Optimal inspection policy for a single-unit system considering two failure modes and production wait time

Hui Xiao, Yuanmin Yan, Gang Kou, Shaomin Wu

***Abstract*—In many real production systems, a system may be stopped due to a lack of demand or exhaustion of raw materials. This is known as production wait and provides a good opportunity for maintenance. Meanwhile, the increased complexity of the modern machines brings new challenges for modeling and analyzing their failure behaviors. To address these real-world problems, this research considers a single-unit system that may fail due to either hard failures or soft failures. The wait time of a system is utilized to conduct inspections and maintenance. The system is replaced when a defect is found during an inspection, a failure occurred or a pre-specified age threshold is reached, whichever comes first. The cost model, which is the long-run maintenance cost per unit of time, is derived. The optimal periodical inspection interval and the threshold age to minimize the long run maintenance cost per unit of time is then obtained. A case study is conducted to demonstrate the proposed maintenance model. The study shows using the proposed maintenance model can reduce the cost. Sensitivity analysis illustrates how each cost parameter affects the optimal inspection interval and the optimal age threshold.**

Keywords: Delay-time concept, inspection, maintenance decision, multiple failure modes, production wait.

I. Introduction

MAINTENANCE is a widely used approach to reduce the operational cost of industrial assets in many sectors such as power grid systems and transportation infrastructure [1]-[4]. Based on how the maintenance action is triggered, we can categorize the strategies as corrective maintenance [5],[6], preventive maintenance [7]-[12], and condition-based maintenance [13]-[17]. The main objectives of maintenance management of technical systems are to minimize maintenance cost and to increase system reliability.

In practice, a system may fail due to multiple types of failures that are independent from each other [18] -[20]. If a failure is instantaneous and has the characteristic of self-announcement, it is referred to as a hard failure. Otherwise, if a failure generates early warning signals or its degradation mode can be detected by inspection or condition monitoring, it is a soft failure [21], [22]. For example, a production system may consist of several subsystems, including electromechanical systems composed of mechanical and electrical components. The mechanical components may be continuously monitored or periodically inspected for wear, while electrical components may suffer suddenly failures [23]. The feed subsystem of a boring machine is composed of the feed screw and other components. The manufacturing accuracy is determined by the feed screw, which the core component of the subsystem. When the error exceeds the predetermined level and cannot meet the quality requirements, it is considered as a soft failure. In addition, hard failures of other components of the subsystem can also cause the subsystem to stop working [24]. These two examples show that the practical system usually fails due to different types of failure modes. Thus, analyzing the different failure modes can provide a more accurate modeling of the system reliability.

The progress of a soft failure may go through multiple stages, including perfect functioning, defects and failure [25]. For example, the increasing number of defective items is an indicative signal of defects in a production system. Inspection is an effective way to detect such signals so that maintenance actions can be taken before system failure. To characterize the failure process with multiple stages, the delay-time concept, is introduced in [26]. In the literature, a substantial amount of works has devoted to modeling and optimizing inspection policies and maintenance policies using the concept delay-time. For example, the two-stage failure process was extended to three stages to distinguish between major and minor defects [27]-[30]. The delay time model was also used to determine optimal maintenance policies in the case of postponed replacement [31]-[33]. Furthermore, the delay time modeling approach was also adopted to optimize the maintenance and spare part inventory jointly [34]-[36]. From the perspective of applications, the delay time concept has been applied to different types of systems such as production systems [37], airport runways [38], critical systems [39], and bus fleets [40].

In production systems, some wait time can be utilized to conduct inspection and maintenance to lower the cost or to

This work was supported by the National Natural Science Foundation of China under grant numbers 71971176, 71725001 and 71910107002, and the Fundamental Research Funds for the Central Universities under grant JBK2103010. Corresponding author: *G. Kou*.

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mitigate the interruption. Production wait, which occurs randomly due to a lack of demand or exhaustion of raw materials, provides a good opportunity for inspection and maintenance [41]-[44]. For example, the inspection maintenance opportunity arises for a machine with excessive buffers in a smoothing production line [45]. Thus, making full use of the opportunity of production wait to identify defects in machines can reduce the disruption time during the normal operation.

However, in the literature, there is little research on analyzing how the production wait time can be utilized to conduct inspection or maintenance, especially for complex systems that subject to multiple failure modes. In this research, a single-unit system that may fail due to two independent failure modes is considered. Production waiting time is utilized to conduct inspections. Whenever a defect is found during an inspection, whether it is a periodical inspection or a production wait inspection, the system is replaced. When the system fails or reaches a threshold age, it is also replaced. Under these assumptions, the long run maintenance cost per unit of time is derived and is minimized to find the optimal periodical inspection intervals and age thresholds.

The contribution of this article is three-fold. Firstly, it develops a maintenance cost model for a single-unit system considering two failure modes and production wait time. The optimal periodical inspection interval and the aged-based replacement time are then sought by minimizing the long run average maintenance cost per unit of time. Secondly, it proves that utilizing the production waits for inspection and maintenance can reduce the long run average maintenance cost per unit of time. Lastly, through sensitivity analysis, it provides useful insights for decision maker to choose the most appropriate maintenance model under different scenarios.

The remainder of this article is arranged as follows. Section 2 provides the system description and the maintenance model. Section 3 derives the cost model for the proposed maintenance policy. Section 4 provides a case study. Finally, we conclude in Section 5.

List of Notations

$X_{1,d}$	duration of the normal state for failure mode 1
$X_{1,f}$	duration of the defect state for failure mode 1
X_2	duration of the operating state for failure mode 2
τ	time at which the defective state of failure mode 1 occurs
t	time of system failure
t_w	time at which the first production wait happens after the system being defective
T_w	random arrival time of the nearest production wait that occurs after the system becomes defective
T	periodic inspection interval
T_p	time of the periodic inspection renewal
$T_{k,fj}$	time of the system failure renewal, where $j=1,2,3$
$T_{k,wj}$	time of the production wait inspection renewal, where $j=1,2,3$
T_{age}	age threshold at which the system is replaced immediately, and is equal to nT
$N(\tau)$	number of production waits that occur during $[0,\tau]$
$f_j(\cdot)$	probability density function of variable j, $j=X_{1,d},X_{1,f},X_2$
$F_j(\cdot)$	cumulative distribution function of variable j, $j=X_{1,d},X_{1,f},X_2$
$R_j(\cdot)$	survival probability of j, and $R_j(\cdot)=1-F_j(\cdot)$, $j=X_{1,d},X_{1,f},X_2$
π_p	inspection cost of the periodical inspection
π_w	inspection cost of the production wait inspection
c_p	replacement cost
c_f	replacement cost in the case of the system failure
$E_p^C(n,T)$	expected renewal cycle cost for the periodical inspection renewal
$E_f^C(n,T)$	expected renewal cycle cost for the failure renewal
$E_w^C(n,T)$	expected renewal cycle cost for the production wait inspection renewal
$E_a^C(n,T)$	expected renewal cycle cost for the age-based renewal
$E_p^L(n,T)$	expected renewal cycle length for the periodical inspection renewal
$E_f^L(n,T)$	expected renewal cycle length for the failure renewal
$E_w^L(n,T)$	expected renewal cycle length for the production wait inspection renewal
$E_a^L(n,T)$	expected renewal cycle length for the age-based renewal
$E^C(n,T)$	average renewal cost of the system
$E^L(n,T)$	average renewal length of the system
$E^{C/L}(n,T)$	expected long run cost per unit of time

II. Problem Formulation

This section first provides a description of the system, and then proposes a maintenance policy for this system. Finally, the maintenance model is developed for the proposed maintenance policy.

A. System description

Consider a single-unit system that is subject to two types of independent failures, both of which can cause the system to fail. Failure mode 1 is referred to as a soft failure. The progress of a soft failure includes three states: normal, defective and failed. The transition from the normal state to the defective state transits the system from new to defective and the transition from the defective state transits the system from defective to failure. Let $X_{1,d}$ and $X_{1,f}$ denote the durations of the normal state and the defective state, respectively, with the corresponding distributions of sojourn times $F_{X_{1,d}}(\cdot)$ and $F_{X_{1,f}}(\cdot)$, and the probability density functions being (PDFs) $f_{X_{1,d}}(\cdot)$ and $f_{X_{1,f}}(\cdot)$. Failure mode 2 is referred to as a hard failure, which occurs without any early warning signals. The time of the system from new to failure in the case of a hard failure is described by a random variable X_2, with a CDF being $F_{X_2}(\cdot)$ and a PDF being $f_{X_2}(\cdot)$.

