_{i-1}]$ respectively for all $i \in \{1,\cdots,n\}$.
Note, $A[x_0;\sigma_0,\delta_0] := A$.

1.3.2 Example. (1) The quantum plane $\mathbb{K}_q[x,y]$, with $xy = qyx$, can be presented as an iterated Ore extension of the form $\mathbb{K}[y][x,\sigma]$, where σ is an automorphism of the polynomial ring $\mathbb{K}[y]$ defined by $\sigma(y) = qy$.

(2) The quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ is generated by E, F and $K^{\pm 1}$; and satisfies the following relations:

$$KE = q^2EK, \qquad FK = q^2KF,$$
$$KK^{-1} = K^{-1}K = 1, \qquad FE = EF + (K^{-1} - K)/(q^{-1} - q).$$

An iterated Ore extension of $U_q(\mathfrak{sl}_2)$ can be written as:

$$U_q(\mathfrak{sl}_2) = \mathbb{K}[E][K^{\pm 1};\sigma_1][F;\sigma_2.\delta_2],$$

where $\sigma_1(E) = q^2E$, $\sigma_2(K) = q^2K$, $\sigma_2(E) = E$, $\delta_2(K) = 0$ and $\delta_2(E) = (K^{-1} - K)/(q^{-1} - q)$. Note, σ_1 and σ_2 are automorphisms of the subalgebras $\mathbb{K}[E]$ and $\mathbb{K}[E][K^{\pm 1};\sigma_1]$ respectively, and δ_2 is a σ_2-derivation of $\mathbb{K}[E][K^{\pm 1};\sigma_1]$.

One can refer to [22, Chapter 2] for more details on Ore and iterated Ore extensions, but we recall Hibert Basis Theorem below.

1.3.3 Theorem. *[7, Lemma I.1.12 & Theorem I.1.13]. Let $R := A[x;\sigma,\delta]$ be a skew polynomial ring, where σ and δ are automorphism and $\sigma-$derivation of the ring A respectively.*

1. *If A is left (right) noetherian then R is also left (right) noetherian [Hibert Basis Theorem].*

2. *If A is a domain then R is also a domain.*

1.3.4 Remark. $U_q^+(\mathfrak{g})$ can be presented as an iterated Ore extension of the form:

$$U_q^+(\mathfrak{g}) = \mathbb{K}[E_{\beta_1}][E_{\beta_2}; \sigma_2, \delta_2] \cdots [E_{\beta_N}; \sigma_N, \delta_N]$$

(see for instance [7, Section I.6.10]), where σ_i is an automorphism of $\mathbb{K}[E_{\beta_1}][E_{\beta_2}; \sigma_2, \delta_2] \cdots [E_{\beta_{i-1}}; \sigma_{i-1}, \delta_{i-1}]$ defined as $\sigma_i(E_{\beta_j}) = q^{-(\beta_j, \beta_i)} E_{\beta_j}$ and δ_i is a σ_i-derivation defined as $\delta_i(E_{\beta_j}) = E_{\beta_i} E_{\beta_j} - q^{-(\beta_j, \beta_i)} E_{\beta_j} E_{\beta_i}$ for all $1 \leq j < i \leq N$. As a result, $U_q^+(\mathfrak{g})$ is a noetherian domain since $\mathbb{K}$ is a noetherian domain. From [32, §1.2], the group of invertible elements of $U_q^+(\mathfrak{g})$ is reduced to $\mathbb{K}^*$.

1.4 Localization and rings of fractions

The materials presented in this section are well known, nonetheless, we follow the presentations in [22, Chapters 6 and 10]. As a result, further details can be read from this reference.

Let S be a subset of a $\mathbb{K}-$algebra A such that $1 \in S$, then S is said to be a *multiplicative set* if it is closed under multiplication. The collection of all non-zero divisors (also known as regular elements) in A forms a multiplicative set. Given a multiplicative set S of A, if for each $s \in S$ and $a \in A$, there exists $t \in S$ and $b \in A$ such that $sb = at$ (or $sA \cap aS \neq \emptyset$), then S is said to satisfy the *right Ore condition.* A multiplicative set S which satisfies the right Ore condition is called *right Ore set.* Symmetrically, one can also define the *left Ore condition* and *left Ore set*. A multiplicative set which satisfies both the right and left Ore condition is called an *Ore set.* In a commutative ring, every multiplicative set is an Ore set. A multiplicative set S is said to be *right reversible* if for $a \in A$ and $s \in S$ with $sa = 0$ then there exists $s' \in S$ such that $as' = 0$. A right reversible right Ore set is called a *right denominator set*. A left denominator set is defined symmetrically. A *denominator set* is any right and left denominator set. Suppose that S is a right Ore set. Then, the set $AS^{-1} := \{as^{-1} \mid s \in S \text{ and } a \in A\}$ is called the *right ring of fractions* or *right Ore localization.* Observe that every element of S is a unit in AS^{-1} and A is a subset of AS^{-1}. The *left ring of fractions* or *left Ore localization* is defined symmetrically.

1.4.1 Example.

1. If A is a noetherian domain, then $S := A \setminus \{0\}$ satisfies the Ore condition, hence, $AS^{-1} = \text{Fract}(A)$ is a right ring of fractions of A.

2. If A is a domain and $x \in A$ is normal (i.e. $xA = Ax$), then $S = \{\lambda x^i \mid \lambda \in \mathbb{K}^*,\ i \in \mathbb{N}\}$ is an Ore set. As a result, one can define $A[x^{\pm 1}, \sigma]$ as $A[x, \sigma]$ localized at S. That is, $A[x^{\pm 1}, \sigma] := A[x, \sigma]S^{-1}$.

1.5 Quantum affine space and quantum torus

Let $N \in \mathbb{N}$. Given a multiplicatively skew-symmetric matrix $\Lambda = (a_{ij}) \in M_N(\mathbb{K}^*)$, one can define a $\mathbb{K}-$algebra $\mathcal{O}_\Lambda(\mathbb{K}^N)$ associated to Λ as follows:

$$\mathcal{O}_\Lambda(\mathbb{K}^N) = \mathbb{K}_\Lambda[Y_1, \cdots, Y_N],$$

where $Y_jY_i = a_{ji}Y_iY_j$ for all $1 \leq i, j \leq N$. The algebra $\mathcal{O}_\Lambda(\mathbb{K}^N)$ is called a *quantum affine space* of rank N associated to Λ. The iterated Ore extension of $\mathcal{O}_\Lambda(\mathbb{K}^N)$ can be written as:

$$\mathcal{O}_\Lambda(\mathbb{K}^N) = \mathbb{K}[Y_1][Y_2; \sigma_2] \cdots [Y_N; \sigma_N],$$

where σ_j is an automorphism of $\mathbb{K}[Y_1][Y_2; \sigma_2] \cdots [Y_{j-1}; \sigma_{j-1}]$ defined by $\sigma_j(Y_i) = a_{ji}Y_i$ for all $1 \leq i < j \leq N$.

Since $Y_1, \cdots, Y_N$ are all normal elements of $\mathcal{O}_\Lambda(\mathbb{K}^N)$, the set $S := \{\lambda Y_1^{k_1} \cdots Y_N^{k_N} \mid \lambda \in \mathbb{K}^* \text{ and } k_1, \cdots, k_N \in \mathbb{N}\}$ is an Ore set of $\mathcal{O}_\Lambda(\mathbb{K}^N)$. One can therefore localize

$\mathcal{O}_\Lambda(\mathbb{K}^N)$ at S as follows:

$$\mathcal{O}_\Lambda(\mathbb{K}^*)^N := \mathcal{O}_\Lambda(\mathbb{K}^N)S^{-1} = \mathbb{K}_\Lambda[Y_1^{\pm 1}, \cdots, Y_N^{\pm 1}].$$

Now, the localization $\mathcal{O}_\Lambda(\mathbb{K}^*)^N$ is called a *quantum torus* of rank N. Furthermore, $\mathcal{O}_\Lambda((\mathbb{K}^*)^N)$ can be written as an iterated Ore extension of the form:

$$\mathcal{O}_\Lambda((\mathbb{K}^*)^N) = \mathbb{K}[Y_1^{\pm 1}][Y_2^{\pm 1}; \sigma_2] \cdots [Y_N^{\pm 1}; \sigma_N],$$

where σ_j is an automorphism of $\mathbb{K}[Y_1^{\pm 1}][Y_2^{\pm 1}; \sigma_2] \cdots [Y_{j-1}^{\pm 1}; \sigma_{j-1}]$ defined by $\sigma_j(Y_i) = a_{ji}Y_i$ for all $1 \leq i < j \leq N$.

A basis of $\mathcal{O}_\Lambda((\mathbb{K}^*)^N)$ is given by the family $(Y_1^{k_1} \cdots Y_N^{k_N})_{k_1, \cdots, k_N \in \mathbb{Z}}$. If we restrict the powers $k_1, \cdots, k_N$ to only $\mathbb{N}$, then $(Y_1^{k_1} \cdots Y_N^{k_N})$ becomes a basis of $\mathcal{O}_\Lambda(\mathbb{K}^N)$.

1.5.1 Remark. Given the finite dimensional complex simple Lie algebra $\mathfrak{g}$, recall from Section 1.1 that one can construct a set $\{\beta_1, \cdots, \beta_N\}$ of positive roots of $\mathfrak{g}$. Set

$$\mu_{ij} := \begin{cases} (\beta_i, \beta_j) & \text{if } i < j \\ 0 & \text{if } i = j \\ -(\beta_j, \beta_i) & \text{if } i > j. \end{cases} \tag{1.5.1}$$

The matrix $\Lambda = (q^{\mu_{ij}}) \in M_N(\mathbb{K}^*)$ is a multiplicatively skew-symmetric matrix (i.e. $q^{\mu_{ij}}q^{\mu_{ji}} = q^{\mu_{ii}} = 1$ for all $1 \leq i, j \leq N$). As a result, one can define a quantum affine space or a quantum torus associated to Λ.

1.6 Prime spectrum

Let P be a proper ideal of a ring A and I_1, I_2 be ideals of A with $I_1I_2 \subseteq P$, the ideal P is called a *prime ideal* provided $P \supseteq I_1$ or $P \supseteq I_2$. A proper ideal P is said to be *completely prime* if $xy \in P$ implies that $x \in P$ or $y \in P$ for all $x, y \in A$. All completely prime ideals are prime ideals (see comments after [31, Theorem 12.5]), however, the converse is not always true. An ideal can be prime but not completely prime. For example, the zero ideal $\langle 0 \rangle$ of $M_2(\mathbb{Z})$ is prime but not completely prime.

Take

$$x = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}.$$

Clearly, $xy \in \langle 0 \rangle$ but $x \notin \langle 0 \rangle$ and $y \notin \langle 0 \rangle$. The collection of all the prime ideals of A is called the *prime spectrum* of A, denoted by Spec(A).

Let M be a right simple $A-$module. The set $\text{ann}_A(M) := \{a \mid ma = 0, \ \forall m \in M\}$ is called the *right annihilator* of M. Now, $\text{ann}_A(M)$ forms an ideal of A called the *right primitive ideal.* The left primitive ideal is defined symmetrically. Finally, an ideal of a ring A is *maximal* if it is not contained in any proper ideal of A. The set Prim(A) is the collection of all the (left/right) primitive ideals of A called the *primitive spectrum* of A, and the set MaxSpec(A) is the collection of all the maximal ideals of A called the *maximal spectrum* of A. We have the following set inclusions: MaxSpec$(A) \subseteq$ Prim(A) $\subseteq$ Spec(A) for any arbitrary ring A [22, Proposition 3.15].

Again, let A be a ring and $P_0, P_1, \cdots, P_n \in$ Spec(A) such that $P_0 \subset P_1 \subset \cdots \subset P_n$ is a chain of prime ideals of A. The number of strict inclusions n that we have in the chain is called the *length* of the chain. That is, n is the number of P_i's in the chain minus 1. A chain is *saturated* if no prime ideal can be included in the chain. Furthermore, let $P \in$ Spec(A). The height of P denoted by ht(P) is the supremum of the lengths of all the chains of prime ideals contained in P. Finally, let P and Q be two distinct prime ideals of A such that $P \subset Q$. Then, A is said to be *catenary* if all saturated chains of prime ideals from P to Q have the same length for any fixed choice of P and Q. For example, $U_q^+(\mathfrak{g})$ is catenary [19, Theorem 4.8].

1.6.1 Remark. [42, Corollary of Theorem 3] If q is not a root of unity, then all prime ideals of $U_q^+(\mathfrak{g})$ are completely prime.

1.6.2 Rational torus action. In this section we define rational torus actions in line with the presentations in [7, Chapter II.2]. Let the group $\mathcal{H}$ act on the $\mathbb{K}-$algebra A via $\mathbb{K}-$algebra automorphisms. Let x be a non-zero element of A. Then, x is said to be a $\mathcal{H}-$*eigenvector* if and only if $h \cdot x \in \mathbb{K}x$ for all $h \in \mathcal{H}$. If a torus $\mathcal{H}$ acts on an algebra A such that the generators of A are simultaneous eigenvectors for all $h \in \mathcal{H}$, then the action of $\mathcal{H}$ on A is said to be *semisimple.*

1.6.3 Example. (1) The torus $\mathcal{H} = (\mathbb{K}^*)^2$ acts on the quantum plane $\mathbb{K}_q[x, y]$ by automorphisms as: $(\alpha, \beta) \cdot x^i y^j = \alpha^i \beta^j x^i y^j$ for all $(\alpha, \beta) \in \mathcal{H}$ and $i, j \geq 0$. This action is semisimple since each monomial $x^i y^j$ is an $\mathcal{H}-$eigenvector for all $i, j \geq 0$.

(2) In general, the torus $\mathcal{H} = (\mathbb{K}^*)^N$ acts on the quantum affine space $\mathbb{K}_\Lambda[Y_1, \cdots, Y_N]$ by automorphisms as follows:

$$(\alpha_1, \cdots, \alpha_N) \cdot Y_i = \alpha_i Y_i,$$

for all $i \in \mathbb{N}$ and $(\alpha_1, \cdots, \alpha_N) \in \mathcal{H}$. This action extends uniquely to an action of the quantum torus, and it is semisimple.

Given an eigenvector, another terminology that obviously comes to mind is an eigenvalue. Therefore, given a $\mathcal{H}-$eigenvector x, the $\mathcal{H}-$eigenvalue of x is a group homomorphism $\lambda : \mathcal{H} \longrightarrow \mathbb{K}^*$ such that $h \cdot x = \lambda(h)x$ for all $h \in \mathcal{H}$. For instance, the group homomorphism $(\alpha, \beta) \longmapsto \alpha^i \beta^j$ is $\mathcal{H}-$eigenvalue of the $\mathcal{H}-$eigenvector $x^i y^j$ in Example 1.6.3. Given an $\mathcal{H}-$eigenvalue λ, the set $A_\lambda := \{x \in A \mid h \cdot x = \lambda(h)x, \forall h \in \mathcal{H}\}$ is called the $\mathcal{H}-$*eigenspace* of A provided it is non-zero.

The $\mathcal{H}-$eigenvalues λ are called *characters* of $\mathcal{H}$. If $\mathcal{H}$ is an algebraic group over $\mathbb{K}$ then a character λ of $\mathcal{H}$ which is also a morphism of algebraic varieties is called a *rational character.* Suppose that $\mathbb{K}$ is infinite, then it follows from [7, Theorem II.2.7] that an action of $\mathcal{H}$ on A is *rational* if and only if it is a semisimple action and the corresponding $\mathcal{H}-$eigenvalues are all rational. One can easily verify that all the torus actions in Example 1.6.3 are rational actions. For the purpose of our studies, this discussion on rational torus action is enough. However, one can read [7, Chapter II.2] for further details. Specifically, see [7, Definition II.2.6] for the general definition of rational torus action. We deduce the remark below from [4, §3.3].

1.6.4 Remark. The torus $\mathcal{H} = (\mathbb{K}^*)^n$ acts rationally on $U_q^+(\mathfrak{g})$ by automorphisms via:

$$(\alpha_1, \cdots, \alpha_n) \cdot E_i = \alpha_i E_i,$$

for all $1 \leq i \leq n$ and $(\alpha_1, \cdots, \alpha_n) \in \mathcal{H}$. Recall; $E_1, \cdots, E_n$ are the standard generators of $U_q^+(\mathfrak{g})$, where n is the rank of $\mathfrak{g}$.

We now present the notion of $\mathcal{H}-$stratification by Goodearl and Letzter [21] which provides a partition of the prime spectrum of a given ring into a disjoint union of strata. More importantly, the $\mathcal{H}-$stratification also helps to study the primitive ideals of a ring.

1.6.5 $\mathcal{H}-$Stratification. Let A be a $\mathbb{K}-$algebra and I be any ideal of A. If $h \cdot I = I$ for all $h \in \mathcal{H}$, then I is said to be $\mathcal{H}-$*invariant ideal* of A. A prime $\mathcal{H}-$invariant ideal is called $\mathcal{H}-$*invariant prime ideal* (or $\mathcal{H}-$*prime ideal* for short). Also, $(I : \mathcal{H}) := \bigcap_{h\in\mathcal{H}} h \cdot I$ is the largest $\mathcal{H}-$invariant ideal contained in I. Let $\mathcal{H}-$Spec(A) represent the collection of all the $\mathcal{H}-$invariant prime ideals of A. For $J \in \mathcal{H}-$Spec(A), the set Spec$_J(A) :=$ $\{P \in$ Spec$(A) \mid (P : \mathcal{H}) = J\}$ is called the $\mathcal{H}-$*stratum* of Spec(A) *associated to* J. One can simply call it the $J-$*stratum* of Spec(A). The collection of all these $J-$strata forms a partition of Spec(A). That is,

$$\text{Spec}(A) = \bigsqcup_{J\in\mathcal{H}-\text{Spec}(A)} \text{Spec}_J(A).$$

This partition is known as the $\mathcal{H}-$*stratification* of Spec(A). Note, given a non-zero algebra A, if the zero ideal and A are the only $\mathcal{H}-$invariant ideals of A, then A is said to be $\mathcal{H}-$*simple.*

For all J in $\mathcal{H}-$Spec(A), we define Prim$_J(A) :=$ Spec$_J(A) \cap$ Prim(A). This gives a corresponding $\mathcal{H}-$stratification of Prim(A). That is,

$$\text{Prim}(A) = \bigsqcup_{J \in \mathcal{H}-\text{Prim}(A)} \text{Prim}_J(A),$$

where $\mathcal{H} - \text{Prim}(A)$ is the collection of all $\mathcal{H}-$invariant primitive ideals of A.

This stratification theory applies to $U_q^+(\mathfrak{g})$, and we have the following lemma.

1.6.6 Lemma. [7, Theorem II.8.4] Let $A = U_q^+(\mathfrak{g})$, the primitive ideals in $\text{Spec}_J(A)$ are just the maximal ideals in $\text{Spec}_J(A)$.

1.6.7 Remark. [24, §6,7] (also see [44, Theorem 3.8]) The poset of $\mathcal{H}-\text{Spec}(U_q^+(\mathfrak{g}))$ (ordered by inclusion) is isomorphic to the poset of the Weyl group (ordered by the Bruhat order) associated to $\mathfrak{g}$. That is, for any element of the Weyl group, one can always associate a unique element of $\mathcal{H}-\text{Spec}(U_q^+(\mathfrak{g}))$.

1.6.8 Example. Let $R = \mathbb{C}_q[x, y]$, with $xy = qyx$, and $\mathcal{H} = (\mathbb{C}^*)^2$. The ideals $\langle 0\rangle$, $\langle x\rangle$, $\langle y\rangle$ and $\langle x, y\rangle$ are all $\mathcal{H}-$invariant prime ideals [7, Example II.2.3]. In fact, they are the only $\mathcal{H}-$invariant prime ideals we have in R. One can easily verify that $R/\langle x, y - \beta\rangle \cong \mathbb{C}$, where $\beta \in \mathbb{C}^*$. Since $\mathbb{C}$ is a domain, $\langle x, y - \beta\rangle$ is a prime ideal. Similarly, $\langle x - \alpha, y\rangle$ with $\alpha \in \mathbb{C}^*$, is also a prime ideal. However, none of these two prime ideals is $\mathcal{H}-$invariant. From [7, Example II.2.3], we have the following strata in $\text{Spec}(R)$:

- $\text{Spec}_{\langle 0\rangle}(R) = \{\langle 0\rangle\}$,
- $\text{Spec}_{\langle x\rangle}(R) = \{\langle x\rangle\} \cup \{\langle x, y - \beta\rangle \mid \beta \in \mathbb{C}^*\}$,
- $\text{Spec}_{\langle y\rangle}(R) = \{\langle y\rangle\} \cup \{\langle x - \alpha, y\rangle \mid \alpha \in \mathbb{C}^*\}$, and
- $\text{Spec}_{\langle x,y\rangle}(R) = \{\langle x, y\rangle\}$.

Consequently, $\text{Spec}(R) = \text{Spec}_{\langle 0\rangle}(R) \cup \text{Spec}_{\langle x\rangle}(R) \cup \text{Spec}_{\langle y\rangle}(R) \cup \text{Spec}_{\langle x,y\rangle}(R)$.

For each $\alpha, \beta \in \mathbb{C}^*$, the poset of $\text{Spec}(R)$ is shown in Figure 1.2. We also have that $\text{ht}(\langle 0\rangle) = 0$, $\text{ht}(\langle x\rangle) = \text{ht}(\langle y\rangle) = 1$, and $\text{ht}(\langle x, y - \beta\rangle) = \text{ht}(\langle x - \alpha, y\rangle) = \text{ht}(\langle x, y\rangle) = 2$. Again; since $\langle 0\rangle$, $\langle x, y - \beta\rangle$, $\langle x - \alpha, y\rangle$ and $\langle x, y\rangle$ are maximal in their respective strata for each $\alpha, \beta \in \mathbb{C}^*$; they are all primitive ideals. Although, $\langle 0\rangle$ is maximal in its strata, it is obvious that it cannot be a maximal ideal in R, since it is strictly contained in other proper ideals of R. However; $\langle x, y-\beta\rangle$, $\langle x-\alpha, y\rangle$ and $\langle x, y\rangle$ are all maximal ideals in R.

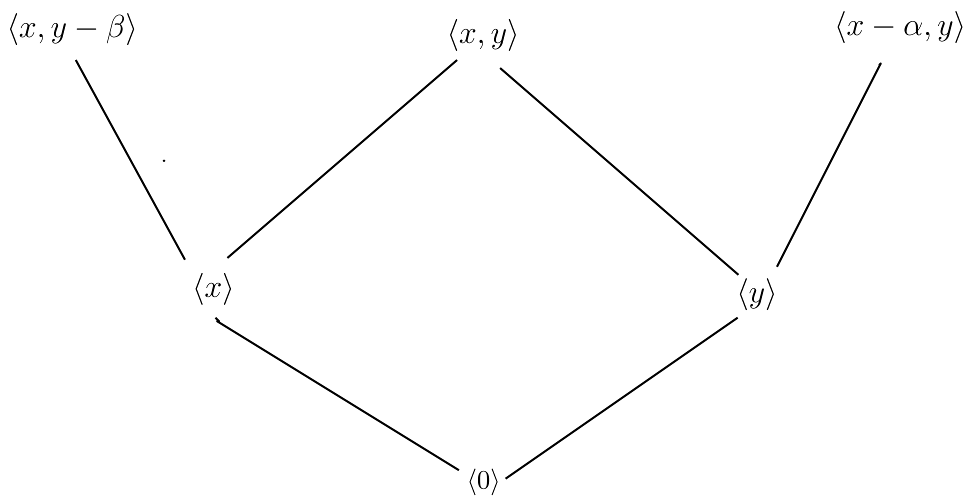


Figure 1.2: Poset of Spec(R)

Following the notations in [4, §2.2], let W be the set of subsets of $I := \{1, \cdots, N\}$. For each $w \in W$, set $K_w := \langle Y_i \mid i \in w\rangle$, where K_w is the two-sided ideal of the quantum affine space $\mathcal{O}_\Lambda(\mathbb{K}^N) = \mathbb{K}_\Lambda[Y_1, \cdots, Y_N]$. From [20, Proposition 2.11] (also, see [4, Proposition 2.1]), we have the following proposition.

1.6.9 Proposition. (1) The $\mathcal{H}-$ invariant prime ideals of $\mathcal{O}_\Lambda(\mathbb{K}^N)$ is the set $\{K_w \mid w \in W\}$. As a result, there are exactly 2^N $\mathcal{H}-$ invariant prime ideals of $\mathcal{O}_\Lambda(\mathbb{K}^N)$.

(2) $\text{Spec}_{K_w}(\mathcal{O}_\Lambda(\mathbb{K}^N)) = \{P \in \text{Spec}(\mathcal{O}_\Lambda(\mathbb{K}^N)) \mid P \cap \{Y_i \mid i \in I\} = \{Y_i \mid i \in w\}\}$ for all $w \in W$.

1.7 Deleting derivations algorithm (DDA)

In this section, we describe the notion of the deleting derivations algorithm (DDA) introduced by Cauchon [8]. We begin with the definition of quantum nilpotent algebras, which are also called Cauchon-Goodearl-Letzter (CGL) extensions. In line with [4, Definition 2], we provide a definition for CGL extension below.

1.7.1 Definition. Suppose that a ring A can be written as an iterated Ore extension as follows:

$$A = \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cdots [X_N; \sigma_N, \delta_N],$$

with $N \in \mathbb{N}$. Then, A is said to be a *quantum nilpotent algebra* or a *CGL extension* if there exists a torus $\mathcal{H} = (\mathbb{K}^*)^m$ that acts rationally by $\mathbb{K}$-automorphism on A, and the following are satisfied:

(a) $X_1, \cdots, X_N$ are $\mathcal{H}-$eigenvectors;

(b) for all $2 \leq i \leq N$, there exists $h_i \in \mathcal{H}$ and $q_i \in \mathbb{K}^*$ (q_i is not a root of unity) such that $h_i \cdot X_i = q_i X_i$ and for all $1 \leq j < i$, there exists $\lambda_{ij} \in \mathbb{K}^*$ such that $h_i \cdot X_j = \lambda_{ij} X_j$;

(c) the set $\{\lambda \in \mathbb{K}^* \mid \exists h \in \mathcal{H},\ h \cdot X_1 = \lambda X_1\}$ is infinite;

(d) for all $2 \leq i \leq N$, we have that δ_i is locally nilpotent, and

(e) for all $1 \leq j < i \leq N$, there exists $\lambda_{ij} \in \mathbb{K}^*$ such that $\sigma_i(X_j) = \lambda_{ij} X_j$ (note, $\sigma_i = (h_i \cdot) \mid_{A_{i-1}}$, where $A_{i-1} := \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cdots [X_{i-1}; \sigma_{i-1}, \delta_{i-1}]$).

Note, from the original definition, we have a condition that states that there exists $q_i \in \mathbb{K}^*$ (q_i is not a root of unity) such that $\sigma_i \circ \delta_i = q_i \delta_i \circ \sigma_i$. However, we did not include this condition in the above definition as it follows from the other conditions (see [23, Equation 3.1] for the necessary details). Suppose that A is a quantum nilpotent algebra, one can conclude from [21, Proposition 4.2] that all $\mathcal{H}-$invariant prime ideals of A are completely prime, and there are at most 2^N of these $\mathcal{H}-$invariant prime ideals.

Deleting derivations algorithm (DDA): let A be a quantum nilpotent algebra. One can use the theory of DDA constructed by Cauchon [8] (see also [4, §2.3]) to describe Spec(A). The algorithm relates Spec(A) to the prime spectrum of the quantum affine space obtained after sequentially 'deleting' all the skew-derivations $\delta_N, \cdots, \delta_2$ from A. We now describe the DDA process below.

Let $j \in \{N+1, \cdots, 2\}$, the algorithm constructs a family $(X_{1,j}, \cdots, X_{N,j})$ of elements of Fract(A) (the right ring of fractions of A) as follows. First, when $j = N+1$, we set $(X_{1,N+1}, \cdots, X_{N,N+1}) := (X_1, \cdots, X_N)$. Second, for $j < N+1$, suppose that the family $(X_{1,j+1}, \cdots, X_{N,j+1})$ has already been constructed. Then, one can construct $X_{1,j}, \cdots, X_{N,j}$ from $X_{1,j+1}, \cdots, X_{N,j+1}$ using the relation below:

$$X_{i,j} := \begin{cases} X_{i,j+1} & \text{if } i \geq j \\ \displaystyle\sum_{k=0}^{+\infty} \frac{(1-q_j)^{-k}}{[k]_{q_j}!} \delta_j^k \circ \sigma_j^{-k}(X_{i,j+1}) X_{j,j+1}^{-k} & \text{if } i < j, \end{cases} \tag{1.7.1}$$

for all $i \in \{1, \cdots, N\}$. Also, $[k]_{q_j}! = [0]_{q_j} \times \cdots \times [k]_{q_j}$ with $[0]_{q_j} = 1$, and $[i]_{q_j} = 1 + q_j + \cdots + q_j^{i-1}$ with $i \geq 1$. Note, from [8, Theorem 3.2.1], $X_{j,j+1} \neq 0$. Moreover, the summation is finite since δ_j are all locally nilpotent and $\delta_j \circ \sigma_j = q_j \sigma_j \circ \delta_j$.

For each $j \in \{2, \cdots, N+1\}$, $A^{(j)}$ is a subalgebra of Fract(A) generated by $X_{1,j}, \cdots, X_{N,j}$. That is, $A^{(j)} := \mathbb{K}\langle X_{1,j}, \cdots, X_{N,j}\rangle$. Since $(X_{1,N+1}, \cdots, X_{N,N+1}) = (X_1, \cdots, X_N)$, we have that $A^{(N+1)} = A$. It follows from [8, Theorem 3.2.1] that

$$A^{(j)} \cong \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cdots [X_{j-1}; \sigma_{j-1}, \delta_{j-1}][X_j; \tau_j] \cdots [X_N; \tau_N],$$

by an isomorphism that maps $X_{i,j}$ to X_i, and $\tau_j, \cdots, \tau_N$ are automorphisms defined by $\tau_l(X_i) = \lambda_{l,i} X_i$ for all $1 \leq i < l \leq N$. With a slight abuse of notation, one can identify $\tau_j, \cdots, \tau_N$ with $\sigma_j, \cdots, \sigma_N$ respectively. With this isomorphism and the slight abuse of notation, we present $A^{(j)}$ as:

$$A^{(j)} = \mathbb{K}[X_{1,j}][X_{2,j}; \sigma_2, \delta_2] \cdots [X_{j-1,j}; \sigma_{j-1}, \delta_{j-1}][X_{j,j}; \sigma_j] \cdots [X_{N,j}; \sigma_N].$$

One can observe that for each $j \in \{2, \cdots, N\}$, the derivations $\delta_j, \cdots, \delta_N$ are all 'deleted' from $A^{(j)}$. For example, if $j = 2$, then $\delta_2, \cdots, \delta_N$ will all be 'deleted' from $A^{(2)}$. As a result, $A^{(2)} = \mathbb{K}[T_1][T_2;\sigma_2]\cdots[T_N;\sigma_N]$, where $T_i := X_{i,2}$ for all $i \in \{1,\cdots,N\}$. Clearly, $\bar{A} := A^{(2)}$ is a quantum affine space. By Theorem 1.3.3, the algebra $A^{(j)}$ is a noetherian domain. Let $j \in \{2,\cdots,N\}$. The set $S_j := \{X_{j,j+1}^n \mid n \in \mathbb{N}\} = \{X_{j,j}^n \mid n \in \mathbb{N}\}$ is a multiplicative system of regular elements of $A^{(j)}$ and $A^{(j+1)}$ that satisfies the Ore condition in $A^{(j)}$ and $A^{(j+1)}$, and $A^{(j)}S_j^{-1} = A^{(j+1)}S_j^{-1}$ [8, Theorem 3.2.1].

Cauchon used DDA to relate Spec(A) to Spec$(\bar{A})$ by constructing an embedding ψ_j : Spec$(A^{(j+1)}) \hookrightarrow$ Spec$(A^{(j)})$ for each $j \in \{2,\cdots,N\}$. Suppose that $P \in$ Spec$(A^{(j+1)})$ and $X_{j,j+1} \notin P$, then ψ_j is defined by

$$\psi_j(P) = PS_j^{-1} \cap A^{(j)} = Q,$$

with $Q \in$ Spec$(A^{(j)})$. The inverse map ψ_j^{-1} is also given by

$$\psi_j^{-1}(Q) = QS_j^{-1} \cap A^{(j+1)} = P,$$

provided $X_{j,j} \notin Q$.

The case where $X_{j,j+1} \in P$ is beyond the scope of this study, however, one can refer to [4, §2.3] and [8, §4.3] for the necessary details.

The map ψ_j is injective but not necessarily bijective. However, ψ_j induces a bijection from $\{P \in$ Spec$(A^{(j+1)}) \mid P \cap S_j = \emptyset\}$ onto $\{Q \in$ Spec$(A^{(j)}) \mid Q \cap S_j = \emptyset\}$. The so-called *canonical embedding* ψ : Spec$(A) \hookrightarrow$ Spec$(\bar{A})$ is obtained by composing all the ψ_j. That is, $\psi := \psi_2 \circ \cdots \circ \psi_N$. This canonical embedding ψ helps to construct a partition of Spec(A) into a disjoint union of strata (known as the *canonical partition*) via the Cauchon diagrams (see [4, §2.3] for further details on this).

1.7.2 Remark. The algebra $U_q^+(\mathfrak{g})$ is a quantum nilpotent algebra [4, §3.1]. One can therefore apply the DDA to study the prime spectrum of $U_q^+(\mathfrak{g})$.

Chapter 2

Quest for height two maximal ideals of $U_q^+(G_2)$

The algebra $A = U_q^+(G_2)$ is a quantum nilpotent algebra, so one can apply the theory of deleting derivations algorithm (DDA) and $\mathcal{H}-$stratification theory to study its prime ideals. Note, from Remark 1.6.1, all prime ideals of A are completely prime. Since A is a quantum nilpotent algebra, it follows from [18, Theorem 7.1] that Tauvel's height formula holds for A. That is, for any prime ideal P of A, we have that

$$\text{GKdim}(A) = \text{ht}(P) + \text{GKdim}(A/P),$$

where GKdim$(*)$ represents the Gelfand-Kirillov dimension of the algebra $*$, and ht(P) is the height of the prime ideal P. For more details on the Gelfand-Kirillov dimension of a ring, we recommend [30]. Since G_2 has 6 positive roots (Section 1.1), it follows from [45, §2.1] that GKdim$(A) = 6$. We are now interested in finding a family of prime ideals such that the quotients of A with those prime ideals are of GKdim 4 and simple. This implies that these prime ideals must be maximal, and using Tauvel's height formula, they must be of height 2. To achieve this result, we use $\mathcal{H}-$stratification theory to study some $\mathcal{H}-$strata of A. Note, the maximal ideals of height 2 can only belong to the $\mathcal{H}-$strata corresponding to an $\mathcal{H}-$invariant prime ideals of height at most 2, so we identify these $\mathcal{H}-$prime ideals first. Of course, the only height 0 prime ideal is the zero ideal $\langle 0\rangle$. In Section 2.3, we proved that the ideals $\langle\Omega_1\rangle$ and $\langle\Omega_2\rangle$ are (completely) prime using Cauchon's theory of DDA, and in Lemma 2.4.2, we proved that they are of height 1. Moreover, studying the poset of $\mathcal{H}-$Spec(A) (as seen in Figure 2.1), no height 2 $\mathcal{H}-$prime ideal is maximal. As a result, the height 2 maximal ideals of A can only come from the $\mathcal{H}-$strata corresponding to $\langle 0\rangle$, $\langle\Omega_1\rangle$ or $\langle\Omega_2\rangle$. In Subsection 2.4.3, we compute explicitly these $\mathcal{H}-$strata. This allows us to study the height 2 maximal ideals of A. We now begin by studying some properties of the algebra $U_q^+(G_2)$.

2.1 The algebra $U_q^+(G_2)$

The background of the materials presented here can be found in Section 1.2. Moreover, the positive part $U_q^+(G_2)$ of $U_q(G_2)$ is going to be of a major interest in this section and beyond. We will begin by finding the defining relations of $U_q(G_2)$, and then proceed to compute the defining relations of the subalgebra $U_q^+(G_2)$.

The algebra $U_q(G_2)$ is a $\mathbb{C}-$algebra generated by $F_1, F_2, K_1^{\pm1}, K_2^{\pm1}, E_1, E_2$ subject to the following relations:

$$
\begin{array}{lll}
K_1E_1 = q^2E_1K_1 & K_1F_1 = q^{-2}F_1K_1 & K_1K_2 = K_2K_1 \\
K_1E_2 = q^{-3}E_2K_1 & K_1F_2 = q^3F_2K_1 & E_2F_2 = F_2E_2 + \dfrac{K_2 - K_2^{-1}}{q^3 - q^{-3}} \\
K_2E_1 = q^{-3}E_1K_2 & K_2F_1 = q^3F_1K_2 & E_1F_1 = F_1E_1 + \dfrac{K_1 - K_1^{-1}}{q - q^{-1}} \\
K_2E_2 = q^6E_2K_2 & K_2F_2 = q^{-6}F_2K_2 & E_1F_2 = F_2E_1 \\
E_2F_1 = F_1E_2, & &
\end{array}
$$

and the quantum Serre relations:

$$(S1)\quad E_1^4E_2 - \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q E_1^3E_2E_1 + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q E_1^2E_2E_1^2 - \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q E_1E_2E_1^3 + E_2E_1^4 = 0,$$

$$(S2)\quad E_2^2E_1 - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{q^3} E_2E_1E_2 + E_1E_2^2 = 0,$$

$$(S3) \quad F_1^4F_2 - \begin{bmatrix}4\\1\end{bmatrix}_q F_1^3F_2F_1 + \begin{bmatrix}4\\2\end{bmatrix}_q F_1^2F_2F_1^2 - \begin{bmatrix}4\\1\end{bmatrix}_q F_1F_2F_1^3 + F_2F_1^4 = 0,$$

$$(S4) \quad F_2^2F_1 - \begin{bmatrix}2\\1\end{bmatrix}_{q^3} F_2F_1F_2 + F_1F_2^2 = 0.$$

Now, $U_q^+(G_2)$ is a subalgebra of $U_q(G_2)$ generated by E_1 and E_2 subject to the quantum Serre relations $(S1)$ and $(S2)$. The actions of the automorphisms T_1 and T_2 of $U_q(G_2)$ on $U_q^+(G_2)$ are as follows:

$$T_2(E_1) = E_2E_1 - q^{-3}E_1E_2 \qquad T_1(E_1) = -F_1K_1$$

$$T_1(E_2) = \frac{E_1^3E_2}{[3]_q[2]_q} - q^{-1}\frac{E_1^2E_2E_1}{[2]_q} + q^{-2}\frac{E_1E_2E_1^2}{[2]_q} - q^{-3}\frac{E_2E_1^3}{[3]_q[2]_q} \qquad T_2(E_2) = -F_2K_2.$$

For the purpose of the computation of the relations of $U_q^+(G_2)$ in Appendix A.1, we include the following inverse automorphism actions: $T_1^{-1}(E_1) = -K_1^{-1}F_1$ and $T_2^{-1}(E_2) = -K_2^{-1}F_2$.

Now, recall the positive roots $\beta_1, \cdots, \beta_6$ of G_2 in Section 1.1. From (1.2.1) and Theorem 1.2.2, the elements E_{β_r} (also called root vectors) of $U_q^+(G_2)$ are as follows:

$$E_{\beta_1} = E_1 \qquad E_{\beta_2} = T_1(E_2) \qquad E_{\beta_3} = T_1T_2(E_1)$$

$$E_{\beta_4} = T_1T_2T_1(E_2) \qquad E_{\beta_5} = T_1T_2T_1T_2(E_1) \qquad E_{\beta_6} = E_2.$$

With a slight abuse of notations, set $E_i := E_{\beta_i}$ for all $1 \leq i \leq 6$. Note, E_2 is no longer E_{β_6} as expected. With these notations, the defining relations of $U_q^+(G_2)$ (see Appendix A.1.1) are as follows:

$$E_2E_1 = q^{-3}E_1E_2 \qquad E_3E_1 = q^{-1}E_1E_3 - (q + q^{-1} + q^{-3})E_2$$

$$E_3E_2 = q^{-3}E_2E_3 \qquad E_4E_1 = E_1E_4 + (1 - q^2)E_3^2$$

$$E_4E_2 = q^{-3}E_2E_4 - \frac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1}E_3^3 \qquad E_4E_3 = q^{-3}E_3E_4$$

$$E_5E_1 = qE_1E_5 - (1 + q^2)E_3 \qquad E_5E_2 = E_2E_5 + (1 - q^2)E_3^2$$

$$E_5E_3 = q^{-1}E_3E_5 - (q + q^{-1} + q^{-3})E_4 \qquad E_5E_4 = q^{-3}E_4E_5$$

$$E_6E_1 = q^3E_1E_6 - q^3E_5 \qquad E_6E_2 = q^3E_2E_6 + (q^4+q^2-1)E_4 + (q^2-q^4)E_3E_5$$

$$E_6E_3 = E_3E_6 + (1-q^2)E_5^2$$

$$E_6E_4 = q^{-3}E_4E_6 - \frac{q^4-2q^2+1}{q^4+q^2+1}E_5^3 \qquad E_6E_5 = q^{-3}E_5E_6.$$

These relations have been confirmed with a mathematical software called GAP [10]. The GAP code can be found in Appendix A.1.2.

2.1.1 Remark. De Graaf [9, Appendix A] has also computed the relations of $U_q^+(G_2)$. If we set $E_1 := E_\alpha,\ E_2 := E_{3\alpha+\beta},\ E_3 := E_{2\alpha+\beta},\ E_4 := E_{3\alpha+2\beta},\ E_5 := E_{\alpha+\beta}$ and $E_6 := E_\beta$, then our relations are the same as his relations. Of course, the only difference is that he used the notations α and β for the simple roots contrary to our notations α_1 and α_2.

From Theorem 1.2.2, the set $\{E_1^{k_1}E_2^{k_2}E_3^{k_3}E_4^{k_4}E_5^{k_5}E_6^{k_6} \mid k_1, \cdots, k_6 \in \mathbb{N}\}$ forms a PBW-basis of $U_q^+(G_2)$. The iterated Ore extension of $U_q^+(G_2)$ is of the form:

$$U_q^+(G_2) = \mathbb{C}[E_1][E_2;\sigma_2][E_3;\sigma_3,\delta_3][E_4;\sigma_4,\delta_4][E_5;\sigma_5,\delta_5][E_6;\sigma_6,\delta_6];$$

where, σ_2 denotes the automorphism of $\mathbb{C}[E_1]$ defined by:

$$\sigma_2(E_1) = q^{-3}E_1,$$

σ_3 denotes the automorphism of $\mathbb{C}[E_1][E_2;\sigma_2]$ defined by:

$$\sigma_3(E_1) = q^{-1}E_1 \qquad \sigma_3(E_2) = q^{-3}E_2,$$

δ_3 denotes the σ_3-derivation of $\mathbb{C}[E_1][E_2;\sigma_2]$ defined by:

$$\delta_3(E_1) = -(q+q^{-1}+q^{-3})E_2 \qquad \delta_3(E_2) = 0,$$

σ_4 denotes the automorphism of $\mathbb{C}[E_1]\cdots[E_3;\sigma_3,\delta_3]$ defined by:

$$\sigma_4(E_1)=E_1 \qquad \sigma_4(E_2)=q^{-3}E_2 \qquad \sigma_4(E_3)=q^{-3}E_3,$$

δ_4 denotes the σ_4-derivation of $\mathbb{C}[E_1]\cdots[E_3;\sigma_3,\delta_3]$ defined by:

$$\delta_4(E_1)=(1-q^2)E_3^2 \qquad \delta_4(E_2)=\frac{-q^4+2q^2-1}{q^4+q^2+1}E_3^3 \qquad \delta_4(E_3)=0,$$

σ_5 denotes the automorphism of $\mathbb{C}[E_1]\cdots[E_4;\sigma_4,\delta_4]$ defined by:

$$\sigma_5(E_1)=qE_1 \qquad \sigma_5(E_2)=E_2 \qquad \sigma_5(E_3)=q^{-1}E_3 \qquad \sigma_5(E_4)=q^{-3}E_4,$$

δ_5 denotes the σ_5-derivation of $\mathbb{C}[E_1]\cdots[E_4;\sigma_4,\delta_4]$ defined by:

$$\delta_5(E_1)=-(1+q^2)E_3 \quad \delta_5(E_2)=(1-q^2)E_3^2 \quad \delta_5(E_3)=-(q+q^{-1}+q^{-3})E_4 \quad \delta_5(E_4)=0,$$

σ_6 denotes the automorphism of $\mathbb{C}[E_1]\cdots[E_5;\sigma_5,\delta_5]$ defined by:

$$\sigma_6(E_1)=q^3E_1 \quad \sigma_6(E_2)=q^3E_2 \quad \sigma_6(E_3)=E_3 \quad \sigma_6(E_4)=q^{-3}E_4 \quad \sigma_6(E_5)=q^{-3}E_5,$$

and δ_6 denotes the σ_6-derivation of $\mathbb{C}[E_1]\cdots[E_5;\sigma_5,\delta_5]$ defined by:

$$\delta_6(E_1)=-q^3E_5 \qquad \delta_6(E_2)=(q^2-q^4)E_3E_5+(q^4+q^2-1)E_4 \qquad \delta_6(E_3)=(1-q^2)E_5^2$$

$$\delta_6(E_4)=\frac{-q^4+2q^2-1}{q^4+q^2+1}E_5^3 \qquad \delta_6(E_5)=0.$$

2.2 DDA and the center of $U_q^+(G_2)$

Henceforth, we set $A:=U_q^+(G_2)$. We are now ready to describe the DDA of A.

2.2.1 DDA of $U_q^+(G_2)$. The algebra A is a quantum nilpotent algebra (Remark 1.7.2), and therefore the theory of DDA studied in Section 1.7 applies to A. We construct the following elements of Fract(A) (computations have been omitted here, but can be found in Appendix A.2):

$$\begin{aligned}
E_{1,6} &= E_1 + rE_5E_6^{-1}\\
E_{2,6} &= E_2 + tE_3E_5E_6^{-1} + uE_4E_6^{-1} + nE_5^3E_6^{-2}\\
E_{3,6} &= E_3 + sE_5^2E_6^{-1}\\
E_{4,6} &= E_4 + bE_5^3E_6^{-1}\\
E_{1,5} &= E_{1,6} + hE_{3,6}E_{5,6}^{-1} + gE_{4,6}E_{5,6}^{-2}\\
E_{2,5} &= E_{2,6} + fE_{3,6}^2E_{5,6}^{-1} + pE_{3,6}E_{4,6}E_{5,6}^{-2} + eE_{4,6}^2E_{5,6}^{-3}\\
E_{3,5} &= E_{3,6} + aE_{4,6}E_{5,6}^{-1}\\
E_{1,4} &= E_{1,5} + sE_{3,5}^2E_{4,5}^{-1}\\
E_{2,4} &= E_{2,5} + bE_{3,5}^3E_{4,5}^{-1}\\
E_{1,3} &= E_{1,4} + aE_{2,4}E_{3,4}^{-1}\\
T_1 &:= E_{1,2} = E_{1,3}\\
T_2 &:= E_{2,2} = E_{2,3} = E_{2,4}\\
T_3 &:= E_{3,2} = E_{3,3} = E_{3,4} = E_{3,5}\\
T_4 &:= E_{4,2} = E_{4,3} = E_{4,4} = E_{4,5} = E_{4,6}\\
T_5 &:= E_{5,2} = E_{5,3} = E_{5,4} = E_{5,5} = E_{5,6} = E_5\\
T_6 &:= E_{6,2} = E_{6,3} = E_{6,4} = E_{6,5} = E_{6,6} = E_6.
\end{aligned}$$

The parameters $a, b, e, f, g, h, n, p, r, s, t, u$ are all defined in Appendix C.

Again, from the theory of the DDA in Section 1.7, we have that for each $j \in \{2, \cdots, 7\}$, the algebra $A^{(j)} := \mathbb{C}\langle E_{i,j} \mid i = 1, \cdots, 6\rangle$ is the subalgebra of Fract(A). Since $(E_{1,7}, \cdots, E_{6,7}) := (E_1, \cdots, E_6)$, it follows that $A^{(7)} = A$.

2.2.2 Remark. Recall that $T_j = E_{j,j} = E_{j,j+1} \in A^{(j)} \cap A^{(j+1)}$. It follows from [8, Theorem 3.2.1] that the set $\Sigma_j := \{T_j^n \mid n \in \mathbb{N}\}$ is an Ore set in both $A^{(j)}$ and $A^{(j+1)}$ for each $1 \leq j \leq 6$, and that $A^{(j)}\Sigma_j^{-1} = A^{(j+1)}\Sigma_j^{-1}$.

2.2.3 The center of $U_q^+(G_2)$. Using (1.5.1), we have the skew-symmetric matrix M below:

$$M := \begin{bmatrix} 0 & (\beta_1,\beta_2) & \cdots & \cdots & (\beta_1,\beta_6) \\ -(\beta_1,\beta_2) & 0 & (\beta_2,\beta_3) & & (\beta_2,\beta_6) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & (\beta_5,\beta_6) \\ -(\beta_1,\beta_6) & \cdots & \cdots & -(\beta_5,\beta_6) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & 1 & 0 & -1 & -3 \\ -3 & 0 & 3 & 3 & 0 & -3 \\ -1 & -3 & 0 & 3 & 1 & 0 \\ 0 & -3 & -3 & 0 & 3 & 3 \\ 1 & 0 & -1 & -3 & 0 & 3 \\ 3 & 3 & 0 & -3 & -3 & 0 \end{bmatrix}.$$

Observe that $\bar{A} := A^{(2)} = \mathbb{C}_{q^M}[T_1, \cdots, T_6]$ is a quantum affine space. Set $\Omega_1 := T_1T_3T_5$ and $\Omega_2 := T_2T_4T_6$. One can verify that Ω_1 and Ω_2 are central elements of $\bar{A}$. That is, $\Omega_iT_j = T_j\Omega_i$ for all $i = 1, 2$, and $1 \leq j \leq 6$.

We now want to successively pull Ω_1 and Ω_2 from the quantum affine space $\bar{A}$ into the algebra A using the data of the DDA of A discussed above. We only summarize the computations here, and provide a sketch of the computations in Appendix A.3.

$$\begin{aligned} \Omega_1 &:= T_1T_3T_5 \\ &= E_{1,4}E_{3,4}E_{5,4} + aE_{2,4}E_{5,4} \\ &= E_{1,5}E_{3,5}E_{5,5} + aE_{2,5}E_{5,5} \\ &= E_{1,6}E_{3,6}E_{5,6} + aE_{1,6}E_{4,6} + aE_{2,6}E_{5,6} + a'E_{3,6}^2 \\ &= E_1E_3E_5 + aE_1E_4 + aE_2E_5 + a'E_3^2, \end{aligned}$$

and

$$\begin{aligned}
\Omega_2 &:= T_2T_4T_6 \\
&= E_{2,5}E_{4,5}E_{6,5} + bE_{3,5}^3E_{6,5} \\
&= E_{2,6}E_{4,6}E_{6,6} + bE_{3,6}^3E_{6,6} \\
&= E_2E_4E_6 + bE_2E_5^3 + bE_3^3E_6 + b'E_3^2E_5^2 + c'E_3E_4E_5 + d'E_4^2.
\end{aligned}$$

The parameters a, b, a', b', c', d' are all defined in Appendix C. Note, Ω_1 and Ω_2 are central elements of $A^{(j)}$ for each $2 \leq j \leq 7$, since Fract$(A^{(j)}) =$ Fract$(\bar{A})$. From the commutation relations of A, one can observe that E_1 and E_6 are enough to generate A. Hence, it is sufficient to confirm that Ω_1 and Ω_2 are central elements of A by showing that they commute with only E_1 and E_6. As a result, we have used GAP to confirm that $\Omega_1 E_i = E_i\Omega_1$ and $\Omega_2 E_i = E_i \Omega_2$ for each $i = 1, 6$. The GAP code can be found in Appendix A.3.1.

We now want to show that the center of A is a polynomial ring generated by Ω_1 and Ω_2 over $\mathbb{C}$. The following discussions will lead us to the proof.

Set $S_j := \{\lambda T_j^{i_j} T_{j+1}^{i_{j+1}} \cdots T_6^{i_6} \mid i_j, \cdots, i_6 \in \mathbb{N}$ and $\lambda \in \mathbb{C}^*\}$ for each $2 \leq j \leq 6$. One can observe that S_j is a multiplicative system of non-zero divisors of $A^{(j)} = \mathbb{C}\langle E_{i,j} \mid$ for all $i = 1, \cdots, 6\rangle$. Furthermore, the elements $T_j, \cdots, T_6$ are all normal in $A^{(j)}$. Hence, S_j is an Ore set in $A^{(j)}$. We can therefore localize $A^{(j)}$ at S_j as follows:

$$R_j := A^{(j)} S_j^{-1}.$$

Recall from Remark 2.2.2 that $\Sigma_j := \{T_j^n \mid n \in \mathbb{N}\}$ is an Ore set in both $A^{(j)}$ and $A^{(j+1)}$ for each $2 \leq j \leq 6$, and that

$$A^{(j)}\Sigma_j^{-1} = A^{(j+1)}\Sigma_j^{-1}.$$

For all $2 \leq j \leq 6$, we have that:

$$R_j = A^{(j)}S_j^{-1} = (A^{(j)}\Sigma_j^{-1})S_{1+j}^{-1} = (A^{(j+1)}\Sigma_j^{-1})S_{1+j}^{-1} = (A^{(j+1)}S_{j+1}^{-1})\Sigma_j^{-1} = R_{j+1}\Sigma_j^{-1}. \tag{2.2.1}$$

Note, $R_7 := A$.

Again, one can also observe that T_1 is normal in R_2. As a result, the localization

$$R_1 := R_2[T_1^{-1}]$$

also holds in R_2. In fact, R_1 is the quantum torus associated to the quantum affine space $\bar{A}$. As a result, $R_1 = \mathbb{C}_{q^M}[T_1^{\pm 1}, \cdots, T_6^{\pm 1}]$, where $T_iT_j = q^{\mu_{ij}}T_jT_i$ for all $1 \leq i, j \leq 6$ and $\mu_{ij} \in M$. Similar to [36, §31], we construct the following tower of algebras:

$$\begin{aligned} A = R_7 \subset R_6 = R_7\Sigma_6^{-1} \subset R_5 = R_6\Sigma_5^{-1} \subset R_4 = R_5\Sigma_4^{-1} \\ \subset R_3 = R_4\Sigma_3^{-1} \subset R_2 = R_3\Sigma_2^{-1} \subset R_1. \end{aligned} \tag{2.2.2}$$

Note, the family $(E_{1,j}^{k_1} \cdots E_{6,j}^{k_6})$, where $k_i \in \mathbb{N}$ if $i < j$ and $k_i \in \mathbb{Z}$ otherwise is a PBW-basis of R_j for all $1 \leq i, j \leq 7$. Therefore, the family $(T_1^{k_1}T_2^{k_2}T_3^{k_3}T_4^{k_4}T_5^{k_5}T_6^{k_6})_{k_1,\cdots,k_6 \in \mathbb{Z}}$ is a basis of R_1.

2.2.4 Lemma. $Z(R_1) = \mathbb{C}[\Omega_1^{\pm 1}, \Omega_2^{\pm 1}]$.

Proof. Obviously, $\mathbb{C}[\Omega_1^{\pm 1}, \Omega_2^{\pm 1}] \subseteq Z(R_1)$. For the reverse inclusion, let $y \in Z(R_1)$. Then, y can be written in terms of the basis of R_1 as:

$$y = \sum_{(i,\cdots,n) \in \mathbb{Z}^6} a_{(i,\cdots,n)} T_1^i T_2^j T_3^k T_4^l T_5^m T_6^n.$$

One can verify that $yT_1 = q^{-3j-k+m+3n}T_1y$. Since $y \in Z(R_1)$, it follows that

$$-3j - k + m + 3n = 0.$$

Similarly, $yT_2 = q^{3i-3k-3l+3n}T_2y$. Since $y \in Z(R_1)$, we have:

$$3i - 3k - 3l + 3n = 0.$$

Following the same pattern for T_3, T_4, T_5 and T_6, one can confirm that

$$\begin{aligned} i + 3j - 3l - m &= 0, \\ 3j + 3k - 3m - 3n &= 0, \\ -i + k + 3l - 3n &= 0, \\ -3i - 3j + 3l + 3m &= 0. \end{aligned}$$

Solving this system of six equations will reveal that $i = k = m$ and $j = l = n$. One can therefore write

$$y = \sum_{(i,j)\in\mathbb{Z}^2} a_{(i,j)} T_1^i T_2^j T_3^i T_4^j T_5^i T_6^j = \sum_{(i,j)\in\mathbb{Z}^2} q^{\bullet} a_{(i,j)} T_1^i T_3^i T_5^i T_2^j T_4^j T_6^j = \sum_{(i,j)\in\mathbb{Z}^2} q^{\bullet} a_{(i,j)} \Omega_1^i \Omega_2^j.$$

This implies that $y \in \mathbb{C}[\Omega_1^{\pm 1}, \Omega_2^{\pm 1}]$ as expected. ■

2.2.5 Corollary. 1. $Z(R_3) = \mathbb{C}[\Omega_1, \Omega_2]$.

2. $Z(\bar{A}) = \mathbb{C}[\Omega_1, \Omega_2]$.

Proof. 1. Clearly, $\mathbb{C}[\Omega_1, \Omega_2] \subseteq Z(R_3)$. For the reverse inclusion, let $y \in Z(R_3)$. Then, y can be written in terms of the basis of R_3 (recall, $T_i = E_{i,3}$) as:

$$y = \sum_{(i,\cdots,n)\in\mathbb{N}^2\times\mathbb{Z}^4} a_{(i,\cdots,n)} T_1^i T_2^j T_3^k T_4^l T_5^m T_6^n.$$

Now; $T_1, \cdots, T_6$ are all normal elements in R_3. In fact, they satisfy the same commutation relations in R_1. Hence, following procedures similar to the lemma above, one will arrive at the conclusion that $i = k = m$ and $j = l = n$. Since $i, j \geq 0$; we have that $y = \sum_{(i,j)\in\mathbb{N}^2} q^{\bullet} a_{(i,j)} T_1^i T_3^i T_5^i T_2^j T_4^j T_6^j = \sum_{(i,j)\in\mathbb{N}^2} q^{\bullet} a_{(i,j)} \Omega_1^i \Omega_2^j$. This implies that

$y \in \mathbb{C}[\Omega_1, \Omega_2]$ as expected.

2. Similar to (1). ■

2.2.6 Lemma. $Z(A) = \mathbb{C}[\Omega_1, \Omega_2]$.

Proof. Since R_i is a localization of R_{i+1}, it follows that $Z(R_{i+1}) \subseteq Z(R_i)$. From (2.2.2), we have that $Z(A) \subseteq Z(R_3)$. Observe that $\mathbb{C}[\Omega_1, \Omega_2] \subseteq Z(A) \subseteq Z(R_3) = \mathbb{C}[\Omega_1, \Omega_2]$. Hence, $Z(A) = \mathbb{C}[\Omega_1, \Omega_2]$. ■

2.2.7 Remark. Since $Z(A) = Z(R_3) = \mathbb{C}[\Omega_1, \Omega_2]$ and $Z(R_{i+1}) \subseteq Z(R_i)$, it follows from (2.2.2) that $Z(A) = Z(R_6) = Z(R_5) = Z(R_4) = Z(R_3) = \mathbb{C}[\Omega_1, \Omega_2]$. One can also deduce from Lemma 2.2.4 that $Z(R_2) = \mathbb{C}[\Omega_1, \Omega_2^{\pm 1}]$.

2.3 Proof of the completely primeness of $\langle\Omega_1\rangle$ and $\langle\Omega_2\rangle$

Since all prime ideals of $A = U_q^+(G_2)$ are completely prime (Remark 1.6.1), it is sufficient to show that $\langle\Omega_1\rangle$ and $\langle\Omega_2\rangle$ are completely prime by showing that they are prime ideals. Note, the data of the DDA of A and the expressions for Ω_1 and Ω_2 in the previous section will be helpful in the proofs of Lemmas 2.3.1 and 2.3.2. Recall, the notation $\langle\Theta\rangle_R$ means an ideal generated by the element Θ in any arbitrary ring R. Where no doubt arises, we will simply write $\langle\Theta\rangle$.

From Section 1.7 we know that there is a bijection between $\{P \in \text{Spec}(A^{(j+1)}) \mid P \cap S_j = \emptyset\}$ and $\{Q \in \text{Spec}(A^{(j)}) \mid Q \cap S_j = \emptyset\}$ via $P = QS_j^{-1} \cap A^{(j+1)}$. Note, $\langle T_1\rangle$ and $\langle T_2\rangle$ are prime ideals of the quantum affine space $\bar{A}$, since each of the factor algebras $\bar{A}/\langle T_1\rangle$ and $\bar{A}/\langle T_2\rangle$ is isomorphic to a quantum affine space of rank 5 which is well known to be a domain. From Section 1.7 we have that $\psi_j : \text{Spec}(A^{(1+j)}) \hookrightarrow \text{Spec}(A^{(j)})$ for $2 \leq j \leq 6$. Hence, $\psi := \psi_6 \circ \cdots \circ \psi_2 : \text{Spec}(A) \hookrightarrow \text{Spec}(\bar{A})$. Recall, $A^{(7)} = A$ and $A^{(2)} = \bar{A}$.

The following result shows that $\langle T_1\rangle \in \text{Im}(\psi)$ (i.e. image of ψ) and that $\langle\Omega_1\rangle$ is the completely prime ideal of A such that $\psi(\langle\Omega_1\rangle) = \langle T_1\rangle$.

2.3.1 Lemma. $\langle\Omega_1\rangle \in \text{Spec}(A)$.

Proof. We will prove this result in several steps by showing that:

1. $\langle T_1\rangle_{A^{(3)}} \in \text{Spec}(A^{(3)})$.

2. $\langle E_{1,4}T_3 + aT_2\rangle = \langle T_1\rangle_{A^{(3)}}[T_3^{-1}] \cap A^{(4)}$, hence $Q_1 := \langle E_{1,4}T_3 + aT_2\rangle \in \text{Spec}(A^{(4)})$.

3. $\langle E_{1,5}T_3 + aE_{2,5}\rangle = Q_1[T_4^{-1}] \cap A^{(5)}$, hence $Q_2 := \langle E_{1,5}T_3 + aE_{2,5}\rangle \in \text{Spec}(A^{(5)})$.

4. $\langle\Omega_1\rangle_{A^{(6)}} = Q_2[T_5^{-1}] \cap A^{(6)}$, hence $\langle\Omega_1\rangle_{A^{(6)}} \in \text{Spec}(A^{(6)})$.

5. $\langle\Omega_1\rangle_A = \langle\Omega_1\rangle_{A^{(6)}}[T_6^{-1}] \cap A$, hence $\langle\Omega_1\rangle_A \in \text{Spec}(A)$.

We are now ready to prove the above claims.

1. One can easily verify that $A^{(3)}/\langle T_1\rangle$ is isomorphic to a quantum affine space of rank 5, which is a domain, hence $\langle T_1\rangle$ is a prime ideal in $A^{(3)}$.

2. Note, $T_1 = E_{1,4}+aT_2T_3^{-1}$. We want to show that $\langle E_{1,4}T_3+aT_2\rangle = \langle T_1\rangle_{A^{(3)}}[T_3^{-1}]\cap A^{(4)}$. Observe that $\langle E_{1,4}T_3 + aT_2\rangle \subseteq \langle T_1\rangle_{A^{(3)}}[T_3^{-1}] \cap A^{(4)}$. We established the reverse inclusion. Let $y \in \langle T_1\rangle_{A^{(3)}}[T_3^{-1}] \cap A^{(4)}$. Then, $y \in \langle T_1\rangle_{A^{(3)}}[T_3^{-1}]$. Therefore, there exists $i \in \mathbb{N}$ such that $yT_3^i \in \langle T_1\rangle_{A^{(3)}}$. This implies that $yT_3^i = T_1v$, for some $v \in A^{(3)}$. Since $A^{(3)}[T_3^{-1}] = A^{(4)}[T_3^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_3^j = v'$, for some $v' \in A^{(4)}$. It follows that $yT_3^{i+j} = T_1vT_3^j = T_1v' = (E_{1,4} + aT_2T_3^{-1})v' = (E_{1,4}T_3 + aT_2)T_3^{-1}v'$. The multiplicative system generated by T_3 satisfies the Ore condition in $A^{(4)}$, hence, there exists $k \in \mathbb{N}$ and $v'' \in A^{(4)}$ such that $T_3^{-1}v' = v''T_3^{-k}$. One can therefore write $yT_3^{i+j} = (E_{1,4}T_3+aT_2)v''T_3^{-k}$. This implies that $yT_3^\delta = \Omega_1'v''$, where $\Omega_1' := E_{1,4}T_3+aT_2$ and $\delta = i + j + k$. Set $S := \{s \in \mathbb{N} \mid \exists v'' \in A^{(4)} : yT_3^s = \Omega_1'v''\}$. Note, $S \neq \emptyset$, since $\delta \in S$. Let $s = s_0$ be the minimum element of S such that $yT_3^{s_0} = \Omega_1'v''$. We want to show that $s_0 = 0$. Remember, $\Omega_1'T_5 = \Omega_1$ in $A^{(4)}$. Since Ω_1 is central in $A^{(4)}$, and T_5 is normal in $A^{(4)}$, we must have Ω_1' to be a normal element in $A^{(4)}$, otherwise, there will be a contradiction. Therefore, there exists $w \in A^{(4)}$ such that $yT_3^{s_0} = \Omega_1'v'' = w\Omega_1'$. Now, $A^{(4)}$ can be viewed as a free left $\mathbb{C}\langle E_{1,4}, T_2, T_4, T_5, T_6\rangle-$module with basis $\left(T_3^\xi\right)_{\xi\in\mathbb{N}}$. One can therefore write $y = \sum_{\xi=0}^n \alpha_\xi T_3^\xi$ and $w = \sum_{\xi=0}^n \beta_\xi T_3^\xi$, where $\alpha_\xi, \beta_\xi \in \mathbb{C}\langle E_{1,4}, T_2, T_4, T_5, T_6\rangle$. This implies that $\sum_{\xi=0}^n \alpha_\xi T_3^{\xi+s_0} = \sum_{\xi=0}^n \beta_\xi T_3^\xi \Omega_1' =$

$\sum_{\xi=0}^{n} q^{\bullet}\beta_\xi\Omega_1' T_3^{\xi}$ (note, $T_3\Omega_1' = q^{-1}\Omega_1' T_3$). Given that $\Omega_1' = E_{1,4}T_3 + aT_2$, we have that $\sum_{\xi=0}^{n}\alpha_\xi T_3^{\xi+s_0} = \sum_{\xi=0}^{n} q^{\bullet}\beta_\xi E_{1,4}T_3^{1+\xi} + \sum_{\xi=0}^{n} q^{\bullet}a\beta_\xi T_2 T_3^{\xi}$. Suppose that $s_0 > 0$. Then, identifying the constant coefficients, we have $q^{\bullet}a\beta_0 T_2 = 0$. As a result, $\beta_0 = 0$, since $q^{\bullet}aT_2 \neq 0$. Hence, w can be written as $w = \sum_{\xi=1}^{n}\beta_\xi T_3^{\xi}$. Returning to $yT_3^{s_0} = w\Omega_1'$, we have that $yT_3^{s_0} = \sum_{\xi=1}^{n}\beta_\xi T_3^{\xi}\Omega_1' = \sum_{\xi=1}^{n} q^{\bullet}\beta_\xi\Omega_1' T_3^{\xi} = \Omega_1'\sum_{\xi=1}^{n} q^{\bullet}\beta_\xi' T_3^{\xi}$. This implies that $yT_3^{s_0-1} = \Omega_1' w'$, where $w' = \sum_{\xi=1}^{n} q^{\bullet}\beta_\xi' T_3^{\xi-1} \in A^{(4)}$, with $\beta_\xi' \in \mathbb{C}\langle E_{1,4}, T_2, T_4, T_5, T_6\rangle$. Consequently, $s_0 - 1 \in S$, a contradiction! Therefore, $s_0 = 0$ and $y = \Omega_1' v'' \in \langle\Omega_1'\rangle = \langle E_{1,4}T_3 + aT_2\rangle$. Hence, $\langle T_1\rangle_{A^{(3)}}[T_3^{-1}] \cap A^{(4)} \subseteq \langle E_{1,4}T_3 + aT_2\rangle$ as desired.

3. We want to show that $\langle E_{1,5}T_3 + aE_{2,5}\rangle = \langle\Omega_1'\rangle_{A^{(4)}}[T_4^{-1}] \cap A^{(5)}$. Note, $\Omega_1' = E_{1,4}T_3 + aT_2 = E_{1,5}T_3 + aE_{2,5}$. Observe that $\langle E_{1,5}T_3 + aE_{2,5}\rangle \subseteq \langle\Omega_1'\rangle_{A^{(4)}}[T_4^{-1}] \cap A^{(5)}$. We establish the reverse inclusion. Let $y \in \langle\Omega_1'\rangle_{A^{(4)}}[T_4^{-1}] \cap A^{(5)}$. Then, $y \in \langle\Omega_1'\rangle_{A^{(4)}}[T_4^{-1}]$. Therefore, there exists $i \in \mathbb{N}$ such that $yT_4^{i} \in \langle\Omega_1'\rangle_{A^{(4)}}$. This implies that $yT_4^{i} = \Omega_1' v$, for some $v \in A^{(4)}$. Since $A^{(4)}[T_4^{-1}] = A^{(5)}[T_4^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_4^{j} = v'$, for some $v' \in A^{(5)}$. It follows that $yT_4^{i+j} = \Omega_1' v T_4^{j} = \Omega_1' v'$. This implies that $yT_4^{\delta} = \Omega_1' v'$, where $\delta = i + j$. Set $S := \{s \in \mathbb{N} \mid \exists v' \in A^{(5)} : yT_4^{s} = \Omega_1' v'\}$. Since $\delta \in S$, we have that $S \neq \emptyset$. Let $s = s_0$ be the minimum element of S such that $yT_4^{s_0} = \Omega_1' v'$. We want to show that $s_0 = 0$. Note, $\Omega_1' T_5 = \Omega_1$ in $A^{(5)}$. Since Ω_1 is central in $A^{(5)}$, and T_5 is normal in $A^{(5)}$, we must have Ω_1' as a normal element in $A^{(5)}$. Therefore, there exists some $v'' \in A^{(5)}$ such that $yT_4^{s_0} = \Omega_1' v' = v''\Omega_1'$. Now, $A^{(5)}$ can be viewed as a free left $\mathbb{C}\langle E_{1,5}, E_{2,5}, T_3, T_5, T_6\rangle$−module with basis $\left(T_4^{\xi}\right)_{\xi\in\mathbb{N}}$. One can write $y = \sum_{\xi=0}^{n}\alpha_\xi T_4^{\xi}$ and $v'' = \sum_{\xi=0}^{n}\beta_\xi T_4^{\xi}$, where $\alpha_\xi, \beta_\xi \in \mathbb{C}\langle E_{1,5}, E_{2,5}, T_3, T_5, T_6\rangle$. This implies that $\sum_{\xi=0}^{n}\alpha_\xi T_4^{\xi+s_0} = \sum_{\xi=0}^{n}\beta_\xi T_4^{\xi}\Omega_1' = \sum_{\xi=0}^{n} q^{\bullet}\beta_\xi\Omega_1' T_4^{\xi}$ (note, $T_4\Omega_1' = q^{-3}\Omega_1' T_4$). Suppose that $s_0 > 0$. Then, identifying the constant coefficients, we have that $q^{\bullet}\beta_0\Omega_1' = 0$. Hence, $\beta_0 = 0$, since $q^{\bullet}\Omega_1' \neq 0$. One can therefore write v'' as $v'' = \sum_{\xi=1}^{n}\beta_\xi T_4^{\xi}$. Returning to $yT_4^{s_0} = v''\Omega_1'$, we have that $yT_4^{s_0} = \sum_{\xi=1}^{n}\beta_\xi T_4^{\xi}\Omega_1' = \sum_{\xi=1}^{n} q^{\bullet}\beta_\xi\Omega_1' T_4^{\xi} = \Omega_1'\sum_{\xi=1}^{n} q^{\bullet}\beta_\xi' T_4^{\xi}$, where $\beta_\xi' \in \mathbb{C}\langle E_{1,5}, E_{2,5}, T_3, T_5, T_6\rangle$. This implies that $yT_4^{s_0-1} = \Omega_1' w$, where $w = \sum_{\xi=1}^{n} q^{\bullet}\beta_\xi' T_4^{\xi-1} \in A^{(5)}$. Consequently, $s_0 - 1 \in S$, a contradiction! Therefore, $s_0 = 0$ and $y = \Omega_1' v' \in \langle\Omega_1'\rangle = \langle E_{1,5}T_3 + aE_{2,5}\rangle$. As a result,

$\langle\Omega_1'\rangle_{A^{(4)}}[T_4^{-1}] \cap A^{(5)} \subseteq \langle E_{1,5}T_3 + aE_{2,5}\rangle$ as desired.

4. Observe that $\Omega_1' = E_{1,5}T_3 + aE_{2,5} = \Omega_1 T_5^{-1}$ in $A^{(6)}[T_5^{-1}]$. We want to show that $\langle\Omega_1'\rangle_{A^{(5)}}[T_5^{-1}] \cap A^{(6)} = \langle\Omega_1\rangle_{A^{(6)}}$. Obviously, $\langle\Omega_1\rangle_{A^{(6)}} \subseteq \langle\Omega_1'\rangle_{A^{(5)}}[T_5^{-1}] \cap A^{(6)}$. We establish the reverse inclusion. Let $y \in \langle\Omega_1'\rangle_{A^{(5)}}[T_5^{-1}] \cap A^{(6)}$. This implies that $y \in \langle\Omega_1'\rangle_{A^{(5)}}[T_5^{-1}]$. There exists $i \in \mathbb{N}$ such that $yT_5^i \in \langle\Omega_1\rangle_{A^{(5)}}$. Hence, $yT_5^i = \Omega_1' v$, for some $v \in A^{(5)}$. Furthermore, since $A^{(5)}[T_5^{-1}] = A^{(6)}[T_5^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_5^j = v'$, for some $v' \in A^{(6)}$. It follows from $yT_5^i = \Omega_1' v$ that $yT_5^{i+j} = \Omega_1' vT_5^j = \Omega_1' v' = \Omega_1 T_5^{-1} v'$ (note, $\Omega_1' T_5 = \Omega_1$ in $A^{(6)}$). The multiplicative system generated by T_5 satisfies the Ore condition in $A^{(6)}$, hence, there exists $k \in \mathbb{N}$ and $v'' \in A^{(6)}$ such that $T_5^{-1}v' = v''T_5^{-k}$. One can therefore write $yT_5^{i+j} = \Omega_1 v'' T_5^k$. Hence, $yT_5^\delta = \Omega_1 v''$, where $\delta = i+j+k$. Set $S := \{s \in \mathbb{N} \mid \exists v'' \in A^{(6)} : yT_5^s = \Omega_1 v''\}$. Since $\delta \in S$, we have that $S \neq \emptyset$. Let $s = s_0$ be the minimum element of S such that $yT_5^{s_0} = \Omega_1 v''$. We want to show that $s_0 = 0$. Now, $A^{(6)}$ can be viewed as a free $\mathbb{C}\langle E_{1,6}, E_{2,6}, E_{3,6}, T_4, T_6\rangle$−module with basis $\left(T_5^\xi\right)_{\xi\in\mathbb{N}}$. One can write $y = \sum_{\xi=0}^n \alpha_\xi T_5^\xi$ and $v'' = \sum_{\xi=0}^n \beta_\xi T_5^\xi$, where $\alpha_\xi, \beta_\xi \in \mathbb{C}\langle E_{1,6}, E_{2,6}, E_{3,6}, T_4, T_6\rangle$. With this, $yT_5^{s_0} = \Omega_1 v''$ implies that $\sum_{\xi=0}^n \alpha_\xi T_5^{\xi+s_0} = \sum_{\xi=0}^n \beta_\xi \Omega_1 T_5^\xi$. Write $\Omega_1 = \gamma_1 T_5 + \gamma_2$, where $\gamma_1 = E_{1,6}E_{3,6} + aE_{2,6}$ and $\gamma_2 = a'E_{3,6}^2 + aE_{1,6}E_{4,6}$. It follows that $\sum_{\xi=0}^n \alpha_\xi T_5^{\xi+s_0} = \sum_{\xi=0}^n \beta_\xi \gamma_1 T_5^{\xi+1} + \sum_{\xi=0}^n \beta_\xi \gamma_2 T_5^\xi$. Suppose that $s_0 > 0$. Then, identifying the constant coefficients, we have that $\beta_0\gamma_2 = 0$. Hence, $\beta_0 = 0$, since $\gamma_2 \neq 0$. One can therefore write $v'' = \sum_{\xi=1}^n \beta_\xi T_5^\xi$. Returning to $yT_5^{s_0} = \Omega_1 v''$, we have that $yT_5^{s_0} = \Omega_1 \sum_{\xi=1}^n \beta_\xi T_5^\xi$. This implies that $yT_5^{s_0-1} = \Omega_1 w$, where $w = \sum_{\xi=1}^n \beta_\xi T_5^{\xi-1} \in A^{(6)}$. As a result, $s_0 - 1 \in S$, a contradiction! Therefore, $s_0 = 0$ and $y = \Omega_1 v'' \in \langle\Omega_1\rangle_{A^{(6)}}$. Consequently, $\langle\Omega_1'\rangle_{A^{(5)}}[T_5^{-1}] \cap A^{(6)} \subseteq \langle\Omega_1\rangle_{A^{(6)}}$ as desired.

