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Geometric processes and its extensions

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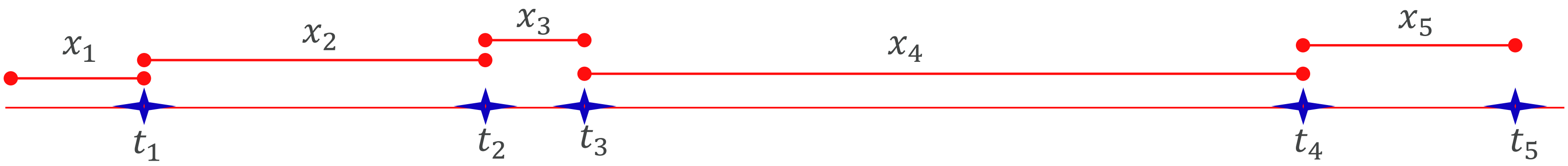
Outline

- Background
- Literature review
- New models and their performance
- Future research

Recurrent event data analysis

- Analysis of recurrent events data:



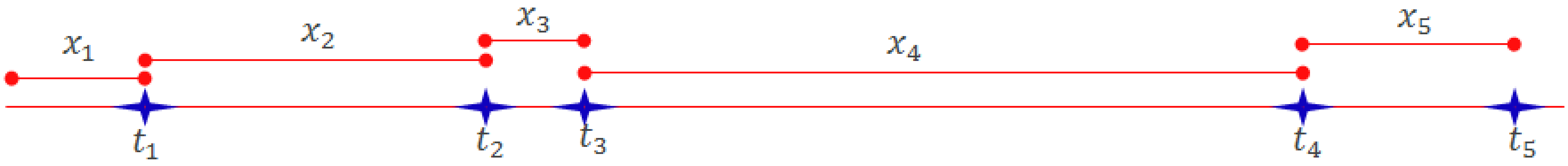
- Applications
  - modelling of insurance claims/warranty claims: times between claims
  - modelling of the outbreak of an epidemic disease: the number of cases
  - modelling of time between failures of technical systems (software or hardware)

Challenges

1. Common difficulties in reliability engineering:
   - Lacks of failures, technical systems are normally reliable and do not have many failure data
   - Censoring: when the observation period ends, not all units have failed - some are survivors
2. Recurrent event data analysis is also widely studied in the healthcare sector, in which
   - the effect of covariates is the focus and
   - they have more data available for modelling

Recurrent event data analysis

- Recurrent events:



- Notations:

$$S_n = \sum_{i=1}^{n} X_i$$

$$Y_t = \sum_{n=1}^{\infty} \chi\{S_n \le t\} = \sup\{n: S_n \le t\}$$

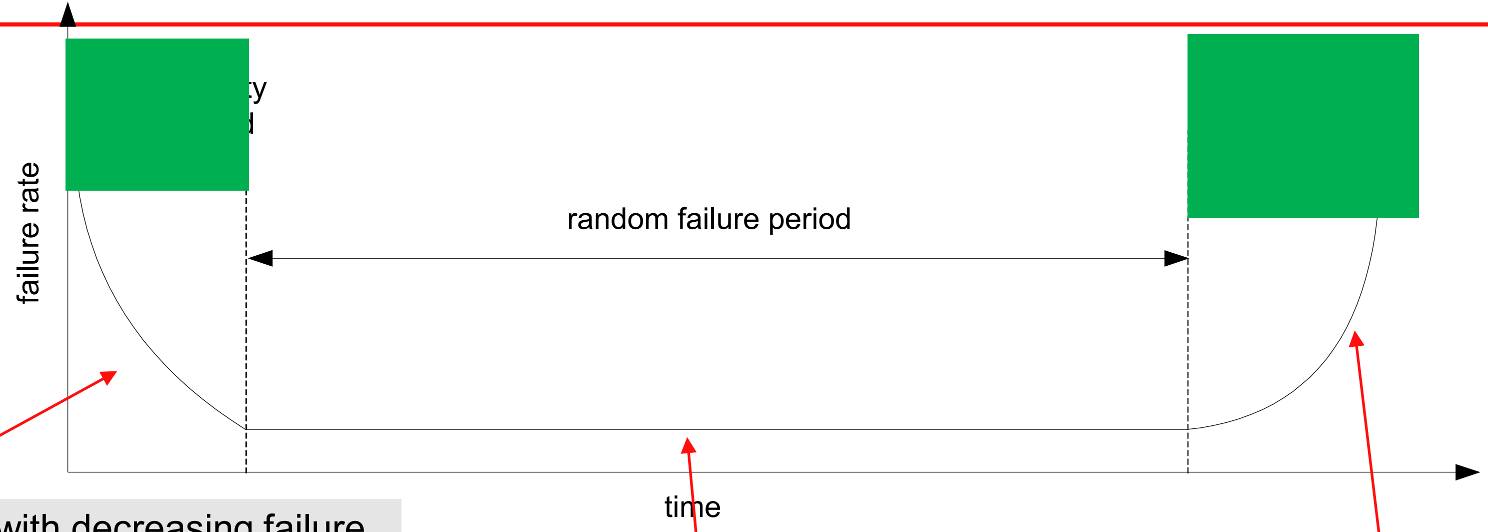
$$m(t) = E[Y_t]$$

Questions in reliability mathematics:

1. What are the distributions of the gap times, i.e., the distributions of X_i?
2. How many events occurred within a given time, i.e., How can we estimate $m(t)$?

