Michailidis, Dimitrios-Xanthos (2021) *Representation Theory of Quiver Hecke Algebras of Type A.* Doctor of Philosophy (PhD) thesis, University of Kent,.

Representation Theory of Quiver Hecke Algebras of Type A

by

Dimitrios-Xanthos Michailidis

A thesis submitted for the degree of

Doctor of Philosophy

School of Mathematics, Statistics and Actuarial Science

University of Kent

29th April, 2021

Declaration

I hereby declare that the work of this dissertation is my own work and where work from outside sources has been used it has been properly and accurately cited.

Dimitrios-Xanthos Michailidis,
April 2021

To my parents Panagiotis and Despoina.

Abstract

In this thesis we study the representation theory of the quiver Hecke algebras of type A. In particular we consider specific quotients which play an important role both in Schur-Weyl dualities and in other areas of mathematics such as statistical mechanics, Lie theory and knot theory.

The thesis is organised in four chapters. Chapter 1 is the introductory chapter and includes an overview of the concept of Schur-Weyl in addition to the basic theory of cellular algebras, which will be central throughout the thesis. Moreover, this is the chapter where we first introduce the quiver Hecke algebras of type A and the quotients of interest, that is the largest possible quasi-hereditary quotients with non-singular Kazhdan-Lusztig theory, denoted by $\mathscr{H}_n^\sigma$ and the Temperley–Lieb algebra of type B also known as "blob algebra", denoted by B_n^σ. Finally, we introduce the main combinatorial objects of the thesis, accompanied by examples which enhance reader's understanding.

Chapter 2 is devoted in the construction of a cellular basis for the quotients $\mathscr{H}_n^\sigma$ different from the well-known results of Hu–Mathas [HM10], as it uses a less familiar order relation. The first section provides combinatorial analogues for the action of the dot-generators of the algebra, which will be essential in the sequel. The second section contains the technical element of this chapter which proves that there exists a chain of two-sided ideals for the algebra $\mathscr{H}_n^\sigma$. The last section utilises that chain of ideals and constructs a cellular basis for $\mathscr{H}_n^\sigma$.

In chapter 3 we encounter the Temperley–Lieb algebra of type B or blob algebra and we endeavour to construct bases for the simple modules of the algebra over a field of characteristic zero. These bases will be indexed by paths in an alcove geometry of type $\hat{A}_1$. We start by defining the concept of alcove geometry which will be important in the chapter's proofs. We also recall known results on the blob algebra and we make appropriate references in the literature. In the third section we construct homomorphisms between cell modules and we calculate the images of these homomorphisms. Over a field of characteristic zero the union of the images

is equal to the radical of the module, hence we have a basis for the (simple) head of the module.

In chapter 4 we construct BGG resolutions associated to any simple module of the blob algebra over a field of characteristic zero. BGG resolutions are very fruitful objects in mathematics with several applications in different areas. In the first section we give a formula for the composition of one-column homomorphisms between cell modules of the blob algebra.

Acknowledgements

First of all I would like to thank my supervisor Dr. Christopher Bowman for his endless support throughout my PhD journey. Always there to help me patiently when I was struggling to cope with the research.

Moreover, I owe gratitude to the University of Kent for the financial support and the excellent working conditions for the duration of my studies.

I would like to thank Claire Carter for her continuous support from day one of my studies. Claire has always been there for a chat and keen to help and guide me with my problems and concerns.

Furthermore, many thanks to Christos for the memories during our first year in Canterbury and the great basketball discussions until today.

I would like to thank my fellow PhDs Nikitas and Sotiris for their friendship and especially for the nice BBQs they organised at their garden. Also, thanks to Michele for being a great office mate.

Many thanks to Artemis for her love, support and patience all these years ♡.

The deepest thanks and gratitude to my parents Panagiotis and Despoina for what they have done for me throughout my life. I would not be able to achieve anything without their selfless support.

Dimitrios-Xanthos Michailidis,

October 2021.

Contents

Chapter 1

Introduction and background

In this chapter we include historical background on the study of the symmetric group and its related algebras. We also fix the notation of the combinatorial and algebraic notions used during the thesis. More precisely we start by an overview of Schur-Weyl duality which can be considered as a standard source of inspiration for studying different algebras. We continue with a section with the basic elements of cellular algebras. This section is slightly "dry" but it will provide the structural framework of the thesis, as all the algebras we consider will be (or be proven to be) cellular algebras. Subsequently, we define our main algebras and their quotients of interest. Finally, we state the basic definitions, followed by relevant examples, for the combinatorial objects which we shall use throughout the thesis.

Note that most of these combinatorial objects are standard tools that can be found in several papers and manuscripts in the area of combinatorial representation theory. As a reference we include [Mic21, BCHM20].

1.1 Schur-Weyl duality overview

We start our story by a brief discussion on the concept of Schur-Weyl duality. This is one of the most well-known results in representation theory and it has motivated the study of various groups and algebras. Let $\mathbb{C}$ be the field of complex numbers and we consider the vector space $V = \mathbb{C}^h$. The symmetric group $\mathfrak{S}_n$ acts naturally on the n-fold tensor power $V^{\otimes n}$, by permuting the tensor positions. We consider the obvious action of the general linear group $\mathrm{GL}_h := \mathrm{GL}_h(\mathbb{C})$, namely the matrix multiplication in each tensor position and let $\mathbb{C}\mathrm{GL}_h$, $\mathbb{C}\mathfrak{S}_n$ be the group algebras of the groups GL_h, $\mathfrak{S}_n$ respectively. By observing that the aforementioned actions commute, we equip the vector space $V^{\otimes n}$ with $\mathbb{C}\mathrm{GL}_h$-$\mathbb{C}\mathfrak{S}_n$-bimodule structure. We recall the classic result by Schur [Sch27], indicating that the image of each group algebra under its representation equals the full centraliser algebra of the other representation. In more detail, if we name the representations as described in [Dot04], namely

$$\mathbb{C}\mathrm{GL}_h \xrightarrow{\quad\rho\quad} \mathrm{End}(V^{\otimes n}) \xleftarrow{\quad\sigma\quad} \mathbb{C}\mathfrak{S}_n$$

we have the following equalities

$$\rho(\mathbb{C}\mathrm{GL}_h) = \mathrm{End}_{\mathfrak{S}_n}(V^{\otimes n})$$

$$\sigma(\mathbb{C}\mathfrak{S}_n) = \mathrm{End}_{\mathrm{GL}_h}(V^{\otimes n}).$$

Later Carter-Lusztig [CL74] and Green [Gre80] have shown that identical results hold over any infinite field $\Bbbk$.

Some of the readers might find the presence of the letter h for denoting the rows and the columns surprising. However, the choice has its origins in Lie theory and it is used to denote the Coxeter number of a Coxeter group. This notion will turn up later in the thesis when we shall discuss the structure and representation theory of certain quotients of the Hecke algebras, which will be governed by this number h. This number will also impose restrictions on the characteristic of the field $\Bbbk$ and this is crucial because the combinatorial algorithms we have built depend very much on this condition.

Schur-Weyl duality is a centerpiece in modern Lie theory. These dualities interrelate reflection and algebraic groups with the symmetric groups, (walled) Brauer, Temperley-Lieb, blob and partition algebras. The aforementioned algebras all form towers of "diagram algebras" which we study in a uniform fashion using ideas from categorical Lie theory. The rest of this section is devoted to the basics on Hecke algebras (quantised deformations of the symmetric groups) and Temperley-Lieb algebras. For the later, we focus our interest on the Temperley-Lieb algebra of type B, which is originally defined in [MS94]. The Temperley-Lieb algebras (of type A and B) are quotients of the Hecke algebras (of type A and B respectively). In modern representation theory, the Khovanov–Lauda–Rouquier (KLR) algebras provide, via the Brundan–Kleschev isomorphism [BK09], a common framework for the study of those algebras.

1.2 Cellular algebras

This section is devoted to cellular algebras. These algebras are of great importance in this thesis as all the algebras we shall consider in the future are cellular.

Cellular algebras have been introduced by Graham and Lehrer [GL96] and they form a class of finite-dimensional algebras with extreme importance in representation theory. Roughly speaking, cellular algebras are algebras with a *cellular basis*, that is a basis which makes them suitable for doing representation theory. One main advantage of cellular algebras is that they provide a framework for constructing the simple modules in terms of certain cellular bilinear forms. The simple modules of an algebra are of great importance and their study is crucial in understanding the structure of the algebra. In addition, analysis of these cellular bilinear forms provides criterion for the algebra to be semisimple. In this section, for the sake of completeness, we present the basic and well-known theory of cellular algebras. We base our presentation on classic textbook [Mat99] by Mathas.

Let $\Bbbk$ a commutative integral domain with unitary element and A be an associative finite-dimensional algebra which is free as $\Bbbk$-module.