5. We want to show that $\langle\Omega_1\rangle_{A^{(6)}}[T_6^{-1}] \cap A = \langle\Omega_1\rangle_A$.

Obviously, $\langle\Omega_1\rangle_A \subseteq \langle\Omega_1\rangle_{A^{(6)}}[T_6^{-1}] \cap A$. We establish the reverse inclusion. Let $y \in \langle\Omega_1\rangle_{A^{(6)}}[T_6^{-1}] \cap A$. This implies that $y \in \langle\Omega_1\rangle_{A^{(6)}}[T_6^{-1}]$. There exists $i \in \mathbb{N}$ such that $yT_6^i \in \langle\Omega_1\rangle_{A^{(6)}}$. Hence, $yT_6^i = \Omega_1 v$, for some $v \in A^{(6)}$. Furthermore, since $A^{(6)}[T_6^{-1}] = A[T_6^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_6^j = v'$, for some $v' \in A$. It follows from $yT_6^i = \Omega_1 v$ that $yT_6^{i+j} = \Omega_1 vT_6^j = \Omega_1 v'$. Hence, $yT_6^\delta = \Omega_1 v'$, where $\delta = i + j$. Set

$S := \{s \in \mathbb{N} \mid \exists v' \in A : yT_6^s = \Omega_1 v'\}$. Note, $\delta \in S$, hence S is non-empty. Let $s = s_0$ be the minimum element of S such that $yT_6^{s_0} = \Omega_1 v'$. We want to show that $s_0 = 0$. Now, A can be viewed as a free $\mathbb{C}\langle E_1, E_2, E_3, E_4, T_5\rangle-$module with basis $\left(T_6^\xi\right)_{\xi \in \mathbb{N}}$. One can write $y = \sum_{\xi=0}^n \alpha_\xi T_6^\xi$ and $v' = \sum_{\xi=0}^n \beta_\xi T_6^\xi$, where $\alpha_\xi, \beta_\xi \in \mathbb{C}\langle E_1, E_2, E_3, E_4, T_5\rangle$. With this, $yT_6^{s_0} = \Omega_1 v'$ implies that $\sum_{\xi=0}^n \alpha_\xi T_6^{\xi+s_0} = \sum_{\xi=0}^n \beta_\xi \Omega_1 T_6^\xi$. Suppose that $s_0 > 0$. Then, identifying constant coefficients, we have that $\beta_0 \Omega_1 = 0$. As a result, $\beta_0 = 0$, since $\Omega_1 \neq 0$. One can therefore write $v' = \sum_{\xi=1}^n \beta_\xi T_6^\xi$. Returning to $yT_6^{s_0} = \Omega_1 v'$, we have that $yT_6^{s_0} = \Omega_1 \sum_{\xi=1}^n \beta_\xi T_6^\xi$. This implies that $yT_6^{s_0-1} = \Omega_1 v''$, where $v'' = \sum_{\xi=1}^n \beta_\xi T_6^{\xi-1} \in A$. Hence, $s_0 - 1 \in S$, a contradiction! Therefore, $s_0 = 0$ and $y = \Omega_1 v' \in \langle \Omega_1 \rangle_A$. Consequently, $\langle \Omega_1 \rangle_{A^{(6)}}[T_6^{-1}] \cap A \subseteq \langle \Omega_1 \rangle_A$ as desired. ∎

The following result also shows that $\langle T_2 \rangle \in \text{Im}(\psi)$ and that $\langle \Omega_2 \rangle$ is the completely prime ideal of A such that $\psi(\langle \Omega_2 \rangle) = \langle T_2 \rangle$.

2.3.2 Lemma. $\langle \Omega_2 \rangle \in \text{Spec}(A)$.

Proof. Similar to the previous lemma, we also prove this result in several steps by showing that:

1. $\langle T_2 \rangle_{A^{(3)}} \in \text{Spec}(A^{(3)})$.
2. $\langle T_2 \rangle_{A^{(4)}} \in \text{Spec}(A^{(4)})$.
3. $\langle E_{2,5}T_4 + bT_3^3 \rangle = \langle T_2 \rangle_{A^{(4)}}[T_4^{-1}] \cap A^{(5)}$, hence $Q_1 := \langle E_{2,5}T_4 + bT_3^3 \rangle \in \text{Spec}(A^{(5)})$.
4. $\langle E_{2,6}T_4 + bE_{3,6}^3 \rangle = Q_1[T_5^{-1}] \cap A^{(6)}$, hence $Q_2 := \langle E_{2,6}T_4 + bE_{3,6}^3 \rangle \in \text{Spec}(A^{(6)})$.
5. $\langle \Omega_2 \rangle = Q_2[T_6^{-1}] \cap A$, hence $\langle \Omega_2 \rangle \in \text{Spec}(A)$.

We are now ready to prove the above claims.

1. The quotient algebra $A^{(3)}/\langle T_2 \rangle$ is isomorphic to a quantum affine space of rank 5, which is a domain, hence $\langle T_2 \rangle$ is a prime ideal in $A^{(3)}$.

2. Similar to (1).

3. Recall from the DDA of A that, $T_2 = E_{2,5} + bT_3^3 T_4^{-1}$. We want to show that $\langle E_{2,5}T_4 + bT_3^3 \rangle = \langle T_2 \rangle_{A^{(4)}}[T_4^{-1}] \cap A^{(5)}$. Observe that $\langle E_{2,5}T_4 + bT_3^3 \rangle \subseteq \langle T_2 \rangle_{A^{(4)}}[T_4^{-1}] \cap A^{(5)}$.

We establish the reverse inclusion. Let $y \in \langle T_2\rangle_{A^{(4)}}[T_4^{-1}] \cap A^{(5)}$. This implies that $y \in \langle T_2\rangle_{A^{(4)}}[T_4^{-1}]$. Therefore, there exists $i \in \mathbb{N}$ such that $yT_4^i \in \langle T_2\rangle_{A^{(4)}}$. This implies that $yT_4^i = T_2v$ for some $v \in A^{(4)}$. Since $A^{(4)}[T_4^{-1}] = A^{(5)}[T_4^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_4^j = v'$ for some $v' \in A^{(5)}$. It follows that $yT_4^{i+j} = T_2vT_4^j = T_2v' = (E_{2,5} + bT_3^3T_4^{-1})v' = (E_{2,5}T_4 + bT_3^3)T_4^{-1}v'$. Note, the multiplicative set generated by T_4 satisfies the Ore condition in $A^{(5)}$, hence, there exist $k \in \mathbb{N}$ and $v'' \in A^{(5)}$ such that $T_4^{-1}v' = v''T_4^{-k}$. One can therefore write $yT_4^{i+j} = (E_{2,5}T_4 + bT_3^3)v''T_4^{-k}$. This implies that $yT_4^{\delta} = \Omega_2'v''$, where $\Omega_2' := E_{2,5}T_4 + bT_3^3$ and $\delta = i + j + k$. Set $S := \{s \in \mathbb{N} \mid \exists v'' \in A^{(5)} : yT_4^s = \Omega_2'v''\}$. Since $\delta \in S$, we have that $S \neq \emptyset$. Let $s = s_0$ be the minimum element of S such that $yT_4^{s_0} = \Omega_2'v''$. Note, $\Omega_2'T_6 = \Omega_2$ in $A^{(5)}$. Given that Ω_2 is central in $A^{(5)}$, and T_6 is normal in $A^{(5)}$, we must have Ω_2' to be a normal element in $A^{(5)}$. Therefore, there exists $w \in A^{(5)}$ such that $yT_4^{s_0} = \Omega_2'v'' = w\Omega_2'$. Now, $A^{(5)}$ can be viewed as a free left $\mathbb{C}\langle E_{1,5}, E_{2,5}, T_3, T_5, T_6\rangle-$module with basis $\left(T_4^{\xi}\right)_{\xi\in\mathbb{N}}$. One can write $y = \sum_{\xi=0}^{n} \alpha_\xi T_4^\xi$ and $w = \sum_{\xi=0}^{n} \beta_\xi T_4^\xi$, where $\alpha_\xi, \beta_\xi \in \mathbb{C}\langle E_{1,5}, E_{2,5}, T_3, T_5, T_6\rangle$. It follows from $yT_4^{s_0} = w\Omega_2'$ that $\sum_{\xi=0}^{n} \alpha_\xi T_4^{\xi+s_0} = \sum_{\xi=0}^{n} \beta_\xi T_4^\xi \Omega_2' = \sum_{\xi=0}^{n} q^{\bullet}\beta_\xi \Omega_2' T_4^\xi$ (note, $T_4\Omega_2' = q^{-3}\Omega_2'T_4$). Since $\Omega_2' = E_{2,5}T_4 + bT_3^3$, we have that $\sum_{\xi=0}^{n} \alpha_\xi T_4^{\xi+s_0} = \sum_{\xi=0}^{n} q^{\bullet}\beta_\xi E_{2,5}T_4^{1+\xi} + \sum_{\xi=0}^{n} q^{\bullet}b\beta_\xi T_3^3T_4^\xi$. Suppose that $s_0 > 0$. Then, identifying the constant coefficients, we have: $q^{\bullet}b\beta_0T_3^3 = 0$. Hence, $\beta_0 = 0$, since $q^{\bullet}bT_3^3 \neq 0$. One can therefore write $w = \sum_{\xi=1}^{n} \beta_\xi T_4^\xi$. Returning to $yT_4^{s_0} = w\Omega_2'$, we have that $yT_4^{s_0} = \sum_{\xi=1}^{n} \beta_\xi T_4^\xi \Omega_2' = \sum_{\xi=1}^{n} q^{\bullet}\beta_\xi \Omega_2' T_4^\xi = \Omega_2' \sum_{\xi=1}^{n} q^{\bullet}\beta_\xi' T_4^\xi$ for some $\beta_\xi' \in \mathbb{C}\langle E_{1,5}, E_{2,5}, T_3, T_5, T_6\rangle$. This implies that $yT_4^{s_0-1} = \Omega_2'w'$, where $w' = \sum_{\xi=1}^{n} q^{\bullet}\beta_\xi' T_4^{\xi-1} \in A^{(5)}$. Consequently, $s_0 - 1 \in S$, a contradiction! Therefore, $s_0 = 0$ and $y = \Omega_2'v'' \in \langle\Omega_2'\rangle = \langle E_{2,5}T_4 + bT_3^3\rangle$. Hence, $\langle T_2\rangle_{A^{(4)}}[T_4^{-1}] \cap A^{(5)} \subseteq \langle E_{2,5}T_4 + bT_3^3\rangle$ as desired.

4. We want to show that $\langle E_{2,6}T_4 + bE_{3,6}^3\rangle = \langle\Omega_2'\rangle_{A^{(5)}}[T_5^{-1}] \cap A^{(6)}$. Observe that $\langle E_{2,6}T_4 + bE_{3,6}^3\rangle \subseteq \langle\Omega_2'\rangle_{A^{(5)}}[T_5^{-1}] \cap A^{(6)}$. We establish the reverse inclusion. Let $y \in \langle\Omega_2'\rangle_{A^{(5)}}[T_5^{-1}] \cap A^{(6)}$. This implies that $y \in \langle\Omega_2'\rangle_{A^{(5)}}[T_5^{-1}]$. Therefore, there exists $i \in \mathbb{N}$ such that $yT_5^i \in \langle\Omega_2'\rangle_{A^{(5)}}$. This implies that $yT_5^i = \Omega_2'v$, for some $v \in A^{(5)}$. Since

$A^{(5)}[T_5^{-1}] = A^{(6)}[T_5^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_5^j = v'$, for some $v' \in A^{(6)}$. It follows that $yT_5^{i+j} = \Omega_2' vT_5^j = \Omega_2' v'$. This implies that $yT_4^{\delta} = \Omega_2' v'$, where $\delta = i + j$. Set $S := \{s \in \mathbb{N} \mid \exists v' \in A^{(6)} : yT_5^s = \Omega_2' v'\}$. Since $\delta \in S$, we have that $S \neq \emptyset$. Let $s = s_0$ be the minimum element of S such that $yT_5^{s_0} = \Omega_2' v'$. Note, $\Omega_2' T_6 = \Omega_2$ in $A^{(6)}$. Given that Ω_2 is central in $A^{(6)}$, and T_6 is normal in $A^{(6)}$, we must have Ω_2' as a normal element in $A^{(6)}$. Therefore, there exists $v'' \in A^{(6)}$ such that $yT_5^{s_0} = \Omega_2' v' = v''\Omega_2'$. Now, $A^{(6)}$ can be viewed as a free left $\mathbb{C}\langle E_{1,6}, E_{2,6}, E_{3,6}, T_4, T_6\rangle-$module with basis $\left(T_5^{\xi}\right)_{\xi\in\mathbb{N}}$. One can write $y = \sum_{\xi=0}^{n} \alpha_\xi T_5^{\xi}$ and $v'' = \sum_{\xi=0}^{n} \beta_\xi T_5^{\xi}$, where $\alpha_\xi, \beta_\xi \in \mathbb{C}\langle E_{1,6}, E_{2,6}, E_{3,6}, E_{4,6}, T_6\rangle$. It follows from $yT_5^{s_0} = v''\Omega_2'$ that $\sum_{\xi=0}^{n} \alpha_\xi T_5^{\xi+s_0} = \sum_{\xi=0}^{n} \beta_\xi T_5^{\xi}\Omega_2' = \sum_{\xi=0}^{n} q^{\bullet}\beta_\xi \Omega_2' T_5^{\xi}$ (note, $T_5\Omega_2' = q^{-3}\Omega_2' T_5$). Suppose that $s_0 > 0$. Then, identifying the constant coefficients, we have that $q^{\bullet}\beta_0\Omega_2' = 0$. Hence, $\beta_0 = 0$, since $q^{\bullet}\Omega_2' \neq 0$. One can therefore write $v'' = \sum_{\xi=1}^{n} \beta_\xi T_5^{\xi}$. Returning to $yT_5^{s_0} = v''\Omega_2'$, we have that $yT_5^{s_0} = \sum_{\xi=1}^{n} \beta_\xi T_5^{\xi}\Omega_2' = \Omega_2' \sum_{\xi=1}^{n} q^{\bullet}\beta_\xi' T_5^{\xi}$. This implies that $yT_5^{s_0-1} = \Omega_2' w$, where $w = \sum_{\xi=1}^{n} q^{\bullet}\beta_\xi' T_5^{\xi-1} \in A^{(6)}$. Consequently, $s_0 - 1 \in S$, a contradiction! Therefore, $s_0 = 0$ and $y = \Omega_2' v' \in \langle\Omega_2'\rangle = \langle E_{2,6}T_4 + bE_{3,6}^3\rangle$. Hence, $\langle\Omega_2'\rangle_{A^{(5)}}[T_5^{-1}] \cap A^{(6)} \subseteq \langle E_{2,6}T_4 + bE_{3,6}^3\rangle$ as desired.

5. Note, $\Omega_2' = E_{2,6}T_4 + bE_{3,6}^3 = \Omega_2 T_6^{-1}$ in $A[T_6^{-1}]$. We want to show that $\langle\Omega_2'\rangle_{A^{(6)}}[T_6^{-1}] \cap A = \langle\Omega_2\rangle_A$. Obviously, $\langle\Omega_2\rangle_A \subseteq \langle\Omega_2'\rangle_{A^{(6)}}[T_6^{-1}] \cap A$. We establish the reverse inclusion. Let $y \in \langle\Omega_2'\rangle_{A^{(6)}}[T_6^{-1}] \cap A$. This implies that $y \in \langle\Omega_2'\rangle_{A^{(6)}}[T_6^{-1}]$. There exists $i \in \mathbb{N}$ such that $yT_6^i \in \langle\Omega_2'\rangle_{A^{(6)}}$. Hence, $yT_6^i = \Omega_2' v$, for some $v \in A^{(6)}$. Furthermore, since $A^{(6)}[T_6^{-1}] = A[T_6^{-1}]$, there exist $j \in \mathbb{N}$ such that $vT_6^j = v'$, for some $v' \in A$. It follows from $yT_6^i = \Omega_2' v$ that $yT_6^{i+j} = \Omega_2' vT_6^j = \Omega_2' v' = \Omega_2 T_6^{-1} v'$. The multiplicative system generated by T_6 satisfies the Ore condition, hence, there exists $k \in \mathbb{N}$ and $v'' \in A$ such that $T_6^{-1}v' = v''T_6^k$. It follows that $yT_6^{i+j} = \Omega_2 T_6^{-1} v' = \Omega_2 v'' T_6^{-k}$. Hence, $yT_6^{\delta} = \Omega_2 v''$, where $\delta = i + j + k$. Set $S := \{s \in \mathbb{N} \mid \exists v'' \in A : yT_6^s = \Omega_2 v''\}$. Since $\delta \in S$, we have that $S \neq \emptyset$. Let $s = s_0$ be the minimum element of S such that $yT_6^{s_0} = \Omega_2 v''$. Now, A can be viewed as a free $\mathbb{C}\langle E_1, E_2, E_3, E_4, T_5\rangle-$module with basis $\left(T_6^{\xi}\right)_{\xi\in\mathbb{N}}$. One can write $y = \sum_{\xi=0}^{n} \alpha_\xi T_6^{\xi}$ and $v'' = \sum_{\xi=0}^{n} \beta_\xi T_6^{\xi}$, where $\alpha_\xi, \beta_\xi \in \mathbb{C}\langle E_1, E_2, E_3, E_4, T_5\rangle$. With this, $yT_6^{s_0} = \Omega_2 v''$ implies that $\sum_{\xi=0}^{n} \alpha_\xi T_6^{\xi+s_0} =$

$\sum_{\xi=0}^{n} \beta_\xi \Omega_2 T_6^{\xi}$. From Subsection 2.2.3, one can write Ω_2 as $\Omega_2 = \gamma_1 T_6 + \gamma_2$, where $\gamma_1 = E_2E_4 + bE_3^3$ and $\gamma_2 = bE_2T_5^3 + b'E_3^2E_5^2 + c'E_3E_4T_5 + d'E_4^2$. It follows that $\sum_{\xi=0}^{n} \alpha_\xi T_6^{\xi+s_0} = \sum_{\xi=0}^{n} \beta_\xi \gamma_1 T_6^{\xi+1} + \sum_{\xi=0}^{n} \beta_\xi \gamma_2 T_6^{\xi}$. Suppose that $s_0 > 0$. Then, identifying the constant coefficients, we have that $\beta_0\gamma_2 = 0$. Hence, $\beta_0 = 0$, since $\gamma_2 \neq 0$. One can therefore write $v'' = \sum_{\xi=1}^{n} \beta_\xi T_6^{\xi}$. Returning to $yT_6^{s_0} = \Omega_2 v''$, we have that $yT_6^{s_0} = \Omega_2 \sum_{\xi=1}^{n} \beta_\xi T_6^{\xi}$. This implies that $yT_6^{s_0-1} = \Omega_2 w$, where $w = \sum_{\xi=1}^{n} \beta_\xi T_6^{\xi-1} \in A$. Consequently, $s_0 - 1 \in S$, a contradiction! Therefore, $s_0 = 0$ and $y = \Omega_2 v'' \in \langle \Omega_2 \rangle_A$. Hence, $\langle \Omega_2' \rangle_{A^{(6)}}[T_6^{-1}] \cap A \subseteq \langle \Omega_2 \rangle_A$ as desired. ■

2.4 Height two maximal ideals of $U_q^+(G_2)$

This section focuses on studying the height two maximal ideals of $A = U_q^+(G_2)$. First and foremost, we will begin by describing the $\mathcal{H}-$invariant prime ideals of A, and then proceed to study the $\mathcal{H}-$strata corresponding to $\langle 0 \rangle$, $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$. We will conclude by describing the height two maximal ideals of A.

2.4.1 $\mathcal{H}-$Spec(A). Recall from Section 2.1 that $A = \mathbb{C}[E_1][E_2;\sigma_2][E_3;\sigma_3,\delta_3]\cdots[E_6;\sigma_6,\delta_6]$. The torus $\mathcal{H} := (\mathbb{C}^*)^2$ acts rationally on A via $(\alpha_1,\alpha_6)\cdot E_i = \alpha_i E_i$; with $i = 1, 6$; for all $(\alpha_1,\alpha_6) \in \mathcal{H}$. Using the defining relations of A, one can easily verify that

$$(\alpha_1,\alpha_6)\cdot E_2 = \alpha_1^3\alpha_6 E_2 \qquad (\alpha_1,\alpha_6)\cdot E_3 = \alpha_1^2\alpha_6 E_3$$

$$(\alpha_1,\alpha_6)\cdot E_4 = \alpha_1^3\alpha_6^2 E_4 \qquad (\alpha_1,\alpha_6)\cdot E_5 = \alpha_1\alpha_6 E_5.$$

We now proceed to describe the set $\mathcal{H}-$Spec(A). In Section 1.1, we studied the Weyl group $\mathscr{W}$ of G_2. It follows from Remark 1.6.7 that the poset of $\mathcal{H}-$Spec(A) is isomorphic to the poset of $\mathscr{W}$. For each $\varepsilon \in \mathscr{W}$, one can associate a unique element of $\mathcal{H}-$Spec(A), which we denote by $K(\varepsilon)$. As a result, $\mathcal{H}-$Spec$(A) := \{K(\varepsilon) \mid \forall \varepsilon \in \mathscr{W}\}$. We have that $|\mathcal{H}-\text{Spec}(A)| = |\mathscr{W}| = 12$. One can therefore partition Spec(A) into a disjoint union of strata as follows:

$$\text{Spec}(A) = \bigsqcup_{\varepsilon \in \mathscr{W}} \text{Spec}_{K(\varepsilon)}(A).$$

Moreover; for $\varepsilon, \varepsilon' \in \mathscr{W}$ with $\varepsilon \preceq \varepsilon'$, we have that $K(\varepsilon) \subset K(\varepsilon')$, where $\preceq$ is the Bruhat order [44, Theorem 3.8]. From Section 1.1, we have that $\omega_0 := s_1s_2s_1s_2s_1s_2 = s_2s_1s_2s_1s_2s_1$ in $\mathscr{W}$. The posets of $\mathscr{W}$ and $\mathcal{H}-\text{Spec}(A)$ are seen in Figure 2.1.

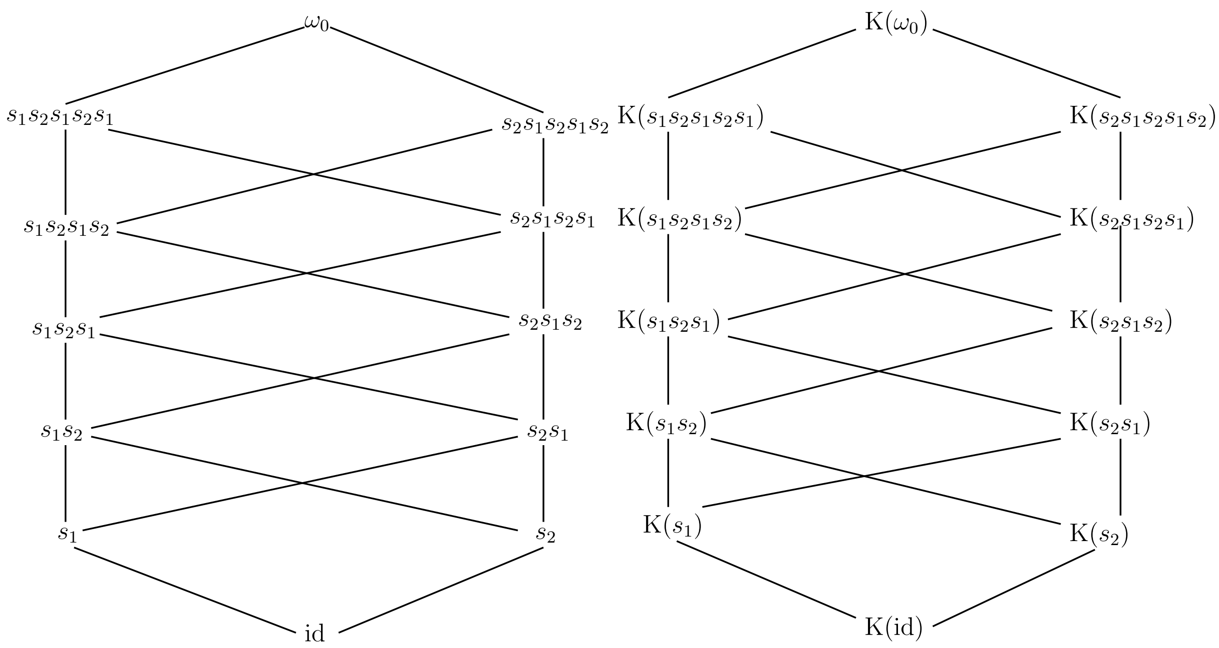


Figure 2.1: Posets of $\mathscr{W}$ and $\mathcal{H}-\text{Spec}(A)$.

Recall from Subsection 2.2.3 that $\bar{A}$ is a quantum affine space. The map $\psi := \psi_2 \circ \cdots \circ \psi_6 : \text{Spec}(A) \hookrightarrow \text{Spec}(\bar{A})$ is a canonical embedding. This canonical embedding restricts to the $\mathcal{H}-$invariant prime spectrum. That is, $\psi : \mathcal{H}-\text{Spec}(A) \hookrightarrow \mathcal{H}-\text{Spec}(\bar{A})$. Let W be the set of all the subsets of $\{1, \cdots, 6\}$. For each $w \in W$, $K_w := \langle T_i \mid i \in w\rangle$ is a $\mathcal{H}-$invariant prime ideal of $\bar{A}$ (Proposition 1.6.9). We therefore have that $\mathcal{H}-\text{Spec}(\bar{A}) := \{K_w \mid w \in W\}$. Observe that the set $\{\langle T_i \rangle \mid 1 \leq i \leq 6\}$ is the set of all the height one $\mathcal{H}-$invariant prime ideals of $\bar{A}$. Since $\psi : \text{Spec}(A) \hookrightarrow \text{Spec}(\bar{A})$, one can easily deduce from Figure 2.1 that there are only two height one $\mathcal{H}-$invariant prime ideals of A. From Lemmas 2.3.1 and 2.3.2, we have that $\langle T_1 \rangle = \psi(\langle \Omega_1 \rangle)$ and $\langle T_2 \rangle = \psi(\langle \Omega_2 \rangle)$. Since ψ preserves the height of a prime ideal, we have the following lemma.

2.4.2 Lemma. 1. $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$ are the only height one $\mathcal{H}-$invariant prime ideals of A.

2. Every non-zero $\mathcal{H}-$invariant prime ideal of A contains either $\langle \Omega_1 \rangle$ or $\langle \Omega_2 \rangle$ as a result of Figure 2.1.

2.4.3 $\mathcal{H}-$strata corresponding to $\langle 0 \rangle$, $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$. In this subsection, we aim to find the $\mathcal{H}-$strata corresponding to $\langle 0 \rangle$, $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$. We state and prove the results in Propositions 2.4.4, 2.4.5 and 2.4.6 using strategies similar to [33, Propositions 2.3 and 2.4]. Note, in this subsection, all ideals in $A = U_q^+(G_2)$ will simply be written as $\langle \Theta \rangle$, where $\Theta \in A$. However, if we want to refer to an ideal in any other algebra, say R, then that ideal will be written as $\langle \Theta \rangle_R$, where in this case, $\Theta \in R$.

2.4.4 Proposition. Let $\mathcal{P}$ be the set of those unitary irreducible polynomials $P(\Omega_1, \Omega_2) \in \mathbb{C}[\Omega_1, \Omega_2]$ with $P(\Omega_1, \Omega_2) \neq \Omega_1$ and $P(\Omega_1, \Omega_2) \neq \Omega_2$. Then, $\text{Spec}_{\langle 0 \rangle}(A) = \{\langle 0 \rangle\} \cup \{\langle P(\Omega_1, \Omega_2) \rangle \mid P(\Omega_1, \Omega_2) \in \mathcal{P}\} \cup \{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \mid \alpha, \beta \in \mathbb{C}^*\}$.

Proof. We claim that $\text{Spec}_{\langle 0 \rangle}(A) = \{Q \in \text{Spec}(A) \mid \Omega_1, \Omega_2 \notin Q\}$. To establish this claim, let us assume that this is not the case. That is, suppose there exists $Q \in \text{Spec}_{\langle 0 \rangle}(A)$ such that $\Omega_1, \Omega_2 \in Q$; then the product $\Omega_1\Omega_2$ which is an $\mathcal{H}-$eigenvector belongs to Q. Consequently, $\Omega_1\Omega_2 \in \bigcap_{h \in \mathcal{H}} h \cdot Q = \langle 0 \rangle$, a contradiction. Hence, we have shown that $\text{Spec}_{\langle 0 \rangle}(A) \subseteq \{Q \in \text{Spec}(A) \mid \Omega_1, \Omega_2 \notin Q\}$. Conversely, suppose that $Q \in \text{Spec}(A)$ such that $\Omega_1, \Omega_2 \notin Q$, then $\bigcap_{h \in \mathcal{H}} h \cdot Q$ is an $\mathcal{H}-$invariant prime ideal of A, which contains neither Ω_1 nor Ω_2. Obviously, the only possibility for $\bigcap_{h \in \mathcal{H}} h \cdot Q$ is $\langle 0 \rangle$ since every non-zero $\mathcal{H}-$invariant prime ideal contains at least Ω_1 or Ω_2. Thus, $\bigcap_{h \in \mathcal{H}} h \cdot Q = \langle 0 \rangle$. Hence, $Q \in \text{Spec}_{\langle 0 \rangle}(A)$. Therefore, $\{Q \in \text{Spec}(A) \mid \Omega_1, \Omega_2 \notin Q\} \subseteq \text{Spec}_{\langle 0 \rangle}(A)$. This confirms our claim.

Since $\Omega_1, \Omega_2 \in Z(A)$, we have that the set $\{\Omega_1^i \Omega_2^j \mid i, j \in \mathbb{N}\}$ is a right denominator set in the noetherian domain A. One can now localize A as $R := A[\Omega_1^{-1}, \Omega_2^{-1}]$. Let $Q \in \text{Spec}_{\langle 0 \rangle}(A)$, the map $\phi : Q \longrightarrow Q[\Omega_1^{-1}, \Omega_2^{-1}]$ is an increasing bijection from $\text{Spec}_{\langle 0 \rangle}(A)$ onto $\text{Spec}(R)$.

Since Ω_1 and Ω_2 are $\mathcal{H}-$eigenvectors, and $\mathcal{H}$ acts on A, we have that $\mathcal{H}$ also acts on R. Let us verify that R is $\mathcal{H}-$simple before we describe Spec(R). Now, ϕ still induces a bijection between the set of those $\mathcal{H}-$invariant prime ideals of Spec$_{\langle 0\rangle}(A)$ and the set of $\mathcal{H}-$invariant prime ideals of Spec(R). It is already known that the set of $\mathcal{H}-$invariant prime ideals of A that neither contains Ω_1 nor Ω_2 consists only of the zero ideal $\{\langle 0\rangle\}$ (Lemma 2.4.2(2)). This implies that $\langle 0\rangle_R$ is the only $\mathcal{H}-$invariant prime ideal of R. Every $\mathcal{H}-$invariant proper ideal of R is contained in an $\mathcal{H}-$invariant prime ideal of R [17, §3.1(v)]. Therefore, $\langle 0\rangle_R$ is the only unique $\mathcal{H}-$invariant proper ideal of R. This confirms that R is $\mathcal{H}-$simple.

We proceed to describe Spec(R) and Spec$_{\langle 0\rangle}(A)$. We deduce from [7, Exercise II.3.A] that the action of $\mathcal{H}$ on R is rational. This rational action coupled with R being $\mathcal{H}-$simple implies that the extension and contraction maps provide a mutually inverse bijection between Spec(R) and Spec$(Z(R))$ [7, Corollary II.3.9]. From Lemma 2.2.6, $Z(A) = \mathbb{C}[\Omega_1, \Omega_2]$, and so $Z(R) = \mathbb{C}[\Omega_1^{\pm 1}, \Omega_2^{\pm 1}]$. Since $\mathbb{C}$ is algebraically closed, we have that Spec$(Z(R)) = \{\langle 0\rangle_{Z(R)}\} \cup \{\langle P(\Omega_1,\Omega_2)\rangle_{Z(R)} \mid P(\Omega_1,\Omega_2) \in \mathcal{P}\} \cup \{\langle \Omega_1-\alpha, \Omega_2-\beta\rangle_{Z(R)} \mid \alpha, \beta \in \mathbb{C}^*\}$. Since there is an inverse bijection between Spec(R) and Spec$(Z(R))$, and also R is $\mathcal{H}-$simple, one can recover Spec(R) from Spec$(Z(R))$ as follows: Spec$(R) = \{\langle 0\rangle_R\} \cup \{\langle P(\Omega_1,\Omega_2)\rangle_R \mid P(\Omega_1,\Omega_2) \in \mathcal{P}\} \cup \{\langle \Omega_1-\alpha, \Omega_2-\beta\rangle_R \mid \alpha, \beta \in \mathbb{C}^*\}$. It follows that Spec$_{\langle 0\rangle}(A) = \{\langle 0\rangle_R \cap A\} \cup \{\langle P(\Omega_1,\Omega_2)\rangle_R \cap A \mid P(\Omega_1,\Omega_2) \in \mathcal{P}\} \cup \{\langle \Omega_1-\alpha, \Omega_2-\beta\rangle_R \cap A \mid \alpha, \beta \in \mathbb{C}^*\}$.

Undoubtedly, $\langle 0\rangle_R \cap A = \langle 0\rangle$. We now have to show that $\langle P(\Omega_1,\Omega_2)\rangle_R \cap A = \langle P(\Omega_1,\Omega_2)\rangle, \forall P(\Omega_1,\Omega_2) \in \mathcal{P}$, and $\langle \Omega_1-\alpha, \Omega_2-\beta\rangle_R \cap A = \langle \Omega_1-\alpha, \Omega_2-\beta\rangle, \forall \alpha, \beta \in \mathbb{C}^*$ to complete the proof.

Fix $P(\Omega_1,\Omega_2) \in \mathcal{P}$. Observe that $\langle P(\Omega_1,\Omega_2)\rangle \subseteq \langle P(\Omega_1,\Omega_2)\rangle_R \cap A$. To show the reverse inclusion, let $y \in \langle P(\Omega_1,\Omega_2)\rangle_R \cap A$. This implies that $y = dP(\Omega_1,\Omega_2)$, where $d \in R$, since $y \in \langle P(\Omega_1,\Omega_2)\rangle_R$. Also, $d \in R$ implies that there exist $i, j \in \mathbb{N}$ such that $d = a\Omega_1^{-i}\Omega_2^{-j}$, where $a \in A$. Therefore, $y = a\Omega_1^{-i}\Omega_2^{-j}P(\Omega_1,\Omega_2)$, which implies that $y\Omega_1^i\Omega_2^j = aP(\Omega_1,\Omega_2)$. Choose $(i,j) \in \mathbb{N}^2$ minimal (in the lexicographic order on $\mathbb{N}^2$) such that the equality holds. Without loss of generality, suppose that $i > 0$, then $aP(\Omega_1, \Omega_2) \in \langle \Omega_1 \rangle$. Given that $\langle \Omega_1 \rangle$ is a completely prime ideal, it implies that $a \in \langle \Omega_1 \rangle$ or $P(\Omega_1, \Omega_2) \in \langle \Omega_1 \rangle$. Since $P(\Omega_1, \Omega_2) \in \mathcal{P}$, it implies that $P(\Omega_1, \Omega_2) \notin \langle \Omega_1 \rangle$, hence $a \in \langle \Omega_1 \rangle$. This further implies that $a = t\Omega_1$, where $t \in A$. Returning to $y\Omega_1^i\Omega_2^j = aP(\Omega_1, \Omega_2)$, we have that $y\Omega_1^i\Omega_2^j = t\Omega_1 P(\Omega_1, \Omega_2)$. Therefore, $y\Omega_1^{i-1}\Omega_2^j = tP(\Omega_1, \Omega_2)$. This clearly contradicts the minimality of (i, j), hence $(i, j) = (0, 0)$, and $y = aP(\Omega_1, \Omega_2) \in \langle P(\Omega_1, \Omega_2) \rangle$. Consequently, $\langle P(\Omega_1, \Omega_2) \rangle_R \cap A = \langle P(\Omega_1, \Omega_2) \rangle$ for all $P(\Omega_1, \Omega_2) \in \mathcal{P}$ as desired.

Similarly, we show that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle_R \cap A = \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle; \forall \alpha, \beta \in \mathbb{C}^*$. Fix $\alpha, \beta \in \mathbb{C}^*$. Observe that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \subseteq \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle_R \cap A$. We establish the reverse inclusion. Let $y \in \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle_R \cap A$. Since $y \in \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle_R$, there exist $i, j \in \mathbb{N}$ such that $y\Omega_1^i\Omega_2^j = m(\Omega_1 - \alpha) + n(\Omega_2 - \beta)$, where $m, n \in A$. Choose $(i, j) \in \mathbb{N}^2$ minimal (in the lexicographic order on $\mathbb{N}^2$) such that the equality holds. Without loss of generality, suppose that $i > 0$ and let $f : A \longrightarrow A/\langle \Omega_2 - \beta \rangle$ be a canonical surjection. We have that $f(y)f(\Omega_1)^i f(\Omega_2)^j = f(m)f(\Omega_1 - \alpha)$. It follows that $f(m)f(\Omega_1 - \alpha) \in \langle f(\Omega_1) \rangle$. Observe that $f(\Omega_1 - \alpha) \notin \langle f(\Omega_1) \rangle$, hence $f(m) \in \langle f(\Omega_1) \rangle$. Therefore, $\exists \lambda \in A$ such that $f(m) = f(\lambda)f(\Omega_1)$. Consequently, $f(y)f(\Omega_1)^i f(\Omega_2)^j = f(\lambda)f(\Omega_1)f(\Omega_1 - \alpha)$. Since $f(\Omega_1) \neq 0$, it implies that $f(y)f(\Omega_1)^{i-1} f(\Omega_2)^j = f(\lambda)f(\Omega_1 - \alpha)$. Therefore, $y\Omega_1^{i-1}\Omega_2^j = \lambda(\Omega_1 - \alpha) + \lambda'(\Omega_2 - \beta)$ for some $\lambda' \in A$. This contradicts the minimality of (i, j). Hence, $(i, j) = (0, 0)$ and so $y = m(\Omega_1 - \alpha) + n(\Omega_2 - \beta) \in \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$. In conclusion, $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle_R \cap A = \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$, $\forall \alpha, \beta \in \mathbb{C}^*$.

■

2.4.5 Proposition. $\text{Spec}_{\langle \Omega_1 \rangle}(A) = \{\langle \Omega_1 \rangle\} \cup \{\langle \Omega_1, \Omega_2 - \beta \rangle \mid \beta \in \mathbb{C}^*\}$.

Proof. We claim that $\text{Spec}_{\langle \Omega_1 \rangle}(A) = \{Q \in \text{Spec}(A) \mid \Omega_1 \in Q \text{ and } \Omega_2 \notin Q\}$. To establish this claim, let us assume that this is not the case. That is, suppose there exists $Q \in \text{Spec}_{\langle \Omega_1 \rangle}(A)$ such that $\Omega_2 \in Q$. Since Ω_2 is an $\mathcal{H}-$eigenvector, $\Omega_2 \in \bigcap_{h \in \mathcal{H}} h \cdot Q = \langle \Omega_1 \rangle$, a contradiction. Hence, we have shown that $\text{Spec}_{\langle \Omega_1 \rangle}(A) \subseteq \{Q \in \text{Spec}(A) \mid \Omega_1 \in Q \text{ and } \Omega_2 \notin Q\}$. Conversely, suppose that $Q \in \text{Spec}(A)$ and $\Omega_1 \in Q$ but $\Omega_2 \notin Q$. Then $\bigcap_{h \in \mathcal{H}} h \cdot Q$ is an $\mathcal{H}-$invariant prime ideal of A, which contains Ω_1 but does not contain

Ω_2. The only possibility for $\bigcap_{h\in\mathcal{H}} h\cdot Q$ is the ideal $\langle\Omega_1\rangle$. Hence, $Q \in \text{Spec}_{\langle\Omega_1\rangle}(A)$. Therefore, $\{Q \in \text{Spec}(A) \mid \Omega_1 \in Q \text{ and } \Omega_2 \notin Q\} \subseteq \text{Spec}_{\langle\Omega_1\rangle}(A)$. This confirms our claim.

Let $\Lambda : A \longrightarrow A/\langle\Omega_1\rangle$ be a canonical surjection. Since A is a noetherian domain and $\langle\Omega_1\rangle$ is a completely prime ideal, one can also deduce that $A/\langle\Omega_1\rangle$ is a noetherian domain. Given that $\Omega_2 \in Z(A)$, we have that $\Lambda(\Omega_2) \neq 0 \in Z(A/\langle\Omega_1\rangle)$. Therefore, the set $\{\Lambda(\Omega_2)^i \mid i \in \mathbb{N}\}$ is a right denominator set in $A/\langle\Omega_1\rangle$. One can therefore localize $A/\langle\Omega_1\rangle$ at this denominator set as $R := \dfrac{A}{\langle\Omega_1\rangle}[\Lambda(\Omega_2)^{-1}]$. Given that $Q \in \text{Spec}_{\langle\Omega_1\rangle}(A)$, the map $\phi : Q \longrightarrow \dfrac{Q}{\langle\Omega_1\rangle}[\Lambda(\Omega_2)^{-1}]$ is an increasing bijection from $\text{Spec}_{\langle\Omega_1\rangle}(A)$ onto $\text{Spec}(R)$.

Since $\langle\Omega_1\rangle$ is an $\mathcal{H}-$invariant prime ideal, $\Lambda(\Omega_2)$ is an $\mathcal{H}-$eigenvector, and $\mathcal{H}$ acts on A, we have that $\mathcal{H}$ also acts on R. Let us verify that R is $\mathcal{H}-$simple before we describe $\text{Spec}(R)$. Now, ϕ still induces a bijection between the set of those $\mathcal{H}-$invariant prime ideals of $\text{Spec}_{\langle\Omega_1\rangle}(A)$ and the set of $\mathcal{H}-$invariant prime ideals of $\text{Spec}(R)$. The set of $\mathcal{H}-$invariant prime ideals of A that contains Ω_1 but does not contain Ω_2 is $\{\langle\Omega_1\rangle\}$, implying that $\langle 0\rangle_R$ is the only $\mathcal{H}-$invariant prime ideal of R (note, $\Omega_1 = 0$ in R). Every $\mathcal{H}-$invariant proper ideal of R is contained in an $\mathcal{H}-$invariant prime ideal of R [17, §3.1(v)]. Therefore, $\langle 0\rangle_R$ is the only unique $\mathcal{H}-$invariant proper ideal of R. This confirms that R is $\mathcal{H}-$simple.

We proceed to describe the $\text{Spec}(R)$ and $\text{Spec}_{\langle\Omega_1\rangle}(A)$. We deduce from [7, Exercise II.3.A] that the action of $\mathcal{H}$ on R is rational. This rational action coupled with R being $\mathcal{H}-$simple implies that the extension and contraction maps provide a mutually inverse bijection between $\text{Spec}(R)$ and $\text{Spec}(Z(R))$ [7, Corollary II.3.9]. One can deduce that $Z(A/\langle\Omega_1\rangle) = \mathbb{C}[\Lambda(\Omega_2)]$, and so $Z(R) = \mathbb{C}[\Lambda(\Omega_2)^{\pm 1}]$. Since $\mathbb{C}$ is algebraically closed, we have that $\text{Spec}(Z(R)) = \{\langle 0\rangle_{Z(R)}\} \cup \{\langle\Lambda(\Omega_2) - \beta\rangle_{Z(R)} \mid \beta \in \mathbb{C}^*\}$. We now recover $\text{Spec}(R)$ from $\text{Spec}(Z(R))$ as follows: $\text{Spec}(R) = \{\langle 0\rangle_R\} \cup \{\langle\Lambda(\Omega_2) - \beta\rangle_R \mid \beta \in \mathbb{C}^*\}$, since there is a mutually inverse bijection between $\text{Spec}(R)$ and $\text{Spec}(Z(R))$, and also R is $\mathcal{H}-$simple. Moreover, since ϕ is an increasing bijection from $\text{Spec}_{\langle\Omega_1\rangle}(A)$ onto

Spec(R), we have that Spec$_{\langle\Omega_1\rangle}(A) = \{\phi^{-1}(\langle 0\rangle_R)\} \cup \{\phi^{-1}(\langle\Lambda(\Omega_2)-\beta\rangle_R) \mid \beta \in \mathbb{C}^*\}$.

Naturally, $\phi^{-1}(\langle 0\rangle_R) = \langle\Omega_1\rangle$. We now show that $\phi^{-1}(\langle\Lambda(\Omega_2)-\beta\rangle_R) = \langle\Omega_1, \Omega_2-\beta\rangle$. Fix $\beta \in \mathbb{C}^*$. Observe that $\langle\Omega_1,\Omega_2-\beta\rangle \subseteq \phi^{-1}(\langle\Lambda(\Omega_2)-\beta\rangle_R)$. To show the reverse inclusion, let $x \in \phi^{-1}(\langle\Lambda(\Omega_2)-\beta\rangle_R)$. Then $\Lambda(x) \in \langle\Lambda(\Omega_2)-\beta\rangle_R$. This implies that $\Lambda(x) = d(\Lambda(\Omega_2)-\beta)$, where $d \in R$. Also, $d \in R$ implies that there exists $t \geq 0$ and $a \in A/\langle\Omega_1\rangle$ such that $d = a\Lambda(\Omega_2)^{-t}$. As a result, $\Lambda(x) = d(\Lambda(\Omega_2)-\beta) = a\Lambda(\Omega_2)^{-t}(\Lambda(\Omega_2)-\beta)$. Hence, $\Lambda(x)\Lambda(\Omega_2)^t = a(\Lambda(\Omega_2)-\beta)$. Choose t minimum such that the equality holds. If $t > 0$, then $\Lambda(x)\Lambda(\Omega_2)^t = a\Lambda(\Omega_2) - a\beta$. One can therefore write $a = \frac{1}{\beta}\Lambda(\Omega_2)(a - \Lambda(x)\Lambda(\Omega_2)^{t-1})$. Set $b := \frac{1}{\beta}(a-\Lambda(x)\Lambda(\Omega_2)^{t-1})$, then $a = b\Lambda(\Omega_2)$. This implies that $\Lambda(x)\Lambda(\Omega_2)^t = a(\Lambda(\Omega_2)-\beta) = b\Lambda(\Omega_2)(\Lambda(\Omega_2)-\beta)$ which further implies that $\Lambda(x)\Lambda(\Omega_2)^{t-1} = b(\Lambda(\Omega_2)-\beta)$. This contradicts the minimality of t, and so $t = 0$. Therefore, $\Lambda(x) = a(\Lambda(\Omega_2)-\beta) \in \langle\Lambda(\Omega_2)-\beta\rangle_R$. Consequently, $x \in \langle\Omega_1,\Omega_2-\beta\rangle$ and $\phi^{-1}(\langle\Lambda(\Omega_2)-\beta\rangle_R) \subseteq \langle\Omega_1,\Omega_2-\beta\rangle$. ∎

2.4.6 Proposition. Spec$_{\langle\Omega_2\rangle}(A) = \{\langle\Omega_2\rangle\} \cup \{\langle\Omega_1-\alpha,\Omega_2\rangle \mid \alpha \in \mathbb{C}^*\}$.

Proof. We claim that Spec$_{\langle\Omega_2\rangle}(A) = \{Q \in \text{Spec}(A) \mid \Omega_1 \notin Q \text{ and } \Omega_2 \in Q\}$. To establish this claim, let us assume that this is not the case. That is, suppose there exists $Q \in$ Spec$_{\langle\Omega_2\rangle}(A)$ such that $\Omega_1 \in Q$. Since Ω_1 is an $\mathcal{H}-$eigenvector, $\Omega_1 \in \bigcap_{h\in\mathcal{H}} h\cdot Q = \langle\Omega_2\rangle$, a contradiction. Hence, we have shown that Spec$_{\langle\Omega_2\rangle}(A) \subseteq \{Q \in \text{Spec}(A) \mid \Omega_1 \notin Q \text{ and } \Omega_2 \in Q\}$. Conversely, suppose that $Q \in$ Spec(A) and $\Omega_2 \in Q$ but $\Omega_1 \notin Q$, then $\bigcap_{h\in\mathcal{H}} h\cdot Q$ is an $\mathcal{H}-$invariant prime ideal of A, which contains Ω_2 but does not contain Ω_1. The only possibility for $\bigcap_{h\in\mathcal{H}} h\cdot Q$ is the ideal $\langle\Omega_2\rangle$. Hence, $Q \in$ Spec$_{\langle\Omega_2\rangle}(A)$. Therefore, $\{Q \in \text{Spec}(A) \mid \Omega_1 \notin Q \text{ and } \Omega_2 \in Q\} \subseteq$ Spec$_{\langle\Omega_2\rangle}(A)$. This confirms our claim.

Let $\Lambda : A \longrightarrow A/\langle\Omega_2\rangle$ be a canonical surjection. Since A is a noetherian domain and $\langle\Omega_2\rangle$ is a completely prime ideal, one can also deduce that $A/\langle\Omega_2\rangle$ is a noetherian domain. Given that $\Omega_1 \in Z(A)$, we have $\Lambda(\Omega_1) \in Z(A/\langle\Omega_2\rangle)$. It follows that the set $\{\Lambda(\Omega_1)^i \mid i \in \mathbb{N}\}$ is a right denominator set in $A/\langle\Omega_2\rangle$. One can therefore localize $A/\langle\Omega_2\rangle$ as $R := \dfrac{A}{\langle\Omega_2\rangle}[\Lambda(\Omega_1)^{-1}]$. Given $Q \in$ Spec$_{\langle\Omega_2\rangle}(A)$, we have a map $\phi : Q \longrightarrow$

$\frac{Q}{\langle\Omega_2\rangle}[\Lambda(\Omega_1)^{-1}]$ which is an increasing bijection from Spec$_{\langle\Omega_2\rangle}(A)$ onto Spec(R).

Similar to the proof of Proposition 2.4.5, one can show that R is $\mathcal{H}-$simple and that the extension and contraction maps provide a mutually inverse bijection between Spec(R) and Spec$(Z(R))$ [7, Corollary II.3.9]. We also have that $Z(A/\langle\Omega_2\rangle) = \mathbb{C}[\Lambda(\Omega_1)]$, and so $Z(R) = \mathbb{C}[\Lambda(\Omega_1)^{\pm 1}]$. Since $\mathbb{C}$ is algebraically closed, we have that Spec$(Z(R)) = \{\langle 0\rangle_{Z(R)}\} \cup \{\langle\Lambda(\Omega_1)-\alpha\rangle_{Z(R)} \mid \alpha \in \mathbb{C}^*\}$. One can recover Spec$(R)$ from Spec$(Z(R))$ as follows: Spec$(R) = \{\langle 0\rangle_R\} \cup \{\langle\Lambda(\Omega_1)-\alpha\rangle_R \mid \alpha \in \mathbb{C}^*\}$, since there is a mutual inverse bijection between Spec(R) and Spec$(Z(R))$, and also R is $\mathcal{H}-$simple. Moreover, since ϕ is an increasing bijection between Spec$_{\langle\Omega_2\rangle}(A)$ and Spec(R), we have that Spec$_{\langle\Omega_2\rangle}(A) = \{\phi^{-1}(\langle 0\rangle_R)\} \cup \{\phi^{-1}(\langle\Lambda(\Omega_1)-\alpha\rangle_R) \mid \alpha \in \mathbb{C}^*\}$.

Naturally, $\phi^{-1}(\langle 0\rangle_R) = \langle\Omega_2\rangle$. We now show that $\phi^{-1}(\langle\Lambda(\Omega_1)-\alpha\rangle_R) = \langle\Omega_1-\alpha, \Omega_2\rangle$. Fix $\alpha \in \mathbb{C}^*$. Observe that $\langle\Omega_1-\alpha,\Omega_2\rangle \subseteq \phi^{-1}(\langle\Lambda(\Omega_1)-\alpha\rangle_R)$. To show the reverse inclusion, let $x \in \phi^{-1}(\langle\Lambda(\Omega_1)-\alpha\rangle_R)$. Then, $\Lambda(x) \in \langle\Lambda(\Omega_1)-\alpha\rangle_R$. Let $d \in R$, then there exists $t \geq 0$ and $a \in A/\langle\Omega_2\rangle$ such that $d = a\Lambda(\Omega_1)^{-t}$. Given that $\Lambda(x) \in \langle\Lambda(\Omega_1)-\alpha\rangle_R$, it follows that $\Lambda(x) = d(\Lambda(\Omega_1)-\alpha) = a\Lambda(\Omega_1)^{-t}(\Lambda(\Omega_1)-\alpha)$ which further implies that $\Lambda(x)\Lambda(\Omega_1)^t = a(\Lambda(\Omega_1)-\alpha)$. Choose t minimum such that the equality holds. Suppose that $t > 0$, then $\Lambda(x)\Lambda(\Omega_1)^t = a\Lambda(\Omega_1) - a\alpha$. We have that $a = \frac{1}{\alpha}\Lambda(\Omega_1)(a - \Lambda(x)\Lambda(\Omega_1)^{t-1})$. Set $b := \frac{1}{\alpha}(a-\Lambda(x)\Lambda(\Omega_1)^{t-1})$. It follows that $a = b\Lambda(\Omega_1)$, with $b \in A/\langle\Omega_2\rangle$. This implies that $\Lambda(x)\Lambda(\Omega_1)^t = a(\Lambda(\Omega_1)-\alpha) = b\Lambda(\Omega_1)(\Lambda(\Omega_1)-\alpha)$, hence $\Lambda(x)\Lambda(\Omega_1)^{t-1} = b(\Lambda(\Omega_1)-\alpha)$. This contradicts the minimality of t, and so $t = 0$. Therefore, $\Lambda(x) = a(\Lambda(\Omega_1)-\alpha)$ belongs to the ideal of $A/\langle\Omega_2\rangle$ generated by $\Lambda(\Omega_1)-\alpha$. Consequently, $x \in \langle\Omega_1-\alpha,\Omega_2\rangle$ and $\phi^{-1}(\langle\Lambda(\Omega_1)-\alpha\rangle_R) \subseteq \langle\Omega_1-\alpha,\Omega_2\rangle$. ∎

2.4.7 Corollary. Let $(\alpha,\beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. The prime ideal $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$ of A is primitive.

Proof. Since Prim$_J(A)$ are just the maximal prime ideals in Spec$_J(A)$ (Lemma 1.6.6), it follows that $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$ is a primitive ideal of A for each $(\alpha,\beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. ∎

2.4.8 Proposition. Let $(\alpha,\beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. The prime ideal $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$ of A is maximal.

Proof. Let $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Suppose that there exists a maximal ideal I of A such that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \subsetneq I \subsetneq A$. Let J be the $\mathcal{H}-$invariant prime ideal in A such that $I \in \text{Spec}_J(A)$. By Propositions 2.4.4, 2.4.5 and 2.4.6, J cannot be $\langle 0 \rangle$, $\langle \Omega_1 \rangle$ or $\langle \Omega_2 \rangle$, since either of these will lead to a contradiction. Every non-zero $\mathcal{H}-$invariant prime ideal contains Ω_1 only or Ω_2 only or both. Since $J \neq \langle \Omega_1 \rangle, \langle \Omega_2 \rangle$, it implies that J contains both Ω_1 and Ω_2. Moreover, since $J \subseteq I$, it implies that $\Omega_1, \Omega_2 \in I$. Given that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \subset I$, we have that $\Omega_1 - \alpha, \Omega_2 - \beta \in I$. It follows that $\alpha, \beta \in I$, hence $I = A$, a contradiction! This confirms that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ is a maximal ideal in A. ■

2.4.9 Remark. Since the algebra A is catenary [19, Theorem 4.8], one can prove that the height of $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ is 2 for all $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ by constructing chains of prime ideals of $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$. Nonetheless, we use a different approach to prove the result in Section 3.1. We do this because the discussion in that section (i.e. Section 3.1) will be useful in the subsequent chapters.

Chapter 3

Simple quotients of $U_q^+(G_2)$ and their relation to the second Weyl algebra

Now that we have found some family of maximal ideals of $A = U_q^+(G_2)$, we are going to study the main algebra of interest in this thesis, namely, the corresponding simple quotients. In view of Dixmier's theorem, we consider these simple quotients as deformations of (a quadratic extension of) the second Weyl algebra $A_2(\mathbb{C})$, and so we compare their properties with some known properties of the Weyl algebras. Recall from Proposition 2.4.8 that $\Omega_1 - \alpha$ and $\Omega_2 - \beta$, where $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$, generate a maximal ideal of A. As a result, the corresponding quotient

$$A_{\alpha,\beta} := \frac{A}{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle}$$

is a simple noetherian domain. Denote the canonical images of E_i in $A_{\alpha,\beta}$ by $e_i := E_i + \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ for all $1 \leq i \leq 6$. The algebra $A_{\alpha,\beta}$ satisfies the following relations:

$$
\begin{aligned}
&e_2e_1 = q^{-3}e_1e_2 && e_3e_1 = q^{-1}e_1e_3 - (q+q^{-1}+q^{-3})e_2 \\
&e_3e_2 = q^{-3}e_2e_3 && e_4e_1 = e_1e_4 + (1-q^2)e_3^2 \\
&e_4e_2 = q^{-3}e_2e_4 - \frac{q^4-2q^2+1}{q^4+q^2+1}e_3^3 && e_4e_3 = q^{-3}e_3e_4 \\
&e_5e_1 = qe_1e_5 - (1+q^2)e_3 && e_5e_2 = e_2e_5 + (1-q^2)e_3^2 \\
&e_5e_3 = q^{-1}e_3e_5 - (q+q^{-1}+q^{-3})e_4 && e_5e_4 = q^{-3}e_4e_5 \\
&e_6e_1 = q^3e_1e_6 - q^3e_5 && e_6e_2 = q^3e_2e_6 + (q^4+q^2-1)e_4 + (q^2-q^4)e_3e_5 \\
&e_6e_3 = e_3e_6 + (1-q^2)e_5^2 && e_6e_4 = q^{-3}e_4e_6 - \frac{q^4-2q^2+1}{q^4+q^2+1}e_5^3 \\
&e_6e_5 = q^{-3}e_5e_6,
\end{aligned}
$$

and

$$e_1e_3e_5 + ae_1e_4 + ae_2e_5 + a'e_3^2 = \alpha, \tag{3.0.1}$$

$$e_2e_4e_6 + be_2e_5^3 + be_3^3e_6 + b'e_3^2e_5^2 + c'e_3e_4e_5 + d'e_4^2 = \beta. \tag{3.0.2}$$

The following additional relations of $A_{\alpha,\beta}$ in the lemma below will be very helpful in this chapter, particularly, in Section 3.2. Note, we put constant coefficients of monomials in a square bracket [ ] in order to distinguish them from monomials easily. These constants are defined in Appendix C.

3.0.1 Lemma.

(1) $e_3^2 = c_1\alpha + [c_2]e_2e_5 + [c_2]e_1e_4 + [c_3]e_1e_3e_5.$

(2)
$$
\begin{aligned}
e_4^2 =& b_1\beta + [b_2]e_2e_4e_6 + [b_3]e_2e_5^3 + [b_4\alpha]e_3e_6 + [b_5]e_2e_3e_5e_6 + [b_6]e_1e_3e_4e_6 \\
&+ [b_7\alpha]e_1e_5e_6 + [b_8]e_1e_2e_5^2e_6 + [b_9]e_1^2e_4e_5e_6 + [b_{10}]e_1^2e_3e_5^2e_6 + [b_{11}\alpha]e_5^2 \\
&+ [b_{12}]e_1e_3e_5^3 + [b_{13}]e_3e_4e_5 + [b_{14}]e_1e_4e_5^2.
\end{aligned}
$$

$$
\begin{aligned}
(3)\ e_3^2e_4 =&[c_1\alpha]e_4 + [q^{-3}c_2]e_2e_4e_5 + [c_2b_4\alpha]e_1e_3e_6 + [b_{15}]e_1e_3e_4e_5 + [\beta b_1c_2]e_1\\
&+ [c_2b_3]e_1e_2e_5^3 + [c_2b_5]e_1e_2e_3e_5e_6 + [c_2b_6]e_1^2e_3e_4e_6 + [c_2b_7\alpha]e_1^2e_5e_6\\
&+ [c_2b_{11}\alpha]e_1e_5^2 + [c_2b_{12}]e_1^2e_3e_5^3 + [c_2b_8]e_1^2e_2e_5^2e_6 + [c_2b_9]e_1^3e_4e_5e_6\\
&+ [c_2b_{10}]e_1^3e_3e_5^2e_6 + [c_2b_{14}]e_1^2e_4e_5^2 + [c_2b_2]e_1e_2e_4e_6.
\end{aligned}
$$

$$
\begin{aligned}
(4)\ e_3e_4^2 =&[\beta b_1]e_3 + [k_1]e_2e_3e_4e_6 + [k_2]e_2e_3e_5^3 + [k_3\alpha^2]e_6 + [k_4\alpha]e_2e_5e_6 + [k_5\alpha]e_1e_4e_6\\
&+ [k_6\alpha]e_1e_3e_5e_6 + [k_7]e_2^2e_5^2e_6 + [k_8\beta]e_1e_5 + [k_9]e_1e_2e_4e_5e_6 + [k_{10}]e_1^3e_2e_5^2e_6^2\\
&+ [k_{11}\alpha]e_4e_5 + [k_{12}\alpha]e_1^3e_5e_6^2 + [k_{13}]e_1^2e_3e_4e_5e_6 + [k_{14}\beta]e_1^2e_6 + [b_{11}\alpha]e_3e_5^2\\
&+ [k_{15}]e_1^2e_2e_4e_6^2 + [k_{16}]e_1^2e_2e_5^3e_6 + [k_{17}\alpha]e_1^2e_3e_6^2 + [k_{18}]e_1^2e_2e_3e_5e_6^2 + [k_{19}]e_1^3e_3e_4e_6^2\\
&+ [k_{20}]e_1^4e_4e_5e_6^2 + [k_{21}\alpha]e_1^2e_5^2e_6 + [k_{22}]e_1^4e_3e_5^2e_6^2 + [k_{23}]e_1^3e_3e_5^3e_6 + [k_{24}]e_1^3e_4e_5^2e_6\\
&+ [k_{25}]e_1e_5^3 + [k_{26}]e_1^2e_4e_5^3 + [k_{27}]e_1^2e_3e_5^4 + [k_{28}]e_2e_4e_5^2 + [k_{29}]e_1e_3e_4e_5^2\\
&+ [k_{30}]e_1e_2e_5^4 + [k_{31}]e_1e_2e_3e_5^2e_6.
\end{aligned}
$$

The rest of the chapter is organized as follows. In Section 3.1, we prove that the GKdim of $A_{\alpha,\beta}$ is 4 and consequently prove that the height of the maximal ideal $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$ is 2 as expected. Section 3.2 focuses on describing a linear basis of $A_{\alpha,\beta}$. Ultimately, we show in Section 3.3 that at appropriate choices of α and β, the algebra $A_{\alpha,\beta}$ is a quadratic extension of the second Weyl algebra $A_2(\mathbb{C})$ at $q=1$. In the next chapter, we study the derivations of $A_{\alpha,\beta}$.

3.1 Gelfand-Kirillov dimension of $A_{\alpha,\beta}$

Let $\alpha,\beta\neq 0$. Recall from Section 2.2 that $R_1=\mathbb{C}_{q^M}[T_1^{\pm 1},\cdots,T_6^{\pm 1}]$ is the quantum torus associated to the quantum affine space $\bar{A}=A^{(2)}$. Also, $\Omega_1=T_1T_3T_5$ and $\Omega_2=T_2T_4T_6$ in $\bar{A}$. It follows from [8, Theorem 5.4.1] that there exists an Ore set $S_{\alpha,\beta}$ in $A_{\alpha,\beta}$ such that $A_{\alpha,\beta}S_{\alpha,\beta}^{-1}\cong R_1/\langle T_1T_3T_5-\alpha,T_2T_4T_6-\beta\rangle$.

Now, set

$$\mathscr{A}_{\alpha,\beta} := \frac{R_1}{\langle T_1T_3T_5 - \alpha, T_2T_4T_6 - \beta\rangle}.$$

Let $t_i := T_i + \langle T_1T_3T_5 - \alpha, T_2T_4T_6 - \beta\rangle$ denote the canonical images of the generators T_i of R_1 in $\mathscr{A}_{\alpha,\beta}$. The algebra $\mathscr{A}_{\alpha,\beta}$ is a quantum torus generated by $t_1^{\pm 1}, \cdots, t_6^{\pm 1}$ subject to the following relations:

$$t_it_j = q^{\mu_{ij}}t_jt_i \qquad t_1 = \alpha t_5^{-1}t_3^{-1} \qquad t_2 = \beta t_6^{-1}t_4^{-1},$$

for all $1 \leq i, j \leq 6$ and $\mu_{ij} \in M$ (the skew-symmetric matrix in Section 2.2). Observe that $\mathscr{A}_{\alpha,\beta} \cong \mathbb{C}_{q^N}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$, where the skew-symmetric matrix N can easily be deduced from M as follows:

$$N := \begin{bmatrix} 0 & 3 & 1 & 0 \\ -3 & 0 & 3 & 3 \\ -1 & -3 & 0 & 3 \\ 0 & -3 & -3 & 0 \end{bmatrix}.$$

Secondly, suppose that $\alpha = 0$ and $\beta \neq 0$.

Then, $A_{0,\beta}S_{0,\beta}^{-1} \cong \mathscr{A}_{0,\beta} = R_1/\langle T_1, T_2T_4T_6 - \beta\rangle$. The algebra $\mathscr{A}_{0,\beta}$ is generated by $t_2^{\pm 1}, \cdots, t_6^{\pm 1}$ subject to the relations

$$t_it_j = q^{\mu_{ij}}t_jt_i \qquad t_1 = 0 \qquad t_2 = \beta t_6^{-1}t_4^{-1},$$

for all $1 \leq i, j \leq 6$ and $\mu_{ij} \in M$. We also have that $\mathscr{A}_{0,\beta} \cong \mathbb{C}_{q^N}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$. Finally, when $\alpha \neq 0$ and $\beta = 0$, then one can also verify that $\mathscr{A}_{\alpha,0} \cong \mathbb{C}_{q^N}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$. As a result, we have that $A_{\alpha,\beta}S_{\alpha,\beta}^{-1} \cong \mathscr{A}_{\alpha,\beta} \cong \mathbb{C}_{q^N}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$. With a slight abuse of notation, we write $A_{\alpha,\beta}S_{\alpha,\beta}^{-1} = \mathscr{A}_{\alpha,\beta} = \mathbb{C}_{q^N}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$ for all $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. It follows from [18, Theorem 6.3] that GKdim$(A_{\alpha,\beta}) =$ GKdim$(A_{\alpha,\beta}S_{\alpha,\beta}^{-1}) =$ GKdim$(\mathscr{A}_{\alpha,\beta}) = 4$. Since Tauvel's height formula holds in $A = U_q^+(G_2)$ (Chapter 2), we have that GKdim$(A) =$ ht$(\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle) +$ GKdim$(A_{\alpha,\beta})$. We already know from Chapter 2 that the GKdim$(A) = 6$. Therefore, ht$(\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle) = 2$ for all $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$.

3.2 Linear basis for $A_{\alpha,\beta}$

Set $A_\beta := A/\langle \Omega_2 - \beta \rangle$, where $\beta \in \mathbb{C}$. Now, denote the canonical images of E_i by $\widehat{e_i} := E_i + \langle \Omega_2 - \beta \rangle$ in A_β. Clearly, $A_{\alpha,\beta} \cong A_\beta/\langle \widehat{\Omega}_1 - \alpha \rangle$. As a result, one can identify $A_{\alpha,\beta}$ with $A_\beta/\langle \widehat{\Omega}_1 - \alpha \rangle$. Moreover, the algebra A_β satisfies the relations of $A = U_q^+(G_2)$ and

$$\widehat{e_2}\widehat{e_4}\widehat{e_6} + b\widehat{e_2}\widehat{e_5}^3 + b\widehat{e_3}^3\widehat{e_6} + b'\widehat{e_3}^2\widehat{e_5}^2 + c'\widehat{e_3}\widehat{e_4}\widehat{e_5} + d'\widehat{e_4}^2 = \beta. \tag{3.2.1}$$

From Propositions 2.4.4 and 2.4.6, one can conclude that $\langle \Omega_2 - \beta \rangle$ is a completely prime ideal (since it is a prime ideal) of A for all $\beta \in \mathbb{C}$. Hence, similar to $A_{\alpha,\beta}$, the algebra A_β is a noetherian domain.

We are now going to find a linear basis for $A_{\alpha,\beta}$, where $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Since $A_{\alpha,\beta}$ is identified with $A_\beta/\langle \widehat{\Omega}_1 - \alpha \rangle$, we will first and foremost find a basis for A_β, and then proceed to find a basis for $A_{\alpha,\beta}$. Note, the relations in Lemma A.1.4 are also valid in A_β and $A_{\alpha,\beta}$, and are going to be very useful in this section.

3.2.1 Proposition. The set $\mathfrak{S} = \{\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^k\widehat{e_4}^\xi\widehat{e_5}^l\widehat{e_6}^m \mid i,j,k,l,m \in \mathbb{N} \text{ and } \xi = 0,1\}$ is a $\mathbb{C}-$basis of A_β.

Proof. Since the family $(\Pi_{s=1}^6 E_s^{i_s})_{i_s \in \mathbb{N}}$ is a PBW-basis of A over $\mathbb{C}$, it follows that the family $(\Pi_{s=1}^6 \widehat{e_s}^{i_s})_{i_s \in \mathbb{N}}$ is a spanning set of A_β over $\mathbb{C}$. We want to show that $\mathfrak{S}$ spans A_β. We do this by showing that $\Pi_{s=1}^6 \widehat{e_s}^{i_s}$ can be written as a finite linear combination of the elements of $\mathfrak{S}$ for all $i_1, \cdots, i_6 \in \mathbb{N}$ by an induction on i_4. The result is obvious when $i_4 = 0$ or 1. For $i_4 \geq 1$, assume that

$$\Pi_{s=1}^6 \widehat{e_s}^{i_s} = \sum_{(\xi,\underline{v}) \in I} a_{(\xi,\underline{v})} \widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^k\widehat{e_4}^\xi\widehat{e_5}^l\widehat{e_6}^m,$$

where $\underline{v} := (i,j,k,l,m) \in \mathbb{N}^5$ and $a_{(\xi,\underline{v})}$ are all complex numbers. Note, I is a finite subset of $\{0,1\}\times\mathbb{N}^5$. It follows from the commutation relations of A_β (see Lemma A.1.4) that

$$\widehat{e_1}^{i_1}\widehat{e_2}^{i_2}\widehat{e_3}^{i_3}\widehat{e_4}^{i_4+1}\widehat{e_5}^{i_5}\widehat{e_6}^{i_6} = q^{\bullet}\Pi_{s=1}^{6}\widehat{e_s}^{i_s}\widehat{e_4} - q^{\bullet}d_1[i_6]\widehat{e_1}^{i_1}\widehat{e_2}^{i_2}\widehat{e_3}^{i_3}\widehat{e_4}^{i_4}\widehat{e_5}^{i_5+3}\widehat{e_6}^{i_6-1}.$$

From the inductive hypothesis, $\widehat{e_1}^{i_1}\widehat{e_2}^{i_2}\widehat{e_3}^{i_3}\widehat{e_4}^{i_4}\widehat{e_5}^{i_5+3}\widehat{e_6}^{i_6-1} \in \text{Span}(\mathfrak{S})$. Hence, we proceed to show that $\Pi_{s=1}^{6}\widehat{e_s}^{i_s}\widehat{e_4}$ is also in the span of $\mathfrak{S}$. From the inductive hypothesis, we have

$$\Pi_{s=1}^{6}\widehat{e_s}^{i_s}\widehat{e_4} = \sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{\xi}\widehat{e_5}^{l}\widehat{e_6}^{m}\widehat{e_4}.$$

Using the commutation relations in Lemma A.1.4, we have that

$$\begin{aligned}\Pi_{s=1}^{6}\widehat{e_s}^{i_s}\widehat{e_4} = &\sum_{(\xi,\underline{v})\in I} q^{\bullet}a_{(\xi,\underline{v})}\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{\xi+1}\widehat{e_5}^{l}\widehat{e_6}^{m}\\ &+\sum_{(\xi,\underline{v})\in I} q^{\bullet}d_1[m]a_{(\xi,\underline{v})}\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{\xi}\widehat{e_5}^{l+3}\widehat{e_6}^{m-1}.\end{aligned}$$

All the terms in the above expression belong to the span of $\mathfrak{S}$ except $\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{2}\widehat{e_5}^{l}\widehat{e_6}^{m}$. From (3.2.1), we have that

$$\widehat{e_4}^2 = \beta_0\widehat{e_2}\widehat{e_4}\widehat{e_6} + b\beta_0\widehat{e_2}\widehat{e_5}^3 + b\beta_0\widehat{e_3}^3\widehat{e_6} + b'\beta_0\widehat{e_3}^2\widehat{e_5}^2 + c'\beta_0\widehat{e_3}\widehat{e_4}\widehat{e_5} - \beta\beta_0, \tag{3.2.2}$$

where $\beta_0 = -1/d'$. Substituting (3.2.2) into $\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{2}\widehat{e_5}^{l}\widehat{e_6}^{m}$, one can easily verify that $\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{2}\widehat{e_5}^{l}\widehat{e_6}^{m} \in \text{Span}(\mathfrak{S})$. Therefore, $\widehat{e_1}^{i_1}\widehat{e_2}^{i_2}\widehat{e_3}^{i_3}\widehat{e_4}^{i_4+1}\widehat{e_5}^{i_5}\widehat{e_6}^{i_6}$ can be written as a finite linear combination of the elements of $\mathfrak{S}$ over $\mathbb{C}$ for all $i_1,\cdots,i_6 \in \mathbb{N}$. By the principle of mathematical induction, $\mathfrak{S}$ is a spanning set of A_β over $\mathbb{C}$.