Literature review

Bathtub curve in reliability engineering



Early-life failures with decreasing failure intensity function

- Poor design
- Defects in the material
- Manufacturing problems

Failures with approximately a constant failure intensity

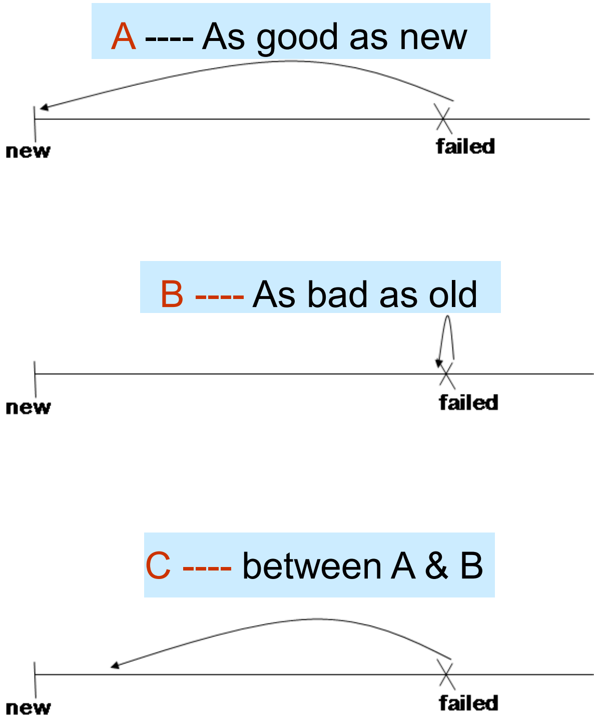
- Maintenance errors
- Human errors
- Failures due to nature

Failures with increasing failure intensity function

- Low cycle fatigue
- Bearing failures
- Corrosion, erosion

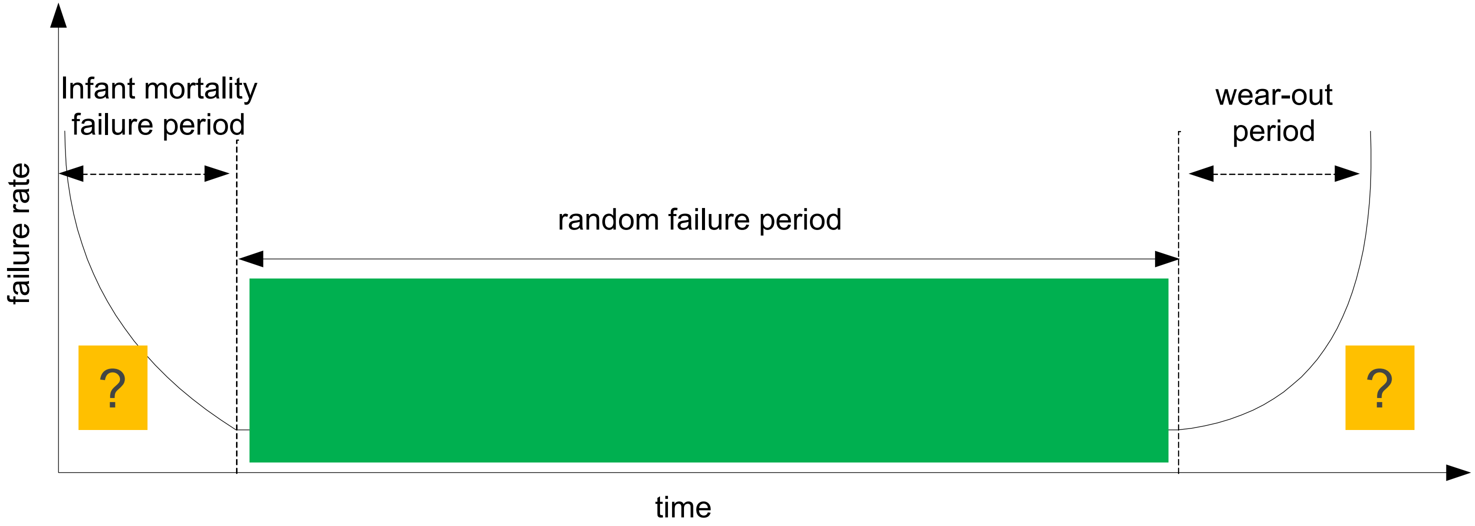
Existing modelling methods

- Parametric methods, for example
  - Renewal process (RP): replacement, repaired as good as new
  - Nonhomogeneous Poisson process (NHPP): minimal repair
  - Virtual age models
  - Geometric process
- Non-parametric methods



Renewal process

- Given a sequence of non-negative random variables $\{X_k, k = 1,2, \ldots\}$, if they are Independent and identically distributed, then $\{X_k, k = 1,2, \cdots\}$ is called a **renewal process** (RP).