When the system is defective, it can continue operating. The defective state of the system can only be identified through two types of inspections. The first type of inspection, which is called periodical inspection, is conducted periodically. The other type of inspection, which is called production wait inspection, is conducted when the system stops for production wait that occur randomly. The production wait happens independently from the failure process of the system. When the system fails due to either failure mode, it can be observed immediately. In order to reduce the probability of failures that may cause a huge cost, the age-based replacement is considered. In this paper, replacement is the same as a system renewal.

B. The maintenance policy

We define the maintenance policy as follows. Starting from new, periodical inspections are conducted with an interval of T. The production wait inspection is also conducted when a production wait happens. The occurrence of production wait follows a Poisson process with rate λ. The system is replaced if an inspection (either the periodical inspection or the production wait inspection) finds the system in a defective state, the system fails, or the age of the system reaches T_{age}, whichever occurs first. The replacement time is negligible. The maintenance policy has two decision variables: the periodic inspection interval T and the time for age-based replacement. In practice, the time for age-based replacement is usually assumed to be a multiple of periodical inspection interval for the convenience of maintenance crew. Thus, we assume the time for age-based replacement is nT, where n is a positive integer. This assumption is also frequently used in the literature of maintenance [23, 33, 37, 44].

The novelty of this maintenance model is the consideration of two types of failure modes and utilizing the production wait time to carry out inspection and maintenance. In this way, the policy takes greater care of industry practice, and model the real-world problem more accurately.

The cost parameters associated with the maintenance policy are explained as follows. The first type of cost is inspection cost. Let π_p and π_w represent the cost of a periodical inspection and a production wait inspection, respectively. Since the periodical inspection interrupts the production process, it is reasonable to assume that $\pi_p > \pi_w$. The second type of cost is the replacement cost. Based on the assumption that the duration of each production wait and the replacement time is negligible compared with the system lifetime, it is reasonable to assume that the replacement cost is the same for: (i) replacing the system during periodical inspection, (ii) replacing the system when the system reaches the age of T_{age}, (iii) replacing the system during production wait inspection. Although the periodical inspection replacement interrupt the production process, the cost associated with interruption is already considered in the inspection cost, where we have assumed that $\pi_p > \pi_w$. Let c_p denote the replacement cost. Let c_f denote the replace cost when the system fails. The replacement cost incurred due to system failure can be much larger than c_p because the sudden failure may bring unpredictable losses for the system. Note the unpredictable loss (penalty cost) for the system failure has already been included in c_f.

C. The maintenance model

Note that the production wait occurs randomly according to a Poison process with arrival rate of λ. Let T_w denote random arrival time of the nearest production wait that occurs immediately after the system becomes defective. The probability that the nearest production wait occurs after t_w for $t_w \in (\tau, \infty)$ is

$$P(T_w > t_w) = P\left\{\left[N(t_w) - N(\tau)\right] = 0\right\} = e^{-\lambda(t_w - \tau)}, \quad (1)$$

where $N(t_w)$ and $N(\tau)$ denote the number of production waits during the time interval $[0, t_w]$ and $[0, \tau]$, respectively, and $E[N(\tau)] = \lambda\tau$.

Thus, the PDF of T_w is

$$f_{T_w}(t_w) = -\frac{dP(T_w > t_w)}{dt_w} = \lambda e^{-\lambda(t_w - \tau)}. \quad (2)$$

The maintenance cost model can be derived as follows.

Let T_f be the system lifetime and $F(t)$ be the survival probability of the system before t. Then, we can derive $F(t)$ as follows.

$$\begin{aligned} F(t) = P(T_f < t) &= P\left[\left(X_{1,d} + X_{1,f} < t\right) \cup \left(X_2 < t\right)\right] \\ &= 1 - \left[1 - F_1(t)\right] \times \left[1 - F_2(t)\right] \\ &= F_1(t) + F_2(t) - F_1(t)F_2(t), \end{aligned} \quad (3)$$

where $F_1(t)$ and $F_2(t)$ denote the probabilities that the system fails before t due to failure mode 1 and failure mode 2, respectively.

The equality in Equation (3) exits since the two failure modes are independent. $F_1(t)$ can be further expressed as

$$\begin{aligned} F_1(t) &= P\left(X_{1,d} + X_{1,f} < t\right) \\ &= P\left[\left(X_{1,d} < t\right) \cup \left(X_{1,f} < t - X_{1,d}\right)\right] \\ &= \int_0^t f_{X_{1,d}}(\tau) F_{X_{1,f}}(t - \tau) d\tau. \end{aligned} \quad (4)$$

Soft and hard failures are often assumed to be independent [21, 23, 46-48]. Based on the maintenance policy defined above, we study the optimal inspection policy for a single-unit system subject to two independent failure modes: the soft failure and the hard failure. Both failure modes can lead to the system failure. The system may be renewed under four different scenarios. When one of the scenarios occurs, the system will update immediately according to our assumptions, so the other three updates will not happen. Therefore, the four different scenarios are independent of each other.

(a) Periodical inspection renewal. The system is renewed when the periodical inspection finds it in the defective state.

(b) Failure renewal. The system is renewed immediately after failure occurs.

(c) Production wait inspection renewal. The system is renewed immediately after the defective state is found by an inspection during production wait.

(d) Age-based renewal. When the system reaches the age of T_{age}, no inspections are performed, even though it is still in the normal, it will be renewed. The threshold age of system T_{age} is an multiple of the periodic inspection interval, i.e., $T_{age}=nT$, where $n\geq 1$ is a positive integer.

Given n and T, let $E_p^C(n,T)$, $E_f^C(n,T)$, $E_w^C(n,T)$ and $E_a^C(n,T)$ denote the expected renewal cycle costs for the periodical inspection renewal, failure renewal, production wait inspection renewal, age-based renewal, respectively. Let $E_p^L(n,T)$, $E_f^L(n,T)$, $E_w^L(n,T)$ and $E_a^L(n,T)$ denote the expected renewal cycle lengths for the periodical inspection renewal, failure renewal, production wait inspection renewal, age-based renewal, respectively. Let $E^C(n,T)$ and $E^L(n,T)$ denote the average renewal cost and renewal length of the system given n and T.

Then, the expected long run cost per unit of time $E^{C/L}(n,T)$ can be expressed as

$$E^{C/L}(n,T)=\frac{E^C(n,T)}{E^L(n,T)}=\frac{E_p^C(n,T)+E_f^C(n,T)+E_w^C(n,T)+E_a^C(n,T)}{E_p^L(n,T)+E_f^L(n,T)+E_w^L(n,T)+E_a^L(n,T)}. \tag{5}$$

The objective of determining the optimal the maintenance policy is to find the best n and T such that $E^{C/T}(n,T)$ can be minimized. This is equivalent to solving the following optimization problem.

$$\min E^{C/L}(n,T),\ s.t.\ T\in(0,\infty),\ n\in\{1,2,...\}. \tag{6}$$

The difficulty associated with solving the optimization problem in Eq. (6) is the derivation of the explicit expression of the objective function $E^{C/L}(n,T)$. In the following sections, we aim to obtain $E^{C/L}(n,T)$ explicitly by analyzing each renewal scenario separately.

III. Cost Model for the Maintenance Policy

Based on the analysis in Section 2.3, we derive the explicit expressions for $E_p^C(n,T)$, $E_f^C(n,T)$, $E_w^C(n,T)$, $E_a^C(n,T)$, $E_p^L(n,T)$, $E_f^L(n,T)$, $E_w^L(n,T)$ and $E_a^L(n,T)$ by analyzing each of the four types of system renewals separately.

A. *Periodical inspection renewal*

In the case of the periodical inspection renewal, it only happens at time kT, where $k=1,2,...,n-1$. Fig. 1 provides a typical graphic representation for the scenario that the kth periodical inspection finds the system is in the defective state, where T is the periodical inspection interval; $X_{1,d}$ is the duration of failure mode 1 in the normal state; $X_{1,f}$ is the duration of at the failure mode 1 defect state; and X_2 is the duration of failure mode 2 at the operating state.

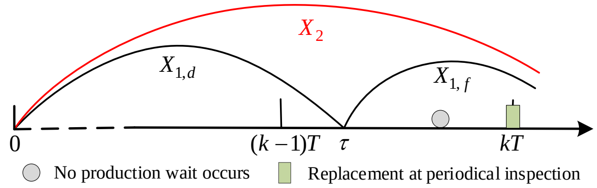

Fig. 1. Graphic representation of periodical renewal.