We further show that $\mathfrak{S}$ is a linearly independent set. Suppose that

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{\xi}\widehat{e_5}^{l}\widehat{e_6}^{m} = 0.$$

Since $A_\beta = A/\langle \Omega_2 - \beta \rangle$, it implies that

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})} E_1^i E_2^j E_3^k E_4^\xi E_5^l E_6^m = (\Omega_2 - \beta)\nu,$$

with $\nu \in A$. Write $\nu = \sum_{(i,\cdots,n)\in J} b_{(i,\cdots,n)} E_1^i E_2^j E_3^k E_4^l E_5^m E_6^n$, where J is a finite subset of $\mathbb{N}^6$ and $b_{(i,\cdots,n)}$ are all complex numbers. It follows that

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})} E_1^i E_2^j E_3^k E_4^\xi E_5^l E_6^m = \sum_{(i,\cdots,n)\in J} b_{(i,\cdots,n)} E_1^i E_2^j E_3^k (\Omega_2 - \beta) E_4^l E_5^m E_6^n. \tag{3.2.3}$$

Before we continue the proof, the following needs to be noted.

♣ Let (i',j',k',l',m',n'), $(i,j,k,l,m,n) \in \mathbb{N}^6$. We say that $(i,j,k,l,m,n) <_4 (i',j',k',l',m',n')$ if $[l < l']$ or $[l = l'$ and $i < i']$ or $[l = l',\ i = i'$ and $j < j']$ or $[l = l',\ i = i',\ j = j'$ and $k < k']$ or $[l = l',\ i = i',\ j = j',\ k = k'$ and $m < m']$ or $[l = l',\ i = i',\ j = j',\ k = k',\ m = m'$ and $n \le n']$. Note, the purpose of the square bracket [ ] is to differentiate the options.

From Section 2.2, we have that $\Omega_2 = E_2E_4E_6 + bE_2E_5^3 + bE_3^3E_6 + b'E_3^2E_5^2 + c'E_3E_4E_5 + d'E_4^2$ in $A = U_q^+(G_2)$. Now,

$$\begin{aligned}\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})} E_1^i E_2^j E_3^k E_4^\xi E_5^l E_6^m &= \sum_{(i,\cdots,n)\in J} b_{(i,\cdots,n)} E_1^i E_2^j E_3^k (\Omega_2 - \beta) E_4^l E_5^m E_6^n \\ &= \sum_{(i,\cdots,n)\in J} d' b_{(i,\cdots,m)} E_1^i E_2^j E_3^k E_4^{l+2} E_5^m E_6^n + \mathrm{LT}_{<_4},\end{aligned}$$

where $\mathrm{LT}_{<_4}$ contains lower order terms with respect to $<_4$ (as in ♣). Moreover, $\mathrm{LT}_{<_4}$ vanishes when $b_{(i,\cdots,n)} = 0$ for all $(i,\cdots,n) \in J$ (one can easily confirm this by fully expanding the right hand side of (3.2.3)).

Now, suppose that there exists $(i,j,k,l,m,n) \in J$ such that $b_{(i,j,k,l,m,n)} \neq 0$. Let (i',j',k',l',m',n') be the greatest element of J with respect to $<_4$ (defined in ♣ above) such that $b_{(i',j',k',l',m',n')} \neq 0$. Note, the family $(E_1^i E_2^j E_3^k E_4^l E_5^m E_6^n)_{(i,\cdots,n)\in\mathbb{N}^6}$ is a basis of A. Since $\mathrm{LT}_{<_4}$ contains lower order terms, identifying the coefficients of $E_1^{i'}E_2^{j'}E_3^{k'}E_4^{l'+2}E_5^{m'}E_6^{n'}$ in the above equality, we have that $d'b_{(i',\cdots,n')} = 0$. Since $b_{(i',j',k',l',m',n')} \neq 0$, it follows that $d' = q^{12}/(q^6-1) = 0$, a contradiction (see Appendix C for the expression of d'). As a result, $b_{(i,j,k,l,m,n)} = 0$ for all $(i,j,k,l,m,n) \in J$. Therefore, $\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}E_1^iE_2^jE_3^kE_4^{\xi}E_5^lE_6^m = 0$. Since $(E_1^iE_2^jE_3^kE_4^lE_5^mE_6^n)_{(i,\cdots,n)\in\mathbb{N}^6}$ is a basis of A, it follows that $a_{(\xi,\underline{v})} = 0$ for all $(\xi,\underline{v}) \in I$. In conclusion, $\mathfrak{S}$ is a linearly independent set and hence forms a basis of A_β as desired. ■

3.2.2 Proposition. Let $(\alpha,\beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. The set $\mathcal{B} = \{e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2}e_5^ke_6^l \mid i,j,k,l \in \mathbb{N}$ and $\epsilon_1,\epsilon_2 \in \{0,1\}\}$ is a $\mathbb{C}-$basis of $A_{\alpha,\beta}$.

Proof. Since the set $\mathfrak{S} = \{\widehat{e_1}^{i_1}\widehat{e_2}^{i_2}\widehat{e_3}^{i_3}\widehat{e_4}^{\xi}\widehat{e_5}^{i_5}\widehat{e_6}^{i_6} \mid i_1,i_2,i_3,i_5,i_6 \in \mathbb{N}$ and $\xi = 0,1\}$ is a $\mathbb{C}-$basis of A_β (Proposition 3.2.1), and $A_{\alpha,\beta}$ is identified with $A_\beta/\langle\widehat{\Omega}_1-\alpha\rangle$, it follows that $(e_1^{i_1}e_2^{i_2}e_3^{i_3}e_4^{\xi}e_5^{i_5}e_6^{i_6})_{i_1,\cdots,i_6\in\mathbb{N}}$, with $\xi \in \{0,1\}$, is a spanning set of $A_{\alpha,\beta}$ over $\mathbb{C}$. We want to show that $\mathcal{B}$ spans $A_{\alpha,\beta}$ by showing that $e_1^{i_1}e_2^{i_2}e_3^{i_3}e_4^{\xi}e_5^{i_5}e_6^{i_6}$ can be written as a finite linear combination of the elements of $\mathcal{B}$ over $\mathbb{C}$ for all $i_1,i_2,i_3,i_5,i_6 \in \mathbb{N}$ and $\xi = 0,1$. By Proposition 3.2.1, it is sufficient to do this by an induction on i_3. The result is obvious when $i_3 = 0$ or 1. For $i_3 \geq 1$, suppose that

$$e_1^{i_1}e_2^{i_2}e_3^{i_3}e_4^{\xi}e_5^{i_5}e_6^{i_6} = \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2}e_5^ke_6^l,$$

where $\underline{v} = (i,j,k,l) \in \mathbb{N}^4$, I is a finite subset of $\{0,1\}^2 \times \mathbb{N}^4$, and $(a_{(\epsilon_1,\epsilon_2,\underline{v})})_{(\epsilon_1,\epsilon_2,\underline{v})\in I}$ is a family of complex numbers. Using the commutation relations in $A_{\alpha,\beta}$ (Lemma A.1.4), we have that:

$$e_1^{i_1}e_2^{i_2}e_3^{i_3+1}e_4^{\xi}e_5^{i_5}e_6^{i_6} = q^{\bullet}e_3e_1^{i_1}e_2^{i_2}e_3^{i_3}e_4^{\xi}e_5^{i_5}e_6^{i_6} - q^{\bullet}d_2[i_1]e_1^{i_1-1}e_2^{1+i_2}e_3^{i_3}e_4^{\xi}e_5^{i_5}e_6^{i_6}.$$

From the inductive hypothesis, $e_1^{i_1-1}e_2^{1+i_2}e_3^{i_3}e_4^{\xi}e_5^{i_5}e_6^{i_6} \in \text{Span}(\mathcal{B})$ for all $i_1 > 0$ (note: $d_2[0] = 0$). As a result, we proceed to show that $e_3e_1^{i_1}e_2^{i_2}e_3^{i_3}e_4^{\xi}e_5^{i_5}e_6^{i_6}$ is also in the span of $\mathcal{B}$. It follows from the inductive hypothesis that

$$\begin{aligned} e_3e_1^{i_1}e_2^{i_2}e_3^{i_3}e_4^{\xi}e_5^{i_5}e_6^{i_6} &= \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}e_3e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2}e_5^ie_6^l \\ &= \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}e_1^ie_2^je_3^{\epsilon_1+1}e_4^{\epsilon_2}e_5^ke_6^l \\ &\quad + \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} d_2[i]a_{(\epsilon_1,\epsilon_2,\underline{v})}e_1^{i-1}e_2^{1+j}e_3^{\epsilon_1}e_4^{\epsilon_2}e_5^ke_6^l. \end{aligned}$$

Clearly, the monomial $e_1^{i-1}e_2^{1+j}e_3^{\epsilon_1}e_4^{\epsilon_2}e_5^ke_6^l$ belongs to the span of $\mathcal{B}$ for all $(\epsilon_1,\epsilon_2,\underline{v})\in I$ (with $i>0$). Again, the monomial $e_1^ie_2^je_3^{\epsilon_1+1}e_4^{\epsilon_2}e_5^ke_6^l$ belongs to the span of $\mathcal{B}$ for all $(\epsilon_1,\epsilon_2,\underline{v})\in I$; with $(\epsilon_1,\epsilon_2)=(0,0),(0,1)$. For $(\epsilon_1,\epsilon_2)=(1,0),(1,1)$; we must show that $e_1^ie_2^je_3^2e_5^ke_6^l$ and $e_1^ie_2^je_3^2e_4e_5^ke_6^l$ belong to the span of $\mathcal{B}$. From Lemma 3.0.1, one can write $e_1^ie_2^je_3^2e_5^ke_6^l$ and $e_1^ie_2^je_3^2e_4e_5^ke_6^l$ as finite linear combinations of the elements of $\mathcal{B}$ over $\mathbb{C}$. Hence, $e_1^ie_2^je_3^2e_5^ke_6^l$ and $e_1^ie_2^je_3^2e_4e_5^ke_6^l$ belong to the span of $\mathcal{B}$ for all $(\epsilon_1,\epsilon_2,\underline{v})\in I$; with $(\epsilon_1,\epsilon_2)=(1,0),(1,1)$. We have therefore established that $e_3e_1^{i_1}e_2^{i_2}e_3^{i_3}e_4^{\xi}e_5^{i_5}e_6^{i_6}\in$ Span($\mathcal{B}$). Consequently, each $e_1^{i_1}e_2^{i_2}e_3^{i_3+1}e_4^{\xi}e_5^{i_5}e_6^{i_6}$ belongs to the span of $\mathcal{B}$. By the principle of mathematical induction, $\mathcal{B}$ is a spanning set of $A_{\alpha,\beta}$ over $\mathbb{C}$ as expected.

Finally, we show that $\mathcal{B}$ is a linearly independent set. Suppose that

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2}e_5^ke_6^l=0.$$

In A_β, we have that

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^{\epsilon_1}\widehat{e_4}^{\epsilon_2}\widehat{e_5}^k\widehat{e_6}^l=(\widehat{\Omega}_1-\alpha)\nu,$$

with $\nu\in A_\beta$. One can write ν in terms of the basis $\mathfrak{S}$ of A_β (Proposition 3.2.1) as:

$$\nu=\sum_{\underline{w}\in J_1} b_{\underline{w}}\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^k\widehat{e_4}\widehat{e_5}^m\widehat{e_6}^n+\sum_{\underline{w}\in J_2} c_{\underline{w}}\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^k\widehat{e_5}^l\widehat{e_6}^m,$$

where $b_{\underline{w}}$ and $c_{\underline{w}}$ are all complex numbers and $w := (i,j,k,l,m)$. Hence,

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^{\epsilon_1}\widehat{e_4}^{\epsilon_2}\widehat{e_5}^k\widehat{e_6}^l = \sum_{\underline{w}\in J_1} b_{\underline{w}}\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^k(\widehat{\Omega}_1-\alpha)\widehat{e_4}\widehat{e_5}^l\widehat{e_6}^m + \sum_{\underline{w}\in J_2} c_{\underline{w}}\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^k(\widehat{\Omega}_1-\alpha)\widehat{e_5}^l\widehat{e_6}^m.$$

Note, $\widehat{\Omega}_1 = \widehat{e_1}\widehat{e_3}\widehat{e_5} + a\widehat{e_1}\widehat{e_4} + a\widehat{e_2}\widehat{e_5} + a'\widehat{e_3}^2$. Using (3.2.2) and the relation $e_3^k e_1 = q^{-k}e_1e_3^k + d_2[k]e_2e_3^{k-1}$ (see Lemma A.1.4), one can verify that the above equality can be written in terms of the basis of A_β (Propositions 3.2.1) as:

$$\begin{aligned}\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^{\epsilon_1}\widehat{e_4}^{\epsilon_2}\widehat{e_5}^k\widehat{e_6}^l &= \sum_{\underline{w}\in J_1} b_{\underline{w}}a'\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^{k+2}\widehat{e_4}\widehat{e_5}^l\widehat{e_6}^m \\ &+ \sum_{\underline{w}\in J_1} q^\bullet b_{\underline{w}}ab\beta_0\widehat{e_1}^{i+1}\widehat{e_2}^j\widehat{e_3}^{k+3}\widehat{e_5}^l\widehat{e_6}^{m+1} \\ &+ \sum_{\underline{w}\in J_2} q^\bullet c_{\underline{w}}a\widehat{e_1}^{i+1}\widehat{e_2}^j\widehat{e_3}^k\widehat{e_4}\widehat{e_5}^l\widehat{e_6}^m \\ &+ \sum_{\underline{w}\in J_2} c_{\underline{w}}a'\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^{k+2}\widehat{e_5}^l\widehat{e_6}^m + \Upsilon, \qquad (3.2.4)\end{aligned}$$

where Υ is defined on the next page.

$$\begin{aligned}
\Upsilon =& \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} \widehat{e_1}^{i+1}\widehat{e_2}^{j}\widehat{e_3}^{k+1}\widehat{e_4}\widehat{e_5}^{l+1}\widehat{e_6}^{m} + \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} d_2[k]\widehat{e_1}^{i}\widehat{e_2}^{1+j}\widehat{e_3}^{k}\widehat{e_4}\widehat{e_5}^{l+1}\widehat{e_6}^{m} \\
&- \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} a\beta\beta_0 \widehat{e_1}^{i+1}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_5}^{l}\widehat{e_6}^{m} + \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} a\beta_0 \widehat{e_1}^{i+1}\widehat{e_2}^{1+j}\widehat{e_3}^{k}\widehat{e_4}\widehat{e_5}^{l}\widehat{e_6}^{m+1} \\
&+ \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} ab\beta_0 \widehat{e_1}^{i+1}\widehat{e_2}^{j+1}\widehat{e_3}^{k}\widehat{e_5}^{l+3}\widehat{e_6}^{m} + \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} ab'\beta_0 \widehat{e_1}^{i+1}\widehat{e_2}^{j}\widehat{e_3}^{k+2}\widehat{e_5}^{l+2}\widehat{e_6}^{m} \\
&+ \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} ac'\beta_0 \widehat{e_1}^{i+1}\widehat{e_2}^{j}\widehat{e_3}^{k+1}\widehat{e_4}\widehat{e_5}^{l+1}\widehat{e_6}^{m} - \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} a\beta\beta_0 d_2[k]\widehat{e_1}^{i}\widehat{e_2}^{1+j}\widehat{e_3}^{k-1}\widehat{e_5}^{l}\widehat{e_6}^{m} \\
&+ \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} a\beta_0 d_2[k]\widehat{e_1}^{i}\widehat{e_2}^{j+2}\widehat{e_3}^{k-1}\widehat{e_4}\widehat{e_5}^{l}\widehat{e_6}^{m+1} + \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} ab\beta_0 d_2[k]\widehat{e_1}^{i}\widehat{e_2}^{j+2}\widehat{e_3}^{k-1}\widehat{e_5}^{l+3}\widehat{e_6}^{m} \\
&+ \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} ab\beta_0 d_2[k]\widehat{e_1}^{i+1}\widehat{e_2}^{j}\widehat{e_3}^{k+2}\widehat{e_5}^{l}\widehat{e_6}^{m+1} + \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} ab'\beta_0 d_2[k]\widehat{e_1}^{i}\widehat{e_2}^{j+1}\widehat{e_3}^{k+1}\widehat{e_5}^{l+2}\widehat{e_6}^{m} \\
&+ \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} ac'\beta_0 d_2[k]\widehat{e_1}^{i}\widehat{e_2}^{j+1}\widehat{e_3}^{k}\widehat{e_4}\widehat{e_5}^{l+1}\widehat{e_6}^{m} + \sum_{\underline{w}\in J_1} q^{\bullet} b_{\underline{w}} a\widehat{e_1}^{i}\widehat{e_2}^{j+1}\widehat{e_3}^{k}\widehat{e_4}\widehat{e_5}^{l+1}\widehat{e_6}^{m} \\
&- \sum_{\underline{w}\in J_1} b_{\underline{w}} \beta\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}\widehat{e_5}^{l}\widehat{e_6}^{m} + \sum_{\underline{w}\in J_2} q^{\bullet} c_{\underline{w}} \widehat{e_1}^{i+1}\widehat{e_2}^{j}\widehat{e_3}^{k+1}\widehat{e_5}^{l+1}\widehat{e_6}^{m} \\
&+ \sum_{\underline{w}\in J_2} q^{\bullet} c_{\underline{w}} d_2[k]\widehat{e_1}^{i}\widehat{e_2}^{1+j}\widehat{e_3}^{k}\widehat{e_5}^{l+1}\widehat{e_6}^{m} + \sum_{\underline{w}\in J_2} q^{\bullet} c_{\underline{w}} ad_2[k]\widehat{e_1}^{i}\widehat{e_2}^{j+1}\widehat{e_3}^{k-1}\widehat{e_4}\widehat{e_5}^{l}\widehat{e_6}^{m} \\
&+ \sum_{\underline{w}\in J_2} q^{\bullet} c_{\underline{w}} a\widehat{e_1}^{i}\widehat{e_2}^{j+1}\widehat{e_3}^{k}\widehat{e_5}^{l+1}\widehat{e_6}^{m} - \sum_{\underline{w}\in J_2} c_{\underline{w}} \alpha\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_5}^{l}\widehat{e_6}^{m}.
\end{aligned}$$

Before we continue the proof, the following point needs to be noted.

- ♠ Let $(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_5,\vartheta_6)$, $(\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_5,\varsigma_6) \in \mathbb{N}^5$. We say that $(\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_5,\varsigma_6) <_3 (\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_5,\vartheta_6)$ if $[\vartheta_3 > \varsigma_3]$ or $[\vartheta_3 = \varsigma_3$ and $\vartheta_1 > \varsigma_1]$ or $[\vartheta_3 = \varsigma_3,\ \vartheta_1 = \varsigma_1$ and $\vartheta_2 > \varsigma_2]$ or $[\vartheta_3 = \varsigma_3,\ \vartheta_1 = \varsigma_1,\ \vartheta_2 = \varsigma_2$ and $\vartheta_5 > \varsigma_5]$ or $[\vartheta_3 = \varsigma_3,\ \vartheta_1 = \varsigma_1,\ \vartheta_2 = \varsigma_2,\ \vartheta_5 = \varsigma_5$ and $\vartheta_6 \geq \varsigma_6]$.

Observe that Υ contains lower order terms with respect to $<_3$ (defined in ♠ above) in each monomial type (note, there are two different types of monomials in the basis of A_β: one with $\widehat{e_4}$ and the other without $\widehat{e_4}$). Now, suppose that there exists $(i,j,k,l,m) \in J_1$ and $(i,j,k,l,m) \in J_2$ such that $b_{(i,j,k,l,m)} \neq 0$ and $c_{(i,j,k,l,m)} \neq 0$. Let (v_1,v_2,v_3,v_5,v_6) and (w_1,w_2,w_3,w_5,w_6) be the greatest elements of J_1 and J_2 respectively with respect to $<_3$ such that $b_{(v_1,v_2,v_3,v_5,v_6)}$ and $c_{(w_1,w_2,w_3,w_5,w_6)}$ are non-zero. Since $\mathfrak{S}$ is a linear basis for A_β, and Υ contains lower order terms with respect to $<_3$, we have the following: if $w_3 - v_3 < 2$, then identifying the coefficients of $\widehat{e_1}^{v_1}\widehat{e_2}^{v_2}\widehat{e_3}^{v_3+2}\widehat{e_4}\widehat{e_5}^{v_5}\widehat{e_6}^{v_6}$ in (3.2.4), we have that $a'b_{(v_1,v_2,v_3,v_5,v_6)} = 0$. But $b_{(v_1,v_2,v_3,v_5,v_6)} \neq 0$, hence $a' = q^6/(q^2-1) = 0$, a contradiction (see Appendix C for the expression of a'). Finally, if $w_3 - v_3 \geq 2$, then identifying the coefficient of $\widehat{e_1}^{w_1}\widehat{e_2}^{w_2}\widehat{e_3}^{w_3+2}\widehat{e_5}^{w_5}\widehat{e_6}^{w_6}$, we have that $a'c_{(w_1,w_2,w_3,w_5,w_6)} = 0$. But $c_{(w_1,w_2,w_3,w_5,w_6)} \neq 0$, hence $a' = 0$, a contradiction! This implies that either all $b_{(i,j,k,l,m)}$ or all $c_{(i,j,k,l,m)}$ are zero. Without loss of generality, suppose that there exists $(i,j,k,l,m) \in J_2$ such that $c_{(i,j,k,l,m)}$ is not zero. Then, $b_{(i,j,k,l,m)}$ are all zero. Let (w_1,w_2,w_3,w_5,w_6) be the greatest element of J_2 such that $c_{(w_1,w_2,w_3,w_5,w_6)} \neq 0$. Identifying the coefficients of $\widehat{e_1}^{w_1}\widehat{e_2}^{w_2}\widehat{e_3}^{w_3+2}\widehat{e_5}^{w_5}\widehat{e_6}^{w_6}$ in the above equality, we have that $a'c_{(w_1,w_2,w_3,w_5,w_6)} = 0$. Since $c_{(w_1,w_2,w_3,w_5,w_6)} \neq 0$, it follows that $a' = 0$, a contradiction! We can therefore conclude that $b_{(i,j,k,l,m)}$ and $c_{(i,j,k,l,m)}$ are all zero. Consequently, $\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{\epsilon_1}\widehat{e_4}^{\epsilon_2}\widehat{e_5}^{k}\widehat{e_6}^{l} = 0$. Since $(\widehat{e_1}^{i_1}\widehat{e_2}^{i_2}\widehat{e_3}^{i_3}\widehat{e_4}^{\xi}\widehat{e_5}^{i_5}\widehat{e_6}^{i_6})_{(\xi,i_1,\cdots,i_6)\in\{0,1\}\times\mathbb{N}^5}$ is a basis of A_β, it implies that $a_{(\epsilon_1,\epsilon_2,\underline{v})} = 0$ for all $(\epsilon_1,\epsilon_2,\underline{v}) \in I$. Therefore, $\mathcal{B}$ is a linearly independent set. ■

3.2.3 Corollary. Let $\underline{v} = (i,j,k,l) \in \mathbb{N}^2 \times \mathbb{Z}^2$, I represent a finite subset of $\{0,1\} \times \mathbb{N}^2 \times \mathbb{Z}^2$ and $(a_{(\epsilon_1,\epsilon_2,\underline{v})})_{(\epsilon_1,\epsilon_2,\underline{v})\in I}$ be a family of complex numbers. If

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} e_1^i e_2^j e_3^{\epsilon_1} e_4^{\epsilon_2} t_5^k t_6^l = 0,$$

then $a_{(\epsilon_1,\epsilon_2,\underline{v})} = 0$ for all $(\epsilon_1,\epsilon_2,\underline{v}) \in I$.

Proof. From Proposition 3.2.2, the result is obvious when $k, l \geq 0$. When k (resp. l) is negative, then one can multiply the above equality enough times (on the right) by t_5 (resp. t_6) to kill all the negative powers and then apply Proposition 3.2.2 to complete the proof. ■

3.2.4 Remark. Given the basis of $A_{\alpha,\beta}$, we have computed the group of units of $A_{\alpha,\beta}$, however, we do not include the details in this thesis due to the voluminous computations involved. We only summarize our findings below. Set

$$h_1 := e_3e_5 + ae_4 \text{ and } h_2 := (q^{-3} - q^{-9})e_2e_4 - (q^4 - 2q^2 + 1)/(q^4 + q^2 + 1)e_3^3.$$

3.2.5 Theorem. *Let* $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ *and* $\mathcal{U}(A_{\alpha,\beta})$ *denote the group of units of* $A_{\alpha,\beta}$. *We have that:*

$$\mathcal{U}(A_{\alpha,\beta}) = \begin{cases} \{\lambda h_1^i \mid \lambda \in \mathbb{C}^*,\ i \in \mathbb{Z}\} & \text{if } \alpha = 0 \\ \{\lambda h_2^i \mid \lambda \in \mathbb{C}^*,\ i \in \mathbb{Z}\} & \text{if } \beta = 0 \\ \mathbb{C}^* & otherwise. \end{cases}$$

3.3 $A_{\alpha,\beta}$ as a q-deformation of a quadratic extension of $A_2(\mathbb{C})$

We are now ready to establish that $A_{\alpha,\beta}$ is a q-deformation of a quadratic extension of $A_2(\mathbb{C})$. Recall that $A_2(\mathbb{C})$ is generated by x_1, x_2, y_1 and y_2 subject to the relations:

$$\begin{array}{llll} y_1y_2 = y_2y_1 & x_2y_1 = y_1x_2 & x_1x_2 = x_2x_1 & x_1y_1 - y_1x_1 = 1 \\ y_1y_2 = y_2y_1 & x_1y_2 = y_2x_1 & x_2y_1 = y_1x_2 & x_2y_2 - y_2x_2 = 1. \end{array}$$

Given the relations of $A_{\alpha,\beta}$ at the onset of this chapter, we have that $A_{1,\frac{1}{9(q^6-1)}}$ satisfies the following relations:

$$\begin{array}{ll} e_2e_1 = q^{-3}e_1e_2 & e_3e_1 = q^{-1}e_1e_3 - (q + q^{-1} + q^{-3})e_2 \\ e_3e_2 = q^{-3}e_2e_3 & e_4e_1 = e_1e_4 + (1 - q^2)e_3^2 \\ e_4e_2 = q^{-3}e_2e_4 - \dfrac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1}e_3^3 & e_4e_3 = q^{-3}e_3e_4 \\ e_5e_1 = qe_1e_5 - (1 + q^2)e_3 & e_5e_2 = e_2e_5 + (1 - q^2)e_3^2 \\ e_5e_3 = q^{-1}e_3e_5 - (q + q^{-1} + q^{-3})e_4 & e_5e_4 = q^{-3}e_4e_5 \end{array}$$

$$e_6e_1 = q^3e_1e_6 - q^3e_5 \qquad e_6e_2 = q^3e_2e_6 + (q^4+q^2-1)e_4 + (q^2-q^4)e_3e_5$$
$$e_6e_3 = e_3e_6 + (1-q^2)e_5^2 \qquad e_6e_4 = q^{-3}e_4e_6 - \frac{q^4-2q^2+1}{q^4+q^2+1}e_5^3$$
$$e_6e_5 = q^{-3}e_5e_6,$$

and

$$(q^{-2}-1)e_1e_3e_5 + (q^2+1+q^{-2})e_1e_4 + (q^2+1+q^{-2})e_2e_5 - q^4e_3^2 = q^{-2}-1,$$

$$(q^6-1)e_2e_4e_6 + \frac{2q^{-1}-q^{-3}-q}{q^4+q^2+1}e_2e_5^3 + \frac{2q^{-1}-q^{-3}-q}{q^4+q^2+1}e_3^3e_6 + \frac{(q^6-1)(q^{13}-q^{11})}{(q^4+q^2+1)^2}e_3^2e_5^2 - \frac{q^9(q^6-1)}{q^4+q^2+1}e_3e_4e_5 + q^{12}e_4^2 = \frac{1}{9}.$$

Note, we have made substitutions for $a,\ a',\ b,\ b',\ c'$ and d' (see Appendix C).

Set $C := \mathbb{C}[z^{\pm 1}]$. One can define a $C[(z^4+z^2+1)^{-1}]$−algebra A_z generated by $e_1, \cdots, e_6$ subject to the following relations:

$$e_2e_1 = z^{-3}e_1e_2 \qquad e_3e_1 = z^{-1}e_1e_3 - (z+z^{-1}+z^{-3})e_2$$
$$e_3e_2 = z^{-3}e_2e_3 \qquad e_4e_1 = e_1e_4 + (1-z^2)e_3^2$$
$$e_4e_2 = z^{-3}e_2e_4 - \frac{z^4-2z^2+1}{z^4+z^2+1}e_3^3 \qquad e_4e_3 = z^{-3}e_3e_4$$
$$e_5e_1 = ze_1e_5 - (1+z^2)e_3 \qquad e_5e_2 = e_2e_5 + (1-z^2)e_3^2$$
$$e_5e_3 = z^{-1}e_3e_5 - (z+z^{-1}+z^{-3})e_4 \qquad e_5e_4 = z^{-3}e_4e_5$$
$$e_6e_1 = z^3e_1e_6 - z^3e_5 \qquad e_6e_2 = z^3e_2e_6 + (z^4+z^2-1)e_4 + (z^2-z^4)e_3e_5$$
$$e_6e_3 = e_3e_6 + (1-z^2)e_5^2 \qquad e_6e_4 = z^{-3}e_4e_6 - \frac{z^4-2z^2+1}{z^4+z^2+1}e_5^3$$
$$e_6e_5 = z^{-3}e_5e_6,$$

$$(z^{-2}-1)e_1e_3e_5 + (z^2+1+z^{-2})e_1e_4 + (z^2+1+z^{-2})e_2e_5 - z^4e_3^2 = z^{-2}-1, \quad \text{and}$$

$$(z^6-1)e_2e_4e_6+\frac{2z^{-1}-z^{-3}-z}{z^4+z^2+1}e_2e_5^3+\frac{2z^{-1}-z^{-3}-z}{z^4+z^2+1}e_3^3e_6$$
$$+\frac{(z^6-1)(z^{13}-z^{11})}{(z^4+z^2+1)^2}e_3^2e_5^2-\frac{z^9(z^6-1)}{z^4+z^2+1}e_3e_4e_5+z^{12}e_4^2=\frac{1}{9}.$$

Observe that A_1 (i.e. when $z=1$) satisfies the following relations:

$$\begin{array}{lll}
e_2e_1=e_1e_2 & e_3e_1=e_1e_3-3e_2 & e_3e_2=e_2e_3 \\
e_4e_1=e_1e_4 & e_4e_2=e_2e_4 & e_4e_3=e_3e_4 \\
e_5e_1=e_1e_5-2e_3 & e_5e_2=e_2e_5 & e_5e_3=e_3e_5-3e_4 \\
e_5e_4=e_4e_5 & e_6e_1=e_1e_6-e_5 & e_6e_2=e_2e_6+e_4 \\
e_6e_3=e_3e_6 & e_6e_4=e_4e_6 & e_6e_5=e_5e_6 \\
e_4^2-1/9=0 & & e_3^2-3e_1e_4-3e_2e_5=0.
\end{array}$$

3.3.1 Lemma. $e_4\in Z(A_1)$ and it is also invertible.

Proof. Since $e_4e_i=e_ie_4$ for all $1\le i\le 6$, we have that $e_4\in Z(A_1)$. Again, from $e_4^2-1/9=0$, we have that $e_4(9e_4)=(9e_4)e_4=1$. Hence e_4 is invertible with $e_4^{-1}=9e_4$. ∎

Given that $e_4^{-1}=9e_4$ and $e_4\in Z(A_1)$, it follows from the relation $e_3^2-3e_1e_4-3e_2e_5=0$ that $e_1=3e_3^2e_4-9e_2e_4e_5$. Therefore, A_1 can be generated by only $e_2,\cdots,e_6$. All these generators commute except

$$e_6e_2=e_2e_6+e_4 \text{ and } e_5e_3=e_3e_5-3e_4.$$

Since e_4 is invertible, one can also verify that $9e_2e_4, 3e_3e_4, e_4, e_5$ and e_6 generate A_1.

Let R be an algebra generated by f_2,f_3,f_4,f_5,f_6 subject to the following defining relations:

$$f_3f_2 = f_2f_3 \qquad f_4f_2 = f_2f_4 \qquad f_4f_3 = f_3f_4$$
$$f_5f_2 = f_2f_5 \qquad f_5f_4 = f_4f_5 \qquad f_6f_3 = f_3f_6$$
$$f_6f_4 = f_4f_6 \qquad f_6f_5 = f_5f_6 \qquad f_4^2 = 1/9$$
$$f_6f_2 = f_2f_6 + 1 \qquad f_5f_3 = f_3f_5 - 1.$$

3.3.2 Proposition. $R \cong A_1$.

Proof. One can define a homomorphism $\phi : R \longrightarrow A_1$ such that

$$\phi(f_2) = 9e_2e_4 \qquad \phi(f_3) = 3e_3e_4 \qquad \phi(f_4) = e_4 \qquad \phi(f_5) = e_5 \qquad \phi(f_6) = e_6.$$

Recall, $e_4^2 = 1/9$. One can confirm that ϕ is indeed a homomorphism by verifying that it is compatible with the relations of R. We check this on the relation $f_6f_2 - f_2f_6 = 1$ and $f_3f_5 - f_5f_3 = 1$, and leave the remaining ones for the reader to verify. We do that as follows: $\phi(f_6)\phi(f_2) - \phi(f_2)\phi(f_6) = 9e_6e_2e_4 - 9e_2e_4e_6 = 9e_4(e_6e_2 - e_2e_6) = 9e_4^2 = 9(1/9) = 1$ as expected. Also, $\phi(f_3)\phi(f_5) - \phi(f_5)\phi(f_3) = 3e_3e_4e_5 - 3e_5e_3e_4 = 3e_4(e_3e_5 - e_5e_3) = 3e_4(3e_4) = 9e_4^2 = 9(1/9) = 1$.

Conversely, one can also define a homomorphism $\varphi : A_1 \longrightarrow R$ such that

$$\varphi(e_1) = 3f_3^2f_4 - f_2f_5 \qquad \varphi(e_2) = f_2f_4 \qquad \varphi(e_3) = 3f_3f_4$$
$$\varphi(e_4) = f_4 \qquad \varphi(e_5) = f_5 \qquad \varphi(e_6) = f_6.$$

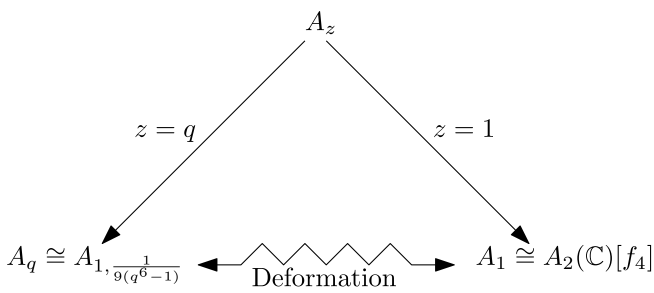
We check this on the relation $e_3^2 - 3e_1e_4 - 3e_2e_5 = 0$, and leave the remaining ones for the reader to verify. We do that as follows: $\varphi(e_3)^2 - 3\varphi(e_1)\varphi(e_4) - 3\varphi(e_2)\varphi(e_5) = (3f_3f_4)^2 - 3(3f_3^2f_4 - f_2f_5)f_4 - 3f_2f_4f_5 = 9f_3^2f_4^2 - 9f_3^2f_4^2 + 3f_2f_4f_5 - 3f_2f_4f_5 = 0$ as expected.

Now, $\phi \circ \varphi = \mathrm{id}_{A_1}$ (one can check this on the generators of A_1) and $\varphi \circ \phi = \mathrm{id}_R$ (one can also check this on the generators of R) (note, id_* means an identity map of the algebra $*$). This confirms that $R \cong A_1$. ■

The corollary below can easily be deduced from the above proposition.

3.3.3 Corollary. Set $\mathbb{F} := \mathbb{C}[f_4]/\langle f_4^2 - 1/9\rangle$, we have that $R \cong A_2(\mathbb{F})$, where $A_2(\mathbb{F})$ is the second Weyl algebra over the ring $\mathbb{F}$.

3.3.4 Remark. Observe that the subalgebra B of R generated by f_2, f_3, f_5, f_6 is isomorphic to $A_2(\mathbb{C})$ and $R \cong B[f_4] \cong A_2(\mathbb{C})[f_4]$. Therefore, R is a *quadratic extension* of $A_2(\mathbb{C})$. Note, $A_{1,\frac{1}{9(q^6-1)}}$ is a q-deformation of $A_1 \cong R \cong A_2(\mathbb{F}) \cong A_2(\mathbb{C})[f_4]$.



3.3.5 Remark. Given the algebra $A_{\alpha,\beta}$, the construction in this section was done for $\alpha = 1$ and $\beta = \dfrac{1}{9(q^6-1)}$. However, any $\alpha \in \mathbb{R}$ and β of the form $\beta = \dfrac{\varsigma}{(q^6-1)}$, where $\varsigma \in \mathbb{R}^*$, will also work.

Chapter 4

Derivations of the simple quotients of $U_q^+(G_2)$

In this chapter, we compute the derivations of the algebra $A_{\alpha,\beta}$. In finding the derivations of $A_{\alpha,\beta}$, we use the theory of deleting derivations algorithm by Cauchon [8] and localization theory to embed $A_{\alpha,\beta}$ into a suitable quantum torus. This is because every derivation of the quantum torus, through the work of Osborn and Passman [40], is known to be the sum of an inner derivation and a central/scalar derivation. Since $A_{\alpha,\beta}$ can be embedded into a quantum torus, we extend every derivation of $A_{\alpha,\beta}$ to a derivation of the quantum torus, and then restrict the derivations of the quantum torus back to a derivation of $A_{\alpha,\beta}$. We conclude that every derivation of $A_{\alpha,\beta}$ is inner when α and β are non-zero. However, when either α or β is zero, we conclude that every derivation of $A_{\alpha,\beta}$ is the sum of an inner and a scalar derivation. In fact, the first Hochschild cohomology group of $A_{\alpha,\beta}$ is of dimension 0 when α and β are non-zero and 1 when either α or β is zero.

4.1 Preliminaries

Let $2 \leq j \leq 7$ and $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Set

$$A_{\alpha,\beta}^{(j)} := \frac{A^{(j)}}{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle},$$

where $A^{(j)}$ is defined in Section 2.2 and, Ω_1 and Ω_2 are the generators of the center of $A^{(j)}$ (Subsection 2.2.3). Recall that $A^{(7)} = A = U_q^+(G_2)$ (Section 2.2). It follows that $A_{\alpha,\beta}^{(7)} = A_{\alpha,\beta}$. For each $2 \leq j \leq 7$, denote the canonical images of the generators $E_{i,j}$ of $A^{(j)}$ in $A_{\alpha,\beta}^{(j)}$ by $e_{i,j}$ for all $1 \leq i \leq 6$. Since the data of the DDA of $A_{\alpha,\beta}$ is going to be crucial in this section, we present them below (note, we deduce them from that of $U_q^+(G_2)$ in Section 2.2):

$$
\begin{aligned}
e_{1,6} &= e_1 + re_5e_6^{-1} \\
e_{2,6} &= e_2 + te_3e_5e_6^{-1} + ue_4e_6^{-1} + ne_5^3e_6^{-2} \\
e_{3,6} &= e_3 + se_5^2e_6^{-1} \\
e_{4,6} &= e_4 + be_5^3e_6^{-1} \\
e_{1,5} &= e_{1,6} + he_{3,6}e_{5,6}^{-1} + ge_{4,6}e_{5,6}^{-2} \\
e_{2,5} &= e_{2,6} + fe_{3,6}^2e_{5,6}^{-1} + pe_{3,6}e_{4,6}e_{5,6}^{-2} + ee_{4,6}^2e_{5,6}^{-3} \\
e_{3,5} &= e_{3,6} + ae_{4,6}e_{5,6}^{-1} \\
e_{1,4} &= e_{1,5} + se_{3,5}^2e_{4,5}^{-1} \\
e_{2,4} &= e_{2,5} + be_{3,5}^3e_{4,5}^{-1} \\
e_{1,3} &= e_{1,4} + ae_{2,4}e_{3,4}^{-1} \\
t_1 &:= e_{1,2} = e_{1,3} \\
t_2 &:= e_{2,2} = e_{2,3} = e_{2,4} \\
t_3 &:= e_{3,2} = e_{3,3} = e_{3,4} = e_{3,5} \\
t_4 &:= e_{4,2} = e_{4,3} = e_{4,4} = e_{4,5} = e_{4,6} \\
t_5 &:= e_{5,2} = e_{5,3} = e_{5,4} = e_{5,5} = e_{5,6} = e_5 \\
t_6 &:= e_{6,2} = e_{6,3} = e_{6,4} = e_{6,5} = e_{6,6} = e_6.
\end{aligned}
$$

Note, t_i is also the canonical image of T_i in $A_{\alpha,\beta}^{(2)}$ for each $1 \leq i \leq 6$. For each $3 \leq j \leq 6$, define $S_j := \left\{ \lambda t_j^{i_j} t_{j+1}^{i_{j+1}} \cdots t_6^{i_6} \mid i_j, \cdots, i_6 \in \mathbb{N} \text{ and } \lambda \in \mathbb{C}^* \right\}$. One can observe that S_j is a multiplicative system of non-zero divisors (or regular elements) of $A_{\alpha,\beta}^{(j)}$. Furthermore;

$t_j, \cdots, t_6$ are all normal elements of $A_{\alpha,\beta}^{(j)}$. It follows from Section 1.7 that S_j is an Ore set in $A_{\alpha,\beta}^{(j)}$. As a result, one can localize $A_{\alpha,\beta}^{(j)}$ at S_j as follows:

$$R_j := A_{\alpha,\beta}^{(j)} S_j^{-1}.$$

Let $3 \leq j \leq 6$, and set $\Sigma_j := \{t_j^k \mid k \in \mathbb{N}\}$. By [8, Theorem 3.2.1], Σ_j is an Ore set in both $A_{\alpha,\beta}^{(j)}$ and $A_{\alpha,\beta}^{(j+1)}$. Consequently,

$$A_{\alpha,\beta}^{(j)} \Sigma_j^{-1} = A_{\alpha,\beta}^{(j+1)} \Sigma_j^{-1}.$$

Similar to (2.2.1), we have that

$$R_j = R_{j+1} \Sigma_j^{-1}, \tag{4.1.1}$$

for all $2 \leq j \leq 6$. By convention, $R_7 := A_{\alpha,\beta}$. We also construct the following tower of algebras in a manner similar to (2.2.2):

$$R_7 = A_{\alpha,\beta} \subset R_6 = R_7 \Sigma_6^{-1} \subset R_5 = R_6 \Sigma_5^{-1} \subset R_4 = R_5 \Sigma_4^{-1} \subset R_3. \tag{4.1.2}$$

Note, $R_3 = A_{\alpha,\beta}^{(3)} S_3^{-1} = R_4 \Sigma_3^{-1}$ is the quantum torus $\mathscr{A}_{\alpha,\beta} = \mathbb{C}_{q^N}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$ studied in Section 3.1.

Linear bases for R_3, R_4 and R_5.

Let $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. We aim to find a basis of R_j for each $j = 3, 4, 5$. Since $R_3 = \mathscr{A}_{\alpha,\beta}$, the set $\{t_3^i t_4^j t_5^k t_6^l \mid i, j, k, l \in \mathbb{Z}\}$ is a $\mathbb{C}-$basis of R_3.

For simplicity, we set

$$f_1 := e_{1,4} \qquad F_1 := E_{1,4}$$

$$z_1 := e_{1,5} \qquad \mathcal{Z}_1 := E_{1,5}$$

$$z_2 := e_{2,5} \qquad \mathcal{Z}_2 := E_{2,5}.$$

Basis for R_4. Observe that

$$A^{(4)}_{\alpha,\beta} = \frac{A^{(4)}}{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle},$$

where $\Omega_1 = F_1T_3T_5 + aT_2T_5$ and $\Omega_2 = T_2T_4T_6$ in $A^{(4)}$ (Subsection 2.2.3). Recall from Section 3.2 that finding a basis for the algebra A_β served as a good ground for finding a basis for $A_{\alpha,\beta}$. In a similar manner, to find a basis for R_4, we will first and foremost find a basis for the algebra

$$A^{(4)}_{\beta}S_4^{-1} = \frac{A^{(4)}S_4^{-1}}{\langle \Omega_2 - \beta \rangle} = \frac{A^{(4)}S_4^{-1}}{\langle T_2T_4T_6 - \beta \rangle},$$

where $\beta \in \mathbb{C}^*$. We will denote the canonical images of $E_{i,4}$ (resp. T_i) in $A^{(4)}_\beta$ by $\widehat{e_{i,4}}$ (resp. $\widehat{t_i}$) for all $1 \leq i \leq 6$. Observe that $\widehat{t_2} = \beta\widehat{t_6}^{-1}\widehat{t_4}^{-1}$ in $A^{(4)}_\beta S_4^{-1}$. Note, when $\beta = 0$, then one can easily deduce that $A^{(4)}_\beta S_4^{-1} = A^{(4)}S_4^{-1}/\langle T_2 \rangle$, hence, $\widehat{t_2} = 0$.

4.1.1 Proposition. The set $\mathfrak{S}_4 = \left\{ \widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} \mid (i_1, i_3, i_4, i_5, i_6) \in \mathbb{N}^2 \times \mathbb{Z}^3 \right\}$ is a $\mathbb{C}-$basis of $A^{(4)}_\beta S_4^{-1}$, where $\beta \in \mathbb{C}$.

Proof. We begin by showing that $\mathfrak{S}_4$ is a spanning set for $A^{(4)}_\beta S_4^{-1}$. It is sufficient to do this by showing that $\widehat{f_1}^{k_1}\widehat{t_2}^{k_2}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}$ can be written as a finite linear combination of the elements of $\mathfrak{S}_4$ for all $(k_1, \cdots, k_6) \in \mathbb{N}^3 \times \mathbb{Z}^3$. We do this by an induction on k_2. The result is clear when $k_2 = 0$. Assume that the statement is true for $k_2 \geq 0$. That is,

$$\widehat{f_1}^{k_1}\widehat{t_2}^{k_2}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6} = \sum_{\underline{i} \in I} a_{\underline{i}}\widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6},$$

where $\underline{i} = (i_1, i_3, i_4, i_5, i_6) \in I \subset \mathbb{N}^2 \times \mathbb{Z}^3$ and $a_{\underline{i}} \in \mathbb{C}$. It follows that

$$\begin{aligned}\widehat{f_1}^{k_1}\widehat{t_2}^{k_2+1}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6} &= q^{\bullet}\left(\widehat{f_1}^{k_1}\widehat{t_2}^{k_2}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}\right)\widehat{t_2} \\ &= \sum_{\underline{i}\in I} q^{\bullet} a_{\underline{i}}\widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}\widehat{t_2} \\ &= \sum_{\underline{i}\in I} q^{\bullet}\beta a_{\underline{i}}\widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4-1}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6-1},\end{aligned}$$

$\left(\text{note, } \widehat{t_2} = \beta\widehat{t_6}^{-1}\widehat{t_4}^{-1}\right)$. By the principle of mathematical induction, $\mathfrak{S}_4$ is a spanning set for $A_\beta^{(4)}S_4^{-1}$ for all $i_2 \in \mathbb{N}$. We now prove that $\mathfrak{S}_4$ is a linearly independent set. Suppose that

$$\sum_{\underline{i}\in I} a_{\underline{i}}\widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} = 0.$$

This implies that

$$\sum_{\underline{i}\in I} a_{\underline{i}}F_1^{i_1}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6} = (\Omega_2 - \beta)\nu,$$

for some $\nu \in A^{(4)}S_4^{-1}$. Write $\nu = \sum_{\underline{j}\in J} b_{\underline{j}}F_1^{i_1}T_2^{i_2}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6}$, where $\underline{j} = (i_1, i_2, i_3, i_4, i_5, i_6) \in J \subset \mathbb{N}^3 \times \mathbb{Z}^3$ and $b_{\underline{j}}$ is a family of complex numbers. Given that $\Omega_2 = T_2T_4T_6$, it follows from the above equality that

$$\sum_{\underline{i}\in I} a_{\underline{i}}F_1^{i_1}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6} = \sum_{\underline{j}\in J} q^{\bullet}b_{\underline{j}}F_1^{i_1}T_2^{i_2+1}T_3^{i_3}T_4^{i_4+1}T_5^{i_5}T_6^{i_6+1} - \sum_{\underline{j}\in J}\beta b_{\underline{j}}F_1^{i_1}T_2^{i_2}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6}.$$

Suppose that there exists $(i_1, \cdots, i_6) \in J$ such that $b_{(i_1,\cdots,i_6)} \neq 0$. Let $(w_1, \cdots, w_6) \in J$ be the greatest element of J with respect to $<_2$ [1] such that $b_{(w_1,\cdots,w_6)} \neq 0$. Note, $\left(F_1^{i_1}T_2^{i_2}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6}\right)_{(i_1,\cdots,i_6)\in J}$ is a basis of $A^{(4)}S_4^{-1}$. Identifying the coefficients of $F_1^{w_1}T_2^{w_2+1}T_3^{w_3}T_4^{w_4+1}T_5^{w_5}T_6^{w_6+1}$, we have that $b_{(w_1,\cdots,w_6)} = 0$. This is a contradiction to our assumption, hence $b_{(i_1,\cdots,i_6)} = 0$ for all $(i_1, \cdots, i_6) \in J$. This implies that

$$\sum_{\underline{i}\in I} a_{\underline{i}}F_1^{i_1}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6} = 0.$$

[1] $(i_1, i_2, i_3, i_4, i_5, i_6) <_2 (w_1, w_2, w_3, w_4, w_5, w_6)$ if $[w_2 > i_2]$ or $[w_2 = i_2,\ w_1 > i_1]$ or $[w_2 = i_2,\ w_1 = i_1,\ w_3 > i_3]$ or $\cdots$ or $[w_l = i_l,\ w_6 \geq t_6,\ l = 2, 1, 3, 4, 5]$ for all $(i_1, \cdots, i_6) \in J$.

Consequently, $a_{\underline{i}} = 0$ for all $\underline{i} \in I$. Therefore, $\mathfrak{S}_4$ is a linearly independent set. ∎

In $R_4 = A_{\alpha,\beta}^{(4)} S_4^{-1}$, we have the following two relations: $f_1 t_3 t_5 + a t_2 t_5 = \alpha$ and $t_2 t_4 t_6 = \beta$. This implies that $f_1 t_3 = \alpha t_5^{-1} - a t_2$ and $t_2 = \beta t_6^{-1} t_4^{-1}$. Putting these two relations together, we have that

$$f_1 t_3 = \alpha t_5^{-1} - a\beta t_6^{-1} t_4^{-1}. \tag{4.1.3}$$

Note, we will usually identify R_4 with $A_\beta^{(4)} S_4^{-1} / \langle \widehat{\Omega}_1 - \alpha \rangle$.

4.1.2 Proposition. The set $\mathcal{B}_4 = \left\{ f_1^{i_1} t_4^{i_4} t_5^{i_5} t_6^{i_6}, t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6} \mid i_1, i_3 \in \mathbb{N} \text{ and } i_4, i_5, i_6 \in \mathbb{Z} \right\}$ is a $\mathbb{C}-$basis of R_4.

Proof. Since $\left(\widehat{f_1}^{k_1} \widehat{t_3}^{k_3} \widehat{t_4}^{k_4} \widehat{t_5}^{k_5} \widehat{t_6}^{k_6} \right)_{(k_1, k_3, \cdots, k_6) \in \mathbb{N}^2 \times \mathbb{Z}^3}$ is a basis of $A_\beta^{(4)} S_4^{-1}$ (Proposition 4.1.1), the set $\left(f_1^{k_1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6} \right)_{(k_1, k_3, \cdots, k_6) \in \mathbb{N}^2 \times \mathbb{Z}^3}$ spans R_4. We show that $\mathcal{B}_4$ is a spanning set of R_4 by showing that $f_1^{k_1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6}$ can be written as a finite linear combination of the elements of $\mathcal{B}_4$ for all $(k_1, k_3, \cdots, k_6) \in \mathbb{N}^2 \times \mathbb{Z}^3$. By Proposition 4.1.1, it is sufficient to do this by induction on k_1. The result is clear when $k_1 = 0$. Assume that the statement is true for $k_1 \geq 0$. That is,

$$f_1^{k_1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6} = \sum_{\underline{i} \in I_1} a_{\underline{i}} f_1^{i_1} t_4^{i_4} t_5^{i_5} t_6^{i_6} + \sum_{\underline{j} \in I_2} b_{\underline{j}} t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6},$$

where $\underline{i} = (i_1, i_4, i_5, i_6) \in I_1 \subset \mathbb{N} \times \mathbb{Z}^3$ and $\underline{j} = (i_3, i_4, i_5, i_6) \in I_2 \subset \mathbb{N} \times \mathbb{Z}^3$. Note, $a_{\underline{i}}$ and $b_{\underline{j}}$ are all complex numbers. It follows that

$$f_1^{k_1+1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6} = f_1 \left(f_1^{k_1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6} \right) = \sum_{\underline{i} \in I_1} a_{\underline{i}} f_1^{i_1+1} t_4^{i_4} t_5^{i_5} t_6^{i_6} + \sum_{\underline{j} \in I_2} b_{\underline{j}} f_1 t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6}.$$

Clearly, the monomial $f_1^{i_1+1} t_4^{i_4} t_5^{i_5} t_6^{i_6} \in \text{Span}(\mathcal{B}_4)$. We have to also show that $f_1 t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6} \in \text{Span}(\mathcal{B}_4)$ for all $i_3 \in \mathbb{N}$ and $i_4, i_5, i_6 \in \mathbb{Z}$. This can easily be achieved by an induction on i_3, and the use of the relation $f_1 t_3 = \alpha t_5^{-1} - a\beta t_6^{-1} t_4^{-1}$. Therefore, by the principle of mathematical induction, $\mathcal{B}_4$ is a spanning set of R_4 over $\mathbb{C}$.

We prove that $\mathcal{B}_4$ is a linearly independent set. Suppose that

$$\sum_{\underline{i}\in I_1} a_{\underline{i}} f_1^{i_1} t_4^{i_4} t_5^{i_5} t_6^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6} = 0.$$

It follows that there exists $\nu \in A_\beta^{(4)} S_4^{-1}$ such that

$$\sum_{\underline{i}\in I_1} a_{\underline{i}} \widehat{f_1}^{i_1} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} \widehat{t_3}^{i_3} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6} = \left(\widehat{\Omega}_1 - \alpha\right)\nu.$$

Write $\nu = \sum_{\underline{l}\in J} c_{\underline{l}} \widehat{f_1}^{i_1} \widehat{t_3}^{i_3} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6}$, where $\underline{l} = (i_1, i_3, i_4, i_5, i_6) \in J \subset \mathbb{N}^2 \times \mathbb{Z}^3$ and $c_{\underline{l}} \in \mathbb{C}$. Note, $\widehat{t_2} = \beta \widehat{t_6}^{-1} \widehat{t_4}^{-1}$. We have that $\widehat{\Omega}_1 = \widehat{f_1}\widehat{t_3}\widehat{t_5} + a\widehat{t_2}\widehat{t_5} = \widehat{f_1}\widehat{t_3}\widehat{t_5} + a\beta \widehat{t_6}^{-1}\widehat{t_4}^{-1}\widehat{t_5}$. Therefore,

$$\begin{aligned}\sum_{\underline{i}\in I_1} a_{\underline{i}} \widehat{f_1}^{i_1} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} \widehat{t_3}^{i_3} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6} =& \sum_{\underline{l}\in J} q^{\bullet} c_{\underline{l}} \widehat{f_1}^{i_1+1} \widehat{t_3}^{i_3+1} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5+1} \widehat{t_6}^{i_6} \\ &+ \sum_{\underline{l}\in J} q^{\bullet} \beta a c_{\underline{l}} \widehat{f_1}^{i_1} \widehat{t_3}^{i_3} \widehat{t_4}^{i_4-1} \widehat{t_5}^{i_5+1} \widehat{t_6}^{i_6-1} \\ &- \sum_{\underline{l}\in J} \alpha c_{\underline{l}} \widehat{f_1}^{i_1} \widehat{t_3}^{i_3} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6}.\end{aligned}$$

Suppose that there exists $(i_1, i_3, i_4, i_5, i_6) \in J$ such that $c_{(i_1,i_3,i_4,i_5,i_6)} \neq 0$.
Let $(w_1, w_3, w_4, w_5, w_6) \in J$ be the greatest element (in the lexicographic order on $\mathbb{N}^2 \times \mathbb{Z}^3$) of J such that $c_{(w_1,w_3,w_4,w_5,w_6)} \neq 0$. Since $\left(\widehat{f_1}^{k_1} \widehat{t_3}^{k_3} \widehat{t_4}^{k_4} \widehat{t_5}^{k_5} \widehat{t_6}^{k_6}\right)_{(k_1,k_3,\cdots,k_6)\in\mathbb{N}^2\times\mathbb{Z}^3}$ is a basis of $A^{(4)}S_4^{-1}$, it implies that the coefficients of $\widehat{f_1}^{w_1+1} \widehat{t_3}^{w_3+1} \widehat{t_4}^{w_4} \widehat{t_5}^{w_5+1} \widehat{t_6}^{w_6}$ in the above equality can be identified as: $q^{\bullet} c_{(w_1,w_3,w_4,w_5,w_6)} = 0$. Hence, $c_{(w_1,w_3,w_4,w_5,w_6)} = 0$, a contradiction! Therefore, $c_{(i_1,i_3,i_4,i_5,i_6)} = 0$ for all $(i_1, i_3, i_4, i_5, i_6) \in J$. This further implies that

$$\sum_{\underline{i}\in I_1} a_{\underline{i}} \widehat{f_1}^{i_1} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} \widehat{t_3}^{i_3} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6} = 0.$$

Consequently, $a_{\underline{i}}$ and $b_{\underline{j}}$ are all zero. In conclusion, $\mathcal{B}_4$ is a linearly independent set. ■

Basis for R_5. We will identify R_5 with $A_\alpha^{(5)} S_5^{-1} / \langle \widehat{\Omega}_2 - \beta \rangle$, where $A_\alpha^{(5)} S_5^{-1} = \dfrac{A^{(5)} S_5^{-1}}{\langle \Omega_1 - \alpha \rangle}$.

Note, the canonical images of $E_{i,j}$ (resp. T_i) in $A_\alpha^{(5)}$ will be denoted by $\widehat{e_{i,j}}$ (resp. $\widehat{t_i}$). We now find a basis for $A_\alpha^{(5)} S_5^{-1}$. Recall from Subsection 2.2.3 that $\Omega_1 = \mathcal{Z}_1 T_3 T_5 + a\mathcal{Z}_2 T_5$ and $\Omega_2 = \mathcal{Z}_2 T_4 T_6 + bT_3^3 T_6$ in $A^{(5)}$ (remember, $\mathcal{Z}_1 := E_{1,5}$ and $\mathcal{Z}_2 := E_{2,5}$). Since $z_2 t_4 t_6 + bt_3^3 t_6 = \beta$ and $\widehat{z_1}\widehat{t_3}\widehat{t_5} + a\widehat{z_2}\widehat{t_5} = \alpha$ in R_5 and $A_\alpha^{(5)} S_5^{-1}$ respectively, we have the relation $\widehat{z_2} = \frac{1}{a}\left(\alpha \widehat{t_5}^{-1} - \widehat{z_1}\widehat{t_3}\right)$ in $A_\alpha^{(5)} S_5^{-1}$ and, in R_5, we have the following two relations:

$$z_2 = \frac{1}{a}\left(\alpha t_5^{-1} - z_1 t_3\right). \tag{4.1.4}$$

$$t_3^3 = \frac{1}{b}\left(\beta t_6^{-1} - z_2 t_4\right) = \frac{\beta}{b} t_6^{-1} - \frac{q^3\alpha}{ab} t_4 t_5^{-1} + \frac{1}{ab} z_1 t_3 t_4. \tag{4.1.5}$$

4.1.3 Proposition. The set $\mathfrak{S}_5 = \left\{ \widehat{z_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} \mid (i_1, i_3, \cdots, i_6) \in \mathbb{N}^3 \times \mathbb{Z}^2 \right\}$ is a $\mathbb{C}-$basis of $A_\alpha^{(5)} S_5^{-1}$, where $\alpha \in \mathbb{C}$.

Proof. The proof is similar to that of Proposition 4.1.1. Hence, we will provide the proof without many details. Assume that the statement is true for $k_2 \geq 0$. That is,

$$\widehat{z_1}^{k_1}\widehat{z_2}^{k_2}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6} = \sum_{\underline{i}\in I} a_{\underline{i}} \widehat{z_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6},$$

where $\underline{i} = (i_1, i_3, i_4, i_5, i_6) \in I \subset \mathbb{N}^3 \times \mathbb{Z}^2$ and $a_{\underline{i}} \in \mathbb{C}$. Given that $\widehat{z_2} = \frac{1}{a}\left(\alpha \widehat{t_5}^{-1} - \widehat{z_1}\widehat{t_3}\right)$, it follows from the inductive hypothesis that

$$\begin{aligned}
\widehat{z_1}^{k_1}\widehat{z_2}^{k_2+1}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6} &= q^\bullet \widehat{z_2}\left(\widehat{z_1}^{k_1}\widehat{z_2}^{k_2}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}\right) \\
&= \sum_{\underline{i}\in I} q^\bullet a_{\underline{i}} \widehat{z_1}^{i_1}\widehat{z_2}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} \\
&= \sum_{\underline{i}\in I} \frac{q^\bullet \alpha}{a} a_{\underline{i}} \widehat{z_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5-1}\widehat{t_6}^{i_6} \\
&\quad - \sum_{\underline{i}\in I} \frac{q^\bullet}{a} a_{\underline{i}} \widehat{z_1}^{i_1+1}\widehat{t_3}^{i_3+1}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5-1}\widehat{t_6}^{i_6},
\end{aligned}$$

where $(k_1, \cdots, k_6) \in \mathbb{N}^4 \times \mathbb{Z}^2$. Hence, $\mathfrak{S}_5$ is a spanning set of $A_\alpha^{(5)} S_5^{-1}$.

Secondly, suppose that

$$\sum_{\underline{i}\in I} a_{\underline{i}} \widehat{z_1}^{i_1} \widehat{t_3}^{i_3} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6} = 0.$$

Then,

$$\sum_{\underline{i}\in I} a_{\underline{i}} Z_1^{i_1} T_3^{i_3} T_4^{i_4} T_5^{i_5} T_6^{i_6} = (\Omega_1 - \alpha)\nu,$$

with $\nu \in A^{(5)} S_5^{-1}$. Write $\nu = \sum_{\underline{j}\in J} b_{\underline{j}} Z_1^{i_1} Z_2^{i_2} T_3^{i_3} T_4^{i_4} T_5^{i_5} T_6^{i_6}$, where $\underline{j} = (i_1, i_3, i_4, i_5, i_6) \in J \subset \mathbb{N}^4 \times \mathbb{Z}^2$ and $b_{\underline{j}}$ is a family of complex numbers. Given that $\Omega_1 = Z_1 T_3 T_5 + a Z_2 T_5$, it follows that

$$\begin{aligned}\sum_{\underline{i}\in I} a_{\underline{i}} Z_1^{i_1} T_3^{i_3} T_4^{i_4} T_5^{i_5} T_6^{i_6} =& \sum_{\underline{j}\in J} q^{\bullet} b_{\underline{j}} Z_1^{i_1+1} Z_2^{i_2} T_3^{i_3+1} T_4^{i_4} T_5^{i_5+1} T_6^{i_6} \\ &- \sum_{\underline{j}\in J} \alpha b_{\underline{j}} Z_1^{i_1} Z_2^{i_2} T_3^{i_3} T_4^{i_4} T_5^{i_5} T_6^{i_6} \\ &+ \sum_{\underline{j}\in J} q^{\bullet} a b_{\underline{j}} Z_1^{i_1} Z_2^{i_2+1} T_3^{i_3} T_4^{i_4} T_5^{i_5+1} T_6^{i_6}.\end{aligned}$$

Suppose that there exists $(i_1, i_2, i_3, i_4, i_5, i_6) \in J$ such that $b_{(i_1,i_2,i_3,i_4,i_5,i_6)} \neq 0$.
Let $(w_1, w_2, w_3, w_4, w_5, w_6) \in J$ be the greatest element (in the lexicographic order on $\mathbb{N}^4 \times \mathbb{Z}^2$) of J such that $b_{(w_1,w_2,w_3,w_4,w_5,w_6)} \neq 0$. Since the family of monomials $\left(Z_1^{i_1} Z_2^{i_2} T_3^{i_3} T_4^{i_4} T_5^{i_5} T_6^{i_6}\right)_{(i_1,\cdots,k_6)\in\mathbb{N}^4\times\mathbb{Z}^2}$ is a basis of $A^{(5)} S_5^{-1}$, the coefficients of $Z_1^{w_1+1} Z_2^{w_2} T_3^{w_3+1} T_4^{w_4} T_5^{w_5+1} T_6^{w_6}$ in the above equality can be identified as: $q^{\bullet} b_{(w_1,w_2,w_3,w_4,w_5,w_6)} = 0$. This implies that $b_{(w_1,w_2,w_3,w_4,w_5,w_6)} = 0$, a contradiction! Hence, $b_{(i_1,\cdots,i_6)} = 0$ for all $(i_1, \cdots, i_6) \in J$. Consequently, $a_{(i_1,i_3,i_4,i_5,i_6)} = 0$ for all $(i_1, i_3, i_4, i_5, i_6) \in I$. Therefore, $\mathfrak{S}_5$ is a linearly independent set. ■

4.1.4 Proposition. The set $\mathcal{B}_5 = \left\{ z_1^{i_1} t_3^{\xi} t_4^{i_4} t_5^{i_5} t_6^{i_6} \mid (\xi, i_1, i_4, i_5, i_6) \in \{0,1,2\} \times \mathbb{N}^2 \times \mathbb{Z}^2 \right\}$ is a $\mathbb{C}-$basis of R_5.

Proof. Since R_5 is identified with $A_\alpha^{(5)} S_5^{-1} / \langle \widehat{\Omega}_2 - \beta \rangle$ and $\mathfrak{S}_5$ is a basis for $A_\alpha^{(5)} S_5^{-1}$

(Proposition 4.1.3), we show that the spanning set $z_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6}$ of R_5 can be written as a finite linear combination of the elements of $\mathcal{B}_5$ for all $(k_1, k_3, \cdots, k_6) \in \mathbb{N}^3 \times \mathbb{Z}^2$. By Proposition 4.1.3, it is sufficient to do this by an induction on k_3. The result is obvious when $k_3 = 0, 1$ or 2. For $k_3 \geq 2$, suppose that

$$z_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6} = \sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})} z_1^{i_1}t_3^{\xi}t_4^{i_4}t_5^{i_5}t_6^{i_6},$$

where I is a finite subset of $\{0,1,2\} \times \mathbb{N}^2 \times \mathbb{Z}^2$ and $a_{(\xi,\underline{i})}$ are all complex numbers. It follows that

$$z_1^{k_1}t_3^{k_3+1}t_4^{k_4}t_5^{k_5}t_6^{k_6} = q^{\bullet}\left(z_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6}\right)t_3 = \sum_{(\xi,\underline{i})\in I} q^{\bullet} a_{(\xi,\underline{i})} z_1^{i_1}t_3^{\xi+1}t_4^{i_4}t_5^{i_5}t_6^{i_6}.$$

Now, $z_1^{i_1}t_3^{\xi+1}t_4^{i_4}t_5^{i_5}t_6^{i_6} \in \text{Span}(\mathcal{B}_5)$ when $\xi = 0, 1$. For $\xi = 2$, one can easily verify that $z_1^{i_1}t_3^{3}t_4^{i_4}t_5^{i_5}t_6^{i_6} \in \text{Span}(\mathcal{B}_5)$ by using the relation (4.1.5). Therefore, by the principle of mathematical induction, $\mathcal{B}_5$ spans R_5.

We now prove that $\mathcal{B}_5$ is a linearly independent set. Suppose that

$$\sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})} z_1^{i_1}t_3^{\xi}t_4^{i_4}t_5^{i_5}t_6^{i_6} = 0.$$

Since R_5 is identified with $A_{\alpha}^{(5)}S_5^{-1}/\langle\widehat{\Omega}_2 - \beta\rangle$, we have that

$$\sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})} \widehat{z}_1^{\,i_1}\widehat{t}_3^{\,\xi}\widehat{t}_4^{\,i_4}\widehat{t}_5^{\,i_5}\widehat{t}_6^{\,i_6} = \langle\widehat{\Omega}_2 - \beta\rangle\nu,$$

where $\nu \in A_{\alpha}^{(5)}S_5^{-1}$. Write $\nu = \sum_{\underline{j}\in J} b_{\underline{j}}\widehat{z}_1^{\,i_1}\widehat{t}_3^{\,i_3}\widehat{t}_4^{\,i_4}\widehat{t}_5^{\,i_5}\widehat{t}_6^{\,i_6}$, with $\underline{j} = (i_1, i_3, i_4, i_5, i_6) \in J \subset \mathbb{N}^3 \times \mathbb{Z}^2$ and $b_{\underline{j}} \in \mathbb{C}$. Given that $\Omega_2 = \mathcal{Z}_2T_4T_6 + bT_3^3T_6$ in $A^{(5)}$ and the relation (4.1.4), one can deduce that

$$\widehat{\Omega}_2 = \widehat{z}_2\widehat{t}_4\widehat{t}_6 + b\widehat{t}_3^{\,3}\widehat{t}_6 = \frac{q^3\alpha}{a}\widehat{t}_4\widehat{t}_5^{\,-1}\widehat{t}_6 - \frac{1}{a}\widehat{z}_1\widehat{t}_3\widehat{t}_4\widehat{t}_6 + b\widehat{t}_3^{\,3}\widehat{t}_6.$$

Therefore,

$$\sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})}\widehat{z_1}^{i_1}\widehat{t_3}^{\xi}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} = \sum_{\underline{j}\in J} \frac{q^{\bullet}\alpha}{a} b_{\underline{j}}\widehat{z_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4+1}\widehat{t_5}^{i_5-1}\widehat{t_6}^{i_6+1}$$
$$+ \sum_{\underline{j}\in J} q^{\bullet}bb_{\underline{j}}\widehat{z_1}^{i_1}\widehat{t_3}^{i_3+3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6+1}$$
$$- \sum_{\underline{j}\in J} \frac{q^{\bullet}}{a} b_{\underline{j}}\widehat{z_1}^{i_1+1}\widehat{t_3}^{i_3+1}\widehat{t_4}^{i_4+1}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6+1}$$
$$- \sum_{\underline{j}\in J} \beta b_{\underline{j}}\widehat{z_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}.$$

Suppose that there exists $(i_1, i_3, i_4, i_5, i_6) \in J$ such that $b_{(i_1,i_3,i_4,i_5,i_6)} \neq 0$.

Let $(w_1, w_3, w_4, w_5, w_6) \in J$ be the greatest element (in the lexicographic order on $\mathbb{N}^3\times\mathbb{Z}^2$) of J such that $b_{(w_1,w_3,w_4,w_5,w_6)} \neq 0$. Since $\left(\widehat{z_1}^{k_1}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}\right)_{(k_1,k_3,\cdots,k_6)\in\mathbb{N}^3\times\mathbb{Z}^2}$ is a basis of $A_{\alpha}^{(5)}S_5^{-1}$, one can identify the coefficients of $\widehat{z_1}^{w_1+1}\widehat{t_3}^{w_3+1}\widehat{t_4}^{w_4+1}\widehat{t_5}^{w_5}\widehat{t_6}^{w_6+1}$ in the above equality as: $\frac{q^{\bullet}}{a}b_{(w_1,w_3,w_4,w_5,w_6)} = 0$. Hence, $b_{(w_1,w_3,w_4,w_5,w_6)} = 0$, a contradiction! Therefore, $b_{(i_1,i_3,i_4,i_5,i_6)} = 0$ for all $(i_1, i_3, i_4, i_5, i_6) \in J$. Consequently,

$$\sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})}\widehat{z_1}^{i_1}\widehat{t_3}^{\xi}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} = 0.$$

It follows that $a_{(\xi,\underline{i})} = 0$ for all $(\xi, \underline{i}) \in I$. As a result, $\mathcal{B}_5$ is a linearly independent set.

■

4.1.5 Corollary. Let I be a finite subset of $\{0, 1, 2\} \times \mathbb{N} \times \mathbb{Z}^3$ and $(a_{(\xi,\underline{i})})_{\underline{i}\in I}$ be a family of complex numbers. If

$$\sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})} z_1^{i_1} t_3^{\xi} t_4^{i_4} t_5^{i_5} t_6^{i_6} = 0,$$

then $a_{(\xi,\underline{i})} = 0$ for all $(\xi, \underline{i}) \in I$.

Proof. When $i_4 \geq 0$, then the result is obvious from the result of Proposition 4.1.4. For $i_4 < 0$, multiply both sides of the equality (on the right) enough times by t_4 to kill all the negative powers of t_4, and then apply Proposition 4.1.4 to complete the proof. ■

4.1.6 Remark. We were not successful in finding a basis for R_6. However, this has no effect on our main result in this chapter. Since $R_7 = A_{\alpha,\beta}$, we already have a basis for R_7 (Proposition 3.2.2).

4.1.7 Lemma. Let $Z(R_i)$ denote the center of R_i, then $Z(R_i) = \mathbb{C}$ for each $3 \leq i \leq 7$.

Proof. One can easily verify that $Z(R_3) = \mathbb{C}$. Note, $R_7 = A_{\alpha,\beta}$. Since R_i is a localization of R_{i+1} (see (4.1.1)), we have that $\mathbb{C} \subseteq Z(R_7) \subseteq Z(R_6) \subseteq Z(R_5) \subseteq Z(R_4) \subseteq Z(R_3) = \mathbb{C}$. Therefore, $Z(R_7) = Z(R_6) = Z(R_5) = Z(R_4) = Z(R_3) = \mathbb{C}$. ■

4.1.8 Remark. Recall the notations:

$$f_1 := e_{1,4} \qquad F_1 := E_{1,4}$$

$$z_1 := e_{1,5} \qquad \not{Z}_1 := E_{1,5}$$

$$z_2 := e_{2,5} \qquad \not{Z}_2 := E_{2,5}.$$

Let $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Given the above notations, we present the following selected data of the DDA of $A_{\alpha,\beta}$, listed at the beginning of this section, in a manner that would be very useful in Subsections 4.2.1 and 4.2.5. They are as follows:

$$f_1 = t_1 - at_2t_3^{-1} \qquad e_{3,6} = t_3 - at_4t_5^{-1}$$

$$z_1 = f_1 - st_3^2t_4^{-1} \qquad e_1 = e_{1,6} - rt_5t_6^{-1}$$

$$z_2 = t_2 - bt_3^3t_4^{-1} \qquad e_3 = e_{3,6} - st_5^2t_6^{-1}$$

$$e_{1,6} = z_1 - he_{3,6}t_5^{-1} - gt_4t_5^{-2} \qquad e_4 = t_4 - bt_5^3t_6^{-1}.$$

4.2 Derivations of $A_{\alpha,\beta}$

We are now going to study the derivations of $A_{\alpha,\beta}$. We will begin with the case where both α and β are non-zero, and then proceed to look at the case where either α or β is zero.

4.2.1 Derivations of $A_{\alpha,\beta}$ $(\alpha, \beta \neq 0)$. Throughout this subsection, we assume that α and β are non-zero. Let $\text{Der}(A_{\alpha,\beta})$ denote the $\mathbb{C}-$derivations of $A_{\alpha,\beta}$ and $D \in \text{Der}(A_{\alpha,\beta})$.