Definition 1.1. [Mat99, Definition 2.1] Suppose that $(\Lambda, \geq)$ is a (finite) poset and for each $\lambda \in \Lambda$ there is a finite indexing set $\mathcal{T}(\lambda)$ and elements $c^{\lambda}_{\mathsf{st}} \in A$ for all

$\mathsf{s},\mathsf{t} \in \mathcal{T}(\lambda)$ such that

$$\mathscr{C} = \{c^\lambda_{\mathsf{st}} \mid \lambda \in \Lambda \text{ and } \mathsf{s},\mathsf{t} \in \mathcal{T}(\lambda)\} \tag{1.1}$$

is a (free) basis of A. For each $\lambda \in \Lambda$ let $A^{>\lambda}$ be the $\Bbbk$-submodule of A with basis $\{c^\mu_{\mathsf{uv}} \mid \mu \in \Lambda, \mu > \lambda \text{ and } \mathsf{u},\mathsf{v} \in \mathcal{T}(\mu)\}$. The pair $(\mathscr{C},\Lambda)$ is a **cellular basis** of A if

(i) the $\Bbbk$-linear map $*\colon A \longrightarrow A$ determined by $(c^\lambda_{\mathsf{st}})^* = c^\lambda_{\mathsf{ts}}$ for all $\lambda \in \Lambda$ and all $\mathsf{s},\mathsf{t} \in \mathcal{T}(\lambda)$, is an algebra anti–isomorphism of A; and,

(ii) for any $\lambda \in \Lambda$, $\mathsf{t} \in \mathcal{T}(\lambda)$ and $a \in A$ there exists $r_{\mathsf{v}} \in \Bbbk$ such that for all $s \in \mathcal{T}(\lambda)$

$$c^\lambda_{\mathsf{st}} a \equiv \sum_{\mathsf{v} \in \mathcal{T}(\lambda)} r_{\mathsf{v}} c^\lambda_{\mathsf{sv}} \mod A^{>\lambda}. \tag{1.2}$$

If A has a cellular basis we say that A is a **cellular algebra**. If in addition there is a function

$$\deg\colon \bigcup_{\lambda\in\Lambda} \mathcal{T}(\lambda) \longrightarrow \mathbb{Z}$$

such that if we define $\deg(c^\lambda_{\mathsf{st}}) = \deg(\mathsf{s}) + \deg(\mathsf{t})$, for $\lambda \in \Lambda$ and $\mathsf{s},\mathsf{t} \in \mathcal{T}(\lambda)$ then A is a graded algebra, we say that A is a **graded cellular algebra** (see [HM10]).

Remark 1.2. We remark that a cellular algebra can have many different cellular bases, where the poset Λ and the indexing sets $\mathcal{T}(\lambda)$ can be completely different. For instance, the size of the poset Λ can be different for different cellular bases of A (see [KX99b]).

Throughout this section we fix a cellular basis $(\mathscr{C},\Lambda)$ of A and we denote by $A^{\geq\lambda}$ the $\Bbbk$-module with basis $\{c^\mu_{\mathsf{uv}} \mid \mu \in \Lambda, \mu \geq \lambda \text{ and } \mathsf{u},\mathsf{v} \in \mathcal{T}(\mu)\}$. It is clear that $A^{>\lambda} \subset A^{\geq\lambda}$ and the quotient $A^{\geq\lambda}/A^{>\lambda}$ has basis

$$\{c^\lambda_{\mathsf{st}} + A^{>\lambda} \mid \mathsf{s},\mathsf{t} \in \mathcal{T}(\lambda)\}. \tag{1.3}$$

By the classical theory of cellular algebras, for each $\lambda \in \Lambda$ there exists a module $\Delta(\lambda)$ with $\Bbbk$-basis

$$\{c^\lambda_{\mathsf{t}} \mid \mathsf{t} \in \mathcal{T}(\lambda)\}$$

and where for each $a \in A$

$$c^\lambda_{\mathsf{t}} a = \sum_{\mathsf{v} \in \mathcal{T}(\lambda)} r_{\mathsf{v}} c^\lambda_{\mathsf{v}}$$

where $r_{\mathsf{v}} \in \Bbbk$ is determined by (1.2). The module $\Delta(\lambda)$ with the basis above is called **cell module**. There is also a unique bilinear form $\langle \cdot, \cdot \rangle \colon \Delta(\lambda) \times \Delta(\lambda) \longrightarrow \Bbbk$ such that $\langle c^\lambda_{\mathsf{s}}, c^\lambda_{\mathsf{t}} \rangle$, for $\mathsf{s}, \mathsf{t} \in \mathcal{T}(\lambda)$, is determined by

$$\langle c^\lambda_{\mathsf{s}}, c^\lambda_{\mathsf{t}} \rangle c^\lambda_{\mathsf{uv}} \equiv c^\lambda_{\mathsf{us}} c^\lambda_{\mathsf{tv}} \mod A^{>\lambda}$$

where u and v are any elements of $\mathcal{T}(\lambda)$. By [Mat99, Proposition 2.10] we have that the bilinear form is symmetric and associative. We define the radical of the cell module $\Delta(\lambda)$ to be the A-submodule of $\Delta(\lambda)$ defined as

$$\operatorname{rad} \Delta(\lambda) = \{x \in \Delta(\lambda) \mid \langle x, y \rangle = 0 \text{ for all } y \in \Delta(\lambda)\}. \tag{1.4}$$

Subsequently, for any $\lambda \in \Lambda$, we define the quotient module $L(\lambda) = \Delta(\lambda)/\operatorname{rad} \Delta(\lambda)$ and let $\Lambda_0 = \{\mu \in \Lambda \mid L(\mu) \neq 0\}$. We have that $\mu \in \Lambda_0$ if and only if the bilinear form $\langle \cdot, \cdot \rangle$ on $\Delta(\mu)$ is non-zero. As we mentioned earlier in this section, one of the advantages of cellular algebras is that we can characterise the their simple modules in a concrete way. The following theorem describes precisely the simple modules of a finite dimensional cellular algebra A.

Theorem 1.3 (Graham–Lehrer)**.** *Suppose that $\Bbbk$ is a field and Λ is finite. Then $\{L(\mu) \mid \mu \in \Lambda_0\}$ is a complete set of pairwise inequivalent irreducible A-modules.*

Proof. See [Mat99, Theorem 2.16]. □

Remark 1.4. It is worth noting that A is semisimple if and only if $\Lambda_0 = \Lambda$ and $L(\mu) = \Delta(\mu)$, for all $\mu \in \Lambda_0$. Also, A is quasi-hereditary if and only if $\Lambda_0 = \Lambda$.

1.3 Algebras of interest

In this section we give a more detailed description of the algebras we study in this thesis. We present the basic definitions but we mostly emphasise on the connections among them and how these are connected with breakthroughs in different (seemingly unrelated) areas of mathematics. More precisely, this section will include elements of Hecke, Temperley–Lieb and Khovanov–Lauda–Rouquier algebras.

1.3.1 Hecke algebras

In the previous section it has been made clear that Schur-Weyl duality tells us that there are connections between the representation theories of different groups and algebras. Information on the representation theory of certain structures gives useful information about the representation theory of their Schur-Weyl dual. Moreover, the concept of Schur-Weyl dualities has initiated studies on the structure of algebras arising as deformations of other algebras.

Let $\Bbbk$ be a (commutative) integral domain. The Hecke algebras are deformations of the group algebras of Coxeter groups and they form families of algebras which depend on a quantum parameter $q \in \Bbbk^\times$. Namely, we recover the group algebra of the Coxeter group when $q = 1$.

One of the most classic and well studied instances of such algebra is the Hecke algebra of the symmetric group $\mathfrak{S}_n$ or *Hecke algebra of type A*. In the literature, for example in [Mat99], the Hecke algebra $\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_n)$ of $\mathfrak{S}_n$ is defined as the unital associative $\Bbbk$-algebra with generators $\{T_1, T_2, \cdots, T_{n-1}\}$ and relations

$$(T_i - q)(T_i + 1) = 0 \qquad \text{for } i = 1, 2, \cdots, n-1 \tag{1.5}$$

$$T_i T_j = T_j T_i \qquad \text{for } 1 \leq i < j - 1 \leq n - 2 \tag{1.6}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad \text{for } i = 1, 2, \cdots, n-2. \tag{1.7}$$

For $q = 1$, relation (1.5) can be written as $T_i^2 = 1$. Hence, we recover the group algebra of the symmetric group $\Bbbk\mathfrak{S}_n$ when $q = 1$.

At this point we shall introduce the Ariki–Koike algebra of the complex reflection groups $(\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_n$ (alternatively the groups of type $G(\ell, 1, n)$ in the Shephard–Todd classification [ST54]). Let $q, Q_1, \cdots, Q_\ell \in \Bbbk$ and $\underline{Q} := \{Q_1, \cdots, Q_\ell\}$. Ariki and Koike [AK94] defined the *Ariki–Koike algebra* to be the unital associative $\Bbbk$-algebra $\mathcal{H}_{\Bbbk,q,\underline{Q}}((\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_n)$ with generators $\{T_0, \cdots, T_{n-1}\}$ and relations

$$(T_0 - Q_1) \cdots (T_0 - Q_\ell) = 1 \tag{1.8}$$

$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0 \tag{1.9}$$

$$(T_i + 1)(T_i - q) = 1 \qquad \text{for } 1 \le i \le n-1 \tag{1.10}$$

$$T_i T_j = T_j T_i \qquad \text{for } 0 \le i < j - 1 \le n-2 \tag{1.11}$$

$$T_{i+1} T_i T_{i+1} = T_i T_{i+1} T_i \qquad \text{for } 1 \le i \le n-2. \tag{1.12}$$

Note that for $\ell = 1$ we get the Hecke algebra of type A. Another instance of Hecke algebra which will be of particular interest in the thesis, is the Hecke algebra of the complex reflection group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n$ or *Hecke algebra of type B*. It is easily understood from the notation that the Hecke algebra of type B can be recovered from the Ariki-Koike algebra for $\ell = 2$. Note that the type B case was defined prior to the Ariki-Koike algebras and in fact was one of the motivations for defining these generalised algebras. In particular, Dipper and James [DJ92], defined the Hecke algebra $\mathcal{H}_{\Bbbk,q,Q}((\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n)$ to be the unital associative $\Bbbk$-algebra with generators $\{T_0, T_1, \cdots, T_{n-1}\}$ and relations

$$(T_0 + 1)(T_0 - Q) = 0 \tag{1.13}$$

$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0 \tag{1.14}$$

$$(T_i + 1)(T_i - q) = 0 \qquad \text{for } 1 \le i \le n-1 \tag{1.15}$$

$$T_i T_j = T_j T_i \qquad \text{for } 0 \le i < j - 1 \le n-2 \tag{1.16}$$

$$T_{i+1} T_i T_{i+1} = T_i T_{i+1} T_i \qquad \text{for } 1 \le i \le n-2. \tag{1.17}$$

For $q = Q = 1$, we recover the group algebra of the complex reflection group $\Bbbk((\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n)$.

