# The Quantum Potential in Time-Dependent Supersymmetric Quantum Mechanics.

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## Abstract

If a wavefunction is written in polar form it becomes possible to write the Schrödinger equation of non-relativistic quantum mechanics in a form analogous to the classical Hamilton-Jacobi equation with an extra term known as the quantum potential. Time-Dependent Supersymmetry is a procedure for finding new solutions of the Schrödinger equation if one solution is known. In this paper a time-dependent supersymmetry transformation is applied to a wavefunction in this polar form and it is shown that the classical potential plus the quantum potential is a conserved quantity under this transformation under certain circumstances. This leads to a modification of our view of the role of the quantum potential and also to a deeper appreciation of the function of a supersymmetry transformation.

## I. INTRODUCTION

In the Madelung-Bohm representation of quantum mechanics[1–4] the wavefunction is written in a particular polar form and the Schrödinger equation can then be rewritten as two coupled equations. One describes the conservation of probability and the other is in a form analogous to the Hamilton-Jacobi equation of classical mechanics. This analogy led Bohm to a generalisation of the pilot-wave interpretation of quantum theory initially put forward by De-Broglie (see references [5–7] and references in [4]). While this interpretation is open to debate, the mathematics underlying it is not. This quantum version of the Hamilton-Jacobi equation contains the standard classical potential we normally come across in both classical and quantum mechanics plus a self-generated term which can be viewed as giving the particle its quantum nature. This new term has been named the 'quantum potential'. In recent times there has been considerable interest in the quantum potential, and it has been found to have physical significance in a number of different areas of physics. Of particular interest are the space-time points where it is zero, these are where the particle being studied should be behaving at its most classical. Among the key examples of this are the work of Rogers *et. al.* who have found applications in non-linear optics [8], Espindola-Ramos *et. al.* [9] have shown that wavefunctions with fold caustics are the most classical because the zeros of the quantum potential coincide with the caustic and the evolution of the caustic is governed by the Hamilton-Jacobi equation. Berry [10] has shown that, for quantum wavepackets, the Bohm potential vanishes on the boundaries of regions where the oscillations become superoscillatory. In a connection with relativistic quantum theory Salesi *et. al.* showed that the quantum potential arises naturally as the kinetic energy associated with the internal "trembling" motion of spin-1/2 particles known as zitterbewegung [11, 12]. The quantum potential is defined in terms of the amplitude of the wavefunction and recently Hojman and Asenjo [13] have taken these ideas further and looked for examples where the particle experiences a classical potential, but where this is cancelled by the quantum potential, so the particle behaves as if it is free.

The Schrödinger equation forms the foundation of non-relativistic quantum mechanics. It has only very few physically meaningful exact solutions. Most of the familiar models in quantum theory are steady-state solutions which means that the space and time dependence of the problem can be separated. These, such as the harmonic oscillator and the one-electron atom, form the basis of much of our understanding of the physics of nature. There are a few known solutions that are not steady state and which have some unusual properties such as being self-accelerating [14–18].

Non-stationary supersymmetric quantum theory has been derived [19–21] and extended [22] and provides a strategy for finding new solutions of the time-dependent Schrödinger equation if we know one solution. This work is a natural extension of the time independent supersymmetric methods discussed in very readable form by Cooper *et. al.* [23].

In this paper we will look at what happens when we apply non-stationary supersymmetry theory to wavefunctions in polar form. While this theory defines a new potential and eigenfunctions for the Schrödinger equation we find that, under certain circumstances, the sum of the classical potential and the quantum potential is conserved by such a supersymmetry transformation. This paper is laid out as follows. In section II we describe the origin of the quantum potential. The mathematical details of non-stationary supersymmetric quantum theory have been written down several times before [19–21] so in section III we discuss this procedure only in sufficient mathematical detail for the new work presented here to be appreciated. Next, in section IV we show that, during a supersymmetric transformation the same quantity is subtracted from the classical potential as is added to the quantum potential, so the sum of these two quantities is conserved. We then go on to illustrate this with several examples in section V. Finally in section VI we discuss the meaning of the results and what conclusions can be drawn from them. Throughout this paper constants are retained in equations, but diagrams are drawn in units where $m = 1/2$ and $\hbar = 1$ and we will work in one-dimension although the theory generalises straightforwardly to higher dimensions.

## II. THE QUANTUM POTENTIAL

Following Bohm's original paper [2], we start by writing the single particle quantum mechanical wavefunction in the form

$$\psi(x,t) = R(x,t)e^{iS(x,t)/\hbar} \tag{1}$$

where both $S(x,t)$ and $R(x,t)$ are real. Substitution into the general time dependent Schrödinger equation shows that $R(x,t)$ and $S(x,t)$ satisfy

$$\begin{aligned} \frac{\partial R(x,t)}{\partial t} &= -\frac{1}{2m}\left(R(x,t)\frac{\partial^2 S(x,t)}{\partial x^2} + 2\frac{\partial R(x,t)}{\partial x}\frac{\partial S(x,t)}{\partial x}\right) \\ \frac{\partial S(x,t)}{\partial t} &= -\left(\frac{1}{2m}\left(\frac{\partial S(x,t)}{\partial x}\right)^2 + V(x,t) + V_B(x,t)\right) \end{aligned} \tag{2}$$