From the relation (4.1.1), D extends uniquely to a derivation of each of the series of algebras in (4.1.2) via localization. Therefore, D extends to a derivation of the quantum torus $R_3 = \mathbb{C}_{q^N}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$. It follows from [40, Corollary 2.3] that D can uniquely be written as:

$$D = \text{ad}_x + \delta,$$

where $x \in R_3$, and δ is a scalar derivation of R_3 defined as $\delta(t_i) = \lambda_i t_i$ for each $i = 3, 4, 5, 6$. Note, $\lambda_i \in Z(R_3) = \mathbb{C}$. Also, ad_x is an inner derivation of R_3 defined as $\text{ad}_x(L) = xL - Lx$ for all $L \in R_3$.

We aim to describe D as a derivation of $A_{\alpha,\beta}$. We do this in several steps. We first describe D as a derivation of R_4.

4.2.2 Lemma. 1. $x \in R_4$.

2. $\lambda_5 = \lambda_4 + \lambda_6$, $\delta(f_1) = -(\lambda_3 + \lambda_5)f_1$ and $\delta(t_2) = -\lambda_5 t_2$.

3. Set $\lambda_1 := -(\lambda_3 + \lambda_5)$ and $\lambda_2 := -\lambda_5$. Then, $D(e_{\kappa,4}) = \text{ad}_x(e_{\kappa,4}) + \lambda_\kappa e_{\kappa,4}$ for all $\kappa \in \{1, \cdots, 6\}$.

Proof. 1. Set $\mathcal{Q}_q := \mathbb{C}_{q^\mu}[t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$, where μ is some skew-symmetric matrix in $M_3(\mathbb{Z})$. Observe that $\mathcal{Q}_q$ is a subalgebra of both R_3 and R_4 with central element

$$z := t_4 t_5^{-1} t_6.$$

Furthermore, since R_3 is a quantum torus, we can present it as a free left $\mathcal{Q}_q-$module with basis $(t_3^s)_{s\in\mathbb{Z}}$. With this presentation, $x \in R_3$ can be written as

$$x = \sum_{s\in\mathbb{Z}} y_s t_3^s,$$

where $y_s \in \mathcal{Q}_q$. Set

$$x_+ := \sum_{s\geq 0} y_s t_3^s \quad \text{and} \quad x_- := \sum_{s<0} y_s t_3^s.$$

Clearly, $x = x_+ + x_-$. Obviously, $x_+ \in R_4$, hence we aim to also show that x_- belongs to R_4 by following a pattern similar to [29, Proposition 7.1.2]. As D is a derivation of R_4, we have that $D(z^j) \in R_4$ for all $j \in \mathbb{N}_{\geq 1}$. Now $D(z^j) = \text{ad}_x(z^j) + \delta(z^j) = \text{ad}_{x_+}(z^j) + \text{ad}_{x_-}(z^j) + \delta(z^j)$. Observe that $\text{ad}_{x_+}(z^j) \in R_4$; since $x_+, z^j \in R_4$. Also, $\delta(z) = \delta(t_4t_5^{-1}t_6) = (\lambda_4 - \lambda_5 + \lambda_6)t_4t_5^{-1}t_6 = (\lambda_4 - \lambda_5 + \lambda_6)z$, where $\lambda_4, \lambda_5, \lambda_6 \in \mathbb{C}$. It follows that $\delta(z^j) = j(\lambda_4 - \lambda_5 + \lambda_6)z^j \in R_4$. We can therefore conclude that each $\text{ad}_{x_-}(z^j)$ belongs to R_4 since $D(z^j), \text{ad}_{x_+}(z^j), \delta(z^j) \in R_4$. We have:

$$\text{ad}_{x_-}(z^j) = x_-z^j - z^jx_- = \sum_{s=-1}^{-n} y_s t_3^s z^j - \sum_{s=-1}^{-n} y_s z^j t_3^s.$$

One can verify that $zt_3 = q^{-2}t_3z$. Therefore,

$$\text{ad}_{x_-}(z^j) = \sum_{s=-1}^{-n} (1 - q^{-2js}) y_s t_3^s z^j, \text{ hence, } \text{ad}_{x_-}(z^j)z^{-j} = \sum_{s=-1}^{-n} (1 - q^{-2js}) y_s t_3^s.$$

Set $\nu_j := \text{ad}_{x_-}(z^j)z^{-j} \in R_4$. It follows that

$$\nu_j = \sum_{s=-1}^{-n} (1 - q^{-2js}) y_s t_3^s,$$

for each $j \in \{1, \cdots, n\}$. One can therefore write the above equality as a matrix equation as follows:

$$\begin{bmatrix} (1-q^2) & (1-q^4) & (1-q^6) & \cdots & (1-q^{2n}) \\ (1-q^4) & (1-q^8) & (1-q^{12}) & \cdots & (1-q^{4n}) \\ (1-q^6) & (1-q^{12}) & (1-q^{18}) & \cdots & (1-q^{6n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-q^{2n}) & (1-q^{4n}) & (1-q^{6n}) & \cdots & (1-q^{2n^2}) \end{bmatrix} \begin{bmatrix} y_{-1}t_3^{-1} \\ y_{-2}t_3^{-2} \\ y_{-3}t_3^{-3} \\ \vdots \\ y_{-n}t_3^{-n} \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \vdots \\ \nu_n \end{bmatrix}.$$

We already know that each ν_j belongs to R_4. We want to show that $y_s t_3^s$ also belongs to R_4 for each $s \in \{-1, \cdots, -n\}$. To establish this, it is sufficient to show that the coefficient matrix of the above matrix equation is invertible. Let U represent this matrix.

Thus,

$$U = \begin{bmatrix} (1-q^2) & (1-q^4) & (1-q^6) & \cdots & (1-q^{2n}) \\ (1-q^4) & (1-q^8) & (1-q^{12}) & \cdots & (1-q^{4n}) \\ (1-q^6) & (1-q^{12}) & (1-q^{18}) & \cdots & (1-q^{6n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-q^{2n}) & (1-q^{4n}) & (1-q^{6n}) & \cdots & (1-q^{2n^2}) \end{bmatrix}.$$

Apply row operations: $-r_{n-1}+r_n \to r_n, \cdots, -r_2+r_3 \to r_3, -r_1+r_2 \to r_2$ to U to obtain:

$$U' = \begin{bmatrix} l_1 & l_2 & l_3 & \cdots & l_n \\ q^2 l_1 & q^4 l_2 & q^6 l_3 & \cdots & q^{2n} l_n \\ q^4 l_1 & q^8 l_2 & q^{12} l_3 & \cdots & q^{4n} l_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q^{2(n-1)} l_1 & q^{4(n-1)} l_2 & q^{6(n-1)} l_3 & \cdots & q^{2n(n-1)} l_n \end{bmatrix},$$

where $l_i := 1-q^{2i}$; $i \in \{1, 2, \cdots, n\}$. Clearly, U' is similar to a Vandermonde matrix (since the terms in each column form a geometric sequence) which is well known to be invertible. This further implies that U is invertible. So each $y_s t_3^s$ is a linear combination of the $\nu_j \in R_4$. As a result, $y_s t_3^s \in R_4$ for all $s \in \{-1, \cdots, -n\}$. Consequently, $x_- = \sum_{s=-1}^{-n} y_s t_3^s \in R_4$ as desired.

2. Recall that $\delta(t_\kappa) = \lambda_\kappa t_\kappa$ for all $\kappa \in \{3, 4, 5, 6\}$ and $\lambda_\kappa \in \mathbb{C}$. From Remark 4.1.8, we have that $f_1 = t_1 - a t_2 t_3^{-1}$. Recall from Section 3.1 that $t_1 = \alpha t_5^{-1} t_3^{-1}$ and $t_2^{-1} = \beta t_6^{-1} t_4^{-1}$ in $R_3 = \mathscr{A}_{\alpha,\beta}$. As a result, $f_1 = \alpha t_5^{-1} t_3^{-1} - a\beta t_6^{-1} t_4^{-1} t_3^{-1}$. Hence,

$$\delta(f_1) = -(\lambda_5 + \lambda_3)\alpha t_5^{-1} t_3^{-1} + (\lambda_6 + \lambda_4 + \lambda_3) a\beta t_6^{-1} t_4^{-1} t_3^{-1}. \tag{4.2.1}$$

From Proposition 4.1.2, the set $\mathcal{B}_4 = \left\{ f_1^{i_1} t_4^{i_4} t_5^{i_5} t_6^{i_6}, t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6} \mid i_1, i_3 \in \mathbb{N} \text{ and } i_4, i_5, i_6 \in \mathbb{Z} \right\}$ is a $\mathbb{C}-$basis of R_4. Since t_4, t_5 and t_6 q-commute with f_1 and t_3, one can also write $\delta(f_1) \in R_4$ in terms of $\mathcal{B}_4$ as follows:

$$\delta(f_1) = \sum_{r>0} a_r f_1^r + \sum_{s\geq 0} b_s t_3^s, \tag{4.2.2}$$

where a_r and b_s belong to $\mathcal{Q}_q = \mathbb{C}_{q^\mu}[t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$.

$$\begin{aligned} f_1^r &= (\alpha t_5^{-1} t_3^{-1} - a\beta t_6^{-1} t_4^{-1} t_3^{-1})^r = \sum_{i=0}^{r} \binom{r}{i}_{q^\bullet} (\alpha t_5^{-1} t_3^{-1})^i (-a\beta t_6^{-1} t_4^{-1} t_3^{-1})^{r-i} \\ &= \sum_{i=0}^{r} \binom{r}{i}_{q^\bullet} \alpha^i (-a\beta)^{r-i} q^{\frac{1}{2}i(i-1)+\frac{3}{2}(r-i)(r-i-1)+3i(i-r)} t_5^{-i} (t_6^{-1} t_4^{-1})^{r-i} t_3^{-r} \\ &= c_r t_3^{-r}, \end{aligned} \tag{4.2.3}$$

where

$$c_r = \sum_{i=0}^{r} \binom{r}{i}_{q^\bullet} q^{\frac{1}{2}i(i-1)+\frac{3}{2}(r-i)(r-i-1)+3i(i-r)} \alpha^i (-a\beta)^{r-i} t_5^{-i} (t_6^{-1} t_4^{-1})^{r-i} \in \mathcal{Q}_q \setminus \{0\}. \tag{4.2.4}$$

Substitute (4.2.3) into (4.2.2) to obtain;

$$\delta(f_1) = \sum_{r>0} a_r c_r t_3^{-r} + \sum_{s\geq 0} b_s t_3^s. \tag{4.2.5}$$

One can rewrite (4.2.1) as

$$\delta(f_1) = d t_3^{-1}, \tag{4.2.6}$$

where $d = -(\lambda_5 + \lambda_3)\alpha t_5^{-1} + (\lambda_6 + \lambda_4 + \lambda_3) a\beta t_6^{-1} t_4^{-1} \in \mathcal{Q}_q$. Comparing (4.2.5) to (4.2.6) shows that $b_s = 0$ for all $s \geq 0$, and $a_r c_r = 0$ for all $r \neq 1$. Therefore $\delta(f_1) = a_1 c_1 t_3^{-1}$. Moreover, from (4.2.4), $c_1 = -a\beta t_6^{-1} t_4^{-1} + \alpha t_5^{-1}$. Hence,

$$\delta(f_1) = a_1 c_1 t_3^{-1} = a_1(-a\beta t_6^{-1} t_4^{-1} + \alpha t_5^{-1}) t_3^{-1} = a_1 \alpha t_5^{-1} t_3^{-1} - a_1 a\beta t_6^{-1} t_4^{-1} t_3^{-1}. \tag{4.2.7}$$

Comparing (4.2.7) to (4.2.1) reveals that $a_1 = -(\lambda_5 + \lambda_3) = -(\lambda_6 + \lambda_4 + \lambda_3)$. Consequently, $\lambda_5 = \lambda_6 + \lambda_4$. Hence, $\delta(f_1) = -(\lambda_5+\lambda_3)\alpha t_5^{-1}t_3^{-1} + (\lambda_5+\lambda_3)a\beta t_6^{-1}t_4^{-1}t_3^{-1} = -(\lambda_5+\lambda_3)f_1$. Finally, since $t_2 = \beta t_6^{-1}t_4^{-1}$ in R_4, it follows that $\delta(t_2) = -(\lambda_6 + \lambda_4)\beta t_6^{-1}t_4^{-1} = -(\lambda_6+\lambda_4)t_2 = -\lambda_5 t_2$.

3. Set $\lambda_1 := -(\lambda_3+\lambda_5)$ and $\lambda_2 := -\lambda_5$. it follows from points (1) and (2) that $D(e_{\kappa,4}) = \text{ad}_x(e_{\kappa,4}) + \delta(e_{\kappa,4}) = \text{ad}_x(e_{\kappa,4}) + \lambda_\kappa e_{\kappa,4}$ for all $\kappa \in \{1,\cdots,6\}$. In conclusion, $D = \text{ad}_x + \delta$, with $x \in R_4$. ■

We proceed to describe D as a derivation of R_5.

4.2.3 Lemma. 1. $x \in R_5$.

2. $\lambda_4 = 3\lambda_3 + \lambda_5,\ \lambda_6 = -3\lambda_3,\ \delta(z_1) = -(\lambda_3+\lambda_5)z_1$ and $\delta(z_2) = -\lambda_5 z_2$.

3. Set $\lambda_1 := -(\lambda_3+\lambda_5),\ \lambda_2 := -\lambda_5$ and $\lambda_6 := -3\lambda_3$. Then, $D(e_{\kappa,5}) = \text{ad}_x(e_{\kappa,5}) + \lambda_\kappa e_{\kappa,5}$ for all $\kappa \in \{1,\cdots,6\}$.

Proof. In this proof, we denote $\underline{v} := (i,j,k,l) \in \mathbb{N}\times\mathbb{Z}^3$.

1. We already know that $x \in R_4 = R_5[t_4^{-1}]$. Given the basis $\mathcal{B}_5$ of R_5 (Proposition 4.1.4), x can be written as $x = \sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l$, where I is a finite subset of $\{0,1,2\}\times\mathbb{N}\times\mathbb{Z}^3$ and $a_{(\xi,\underline{v})}$ are complex numbers. Write $x = x_- + x_+$, where

$$x_+ = \sum_{\substack{(\xi,\underline{v})\in I\\ j\geq 0}} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l \quad \text{and} \quad x_- = \sum_{\substack{(\xi,\underline{v})\in I\\ j<0}} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

Suppose that there exists a minimum $j_0 < 0$ such that $a_{(\xi,i,j_0,k,l)} \neq 0$ for some $(\xi,i,j_0,k,l) \in I$ and $a_{(\xi,i,j,k,l)} = 0$ for all $(\xi,i,j_0,k,l)\in I$ with $j<j_0$. Given this assumption, write

$$x_- = \sum_{\substack{(\xi,\underline{v})\in I\\ j_0\leq j\leq -1}} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

Now, $D(t_6) = \text{ad}_{x_+}(t_6) + \text{ad}_{x_-}(t_6) + \delta(t_6) \in R_5$. This implies that $\text{ad}_{x_-}(t_6) \in R_5$, since $\text{ad}_{x_+}(t_6) + \delta(t_6) = \text{ad}_{x_+}(t_6) + \lambda_6 t_6 \in R_5$. We aim to show that $x_- = 0$. Since t_6 is normal in R_5, one can easily verify that

$$\text{ad}_{x_-}(t_6) = \sum_{\substack{(\xi,\underline{v})\in I\\ j_0\le j\le -1}} \left(q^{3(i-j-k)}-1\right) a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^{l+1}.$$

Set $\underline{w} := (i,j,k,l) \in \mathbb{N}^2 \times \mathbb{Z}^2$. One can equally write $\text{ad}_{x_-}(t_6) \in R_5$ in terms of the basis $\mathcal{B}_5$ of R_5 (Proposition 4.1.4) as:

$$\text{ad}_{x_-}(t_6) = \sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l,$$

where J is a finite subset of $\{0,1,2\} \times \mathbb{N}^2 \times \mathbb{Z}^2$ and $b_{(\xi,\underline{w})}$ are all complex numbers. It follows that

$$\sum_{\substack{(\xi,\underline{v})\in I\\ j_0\le j\le -1}} \left(q^{3(i-j-k)}-1\right) a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^{l+1} = \sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

As $\mathcal{B}_5$ is a basis for R_5, we deduce from Corollary 4.1.5 that $\left(z_1^i t_3^\xi t_4^j t_5^k t_6^l\right)_{(i\in\mathbb{N};j,k,l\in\mathbb{Z};\xi\in\{0,1,2\})}$ is a basis for $R_5[t_4^{-1}]$. Now, at $j = j_0$, denote $\underline{v} = (i,j,k,l)$ by $\underline{v}_0 := (i,j_0,k,l)$. Since $\underline{v}_0 \in \mathbb{N}\times\mathbb{Z}^3$ (with $j_0 < 0$) and $\underline{w} = (i,j,k,l) \in \mathbb{N}^2\times\mathbb{Z}^2$ (with $j \ge 0$), it follows from the above equality that, at $\underline{v}_0$, we must have

$$\left(q^{3(i-j_0-k)}-1\right) a_{(\xi,\underline{v}_0)} = 0.$$

From our initial assumption, the coefficients $a_{(\xi,\underline{v}_0)}$ are all not zero, therefore $q^{3(i-j_0-k)} - 1 = 0$. This implies that

$$k = i - j_0, \tag{4.2.8}$$

for some $(\xi,\underline{v}_0) \in I$.

In a similar manner, $D(t_3) = \text{ad}_{x_+}(t_3) + \text{ad}_{x_-}(t_3) + \delta(t_3) \in R_5$. This implies that

$\mathrm{ad}_{x_-}(t_3) \in R_5$, since $\mathrm{ad}_{x_+}(t_3) + \delta(t_3) = \mathrm{ad}_{x_+}(t_3) + \lambda_3 t_3 \in R_5$. We have that

$$\mathrm{ad}_{x_-}(t_3) = \sum_{\substack{(\xi,\underline{v})\in I\\ j_0\leq j\leq -1}} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l t_3 - \sum_{\substack{(\xi,\underline{v})\in I\\ j_0\leq j\leq -1}} a_{(\xi,\underline{v})} t_3 z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

One can deduce from Lemma A.1.4(3a) that

$$t_3 z_1^i = q^{-i} z_1^i t_3 + d_2[i] z_1^{i-1} z_2,$$

where $d_2[i] = q^{1-i} d_2[1]\left(\frac{1-q^{-2i}}{1-q^{-2}}\right)$, $d_2[1] = -(q+q^{-1}+q^{-3})$ and $d_2[0] = 0$. Therefore, the above expression for $\mathrm{ad}_{x_-}(t_3)$ can be expressed as:

$$\begin{aligned}\mathrm{ad}_{x_-}(t_3) = &\sum_{\substack{(0,\underline{v})\in I\\ j_0\leq j\leq -1}} f[i,j,k] a_{(0,\underline{v})} z_1^i t_3 t_4^j t_5^k t_6^l + \sum_{\substack{(1,\underline{v})\in I\\ j_0\leq j\leq -1}} f[i,j,k] a_{(1,\underline{v})} z_1^i t_3^2 t_4^j t_5^k t_6^l + \\ &+ \sum_{\substack{(2,\underline{v})\in I\\ j_0\leq j\leq -1}} f[i,j,k] a_{(2,\underline{v})} z_1^i t_3^3 t_4^j t_5^k t_6^l - \sum_{\substack{(\xi,\underline{v})\in I\\ j_0\leq j\leq -1}} a_{(\xi,\underline{v})} d_2[i] z_1^{i-1} z_2 t_3^\xi t_4^j t_5^k t_6^l,\end{aligned}$$

where $f[i,j,k] := q^{-(k+3j)} - q^{-i}$. Recall from (4.1.4) and (4.1.5) that

$$z_2 = \frac{1}{a}\left(\alpha t_5^{-1} - z_1 t_3\right) \quad \text{and} \quad t_3^3 = \frac{\beta}{b} t_6^{-1} - \frac{q^3\alpha}{ab} t_4 t_5^{-1} + \frac{1}{ab} z_1 t_3 t_4,$$

where a and b are non-zero scalars (Appendix C). Using these two expressions, one can write $\mathrm{ad}_{x_-}(t_3)$ in terms of the basis of R_5 as:

$$\begin{aligned}\mathrm{ad}_{x_-}(t_3) =&\mathcal{K} + \sum_{(0,\underline{v}_0)\in I} g[i,j_0,k]a_{(0,\underline{v}_0)} z_1^i t_3 t_4^{j_0} t_5^k t_6^l + \sum_{(1,\underline{v}_0)\in I} g[i,j_0,k]a_{(1,\underline{v}_0)} z_1^i t_3^2 t_4^{j_0} t_5^k t_6^l \\ &+ \sum_{(2,\underline{v}_0)\in I} \frac{q^\bullet \beta}{b} a_{(2,\underline{v}_0)} g[i,j_0,k] z_1^i t_4^{j_0} t_5^k t_6^{l-1} - \sum_{(\xi,\underline{v}_0)\in I} \frac{q^\bullet \alpha}{a} d_2[i] a_{(\xi,\underline{v}_0)} z_1^{i-1} t_3^{\xi} t_4^{j_0} t_5^{k-1} t_6^l \\ =&\sum 1/b \left(q^\bullet \beta g[i,j_0,k]a_{(2,i,j_0,k,l+1)} + (q^\bullet \alpha b d_2[i+1]/a) a_{(0,i+1,j_0,k+1,l)}\right) z_1^i t_4^{j_0} t_5^k t_6^l \\ &+ \sum \left(g[i,j_0,k]a_{(0,i,j_0,k,l)} + (q^\bullet \alpha d_2[i+1]/a) a_{(1,i+1,j_0,k+1,l)}\right) z_1^i t_3 t_4^{j_0} t_5^k t_6^l \\ &+ \sum \left(g[i,j_0,k]a_{(1,i,j_0,k,l)} + (q^\bullet \alpha d_2[i+1]/a) a_{(2,i+1,j_0,k+1,l)}\right) z_1^i t_3^2 t_4^{j_0} t_5^k t_6^l + \mathcal{K},\end{aligned} \tag{4.2.9}$$

where $g[i,j_0,k] := q^{-(k+3j_0)} - q^{-i} + d_2[i]/a$ and

$$\mathcal{K} \in \mathrm{Span}\left(\mathcal{B}_5 \setminus \{z_1^i t_3^\xi t_4^{j_0} t_5^k t_6^l \mid (\xi,i,j_0,k,l) \in \{0,1,2\} \times \mathbb{N} \times \mathbb{Z}^3\}\right).$$

One can also write $\mathrm{ad}_{x_-}(t_3) \in R_5$ in terms of the basis $\mathcal{B}_5$ of R_5 (Proposition 4.1.4) as:

$$\mathrm{ad}_{x_-}(t_3) = \sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l, \tag{4.2.10}$$

where J is a finite subset of $\{0,1,2\} \times \mathbb{N}^2 \times \mathbb{Z}^2$, and $b_{(\xi,\underline{w})} \in \mathbb{C}$. Recall: $\underline{w} = (i,j,k,l) \in \mathbb{N}^2 \times \mathbb{Z}^2$. Now, (4.2.9) and (4.2.10) imply that

$$\begin{aligned}&\sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l \\ &\quad= \sum 1/b \left(q^\bullet \beta g[i,j_0,k]a_{(2,i,j_0,k,l+1)} + (q^\bullet \alpha b d_2[i+1]/a) a_{(0,i+1,j_0,k+1,l)}\right) z_1^i t_4^{j_0} t_5^k t_6^l \\ &\quad+ \sum \left(g[i,j_0,k]a_{(0,i,j_0,k,l)} + (q^\bullet \alpha d_2[i+1]/a) a_{(1,i+1,j_0,k+1,l)}\right) z_1^i t_3 t_4^{j_0} t_5^k t_6^l \\ &\quad+ \sum \left(g[i,j_0,k]a_{(1,i,j_0,k,l)} + (q^\bullet \alpha d_2[i+1]/a) a_{(2,i+1,j_0,k+1,l)}\right) z_1^i t_3^2 t_4^{j_0} t_5^k t_6^l + \mathcal{K}.\end{aligned}$$

We have already established that $\left(z_1^i t_3^\xi t_4^j t_5^k t_6^l\right)_{(i\in\mathbb{N};j,k,l\in\mathbb{Z};\xi\in\{0,1,2\})}$ is a basis for $R_5[t_4^{-1}]$. Given that $\underline{v}_0 = (i,j_0,k,l) \in \mathbb{N} \times \mathbb{Z}^3$ (with $j_0 < 0$) and $\underline{w} = (i,j,k,l) \in \mathbb{N}^2 \times \mathbb{Z}^2$ (with

$j \geq 0$), it follows that

$$q^{\bullet}\beta g[i, j_0, k]a_{(2,i,j_0,k,l+1)} + (q^{\bullet}\alpha b d_2[i+1]/a)a_{(0,i+1,j_0,k+1,l)} = 0. \tag{4.2.11}$$

$$g[i, j_0, k]a_{(0,i,j_0,k,l)} + (q^{\bullet}\alpha d_2[i+1]/a)a_{(1,i+1,j_0,k+1,l)} = 0. \tag{4.2.12}$$

$$g[i, j_0, k]a_{(1,i,j_0,k,l)} + (q^{\bullet}\alpha d_2[i+1]/a)a_{(2,i+1,j_0,k+1,l)} = 0. \tag{4.2.13}$$

Suppose that there exists $(\xi, i, j_0, k, l) \in I$ such that $g[i, j_0, k] = 0$. Then,

$$g[i, j_0, k] = q^{-(k+3j_0)} - q^{-i} + d_2[i]/a = 0.$$

Note, $d_2[i] = d_2[1]q^{1-i}\left(\dfrac{1-q^{-2i}}{1-q^{-2}}\right)$, where $d_2[1] = -(q + q^{-1} + q^{-3})$ and $d_2[0] = 0$. Again, recall from Appendix C that $a = (q^2+1+q^{-2})/(q^{-2}-1) = \dfrac{qd_2[1]}{1-q^{-2}}$. Given these expressions for $d_2[i]$ and a, we have that

$$g[i, j_0, k] = q^{-(k+3j_0)} - q^{-i} + d_2[i]/a = q^{-3j_0-k} - q^{-3i} = 0.$$

Since q is not a root of unity, we get

$$k = 3(i - j_0). \tag{4.2.14}$$

Comparing (4.2.14) to (4.2.8) shows that $i - j_0 = 0$ which implies that $i = j_0 < 0$, a contradiction (note, $i \geq 0$). Therefore, $g[i, j_0, k] \neq 0$ for all $(\xi, i, j, k, l) \in I$.

Now, observe that if there exists $\xi \in \{0, 1, 2\}$ such that $a_{(\xi,i,j_0,k,l)} = 0$ for all $(i, j_0, k, l) \in \mathbb{N} \times \mathbb{Z}^3$, then one can easily deduce from equations (4.2.11), (4.2.12) and (4.2.13) that $a_{(\xi,i,j_0,k,l)} = 0$ for all $(\xi, i, j_0, k, l) \in I$. This will contradict our initial assumption. Therefore, there exists some $(i, j_0, k, l) \in \mathbb{N} \times \mathbb{Z}^3$ such that $a_{(\xi,i,j_0,k,l)} \neq 0$ for each $\xi \in \{0, 1, 2\}$. Without loss of generality, let (u, j_0, v, w) be the greatest element in the lexicographic order on $\mathbb{N} \times \mathbb{Z}^3$ such that $a_{(0,u,j_0,v,w)} \neq 0$ and $a_{(0,i,j_0,k,l)} = 0$ for all $i > u$.

From (4.2.12), at $(i, j_0, k, l) = (u, j_0, v, w)$, we have:

$$g[u, j_0, v]a_{(0,u,j_0,v,w)} + (q^\bullet \alpha d_2[u + 1]/a)a_{(1,u+1,j_0,v+1,w)} = 0.$$

From (4.2.13), at $(i, j_0, k, l) = (u + 1, j_0, v + 1, w)$, we have:

$$g[u + 1, j_0, v + 1]a_{(1,u+1,j_0,v+1,w)} + (q^\bullet \alpha d_2[u + 2]/a)a_{(2,u+2,j_0,v+2,w)} = 0.$$

Finally, from (4.2.11), at $(i, j_0, k, l) = (u + 2, j_0, v + 2, w - 1)$, we have:

$$q^\bullet \beta g[u + 2, j_0, v + 2]a_{(2,u+2,j_0,v+2,w)} + (q^\bullet \alpha b d_2[u + 3]/a)a_{(0,u+3,j_0,v+3,w-1)} = 0.$$

Note: $a, b, \alpha, \beta, q^\bullet \neq 0$; $g[i, j_0, k] \neq 0$ for all $(\xi, i, j_0, k, l) \in I$; and $d_2[i] \neq 0$ for $i > 0$. Since $u + 3 > u$, it follows from the above list of equations (starting from the last one) that

$$a_{(0,u+3,j_0,v+3,w-1)} = 0 \Rightarrow a_{(2,u+2,j_0,v+2,w)} = 0 \Rightarrow a_{(1,u+1,j_0,v+1,w)} = 0 \Rightarrow a_{(0,u,j_0,v,w)} = 0,$$

a contradiction! Hence, $a_{(0,i,j_0,k,l)} = 0$ for all $(i, j_0, k, l) \in \mathbb{N}\times\mathbb{Z}^3$. From (4.2.11), (4.2.12) and (4.2.13), one can easily conclude that $a_{(\xi,i,j_0,k,l)} = 0$ for all $(\xi, i, j_0, k, l) \in I$. This contradicts our initial assumption, hence $x_- = 0$. Consequently, $x = x_+ \in R_5$ as desired.

2. From Remark 4.1.8, we have $z_2 = t_2 - bt_3^3t_4^{-1}$. Since $\delta(t_\kappa) = \lambda_\kappa t_\kappa$, $\kappa \in \{2, \cdots, 6\}$, with $\lambda_2 := -\lambda_5$ (see Lemma 4.2.2), it follows that

$$\delta(z_2) = -\lambda_5 t_2 - b(3\lambda_3 - \lambda_4)t_3^3t_4^{-1} = -\lambda_5 z_2 - b(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3t_4^{-1}.$$

Furthermore,

$$D(z_2) = \text{ad}_x(z_2) + \delta(z_2) = \text{ad}_x(z_2) - \lambda_5 z_2 - b(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3t_4^{-1} \in R_5.$$

Hence $b(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3t_4^{-1} \in R_5$, since $\text{ad}_x(z_2) - \lambda_5 z_2 \in R_5$. This implies that

$b(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3 \in R_5t_4$ (note, from Appendix C, $b \neq 0$). Set $w := 3\lambda_3 - \lambda_4 + \lambda_5$. Suppose that $w \neq 0$. From (4.1.5), we have:

$$t_3^3 = \frac{\beta}{b}t_6^{-1} - \frac{q^3\alpha}{ab}t_4t_5^{-1} + \frac{1}{ab}z_1t_3t_4.$$

It follows that

$$wbt_3^3 = w\beta t_6^{-1} - \frac{q^3w\alpha}{a}t_4t_5^{-1} + \frac{w}{a}z_1t_3t_4 \in R_5t_4.$$

Since t_3^3, $t_4t_5^{-1}$ and $z_1t_3t_4$ are all elements of R_5t_4, it implies that $t_6^{-1} \in R_5t_4$. Hence, $1 \in R_5t_4t_6$. Using the basis $\mathcal{B}_5$ of R_5 (Proposition 4.1.4), this leads to contradiction. Therefore, $w = 0$. That is, $3\lambda_3 - \lambda_4 + \lambda_5 = 0$, and so $\lambda_4 = 3\lambda_3 + \lambda_5$. This further implies that $\delta(z_2) = -\lambda_5 z_2$ as desired.

Again, from Lemma 4.2.2, we have that $\delta(f_1) = -(\lambda_3 + \lambda_5)f_1$. Recall from Remark 4.1.8 that $z_1 = f_1 - st_3^2t_4^{-1}$. It follows that

$$\begin{aligned}\delta(z_1) =& -(\lambda_3 + \lambda_5)f_1 - s(2\lambda_3 - \lambda_4)t_3^2t_4^{-1} = -(\lambda_3 + \lambda_5)z_1 - s(3\lambda_3 - \lambda_4 + \lambda_5)t_3^2t_4^{-1}\\ =& -(\lambda_3 + \lambda_5)z_1 - s(3\lambda_3 - (3\lambda_3 + \lambda_5) + \lambda_5)t_3^2t_4^{-1} = -(\lambda_3 + \lambda_5)z_1.\end{aligned}$$

Finally, we know that $\delta(t_6) = \lambda_6 t_6$. This implies that $\delta(t_6^{-1}) = -\lambda_6 t_6^{-1}$. From (4.1.5), we have that

$$t_3^3 = \frac{\beta}{b}t_6^{-1} - \frac{q^3\alpha}{ab}t_4t_5^{-1} + \frac{1}{ab}z_1t_3t_4,$$

where a and b are non-zero scalars (Appendix C). This implies that

$$t_6^{-1} = \frac{b}{\beta}t_3^3 + \frac{q^3\alpha}{a\beta}t_4t_5^{-1} - \frac{1}{a\beta}z_1t_3t_4.$$

Given that $\delta(z_1) = -(\lambda_3 + \lambda_5)z_1$, $\delta(t_3) = \lambda_3 t_3$, $\delta(t_4) = (3\lambda_3 + \lambda_5)t_4$ and $\delta(t_5) = \lambda_5 t_5$, applying δ to the above relation gives

$$-\lambda_6 t_6^{-1} = 3\lambda_3\left(\frac{b}{\beta}t_3^3 + \frac{q^3\alpha}{a\beta}t_4t_5^{-1} - \frac{1}{a\beta}z_1t_3t_4\right).$$

It follows that $\lambda_6 = -3\lambda_3$ as desired.

3. Set $\lambda_1 := -(\lambda_3 + \lambda_5)$ and $\lambda_2 := -\lambda_5$. It follows from points (1) and (2) that $D(e_{\kappa,5}) = \text{ad}_x(e_{\kappa,5}) + \delta(e_{\kappa,5}) = \text{ad}_x(e_{\kappa,5}) + \lambda_\kappa e_{\kappa,5}$ for all $\kappa \in \{1, \cdots, 6\}$. In conclusion, $D = \text{ad}_x + \delta$ with $x \in R_5$. ■

We are now ready to describe D as a derivation of $A_{\alpha,\beta}$.

4.2.4 Lemma. 1. $x \in A_{\alpha,\beta}$.

2. $\delta(e_\kappa) = 0$ for all $\kappa \in \{1, \cdots, 6\}$.

3. $D = \text{ad}_x$.

Proof. In this proof, we denote $\underline{v} := (i, j, k, l) \in \mathbb{N}^2 \times \mathbb{Z}^2$. Also, recall from the DDA of $A_{\alpha,\beta}$ at the beginning of this section that $t_5 = e_5$ and $t_6 = e_6$.

1. Given the basis $\mathcal{B}$ of $A_{\alpha,\beta}$ (Proposition 3.2.2), one can write $x \in R_5 = A_{\alpha,\beta}[t_5^{-1}, t_6^{-1}]$ as:

$$x = \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} e_1^i e_2^j e_3^{\epsilon_1} e_4^{\epsilon_2} t_5^k t_6^l,$$

where I is a finite subset of $\{0, 1\}^2 \times \mathbb{N}^2 \times \mathbb{Z}^2$, and $a_{(\epsilon_1,\epsilon_2,\underline{v})}$ are complex numbers. Write $x = x_- + x_+$, where

$$x_+ = \sum_{\substack{(\epsilon_1,\epsilon_2,\underline{v})\in I \\ k,\ l\geq 0}} a_{(\epsilon_1,\epsilon_2,\underline{v})} e_1^i e_2^j e_3^{\epsilon_1} e_4^{\epsilon_2} t_5^k t_6^l,$$

and

$$x_- = \sum_{\substack{(\epsilon_1,\epsilon_2,\underline{v})\in I \\ k<0 \text{ or } l<0}} a_{(\epsilon_1,\epsilon_2,\underline{v})} e_1^i e_2^j e_3^{\epsilon_1} e_4^{\epsilon_2} t_5^k t_6^l.$$

Suppose that there exists a minimum negative integer k_0 or l_0 such that $a_{(\epsilon_1,\epsilon_2,i,j,k_0,l)} \neq 0$ or $a_{(\epsilon_1,\epsilon_2,i,j,k,l_0)} \neq 0$ for some $(\epsilon_1, \epsilon_2, i, j, k_0, l), (\epsilon_1, \epsilon_2, i, j, k, l_0) \in I$, and $a_{(\epsilon_1,\epsilon_2,i,j,k,l)} = 0$ whenever $k < k_0$ or $l < l_0$. Write

$$x_- = \sum_{\substack{(\epsilon_1,\epsilon_2,\underline{v})\in I \\ k_0\leq k\leq -1 \text{ or } l_0\leq l\leq -1}} a_{(\epsilon_1,\epsilon_2,\underline{v})} e_1^i e_2^j e_3^{\epsilon_1} e_4^{\epsilon_2} t_5^k t_6^l.$$

Now, $D(e_3) = \mathrm{ad}_{x_+}(e_3) + \mathrm{ad}_{x_-}(e_3) + \delta(e_3) \in A_{\alpha,\beta}$. From Remark 4.1.8, we have that $e_3 = e_{3,6} - st_5^2t_6^{-1}$ and $e_{3,6} = t_3 - at_4t_5^{-1}$. Putting these two together gives

$$e_3 = t_3 - at_4t_5^{-1} - st_5^2t_6^{-1}.$$

Again, from Remark 4.1.8, we also have that $t_4 = e_4 + bt_5^3t_6^{-1}$. Note, $\delta(t_\kappa) = \lambda_\kappa t_\kappa,\ \kappa \in \{3,4,5,6\}$. Now,

$$\begin{aligned}\delta(e_3) &= \lambda_3 t_3 - a(\lambda_4 - \lambda_5)t_4t_5^{-1} - s(2\lambda_5 - \lambda_6)t_5^2t_6^{-1}\\ &= \lambda_3(e_{3,6} + at_4t_5^{-1}) + a(\lambda_5 - \lambda_4)t_4t_5^{-1} + s(\lambda_6 - 2\lambda_5)t_5^2t_6^{-1}\\ &= \lambda_3 e_{3,6} + a(\lambda_3 - \lambda_4 + \lambda_5)t_4t_5^{-1} + s(\lambda_6 - 2\lambda_5)t_5^2t_6^{-1}\\ &= \lambda_3(e_3 + st_5^2t_6^{-1}) + a(\lambda_3 - \lambda_4 + \lambda_5)(e_4 + bt_5^3t_6^{-1})t_5^{-1} + s(\lambda_6 - 2\lambda_5)t_5^2t_6^{-1}\\ &= \lambda_3 e_3 + \alpha_1 e_4 t_5^{-1} + \alpha_2 t_5^2t_6^{-1}, \end{aligned} \tag{4.2.15}$$

where $\alpha_1 = a(\lambda_3 - \lambda_4 + \lambda_5)$ and $\alpha_2 = s(\lambda_3 - 2\lambda_5 + \lambda_6) + q^{-3}ab(\lambda_3 - \lambda_4 + \lambda_5)$. Therefore, $D(e_3) = \mathrm{ad}_{x_+}(e_3) + \mathrm{ad}_{x_-}(e_3) + \lambda_3 e_3 + \alpha_1 e_4 t_5^{-1} + \alpha_2 t_5^2 t_6^{-1} \in A_{\alpha,\beta}$. It follows that $D(e_3)t_5t_6 = \mathrm{ad}_{x_+}(e_3)t_5t_6 + \mathrm{ad}_{x_-}(e_3)t_5t_6 + \lambda_3 e_3 t_5 t_6 + \alpha_1 e_4 t_6 + q^3\alpha_2 t_5^3 \in A_{\alpha,\beta}$. Hence, $\mathrm{ad}_{x_-}(e_3)t_5t_6 \in A_{\alpha,\beta}$, since $\mathrm{ad}_{x_+}(e_3)t_5t_6 + \lambda_3 e_3t_5t_6 + \alpha_1 e_4 t_6 + q^3\alpha_2 t_5^3 \in A_{\alpha,\beta}$.

Now,

$$\mathrm{ad}_{x_-}(e_3) = \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} e_1^i e_2^j e_3^{\epsilon_1} e_4^{\epsilon_2} t_5^k t_6^l e_3 - \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} e_3 e_1^i e_2^j e_3^{\epsilon_1} e_4^{\epsilon_2} t_5^k t_6^l. \tag{4.2.16}$$

Using Lemma A.1.4, we have the following:

$$t_5^k t_6^l e_3 = q^{-k} e_3 t_5^k t_6^l + d_2[k] e_4 t_5^{k-1} t_6^l + d_3[l] t_5^{k+2} t_6^{l-1}, \tag{4.2.17}$$

$$e_3 e_1^i e_2^j = q^{-i-3j} e_1^i e_2^j e_3 + d_2[i] e_1^{i-1} e_2^{j+1}, \tag{4.2.18}$$

(note: $d_2[i], d_2[k]$ and $d_3[l]$ are defined in Lemma A.1.4). Substitute (4.2.17) and (4.2.18) into (4.2.16), simplify and multiply (on the right) by t_5t_6 to obtain

$$\begin{aligned}\text{ad}_{x_-}(e_3)t_5t_6 = \\ &\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}\left(g[i,j,\epsilon_2,l]e_1^ie_2^je_3^{\epsilon_1+1}e_4^{\epsilon_2}t_5^{k+1}t_6^{l+1} + q^{-3l}d_2[k]e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2+1}t_5^kt_6^{l+1}\right. \\ &\left. + q^{-3(l-1)}d_3[l]e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2}t_5^{k+3}t_6^{l} - q^{-3l}d_2[i]e_1^{i-1}e_2^{j+1}e_3^{\epsilon_1}e_4^{\epsilon_2}t_5^{k+1}t_6^{l+1}\right),\end{aligned} \tag{4.2.19}$$

where $g[i,j,\epsilon_2,l] := q^{-k-3\epsilon_2-3l} - q^{-i-3j-3l}$.

Assume that there exists $l < 0$ such that $a_{(\epsilon_1,\epsilon_2,\underline{v})} \neq 0$. It follows from our initial assumption that $a_{(\epsilon_1,\epsilon_2,i,j,k,l_0)} \neq 0$. Now, at $l = l_0$, denote $\underline{v} = (i,j,k,l)$ by $\underline{v}_0 := (i,j,k,l_0)$. From (4.2.19), we have that

$$\text{ad}_{x_-}(e_3)t_5t_6 = \sum_{(\epsilon_1,\epsilon_2,\underline{v}_0)\in I} q^{-3(l_0-1)}a_{(\epsilon_1,\epsilon_2,\underline{v}_0)}d_3[l_0]e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2}t_5^{k+3}t_6^{l_0} + \mathcal{J}_1,$$

where $\mathcal{J}_1 \in \text{Span}\left(\mathcal{B}\setminus\{e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2}t_5^kt_6^{l_0} \mid \epsilon_1,\epsilon_2\in\{0,1\},\ k\in\mathbb{Z} \text{ and } i,j\in\mathbb{N}\}\right)$.

Set $\underline{w} := (i,j,k,l) \in \mathbb{N}^4$. One can also write $\text{ad}_{x_-}(e_3)t_5t_6 \in A_{\alpha,\beta}$ in terms of the basis $\mathcal{B}$ of $A_{\alpha,\beta}$ (Proposition 3.2.2) as:

$$\text{ad}_{x_-}(e_3)t_5t_6 = \sum_{(\epsilon_1,\epsilon_2,\underline{w})\in J} b_{(\epsilon_1,\epsilon_2,\underline{w})}e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2}t_5^kt_6^l, \tag{4.2.20}$$

where J is a finite subset of $\{0,1\}^2\times\mathbb{N}^4$, and $b_{(\epsilon_1,\epsilon_2,\underline{w})} \in \mathbb{C}$. It follows that

$$\sum_{(\epsilon_1,\epsilon_2,\underline{w})\in J} b_{(\epsilon_1,\epsilon_2,\underline{w})}e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2}t_5^kt_6^l = \sum_{(\epsilon_1,\epsilon_2,\underline{v}_0)\in I} q^{-3(l_0-1)}a_{(\epsilon_1,\epsilon_2,\underline{v}_0)}d_3[l_0]e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2}t_5^{k+3}t_6^{l_0} + \mathcal{J}_1.$$

Since $\mathcal{B}$ is a basis for $A_{\alpha,\beta}$, we deduce from Corollary 3.2.3 that $\left(e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2}t_5^kt_6^l\right)_{((\epsilon_1,\epsilon_2,\underline{v})\in\{0,1\}^2\times\mathbb{N}^2\times\mathbb{Z}^2)}$ is also a basis for $A_{\alpha,\beta}[t_5^{-1},t_6^{-1}]$. Since $\underline{v}_0 = (i,j,k,l_0) \in \mathbb{N}^2\times\mathbb{Z}^2$ (with $l_0 < 0$) and $\underline{w} = (i,j,k,l) \in \mathbb{N}^4$ (with $l \geq 0$) in the above equality, we must have

$$q^{-3(l_0-1)}a_{(\epsilon_1,\epsilon_2,\underline{v}_0)}d_3[l_0] = 0.$$

Given that $q^{-3(l_0-1)}d_3[l_0] \neq 0$, it follows that $a_{(\epsilon_1,\epsilon_2,\underline{v}_0)} = a_{(\epsilon_1,\epsilon_2,i,j,k,l_0)}$ are all zero. This is a contradiction. Therefore, $l \geq 0$ (i.e. there is no negative exponent for t_6).

Since $l \geq 0$, it follows from our initial assumption that there exists $k = k_0 < 0$ such that $a_{(\epsilon_1,\epsilon_2,i,j,k_0,l)} \neq 0$. The rest of the proof will show that this assumption cannot also hold. Set $\underline{v}_0 := (i,j,k_0,l) \in \mathbb{N}^2 \times \mathbb{Z} \times \mathbb{N}$. From (4.2.19), we have that

$$\text{ad}_{x_-}(e_3)t_5t_6 = \sum_{(\epsilon_1,\epsilon_2,\underline{v}_0)\in I} q^{-3l} a_{(\epsilon_1,\epsilon_2,\underline{v}_0)} d_2[k_0] e_1^i e_2^j e_3^{\epsilon_1} e_4^{\epsilon_2+1} t_5^{k_0} t_6^{l+1} + V,$$

where $V \in \mathcal{J}_2 := \text{Span}\left(\mathcal{B} \setminus \{e_1^i e_2^j e_3^{\epsilon_1} e_4^{\epsilon_2} t_5^{k_0} t_6^l \mid \epsilon_1, \epsilon_2 \in \{0,1\} \text{ and } i,j,l \in \mathbb{N}\}\right)$. It follows that:

$$\begin{aligned}
&\text{ad}_{x_-}(e_3)t_5t_6 = \\
&\quad \sum_{(0,0,\underline{v}_0)\in I} q^{-3l} a_{(0,0,\underline{v}_0)} d_2[k_0] e_1^i e_2^j e_4 t_5^{k_0} t_6^{l+1} + \sum_{(1,0,\underline{v}_0)\in I} a_{(1,0,\underline{v}_0)} d_2[k_0] e_1^i e_2^j e_3 e_4 t_5^{k_0} t_6^{l+1} \\
&\quad + \sum_{(0,1,\underline{v}_0)\in I} q^{-3l} a_{(0,1,\underline{v}_0)} d_2[k_0] e_1^i e_2^j e_4^2 t_5^{k_0} t_6^{l+1} \sum_{(1,1,\underline{v}_0)\in I} a_{(1,1,\underline{v}_0)} d_2[k_0] e_1^i e_2^j e_3 e_4^2 t_5^{k_0} t_6^{l+1} + V.
\end{aligned} \tag{4.2.21}$$

Write the relations in Lemma 3.0.1(2),(4) as:

$$e_4^2 = b_1\beta + b_2 e_2 e_4 e_6 + b_4 \alpha e_3 e_6 + b_6 e_1 e_3 e_4 e_6 + L_1, \tag{4.2.22}$$

$$\begin{aligned}
e_3 e_4^2 =& \beta b_1 e_3 + k_1 e_2 e_3 e_4 e_6 + k_3 \alpha^2 e_6 + k_5 \alpha e_1 e_4 e_6 + k_{14} \beta e_1^2 e_6 \\
&+ k_{15} e_1^2 e_2 e_4 e_6^2 + k_{17} \alpha e_1^2 e_3 e_6^2 + k_{19} e_1^3 e_3 e_4 e_6^2 + L_2,
\end{aligned} \tag{4.2.23}$$

where L_1 and L_2 are some elements of the left ideal $A_{\alpha,\beta}t_5 \subseteq \mathcal{J}_2$. Substitute (4.2.22) and (4.2.23) into (4.2.21), and simplify to obtain:

$$\begin{aligned}
\mathsf{ad}_{x_-}(e_3)t_5t_6 = &\sum[\lambda_{1,1}\beta a_{(0,1,i,j,k_0,l-1)} + \lambda_{1,2}\alpha^2 a_{(1,1,i,j,k_0,l-2)} \\
&\quad + \lambda_{1,3}\beta a_{(1,1,i-2,j,k_0,l-2)}]e_1^i e_2^j t_5^{k_0} t_6^l \\
&+ \sum[\lambda_{2,1}\alpha a_{(0,1,i,j,k_0,l-2)} + \lambda_{2,2}\beta a_{(1,1,i,j,k_0,l-1)} \\
&\quad + \lambda_{2,3}\alpha a_{(1,1,i-2,j,k_0,l-3)}]e_1^i e_2^j e_3 t_5^{k_0} t_6^l \\
&+ \sum[\lambda_{3,1} a_{(0,1,i,j-1,k_0,l-2)} + \lambda_{3,2}\alpha a_{(1,1,i-1,j,k_0,l-2)} \\
&\quad + \lambda_{3,3} a_{(1,1,i-2,j-1,k_0,l-3)} + \lambda_{3,4} a_{(0,0,i,j,k_0,l-1)}]e_1^i e_2^j e_4 t_5^{k_0} t_6^l \\
&+ \sum[\lambda_{4,1} a_{(0,1,i-1,j,k_0,l-2)} + \lambda_{4,2} a_{(1,1,i,j-1,k_0,l-2)} \\
&\quad + \lambda_{4,3} a_{(1,1,i-3,j,k_0,l-3)} + \lambda_{4,4} a_{(1,0,i,j,k_0,l-1)}]e_1^i e_2^j e_3 e_4 t_5^{k_0} t_6^l + V',
\end{aligned} \tag{4.2.24}$$

where $V' \in \mathcal{J}_2$. Also, $\lambda_{s,t} := \lambda_{s,t}(j, k_0, l)$ are some families of complex numbers which are non-zero for all $s, t \in \{1, 2, 3, 4\}$ and $j, l \in \mathbb{N}$, except $\lambda_{1,4}$ and $\lambda_{2,4}$ which are assumed to be zero since they do not exist in the above expression. Note, although each $\lambda_{s,t}$ depends on j, k_0, l, we have not made this dependency explicit in the above expression since the minimum requirement we need to complete the proof is for all the $\lambda_{s,t}$ existing in the above expression to be non-zero, which we already have.

Observe that (4.2.24) and (4.2.20) are equal, hence,

$$\sum_{(\epsilon_1,\epsilon_2,\underline{w})\in J} b_{(\epsilon_1,\epsilon_2,\underline{w})} e_1^i e_2^j e_3^{\epsilon_1} e_4^{\epsilon_2} t_5^k t_6^l = \sum [\lambda_{1,1}\beta a_{(0,1,i,j,k_0,l-1)} + \lambda_{1,2}\alpha^2 a_{(1,1,i,j,k_0,l-2)}$$
$$+ \lambda_{1,3}\beta a_{(1,1,i-2,j,k_0,l-2)}] e_1^i e_2^j t_5^{k_0} t_6^l$$
$$+ \sum [\lambda_{2,1}\alpha a_{(0,1,i,j,k_0,l-2)} + \lambda_{2,2}\beta a_{(1,1,i,j,k_0,l-1)}$$
$$+ \lambda_{2,3}\alpha a_{(1,1,i-2,j,k_0,l-3)}] e_1^i e_2^j e_3 t_5^{k_0} t_6^l$$
$$+ \sum [\lambda_{3,1} a_{(0,1,i,j-1,k_0,l-2)} + \lambda_{3,2}\alpha a_{(1,1,i-1,j,k_0,l-2)}$$
$$+ \lambda_{3,3} a_{(1,1,i-2,j-1,k_0,l-3)} + \lambda_{3,4} a_{(0,0,i,j,k_0,l-1)}] e_1^i e_2^j e_4 t_5^{k_0} t_6^l$$
$$+ \sum [\lambda_{4,1} a_{(0,1,i-1,j,k_0,l-2)} + \lambda_{4,2} a_{(1,1,i,j-1,k_0,l-2)}$$
$$+ \lambda_{4,3} a_{(1,1,i-3,j,k_0,l-3)} + \lambda_{4,4} a_{(1,0,i,j,k_0,l-1)}] e_1^i e_2^j e_3 e_4 t_5^{k_0} t_6^l + V'.$$

We have previously established that $(e_1^i e_2^j e_3^{\epsilon_1} e_4^{\epsilon_2} t_5^k t_6^l)_{((\epsilon_1,\epsilon_2,\underline{v})\in\{0,1\}^2\times\mathbb{N}^2\times\mathbb{Z}^2)}$ is a basis for $A_{\alpha,\beta}[t_5^{-1}, t_6^{-1}]$ (note, in this part of the proof $l \geq 0$). Since $\underline{v}_0 = (i,j,k_0,l) \in \mathbb{N}^2 \times \mathbb{Z} \times \mathbb{N}$ (with $k_0 < 0$) and $\underline{w} = (i,j,k,l) \in \mathbb{N}^4$ (with $k \geq 0$) in the above equality, it follows that

$$\lambda_{1,1}\beta a_{(0,1,i,j,k_0,l-1)} + \lambda_{1,2}\alpha^2 a_{(1,1,i,j,k_0,l-2)} + \lambda_{1,3}\beta a_{(1,1,i-2,j,k_0,l-2)} = 0, \quad (4.2.25)$$
$$\lambda_{2,1}\alpha a_{(0,1,i,j,k_0,l-2)} + \lambda_{2,2}\beta a_{(1,1,i,j,k_0,l-1)} + \lambda_{2,3}\alpha a_{(1,1,i-2,j,k_0,l-3)} = 0, \quad (4.2.26)$$
$$\lambda_{3,1} a_{(0,1,i,j-1,k_0,l-2)} + \lambda_{3,2}\alpha a_{(1,1,i-1,j,k_0,l-2)} + \lambda_{3,3} a_{(1,1,i-2,j-1,k_0,l-3)}$$
$$+ \lambda_{3,4} a_{(0,0,i,j,k_0,l-1)} = 0, \quad (4.2.27)$$
$$\lambda_{4,1} a_{(0,1,i-1,j,k_0,l-2)} + \lambda_{4,2} a_{(1,1,i,j-1,k_0,l-2)} + \lambda_{4,3} a_{(1,1,i-3,j,k_0,l-3)}$$
$$+ \lambda_{4,4} a_{(1,0,i,j,k_0,l-1)} = 0. \quad (4.2.28)$$

From (4.2.25) and (4.2.26), one can easily deduce that

$$a_{(0,1,i,j,k_0,l)} = -\frac{\alpha^2\lambda_{1,2}}{\beta\lambda_{1,1}} a_{(1,1,i,j,k_0,l-1)} - \frac{\lambda_{1,3}}{\lambda_{1,1}} a_{(1,1,i-2,j,k_0,l-1)}, \quad (4.2.29)$$
$$a_{(1,1,i,j,k_0,l)} = -\frac{\alpha\lambda_{2,1}}{\beta\lambda_{2,2}} a_{(0,1,i,j,k_0,l-1)} - \frac{\alpha\lambda_{2,3}}{\beta\lambda_{2,2}} a_{(1,1,i-2,j,k_0,l-2)}. \quad (4.2.30)$$

Note, $a_{(\epsilon_1,\epsilon_2,i,j,k_0,l)} := 0$ whenever $i < 0$ or $j < 0$ or $l < 0$ for all $\epsilon_1, \epsilon_2 \in \{0,1\}$.

Claim. The coefficients $a_{(0,1,i,j,k_0,l)}$ and $a_{(1,1,i,j,k_0,l)}$ are all zero for all $l \geq 0$. We now justify the claim by an induction on l. From (4.2.29) and (4.2.30), the result is obviously true when $l = 0$. For $l \geq 0$, assume that $a_{(0,1,i,j,k_0,l)} = a_{(1,1,i,j,k_0,l)} = 0$. Then, it follows from (4.2.29) and (4.2.30) that

$$a_{(0,1,i,j,k_0,l+1)} = -\frac{\alpha^2\lambda_{1,2}}{\beta\lambda_{1,1}}a_{(1,1,i,j,k_0,l)} - \frac{\lambda_{1,3}}{\lambda_{1,1}}a_{(1,1,i-2,j,k_0,l)},$$

$$a_{(1,1,i,j,k_0,l+1)} = -\frac{\alpha\lambda_{2,1}}{\beta\lambda_{2,2}}a_{(0,1,i,j,k_0,l)} - \frac{\alpha\lambda_{2,3}}{\beta\lambda_{2,2}}a_{(1,1,i-2,j,k_0,l-1)}.$$

From the inductive hypothesis, $a_{(1,1,i,j,k_0,l)} = a_{(1,1,i-2,j,k_0,l)} = a_{(0,1,i,j,k_0,l)} = a_{(1,1,i-2,j,k_0,l-1)}$ $= 0$. Hence, $a_{(1,1,i,j,k_0,l+1)} = a_{(0,1,i,j,k_0,l+1)} = 0$. By the principle of mathematical induction, $a_{(0,1,i,j,k_0,l)} = a_{(1,1,i,j,k_0,l)} = 0$ for all $l \geq 0$ as desired. Given that the families $a_{(0,1,i,j,k_0,l)}$ and $a_{(1,1,i,j,k_0,l)}$ are all zero, it follows from (4.2.27) and (4.2.28) that $a_{(0,0,i,j,k_0,l)}$ and $a_{(1,0,i,j,k_0,l)}$ are also zero for all $(i,j,k_0,l) \in \mathbb{N}^2 \times \mathbb{Z} \times \mathbb{N}$. Since $a_{(\epsilon_1,\epsilon_2,i,j,k_0,l)}$ are all zero, it contradicts our assumption. Hence, $x_- = 0$. Consequently, $x = x_+ \in A_{\alpha,\beta}$ as desired.

2. From Remark 4.1.8, we have $e_4 = t_4 - bt_5^3t_6^{-1}$. Again, from Lemma 4.2.3, we have that $\lambda_4 = 3\lambda_3 + \lambda_5$ and $\lambda_6 = -3\lambda_3$. Therefore,

$$\begin{aligned}
\delta(e_4) &= \lambda_4 t_4 - b(3\lambda_5 - \lambda_6)t_5^3t_6^{-1} \\
&= (3\lambda_3 + \lambda_5)e_{4,6} - 3b(\lambda_3 + \lambda_5)t_5^3t_6^{-1} \\
&= (3\lambda_3 + \lambda_5)(e_4 + bt_5^3t_6^{-1}) - 3b(\lambda_3 + \lambda_5)t_5^3t_6^{-1} \\
&= (3\lambda_3 + \lambda_5)e_4 - 2b\lambda_5t_5^3t_6^{-1}.
\end{aligned}$$

Moreover, $D(e_4) = \text{ad}_x(e_4) + \delta(e_4) = \text{ad}_x(e_4) + (3\lambda_3 + \lambda_5)e_4 - 2b\lambda_5t_5^3t_6^{-1} \in A_{\alpha,\beta}$. It follows that $b\lambda_5t_5^3t_6^{-1} \in A_{\alpha,\beta}$, since $\text{ad}_x(e_4) + (3\lambda_3 + \lambda_5)e_4 \in A_{\alpha,\beta}$. Consequently, $b\lambda_5t_5^3 \in A_{\alpha,\beta}t_6$. Since $b \neq 0$ (Appendix C), we must have $\lambda_5 = 0$, otherwise, there will be a contradiction using the basis of $A_{\alpha,\beta}$ (Proposition 3.2.2). Therefore, $\delta(e_4) =$

$3\lambda_3 e_4$ and $\delta(t_5) = 0$. We already know from Lemma 4.2.3 that $\delta(t_6) = -3\lambda_3 t_6$. From (4.2.15), we have that $\delta(e_3) = \lambda_3 e_3 + a(\lambda_3 - \lambda_4 + \lambda_5)e_4 t_5^{-1} + [s(\lambda_3 - 2\lambda_5 + \lambda_6) + q^{-3}ab(\lambda_3 - \lambda_4 + \lambda_5)]t_5^2 t_6^{-1}$. Given that $\lambda_4 = 3\lambda_3,\ \lambda_5 = 0$ and $\lambda_6 = -3\lambda_3$, we have that $\delta(e_3) = \lambda_3 e_3 - 2a\lambda_3 e_4 t_5^{-1}$ (note, from Appendix C, one can confirm that $q^{-3}ab + s = 0$). Now, $D(e_3) = \text{ad}_x(e_3) + \delta(e_3) = \text{ad}_x(e_3) + \lambda_3 e_3 - 2a\lambda_3 e_4 t_5^{-1} \in A_{\alpha,\beta}$. Observe that $\text{ad}_x(e_3) + \lambda_3 e_3 \in A_{\alpha,\beta}$. Hence, $2a\lambda_3 e_4 t_5^{-1} \in A_{\alpha,\beta}$, and so $2a\lambda_3 e_4 \in A_{\alpha,\beta} t_5$. Since $a \neq 0$, it implies that $\lambda_3 = 0$, otherwise, there will be a contradiction using the basis of $A_{\alpha,\beta}$. We now have that $\delta(e_3) = \delta(e_4) = \delta(e_5) = \delta(e_6) = 0$. We finish the proof by showing that $\delta(e_1) = \delta(e_2) = 0$. Recall from (3.0.2) that

$$e_2 e_4 e_6 + b e_2 e_5^3 + b e_3^3 e_6 + b' e_3^2 e_5^2 + c' e_3 e_4 e_5 + d' e_4^2 = \beta.$$

Apply δ to this relation to obtain $\delta(e_2)e_4 e_6 + b\delta(e_2)e_5^3 = 0$. This implies that $\delta(e_2)(e_4 e_6 + b e_5^3) = 0$. Since $e_4 e_6 + b e_5^3 \neq 0$, it follows that $\delta(e_2) = 0$. Similarly, from (3.0.1), we have that

$$e_1 e_3 e_5 + a e_1 e_4 + a e_2 e_5 + a' e_3^2 = \alpha.$$

Apply δ to this relation to obtain $\delta(e_1)(e_3 e_5 + a e_4) = 0$. Since $e_3 e_5 + a e_4 \neq 0$, we have that $\delta(e_1) = 0$. In conclusion, $\delta(e_\kappa) = 0$ for all $\kappa \in \{1, \cdots, 6\}$.

3. As a result of (1) and (2), we have that $D(e_\kappa) = \text{ad}_x(e_\kappa)$. Therefore, $D = \text{ad}_x$ as desired. ■

4.2.5 Derivations of $A_{\alpha,0}$ and $A_{0,\beta}$. In this subsection, we explore the derivations of $A_{\alpha,\beta}$ when either α or β is zero (but not both). We are going to follow the same pattern used to compute the derivations of $A_{\alpha,\beta}$ $(\alpha, \beta \neq 0)$ in the previous subsection. Of course, results that can easily be obtained from the previous subsection are not going to be repeated here. In this case, the appropriate references shall be made. We will begin with the derivations of $A_{\alpha,0}$.

Derivations of $A_{\alpha,0}$. Let Der$(A_{\alpha,0})$ denote the $\mathbb{C}-$derivations of $A_{\alpha,0}$ and $D \in$ Der$(A_{\alpha,0})$. We already know from the previous subsection that D extends uniquely to a derivation of each of the series of algebras in (4.1.2) via localization. Therefore, D extends to a derivation of the quantum torus $R_3 = \mathbb{C}_{q^N}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$. It follows from [40, Corollary 2.3] that D can uniquely be written as:

$$D = \text{ad}_x + \delta,$$

where $x \in R_3$, and δ is a scalar derivation of R_3 defined as $\delta(t_i) = \lambda_i t_i$ for each $i = 3, 4, 5, 6$. Note, $\lambda_i \in Z(R_3) = \mathbb{C}$ (Lemma 4.1.7). Also, ad_x is an inner derivation of R_3 defined as $\text{ad}_x(L) = xL - Lx$ for all $L \in R_3$.

We will describe D as a derivation of $A_{\alpha,0}$. We first describe D as a derivation of R_4.

4.2.6 Lemma. 1. $x \in R_4$.

2. $\delta(f_1) = -(\lambda_3 + \lambda_5) f_1$ and $\delta(t_2) = 0$.

3. Set $\lambda_1 := -(\lambda_3 + \lambda_5)$ and $\lambda_2 := 0$. Then, $D(e_{\kappa,4}) = \text{ad}_x(e_{\kappa,4}) + \lambda_\kappa e_{\kappa,4}$ for all $\kappa \in \{1, \cdots, 6\}$.

Proof. 1. Similar to that of Lemma 4.2.2(1).

2. From Section 3.1, we have that $t_1 = \alpha t_5^{-1} t_3^{-1}$ and $t_2 = \beta t_6^{-1} t_4^{-1}$ in $R_3 = \mathscr{A}_{\alpha,\beta}$. Clearly, $t_2 = 0$ since $\beta = 0$. From Remark 4.1.8, we have that $f_1 = t_1 - a t_2 t_3^{-1}$. This simplifies to $f_1 = t_1 = \alpha t_5^{-1} t_3^{-1}$. It follows that $\delta(f_1) = -(\lambda_3 + \lambda_5) f_1$. Obviously, $\delta(t_2) = 0$.

3. As a result of (1) and (2), $D(e_{\kappa,4}) = \text{ad}_x(e_{\kappa,4}) + \lambda_\kappa e_{\kappa,4}$ for all $\kappa \in \{1, \cdots, 6\}$. ■

We proceed to also describe D as a derivation of R_5.

4.2.7 Lemma. 1. $x \in R_5$.

2. $\lambda_5 = \lambda_4 - 3\lambda_3$, $\delta(z_1) = -(\lambda_3 + \lambda_5) z_1$ and $\delta(z_2) = -\lambda_5 z_2$.

3. Set $\lambda_1 := -(\lambda_3 + \lambda_5)$ and $\lambda_2 := -\lambda_5$. Then, $D(e_{\kappa,5}) = \text{ad}_x(e_{\kappa,5}) + \lambda_\kappa e_{\kappa,5}$ for all $\kappa \in \{1, \cdots, 6\}$.

Proof. 1. Similar to that of Lemma 4.2.3(1).

2. Recall that $\delta(t_i) = \lambda_i t_i$, $i = 3, 4, 5, 6$, and $\delta(f_1) = -(\lambda_3 + \lambda_5) f_1$. From Remark 4.1.8, we have that $z_1 = f_1 - s t_3^2 t_4^{-1}$. One can confirm that $\delta(z_1) = -(\lambda_3+\lambda_5) z_1 + s(\lambda_4 - 3\lambda_3 - \lambda_5) t_3^2 t_4^{-1}$. Now, $D(z_1) = \text{ad}_x + \delta(z_1) = \text{ad}_x - (\lambda_3+\lambda_5) z_1 + s(\lambda_4 - 3\lambda_3 - \lambda_5) t_3^2 t_4^{-1} \in R_5$. Since $\text{ad}_x - (\lambda_3 + \lambda_5) z_1 \in R_5$, we have that $s(\lambda_4 - 3\lambda_3 - \lambda_5) t_3^2 t_4^{-1} \in R_5$. Consequently, $s(\lambda_4 - 3\lambda_3 - \lambda_5) t_3^2 \in R_5 t_4$. Since $s \neq 0$ (Appendix C), it follows that $\lambda_4 - 3\lambda_3 - \lambda_5 = 0$, and so $\lambda_5 = \lambda_4 - 3\lambda_3$. Otherwise, there will be a contradiction using the basis of R_5 (Proposition 4.1.4). We now have $\delta(z_1) = -(\lambda_3 + \lambda_5) z_1$. From the previous lemma, we have that $t_2 = 0$. As a result, the relation $z_2 = t_2 - b t_3^3 t_4^{-1}$ in Remark 4.1.8 becomes $z_2 = -b t_3^3 t_4^{-1}$. It follows that $\delta(z_2) = (3\lambda_3 - \lambda_4) z_2 = -\lambda_5 z_2$.

3. Set $\lambda_1 := -(\lambda_3 + \lambda_5)$ and $\lambda_2 := -\lambda_5$. It follows from points (1) and (2) that $D(e_{\kappa,5}) = \text{ad}_x(e_{\kappa,5}) + \lambda_\kappa e_{\kappa,5}$ for all $\kappa \in \{1, \cdots, 6\}$.

■

We now describe D as a derivation of $A_{\alpha,0}$.

4.2.8 Lemma. 1. $x \in A_{\alpha,0}$.

2. $\lambda_6 = 2\lambda_5$, $\delta(e_1) = -\lambda_5 e_1$, $\delta(e_2) = -\lambda_5 e_2$, $\delta(e_3) = 0$ and $\delta(e_4) = \lambda_5 e_4$.

3. Set $\lambda_1 := -\lambda_5$, $\lambda_2 := -\lambda_5$, $\lambda_3 := 0$ and $\lambda_4 := \lambda_5$. Then, $D(e_\kappa) = \text{ad}_x(e_\kappa) + \lambda_\kappa e_\kappa$ for all $\kappa \in \{1, \cdots, 6\}$.

Proof. 1. Similar to the proof of Lemma 4.2.4(1).

2. From Lemma 4.2.7, we have $\lambda_5 = \lambda_4 - 3\lambda_3$, which implies that $\lambda_4 = 3\lambda_3 + \lambda_5$. Furthermore, from Remark 4.1.8, we have $e_4 = t_4 - b t_5^3 t_6^{-1}$. One can confirm that $\delta(e_4) = (3\lambda_3 + \lambda_5) e_4 + b(3\lambda_3 - 2\lambda_5 + \lambda_6) t_5^3 t_6^{-1}$. Now, $D(e_4) = \text{ad}_x(e_4) + \delta(e_4) = \text{ad}_x(e_4) + (3\lambda_3 + \lambda_5) e_4 + b(3\lambda_3 - 2\lambda_5 + \lambda_6) t_5^3 t_6^{-1} \in A_{\alpha,0}$. Since $\text{ad}_x(e_4) + (3\lambda_3 + \lambda_5) e_4 \in A_{\alpha,0}$, it follows that $b(3\lambda_3 - 2\lambda_5 + \lambda_6) t_5^3 t_6^{-1} \in A_{\alpha,0}$. This implies that $b(3\lambda_3 - 2\lambda_5 + \lambda_6) t_5^3 \in A_{\alpha,0} t_6$. Note, $b \neq 0$ (Appendix C). As a result, $3\lambda_3 - 2\lambda_5 + \lambda_6 = 0$, otherwise, we will have a contradiction using the basis of $A_{\alpha,0}$ (Proposition 3.2.2). Consequently, $\lambda_6 = 2\lambda_5 - 3\lambda_3$ and $\delta(e_4) = (3\lambda_3 + \lambda_5) e_4$.

The relation $\delta(e_3) = \lambda_3 e_3 + a(\lambda_3 - \lambda_4 + \lambda_5)e_4 t_5^{-1} + [s(\lambda_3 - 2\lambda_5 + \lambda_6) + q^{-3}ab(\lambda_3 - \lambda_4 + \lambda_5)]t_5^2 t_6^{-1}$ (see (4.2.15)) is also valid in $A_{\alpha,0}$. Given that $\lambda_4 = 3\lambda_3 + \lambda_5$ and $\lambda_6 = 2\lambda_5 - 3\lambda_3$, one can easily verify that $\delta(e_3) = \lambda_3 e_3 - 2a\lambda_3 e_4 t_5^{-1}$ (note, from Appendix C, we have that $q^{-3}ab + s = 0$). Now, $D(e_3) = \text{ad}_x(e_3) + \delta(e_3) = \text{ad}_x(e_3) + \lambda_3 e_3 - 2a\lambda_3 e_4 t_5^{-1} \in A_{\alpha,0}$. Since $\text{ad}_x(e_3) + \lambda_3 e_3 \in A_{\alpha,0}$, we have that $2a\lambda_3 e_4 t_5^{-1} \in A_{\alpha,0}$, which implies that $2a\lambda_3 e_4 \in A_{\alpha,0} t_5$. Hence, $\lambda_3 = 0$, otherwise, we will have a contradiction using the basis of $A_{\alpha,0}$ (note, from Appendix C, $a \neq 0$). Since $\lambda_3 = 0$, we have: $\lambda_6 = 2\lambda_5$, $\delta(e_4) = \lambda_5 e_4$ and $\delta(e_3) = 0$. From Remark 4.1.8, we have: $e_1 = e_{1,6} - rt_5 t_6^{-1}$, $e_{1,6} = z_1 - he_{3,6} t_5^{-1} - gt_4 t_5^{-2}$ and $e_{3,6} = t_3 - at_4 t_5^{-1}$. Putting these three relations together gives

$$e_1 = z_1 - rt_5 t_6^{-1} - ht_3 t_5^{-1} + (ah - g)t_4 t_5^{-2}. \tag{4.2.31}$$

From the previous lemma, we have that $\delta(z_1) = -(\lambda_3 + \lambda_5)z_1$. Given that $\lambda_6 = 2\lambda_5$, $\lambda_4 = \lambda_5$, $\lambda_3 = 0$ and $\delta(t_5) = \lambda_5 t_5$, one can verify that $\delta(e_1) = -\lambda_5 e_1$. Finally, from the commutation relations of $A_{\alpha,0}$ in Chapter 3, we have that $e_3 e_1 = q^{-1} e_1 e_3 - (q + q^{-1} + q^{-3})e_2$. Since $\delta(e_1) = -\lambda_5 e_1$ and $\delta(e_3) = 0$, applying δ to this relation gives $\delta(e_2) = -\lambda_5 e_2$.

3. Set $\lambda_1 := -\lambda_5$, $\lambda_2 := -\lambda_5$, $\lambda_3 := 0$ and $\lambda_4 := \lambda_5$, it follows from points (1) and (2) that $D(e_\kappa) = \text{ad}_x(e_\kappa) + \lambda_\kappa e_\kappa$ for all $\kappa \in \{1, \cdots, 6\}$. ■

Derivations of $A_{0,\beta}$. Every derivation D of $A_{0,\beta}$ extends uniquely to a derivation of each of the series of algebras in (4.1.2). Therefore, D extends to a derivation of the quantum torus $R_3 = \mathbb{C}_{q^N}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$. It follows from [40, Corollary 2.3] that D can uniquely be written as

$$D = \text{ad}_x + \delta,$$

where $x \in R_3$, and δ is a scalar derivation of R_3 defined as $\delta(t_i) = \lambda_i t_i$ for each $i = 3, 4, 5, 6$. Note, $\lambda_i \in Z(R_3) = \mathbb{C}$ (Lemma 4.1.7). Also, ad_x is an inner derivation of R_3 defined as $\text{ad}_x(L) = xL - Lx$ for all $L \in R_3$.

We want to describe D as a derivation of $A_{0,\beta}$. We begin by describing D as a derivation of R_4.

4.2.9 Lemma. 1. $x \in R_4$.

2. $\delta(f_1) = -(\lambda_3 + \lambda_4 + \lambda_6)f_1$ and $\delta(t_2) = -(\lambda_6 + \lambda_4)t_2$.

3. Set $\lambda_1 := -(\lambda_3+\lambda_4+\lambda_6)$ and $\lambda_2 := -(\lambda_6+\lambda_4)$. Then, $D(e_{\kappa,4}) = \text{ad}_x(e_{\kappa,4})+\lambda_\kappa e_{\kappa,4}$ for all $\kappa \in \{1, \cdots, 6\}$.