1.3.2 Khovanov-Lauda-Rouquier algebras

The Khovanov-Lauda-Rouquier algebras, most commonly known as KLR algebras, were introduced by Khovanov and Lauda [KL09] and independently by Rouquier [Rou]. In their pioneering work, Brundan and Kleshchev [BK09] proved that the Hecke algebras from above, are isomorphic to the KLR algebras. This discovery opened brand new routes in the study of Hecke algebras, since it gives the option of utilising more advanced combinatorics and the diagrammatic presentation of the KLR algebras.

Recall that we denote by $\mathfrak{S}_n$ the symmetric group in n letters and let $e \in \{2, 3, \cdots\}$. Given $\underline{i} = (i_1, i_2, \cdots, i_n) \in (\mathbb{Z}/e\mathbb{Z})^n$ and $s_r = (r, r+1) \in \mathfrak{S}_n$ we set $s_r(\underline{i}) = (i_1, \cdots, i_{r-1}, i_{r+1}, i_r, i_{r+2}, \cdots, i_n)$.

Definition 1.5. [BK09] Fix $e > 2$. The quiver Hecke algebra or Khovanov-Lauda-Rouquier (KLR) algebra $\mathcal{H}_n$, is defined to be the associative $\mathbb{Z}$-algebra with generators

$$\{e_{\underline{i}} \mid \underline{i} = (i_1, \cdots, i_n) \in (\mathbb{Z}/e\mathbb{Z})^n\} \cup \{y_1, \cdots, y_n\} \cup \{\psi_1, \cdots, \psi_{n-1}\},$$

subject to the relations

$$e_{\underline{i}} e_{\underline{j}} = \delta_{\underline{i},\underline{j}} e_{\underline{i}} \qquad \sum_{\underline{i} \in (\mathbb{Z}/e\mathbb{Z})^n} e_{\underline{i}} = 1_{\mathcal{H}_n} \qquad \psi_r e_{\underline{i}} = e_{s_r(\underline{i})} \psi_r \tag{R1}$$

$$y_r e_{\underline{i}} = e_{\underline{i}} y_r \qquad y_r y_s = y_s y_r \tag{R2}$$

for all $r, s, \underline{i}, \underline{j}$ and

$$\psi_r y_s = y_s \psi_r \text{ for } s \neq r, r+1 \qquad \psi_r \psi_s = \psi_s \psi_r \text{ for } |r - s| > 1 \tag{R3}$$

$$y_r \psi_r e_{\underline{i}} = (\psi_r y_{r+1} - \delta_{i_r, i_{r+1}}) e_{\underline{i}} \qquad y_{r+1} \psi_r e_{\underline{i}} = (\psi_r y_r + \delta_{i_r, i_{r+1}}) e_{\underline{i}} \tag{R4}$$

$$\psi_r^2 e_{\underline{i}} = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} + 1)e_{\underline{i}} & \text{if } i_r = i_{r+1}, \\ e_{\underline{i}} & \text{if } i_{r+1} \neq i_r, i_r \pm 1, \\ (y_{r+1} - y_r)e_{\underline{i}} & \text{if } i_{r+1} = i_r + 1, \\ (y_r - y_{r+1})e_{\underline{i}} & \text{if } i_{r+1} = i_r - 1 \end{cases} \tag{R5}$$

$$\psi_r\psi_{r+1}\psi_r e_{\underline{i}} = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} - 1)e_{\underline{i}} & \text{if } i_r = i_{r+2} = i_{r+1} + 1, \\ (\psi_{r+1}\psi_r\psi_{r+1} + 1)e_{\underline{i}} & \text{if } i_r = i_{r+2} = i_{r+1} - 1 \\ \psi_{r+1}\psi_r\psi_{r+1}e_{\underline{i}} & \text{otherwise} \end{cases} \tag{R6}$$

for all permitted $r, s, \underline{i}, \underline{j}$. We identify such elements with decorated permutations and the multiplication with vertical concatenation, $\circ$, of these diagrams in the standard fashion of [BK09, Section 1]. We let $*$ denote the anti-involution which fixes the generators (this can be visualised as a flip through the horizontal axis of the diagram).

The cyclotomic quiver Hecke algebra is defined as quotient of the quiver Hecke algebra. Let $\ell \geq 1$ be an integer and $\sigma = (\sigma_0, \sigma_1, \cdots, \sigma_{\ell-1}) \in \mathbb{Z}^\ell$ be an ℓ-tuple.

Definition 1.6. Fix $e > 2$ and $\sigma \in \mathbb{Z}^\ell$. The cyclotomic quiver Hecke algebra, $\mathcal{H}_n^\sigma$, is defined to be the quotient of $\mathcal{H}_n$ by the relation

$$y_1^{\sharp\{\sigma_m | \sigma_m = i_1 (\operatorname{mod} e), 0 \leq m < \ell\}} e_{\underline{i}} = 0 \qquad \text{for } \underline{i} \in (\mathbb{Z}/e\mathbb{Z})^n. \tag{1.18}$$

We refer to relation (1.18) as *cyclotomic relation*.

We remark that the algebra of Definition 1.5 is isomorphic to the *affine Hecke algebra*. The extra relation of Definition 1.6 gives a family of quotient algebras, which are isomorphic to the *cyclotomic Hecke algebra*.

Given $p < q$ we set

$$w_q^p = s_p s_{p+1} \dots s_{q-1}, \qquad w_p^q = s_{q-1} \dots s_{p+1} s_p \tag{1.19}$$

$$\psi_q^p = \psi_p \psi_{p+1} \cdots \psi_{q-1}, \qquad \psi_p^q = \psi_{q-1} \cdots \psi_{p+1} \psi_p \tag{1.20}$$

and given an expression $\underline{w} = s_{i_1} \cdots s_{i_p} \in \mathfrak{S}_n$ we set $\psi_{\underline{w}} = \psi_{i_1} \cdots \psi_{i_p} \in \mathcal{H}_n$. We note that the element $\psi_{\underline{w}}$ depends on the expression $s_{i_1} \cdots s_{i_p}$, not just on $\underline{w} \in \mathfrak{S}_n$ Finally, we define the degree function on $\mathcal{H}_n^\sigma$ as follows.

Definition 1.7. We define the function $\deg\colon \mathcal{H}_n^\sigma \longrightarrow \mathbb{Z}$ determined by

$$\deg(e_{\underline{i}}) = 0 \quad \deg(y_r e_{\underline{i}}) = 2 \quad \deg(\psi_s e_{\underline{i}}) = \begin{cases} -2 & \text{if } i_s = i_{s+1} \\ 1 & \text{if } i_s = i_{s+1} \pm 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.21}$$

for $1 \leq r \leq n$ and $1 \leq s \leq n-1$. This is a degree function on $\mathcal{H}_n^\sigma$, hence the cyclotomic quiver Hecke algebra is a $\mathbb{Z}$-graded algebra with degree function deg.

One of the biggest advantages with KLR algebras is that we have a diagrammatic presentation and we can view the generators and the elements as planar diagrams of decorated strands. For a more detailed description of the diagrammatic presentation, the reader may refer to [LP] and [HMP18]. Each KLR diagram of the quiver Hecke algebra $\mathcal{H}_n$ consists of n strings and each string carries an integer $i \in \mathbb{Z}/e\mathbb{Z}$. The bottom and the top of the KLR diagram are sequences of integers. The product of two KLR diagrams is given by vertical concatenation. If $\underline{i} = (i_1, i_2, \cdots, i_d) \in (\mathbb{Z}/e\mathbb{Z})^n$ the correspondence between the generators of the KLR algebra and the diagrammatic presentation can be seen in Figure 1.1.

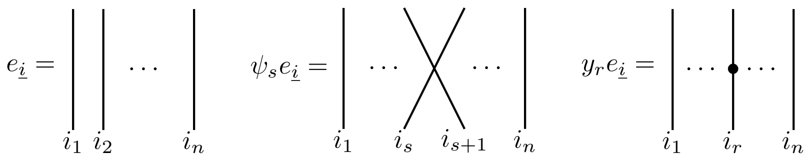


Figure 1.1: The correspondence between the algebraic and diagrammatic generators of the quiver Hecke algerba.

There are also some rules and conventions which must satisfied by the KLR diagrams. These conventions are listed in [HMP18] and we outline them at this point for the sake of completeness. In particular, in a KLR diagram:

- all intersections are transversal;

- there are no triple intersections;
- the strings can be decorated with a finite number of dots at non-intersection points.

As we see in Figure 1.1, the idempotent $e_{\underline{i}}$ labelled by the sequence $\underline{i} \in (\mathbb{Z}/e\mathbb{Z})^n$ is visualised as straight strands with each one carrying the integers $i_1, \cdots, i_n$. The ψ_s elements can be seen as a single crossing of the strands labelled by the integers $i_s, i_{s+1} \in \mathbb{Z}/e\mathbb{Z}$. We refer to them as KLR crossings or simply crossings. Finally, the y_r elements are visualised as dots on strands; we hence refer to them as KLR dots or simply dots.