As shown in Fig. 1, the system is renewed at the kth periodical inspection when the following conditions are met: (i) the system is still in the normal state up to time $(k-1)T$; (ii) the system becomes defective before time kT; (iii) no production wait occurs after the system being defective; and (iv) the system has not failed.

Let τ denote the time when the system becomes defective. According to the description above, τ must be between $(k-1)T$ and kT. Let T_p denote the length of the periodical renewal. The probability that T_p is equal to kT is

$$\begin{aligned}
&P(T_p=kT)\\
&=P\left\{\begin{aligned}&[(k-1)T<\tau<kT]\cap[N(kT)-N(\tau)=0]\\&\cap[(X_{1,d}+X_{1,f})>kT]\cap[X_2>kT]\end{aligned}\right\}\\
&=P[(k-1)T<\tau<kT]\times P[N(kT)-N(\tau)=0]\\
&\quad\times P(X_2>kT)\times P[(X_{1,d}+X_{1,f})>kT]\\
&=R_{X_2}(kT)\int_{(k-1)T}^{kT}e^{-\lambda(kT-\tau)}f_{X_{1,d}}(\tau)R_{X_{1,f}}(kT-\tau)d\tau,
\end{aligned} \tag{7}$$

where $R_{X_{1,f}}(\cdot)=1-F_{X_{1,f}}(\cdot)$ and $R_{X_2}(\cdot)=1-F_{X_2}(\cdot)$ denote the survival functions.

The last equality of Eq. (7) follows because

$$P[N(kT)-N(\tau)=0]=e^{-\lambda(kT-\tau)}, \tag{8}$$

where the equality in Eq. (8) comes from Eq. (1) and Eq. (2).

Given the probability function that the length of the periodical renewal T_p is equal to kT in Eq. (7), the expected renewal cycle length for the periodical inspection renewal $E_p^L(n,T)$ can be obtained by using a weighted average as

$$\begin{aligned}
E_p^L(n,T)&=\sum_{k=1}^{n-1}kT\times P(T_p=kT)\\
&=\sum_{k=1}^{n-1}kT\times R_{X_2}(kT)\int_{(k-1)T}^{kT}e^{-\lambda(kT-\tau)}f_{X_{1,d}}(\tau)R_{X_{1,f}}(kT-\tau)d\tau.
\end{aligned} \tag{9}$$

Similarly, we can derive the expected renewal cycle cost for the periodical inspection renewal $E_p^C(n,T)$ as follows.

$$\begin{aligned}
E_p^C(n,T)&=\sum_{k=1}^{n-1}[\pi_p k+\pi_w N(\tau)+c_p]\times P(T_p=kT)\\
&=\sum_{k=1}^{n-1}[\pi_p k+\pi_w N(\tau)+c_p]\\
&\quad\times R_{X_2}(kT)\int_{(k-1)T}^{kT}e^{-\lambda(kT-\tau)}f_{X_{1,d}}(\tau)R_{X_{1,f}}(kT-\tau)d\tau,
\end{aligned} \tag{10}$$

where π_p and π_w represent the inspection cost of a periodical inspection and a production wait inspection, respectively; c_p denotes the replacement cost.

B. Failure renewal

In the case of the failure renewal, the system may fail due to either failure mode 1 or failure mode 2. If the system fails due to failure mode 2, it may fail before or after the system becomes defective. The three different scenarios are shown in Fig. 2. We analyze each scenario one by one as follows.

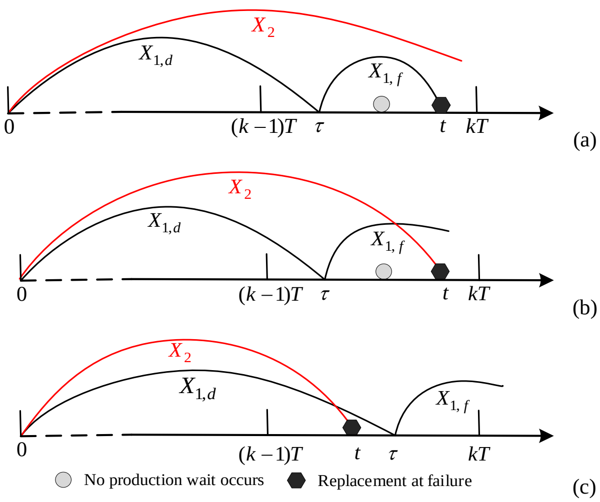

Fig. 2. Graphic representation of failure renewal.

Scenario 1.

A typical case of Scenario 1 is graphically shown in Fig. 2(a). Under this scenario, the system fails due to failure mode 1, and the failure happens between the $(k-1)$th and the kth inspections, where $k=1,2,...,n$. Let t denote the time when the system fails and τ denote the time when the system becomes defective.

Scenario 1 happens when the following conditions are met: (i) the system is still in the normal state up to time $(k-1)T$; (ii) the system becomes defective before time kT; (iii) no production wait occurs during the period from being defective to failure, i.e., in the time interval $[\tau,t)$; and (iv) the system fails after being defective due to failure mode 1 while the failure due to failure mode 2 does not occur.

Based on four conditions, $t\in((k-1)T,kT)$ and $\tau\in((k-1)T,t)$. Let $T_{k,f1}$ be the random variable that denotes the failure time of the system in the case of Scenario 1. Then, the probability that T_{f1} is between $(k-1)T$ and kT is

$$P\{(k-1)T<T_{k,f1}<kT\}$$
$$=\int_{(k-1)T}^{kT}\lim_{\Delta t\to 0}\frac{P\left\{\begin{matrix}\left[\left(X_{1,d}+X_{1,f}\right)\in(t,t+\Delta t)\right]\\ \cap\left[(k-1)T<X_{1,d}<t\right]\\ \cap\left(X_2>t\right)\cap\left[N(t)-N(\tau)=0\right]\end{matrix}\right\}}{\Delta t}dt$$
$$=\int_{(k-1)T}^{kT}R_{X_2}(t)\int_{(k-1)T}^{t}e^{-\lambda(t-\tau)}\times f_{X_{1,d}}(\tau)\times f_{X_{1,f}}(t-\tau)d\tau dt. \quad (11)$$

Scenario 2.

A typical case of Scenario 2 is graphically shown in Fig. 2(b). Under this scenario, the system fails due to failure mode 2, and the failure happens between the $(k-1)$th and the kth inspections, where $k=1,2,...,n$. Let t denote the time when the system fails and τ denote the time when the system becomes defective.

Scenario 2 happens when the following conditions are met: (i) the system is still in the normal state up to time $(k-1)T$; (ii) the system fails at time t due to failure mode 2, where $t\in((k-1)T,kT)$; (iii) the system becomes defective at time τ, where $\tau\in((k-1)T,t)$, and it remains defective up to t; (iv) no production wait occurs during the period from being defective to failure, i.e., in the time interval $[\tau,t)$.

Let $T_{k,f2}$ be the random variable that denotes the failure time of the system in the case of Scenario 2. Then, the probability that $T_{k,f2}$ is between $(k-1)T$ and kT is

$$P\{(k-1)T<T_{k,f2}<kT\}$$
$$=\int_{(k-1)T}^{kT}\lim_{\Delta t\to 0}\frac{P\left\{\begin{matrix}\left[X_2\in(t,t+\Delta t)\right]\cap\left[(k-1)T<X_{1,d}<t\right]\\ \cap\left[\left(X_{1,d}+X_{1,f}\right)>t\right]\cap\left[N(t)-N(\tau)=0\right]\end{matrix}\right\}}{\Delta t}dt$$
$$=\int_{(k-1)T}^{kT}f_{X_2}(t)\int_{(k-1)T}^{t}e^{-\lambda(t-\tau)}f_{X_{1,d}}(\tau)R_{X_{1,f}}(t-\tau)d\tau dt. \quad (12)$$

Scenario 3.