Renewal process

Geometric process

- Given a sequence of non-negative random variables $\{X_k, k = 1,2, \dots\}$, if they are independent and the cdf of X_k is given by $\boldsymbol{F(a^{k-1}x)}$ for $k = 1,2, \dots$, where a is a positive constant, then $\{X_k, k = 1,2, \cdots\}$ is called a **geometric process** (GP) (Lam, 1988).

- Remarks
  - If $a > 1$, then $\{X_k, k = 1, 2, \cdots\}$ is stochastically decreasing.
  - If $a < 1$, then $\{X_k, k = 1, 2, \cdots\}$ is stochastically increasing.
  - If $a = 1$, then $\{X_k, k = 1, 2, \cdots\}$ is a renewal process (RP).
  - If $\{X_k, k = 1,2, \dots\}$ is a GP and X_1 follows the Weibull distribution, then the shape parameter of X_k for $k = 2,3, \dots$ remains the same as that of X_1.

Literature review

- The GP has been extensively studied since its introduction in 1988 (Lam, 1988), mainly due to its elegance and convenience in deriving mathematical properties in applications

Geometric Process (GP)

$X_k \sim F(a_k x)$,
$1 = a_1 \le \cdots \le a_k \ldots$
(Finkelstein, 1993)

$X_k = a^{k-1} W_k$,
W_k are iid
(Wang, et. al. 1996)

$X_k \sim F(k^{-a} x)$,
(Braun, et. al. 2005)

$X_k \sim F\big((\alpha a^{k-1} +$

$\{a_i^{n-M_i} Z_n, M_i \le n < M_{i+1}\}$
forms a renewal process
(Chan, et. al., 2006)

$X_k \sim F\left(a^{-b_k} x\right)$,
(Bordes, et. al. 2013)

Geometric Process (GP)

$X_k \sim F(g(k)x)$
$g(k)$ is a function of k

Motivation for the doubly geometric process (DGP)

The process $\{X_k, k = 1,2,\ldots\}$ with $X_k \sim F(g(k)x)$ has the following restrictive implications

- **Invariance of the CV** (coefficient of variation): Given a GP $\{X_1, X_2, \ldots\}$, then

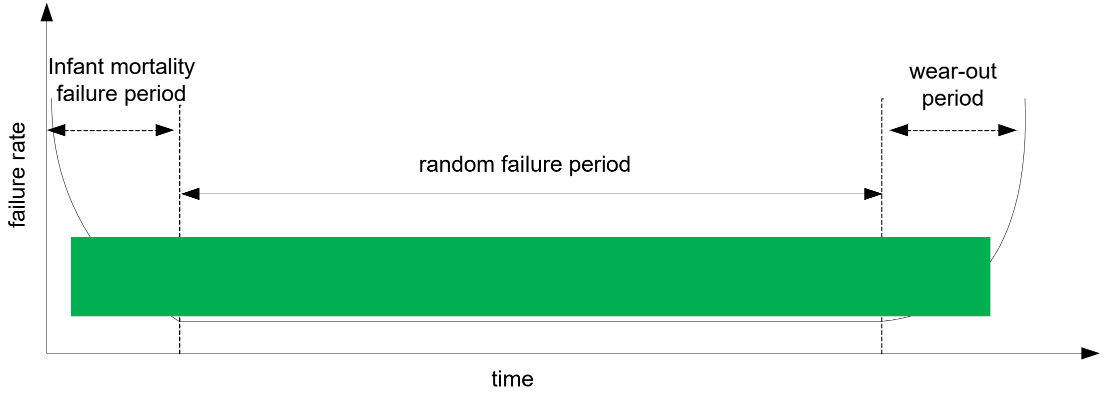
$$CV_k = \frac{\sqrt{V(X_k)}}{E(X_k)} = \frac{\sqrt{E(X_1^2) - \left(E(X_1)\right)^2}}{E(X_1)}$$

- **Invariance of the shape parameter**: Suppose $X_1 \sim 1 - \exp\{-\left(\frac{x}{\theta_1}\right)^{\theta_2}\}$, then

$$F(g(k)x) = 1 - \exp\{-\left(\frac{x}{\theta_1\left(g(k)\right)^{-1}}\right)^{\theta_2}\}$$

Literature review

- **Monotonicity of the GP**: from the remarks, the GP $\{X_k, k = 1,2,\ldots\}$ change monotonously, either increasing or decreasing



Geometric Process (GP)

$X_k \sim F\big((\alpha a^{k-1} +$

$\{a_i^{n-M_i} Z_n, M_i \leq n < M_{i+1}\}$ forms a renewal process (Chan, et. al., 2006)

Doubly geometric process

- Given a sequence of non-negative random variables $\{X_k, k = 1,2, \ldots\}$, if they are independent and the cdf of X_k is given by $\boldsymbol{F(a^{k-1}x^{h(k)})}$ for $k = 1,2, \ldots$, where a is a positive constant, then $\{X_k, k = 1,2, \cdots\}$ is called a **doubly geometric process** (GP) (Wu, 2018).

