Comparing with the joint importance under consideration of consecutive-k-out-of-n system structure changes

Hongyan Dui [a,1] , Tianzi Tian [b], Jiangbin Zhao [c], Shaomin Wu [d]

[a] *School of Management Engineering, Zhengzhou University, Zhengzhou 450001, China*

[b] *School of Reliability and Systems Engineering, Beihang University, Beijing, 100191, China*

[c] *School of Mechanical Engineering, Xi'an University of science and technology, Xi'an, 710054, China*

[d] *Kent Business School, University of Kent, Canterbury, Ken,t CT2 7FS, UK*

A B S T R A C T

For a multi-component system, the impact of the alteration of components on the system reliability often needs assessment. Existing importance measures, however, do not consider the impact of the possible change of the system structure during its life cycle. Therefore, relevant factors should be considered to better reflect the changes in system reliability. To this end, this paper proposes joint importance measures for the optimal component sequence of a consecutive-*k*-out-of-*n* system. Incorporating the generalized measures, the paper obtains the joint integrated importance measure and the joint differential importance measure for the optimal component sequence in the binary and multistate consecutive-*k*-out-of-*n* systems. Then some properties of the proposed joint importance measures for optimal component sequence are analyzed. Furthermore, this measure reveals the relationship between component reliability and joint importance measure under consideration of consecutive-*k*-out-of-*n* system structure changes. Finally, numerical examples are given to demonstrate the applicability of the proposed measures.

Keywords: Importance measure, System reliability, Consecutive-k-out-of-n system, Component sequence

1. Introduction

In actual engineering applications, generally engineers can increase the reliability of the system by adding more redundant components to critical components [9, 11]. An example of such a redundant system is a linear consecutive *k*-out-of-*n*: F(G) system, which is a system defined as: If *k* consecutive components fail (work), the system will fail (work). Lin, Cui, Coit, and Lv [9] studied the reliability model of the linear consecutive *k*-out-of-*n*: F system connected with shared nodes. Based on this reliability model, Dui Si and Yam [3] proposed the method of using importance measure to determine the critical components and weak links to optimize the topological structure of the systems. In terms of reliability assessment, Mo, Xing, Cui, and Si [15] investigated

[1] Corresponding author

Suggested citation: Hongyan Dui, Tianzi Tian, Jiangbin Zhao, Shaomin Wu, Comparing with the joint importance under consideration of consecutive-k-out-of-n system structure changes, Reliability Engineering & System Safety, Volume 219, 2022, 108255, https://doi.org/10.1016/j.ress.2021.108255.

a reliability modeling method based on multi-value decision diagram (MDD) for multi-state linear consecutive-*k*-out-of-*n*: F system.

In addition, the *k*-out-of-*n* system is a good example of a reconfigurable system, which means that the reliability of the system can be improved through the rearrangement of components. (Zhao, Si, and Cai, [26]). And it is widely used in specific practical scenarios, such as weighted voting system [5], multi-state windowing system [24], aerospace repeater system [21] etc. Recently, Ma, Wang, Cai, Si, and Zha [13] studied the production monitoring system, that is, when at least *k* consecutive monitors fail, there will be a blind zone, in which monitors with different reliability can be interchanged in use. Several researchers have studied that for consecutive *k*-out-of-*n* systems, there is an optimal component allocation arrangement strategy to maximize system performance [14], or minimize the expected total cost of the system [8]. Ling, Wei and Si [10] further studied an optimal allocation strategy of components under the maximum allowable constraints in the system.

When the reliability of a component changes, optimal configuration of the system may need restructuring. At the same time, the importance measure of the corresponding components can be determined by the selected system structure. When the optimal layout remains unchanged, heuristic algorithms can be used to calculate the approximate optimal allocation of the system[3]. Cai, Si, Sun, and Li [2] proposed a Birnbaum importance measure-based genetic algorithm to search the near global optimal solution for linear consecutive-*k*-out-of-*n* systems. Yi, Kou, Gao, and Xiao [24] used particle swarm algorithm to give the optimal system structure for the linear sliding window system of phased tasks, and carried out the reliability evaluation of the system by extending the general generating function. Ma, Wang, Cai, Si and Zhao [13] proposed

Nomenclature

n	Number of components in the system.
k	Minimum number of consecutive failed components, which cause system failure.
s_i	A random variable representing the state of component i. In a binary system: $s_i = 1$ if component i is working at time t, $s_i = 0$ otherwise.
$(s_i, \boldsymbol{p_i(t)})$	$(p_1(t), p_2(t), \dots, p_{i-1}(t), s_i(t), p_1(t), p_{i+1}(t), \dots, p_n(t))$
$\phi(\cdot)$	Non-decreasing structure function of the system
$R(\cdot)$	Reliability function of the system with $R(\boldsymbol{p(t)}) = E[\phi(s)]$
$R_i(t)$	The reliability of component i.
$\lambda_i(t)$	Failure rate of component i
$p_i(t)$	Probability that component i is worked

$q_i(t)$	Probability that component i is failed
$\boldsymbol{p}(t)$	$(p_1(t), \dots, p_n(t))$

a Delta Importance (DI) heuristic algorithm for redistributing components to improve system reliability.

The importance measures can be used to determine the key components in the system, and then the topology can be optimized [4, 6, 7, 20]. Importance measures have a wide range of applications, such as identifying the critical components of the system [18], guiding preventive maintenance [22], and others. Dui Si and Yam [3] gave the definition of system optimization topology based on importance measures and analyzed the relationship of optimal topology under different importance measures. Borgonovo and Apostolakis [1] proposed the differential importance measure, conducted case studies on these important measures and suggested their applications. Zio and Podofillini [29] further studied the differential importance measure. When evaluating the improvement of system performance due to the alteration of component reliability, the second-order extension of the differential importance measure is proposed to evaluate the interaction between component pairs. Zhu, Boushaba, and Reghioua [28] studied the reliability and joint reliability importance of components with Markov dependence and presented a practical application in the area of quality control. Moreover, a generalized Birnbaum importance measure was proposed to quantify the contribution of individual components to the improvement of system reliability [19].

Importance measure has been widely adopted in the field of k-out-of-n systems. Lyu and Si [12] proposed a dynamic importance measure for k-out-of-n: G systems under repeated random loads to identify dynamic weaknesses of systems subjected to repeated and random loads. Wu, Tang, Yu, Jiang [23] used the Laplace transform to define the expression of several important reliability indexes in the k-out-of-n system, and then studied the influence of the parameters on the indexes. Zhang, and Wu [25] derived the reliability index of the system in the k-out-of-n system considering the replacement and maintenance policy. Zhao, Wu, Wang, and Wang [27] investigated the system reliability evaluation method based on stochastic process for k-out-of-n: G system. Zhu, Boushaba, and Reghioua [28] focused on the joint reliability importance of components in a consecutive-k-out-of-n: F system, and a m-consecutive-k-out-of-n: F system, respectively, both with Markov-dependent components. Shen and Cui [16] used a finite Markov chain embedding method to assess system reliability and obtain Birnbaum's importance considering a circular consecutive-k-out-of-n system.

The change of the optimal component sequence and the improvement of the remaining useful life of a system are closely related to the importance measures. The main reason is that the importance measure can well identify the weak links and critical components of a system. In particular, the joint importance measure is very helpful for evaluating the impact of two component repairs on the system and understanding the interaction between components. Moreover, the linear consecutive-k-out-of-n systems are used as a typical application of a reconfigurable system. Changing the sequence of component layouts have complex effect on system reliability. The joint importance measure for the optimal structure may better reflect the importance degrees of the components and the joint effect between the components under the actual reconfigurable system. Therefore, in order to further study the optimal component sequence of the system, it is meaningful to investigate the joint importance. As such, there is a need to rethink about the following points.

- With the changes in multiple component reliabilities, how does the joint importance measure under the optimal component sequence change? How does the change of the optimal structure affect the joint importance?
- Under different optimal component sequence changes, does the location of the component affect the joint importance measure? If so, how?
- In the actual application, what are the differences of the generalized joint importance measures between the large system and the small system? How does the shape parameter of the lifetime distribution affect the joint importance measure?

In order to answer the above questions, this paper considers possible changes of the optimal component sequence of a system in its life cycle and proposes joint importance measures under the optimal component sequence to describe the component joint influence on the system reliability. Then the joint integrated importance and joint differential importance from the second-order interactions are extended to multistate situations. In addition, the paper carries out simulation analysis of the generalized measures for consecutive-k-out-of-n systems.

The remainder of this paper is arranged as follows. Section 2 discusses the joint importance measures for optimal structural component sequence in binary and multi-state systems. Section 3 gives some properties of the joint importance measure of the optimal component sequence. Section 4 presents the changes of the proposed measures with respect to the optimal system structure through numerical examples. Section 5 closes the paper.

2. Joint importance model for optimal component sequence

2.1. Existing importance measures

The Birnbaum importance of component i is defined by $R(1_i, \boldsymbol{p_i}(t)) - R(0_i, \boldsymbol{p_i}(\boldsymbol{t}))$, where $R(\cdot)$ is the reliability function of the system with $R(\boldsymbol{p}(\boldsymbol{t})) = E[\phi(x)]$. Si et al. **Error! Reference source not found.** propose the integrated importance measure (IIM) of component i, as shown in

$$I_i^{IIM}(t) = R_i(t)\lambda_i(t)\frac{\partial R(t)}{\partial R_i(t)}, \tag{1}$$

where $\lambda_i(t)$ is the failure rate of component i.

The joint reliability importance of components i and l can be denoted as **Error! Reference source not found.**

$$JRI(i,l) = R(1_i, 1_l, \boldsymbol{p_{il}}(\boldsymbol{t})) + R(0_i, 0_l, \boldsymbol{p_{il}}(\boldsymbol{t})) - R(1_i, 0_l, \boldsymbol{p_{il}}(\boldsymbol{t})) - R(0_i, 1_l, \boldsymbol{p_{il}}(\boldsymbol{t})).$$

Then in unit time, the change of the system reliability made by repairing component i is

$$\frac{d[R(1_i,\boldsymbol{p_i}(t))-R(0_i,\boldsymbol{p_i}(t))]}{dt}=\sum_{l=1,l\neq i}^{n}\frac{dR_l(t)}{dt}\frac{\partial R(1_i,\boldsymbol{p_i}(t))}{\partial R_l(t)}-\sum_{l=1,l\neq i}^{n}\frac{dR_l(t)}{dt}\frac{\partial R(0_i,\boldsymbol{p_i}(t))}{\partial R_l(t)}$$

$$=-\sum_{l=1,l\neq i}^{n}\lambda_l(t)R_l(t)[R(1_i,1_l,\boldsymbol{p_{il}(t)})+R(0_i,0_l,\boldsymbol{p_{il}(t)})-R(1_i,0_l,\boldsymbol{p_{il}(t)})-R(0_i,1_l,\boldsymbol{p_{il}(t)})]$$

$$=-\sum_{l=1,l\neq i}^{n}\lambda_l(t)R_l(t)JRI(i,l). \tag{2}$$

In equation (2), $-\sum_{l=1,l\neq i}^{n}\lambda_l(t)R_l(t)JRI(i,l)$ suggests that when component i is repaired, the change of system reliability in a unit of time is the sum of $\lambda_l(t)R_l(t)JRI(i,l)$ of all other system component l, in which $\lambda_l(t)R_l(t)JRI(i,l)$ is the contribution of component l to the change of the system reliability in a unit of time after component i is repaired. $R_i(t)\lambda_i(t)[R(1_i,\boldsymbol{p_i(t)})-R(0_i,\boldsymbol{p_i(t)})]$ represents the change of system reliability in a unit of time due to component i. The joint integrated importance of components i and l is $JIIM_{i,l}(II)=\lambda_i(t)\lambda_l(t)R_i(t)R_l(t)JRI(i,l)$, which represents the joint contribution of components i and l to the change of system reliability in a unit of time. For component i, when the effect of the sum of all other system component is considered, we have $JIIM_i(I)=\sum_{l=1,l\neq i}^{n}\lambda_i(t)\lambda_l(t)R_i(t)R_l(t)JRI(i,l)$.