A typical case of Scenario 3 is graphically shown in Fig. 2(c). Under this scenario, the system fails due to failure mode 2, and the failure happens between the $(k-1)$th and the kth inspections, where $k=1,2,...,n$. The system fails suddenly when it is still in the normal state. Let t denote the time when the system fails, where $t\in((k-1)T,kT)$. Let $T_{k,f3}$ be the random variable that denotes the failure time of the system in the case of Scenario 3. Then, the probability that $T_{k,f3}$ is between $(k-1)T$ and kT is

$$P\{(k-1)T<T_{k,f3}<kT\}$$
$$=\int_{(k-1)T}^{kT}\lim_{\Delta t\to 0}\frac{P\{\left[X_2\in(t,t+\Delta t)\right]\cap\left(X_{1,d}>t\right)\}}{\Delta t}dt$$
$$=\int_{(k-1)T}^{kT}R_{X_{1,d}}(t)f_{X_2}(t)dt. \quad (13)$$

Since the three scenarios are independent, the probability that a failure renewal happens between the $(k-1)$th and the kth inspections is the sum of the three probabilities. Let $T_{k,f}$ denote the time of the system that fails between the $(k-1)$th and the kth inspections. Then,

$$P\{(k-1)T<T_{k,f}<kT\}$$
$$=\int_{(k-1)T}^{kT}R_{X_2}(t)\int_{(k-1)T}^{t}e^{-\lambda(t-\tau)}\times f_{X_{1,d}}(\tau)\times f_{X_{1,f}}(t-\tau)d\tau dt$$
$$+\int_{(k-1)T}^{kT}f_{X_2}(t)\int_{(k-1)T}^{t}e^{-\lambda(t-\tau)}f_{X_{1,d}}(\tau)R_{X_{1,f}}(t-\tau)d\tau dt$$
$$+\int_{(k-1)T}^{kT}R_{X_{1,d}}(t)f_{X_2}(t)dt. \quad (14)$$

The probability density function of $T_{k,f}$ can be obtained by differentiating Eq. (14) with respect to t. Let $f_{T_{k,f}}(t)$ denote this PDF. Then we have

$$f_{T_{k,f}}(t)=R_{X_2}(t)\int_{(k-1)T}^{t}e^{-\lambda(t-\tau)}\times f_{X_{1,d}}(\tau)\times f_{X_{1,f}}(t-\tau)d\tau + f_{X_2}(t)\int_{(k-1)T}^{t}e^{-\lambda(t-\tau)}f_{X_{1,d}}(\tau)R_{X_{1,f}}(t-\tau)d\tau + R_{X_{1,d}}(t)f_{X_2}(t). \tag{15}$$

Then, the expected length for the failure renewal, denoted by $E_f^L(n,T)$, can be obtained as

$$E_f^L(n,T)=\sum_{k=1}^{n}\int_{(k-1)T}^{kT}t\times f_{T_{k,f}}(t)dt. \tag{16}$$

The expected cost for the failure renewal, denoted by $E_f^C(n,T)$, can be obtained as

$$\begin{aligned}E_f^C(n,T) &=\sum_{k=1}^{n}\left(\left[\pi_p(k-1)+\pi_w N(\tau)+c_f\right]\times P\left[(k-1)T<T_{k,f1}<kT\right]\right)\\&+\sum_{k=1}^{n}\left(\left[\pi_p(k-1)+\pi_w N(\tau)+c_f\right]\times P\left[(k-1)T<T_{k,f2}<kT\right]\right)\\&+\sum_{k=1}^{n}\left(\left[\pi_p(k-1)+\pi_w N(t)+c_f\right]\times P\left[(k-1)T<T_{k,f3}<kT\right]\right).\end{aligned} \tag{17}$$

C. *Production wait inspection renewal*

During the production waits, inspections are also carried out. If an inspection finds out that the system is at the defective state, the system is replaced. A production wait inspection renewal happens when a production wait is needed after the system being defective before failure. Fig. 3 shows an example of the production wait renewal happen between the $(k-1)$th and the kth inspections. Note that the three different cases in Fig. 3(a), Fig. 3(b) and Fig. 3(c) can be considered together when we derive the analytical model.

Suppose that a production wait inspection renewal happens between the $(k-1)$th and the kth inspections. As shown in Fig. 3, the system is still in the normal state up to time $(k-1)T$. Let τ denote the time when the system becomes defective and t_w be the time that the first production wait happens after the system being defective. As shown in Fig. 3, we know that $t_w\in((k-1)T,kT)$ and $\tau\in((k-1)T,t_w)$. In order to make the production wait inspection renewal happens, the system must have not failed yet up to time t_w. Let $T_{k,w}$ denote the time of the production wait renewal that happens between the $(k-1)$th and the kth inspections. Then, we can obtain the probability density function of $T_{k,w}$ as

$$\begin{aligned}f_{T_{k,w}}(t_w) &=\lim_{\Delta t_w\to 0}\frac{P\left\{\begin{matrix}\left[(k-1)T<X_{1,d}<t_w\right]\cap\left[T_{k,w}\in(t_w,t_w+\Delta t_w)\right]\\ \cap(X_2>t_w)\cap\left[(X_{1,d}+X_{1,f})>t_w\right]\end{matrix}\right\}}{\Delta t_w}\\&=R_{X_2}(t_w)\int_{(k-1)T}^{t_w}f_{T_w}(t_w)f_{X_{1,d}}(\tau)R_{X_{1,f}}(t_w-\tau)d\tau.\end{aligned} \tag{18}$$

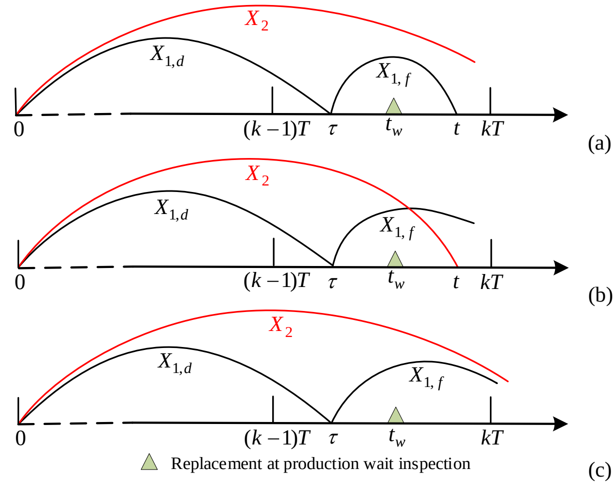

Fig. 3. Graphic representation of production wait inspection renewal.

Thus, the probability of the production wait renewal between the $(k-1)$th and the kth inspections is

$$P\{(k-1)T<T_{k,w}<kT\}=\int_{(k-1)T}^{kT}f_{T_{k,w}}(t_w)dt_w. \tag{19}$$

Then, the expected length for the production wait inspection renewal, denoted by $E_w^L(n,T)$, can be obtained as

$$E_w^L(n,T)=\sum_{k=1}^{n}\int_{(k-1)T}^{kT}t_w\times f_{T_{k,w}}(t_w)dt_w. \tag{20}$$

The expected cost for the production wait inspection renewal, denoted by $E_w^C(n,T)$, can be obtained as

$$E_w^C(n,T)=\sum_{k=1}^{n}\left\{\begin{matrix}\left[\pi_p(k-1)+\pi_w N(t_w)+c_p\right]\\ \times P\{(k-1)T<T_{k,w}<kT\}\end{matrix}\right\}. \tag{21}$$

D. *Age-based renewal*

When the system reaches the age of nT, the system is replaced without any inspection. The age-based replacement happens if the system is still in the normal state when it reaches the age of nT or if the system becomes defective at some time between $(n-1)T$ and nT, and there is no production wait after the system being defective. The two scenarios are shown in Fig. 4 (a) and Fig. 4(b) respectively.

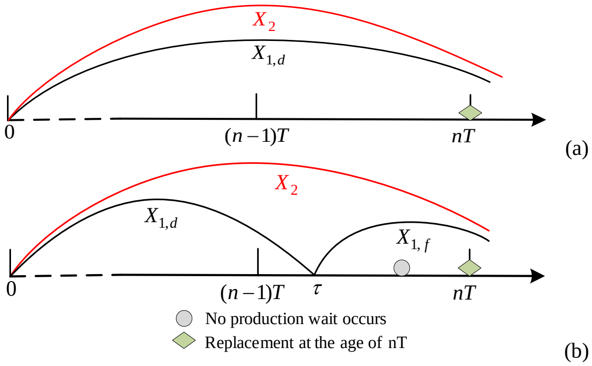

Fig. 4. Graphic representation of age-based renewal.

Scenario 1.

In this scenario, the system is in the normal state at time nT. Thus, $X_{1,d}$ and X_2 must be larger than nT. The probability that the system is in the normal, denoted by $P_{T_{age},1}$ is

$$P_{T_{age},1} = P\left\{\left(X_{1,d} > nT\right)\cap\left(X_2 > nT\right)\right\} = R_{X_{1,d}}\left(nT\right)R_{X_2}\left(nT\right). \tag{22}$$

Scenario 2.