Care must be taken in the application of these equations because at a node in the wavefunction $R(x,t) = 0$, but then $S(x,t)$ is undefined and may be discontinuous. The first of equations (2) can easily be shown to be equivalent to the conservation of probability provided we make the following identification for the velocity

$$v = \frac{dx}{dt} = \frac{1}{m}\frac{\partial S(x,t)}{\partial x}. \tag{3}$$

So, it is $S(x,t)$ which determines the dynamics of the particle. We simply have to define the initial conditions to solve for $x(t)$. For consistency with the postulates of quantum mechanics we require that the probability that a particle lies between the points $x$ and $x+dx$ at time $t$ is $P(x,t)dx = R^2(x,t)dx$ which means $R(x,t)$ plays the role of a probability amplitude.

In the second of equations (2) we have defined

$$V_B(R(x,t)) = -\frac{\hbar^2}{2m}\frac{1}{R(x,t)}\frac{\partial^2 R(x,t)}{\partial x^2} \tag{4}$$

which is known as the quantum potential[1, 2, 24]. If we omit the quantum potential from the second of equations (2) it is simply the Hamilton-Jacobi equation of classical mechanics with a familiar interpretation in terms of massive point particles. The Hamilton-Jacobi equation is a way of writing the equations of motion for a system of particles which is an alternative to Newton's laws. It is clearly completely classical. The quantum potential would be zero if the universe were classical (i.e. if $\hbar = 0$). However Planck's constant is very small, but not zero, and this term can be regarded as an additional self-generated potential that the classical particle experiences to give it its quantum nature. Furthermore, a number of authors have argued that a quantum particle behaves at its most classical in places where the quantum potential is zero [8–10]. From the point of view of the current work, the key thing to note from this formalism can be seen in the second of equations (2). If we have two wavefunctions with the same value of $S(x,t)$ then the sum $V(x,t) + V_B(x,t)$ must also be the same for both of them.

## III. NON-STATIONARY SUPERSYMMETRIC QUANTUM MECHANICS

Non-stationary supersymmetric quantum mechanics essentially involves employing a time-dependent Darboux transformation to reconstruct the Schrödinger equation. In this procedure we start with a potential and eigenfunctions that satisfy the Schrödinger equation and from these we generate a new potential and eigenfunctions of the Schrödinger equation. This approach defines a hierarchy of solutions. Once we have found the new potential and wavefunction we can use them as the input to a subsequent transformation. Here we outline the method, but refer the reader to the original literature for the calculational details [19–23]. Non-stationary supersymmetry is a powerful technique but its implementation has been limited so far. Bagrov *et. al.* performed a number of examples in their papers [19–21] deriving the method, although these contain little physical interpretation of the results. Both Zelaya and Rosas-Ortiz [25] and Contreras-Astorga [26] have found interesting new potentials starting from the harmonic oscillator. Rasinskaitė and Strange [27] have recently used the technique to describe surfing on a quantum level. The method has recently been extended to nonlinear equations by Hayward and Biancalana[28].

Consider two different time-dependent one-dimensional Schrödinger equations

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}_0\right)\psi(x,t) = 0$$
$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}_1\right)\phi(x,t) = 0 \tag{5}$$

with

$$\hat{H}_i = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_i(x,t) \tag{6}$$

with $i = 0$ or 1. Let us postulate that there exists an operator $\hat{A}$ such that

$$\hat{A}\left(i\hbar\frac{\partial}{\partial t} - \hat{H}_0\right)\psi(x,t) = \left(i\hbar\frac{\partial}{\partial t} - \hat{H}_1\right)\hat{A}\psi(x,t) \tag{7}$$

From equation (5) the left hand side of this is zero and so the right hand side must also be zero, which implies

$$\phi(x,t) = \hat{A}\psi(x,t) \tag{8}$$

It has been shown that such an operator does exist and it is written as a function of $x$ and $t$ as

$$\hat{A} = \hat{A}_0(x,t) + \hat{A}_1(t)\frac{\partial}{\partial x} \tag{9}$$

Here we find that $\hat{A}_1(t)$ has units of distance (although it is only a function of time, not $x$), while $A_0(x,t)$ is dimensionless and is given by

$$\hat{A}_0(x,t) = -\frac{1}{u(x,t)}\frac{\partial u(x,t)}{\partial x}\hat{A}_1(t). \tag{10}$$

Here $u(x,t)$ is known as a transfer function and is a distinct solution of the same Schrödinger equation as $\psi(x,t)$. So for the new wavefunction we have

$$\phi(x,t) = \hat{A}_1(t)\left(\frac{\partial}{\partial x} - \frac{1}{u(x,t)}\frac{\partial u(x,t)}{\partial x}\right)\psi(x,t) \tag{11}$$

and the new potential is given by

$$V_1(x,t) = V_0(x,t) + i\hbar\frac{1}{\hat{A}_1(t)}\frac{\partial \hat{A}_1(t)}{\partial t} - \frac{\hbar^2}{m}\frac{\partial^2}{\partial x^2}(\log u). \tag{12}$$

$\hat{A}_1(t)$ is essentially arbitrary, but can be chosen to find the representation in which $V_1(x,t)$ is real if such a representation exists. Then

$$V_1(x,t) = V_0(x,t) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(\log|u(x,t)|^2) \tag{13}$$