Considering combinations or groups that can be applied to components or basic events, we introduce a Joint Differential Importance Measure ($JDIM$), which considers the interaction between pairs of components when evaluating system performance changes due to changes in component reliability parameters.

The $JDIM$ of components i and l is (a second-order DIM)

$$JDIM_i(I)=\frac{\Delta R_i^{II}}{\Delta R^{II}}=\frac{\frac{\partial R}{\partial R_i}\Delta R_i+\sum_{l=1,l\neq i}^{n}\frac{\partial^2 R}{\partial R_i\partial R_l}\Delta R_i\Delta R_l}{\sum_{i=1}^{n}\frac{\partial R}{\partial R_i}\Delta R_i+\sum_{i=1}^{n}\sum_{l>i=1}^{n}\frac{\partial^2 R}{\partial R_i\partial R_l}\Delta R_i\Delta R_l}, \tag{3}$$

and

$$JDIM_{i,l}(II)=\frac{\Delta R_{il}^{II}}{\Delta R^{II}}=\frac{\frac{\partial R}{\partial R_i}\Delta R_i+\frac{\partial R}{\partial R_l}\Delta R_l+\frac{\partial^2 R}{\partial R_i\partial R_l}\Delta R_i\Delta R_l}{\sum_{i=1}^{n}\frac{\partial R}{\partial R_i}\Delta R_i+\sum_{i=1}^{n}\sum_{l>i=1}^{n}\frac{\partial^2 R}{\partial R_i\partial R_l}\Delta R_i\Delta R_l}. \tag{4}$$

By this definition, $JDIM_{i,l}(II)$ captures the second order interaction effects in determining component i importance.

2.2 Binary systems

By using the definition of the consecutive-*k*-out-of-*n*: F system, we have

$$R(1_i,\boldsymbol{p_i(t)})=R(p_1,\dots,p_{i-1},1,p_{i+1},\dots,p_n)=R(p_1,\dots,p_{i-1})R(p_{i+1},\dots,p_n)=R([1]\to[i-1])R([i+1]\to[n]),$$

$$R(0_i,\boldsymbol{p_i(t)})=R(p_1,\dots,p_{i-1},0,p_{i+1},\dots,p_n)$$

$$= R([1] \to [i-1]) R(p_{i-k+1}, \dots, p_{i-1}, 0, p_{i+1}, \dots, p_{i+k-1}) R([i+1] \to [n])$$

$$= R([1] \to [i-1]) R'([i-k+1] \to [i+k-1]) R([i+1] \to [n]),$$

and $R(n) = p_i R(1_i, \boldsymbol{p_i(t)}) + q_i R(0_i, \boldsymbol{p_i(t)})$.

By using the definition of the Birnbaum importance, we can obtain

$$R(BM_i) = R\big(1_i, \boldsymbol{p_i(t)}\big) - R\big(0_i, \boldsymbol{p_i(t)}\big) = R\big(1_i, \boldsymbol{p_i(t)}\big) - \frac{R(n) - p_i R\big(1_i, \boldsymbol{p_i(t)}\big)}{q_i}$$

$$= \frac{R([1] \to [i-1]) R([i+1] \to [n]) - R(n)}{q_i}.$$

Imposing $i + 1 < l$, we can derive the following definition：

$$R(1_i, 1_l, \boldsymbol{p_i(t)}) = R([1] \to [i-1]) R([i+1] \to [l-1]) R([l+1] \to [n]),$$

and

$$R(1_i, 0_l, \boldsymbol{p_i(t)}) = R(p_1, \dots, p_{i-1}, 1, p_{i+1}, \dots, p_{l-1}, 0, p_{l+1}, \dots, p_n)$$

$$= R(p_1, \dots, p_{i-1}) R(p_{i+1}, \dots, p_{l-1}, 0, p_{l+1}, \dots, p_n)$$

$$= R([1] \to [i-1]) R([i+1] \to [l-1]) R(p_{l-k+1}, \dots, p_{l-1}, 0, p_{l+1}, \dots, p_{l+k-1}) R([l+1] \to [n])$$

$$= R([1] \to [i-1]) R([i+1] \to [l-1]) R'([l-k+1] \to [l+k-1]) R([l+1] \to [n]).$$

Since the above formula helps us to derive the specific expansion of $JRI(i,l)$, we can obtain

$$R(1_i, 1_l, \boldsymbol{p_i(t)}) - R(0_i, 1_l, \boldsymbol{p_i(t)})$$

$$= \frac{\big(q_i(t)p_l(t) - p_i(t)p_l(t)\big) R([1] \to [i-1]) R([i+1] \to [l-1]) R([l+1] \to [n]) + R\big(1_l, \boldsymbol{p(t)}\big)}{q_i(t)p_l(t)}$$

$$= \frac{\big(q_i(t)p_l(t) - p_i(t)p_l(t)\big) R([1] \to [i-1]) R([i+1] \to [l-1]) R([l+1] \to [n]) + R([1] \to [l-1]) R([l+1] \to [n])}{q_i(t)p_l(t)}$$

$$= \frac{\{(q_i(t)p_l(t) - p_i(t)p_l(t)) R([1] \to [i-1]) R([i+1] \to [l-1]) + R([1] \to [l-1])\} R([l+1] \to [n])}{q_i(t)p_l(t)},$$

and

$$R\big(1_i, 0_l, \boldsymbol{p_{il}(t)}\big) - R\big(0_i, 0_l, \boldsymbol{p_{il}(t)}\big)$$

$$= \frac{\left\{\begin{matrix}\big(q_i(t)q_l(t) - p_i(t)q_l(t)\big) R([1] \to [i-1]) R([i+1] \to [l-1]) \\ \cdot R'([l-k+1] \to [l+k-1]) R([l+1] \to [n])\end{matrix}\right\} + R\big(0_l, \boldsymbol{p_{il}(t)}\big)}{q_i(t)q_l(t)}$$

$$= \frac{\begin{matrix}\big(q_i(t)q_l(t) - p_i(t)q_l(t)\big) R([1] \to [i-1]) R([i+1] \to [l-1]) R'([l-k+1] \to [l+k-1]) R([l+1] \to [n]) \\ + R([1] \to [l-1]) R'([l-k+1] \to [l+k-1]) R([l+1] \to [n])\end{matrix}}{q_i(t)q_l(t)}$$

$$= \frac{\left\{\left(q_i(t)q_l(t) - p_i(t)q_l(t)\right)R([1] \to [i-1])R([i+1] \to [l-1]) + R([1] \to [l-1])\right\} \cdot R'([l-k+1] \to [l+k-1])R([l+1] \to [n])}{q_i(t)q_l(t)}$$

Then we have joint reliability importance for a consecutive-*k*-out-of-*n*: F system,

$$R(JRI(i,l)) = R(1_i, 1_l, \boldsymbol{p_{il}(t)}) - R(0_i, 1_l, \boldsymbol{p_{il}(t)}) - (R(1_i, 0_l, \boldsymbol{p_{il}(t)}) - R(0_i, 0_l, \boldsymbol{p_{il}(t)}))$$

$$= \frac{\left\{\left(q_i(t)p_l(t) - p_i(t)p_l(t)\right)R([1] \to [i-1])R([i+1] \to [l-1]) + R([1] \to [l-1])\right\}R([l+1] \to [n])}{q_i(t)p_l(t)}$$

$$- \frac{\left\{\begin{matrix}\left(q_i(t)q_l(t) - p_i(t)q_l(t)\right)R([1] \to [i-1])R([i+1] \to [l-1]) \\ +R([1] \to [l-1])\end{matrix}\right\}R'([l-k+1] \to [l+k-1])R([l+1] \to [n])}{q_i(t)q_l(t)}$$

$$= \frac{q_l(t)\left\{\begin{matrix}\left(q_i(t)p_l(t) - p_i(t)p_l(t)\right)R([1] \to [i-1])R([i+1] \to [l-1]) \\ +R([1] \to [l-1])\end{matrix}\right\}R([l+1] \to [n])}{p_l(t)q_i(t)q_l(t)}$$

$$- \frac{p_l(t)\left\{\begin{matrix}\left(q_i(t)q_l(t) - p_i(t)q_l(t)\right)R([1] \to [i-1])R([i+1] \to [l-1]) \\ +R([1] \to [l-1])\end{matrix}\right\}R'([l-k+1] \to [l+k-1])R([l+1] \to [n])}{q_l(t)q_i(t)p_l(t)}$$

$$= \frac{\left\{\begin{matrix}\left\{\begin{matrix} q_l(t)\left(q_i(t)p_l(t) - p_i(t)p_l(t)\right) \\ -p_l(t)\left(q_i(t)q_l(t) - p_i(t)q_l(t)\right) \\ \cdot R'([l-k+1] \to [l+k-1]) \end{matrix}\right\}R([1] \to [i-1])R([i+1] \to [l-1]) \\ +\{q_l(t) - p_l(t)R'([l-k+1] \to [l+k-1])\}R([1] \to [l-1])\end{matrix}\right\}R([l+1] \to [n])}{q_i(t)p_l(t)q_l(t)}.$$

With component i, when considering the effect of the sum of all other system component, the joint integrated importance for the consecutive-*k*-out-of-*n*: F system is

$$JIIM_i(I) = \sum_{l=1, l \neq i}^{n} \lambda_i(t)\lambda_l(t)R_i(t)R_l(t)\, R\big(JRI(i,l)\big). \tag{5}$$

The joint integrated importance of components i and l for the consecutive-*k*-out-of-*n*: F system is

$$JIIM_{i,l}(II) = \lambda_i(t)\lambda_l(t)R_i(t)R_l(t)\, R\big(JRI(i,l)\big). \tag{6}$$

For component i, when the effect of the sum of all other system component is considered, the joint DIM for the consecutive-*k*-out-of-*n*: F system is

$$JDIM_i(I) = \frac{\Delta R_i^{II}}{\Delta R^{II}} = \frac{R(BM_i)\Delta R_i + \sum_{l=1, l \neq i}^{n} R\big(JRI(i,l)\big)\Delta R_i \Delta R_l}{\sum_{i=1}^{n} R(BM_i)\Delta R_i + \sum_{i=1}^{n}\sum_{l>i=1}^{n} R\big(JRI(i,l)\big)\Delta R_i \Delta R_l}. \tag{7}$$

The joint DIM of components i and l for the consecutive-*k*-out-of-*n*: F system is

$$JDIM_{i,l}(II) = \frac{R(BM_i)\Delta R_i + R(BM_l)\Delta R_l + R\big(JRI(i,l)\big)\Delta R_i \Delta R_l}{\sum_{i=1}^{n} R(BM_i)\Delta R_i + \sum_{i=1}^{n}\sum_{l>i=1}^{n} R\big(JRI(i,l)\big)\Delta R_i \Delta R_l}. \tag{8}$$