Geometric Process (GP)

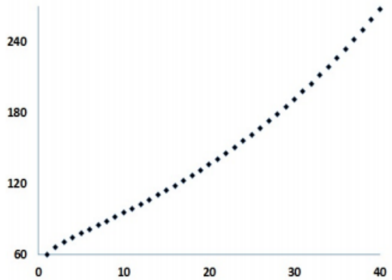
$\downarrow$

Doubly geometric process (DGP)

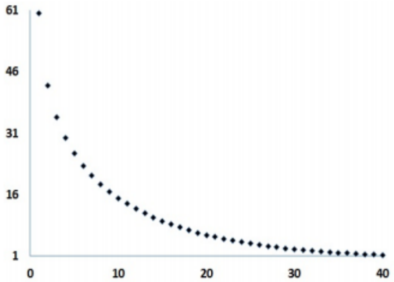
$X_k \sim F\left(a^{k-1}x^{h(k)}\right)$

$h(k)$ is a function of k

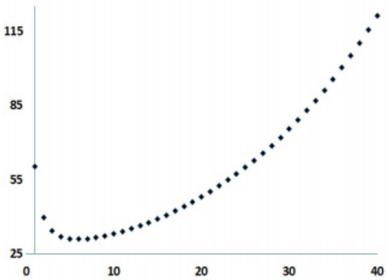
Examples



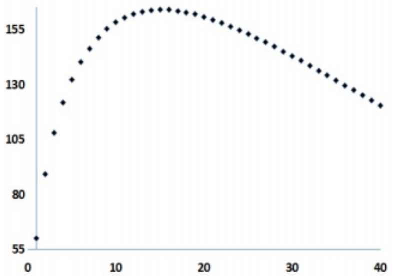
(a) $a = 0.97, b = -0.05, \theta_1 = 40$ and $\theta_2 = 0.6$.



(b) $a = 1.1, b = 0.2, \theta_1 = 40$ and $\theta_2 = 0.6$.



(c) $a = 0.92, b = 0.4, \theta_1 = 40$ and $\theta_2 = 0.6$.



(d) $a = 1.02, b = -0.3, \theta_1 = 40$ and $\theta_2 = 0.6$.

Figure 1: DGPs with different parameter settings: the Y-axes are $\mathbb{E}[X_k]$ and the X axes are k, $F(x) = 1 - e^{-(\frac{x}{\theta_1})^{\theta_2}}$, $h(k) = (1 + \log(k))^b$.

- If X_1 follows the exponential distribution and
  - if $\{X_k, k = 1,2, \ldots\}$ follows the GP, then X_k (for $k = 2,3, \ldots$) follows the exponential distribution with different rate parameters from that of X_1,
  - if $\{X_k, k = 1,2, \ldots\}$ follows the DGP, then X_k (for $k = 2,3, \ldots$) follows the Weibull distribution,
- If $\{X_k, k = 1,2, \ldots\}$ follows the DGP and X_1 follows the Weibull distribution, then X_k (for $k > 1$) follows the Weibull distribution with different shape and scale parameters from those of X_1
- Suppose that $\{X_k, k = 1,2, \ldots\}$ is a DGP, then the coefficient of variation (CV) of X_k changes over k's.

Parameter estimation

- The least square method

$$(\hat{\mu}, \hat{a}, \hat{b}) = \arg\min_{\mu,a,b} \sum_{k=1}^{N_0} \left(x_k - (\mu a^{1-k})^{(1+\log(k))^{-b}} \right)^2$$

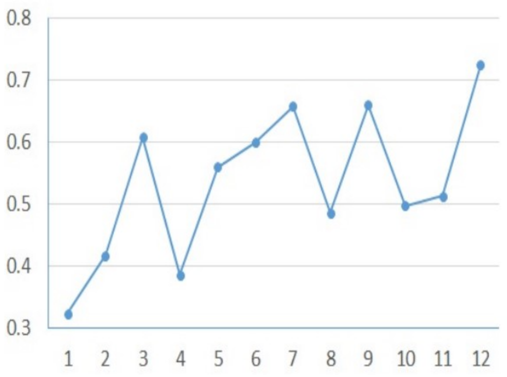
- The maximum likelihood method

$$\begin{aligned} L(a, b, \boldsymbol{\theta}) &= \prod_{j=1}^{N} \left\{ \left[1 - F(a^{N_j}(x_{j,N_j})^{(1+\log(N_j+1))^b}) \right] \prod_{k=1}^{N_j} f_k(x_{j,i}) \right\} \\ &= \prod_{j=1}^{N} \left\{ \left[1 - F(a^{N_j}(x_{j,N_j})^{(1+\log(N_j+1))^b}) \right] \right. \\ &\quad \left. \times \prod_{k=1}^{N_j} \left[a^{k-1}(1+\log(k))^b (x_{j,i})^{(1+\log(k))^b - 1} f(a^{k-1}(x_{j,i})^{(1+\log(k))^b}) \right] \right\} \end{aligned}$$