This clearly contains only real terms. We now have all we need to calculate new solutions of the Schrödinger equation from known solutions. The procedure is as follows. We choose two known solutions of the upper of equations (5) as $\psi(x,t)$ and the transfer function $u(x,t)$, and we also know the corresponding potential $V_0(x,t)$. First we calculate $V_1(x,t)$ from equation (12) with $A_1(t) = 1$ and if it is not real we choose an expression for $\hat{A}_1(t)$ to make it real. The details of how to do this are given in the original literature. If this is not possible the calculation may be mathematically interesting, but there is very unlikely to be any physical applications of the results. Next we calculate $\hat{A}_0(x,t)$ from equation (10) and then $\hat{A}(x,t)$ from equation (9). Finally we find $\phi(x,t)$ from equation (8) or (11) and that is the wavefunction corresponding to the potential $V_1(x,t)$. This completes the calculation because that $V_1(x,t)$ and $\phi(x,t)$ are the potential and solutions of the lower of equations (5).

## IV. POLAR REPRESENTATIONS OF WAVEFUNCTIONS AND SUPERSYMMETRIC QUANTUM MECHANICS

In this section we will use wavefunctions in the form of equation (1) in an implementation of the time-dependent supersymmetry method. To this end we write the transfer function , the initial wavefunction and the final wavefunction respectively as

$$u(x,t) = R_u(x,t)e^{iS_u(x,t)/\hbar} \qquad \psi(x,t) = R_\psi(x,t)e^{iS_\psi(x,t)/\hbar} \qquad \phi(x,t) = R_\phi(x,t)e^{iS_\phi(x,t)/\hbar} \tag{14}$$

Henceforth in this section we will drop the explicit $x$ and $t$ dependence of these quantities for clarity. Next we substitute these forms into equation (11). It turns out that if we insist that

$$\frac{\partial S_\psi}{\partial x} = \frac{\partial S_u}{\partial x} \tag{15}$$

then $S_\phi = S_\psi$, and

$$R_\phi = \hat{A}(t)\left(\frac{\partial R_\psi}{\partial x} - \frac{R_\psi}{R_u}\frac{\partial R_u}{\partial x}\right) \tag{16}$$

As it is the expression for $S$ that determines the dynamics of the particle, this means that the new solution of the Schrödinger equation obeys the same dynamical equations as the old solution (up to a constant of integration). Now let us put the first of equations (14) into equation (13). After some manipulation this yields

$$V_1 = V_0 - \frac{\hbar^2}{m}\left(\frac{1}{R_u}\frac{\partial^2 R_u}{\partial x^2} - \frac{1}{R_u^2}\left(\frac{\partial R_u}{\partial x}\right)^2\right) \tag{17}$$

Prior to the first transformation the quantum potential is given by

$$V_B(R_\psi(x,t)) = -\frac{\hbar^2}{2m}\frac{1}{R_\psi(x,t)}\frac{\partial^2 R_\psi(x,t)}{\partial x^2} \tag{18}$$

and after the first transformation it is given by

$$V_B(R_\phi(x,t)) = -\frac{\hbar^2}{2m}\frac{1}{R_\phi(x,t)}\frac{\partial^2 R_\phi(x,t)}{\partial x^2} \tag{19}$$

Building equation (19) from equation (16) and making use of both (15) and (2) it can be shown that

$$V_B(R_\phi) = V_B(R_\psi) + \frac{\hbar^2}{m}\left(\frac{1}{R_u}\frac{\partial^2 R_u}{\partial x^2} - \frac{1}{R_u^2}\left(\frac{\partial R_u}{\partial x}\right)^2\right) \tag{20}$$

Now adding equations (17) and (20) yields our key result that

$$V_1 + V_B(R_\phi) = V_0 + V_B(R_\psi) \tag{21}$$

i.e. the sum of the usual classical potential and the quantum potential is conserved by a supersymmetry transformation provided equation (15) is satisfied. This is just what we observed at the end of section II, that $V(x,t) + V_B(x,t)$ is conserved if $S(x,t)$ is the same for the initial and final wavefunction. In Equations (17) and (20) we have shown mathematically that the same quantity is subtracted from $V_0$ as is added to $V_B(R_\psi)$ and provided an explicit expression for that quantity. Equation (21) should be regarded as a mathematical condition on the wavefunctions and potentials. That this can be written in terms of the quantum potential enables us to discuss this result in terms of the De-Broglie-Bohm model.

In fact when we have an initial wavefunction for which the value of $S$ satisfies equation (15) we can perform any number of supersymmetry transformations on it and each wavefunction will have the same expression for $S$. The classical and quantum potentials are not conserved individually by a time dependent supersymmetry transformation (there would be no point in it if they were), so an interesting way of regarding such a transformation is as transferring potential between the classical potential and the quantum potential. In the following section we illustrate this with a number of examples.

## V. EXAMPLES

In this section we display a number of examples of transferring weight between the classical and quantum potentials. As part of this we calculate some expectation values. Because of the symmetry these are all zero if calculated in the usual manner. Here the wavefunctions displayed are equal to zero at $x = 0$ at all times, so it is legitimate to calculate the expectation value over just one half of the space, which we have done.