Proof. 1. Similar to that of Lemma 4.2.2(1).

2. Recall from Section 3.1 that $t_1 = 0$ and $t_2 = \beta t_6^{-1}t_4^{-1}$ in $R_3 = \mathscr{A}_{0,\beta}$. Observe that $\delta(t_2) = -(\lambda_4 + \lambda_6)t_2$. From Remark 4.1.8, the relation $f_1 = t_1 - at_2t_3^{-1}$ becomes $f_1 = -at_2t_3^{-1} = -a\beta t_6^{-1}t_4^{-1}t_3^{-1}$. It follows that $\delta(f_1) = -(\lambda_3 + \lambda_4 + \lambda_6)f_1$.

3. Set $\lambda_1 := -(\lambda_3 + \lambda_4 + \lambda_6)$ and $\lambda_2 := -(\lambda_6 + \lambda_4)$, it follows from points (1) and (2) that $D(e_{\kappa,4}) = \text{ad}_x(e_{\kappa,4}) + \lambda_\kappa e_{\kappa,4}$ for all $\kappa \in \{1, \cdots, 6\}$. ■

We proceed to describe D as a derivation of R_5.

4.2.10 Lemma. 1. $x \in R_5$.

2. $\lambda_6 = -3\lambda_3,\ \delta(z_1) = (2\lambda_3 - \lambda_4)z_1$ and $\delta(z_2) = (3\lambda_3 - \lambda_4)z_2$.

3. Set $\lambda_1 := (2\lambda_3 - \lambda_4)$ and $\lambda_2 := 3\lambda_3 - \lambda_4$. Then, $D(e_{\kappa,5}) = \text{ad}_x(e_{\kappa,5}) + \lambda_\kappa e_{\kappa,5}$ for all $\kappa \in \{1, \cdots, 6\}$.

Proof. 1. Similar to the proof of Lemma 4.2.3(1).

2. Recall that $\delta(t_i) = \lambda_i t_i$, $i = 3, 4, 5, 6$, and $\delta(t_2) = -(\lambda_4 + \lambda_6)t_2$. From Remark 4.1.8, we have that $z_2 = t_2 - bt_3^3t_4^{-1}$. This implies that $\delta(z_2) = -(\lambda_4 + \lambda_6)z_2 - b(\lambda_6 + 3\lambda_3)t_3^3t_4^{-1}$. Now, $D(z_2) = \text{ad}_x(z_2)+\delta(z_2) = \text{ad}_x(z_2)-(\lambda_4+\lambda_6)z_2-b(\lambda_6+3\lambda_3)t_3^3t_4^{-1} \in R_5$. Since $\text{ad}_x(z_2) - (\lambda_4 + \lambda_6)z_2 \in R_5$, we have that $b(\lambda_6 + 3\lambda_3)t_3^3t_4^{-1} \in R_5$. This implies that $b(\lambda_6 + 3\lambda_3)t_3^3 \in R_5t_4$. Since $\alpha = 0$, from (4.1.5), we have that $t_3^3 = \dfrac{\beta}{b}t_6^{-1} + \dfrac{1}{ab}z_1t_3t_4$. Therefore, $b(\lambda_6 + 3\lambda_3)t_3^3 = (\lambda_6 + 3\lambda_3)\left(\beta t_6^{-1} + \dfrac{1}{a}z_1t_3t_4\right) = \beta(\lambda_6 + 3\lambda_3)t_6^{-1} + \dfrac{1}{a}(\lambda_6 + 3\lambda_3)z_1t_3t_4 \in R_5t_4$. Observe that $z_1t_3t_4 \in R_5t_4$. This implies that $\beta(\lambda_6 + 3\lambda_3)t_6^{-1} \in R_5t_5$. Hence, $\beta(\lambda_6 + 3\lambda_3) \in R_5t_5t_6$. Since $\beta \neq 0$, we must have: $\lambda_6 + 3\lambda_3 = 0$, and so $\lambda_6 =$

$-3\lambda_3$. Otherwise, there will be a contradiction using the basis of R_5 (Proposition 4.1.4). As a result, $\delta(z_2) = -(\lambda_4 + \lambda_6)z_2 = (3\lambda_3 - \lambda_4)z_2$.

Again, from Remark 4.1.8, we have that $z_1 = f_1 - st_3^2t_4^{-1}$. From the previous lemma, we also have $\delta(f_1) = -(\lambda_3 + \lambda_4 + \lambda_6)f_1$. Given that $\lambda_6 = -3\lambda_3$, one can easily verify that $\delta(z_1) = (2\lambda_3 - \lambda_4)z_1$.

3. Set $\lambda_1 := (2\lambda_3 - \lambda_4)$ and $\lambda_2 := 3\lambda_3 - \lambda_4$, it follows from points (1) and (2) that $D(e_{\kappa,5}) = \text{ad}_x(e_{\kappa,5}) + \lambda_\kappa e_{\kappa,5}$ for all $\kappa \in \{1, \cdots, 6\}$.

■

We now describe D as a derivation of $A_{0,\beta}$.

4.2.11 Lemma. 1. $x \in A_{0,\beta}$.

2. $\lambda_6 = 3\lambda_5,\ \delta(e_1) = -2\lambda_5 e_1,\ \delta(e_2) = -3\lambda_5 e_2,\ \delta(e_3) = -\lambda_5 e_3$ and $\delta(e_4) = 0$.

3. Set $\lambda_1 := -2\lambda_5,\ \lambda_2 := -3\lambda_5,\ \lambda_3 := -\lambda_5$ and $\lambda_4 := 0$. Then, $D(e_\kappa) = \text{ad}_x(e_\kappa) + \lambda_\kappa e_\kappa$ for all $\kappa \in \{1, \cdots, 6\}$.

Proof. 1. Similar to the proof of Lemma 4.2.4(1).

2. From Remark 4.1.8, we have $e_4 = t_4 - bt_5^3t_6^{-1}$. One can confirm that $\delta(e_4) = \lambda_4 e_4 + b(\lambda_4 - 3\lambda_5 - 3\lambda_3)t_5^3t_6^{-1}$. Now, $D(e_4) = \text{ad}_x(e_4) + \delta(e_4) = \text{ad}_x(e_4) + \lambda_4 e_4 + b(\lambda_4 - 3\lambda_5 - 3\lambda_3)t_5^3t_6^{-1} \in A_{0,\beta}$. Since $\text{ad}_x(e_4) + \lambda_4 e_4 \in A_{0,\beta}$, it follows that $b(\lambda_4 - 3\lambda_5 - 3\lambda_3)t_5^3t_6^{-1} \in A_{0,\beta}$. This implies that $b(\lambda_4 - 3\lambda_5 - 3\lambda_3)t_5^3 \in A_{0,\beta}t_6$. Note, from Appendix C, $b \neq 0$. Therefore, $\lambda_4 - 3\lambda_5 - 3\lambda_3 = 0$, otherwise, there will be a contradiction using the basis of $A_{0,\beta}$ (Proposition 3.2.2). Hence, $\lambda_4 = 3(\lambda_3 + \lambda_5)$, and $\delta(e_4) = 3(\lambda_3 + \lambda_5)e_4$. The relation $\delta(e_3) = \lambda_3 e_3 + a(\lambda_3 - \lambda_4 + \lambda_5)e_4t_5^{-1} + [s(\lambda_3 - 2\lambda_5 + \lambda_6) + q^{-3}ab(\lambda_3 - \lambda_4 + \lambda_5)]t_5^2t_6^{-1}$ (see (4.2.15)) is also valid in $A_{0,\beta}$. From Lemma 4.2.10, we have that $\lambda_6 = -3\lambda_3$. Given that $\lambda_4 = 3(\lambda_3 + \lambda_5)$ and $\lambda_6 = -3\lambda_3$, one can confirm that $\delta(e_3) = \lambda_3 e_3 - 2a(\lambda_3 + \lambda_5)e_4t_5^{-1}$ (note, from Appendix C, $q^{-3}ab + s = 0$). Now, $D(e_3) = \text{ad}_x(e_3) + \delta(e_3) = \text{ad}_x(e_3) + \lambda_3 e_3 - 2a(\lambda_3 + \lambda_5)e_4t_5^{-1} \in A_{0,\beta}$. Since $\text{ad}_x(e_3) + \lambda_3 e_3 \in A_{0,\beta}$, it follows that $2a(\lambda_3 + \lambda_5)e_4t_5^{-1} \in A_{0,\beta}$. This implies that $2a(\lambda_3 + \lambda_5)e_4 \in A_{0,\beta}t_5$. We must have $\lambda_3 + \lambda_5 = 0$, and so $\lambda_3 = -\lambda_5$ (note, $a \neq 0$). Otherwise, there will be a contradiction

using the basis of $A_{0,\beta}$ (Proposition 3.2.2). As a result, $\delta(e_3) = -\lambda_5 e_3$, $\delta(e_4) = 0$ and $\lambda_6 = 3\lambda_5$.

From (4.2.31), we have $e_1 = z_1 - rt_5t_6^{-1} - ht_3t_5^{-1} + (ah - g)t_4t_5^{-2}$. Recall from Lemma 4.2.10 that $\delta(z_1) = (2\lambda_3 - \lambda_4)z_1$. Given that $\lambda_6 = 3\lambda_5,\ \lambda_4 = 0,\ \lambda_3 = -\lambda_5$ and $\delta(t_5) = \lambda_5 t_5$, one can verify that $\delta(e_1) = -2\lambda_5 e_1$. From the commutation relations of $A_{0,\beta}$ in Chapter 3, we have that $e_3e_1 = q^{-1}e_1e_3 - (q+q^{-1}+q^{-3})e_2$. Since $\delta(e_1) = -2\lambda_5 e_1$ and $\delta(e_3) = -\lambda_5 e_3$, applying δ to this relation gives $\delta(e_2) = -3\lambda_5 e_2$. ■

In the next lemma, we prove that the first Hochschild cohomology groups of $A_{0,\beta}$ and $A_{\alpha,0}$ are non-trivial.

4.2.12 Lemma. Let $(\alpha,\beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and, θ and $\tilde{\theta}$ be linear maps of $A_{\alpha,0}$ and $A_{0,\beta}$ respectively defined by

$$\theta(e_1) = -e_1,\quad \theta(e_2) = -e_2,\quad \theta(e_3) = 0,\quad \theta(e_4) = e_4,\quad \theta(e_5) = e_5,\quad \theta(e_6) = 2e_6,$$

and

$$\tilde{\theta}(e_1) = -2e_1,\quad \tilde{\theta}(e_2) = -3e_2,\quad \tilde{\theta}(e_3) = -e_3,\quad \tilde{\theta}(e_4) = 0,\quad \tilde{\theta}(e_5) = e_5,\quad \tilde{\theta}(e_6) = 3e_6.$$

Then, θ and $\tilde{\theta}$ are $\mathbb{C}-$derivations of $A_{\alpha,0}$ and$A_{0,\beta}$ respectively.

Proof. Given the algebra relations of $A_{\alpha,\beta}$ in Chapter 3, we verify that θ satisfies the relations of $A_{\alpha,\beta}$ when $\alpha \neq 0$ and $\beta = 0$, and $\tilde{\theta}$ satisfies the relations of $A_{\alpha,\beta}$ when $\alpha = 0$ and $\beta \neq 0$. We will verify this for only one of the relations, and leave the remaining ones for the reader to check. From (3.0.1), we have that

$$e_1e_3e_5 + ae_1e_4 + ae_2e_5 + a'e_3^2 = \alpha$$

in $A_{\alpha,0}$.

Now,

$$\theta(e_1)e_3e_5 + e_1\theta(e_3)e_5 + e_1e_3\theta(e_5) + a\theta(e_1)e_4 + ae_1\theta(e_4) + a\theta(e_2)e_5 + ae_2\theta(e_5) + 2a'\theta(e_3)e_3$$
$$= 0$$

as expected. In $A_{0,\beta}$, we have that

$$e_1e_3e_5 + ae_1e_4 + ae_2e_5 + a'e_3^2 = 0.$$

Observe that

$$\tilde{\theta}(e_1)e_3e_5 + e_1\tilde{\theta}(e_3)e_5 + e_1e_3\tilde{\theta}(e_5) + a\tilde{\theta}(e_1)e_4 + ae_1\tilde{\theta}(e_4) + a\tilde{\theta}(e_2)e_5 + ae_2\tilde{\theta}(e_5) + 2a'\tilde{\theta}(e_3)e_3$$
$$= -2(e_1e_3e_5 + ae_1e_4 + ae_2e_5 + a'e_3^2) = 0,$$

as expected.

■

4.2.13 Remark. From Lemma 4.1.7, $Z(A_{\alpha,\beta}) = \mathbb{C}$. Let $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and Der$(A_{\alpha,\beta})$ represent the $\mathbb{C}-$derivations of $A_{\alpha,\beta}$. Then, the first Hochschild cohomology group of $A_{\alpha,\beta}$ (denoted by $HH^1(A_{\alpha,\beta})$) defined by

$$HH^1(A_{\alpha,\beta}) := \frac{\text{Der}(A_{\alpha,\beta})}{\text{InnDer}(A_{\alpha,\beta})}$$

is a free module over $Z(A_{\alpha,\beta}) = \mathbb{C}$, where InnDer$(A_{\alpha,\beta}) := \{\text{ad}_x \mid x \in A_{\alpha,\beta}\}$ is the set of inner derivations of $A_{\alpha,\beta}$.

We summarize our main results in this chapter in the theorem below.

4.2.14 Theorem. *Given that* $A_{\alpha,\beta} = U_q^+(G_2)/\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$, *with* $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$, *we have the following results:*

1. *if* $\alpha, \beta \neq 0$; *then every derivation* D *of* $A_{\alpha,\beta}$ *can uniquely be written as* $D = ad_x$, *where* $x \in A_{\alpha,\beta}$.

2. *if* $\alpha \neq 0$ *and* $\beta = 0$, *then every derivation* D *of* $A_{\alpha,0}$ *can uniquely be written as* $D = ad_x + \lambda\theta$, *where* $\lambda \in \mathbb{C}$ *and* $x \in A_{\alpha,0}$.

3. *if* $\alpha = 0$ *and* $\beta \neq 0$, *then every derivation* D *of* $A_{0,\beta}$ *can uniquely be written as* $D = ad_x + \lambda\tilde{\theta}$, *where* $\lambda \in \mathbb{C}$ *and* $x \in A_{0,\beta}$.

4. $HH^1(A_{\alpha,0}) = \mathbb{C}[\theta]$ *and* $HH^1(A_{0,\beta}) = \mathbb{C}[\tilde{\theta}]$, *where* $[\theta]$ *and* $[\tilde{\theta}]$ *respectively denote the classes of* θ *and* $\tilde{\theta}$ *modulo the space of inner derivations.*

5. *if* $\alpha, \beta \neq 0$; *then* $HH^1(A_{\alpha,\beta}) = \{[0]\}$, *where* $[0]$ *denotes the class of* 0 *modulo the space of inner derivations.*

Proof. Points (1), (2) and (3) are as a result of Lemmas 4.2.4, 4.2.8 and 4.2.11 respectively. Point (4) is a consequence of Lemma 4.2.12, and (5) is a consequence of (1). ∎

Chapter 5

Semiclassical limit of the simple quotients of $U_q^+(G_2)$

In this chapter, we study a semiclassical limit of $U_q^+(G_2)$, and its simple quotients. Unless stated explicitly, we do not transfer notations used in the previous chapters to this and the subsequent chapter. Given a non-commutative algebra A, one can obtain a Poisson algebra $\mathcal{A}$ from A through a process called *semiclassical limit.* Conversely, given a Poisson algebra $\mathcal{A}$, one can obtain a non-commutative algebra A from $\mathcal{A}$ through a process called *quantization*. We briefly explain these concepts in line with the presentations in [14] in the next section. Since $U_q^+(G_2)$ is a non-commutative algebra, we study its semiclassical limit in Section 5.2. Following strategies similar to that of $U_q^+(G_2)$ in Chapter 2, we study the Poisson maximal ideals (of height 2) of the semiclassical limit of $U_q^+(G_2)$ using $\mathcal{H}-$stratification by Goodearl [16], and Poisson deleting derivations algorithm by Launois and Lecoutre [34]. We finally conclude in Section 5.3 that the simple quotients of the semiclassical limit of $U_q^+(G_2)$ are the semiclassical limits of the non-commutative algebra $A_{\alpha,\beta}$. Most of the results in this and the subsequent chapters are analogues to their non-commutative counterparts.

We begin with some preliminaries on Poisson algebras.

5.1 Preliminaries

This section studies some preliminary materials such as semiclassical limit, Poisson prime spectrum and Poisson deleting derivations algorithm. We begin with the following definitions and examples.

5.1.1 Definitions and examples. A Poisson algebra $\mathcal{A}$ is a commutative algebra over a field $\mathbb{K}$ endowed with a skew-symmetric $\mathbb{K}-$bilinear map $\{-,-\} : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$ which satisfies the following properties:

- $\{x, yz\} = \{x, y\}z + y\{x, z\}$ for all $x, y, z \in \mathcal{A}$ (Leibniz rule).
- $\{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} = 0$ for all $x, y, z \in \mathcal{A}$ (Jacobi identity).

Note, the skew-symmetric $\mathbb{K}-$bilinear map $\{-,-\}$ is called the *Poisson bracket.* Every Poisson algebra is also a Lie algebra since they both satisfy the Jacobi identity. Since the Poisson bracket also satisfies the Leibniz rule, it is well known that for each $x \in \mathcal{A}$ the $\mathbb{K}-$linear map $\text{ham}_x := \{x, -\} : \mathcal{A} \longrightarrow \mathcal{A}$ is a derivation known as the *Hamiltonian derivation* associated to x. A *Poisson ideal* I of $\mathcal{A}$ is any ideal such that $\{x, u\} \in I$ for all $x \in \mathcal{A}$ and $u \in I$. The set $Z_P(\mathcal{A}) := \{a \in \mathcal{A} \mid \{a, x\} = 0,\ \forall x \in \mathcal{A}\}$ is a Poisson subalgebra of $\mathcal{A}$ called the *Poisson center* of $\mathcal{A}$.

5.1.2 Remark. If $\mathcal{A}$ is a Poisson algebra and $\{x_1, \cdots, x_n\}$ is a generating set for $\mathcal{A}$ (as an algebra), then it is enough to define a Poisson bracket $\{-,-\}$ on $\mathcal{A}$ by defining it on only the generating set.

5.1.3 Example. The following are some examples of Poisson algebras.

1. The polynomial ring $\mathbb{C}[x, y]$ is a Poisson algebra via $\{x, y\} = a$, where $a \in \mathbb{C}$. When $a = 0$, then we have a trivial Poisson structure on $\mathbb{C}[x, y]$. In fact, any commutative algebra can be endowed with the trivial Poisson structure to form a Poisson algebra.

2. Let $\mathcal{R} = \mathbb{K}[y_1, \cdots, y_N]$ be a polynomial algebra over the field $\mathbb{K}$, with $\{y_i, y_j\} = \mu_{ij}y_iy_j,\ i, j \in \mathbb{N}_{\geq 1}$, where $M = (\mu_{ij})$ is an $N \times N$ skew-symmetric matrix over

the field $\mathbb{K}$. Then, $\mathcal{R}$ is a Poisson algebra known as the *Poisson affine space.* Note, for all $f, g \in \mathcal{R}$, we have that:

$$\{f, g\} = \sum_{i,j=1}^{n} \mu_{ij} y_i y_j \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_j}.$$

The set $S := \{\lambda y_1^{i_1} \cdots y_N^{i_N} \mid i_1, \cdots, i_N \in \mathbb{N} \text{ and } \lambda \in \mathbb{C}^*\}$ is a multiplicative set of $\mathcal{R}$. Hence, one can localize $\mathcal{R}$ at S as $\mathcal{R}S^{-1} = \mathbb{K}[y_1^{\pm 1}, \cdots, y_N^{\pm 1}]$. Note, the Poisson bracket of $\mathcal{R}$ extends uniquely to a Poisson bracket of $\mathcal{R}S^{-1}$. Therefore, $\mathcal{R}S^{-1}$ is a Poisson algebra known as the *Poisson torus* associated to the Poisson affine space $\mathcal{R}$.

In general, let $S \ni 1$ be a multiplicative set of a Poisson algebra $\mathcal{A}$. Then, the localization $\mathcal{A}S^{-1}$ admits a Poisson bracket extended uniquely from $\mathcal{A}$ as follows:

$$\{xs^{-1}, yt^{-1}\} = \{x, y\}s^{-1}t^{-1} - \{x, t\}ys^{-1}t^{-2} - \{s, y\}xs^{-2}t^{-1} + \{s, t\}xys^{-2}t^{-2},$$

for all $x, y \in \mathcal{A}$ and all $s, t \in S$.

In a special case where $\mathcal{A}$ is a domain, then the Poisson structure of $\mathcal{A}$ extends uniquely to its field of fractions.

3. Set $\mathcal{R} := \mathcal{O}(M_2(\mathbb{C})) = \mathbb{C}\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Then, $\mathcal{R}$ is a Poisson algebra called 2×2 *Poisson matrix algebra* with Poisson bracket defined as follows:

$$\{a, b\} = ab, \quad \{a, c\} = ac, \quad \{b, c\} = 0, \quad \{b, d\} = bd, \quad \{c, d\} = cd, \quad \{a, d\} = 2bc.$$

4. Take any arbitrary Poisson algebra $\mathcal{A}$ and a Poission ideal I of $\mathcal{A}$. Then, it is well known that the quotient algebra $\mathcal{A}/I$ is a Poisson algebra with an induced Poisson bracket defined as $\{\bar{x}, \bar{y}\} = \overline{\{x, y\}}$, where $\bar{x} := x + I$ and $\bar{y} := y + I$, with $x, y \in \mathcal{A}$.

5.1.4 Semiclassical limit. As already stated, given a non-commutative algebra A, one can obtain a Poisson algebra $\mathcal{A}$ from A through a process called *semiclassical limit.*

Conversely, given a Poisson algebra $\mathcal{A}$, one can obtain a non-commutative algebra A from $\mathcal{A}$ through a process called *quantization*. That is, one can either view A as a quantization of $\mathcal{A}$ or $\mathcal{A}$ as the semiclassical limit of A under certain conditions, which we present in the next paragraph. Unless stated otherwise, we present ideas similar to [14, §1.1.3].

Let R be a commutative principal ideal domain containing the field $\mathbb{K}$ and hR be a maximal ideal of R for a fix $h \in R$. According to [7, Chapter III.5], in practice, R will always be either a polynomial or Laurent polynomial ring over $\mathbb{K}$ in one variable, and h will be a linear polynomial. Take an algebra A (which is not necessarily commutative torsion-free $R-$algebra) such that the factor algebra $\mathcal{A} := A/hA$ is commutative. Let $u, v \in A$. Then, $\bar{u} := u + hA$ and $\bar{v} := v + hA$ are the canonical images of u and v in $\mathcal{A}$ respectively. Since $\bar{u}\bar{v} = \bar{v}\bar{u}$, we have that $[u, v] := uv - vu \in hA$. There exists a unique element $\gamma(u, v)$ of A such that $[u, v] = h\gamma(u, v)$. It follows that

$$\{\bar{u}, \bar{v}\} := \gamma(u, v) + hA = \frac{[u, v]}{h} + hA$$

defines a Poisson bracket on $\mathcal{A}$. One can prove that the above definition defines a Poisson bracket on $\mathcal{A}$ by showing that it is well-defined, the Jacobi identity and the Leibniz rule hold. The details of the proof can be found in [14, §1.1.3]. Given the above presentation, A is said to be a *quantization* of $\mathcal{A}$, and $\mathcal{A}$ is termed as the *semiclassical limit* of A. Let $\lambda \in \mathbb{K}$, then the algebra $\mathcal{A}_\lambda := A/(h - \lambda)A$ is a *deformation* of the Poisson algebra $\mathcal{A} = \mathcal{A}_0$ provided the central element $h - \lambda$ is not invertible in A.

5.1.5 Example. 1. From Remark 1.5.1, we can define a quantum affine space of the form $\mathcal{O}_\Lambda(\mathbb{K}^N) = \mathbb{K}_\Lambda[Y_1, \cdots, Y_N]$, with $Y_jY_i = q^{\mu_{ji}}Y_iY_j$, where $\Lambda = (q^{\mu_{ij}})$ is a skew-symmetric matrix defined in Remark 1.5.1. Set $R := \mathbb{K}[z^{\pm 1}]\langle Y_1, \cdots, Y_N \mid Y_jY_i = z^{\mu_{ji}}Y_iY_j \text{ for all } 1 \leq i, j \leq N\rangle$. The element $z-1$ is central and not invertible in R. Hence, $\mathcal{R} := R/(z-1)R = \mathbb{K}[y_1, \cdots, y_N]$, with $y_i := Y_i + (z-1)R,\ 1 \leq i \leq N$. Now, $y_iy_j = y_jy_i$ for all $1 \leq i, j \leq N$, hence, $\mathcal{R}$ is a commutative algebra.

A Poisson structure is defined on $\mathcal{R}$ as follows:

$$\{y_j, y_i\} = \frac{[Y_j, Y_i]}{z-1} + (z-1)R = \frac{(z^{\mu_{ji}}-1)Y_iY_j}{z-1} + (z-1)R = \mu_{ji}y_iy_j,$$

for all $1 \leq i, j \leq n$. Therefore, $\mathcal{R}$ is a Poisson affine space and it is the semiclassical limit of the algebra R. Moreover, $R/(z-q)R = \mathcal{O}_\Lambda(\mathbb{K}^N)$ is a deformation of the Poisson affine space $\mathcal{R}$. Since a Poisson bracket on a Poisson affine space extends uniquely to a Poission bracket of the associated Poisson torus, one can also conclude that the Poisson torus $\mathbb{K}[y_1^{\pm 1}, \cdots, y_N^{\pm 1}]$ is the semiclassical limit of the quantum torus $\mathbb{K}_\Lambda[Y_1^{\pm 1}, \cdots, Y_N^{\pm 1}]$.

2. Recall that the 2×2 quantum matrices $\mathcal{O}_q(M_2(\mathbb{C})) = \mathbb{C}\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$ satisfies the relation:

$$ab = qba, \quad ac = qca, \quad bc = cb, \quad bd = qdb, \quad cd = qdc, \quad ad - da = (q-q^{-1})bc$$

(see [7, Example 1.1.7]). Now, the 2×2 Poisson matrix algebra in Example 5.1.3(3) is the semiclassical limit of the non-commutative algebra $\mathcal{O}_q(M_2(\mathbb{C}))$.

5.1.6 Poisson prime spectrum. In Section 1.6 we discussed the prime spectrum Spec(A) of an algebra A. In this section, we discuss the Poisson analogue called the Poisson prime spectrum. Let P be a proper Poisson ideal of a Poisson algebra $\mathcal{A}$ and I_1, I_2 be Poisson ideals of $\mathcal{A}$ such that $P \supseteq I_1I_2$, the ideal P is called a *Poisson prime ideal* provided $P \supseteq I_1$ or $P \supseteq I_2$. It is well known that a Poisson ideal which is also a prime ideal is a Poisson prime ideal, however, the converse is not always true, except the case where $\mathcal{A}$ is noetherian (see [16, Lemma 1.1]). The collection of all Poisson prime ideals of $\mathcal{A}$ is called the *Poisson prime spectrum* of $\mathcal{A}$, denoted by P.Spec$(\mathcal{A})$. The largest Poisson prime ideal contained in a given maximal ideal of $\mathcal{A}$ is called a *Poisson primitive ideal* of $\mathcal{A}$. The collection of all these primitive ideals is also called *Poisson primitive spectrum* of $\mathcal{A}$, denoted by P.Prim$(\mathcal{A})$.

Let A be a $\mathbb{K}-$algebra. Recall from Section 1.6 that the prime spectrum Spec(A) of A endowed with a suitable torus action can be partitioned into a disjoint union of strata, a partition known as the $\mathcal{H}-$*Stratification* of Spec(A). In a similar manner, P.Spec$(\mathcal{A})$ can also be partitioned into a disjoint union of strata in the presence of a suitable torus. We discuss this partition in the next paragraph. In [16, §3], Goodearl described this process for a commutative noetherian differential $\mathbb{K}-$algebras and, in [35, §2.2], Launois and Lecoutre described the process for noetherian Poisson algebras via the Cauchon diagrams. Since we have already described this process for any arbitrary $\mathbb{K}-$algebra in Section 1.6, we will only provide a summary for the Poisson version.

Let $\mathcal{A}$ be a noetherian Poisson algebra and $\mathcal{H}$ be an algebraic torus acting rationally on $\mathcal{A}$ by Poisson automorphism (an automorphism that preserves the Poisson bracket). A Poisson prime ideal P is $\mathcal{H}-$*invariant* if $h \cdot P = P$ for all $h \in \mathcal{H}$. The set

$$\text{P.Spec}_J(\mathcal{A}) := \{P \in \text{P.Spec}(\mathcal{A}) \mid (P : \mathcal{H}) = J\}$$

is called the $J-$*stratum* of P.Spec$(\mathcal{A})$. Note, $J := \bigcap_{h\in\mathcal{H}} h \cdot P$ is the largest Poisson $\mathcal{H}-$invariant prime ideal contained in P. The $\mathcal{H}-$strata P.Spec$_J(\mathcal{A})$ partitioned P.Spec$(\mathcal{A})$ into a disjoint union of strata. Hence,

$$\text{P.Spec}(\mathcal{A}) = \bigsqcup_{J\in\mathcal{H}-\text{P.Spec}(\mathcal{A})} \text{P.Spec}_J(A),$$

where $\mathcal{H} -$ P.Spec$(\mathcal{A})$ is the collection of all the Poisson $\mathcal{H}-$invariant prime ideals of $\mathcal{A}$. This process is called $\mathcal{H}-$*stratification* of P.Spec$(\mathcal{A})$. In a similar manner, a $\mathcal{H}-$stratification of P.Prim$(\mathcal{A})$ is obtained as follows:

$$\text{P.Prim}(\mathcal{A}) = \bigsqcup_{J\in\mathcal{H}-\text{P.Prim}(\mathcal{A})} \text{P.Prim}_J(\mathcal{A}),$$

where P.Prim$_J(\mathcal{A})$ = P.Spec$_J(\mathcal{A}) \cap$ P.Prim$(\mathcal{A})$ and $\mathcal{H} -$ P.Prim$(\mathcal{A})$ is the collection of all the Poisson $\mathcal{H}-$invariant primitive ideals of $\mathcal{A}$.

5.1.7 Proposition. [16, Theorem 4.3] Let $P \in$ P.Spec$_J(\mathcal{A})$, P is Poisson primitive if and only if P is maximal in P.Spec$_J(\mathcal{A})$.

5.1.8 Poisson deleting derivation algorithm (PDDA). We have already studied Cauchon's theory of deleting derivation algorithm for non-commutative algebras in Section 1.7. In [35], Launois and Lecoutre studied the Poisson version of the deleting derivation algorithm called the *Poisson deleting derivation algorithm (PDDA)* for some class of Poisson algebras with a base field of characteristic zero and prime characteristic. In this section, we will only discuss the algorithm for the characteristic zero case. We begin with the following theorem.

5.1.9 Theorem. *[39, Theorem 1.1] Let $\mathcal{A}$ be a Poisson $\mathbb{K}-$algebra and $\sigma, \delta : \mathcal{A} \to \mathcal{A}$ be $\mathbb{K}-$linear maps. Then, the polynomial algebra $\mathcal{R} = \mathcal{A}[X]$ endowed with a Poisson bracket (i.e. $\{X, a\} = \sigma(a)X + \delta(a)$ for all $a \in \mathcal{A}$) extending from the Poisson bracket of $\mathcal{A}$ is a Poisson algebra if and only if:*

1. *σ is a Poisson derivation of $\mathcal{A}$. That is, σ is a $\mathbb{K}-$derivation of $\mathcal{A}$ with*

$$\sigma(\{a, b\}) = \{\sigma(a), b\} + \{a, \sigma(b)\} \text{ for all } a, b \in \mathcal{A}.$$

2. *δ is a Poisson $\sigma-$derivation of $\mathcal{A}$. That is, δ is a $\mathbb{K}-$derivation of $\mathcal{A}$ with*

$$\delta(\{a, b\}) = \{\delta(a), b\} + \{a, \delta(b)\} + \sigma(a)\delta(b) - \delta(a)\sigma(b) \text{ for all } a, b \in \mathcal{A}.$$

Given a Poisson algebra $\mathcal{R} = \mathcal{A}[X]$ satisfying the above conditions, one can simply write $\mathcal{R}$ as $\mathcal{R} = \mathcal{A}[X, \sigma, \delta]_P$. This is called the *Poisson-Ore extension* of $\mathcal{A}$. If $\delta = 0$, then $\mathcal{A}[X, \sigma, 0]_P$ is simply written as $\mathcal{A}[X, \sigma]_P$. Furthermore, $\mathcal{R}$ is called an *iterated Poisson-Ore extension over $\mathcal{A}$* if there exists a set $\{\sigma_i \mid i = 1, \cdots, N\}$ of Poisson derivations and a set $\{\delta_i \mid i = 1, \cdots, N\}$ of Poisson σ_i- derivations such that

$$\mathcal{R} = \mathcal{A}[X_1; \sigma_1, \delta_1]_p \cdots [X_n; \sigma_n, \delta_n]_P.$$

5.1.10 Example. In Example 5.1.3(3), we have that the matrix Poisson algebra

$\mathcal{R} = \mathbb{C}\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ satisfies the relations:

$$\{a, b\} = ab, \quad \{a, c\} = ac, \ \{b, c\} = 0, \quad \{b, d\} = bd, \quad \{c, d\} = cd, \quad \{a, d\} = 2bc.$$

Now, set $X_1 := a, \ X_2 := b, \ X_3 := c$ and $X_4 := d$. Then, $\mathcal{R}$ can be written as an iterated Poisson-Ore extension as:

$$\mathcal{R} = \mathbb{C}[X_1][X_2; \sigma_2]_P[X_3; \sigma_3]_P[X_4; \sigma_4, \delta_4]_P,$$

where $\sigma_2(X_1) = X_1$, $\sigma_3(X_1) = X_1$, $\sigma_3(X_2) = 0$, $\sigma_4(X_1) = 0$, $\sigma_4(X_2) = X_2$, $\sigma_4(X_3) = X_3$, and $\delta_4(X_1) = 2X_2X_3$, $\delta_4(X_2) = \delta_4(X_3) = 0$. Note, σ_j is a Poisson derivation and δ_j is a Poisson σ_j-derivation of $\mathbb{C}[X_1][X_2; \sigma_2]_P \cdots [X_{j-1}; \sigma_{j-1}, \delta_{j-1}]_P$ for each $2 \leq j \leq 4$ (note, $\delta_2 = \delta_3 = \delta_4 = 0$).

We are now going to discuss the PDDA process.

Let $\mathbb{K}$ be a field with characteristic zero, and $\mathcal{A} = \mathbb{K}[X_1][X_2; \sigma_2, \delta_2]_p \cdots [X_N; \sigma_N, \delta_N]_P$ be an iterated Poisson polynomial algebra over $\mathbb{K}$. Suppose that $\mathcal{A}$ satisfies the conditions in the hypothesis below.

5.1.11 Hypothesis. (H1) For all $1 \leq j < i \leq N$, there exists $\mu_{ij} \in \mathbb{K}$, with $\mu_{ji} := -\mu_{ij}$, such that $\sigma_i(X_j) = \mu_{ij}X_j$.

(H2) The derivation δ_i is locally nilpotent and $\delta_i\alpha - \alpha\delta_i = \eta_i\delta_i$ for some non-zero scalar η_i for all $2 \leq i \leq N$.

Then, the PDDA can be used to study the Poisson prime spectrum of $\mathcal{A}$. Let $j \in \{N+1, \cdots, 2\}$. The algorithm (PDDA) constructs a family $(X_{1,j}, \cdots, X_{N,j})$ of elements of Fract($\mathcal{A}$) as follows. For $j = N+1$, set $(X_{1,N+1}, \cdots, X_{N,N+1}) := (X_1, \cdots, X_N)$. Now, suppose that the family $(X_{1,j+1}, \cdots, X_{N,j+1})$ has already been constructed. Then, for $j < N+1$, construct $X_{1,j}, \cdots, X_{N,j}$ from $X_{1,j+1}, \cdots, X_{N,j+1}$ using the relation

$$X_{i,j} := \begin{cases} X_{i,j+1} & \text{if } i \geq j \\ \sum_{k=0}^{+\infty} \frac{1}{\eta_j^k k!} \delta_j^k(X_{i,j+1}) X_{j,j+1}^{-k} & \text{if } i < j, \end{cases}$$

for all $i \in \{1, \cdots, N\}$. Since δ_j is locally nilpotent, the summation is finite.

For each $j \in \{2, \cdots, N+1\}$, the algebra $\mathcal{A}^{(j)}$ represents the subalgebra of Fract($\mathcal{A}$) generated by all the $X_{i,j}$. That is, $\mathcal{A}^{(j)} = \mathbb{K}[X_{1,j}, \cdots, X_{N,j}]$. Note, $\mathcal{A}^{(N+1)} = \mathcal{A}$. It follows from [35, Proposition 1.11] that

$$\mathcal{A}^{(j)} \cong \mathbb{K}[X_1][X_2; \sigma_2, \delta_2]_P \cdots [X_{j-1}; \sigma_{j-1}, \delta_{j-1}]_P [X_j; \tau_j]_P \cdots [X_N; \tau_N]_P,$$

by an isomorphism that maps $X_{i,j}$ to X_i, and $\tau_j, \cdots, \tau_N$ denote the Poisson derivations defined by $\tau_l(X_i) = \mu_{li} X_i$ for all $1 \leq i < l \leq N$. With a slight abuse of notation, one can identify $\tau_j, \cdots, \tau_N$ with $\sigma_j, \cdots, \sigma_N$ respectively.

Moreover, the set $S_j := \{X_{j,j+1}^n \mid n \in \mathbb{N}\} = \{X_{j,j}^n \mid n \in \mathbb{N}\}$ is a multiplicative system of regular elements of $\mathcal{A}^{(j)}$ and $\mathcal{A}^{(j+1)}$, and $\mathcal{A}^{(j)} S_j^{-1} = \mathcal{A}^{(j+1)} S_j^{-1}$ [35, Proposition 1.11]. Launois and Lecoutre [35] used the PDDA to relate P.Spec($\mathcal{A}$) to P.Spec($\bar{\mathcal{A}}$), where $\bar{\mathcal{A}} := \mathcal{A}^{(2)}$, by constructing an embedding ψ_j : P.Spec($\mathcal{A}^{(j+1)}$) $\hookrightarrow$ P.Spec($\mathcal{A}^{(j)}$) for each $j \in \{2, \cdots, N\}$. Suppose that $P \in$ P.Spec($\mathcal{A}^{(j+1)}$) and $X_{j,j+1} \notin P$, then ψ_j is defined by

$$\psi_j(P) = PS_j^{-1} \cap \mathcal{A}^{(j)} = Q,$$

with $Q \in$ P.Spec($\mathcal{A}^{(j)}$). In the case where $X_{j,j} \notin Q$, then the inverse map ψ_j^{-1} is also given by

$$\psi_j^{-1}(Q) = QS_j^{-1} \cap \mathcal{A}^{(j+1)} = P.$$

The map ψ_j is injective but not necessarily bijective, however, ψ_j induces a bijection from $\{P \in \text{P.Spec}(\mathcal{A}^{(j+1)}) \mid P \cap S_j = \emptyset\}$ onto $\{Q \in \text{P.Spec}(\mathcal{A}^{(j)}) \mid Q \cap S_j = \emptyset\}$ [35, Section 2.1]. The so-called *canonical embedding* ψ : P.Spec($\mathcal{A}$) $\hookrightarrow$ P.Spec($\bar{\mathcal{A}}$) is obtained by composing all the ψ_j. That is, $\psi := \psi_2 \circ \cdots \circ \psi_N$. This canonical embedding ψ helps to construct a partition of P.Spec($\mathcal{A}$) into a disjoint union of strata known as the *canonical partition* via the Cauchon diagrams. See [35] for further details on this, and how the map ψ_j is defined when $X_{j,j+1} \in P$ (we omit this case).

5.2 Semiclassical limit of $U_q^+(G_2)$

We are now ready to study the semiclassical limit of the algebra $A := U_q^+(G_2)$ studied in Chapter 2. Recall that A satisfies the following relations:

$$
\begin{aligned}
E_2E_1 &= q^{-3}E_1E_2 & E_3E_1 &= q^{-1}E_1E_3 - (q + q^{-1} + q^{-3})E_2 \\
E_3E_2 &= q^{-3}E_2E_3 & E_4E_1 &= E_1E_4 + (1 - q^2)E_3^2 \\
E_4E_2 &= q^{-3}E_2E_4 - \frac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1}E_3^3 & E_4E_3 &= q^{-3}E_3E_4 \\
E_5E_1 &= qE_1E_5 - (1 + q^2)E_3 & E_5E_2 &= E_2E_5 + (1 - q^2)E_3^2 \\
E_5E_3 &= q^{-1}E_3E_5 - (q + q^{-1} + q^{-3})E_4 & E_5E_4 &= q^{-3}E_4E_5 \\
E_6E_1 &= q^3E_1E_6 - q^3E_5 & E_6E_2 &= q^3E_2E_6 + (q^4 + q^2 - 1)E_4 + \\
E_6E_3 &= E_3E_6 + (1 - q^2)E_5^2 & &\qquad (q^2 - q^4)E_3E_5 \\
E_6E_4 &= q^{-3}E_4E_6 - \frac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1}E_5^3 & E_6E_5 &= q^{-3}E_5E_6.
\end{aligned}
$$

Set $U_i := (q-1)E_i$ for $i = 1, 3, 4, 5$, and $U_i := f(q)(q-1)E_i$ for $i = 2, 6$; where $f(q) = q^4 + q^2 + 1$. Then, A is now generated by $U_1, \cdots, U_6$ subject to the relations:

$$
\begin{aligned}
U_2U_1 &= q^{-3}U_1U_2 & U_3U_2 &= q^{-3}U_2U_3 \\
U_3U_1 &= q^{-1}U_1U_3 - q^{-3}(q-1)U_2 & U_4U_1 &= U_1U_4 + (1 - q^2)U_3^2 \\
U_4U_2 &= q^{-3}U_2U_4 - (q+1)^2(q-1)U_3^3 & U_4U_3 &= q^{-3}U_3U_4 \\
U_5U_1 &= qU_1U_5 - (1 + q^2)(q-1)U_3 & U_5U_2 &= U_2U_5 + f(q)(1 - q^2)U_3^2 \\
U_5U_3 &= q^{-1}U_3U_5 - f(q)(q^{-2} - q^{-3})U_4 & U_5U_4 &= q^{-3}U_4U_5
\end{aligned}
$$

$$U_6U_1 = q^3U_1U_6 - f(q)(q^4 - q^3)U_5 \qquad U_6U_2 = q^3U_2U_6 + f(q)^2(q^2 - q^4)U_3U_5 + f(q)^2(q^4 + q^2 - 1)(q - 1)U_4$$
$$U_6U_3 = U_3U_6 + f(q)(1 - q^2)U_5^2$$
$$U_6U_4 = q^{-3}U_4U_6 - (q + 1)^2(q - 1)U_5^3 \qquad U_6U_5 = q^{-3}U_5U_6.$$

We are now going to find a "new" presentation for A that allows us to introduce a quantisation of A. We discuss this as follows. Let $\widehat{A}$ be a $\mathbb{C}[z^{\pm 1}]-$algebra generated by $\widehat{U}_1, \cdots, \widehat{U}_6$ subject to the relations:

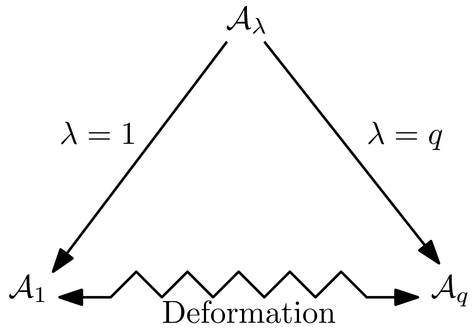
$$\widehat{U}_2\widehat{U}_1 = z^{-3}\widehat{U}_1\widehat{U}_2 \qquad \widehat{U}_3\widehat{U}_2 = z^{-3}\widehat{U}_2\widehat{U}_3$$
$$\widehat{U}_3\widehat{U}_1 = z^{-1}\widehat{U}_1\widehat{U}_3 - z^{-3}(z - 1)\widehat{U}_2 \qquad \widehat{U}_4\widehat{U}_1 = \widehat{U}_1\widehat{U}_4 + (1 - z^2)\widehat{U}_3^2$$
$$\widehat{U}_4\widehat{U}_2 = z^{-3}\widehat{U}_2\widehat{U}_4 - (z + 1)^2(z - 1)\widehat{U}_3^3 \qquad \widehat{U}_4\widehat{U}_3 = z^{-3}\widehat{U}_3\widehat{U}_4$$
$$\widehat{U}_5\widehat{U}_1 = z\widehat{U}_1\widehat{U}_5 - (1 + z^2)(z - 1)\widehat{U}_3 \qquad \widehat{U}_5\widehat{U}_2 = \widehat{U}_2\widehat{U}_5 + f(z)(1 - z^2)\widehat{U}_3^2$$
$$\widehat{U}_5\widehat{U}_3 = z^{-1}\widehat{U}_3\widehat{U}_5 - f(z)(z^{-2} - z^{-3})\widehat{U}_4 \qquad \widehat{U}_5\widehat{U}_4 = z^{-3}\widehat{U}_4\widehat{U}_5$$
$$\widehat{U}_6\widehat{U}_1 = z^3\widehat{U}_1\widehat{U}_6 - f(z)(z^4 - z^3)\widehat{U}_5 \qquad \widehat{U}_6\widehat{U}_2 = z^3\widehat{U}_2\widehat{U}_6 + f(z)^2(z^2 - z^4)\widehat{U}_3\widehat{U}_5 + f(z)^2(z^4 + z^2 - 1)(z - 1)\widehat{U}_4$$
$$\widehat{U}_6\widehat{U}_3 = \widehat{U}_3\widehat{U}_6 + f(z)(1 - z^2)\widehat{U}_5^2$$
$$\widehat{U}_6\widehat{U}_4 = z^{-3}\widehat{U}_4\widehat{U}_6 - (z + 1)^2(z - 1)\widehat{U}_5^3 \qquad \widehat{U}_6\widehat{U}_5 = z^{-3}\widehat{U}_5\widehat{U}_6,$$

where $f(z) = z^4 + z^2 + 1$. For $\lambda \in \mathbb{C}^*$, observe that the element $z - \lambda$ is central and not invertible in $\widehat{A}$. Hence, we can set $\mathcal{A}_\lambda := \widehat{A}/(z - \lambda)\widehat{A}$. Now, $\mathcal{A}_q$ is the non-commutative algebra A, and $\mathcal{A}_1 = \mathbb{C}[X_1, \cdots, X_6]$ with $X_i := \widehat{U}_i + (z - 1)\widehat{A}$ is a Poisson algebra with the Poisson bracket defined as follows (see Appendix B.1 for the necessary details):

$$\{X_2, X_1\} = -3X_1X_2 \qquad \{X_3, X_1\} = -X_1X_3 - X_2$$
$$\{X_3, X_2\} = -3X_2X_3 \qquad \{X_4, X_1\} = -2X_3^2$$
$$\{X_4, X_2\} = -3X_2X_4 - 4X_3^3 \qquad \{X_4, X_3\} = -3X_3X_4$$
$$\{X_5, X_1\} = X_1X_5 - 2X_3 \qquad \{X_5, X_2\} = -6X_3^2$$
$$\{X_5, X_3\} = -X_3X_5 - 3X_4 \qquad \{X_5, X_4\} = -3X_4X_5$$

$$\{X_6, X_1\} = 3X_1X_6 - 3X_5 \qquad \{X_6, X_2\} = 3X_2X_6 + 9X_4 - 18X_3X_5$$
$$\{X_6, X_3\} = -6X_5^2 \qquad \{X_6, X_4\} = -3X_4X_6 - 4X_5^3$$
$$\{X_6, X_5\} = -3X_5X_6.$$

Therefore, $\mathcal{A}_1$ is the semiclassical limit of the non-commutative algebra $\widehat{A}$, and $\mathcal{A}_q$ is a deformation of the Poisson algebra $\mathcal{A}_1$.



Henceforth, for simplicity, we set

$$\mathcal{A} := \mathcal{A}_1 = \mathbb{C}[X_1, \cdots, X_6].$$

One can write the Poisson algebra $\mathcal{A}$ as an iterated Poisson-Ore extension as:

$$\mathcal{A} = \mathbb{C}[X_1][X_2; \sigma_2]_P[X_3; \sigma_3, \delta_3]_P[X_4; \sigma_4, \delta_4]_P[X_5; \sigma_5, \delta_5]_P[X_6; \sigma_6, \delta_6]_P;$$

where,

$$\sigma_2(X_1) = -3X_1 \qquad \sigma_3(X_1) = -X_1 \qquad \sigma_3(X_2) = -3X_2 \qquad \sigma_4(X_1) = 0$$
$$\sigma_4(X_2) = -3X_2 \qquad \sigma_4(X_3) = -3X_3 \qquad \sigma_5(X_1) = X_1 \qquad \sigma_5(X_2) = 0$$
$$\sigma_5(X_3) = -X_3 \qquad \sigma_5(X_4) = -3X_4 \qquad \sigma_6(X_1) = 3X_1 \qquad \sigma_6(X_2) = 3X_2$$
$$\sigma_6(X_3) = 0 \qquad \sigma_6(X_4) = -3X_4 \qquad \sigma_6(X_5) = -3X_5,$$

and

$$\delta_3(X_1) = -X_2 \quad \delta_3(X_2) = 0 \quad \delta_4(X_1) = -2X_3^2 \quad \delta_4(X_2) = -4X_3^3$$
$$\delta_4(X_3) = 0 \quad \delta_5(X_1) = -2X_3 \quad \delta_5(X_2) = -6X_3^2 \quad \delta_5(X_3) = -3X_4$$
$$\delta_5(X_4) = 0 \quad \delta_6(X_1) = -3X_5 \quad \delta_6(X_2) = 9X_4 - 18X_3X_5 \quad \delta_6(X_3) = -6X_5^2$$
$$\delta_6(X_4) = -4X_5^3 \quad \delta_6(X_5) = 0.$$

Note, σ_j is a Poisson derivation and δ_j is a Poisson σ_j−derivation of $\mathbb{C}[X_1][X_2;\sigma_2]_P \cdots [X_{j-1};\sigma_{j-1},\delta_{j-1}]_P$ for each $2 \leq j \leq 6$ (note, $\delta_2 = 0$).

5.2.1 PDDA of $\mathcal{A}$. The algebra

$$\mathcal{A} = \mathbb{C}[X_1][X_2;\sigma_2]_P[X_3;\sigma_3,\delta_3]_P[X_4;\sigma_4,\delta_4]_P[X_5;\sigma_5,\delta_5]_P[X_6;\sigma_6,\delta_6]_P$$

satisfies H1 and H2 in Hypothesis 5.1.11, and so the theory of PDDA applies to $\mathcal{A}$. We construct the following elements of Fract($\mathcal{A}$) (computations have been omitted here, but can be found in Appendix B.2):

$$X_{1,6} = X_1 - \frac{1}{2}X_5X_6^{-1}$$
$$X_{2,6} = X_2 + \frac{3}{2}X_4X_6^{-1} - 3X_3X_5X_6^{-1} + X_5^3X_6^{-2}$$
$$X_{3,6} = X_3 - X_5^2X_6^{-1}$$
$$X_{4,6} = X_4 - \frac{2}{3}X_5^3X_6^{-1}$$
$$X_{1,5} = X_{1,6} - X_{3,6}X_{5,6}^{-1} + \frac{3}{4}X_{4,6}X_{5,6}^{-2}$$
$$X_{2,5} = X_{2,6} - 3X_{3,6}^2X_{5,6}^{-1} + \frac{9}{2}X_{3,6}X_{4,6}X_{5,6}^{-2} - \frac{9}{4}X_{4,6}^2X_{5,6}^{-3}$$
$$X_{3,5} = X_{3,6} - \frac{3}{2}X_{4,6}X_{5,6}^{-1}$$
$$X_{1,4} = X_{1,5} - \frac{1}{3}X_{3,5}^2X_{4,5}^{-1}$$
$$X_{2,4} = X_{2,5} - \frac{2}{3}X_{3,5}^3X_{4,5}^{-1}$$
$$X_{1,3} = X_{1,4} - \frac{1}{2}X_{2,4}X_{3,4}^{-1}$$

$$T_1 := X_{1,2} = X_{1,3}$$

$$T_2 := X_{2,2} = X_{2,3} = X_{2,4}$$

$$T_3 := X_{3,2} = X_{3,3} = X_{3,4} = X_{3,5}$$

$$T_4 := X_{4,2} = X_{4,3} = X_{4,4} = X_{4,5} = X_{4,6}$$

$$T_5 := X_{5,2} = X_{5,3} = X_{5,4} = X_{5,5} = X_{5,6} = X_5$$

$$T_6 := X_{6,2} = X_{6,3} = X_{6,4} = X_{6,5} = X_{6,6} = X_6.$$

From the theory of the PDDA, we have that for each $j \in \{2, \cdots, 7\}$, the algebra $\mathcal{A}^{(j)} := \mathbb{C}\langle X_{i,j} \mid i = 1, \cdots, 6\rangle$ is a subalgebra of Fract($\mathcal{A}$). Since $(X_{1,7}, \cdots, X_{6,7}) := (X_1, \cdots, X_6)$, it follows that $\mathcal{A}^{(7)} = \mathcal{A}$.

We recall the skew-symmetric matrix in Subsection 2.2.3 below:

$$M = \begin{bmatrix} 0 & 3 & 1 & 0 & -1 & -3 \\ -3 & 0 & 3 & 3 & 0 & -3 \\ -1 & -3 & 0 & 3 & 1 & 0 \\ 0 & -3 & -3 & 0 & 3 & 3 \\ 1 & 0 & -1 & -3 & 0 & 3 \\ 3 & 3 & 0 & -3 & -3 & 0 \end{bmatrix}.$$

Observe that $\bar{A} := \mathcal{A}^{(2)} = \mathbb{C}[T_1, \cdots, T_6]$ satisfies the relation $T_iT_j = \mu_{ij}T_jT_i$ for all $1 \leq i, j \leq 6$, with $\mu_{ij} \in M$, hence $\mathcal{A}$ is a Poisson affine space. Set $\Omega_1 := T_1T_3T_5$ and $\Omega_2 := T_2T_4T_6$. One can verify that Ω_1 and Ω_2 are Poisson central elements of $\bar{\mathcal{A}}$. That is, $\{\Omega_i, T_j\} = 0$ for all $i = 1, 2$, and $1 \leq j \leq 6$. We now want to successively pull Ω_1 and Ω_2 from the Poisson affine space $\bar{\mathcal{A}}$ into the algebra $\mathcal{A}$ using the PDDA of $\mathcal{A}$ discussed above. Through a direct computation, one can confirm that:

$$\begin{aligned}
\Omega_1 &:= T_1T_3T_5 \\
&= X_{1,3}X_{3,3}X_{5,3} \\
&= X_{1,4}X_{3,4}X_{5,4} - \frac{1}{2}X_{2,4}X_{5,4} \\
&= X_{1,5}X_{3,5}X_{5,5} - \frac{1}{2}X_{2,5}X_{5,5} \\
&= X_{1,6}X_{3,6}X_{5,6} - \frac{3}{2}X_{1,6}X_{4,6} - \frac{1}{2}X_{2,6}X_{5,6} + \frac{1}{2}X_{3,6}^2 \\
&= X_1X_3X_5 - \frac{3}{2}X_1X_4 - \frac{1}{2}X_2X_5 + \frac{1}{2}X_3^2,
\end{aligned}$$

and

$$\begin{aligned}
\Omega_2 &:= T_2T_4T_6 \\
&= X_{2,4}X_{4,4}X_{6,4} \\
&= X_{2,5}X_{4,5}X_{6,5} - \frac{2}{3}X_{3,5}^3X_{6,5} \\
&= X_{2,6}X_{4,6}X_{6,6} - \frac{2}{3}X_{3,6}^3X_{6,6} \\
&= X_2X_4X_6 - \frac{2}{3}X_3^3X_6 - \frac{2}{3}X_2X_5^3 + 2X_3^2X_5^2 - 3X_3X_4X_5 + \frac{3}{2}X_4^2.
\end{aligned}$$

As a result, Ω_1 and Ω_2 are also Poisson central elements of $\mathcal{A}^{(j)}$ for each $2 \le j \le 7$ since Fract$(\mathcal{A}^{(j)}) =$ Fract$(\bar{\mathcal{A}})$. We now want to show that the Poisson center of $\mathcal{A}$ is a polynomial ring generated by Ω_1 and Ω_2 over $\mathbb{C}$. The following discussions (similar to that of the non-commutative algebra $U_q^+(G_2)$ in Subsection 2.2.3) will lead us to the proof.

Set $S_j := \{\lambda T_j^{i_j}T_{j+1}^{i_{j+1}}\cdots T_6^{i_6} \mid i_j, \cdots, i_6 \in \mathbb{N}\}$ for each $2 \le j \le 6$. One can observe that S_j is a multiplicative system of non-zero divisors of $\mathcal{A}^{(j)} = \mathbb{C}\langle X_{i,j} \mid$ for all $i = 1, \cdots, 6\rangle$. One can therefore localize $\mathcal{A}^{(j)}$ at S_j as follows:

$$\mathfrak{R}_j := \mathcal{A}^{(j)}S_j^{-1}.$$

Note, the set $\Sigma_j := \{T_j^n \mid n \in \mathbb{N}\}$ is a multiplicative set in both $\mathcal{A}^{(j)}$ and $\mathcal{A}^{(j+1)}$ for each

$2 \leq j \leq 6$. It follows from the discussions in Subsection 5.1.8 that

$$\mathcal{A}^{(j)}\Sigma_j^{-1} = \mathcal{A}^{(j+1)}\Sigma_j^{-1}.$$

Similar to (2.2.1), one can verify that

$$\mathfrak{R}_j = \mathfrak{R}_{j+1}\Sigma_j^{-1}, \quad \text{for all} \quad 2 \leq j \leq 6.$$

Again, the localization

$$\mathfrak{R}_1 := \mathfrak{R}_2[T_1^{-1}]$$

also holds in $\mathfrak{R}_2$ since T_1 generates a multiplicative system in $\mathfrak{R}_2$. In fact, $\mathfrak{R}_1$ is the Poisson torus associated to the Poisson affine space $\bar{\mathcal{A}}$. As a result, $\mathfrak{R}_1 = \mathbb{C}[T_1^{\pm 1}, \cdots, T_6^{\pm 1}]$, where $T_iT_j = \mu_{ij}T_jT_i$ for all $1 \leq i, j \leq 6$ and $\mu_{ij} \in M$. Similar to (2.2.2), we construct the following embeddings:

$$\begin{aligned} \mathcal{A} := \mathfrak{R}_7 \subset \mathfrak{R}_6 = \mathfrak{R}_7\Sigma_6^{-1} \subset \mathfrak{R}_5 = \mathfrak{R}_6\Sigma_5^{-1} \subset \mathfrak{R}_4 = \mathfrak{R}_5\Sigma_4^{-1} \\ \subset \mathfrak{R}_3 = \mathfrak{R}_4\Sigma_3^{-1} \subset \mathfrak{R}_2 = \mathfrak{R}_3\Sigma_2^{-1} \subset \mathfrak{R}_1. \end{aligned} \tag{5.2.1}$$

Note, the family $(X_{1,j}^{k_1} \cdots X_{6,j}^{k_6})$, where $k_i \in \mathbb{N}$ if $i < j$ and $k_i \in \mathbb{Z}$ otherwise is a PBW-basis of $\mathfrak{R}_j$ for all $1 \leq i, j \leq 7$. Therefore, the family $(T_1^{k_1}T_2^{k_2}T_3^{k_3}T_4^{k_4}T_5^{k_5}T_6^{k_6})_{k_1,\cdots,k_6 \in \mathbb{Z}}$ is a basis of $\mathfrak{R}_1$.

5.2.2 Lemma. $Z_P(\mathfrak{R}_1) = \mathbb{C}[\Omega_1^{\pm 1}, \Omega_2^{\pm 1}]$.

Proof. Obviously, $\mathbb{C}[\Omega_1^{\pm 1}, \Omega_2^{\pm 1}] \subseteq Z_P(\mathfrak{R}_1)$. For the reverse inclusion, let $y \in Z_P(\mathfrak{R}_1)$. Then, y can be written in terms of the basis of $\mathfrak{R}_1$ as:

$$y = \sum_{(i,\cdots,n) \in \mathbb{Z}^6} a_{(i,\cdots,n)} T_1^i T_2^j T_3^k T_4^l T_5^m T_6^n.$$

One can verify that $\{y, T_1\} = (-3j-k+m+3n)yT_1$. Since $y \in Z_P(\mathfrak{R}_1)$, it follows that

$$-3j-k+m+3n=0.$$

Similarly, $\{y, T_2\} = (3i-3k-3l+3n)yT_2$. Since $y \in Z_P(\mathfrak{R}_1)$, we have:

$$3i-3k-3l+3n=0.$$

Following the same pattern for T_3, T_4, T_5 and T_6, one can confirm that

$$\begin{aligned} i+3j-3l-m =&0,\\ 3j+3k-3m-3n =&0,\\ -i+k+3l-3n =&0,\\ -3i-3j+3l+3m =&0. \end{aligned}$$

We already know the solution to this system of equations as $i=k=m$ and $j=l=n$ (see the proof of Lemma 2.2.4). One can therefore write

$$y = \sum_{(i,j)\in\mathbb{Z}^2} a_{(i,j)} T_1^i T_2^j T_3^i T_4^j T_5^i T_6^j = \sum_{(i,j)\in\mathbb{Z}^2} q^{\bullet} a_{(i,j)} T_1^i T_3^i T_5^i T_2^j T_4^j T_6^j = \sum_{(i,j)\in\mathbb{Z}^2} q^{\bullet} a_{(i,j)} \Omega_1^i \Omega_2^j.$$

This implies that $y \in \mathbb{C}[\Omega_1^{\pm 1}, \Omega_2^{\pm 1}]$ as expected. ■

5.2.3 Corollary. 1. $Z_P(\mathfrak{R}_3) = \mathbb{C}[\Omega_1, \Omega_2]$.

2. $Z_P(\bar{A}) = \mathbb{C}[\Omega_1, \Omega_2]$.

Proof. 1. Clearly, $\mathbb{C}[\Omega_1, \Omega_2] \subseteq Z_P(\mathfrak{R}_3)$. For the reverse inclusion, let $y \in Z_P(\mathfrak{R}_3)$. Then, y can be written in terms of the basis of $\mathfrak{R}_3$ (recall, $T_i = X_{i,3}$) as:

$$y = \sum_{(i,\cdots,n)\in\mathbb{N}^2\times\mathbb{Z}^4} a_{(i,\cdots,n)} T_1^i T_2^j T_3^k T_4^l T_5^m T_6^n.$$

Note, the generators $T_1, \cdots, T_6$ of $\mathfrak{R}_3$ satisfy the Poisson bracket of $\mathfrak{R}_1$. Hence, following procedures similar to the lemma above, one will arrive at the conclusion that $i = k = m$ and $j = l = n$. Since $i, j \geq 0$; it follows that $y = \sum_{(i,j)\in\mathbb{N}^2} a_{(i,j)} T_1^i T_3^i T_5^i T_2^j T_4^j T_6^j = \sum_{(i,j)\in\mathbb{N}^2} a_{(i,j)} \Omega_1^i \Omega_2^j$. This implies that $y \in \mathbb{C}[\Omega_1, \Omega_2]$ as expected.

2. Similar to (1). ■

5.2.4 Lemma. $Z_P(\mathcal{A}) = \mathbb{C}[\Omega_1, \Omega_2]$.

Proof. Since $\mathfrak{R}_i$ is a localization of $\mathfrak{R}_{i+1}$, it follows that $Z_P(\mathfrak{R}_{i+1}) \subseteq Z_P(\mathfrak{R}_i)$. From (5.2.1), we have that $Z_P(\mathcal{A}) \subseteq Z_P(\mathfrak{R}_3)$. Observe that $\mathbb{C}[\Omega_1, \Omega_2] \subseteq Z_P(\mathcal{A}) \subseteq Z_P(\mathfrak{R}_3) = \mathbb{C}[\Omega_1, \Omega_2]$. Hence, $Z_P(\mathcal{A}) = \mathbb{C}[\Omega_1, \Omega_2]$. ■

5.2.5 Poisson $\mathcal{H}-$invariant prime ideals of $\mathcal{A}$ of at most height 2. We begin this section by showing that $\langle\Omega_1\rangle$ and $\langle\Omega_2\rangle$ are Poisson prime ideals. Note, the data of the PDDA of $\mathcal{A}$ and expressions for Ω_1 and Ω_2 in the previous subsection will be very helpful in the proofs of Lemmas 5.2.6 and 5.2.7. In fact, Lemmas 5.2.6 and 5.2.7 are similar to Lemmas 2.3.1 and 2.3.2 respectively, as a result, the strategies of their proofs are also similar. Recall that $\langle\Theta\rangle_R$ denotes an ideal generated by the element Θ in any arbitrary ring R. Where no doubt arises, we will simply write $\langle\Theta\rangle$.

Recall that ψ_j : P.Spec$(\mathcal{A}^{(j+1)}) \hookrightarrow$ P.Spec$(\mathcal{A}^{(j)})$ for all $j \in \{2, \cdots, 7\}$, and $\psi := \psi_2 \circ \cdots \circ \psi_6$: P.Spec$(\mathcal{A}) \hookrightarrow$ P.Spec$(\bar{\mathcal{A}})$. Let $Q \in$ P.Spec$(\mathcal{A}^{(j)})$. If $T_j = X_{j,j} \notin Q$, then we already know that $\psi_j^{-1}(Q) = QS_j^{-1} \cap \mathcal{A}^{(j+1)} = P \in$ P.Spec$(\mathcal{A}^{(j+1)})$ for all $j \in \{2, \cdots, 7\}$ (see Subsection 5.1.8). Observe that $\langle T_1\rangle$ and $\langle T_2\rangle$ are both elements of P.Spec$(\bar{\mathcal{A}})$. Note, $\mathcal{A}^{(7)} = \mathcal{A}$ and $\mathcal{A}^{(2)} = \bar{\mathcal{A}}$.

The following result shows that $\langle T_1\rangle \in$ Im(ψ) and that $\langle\Omega_1\rangle$ is the Poisson prime ideal of $\mathcal{A}$ such that $\psi(\langle\Omega_1\rangle) = \langle T_1\rangle$.

5.2.6 Lemma. $\langle\Omega_1\rangle \in$ P.Spec$(\mathcal{A})$.

Proof. We will prove this result in several steps by showing that:

1. $\langle T_1\rangle \in$ P.Spec$(\mathcal{A}^{(3)})$.

2. $\langle X_{1,4}T_3 - \frac{1}{2}T_2\rangle \in$ P.Spec$(\mathcal{A}^{(4)})$ [note, $\langle T_1\rangle[T_3^{-1}] \cap \mathcal{A}^{(4)} = \langle X_{1,4}T_3 - \frac{1}{2}T_2\rangle$].

3. $\langle X_{1,5}T_3 - \frac{1}{2}X_{2,5}\rangle \in$ P.Spec$(\mathcal{A}^{(5)})$ [note, $\langle X_{1,4}T_3 - \frac{1}{2}T_2\rangle[T_4^{-1}] \cap \mathcal{A}^{(5)} = \langle X_{1,5}T_3 - \frac{1}{2}X_{2,5}\rangle$].

4. $\langle X_{1,5}T_3 - \frac{1}{2}X_{2,5}\rangle[T_5^{-1}] \cap \mathcal{A}^{(6)} = \langle \Omega_1\rangle_{\mathcal{A}^{(6)}}$, hence $\langle \Omega_1\rangle_{\mathcal{A}^{(6)}} \in$ P.Spec$(\mathcal{A}^{(6)})$.

5. $\langle \Omega_1\rangle_{\mathcal{A}^{(6)}}[T_6^{-1}] \cap \mathcal{A} = \langle \Omega_1\rangle_{\mathcal{A}}$, hence $\langle \Omega_1\rangle_{\mathcal{A}} \in$ P.Spec$(\mathcal{A})$.

We proceed to prove the above claims.

1. One can easily verify that $\mathcal{A}^{(3)}/\langle T_1\rangle$ is isomorphic to a Poisson affine space of rank 5. Hence, $\langle T_1\rangle$ is a Poisson prime ideal in $\mathcal{A}^{(3)}$.

2. Set $I := \langle X_{1,4}T_3 - \frac{1}{2}T_2\rangle$. One can verify that $\{X_{i,4}, I\} \subseteq I$ for all $i = 1, \cdots, 6$. Therefore, I is a Poisson ideal in $\mathcal{A}^{(4)}$. In addition, $\mathcal{A}^{(4)}/I$ is isomorphic to a polynomial ring in five variables which is a domain, hence, I is a prime ideal. Since I is both Poisson and prime ideal, it is a Poisson prime ideal in $\mathcal{A}^{(4)}$.

3. Similar to (2).

4. Observe that $\Omega_1' := X_{1,5}T_3 - \frac{1}{2}X_{2,5} = \Omega_1 T_5^{-1}$ in $\mathcal{A}^{(6)}[T_5^{-1}]$. Since $\langle \Omega_1'\rangle \in$ P.Spec$(\mathcal{A}^{(5)})$, we want to show that $\langle \Omega_1'\rangle[T_5^{-1}] \cap \mathcal{A}^{(6)} = \langle \Omega_1\rangle_{\mathcal{A}^{(6)}}$. Observe that $\langle \Omega_1\rangle_{\mathcal{A}^{(6)}} \subseteq \langle \Omega_1'\rangle[T_5^{-1}] \cap \mathcal{A}^{(6)}$. We establish the reverse inclusion. Let $y \in \langle \Omega_1'\rangle[T_5^{-1}] \cap \mathcal{A}^{(6)}$. Then, $y \in \langle \Omega_1'\rangle[T_5^{-1}]$. There exists $i \in \mathbb{N}$ such that $yT_5^i \in \langle \Omega_1'\rangle$. Hence, $yT_5^i = \Omega_1' v$, for some $v \in \mathcal{A}^{(5)}$. Furthermore, since $\mathcal{A}^{(5)}[T_5^{-1}] = \mathcal{A}^{(6)}[T_5^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_5^j = v'$ for some $v' \in \mathcal{A}^{(6)}$. It follows from $yT_5^i = \Omega_1' v$ that $yT_5^{i+j} = \Omega_1' vT_5^j = \Omega_1' v'$. Hence, $yT_5^{\delta} = \Omega_1' T_5 v' = \Omega_1 v'$, where $\delta = i + j + 1$ (note, $\Omega_1' T_5 = \Omega_1$ in $\mathcal{A}^{(6)}$). Let $S = \{s \in \mathbb{N} \mid \exists v' \in \mathcal{A}^{(6)} : yT_5^s = \Omega_1 v'\}$. Since $\delta \in S$, we have that $S \neq \emptyset$. Let $s = s_0$ be the minimum element of S such that $yT_5^{s_0} = \Omega_1 v'$ for some $v' \in \mathcal{A}^{(6)}$. We want to show that $s_0 = 0$. Suppose that $s_0 > 0$. Since T_5 is irreducible, $yT_5^{s_0} = \Omega_1 v'$ implies that T_5 is a factor of Ω_1 or v'. Clearly, T_5 is not a factor of Ω_1, hence, it must be a factor of v'. Now $\mathcal{A}^{(6)}$ can be viewed as a free $\mathbb{C}\langle X_{1,6}, X_{2,6}, X_{3,6}, T_4, T_6\rangle$−module with basis $\left(T_5^{\xi}\right)_{\xi\in\mathbb{N}}$. One can therefore write $v' = \sum_{\xi=1}^{n} \beta_\xi T_5^{\xi}$, where $\beta_\xi \in \mathbb{C}\langle X_{1,6}, X_{2,6}, X_{3,6}, T_4, T_6\rangle$. Returning to $yT_5^{s_0} = \Omega_1 v'$, we have that $yT_5^{s_0} = \Omega_1 \sum_{\xi=1}^{n} \beta_\xi T_5^{\xi}$. This implies that $yT_5^{s_0-1} = \Omega_1 v''$, where $v'' = \sum_{\xi=1}^{n} \beta_\xi T_5^{\xi-1} \in \mathcal{A}^{(6)}$. Consequently, $s_0 - 1 \in S$, a contradiction! Therefore, $s_0 = 0$ and $y = \Omega_1 v' \in \langle \Omega_1\rangle_{\mathcal{A}^{(6)}}$. Hence, $\langle \Omega_1'\rangle[T_5^{-1}] \cap \mathcal{A}^{(6)} \subseteq \langle \Omega_1\rangle_{\mathcal{A}^{(6)}}$ as expected.

5. The proof is similar to (4). We want to show that $\langle \Omega_1\rangle_{\mathcal{A}} = \langle \Omega_1\rangle_{\mathcal{A}^{(6)}}[T_6^{-1}] \cap \mathcal{A}$. Note, $\langle \Omega_1\rangle_{\mathcal{A}} \subseteq \langle \Omega_1\rangle_{\mathcal{A}^{(6)}}[T_6^{-1}] \cap \mathcal{A}$ is trivial. Let $y \in \langle \Omega_1\rangle_{\mathcal{A}^{(6)}}[T_6^{-1}] \cap \mathcal{A}$. Then, there exists $i \in \mathbb{N}$ such that $yT_6^i = \Omega_1 v$ for some $v \in \mathcal{A}^{(6)}$. Furthermore, since $\mathcal{A}^{(6)}[T_6^{-1}] = \mathcal{A}[T_6^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_6^j = v'$ for some $v' \in \mathcal{A}$. It follows from $yT_6^i = \Omega_1 v$ that $yT_6^\delta = \Omega_1 v'$, where $\delta = i + j$. Similar to (4), one can easily verify that $\delta = 0$. Hence, $y = \Omega_1 v' \in \langle \Omega_1\rangle_{\mathcal{A}}$. Consequently, $\langle \Omega_1\rangle_{\mathcal{A}^{(6)}}[T_6^{-1}] \cap \mathcal{A} \subseteq \langle \Omega_1\rangle_{\mathcal{A}}$. ■

Similarly, the following result also shows that $\langle T_2\rangle \in \text{Im}(\psi)$, and that $\langle \Omega_2\rangle$ is the Poisson prime ideal of $\mathcal{A}$ such that $\psi(\langle \Omega_2\rangle) = \langle T_2\rangle$.

5.2.7 Lemma. $\langle \Omega_2\rangle \in \text{P.Spec}(\mathcal{A})$.

Proof. We will also prove this result in several steps by showing that:

1. $\langle T_2\rangle_{\mathcal{A}^{(3)}} \in \text{P.Spec}(\mathcal{A}^{(3)})$.
2. $\langle T_2\rangle_{\mathcal{A}^{(4)}} \in \text{P.Spec}(\mathcal{A}^{(4)})$.
3. $\langle T_2\rangle_{\mathcal{A}^{(4)}}[T_4^{-1}] \cap \mathcal{A}^{(5)} = \langle X_{2,5}T_4 - \frac{2}{3}T_3^3\rangle$, hence $\langle X_{2,5}T_4 - \frac{2}{3}T_3^3\rangle \in \text{P.Spec}(\mathcal{A}^{(5)})$.
4. $\langle X_{2,5}T_4 - \frac{2}{3}T_3^3\rangle[T_5^{-1}] \cap \mathcal{A}^{(6)} = \langle \Omega_2'\rangle_{\mathcal{A}^{(6)}}$, hence $\langle X_{2,6}T_4 - \frac{2}{3}X_{3,6}^3\rangle \in \text{P.Spec}(\mathcal{A}^{(6)})$.
5. $\langle X_{2,6}T_4 - \frac{2}{3}X_{3,6}^3\rangle[T_6^{-1}] \cap \mathcal{A} = \langle \Omega_2\rangle$, hence $\langle \Omega_2\rangle \in \text{P.Spec}(\mathcal{A})$.