In this scenario, the system is in the normal state at the $(n-1)$th periodical inspection. The system becomes defective at time τ, which is between $(n-1)T$ and nT. No production wait happens during (τ, nT). The system has not failed up to nT. the probability that this scenario happens, denoted by $P_{T_{age},2}$, is

$$\begin{aligned}P_{T_{age},2} &= P\left\{\begin{matrix}\left[(n-1)T < X_{1,d} < nT\right]\cap\left[\left(X_{1,d}+X_{1,f}\right) > nT\right] \\ \cap\left(X_2 > nT\right)\cap\left[N\left(nT\right)-N\left(\tau\right)=0\right]\end{matrix}\right\} \\ &= R_{X_2}\left(nT\right)\int_{(n-1)T}^{nT} e^{-\lambda(nT-\tau)} f_{X_{1,d}}\left(\tau\right)R_{X_{1,f}}\left(nT-\tau\right)d\tau.\end{aligned} \tag{23}$$

Then, the expected length for the age-based renewal, denoted by $E_a^L(n,T)$, can be obtained as

$$E_a^L\left(n,T\right) = nT \times \left(P_{T_{age},1} + P_{T_{age},2}\right). \tag{24}$$

The expected cost for the age-based renewal, denoted by $E_w^C(n,T)$, can be obtained as

$$\begin{aligned}E_a^C\left(n,T\right) &= \left[\pi_p\left(n-1\right)+\pi_w N\left(nT\right)+c_p\right]\times P_{T_{age},1} \\ &+\left[\pi_p\left(n-1\right)+\pi_w N\left(\tau\right)+c_p\right]\times P_{T_{age},2},\end{aligned} \tag{25}$$

where $N(nT)$ and $N(\tau)$ denote the number of production waits within time nT and τ.

In summary, we have derived the $E_p^C(n,T)$, $E_f^C(n,T)$, $E_w^C(n,T)$, $E_a^C(n,T)$ and $E_p^L(n,T)$, $E_f^L(n,T)$, $E_w^L(n,T)$, $E_w^L(n,T)$ given n and T. The optimal value of n and T can be obtained by solving the optimization model defined in Eq. (6).

IV. Case Study

The numerical study is adapted from the single-component steel convertor plant in a steel mill [37]. The production machine is a key plant in the process of steel making. This machine converts molten iron into steel by removing impurities from the molten iron through the oxidation process. Manual periodical inspections are conducted regularly. The machine operates 24 hours a day for 7 days a week and may stop due to insufficient supply of molten iron. This is referred to as the production wait time, which can be utilized for inspection and maintenance. The electromechanical system of this machine consists of mechanical and electrical components, which may suffer soft failures or hard failures. Defects of the machine such as fatigue cracks, pitting corrosion and the decrease of the steel quality can be found during inspections. Besides, the system is overhauled when it reaches the age of T_{age} in order to reduce the probability of system failures that may cause a significant loss. The time of overhaul can be treated as a renewal point.

Weibull distribution is commonly used for modeling the failure process of engineering systems. Since no real data are collected from the system, we provide a rough estimation for the parameters based on the two related work [23, 37]. Although the parameters are not collected from real systems, the numerical analysis provides useful insights for analyzing production systems with two failure modes and production wait time. The failure process of this system follows a Weibull distribution as

$$f_{X_{1,d}}\left(x\right) = (1.5/5.61)(x/5.61)^{0.5}\exp(-(x/5.61)^{1.5})$$

$$f_{X_{1,f}}\left(x\right) = (1.2/2.02)(x/2.02)^{0.2}\exp(-(x/2.02)^{1.2})$$

$$f_{X_2}\left(x\right) = (2/10.83)(x/10.83)^{1}\exp(-(x/10.83)^{2})$$

We assume the unit cost for the periodical inspection is 800, the unit cost for the production wait inspection is 50, the cost for the inspection replacement is 10000, and the cost for the failure replacement is 70000. The arrival of the production wait follows a Poisson distribution with a rate of 0.8.

A. The proposed maintenance model

Fig. 5 shows the expected long run cost per unit of time under different values of n and T.

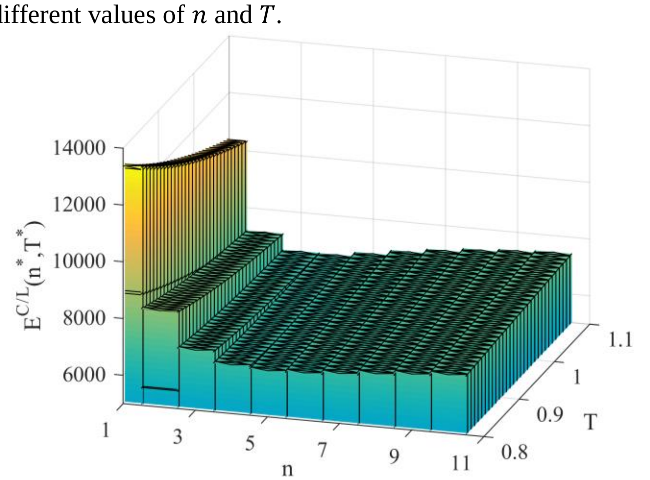

Fig. 5. The expected long run cost per unit of time under different values of n and T.

From the figure we can see that finding the optimal values of n and T can reduce the cost significantly. The minimized cost is 6599 with $n = 4$ and $T = 0.98$, while the cost can reach nearly 14000 in the case of $n = 1$ and $T = 0.8$. This demonstrates the benefit of finding the optimal periodical inspection interval and the optimal time for age-based replacement.

Given that the data is not directly collected from the real system, it would be important for us to carry out sensitivity analysis on each cost parameter. We will analyze how the change of the cost parameters π_p, π_w, c_p and c_f affect the expected long run cost per unit of time.

Fig. 6 and Fig. 7 show the effect of π_p (periodical inspection cost) and π_w (production wait inspection cost) on $E^{C/L}(n^*,T^*)$ (the optimal expected long run cost per unit of time).

As shown in Fig. 6 and Fig. 7, the optimal expected long run cost per unit of time $E^{C/L}(n^*,T^*)$ increases almost linearly with increasing cost of a periodical inspection or increasing cost of production inspection. This suggests that the inspection cost has an approximately linear relationship with $E^{C/L}(n^*,T^*)$.

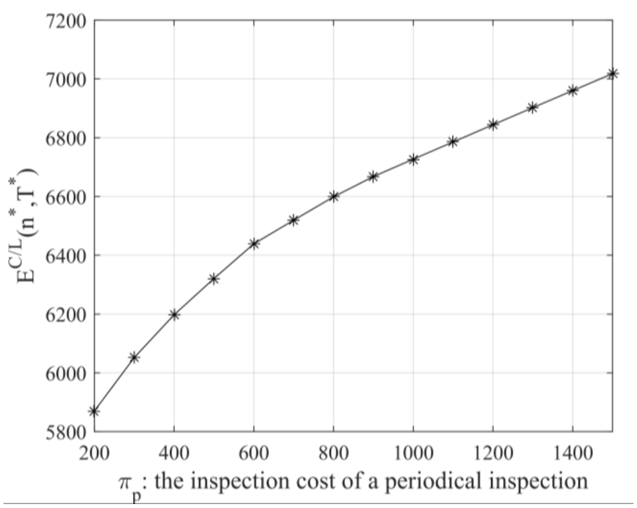

Fig. 6. The effect of periodic inspection cost on $E^{C/L}(n^*,T^*)$.

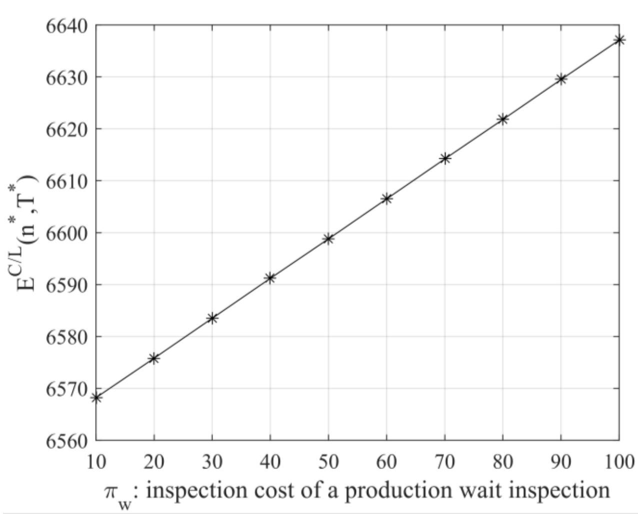

Fig. 7. The effect of production wait inspection cost on $E^{C/L}(n^*,T^*)$.