### A. The Free-Particle Hermite Wavefunction.

In this first example we consider a known free particle wavefunction. This will form the starting point for subsequent examples where it is used to initiate a number of supersymmetry transformations. A solution of the time-dependent free particle Schrödinger equation with a form suitable for illustrating the above theory is:

$$\Psi(x,t) = \sqrt{\frac{1}{n!}}\left(\frac{m}{\hbar\tau\pi}\right)^{1/4}\frac{2^{-n/2}}{(1+t^2/\tau^2)^{1/4}}e^{\left(-\frac{m\tau x^2}{(2\hbar(t^2+\tau^2)}\right)}e^{\left(-i(n+1/2)\arctan\left(\frac{t}{\tau}\right)\right)}e^{\left(\frac{imx^2t}{2\hbar(t^2+\tau^2)}\right)}H_n\left(\left(\frac{m\tau}{\hbar(t^2+\tau^2)}\right)^{1/2}x\right). \tag{22}$$

Here the symbols have their usual meanings. $\tau$ is a positive constant with the dimensions of time. $\tau = 1$ has been chosen throughout this paper unless otherwise stated. $H_n$ is a Hermite polynomial and $n$ is a non-negative integer quantum number. To our knowledge this wavefunction was first found by Miller [15]. It has a number of interesting properties which have been discussed since by Bagrov *et. al* [20], Guerrero *et. al.* [16, 17] and Strange [18] for example. The probability density associated with this wavefunction is displayed in Figure 1. For this case

$$S(x,t) = \frac{mx^2t}{2(t^2+\tau^2)} - (n+1/2)\hbar\arctan\left(\frac{t}{\tau}\right) \tag{23}$$

and

$$R(x,t) = \sqrt{\frac{1}{n!}} \left(\frac{m}{\hbar\tau\pi}\right)^{1/4} \frac{2^{-n/2}}{(1+t^2/\tau^2)^{1/4}} e^{\left(-\frac{m\tau x^2}{(2\hbar(t^2+\tau^2)}\right)} H_n\left(\left(\frac{m\tau}{\hbar(t^2+\tau^2)}\right)^{1/2} x\right). \tag{24}$$

Then equation (3) yields

$$\frac{dx}{dt} = \frac{xt}{t^2+\tau^2} \tag{25}$$

which can be solved trivially to give

$$x = x_0 \frac{\sqrt{t^2+\tau^2}}{\tau} \tag{26}$$

where $x_0$ is a constant of integration. The division by $\tau$ here is not necessary, but is done to give $x_0$ units of distance. This result has also been obtained using semiclassical methods in reference [18]. For the values of the parameters used we find the expectation value of position $\langle \hat{x} \rangle$ has $x_0 = 2.394$. We can use equation (24) in (4) to find the quantum potential for this wavefunction:

$$V_B(R(x,t)) = \frac{\tau((n+1/2)(t^2+\tau^2)\hbar - mx^2\tau)}{2(t^2+\tau^2)^2} \tag{27}$$

The wavefunction (22) is one member of a family of wavefunctions that all have the $S(x,t)$ given by equation (23) for any particular value of $n$. This wavefunction is in some sense maximally quantal because the entirety of the potential it experiences is the quantum potential. In figure 1 we plot this potential for several times for $n = 3$. We note that this potential depends on $t^2$ as opposed to $t$. meaning that Figure 1 would look the same if we replaced the values of $t$ by $-t$. Furthermore any properties of a particle experiencing this potential should be of identical magnitude at $\pm t$. The potential is an inverted parabola at all times, but it becomes flatter very rapidly. As this corresponds to zero

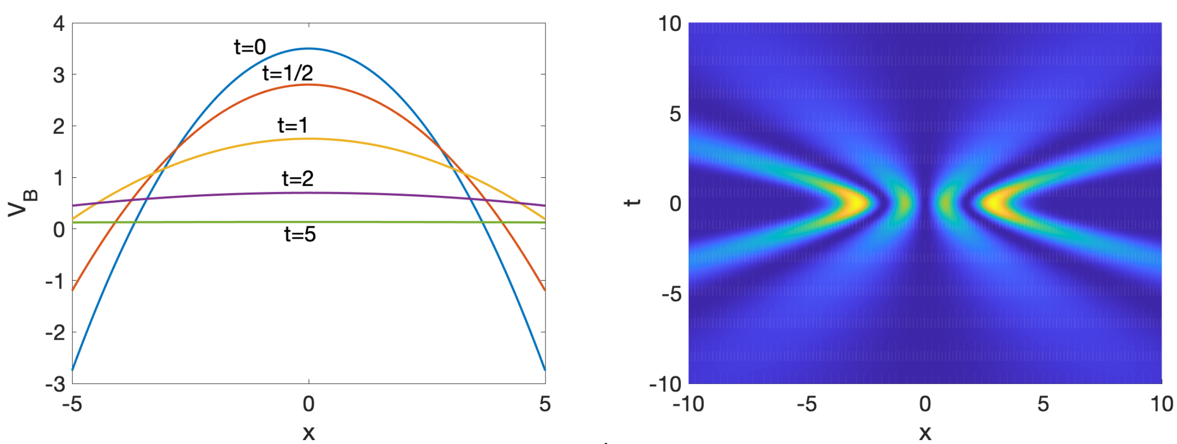


FIG. 1: *Left*: The quantum potential of equation (27) as a function of position at several different times for $n = 3$ and $\tau = 1$. *Right* A space time map for the probability density associated with the wavefunction of equation (22) with $n = 3$

conventional potential this is also $V + V_B$ which is constant for all wavefunctions found from equation (22) using a supersymmetry transformation.