If the component reliability changes, the optimal component sequence may change. It is not difficult to know that the joint importance of the corresponding components changes with the change of the optimal system structure. We use the genetic algorithm to traverse the reliability of the system after different sequence of components. Then we select the arrangement that maximizes the reliability of the system as the optimal component sequence. Assume that $R_{opt}(BM_i)$ and $R_{opt}(JRI(i,l))$ are the system or subsystem reliabilities obtained for the optimal component sequence at time *t*, i.e., bring the optimal component sequence into the $R(BM_i)$and$R(JRI(i,l))$ in consecutive-*k*-out-of-*n*: F system proposed above. Then $JIIM_i(I)$, $JIIM_{i,l}(II)$, $JDIM_i(I)$, and $JDIM_{i,l}(II)$ for the optimal component sequence of the consecutive-*k*-out-of-*n*: F system are

$$JIIM_i^{opt}(I) = \sum_{l=1,l\neq i}^{n} \lambda_i(t)\lambda_l(t)R_i(t)R_l(t)\, R_{opt}\big(JRI(i,l)\big), \tag{9}$$

$$JIIM_{i,l}^{opt}(II) = \lambda_i(t)\lambda_l(t)R_i(t)R_l(t)\, R_{opt}\big(JRI(i,l)\big), \tag{10}$$

$$JDIM_i^{opt}(I) = \frac{\Delta R_i^{II}}{\Delta R^{II}} = \frac{R_{opt}(BM_i)\Delta R_i + \sum_{l=1,l\neq i}^{n} R_{opt}\big(JRI(i,l)\big)\Delta R_i \Delta R_l}{\sum_{i=1}^{n} R_{opt}(BM_i)\Delta R_i + \sum_{i=1}^{n}\sum_{l>i=1}^{n} R_{opt}\big(JRI(i,l)\big)\Delta R_i \Delta R_l}, \tag{11}$$

and

$$JDIM_{i,l}^{opt}(II) = \frac{R(BM_i)\Delta R_i + R(BM_l)\Delta R_l + R\big(JRI(i,l)\big)\Delta R_i \Delta R_l}{\sum_{i=1}^{n} R(BM_i)\Delta R_i + \sum_{i=1}^{n}\sum_{l>i=1}^{n} R\big(JRI(i,l)\big)\Delta R_i \Delta R_l}. \tag{12}$$

2.3 Multistate systems

Assume that the components in a multistate system are independent of each other. Assume that a component can only degrade for one state at a time, but the system may degrade for multiple states. We stipulate those components and system have $M+1$ states: 0,1,.., M. We give $S = \{s_1, s_2, \ldots, s_n\}$, s_i is the state of component $i(0 \le s_i \le M)$. $\boldsymbol{S_i} = \{s_1, s_{i-1}, \cdot, s_{i+1} \ldots, s_n\}$. The component *i* in the state *v* can be expressed as a vector $(v_i, \boldsymbol{S_i})$. When the system state is *m*, $\phi(v_i, \boldsymbol{S_i}) = m$, in order to enhance its applicability, we extend the two joint importance measures of reconfigurable systems to multistate systems. We stipulate $R_i(v) = \Pr[s_i \ge v]$, $R(v_i, \boldsymbol{S_i}; m) = Pr[\phi(v_i, \boldsymbol{S_i}) \ge m]$. Then,

$$\frac{d[R(v_i,\boldsymbol{S_i}) - R((v-1)_i,\boldsymbol{S_i})]}{dt} = \sum_{m=1}^{M}\left\{\sum_{l=1,l\neq i}^{n} \frac{dR_l(k)}{dt}\frac{\partial R(v_i,\boldsymbol{S_i};m)}{\partial R_l(k)} - \sum_{l=1,l\neq i}^{n} \frac{dR_l(k)}{dt}\frac{\partial R((v-1)_i,\boldsymbol{S_{il}};m)}{\partial R_l(k)}\right\}$$

$$= \sum_{m=0}^{M+1} \left\{ \begin{array}{c} \sum_{l=1,l\neq i}^{n} \frac{dR_l(k)}{dt} [R(v_i, k_l, \boldsymbol{S_{il}}; m) - R(v_i, (k-1)_l, \boldsymbol{S_{il}}; m)] \\ - \sum_{l=1,l\neq i}^{n} \frac{dR_l(k)}{dt} [R((v-1)_i, k_l, \boldsymbol{S_{il}}; m) - R((v-1)_i, (k-1)_l, \boldsymbol{S_{il}}; m)] \end{array} \right\}$$

$$= \sum_{m=0}^{M+1} \left\{ \sum_{l=1,l\neq i}^{n} \frac{dR_l(k)}{dt} \begin{bmatrix} R(v_i, k_l, \boldsymbol{S_{il}}; m) + R((v-1)_i, (k-1)_l, \boldsymbol{S_{il}}; m) \\ -R(v_i, (k-1)_l, \boldsymbol{S_{il}}; m) - R((v-1)_i, k_l, \boldsymbol{S_{il}}; m) \end{bmatrix} \right\}$$

$$= - \sum_{l=1,l\neq i}^{n} \sum_{m=0}^{M+1} \lambda_l(k) R_l(k) \begin{bmatrix} R(v_i, k_l, \boldsymbol{S_{il}}; m) + R((v-1)_i, (k-1)_l, \boldsymbol{S_{il}}; m) \\ -R(v_i, (k-1)_l, \boldsymbol{S_{il}}; m) - R((v-1)_i, k_l, \boldsymbol{S_{il}}; m) \end{bmatrix}.$$

$(i, l; v, k)$ indicates that the state of component i is v, and the state of component l is k. We can obtain,

$$R^m\big(JRI(i, l; v, k)\big) = \sum_{m=0}^{M+1} \begin{bmatrix} R(v_i, k_l, \boldsymbol{S_{il}}; m) + R((v-1)_i, (k-1)_l, \boldsymbol{S_{il}}; m) \\ -R(v_i, (k-1)_l, \boldsymbol{S_{il}}; m) - R((v-1)_i, k_l, \boldsymbol{S_{il}}; m) \end{bmatrix}.$$

The Birnbaum importance measure in multistate systems can be expressed as

$$R^m\big(BM_{i,v}\big) = \sum_{m=0}^{M+1} R(v_i, \boldsymbol{S_i}; m) - R((v-1)_i, \boldsymbol{S_i}; m)].$$

For linear consecutive-*k*-out-of-*n* systems, we assume that ξ_i is the threshold state of component i. Once the component state is degraded below the threshold state, it will fail. Similarly, we can obtain

$R((\geq \xi_i)_i, (\geq \xi_l)_l, \boldsymbol{S_{il}}; m) = R([1] \to [i-1]) R([i+1] \to [l-1]) R([l+1] \to [n]),$

and

$R((\geq \xi_i)_i, (< \xi_l)_l, \boldsymbol{S_{il}}; m) = R([1] \to [i-1]) R([i+1] \to [l-1]) \cdot$

$R((\geq \xi_{l-k+1})_{l-k+1}, \dots, (\geq \xi_{l-1})_{l-1}, (< \xi_l)_l, (\geq \xi_{l+1})_{l+1}, \dots, (\geq \xi_{l+k-1})_{l+k-1}) R(l+1 \to n)$

$$= R([1] \to [i-1]) R([i+1] \to [l-1]) R'([i-k+1] \to [i+k-1]) R([l+1] \to [n]).$$

$R((\geq \xi_l)_l, \boldsymbol{S_l}) = R_i(\geq \xi_i) R_l(\geq \xi_l) R((\geq \xi_i)_i, (\geq \xi_l)_l, \boldsymbol{S_{il}}) + R_i(< \xi_i) R_l(\geq \xi_l) R((< \xi_i)_i, (\geq \xi_l)_l, \boldsymbol{S_{il}}),$

$R((< \xi_l)_l, \boldsymbol{S_l}) = R_i(\geq \xi_i) R_l(< \xi_l) R((\geq \xi_i)_i, (< \xi_l)_l, \boldsymbol{S_{il}}) + R_i(< \xi_i) R_l(< \xi_l) R((< \xi_i)_i, (< \xi_l)_l, \boldsymbol{S_{il}}).$

According to the above formula, for different states of components, we can calculate the corresponding value of $R(v_i, k_l, \boldsymbol{S_{il}}; m) + R((v-1)_i, (k-1)_l, \boldsymbol{S_{il}}; m) - R(v_i, (k-1)_l, \boldsymbol{S_{il}}; m) - R((v-1)_i, k_l, \boldsymbol{S_{il}}; m).$

We stipulate that $R^m_{opt}\big(BM_{i,v}\big)$ and $R^m_{opt}\big(JRI(i, l; v, k)\big)$ are the reliability of systems or subsystems obtained for the best configuration at time *t* in multistate systems.

Subsequently, $JIIM^{opt}_{i;v}(I)$, $JIIM^{opt}_{i,l;v,k}(II)$, $JDIM^{opt}_{i;v}(I)$, and$JDIM^{opt}_{i,l;v,k}(II)$ for the optimal component sequence of the consecutive-k-out-of-n: F system in multistate systems can respectively be obtained

$$JIIM_{i;v}^{opt}(I) = \sum_{l=1,l\neq i}^{n} \lambda_i(v)\lambda_l(k)R_i(v)R_l(k)\, R_{opt}^{m}\big(JRI(i,l;v,k)\big), \tag{13}$$

$$JIIM_{i,l;v,k}^{opt}(II) = \lambda_i(v)\lambda_l(k)R_i(v)R_l(k)\, R_{opt}^{m}\big(JRI(i,l;v,k)\big), \tag{14}$$

$$JDIM_{i;v}^{opt}(I) = \frac{\Delta R_i^{II}}{\Delta R^{II}} = \frac{R_{opt}^{m}\big(BM_{i,v}\big)\Delta R_i + \sum_{l=1,l\neq i}^{n} R_{opt}^{m}\big(JRI(i,l;v,k)\big)\Delta R_i \Delta R_l}{\sum_{i=1}^{n} R_{opt}^{m}\big(BM_{i,v}\big)\Delta R_i + \sum_{i=1}^{n}\sum_{l>i=1}^{n} R_{opt}^{m}\big(JRI(i,l;v,k)\big)\Delta R_i \Delta R_l}, \tag{15}$$

$$JDIM_{i,l;v,k}^{opt}(II) == \frac{R_{opt}^{m}\big(BM_{i,v}\big)\Delta R_i + R_{opt}^{m}\big(BM_{l,k}\big)\Delta R_l + R_{opt}^{m}\big(JRI(i,l;v,k)\big)\Delta R_i \Delta R_l}{\sum_{i=1}^{n} R_{opt}^{m}\big(BM_{i,v}\big)\Delta R_i + \sum_{i=1}^{n}\sum_{l>i=1}^{n} R_{opt}^{m}\big(JRI(i,l;v,k)\big)\Delta R_i \Delta R_l}. \tag{16}$$

3 Properties on the proposed measures

In many actual application scenarios, the components (such as telecommunication and pipe-line networks, etc.) in a consecutive n-th system may be identical. When the lifetime of all components are independently and identically distributed, the characteristics of the joint importance measure of the optimal component sequence appear particularly important. We have the following discussions.

We will then give some properties of joint importance measures for an optimal component sequence in a linear consecutive-k-out-of-n: F system as follows. The following theorems are all based on the hypothesis that the components are independent and identically distributed.

Theorem 1 For the optimal component sequence of a linear consecutive-k-out-of-n: F system, when all components are independently and identically distributed, and $p_1(t) = p_2(t) = \cdots = p_n(t) = p(t)$. When k is determined, $R'([l-k+1] \to [l+k-1])$ can be regarded as a constant.

The proof of this theorem and those of the other theorems are given in Appendix.

Theorem 2 For the optimal component sequence of a linear consecutive-k-out-of-n: F system, assume that when all components are independently and identically distributed, and $p_1(t) = p_2(t) = \cdots = p_n(t) = p(t)$.

- If $\lambda_i(t) \leq \lambda_{i+1}(t)$, then $JIIM_{i,l}^{opt}(II) < JIIM_{i+1,l}^{opt}(II)$, for $i < k$;
- If $\lambda_l(t) \leq \lambda_{l+1}(t)$, then $JIIM_{i,l}^{opt}(II) < JIIM_{i,l+1}^{opt}(II)$, for $l < k$;
- If $\lambda_i(t) \leq \lambda_{i+1}(t), \lambda_l(t) \leq \lambda_{l+1}(t)$, then $JIIM_{i,l}^{opt}(II) < JIIM_{i+1,l+1}^{opt}(II)$, for $i < l < k$;
- For $n-k < i < l < k$, $R_{opt}(JRI(i,l)) = R_{opt}(JRI(i+1,l)) = R_{opt}(JRI(i,l+1)) = R_{opt}(JRI(i+1,l+1))$

Similarly, we can draw the following inference.