In Figure 1.1 we can see a KLR diagram with KLR dots and crossings. Note that the element ψ_q^p is the element defined in (1.20) and the bottom of the diagram is labelled by $\underline{i} = (i_1, \cdots, i_n) \in (\mathbb{Z}/e\mathbb{Z})^n$.

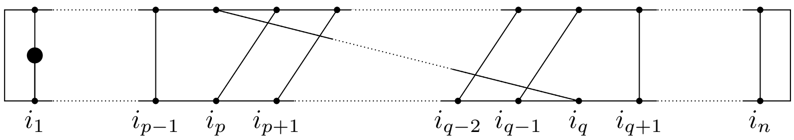


FIGURE 1.2: The element $y_1\psi_q^p e_{\underline{i}}$ for $1 < p < q \leq n$.

We mentioned above that the product of two or more KLR diagrams can be thought of as vertical concatenation. We denote the vertical concatenation by $\circ$ and in the following example we visualise vertical concatenation of diagrams.

Example 1.8. *Consider the elements $y_1\psi_4^2 e_{(0,1,2,3)}, y_1y_2\psi_2^4 e_{(0,3,1,2)} \in \mathcal{H}_4$. The KLR diagrams of these elements are depicted in the left-hand side of the equation of Figure 1.3. The vertical concatenation of the diagrams is depicted in the right-hand side of Figure 1.3. We also remark that the diagram on the right-hand side of Figure 1.3 can be simplified further, as it is not reduced. The simplifications can be performed by applying the appropriate relations of Definitions 1.5, 1.6.*

We denote by $\boxtimes$ the horizontal concatenation of both KLR diagrams and residue sequences. In the following example we describe a horizontal concatenation of

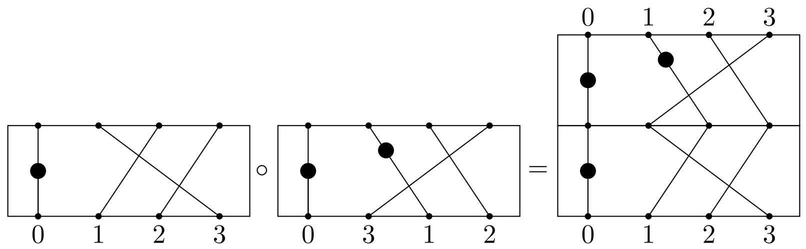


FIGURE 1.3: The vertical concatenation of the KLR diagrams of $y_1\psi_4^2 e_{(0,1,2,3)}, y_1y_2\psi_2^4 e_{(0,3,1,2)} \in \mathcal{H}_4$.

KLR diagrams and their residue sequences. Note that the result of the horizontal concatenation will be 0 unless the residues are compatible. To see that the reader can refer to the first relation of (R1).

Example 1.9. *Consider the elements* $y_1\psi_4^2 e_{(0,1,2,3)}$*,* $y_1\psi_5^2 e_{(0,1,2,3,4)}$ *and* $y_1\psi_2^4 e_{(0,1,2,3)}$*. The KLR diagrams of those elements are the ones displayed in Figure 1.4. The horizontal concatenation is the element of* $\mathcal{H}_{13}$ *illustrated in Figure 1.5.*

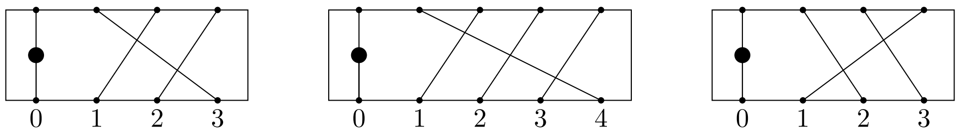


FIGURE 1.4: The elements $y_1\psi_4^2 e_{(0,1,2,3)}, y_1\psi_5^2 e_{(0,1,2,3,4)}, y_1\psi_2^4 e_{(0,1,2,3)}$.

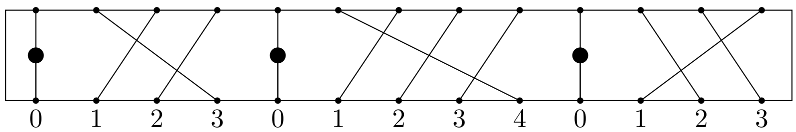


FIGURE 1.5: The element $y_1\psi_4^2 e_{(0,1,2,3)} \boxtimes y_1\psi_5^2 e_{(0,1,2,3,4)} \boxtimes y_1\psi_2^4 e_{(0,1,2,3)}$.

The concept of vertical concatenation of KLR diagrams will be widely utilised throughout the proofs of chapter 3, as it simplifies our calculations massively. Horizontal concatenation will be a useful tool throughout chapter 2 and it will provide a useful language which makes our notation simpler and easier for the reader.

1.3.3 Quasi-hereditary quotients of quiver Hecke algebras

The symmetric group lies at the intersection of two great categorical theories: Khovanov–Lauda and Rouquier's categorification of quantum groups and their knot invariants [KL09, Rou] and Elias–Williamson's diagrammatic categorification in terms of endomorphisms of Bott–Samelson bimodules. The objects of this section are the main objects of the two companion papers [BCHM20, BCH]. The purpose of those papers is to construct an explicit isomorphism between these two diagrammatic worlds. The backbone of this isomorphism is provided by the "light-leaves" bases of these algebras.

The light leaves bases of diagrammatic Bott–Samelson endomorphism algebras were crucial in the calculation of counterexamples to the expected bounds of Lusztig's and James' conjectures [Wil17]. These bases are structurally far richer than any known basis of the quiver Hecke algebra — they vary with respect to each possible choice of reduced word/path-vector in the alcove geometry — this richer structure is necessary in order to construct a basis in terms of the "Soergel 2-generators" of these algebras. We note that for these algebras, the path-theoretic light-leaves basis is out of the scope of this thesis and the interested reader may refer to [BCHM20, Section 2] for more details. In particular, we shall construct a "classical-type" cellular basis for specific quasi-hereditary quotients of the cyclotomic quiver Hecke algebra of Definition 1.6.

A long-standing belief in modular Lie theory is that we should (first) restrict our attention to fields whose characteristic, p, is greater than the Coxeter number, h, of the algebraic group we are studying. This allows one to consider a "regular" or "principal block" of the algebraic group in question. For $p > h + 1$ we consider the idempotent

$$\mathsf{e}_h = \sum_{\substack{i_{k+1}=i_k+1 \\ 1\leq k\leq h}} e_{(i_1,\dots,i_n)} \tag{1.22}$$

modulo "more dominant terms". We recall Definitions 1.5, 1.6 of quiver Hecke algebras $\mathcal{H}_n$ and their cyclotomic quotients $\mathcal{H}_n^\sigma$, for $\sigma \in \mathbb{Z}^\ell$. In order to define the quasi-hereditary quotients of interest, we need a few definitions along with an important long-standing convention.

Definition 1.10. Fix integers $h, \ell \in \mathbb{Z}_{>0}$ and $e \geq (h+1)\ell$. An ℓ-tuple $\sigma \in \mathbb{Z}^\ell$ such that $h < |\sigma_i - \sigma_j| < e - h$ for $0 \leq i \neq j \leq \ell - 1$, is called (h, e)-admissible charge.

Using the work and definitions from above, we define another algebra of interest.

Definition 1.11. For an (h, e)-admissible charge $\sigma \in \mathbb{Z}^\ell$, we define the following quotient of the cyclotomic quiver Hecke algebra $\mathscr{H}_n^\sigma := \mathcal{H}_n^\sigma / \mathcal{H}_n^\sigma \mathsf{e}_h \mathcal{H}_n^\sigma$.

For ease of notation, we assume that the (h, e)-admissible charge $\sigma \in \mathbb{Z}^\ell$ is increasing. This assumption does not restrict the definition of the algebra as it is independent of the ordering. It is only a convention which will make our combinatorics easier throughout the thesis.

Convention. Throughout this section and the second chapter of the thesis, when we refer to the *e-charge* or *charge* $\sigma \in \mathbb{Z}^\ell$ we will assume that it is (h, e)-admissible and increasing.

Constructing a cellular basis for the algebra of Definition 1.11 will be one of the main results of this thesis. More details on the motivation and the applications using such cellular basis can be found in the introduction of the related chapter.

1.3.4 Blob algebra

We will now discuss the "blob algebra", one of the algebras that can be regarded as a special case of the quasi-hereditary quotients of the Hecke algebra. Despite being just a special case of the above, the definition of the blob algebras was motivated by studies in physics and in particular in statistical mechanics.

Statistical mechanics aims to understand the large-scale observables (temperature, pressure) of physical systems in terms of microscopic fluctuations of the system:

water in a kettle, gas within a container, or the atomic structure of magnets. The Temperley–Lieb algebras of type A and B (the latter of which is also known as the "blob" algebra) first arose in the context of the 2-dimensional ferromagnetic Ising and Potts models. The Temperley–Lieb of type A played central role for Vaughan Jones in the discovery of new polynomial invariant of knots and links [Jon97]. Jones had an algebraic approach and in his work the Temperley–Lieb algebra was presented in terms of generators and relations. This presentation is rather restrictive and later Louis Kauffman [Kau90] gave a handy description as *planar diagram algebra*. The characters of simple modules of the Temperley–Lieb algebras of type A, over a field of arbitrary characteristic, were computed by de Boeck, Evseev, Lyle and Speyer in [dBELS18].