### B. A Wavefunction generated from a single supersymmetry transformation

In this example we have chosen a simple case $u(x,t) = \Psi(x,t)$ above with $n = 1$, and $\psi(x,t) = \Psi(x,t)$ with $n = 3$. $V_0 = 0$ of course and then after one supersymmetry procedure

$$V_1(x,t) = \hbar\left(\frac{\tau}{t^2+\tau^2} + \frac{\hbar}{mx^2}\right) \tag{28}$$

and

$$V_B(x,t) = -\frac{mx^2\tau^2}{2(t^2+\tau^2)^2} + \frac{5\tau\hbar}{2(t^2+\tau^2)} - \frac{\hbar^2}{mx^2} \tag{29}$$

$V_1(x,t)$ and $V_B(x,t)$ at $t=0$ are shown as the blue dashed and red dotted lines respectively on the left hand side of Figure 2. Clearly the supersymmetry transformation has introduced an infinity into the potential. This comes about because $R_u$ passes through zero at the origin. The wavefunction is given by

$$\phi(x,t) = \left(\frac{m}{\hbar}\right)^{7/4} \frac{4x^2\tau^{5/4}}{\sqrt{3}\left(\pi+\pi t^2/\tau^2\right)^{1/4}(t^2+\tau^2)} e^{\frac{i}{2}\left(\frac{mx^2 t}{\hbar(t^2+\tau^2)} - 7\arctan\left(\frac{t}{\tau}\right)\right)} e^{-\frac{mx^2\tau}{2\hbar(t^2+\tau^2)}} \tag{30}$$

and so

$$R(x,t) = \left(\frac{m}{\hbar}\right)^{7/4} \frac{4x^2\tau^{5/4}}{\sqrt{3}\left(\pi+\pi t^2/\tau^2\right)^{1/4}(t^2+\tau^2)} e^{-\frac{mx^2\tau}{2\hbar(t^2+\tau^2)}} \tag{31}$$

and

$$S(x,t) = \frac{mx^2 t}{2(t^2+\tau^2)} - \frac{7}{2}\hbar\arctan\left(\frac{t}{\tau}\right) \tag{32}$$

$S(x,t)$ here is in the same form as equation (23) which implies that the dynamics is also the same (to within a constant). It is straightforward to verify that the expectation value $\langle\hat{x}\rangle$ obeys equation (26) with $x_0 = 2.128$. In

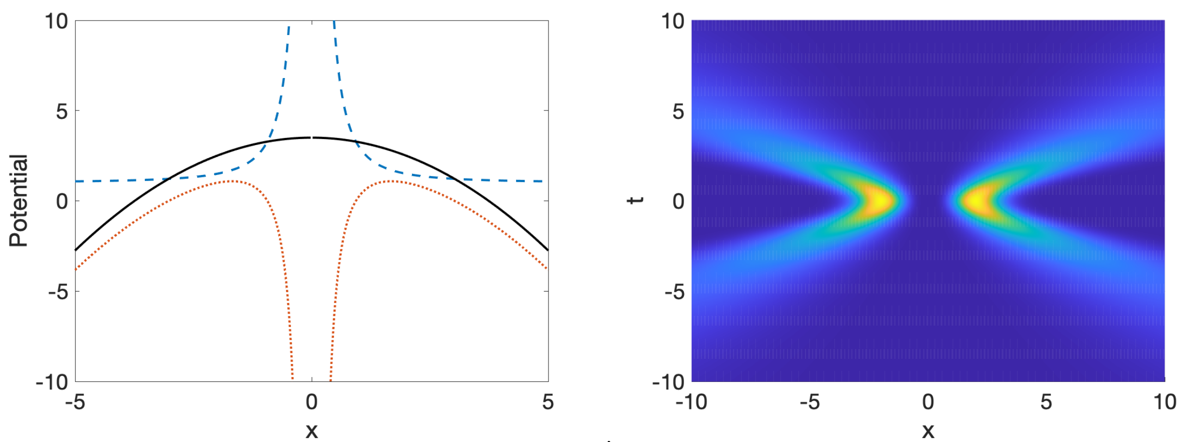


FIG. 2: *Left*: The potentials of equation (28) (blue, dashed) and (29) (red, dotted) and the sum $V(x,t)+V_B(x,t)$ (black, full) as a function of position at $t=0$. The black full line is identical to the blue $t=0$ line in Figure 1. *Right*: A space-time map of the probability density for this wavefunction. (blue (dark) = a very low probability density and yellow/white = a high probability density.)

this case it is easy to see why the two branches of the probability density are separate, the classical potential has an infinity at the origin meaning that any particle experiencing this potential will be unable to pass through it and hence will be trapped on one side of the potential for all times.

### C. Quantum Surfing Wavefunction.

As a further example we perform the two successive supersymmetry transformations described by Strange and Rasinskaitė [27] to obtain a solution of the Schrödinger equation where the particle appears to 'surf' on the time-dependent potential.