Theorem 3 For the optimal component sequence of a linear consecutive-*k*-out-of-*n*: F system, assume that $p_1(t) = p_2(t) = \cdots = p_n(t) = p(t)$.

- For $n - k < i < k$， $R_{opt}(BM_i) = R_{opt}(BM_{i+1})$;
- If $\Delta R_i \leq \Delta R_{i+1}$ then $JDIM_{i,l}^{opt}(II) < JDIM_{i+1,l}^{opt}(II)$， for $i < k$;
- If $\Delta R_l \leq \Delta R_{l+1}$, , then $JDIM_{i,l}^{opt}(II) < JDIM_{i,l+1}^{opt}(II)$, for $l < k$;
- If $\Delta R_i \leq \Delta R_{i+1}, \Delta R_l \leq \Delta R_{l+1}$， then $JDIM_{i,l}^{opt}(II) < JDIM_{i+1,l+1}^{opt}(II)$, for $i < l < k$.

Under hypothetical conditions, when adjacent components i and $i+1$ fail respectively, $JIIM_{i,l}^{opt}(II)$ relates to the failure rate of components i and $i+1$. When component i fails, $JIIM_{i,l}^{opt}(II)$ relates to the failure rate of components l and $l+1$. The same properties are also shown in $JDIM_{i,l}^{opt}(II)$.

According to Theorems 2 and 3, we can know from it, when considering the impact of one component on the system reliability and another component fails, the component that has the greatest impact on system reliability can be selected for preventive maintenance. This provides information on which component is the more important and improve it first. Under this strategy, we can improve system reliability as high as possible. $JIIM_i^{opt}(I)$ and $JDIM_i^{opt}(I)$ are the influence of component *i* on the system reliability when considering the sum of all other system components. Moreover, we propose Theorem 4 for the change of component i.

Theorem 4 For the optimal component sequence of a linear consecutive-*k*-out-of-*n*: F system, assume that $p_1(t) = p_2(t) = \cdots = p_n(t) = p(t)$.

- If $\lambda_i(t) \leq \lambda_{i+1}(t)$, then $JIIM_i^{opt}(I) < JIIM_{i+1}^{opt}(I)$, for $i < k$;
- If $\Delta R_i \leq \Delta R_{i+1}$, then $JDIM_i^{opt}(I) < JDIM_{i+1}^{opt}(I)$ for $i < k$

According to Theorem 4, for component i, the effect of component i on system reliability is less than that of component $i+1$ when the influence of the sum of all other system components is considered and the failure rate function of component i is smaller than that of component $i+1$. Then, when the impact of component i on system reliability is considered, the most important components can be determined.

4 Numerical examples

In this section, we analyze the value changes of $JIIM_i^{opt}(I)$,$JIIM_{i,l}^{opt}(II)$, $JDIM_i^{opt}(I)$, and $JDIM_{i,l}^{opt}(II)$ for the optimal system structure when *n* = 5 and *n* = 10 in linear consecutive-*k*-out-of-*n*: F(G) systems. We assume that the target component *n* follows the Weibull distribution so that the influence of the shape parameters of the components on the joint importance measure can be analyzed. . Denote the Weibull distribution by $F(t) = 1 - exp[-(t/\beta)^{\alpha}]$, β, α>0, where α and β are the shape and scale parameters, respectively. Meanwhile, for easy calculation, the other components (components 1~n-1) are assumed to follow the exponential distribution with failure rate: $0.2: 0.2: 0.2(i-1)$. That is, the failure rate of component reliability gradually increases with a common difference of 0.2. As we all know, the life of the system is generally described as three periods in a bathtub curve, and the components will have three situations of decreasing, constant and increasing failure rate, so we use α=0.1, α=1, and α=3 to describe these three periods respectively.

For simplicity, the simulation of $JIIM_i{}^{opt}(I)$ and $JDIM_i{}^{opt}(I)$ are carried out for target component n, and the following description is abbreviated as $JIIM^{opt}(I)$ and $JDIM^{opt}(II)$. In particular, when calculating the joint importance of the second-order interaction, the importance is the joint measure of component n and component n-1. The following description of $JIIM_{i,l}{}^{opt}(II)$ and $JDIM_{i,l}{}^{opt}(II)$ is abbreviated as $JIIM^{opt}(II)$ and $JDIM^{opt}(II)$.

4.1 Linear consecutive-k-out-of-5 system

The optimal component sequence in a linear consecutive-*2*-out-of-*5*: F(G) system changes with time. Table 1 and Table 2 present the optimal component sequence of a linear consecutive-*2*-out-of-*5*: F(G) system at different moments, respectively. The simulation of $JIIM_i{}^{opt}(I)$ and $JDIM_i{}^{opt}(I)$ are carried out for component 5, and the following description is abbreviated as $JIIM^{opt}(I)$ and $JDIM^{opt}(II)$. The simulation of $JIIM_{i,l}{}^{opt}(II)$ and $JDIM_{i,l}{}^{opt}(II)$ are carried out for component 5 and component 4. The joint importance of component 5 for a linear consecutive-2-out-of-5 system is given in Fig. 1.

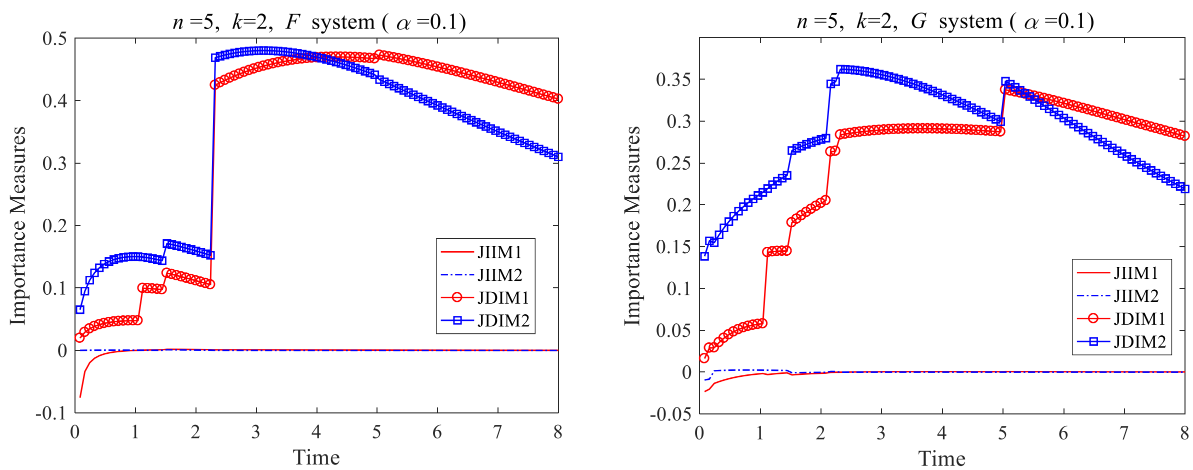
Table 1. Optimal component sequence in linear consecutive-2-out-of-5: F systems when shape parameter is 0.1.

T	optimal configuration
0	5 1 3 4 2
1.1163	4 1 3 2 5
1.5548	4 1 5 2 3
2.4319	4 1 2 5 3
5.2824	4 5 2 1 3

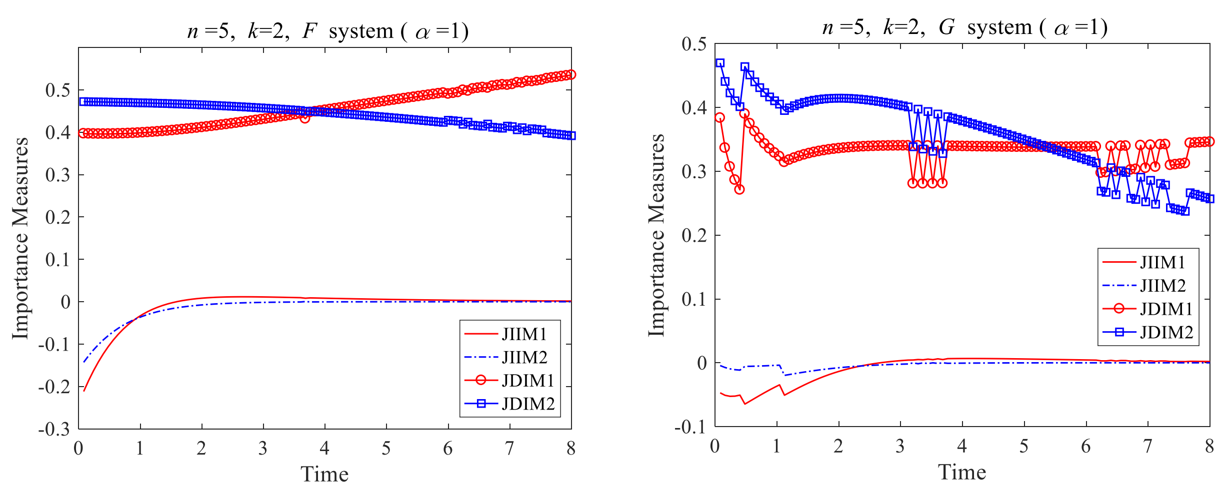
Table 2. Optimal component sequence in linear consecutive-2-out-of-5: G systems when shape parameter is 0.1.

t	optimal configuration

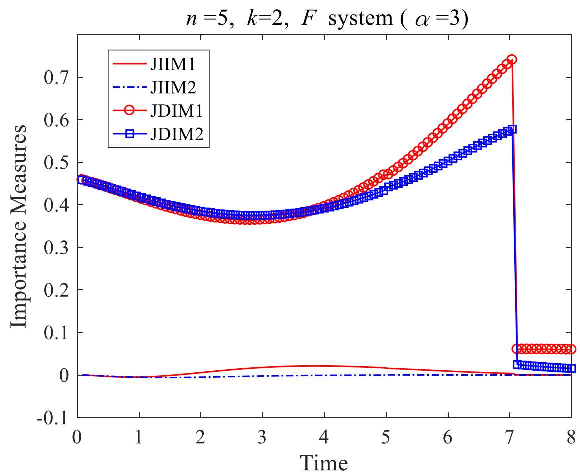
0	5 1 3 4 2
1.1163	4 1 3 2 5
1.5548	4 1 5 2 3
2.4319	4 1 2 5 3
5.2824	4 5 2 1 3

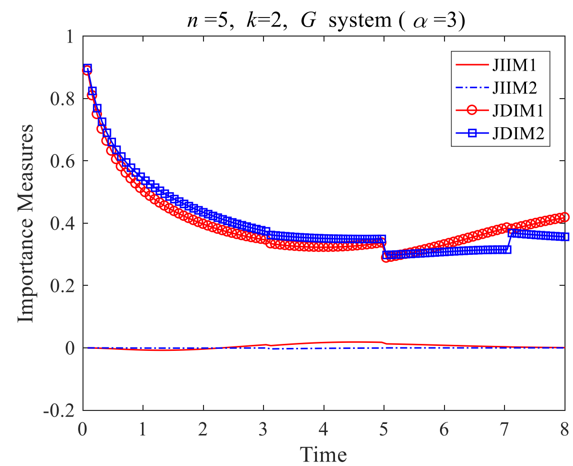


(a) Joint importance of component 5 when shape parameter is 0.1 for F system (b) Joint importance of component 5 when shape parameter is 0.1 for G system



(c) Joint importance of component 5 when shape parameter is 1 for F system (d) Joint importance of component 5 when shape parameter is 1 for G system



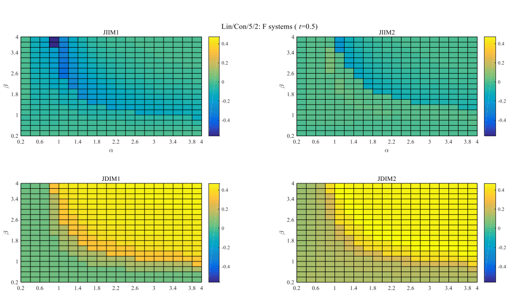


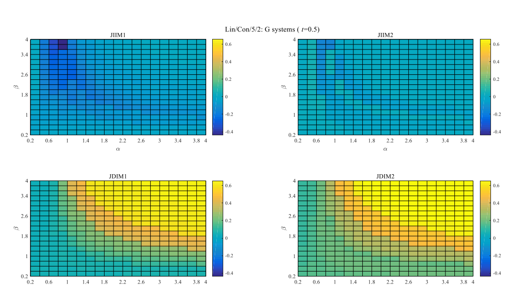
(e) Joint importance of component 5 when shape parameter is 3 for F system (f) Joint importance of component 5 when shape parameter is 3 for G system

Fig. 1 Joint importance of component 5 with optimal component sequence when *n*=5, *k*=2.