The effects of c_p (the replacement cost) on $E^{C/L}(n^*,T^*)$ and T^*_{age} (the optimal time for the age-based replacement) are shown in Fig. 8, which indicates that both $E^{C/L}(n^*,T^*)$ and T^*_{age} increase with increasing c_p. This conclusion suggests that the system should be used for a longer time if the replacement cost becomes higher.

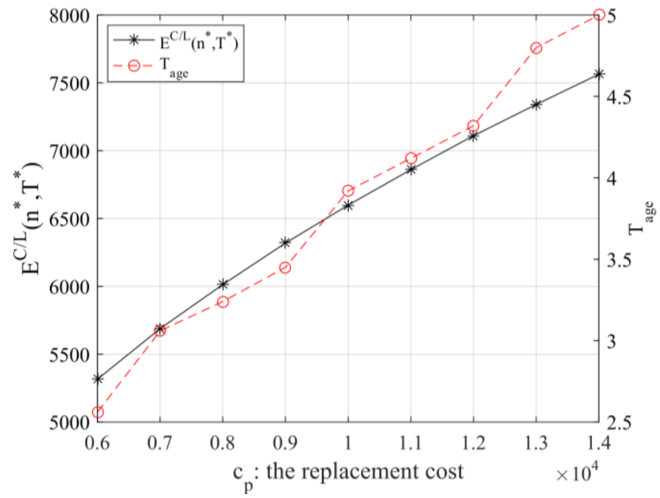

Fig. 8. The effect of c_p (the replacement cost) on $E^{C/L}(n^*,T^*)$ and T^*_{age}.

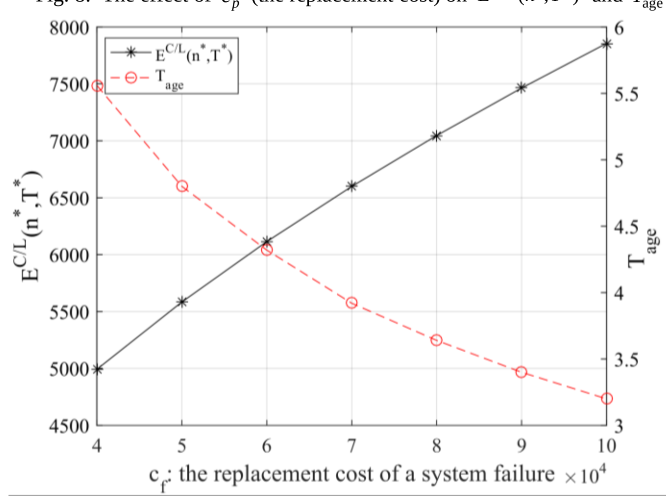

Fig. 9. The effect of c_f (the failure replacement cost) on $E^{C/L}(n^*,T^*)$ and T^*_{age}.

Fig. 9 shows the effect of c_f (the failure replacement cost) on $E^{C/L}(n^*,T^*)$ and T^*_{age}. $E^{C/L}(n^*,T^*)$ still increases with increasing c_f. Different from Fig.8, T^*_{age} decreases with increasing c_f. Due to higher cost of failure replacement, the system will be used for a shorter time such that the probability of failure decreases. This reduces the expected cost of failure replacement. This suggests that high reliability system should be replaced more frequently.

In this numerical study, the arrival of production is assumed to follow a Poisson process with a rate of $\lambda = 0.8$. The effect of λ on $E^{C/L}(n^*,T^*)$ and T^*_{age} are shown in Fig. 10. Increasing the arrival rate can results in longer time for periodical inspection. This result clearly shows that utilizing the production wait time can reduce the long run maintenance cost per unit of time.

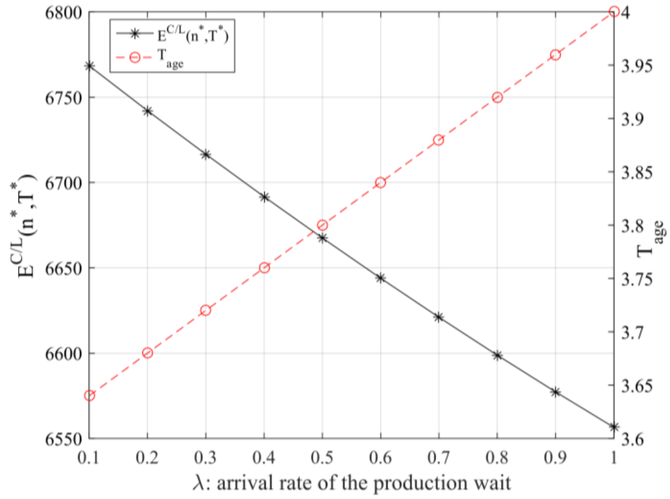

Fig. 10. The effect of λ (the arrival rate of production wait) on $E^{C/L}(n^*, T^*)$ and T^*_{age}.

B. Comparison with existing models

To further justify the effectiveness of the proposed maintenance model, we compare our model with some existing models. The four models for comparison are summarized as follows.

i. Policy 1 is the proposed maintenance in this research that considers both the periodical inspection and production wait inspection. The system is also replaced when it reaches a threshold age.

ii. Policy 2 is a maintenance model studied in [23], which considers the periodical inspection and age-based replacement.

iii. Policy 3 is a variant of our proposed maintenance model that only carries out inspections during production wait time. No periodical inspection is scheduled. The system is also replaced when it reaches a threshold age.

iv. Policy 4 is a variant of our proposed maintenance model that does not perform age-based replacement. Inspection and maintenance are carried out periodically and during the production wait time.

Since policy 2 is the maintenance model studied in [23], we can use their maintenance cost model directly. Policy 4 can be obtain easily using policy 1 (the proposed policy in this article) by letting n be infinity. Thus, in order to avoid repetition, we only provide the cost model for policy 3 at *Appendix A*.

To demonstrate the robustness of the proposed maintenance model, we perform sensitivity analyses of each cost parameter through comparing the above four models. These sensitivity analyses include: (a) Sensitivity analysis of the periodical inspection cost (π_p), (b) Sensitivity analysis of the production wait inspection cost (π_w), (c) Sensitivity analysis of the inspection replacement cost (c_p), and (d) Sensitivity analysis of the failure replacement cost (c_f).

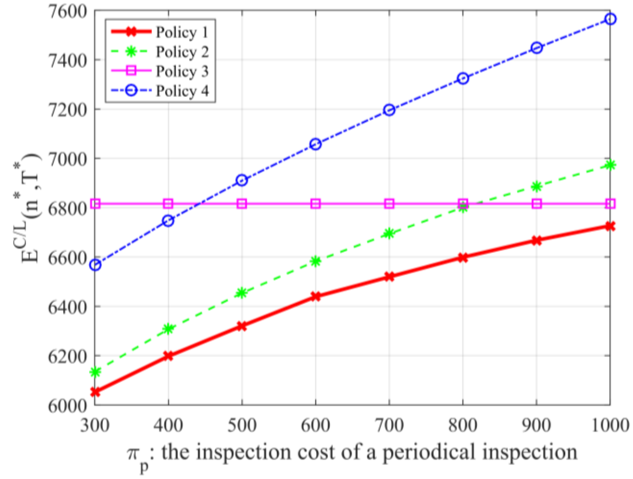

Fig. 11. The impact of the periodical inspection cost for different maintenance models.

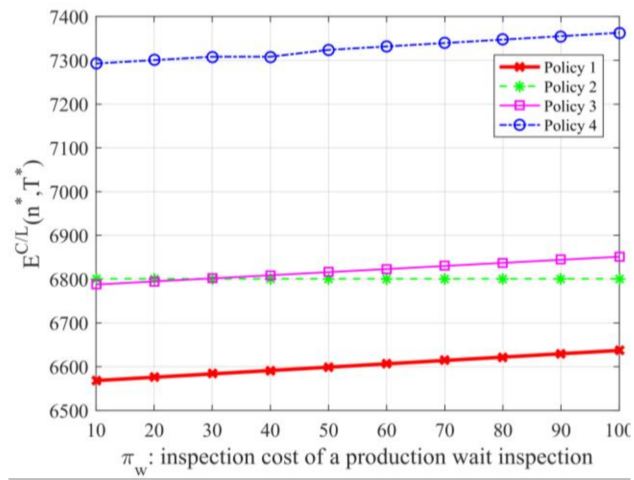

Fig. 12. The impact of the production wait inspection cost for different maintenance models.