For this example we have taken the wavefunction from equation (30) above as $\psi(x,t)$ for a second supersymmetry transformation. Then we have performed another supersymmetry calculation analogous to that of example **B**, starting from the same eigenfunction $\Psi(x,t)$, but with $n=1$ for $u(x,t)$ and $n=2$ for $\psi(x,t)$ to get a new $u(x,t)$ for input to the second supersymmetry transformation. Note that we have used the same quantum number for $u(x,t)$ in both our first round of supersymmetry transformations. This is because they then both generate the same new potential. Using this potential generated from the first transformation as $V_0$ in the second transformation we end up with the following output potential

$$V_1(x,t) = \frac{2\tau\hbar(4m^2x^4\tau^2 + 8mx^2\tau(t^2+\tau^2)\hbar - (t^2+\tau^2)^2\hbar^2)}{(t^2+\tau^2)(2mx^2\tau + (t^2+\tau^2)\hbar)^2}. \tag{33}$$

and the quantum potential is

$$V_B(x,t) = \frac{\tau\left(-4m^3x^6\tau^3 + 8m^2x^4\tau^2(t^2+\tau^2)\hbar - 5mx^2\tau(t^2+\tau^2)^2\hbar^2 + 11(t^2+\tau^2)^3\hbar^3\right)}{2(t^2+\tau^2)^2(2mx^2\tau + (t^2+\tau^2)\hbar)^2} \tag{34}$$

The classical potential at $t = 0$ is shown as the blue dashed line on the left diagram of Figure 3. It retains that

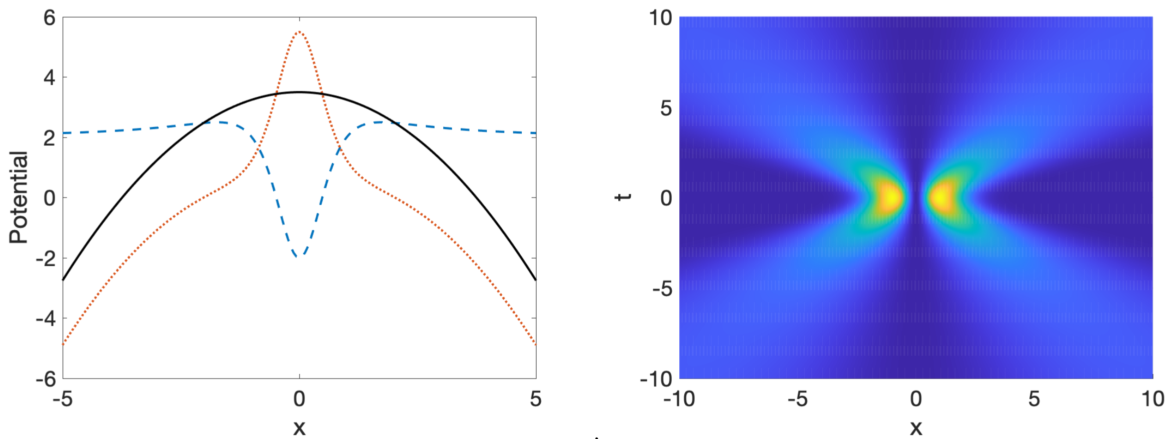


FIG. 3: *Left*: The potentials of equation (33) (blue, dashed) and (34) (red, dotted) and the sum $V(x,t) + V_B(x,t)$ (black) as a function of position at $t = 0$. The black full line is identical to the blue $t = 0$ line in Figure 1. *Right*: A space-time map of the probability density for this wavefunction. (blue (dark) = a very low probability density and yellow/white = a high probability density.)

shape for all times, but for $t << 0$ it is very stretched and shallow. As time increases from $t = -\infty$ the potential contracts towards zero, the maxima of the potential get closer to zero, and the central potential well deepens until it takes on the form shown in Figure 3. For $t > 0$ the motion of the potential reverses and it becomes very stretched and shallow again. The quantum potential is given by the red dotted line on the left of Figure 3. For $t << 0$ and $t >> 0$ it is close to an inverted parabola. Around $t = 0$ a more well-defined peak appears symmetric about $x = 0$ and the rest of the parabola decreases in size. This peak is an exact 'counterbalance' to the potential well formed in the classical potential. The right hand side of Figure 3 is a space-time map of the probability density determined from the wavefunction that is output from the supersymmetry transformation. This is a single particle wavefunction and, although there are two branches to the probability density, the particle will only inhabit one of them at any one time. This is determined by the boundary conditions, Clearly the particle is fairly well localised at $t = 0$ but the probability density spreads out rapidly either side of $t = 0$. If we evaluate the expectation value of position as a function of time we find it obeys equation (26) with $x_0 \approx 1.305$. There is a region between the two peaks at $t = 0$ where the probability density is close to zero, so there is very little probability of the particle being there. However different values of the free quantum number associated with $\psi$ yield eigenfunctions where the probability density is high at the bottom of the well. Comparing the right hand sides of figures 2 and 3 we see that the quantum surfing probability density has superficially the same shape the that shown in figure 2, but it broadens considerably more rapidly.