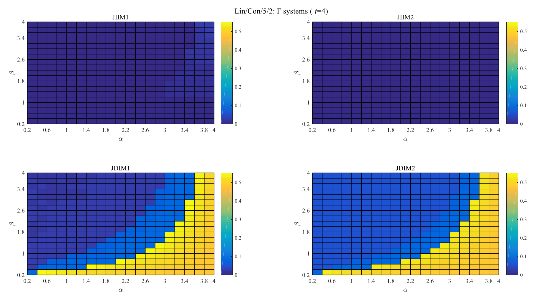
For linear consecutive-2-out-of-5: F(G) systems, when the shape parameter is 0.1 and the optimal component sequence changes, the two proposed joint importance measures are most obviously affected by the optimal ranking of components. $JDIM^{opt}(I)$ and $JDIM^{opt}(II)$ of the corresponding components sometimes jumps upwards. Specifically, it can be identified from the Figs. 1(a) and 1(b). In Figs. 1(b), 1(d), $JIIM^{opt}(I), and\ JIIM^{opt}(II)$ have relatively large fluctuations. But like $JIIM^{opt}(I),$ and $JIIM^{opt}(II)$ under other different parameters, it gradually approaches 0 as time evolves. The importance of the target component changes irregularly with time. The change in the reliability of the target component may cause the rearrangement of the optimal component sequence, and the importance of the new optimal component sequence will be determined again. This also means that the reliability of components will affect the joint importance. Furthermore, its influence on $JDIM^{opt}$is more significant

Fig. 2 shows the change of the corresponding joint importance of the target component under different scale parameters and shape parameters.

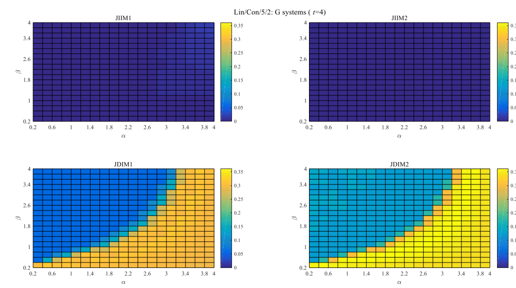




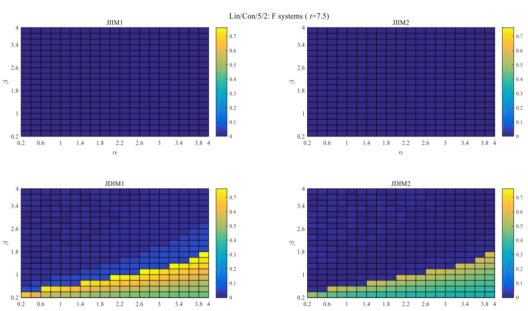
(a) Joint importance of component 5 for F system at t=0.5 (b) Joint importance of component 5 for G system at t=0.5



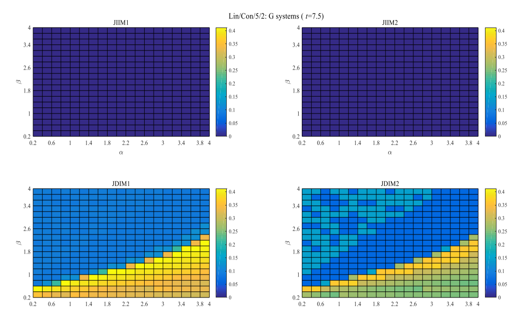
(c) Joint importance of component 5 for F system at t=4



(d) Joint importance of component 5 for G system at t=4



(e) Joint importance of component 5 for F system at t=7.5

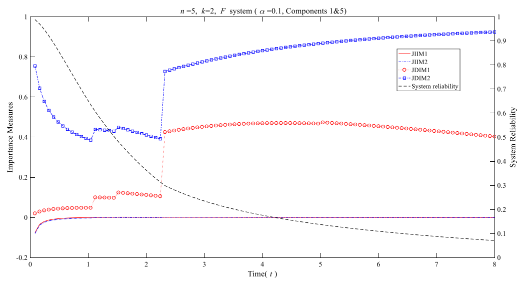


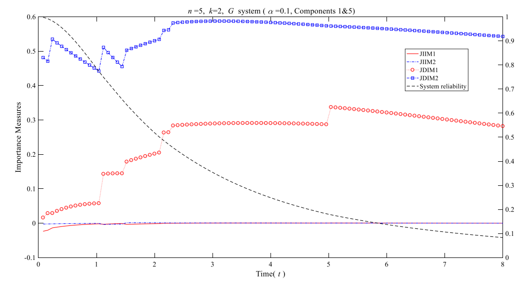
(f) Joint importance of component 5 for G system at t=7.5

Fig. 2 The joint importance of component 5 under different combinations of α and β when *n*=5, *k*=2 (t=0.5, t=4, t=7.5)

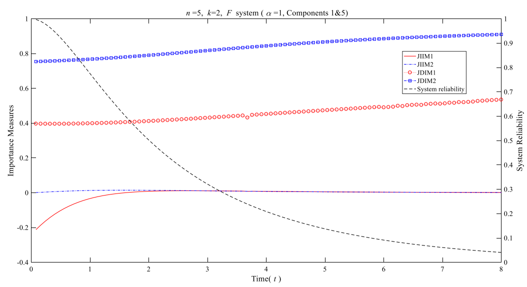
According to Fig. 2, we can observe that under different shape parameters and scale parameters, the corresponding four joint importance measures have different degrees of change. For the F and G systems at *t*=0.5, i.e., Figs. 2(a) and 2(b), with the simultaneous increase of α and β, $JDIM^{opt}(I)$, and $JDIM^{opt}(II)$ have an increasing trend. At t=4, i.e., Figs. 2 (c) and 2(d), α increases whereas β decreases, $JDIM^{opt}(I)$ and $JDIM^{opt}(II)$ ncrease. Compared with $JDIM^{opt}$, $JIIM^{opt}$ changes irregularly at *t*=0.5. $JIIM^{opt}$ gradually approaches 0 as time increases. In other words, the optimal configuration has less effect on $JIIM^{opt}$ in linear consecutive-2-out-of-5 systems.

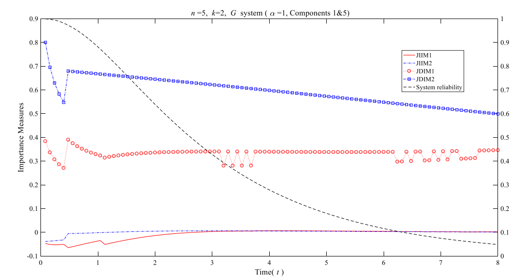
Combining component 1 and component 5 to calculate the joint importance measure when shape parameter is 0.1, 1, and 3 in linear consecutive-2-out-of-5: F(G) system, respectively, then the results are shown in Fig. 3. With the increase of time, the reliability of the system shows a downward trend. In addition, we can find the behavior of joint importance measure of component 1 and component 5 and the joint importance measure of component 5 and component 4 (The simulation above is the joint importance measure between components *n* and *n*-1) are different. Similarly, we can calculate the joint importance of other components and the target component. When applied to the preventive maintenance process, if the target component fails, other components can be sorted according to the joint importance, so as to find the best solution to improve system reliability.



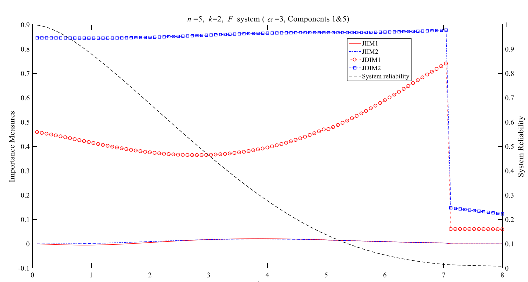


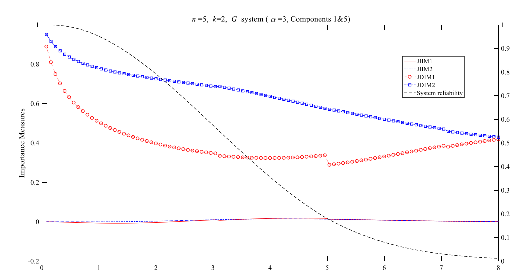
(a) Joint importance of component 5 when shape parameter is 0.1 for F system (b) Joint importance of component 5 when shape parameter is 0.1 for G system





(c) Joint importance of component 5 when shape parameter is 1 for F system (d) Joint importance of component 5 when shape parameter is 1 for G system

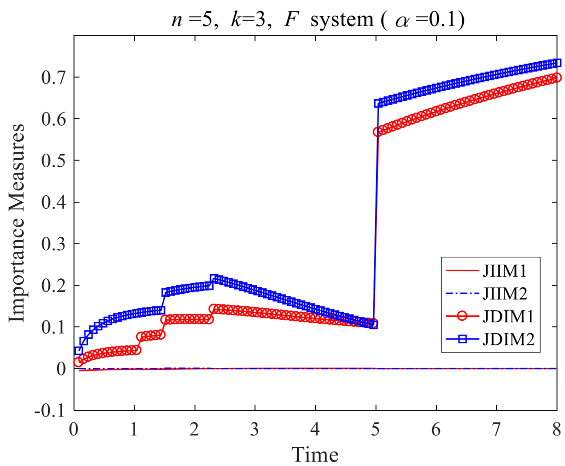


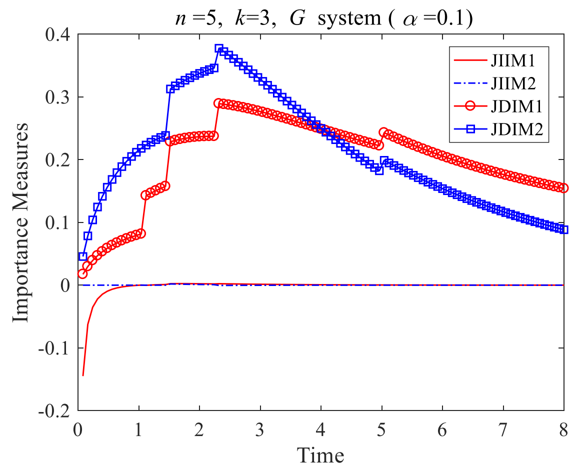


(e) Joint importance of component 5 when shape parameter is 3 for F system (f) Joint importance of component 5 when shape parameter is 3 for G system

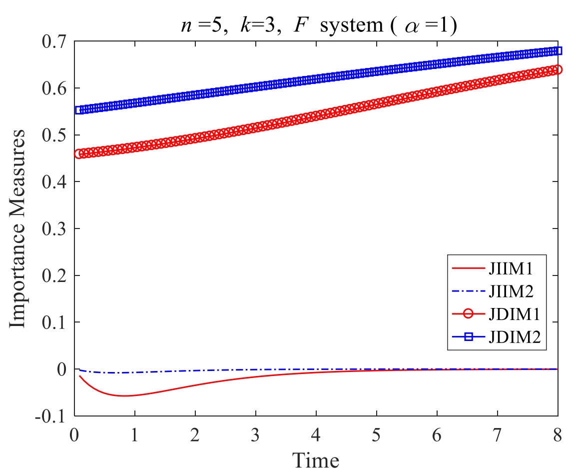
Fig. 3 Joint importance of component 1 and component 5 with optimal component sequence when n=5, k=2

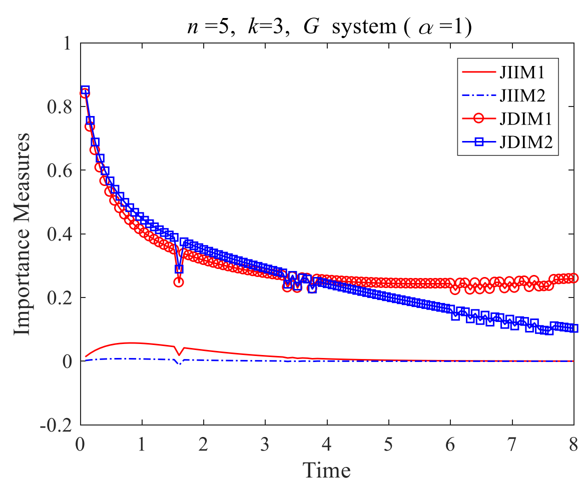
Figs. 4 and 5 show the changes in the joint importance measure of components 5 when n=5, k=3 and n=5, k=4 with optimal component sequence.