The Temperley–Lieb algebra of type B was defined by Martin and Saleur [MS94] as a two parameter generalisation of the Temperley–Lieb algebra of type A. In fact, they originally named this algebra "blob algebra" due to the fact that the planar diagrams generating it are the planar diagrams for the Temperley–Lieb algebra of type A decorated with blobs. The presentation of the blob algebra in terms of planar diagrams, is out of the scope of this thesis. The interested reader can refer to [MS94], where there is a recap on the diagrammatic presentation of the Temperley–Lieb algebra of type A and the "blob generalisation" with the aforementioned decorated diagrams.

One might raise the question, why the Temperley–Lieb algebras are characterised by their type. The answer is hidden in the connection of the Temperley–Lieb algebras of type A and B with the Hecke algebras of type A and B respectively. The isomorphism between the Hecke algebra of type B and the blob algebra was constructed by Martin and Woodcock in [MW03, Proposition 4.4]. The reader can find further information -closer to our notation- on the connection between these algebras in [PRH14].

We shall now give the exact definition of the blob algebra, as a quotient of the cyclotomic quiver Hecke algebra of level 2. Recall that level 2 implies that in

Definition 1.6, $\ell = 2$ and we our quotient depends on an e-bicharge $\sigma = (\sigma_0, \sigma_1) \in \mathbb{Z}^2$. Moreover let $\Bbbk$ be a field of any characteristic.

Definition 1.12. [PRH14, Corollary 3.6] Fix $e > 2$ and $\sigma = (\sigma_0, \sigma_1)$ an e-bicharge. The blob algebra B_n^σ is the $\Bbbk$-algebra with generators

$$\{e_{\underline{i}} \mid \underline{i} = (i_1, \cdots, i_n) \in (\mathbb{Z}/e\mathbb{Z})^n\} \cup \{y_1, \cdots, y_n\} \cup \{\psi_1, \cdots, \psi_{n-1}\}$$

subject to the relations of the cyclotomic quiver Hecke algebra of Definition 1.6, modulo the additional relation

$$e_{\underline{i}} = 0, \qquad \text{if } i_2 = i_1 + 1. \tag{1.23}$$

We refer to relation (1.23) as the *blob relation*.

Note that the original definition of the blob algebra was in terms of generators and relations similar to the ones of the Temperley–Lieb algebras. The presentation of Definition 1.12, which we will be using throughout the thesis, was proven to exist in the work of Plaza and Ryom–Hansen [PRH14].

Remark 1.13. In order to further connect the blob algebra with the quasi-hereditary quotients of the Hecke algebra, we shall re-write the blob relation (1.23) in terms of the idempotent (1.22). In particular, the blob relation gives a quotient by the idempotent

$$\mathsf{e}_1 = \sum_{i_2 = i_1 + 1} e_{(i_1, \cdots, i_n)}.$$

As a direct consequence of Definition 1.12, the blob algebra has the structure of a graded algebra. The result that the blob algebra admits a $\mathbb{Z}$-grading is due to Plaza and Ryom–Hansen [PRH14] and the definition if its degree function follows.

Definition 1.14. We define the function $\deg \colon \mathsf{B}_n^\sigma \longrightarrow \mathbb{Z}$ determined by

$$\deg(e_{\underline{i}}) = 0, \quad \deg(y_r e_{\underline{i}}) = 2, \quad \deg(\psi_s e_{\underline{i}}) = \begin{cases} -2 & \text{if } i_s = i_{s+1} \\ 1 & \text{if } i_s = i_{s+1} \pm 1 \\ 0 & \text{if } i_s \neq i_{s+1} \pm 1 \end{cases}$$

for $1 \leq r \leq n$ and $1 \leq s \leq n-1$. This is a degree function on B_n^σ, hence the blob algebra is a $\mathbb{Z}$-graded algebra with degree function deg.

Note that the equations which determine the degree function are those of relation (1.21) in Definition 1.7, i.e. the equations determining the degree function of the cyclotomic Hecke algebras. The grading of the blob algebra has opened new horizons in its study and some of the most recent results are due to this grading. Plaza in [Pla13] calculated the graded decomposition numbers of the blob algebra over a field of characteristic 0. Moreover, Hazi, Martin and Parker [HMP18] determined the structure of the indecomposable tilting modules of the blob algebra over $\mathbb{C}$, again by using the graded structure.

In the chapter dedicated to the blob algebra, we shall state some of the aforementioned results as they will be essential for our proofs and findings. Hence, more details on the structure and combinatorics of the blob algebra can be found in chapter 3. This section serves as an overview of the work that has been done on the blob algebra and is mainly used in order to fix the notation that we shall use later on in the thesis.

1.4 Combinatorics

In this section we present the basic combinatorial concepts which will be useful in formalising our ideas towards the study of the algebras defined in previous sections. Note that the main purpose of the first subsections is to give a general flavour of the combinatorics and also fix some notation. The particularities, assumptions and constrains of each combinatorial theory will be included in the last two subsections. In particular, we shall summarise the combinatorics appearing in the study of the quasi-hereditary quotients of the Hecke algebra and the combinatorics appearing in the blob algebras. The key idea in both combinatorial theories is that we restrict the number of columns and/or the number of components of the partitions and compositions.

1.4.1 Partitions and residues

We define a box-configuration to be a subset of

$$\{[i,j,m] \mid 0 \leq m < \ell, 1 \leq i,j \leq n\}$$

and we let $\mathcal{B}_\ell(n)$ denote the set of all box-configurations with n boxes. We refer to a box $[i,j,m] \in \mathcal{B}_\ell(n)$ as being in the ith row and jth column of the mth component of the configuration. Given a box, $[i,j,m]$, we define the content of this box to be $\mathrm{ct}[i,j,m] = \sigma_m + j - i$ and we define its residue to be $\mathrm{res}[i,j,m] = \mathrm{ct}[i,j,m] \pmod{e}$. We refer to a box of residue $r \in \mathbb{Z}/e\mathbb{Z}$ as an r-box. We define a composition, λ, of n to be a finite sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots)$ whose sum, $|\lambda| = \lambda_1 + \lambda_2 + \cdots$, equals n. We say that λ is a partition if, in addition, this sequence is weakly decreasing. An ℓ-composition (respectively ℓ-partition) $\lambda = (\lambda^{(0)}, \cdots, \lambda^{(\ell-1)})$ of n is an ℓ-tuple of compositions (respectively partitions) such that $|\lambda^{(0)}| + \cdots + |\lambda^{(\ell-1)}| = n$. We denote by $\mathcal{C}_\ell(n)$ and $\mathscr{P}_\ell(n)$ the set of ℓ-compositions and ℓ-partitions of n, respectively. Given $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(\ell-1)}) \in \mathscr{P}_\ell(n)$, the Young diagram of λ is defined to be the box configuration,

$$\{[i,j,m] \mid 1 \leq j \leq \lambda_i^{(m)}, 0 \leq m < \ell\}.$$

We do not distinguish between the ℓ-partition and its Young diagram.

Given $\lambda \in \mathscr{P}_\ell(n)$, we let $\mathrm{Rem}(\lambda)$ (respectively $\mathrm{Add}(\lambda)$) denote the set of all removable (respectively addable) boxes of the Young diagram of λ so that the resulting diagram is the Young diagram of an ℓ-partition. We let $\mathrm{Rem}_r(\lambda) \subseteq \mathrm{Rem}(\lambda)$ and $\mathrm{Add}_r(\lambda) \subseteq \mathrm{Add}(\lambda)$ denote the subsets of boxes of residue $r \in \mathbb{Z}/e\mathbb{Z}$.

Example 1.15. *Let $n = 25$, $\ell = 3$, $\sigma = (0,2,4) \in \mathbb{Z}^3$ and $e = 5$. Consider the box configurations $\lambda = ((4,3,1),(3,1^2),(4,3^2,2)) \in \mathscr{P}_3(25)$ and $\mu = ((2,3^3,2),(4^2,3,1)) \in \mathcal{C}_2(25)$. The box configurations corresponding to λ and μ*

are the following

$$\lambda = \left(\ydiagram{4,3,1}\,,\ \ydiagram{3,1,1}\,,\ \ydiagram{4,3,3,2} \right)$$

$$\mu = \left(\ydiagram{2,3,3,3,2}\,,\ \ydiagram{4,4,3,1} \right).$$

We notice that all three components of λ *are partitions, whereas the first component of* μ *is a composition. We consider the boxes* $[2,1,0]$, $[3,2,1]$ *and we calculate their content and residue. We have that*

$$\begin{aligned} \mathrm{ct}[2,1,0] &= 0+1-2=-1 \\ \mathrm{ct}[3,2,1] &= 2+2-3=4 \end{aligned}$$

and

$$\mathrm{res}[2,1,0] = \mathrm{res}[3,2,1] = 4$$

as we consider the contents modulo $e = 5$*. We focus on the* 3*-partition* λ *and one can easily observe that the residues of the boxes are the same in the diagonals of the box-configurations. We colour the boxes of the same residue with the same colour and then our partition looks as follows.*

$$\lambda = \left(\begin{ytableau} 0 & 1 & 2 & 3 \\ 4 & 0 & 1 \\ 3 \end{ytableau}\,,\ \begin{ytableau} 2 & 3 & 4 \\ 1 \\ 0 \end{ytableau}\,,\ \begin{ytableau} 4 & 0 & 1 & 2 \\ 3 & 4 & 0 \\ 2 & 3 & 4 \\ 1 & 2 \end{ytableau} \right)$$

The numbers in the boxes are the residues of each box.

1.4.2 Tableaux

The combinatorial objects arising naturally from partitions, compositions and their Young diagrams are tableaux. They have been in the centre of the representation theory of the symmetric groups and their related algebras from the early days of their study by James [Jam77]. Ever since tableaux (and indeed the special case of standard tableaux) are of extreme importance in combinatorial representation theory and they are used to encode useful information for the structure of the algebra.