### D. A highly localised wavefunction.

We present this example because, while in some ways apparently unphysical, it also exhibits some noteworthy properties. Here we have repeated the calculation in example **B** with $n = 1$ for $u(x,t)$ and $n = 4$ for $\psi(x,t)$. The output wavefunction from this calculation is our new $u(x,t)$. We have repeated this procedure with $n=1$ and $n=2$ respectively and the output from that calculation is our new $\psi(x,t)$. We then perform the further supersymmetry transformation that results in the more complicated resultant classical potential

$$V_1(x,t) = \frac{2\tau\hbar(16m^4x^8\tau^4 - 8m^2x^4\tau^2\hbar^2(t^2+\tau^2)^2 + 48mx^2\tau\hbar^3(t^2+\tau^2)^3 - 3\hbar^4(t^2+\tau^2)^4}{(t^2+\tau^2)(-4m^2x^4\tau^2 + 4mx^2\tau\hbar(t^2+\tau^2) + \hbar^2(t^2+\tau^2)^2)^2} \tag{35}$$

and a quantum potential given by

$$V_B(x,t) = \frac{\tau(-16m^5x^{10}\tau^5 + 80m^4x^8\tau^4\hbar(t^2+\tau^2) - 232m^3x^6\tau^3\hbar^2(t^2+\tau^2)^2 + 80m^2x^4\tau^2\hbar^3(t^2+\tau^2)^3)}{2(t^2+\tau^2)^2(-4m^2x^4\tau^2 + 4mx^2\tau\hbar(t^2+\tau^2) + \hbar^2(t^2+\tau^2)^2)^2}$$

$$+ \frac{\tau(-137mx^2\tau\hbar^4(t^2+\tau^2)^4 + 19(\hbar^5(t^2+\tau^2)^5)}{2(t^2+\tau^2)^2(-4m^2x^4\tau^2 + 4mx^2\tau\hbar(t^2+\tau^2) + \hbar^2(t^2+\tau^2)^2)^2}. \tag{36}$$

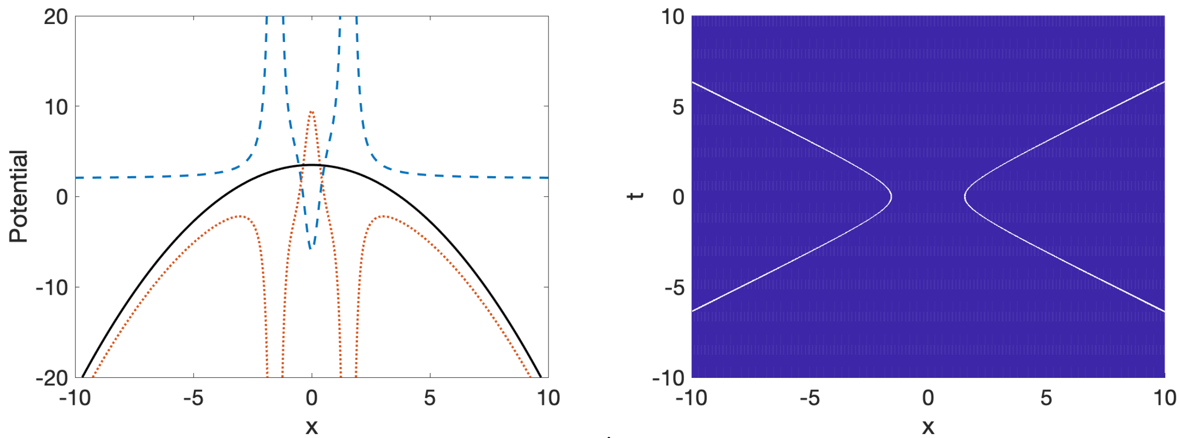


FIG. 4: *Left*: The potentials of equation (35) (blue, dashed) and (36) (red, dotted) and the sum $V(x,t) + V_B(x,t)$ (black, full) as a function of position at $t=0$. The black full line is identical to the blue $t=0$ line in Figure 1. *Right*: A space-time map of the probability density for this wavefunction. (blue (dark) = a very low probability density and yellow/white = a high probability density.)

These potentials are shown at $t=0$ in left picture in Figure 4 along with their total which is the black full line and is identical to the equivalent lines in the previous figures. The potential here has a new characteristic in that it contains a pair of infinities and these move symmetrically towards the origin for $t<0$ and away from the origin for $t>0$. The origin of these infinities is easy to see. In equations (17) and (18) we see that an infinity will arise in the classical potential and in the quantum potential if $R_u = 0$ and that is indeed the case here. Furthermore, a zero in $R_u$ will arise from a zero in $u(x,t)$. As can be seen from equation (11) a zero in $u(x,t)$ will lead to an infinity in the new wavefunction $\phi(x,t)$. This means that infinities in both the potential and probability density will coincide in space and time. This is shown in the right hand picture in Figure 4, where the infinity dominates the probability density, which makes it very highly localised. Unusually, we have here a localised wavefunction that does not broaden. It is tied to a particular position by the zero in $u(x,t)$. The expectation value $\langle\hat{x}\rangle$ cannot be determined numerically because of the infinities in the probability density. However the probability density peak must correspond with the expectation value of position and so the right hand picture in Figure 4 also shows the expectation value of position as a function of time. We can estimate graphically that for this case $x_0 = 1.554$. The non-spreading nature of the probability density continues indefinitely in both time and space. While the picture this solution gives us, of a particle trapped at an infinite potential peak, is unphysical, the lack of broadening it produces is worth noting.