Data analysis

Table 1: Time between warranty claims of 22 identical items (unit: day).

Months \ Shipments	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	CV
1	10	8	13	7	8	16	9	6	7	15	11	9	13	7	9	6	13	10	9	5	0.323
2	7	4	8	6	9	6	1	8	8	9	11	10	10	9	7	8	1	3	9	12	0.417
3	11	7	15	3	4	3	3	13	9	13	6	4	3	5	5	6	3	2	8	5	0.607
4	8	3	12	6	7	6	11	9	9	7	10	7	8	11	6	5	8	5	6	17	0.385
5	4	3	4	2	8	6	7	15	7	9	10	5	2	6	4	14	3	7	10	13	0.559
6	11	8	5	10	4	5	7	8	1	6	11	1	3	4	3	9	4	5	16	13	0.599
7	7	7	22	3	5	14	12	5	4	7	9	4	4	6	17	4	13	3	6	5	0.658
8	11	8	4	5	4	12	6	10	3	4	8	3	5	12	9	10	3	11	4	4	0.486
8	4	3	16	7	1	8	3	6	1	5	6	4	4	12	5	2	4	5	5	6	0.660
10	2	5	9	4	3	10	11	8	1	12	8	6	10	7	2	3	9	10	6	9	0.497
11	5	4	8	4	7	12	1	9	5	8	4	7	3	2	3	5	13	8	7	6	0.513
12	4	5	2	6	1	7	6	10	4	3	12	2	2	17	4	13	6	1	9	5	0.724



- The AICc of DGP is 630.090 and that of GP is 630.242

Reliability data: Time between failure data

Reliability data

Table 2: The datasets, including TBF(Time between failures).

No.	Dataset	N_0	References
1	Hydraulic system (LHD3)	25	Kumar and Klefsjö (1992)
2	Propulsion diesel engine failure data	71	Ascher and Feingold (1984)

The least squares method

Table 3 Comparison of the performance of the GP and the DGP based on the least squares method

No.	*Parameters of the DGP*			*Parameters of the GP*		$RMSE_{DGP}$	$RMSE_{GP}$
	$\hat{a}$	$\hat{b}$	$\hat{\mu}$	$\hat{a}$	$\hat{\mu}$		
1	0.944 (0.0559)	0.499 (0.174)	531.406 (109.390)	1.0382 (0.0315)	209.841 (67.652)	*111.729*	144.431
2	0.909 (0.0607)	0.488 (0.280)	147.624 (62.664)	0.972 (0.0181)	56.702 (20.486)	*65.670*	69.810

The maximum likelihood method

Table 4 Comparison of the performance of the GP and the DGP based on the maximum likelihood method

No.	*Estimated parameters of the DGP*				*Estimated parameters of the GP*			$AICc_{DGP}$	$AICc_{GP}$	$AICc_{PL}$
	$\hat{a}$	$\hat{b}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{a}$	$\hat{\theta}_1$	$\hat{\theta}_2$			
1	0.884 (0.0938)	0.638 (0.352)	449.165 (337.92)	0.789 (0.227)	1.0147 (0.0230)	168.807 (58.139)	1.0287 (0.159)	*301.376*	304.182	311.851
2	0.899 (0.0714)	0.502 (0.349)	147.636 (103.569)	0.964 (0.281)	0.983 (0.0151)	73.070 (19.461)	1.295 (0.182)	*318.030*	319.445	323.094

GP-like models that can describe non-monotonic trends

- There are GP-like models that can describe non-monotonic trends

Geometric Process (GP)

$X_k \sim F\big((\alpha a^{k-1} +$	$\{a_i^{n-M_i} Z_n, M_i \le n < M_{i+1}\}$ forms a renewal process (Chan, et. al., 2006)	$X_k \sim F\big(a^{-b_k} x^{h(k)}\big)$, (Wu, 2018)
4 parameters	Flexible, but many parameters	2 parameters, the shape parameters changes over k

- **Invariance of the shape parameter**: S[illegible]pose $X_1 \sim 1 - \exp\{$[illegible]$\left(\frac{x}{\theta_1}\right)^{\theta_2}\}$, then

$$F(g([illegible] \exp\{ [illegible] \left(\frac{x}{\theta_1(g([illegible]))^{-1}}\right)^{\theta_2}\}$$