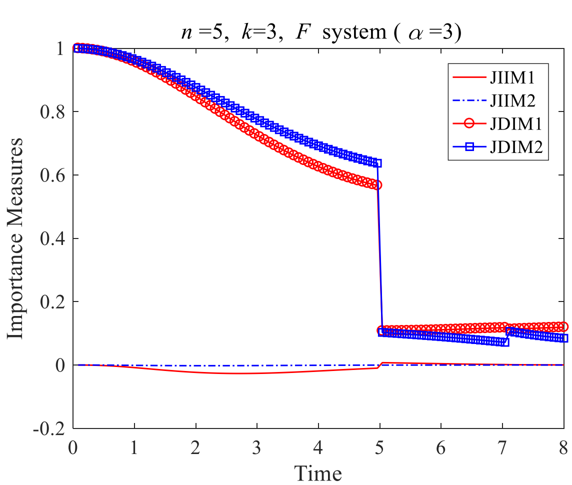


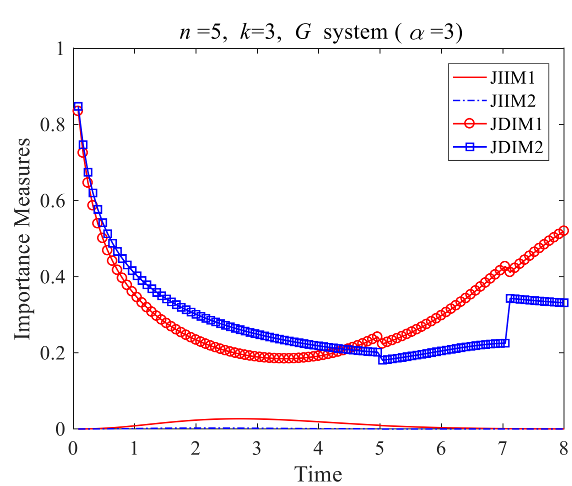
(a) Joint importance of component 5 when shape parameter is 0.1 for F system (b) Joint importance of component 5 when shape parameter is 0.1 for G system





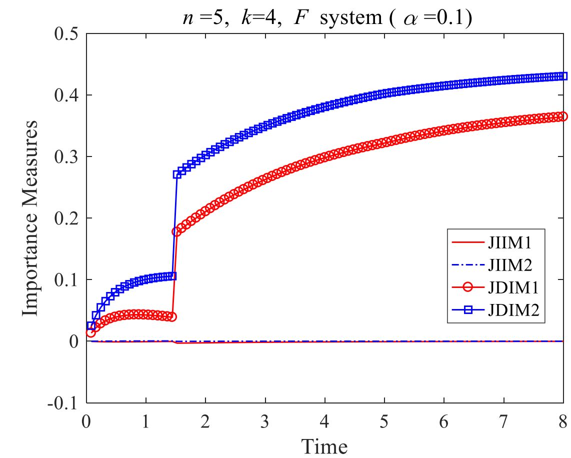
(c) Joint importance of component 5 when shape parameter is 1 for F system (d) Joint importance of component 5 when shape parameter is 1 for G system

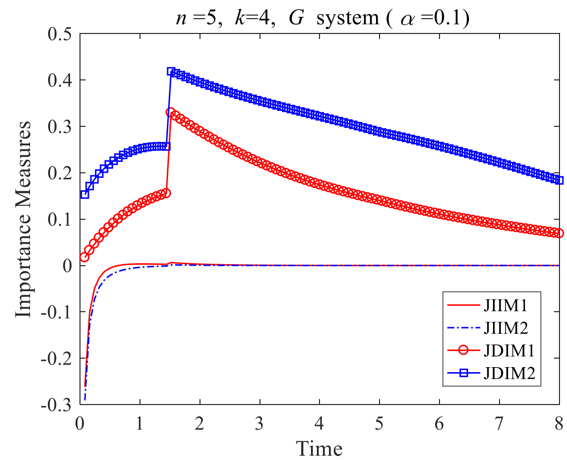




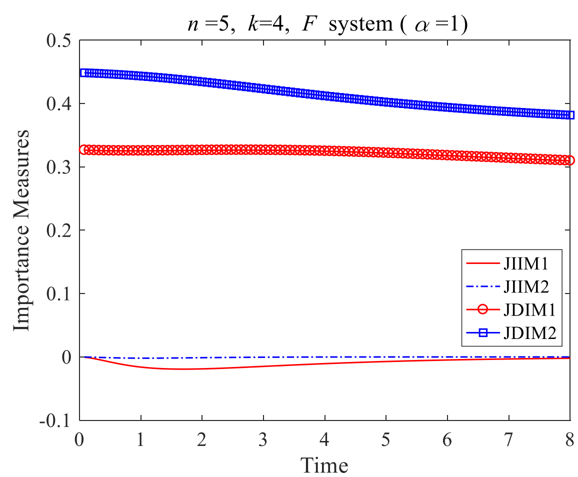
(e) Importance of component 5 when shape parameter is 3 for F system (f) Importance of component 5 when shape parameter is 3 for G system

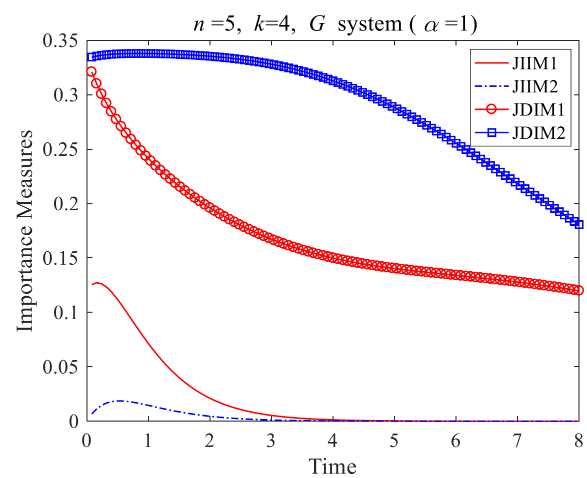
Fig. 4 Joint importance of component 5 with optimal component sequence when n=5, k=3.



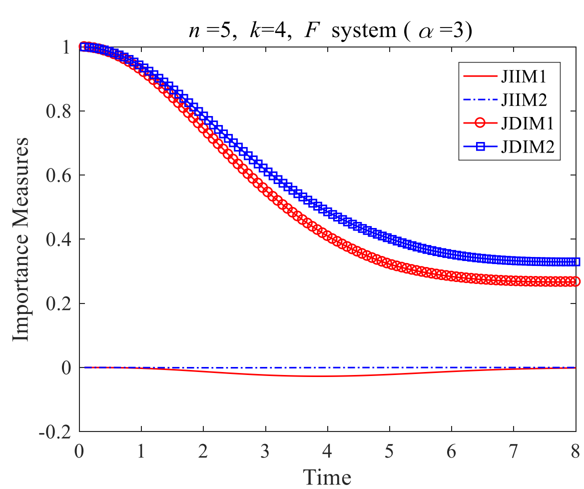


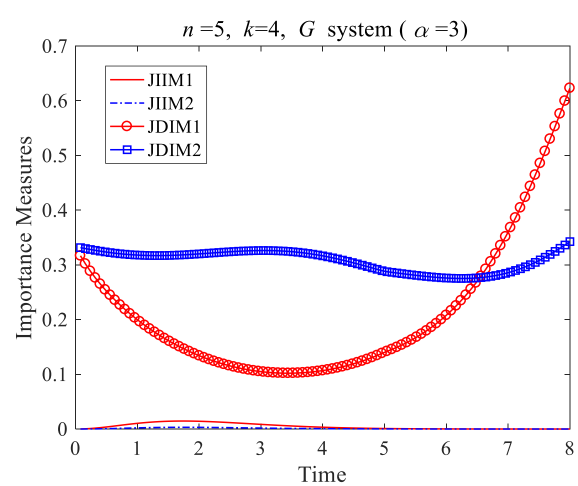
(a) Joint importance of component 5 when shape parameter is 0.1 for F system (b) Joint importance of component 5 when shape parameter is 0.1 for G system





(c) Joint importance of component 5 when shape parameter is 1 for F system (d) Joint importance of component 5 when shape parameter is 1 for G system



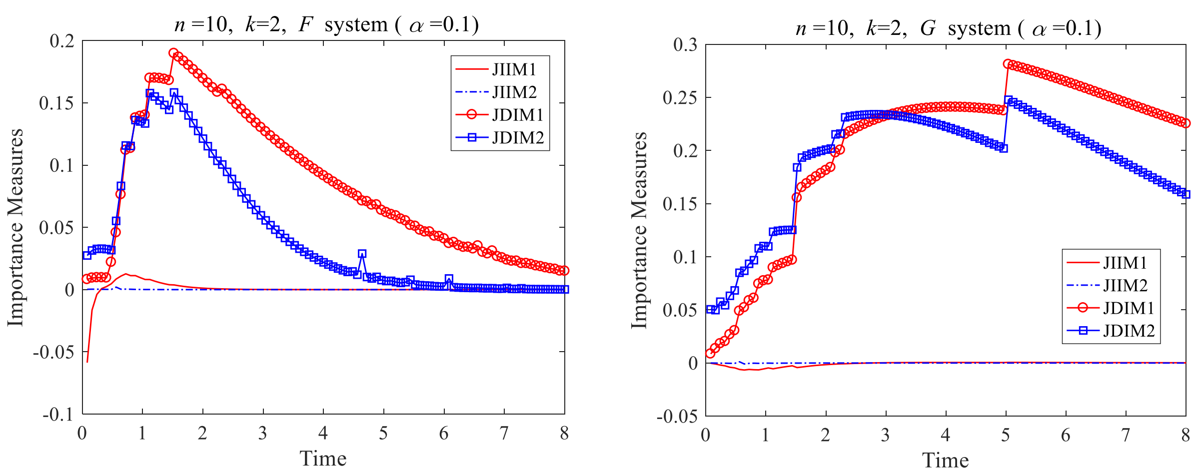


(e) Joint importance of component 5 when shape parameter is 3 for F system (f) I Joint importance of component 5 when shape parameter is 3 for G system