### E. A Final Wavefunction

We have seen that there exists families of potentials and solutions of the Schrödinger equation that have identical values of the quantum number $n$ and identical values of $S(x,t)$, but differing potentials and values of $R(x,t)$, such that $V(x,t)+V_B(x,t)$ is always the same. One 'end' of this series is the example in section **A** where the classical potential is zero and the quantum potential takes on the value of equation (27). At the other end of the scale is the case where the classical potential is given by equation (27)

$$V(x,t)=\frac{(\frac{7}{2}(t^2+\tau^2)\hbar-mx^2\tau)\tau}{2(t^2+\tau^2)^2} \tag{37}$$

for $n=3$, and the quantum potential is zero. A solution to the time-dependent Schrödinger equation for this potential can be found by inspection as

$$\psi(x,t)=\frac{B}{(t^2+\tau^2)^{1/4}}\exp\left[i\left(\frac{mx^2t}{2\hbar(t^2+\tau^2)}-\frac{7}{2}\arctan\left(\frac{t}{\tau}\right)\right)\right] \tag{38}$$

where $B$ is a constant. Clearly this eigenfunction is a member of the same family as the previous examples because it has the same expression for $S(x,t)$. It is easy to verify that the quantum potential associated with this wavefunction is zero. This potential and wavefunction represent two things. Firstly the action

$$S(x,t)=\frac{mx^2t}{2(t^2+\tau^2)}-\frac{7\hbar}{2}\arctan\left(\frac{t}{\tau}\right) \tag{39}$$

is a solution of equation (2) with the quantum potential equal to zero, i.e. the classical Hamiltonian-Jacobi equation. Standard classical mechanics leads to this action describing motion of a classical particle according to equation (26). So, in one sense the result of Equation (38) describes a fairly simple classical particle. On the other hand, the potential of equation (37) is the potential that gives the solution in example **A** its 'quantumness'. The wavefunction it generates, shown in equation (38), is unnormalizable. and the probability density it produces is independent of $x$, so constant over all space. We can calculate the probability and probability current density easily using the standard prescriptions giving

$$\rho(x,t)=\frac{B^2}{(t^2+\tau^2)^{1/2}}, \qquad J(x,t)=\frac{t}{\tau}\frac{B^2x_0t}{t^2+\tau^2} \tag{40}$$

and if we define velocity as current density divided by probability density we find

$$v=\frac{\mathbf{J}}{\rho}=\frac{x_0}{\tau}\frac{t}{(t^2+\tau^2)^{1/2}} \tag{41}$$

which, with $x=x_0\sqrt{t^2+\tau^2)}/\tau$ is equivalent to equation (25), hence showing that a particle described by this wavefunction does obey the same dynamical equations as the other examples.

## VI. DISCUSSION

This work leads to a new perspective on both supersymmetric quantum mechanics and on the Madelung-Bohm representation of the Schrödinger equation.

We have seen that one way of viewing a supersymmetry transformation is as a procedure to transfer potential between the quantum potential and the classical potential function provided simple restrictions are placed on the quantum mechanical action. This enables us to create families of wavefunctions which have differing values of $R(x,t)$ but the same $S(x,t)$. Members of these families all have the same basic dynamics, but differing initial probability distributions. The transfer function $u(x,t)$ is required by the method to be a solution of the same Schrödinger equation as the wavefunction before the supersymmetry transformation $\psi(x,t)$, but within that limitation we still have some freedom to choose the nature of the transferred potential. In turn this gives us the capability to influence the properties of the resulting wavefunction. The formalism generalises straightforwardly to higher dimensions.

This has been illustrated in a set of examples where we have found several members of the same family of wavefunctions, each of which has the same basic dynamics (apart from an arbitrary constant of integration), but have very different probability densities. One noteworthy case is where we can add infinities to the classical potential and subtract them from the quantum potential which leads to particles whose position is localised and does not broaden with time.

Equations (2) are an alternative, less general, form of the Schrödinger equation. Because they are a pair of coupled equations for $R(x,t)$ and $S(x,t)$ it is often stated that these two quantities determine each other. In this work we have shown that, rigorously, this is not the case. There is a potentially infinite set of different expressions for the probability amplitude $R(x,t)$ for a given $S(x,t)$. This is consistent with the interpretation that of all the possible dynamics defined by equation (3) we are choosing those that are compatible with the initial probability distribution $R^2(x,0)$ [4]. In some sense the quantum potential contains the quantum nature of the particle. Therefore we can regard the supersymmetry procedure as adding or subtracting 'quantumness' to the wavefunction. The examples used to illustrate the theory are all members of the same family. Example **A** has zero classical potential and non-zero quantum potential and so may be regarded as the most quantum mechanical case. Example **E** is the most classical in the same sense because the particle experiences the full classical potential, but zero quantum potential. The example that actually behaves most classically is example **D** as it does not broaden.

We conclude by pointing out that the procedure described here provides a means of investigating the effects of the quantum potential in many more cases and provides a route to a deeper understanding of the relationship between classical and quantum mechanics.


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