We now proceed to prove the above claims.

1. Observe that $\mathcal{A}^{(3)}/\langle T_2\rangle$ is isomorphic to a Poisson affine space of rank 5 which is a domain. Hence, $\langle T_2\rangle$ is a Poisson prime ideal in $\mathcal{A}^{(3)}$.

2. Similar to (1).

3. Recall from the PDDA that $T_2 = X_{2,5} - \frac{2}{3}T_3^3T_4^{-1}$. We want to show that $\langle T_2\rangle_{\mathcal{A}^{(4)}}[T_4^{-1}] \cap \mathcal{A}^{(5)} = \langle X_{2,5}T_4 - \frac{2}{3}T_3^3\rangle$. Clearly, $\langle X_{2,5}T_4 - \frac{2}{3}T_3^3\rangle \subseteq \langle T_2\rangle_{\mathcal{A}^{(4)}}[T_4^{-1}] \cap \mathcal{A}^{(5)}$. For the reverse inclusion, let $y \in \langle T_2\rangle_{\mathcal{A}^{(4)}}[T_4^{-1}] \cap \mathcal{A}^{(5)}$. Then, $y \in \langle T_2\rangle_{\mathcal{A}^{(4)}}[T_4^{-1}]$. Hence, there exists $i \in \mathbb{N}$ such that $yT_4^i = vT_2$, for some $v \in \mathcal{A}^{(4)}$. Furthermore, since $\mathcal{A}^{(4)}[T_4^{-1}] = \mathcal{A}^{(5)}[T_4^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_4^j = v'$ for some $v' \in \mathcal{A}^{(5)}$. It follows from $yT_4^i = vT_2$ that $yT_4^{i+j} = vT_4^jT_2 = v'\left(X_{2,5} - \frac{2}{3}T_3^3T_4^{-1}\right)$. Consequently,

$yT_4^{\delta} = \Omega_2'v'$, where $\Omega_2' := X_{2,5}T_4 - \frac{2}{3}T_3^3$ and $\delta = i+j+1$. Let $S = \{s \in \mathbb{N} \mid \exists v' \in \mathcal{A}^{(5)} : yT_4^s = \Omega_2'v'\}$. Since $\delta \in S$, we have that $S \neq \emptyset$. Let $s = s_0$ be the minimum element of S such that $yT_4^{s_0} = \Omega_2'v'$ for some $v' \in \mathcal{A}^{(5)}$. Suppose that $s_0 > 0$. We want to show that $s_0 = 0$. Since T_4 is irreducible, $yT_4^{s_0} = \Omega_2'v'$ implies that T_4 is a factor of Ω_2' or v'. Clearly, T_4 is not a factor of Ω_2', hence, it must be a factor of v'. Now $\mathcal{A}^{(5)}$ can be viewed as a free $\mathbb{C}\langle X_{1,5}, X_{2,5}, T_3, T_5, T_6\rangle-$module with basis $\left(T_4^{\xi}\right)_{\xi\in\mathbb{N}}$. One can therefore write $v' = \sum_{\xi=1}^{n}\beta_{\xi}T_4^{\xi}$, where $\beta_{\xi} \in \mathbb{C}\langle X_{1,5}, X_{2,5}, T_3, T_5, T_6\rangle$. Given $yT_4^{s_0} = \Omega_2'v'$, we have that $yT_4^{s_0} = \Omega_2'\sum_{\xi=1}^{n}\beta_{\xi}T_4^{\xi}$. This implies that $yT_4^{s_0-1} = \Omega_2'v''$, where $v'' = \sum_{\xi=1}^{n}\beta_{\xi}T_4^{\xi-1} \in \mathcal{A}^{(5)}$. Consequently, $s_0 - 1 \in S$, a contradiction! Therefore, $s_0 = 0$ and $y = \Omega_2'v' \in \langle\Omega_2'\rangle$. Hence, $\langle T_2\rangle_{\mathcal{A}^{(4)}}[T_4^{-1}] \cap \mathcal{A}^{(5)} \subseteq \langle\Omega_2'\rangle$ as desired.

4. The proof is similar to point (3). Note, $\Omega_2' = X_{2,5}T_4 - \frac{2}{3}T_3^3 = X_{2,6}T_4 - \frac{2}{3}X_{3,6}^3$. We want to show that $\langle\Omega_2'\rangle_{\mathcal{A}^{(5)}}[T_5^{-1}] \cap \mathcal{A}^{(6)} = \langle\Omega_2'\rangle_{\mathcal{A}^{(6)}}$. The inclusion $\langle\Omega_2'\rangle_{\mathcal{A}^{(6)}} \subseteq \langle\Omega_2'\rangle_{\mathcal{A}^{(5)}}[T_5^{-1}] \cap \mathcal{A}^{(6)}$ is trivial. For the reverse inclusion, let $y \in \langle\Omega_2'\rangle_{\mathcal{A}^{(5)}}[T_5^{-1}] \cap \mathcal{A}^{(6)}$. This implies that there exists $i \in \mathbb{N}$ such that $yT_5^i = v\Omega_2'$ for some $v \in \mathcal{A}^{(5)}$. Since $\mathcal{A}^{(5)}[T_5^{-1}] = \mathcal{A}^{(6)}[T_5^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_5^j = v'$ for some $v' \in \mathcal{A}^{(6)}$. Therefore, $yT_5^{\delta} = v'\Omega_2'$, where $\delta = i + j$. An argument similar to point (3) above will reveal that $\delta = 0$. As a result, $y = v'\Omega_2' = \langle\Omega_2'\rangle_{\mathcal{A}^{(6)}}$. Consequently, $\langle\Omega_2'\rangle_{\mathcal{A}^{(5)}}[T_5^{-1}]\cap\mathcal{A}^{(6)} \subseteq \langle\Omega_2'\rangle_{\mathcal{A}^{(6)}}$ as expected.

5. Note, $\Omega_2' = X_{2,5}T_4 - \frac{2}{3}T_3^3 = \Omega_2T_6^{-1}$ in $\mathcal{A}[T_6^{-1}]$. We want to show that $\langle\Omega_2'\rangle_{\mathcal{A}^{(6)}}[T_6^{-1}]\cap \mathcal{A} = \langle\Omega_2\rangle$. Clearly, $\langle\Omega_2\rangle \subseteq \langle\Omega_2'\rangle_{\mathcal{A}^{(6)}}[T_6^{-1}] \cap \mathcal{A}$. For the reverse inclusion, let $y \in \langle\Omega_2'\rangle_{\mathcal{A}^{(6)}}[T_6^{-1}] \cap \mathcal{A}$. Then, $y \in \langle\Omega_2'\rangle_{\mathcal{A}^{(6)}}[T_6^{-1}]$. Hence, there exists $i \in \mathbb{N}$ such that $yT_6^i = \Omega_2'v$, for some $v \in \mathcal{A}^{(6)}$. Furthermore, since $\mathcal{A}^{(6)}[T_6^{-1}] = \mathcal{A}[T_6^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_6^j = v'$ for some $v' \in \mathcal{A}$. Now, $yT_6^i = \Omega_2'v$ implies that $yT_6^{s_0} = \Omega_2'v'T_6 = \Omega_2v'$, where $s_0 = i + j + 1$. Note, $\Omega_2 = \Omega_2'T_6$ in $\mathcal{A}$. Similar to point (3), one can easily show that $s_0 = 0$ and $y = \Omega_2v' \in \langle\Omega_2\rangle$. As a result, $\langle\Omega_2'\rangle_{\mathcal{A}^{(6)}}[T_6^{-1}] \cap \mathcal{A} \subseteq \langle\Omega_2\rangle$ as desired. ■

Observe that $\langle T_1, T_2\rangle$ and $\langle T_2, T_3\rangle$ are Poisson prime ideals of $\bar{A}$. In the next lemma, we will show that $\langle T_1, T_2\rangle, \langle T_2, T_3\rangle \in \psi(\text{P.Spec}(\mathcal{A}))$.

5.2.8 Lemma. $\langle T_1, T_2\rangle, \langle T_2, T_3\rangle \in \psi(\text{P.Spec}(\mathcal{A}))$.

Proof. Let $J^{(j)} \in \text{P.Spec}(\mathcal{A}^{(j)})$ for all $2 \leq j \leq 6$. We already know that $J^{(j+1)} = J^{(j)}[T_j^{-1}] \cap \mathcal{A}^{(j+1)} \in \text{P.Spec}(\mathcal{A}^{(1+j)})$ provided $T_j \notin J^{(j)}$. Note, $J^{(7)} = J$ and $\mathcal{A}^{(7)} = \mathcal{A}$.

We begin by showing that $\langle T_1, T_2\rangle \in \psi(\text{P.Spec}(\mathcal{A}))$.

Set $J_{1,2}^{(3)} := \langle T_1, T_2\rangle \in \text{P.Spec}(\mathcal{A}^{(3)})$. Observe that $T_3 \notin J_{1,2}^{(3)}$. Therefore, $J_{1,2}^{(3)}[T_3^{-1}] \cap \mathcal{A}^{(4)} = J_{1,2}^{(4)}$. Suppose that $T_4 \in J_{1,2}^{(4)}$. Then, since $J_{1,2}^{(3)}[T_3^{-1}] = J_{1,2}^{(4)}[T_3^{-1}]$, we have that $T_4 \in J_{1,2}^{(3)}[T_3^{-1}] \cap \mathcal{A}^{(4)} = J_{1,2}^{(4)}[T_3^{-1}] \cap \mathcal{A}^{(3)} = J_{1,2}^{(3)}$, a contradiction! Therefore, $T_4 \notin J_{1,2}^{(4)}$. Hence, $J_{1,2}^{(4)}[T_4^{-1}] \cap \mathcal{A}^{(5)} = J_{1,2}^{(5)}$. Suppose that $T_5 \in J_{1,2}^{(5)}$. Then, $T_5 \in J_{1,2}^{(4)}[T_4^{-1}] \cap \mathcal{A}^{(5)} = J_{1,2}^{(5)}[T_4^{-1}] \cap \mathcal{A}^{(4)} = J_{1,2}^{(4)}$, a contradiction! Therefore, $T_5 \notin J_{1,2}^{(5)}$. Hence, $J_{1,2}^{(5)}[T_5^{-1}] \cap \mathcal{A}^{(6)} = J_{1,2}^{(6)}$. Similarly, one can show that $T_6 \notin J_{1,2}^{(6)}$. Hence, $J_{1,2}^{(6)}[T_6^{-1}] \cap \mathcal{A} = J_{1,2}$. Therefore, there exists $J_{1,2} \in \text{P.Spec}(\mathcal{A})$ such that $\psi(J_{1,2}) = \langle T_1, T_2\rangle$.

We finally show that $\langle T_2, T_3\rangle \in \psi(\text{P.Spec}(\mathcal{A}))$.

Set $J_{2,3}^{(4)} := \langle T_2, T_3\rangle \in \text{P.Spec}(\mathcal{A}^{(4)})$. Similar to (1), one can verify that $T_j \notin J_{2,3}^{(j)}$ for all $j = 4, 5, 6$. By an induction on j (note, $j = 4, 5, 6$), we have that $J_{2,3}^{(j)}[T_j^{-1}] \cap \mathcal{A}^{(j+1)} = J_{2,3}^{(j+1)}$. Therefore, there exists $J_{2,3}^{(7)} = J_{2,3} \in \text{P.Spec}(\mathcal{A})$ such that $\psi(J_{2,3}) = \langle T_2, T_3\rangle$. ■

In $\bar{\mathcal{A}}$, recall that $\Omega_1 = T_1T_3T_5$ and $\Omega_2 = T_2T_4T_6$. Observe that Ω_1, Ω_2 are both elements of $\langle T_1, T_2\rangle$ and $\langle T_2, T_3\rangle$. In the above lemma, we know that $J_{1,2}$ and $J_{2,3}$ are the elements of $\text{P.Spec}(\mathcal{A})$ such that $\psi(J_{1,2}) = \langle T_1, T_2\rangle$ and $\psi(J_{2,3}) = \langle T_2, T_3\rangle$. In the next lemma, we show that $J_{1,2}$ and $J_{2,3}$ contain Ω_1 and Ω_2.

5.2.9 Lemma. Ω_1 and Ω_2 are elements of $J_{1,2}$ and $J_{2,3}$.

Proof. Recall that Ω_1 and Ω_2 are central elements of $\mathcal{A}^{(j)}$ for all $2 \leq j \leq 7$. Given the set-up in the proof of Lemma 5.2.8, we know that $\Omega_1, \Omega_2 \in J_{1,2}^{(3)} = \langle T_1, T_2\rangle$. By an induction on j (where $j = 3, 4, 5, 6$), one can confirm that $\Omega_1, \Omega_2 \in J_{1,2}^{(j)}[T_j^{-1}] \cap \mathcal{A}^{(j+1)} = J_{1,2}^{(j+1)}$. Therefore; $\Omega_1, \Omega_2 \in J_{1,2}$.

Similarly; $\Omega_1, \Omega_2 \in J_{2,3}^{(4)} = \langle T_2, T_3\rangle$. By an induction on j (where $j = 4, 5, 6$), one can confirm that $\Omega_1, \Omega_2 \in J_{2,3}^{(j)}[T_j^{-1}] \cap \mathcal{A}^{(j+1)} = J_{2,3}^{(j+1)}$. Therefore, $\Omega_1, \Omega_2 \in J_{2,3}$. ■

We now want to find the height one Poisson $\mathcal{H}-$invariant prime ideals of $\mathcal{A}$, and show that the height 2 Poisson $\mathcal{H}-$invariant prime ideals of $\mathcal{A}$ contain those of height one. Of course, the zero ideal $\langle 0\rangle$ is the only height zero Poisson $\mathcal{H}-$invariant prime ideal.

Let the torus $\mathcal{H} := (\mathbb{C}^*)^2$ acts by Poisson automorphisms on $\mathcal{A}$ via:

$$h\cdot X_1=\alpha_1X_1 \qquad h\cdot X_2=\alpha_1^3\alpha_6X_2 \qquad h\cdot X_3=\alpha_1^2\alpha_6X_3$$
$$h\cdot X_4=\alpha_1^3\alpha_6^2X_4 \qquad h\cdot X_5=\alpha_1\alpha_6X_5 \qquad h\cdot X_6=\alpha_6X_6,$$

for all $h:=(\alpha_1,\alpha_6)\in\mathcal{H}$. This $\mathcal{H}-$action is rational. Furthermore, Ω_1 and Ω_2 are Poisson $\mathcal{H}-$eigenvectors. Hence, $\langle\Omega_1\rangle$ and $\langle\Omega_2\rangle$ are Poisson $\mathcal{H}-$invariant prime ideals.

Set $\theta_1 := \text{id}_{\text{P.Spec}(\bar{\mathcal{A}})}$. For all $2\leq j\leq 6$, define $\theta_j := \theta_{j-1}\circ\psi_j$. Then, θ_j : P.Spec$(\mathcal{A}^{(j+1)})\hookrightarrow$ P.Spec$(\bar{\mathcal{A}})$. The map θ_j is injective [35, Section 2.3]. Let $\langle X_{j,j+1}\rangle_P$ denote the smallest Poisson ideal in $\mathcal{A}^{(1+j)}$ containing $X_{j,j+1}$. It follows from [35, Lemma 2.2] that there is a surjective Poisson algebra homomorphism $g_j:\mathcal{A}^{(j)}\longrightarrow\mathcal{A}^{(j+1)}/\langle X_{j,j+1}\rangle_P$ given by $g_j(X_{i,j})=\overline{X_{i,j+1}}:=X_{i,j+1}+\langle X_{j,j+1}\rangle_P$ for all $1\leq i\leq 6$. Denote the kernel of g_j by ker(g_j) and the image of ψ by Im(ψ). We have the following lemma.

5.2.10 Lemma. [35, Proposition 2.8] Let $P\in$ P.Spec$(\bar{\mathcal{A}})$. The following are equivalent:

- $P\in$ Im(ψ),
- for all $2\leq j\leq 6$, we have $P\in$ Im(θ_{j-1}) and either $X_{j,j}=X_{j,j+1}\notin\theta_{j-1}^{-1}(P)$ or ker$(g_j)\subseteq\theta_{j-1}^{-1}(P)$.

Note, the map ψ induces a canonical embedding from $\mathcal{H}-$P.Spec$(\mathcal{A})$ to $\mathcal{H}-$P.Spec$(\bar{\mathcal{A}})$. Observe that $\{\langle T_i\rangle \mid i=1,\cdots,6\}$ is the set of only height one Poisson $\mathcal{H}-$invariant prime ideals we have in $\bar{\mathcal{A}}$. Since ψ preserves the height of a prime ideal, if $\psi^{-1}(\langle T_i\rangle)\in$ P.Spec$(\mathcal{A})$, then it is a height one Poisson $\mathcal{H}-$invariant prime ideal in $\mathcal{A}$ for all $1\leq i\leq 6$. For example, we already know that $\psi^{-1}(\langle T_1\rangle)=\langle\Omega_1\rangle$ and $\psi^{-1}(\langle T_2\rangle)=\langle\Omega_2\rangle$. Therefore, $\langle\Omega_1\rangle$ and $\langle\Omega_2\rangle$ are height one Poisson $\mathcal{H}-$invariant prime ideals in $\mathcal{A}$. We will show in the next lemma that $\langle\Omega_1\rangle$ and $\langle\Omega_2\rangle$ are the only height one Poisson $\mathcal{H}-$invariant prime ideals we have in $\mathcal{A}$.

5.2.11 Lemma. For each $j \in \{3,4,5,6\}$, $\langle T_j\rangle \notin \psi(\text{P.Spec}(\mathcal{A}))$.

Proof. Suppose that there exists $j \in \{3,4,5,6\}$ such that $\langle T_j\rangle \in \psi(\text{P.Spec}(\mathcal{A}))$. Then, there exists $P \in \text{P.Spec}(\mathcal{A})$ such that $\psi(P) = \langle T_j\rangle_{\mathcal{A}^{(j)}}$, where $\psi := \psi_j \circ \cdots \psi_6$. Set $P^{(j)} := \langle T_j\rangle_{\mathcal{A}^{(j)}}$. Since $T_j \in P^{(j)}$, it follows from Theorem 5.2.10 that $\ker(g_j) \subseteq P^{(j)}$. The rest follows in cases.

- When $j = 3$, then $\ker(g_3) \subseteq P^{(3)} = \langle T_3\rangle_{\mathcal{A}^{(3)}}$. One can easily deduce from the Poisson bracket of $\mathcal{A}$ that $\{X_{3,4}, X_{1,4}\} = -X_{1,4}X_{3,4} - X_{2,4}$. This implies that $X_{2,4} = -\{X_{3,4}, X_{1,4}\} - X_{1,4}X_{3,4} \in \langle X_{3,4}\rangle_P = \langle T_3\rangle_P$. It follows that $g_3(X_{2,3}) = X_{2,4} + \langle T_3\rangle_P = \overline{0}$. Hence, $X_{2,3} = T_2 \in \ker(g_3) \subseteq \langle T_3\rangle_{\mathcal{A}^{(3)}}$, a contradiction! Consequently, $\langle T_3\rangle \notin \psi(\text{P.Spec}(\mathcal{A}))$.

- When $j = 4$, then $\ker(g_4) \subseteq P^{(4)} = \langle T_4\rangle_{\mathcal{A}^{(4)}}$. From the Poisson bracket of $\mathcal{A}$, we have that $X_{3,5}^2 = -\frac{1}{2}\{X_{4,5}, X_{1,5}\} \in \langle X_{4,5}\rangle_P = \langle T_4\rangle_P$. Since g_4 is a homomorphism, it follows that $g_4(X_{3,4}^2) = (g_4(X_{3,4}))^2 = X_{3,5}^2 + \langle T_4\rangle_P = \overline{0}$. Therefore, $X_{3,4}^2 = T_3^2 \in \ker(g_4) \subseteq \langle T_4\rangle_{\mathcal{A}^{(4)}}$, a contradiction! Hence, $\langle T_4\rangle \notin \psi(\text{P.Spec}(\mathcal{A}))$.

- When $j = 5$, then $\ker(g_5) \subseteq P^{(5)} = \langle T_5\rangle_{\mathcal{A}^{(5)}}$. One can deduce from the Poisson bracket of $\mathcal{A}$ that $X_{4,6} = -\frac{1}{3}X_{3,6}X_{5,6} - \frac{1}{3}\{X_{5,6}, X_{3,6}\} \in \langle X_{5,6}\rangle_P = \langle T_5\rangle_P$. It follows that $g_5(X_{4,5}) = X_{4,6} + \langle T_5\rangle_P = \overline{0}$. Hence, $X_{4,6} = T_4 \in \ker(g_5) \subseteq \langle T_5\rangle_{\mathcal{A}^{(5)}}$, a contradiction! Therefore, $\langle T_5\rangle \notin \psi(\text{P.Spec}(\mathcal{A}))$.

- Finally, when $j = 6$, then $\ker(g_6) \subseteq P^{(6)} = \langle T_6\rangle_{\mathcal{A}^{(6)}}$. Similarly, one can verify that $X_5 = X_1X_6 - \frac{1}{3}\{X_6, X_1\} \in \langle X_6\rangle_P = \langle T_6\rangle_P$. Now, $g_6(X_{5,6}) = X_5 + \langle T_6\rangle_P = \overline{0}$. Therefore, $X_{5,6} = T_5 \in \ker(g_6) \subseteq \langle T_6\rangle_{\mathcal{A}^{(6)}}$, a contradiction! Hence, $\langle T_6\rangle \notin \psi(\text{P.Spec}(\mathcal{A}))$.

In conclusion, $\langle T_j\rangle \notin \psi(\text{P.Spec}(\mathcal{A}))$ for all $j \in \{3,4,5,6\}$ as expected.

The corollary below is deduced from the above proof (i.e. the proof of Lemma 5.2.11). ∎

5.2.12 Corollary. Let $P \in \psi(\text{P.Spec}(\mathcal{A}))$. If $T_j \in P$, then $T_{j-1}, \cdots, T_2 \in P$ for all $3 \leq j \leq 6$.

Recall from Lemma 5.2.8 that there exist $J_{1,2}$ and $J_{2,3}$ of P.Spec$(\mathcal{A})$ such that $\psi(J_{1,2}) = \langle T_1, T_2 \rangle$ and $\psi(J_{2,3}) = \langle T_2, T_3 \rangle$. As a result of Corollary 5.2.12, the Poisson ideals $\langle T_1, T_2 \rangle$ and $\langle T_2, T_3 \rangle$ are the only height two Poisson $\mathcal{H}-$invariant prime ideals of $\psi(\text{P.Spec}(\mathcal{A}))$. Since ψ preserves Poisson $\mathcal{H}-$invariant prime ideals and the height of a Poisson prime ideal, it implies that $J_{1,2}$ and $J_{2,3}$ are the only height two Poisson $\mathcal{H}-$invariant prime ideals of $\mathcal{A}$. It follows from Lemma 5.2.9 that the height two Poisson $\mathcal{H}-$invariant prime ideals of $\mathcal{A}$ contain Ω_1 and Ω_2.

5.2.13 Remark. Since the height two Poisson $\mathcal{H}-$invariant prime ideals of $\mathcal{A}$ contain Ω_1 and Ω_2, every non-zero Poisson $\mathcal{H}-$invariant prime ideal of $\mathcal{A}$ will contain either Ω_1 or Ω_2. Note, those Poisson $\mathcal{H}-$invariant primes of at least height 2 will contain both Ω_1 and Ω_2.

5.2.14 $\mathcal{H}-$strata corresponding $\langle 0 \rangle$, $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$. In this subsection, we aim to find the $\mathcal{H}-$strata corresponding to $\langle 0 \rangle$, $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$. We state the results in Propositions 5.2.15, 5.2.16 and 5.2.17. The proofs of these three propositions are similar to those of Propositions 2.4.4, 2.4.5 and 2.4.6 respectively. As a result, we will only prove Proposition 5.2.15, and leave the remaining ones for the reader to verify. Note, all Poisson ideals in $\mathcal{A}$ shall be written as $\langle \Theta \rangle$, where $\Theta \in \mathcal{A}$. However, if we want to refer to a Poisson ideal in any other Poisson algebra, say R, then that Poisson ideal shall be written as $\langle \Theta \rangle_R$, where in this case, $\Theta \in R$.

5.2.15 Proposition. Let $\mathcal{P}$ be the set of those unitary irreducible polynomials $P(\Omega_1, \Omega_2) \in \mathbb{C}[\Omega_1, \Omega_2]$ with $P(\Omega_1, \Omega_2) \neq \Omega_1$ and $P(\Omega_1, \Omega_2) \neq \Omega_2$. Then $\text{P.Spec}_{\langle 0 \rangle}(\mathcal{A}) = \{\langle 0 \rangle\} \cup \{\langle P(\Omega_1, \Omega_2) \rangle \mid P(\Omega_1, \Omega_2) \in \mathcal{P}\} \cup \{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \mid \alpha, \beta \in \mathbb{C}^*\}$.

Proof. We claim that $\text{P.Spec}_{\langle 0 \rangle}(\mathcal{A}) = \{Q \in \text{P.Spec}(\mathcal{A}) \mid \Omega_1, \Omega_2 \notin Q\}$. To establish this claim, let us assume that this is not the case. That is, suppose there exists $Q \in \text{P.Spec}_{\langle 0 \rangle}(\mathcal{A})$ such that $\Omega_1, \Omega_2 \in Q$, then the product $\Omega_1 \Omega_2$ which is Poisson

$\mathcal{H}-$eigenvector belongs to Q. Consequently, $\Omega_1\Omega_2 \in \bigcap_{h\in\mathcal{H}} h\cdot Q = \langle 0\rangle$, a contradiction! Hence, we have shown that P.Spec$_{\langle 0\rangle}(\mathcal{A}) \subseteq \{Q \in$ P.Spec$(\mathcal{A}) \mid \Omega_1, \Omega_2 \notin Q\}$. Conversely, suppose that $Q \in$ P.Spec$(\mathcal{A})$ such that $\Omega_1, \Omega_2 \notin Q$, then $\bigcap_{h\in\mathcal{H}} h\cdot Q$ is a Poisson $\mathcal{H}-$invariant prime ideal of $\mathcal{A}$, which contains neither Ω_1 nor Ω_2. The only possibility for $\bigcap_{h\in\mathcal{H}} h\cdot Q$ is the zero ideal since every non-zero Poisson $\mathcal{H}-$invariant prime ideal of $\mathcal{A}$ contains Ω_1 or Ω_2 (Remark 5.2.13). Thus, $\bigcap_{h\in\mathcal{H}} h\cdot Q = \langle 0\rangle$. Hence, $Q \in$ P.Spec$_{\langle 0\rangle}(\mathcal{A})$. Therefore, $\{Q \in$ P.Spec$(\mathcal{A}) \mid \Omega_1, \Omega_2 \notin Q\} \subseteq$ P.Spec$_{\langle 0\rangle}(\mathcal{A})$. This confirms our claim.

Since $\Omega_1, \Omega_2 \in Z_P(\mathcal{A})$, we have that the set $\{\Omega_1^i\Omega_2^j \mid i, j \in \mathbb{N}\}$ is a multiplicative set in $\mathcal{A}$. We can now localize $\mathcal{A}$ as $R := \mathcal{A}[\Omega_1^{-1}, \Omega_2^{-1}]$. Given $Q \in$ P.Spec$_{\langle 0\rangle}(\mathcal{A})$, the map $\phi : Q \longrightarrow Q[\Omega_1^{-1}, \Omega_2^{-1}]$ is an increasing bijection from P.Spec$_{\langle 0\rangle}(\mathcal{A})$ onto P.Spec(R).

Let us verify that R is Poisson $\mathcal{H}-$simple before we describe P.Spec(R). Now, ϕ still induces a bijection between the set of those Poisson $\mathcal{H}-$invariant prime ideals of P.Spec$_{\langle 0\rangle}(\mathcal{A})$ and the set of Poisson $\mathcal{H}-$invariant prime ideals of P.Spec(R). It is already known that the set of Poisson $\mathcal{H}-$invariant prime ideals of $\mathcal{A}$ that contains neither Ω_1 nor Ω_2 consists only of the zero ideal $\{\langle 0\rangle\}$ (Remark 5.2.13). This implies that $\langle 0\rangle_R$ is the only Poisson $\mathcal{H}-$invariant prime ideal of R. Every Poisson $\mathcal{H}-$invariant proper ideal of R is contained in a Poisson $\mathcal{H}-$invariant prime ideal of R. Therefore, $\langle 0\rangle_R$ is the only unique Poisson $\mathcal{H}-$invariant proper ideal of R. This confirms that R is Poisson $\mathcal{H}-$simple. It follows from [16, Theorem 4.2] that the extension and contraction maps provide a mutually inverse bijection between P.Spec(R) and Spec$(Z_P(R))$. From Lemma 5.2.4, $Z_P(\mathcal{A}) = \mathbb{C}[\Omega_1, \Omega_2]$, and so $Z_P(R) = \mathbb{C}[\Omega_1^{\pm 1}, \Omega_2^{\pm 1}]$. Since $\mathbb{C}$ is algebraically closed, we have that Spec$(Z_P(R)) = \{\langle 0\rangle_{Z_P(R)}\} \cup \{\langle P(\Omega_1, \Omega_2)\rangle_{Z_P(R)} \mid P(\Omega_1, \Omega_2) \in \mathcal{P}\} \cup \{\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_{Z_P(R)} \mid \alpha, \beta \in \mathbb{C}^*\}$. One can now recover P.Spec(R) from Spec$(Z_P(R))$ as follows: P.Spec(R)= $\{\langle 0\rangle_R\} \cup \{\langle P(\Omega_1, \Omega_2)\rangle_R \mid P(\Omega_1, \Omega_2) \in \mathcal{P}\} \cup \{\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_R \mid \alpha, \beta \in \mathbb{C}^*\}$. It follows that P.Spec$_{\langle 0\rangle}(\mathcal{A}) = \{\langle 0\rangle_R \cap \mathcal{A}\} \cup \{\langle P(\Omega_1, \Omega_2)\rangle_R \cap \mathcal{A} \mid P(\Omega_1, \Omega_2) \in \mathcal{P}\} \cup \{\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_R \cap \mathcal{A} \mid \alpha, \beta \in \mathbb{C}^*\}$.

Undoubtedly, $\langle 0\rangle_R \cap \mathcal{A} = \langle 0\rangle$. We now have to show that $\langle P(\Omega_1, \Omega_2)\rangle_R \cap \mathcal{A} = \langle P(\Omega_1, \Omega_2)\rangle, \forall P(\Omega_1, \Omega_2) \in \mathcal{P}$, and $\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_R \cap \mathcal{A} = \langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle, \forall \alpha, \beta \in \mathbb{C}^*$, in order to complete the proof.

Fix $P(\Omega_1, \Omega_2) \in \mathcal{P}$. Clearly, $\langle P(\Omega_1, \Omega_2)\rangle \subseteq \langle P(\Omega_1, \Omega_2)\rangle_R \cap \mathcal{A}$. To show the reverse inclusion, let $y \in \langle P(\Omega_1, \Omega_2)\rangle_R \cap \mathcal{A}$. Since $y \in \langle P(\Omega_1, \Omega_2)\rangle_R$, it implies that $y = dP(\Omega_1, \Omega_2)$ for some $d \in R$. Also, $d \in R$ implies that there exists $i, j \in \mathbb{N}$ such that $d = a\Omega_1^{-i}\Omega_2^{-j}$, where $a \in \mathcal{A}$. Therefore, $y = a\Omega_1^{-i}\Omega_2^{-j}P(\Omega_1, \Omega_2)$, which implies that $y\Omega_1^i\Omega_2^j = aP(\Omega_1, \Omega_2)$. Choose $(i, j) \in \mathbb{N}^2$ minimal (in the lexicographic order on $\mathbb{N}^2$) such that the equality holds. Without loss of generality, let us suppose that $i > 0$, then $aP(\Omega_1, \Omega_2) \in \langle \Omega_1\rangle$. Since $\langle \Omega_1\rangle$ is a prime ideal, it implies that $a \in \langle \Omega_1\rangle$ or $P(\Omega_1, \Omega_2) \in \langle \Omega_1\rangle$. Since $P(\Omega_1, \Omega_2) \in \mathcal{P}$, it implies that $P(\Omega_1, \Omega_2) \notin \langle \Omega_1\rangle$. Hence, $a \in \langle \Omega_1\rangle$, which implies that $a = t\Omega_1$ for some $t \in \mathcal{A}$. Returning to $y\Omega_1^i\Omega_2^j = aP(\Omega_1, \Omega_2)$, we have that $y\Omega_1^i\Omega_2^j = t\Omega_1P(\Omega_1, \Omega_2)$. Finally, $y\Omega_1^{i-1}\Omega_2^j = tP(\Omega_1, \Omega_2)$. This clearly contradicts the minimality of (i, j), hence $(i, j) = (0, 0)$. As a result, $y = aP(\Omega_1, \Omega_2) = \langle P(\Omega_1, \Omega_2)\rangle$. Consequently, $\langle P(\Omega_1, \Omega_2)\rangle_R \cap \mathcal{A} = \langle P(\Omega_1, \Omega_2)\rangle$ for all $P(\Omega_1, \Omega_2) \in \mathcal{P}$ as desired.

Similarly, we show that $\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_R \cap \mathcal{A} = \langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle$; $\forall \alpha, \beta \in \mathbb{C}^*$. Fix $\alpha, \beta \in \mathbb{C}^*$. Obviously, $\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle \subseteq \langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_R \cap \mathcal{A}$. We establish the reverse inclusion. Let $y \in \langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_R \cap \mathcal{A}$. Since $y \in \langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_R$, we have that $y = m_0(\Omega_1 - \alpha) + n_0(\Omega_2 - \beta)$, where $m_0, n_0 \in R$. Also, $m_0, n_0 \in R$ implies that there exists $i, j \in \mathbb{N}$ such that $m_0 = m\Omega_1^{-i}\Omega_2^{-j}$ and $n_0 = n\Omega_1^{-i}\Omega_2^{-j}$ for some $m, n \in \mathcal{A}$. Therefore, $y = m\Omega_1^{-i}\Omega_2^{-j}(\Omega_1 - \alpha) + n\Omega_1^{-i}\Omega_2^{-j}(\Omega_2 - \beta)$, which implies that $y\Omega_1^i\Omega_2^j = m(\Omega_1 - \alpha) + n(\Omega_2 - \beta)$. Choose $(i, j) \in \mathbb{N}^2$ minimal (in the lexicographic order on $\mathbb{N}^2$) such that the equality holds. Without loss of generality, suppose that $i > 0$ and let $f : \mathcal{A} \longrightarrow \mathcal{A}/\langle \Omega_2 - \beta\rangle$ be a canonical surjection. We have that $f(y)f(\Omega_1)^if(\Omega_2)^j = f(m)f(\Omega_1 - \alpha)$. It follows that $f(m)f(\Omega_1 - \alpha) \in \langle f(\Omega_1)\rangle$. Note, $f(\Omega_1 - \alpha) \notin \langle f(\Omega_1)\rangle$. It implies that $f(m) \in \langle f(\Omega_1)\rangle$. Therefore, $\exists \lambda \in \mathcal{A}$ such that $f(m) = f(\lambda)f(\Omega_1)$. Consequently, $f(y)f(\Omega_1)^if(\Omega_2)^j = f(\lambda)f(\Omega_1)f(\Omega_1 - \alpha)$. Since $f(\Omega_1) \neq 0$, it implies that $f(y)f(\Omega_1)^{i-1}f(\Omega_2)^j = f(\lambda)f(\Omega_1 - \alpha)$. Therefore, $y\Omega_1^{i-1}\Omega_2^j = \lambda(\Omega_1 - \alpha) + \lambda'(\Omega_2 - \beta)$, for some $\lambda' \in \mathcal{A}$. This contradicts the minimality of (i, j). Hence, $(i, j) = (0, 0)$ and so $y = m(\Omega_1 - \alpha) + n(\Omega_2 - \beta) \in \langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle$.

In conclusion, $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle_R \cap \mathcal{A} = \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$, $\forall \alpha, \beta \in \mathbb{C}^*$. ■

5.2.16 Proposition. $\text{P.Spec}_{\langle \Omega_1 \rangle}(\mathcal{A}) = \{\langle \Omega_1 \rangle\} \cup \{\langle \Omega_1, \Omega_2 - \beta \rangle \mid \beta \in \mathbb{C}^*\}$.

Proof. The proof is similar to that of Proposition 2.4.5. ■

5.2.17 Proposition. $\text{P.Spec}_{\langle \Omega_2 \rangle}(\mathcal{A}) = \{\langle \Omega_2 \rangle\} \cup \{\langle \Omega_1 - \alpha, \Omega_2 \rangle \mid \alpha \in \mathbb{C}^*\}$.

Proof. The proof is similar to that of Proposition 2.4.6. ■

5.2.18 Corollary. The Poisson ideal $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ is primitive in $\mathcal{A}$ for each $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.

Proof. Since the Poisson ideal $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ is maximal in its respective strata for each $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, it is also primitive (see Proposition 5.1.7). ■

5.2.19 Proposition. Let $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. The Poisson prime ideal $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ is maximal in $\mathcal{A}$.

Proof. The proof is similar to that of Proposition 2.4.8. Suppose that there exists a maximal Poisson ideal I of $\mathcal{A}$ such that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \subsetneq I \subsetneq \mathcal{A}$. Let J be the Poisson $\mathcal{H}-$invariant prime ideal in $\mathcal{A}$ such that $I \in \text{P.Spec}_J(\mathcal{A})$. By Propositions 5.2.15, 5.2.16 and 5.2.17, J cannot be $\langle 0 \rangle$, $\langle \Omega_1 \rangle$ or $\langle \Omega_2 \rangle$, since either of these will lead to a contradiction. Every non-zero Poisson $\mathcal{H}-$invariant prime ideal contains Ω_1 only or Ω_2 only or both (Remark 5.2.13). Since $J \neq \langle \Omega_1 \rangle, \langle \Omega_2 \rangle$, it implies that J contains both Ω_1 and Ω_2. Moreover, since $J \subseteq I$, it implies that $\Omega_1, \Omega_2 \in I$. Given $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \subset I$, we have that $\Omega_1 - \alpha, \Omega_2 - \beta \in I$. It follows that $\alpha, \beta \in I$, hence $I = \mathcal{A}$, a contradiction! This confirms that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ is a maximal Poisson ideal in $\mathcal{A}$. ■

5.3 Semiclassical limit of $A_{\alpha,\beta}$

In Chapter 3, we studied the simple quotients $A_{\alpha,\beta}$. Similarly, in this section, we will also study the simple quotients of $\mathcal{A} = \mathbb{C}[X_1, \cdots, X_6]$, which we denote by $\mathcal{A}_{\alpha,\beta}$ (note,

$\alpha, \beta \in \mathbb{C}^2 \setminus \{(0,0)\}$). We will conclude that $\mathcal{A}_{\alpha,\beta}$ is the semiclassical of the non-commutative algebra $A_{\alpha,\beta}$, and close this section by finding a $\mathbb{C}-$basis for $\mathcal{A}_{\alpha,\beta}$.

Let $\alpha, \beta \in \mathbb{C}^2 \setminus \{(0,0)\}$, and set

$$\mathcal{A}_{\alpha,\beta} := \frac{\mathcal{A}}{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle}.$$

We denote the canonical image of X_i by $x_i := X_i + \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ for each $i \in \{1, \cdots, 6\}$. The algebra $\mathcal{A}_{\alpha,\beta}$ is commutative, and satisfies the following two relations:

$$x_1x_3x_5 - \frac{3}{2}x_1x_4 - \frac{1}{2}x_2x_5 + \frac{1}{2}x_3^2 = \alpha, \tag{5.3.1}$$

$$x_2x_4x_6 - \frac{2}{3}x_3^3x_6 - \frac{2}{3}x_2x_5^3 + 2x_3^2x_5^2 - 3x_3x_4x_5 + \frac{3}{2}x_4^2 = \beta. \tag{5.3.2}$$

We also have the following extra relations in $\mathcal{A}_{\alpha,\beta}$.

5.3.1 Lemma.

(1) $x_3^2 = 2\alpha + 3x_1x_4 + x_2x_5 - 2x_1x_3x_5.$

(2) $$x_4^2 = \frac{2}{3}\beta + \frac{8}{9}\alpha x_3x_6 + \frac{4}{3}x_1x_3x_4x_6 + \frac{4}{9}x_2x_3x_5x_6 - \frac{16}{9}\alpha x_1x_5x_6 - \frac{8}{3}x_1^2x_4x_5x_6 + \frac{16}{9}x_1^2x_3x_5^2x_6 - \frac{8}{9}x_2x_5^3 - \frac{8}{3}\alpha x_5^2 - 4x_1x_4x_5^2 + \frac{8}{3}x_1x_3x_5^3 + 2x_3x_4x_5 - \frac{2}{3}x_2x_4x_6 - \frac{8}{9}x_1x_2x_5^2x_6.$$

(3) $$x_3^2x_4 = 2\alpha x_4 + x_2x_4x_5 + 2\beta x_1 + \frac{8}{3}\alpha x_1x_3x_6 + 4x_1^2x_3x_4x_6 + \frac{4}{3}x_1x_2x_3x_5x_6 - 8x_1^3x_4x_5x_6 - \frac{8}{3}x_1^2x_2x_5^2x_6 + \frac{16}{3}x_1^3x_3x_5^2x_6 - \frac{8}{3}x_1x_2x_5^3 - 8\alpha x_1x_5^2 - 12x_1^2x_4x_5^2 + 8x_1^2x_3x_5^3 + 4x_1x_3x_4x_5 - 2x_1x_2x_4x_6 - \frac{16}{3}\alpha x_1^2x_5x_6.$$

$$\begin{aligned}(4)\ x_3x_4^2 = &\frac{2}{3}\beta x_3 + \frac{16}{9}\alpha^2 x_6 + \frac{16}{3}\alpha x_1x_4x_6 + \frac{16}{9}\alpha x_2x_5x_6 + \frac{16}{9}\alpha x_1x_3x_5x_6 + \frac{4}{9}x_2^2x_5^2x_6\\ &+\frac{8}{9}x_1x_2x_3x_5^2x_6 - \frac{64}{9}\alpha x_1^3x_5x_6^2 - \frac{160}{9}\alpha x_1^2x_5^2x_6 - \frac{80}{3}x_1^3x_4x_5^2x_6 - \frac{64}{9}x_1^2x_2x_5^3x_6\\ &-\frac{8}{9}x_2x_3x_5^3 - \frac{8}{3}\alpha x_3x_5^2 + 4x_1x_3x_4x_5^2 + \frac{160}{9}x_1^3x_3x_5^3x_6 - 16x_1^2x_4x_5^3 - \frac{8}{3}x_1x_2x_5^4\\ &-\frac{4}{3}x_1x_2x_4x_5x_6 + \frac{8}{3}\beta x_1^2x_6 + \frac{32}{9}\alpha x_1^2x_3x_6^2 + \frac{16}{3}x_1^3x_3x_4x_6^2 + \frac{16}{9}x_1^2x_2x_3x_5x_6^2\\ &-\frac{32}{3}x_1^4x_4x_5x_6^2 - \frac{8}{3}x_1^2x_2x_4x_6^2 + 4\alpha x_4x_5 + 2x_2x_4x_5^2 + 4\beta x_1x_5 + \frac{64}{9}x_1^4x_3x_5^2x_6^2\\ &-\frac{2}{3}x_2x_3x_4x_6 - \frac{32}{3}\alpha x_1x_5^3 + \frac{32}{3}x_1^2x_3x_5^4 + \frac{32}{3}x_1^2x_3x_4x_5x_6 - \frac{32}{9}x_1^3x_2x_5^2x_6^2.\end{aligned}$$

Now, the commutative algebra $\mathcal{A}_{\alpha,\beta}$ is a Poisson $\mathbb{C}-$algebra with the Poisson bracket defined as follows:

$$\begin{array}{lll}\{x_2,x_1\} = -3x_1x_2 & \{x_3,x_1\} = -x_1x_3 - x_2 & \{x_3,x_2\} = -3x_2x_3\\ \{x_4,x_1\} = -2x_3^2 & \{x_4,x_2\} = -3x_2x_4 - 4x_3^3 & \{x_4,x_3\} = -3x_3x_4\\ \{x_5,x_1\} = x_1x_5 - 2x_3 & \{x_5,x_2\} = -6x_3^2 & \{x_5,x_3\} = -x_3x_5 - 3x_4\\ \{x_5,x_4\} = -3x_4x_5 & \{x_6,x_1\} = 3x_1x_6 - 3x_5 & \{x_6,x_2\} = 3x_2x_6 + 9x_4 - 18x_3x_5\\ \{x_6,x_3\} = -6x_5^2 & \{x_6,x_4\} = -3x_4x_6 - 4x_5^3 & \{x_6,x_5\} = -3x_5x_6.\end{array}$$

Since $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$ is a maximal Poisson prime ideal of $\mathcal{A}$ (Proposition 5.2.19), $\mathcal{A}_{\alpha,\beta}$ is a simple domain. Moreover, $\mathcal{A}_{\alpha,\beta}$ is noetherian since it is a factor algebra of a noetherian ring $\mathcal{A}$.

5.3.2 Remark. The Poisson algebra $\mathcal{A}_{\alpha,\beta}$ is the semiclassical limit of the non-commutative algebra $A_{\alpha,\beta}$.

Let $\alpha,\beta\neq 0$. Recall that $\Omega_1 = T_1T_3T_5$ and $\Omega_2 = T_2T_4T_6$ in $\bar{\mathcal{A}}$. There exists a multiplicative set $S_{\alpha,\beta}$ such that

$$\mathcal{A}_{\alpha,\beta}S_{\alpha,\beta}^{-1} \cong \mathscr{P}_{\alpha,\beta} := \frac{\mathfrak{R}_1}{\langle T_1T_3T_5-\alpha, T_2T_4T_6-\beta\rangle},$$

where $\mathfrak{R}_1 = \mathbb{C}[T_1^{\pm1},\cdots,T_6^{\pm1}]$ is a Poisson torus associated to the Poisson affine space

$\bar{\mathcal{A}}$. Let $t_i := T_i + \langle T_1T_3T_5 - \alpha, T_2T_4T_6 - \beta\rangle$ denote the canonical image of T_i in $\mathscr{P}_{\alpha,\beta}$ for each $1 \leq i \leq 6$. The algebra $\mathscr{P}_{\alpha,\beta}$ is a Poisson torus generated by $t_1^{\pm 1}, \cdots, t_6^{\pm 1}$ subject to the relations:

$$t_1 = \alpha t_3^{-1} t_5^{-1} \text{ and } t_2 = \beta t_4^{-1} t_5^{-1}.$$

One can verify that $\mathscr{P}_{\alpha,\beta} \cong \mathbb{C}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$. One can also verify that this isomorphism holds when either $\alpha = 0$ or $\beta = 0$ (see Section 3.1 for a similar construction). Henceforth, we will identify $\mathscr{P}_{\alpha,\beta}$ with $\mathbb{C}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$ for all $(\alpha, \beta) \in \mathbb{C} \setminus \{(0,0)\}$.

Set $\mathcal{A}_\beta := \mathcal{A}/\langle \Omega_2 - \beta\rangle$, $\beta \in \mathbb{C}$. Denote the canonical image of X_i in $\mathcal{A}_\beta$ by $\widehat{x}_i := X_i + \langle \Omega_2 - \beta\rangle$ for each $1 \leq i \leq 6$. It can be verified that $\mathcal{A}_{\alpha,\beta} \cong \mathcal{A}_\beta/\langle \widehat{\Omega}_1 - \alpha\rangle$. Note, $\mathcal{A}_\beta$ satisfies the relation:

$$\widehat{x}_4{}^2 = \frac{2}{3}\beta - \frac{2}{3}\widehat{x}_2\widehat{x}_4\widehat{x}_6 + \frac{4}{9}\widehat{x}_3{}^3\widehat{x}_6 + \frac{4}{9}\widehat{x}_2\widehat{x}_5{}^3 - \frac{4}{3}\widehat{x}_3{}^2\widehat{x}_5{}^2 + 2\widehat{x}_3\widehat{x}_4\widehat{x}_5. \tag{5.3.3}$$

5.3.3 Proposition. The set $\mathfrak{F} = \{\widehat{x}_1{}^i\widehat{x}_2{}^j\widehat{x}_3{}^k\widehat{x}_4{}^\xi\widehat{x}_5{}^l\widehat{x}_6{}^m \mid (\xi, i, j, k, l, m) \in \{0,1\} \times \mathbb{N}^5\}$ is a $\mathbb{C}-$basis of $\mathcal{A}_\beta$.

Proof. Since $(\Pi_{s=1}^6 X_s^{i_s})_{i_s \in \mathbb{N}}$ is a basis of $\mathcal{A}$ over $\mathbb{C}$, we have that $(\Pi_{s=1}^6 \widehat{x}_s{}^{i_s})_{i_s \in \mathbb{N}}$ is a spanning set of $\mathcal{A}_\beta$ over $\mathbb{C}$. We want to show that $\mathfrak{F}$ is a spanning set of $\mathcal{A}_\beta$. It is sufficient to do that by showing that $\widehat{x}_1{}^{i_1}\widehat{x}_2{}^{i_2}\widehat{x}_3{}^{i_3}\widehat{x}_4{}^{i_4}\widehat{x}_5{}^{i_5}\widehat{x}_6{}^{i_6}$ can be written as a finite linear combination of the elements of $\mathfrak{F}$ over $\mathbb{C}$ for all $i_1, \cdots, i_6 \in \mathbb{N}$. We do this by an induction on i_4. The result is clear when $i_4 = 0$. For $i_4 \geq 0$, suppose that

$$\widehat{x}_1{}^{i_1}\widehat{x}_2{}^{i_2}\widehat{x}_3{}^{i_3}\widehat{x}_4{}^{i_4}\widehat{x}_5{}^{i_5}\widehat{x}_6{}^{i_6} = \sum_{(\xi,\underline{v}) \in I} a_{(\xi,\underline{v})}\widehat{x}_1{}^i\widehat{x}_2{}^j\widehat{x}_3{}^k\widehat{x}_4{}^\xi\widehat{x}_5{}^l\widehat{x}_6{}^m,$$

where $\underline{v} := (i, j, k, l, m) \in \mathbb{N}^5$, I is a finite subset of $\{0,1\} \times \mathbb{N}^5$, and $a_{(\xi,\underline{v})}$ are complex numbers. It follows that

$$\widehat{x}_1{}^{i_1}\widehat{x}_2{}^{i_2}\widehat{x}_3{}^{i_3}\widehat{x}_4{}^{i_4+1}\widehat{x}_5{}^{i_5}\widehat{x}_6{}^{i_6} = \sum_{(\xi,\underline{v}) \in I} a_{(\xi,\underline{v})}\widehat{x}_1{}^i\widehat{x}_2{}^j\widehat{x}_3{}^k\widehat{x}_4{}^{\xi+1}\widehat{x}_5{}^l\widehat{x}_6{}^m.$$

We have to show that $\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}^{\xi+1}\widehat{x_5}^l\widehat{x_6}^m \in \text{Span}(\mathfrak{F})$ for all $(\xi, \underline{v}) \in I$. The result is obvious when $\xi = 0$. For $\xi = 1$, then using (5.3.3), one can verify that $\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}^2\widehat{x_5}^l\widehat{x_6}^m \in \text{Span}(\mathfrak{F})$. Consequently, $\widehat{x_1}^{i_1}\widehat{x_2}^{i_2}\widehat{x_3}^{i_3}\widehat{x_4}^{i_4+1}\widehat{x_5}^{i_5}\widehat{x_6}^{i_6} \in \text{Span}(\mathfrak{F})$. Therefore, $\mathfrak{F}$ spans $\mathcal{A}_\beta$.

We proceed to show that $\mathfrak{F}$ is a linearly independent set. Note, the ordering $<_4$ in the proof of Proposition 3.2.1 (see item ♣) will be helpful in this part of the proof. Suppose that

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}^\xi\widehat{x_5}^l\widehat{x_6}^m = 0.$$

It follows that

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}X_1^iX_2^jX_3^kX_4^\xi X_5^lX_6^m = \nu(\Omega_2 - \beta),$$

where $\nu \in \mathcal{A}$. Write $\nu = \sum_{(i,\cdots,n)\in J} b_{(i,\cdots,n)}X_1^iX_2^jX_3^kX_4^lX_5^mX_6^n$, where J is a finite subset of $\mathbb{N}^6$, and $b_{(i,\cdots,n)}$ are complex numbers. From Subsection 5.2.1, we have that $\Omega_2 = X_2X_4X_6 - \frac{2}{3}X_3^3X_6 - \frac{2}{3}X_2X_5^3 + 2X_3^2X_5^2 - 3X_3X_4X_5 + \frac{3}{2}X_4^2$. It follows that:

$$\begin{aligned}\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}X_1^iX_2^jX_3^kX_4^\xi X_5^lX_6^m &= \sum_{(i,\cdots,n)\in J} b_{(i,\cdots,n)}X_1^iX_2^jX_3^kX_4^lX_5^mX_6^n(\Omega_2-\beta)\\ &= \sum_{(i,\cdots,n)\in J} \frac{3}{2}b_{(i,\cdots,n)}X_1^iX_2^jX_3^kX_4^{l+2}X_5^mX_6^n + \text{LT}_{<_4},\end{aligned}$$

where $\text{LT}_{<_4}$ contains lower order terms with respect to $<_4$ (see item ♣ in the proof of Proposition 3.2.1). Moreover, $\text{LT}_{<_4}$ vanishes when $b_{(i,\cdots,n)} = 0$ for all $(i,\cdots,n) \in J$. One can easily confirm this when the previous line of equality (right hand side) is fully expanded.

Suppose that there exists $(i,j,k,l,m,n) \in J$ such that $b_{(i,j,k,l,m,n)} \neq 0$. Let (i',j',k',l',m',n') be the greatest element of J with respect to $<_4$ such that $b_{(i',j',k',l',m',n')} \neq 0$. Note, the family $(X_1^iX_2^jX_3^kX_4^lX_5^mX_6^n)_{i,\cdots,n\in\mathbb{N}}$ is a basis for $\mathcal{A}$ and $\text{LT}_{<_4}$ contains lower order terms. Hence, identifying the coefficients of $X_1^{i'}X_2^{j'}X_3^{k'}X_4^{l'+2}X_5^{m'}X_6^{n'}$, we have $\frac{3}{2}b_{(i',j',k',l',m',n')} = 0$. Therefore, $b_{(i',j',k',l',m',n')} = 0$, a contradiction! As a result, $b_{(i,j,k,l,m,n)} = 0$ for all $(i,j,k,l,m,n) \in J$, and $\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}X_1^iX_2^jX_3^kX_4^\xi X_5^lX_6^m = 0$. Consequently, $a_{(\xi,i,j,k,l)} = 0$ for all $(\xi,i,j,k,l) \in I$. ∎

We are now ready to find a basis for $\mathcal{A}_{\alpha,\beta}$.

5.3.4 Proposition. The set $\mathfrak{P} = \{x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} x_5^k x_6^l \mid (\epsilon_1, \epsilon_2, i, j, k, l) \in \{0,1\}^2 \times \mathbb{N}^4\}$ is a $\mathbb{C}-$basis of $\mathcal{A}_{\alpha,\beta}$.

Proof. Since the set $\mathfrak{F} = \{\widehat{x_1}^{i_1}\widehat{x_2}^{i_2}\widehat{x_3}^{i_3}\widehat{x_4}^{\xi}\widehat{x_5}^{i_5}\widehat{x_6}^{i_6} \mid (i_1, i_2, i_3, \xi, i_5, i_6) \in \{0,1\} \times \mathbb{N}^4\}$ is a $\mathbb{C}-$basis of $\mathcal{A}_\beta$ over $\mathbb{C}$ (Proposition 5.3.3) and $\mathcal{A}_{\alpha,\beta}$ is identified with $\mathcal{A}_\beta/\langle\widehat{\Omega}_1 - \alpha\rangle$, it follows that $(x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{\xi}x_5^{i_5}x_6^{i_6})_{(i_1,i_2,i_3,\xi,i_5,i_6)\in\{0,1\}\times\mathbb{N}^5}$ is a spanning set of $\mathcal{A}_{\alpha,\beta}$ over $\mathbb{C}$. We want to show that $\mathfrak{P}$ spans $\mathcal{A}_{\alpha,\beta}$ by showing that $e_1^{i_1}e_2^{i_2}e_3^{i_3}e_4^{\xi}e_5^{i_5}e_6^{i_6}$ can be written as a finite linear combination of the elements of $\mathfrak{P}$ over $\mathbb{C}$ for all $(i_1, i_2, i_3, \xi, i_5, i_6) \in \{0,1\} \times \mathbb{N}^5$. By Proposition 5.3.3, it is sufficient to do this by an induction on i_3. The result is obvious when $i_3 = 0$ or 1. For $i_3 \geq 1$, suppose that

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{\xi}x_5^{i_5}x_6^{i_6} = \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} x_5^k x_6^l,$$

where $\underline{v} := (i,j,k,l) \in \mathbb{N}^4$, $a_{(\epsilon_1,\epsilon_2,\underline{v})}$ are complex numbers, and I is a finite subset of $\{0,1\}^2 \times \mathbb{N}^4$. It follows from the inductive hypothesis that

$$x_1^{i_1}x_2^{i_2}x_3^{i_3+1}x_4^{\xi}x_5^{i_5}x_6^{i_6} = \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} x_1^i x_2^j x_3^{\epsilon_1+1} x_4^{\epsilon_2} x_5^k x_6^l.$$

We need to show that $x_1^i x_2^j x_3^{\epsilon_1+1} x_4^{\epsilon_2} x_5^k x_6^l \in \text{Span}(\mathfrak{P})$ for all $(\epsilon_1, \epsilon_2, \underline{v}) \in I$. The result is obvious when $(\epsilon_1, \epsilon_2) = (0,0), (0,1)$. Using Lemma 5.3.1(1),(3), one can also show that $x_1^i x_2^j x_3^{\epsilon_1+1} x_4^{\epsilon_2} x_5^k x_6^l \in \text{Span}(\mathfrak{P})$ for all $(\epsilon_1, \epsilon_2) = (1,0), (1,1)$; and $i,j,k,l \in \mathbb{N}$. Therefore, $x_1^{i_1}x_2^{i_2}x_3^{i_3+1}x_4^{\xi}x_5^{i_5}x_6^{i_6} \in \text{Span}(\mathfrak{P})$. As a result, $\mathfrak{P}$ spans $\mathcal{A}_{\alpha,\beta}$.

We proceed to show that $\mathfrak{F}$ is a linearly independent set. Note, the ordering $<_3$ in the proof of Proposition 3.2.2 (see item ♠) will be helpful in this part of the proof. Suppose that

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} x_5^k x_6^l = 0.$$

Then, we have that

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^{\epsilon_1}\widehat{x_4}^{\epsilon_2}\widehat{e_5}^k\widehat{e_6}^l = (\widehat{\Omega}_1-\alpha)\nu$$

in $\mathcal{A}_\beta$, where $\nu \in \mathcal{A}_\beta$. Set $\underline{w} := (i,j,k,l,m) \in \mathbb{N}^5$, and let J_1 and J_2 be finite subsets of $\mathbb{N}^5$. One can write ν in terms of the basis $\mathfrak{F}$ of $\mathcal{A}_\beta$ as

$$\nu = \sum_{\underline{w}\in J_1} b_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}\widehat{x_5}^l\widehat{x_6}^m + \sum_{\underline{w}\in J_2} c_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_5}^l\widehat{x_6}^m,$$

where $b_{\underline{w}}$ and $c_{\underline{w}}$ are complex numbers. Note, $\widehat{\Omega}_1 = \widehat{x_1}\widehat{x_3}\widehat{x_5} - \frac{3}{2}\widehat{x_1}\widehat{x_4} - \frac{1}{2}\widehat{x_2}\widehat{x_5} + \frac{1}{2}\widehat{x_3}^2$. Given this expression, and the relation (5.3.3), one can express the above equality as follows:

$$\begin{aligned}
\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^{\epsilon_1}\widehat{x_4}^{\epsilon_2}\widehat{e_5}^k\widehat{e_6}^l &= \sum_{\underline{w}\in J_1} b_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}\widehat{x_5}^l\widehat{x_6}^m(\widehat{\Omega}_1-\alpha)\\
&\quad + \sum_{\underline{w}\in J_2} c_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_5}^l\widehat{x_6}^m(\widehat{\Omega}_1-\alpha)\\
&= \sum_{\underline{w}\in J_1} \frac{1}{2}b_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^{k+2}\widehat{x_4}\widehat{x_5}^l\widehat{x_6}^m\\
&\quad - \sum_{\underline{w}\in J_1} \frac{2}{3}b_{\underline{w}}\widehat{x_1}^{i+1}\widehat{x_2}^j\widehat{x_3}^{k+3}\widehat{x_5}^l\widehat{x_6}^{m+1}\\
&\quad - \sum_{\underline{w}\in J_2} \frac{3}{2}c_{\underline{w}}\widehat{x_1}^{i+1}\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}\widehat{x_5}^l\widehat{x_6}^m\\
&\quad + \sum_{\underline{w}\in J_2} \frac{1}{2}c_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^{k+2}\widehat{x_5}^l\widehat{x_6}^m + \Upsilon,
\end{aligned}$$

where Υ is defined on the next page.

$$\begin{aligned}\Upsilon = &\sum_{\underline{w}\in J_1} r_1 b_{\underline{w}} \widehat{x_1}^{i+1}\widehat{x_2}^{j}\widehat{x_3}^{k}\widehat{x_5}^{l}\widehat{x_6}^{m} + \sum_{\underline{w}\in J_1} r_2 b_{\underline{w}} \widehat{x_1}^{i+1}\widehat{x_2}^{1+j}\widehat{x_3}^{k}\widehat{x_4}\widehat{x_5}^{l}\widehat{x_6}^{m+1} \\ &+ \sum_{\underline{w}\in J_1} r_3 b_{\underline{w}} \widehat{x_1}^{i+1}\widehat{x_2}^{j+1}\widehat{x_3}^{k}\widehat{x_5}^{l+3}\widehat{x_6}^{m} + \sum_{\underline{w}\in J_1} r_4 b_{\underline{w}} \widehat{x_1}^{i+1}\widehat{x_2}^{j}\widehat{x_3}^{k+2}\widehat{x_5}^{l+2}\widehat{x_6}^{m} \\ &+ \sum_{\underline{w}\in J_1} r_5 b_{\underline{w}} \widehat{x_1}^{i+1}\widehat{x_2}^{j}\widehat{x_3}^{k+1}\widehat{x_4}\widehat{x_5}^{l+1}\widehat{x_6}^{m} + \sum_{\underline{w}\in J_1} r_6 b_{\underline{w}} \widehat{x_1}^{i}\widehat{x_2}^{j+1}\widehat{x_3}^{k}\widehat{x_4}\widehat{x_5}^{l+1}\widehat{x_6}^{m} \\ &+ \sum_{\underline{w}\in J_1} r_7 b_{\underline{w}} \beta\widehat{x_1}^{i}\widehat{x_2}^{j}\widehat{x_3}^{k}\widehat{x_4}\widehat{x_5}^{l}\widehat{x_6}^{m} + \sum_{\underline{w}\in J_1} r_8 b_{\underline{w}} \widehat{x_1}^{i+1}\widehat{x_2}^{j}\widehat{x_3}^{k+1}\widehat{x_4}\widehat{x_5}^{l+1}\widehat{x_6}^{m} \\ &+ \sum_{\underline{w}\in J_2} r_9 c_{\underline{w}} \widehat{x_1}^{i+1}\widehat{x_2}^{j}\widehat{x_3}^{k+1}\widehat{x_5}^{l+1}\widehat{x_6}^{m} + \sum_{\underline{w}\in J_2} r_{10} c_{\underline{w}} \widehat{x_1}^{i}\widehat{x_2}^{j+1}\widehat{x_3}^{k}\widehat{x_5}^{l+1}\widehat{x_6}^{m} \\ &+ \sum_{\underline{w}\in J_2} r_{11} c_{\underline{w}} \alpha\widehat{x_1}^{i}\widehat{x_2}^{j}\widehat{x_3}^{k}\widehat{x_5}^{l}\widehat{x_6}^{m}.\end{aligned}$$

Note: $r_1, \cdots, r_{11}$ are some non-zero rational numbers.

Observe that Υ contains lower order terms with respect to $<_3$ (defined in ♠, see the proof of Proposition 3.2.2) in each monomial type (note, there are two different types of monomials in the basis of $\mathcal{A}_\beta$; one with $\widehat{x_4}$ and the other without $\widehat{x_4}$). Now, suppose that there exists $(i,j,k,l,m) \in J_1$ and $(i,j,k,l,m) \in J_2$ such that $b_{(i,j,k,l,m)} \neq 0$ and $c_{(i,j,k,l,m)} \neq 0$. Let (v_1,v_2,v_3,v_5,v_6) and (w_1,w_2,w_3,w_5,w_6) be the greatest elements of J_1 and J_2 respectively with respect to $<_3$ such that $b_{(v_1,v_2,v_3,v_5,v_6)}$ and $c_{(w_1,w_2,w_3,w_5,w_6)}$ are non-zero. Since $\mathfrak{F}$ is a linear basis for $\mathcal{A}_\beta$ and Υ contains lower order terms with respect to $<_3$, we have the following: if $w_3 - v_3 < 2$, then identifying the coefficients of $\widehat{x_1}^{v_1}\widehat{x_2}^{v_2}\widehat{x_3}^{v_3+2}\widehat{x_4}\widehat{x_5}^{v_5}\widehat{x_6}^{v_6}$, we have $\frac{1}{2}b_{(v_1,v_2,v_3,v_5,v_6)} = 0$, a contradiction! Finally, if $w_3 - v_3 \geq 2$, then identifying the coefficients of $\widehat{x_1}^{w_1}\widehat{x_2}^{w_2}\widehat{x_3}^{w_3+2}\widehat{x_5}^{w_5}\widehat{x_6}^{w_6}$, we have $\frac{1}{2}c_{(w_1,w_2,w_3,w_5,w_6)} = 0$, another contradiction! This implies that either all $b_{(i,j,k,m,n)}$ or all $c_{(i,j,k,m,n)}$ are zero. Without loss of generality, suppose that there exists $(i,j,k,m,n) \in J_2$ such that $c_{(i,j,k,m,n)}$ is not zero. Then, $b_{(i,j,k,m,n)}$ are all zero. Let (w_1,w_2,w_3,w_5,w_6) be the greatest element of J_2 such that $c_{(w_1,w_2,w_3,w_5,w_6)} \neq 0$. Identifying the coefficients of $\widehat{x_1}^{w_1}\widehat{x_2}^{w_2}\widehat{x_3}^{w_3+2}\widehat{x_5}^{w_5}\widehat{x_6}^{w_6}$ in the above equality, we have that $\frac{1}{2}c_{(w_1,w_2,w_3,w_5,w_6)} = 0$, a contradiction! We can therefore conclude that $b_{(i,j,k,m,n)}$ and $c_{(i,j,k,m,n)}$ are all zero.

Consequently,

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^{\epsilon_1}\widehat{x_4}^{\epsilon_2}\widehat{x_5}^k\widehat{x_6}^l = 0.$$

Since $\mathfrak{F}$ is a basis for $\mathcal{A}_\beta$, it implies that $a_{(\epsilon_1,\epsilon_2,\underline{v})} = 0$ for all $(\epsilon_1, \epsilon_2, \underline{v}) \in I$. Therefore, $\mathfrak{P}$ is a linearly independent set. ■

5.3.5 Corollary. Let $\underline{v} = (i, j, k, l) \in \mathbb{N}^2 \times \mathbb{Z}^2$, I represent a finite subset of $\{0, 1\}^2 \times \mathbb{N}^2 \times \mathbb{Z}^2$, and $(a_{(\epsilon_1,\epsilon_2,\underline{v})})_{(\epsilon_1,\epsilon_2,\underline{v})\in I}$ be a family of complex numbers. If

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}x_1^ix_2^je_3^{\epsilon_1}x_4^{\epsilon_2}t_5^kt_6^l = 0,$$

then $a_{(\epsilon_1,\epsilon_2,\underline{v})} = 0$ for all $(\epsilon_1, \epsilon_2, \underline{v}) \in I$.

Proof. The result is obvious when $k, l \geq 0$ due to Proposition 5.3.4. When k (resp. l) is negative, then one can multiply the above equality enough times by t_5 (resp. t_6) to kill all the negative powers and then apply Proposition 5.3.4 to complete the proof. ■

Chapter 6

Poisson derivations of the semiclassical limits of $A_{\alpha,\beta}$

In this chapter, we compute the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$. We employ the same strategy used in computing the derivations of the non-commutative algebra $A_{\alpha,\beta}$ in Chapter 4. We begin by finding the Poisson derivations of a Poisson group algebra by following a pattern similar to [40]. We conclude that every Poisson derivation of the Poisson group algebra is the sum of an inner Poisson derivation and a scalar Poisson derivation. Since the Poisson torus is a Poisson group algebra, we have that every Poisson derivation of a Poisson torus is the sum of an inner and a central/scalar Poisson derivation. Knowing the Poisson derivations of the Poisson torus, we embed each of the Poisson algebra $\mathcal{A}_{\alpha,\beta}$ into a suitable Poisson torus, and then extend every Poisson derivation of $\mathcal{A}_{\alpha,\beta}$ uniquely to the Poisson torus. We then restrict the Poisson derivations of the Poisson torus back to $\mathcal{A}_{\alpha,\beta}$, and conclude that the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$ are all inner when α and β are non-zero. However, when either α or β is zero, we conclude that every Poisson derivation of $\mathcal{A}_{\alpha,\beta}$ is the sum of an inner and a scalar Poisson derivation. More precisely, the first Hochschild cohomology group of $\mathcal{A}_{\alpha,\beta}$ is of dimension 0 when α and β are non-zero and 1 when either α or β is zero. The results in this chapter are congruent to their non-commutative counterparts.

6.1 Poisson derivations of Poisson group algebras

Osborn and Passman have studied the derivations of the twisted group algebras [40, §1&2]. In this section, we produce the Poisson version of their results. That is, we will study the Poisson derivations of the Poisson group algebras. Where applicable, we will maintain their notations. Let G be a finitely generated abelian group and K be a field with characteristic zero. A Poisson group algebra $K_P^{\lambda}[G]$ is a commutative $K-$algebra which has a copy $\overline{G} := \{\bar{g} \mid g \in G\}$ of G as a basis and satisfies the Poisson bracket via $\{\bar{x}, \bar{y}\} = \lambda(x, y)\bar{x}\bar{y} = \lambda(x, y)\overline{xy}$ for all $x, y \in G$; and $\lambda : G \times G \longrightarrow K$ (note, $\bar{x}\bar{y} = \overline{xy}$). The map λ satisfies the following properties: $\lambda(y, x) = -\lambda(x, y)$ and $\lambda(x, yz) = \lambda(x, y) + \lambda(x, z)$. Obviously, $\lambda(x, y) = 0$ if and only if $\{\bar{x}, \bar{y}\} = 0$.

For example, take the additive group $G = \mathbb{Z}^2$, and let $\mathbb{C}[G]$ be a group algebra generated by $x^{\pm 1}, y^{\pm 1}$ over $\mathbb{C}$ with a basis $\overline{G} = \{x^i y^j \mid (i, j) \in G\}$. One can define a Poisson structure on $\mathbb{C}[G]$ via $\{x^i y^j, x^k y^l\} = \lambda((i, j), (k, l))x^{i+k}y^{j+l}$, where $\lambda((i, j), (k, l)) := il - jk$. In general, $K_P^{\lambda}[\mathbb{Z}^n]$ is a Poisson torus of rank n over the field K for some $\lambda : \mathbb{Z}^n \times \mathbb{Z}^n \to K$, where $\mathbb{Z}^n$ is the usual additive group.

For any $\gamma \in K_P^{\lambda}[G]$, one can write γ as $\gamma = \sum_g c_g \bar{g}$ with $g \in G$ and $c_g \in K$. Note, $c_g = 0$ for almost all $c_g \in K$. The set supp$(\gamma) := \{g \in G \mid c_g \neq 0 \text{ in } \gamma\}$ is called the support of γ. Furthermore, set $C := \{g \in G \mid \{\bar{g}, \bar{x}\} = 0 \text{ for all } x \in G\}$ and $\Delta(\bar{x}) := \{g \in G \mid \{\bar{g}, \bar{x}\} = 0\}$. One can observe that C and $\Delta(\bar{x})$ are both subgroups of G. If $g_1, \cdots, g_n$ are the generators of G, then $C = \bigcap_{i=1}^{n} \Delta(g_i)$.

6.1.1 Lemma. The Poisson center $Z_P(K_P^{\lambda}[G])$ of $K_P^{\lambda}[G]$ is $K_P^{\lambda}[C]$.

Proof. Clearly, $K_P^{\lambda}[C] \subseteq Z_P(K_P^{\lambda}[G])$. For the reverse inclusion, take $\gamma = \sum_g c_g \bar{g} \in Z_P(K_P^{\lambda}[G])$. It follows that $0 = \{\gamma, \bar{x}\} = \sum_g c_g\{\bar{g}, \bar{x}\} = \sum_g c_g \lambda(g, x)\bar{g}\bar{x}$ for any $x \in G$. Consequently, $\lambda(g, x) = 0$ for all $g \in$ supp(γ). This implies that supp$(\gamma) \subseteq C$, hence, $\gamma \in K_P^{\lambda}[C]$. ■

6.1.2 Remark. Let e be the identity element of G. One can easily observe that $Z_P(K_P^{\lambda}[G]) = K$ if and only if $C = \{e\}$.

Central and inner Poisson derivations. Let $\theta : (G,\cdot) \longrightarrow (K_P^\lambda[C],+)$ be a group homomorphism. That is, $\theta(xy) = \theta(x)+\theta(y)$ for all $x,y \in G$. Define the $K-$linear operator $\mathcal{D} := \mathcal{D}_\theta$ by $\mathcal{D}(\bar{x}) = \theta(x)\bar{x}$ for all $x \in G$. We claim that $\mathcal{D}$ is a Poisson derivation of $K_P^\lambda[G]$. To establish this claim, we need to show that $\mathcal{D}(\bar{x}\bar{y}) = \mathcal{D}(\bar{x})\bar{y} + \bar{x}\mathcal{D}(\bar{y})$ and $\mathcal{D}(\{\bar{x},\bar{y}\}) = \{\mathcal{D}(\bar{x}),\bar{y}\} + \{\bar{x},\mathcal{D}(\bar{y})\}$ for all $x,y \in G$. Now, $\mathcal{D}(\bar{x}\bar{y}) = \theta(xy)\overline{xy} = \theta(xy)\bar{x}\bar{y} = \theta(x)\bar{x}\bar{y} + \theta(y)\bar{x}\bar{y} = \mathcal{D}(\bar{x})\bar{y} + \bar{x}\mathcal{D}(\bar{y})$. Secondly, $\mathcal{D}(\{\bar{x},\bar{y}\}) = \lambda(x,y)\mathcal{D}(\bar{x}\bar{y}) = \theta(xy)\lambda(x,y)\bar{x}\bar{y} = \theta(xy)\{\bar{x},\bar{y}\} = [\theta(x)+\theta(y)]\{\bar{x},\bar{y}\} = \theta(x)\{\bar{x},\bar{y}\} + \theta(y)\{\bar{x},\bar{y}\} = \{\theta(x)\bar{x},\bar{y}\} + \{\bar{x},\theta(y)\bar{y}\} = \{\mathcal{D}(\bar{x}),\bar{y}\} + \{\bar{x},\mathcal{D}(\bar{y})\}$ (note: $\{\theta(x),\bar{y}\} = \{\bar{x},\theta(y)\} = 0$, since $\theta(x)$ and $\theta(y)$ are Poisson central elements). This establishes our claim. Call any derivation $\mathcal{D}$ in line with this construction a *central Poisson derivation*. If $Z_P(K_P^\lambda[G]) = K$, then call $\mathcal{D}$ a *scalar Poisson derivation*. Observe that $\mathcal{D}(\bar{x}) \in K_P^\lambda[Cx]$ for all $x \in G$.