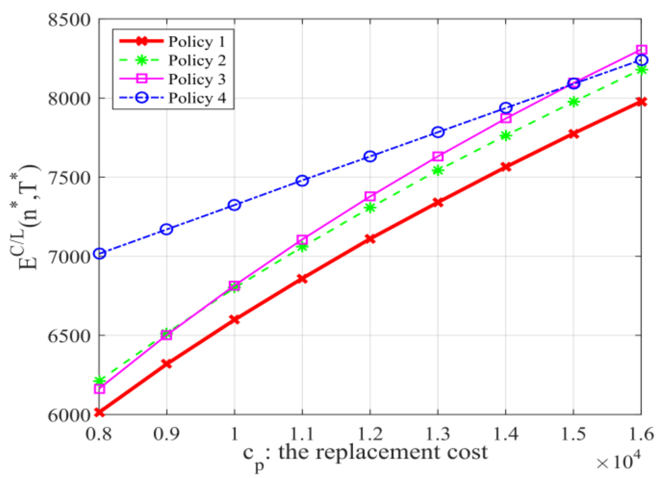

Fig. 13. The impact of the replacement cost for different maintenance models.

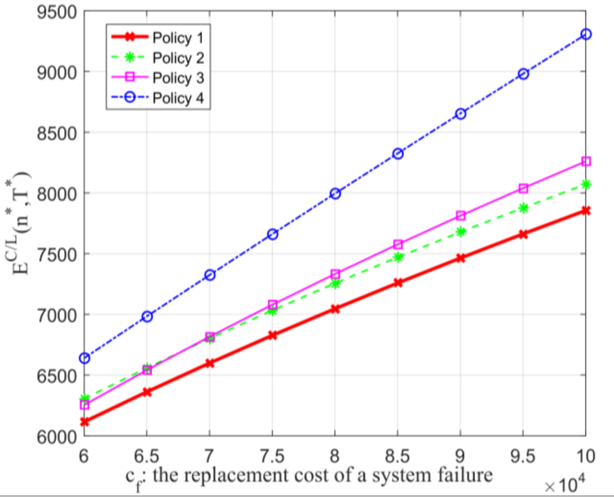

Fig. 14. The impact of the failure replacement cost for different maintenance.

Figs. 11 – 14 show the impact of the cost parameters π_p, π_w, c_p and c_f towards the long time cost per unit of time for all the four maintenance models. From the figures, we can see that the total cost per unit of time increases with the increasing of π_p, π_w, c_p or c_f for each maintenance model. In all figures, policy 1 outperforms any other maintenance models.

In general, policy 4, which does not consider the age-based replacement, underperforms in most of the scenarios. This indicates that the age-based replacement policy is important to reduce the potential risk of hard failure, thereby helping to reduce the long run maintenance cost per unit of time.

V. Conclusion

Motivated by real-world applications, this paper considered two types of failures in single-unit systems: a traditional catastrophic failure and a two-stage delay failure. Besides periodical inspections, the production wait occurred due to a lack of demand or a shortage of materials is utilized as an opportunity to conduct inspection and maintenance. The system is replaced when any of the following conditions is met (i) the system fails, (ii) the age of the system reaches a pre-specified threshold, (iii) a defect is found during inspection. We developed a cost model for the proposed maintenance policy and analyzed the effect of each cost parameter on the expected long run cost per unit of time. Further, we gave some special cases as alternative maintenance polices, and compared the proposed maintenance policy with these alternative policies. The case study showed that the proposed maintenance policy reduced the expected long run maintenance cost per unit of time. Through the study, we also concluded that how each cost parameter affected the optimal periodical inspection interval and the optimal time for the age-based replacement.

Although this work analyzes a production system subject to two failure modes, the modeling and analyzing methods can be easily extended to system with multiple failure modes. In future, the optimal inspection and maintenance policy for a multi-component system with multiple failure modes can be studied.

Appendix A.

Before we derive the cost model for policy 3, we first define the following variables that are different from those used in the main part of this article.

T_{fj}	system failure renewal time of policy 3 under scenario, j, $j=1,2,3$
$E_f^C(T_{age})$	expected renewal cycle cost for the failure renewal of policy 3
$E_w^C(T_{age})$	expected renewal cycle costs for the production wait inspection renewal of policy 3
$E_a^C(T_{age})$	expected renewal cycle costs for the age-based renewal of policy 3
$E_f^L(T_{age})$	expected renewal cycle length for the failure renewal of policy 3
$E_w^L(T_{age})$	expected renewal cycle length for the production wait inspection renewal of policy 3
$E_a^L(T_{age})$	expected renewal cycle length for the age-based renewal of policy 3
Γ_w	time of the production wait inspection renewal of policy 3

In the case of policy 3, the system is only inspected during the production wait time. There is no periodical inspection. Thus, there exist three types of system renewal including failure renewal, production wait inspection renewal and age-based renewal.

Failure renewal

A typical scenario of the failure renewal for policy 3 is shown in Fig. A1 (a), (b) and (c). Part (a) denotes the scenario that the failure is caused by the first failure mode. It requires that there should be no production wait occurs after the system becomes defective at the time τ. Part (b) denotes the scenario that the failure is caused by the second failure mode, but the system is already defective before failure. This scenario also requires there should be no production wait occurs after the system becomes defective at the time τ. Part (c) denotes the scenario that the failure is caused by the second failure mode. The system does not become defective.

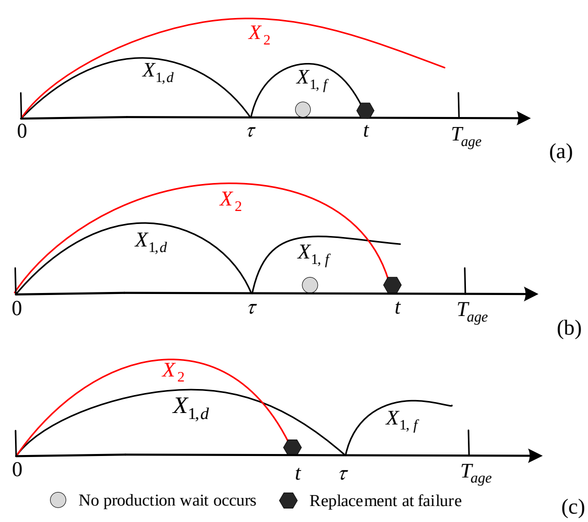

Fig. A1. Graphic representation of failure renewal of the policy 3.

Let $P\{0 < T_{fj} < T_{age}\}$ denote the probability the jth scenario of the failure renewal happens before the age threshold, where $j = 1,2,3$ denotes the three scenarios in Fig. A1. Based on the discussion above, we have

$$P\left\{0<T_{f1}<T_{age}\right\} = \int_0^{T_{age}} \lim_{\Delta t\to 0} \frac{P\left\{\begin{matrix}\left[\left(X_{1,d}+X_{1,f}\right)\in\left(t,t+\Delta t\right)\right]\\ \cap\left(0<X_{1,d}<t\right)\cap\left(X_2>t\right)\cap\left[N(t)-N(\tau)=0\right]\end{matrix}\right\}}{\Delta t}dt = \int_0^{T_{age}} R_{X_2}(t)\int_0^t e^{-\lambda(t-\tau)}\times f_{X_{1,d}}(\tau)\times f_{X_{1,f}}(t-\tau)d\tau dt. \tag{A1}$$

$$P\left\{0<T_{f2}<T_{age}\right\} = \int_0^{T_{age}} \lim_{\Delta t\to 0} \frac{P\left\{\begin{matrix}\left[X_2\in\left(t,t+\Delta t\right)\right]\cap\left(0<X_{1,d}<t\right)\\ \cap\left[\left(X_{1,d}+X_{1,f}\right)>t\right]\cap\left[N(t)-N(\tau)=0\right]\end{matrix}\right\}}{\Delta t}dt = \int_0^{T_{age}} f_{X_2}(t)\int_0^t e^{-\lambda(t-\tau)} f_{X_{1,d}}(\tau)R_{X_{1,f}}(t-\tau)d\tau dt. \tag{A2}$$

$$P\left\{0<T_{f3}<T_{age}\right\} = \int_0^{T_{age}} \lim_{\Delta t\to 0} \frac{P\left\{\left[X_2\in\left(t,t+\Delta t\right)\right]\cap\left(X_{1,d}>t\right)\right\}}{\Delta t}dt = \int_0^{T_{age}} R_{X_{1,d}}(t) f_{X_2}(t)dt. \tag{A3}$$