Given $\lambda \in \mathscr{P}_\ell(n)$, we define a λ-tableau to be a filling of the boxes of $[\lambda]$ with the numbers $\{1, \cdots, n\}$ without repeated entries. A tableau is called row-standard if the entries increase along the rows in each component, and column-standard if the entries increase down the columns in each component. A tableau is called standard if it is both row and column standard. We denote the set of all tableau of shape λ by $\mathrm{Tab}(\lambda)$. We denote by $\mathrm{RStd}(\lambda)$, $\mathrm{CStd}(\lambda)$, $\mathrm{Std}(\lambda) \subset \mathrm{Tab}(\lambda)$ the subsets of row-standard, column-standard and standard tableau, respectively. Moreover, for $n \in \mathbb{Z}$ we define $\mathrm{Std}(n) = \bigcup_{\lambda \in \mathscr{P}_\ell(n)} \mathrm{Std}(\lambda)$.

One can view a λ-tableau as a bijection $\mathsf{t} \colon [\lambda] \longrightarrow \{1, \cdots, n\}$ and we say that the tableau t has shape λ and we write $\mathrm{Shape}(\mathsf{t}) = \lambda$. We denote by $\mathsf{t}^{-1}(k)$ the box occupied by the integer $k \in \{1, \cdots, n\}$ and by $\mathsf{t}[r, c, m]$ the integer occupying the box $[r, c, m] \in [\lambda]$. For a tableau t we write $\mathsf{t}{\downarrow}_{\leq k}$, $\mathsf{t}{\downarrow}_{\geq k}$ for the subtableaux of t containing the entries $\{1, \cdots, k\}$, $\{k, \cdots, n\}$, for $1 \leq k \leq n$ respectively. Sometimes, for ease of notation we shall denote the above subtableaux simply by $\mathsf{t}_{\leq k}$ and $\mathsf{t}_{\geq k}$.

Definition 1.16. Given two λ-tableaux $\mathsf{s}, \mathsf{t} \in \mathrm{Tab}(\lambda)$, we let $w^{\mathsf{s}}_{\mathsf{t}} \in \mathfrak{S}_n$ be the permutation such that $w^{\mathsf{s}}_{\mathsf{t}}(\mathsf{s}) = \mathsf{t}$.

Remark 1.17. The symmetric group $\mathfrak{S}_n$ acts in a natural way on the set of tableaux. In particular if t is a tableau and s_i is a simple transposition, the tableau $s_i\mathsf{t}$ obtained by interchanging the entries i, $i + 1$.

In the following example we shall see examples of standard, row-standard, column-standard and non-standard tableaux.

Example 1.18. *Let* $n = 14$, $e = 5$ *and* $\lambda = ((3, 2^2, 1), (2^2, 1^2)) \in \mathscr{P}_2(14)$. *The tableaux*

$$\mathsf{t}_\lambda = \left(\begin{ytableau} 1 & 2 & 3 \\ 6 & 7 \\ 10 & 11 \\ 13 \end{ytableau} \ , \ \begin{ytableau} 4 & 5 \\ 8 & 9 \\ 12 \\ 14 \end{ytableau} \right), \quad \mathsf{s} = \left(\begin{ytableau} 1 & 2 & 4 \\ 6 & 7 \\ 10 & 11 \\ 14 \end{ytableau} \ , \ \begin{ytableau} 3 & 5 \\ 8 & 12 \\ 9 \\ 13 \end{ytableau} \right)$$

are standard λ*-tableaux. On the other hand the* λ*-tableaux*

$$\mathsf{u}_1 = \left(\begin{ytableau} 1 & 2 & 3 \\ 6 & 7 \\ 11 & 10 \\ 13 \end{ytableau} \ , \ \begin{ytableau} 4 & 5 \\ 8 & 9 \\ 14 \\ 12 \end{ytableau} \right), \quad \mathsf{u}_2 = \left(\begin{ytableau} 1 & 2 & 3 \\ 6 & 7 \\ 10 & 11 \\ 13 \end{ytableau} \ , \ \begin{ytableau} 4 & 5 \\ 8 & 9 \\ 12 \\ 14 \end{ytableau} \right), \quad \mathsf{u}_3 = \left(\begin{ytableau} 1 & 2 & 3 \\ 6 & 11 \\ 10 & 7 \\ 13 \end{ytableau} \ , \ \begin{ytableau} 4 & 5 \\ 8 & 9 \\ 12 \\ 14 \end{ytableau} \right)$$

are non-standard. In particular, u_1 *is column-standard but not row-standard, while* u_2 *is row-standard but not column-standard and* u_3 *is neither row nor column standard.*

Remark 1.19. We remark that tableau t_λ of the Example 1.18 is a special tableau which will be of huge importance later in the thesis. Sometimes it can be found in the literature as *superstandard* or *initial* tableau.

Example 1.20. *We continue from Example 1.18 and we have that* $w_{\mathsf{s}}^{\mathsf{t}_\lambda} = (3,4)(9,12)(13,14) \in \mathfrak{S}_{14}$ *is the permutation defined above. We can give the corresponding permutation for any pair of tableaux, but the permutations related to the tableau* t_λ *are of particular interest, since they help to index specific elements which form bases of our algebras.*

Definition 1.21. Let $\lambda \in \mathscr{P}_\ell(n)$ and t be a λ-tableau. We define the $\mathsf{residue\ sequence}$ of t to be the n-tuple:

$$\mathrm{res}(\mathsf{t}) := (\mathrm{res}(\mathsf{t}^{-1}(1)), \cdots, \mathrm{res}(\mathsf{t}^{-1}(n))) \in I^n.$$

Moreover, we set $e_\mathsf{t} := e_{\mathrm{res}(\mathsf{t})} \in \mathscr{H}_n^\sigma$.

We shall now define the degree of a tableau in the usual fashion, in terms of the number of addable and removable nodes of a certain residue. For this we will first need to define an order relation on the set of boxes.

Definition 1.22. We define the reverse lexicographic order on boxes as follows. Let $1 \leq i, i', j, j' \leq n$ and $1 \leq m, m' \leq \ell$. We write $[i, j, m] \succ [i', j', m']$ if

(i) $i < i'$, or

(ii) $i = i'$ and $m < m'$, or

(iii) $i = i'$ and $m = m'$ and $j < j'$.

For a λ-tableau t we denote by $\mathsf{Add}_{\mathsf{t}}(k)$ and $\mathsf{Rem}_{\mathsf{t}}(k)$ the following sets:

$$\mathsf{Add}_{\mathsf{t}}(k) := \{A \in \mathsf{Add}(\mathrm{Shape}(\mathsf{t}_{\leq k})) \mid A \preceq \mathsf{t}^{-1}(k), \mathrm{res}(A) = \mathrm{res}(\mathsf{t}^{-1}(k))\} \qquad (1.24)$$

and

$$\mathsf{Rem}_{\mathsf{t}}(k) := \{A \in \mathsf{Rem}(\mathrm{Shape}(\mathsf{t}_{\leq k})) \mid A \preceq \mathsf{t}^{-1}(k), \mathrm{res}(A) = \mathrm{res}(\mathsf{t}^{-1}(k))\} \qquad (1.25)$$

for all $1 \leq k \leq n$. By using (1.24), (1.25) we define the degree of the tableau t.

Definition 1.23. Let $\mathsf{t} \in \mathrm{Std}(n)$ be a standard tableau. We define the degree of the node $\mathsf{t}^{-1}(k)$ to be

$$\deg(\mathsf{t}^{-1}(k)) := |\mathsf{Add}_{\mathsf{t}}(k)| - |\mathsf{Rem}_{\mathsf{t}}(k)|.$$

The degree of the tableau t is the sum of the degrees of its nodes, namely

$$\deg(\mathsf{t}) = \sum_{k=1}^{n} \deg(\mathsf{t}^{-1}(k)).$$

1.4.3 Combinatorics of quasi-hereditary quotients

We fix two positive integers $h, \ell \in \mathbb{Z}_{>0}$, $e \geq (h+1)\ell$ and let $\sigma = (\sigma_0, \cdots, \sigma_{\ell-1}) \in \mathbb{Z}^{\ell}$ be a (h, e)-admissible charge. The order relations we shall use later in the thesis, arise from the aforementioned charge. Note that these charges are of different flavour compared to the ones used by Dipper–James–Mathas [DJM98]. Recall from previous subsections that we denote by $\mathscr{P}_\ell(n)$ the set of ℓ-partitions of n. We denote by $\mathscr{P}_{h,\ell}(n) \subset \mathscr{P}_\ell(n)$ the subset of ℓ-partitions of n with *at most* h-columns in each component. By using the usual conventions, we shall denote by $\mathrm{Tab}_h(n) \subset \mathrm{Tab}(n)$ the set of tableaux of shape with at most h columns in each component. For $\lambda \in \mathrm{Tab}_h(n)$, we denote by $\mathrm{Std}_h(\lambda)$, $\mathrm{RStd}_h(\lambda)$ and $\mathrm{CStd}_h(\lambda)$ the set of standard, row-standard and column-standard tableaux of shape with at most h columns in each component. Moreover, we set by $\mathrm{Std}_h(n) := \bigcup_{\lambda \in \mathscr{P}_{h,\ell}(n)} \mathrm{Std}_h(\lambda)$. Note that when the subscript h is clear from the context we will omit it.