Let $\gamma = \sum_g c_g\bar{g} \in K_P^\lambda[G]$, where $c_g \in K$, and $\text{ham}_\gamma := \{\gamma, -\}$. It is well known that $\text{ham}_\gamma : K_P^\lambda[G] \longrightarrow K_P^\lambda[G]$ is a derivation called the *hamiltonian derivation associated to* γ. Moreover, $\text{ham}_\gamma(\bar{x}) = \{\gamma,\bar{x}\} = \sum_g \lambda(g,x)c_g\bar{g}\bar{x} \in K_P^\lambda[Gx]$ for all $x \in G$. Observe that the elements of $C \cap \text{supp}(\gamma)$ do not have any effect on the map ham_γ. That is, $\text{ham}_\gamma = \text{ham}_{\gamma+\mu\bar{t}}$ for all $t \in C \cap \text{supp}(\gamma)$ and $\mu \in K$. As a result, one can always assume that $C \cap \text{supp}(\gamma) = \emptyset$. Therefore, $\text{ham}_\gamma(\bar{x}) \in K_P^\lambda[(G\setminus C)x]$ for all $x \in G$. Let us call the hamiltonian derivation ham_γ an *inner Poisson derivation*. We have the following theorem.

6.1.3 Theorem. *Every Poisson derivation of $K_P^\lambda[G]$ is uniquely the sum of an inner Poisson derivation and a central Poisson derivation.*

Proof. Let $\mathcal{D}$ be a Poisson derivation of $K_P^\lambda[G]$. Then, for $x \in G$, we have that $\mathcal{D}(\bar{x}) \in K_P^\lambda[G]$. Hence, $\mathcal{D}(\bar{x}) = \sum_{h\in G} b_h(x)\bar{h} = \sum_{h\in G} b_h(x)\bar{h}\bar{x}^{-1}\bar{x}$. Now, the map $G \to G$ with $h \mapsto hx^{-1}$ is bijective, and so

$$\mathcal{D}(\bar{x}) = \sum_g a_g(x)\bar{g}\bar{x},$$

where $g := hx^{-1}$ and $a_g(x) := b_{gx}(x)$. Note; $a_g : G \longrightarrow K$ and $a_g(x) = 0$ for almost all

$x \in G$.

Since $\mathcal{D}$ is a Poisson derivation, we have that $\mathcal{D}(\bar{x}\bar{y}) = \mathcal{D}(\bar{x})\bar{y} + \bar{x}\mathcal{D}(\bar{y})$ for all $x, y \in G$. As a result,

$$\sum_g a_g(xy)\bar{g}\bar{x}\bar{y} = \sum_g a_g(x)\bar{g}\bar{x}\bar{y} + \sum_g a_g(y)\bar{g}\bar{x}\bar{y} = \sum_g [a_g(x) + a_g(y)]\bar{g}\bar{x}\bar{y}.$$

Identifying the coefficients in the above equality reveals that

$$a_g(xy) = a_g(x) + a_g(y).$$

Secondly, $\mathcal{D}(\{\bar{x}, \bar{y}\}) = \{\mathcal{D}(\bar{x}), \bar{y}\} + \{\bar{x}, \mathcal{D}(\bar{y})\}$. Now,

$$\mathcal{D}(\{\bar{x}, \bar{y}\}) = \lambda(x, y)\mathcal{D}(\bar{x}\bar{y}) = \sum_g \lambda(x, y)a_g(xy)\bar{g}\bar{x}\bar{y}. \tag{6.1.1}$$

On the other hand,

$$\begin{aligned}
\{\mathcal{D}(\bar{x}), \bar{y}\} + \{\bar{x}, \mathcal{D}(\bar{y})\} &= \sum_g a_g(x)\{\bar{g}\bar{x}, \bar{y}\} + \sum_g a_g(y)\{\bar{x}, \bar{g}\bar{y}\} \\
&= \sum_g a_g(x)\{\overline{gx}, \bar{y}\} + \sum_g a_g(y)\{\bar{x}, \overline{gy}\} \\
&= \sum_g [a_g(x)\lambda(gx, y) + a_g(y)\lambda(x, gy)]\bar{g}\bar{x}\bar{y} \\
&= \sum_g a_g(x)[\lambda(g, y) + \lambda(x, y)] + a_g(y)[\lambda(x, g) + \lambda(x, y)]\bar{g}\bar{x}\bar{y} \\
&= \sum_g [\lambda(x, y)a_g(xy) + a_g(x)\lambda(g, y) - a_g(y)\lambda(g, x)]\bar{g}\bar{x}\bar{y}.
\end{aligned} \tag{6.1.2}$$

Since $\mathcal{D}(\{\bar{x}, \bar{y}\} = \{\mathcal{D}(\bar{x}), \bar{y}\} + \{\bar{x}, \mathcal{D}(\bar{y})\}$, comparing (6.1.1) to (6.1.2) reveals that

$$\lambda(x, y)a_g(xy) = \lambda(x, y)a_g(xy) + a_g(x)\lambda(g, y) - a_g(y)\lambda(g, x).$$

This implies that

$$a_g(x)\lambda(g,y) = a_g(y)\lambda(g,x). \tag{6.1.3}$$

Suppose that $g \in C$, it follows that $\lambda(g,y) = \lambda(g,x) = 0$ for all $x, y \in G$. Since $a_g(xy) = a_g(x)+a_g(y)$, the map $\theta : (G,\cdot) \longrightarrow (K_P^\lambda[C],+)$ given by $\theta(x) = \sum_{g\in C} a_g(x)\bar{g}$ is a group homomorphism. Hence, θ defines a central Poisson derivation $\mathcal{D}_\theta$ of $K_P^\lambda[G]$, where

$$\mathcal{D}_\theta(\bar{x}) = \sum_{g\in C} a_g(x)\bar{g}\bar{x}. \tag{6.1.4}$$

Now, let $g \notin C$. There exists $y \in G$ such that $\lambda(g,y) \neq 0$. Fix y and define

$$c_g := \frac{a_g(y)}{\lambda(g,y)}.$$

Take any arbitrary element $x \in G$. It follows that

$$c_g\lambda(g,x) = \frac{a_g(y)\lambda(g,x)}{\lambda(g,y)}.$$

From (6.1.3), we have that

$$c_g\lambda(g,x) = \frac{a_g(y)\lambda(g,x)}{\lambda(g,y)} = \frac{a_g(x)\lambda(g,y)}{\lambda(g,y)} = a_g(x),$$

for all $x \in G$.

Define $\gamma \in K_P^\lambda[G]$ as $\gamma := \sum_{g\notin C} c_g\bar{g}$. Then,

$$\text{ham}_\gamma(\bar{x}) = \{\gamma, \bar{x}\} = \sum_{g\notin C} c_g\lambda(g,x)\bar{g}\bar{x} = \sum_{g\notin C} a_g(x)\bar{g}\bar{x}. \tag{6.1.5}$$

From (6.1.4) and (6.1.5), one can conclude that every Poisson derivation $\mathcal{D}$ of $K_P^\lambda[G]$ can be written as $\mathcal{D} = \mathcal{D}_\theta + \text{ham}_\gamma$. This decomposition of $\mathcal{D}$ into an inner Poisson derivation (ham_γ) and a central Poisson derivation ($\mathcal{D}_\theta$) is actually unique. This is because $K_P^\lambda[Gx]$ can be decomposed as $K_P^\lambda[Gx] = K_P^\lambda[Cx] \oplus K_P^\lambda[(G \setminus C)x]$. Now, every central Poisson derivation maps $\bar{x}$ to an element of the subspace $K_P^\lambda[Cx]$, and every inner Poisson derivation maps $\bar{x}$ to an element of the subspace $K_P^\lambda[(G \setminus C)x]$. ■

6.1.4 Corollary. Suppose that $C = \{e\}$ (equivalently, $Z_P(K_P^\lambda[G]) = K$). Then, every Poisson derivation of $K_P^\lambda[G]$ is uniquely the sum of an inner and a scalar Poisson derivation.

6.2 Preliminaries on the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$

Let $2 \leq j \leq 7$ and $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Set

$$\mathcal{A}_{\alpha,\beta}^{(j)} := \frac{\mathcal{A}^{(j)}}{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle},$$

where $\mathcal{A}^{(j)}$ is defined in Subsection 5.2.1, and Ω_1 and Ω_2 are the generators of the center of $\mathcal{A}^{(j)}$ (Subsection 5.2.1). Recall that $\mathcal{A}^{(7)} = \mathcal{A} = \mathbb{C}[X_1, \cdots, X_6]$. It follows that $\mathcal{A}_{\alpha,\beta}^{(7)} = \mathcal{A}_{\alpha,\beta}$. For each $2 \leq j \leq 7$, denote the canonical images of the generators $X_{i,j}$ of $\mathcal{A}^{(j)}$ in $\mathcal{A}_{\alpha,\beta}^{(j)}$ by $x_{i,j}$ for all $1 \leq i \leq 6$. Since the data of the PDDA of $\mathcal{A}_{\alpha,\beta}$ is going to be useful in this section, we present them below (note, we deduce them from that of $\mathcal{A}$ in Subsection 5.2.1):

$$\begin{aligned}
x_{1,6} &= x_1 - \frac{1}{2}x_5x_6^{-1}\\
x_{2,6} &= x_2 + \frac{3}{2}x_4x_6^{-1} - 3x_3x_5x_6^{-1} + x_5^3x_6^{-2}\\
x_{3,6} &= x_3 - x_5^2x_6^{-1}\\
x_{4,6} &= x_4 - \frac{2}{3}x_5^3x_6^{-1}\\
x_{1,5} &= x_{1,6} - x_{3,6}x_{5,6}^{-1} + \frac{3}{4}x_{4,6}x_{5,6}^{-2}\\
x_{2,5} &= x_{2,6} - 3x_{3,6}^2x_{5,6}^{-1} + \frac{9}{2}x_{3,6}x_{4,6}x_{5,6}^{-2} - \frac{9}{4}x_{4,6}^2x_{5,6}^{-3}\\
x_{3,5} &= x_{3,6} - \frac{3}{2}x_{4,6}x_{5,6}^{-1}
\end{aligned}$$

$$x_{1,4} = x_{1,5} - \frac{1}{3}x_{3,5}^2 x_{4,5}^{-1}$$
$$x_{2,4} = x_{2,5} - \frac{2}{3}x_{3,5}^3 x_{4,5}^{-1}$$
$$x_{1,3} = x_{1,4} - \frac{1}{2}x_{2,4} x_{3,4}^{-1}$$
$$t_1 := x_{1,2} = x_{1,3}$$
$$t_2 := x_{2,2} = x_{2,3} = x_{2,4}$$
$$t_3 := x_{3,2} = x_{3,3} = x_{3,4} = x_{3,5}$$
$$t_4 := x_{4,2} = x_{4,3} = x_{4,4} = x_{4,5} = x_{4,6}$$
$$t_5 := x_{5,2} = x_{5,3} = x_{5,4} = x_{5,5} = x_{5,6} = x_5$$
$$t_6 := x_{6,2} = x_{6,3} = x_{6,4} = x_{6,5} = x_{6,6} = x_6.$$

Note, the t_i are the canonical images of T_i in $\mathcal{A}_{\alpha,\beta}^{(2)}$ for all $1 \leq i \leq 6$. For each $2 \leq j < 7$, define $S_j := \{\lambda t_j^{i_j} t_{j+1}^{i_{j+1}} \cdots t_6^{i_6} \mid i_j, \cdots, i_6 \in \mathbb{N},\ \lambda \in \mathbb{C}^*\}$. One can observe that S_j is a multiplicative system of non-zero divisors (or regular elements) of $\mathcal{A}_{\alpha,\beta}^{(j)}$. As a result, one can localize $\mathcal{A}_{\alpha,\beta}^{(j)}$ at S_j. Let us denote this localization by $\mathcal{R}_j$. That is,

$$\mathcal{R}_j := \mathcal{A}_{\alpha,\beta}^{(j)} S_j^{-1}.$$

Again, set $\Sigma_j := \{t_j^k \mid k \in \mathbb{N}\}$, with $2 \leq j \leq 6$. Then, Σ_j is a multiplicative set in both $\mathcal{A}_{\alpha,\beta}^{(j)}$ and $\mathcal{A}_{\alpha,\beta}^{(j+1)}$. Therefore,

$$\mathcal{A}_{\alpha,\beta}^{(j)} \Sigma_j^{-1} = \mathcal{A}_{\alpha,\beta}^{(j+1)} \Sigma_j^{-1}.$$

One can also verify that (similar to (2.2.1)):

$$\mathcal{R}_j = \mathcal{R}_{j+1} \Sigma_j^{-1}, \tag{6.2.1}$$

for all $2 \leq j \leq 6$, with the convention that $\mathcal{R}_7 := \mathcal{A}_{\alpha,\beta}$.

Similar to (4.1.2), we have the following tower of algebras:

$$\mathcal{R}_7 = \mathcal{A}_{\alpha,\beta} \subset \mathcal{R}_6 = \mathcal{R}_7\Sigma_6^{-1} \subset \mathcal{R}_5 = \mathcal{R}_6\Sigma_5^{-1} \subset \mathcal{R}_4 = \mathcal{R}_5\Sigma_4^{-1} \subset \mathcal{R}_3. \tag{6.2.2}$$

Observe that $\mathcal{R}_3 = \mathcal{A}_{\alpha,\beta}^{(3)}S_3^{-1} = \mathcal{R}_4\Sigma_3^{-1}$ is the Poisson torus $\mathscr{P}_{\alpha,\beta} = \mathbb{C}[t_3^{\pm1}, t_4^{\pm1}, t_5^{\pm1}, t_6^{\pm1}]$ in Section 5.3.

Linear bases for $\mathcal{R}_3, \mathcal{R}_4$ and $\mathcal{R}_5$.

We aim to find a basis for $\mathcal{R}_j$ for each $j = 3, 4, 5$. Since $\mathcal{R}_3$ is a Poisson torus generated by $t_3^{\pm1}, \cdots, t_6^{\pm1}$ over $\mathbb{C}$, the set $\{t_3^i t_4^j t_5^k t_6^l \mid i, j, k, l \in \mathbb{Z}\}$ is a basis of $\mathcal{R}_3$.

For simplicity, we set

$$\begin{aligned} f_1 &:= x_{1,4} & F_1 &:= X_{1,4} \\ z_1 &:= x_{1,5} & \mathcal{Z}_1 &:= X_{1,5} \\ z_2 &:= x_{2,5} & \mathcal{Z}_2 &:= X_{2,5}. \end{aligned}$$

Basis for $\mathcal{R}_4$. Observe that

$$\mathcal{A}_{\alpha,\beta}^{(4)} = \frac{\mathcal{A}^{(4)}}{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle},$$

where $\Omega_1 = F_1T_3T_5 - \frac{1}{2}T_2T_5$ and $\Omega_2 = T_2T_4T_6$ in $\mathcal{A}^{(4)}$ (Subsection 5.2.1). Set

$$\mathcal{A}_{\beta}^{(4)}S_4^{-1} := \frac{\mathcal{A}^{(4)}S_4^{-1}}{\langle \Omega_2 - \beta \rangle},$$

where $\beta \in \mathbb{C}$. We will denote the canonical images of $X_{i,4}$ (resp. T_i) in $\mathcal{A}_{\beta}^{(4)}$ by $\widehat{x_{i,4}}$ (resp. $\widehat{t_i}$) for all $1 \leq i \leq 6$. Observe that $\widehat{t_2} = \beta\widehat{t_6}^{-1}\widehat{t_4}^{-1}$ in $\mathcal{A}_{\beta}^{(4)}S_4^{-1}$. As usual, one can identify $\mathcal{R}_4$ with $\mathcal{A}_{\beta}^{(4)}S_4^{-1}/\langle \Omega_1 - \alpha \rangle$.

The proofs of the following two propositions are similar to that of Propositions 4.1.2 and 4.1.4, nonetheless, we will still prove the results.

6.2.1 Proposition. The set $\mathfrak{P}_4 = \{f_1^{i_1}t_4^{i_4}t_5^{i_5}t_6^{i_6}, t_3^{i_3}t_4^{i_4}t_5^{i_5}t_6^{i_6} \mid (i_1, i_3, i_4, i_5, i_6) \in \mathbb{N}^2 \times \mathbb{Z}^3\}$ is a $\mathbb{C}-$basis of $\mathcal{R}_4$.

Proof. One can verify that $\left(\widehat{f_1}^{k_1}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}\right)_{(k_1,k_3,\cdots,k_6)\in\mathbb{N}^2\times\mathbb{Z}^3}$ is a basis of $\mathcal{A}_\beta^{(4)}S_4^{-1}$ (the proof is similar to that of Proposition 4.1.1). Since $\mathcal{A}_\beta^{(4)}S_4^{-1} = \mathcal{A}^{(4)}S_4^{-1}/\langle\Omega_2 - \beta\rangle$, the family $(f_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6})_{(k_1,k_3,\cdots,k_6)\in\mathbb{N}^2\times\mathbb{Z}^3}$ spans $\mathcal{R}_4$. We show that $\mathfrak{P}_4$ is a spanning set of $\mathcal{R}_4$ by showing that $f_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6}$ can be written as a finite linear combination of the elements of $\mathfrak{P}_4$ for all $(k_1, k_3, \cdots, k_6) \in \mathbb{N}^2 \times \mathbb{Z}^3$. It is sufficient to do this by an induction on k_1. The result is clear when $k_1 = 0$. Assume that the statement is true for $k_1 \geq 0$. That is,

$$f_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6} = \sum_{\underline{i}\in I_1} a_{\underline{i}}f_1^{i_1}t_4^{i_4}t_5^{i_5}t_6^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}}t_3^{i_3}t_4^{i_4}t_5^{i_5}t_6^{i_6},$$

where $\underline{i} = (i_1, i_4, i_5, i_6) \in I_1 \subset \mathbb{N} \times \mathbb{Z}^3$ and $\underline{j} = (i_3, i_4, i_5, i_6) \in I_2 \subset \mathbb{N} \times \mathbb{Z}^3$. Note, $a_{\underline{i}}$ and $b_{\underline{j}}$ are all complex numbers.

$$f_1^{k_1+1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6} = f_1(f_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6}) = \sum_{\underline{i}\in I_1} a_{\underline{i}}f_1^{i_1+1}t_4^{i_4}t_5^{i_5}t_6^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}}f_1t_3^{i_3}t_4^{i_4}t_5^{i_5}t_6^{i_6}.$$

Clearly, the monomial $f_1^{i_1+1}t_4^{i_4}t_5^{i_5}t_6^{i_6} \in \text{Span}(\mathfrak{P}_4)$. We have to also show that $f_1t_3^{i_3}t_4^{i_4}t_5^{i_5}t_6^{i_6} \in \text{Span}(\mathfrak{P}_4)$ for all $i_3 \in \mathbb{N}$ and $i_4, i_5, i_6 \in \mathbb{Z}$. This can easily be achieved by induction on i_3, and the use of the relation $f_1t_3 = \alpha t_5^{-1} + \frac{1}{2}\beta t_4^{-1}t_6^{-1}$. Therefore, by the principle of mathematical induction, $\mathfrak{P}_4$ is a spanning set of $\mathcal{R}_4$ over $\mathbb{C}$.

We prove that $\mathfrak{P}_4$ is a linearly independent set. Suppose that

$$\sum_{\underline{i}\in I_1} a_{\underline{i}}f_1^{i_1}t_4^{i_4}t_5^{i_5}t_6^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}}t_3^{i_3}t_4^{i_4}t_5^{i_5}t_6^{i_6} = 0.$$

It follows that there exists $\nu \in \mathcal{A}_\beta^{(4)}S_4^{-1}$ such that

$$\sum_{\underline{i}\in I_1} a_{\underline{i}}\widehat{f_1}^{i_1}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} = \left(\widehat{\Omega}_1 - \alpha\right)\nu.$$

Write $\nu = \sum_{\underline{l}\in J} c_{\underline{l}} \widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}$, with $\underline{l} = (i_1, i_3, i_4, i_5, i_6) \in J \subset \mathbb{N}^2 \times \mathbb{Z}^3$, and $c_{\underline{l}} \in \mathbb{C}$. One can easily deduce that $\widehat{\Omega}_1 = \widehat{f_1}\widehat{t_3}\widehat{t_5} - \frac{1}{2}\widehat{t_2}\widehat{t_5} = \widehat{f_1}\widehat{t_3}\widehat{t_5} - \frac{1}{2}\beta\widehat{t_6}^{-1}\widehat{t_4}^{-1}\widehat{t_5}$ (note, $\widehat{t_2} = \beta\widehat{t_6}^{-1}\widehat{t_4}^{-1}$). It follows that

$$\begin{aligned}\sum_{\underline{i}\in I_1} a_{\underline{i}} \widehat{f_1}^{i_1}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} \widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} &= \sum_{\underline{l}\in J} c_{\underline{l}} \widehat{f_1}^{i_1+1}\widehat{t_3}^{i_3+1}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5+1}\widehat{t_6}^{i_6} \\ &\quad - \sum_{\underline{l}\in J} \frac{1}{2}\beta c_{\underline{l}} \widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4-1}\widehat{t_5}^{i_5+1}\widehat{t_6}^{i_6-1} \\ &\quad - \sum_{\underline{l}\in J} \alpha c_{\underline{l}} \widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}.\end{aligned}$$

Suppose that there exists $(i_1, i_3, i_4, i_5, i_6) \in J$ such that $c_{(i_1,i_3,i_4,i_5,i_6)} \neq 0$.

Let $(w_1, w_3, w_4, w_5, w_6) \in J$ be the greatest element (in the lexicographic order on $\mathbb{N}^2 \times \mathbb{Z}^3$) of J such that $c_{(w_1,w_3,w_4,w_5,w_6)} \neq 0$. Since $\left(\widehat{f_1}^{k_1}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}\right)_{(k_1,k_3,\cdots,k_6)\in\mathbb{N}^2\times\mathbb{Z}^3}$ is a basis of $\mathcal{A}^{(4)}S_4^{-1}$, it implies that the coefficients of $\widehat{f_1}^{w_1+1}\widehat{t_3}^{w_3+1}\widehat{t_4}^{w_4}\widehat{t_5}^{w_5+1}\widehat{t_6}^{w_6}$ in the above equality can be identified as: $c_{(w_1,w_3,w_4,w_5,w_6)} = 0$. Hence, $c_{(w_1,w_3,w_4,w_5,w_6)} = 0$, a contradiction! Therefore, $c_{(i_1,i_3,i_4,i_5,i_6)} = 0$ for all $(i_1, i_3, i_4, i_5, i_6) \in J$. This further implies that

$$\sum_{\underline{i}\in I_1} a_{\underline{i}} \widehat{f_1}^{i_1}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} \widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} = 0.$$

Consequently, $a_{\underline{i}}$ and $b_{\underline{j}}$ are all zero. In conclusion, $\mathfrak{P}_4$ is a linearly independent set. ∎

Basis for $\mathcal{R}_5$. We will identify $\mathcal{R}_5$ with $\mathcal{A}_\alpha^{(5)}S_5^{-1}/\langle\widehat{\Omega}_2 - \beta\rangle$, where $\mathcal{A}_\alpha^{(5)}S_5^{-1} = \mathcal{A}^{(5)}S_5^{-1}/\langle\Omega_1 - \alpha\rangle$. Note, the canonical images of $X_{i,5}$ (resp. T_i) in $\mathcal{A}_\alpha^{(5)}$ will be denoted by $\widehat{x_{i,5}}$ (resp. $\widehat{t_i}$) for all $1 \leq i \leq 6$. We now find a basis for $\mathcal{A}_\alpha^{(5)}S_5^{-1}$. Recall from Subsection 5.2.1 that $\Omega_1 = Z_1T_3T_5 - \frac{1}{2}Z_2T_5$ and $\Omega_2 = Z_2T_4T_6 - \frac{2}{3}T_3^3T_6$ in $\mathcal{A}^{(5)}$ (remember, $Z_1 := X_{1,5}$ and $Z_2 := X_{2,5}$). Since $z_2t_4t_6 - \frac{2}{3}t_3^3t_6 = \beta$ and $\widehat{z_1}\widehat{t_3}\widehat{t_5} - \frac{1}{2}\widehat{z_2}\widehat{t_5} = \alpha$ in $\mathcal{R}_5$ and $\mathcal{A}_\alpha^{(5)}S_5^{-1}$ respectively, we have the relation $\widehat{z_2} = 2\left(\widehat{z_1}\widehat{t_3} - \alpha\widehat{t_5}^{-1}\right)$ in $A_\alpha^{(5)}S_5^{-1}$ and, in R_5, we have the following two relations:

$$z_2 = 2(z_1t_3 - \alpha t_5^{-1}). \tag{6.2.3}$$

$$t_3^3 = \frac{3}{2}(z_2t_4 - \beta t_6^{-1}) = 3z_1t_3t_4 - \frac{3}{2}\beta t_6^{-1} - 3\alpha t_4t_5^{-1}. \tag{6.2.4}$$

6.2.2 Proposition. The set $\mathfrak{P}_5 = \left\{ z_1^{i_1}t_3^{\xi}t_4^{i_4}t_5^{i_5}t_6^{i_6} \mid (\xi, i_1, i_4, i_5, i_6) \in \{0,1,2\} \times \mathbb{N}^2 \times \mathbb{Z}^2 \right\}$ is a basis of $\mathcal{R}_5$.

Proof. One can easily show that the family $\left(\widehat{z_1}^{k_1}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}\right)_{(k_1,k_3,k_4,k_5,k_6)\in\mathbb{N}^2\times\mathbb{Z}^2}$ is a basis of $\mathcal{A}_\alpha^{(5)}S_5^{-1}/\langle\widehat{\Omega}_2 - \beta\rangle$ (the proof is similar to that of Proposition 4.1.3). Since $\mathcal{R}_5$ is identified with $\mathcal{A}_\alpha^{(5)}S_5^{-1}/\langle\widehat{\Omega}_2 - \beta\rangle$, we show that $z_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6}$ can be written as a finite linear combination of the elements of $\mathfrak{P}_5$ for all $(k_1, k_3, k_4, k_5, k_6) \in \mathbb{N}^3 \times \mathbb{Z}^2$. It is sufficient to do this by an induction on k_3. The result is obvious when $k_3 = 0, 1$ or 2. For $k_3 \geq 2$, suppose that

$$z_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6} = \sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})} z_1^{i_1}t_3^{\xi}t_4^{i_4}t_5^{i_5}t_6^{i_6},$$

where I is a finite subset of $\{0,1,2\} \times \mathbb{N}^2 \times \mathbb{Z}^2$, and $a_{(\xi,\underline{i})}$ are all complex numbers. It follows that

$$z_1^{k_1}t_3^{k_3+1}t_4^{k_4}t_5^{k_5}t_6^{k_6} = \left(z_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6}\right)t_3 = \sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})} z_1^{i_1}t_3^{\xi+1}t_4^{i_4}t_5^{i_5}t_6^{i_6}.$$

Now, $z_1^{i_1}t_3^{\xi+1}t_4^{i_4}t_5^{i_5}t_6^{i_6} \in \text{Span}(\mathfrak{P}_5)$ when $\xi = 0, 1$. For $\xi = 2$, one can easily verify that $z_1^{i_1}t_3^{3}t_4^{i_4}t_5^{i_5}t_6^{i_6} \in \text{Span}(\mathfrak{P}_5)$ by using the relation in (6.2.4). Therefore, by the principle of mathematical induction, $\mathfrak{P}_5$ spans $\mathcal{R}_5$.

We now prove that $\mathfrak{P}_5$ is a linearly independent set. Suppose that

$$\sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})} z_1^{i_1}t_3^{\xi}t_4^{i_4}t_5^{i_5}t_6^{i_6} = 0.$$

Since $\mathcal{R}_5$ is identified with $\mathcal{A}_\alpha^{(5)}S_5^{-1}/\langle\widehat{\Omega}_2-\beta\rangle$, we have that

$$\sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})}\widehat{z}_1^{\,i_1}\widehat{t}_3^{\,\xi}\widehat{t}_4^{\,i_4}\widehat{t}_5^{\,i_5}\widehat{t}_6^{\,i_6}=\langle\widehat{\Omega}_2-\beta\rangle\nu,$$

where $\nu\in\mathcal{A}_\alpha^{(5)}S_5^{-1}$. Write $\nu=\sum_{\underline{j}\in J} b_{\underline{j}}\widehat{z}_1^{\,i_1}\widehat{t}_3^{\,i_3}\widehat{t}_4^{\,i_4}\widehat{t}_5^{\,i_5}\widehat{t}_6^{\,i_6}$, with $\underline{j}=(i_1,i_3,i_4,i_5,i_6)\in J\subset\mathbb{N}^3\times\mathbb{Z}^2$ and $b_{\underline{j}}\in\mathbb{C}$. Given that $\Omega_2=\not{Z}_2T_4T_6-\frac{2}{3}T_3^3T_6$ in $\mathcal{A}^{(5)}$ and the relation (6.2.3), one can deduce that

$$\widehat{\Omega}_2=\widehat{z}_2\widehat{t}_4\widehat{t}_6-\frac{2}{3}\widehat{t}_3^{\,3}\widehat{t}_6=2\widehat{z}_1\widehat{t}_3\widehat{t}_4\widehat{t}_6-2\alpha\widehat{t}_4\widehat{t}_5^{\,-1}\widehat{t}_6-\frac{2}{3}\widehat{t}_3^{\,3}\widehat{t}_6.$$

Therefore,

$$\begin{aligned}\sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})}\widehat{z}_1^{\,i_1}\widehat{t}_3^{\,\xi}\widehat{t}_4^{\,i_4}\widehat{t}_5^{\,i_5}\widehat{t}_6^{\,i_6} =&\sum_{\underline{j}\in J} 2b_{\underline{j}}\widehat{z}_1^{\,i_1+1}\widehat{t}_3^{\,i_3+1}\widehat{t}_4^{\,i_4+1}\widehat{t}_5^{\,i_5}\widehat{t}_6^{\,i_6+1}\\ &-\sum_{\underline{j}\in J}\frac{2}{3}b_{\underline{j}}\widehat{z}_1^{\,i_1}\widehat{t}_3^{\,i_3+3}\widehat{t}_4^{\,i_4}\widehat{t}_5^{\,i_5}\widehat{t}_6^{\,i_6+1}\\ &-\sum_{\underline{j}\in J}2\alpha b_{\underline{j}}\widehat{z}_1^{\,i_1}\widehat{t}_3^{\,i_3}\widehat{t}_4^{\,i_4+1}\widehat{t}_5^{\,i_5-1}\widehat{t}_6^{\,i_6+1}\\ &-\sum_{\underline{j}\in J}\beta b_{\underline{j}}\widehat{z}_1^{\,i_1}\widehat{t}_3^{\,i_3}\widehat{t}_4^{\,i_4}\widehat{t}_5^{\,i_5}\widehat{t}_6^{\,i_6}.\end{aligned}$$

Suppose that there exists $(i_1,i_3,i_4,i_5,i_6)\in J$ such that $b_{(i_1,i_3,i_4,i_5,i_6)}\neq 0$.

Let $(w_1,w_3,w_4,w_5,w_6)\in J$ be the greatest element (in the lexicographic order on $\mathbb{N}^3\times\mathbb{Z}^2$) of J such that $b_{(w_1,w_3,w_4,w_5,w_6)}\neq 0$. Given that $\left(\widehat{z}_1^{\,k_1}\widehat{t}_3^{\,k_3}\widehat{t}_4^{\,k_4}\widehat{t}_5^{\,k_5}\widehat{t}_6^{\,k_6}\right)_{(k_1,k_3,\cdots,k_6)\in\mathbb{N}^3\times\mathbb{Z}^2}$ is a basis of $\mathcal{A}_\alpha^{(5)}S_5^{-1}$, one can identify the coefficients of $\widehat{z}_1^{\,w_1+1}\widehat{t}_3^{\,w_3+1}\widehat{t}_4^{\,w_4+1}\widehat{t}_5^{\,w_5}\widehat{t}_6^{\,w_6+1}$ in the above equality as: $2b_{(w_1,w_3,w_4,w_5,w_6)}=0$. Hence, $b_{(w_1,w_3,w_4,w_5,w_6)}=0$, a contradiction! Therefore, $b_{(i_1,i_3,i_4,i_5,i_6)}=0$ for all $(i_1,i_3,i_4,i_5,i_6)\in J$. Consequently,

$$\sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})}\widehat{z}_1^{\,i_1}\widehat{t}_3^{\,\xi}\widehat{t}_4^{\,i_4}\widehat{t}_5^{\,i_5}\widehat{t}_6^{\,i_6}=0.$$

It follows that $a_{(\xi,\underline{i})}=0$ for all $(\xi,\underline{i})\in I$. As a result, $\mathfrak{P}_5$ is a linearly independent set. ■

6.2.3 Corollary. Let I be a finite subset of $\{0,1,2\}\times\mathbb{N}\times\mathbb{Z}^3$ and $(a_{(\xi,\underline{i})})_{\underline{i}\in I}$ be a family of complex numbers. If

$$\sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})} z_1^{i_1} t_3^{\xi} t_4^{i_4} t_5^{i_5} t_6^{i_6} = 0,$$

then $a_{(\xi,\underline{i})} = 0$ for all $(\xi,\underline{i})\in I$.

Proof. When $i_4 \geq 0$, then the result is obvious as a result of Proposition 6.2.2. For $i_4 < 0$, multiply both sides of the equality enough times by t_4 to kill all the negative powers of t_4, and then apply Proposition 6.2.2 to complete the proof. ■

6.2.4 Remark. We were not successful in finding a basis for $\mathcal{R}_6$. However, this has no effect on our main result in this chapter. Since $\mathcal{R}_7 = \mathcal{A}_{\alpha,\beta}$, we already have a basis for $\mathcal{R}_7$ (Proposition 5.3.4).

6.2.5 Lemma. $Z_P(\mathcal{R}_i) = \mathbb{C}$ for each $3 \leq i \leq 7$.

Proof. Similar to that of Lemma 4.1.7. ■

6.2.6 Remark. Recall the notations:

$$f_1 := x_{1,4} \qquad F_1 := X_{1,4}$$
$$z_1 := x_{1,5} \qquad Z_1 := X_{1,5}$$
$$z_2 := x_{2,5} \qquad Z_2 := X_{2,5}.$$

Let $(\alpha,\beta)\in\mathbb{C}^2\setminus\{(0,0)\}$. Given the above notations, we present the following selected data of the PDDA of $\mathcal{A}_{\alpha,\beta}$, listed at the beginning of this section, in a manner that would be very useful in Subsections 6.3.1. They are as follows:

$$f_1 = t_1 + \frac{1}{2}t_2t_3^{-1} \qquad x_{3,6} = t_3 + \frac{3}{2}t_4t_5^{-1}$$
$$z_1 = f_1 + \frac{1}{3}t_3^2t_4^{-1} \qquad x_1 = x_{1,6} + \frac{1}{2}t_5t_6^{-1}$$

$$z_2 = t_2 + \frac{2}{3}t_3^3t_4^{-1} \qquad x_3 = x_{3,6} + t_5^2t_6^{-1}$$
$$x_{1,6} = z_1 + x_{3,6}t_5^{-1} - \frac{3}{4}t_4t_5^{-2} \qquad x_4 = t_4 + \frac{2}{3}t_5^3t_6^{-1}.$$

The remark below will also be helpful in Subsection 6.3.1.

6.2.7 Remark. Fix $n \in \mathbb{N}$. Let $R_s = \mathbb{K}\langle u_1, \cdots, u_s^{\pm 1}, u_{s+1}^{\pm 1}, \cdots, u_n^{\pm 1}\rangle$ be a commutative algebra generated by the elements $u_1, \cdots, u_s^{\pm 1}, u_{s+1}^{\pm 1}, \cdots, u_n^{\pm 1}$ over the field $\mathbb{K}$, with $1 \leq s \leq n+1$ (note, $R_{n+1} := \mathbb{K}\langle u_1, \cdots, u_n\rangle$). Then, for all $f, g \in R_s$, we have that

$$\{f, g\} = \sum_{i,j=1}^{n} \{u_i, u_j\} \frac{\partial f}{\partial u_i} \frac{\partial g}{\partial u_j}.$$

As a result,

$$\{u_1^{i_1} \cdots u_n^{i_n}, u_j\} = i_1 u_1^{i_1-1} u_2^{i_2} \cdots u_n^{i_n}\{u_1, u_j\} + i_2 u_1^{i_1} u_2^{i_2-1} u_3^{i_3} \cdots u_n^{i_n}\{u_2, u_j\}$$
$$+ \cdots + i_n u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{i_n-1}\{u_n, u_j\},$$

for all $(i_1, \cdots, i_n) \in \mathbb{Z}^n$. Note, $\{u_i, u_j\} = 0$ whenever $i = j$.

6.3 Poisson derivations of $\mathcal{A}_{\alpha,\beta}$

We are now going to study the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$. We will begin with the case where both α and β are non-zero, and then proceed to look at the case where either α or β is zero.

6.3.1 Poisson derivations of $\mathcal{A}_{\alpha,\beta}$ $(\alpha, \beta \neq 0)$. Throughout this subsection, we assume that α and β are non-zero. Let $\text{Der}_P(\mathcal{A})$ be the collection of all the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$ and $\mathcal{D} \in \text{Der}_P(\mathcal{A})$. Then, it follows from (6.2.1) that $\mathcal{D}$ extends uniquely to a Poisson derivation of each of the series of algebras in (6.2.2) via localization. Hence, $\mathcal{D}$ is a Poisson derivation of the Poisson torus $\mathcal{R}_3 = \mathbb{C}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$. As a result, $\mathcal{D}$ can be written as

$$\mathcal{D} = \text{ham}_x + \rho,$$

where ρ is a scalar Poisson derivation of $\mathcal{R}_3$ defined as $\rho(t_i) = \lambda_i t_i,\ i = 3,4,5,6$. Note, $\lambda_i \in Z_P(\mathcal{R}_3) = \mathbb{C}$ and $\text{ham}_x := \{x, -\} : \mathcal{R}_3 \to \mathcal{R}_3$ with $x \in \mathcal{R}_3$ (see Corollary 6.1.4).

We aim to describe $\mathcal{D}$ as a Poisson derivation of $\mathcal{A}_{\alpha,\beta}$. We do this in several steps. We first describe $\mathcal{D}$ as a Poisson derivation of $\mathcal{R}_4$.

6.3.2 Lemma. 1. $x \in \mathcal{R}_4$.

2. $\lambda_5 = \lambda_4 + \lambda_6$, $\rho(f_1) = -(\lambda_3 + \lambda_5)f_1$ and $\rho(t_2) = -\lambda_5 t_2$.

3. $\mathcal{D}(x_{\kappa,4}) = \text{ham}_x(x_{\kappa,4}) + \lambda_\kappa x_{\kappa,4}$ for all $\kappa \in \{1, \cdots, 6\}$, where $\lambda_1 := -(\lambda_3 + \lambda_5)$ and $\lambda_2 := -\lambda_5$.

Proof. 1. Observe that $\mathcal{Q} := \mathbb{C}[t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$ is a subalgebra of both $\mathcal{R}_3$ and $\mathcal{R}_4$. Furthermore, $\mathcal{R}_3 = \mathcal{R}_4[t_3^{-1}]$. One can easily verify that $z := t_4 t_5^{-1} t_6$ is a Poisson central element of $\mathcal{Q}$. Since $\mathcal{R}_3$ is a Poisson torus, it can be presented as a free $\mathcal{Q}-$module with basis $(t_3^j)_{j\in\mathbb{Z}}$. One can therefore write $x \in \mathcal{R}_3$ as: $x = \sum_{j\in\mathbb{Z}} b_j t_3^j$, where $b_j \in \mathcal{Q}$. Decompose x as follows: $x = x_- + x_+$, where $x_+ := \sum_{j\geq 0} b_j t_3^j$ and $x_- := \sum_{j<0} b_j t_3^j$. Clearly, $x_+ \in \mathcal{R}_4$. We now want to show that $x_- \in \mathcal{R}_4$. Write $x_- = \sum_{j=-1}^{-m} b_j t_3^j$ for some $m \in \mathbb{N}_{>0}$.

Now, $\mathcal{D}(z) = \text{ham}_x(z) + \rho(z) = \text{ham}_{x_-}(z) + \text{ham}_{x_+}(z) + (\lambda_4 - \lambda_5 + \lambda_6)z \in \mathcal{R}_4$. We have that $\text{ham}_{x_+}(z) + (\lambda_4 - \lambda_5 + \lambda_6)z \in \mathcal{R}_4$, hence, $\text{ham}_{x_-}(z) \in \mathcal{R}_4$. Note: $\{t_3, z\} = 2zt_3$, and $\{\gamma, z\} = 0$ for all $\gamma \in \mathcal{Q}$ since z is Poisson central in $\mathcal{Q}$. One can therefore express $\text{ham}_{x_-}(z)$ as follows:

$$\text{ham}_{x_-}(z) = \{x_-, z\} = \sum_{j=-1}^{-m} b_j \{t_3^j, z\} = \sum_{j=-1}^{-m} 2jb_j z t_3^j \in \mathcal{R}_4.$$

Let $n \in \mathbb{N}_{>0}$, and set

$$\ell^{(n)} := \underbrace{\{\{\cdots\{}_{n-\text{times}} x_-, \underbrace{z\}, z\}, \cdots, z\}}_{n-\text{times}} \in \mathcal{R}_4.$$

We claim that

$$\ell^{(n)} = \sum_{j=-1}^{-m} (2j)^n z^n b_j t_3^j,$$

for all $n \in \mathbb{N}_{>0}$. Observe that

$$\ell^{(1)} = \mathsf{ham}_{x_-}(z) = \sum_{j=-1}^{-m} 2jb_j z t_3^j,$$

hence, the result is true for $n = 1$. Suppose that the result is true for $n \geq 1$. Then,

$$\ell^{(n+1)} = \{\ell^{(n)}, z\} = \sum_{j=-1}^{-m} (2j)^n z^n b_j \{t_3^j, z\} = \sum_{j=-1}^{-m} (2j)^{n+1} z^{n+1} b_j t_3^j$$

as expected. By the principle of mathematical induction, the claim is proved.

Given that $\ell^{(n)} = \sum_{j=-1}^{-m} (2j)^n z^n b_j t_3^j$, it follows that

$$\mu_n = \sum_{j=-1}^{-m} (2j)^n b_j t_3^j, \quad \text{where} \quad \mu_n := \ell^{(n)} z^{-n} \in \mathcal{R}_4.$$

The above equality can be written as a matrix equation in the form:

$$\begin{bmatrix} -2 & -4 & -6 & \cdots & -2m \\ (-2)^2 & (-4)^2 & (-6)^2 & \cdots & (-2m)^2 \\ (-2)^3 & (-4)^3 & (-6)^3 & \cdots & (-2m)^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-2)^m & (-4)^m & (-6)^m & \cdots & (-2m)^m \end{bmatrix} \begin{bmatrix} b_{-1}t_3^{-1} \\ b_{-2}t_3^{-2} \\ b_{-3}t_3^{-3} \\ \vdots \\ b_{-m}t_3^{-m} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_m \end{bmatrix}.$$

One can observe that the coefficient matrix

$$\begin{bmatrix} -2 & -4 & -6 & \cdots & -2m \\ (-2)^2 & (-4)^2 & (-6)^2 & \cdots & (-2m)^2 \\ (-2)^3 & (-4)^3 & (-6)^3 & \cdots & (-2m)^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-2)^m & (-4)^m & (-6)^m & \cdots & (-2m)^m \end{bmatrix}$$

is similar to a Vandermonde matrix (since the terms in each column form a geometric sequence) which is well known to be invertible. This therefore implies that each $b_jt_3^j$ is a linear combination of the $\mu_n \in \mathcal{R}_4$. As a result, $b_jt_3^j \in \mathcal{R}_4$ for all $j \in \{-1, \cdots, -m\}$. Consequently, $x_- = \sum_{j=-1}^{-m} b_jt_3^j \in \mathcal{R}_4$ as desired.

2. Recall that $\rho(t_\kappa) = \lambda_\kappa t_\kappa$ for all $\kappa \in \{3, 4, 5, 6\}$ and $\lambda_\kappa \in \mathbb{C}$. From Remark 6.2.6, we have that $f_1 = t_1 + \frac{1}{2}t_2t_3^{-1}$. Again, recall from Section 5.3 that $t_1 = \alpha t_3^{-1}t_5^{-1}$ and $t_2 = \beta t_4^{-1}t_6^{-1}$ in $\mathcal{R}_3 = \mathscr{P}_{\alpha,\beta}$. As a result, $f_1 = \alpha t_3^{-1}t_5^{-1} + \frac{1}{2}\beta t_3^{-1}t_4^{-1}t_6^{-1}$. Therefore,

$$\rho(f_1) = -(\lambda_3 + \lambda_5)\alpha t_3^{-1}t_5^{-1} - \frac{1}{2}(\lambda_3 + \lambda_4 + \lambda_6)\beta t_3^{-1}t_4^{-1}t_6^{-1}. \tag{6.3.1}$$

Also, $\rho(f_1) \in \mathcal{R}_4$ implies that $\rho(f_1)$ can be written in terms of the basis $\mathfrak{P}_4$ of $\mathcal{R}_4$ (Proposition 6.2.1) as:

$$\rho(f_1) = \sum_{r>0} a_r f_1^r + \sum_{s\geq 0} b_s t_3^s, \tag{6.3.2}$$

where a_r and b_s belong to $\mathcal{Q} = \mathbb{C}[t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$.

$$\begin{aligned} f_1^r &= \left(\alpha t_3^{-1}t_5^{-1} + \frac{1}{2}\beta t_3^{-1}t_4^{-1}t_6^{-1}\right)^r = \sum_{i=0}^{r} \binom{r}{i} (\alpha)^i (\beta/2)^{r-1} t_3^{-r} t_4^{i-r} t_5^{-i} t_6^{i-r} \\ &= c_r t_3^{-r}, \end{aligned} \tag{6.3.3}$$

where

$$c_r = \sum_{i=0}^{r} \binom{r}{i} (\alpha)^i (\beta/2)^{r-i} t_4^{i-r} t_5^{-i} t_6^{i-r} \in \mathcal{Q} \setminus \{0\}. \tag{6.3.4}$$

Substitute (6.3.3) into (6.3.2) to obtain

$$\rho(f_1) = \sum_{r>0} a_r c_r t_3^{-r} + \sum_{s\geq 0} b_s t_3^s. \tag{6.3.5}$$

One can rewrite (6.3.1) as

$$\rho(f_1) = dt_3^{-1}, \tag{6.3.6}$$

where $d = -(\lambda_5+\lambda_3)\alpha t_5^{-1} - \frac{1}{2}(\lambda_6+\lambda_4+\lambda_3)\beta t_4^{-1}t_6^{-1} \in \mathcal{Q}$. Comparing (6.3.5) to (6.3.6) shows that $b_s = 0$ for all $s \geq 0$, and $a_r c_r = 0$ for all $r \neq 1$. Therefore, $\rho(f_1) = a_1 c_1 t_3^{-1}$. Moreover, from (6.3.4), $c_1 = \frac{1}{2}\beta t_4^{-1}t_6^{-1} + \alpha t_5^{-1}$. Hence,

$$\rho(f_1) = a_1c_1t_3^{-1} = a_1\left(\frac{1}{2}\beta t_4^{-1}t_6^{-1} + \alpha t_5^{-1}\right)t_3^{-1} = a_1\alpha t_3^{-1}t_5^{-1} + \frac{1}{2}a_1\beta t_3^{-1}t_4^{-1}t_6^{-1}. \tag{6.3.7}$$

Comparing (6.3.7) to (6.3.1) reveals that $a_1 = -(\lambda_5+\lambda_3) = -(\lambda_6+\lambda_4+\lambda_3)$. Consequently, $\lambda_5 = \lambda_6 + \lambda_4$. Hence, $\rho(f_1) = -(\lambda_5+\lambda_3)\alpha t_3^{-1}t_5^{-1} - \frac{1}{2}(\lambda_5+\lambda_3)\beta t_3^{-1}t_4^{-1}t_6^{-1} = -(\lambda_5+\lambda_3)f_1$. Finally, since $t_2 = \beta t_4^{-1}t_6^{-1}$ in $\mathcal{R}_4$, it follows that

$$\rho(t_2) = -(\lambda_6+\lambda_4)\beta t_4^{-1}t_6^{-1} = -(\lambda_6+\lambda_4)t_2 = -\lambda_5 t_2.$$

3. Set $\lambda_1 := -(\lambda_3+\lambda_5)$ and $\lambda_2 := -\lambda_5$, it follows from points (1) and (2) that $\mathcal{D}(x_{\kappa,4}) = \text{ham}_x(x_{\kappa,4}) + \rho(x_{\kappa,4}) = \text{ham}_x(x_{\kappa,4}) + \lambda_\kappa x_{\kappa,4}$ for all $\kappa \in \{1,\cdots,6\}$. In conclusion, $\mathcal{D} = \text{ham}_x + \rho$, with $x \in \mathcal{R}_4$. ■

We now proceed to describe $\mathcal{D}$ as a Poisson derivation of $\mathcal{R}_5$.

6.3.3 Lemma. 1. $x \in \mathcal{R}_5$.

2. $\lambda_4 = 3\lambda_3 + \lambda_5$, $\lambda_6 = -3\lambda_3$, $\rho(z_1) = -(\lambda_3+\lambda_5)z_1$ and $\rho(z_2) = -\lambda_5 z_2$.

3. Set $\lambda_1 := -(\lambda_3+\lambda_5)$ and $\lambda_2 := -\lambda_5$, then $\mathcal{D}(x_{\kappa,5}) = \text{ham}_x(x_{\kappa,5}) + \lambda_\kappa x_{\kappa,5}$ for all $\kappa \in \{1,\cdots,6\}$.

Proof. In this proof, we denote $\underline{v} := (i,j,k,l) \in \mathbb{N}\times\mathbb{Z}^3$.

1. We already know that $x \in \mathcal{R}_4 = \mathcal{R}_5[t_4^{-1}]$. Given the basis $\mathfrak{P}_5$ of $\mathcal{R}_5$ (Proposition 6.2.2), x can be written as $x = \sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l$, where I is a finite subset of

$\{0,1,2\} \times \mathbb{N} \times \mathbb{Z}^3$ and $a_{(\xi,\underline{v})}$ are complex numbers. Write $x = x_- + x_+$, where

$$x_+ = \sum_{\substack{(\xi,\underline{v})\in I \\ j\geq 0}} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l \quad \text{and} \quad x_- = \sum_{\substack{(\xi,\underline{v})\in I \\ j<0}} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

Suppose that there exists a minimum $j_0 < 0$ such that $a_{(\xi,i,j_0,k,l)} \neq 0$ for some $(\xi, i, j_0, k, l) \in I$ and $a_{(\xi,i,j,k,l)} = 0$ for all $(\xi, i, j_0, k, l) \in I$ with $j < j_0$. Given this assumption, write

$$x_- = \sum_{\substack{(\xi,\underline{v})\in I \\ j_0\leq j\leq -1}} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

Let $s = 3, 6$. Then, $\mathcal{D}(t_s) = \text{ham}_{x_+}(t_s) + \text{ham}_{x_-}(t_s) + \rho(t_s) \in \mathcal{R}_5$ for each $s = 3, 6$. This implies that $\text{ham}_{x_-}(t_s) \in \mathcal{R}_5$, since $\text{ham}_{x_+}(t_s) + \rho(t_s) = \text{ham}_{x_+}(t_s) + \lambda_s t_s \in \mathcal{R}_5$. Set $\underline{w} := (i, j, k, l) \in \mathbb{N}^2 \times \mathbb{Z}^2$. One can therefore write $\text{ham}_{x_-}(t_s) \in \mathcal{R}_5$ in terms of the basis $\mathfrak{P}_5$ of $\mathcal{R}_5$ as:

$$\text{ham}_{x_-}(t_s) = \sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l, \tag{6.3.8}$$

where J is a finite subset of $\{0,1,2\} \times \mathbb{N}^2 \times \mathbb{Z}^2$ and $b_{(\xi,\underline{w})}$ are all complex numbers.

When $s = 6$, then using Remark 6.2.7, one can also express $\text{ham}_{x_-}(t_6)$ as:

$$\text{ham}_{x_-}(t_6) = \sum_{\substack{(\xi,\underline{v})\in I \\ j_0\leq j\leq -1}} 3(k+j-i) a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^{l+1}.$$

Comparing this expression for $\text{ham}_{x_-}(t_6)$ to (6.3.8) (when $s = 6$), we have that

$$\sum_{\substack{(\xi,\underline{v})\in I \\ j_0\leq j\leq -1}} 3(k+j-i) a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^{l+1} = \sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

As $\mathfrak{P}_5$ is a basis for $\mathcal{R}_5$ (Proposition 6.2.2), we deduce from Corollary 6.2.3 that $\left(z_1^i t_3^\xi t_4^j t_5^k t_6^l\right)_{(i\in\mathbb{N};j,k,l\in\mathbb{Z};\xi\in\{0,1,2\})}$ is a basis for $\mathcal{R}_5[t_4^{-1}]$. Now, at $j = j_0$, denote $\underline{v} = (i, j, k, l)$ by $\underline{v}_0 := (i, j_0, k, l)$. Since $\underline{v}_0 \in \mathbb{N} \times \mathbb{Z}^3$ (with $j_0 < 0$) and $\underline{w} = (i, j, k, l) \in$

$\mathbb{N}^2 \times \mathbb{Z}^2$ (with $j \geq 0$), it follows from the above equality that, at $\underline{v}_0$, we must have

$$k = i - j_0, \tag{6.3.9}$$

for some $(\xi, \underline{v}_0) \in I$.

Similarly, when $s = 3$, then using Remark 6.2.7, one can also express $\text{ham}_{x_-}(t_3)$ as:

$$\begin{aligned}\text{ham}_{x_-}(t_3) = &-\sum \left[\frac{3}{2}\beta(3i - k - 3j_0)a_{2,i,j_0,k,l+1} + 2(i+1)\alpha a_{(0,i+1,j_0,k+1,l)}\right] z_1^i t_4^{j_0} t_5^k t_6^l \\ &+ \sum \left[(3i - k - 3j_0)a_{(0,i,j_0,k,l)} - 2(i+1)\alpha a_{(1,i+1,j_0,k+1,l)}\right] z_1^i t_3 t_4^{j_0} t_5^k t_6^l \\ &+ \sum \left[(3i - k - 3j_0)a_{(1,i,j_0,k,l)} - 2(i+1)\alpha a_{(2,i+1,j_0,k+1,l)}\right] z_1^i t_3^2 t_4^{j_0} t_5^k t_6^l + \mathcal{K},\end{aligned}$$

where $\mathcal{K} \in \text{Span}\left(\mathfrak{P}_5 \setminus \{z_1^i t_3^\xi t_4^{j_0} t_5^k t_6^l \mid (\xi, i, j_0, k, l) \in \{0,1,2\} \times \mathbb{N} \times \mathbb{Z}^3\}\right)$ (note, one will need the following two expressions $z_2 = 2(z_1 t_3 - \alpha t_5^{-1})$ and $t_3^3 = 3z_1 t_3 t_4 - 3\alpha t_4 t_5^{-1} - \frac{3\beta}{2} t_6^{-1}$ from (6.2.3) and (6.2.4) to express some of the monomials in terms of the basis $\mathfrak{P}_5$ of $\mathcal{R}_5$). Comparing this expression for $\text{ham}_{x_-}(t_3)$ to (6.3.8) (when $s = 3$) reveals that:

$$\begin{aligned}&\sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l \\ &= -\sum \left[\frac{3}{2}\beta(3i - k - 3j_0)a_{(2,i,j_0,k,l+1} + 2(i+1)\alpha a_{(0,i+1,j_0,k+1,l)}\right] z_1^i t_4^{j_0} t_5^k t_6^l \\ &\quad + \sum \left[(3i - k - 3j_0)a_{(0,i,j_0,k,l)} - 2(i+1)\alpha a_{(1,i+1,j_0,k+1,l)}\right] z_1^i t_3 t_4^{j_0} t_5^k t_6^l \\ &\quad + \sum \left[(3i - k - 3j_0)a_{(1,i,j_0,k,l)} - 2(i+1)\alpha a_{(2,i+1,j_0,k+1,l)}\right] z_1^i t_3^2 t_4^{j_0} t_5^k t_6^l + \mathcal{K}.\end{aligned}$$

We have already established that $\left(z_1^i t_3^\xi t_4^j t_5^k t_6^l\right)_{(i\in\mathbb{N};j,k,l\in\mathbb{Z};\xi\in\{0,1,2\})}$ is a basis for $\mathcal{R}_5[t_4^{-1}]$. Since $\underline{v}_0 = (i, j_0, k, l) \in \mathbb{N} \times \mathbb{Z}^3$ (with $j_0 < 0$) and $\underline{w} = (i, j, k, l) \in \mathbb{N}^2 \times \mathbb{Z}^2$ (with

$j \geq 0$), it follows from the above equality that, at $\underline{v}_0$, we must have

$$\frac{3}{2}\beta(3i-k-3j_0)a_{(2,i,j_0,k,l+1)} + 2(i+1)\alpha a_{(0,i+1,j_0,k+1,l)} = 0, \quad (6.3.10)$$

$$(3i-k-3j_0)a_{(0,i,j_0,k,l)} - 2(i+1)\alpha a_{(1,i+1,j_0,k+1,l)} = 0, \quad (6.3.11)$$

$$(3i-k-3j_0)a_{(1,i,j_0,k,l)} - 2(i+1)\alpha a_{(2,i+1,j_0,k+1,l)} = 0. \quad (6.3.12)$$

Suppose that there exists $(\xi, i, j_0, k, l) \in I$ such that $3i - k - 3j_0 = 0$. Then,

$$k = 3(i - j_0). \quad (6.3.13)$$

Comparing (6.3.13) to (6.3.9) clearly shows that $i - j_0 = 0$ which implies that $i = j_0 < 0$, a contradiction (note: $i \geq 0$). Therefore, $3i - k - 3j_0 \neq 0$ for all $(\xi, i, j, k) \in I$.

Now, observe that if there exists $\xi \in \{0, 1, 2\}$ such that $a_{(\xi,i,j_0,k,l)} = 0$ for all $(i, j_0, k, l) \in \mathbb{N} \times \mathbb{Z}^3$, then one can easily deduce from equations (6.3.10), (6.3.11) and (6.3.12) that $a_{(\xi,i,j_0,k,l)} = 0$ for all $(\xi, i, j_0, k, l) \in I$. This contradicts our initial assumption. Therefore, there exists some $(i, j_0, k, l) \in \mathbb{N} \times \mathbb{Z}^3$ such that $a_{(\xi,i,j_0,k,l)} \neq 0$ for each $\xi \in \{0, 1, 2\}$. Without loss of generality, let (u, j_0, v, w) be the greatest element in the lexicographic order on $\mathbb{N} \times \mathbb{Z}^3$ such that $a_{(0,u,j_0,v,w)} \neq 0$ and $a_{(0,i,j_0,k,l)} = 0$ for all $i > u$.

From (6.3.11), at $(i, j_0, k, l) = (u, j_0, v, w)$, we have:

$$(3u - v - 3j_0)a_{(0,u,j_0,v,w)} - 2(u+1)\alpha a_{(1,u+1,j_0,v+1,w)} = 0.$$

From (6.3.12), at $(i, j_0, k, l) = (u + 1, j_0, v + 1, w)$, we have:

$$(3u - v - 3j_0)a_{(1,u+1,j_0,v+1,w)} - 2(u+1)\alpha a_{(2,u+2,j_0,v+2,w)} = 0.$$

Finally, from (6.3.10), at $(i, j_0, k, l) = (u + 2, j_0, v + 2, w - 1)$, we have:

$$\frac{3}{2}\beta(3u - v - 3j_0)a_{(2,u+2,j_0,v+2,w)} + 2(u + 1)\alpha a_{(0,u+3,j_0,v+3,w-1)} = 0.$$

Since $3i - k - 3j_0 \neq 0$ for all $(i, j_0, k, l) \in I$ and $u + 3 > u$, it follows from the above list of equations (starting from the last one) that

$$a_{(0,u+3,j_0,v+3,w-1)} = 0 \Rightarrow a_{(2,u+2,j_0,v+2,w)} = 0 \Rightarrow a_{(1,u+1,j_0,v+1,w)} = 0 \Rightarrow a_{(0,u,j_0,v,w)} = 0,$$

a contradiction! Hence, $a_{(0,i,j_0,k,l)} = 0$ for all $(i, j_0, k, l) \in \mathbb{N}\times\mathbb{Z}^3$. From (6.3.10), (6.3.11) and (6.3.12), one can easily conclude that $a_{(\xi,i,j_0,k,l)} = 0$ for all $(\xi, i, j_0, k, l) \in I$. This contradicts our initial assumption, hence $x_- = 0$. Consequently, $x = x_+ \in \mathcal{R}_5$ as desired.

2. From Remark 6.2.6, we have that $z_2 = t_2 + \frac{2}{3}t_3^3t_4^{-1}$. Since $\rho(t_\kappa) = \lambda_\kappa t_\kappa$, $\kappa \in \{2, 3, 4, 5, 6\}$, with $\lambda_2 := -\lambda_5$ (see Lemma 6.3.2), it follows that

$$\rho(z_2) = -\lambda_5 t_2 + \frac{2}{3}(3\lambda_3 - \lambda_4)t_3^3t_4^{-1} = -\lambda_5 z_2 + \frac{2}{3}(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3t_4^{-1}.$$

Furthermore,

$$\mathcal{D}(z_2) = \text{ham}_x(z_2) + \rho(z_2) = \text{ham}_x(z_2) - \lambda_5 z_2 + \frac{2}{3}(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3t_4^{-1} \in \mathcal{R}_5.$$

We have that $(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3t_4^{-1} \in \mathcal{R}_5$, since $\text{ham}_x(z_2) - \lambda_5 z_2 \in \mathcal{R}_5$. This implies that $(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3 \in \mathcal{R}_5 t_4$. Set $w := 3\lambda_3 - \lambda_4 + \lambda_5$. Suppose that $w \neq 0$. From (6.2.4), we have:

$$t_3^3 = 3z_1t_3t_4 - \frac{3}{2}\beta t_6^{-1} - 3\alpha t_4t_5^{-1}.$$

It follows that

$$wt_3^3 = 3wz_1t_3t_4 - 3w\alpha t_4t_5^{-1} - \frac{3}{2}w\beta t_6^{-1} \in \mathcal{R}_5t_4.$$

Since t_3^3, $t_4t_5^{-1}$ and $z_1t_3t_4$ are all elements of $\mathcal{R}_5t_4$, it implies that $t_6^{-1} \in \mathcal{R}_5t_4$. Hence, $1 \in \mathcal{R}_5t_4t_6$. Using the basis $\mathfrak{P}_5$ of $\mathcal{R}_5$ (Proposition 6.2.2), this leads to a contradiction.

Therefore, $w = 0$. That is, $w = 3\lambda_3 - \lambda_4 + \lambda_5 = 0$, and so $\lambda_4 = 3\lambda_3 + \lambda_5$. This further implies that $\rho(z_2) = -\lambda_5 z_2$ as desired.

Again, from Lemma 6.3.2, we have that $\rho(f_1) = -(\lambda_3 + \lambda_5)f_1$. Recall from Remark 6.2.6 that $z_1 = f_1 + \frac{1}{3}t_3^2 t_4^{-1}$. It follows that

$$\begin{aligned}\rho(z_1) &= -(\lambda_3+\lambda_5)f_1 + \frac{1}{3}(2\lambda_3 - \lambda_4)t_3^2t_4^{-1} = -(\lambda_3+\lambda_5)z_1 + \frac{1}{3}(3\lambda_3 - \lambda_4 + \lambda_5)t_3^2t_4^{-1}\\ &= -(\lambda_3+\lambda_5)z_1 + \frac{1}{3}(3\lambda_3 - (3\lambda_3+\lambda_5) + \lambda_5)t_3^2t_4^{-1} = -(\lambda_3+\lambda_5)z_1.\end{aligned}$$

Finally, we know that $\rho(t_6) = \lambda_6 t_6$. From the relation (6.2.4), we have:

$$t_3^3 = 3z_1t_3t_4 - 3\alpha t_4 t_5^{-1} - \frac{3\beta}{2}t_6^{-1}.$$

This implies that

$$t_6^{-1} = \frac{2}{3\beta}(3z_1t_3t_4 - 3\alpha t_4t_5^{-1} - t_3^3).$$

Apply ρ to this relation to obtain

$$-\lambda_6 t_6^{-1} = 3\lambda_3\left(\frac{2}{3\beta}\left(3z_1t_3t_4 - 3\alpha t_4t_5^{-1} - t_3^3\right)\right).$$

Clearly, $\lambda_6 = -3\lambda_3$ as desired.

3. Set $\lambda_1 := -(\lambda_3 + \lambda_5)$ and $\lambda_2 := -\lambda_5$. It follows from points (1) and (2) that $\mathcal{D}(x_{\kappa,5}) = \text{ham}_x(x_{\kappa,5}) + \rho(x_{\kappa,5}) = \text{ham}_x(x_{\kappa,5}) + \lambda_\kappa x_{\kappa,5}$ for all $\kappa \in \{1, \cdots, 6\}$. In conclusion, $\mathcal{D} = \text{ham}_x + \rho$ with $x \in \mathcal{R}_5$. ■

We are now ready to describe $\mathcal{D}$ as a Poisson derivation of $\mathcal{A}_{\alpha,\beta}$.

6.3.4 Lemma. 1. $x \in \mathcal{A}_{\alpha,\beta}$.

2. $\rho(x_\kappa) = 0$ for all $\kappa \in \{1, \cdots, 6\}$.

3. $\mathcal{D} = \text{ham}_x$.

Proof. In this proof, we denote $\underline{v} := (i, j, k, l) \in \mathbb{N}^2 \times \mathbb{Z}^2$. Also, recall from the PDDA of $\mathcal{A}_{\alpha,\beta}$ at the beginning of this section that $t_5 = x_5$ and $t_6 = x_6$.

1. Given the basis $\mathfrak{P}$ of $\mathcal{A}_{\alpha,\beta}$ (Proposition 5.3.4), one can write $x \in \mathcal{R}_5 = \mathcal{A}_{\alpha,\beta}[t_5^{-1}, t_6^{-1}]$ as

$$x = \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l,$$

where I is a finite subset of $\{0,1\}^2 \times \mathbb{N}^2 \times \mathbb{Z}^2$ and $a_{(\epsilon_1,\epsilon_2,\underline{v})}$ are complex numbers. Write $x = x_- + x_+$, where

$$x_+ = \sum_{\substack{(\epsilon_1,\epsilon_2,\underline{v})\in I \\ k,\ l\geq 0}} a_{(\epsilon_1,\epsilon_2,\underline{v})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l,$$

and

$$x_- = \sum_{\substack{(\epsilon_1,\epsilon_2,\underline{v})\in I \\ k<0 \text{ or } l<0}} a_{(\epsilon_1,\epsilon_2,\underline{v})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l.$$

Suppose that there exists a minimum negative integer k_0 or l_0 such that $a_{(\epsilon_1,\epsilon_2,i,j,k_0,l)} \neq 0$ or $a_{(\epsilon_1,\epsilon_2,i,j,k,l_0)} \neq 0$ for some $(\epsilon_1,\epsilon_2,i,j,k_0,l), (\epsilon_1,\epsilon_2,i,j,k,l_0) \in I$; and $a_{(\epsilon_1,\epsilon_2,i,j,k,l)} = 0$ whenever $k < k_0$ or $l < l_0$. Write

$$x_- = \sum_{\substack{(\epsilon_1,\epsilon_2,\underline{v})\in I \\ k_0\leq k\leq -1 \text{ or } l_0\leq l\leq -1}} a_{(\epsilon_1,\epsilon_2,\underline{v})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l,$$

Now, $\mathcal{D}(x_3) = \text{ham}_{x_+}(x_3) + \text{ham}_{x_-}(x_3) + \rho(x_3) \in \mathcal{A}_{\alpha,\beta}$. From Remark 6.2.6, we have that $x_3 = x_{3,6} + t_5^2 t_6^{-1}$ and $x_{3,6} = t_3 + \frac{3}{2} t_4 t_5^{-1}$. Putting these two together gives

$$x_3 = t_3 + \frac{3}{2} t_4 t_5^{-1} + t_5^2 t_6^{-1}.$$

Again, from Remark 6.2.6, we also have that $t_4 = x_4 - \frac{2}{3} t_5^3 t_6^{-1}$. Note, $\rho(t_\kappa) = \lambda_\kappa t_\kappa,\ \kappa = 3, 4, 5, 6.$

Now,

$$\begin{aligned}
\rho(x_3) &= \lambda_3 t_3 + \frac{3}{2}(\lambda_4-\lambda_5)t_4t_5^{-1} + (2\lambda_5-\lambda_6)t_5^2t_6^{-1}\\
&= \lambda_3\left(x_{3,6} - \frac{3}{2}t_4t_5^{-1}\right) + \frac{3}{2}(\lambda_4-\lambda_5)t_4t_5^{-1} + (2\lambda_5-\lambda_6)t_5^2t_6^{-1}\\
&= \lambda_3 x_{3,6} - \frac{3}{2}(\lambda_3-\lambda_4+\lambda_5)t_4t_5^{-1} + (2\lambda_5-\lambda_6)t_5^2t_6^{-1}\\
&= \lambda_3(x_3 - t_5^2t_6^{-1}) - \frac{3}{2}(\lambda_3-\lambda_4+\lambda_5)\left(x_4 - \frac{2}{3}t_5^3t_6^{-1}\right)t_5^{-1} + (2\lambda_5-\lambda_6)t_5^2t_6^{-1}\\
&= \lambda_3x_3 + \alpha_1x_4t_5^{-1} + \alpha_2t_5^2t_6^{-1}, \qquad (6.3.14)
\end{aligned}$$

where $\alpha_1 = \frac{3}{2}(\lambda_4-\lambda_3-\lambda_5)$ and $\alpha_2 = (3\lambda_5-\lambda_4-\lambda_6)$. Therefore, $\mathcal{D}(x_3) = \text{ham}_{x_+}(x_3) + \text{ham}_{x_-}(x_3) + \lambda_3x_3 + \alpha_1x_4t_5^{-1} + \alpha_2t_5^2t_6^{-1} \in \mathcal{A}_{\alpha,\beta}$. It follows that $\mathcal{D}(x_3)t_5t_6 = \text{ham}_{x_+}(x_3)t_5t_6 + \text{ham}_{x_-}(x_3)t_5t_6 + \lambda_3x_3t_5t_6 + \alpha_1x_4t_6 + \alpha_2t_5^3 \in \mathcal{A}_{\alpha,\beta}$. Hence, $\text{ham}_{x_-}(x_3)t_5t_6 \in \mathcal{A}_{\alpha,\beta}$, since $\text{ham}_{x_+}(x_3)t_5t_6 + \lambda_3x_3t_5t_6 + \alpha_1x_4t_6 + \alpha_2t_5^3 \in \mathcal{A}_{\alpha,\beta}$.

Using Remark 6.2.7, one can verify that

$$\begin{aligned}
\text{ham}_{x_-}(x_3)t_5t_6 = \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}\Big(&(i+3j-3\epsilon_2-k)x_1^ix_2^jx_3^{\epsilon_1+1}x_4^{\epsilon_2}t_5^{k+1}t_6^{l+1}\\
&- 3kx_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2+1}t_5^kt_6^{l+1} + ix_1^{i-1}x_2^{j+1}x_3^{\epsilon_1}x_4^{\epsilon_2}t_5^{k+1}t_6^{l+1}\\
&-6lx_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2}t_5^{k+3}t_6^{l}\Big). \qquad (6.3.15)
\end{aligned}$$

Assume that there exists $l<0$ such that $a_{(\epsilon_1,\epsilon_2,i,j,k,l)} \neq 0$. It follows from our initial assumption that $a_{(\epsilon_1,\epsilon_2,i,j,k,l_0)} \neq 0$. Now, at $l = l_0$, denote $\underline{v} = (i,j,k,l)$ by $\underline{v}_0 := (i,j,k,l_0)$. From (6.3.15), we have that

$$\text{ham}_{x_-}(x_3)t_5t_6 = -\sum_{(\epsilon_1,\epsilon_2,\underline{v}_0)\in I} 6l_0a_{(\epsilon_1,\epsilon_2,\underline{v}_0)}x_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2}t_5^{k+3}t_6^{l_0} + \mathcal{J}_1,$$

where $\mathcal{J}_1 \in \text{Span}\left(\mathfrak{P}\setminus\{x_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2}t_5^kt_6^{l_0} \mid \epsilon_1,\epsilon_2\in\{0,1\},\ k\in\mathbb{Z} \text{ and } i,j\in\mathbb{N}\}\right)$.

Set $\underline{w} := (i,j,k,l) \in \mathbb{N}^4$. One can also write $\text{ham}_{x_-}(x_3)t_5t_6 \in \mathcal{A}_{\alpha,\beta}$ in terms of the basis $\mathfrak{P}$ of $\mathcal{A}_{\alpha,\beta}$ (Proposition 5.3.4) as:

$$\text{ham}_{x_-}(x_3)t_5t_6 = \sum_{(\epsilon_1,\epsilon_2,\underline{w})\in J} b_{(\epsilon_1,\epsilon_2,\underline{w})}x_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2}t_5^kt_6^l, \tag{6.3.16}$$

where J is a finite subset of $\{0,1\}^2\times\mathbb{N}^4$ and $b_{(\epsilon_1,\epsilon_2,\underline{w})}\in\mathbb{C}$. It follows that

$$\sum_{(\epsilon_1,\epsilon_2,\underline{w})\in J} b_{(\epsilon_1,\epsilon_2,\underline{w})}x_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2}t_5^kt_6^l = -\sum_{(\epsilon_1,\epsilon_2,\underline{v}_0)\in I} 6l_0a_{(\epsilon_1,\epsilon_2,\underline{v}_0)}x_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2}t_5^{k+3}t_6^{l_0}+\mathcal{J}_1.$$

As $\mathfrak{P}$ is a basis for $\mathcal{A}_{\alpha,\beta}$, we deduce from Corollary 5.3.5 that $\left(x_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2}t_5^kt_6^l\right)_{((\epsilon_1,\epsilon_2,\underline{v})\in\{0,1\}^2\times\mathbb{N}^2\times\mathbb{Z}^2)}$ is a basis for $\mathcal{A}_{\alpha,\beta}[t_5^{-1},t_6^{-1}]$. Since $\underline{v}_0=(i,j,k,l_0)\in\mathbb{N}^2\times\mathbb{Z}^2$ (with $l_0<0$) and $\underline{w}=(i,j,k,l)\in\mathbb{N}^4$ (with $l\geq 0$) in the above equality, we must have

$$6l_0a_{(\epsilon_1,\epsilon_2,\underline{v}_0)}=0.$$

Note, $l_0\neq 0$, it follows that $a_{(\epsilon_1,\epsilon_2,\underline{v}_0)}=a_{(\epsilon_1,\epsilon_2,i,j,k,l_0)}$ are all zero. This is a contradiction. Therefore, $l\geq 0$ (i.e. there is no negative exponent for t_6).