Since the three scenarios are independent, the probability that a failure renewal happens before the age threshold is

$$\begin{aligned}P\left\{0<T_f<T_{age}\right\} &= P\left\{0<T_{f1}<T_{age}\right\}+P\left\{0<T_{f2}<T_{age}\right\}+P\left\{0<T_{f3}<T_{age}\right\}\\ &= \int_0^{T_{age}} R_{X_2}(t)\int_0^t e^{-\lambda(t-\tau)}\times f_{X_{1,d}}(\tau)\times f_{X_{1,f}}(t-\tau)d\tau dt\\ &\quad+\int_0^{T_{age}} f_{X_2}(t)\int_0^t e^{-\lambda(t-\tau)} f_{X_{1,d}}(\tau)R_{X_{1,f}}(t-\tau)d\tau dt\\ &\quad+\int_0^{T_{age}} R_{X_{1,d}}(t) f_{X_2}(t)dt.\end{aligned} \tag{A4}$$

The probability density function of $T_{k,f}$ can be obtained by differentiating Eq. (A4) with respect to t. Let $f_{T_{k,f}}(t)$ denote this probability density function. Then we have

$$\begin{aligned}f_{T_f}(t) &= R_{X_2}(t)\int_0^t e^{-\lambda(t-\tau)}\times f_{X_{1,d}}(\tau)\times f_{X_{1,f}}(t-\tau)d\tau\\ &\quad+ f_{X_2}(t)\int_0^t e^{-\lambda(t-\tau)} f_{X_{1,d}}(\tau)R_{X_{1,f}}(t-\tau)d\tau\\ &\quad+R_{X_{1,d}}(t) f_{X_2}(t).\end{aligned} \tag{A5}$$

Then, the expected length for the failure renewal, denoted by $E_f^L(T_{age})$, can be obtained as

$$E_f^L\left(T_{age}\right)=\int_0^{T_{age}} t\times f_{T_f}(t)dt. \tag{A6}$$

The expected cost for the failure renewal, denoted by $E_f^C(T_{age})$, can be obtained as

$$\begin{aligned}E_f^C\left(T_{age}\right) &= \left[\pi_w N(\tau)+c_f\right]\times P\left\{0<T_{f1}<T_{age}\right\}\\ &\quad+\left[\pi_w N(\tau)+c_f\right]\times P\left\{0<T_{f2}<T_{age}\right\}\\ &\quad+\left[\pi_w N(t)+c_f\right]\times P\left\{0<T_{f3}<T_{age}\right\}.\end{aligned} \tag{A7}$$

Production wait inspection renewal

Fig. A2 shows the case when production wait inspection renewal happens for policy 3. In any of the scenarios (a) (b) (c), it requires a production wait occurs after the system become defective at time τ and this production wait occurs before the system fails.

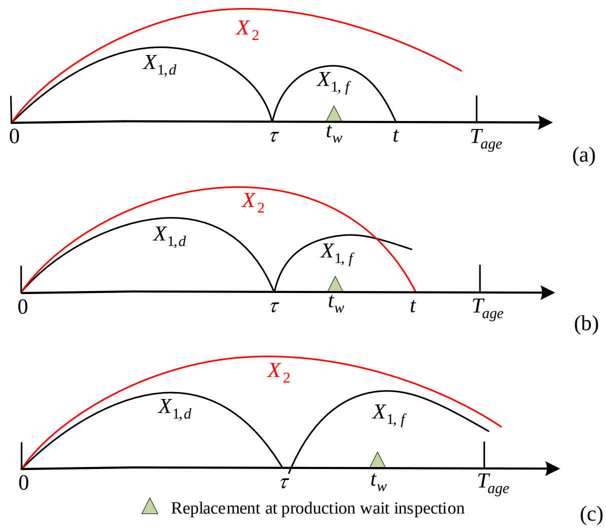

Fig. A2. Graphic representation of production wait inspection renewal of policy 3.

Let Γ_w be the random variable that denotes the time of the production wait renewal

$$f_{\Gamma_w}\left(t_w\right)=\lim_{\Delta t_w\to 0}\frac{P\left\{\begin{matrix}\left(0<X_{1,d}<T_{age}\right)\cap\left[\Gamma_w\in\left(t_w,t_w+\Delta t_w\right)\right]\\ \cap\left(X_2>t_w\right)\cap\left[\left(X_{1,d}+X_{1,f}\right)>t_w\right]\end{matrix}\right\}}{\Delta t_w} = R_{X_2}\left(t_w\right)\int_0^{t_w} f_{Tw}\left(t_w\right) f_{X_{1,d}}(\tau)R_{X_{1,f}}\left(t_w-\tau\right)d\tau. \tag{A8}$$

Thus, the probability of the production wait renewal of policy 3 is

$$P\left\{0<\Gamma_w<T_{age}\right\}=\int_0^{T_{age}} f_{\Gamma_w}\left(t_w\right)dt_w. \tag{A9}$$

Then, the expected length for the production wait inspection renewal, denoted by $E_w^L(n,T)$, can be obtained as

$$E_w^L\left(T_{age}\right)=\int_0^{T_{age}} t_w\times f_{\Gamma_w}\left(t_w\right)dt_w. \tag{A10}$$

The expected cost for the production wait inspection renewal, denoted by $E_w^C(n,T)$, can be obtained as

$$E_w^C\left(T_{age}\right)=\left[\pi_w N\left(t_w\right)+c_p\right]\times P\left\{0<\Gamma_w<T_{age}\right\}. \tag{A11}$$

Age-based renewal

Age-based renewal happens if either the scenario (a) or scenario (b) of Fig. A3 occurs. Scenario (a) means the system is still in the normal state after it reaches the age of T_{age}. Scenario (b) means the system is defective when it reaches the age of T_{age}, but there is no production wait occurs after the system become defective.

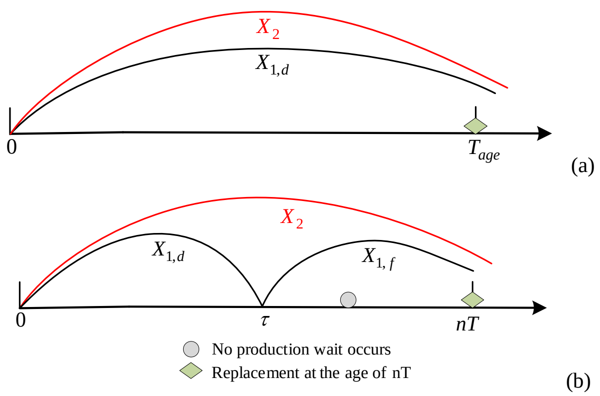

Fig. A3. Graphic representation of age-based renewal of policy 3.

Let $P_{T_{age},1}$ and $P_{T_{age},2}$ denote the probability of Scenario (a) and Scenario (b) happen respectively. We have

$$\begin{aligned} P_{T_{age},1} &= P\left\{\left(X_{1,d} > T_{age}\right)\cap\left(X_2 > T_{age}\right)\right\} \\ &= R_{X_{1,d}}\left(T_{age}\right)R_{X_2}\left(T_{age}\right). \end{aligned} \quad (A12)$$

$$\begin{aligned} P_{T_{age},2} &= P\left\{\begin{aligned}&\left[0 < X_{1,d} < T_{age}\right]\cap\left[\left(X_{1,d} + X_{1,f}\right) > T_{age}\right] \\ &\cap\left(X_2 > T_{age}\right)\cap\left[N\left(T_{age}\right) - N\left(\tau\right) = 0\right]\end{aligned}\right\} \\ &= R_{X_2}\left(T_{age}\right)\int_0^{T_{age}} e^{-\lambda\left(T_{age}-\tau\right)} f_{X_{1,d}}\left(\tau\right)R_{X_{1,f}}\left(T_{age}-\tau\right)d\tau. \end{aligned} \quad (A13)$$

Then, the expected length for the age-based renewal, denoted by $E_a^L(T_{age})$, can be obtained as

$$E_a^L\left(T_{age}\right) = T_{age} \times \left(P_{T_{age},1} + P_{T_{age},2}\right). \quad (A14)$$

The expected cost for the age-based renewal, denoted by $E_w^C(n,T)$, can be obtained as

$$\begin{aligned} E_a^C\left(T_{age}\right) &= \left[\pi_w N\left(nT\right) + c_p\right] \times P_{T_{age},1} \\ &\quad + \left[\pi_w N\left(\tau\right) + c_p\right] \times P_{T_{age},2}, \end{aligned} \quad (A15)$$

where $N\left(nT\right)$ and $N\left(\tau\right)$ denote the number of production waits within time nT and τ.

Summarizing the three different types of system renewal, the long-run maintenance cost per unit of time is

$$\frac{E_f^C(T_{age}) + E_w^C(T_{age}) + E_a^C(T_{age})}{E_f^L(T_{age}) + E_w^L(T_{age}) + E_a^L(T_{age})}. \quad (A16)$$

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