At this point we shall introduce the order relation, which will be used in the construction of a filtration for the quasi-hereditary quotients of the cyclotomic Hecke algebras. Recall that we have already defined the reverse lexicographic order on the set of boxes/nodes (see Definition 1.22) and we, naturally, define the reverse lexicographic order on the set of box-configurations.

Definition 1.24. We define the reverse lexicographic order on $\mathcal{B}_\ell(n)$ as follows. Given $\lambda, \mu \in \mathcal{B}_\ell(n)$, $\lambda \neq \mu$, we write $\lambda \succ \mu$ if the lexicographically minimal box $\square \in (\lambda \cup \mu) \setminus (\lambda \cap \mu)$ belongs to μ.

In the following example, we see how the reverse lexicographic order can be viewed in terms of the lexicographic order of the symmetric group, as the reader might be more familiar with that one.

Example 1.25. *For the symmetric group, the reverse lexicographic ordering is equal to the transpose of the usual lexicographic ordering. In other words $\lambda \succ \mu$ if there exists some $t \geq 1$ such that*

$$\sum_{1 \leq i \leq t} \lambda_i^T < \sum_{1 \leq i \leq t} \mu_i^T \quad \textit{and} \quad \sum_{1 \leq i \leq k} \lambda_i^T = \sum_{1 \leq i \leq k} \mu_i^T$$

for all $1 \leq k \leq t$ where T denotes the transpose partition. More generally, $\succ$ is a total refinement of the so-called "FLOTW" dominance order on $\mathscr{P}_{h,\ell}(n)$ in [Bow, BC18, LP].

Definition 1.26. Let $\lambda \in \mathscr{P}_\ell(n)$ be an ℓ-partition. We define the standard tableau $\mathsf{t}_\lambda \in \mathrm{Std}(\lambda)$ to be the tableaux in which we place the entry n in the minimal $\succ$-node of λ, then continue in this fashion inductively.

Example 1.27. *Let* $\lambda = (3, 2^2, 1^6) \in \mathscr{P}_{3,1}(13)$ *and* $\mu = (3, 2, 1^7) \cup \{[2, 6, 1]\} \in \mathscr{B}_1(13)$*. Note that the node* $[3, 2, 0]$ *is least in the lexicographic order and* $[3, 2, 0] \notin \lambda \cap \mu$ *and furthermore we have that* $[3, 2, 0] \in \lambda$*. Hence, according to the definition of the reverse lexicographic order on box-configurations, we have that* $\mu \succ \lambda$*. The box configurations of the example can be seen in Figure 1.6.*

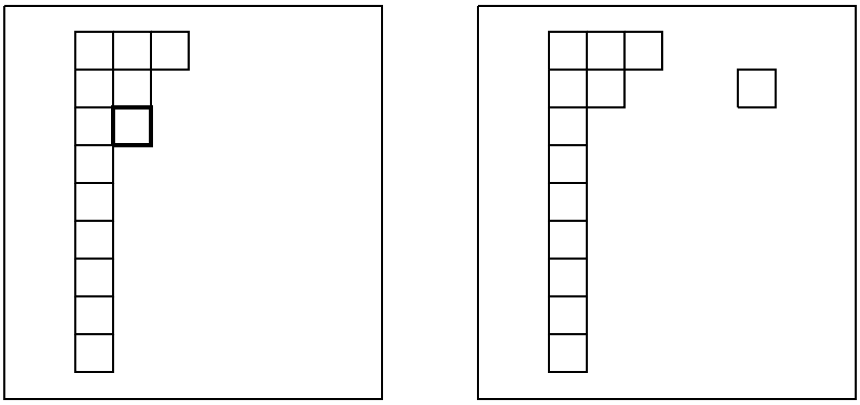


FIGURE 1.6: The left-hand side box-configuration (partition) is $\lambda = (3, 2^2, 1^6)$ and the right-hand side box-configuration is $\mu = (3, 2, 1^7) \cup [2, 6, 1]$. The node $[3, 2, 0] \in \lambda$, which results to $\mu \succ \lambda$, is highlighted in the first box-configuration.

Example 1.28. *Let* $n = 14$*,* $\ell = 2$*,* $h = 3$ *and* $\lambda = ((3, 2^2, 1), (2^2, 1^2)) \in \mathscr{P}_{3,2}(14)$*. Note that the parameters in this example are similar to those in Example 1.18. However, in order to be consistent with the assumptions and conventions of this section we have picked the integer* $h = 3$ *and now* $e = 9$*, so that* $e \geq (h + 1)\ell$*. Below we have two examples of standard tableaux of shape* λ*. Note that the tableau* t_λ *is the tableau of Definition 1.26.*

$$\mathsf{t}_\lambda = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 6 & 7 \\ \cline{1-2} 10 & 11 \\ \cline{1-2} 13 \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|} \hline 4 & 5 \\ \hline 8 & 9 \\ \hline 12 \\ \cline{1-1} 14 \\ \cline{1-1} \end{array} \right), \quad \mathsf{s} = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 6 & 7 \\ \cline{1-2} 10 & 11 \\ \cline{1-2} 14 \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|} \hline 3 & 5 \\ \hline 8 & 12 \\ \hline 9 \\ \cline{1-1} 13 \\ \cline{1-1} \end{array} \right)$$

We shall now give the definition of the Garnir tableaux associated to a node of a ℓ-partition of the set $\mathscr{P}_{h,\ell}(n)$. We note that the definition can be extended to any partition of the set $\mathscr{P}_\ell(n)$, but we choose to present it here as it will only be used in the combinatorics of the principal block of the Hecke algebra. For the study of the blob algebra we shall introduce Garnir tableaux of different combinatorial flavour, which will be defined later in the thesis.

Definition 1.29. Given any node $A = [r, c, m] \in \lambda$ with $r \neq 1$, we define the associated Garnir belt B^A to be the collection of boxes

$$\{[r, j, k] \mid j \geq 1, 1 \leq k < m\} \cup \{[r, j, m] \mid 1 \leq j \leq c\} \cup$$

$$\cup \{[r-1, j, m] \mid c \leq j\} \cup \{[r-1, j, k] \mid j \geq 1, k > m\}.$$

Remark 1.30. We emphasise that, by definition, we don't obtain Garnir belts for the boxes in the first row of each component. This happens because we use Garnir belts because of our need to have a useful language for describing the movement of a box in positions higher in the $\succ$-order.

Example 1.31. *Let $n = 35$, $\sigma = (0, 5, 10) \in \mathbb{Z}^3$ and $e = 16$. We consider the 3-partition $\lambda = ((3^2, 2^2), (4^2, 3, 2), (4^2, 3, 1)) \in \mathscr{P}_{4,3}(35)$ and the node $[3, 3, 1]$. The associated Garnir belt is given by*

$$\lambda = \left(\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 15 & 0 & 1 \\ \hline 14 & 15 \\ \cline{1-2} 13 & 14 \\ \cline{1-2} \end{array} \;,\; \begin{array}{|c|c|c|c|} \hline 5 & 6 & 7 & 8 \\ \hline 4 & 5 & 6 & 7 \\ \hline 3 & 4 & 5 \\ \cline{1-3} 2 & 3 \\ \cline{1-2} \end{array} \;,\; \begin{array}{|c|c|c|c|} \hline 10 & 11 & 12 & 13 \\ \hline 9 & 10 & 11 & 12 \\ \hline 8 & 9 & 10 \\ \cline{1-3} 7 \\ \cline{1-1} \end{array} \right)$$

where here we have coloured the Garnir belt in yellow. We note that this is of a different combinatorial flavour to the Garnir belts of [Mat99] as we are working with a different weighting on our Hecke algebra, or equivalently a "twisted" Fock-Uglov-space ordering, or equivalently a Cherednik algebra which is not Morita equivalent to the cyclotomic q-Schur algebra. See [BC18, LP, LPRH] for more details.

We also note that the Garnir belt is independent of the parameters e, σ, however in this example we use them since we want to show how the residue pattern works within the Garnir belt.

Remark 1.32. Note that the Garnir belt is independent of e, σ. The only reason we mention them in Example 1.31 is because we want to provide another example of how the residues look like. In particular, we want the reader to observe that within the Garnir belt, each residue appears with multiplicity at most 1.

1.4.4 Blob-type combinatorics

For the study of the blob algebra we use combinatorics that can be viewed as special cases of the combinatorics described earlier in this chapter. For a positive integer $n \in \mathbb{Z}_{>0}$ we shall work with ℓ-partitions of n consisting of two components, with at most one column in each component. We refer to those partitions as one-column bipartitions or simply bipartitions of n and we denote the set consisting of those bipartitions by $\mathrm{Bip}_1(n)$. Note that such bipartitions are of the form $\lambda = ((1^{\lambda_1}), (1^{\lambda_2}))$ while the nodes of such bipartitions are of the form $[i, 1, m]$

Remark 1.33. In order to become clear that the combinatorial objects (partitions, tableaux) that we use in the case of the blob algebra are special cases of the combinatorics defined previously, we remark that the set $\mathrm{Bip}_1(n)$ is another notation for the set $\mathscr{P}_{1,2}(n)$. We keep the former because we choose to be aligned with the notation in [Mic21].

Remark 1.34. Consider the set $\Lambda_n = \{-n, -n+2, \cdots, n-2, n\}$. There is an obvious bijective map between Λ_n and the set $\mathrm{Bip}_1(n)$ of bipartitions of n, given by

$$\mathrm{Bip}_1(n) \longrightarrow \Lambda_n, \ \ ((1^{\lambda_1}), (1^{\lambda_2})) \longmapsto \lambda_1 - \lambda_2.$$

In other words we can identify each bipartition with an integer in the set Λ_n. Using the above bijection we freely identify a bipartition $((1^{\lambda_1}), (1^{\lambda_2}))$ and the integer $\lambda_1 - \lambda_2$.