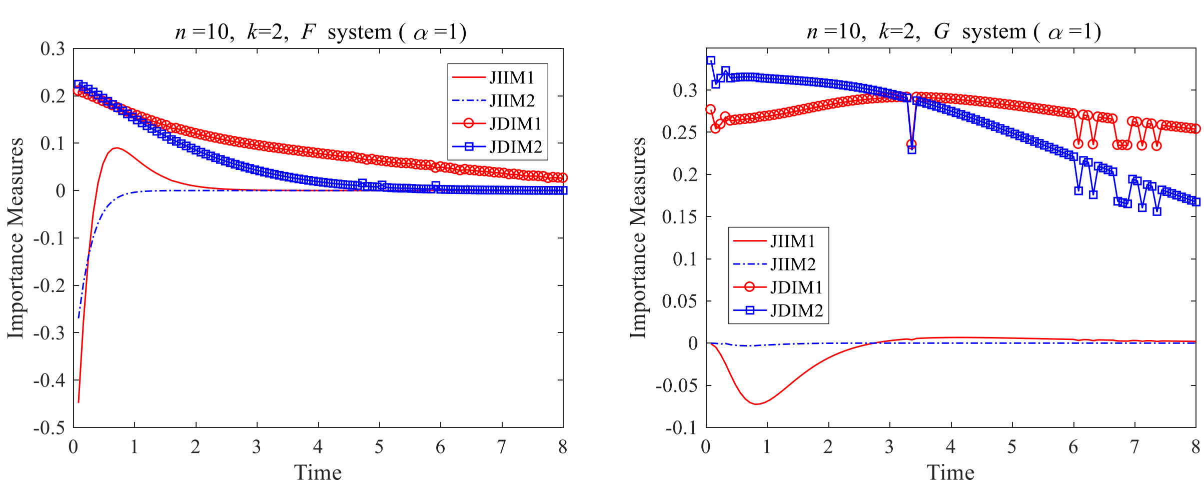
Fig. 5 Joint importance of component 5 with optimal component sequence when n=5, k=4

4.2 Linear consecutive-k-out-of-10 system

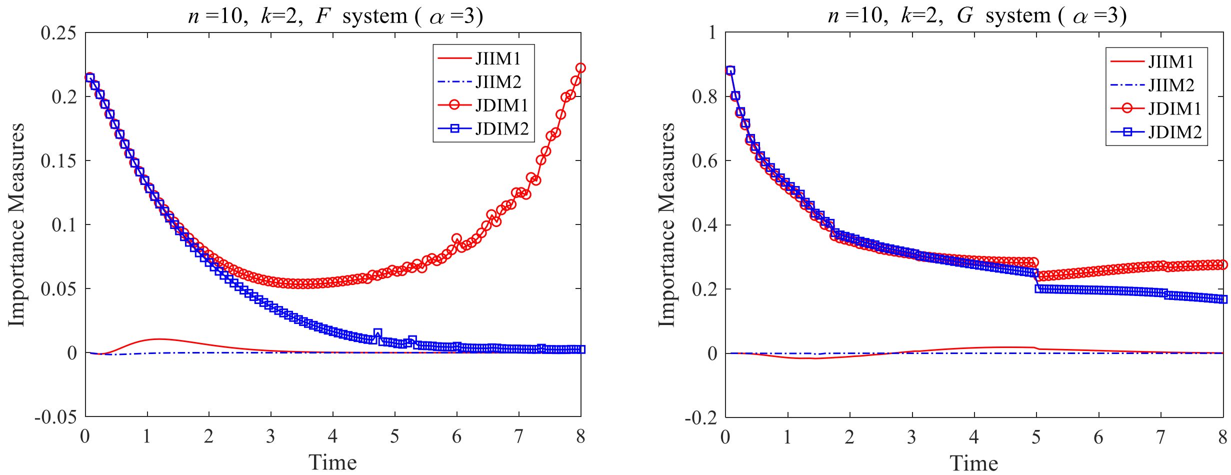
Fig. 6 shows the four joint importance measures of component 10 with different shape parameters in a linear consecutive-2-out-of-10 system with the optimal component sequence. When the shape parameter of the Weibull distribution of component 10 is 0.1 for the F system and 0.1 for the G system, there are many optimal component sequences in these systems, and the $JDIM^{opt}(I)$ and $JDIM^{opt}(II)$ values have changed greatly. Obviously, the curve of the $JDIM^{opt}$ of component 10 shows an increasing trend after the jump. At this time, the corresponding $JDIM^{opt}$ is undergoing a non-monotonous and unsmooth change. Furthermore, we can see that no matter what kind of system and shape parameter is $JIIM^{opt}$ for, it remains at 0 with relatively smooth fluctuations.



(a)Joint importance of component 10 when shape parameter is 0.1 for F system (b) Joint importance of component 10 when shape parameter is 0.1 for G system



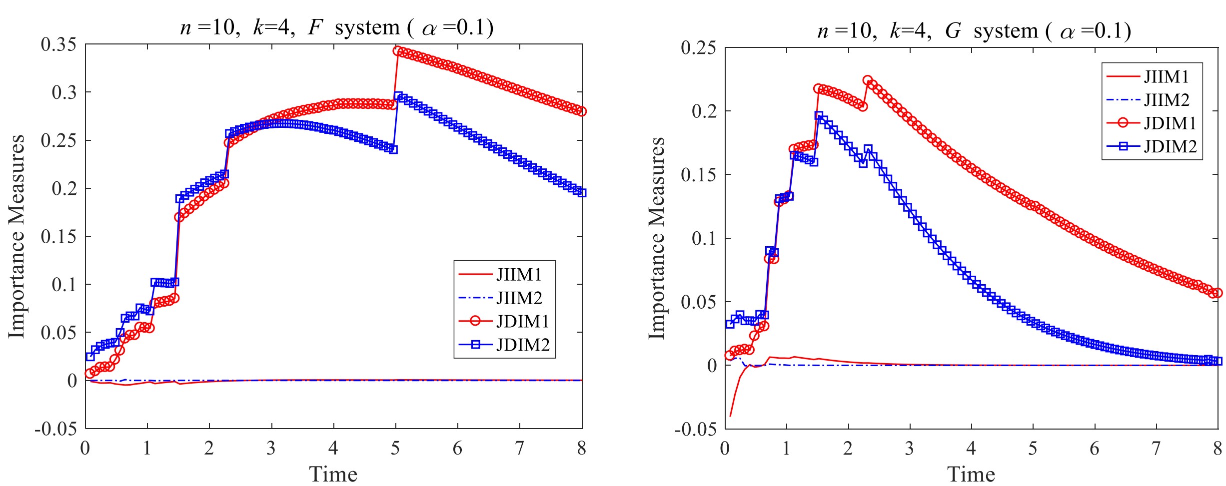
(c) Joint importance of component 10 when shape parameter is 1 for F system (d) Joint importance of component 10 when shape parameter is 1 for G system



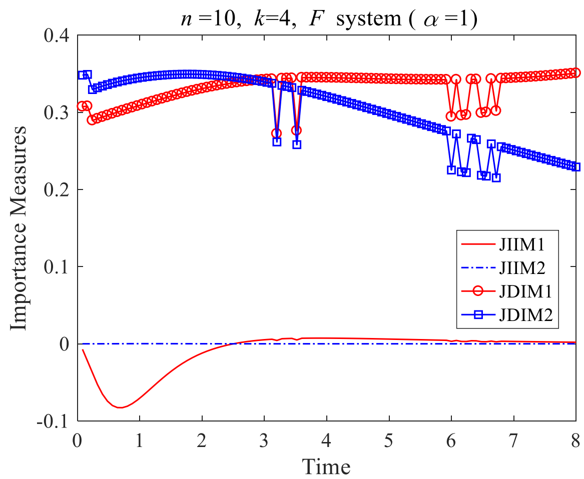
(e) Joint importance of component 10 when shape parameter is 3 for F system (f) Joint importance of component 10 when shape parameter is 3 for G system

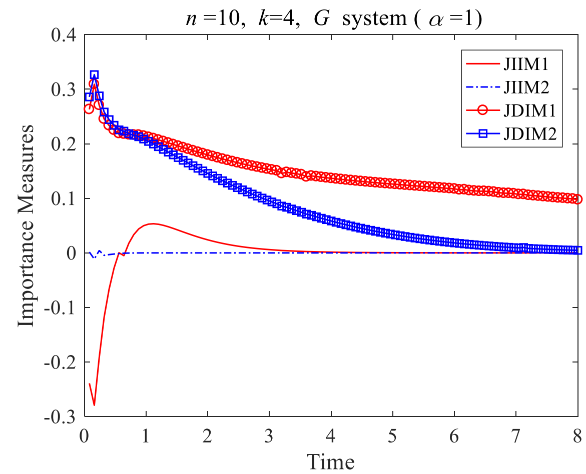
Fig. 6 Joint importance of component 10 with optimal component sequence when *n*=10, *k*=2.

Figs. 7 and 8 give different joint importance measures of the component 10 with different shape parameters in a linear consecutive-4-out-of-10 system and in a linear consecutive-8-out-of-10 system with the optimal component sequence Similarly, we can find the characteristics. Therefore, the change in $JDIM^{opt}$ of the component 10 is closely related to the optimal system structure.

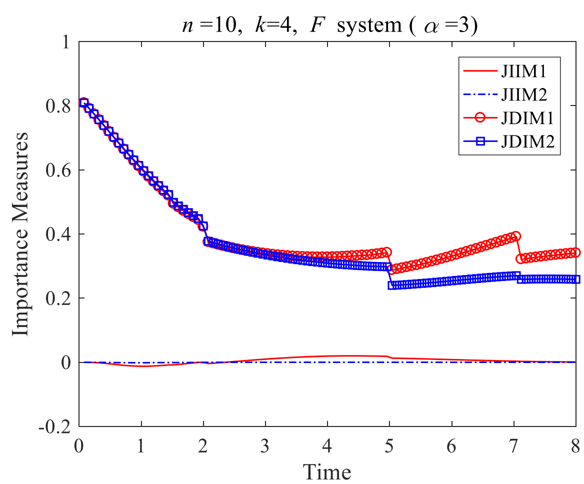


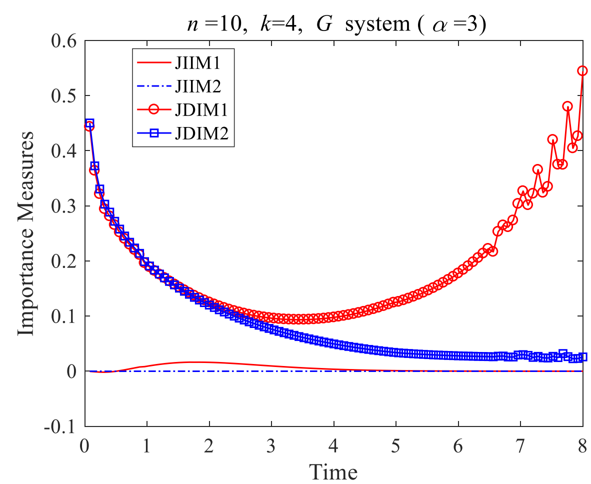
(a) Joint importance of component 10 when shape parameter is 0.1 for F system(b) Joint importance of component 10 when shape parameter is 0.1 for G system





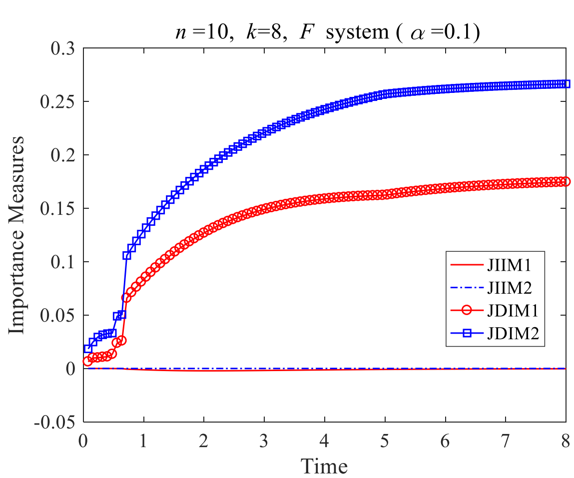
(c) Joint importance of component 10 when shape parameter is 1 for F system (d) Joint importance of component 10 when shape parameter is 1 for G system