Requirements of a new GP-like model

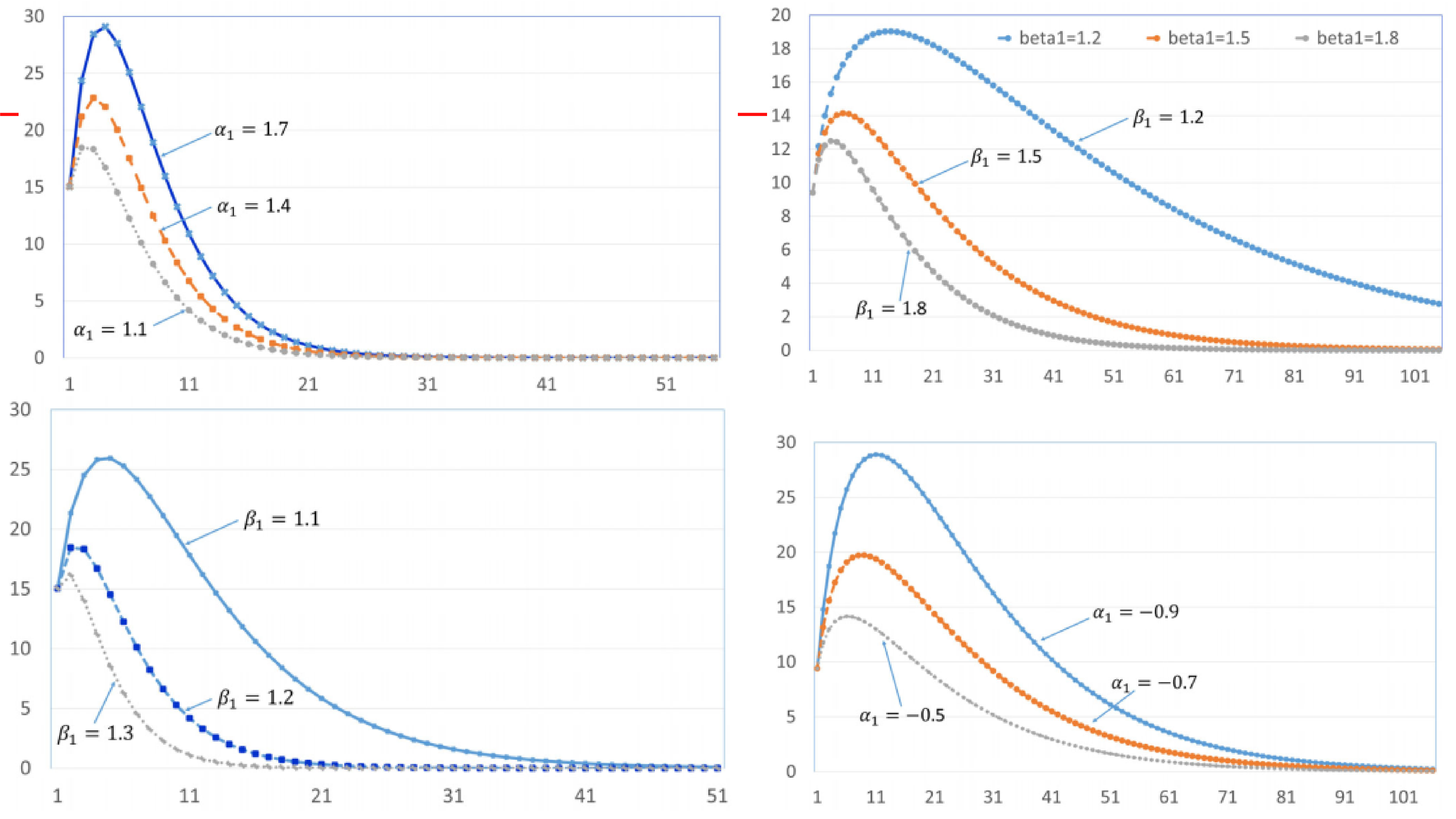
- We require a model satisfying three conditions
  - A parsimonious model: with fewer parameters
  - Be able to describe non-monotonic trends
  - If X_1 follows the Weibull distribution, the distributions of X_k (for $k = 2,3, \ldots$) have the same shape parameter as that of X_1

- DRGP: Given a sequence of non-negative random variables $\{Z_k^D, k = 1,2, \ldots\}$, if they are independent and the cdf of Z_k^D is given by $F_k^D(t) = 1 - \exp\{-\int_0^t b_k h(a_k u) du\}$ for $k = 1,2, \ldots$, where a_k and b_k are positive parameters (or ratios) and $a_1 = b_1 = 1$. $\{Z_k^D, k = 1,2, \ldots\}$ is called the **double-ratio geometric process** (DRGP) (Wu, 2021)

An equivalent definition

- Lemma: Given a sequence of non-negative random variables $\{Z_k^D, k = 1,2, \dots\}$, if they are independent and the cdf of $a_k^{-1} Z_k^D$ is given by $F_k^D(t) = 1 - \left(1 - F_1^D(a_k t)\right)^{\frac{b_k}{a_k}}$ for $k = 1,2, \dots$, where a_k and b_k are positive parameters (or ratios), and $a_1 = b_1 = 1$. Then $\{Z_k^D, k = 1,2, \dots\}$ is a double-ratio geometric process.