Given that $l\geq 0$, it follows from our initial assumption that there exists $k=k_0<0$ such that $a_{(\epsilon_1,\epsilon_2,i,j,k_0,l)}\neq 0$. The rest of the proof will show that this assumption cannot also hold. Set $\underline{v}_0:=(i,j,k_0,l)\in\mathbb{N}^2\times\mathbb{Z}\times\mathbb{N}$. From (6.3.15), we have that

$$\text{ham}_{x_-}(x_3)t_5t_6 = -\sum_{(\epsilon_1,\epsilon_2,\underline{v}_0)\in I} 3ka_{(\epsilon_1,\epsilon_2,\underline{v}_0)}x_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2+1}t_5^{k_0}t_6^{l+1}+V,$$

where $V\in\mathcal{J}_2:=\text{Span}\left(\mathfrak{P}\setminus\{x_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2}t_5^{k_0}t_6^l\mid \epsilon_1,\epsilon_2\in\{0,1\}\text{ and }i,j,l\in\mathbb{N}\}\right)$. It follows that:

$$\begin{aligned}
&\mathrm{ham}_{x_-}(x_3)t_5t_6 = \\
&\quad - \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} 3k_0 a_{(0,0,\underline{v})} x_1^i x_2^j x_4 t_5^{k_0} t_6^{l+1} - \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} 3k_0 a_{(1,0,\underline{v})} x_1^i x_2^j x_3 x_4 t_5^{k_0} t_6^{l+1} \\
&\quad - \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} 3k_0 a_{(0,1,\underline{v})} x_1^i x_2^j x_4^2 t_5^{k_0} t_6^{l+1} - \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} 3k_0 a_{(1,1,\underline{v})} x_1^i x_2^j x_3 x_4^2 t_5^{k_0} t_6^{l+1} + V.
\end{aligned} \tag{6.3.17}$$

Write the relations in Lemma 5.3.1(2),(4) as follows:

$$x_4^2 = \frac{2}{3}\beta - \frac{2}{3}x_2x_4x_6 + \frac{8}{9}\alpha x_3x_6 + \frac{4}{3}x_1x_3x_4x_6 + L_1, \tag{6.3.18}$$

$$\begin{aligned}
x_3x_4^2 =& \frac{2}{3}\beta x_3 - \frac{2}{3}x_2x_3x_4x_6 + \frac{16}{9}\alpha^2x_6 + \frac{16}{3}\alpha x_1x_4x_6 + \frac{8}{3}\beta x_1^2x_6 \\
&- \frac{8}{3}x_1^2x_2x_4x_6^2 + \frac{32}{9}\alpha x_1^2x_3x_6^2 + \frac{16}{3}x_1^3x_3x_4x_6^2 + L_2,
\end{aligned} \tag{6.3.19}$$

where L_1 and L_2 are some elements of the ideal $\mathcal{A}_{\alpha,\beta}t_5 \subseteq \mathcal{J}_2$. Substitute (6.3.18) and (6.3.19) into (6.3.17) and simplify to obtain:

$$\begin{aligned}
\mathrm{ham}_{x_-}(e_3)t_5t_6 = &\sum[\lambda_{1,1}\beta a_{(0,1,i,j,k_0,l-1)} + \lambda_{1,2}\alpha^2 a_{(1,1,i,j,k_0,l-2)} \\
&\quad + \lambda_{1,3}\beta a_{(1,1,i-2,j,k_0,l-2)}] x_1^i x_2^j t_5^{k_0} t_6^l \\
&+ \sum[\lambda_{2,1}\alpha a_{(0,1,i,j,k_0,l-2)} + \lambda_{2,2}\beta a_{(1,1,i,j,k_0,l-1)} \\
&\quad + \lambda_{2,3}\alpha a_{(1,1,i-2,j,k_0,l-3)}] x_1^i x_2^j x_3 t_5^{k_0} t_6^l \\
&+ \sum[\lambda_{3,1} a_{(0,1,i,j-1,k_0,l-2)} + \lambda_{3,2}\alpha a_{(1,1,i-1,j,k_0,l-2)} \\
&\quad + \lambda_{3,3} a_{(1,1,i-2,j-1,k_0,l-3)} + \lambda_{3,4} a_{(0,0,i,j,k_0,l-1)}] x_1^i x_2^j x_4 t_5^{k_0} t_6^l \\
&+ \sum[\lambda_{4,1} a_{(0,1,i-1,j,k_0,l-2)} + \lambda_{4,2} a_{(1,1,i,j-1,k_0,l-2)} \\
&\quad + \lambda_{4,3} a_{(1,1,i-3,j,k_0,l-3)} + \lambda_{4,4} a_{(1,0,i,j,k_0,l-1)}] x_1^i x_2^j x_3 x_4 t_5^{k_0} t_6^l + V',
\end{aligned} \tag{6.3.20}$$

where $V' \in \mathcal{J}_2$. Also, $\lambda_{s,t} := \lambda_{s,t}(j, k_0, l)$ are some families of complex numbers which are non-zero for all $s, t \in \{1, 2, 3, 4\}$ and $j, l \in \mathbb{N}$, except $\lambda_{1,4}$ and $\lambda_{2,4}$ which are assumed to be zero since they do not exist in the above expression. Note, although each $\lambda_{s,t}$ depends on j, k_0, l, we have not made this dependency explicit in the above expression since the minimum requirement we need to complete the proof is for all the $\lambda_{s,t}$ existing in the above expression to be non-zero, which we already have.

Observe that (6.3.20) and (6.3.16) are equal, hence,

$$\begin{aligned}\sum_{(\epsilon_1,\epsilon_2,\underline{w})\in J} b_{(\epsilon_1,\epsilon_2,\underline{w})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l = &\sum [\lambda_{1,1}\beta a_{(0,1,i,j,k_0,l-1)} + \lambda_{1,2}\alpha^2 a_{(1,1,i,j,k_0,l-2)} \\ &+ \lambda_{1,3}\beta a_{(1,1,i-2,j,k_0,l-2)}] x_1^i x_2^j t_5^{k_0} t_6^l \\ &+ \sum [\lambda_{2,1}\alpha a_{(0,1,i,j,k_0,l-2)} + \lambda_{2,2}\beta a_{(1,1,i,j,k_0,l-1)} \\ &+ \lambda_{2,3}\alpha a_{(1,1,i-2,j,k_0,l-3)}] x_1^i x_2^j x_3 t_5^{k_0} t_6^l \\ &+ \sum [\lambda_{3,1} a_{(0,1,i,j-1,k_0,l-2)} + \lambda_{3,2}\alpha a_{(1,1,i-1,j,k_0,l-2)} \\ &+ \lambda_{3,3} a_{(1,1,i-2,j-1,k_0,l-3)} + \lambda_{3,4} a_{(0,0,i,j,k_0,l-1)}] x_1^i x_2^j x_4 t_5^{k_0} t_6^l \\ &+ \sum [\lambda_{4,1} a_{(0,1,i-1,j,k_0,l-2)} + \lambda_{4,2} a_{(1,1,i,j-1,k_0,l-2)} \\ &+ \lambda_{4,3} a_{(1,1,i-3,j,k_0,l-3)} + \lambda_{4,4} a_{(1,0,i,j,k_0,l-1)}] x_1^i x_2^j x_3 x_4 t_5^{k_0} t_6^l + V'.\end{aligned}$$

We have previously established that $(x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l)_{((\epsilon_1,\epsilon_2,\underline{v})\in\{0,1\}^2\times\mathbb{N}^2\times\mathbb{Z}^2)}$ is a basis for $\mathcal{A}_{\alpha,\beta}[t_5^{-1}, t_6^{-1}]$ (note, in this part of the proof $l \geq 0$). Since $\underline{v}_0 = (i, j, k_0, l) \in \mathbb{N}^2 \times \mathbb{Z} \times \mathbb{N}$ (with $k_0 < 0$) and $\underline{w} = (i, j, k, l) \in \mathbb{N}^4$ (with $k \geq 0$) in the above equality, it follows that

$$\lambda_{1,1}\beta a_{(0,1,i,j,k_0,l-1)} + \lambda_{1,2}\alpha^2 a_{(1,1,i,j,k_0,l-2)} + \lambda_{1,3}\beta a_{(1,1,i-2,j,k_0,l-2)} = 0, \tag{6.3.21}$$

$$\lambda_{2,1}\alpha a_{(0,1,i,j,k_0,l-2)} + \lambda_{2,2}\beta a_{(1,1,i,j,k_0,l-1)} + \lambda_{2,3}\alpha a_{(1,1,i-2,j,k_0,l-3)} = 0, \tag{6.3.22}$$

$$\begin{aligned}&\lambda_{3,1} a_{(0,1,i,j-1,k_0,l-2)} + \lambda_{3,2}\alpha a_{(1,1,i-1,j,k_0,l-2)} + \lambda_{3,3} a_{(1,1,i-2,j-1,k_0,l-3)} \\ &\quad + \lambda_{3,4} a_{(0,0,i,j,k_0,l-1)} = 0,\end{aligned} \tag{6.3.23}$$

$$\begin{aligned}&\lambda_{4,1} a_{(0,1,i-1,j,k_0,l-2)} + \lambda_{4,2} a_{(1,1,i,j-1,k_0,l-2)} + \lambda_{4,3} a_{(1,1,i-3,j,k_0,l-3)} \\ &\quad + \lambda_{4,4} a_{(1,0,i,j,k_0,l-1)} = 0.\end{aligned} \tag{6.3.24}$$

From (6.3.21) and (6.3.22), one can easily deduce that

$$a_{(0,1,i,j,k_0,l)} = -\frac{\alpha^2\lambda_{1,2}}{\beta\lambda_{1,1}}a_{(1,1,i,j,k_0,l-1)} - \frac{\lambda_{1,3}}{\lambda_{1,1}}a_{(1,1,i-2,j,k_0,l-1)}, \quad (6.3.25)$$

$$a_{(1,1,i,j,k_0,l)} = -\frac{\alpha\lambda_{2,1}}{\beta\lambda_{2,2}}a_{(0,1,i,j,k_0,l-1)} - \frac{\alpha\lambda_{2,3}}{\beta\lambda_{2,2}}a_{(1,1,i-2,j,k_0,l-2)}. \quad (6.3.26)$$

Note, $a_{(\epsilon_1,\epsilon_2,i,j,k_0,l)} := 0$ whenever $i < 0$ or $j < 0$ or $l < 0$ for all $\epsilon_1, \epsilon_2 \in \{0, 1\}$.

Claim. The coefficients $a_{(0,1,i,j,k_0,l)}$ and $a_{(1,1,i,j,k_0,l)}$ are all zero for all $l \geq 0$. We now justify the claim by an induction on l. From (6.3.25) and (6.3.26), the result is obviously true when $l = 0$. For $l \geq 0$, assume that $a_{(0,1,i,j,k_0,l)} = a_{(1,1,i,j,k_0,l)} = 0$. Then, it follows from (6.3.25) and (6.3.26) that

$$a_{(0,1,i,j,k_0,l+1)} = -\frac{\alpha^2\lambda_{1,2}}{\beta\lambda_{1,1}}a_{(1,1,i,j,k_0,l)} - \frac{\lambda_{1,3}}{\lambda_{1,1}}a_{(1,1,i-2,j,k_0,l)},$$

$$a_{(1,1,i,j,k_0,l+1)} = -\frac{\alpha\lambda_{2,1}}{\beta\lambda_{2,2}}a_{(0,1,i,j,k_0,l)} - \frac{\alpha\lambda_{2,3}}{\beta\lambda_{2,2}}a_{(1,1,i-2,j,k_0,l-1)}.$$

From the inductive hypothesis, $a_{(1,1,i,j,k_0,l)} = a_{(1,1,i-2,j,k_0,l)} = a_{(0,1,i,j,k_0,l)} = a_{(1,1,i-2,j,k_0,l-1)} = 0$. Hence, $a_{(1,1,i,j,k_0,l+1)} = a_{(0,1,i,j,k_0,l+1)} = 0$. By the principle of mathematical induction, $a_{(0,1,i,j,k_0,l)} = a_{(1,1,i,j,k_0,l)} = 0$ for all $l \geq 0$ as desired. Given that the families $a_{(0,1,i,j,k_0,l)}$ and $a_{(1,1,i,j,k_0,l)}$ are all zero, it follows from (6.3.23) and (6.3.24) that $a_{(0,0,i,j,k_0,l)}$ and $a_{(1,0,i,j,k_0,l)}$ are also zero for all $(i, j, k_0, l) \in \mathbb{N}^2 \times \mathbb{Z} \times \mathbb{N}$. Since $a_{(\epsilon_1,\epsilon_2,i,j,k_0,l)}$ are all zero, it contradicts our assumption. Hence, $x_- = 0$. Consequently, $x = x_+ \in \mathcal{A}_{\alpha,\beta}$ as desired.

2. From Remark 6.2.6, we have that $x_4 = x_{4,6} + \frac{2}{3}t_5^3t_6^{-1} = t_4 + \frac{2}{3}t_5^3t_6^{-1}$. Again, from Lemma 6.3.4, we have that $\lambda_4 = 3\lambda_3 + \lambda_5$ and $\lambda_6 = -3\lambda_3$. Therefore,

$$\begin{aligned}\rho(x_4) &= \lambda_4 t_4 + \frac{2}{3}(3\lambda_5 - \lambda_6)t_5^3 t_6^{-1}\\ &= (3\lambda_3 + \lambda_5)x_{4,6} + 2(\lambda_3 + \lambda_5)t_5^3 t_6^{-1}\\ &= (3\lambda_3 + \lambda_5)\left(x_4 - \frac{2}{3}t_5^3 t_6^{-1}\right) + 2(\lambda_3 + \lambda_5)t_5^3 t_6^{-1}\\ &= (3\lambda_3 + \lambda_5)x_4 + \frac{4}{3}\lambda_5 t_5^3 t_6^{-1}.\end{aligned}$$

Hence,

$$\mathcal{D}(x_4) = \mathsf{ham}_x(x_4) + \rho(x_4) = \mathsf{ham}_x(x_4) + (3\lambda_3 + \lambda_5)x_4 + \frac{4}{3}\lambda_5 t_5^3 t_6^{-1} \in \mathcal{A}_{\alpha,\beta}.$$

It follows that $\lambda_5 t_5^3 t_6^{-1} \in \mathcal{A}_{\alpha,\beta}$, since $\mathsf{ham}_x(x_4) + (3\lambda_3 + \lambda_5)x_4 \in \mathcal{A}_{\alpha,\beta}$. Consequently, $\lambda_5 t_5^3 \in \mathcal{A}_{\alpha,\beta} t_6$. Clearly, $\lambda_5 = 0$, otherwise, there will be a contradiction using the basis of $\mathcal{A}_{\alpha,\beta}$ (Proposition 5.3.4). Therefore, $\rho(x_4) = 3\lambda_3 x_4$ and $\rho(t_5) = 0$. We already know from Lemma 6.3.4 that $\rho(t_6) = -3\lambda_3 t_6$. From (6.3.14), we have $\rho(x_3) = \lambda_3 x_3 + \frac{3}{2}(\lambda_4 - \lambda_3 - \lambda_5)x_4 t_5^{-1} + (3\lambda_5 - \lambda_4 - \lambda_6)t_5^2 t_6^{-1}$. Given that $\lambda_4 = 3\lambda_3,\ \lambda_5 = 0$ and $\lambda_6 = -3\lambda_3$, we have that $\rho(x_3) = \lambda_3 x_3 + 3\lambda_3 x_4 t_5^{-1}$. Now, $\mathcal{D}(x_3) = \mathsf{ham}_x(x_3) + \rho(x_3) = \mathsf{ham}_x(x_3) + \lambda_3 x_3 + 3\lambda_3 x_4 t_5^{-1} \in \mathcal{A}_{\alpha,\beta}$. Observe that $\mathsf{ham}_x(x_3), \lambda_3 x_3 \in \mathcal{A}_{\alpha,\beta}$. Hence, $\lambda_3 x_4 t_5^{-1} \in \mathcal{A}_{\alpha,\beta}$ which implies that $\lambda_3 x_4 \in \mathcal{A}_{\alpha,\beta} t_5$. Therefore, $\lambda_3 = 0$, otherwise, there will be a contradiction using the basis of $\mathcal{A}_{\alpha,\beta}$. We now have that $\rho(x_3) = \rho(x_4) = \rho(x_5) = \rho(x_6) = 0$. We finish the proof by showing that $\rho(x_1) = \rho(x_2) = 0$. Recall from (5.3.2) that

$$x_2 x_4 x_6 - \frac{2}{3}x_3^3 x_6 - \frac{2}{3}x_2 x_5^3 + 2x_3^2 x_5^2 - 3x_3 x_4 x_5 + \frac{3}{2}x_4^2 = \beta.$$

Apply ρ to this relation to obtain $\rho(x_2)x_4 x_6 - \frac{2}{3}\rho(x_2)x_5^3 = 0$. This implies that $\rho(x_2)\left(x_4 x_6 - \frac{2}{3}x_5^3\right) = 0$. Since $x_4 x_6 - \frac{2}{3}x_5^3 \neq 0$, it follows that $\rho(x_2) = 0$. Similarly, from (5.3.1), we have that

$$x_1 x_3 x_5 - \frac{3}{2}x_1 x_4 - \frac{1}{2}x_2 x_5 + \frac{1}{2}x_3^2 = \alpha.$$

Apply ρ to this relation to obtain $\rho(x_1)\left(x_3x_5 - \frac{3}{2}x_4\right) = 0$. Since $x_3x_5 - \frac{3}{2}x_4 \neq 0$, we must have $\rho(x_1) = 0$. In conclusion, $\rho(x_\kappa) = 0$ for all $\kappa \in \{1, \cdots, 6\}$.

3. As a result of (1) and (2), we have that $\mathcal{D}(x_\kappa) = \text{ham}_x(x_\kappa)$. Consequently, $\mathcal{D} = \text{ham}_x$ as desired. ■

6.3.5 Poisson derivations of $\mathcal{A}_{\alpha,0}$ and $\mathcal{A}_{0,\beta}$. One can observe that the process we went through to compute the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$ (when $\alpha, \beta \neq 0$) is similar to the process we went through to compute the derivations of the non-commutative analogue $A_{\alpha,\beta}$ (see Subsection 4.2.1). Similarly, when α or β is zero, then one can follow procedures similar to Subsection 4.2.5 to compute the Poisson derivations of $\mathcal{A}_{\alpha,0}$ and $\mathcal{A}_{0,\beta}$. The computations have been done, however, for the avoidance of redundancy, we are not going to include them here. We only summarize the results. Before we do that, we compute explicitly the scalar Poisson derivations of $\mathcal{A}_{\alpha,0}$ and $\mathcal{A}_{0,\beta}$.

6.3.6 Lemma. Let $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Suppose that ϑ and $\tilde{\vartheta}$ are linear maps of $\mathcal{A}_{\alpha,0}$ and $\mathcal{A}_{0,\beta}$ respectively, and, are defined by:

$$\vartheta(x_1) = -x_1, \quad \vartheta(x_2) = -x_2, \quad \vartheta(x_3) = 0, \quad \vartheta(x_4) = x_4, \quad \vartheta(x_5) = x_5, \quad \vartheta(x_6) = 2x_6,$$

and

$$\tilde{\vartheta}(x_1) = -2x_1, \quad \tilde{\vartheta}(x_2) = -3x_2, \quad \tilde{\vartheta}(x_3) = -x_3, \quad \tilde{\vartheta}(x_4) = 0, \quad \tilde{\vartheta}(x_5) = x_5, \quad \tilde{\vartheta}(x_6) = 3x_6.$$

Then, ϑ and $\tilde{\vartheta}$ are $\mathbb{C}-$Poisson derivations of $\mathcal{A}_{\alpha,0}$ and $\mathcal{A}_{0,\beta}$ respectively.

Proof. We need to show that ϑ satisfies the following two relations (see (5.3.1) and (5.3.2)):

$$\begin{aligned}
&x_1x_3x_5 - \frac{3}{2}x_1x_4 - \frac{1}{2}x_2x_5 + \frac{1}{2}x_3^2 = \alpha, \\
&x_2x_4x_6 - \frac{2}{3}x_3^3x_6 - \frac{2}{3}x_2x_5^3 + 2x_3^2x_5^2 - 3x_3x_4x_5 + \frac{3}{2}x_4^2 = \beta,
\end{aligned}$$

and the Poisson bracket of $\mathcal{A}_{\alpha,\beta}$ (see Section 5.3) when $\alpha \neq 0$ and $\beta = 0$, and do the same for $\tilde{\vartheta}$ when $\alpha = 0$ and $\beta \neq 0$. We will only do this for the relation $x_1x_3x_5 - \frac{3}{2}x_1x_4 - \frac{1}{2}x_2x_5 + \frac{1}{2}x_3^2 = \alpha$ and the Poisson bracket $\{x_6, x_2\} = 3x_2x_6 + 9x_4 - 18x_3x_5$ in both cases, and leave the remaining ones for the reader to verify. We have:

$$\begin{aligned}\vartheta(x_1)x_3x_5 + x_1\vartheta(x_3)x_5 + x_1x_3\vartheta(x_5) - \frac{3}{2}[\vartheta(x_1)x_4 + x_1\vartheta(x_4)]\\ -\frac{1}{2}[\vartheta(x_2)x_5 + x_2\vartheta(x_5)] + \vartheta(x_3)x_3\\ = 0,\end{aligned}$$

and

$$\begin{aligned}\vartheta(\{x_6, x_2\}) &= \vartheta(3x_2x_6 + 9x_4 - 18x_3x_5)\\ &= 3[\vartheta(x_2)x_6 + x_2\vartheta(x_6)] + 9\vartheta(x_4) - 18[\vartheta(x_3)x_5 + x_3\vartheta(x_5)]\\ &= 3(-x_2x_6 + 2x_2x_6) + 9x_4 - 18x_3x_5\\ &= 3x_2x_6 + 9x_4 - 18x_3x_5\\ &= \{x_6, x_2\}\\ &= 2\{x_6, x_2\} - \{x_6, x_2\}\\ &= \{2x_6, x_2\} + \{x_6, -x_2\}\\ &= \{\vartheta(x_6), x_2\} + \{x_6, \vartheta(x_2)\}.\end{aligned}$$

When $\alpha = 0$ and $\beta \neq 0$, we show that $\tilde{\vartheta}$ satisfies the same relations as follows:

$$\begin{aligned}\tilde{\vartheta}(x_1)x_3x_5 + x_1\tilde{\vartheta}(x_3)x_5 + x_1x_3\tilde{\vartheta}(x_5) - \frac{3}{2}[\tilde{\vartheta}(x_1)x_4 + x_1\tilde{\vartheta}(x_4)]\\ -\frac{1}{2}[\tilde{\vartheta}(x_2)x_5 + x_2\tilde{\vartheta}(x_5)] + \tilde{\vartheta}(x_3)x_3\\ = -2\left(x_1x_3x_5 - \frac{3}{2}x_1x_4 - \frac{1}{2}x_2x_5 + \frac{1}{2}x_3^2\right)\\ = 0,\end{aligned}$$

and

$$
\begin{aligned}
\tilde{\vartheta}(\{x_6, x_2\}) &= \tilde{\vartheta}(3x_2x_6 + 9x_4 - 18x_3x_5) \\
&= 3[\tilde{\vartheta}(x_2)x_6 + x_2\tilde{\vartheta}(x_6)] + 9\tilde{\vartheta}(x_4) - 18[\tilde{\vartheta}(x_3)x_5 + x_3\tilde{\vartheta}(x_5)] \\
&= 0 \\
&= 3\{x_6, x_2\} - 3\{x_6, x_2\} \\
&= \{3x_6, x_2\} + \{x_6, -3x_2\} \\
&= \{\tilde{\vartheta}(x_6), x_2\} + \{x_6, \tilde{\vartheta}(x_2)\}.
\end{aligned}
$$

■

6.3.7 Remark. Let $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $\lambda \in \mathbb{C}$.

1. Every Poisson derivation $\mathcal{D}$ of $\mathcal{A}_{\alpha,0}$ is of the form $\mathcal{D} = \text{ham}_x + \rho_\lambda$, where ham_x is an inner Poisson derivation of $\mathcal{A}_{\alpha,0}$, and ρ is a scalar Poisson derivation of $\mathcal{A}_{\alpha,0}$ defined as follows:

$$
\begin{aligned}
\rho_\lambda(x_1) &= -\lambda x_1 & \rho_\lambda(x_3) &= 0 & \rho_\lambda(x_5) &= \lambda x_5 \\
\rho_\lambda(x_2) &= -\lambda x_2 & \rho_\lambda(x_4) &= \lambda x_4 & \rho_\lambda(x_6) &= 2\lambda x_6.
\end{aligned}
$$

2. Every Poisson derivation $\mathcal{D}$ of $\mathcal{A}_{0,\beta}$ is of the form $\mathcal{D} = \text{ham}_x + \rho_\lambda$, where ham_x is an inner Poisson derivation of $\mathcal{A}_{0,\beta}$, and ρ is a scalar Poisson derivation of $\mathcal{A}_{0,\beta}$ defined as follows:

$$
\begin{aligned}
\rho_\lambda(x_1) &= -2\lambda x_1 & \rho_\lambda(x_3) &= -\lambda x_3 & \rho_\lambda(x_5) &= \lambda x_5 \\
\rho_\lambda(x_2) &= -3\lambda x_2 & \rho_\lambda(x_4) &= 0 & \rho_\lambda(x_6) &= 3\lambda x_6.
\end{aligned}
$$

6.3.8 Theorem. *Given that* $\mathcal{A}_{\alpha,\beta} = \mathbb{C}[X_1, \cdots, X_6]/\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle$, *with* $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$, *we have the following results:*

1. *if* $\alpha, \beta \neq 0$; *then every Poisson derivation* $\mathcal{D}$ *of* $\mathcal{A}_{\alpha,\beta}$ *can uniquely be written as* $\mathcal{D} = ham_x$, *where* $x \in \mathcal{A}_{\alpha,\beta}$.

2. *if* $\alpha \neq 0$ *and* $\beta = 0$, *then every Poisson derivation* $\mathcal{D}$ *of* $\mathcal{A}_{\alpha,0}$ *can uniquely be written as* $\mathcal{D} = ham_x + \lambda\vartheta$, *where* $\lambda \in \mathbb{C}$ *and* $x \in \mathcal{A}_{\alpha,0}$.

3. *if* $\alpha = 0$ *and* $\beta \neq 0$, *then every Poisson derivation* $\mathcal{D}$ *of* $\mathcal{A}_{0,\beta}$ *can uniquely be written as* $\mathcal{D} = ham_x + \lambda\tilde{\vartheta}$, *where* $\lambda \in \mathbb{C}$ *and* $x \in \mathcal{A}_{0,\beta}$.

4. $HH^1(\mathcal{A}_{\alpha,0}) = \mathbb{C}[\vartheta]$ *and* $HH^1(\mathcal{A}_{0,\beta}) = \mathbb{C}[\tilde{\vartheta}]$, *where* $[\vartheta]$ *and* $[\tilde{\vartheta}]$ *respectively denote the classes of* ϑ *and* $\tilde{\vartheta}$ *modulo the space of inner Poisson derivations.*

5. *if* $\alpha, \beta \neq 0$; *then* $HH^1(\mathcal{A}_{\alpha,\beta}) = \{[0]\}$, *where* $[0]$ *denotes the class of* 0 *modulo the space of inner Poisson derivations.*

Proof. Points (1) is as a result of Lemma 6.3.4. Points (2) and (3) are as a result of Remark 6.3.7. Point (4) is a consequence of Lemma 6.3.6, and (5) is a consequence of (1). ∎

Conclusion

This thesis studied a q-deformation $A_{\alpha,\beta}$ $((\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\})$ of a quadratic extension of $A_2(\mathbb{C})$ and compared some properties of $A_{\alpha,\beta}$ to those of $A_2(\mathbb{C})$. Similar to $A_2(\mathbb{C})$, the algebra $A_{\alpha,\beta}$ is simple, noetherian domain, has GKdim of 4 and the central elements are all scalars. In addition, if $\alpha, \beta \neq 0$; then the group of units of $A_{\alpha,\beta}$ is the set of non-zero complex numbers and the derivations of $A_{\alpha,\beta}$ are all inner. However, when either α or β is zero, then the group of units of $A_{\alpha,\beta}$ are non-trivial, and the derivations are the sum of inner and scalar derivations. Therefore, when α and β are non-zero, then the properties of $A_{\alpha,\beta}$ largely reflect those of $A_2(\mathbb{C})$. As a result, $A_{\alpha,\beta}$ gives a better deformation of a quadratic extension of $A_2(\mathbb{C})$ when $\alpha\beta \neq 0$. We also compared the derivations of $A_{\alpha,\beta}$ to the Poisson derivations of a semiclassical limit $\mathcal{A}_{\alpha,\beta}$ of $A_{\alpha,\beta}$. In fact, the derivations of $A_{\alpha,\beta}$ and the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$ are congruent.

All efforts to compute the automorphism group of $A_{\alpha,\beta}$ (i.e. $\mathsf{Aut}(A_{\alpha,\beta})$) were not successful. Nevertheless, we realized some automorphism subgroups of $\mathsf{Aut}(A_{\alpha,\beta})$. Let $\mu, \lambda \in \mathbb{C}^*$; and define $\phi_{\mu,\lambda} : A_{\alpha,\beta} \to A_{\alpha,\beta}$ by $\phi_{\mu,\lambda}(e_1) = \mu e_1$ and $\phi_{\mu,\lambda}(e_6) = \lambda e_6$. We have the following:

- $\{\phi_{\mu,\lambda} \mid \mu^2\lambda = 1;\ \forall \mu, \lambda \in \mathbb{C}^*\} \subseteq \mathsf{Aut}(A_{\alpha,0})$, where $\alpha \neq 0$.
- $\{\phi_{\mu,\lambda} \mid \mu^3\lambda^2 = 1;\ \forall \mu, \lambda \in \mathbb{C}^*\} \subseteq \mathsf{Aut}(A_{0,\beta})$, where $\beta \neq 0$.
- $\{\phi_{\mu,\lambda} \mid \mu, \lambda \in \{-1, 1\}\} \subseteq \mathsf{Aut}(A_{\alpha,\beta})$, where $\alpha, \beta \neq 0$.

The following questions are worth considering, and are opened for further studies.

Questions. Let $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Are there any automorphisms of $A_{\alpha,\beta}$? Is every endomorphism of $A_{\alpha,\beta}$ an automorphism?

Appendix A

Computations in $U_q^+(G_2)$

In this appendix, we provide the computations of the algebra relations of $U_q^+(G_2)$ in Section A.1, computations of the data of the deleting derivations algorithm (DDA) of $U_q^+(G_2)$ in Section A.2, and finally, provide computations of the generators of the central elements of $U_q^+(G_2)$ in Section A.3.

A.1 Algebra relations of $U_q^+(G_2)$

This section focuses on providing a summary of the computations of the defining relations of $U_q^+(G_2)$ in Subsection A.1.1, and confirming these relations with GAP in Subsection A.1.2. We finally provide some general relations of $U_q^+(G_2)$ in Subsection A.1.3.

A.1.1 Summary of the computations of the defining relations of $U_q^+(G_2)$. Note, the actions of T_1 and T_2 on $U_q^+(G_2)$ are defined in Section 2.1.

(1) $E_{\beta_1}E_{\beta_6}-q^{(\beta_1,\beta_6)}E_{\beta_6}E_{\beta_1}=\lambda E_{\beta_5}$. This implies that $E_1E_2-q^{-3}E_2E_1=\lambda T_1T_2T_1T_2(E_1)$ $=\lambda T_2^{-1}(T_2T_1T_2T_1T_2(E_1))=\lambda T_2^{-1}(E_1)$. Hence, $T_2(E_1)T_2(E_2)-q^{-3}T_2(E_2)T_2(E_1)=$ λE_1. One can verify that $T_2(E_1)T_2(E_2)-q^{-3}T_2(E_2)T_2(E_1)=E_1$. As a result, $\lambda=1$. Therefore, $\boxed{E_{\beta_1}E_{\beta_6}-q^{-3}E_{\beta_6}E_{\beta_1}=E_{\beta_5}}$.

(2) $E_{\beta_1}E_{\beta_3}-q^{(\beta_1,\beta_3)}E_{\beta_3}E_{\beta_1}=\lambda E_{\beta_2}$. This implies that $E_1T_1T_2(E_1)-qT_1T_2(E_1)E_1=$ $\lambda T_1(E_2)$. Hence, $T_1^{-1}(E_1)T_2(E_1)-qT_2(E_1)T_1^{-1}(E_1)=\lambda E_2$. One can verify that

$T_1^{-1}(E_1)T_2(E_1) - qT_2(E_1)T_1^{-1}(E_1) = (q^2+1+q^{-2})E_2$. As a result, $\lambda = (q^2+1+q^{-2})$, and $\boxed{E_{\beta_1}E_{\beta_3} - qE_{\beta_3}E_{\beta_1} = (q^2+1+q^{-2})E_{\beta_2}}$.

(3) $E_{\beta_3}E_{\beta_5} - q^{(\beta_3,\beta_5)}E_{\beta_5}E_{\beta_3} = \lambda E_{\beta_4}$. This implies that $E_{\beta_3}E_{\beta_5} - qE_{\beta_5}E_{\beta_3} = \lambda E_{\beta_4}$.

This relation is similar to the relation in point (2). Hence, $\lambda = (q^2+1+q^{-2})$, and $\boxed{E_{\beta_3}E_{\beta_5} - qE_{\beta_5}E_{\beta_3} = (q^2+1+q^{-2})E_{\beta_4}}$.

(4) $E_{\beta_2}E_{\beta_4} - q^{(\beta_2,\beta_4)}E_{\beta_4}E_{\beta_2} = \lambda E_{\beta_3}^3$. This implies that $T_1(E_2)T_1T_2T_1(E_2) - q^3T_1T_2T_1(E_2)T_1(E_2) = \lambda T_1T_2(E_1^3)$. Hence, $T_2^{-1}(E_2)T_1(E_2) - q^3T_1(E_2)T_2^{-1}(E_2) = \lambda E_1^3$. One can verify that $T_2^{-1}(E_2)T_1(E_2) - q^3T_1(E_2)T_2^{-1}(E_2) = \dfrac{q^7-2q^5+q^3}{q^4+q^2+1}E_1^3$. As a result, $\lambda = \dfrac{q^7-2q^5+q^3}{q^4+q^2+1}$, and $\boxed{E_{\beta_2}E_{\beta_4} - q^3E_{\beta_4}E_{\beta_2} = \dfrac{q^7-2q^5+q^3}{q^4+q^2+1}E_{\beta_3}^3}$.

(5) $E_{\beta_4}E_{\beta_6} - q^{(\beta_4,\beta_6)}E_{\beta_6}E_{\beta_4} = \lambda E_{\beta_5}^3$. This implies that $E_{\beta_4}E_{\beta_6} - q^3E_{\beta_6}E_{\beta_4} = \lambda E_{\beta_5}^3$.

This relation is similar to point (4). Hence, $\lambda = \dfrac{q^7-2q^5+q^3}{q^4+q^2+1}$, and $\boxed{E_{\beta_4}E_{\beta_6} - q^3E_{\beta_6}E_{\beta_4} = \dfrac{q^7-2q^5+q^3}{q^4+q^2+1}E_{\beta_5}^3}$.

(6) $E_{\beta_1}E_{\beta_5} - q^{(\beta_1,\beta_5)}E_{\beta_5}E_{\beta_1} = \lambda E_{\beta_3}$. This implies that $E_{\beta_1}E_{\beta_5} - q^{-1}E_{\beta_5}E_{\beta_1} = \lambda E_{\beta_3}$.

Now, $E_{\beta_3} = T_1T_2(E_1) = T_1(E_1E_2 - q^{-3}E_2E_1) = T_1(E_2)T_1(E_1) - q^{-3}T_1(E_1)T_1(E_2)$.

Further simplification shows that $E_{\beta_3} = T_1(E_2)T_1(E_1) - q_2^{-1}T_1(E_1)T_1(E_2) = \dfrac{-1}{q+q^{-1}}E_1^2E_2 + \dfrac{q^{-3}+q^{-1}}{q+q^{-1}}E_1E_2E_1 - \dfrac{q^{-4}}{q+q^{-1}}E_2E_1^2$. Furthermore, from point (1), $E_{\beta_5} = E_{\beta_1}E_{\beta_6} - q^{-3}E_{\beta_6}E_{\beta_1} = E_1E_2 - q^{-3}E_2E_1$. It follows that:

$$\begin{aligned}
E_{\beta_1}E_{\beta_5} - q^{-1}E_{\beta_5}E_{\beta_1} &= E_1(E_1E_2 - q^{-3}E_2E_1) - q^{-1}(E_1E_2 - q^{-3}E_2E_1)E_1 \\
&= E_1^2E_2 - (q^{-3}+q^{-1})E_1E_2E_1 + q^{-4}E_2E_1^2 \\
&= -E_1^2E_2 + (q^{-3}+q^{-1})E_1E_2E_1 - q^{-4}E_2E_1^2 \\
&= (q+q^{-1})E_{\beta_3}.
\end{aligned}$$

Consequently, $\boxed{E_{\beta_1}E_{\beta_5} - q^{-1}E_{\beta_5}E_{\beta_1} = (q+q^{-1})E_{\beta_3}}$.

(7) $E_{\beta_3}E_{\beta_6} - q^{(\beta_3,\beta_6)}E_{\beta_6}E_{\beta_3} = \lambda E_{\beta_5}^2$. This implies that $E_{\beta_3}E_2 - E_2E_{\beta_3} = \lambda E_{\beta_5}^2$. From point (6), one can make substitutions for E_{β_3} and E_{β_5}, and simplify to obtain $\lambda = q^2 - 1$. Therefore, $\boxed{E_{\beta_3}E_{\beta_6} - E_{\beta_6}E_{\beta_3} = (q^2-1)E_{\beta_5}^2}$.

(8) $E_{\beta_1}E_{\beta_4} - q^{(\beta_1,\beta_4)}E_{\beta_4}E_{\beta_1} = \lambda E_{\beta_3}^2$. This implies that $E_{\beta_1}E_{\beta_4} - E_{\beta_4}E_{\beta_1} = \lambda E_{\beta_3}^2$. This relation is similar to (7). Hence, $\lambda = q^2-1$, and $\boxed{E_{\beta_1}E_{\beta_4} - E_{\beta_4}E_{\beta_1} = (q^2-1)E_{\beta_3}^2}$.

(9) $E_{\beta_2}E_{\beta_5} - q^{(\beta_2,\beta_5)}E_{\beta_5}E_{\beta_2} = \lambda E_{\beta_3}^2$. This implies that $T_1(E_2)E_{\beta_5} - E_{\beta_5}T_1(E_2) = \lambda E_{\beta_3}^2$. Similarly, from points (6), we can make substitutions for E_{β_3} and E_{β_5}. Further simplification shows that $\lambda = q^2-1$. As a result, $\boxed{E_{\beta_2}E_{\beta_5} - E_{\beta_5}E_{\beta_2} = (q^2-1)E_{\beta_3}^2}$.

(10) $E_{\beta_2}E_{\beta_6} - q^{(\beta_2,\beta_6)}E_{\beta_6}E_{\beta_2} = \lambda E_{\beta_4} + \mu E_{\beta_3}E_{\beta_5}$. This implies that $T_1(E_2)E_2 - q^{-3}E_2T_1(E_2)$ $= \lambda E_{\beta_4} + \mu E_{\beta_3}E_{\beta_5}$. Using the expressions for E_{β_3} and E_{β_5} in point (6), and the relation: $E_{\beta_4} = (q^2+1+q^{-2})^{-1}(E_{\beta_3}E_{\beta_5} - qE_{\beta_5}E_{\beta_3})$, one can simplify $T_1(E_2)E_2 - q^{-3}E_2T_1(E_2) = \lambda E_{\beta_4} + \mu E_{\beta_3}E_{\beta_5}$ to obtain $\lambda = -(q+q^{-1}-q^{-3})$ and $\mu = q - q^{-1}$. Consequently, $\boxed{E_{\beta_2}E_{\beta_6} - q^{-3}E_{\beta_6}E_{\beta_2} = -(q+q^{-1}-q^{-3})E_{\beta_4} + (q-q^{-1})E_{\beta_3}E_{\beta_5}}$.

(11) $\boxed{E_{\beta_i}E_{\beta_j} = q^3E_{\beta_j}E_{\beta_i}}$, for all $1 \le i, j \le 6$ with $j - i = 1$.

A.1.2 Algebra relations of $U_q^+(G_2)$: GAP Code. The code below confirms the defining relations of $U_q^+(G_2)$. Recall that $E_i = E_{\beta_i}$ (see the comments before the relations of $U_q^+(G_2)$ in Section 2.1). Moreover, $E_3^{(r)} = \dfrac{E_3^r}{[r]_q!}$ and $E_5^{(r)} = \dfrac{E_5^r}{[r]_q!}$ for all $r \in \mathbb{N}_{>1}$.

```
brk> U:=QuantizedUEA(RootSystem("G",2));;
U:=QuantizedUEA(RootSystem("G",2));;
^

brk> T:=GeneratorsOfAlgebra(U);
T:=GeneratorsOfAlgebra(U);
^

[ F1, F2, F3, F4, F5, F6, K1, (-q+q^-1)*[ K1 ; 1 ]+K1, K2,
(-q^3+q^-3)*[ K2 ; 1 ]+K2, E1, E2, E3, E4, E5, E6 ]
brk> g:=[T[11],T[12],T[13],T[14],T[15],T[16]];
#The generators of U_q^+(G_2) are;
g:=[T[11],T[12],T[13],T[14],T[15],T[16]];
^

[ E1, E2, E3, E4, E5, E6 ]
```

```
brk> #We now compute the algebra relations for U_q^+(G_2)
brk> g[2]*g[1];
(q^-3)*E1*E2
brk> g[3]*g[1];
(q^-1)*E1*E3+(-q-q^-1-q^-3)*E2
brk> g[3]*g[2];
(q^-3)*E2*E3
brk> g[4]*g[1];
E1*E4+(-q^3+q^-1)*E3^(2)
brk> g[4]*g[2];
(q^-3)*E2*E4+(-q^3+q+q^-1-q^-3)*E3^(3)
brk> g[4]*g[3];
(q^-3)*E3*E4
brk> g[5]*g[1];
(q)*E1*E5+(-q^2-1)*E3
brk> g[5]*g[2];
E2*E5+(-q^3+q^-1)*E3^(2)
brk> g[5]*g[3];
(q^-1)*E3*E5+(-q-q^-1-q^-3)*E4
brk> g[5]*g[4];
(q^-3)*E4*E5Page 122
brk> g[6]*g[1];
(q^3)*E1*E6+(-q^3)*E5
brk> g[6]*g[2];
(q^3)*E2*E6+(-q^4+q^2)*E3*E5+(q^4+q^2-1)*E4
brk> g[6]*g[3];
E3*E6+(-q^3+q^-1)*E5^(2)
brk> g[6]*g[4];
(q^-3)*E4*E6+(-q^3+q+q^-1-q^-3)*E5^(3)
brk> g[6]*g[5];
(q^-3)*E5*E6
```

A.1.3 Some general relations of $U_q^+(G_2)$. We have the following selected general relations of $U_q^+(G_2)$.

A.1.4 Lemma. For any $n \in \mathbb{N}$, we have that:

1(a) $E_jE_i^n = q^{-3n}E_i^nE_j$ (b) $E_j^nE_i = q^{-3n}E_iE_j^n$ for all $1 \leq i, j \leq 6$, with $j-i = 1$.

2(a) $E_6E_4^n = q^{-3n}E_4^nE_6 + d_1[n]E_4^{n-1}E_5^3$ (b) $E_6^nE_4 = q^{-3n}E_4E_6^n + d_1[n]E_5^3E_6^{n-1}$

(c) $E_4E_2^n = q^{-3n}E_2^nE_4 + d_1[n]E_2^{n-1}E_3^3$ (d) $E_4^nE_2 = q^{-3n}E_2E_4^n + d_1[n]E_3^3E_4^{n-1}$,

where $d_1[n] = q^{3(1-n)}d_1[1]\left(\dfrac{1-q^{-6n}}{1-q^{-6}}\right)$; $d_1[1] = -\dfrac{q^4-2q^2+1}{q^4+q^2+1}$ and $d_1[0] := 0$.

3(a) $E_3E_1^n = q^{-n}E_1^nE_3 + d_2[n]E_1^{n-1}E_2$ (b) $E_3^nE_1 = q^{-n}E_1E_3^n + d_2[n]E_2E_3^{n-1}$

(c) $E_5E_3^n = q^{-n}E_3^nE_5 + d_2[n]E_3^{n-1}E_4$ (d) $E_5^nE_3 = q^{-n}E_3E_5^n + d_2[n]E_4E_5^{n-1}$,

where $d_2[n] = q^{1-n}d_2[1]\left(\dfrac{1-q^{-2n}}{1-q^{-2}}\right)$; $d_2[1] = -(q+q^{-1}+q^{-3})$ and $d_2[0] := 0$.

4(a) $E_6^nE_3 = E_3E_6^n + d_3[n]E_5^2E_6^{n-1}$ (b) $E_5E_2^n = E_2^nE_5 + d_3[n]E_2^{n-1}E_3^2$,

where $d_3[n] = d_3[1]\left(\dfrac{1-q^{-6n}}{1-q^{-6}}\right)$; $d_3[1] = 1-q^2$ and $d_3[0] := 0$.

Proof. We prove the lemma by an induction on n. We will only prove for the first non-trivial case (i.e. 2(a)), and leave the remaining ones for the reader to verify.

2(a). The result is clear when $n = 0$. For $n \geq 0$, suppose that $E_6E_4^n = q^{-3n}E_4^nE_6 + d_1[n]E_4^{n-1}E_5^3$, where $d_1[n] = q^{3(1-n)}d_1[1](1-q^{-6n})/(1-q^{-6})$; $d_1[1] = -(q^4-2q^2+1)/(q^4+q^2+1)$ and $d_1[0] := 0$. Then, $E_6E_4^{n+1} = (q^{-3n}E_4^nE_6 + d_1[n]E_4^{n-1}E_5^3)E_4 = q^{-3(n+1)}E_4^{n+1}E_6 + (q^{-3n}d_1[1] + q^{-9}d_1[n])E_4^nE_5^3$. Note, $E_6E_4 = q^{-3}E_4E_6 + d_1[1]E_5^3$. Now, $q^{-3n}d_1[1] + q^{-9}d_1[n] = q^{-3n}d_1[1] + q^{3(1-n)-9}d_1[1](1-q^{-6n})/(1-q^{-6}) = q^{-3n}d_1[1](1-q^{-6(n+1)})/(1-q^{-6}) = d_1[n+1]$ as expected. ∎

A.2 Deleting derivations algorithm of $U_q^+(G_2)$

Given that $\sigma_j \circ \delta_j = q_j\delta_j \circ \sigma_j$ (see the comments after Definition 1.7.1), we have that

- $\sigma_3 \circ \delta_3(E_1) = q^{-2}\delta_3 \circ \sigma_3(E_1)$ hence $q_3 = q^{-2}$,
- $\sigma_4 \circ \delta_4(E_1) = q^{-6}\delta_4 \circ \sigma_4(E_1)$ hence $q_4 = q^{-6}$,
- $\sigma_5 \circ \delta_5(E_1) = q^{-2}\delta_5 \circ \sigma_5(E_1)$ hence $q_5 = q^{-2}$,
- $\sigma_6 \circ \delta_6(E_1) = q^{-6}\delta_6 \circ \sigma_6(E_1)$ hence $q_6 = q^{-6}$.

From (1.7.1), we have the relation:

$$E_{i,j} = \begin{cases} E_{i,j+1} & \text{if } i \geq j \\ \sum_{k=0}^{+\infty} \frac{(1-q_j)^{-k}}{[k]_{q_j}!} \delta_j^k \circ \sigma_j^{-k}(E_{i,j+1}) E_{j,j+1}^{-k} & \text{if } i < j. \end{cases}$$

Given this relation, one can compute the elements of Fract($U_q^+(G_2)$) as follows:

$$\begin{aligned}
E_{1,6} &= E_1 + (1-q_6)^{-1}\delta_6 \circ \sigma_6^{-1}(E_1)E_6^{-1} \\
&= E_1 + (1-q^{-6})^{-1}E_5E_6^{-1} \\
&= E_1 + rE_5E_6^{-1}. \\
E_{2,6} &= E_2 + (1-q_6)^{-1}\delta_6 \circ \sigma_6^{-1}(E_2)E_6^{-1} + \frac{(1-q_6)^{-2}}{[2]_{q_6}!}\delta_6^2 \circ \sigma_6^{-2}(E_2)E_6^{-2} \\
&= E_2 + (1-q^{-6})^{-1}[(q^{-1}-q)E_3E_5 + (q+q^{-1}-q^{-3})E_4]E_6^{-1} \\
&+ \frac{(1-q^{-6})^{-2}}{1+q^{-6}}\left[(q^{-4}-q^{-2})(1-q^2) + \frac{(q^{-2}+q^{-4}-q^{-6})(2q^2-q^4-1)}{(q^4+q^2+1)}\right]E_5^3E_6^{-2} \\
&= E_2 + \frac{q^{-1}-q}{1-q^{-6}}E_3E_5E_6^{-1} + \frac{q+q^{-1}-q^{-3}}{1-q^{-6}}E_4E_6^{-1} \\
&+ \frac{(q^{-4}-q^{-2})(1-q^2)(q^4+q^2+1) + (q^{-2}+q^{-4}-q^{-6})(2q^2-q^4-1)}{(1-q^{-6})^2(1+q^{-6})(q^4+q^2+1)}E_5^3E_6^{-2} \\
&= E_2 + tE_3E_5E_6^{-1} + uE_4E_6^{-1} + nE_5^3E_6^{-2}. \\
E_{3,6} &= E_3 + (1-q_6)^{-1}\delta_6 \circ \sigma_6^{-1}(E_3)E_6^{-1} \\
&= E_3 + \frac{1-q^2}{1-q^{-6}}E_5^2E_6^{-1} \\
&= E_3 + sE_5^2E_6^{-1}. \\
E_{4,6} &= E_4 + (1-q_6)^{-1}\delta_6 \circ \sigma_6^{-1}(E_4)E_6^{-1} \\
&= E_4 + \frac{-q^7+2q^5-q^3}{(1-q^{-6})(q^4+q^2+1)}E_5^3E_6^{-1} \\
&= E_4 + bE_5^3E_6^{-1}.
\end{aligned}$$

$$\begin{aligned}
E_{1,5} &= E_{1,6} + (1-q_5)^{-1}\delta_5 \circ \sigma_5^{-1}(E_{1,6})E_{5,6}^{-1} + \frac{(1-q_5)^{-2}}{[2]_{q_5}!}\delta_5^2 \circ \sigma_5^{-2}(E_{1,6})E_{5,6}^{-2} \\
&= E_{1,6} + \frac{q+q^{-1}}{q^{-2}-1}E_{3,6}E_{5,6}^{-1} + \frac{q+q^{-1}+q^{-3}}{(1-q^{-2})^2}E_{4,6}E_{5,6}^{-2} \\
&= E_{1,6} + hE_{3,6}E_{5,6}^{-1} + gE_{4,6}E_{5,6}^{-2}. \\
E_{2,5} &= E_{2,6} + (1-q_5)^{-1}\delta_5 \circ \sigma_5^{-1}(E_{2,6})E_{5,6}^{-1} + \frac{(1-q_5)^{-2}}{[2]_{q_5}!}\delta_5^2 \circ \sigma_5^{-2}(E_{2,6})E_{5,6}^{-2} \\
&+ \frac{(1-q_5)^{-3}}{[3]_{q_5}!}\delta_5^3 \circ \sigma_5^{-3}(E_{2,6})E_{5,6}^{-3} \\
&= E_{2,6} + \frac{1-q^2}{1-q^{-2}}E_{3,6}^2E_{5,6}^{-1} + \frac{(q^{-1}+q^{-3})(q^2-1)(q+q^{-1}+q^{-3})}{(1-q^{-2})^2(1+q^{-2})}E_{3,6}E_{4,6}E_{5,6}^{-2} \\
&+ \frac{(q^{-1}+q^{-3})(1-q^2)(q+q^{-1}+q^{-3})^2}{(1-q^{-2})^3(1+q^{-2})(1+q^{-2}+q^{-4})}E_{4,6}^2E_{5,6}^{-3} \\
&= E_{2,6} + fE_{3,6}^2E_{5,6}^{-1} + pE_{3,6}E_{4,6}E_{5,6}^{-2} + eE_{4,6}^2E_{5,6}^{-3}. \\
E_{3,5} &= E_{3,6} + (1-q_5)^{-1}\delta_5 \circ \sigma_5^{-1}(E_{3,6})E_{3,6}^{-1} \\
&= E_{3,6} + \frac{q^2+1+q^{-2}}{q^{-2}-1}E_{4,6}E_{5,6}^{-1} \\
&= E_{3,6} + aE_{4,6}E_{5,6}^{-1}. \\
E_{1,4} &= E_{1,5} + (1-q_4)^{-1}\delta_4 \circ \sigma_4^{-1}(E_{1,5})E_{4,5}^{-1} \\
&= E_{1,5} + \frac{1-q^2}{1-q^{-6}}E_{3,5}^2E_{4,5}^{-1} \\
&= E_{1,5} + sE_{3,5}^2E_{4,5}^{-1}. \\
E_{2,4} &= E_{2,5} + (1-q_4)^{-1}\delta_4 \circ \sigma_4^{-1}(E_{2,5})E_{4,5}^{-1} \\
&= E_{2,5} + \frac{-q^7+2q^5-q^3}{(1-q^{-6})(q^4+q^2+1)}E_{3,5}^3E_{4,5}^{-1} \\
&= E_{2,5} + bE_{3,5}^3E_{4,5}^{-1}. \\
E_{1,3} &= E_{1,4} + (1-q_3)^{-1}\delta_3 \circ \sigma_3^{-1}(E_{1,4})E_{3,4}^{-1} \\
&= E_{1,4} + \frac{q^2+1+q^{-2}}{q^{-2}-1}E_{2,4}E_{3,4}^{-1} \\
&= E_{1,4} + aE_{2,4}E_{3,4}^{-1}.
\end{aligned}$$

Otherwise, $E_{i,j} = E_{i,j+1}$, where $1 \leq i,j \leq 6$. Note, all the *constant coefficients* $(a,b,e,f,g,h,n,p,r,s,t,u)$ are defined in Appendix C.

A.3 Generators of the center of $U_q^+(G_2)$

Recall from Section 2.2 that $T_i = E_{i,2}$ for all $1 \leq i \leq 6$, $\Omega_1 = T_1T_3T_5$ and $\Omega_2 = T_2T_4T_6$. Now, using the data of the DDA of $U_q^+(G_2)$, we have the following:

$$\begin{aligned}
\Omega_1 = T_1T_3T_5 &= E_{1,2}E_{3,2}E_{5,2} = E_{1,3}E_{3,3}E_{5,3} \\
&= (E_{1,4} + aE_{2,4}E_{3,4}^{-1})E_{3,4}E_{5,4} \\
&= (E_{1,4}E_{3,4} + aE_{2,4})E_{5,4} \\
&= [(E_{1,5} + sE_{3,5}^2E_{4,5}^{-1})E_{3,5} + a(E_{2,5} + bE_{3,5}^3E_{4,5}^{-1})]E_{5,4} \\
&= (E_{1,5}E_{3,5} + q^3sE_{3,5}^3E_{4,5}^{-1} + abE_{3,5}^3E_{4,5}^{-1} + aE_{2,5})E_{5,5} \\
&= (E_{1,5}E_{3,5} + aE_{2,5})E_{5,5} \text{ (Note, } q^3s + ab = 0) \\
&= [(E_{1,6} + hE_{3,6}E_{5,6}^{-1} + gE_{4,6}E_{5,6}^{-2})(E_{3,6} + aE_{4,6}E_{5,6}^{-1}) + aE_{2,6} + afE_{3,6}^2E_{5,6}^{-1} \\
&\quad + apE_{3,6}E_{4,6}E_{5,6}^{-2} + aeE_{4,6}^2E_{5,6}^{-3}]E_{5,6} \\
&= E_{1,6}E_{3,6}E_{5,6} + (hq + af)E_{3,6}^2 + aE_{1,6}E_{4,6} + aE_{2,6}E_{5,6}. \qquad \text{(A.3.1)}
\end{aligned}$$

From the data of the DDA of $U_q^+(G_2)$, one can make the necessary substitutions for $E_{i,6}$ (with $1 \leq i \leq 6$) in (A.3.1), and simplify to obtain

$$\Omega_1 = T_1T_3T_5 = E_1E_3E_5 + aE_1E_4 + aE_2E_5 + a'E_3^2$$

as desired. In a similar manner, we have that:

$$\begin{aligned}
\Omega_2 = T_2T_4T_6 &= E_{2,2}E_{4,2}E_{6,2} = E_{2,3}E_{4,3}E_{6,3} = E_{2,4}E_{4,4}E_{6,4} \\
&= (E_{2,5} + bE_{3,5}^3E_{4,5}^{-1})E_{4,5}E_{6,5} \\
&= E_{2,5}E_{4,5}E_{6,5} + bE_{3,5}^3E_{6,5} \\
&= (E_{2,6} + fE_{3,6}^2E_{5,6}^{-1} + pE_{3,6}E_{4,6}E_{5,6}^{-2} + eE_{4,6}^2E_{5,6}^{-3})E_{4,6}E_{6,6} \\
&\quad + b(E_{3,6} + aE_{4,6}^3E_{5,6}^{-1})^3E_{6,6} \\
&= E_{2,6}E_{4,6}E_{6,6} + bE_{3,6}^3E_{6,6}. \qquad \text{(A.3.2)}
\end{aligned}$$

Again, from the data of the DDA, one can make the necessary substitutions for $E_{i,6}$ (with $1 \leq i \leq 6$) in (A.3.2), and simplify to obtain:

$$\Omega_2 = T_2T_4T_6 = E_2E_4E_6 + bE_2E_5^3 + bE_3^3E_6 + b'E_3^2E_5^2 + c'E_3E_4E_5 + d'E_4^2$$

as desired.

Note, the constants a, a', b, b', c' and d' are defined in Appendix C.

A.3.1 The generators of the center of $U_q^+(G_2)$: GAP Code. The code below confirms that Ω_1 and Ω_2 commute with E_1 and E_6 as expected.

```
brk> U:=QuantizedUEA(RootSystem("G",2));;
U:=QuantizedUEA(RootSystem("G",2));;
^
brk> T:=GeneratorsOfAlgebra(U);
T:=GeneratorsOfAlgebra(U);
^
[ F1, F2, F3, F4, F5, F6, K1, (-q+q^-1)*[ K1 ; 1 ]+K1, K2,
(-q^3+q^-3)*[ K2 ; 1 ]+K2, E1, E2, E3, E4, E5, E6 ]
#The generators of U_q^+(G_2) are;
g:=[T[11],T[12],T[13],T[14],T[15],T[16]];
^
[ E1, E2, E3, E4, E5, E6 ]
brk> a:=-(_q^2+1+_q^-2)/(1-_q^-2);
a:=-(_q^2+1+_q^-2)/(1-_q^-2);
^
(-q^4-q^2-1)/(q^2-1)
brk> a1:=-_q^6/(1-_q^2);
a1:=-_q^6/(1-_q^2);
^
(-q^6)/(-q^2+1)Page 123
brk> b:=(_q^7-2*_q^5+_q^3)/((1-_q^-6)*(_q^4+_q^2+1));
b:=-(_q^7-2*_q^5+_q^3)/((1-_q^-6)*(_q^4+_q^2+1));
^
(q^11-q^9)/(q^8+2*q^6+3*q^4+2*q^2+1)
brk> b1:=(_q^11-_q^13)/(_q^8+2*_q^6+3*_q^4+2*_q^2+1);
b1:=-(_q^11-_q^13)/(_q^8+2*_q^6+3*_q^4+2*_q^2+1);
^
```

```
(-q^13+q^11)/(q^8+2*q^6+3*q^4+2*q^2+1)
brk> c1:=-_q^9/(_q^4+_q^2+1);
c1:=-_q^9/(_q^4+_q^2+1);
^
(-q^9)/(q^4+q^2+1)
brk> d1:=-_q^12/(1-_q^6);
d1:=-_q^12/(1-_q^6);
^
(-q^12)/(-q^6+1)
brk> Omega_1:=g[1]*g[3]*g[5]+a*g[1]*g[4]+a*g[2]*g[5]+a1*g[3]^2;
E1*E3*E5+((-q^4-q^2-1)/(q^2-1))*E1*E4+((-q^4-q^2-1)/(q^2-1))*E2*E5+
((-q^7-q^5)/(-q^2+1))*E3^(2)
brk> Omega_1*g[1]=g[1]*Omega_1;
true
brk> Omega_1*g[6]=g[6]*Omega_1;
true
brk>Omega_2:=g[2]*g[4]*g[6]+b*g[2]*g[5]^3+b*g[3]^3*g[6]+b1*g[3]^2*g[5]^2
+c1*g[3]*g[4]*g[5]+d1*g[4]^2;
E2*E4*E6+((-q^10+q^6)/(q^4+q^2+1))*E2*E5^(3)+((-q^9)/(q^4+q^2+1))*E3*E4*E5
+((q^15+q^13-q^11-q^9)/(q^8+2*q^6+3*q^4+2*q^2+1))*E3^(2)*E5^(2)
+((-q^10+q^6)/(q^4+q^2+1))*E3^(3)*E6+((-q^15-q^9)/(-q^6+1))*E4^(2)
brk> Omega_2*g[1]=g[1]*Omega_2;
true
brk> Omega_2*g[6]=g[6]*Omega_2;
true
```

Appendix B

Computations in $\mathcal{A} = \mathbb{C}[X_1, \cdots, X_6]$

Recall that the Poisson algebra $\mathcal{A} = \mathbb{C}[X_1, \cdots, X_6]$ is the semiclassical limit of $U_q^+(G_2)$ (Section 5.2). In this appendix, we define the Poisson bracket of $\mathcal{A}$ in Section B.1, and the data of the Poisson deleting derivations algorithm (PDDA) of $\mathcal{A}$ in Section B.2.

B.1 Poisson bracket of $\mathcal{A}$

Note, $f(z) = z^4 + z^2 + 1$ (see Section 5.2). In addition, set $g(z) := z^2 + z + 1$. We have the following:

$$\begin{aligned}
\{X_3, X_1\} &= \frac{\widehat{U}_3\widehat{U}_1 - \widehat{U}_1\widehat{U}_3}{z-1} + (z-1)\widehat{A} = -z^{-1}\widehat{U}_1\widehat{U}_3 - z^{-3}\widehat{U}_2 + (z-1)\widehat{A} \\
&= -X_1X_3 - X_2. \\
\{X_4, X_1\} &= \frac{\widehat{U}_4\widehat{U}_1 - \widehat{U}_1\widehat{U}_4}{z-1} + (z-1)\widehat{A} = -(z+1)\widehat{U}_3^2 + (z-1)\widehat{A} = -2X_3^2. \\
\{X_4, X_2\} &= \frac{\widehat{U}_4\widehat{U}_2 - \widehat{U}_2\widehat{U}_4}{z-1} + (z-1)\widehat{A} = -z^{-3}g(z)\widehat{U}_2\widehat{U}_4 - (z+1)^2\widehat{U}_3^3 + (z-1)\widehat{A} \\
&= -3X_2X_4 - 4X_3^3. \\
\{X_5, X_1\} &= \frac{\widehat{U}_5\widehat{U}_1 - \widehat{U}_1\widehat{U}_5}{z-1} + (z-1)\widehat{A} = \widehat{U}_1\widehat{U}_5 - (z^2+1)\widehat{U}_3 + (z-1)\widehat{A} \\
&= X_1X_5 - 2X_3. \\
\{X_5, X_2\} &= \frac{\widehat{U}_5\widehat{U}_2 - \widehat{U}_2\widehat{U}_5}{z-1} + (z-1)\widehat{A} = -(z+1)f(z)\widehat{U}_3^2 + (z-1)\widehat{A} = -6X_3^2.
\end{aligned}$$

$$\{X_5, X_3\} = \frac{\widehat{U}_5\widehat{U}_3 - \widehat{U}_3\widehat{U}_5}{z-1} + (z-1)\widehat{A} = -z^{-1}\widehat{U}_3\widehat{U}_5 - z^{-3}f(z)\widehat{U}_4 + (z-1)\widehat{A}$$
$$= -X_3X_5 - 3X_4.$$
$$\{X_6, X_1\} = \frac{\widehat{U}_6\widehat{U}_1 - \widehat{U}_1\widehat{U}_6}{z-1} + (z-1)\widehat{A} = g(z)\widehat{U}_1\widehat{U}_6 - z^3f(z)\widehat{U}_5 + (z-1)\widehat{A}$$
$$= 3X_1X_6 - 3X_5.$$
$$\{X_6, X_2\} = \frac{\widehat{U}_6\widehat{U}_2 - \widehat{U}_2\widehat{U}_6}{z-1} + (z-1)\widehat{A} = g(z)\widehat{U}_2\widehat{U}_6 + (z^4+z^2-1)f(z)^2\widehat{U}_4$$
$$- f(z)^2(z^2+z^3)\widehat{U}_3\widehat{U}_5 + (z-1)\widehat{A}$$
$$= 3X_2X_6 + 9X_4 - 18X_3X_5.$$
$$\{X_6, X_3\} = \frac{\widehat{U}_6\widehat{U}_3 - \widehat{U}_3\widehat{U}_6}{z-1} + (z-1)\widehat{A} = -f(z)(1+z)\widehat{U}_5^2 + (z-1)\widehat{A} = -6X_5^2.$$
$$\{X_6, X_4\} = \frac{\widehat{U}_6\widehat{U}_4 - \widehat{U}_4\widehat{U}_6}{z-1} + (z-1)\widehat{A} = -z^{-3}g(z)\widehat{U}_4\widehat{U}_6 - (z+1)^2\widehat{U}_5^3 + (z-1)\widehat{A}$$
$$= -3X_4X_6 - 4X_5^3.$$
$$\{X_j, X_i\} = \frac{\widehat{U}_j\widehat{U}_i - \widehat{U}_i\widehat{U}_j}{z-1} + (z-1)\widehat{A} = -z^{-3}g(z)\widehat{U}_i\widehat{U}_j + (z-1)\widehat{A} = -3X_iX_j,$$

for all $1 \leq i < j \leq 6$ with $j - i = 1$.

B.2 PDDA of the semiclassical limit of $U_q^+(G_2)$

Given that $\delta_j\alpha_j - \alpha_j\delta_j = \eta_j\delta_j$ (Hypothesis 5.1.11), we have the following:

- $(\delta_3\sigma_3 - \sigma_3\delta_3)(X_1) = -2X_2$ and $\delta_3(X_1) = -X_2$, hence $\eta_3 = 2$;
- $(\delta_4\sigma_4 - \sigma_4\delta_4)(X_1) = -12X_3^2$ and $\delta_4(X_1) = -2X_3^2$, hence $\eta_4 = 6$;
- $(\delta_5\sigma_5 - \sigma_5\delta_5)(X_2) = -12X_3^2$ and $\delta_5(X_2) = -6X_3^2$, hence $\eta_5 = 2$;
- $(\delta_6\sigma_6 - \sigma_6\delta_6)(X_1) = -18X_5$ and $\delta_6(X_1) = -3X_5$, hence $\eta_6 = 6$.

We now compute the data of the PDDA of $\mathcal{A}$ using the relation

$$X_{i,j} = \begin{cases} X_{i,j+1} & \text{if } i \geq j \\ \sum_{k=0}^{+\infty} \frac{1}{\eta_j^k k!} \delta_j^k(X_{i,j+1}) X_{j,j+1}^{-k} & \text{if } i < j \end{cases}$$

(which can be found in Subsection 5.1.8) as follows:

- $X_{1,6} = X_1 + \frac{1}{\eta_6}\delta_6(X_1)X_6^{-1} = X_1 - \frac{1}{2}X_5X_6^{-1}$.
- $X_{2,6} = X_2 + \frac{1}{\eta_6}\delta_6(X_2)X_6^{-1} + \frac{1}{2\eta_6^2}\delta_6^2(X_2)X_6^{-2} = X_2 + \frac{3}{2}X_4X_6^{-1} - 3X_3X_5X_6^{-1} + X_5^3X_6^{-2}$.
- $X_{3,6} = X_3 + \frac{1}{\eta_6}\delta_6(X_3)X_6^{-1} = X_3 - X_5^2X_6^{-1}$.
- $X_{4,6} = X_4 + \frac{1}{\eta_6}\delta_6(X_4)X_6^{-1} = X_4 - \frac{2}{3}X_5^3X_6^{-1}$.
- $X_{1,5} = X_{1,6} + \frac{1}{\eta_5}\delta_5(X_{1,6})X_{5,6}^{-1} + \frac{1}{2\eta_5^2}\delta_5^2(X_{1,6})X_{5,6}^{-2} = X_{1,6} - X_{3,6}X_{5,6}^{-1} + \frac{3}{4}X_{4,6}X_{5,6}^{-2}$.
- $X_{2,5} = X_{2,6} + \frac{1}{\eta_5}\delta_5(X_{2,6})X_{5,6}^{-1} + \frac{1}{2\eta_5^2}\delta_5^2(X_{2,6})X_{5,6}^{-2} + \frac{1}{3!\eta_5^3}\delta_5^3(X_{2,6})X_{5,6}^{-3}$
$= X_{2,6} - 3X_{3,6}^2X_{5,6}^{-1} + \frac{9}{2}X_{3,6}X_{4,6}X_{5,6}^{-2} - \frac{9}{4}X_{4,6}^2X_{5,6}^{-3}$.
- $X_{3,5} = X_{3,6} + \frac{1}{\eta_5}\delta_5(X_{3,6})X_{5,6}^{-1} = X_{3,6} - \frac{3}{2}X_{4,6}X_{5,6}^{-1}$.
- $X_{1,4} = X_{1,5} + \frac{1}{\eta_4}\delta_4(X_{1,5})X_{4,5}^{-1} = X_{1,5} - \frac{1}{3}X_{3,5}^2X_{4,5}^{-1}$.
- $X_{2,4} = X_{2,5} + \frac{1}{\eta_4}\delta_4(X_{2,5})X_{4,5}^{-1} = X_{2,5} - \frac{2}{3}X_{3,5}^3X_{4,5}^{-1}$.
- $X_{1,3} = X_{1,4} + \frac{1}{\eta_3}\delta_3(X_{1,4})X_{3,4}^{-1} = X_{1,4} - \frac{1}{2}X_{2,4}X_{3,4}^{-1}$.
- Otherwise, $X_{i,j} = X_{i,j+1}$, where $1 \leq i, j \leq 6$.

Appendix C

Definition of scalars used

In this appendix, we define some scalars used in the thesis. Note, for all $n \in \mathbb{N}$, we have already defined the scalars $d_2[n]$ in Lemma A.1.4, hence, we are not going to repeat them here. Any other scalars not defined here must be defined in/before the context in which it is found.

$$a = \frac{q^2 + 1 + q^{-2}}{q^{-2} - 1} \qquad b = -\frac{q^7 - 2q^5 + q^3}{(q^4 + q^2 + 1)(1 - q^{-6})}$$

$$g = \frac{q + q^{-1} + q^{-3}}{(1 - q^{-2})^2} \qquad f = \frac{1 - q^2}{1 - q^{-2}}$$

$$h = \frac{q + q^{-1}}{q^{-2} - 1} \qquad s = \frac{1 - q^2}{1 - q^{-6}}$$

$$t = \frac{q^{-1} - q}{1 - q^{-6}} \qquad u = \frac{q + q^{-1} - q^{-3}}{1 - q^{-6}}$$

$$p = \frac{q^4 + q^2 + 1}{q^2 - 1} \qquad r = \frac{-1}{1 - q^{-6}}$$

$$e = \frac{-(q^7 + q^5 + q^3)}{q^4 - 2q^2 + 1} \qquad q'' = \frac{q^7 - 2q^5 + q^3}{q^4 + q^2 + 1}$$

$$n = \frac{q^{12}}{(q^4 + q^2 + 1)^3} \qquad q' = -(q^2 + 1 + q^{-2})$$

$$k_1 = q^{-3}b_2 + b_6 d_2[1] \qquad a' = af + hq = \frac{q^6}{q^2 - 1}$$

$$k_2 = q^{-3}b_3 + b_{12}d_2[1] \qquad b' = \frac{q^{13} - q^{11}}{(q^4 + q^2 + 1)^2}$$

$$k_3 = b_4c_1 \qquad d' = \frac{q^{12}}{q^6 - 1}$$

$$k_4 = b_4c_2 + q^{-3}b_5c_1 + b_7d_2[1] \qquad c' = -\frac{q^9}{q^4 + q^2 + 1}$$

$$k_5 = b_4c_2 + q^{-1}b_6c_1 \qquad c_1 = \frac{1}{a'}$$

$$k_6 = c_3b_4 + q^{-1}b_7 + q^{-3}b_4b_{13}c_2 \qquad c_2 = -ac_1$$

$$k_7 = q^{-3}c_2b_5 + b_8d_2[1] \qquad c_3 = -c_1$$

$$k_8 = b_1b_{13}c_2 \qquad b_1 = \frac{1}{d'}$$

$$k_9 = q^{-4}b_6c_2 + b_9d_2[2] + q^{-3}b_2c_2b_{13} + q^{-3}b_5c_2 \qquad b_2 = b_1bc_2(q + q^{-1} + q^{-3}) - b_1$$

$$k_{10} = q^{-1}b_6b_8c_2 \qquad b_3 = -b'b_1c_2 - bb_1$$

$$k_{11} = b_{13}c_1 \qquad b_4 = -b_1bc_1$$

$$k_{12} = q^{-1}b_6b_7c_2 \qquad b_5 = b_1b(c_3(q + q^{-1} + q^{-3}) - q^{-3}c_2)$$

$$k_{13} = q^{-4}b_6c_3 + q^{-2}b_9 + q^{-3}b_6b_{13}c_2 + q^{-1}b_6b_{13}c_2 \qquad b_6 = -q^{-1}c_2b_1b$$

$$k_{14} = q^{-1}b_1b_6c_2 \qquad b_7 = -q^{-1}b_1bc_1c_3$$

$$k_{15} = q^{-1}b_2b_6c_2 \qquad b_8 = b_9 = -q^{-1}b_1bc_2c_3$$

$$k_{16} = q^{-1}b_3b_6c_2 + q^{-2}b_{10}c_2 + q^{-3}b_8b_{13}c_2 \qquad b_{10} = -q^{-1}c_3^2b_1b$$

$$k_{17} = q^{-1}b_4b_6c_2 \qquad b_{11} = -b'b_1c_1$$

$$k_{18} = q^{-1}b_5b_6c_2 \qquad b_{12} = -b'b_1c_3$$

$$k_{19} = q^{-1}b_6^2c_2 \qquad b_{13} = -b_1c'$$

$$k_{20} = q^{-1}b_6b_9c_2 \qquad b_{14} = -b'b_1c_2$$

$$k_{21} = q^{-1}b_6b_{11}c_2 + q^{-2}b_{10}c_1 + q^{-3}b_7b_{13}c_2 \qquad b_{15} = q^{-3}c_3 + c_2b_{13}$$

$$k_{23} = q^{-1}b_6b_{12}c_2 + q^{-2}b_{10}c_3 + q^{-3}b_{10}b_{13}c_2 \qquad k_{22} = q^{-1}b_6b_{10}c_2$$

$$k_{25} = q^{-1}b_{12}c_1 + b_{11}b_{13}c_2 \qquad k_{24} = q^{-1}b_6b_{14}c_2 + q^{-2}b_{10}c_2 + q^{-3}b_9b_{13}c_2$$

$$k_{27} = q^{-1}b_{12}c_3 + b_{12}b_{13}c_2 \qquad k_{26} = q^{-1}b_{12}c_2 + b_{13}b_{14}c_2$$

$$k_{29} = b_{13}b_{15} + q^{-1}b_{14} \qquad k_{28} = q^{-3}b_{13}c_2 + b_{14}d_2[1]$$

$$k_{31} = q^{-3}b_5b_{13}c_2 + q^{-3}b_5c_3 + q^{-4}b_8 + b_{10}d_2[2] \qquad k_{30} = b_3b_{13}c_2 + q^{-1}b_{12}c_2$$

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