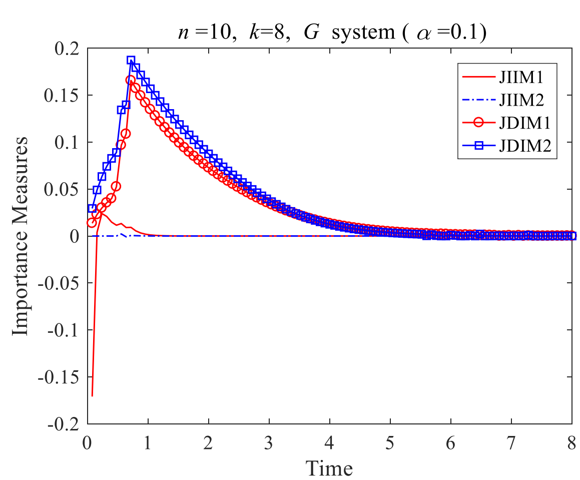




(e) Joint importance of component 10 when shape parameter is 3 for F system (f) Joint importance of component 10 when shape parameter is 3 for G system

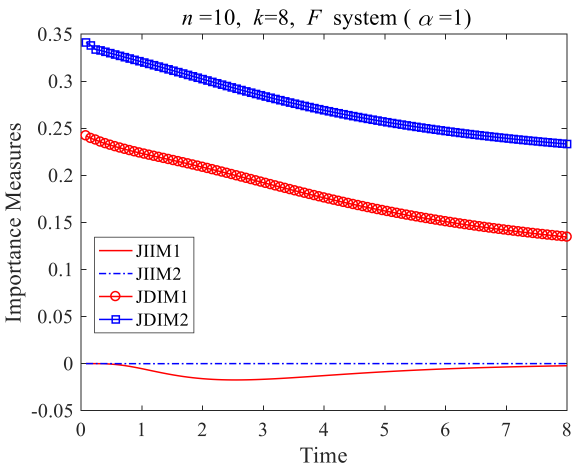
Fig. 7 Joint importance of component 10 with optimal component sequence when n=10, k=4

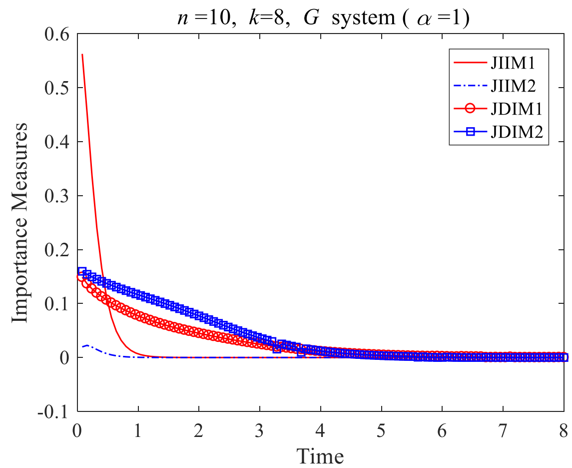




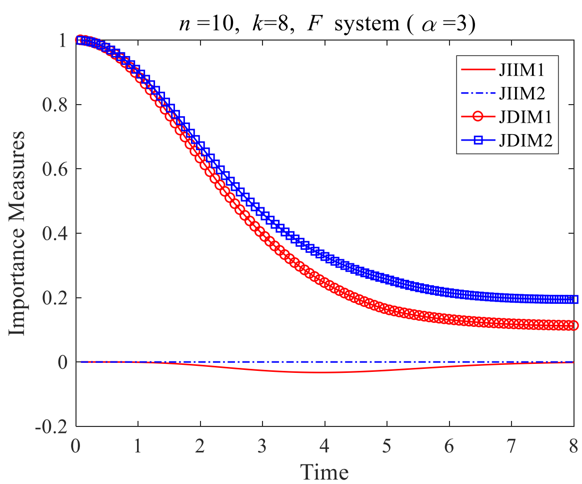
(a) (a) Joint

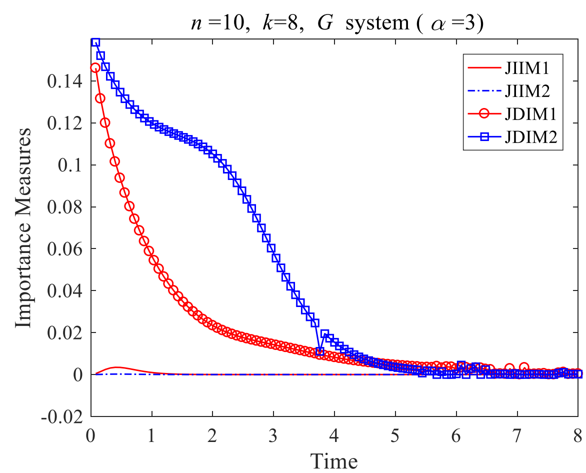
importance of component 10 when shape parameter is 0.1 for F system(b) Joint importance of component 10 when shape parameter is 0.1 for G system





(c) (c) Joint importance of component 10 when shape parameter is 1 for F system (d) Joint importance of component 10 when shape parameter is 1 for G system





(e) Joint importance of component 10 when shape parameter is 3 for F system (f) Joint importance of component 10 when shape parameter is 3 for G system Fig. 8 Fig. 8 Joint importance of component 10 with optimal component sequence when n=10, k=8

4. Conclusions

This paper derived a joint integrated importance measure and a joint differential importance measure for linear consecutive-k-out-of-n systems and proposed the definition under the optimal component sequence. It also investigated some properties of the proposed measures and performed analysis of the four joint importance measures for the optimal system structure of the multistate linear consecutive-k-out-of-n systems. The original optimal component sequence can be rearranged after the reliability of the target component changes, and it is reflected in the corresponding joint importance, which also means that the joint importance of the components is related to the optimal system structure. The shape parameters of the Weibull distribution followed the components are different, that is, the trend of component failure rate is different, then the behavior of the joint importance measures is different. When the shape parameter is 0.1, the joint importance measure at this time is most obviously affected by the optimal allocation of components.

Acknowledgements

The authors gratefully acknowledge the financial support for this research from the National Natural Science Foundation of China (Nos. 72071182, U1904211), the ministry of education's humanities and social sciences planning fund (No. 20YJA630012), the Program for young

References

[1] Borgonovo E, Apostolakis GE. A new importance measure for risk-informed decision making. Reliab Eng Syst Saf 2001;72:193–212.

[2] Cai Z, Si S, Sun S, Li C. Optimization of linear consecutive-k-out-of-n system with a Birnbaum importance-based genetic algorithm. Reliab Eng Syst Saf 2016;152: 248–58.

[3] Dui H, Si S, Yam RCM. Importance measures for optimal structure in linear consecutive-k-out-of-n systems. Reliab Eng Syst Saf 2018;169:339–50.

[4] Gao XL, Cui LR, Li JL. Analysis for joint importance of components in a coherent system. Eur J Oper Res 2007;182(1):282–99.

[5] Levitin G, Lisnianski A. Reliability optimization for weighted voting system. Reliab Eng Syst Saf 2001;71:131–8.

[6] Levitin G, Podofillini L, Zio E. Generalised importance measures for multi-state elements based on performance level restrictions. Reliab Eng Syst Saf 2003;82: 287–98.

[7] Levitin G, Xing L, Dai Y. Linear multistate consecutively-connected systems subject to a constrained number of gaps. Reliab Eng Syst Saf 2015;133:246–52.

[8] Levitin G, Xing L, Dai Y. Sequencing Optimization in k-out-of-n Cold-Standby Systems Considering Mission Cost. Int J Gen Syst 2013;42(8):870–82.

[9] Lin C, Cui L, Coit DW, Lv M. Reliability modeling on consecutive-kr -out-of-nr: f linear zigzag structure and circular polygon structure. IEEE Trans Reliab 2016;65 (3):1509–21.

[10] Ling X, Wei Y, Si S. Reliability optimization of k-out-of-n system with random selection of allocative components. Reliab Eng Syst Saf 2019;186:186–93.

[11] Liu Y, Zuo MJ, Li Y, Huang H. Dynamic reliability assessment for multi-state systems utilizing system-level inspection Data. IEEE Trans Reliab 2015;64(4): 1287–99.

[12] Lyu D, Si S. Dynamic importance measure for the K-out-of-n: g system under repeated random load. Reliab Eng Syst Saf 2020;195:106720.

[13] Ma C, Wang Q, Cai Z, Si S, Zhao J. Component reassignment for reliability optimization of reconfigurable systems considering component degradation. Reliab Eng Syst Saf 2021;215:107867.

[14] Mannai N, Gasmi S. Optimal design of k-out-of-n system under first and last replacement in reliability theory. Oper Res 2018;20(3):1353–68.

[15] Mo Y, Xing L, Cui L, Si S. MDD-based performability analysis of multi-state linear consecutive-k-out-of-n: f systems. Reliab Eng Syst Saf 2017;166:124–31.

[16] Shen J, Cui L. Reliability and birnbaum importance for sparsely connected circular consecutive-*k* systems. IEEE Trans Reliab 2015;64(4):1140–57.

[17] Si S, Zhao X, Zhang S, Sun S. Integrated importance measure of component states based on loss of system performance. IEEE Trans Reliab 2012;61:192–202.

[18] Si S, Levitin G, Dui H, Sun S. Importance analysis for reconfigurable systems. Reliab Eng Syst Saf 2014;126:72–80.

[19] Si S, Liu M, Jiang Z, Jin T, Cai Z. System reliability allocation and optimization based on generalized birnbaum importance measure. IEEE Trans Reliab 2019;68 (3):831–43.

[20] Si S, Zhao J, Cai Z. Recent advances in system reliability optimization driven by importance measures. Fron Eng Manag 2020;7(3):335–58. 2020.

[21] Wang D, Si S, Cai Z, Zhao J. Reliability optimization of linear consecutive-k-out-of- n: f systems driven by reconfigurable importance. Reliab Eng Syst Saf 2021;216: 107994.

[22] Wu S, Chen Y, Wu Q, Wang Z. Linking component importance to optimisation of preventive maintenance policy. Reliab Eng Syst Saf 2016;146:26–32.

[23] Wu W, Tang Y, Yu M, Jiang Y. Reliability analysis of a *k* - out - of - *n*: g repairable system with single vacation. Appl Math Model 2014;38(24):6075–97.

[24] Yi K, Kou G, Gao K, Xiao H. Optimal allocation of multi-state elements in a sliding window system with phased missions. Proc Inst Mech Eng, Part O: J Risk Reliab 2020;235:50–62.

[25] Zhang YL, Wu S. Reliability analysis for a *k*/*n*(F) system with repairable repair- equipment. Appl Math Model 2009;33(7):3052–67.

[26] Zhao J, Si S, Cai Z. A multi-objective reliability optimization for reconfigurable systems considering components degradation. Reliab Eng Syst Saf 2019;183: 104–15.

[27] Zhao X, Wu C, Wang S, Wang X. Reliability analysis of multi-state k-out-of-n: g system with common bus performance sharing. Comput Ind Eng 2018;124:359–69.

[28] Zhu X, Boushaba M, Reghioua M. Reliability and joint reliability importance in a consecutive- *k* -Within- *m* -out-of- *n*: f system with markov-dependent components. IEEE Trans Reliab 2016;65(2):802–15.

[29] Zio E, Podofillini L. Accounting for components interactions in the differential importance measure. Reliab Eng Syst Saf 2006;91:1163–74.