- [Monotonicity] Suppose $h(t)$ is a monotonously increasing function in t, $\{Z_k^D, k = 1,2, \dots\}$ is a DRGP, then
- If both a_k and b_k are increasing in k, then the DRGP is stochastically decreasing;
- If both a_k and b_k are decreasing in k, then the DRGP is stochastically increasing; and
- If both a_k (or b_k) is increasing in k and b_k (or a_k) is decreasing, then the DRGP may not be stochastically monotonic.



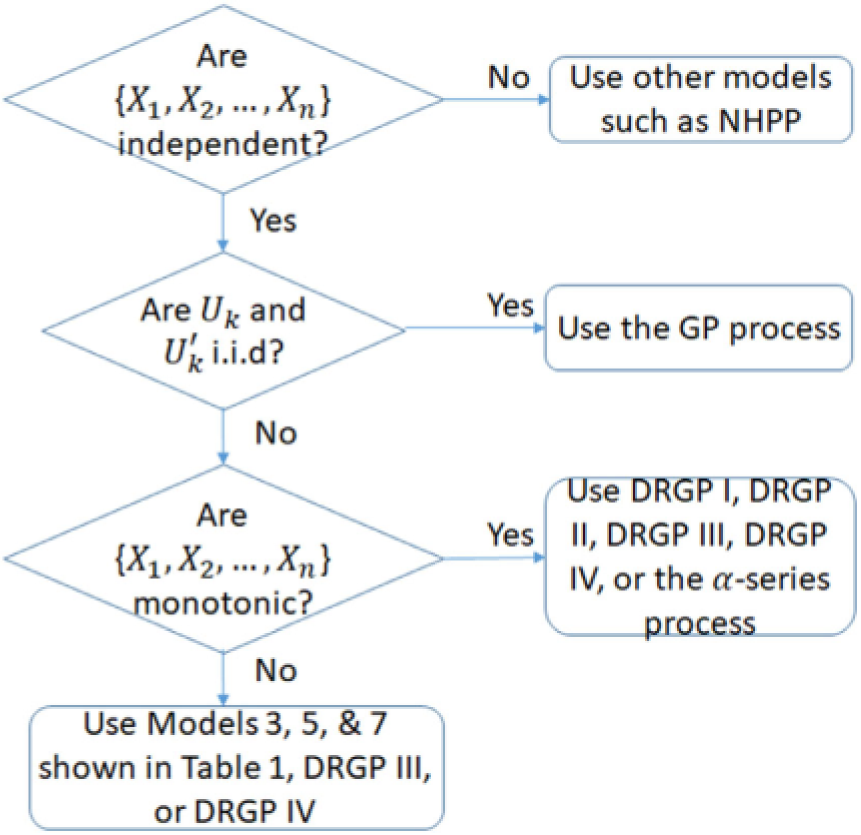
$\alpha_1 = 1.7$
$\alpha_1 = 1.4$
$\alpha_1 = 1.1$
beta1=1.2
beta1=1.5
beta1=1.8
$\beta_1 = 1.2$
$\beta_1 = 1.5$
$\beta_1 = 1.8$
$\beta_1 = 1.1$
$\beta_1 = 1.2$
$\beta_1 = 1.3$
$\alpha_1 = -0.9$
$\alpha_1 = -0.7$
$\alpha_1 = -0.5$

cdf of Z_k^D

- $F_k^D(t) = 1 - \left(1 - F_1^D(a_k\, t)\right)^{\frac{b_k}{a_k}}$,
- Assume Z_1^D follows the exponential distribution with hazard function $h(u) = \lambda$, then $F_k^D(t) = 1 - \mathrm{e}^{-b_k \lambda t}$. That is, a_k does not play a role in DRGP. Below are two special cases.
  - If $b_k = b^{k-1}$, regardless of the form of a_k, then $\{Z_k^D, k = 1,2, \ldots\}$ is a GP with the cdf of X_k being $F_k^D(t) = 1 - \exp(-b^{k-1}\lambda t)$, and
  - If $b_k = k^{\alpha}$, regardless of the form of a_k, then $\{Z_k^D, k = 1,2, \ldots\}$ is an α-series process with the cdf of X_k being $F_k^D(t) = 1 - \exp(-k^{\alpha}\lambda t)$.

Equivalence

- Let Z_1^D follow the Weibull distribution $F_1^D(t) = 1 - \exp\{-\left(\frac{a_k t}{\theta_2}\right)^{\theta_1}\}$, where $a_k = k^{\alpha_1}$ and $b_k = \beta_1^{k-1}$ for model DRGP-III, and $a_k = \beta_2^{k-1}$ and $b_k = k^{\alpha_2}$ for model DRGP-IV. Denote the maximum log-likelihood estimates for model DRGP-III and model DRGP-IV by $\hat{l}_1$ and $\hat{l}_2$, respectively. Then the two models DRGP-III and DRGP-IV are equivalent with respect of modelling a given dataset based on the maximum likelihood estimation from the following perspectives.
  - $\hat{l}_1 = \hat{l}_2$;
  - $\hat{\theta}_1$ and $\hat{\theta}_2$ from model DRGP-III equal $\hat{\theta}_1$ and $\hat{\theta}_2$ from model DRGP-IV, respectively; and
  - $\hat{\beta}_1 = \hat{\beta}_2^{\hat{\theta}_1 - 1}$ and $\hat{\alpha}_1 = \frac{\hat{\alpha}_2}{\hat{\theta}_1 - 1}$.


Are $\{X_1, X_2, \ldots, X_n\}$ independent?
No
Use other models such as NHPP
Yes
Are U_k and U_k' i.i.d?
Yes
Use the GP process
No
Are $\{X_1, X_2, \ldots, X_n\}$ monotonic?
Yes
Use DRGP I, DRGP II, DRGP III, DRGP IV, or the α-series process
No
Use Models 3, 5, & 7 shown in Table 1, DRGP III, or DRGP IV

Datasets for testing

No.	Dataset	n	Description	Reference
1	LHD3	25	failures of a load-haul-dump (LHD) machine deployed at Kiruna mine, Sweden	Kumar and Klefsjö (1992)
2	LHD11	28	failures of a load-haul-dump (LHD) machine deployed at Kiruna mine, Sweden	Kumar and Klefsjö (1992)
3	Calvert Cliffs	23	diesel generator failure data from power plant "Calvert Cliffs"	Kvam et al. (2002)
4	Pump D	30	reliability data collected from a main pump (A) at an oil refinery	Percy and Alkali (2007)

Test for non-monotonic trend.

No.	p-value from the Ljung-Box test	p-value from V_1	p-value from V_4
1	0.886	0.00866	0.0216
2	0.899	0.0159	0.0114
3	0.509	0.0194	0.0276
4	0.107	0.00874	0.00180

Table 5: AICc values of the models

No.	EGP	DGP	TGP	DRGP-I	DRGP-II	DRGP-III	DRGP-IV
1	310.198	301.345	309.431	307.040	305.281	300.228	300.228
2	328.790	323.711	324.103	325.802	324.983	321.312	321.312
3	224.545	221.461	222.055	221.490	222.299	219.632	219.632
4	297.200	300.939	298.127	306.622	308.924	295.031	295.031

Table 6: Parameters

No.	DRGP-III ($\hat{a}_k = k^{\hat{\alpha}_1}$ and $\hat{b}_k = \hat{\beta}_1^{k-1}$)				DRGP-IV ($\hat{a}_k = \hat{\beta}_2^{k-1}$ and $\hat{b}_k = k^{\hat{\alpha}_2}$)			
	$\widehat{\alpha}_1$	$\widehat{\beta}_1$	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\alpha}_2$	$\widehat{\beta}_2$	$\widehat{\theta}_1$	$\widehat{\theta}_2$
1	6.942	0.848	1.231	553.510	1.603	0.490	1.231	553.510
	(5.502)	(0.0621)	(0.200)	(308.283)	(0.671)	(0.287)	(0.200)	(340.652)
2	14.233	0.891	1.085	347.046	1.205	0.254	1.085	347.046
	(27.230)	(0.0555)	(0.166)	(251.771)	(0.617)	(0.660)	(0.165)	(251.777)
3	-29.898	0.855	0.960	92.937	1.192	50.442	0.960	92.937
	(144.543)	(0.0861)	(0.182)	(106.597)	(0.935)	(920.240)	(0.176)	(106.516)
4	-8.996	1.281	1.297	4.629	-2.673	2.304	1.297	4.629
	(4.827)	(0.0787)	(0.185)	(2.985)	(0.689)	(1.0352)	(0.186)	(2.985)

Contributions and further research

- **Contributions**
  - the DGP can model a stochastic process with varying CV
  - the DGP can model recurrent event processes where $F_k(x)$'s have different shape parameters over k's, which can be done by neither the GP-like models nor other repair models such as reduction of age models
  - the DGP and DRGP can model not only monotonously increasing or decreasing stochastic processes, but also processes with complicated failure intensity functions such as the bathtub shaped curves and the upside-down bathtub shaped curves
- **Further research**
  - Both the DGP and DRGP have the limitation that they are parametric models, a Bayesian nonparametric method is under development

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Questions?