The following remark will adjust the reverse lexicographic order on box-configurations (see Definition 1.24) in the context of one-column bipartitions. This will provide an easy criterion about determining the order between one-column bipartitions. Based on that adjustment, we shall define an order on the set of standard tableaux which are of shape $\lambda \in \mathrm{Bip}_1(n)$.

Definition 1.35. Let $\lambda = ((1^{\lambda_1}), (1^{\lambda_2})), \mu = ((1^{\mu_1}), (1^{\mu_2})) \in \mathrm{Bip}_1(n)$ with $\lambda \neq \mu$ be two bipartitions of n. Then

$$\lambda \succ \mu \text{ if and only if } |\lambda_1 - \lambda_2| < |\mu_1 - \mu_2|.$$

If $|\lambda_1 - \lambda_2| = |\mu_1 - \mu_2|$ and $\lambda \neq \mu$ then λ and μ are incomparable. We say that λ is more dominant than μ and we write $\lambda \trianglerighteq \mu$, if additionally λ, μ have the same multiset of residues.

Remark 1.36. We note that in Definition 1.35 the only case that the equality in the order relation holds is when we compare a bipartition with itself. This is the only difference of the above partial order with the order of Definition 1.24. For consistency, we shall keep the same notation as it will be clear from the context when we refer to the blob algebra.

Definition 1.37. Let $\mathsf{t}, \mathsf{s} \in \mathrm{Std}(\lambda)$. We write $\mathsf{t} \preceq \mathsf{s}$ if and only if

$$\mathrm{Shape}(\mathsf{t}_{\leq k}) \preceq \mathrm{Shape}(\mathsf{s}_{\leq k})$$

for $1 \leq k \leq n$. In addition if $\mathrm{res}(\mathsf{t}) = \mathrm{res}(\mathsf{s})$, we write $\mathsf{t} \trianglelefteq \mathsf{s}$ and we say that t is less dominant than s.

Note that if $\mathsf{t} \in \mathrm{Std}(n)$ is a standard tableau, we denote by t^t the transpose of t. For ease of notation we will often use the transpose tableau.

Example 1.38. *Let $\lambda = ((1), (1^9)) \in \mathrm{Bip}_1(10)$, $\sigma = (0, 2)$ and $e = 4$ and we consider the standard λ-tableaux*

$$\mathsf{t}^t = \left(\boxed{10}\,, \ \boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{7}\boxed{8}\boxed{9} \right)$$

and

$$\mathsf{s}^t = \left(\boxed{8}, \ \boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{7}\boxed{9}\boxed{10} \right).$$

We remark that $\mathrm{res}(\mathsf{t}) \neq \mathrm{res}(\mathsf{s})$ *and* $\mathrm{Shape}(\mathsf{t}_{\leq k}) = \mathrm{Shape}(\mathsf{s}_{\leq k})$, *for* $1 \leq k \leq 7$. *However we have that*

$$\mathrm{Shape}(\mathsf{t}_{\leq k}) \preceq \mathrm{Shape}(\mathsf{s}_{\leq k})$$

for $8 \leq k \leq 10$, *hence* t *precedes* s *in the order of Definition 1.37 and we write* $\mathsf{t} \prec \mathsf{s}$. *Now we consider the standard* λ*-tableau*

$$\mathsf{u}^t = \left(\boxed{7},\ \boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{8}\boxed{9}\boxed{10}\right)$$

and we note that $\mathrm{res}(\mathsf{u}) = \mathrm{res}(\mathsf{s})$. *Since* $\mathrm{Shape}(\mathsf{s}_{\leq k}) = \mathrm{Shape}(\mathsf{u}_{\leq k})$, *for* $1 \leq k \leq 6$, *and*

$$\mathrm{Shape}(\mathsf{s}_{\leq k}) \prec \mathrm{Shape}(\mathsf{u}_{\leq k})$$

for $7 \leq k \leq 10$, *we have that* $\mathsf{s} \trianglelefteq \mathsf{u}$, *i.e* s *is less dominant than* u.

Definition 1.39. [Pla13, Section 3] Let $\lambda = ((1^{\lambda_1}), (1^{\lambda_2})) \in \mathrm{Bip}_1(n)$ and $m = \min\{\lambda_1, \lambda_2\}$. We define the initial tableau $\mathsf{t}^\lambda \in \mathrm{Std}(\lambda)$ to be the tableau obtained by filling the nodes increasingly down to columns as follows:

1. even numbers less than or equal to $2m$ in the first component,
2. odd numbers less than $2m$ in the second component,
3. numbers greater than $2m$ in the remaining nodes.

For a given bipartition $\lambda = ((1^{\lambda_1}), (1^{\lambda_2})) \in \mathrm{Bip}_1(n)$ the standard tableau $\mathsf{t}^\lambda \in \mathrm{Std}(\lambda)$ is maximal under the order of Definition 1.37. Note that the tableau t_λ of Definition 1.26 is also maximal but for historic reasons and in order to be aligned with [Mic21], in the context of the blob algebra we shall use t^λ. In order to simplify the notation, in later sections we shall write $\underline{i}^\lambda = (i_1^\lambda, \cdots, i_n^\lambda) \in I^n$ instead of $\mathrm{res}(\mathsf{t}^\lambda)$ for the residue sequence of the tableau t^λ.

Remark 1.40. For any λ-tableau t we define $w_\mathsf{t} := w_\mathsf{t}^{\mathsf{t}^\lambda} \in \mathfrak{S}_n$, in the sense of Definition 1.16. We refer to $w_\mathsf{t} = s_{i_1} \cdots s_{i_k}$, where $s_{i_j}.1 \leq j \leq k$ are simple transpositions, as the reduced expression of t.

The following remark is of particular importance, as many of the results of the blob algebra are based on this fact.

Remark 1.41. Let $\lambda \in \mathrm{Bip}_1(n)$ be a bipartition of n and $r, r+1, r+2$, $1 \leq r \leq n-2$, be three successive positive integers. There are eight different cases for a standard λ-tableau and four of them are depicted in Figure 1.7 and we denote them (T1)-(T4) respectively. The rest four standard tableaux are the ones obtained by interchanging the numbers between the components and we denote them (T1') - (T4'). For instance the tableau (T1') is the tableau with $r+2$ in the first component and $r, r+1$ in the second component.

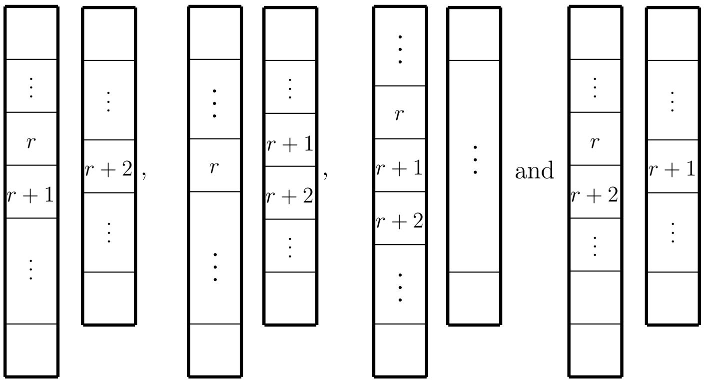


FIGURE 1.7: We depict the four out of eight different cases of standard tableaux for three successive integers $r, r+1, r+2$ and we denote them (T1)-(T4).

The subword $s_r s_{r+1} s_r$ cannot appear in the reduced expression of any of the above tableaux, as if we apply it to any standard tableau we get a non-standard tableau. In particular if $\mathsf{t} \in \mathrm{Std}(\lambda)$ is the initial tableau, the non-standard tableau would be the one coming from the interchange of the nodes occupied by the entries $r, r+2$, which can be denoted by $\mathsf{t}_{r \leftrightarrow r+2}$. Hence the reduced expression for each tableau is unique up to the commuting relations of the symmetric group.

Definition 1.42. Let $n, n' \in \mathbb{Z}$ be two positive integers with $n' < n$. If $\lambda \in \mathrm{Bip}_1(n)$ and $\nu \in \mathrm{Bip}_1(n')$ with $[\nu] \subset [\lambda]$, we define the skew bipartition $\lambda \setminus \nu$ to be the bipartition with diagram the set difference $[\lambda] - [\nu]$.

Definition 1.43. Let $n, n' \in \mathbb{Z}$ with $n' < n$, $\lambda \in \mathrm{Bip}_1(n)$, $\nu \in \mathrm{Bip}_1(n')$ and let $\lambda \setminus \nu$ be the skew bipartition. If $\mathsf{t} \in \mathrm{Std}(\nu)$ and $\mathsf{s} \in \mathrm{Std}(\lambda \setminus \nu)$ then the λ-tableau with entries $\{1, 2, \cdots, n\}$ in the nodes

$$(\mathsf{t}^{-1}(1), \cdots, \mathsf{t}^{-1}(n'), \mathsf{s}^{-1}(1), \cdots, \mathsf{s}^{-1}(n-n'))$$

respectively, is the composition $\mathsf{t} \circ \mathsf{s} \in \mathrm{Std}(\lambda)$ of the tableaux t and s.

The following example will clarify the notions of skew bipartitions and composition of tableaux.

Example 1.44. *Let $n = 10$ and $n' = 4$ and consider the bipartitions $\lambda = ((1^3), (1^7)) \in \mathrm{Bip}_1(10)$, $\nu = ((1), (1^3)) \in \mathrm{Bip}_1(4)$. The skew bipartition $\lambda \setminus \nu$ is the box-configuration consisting of the following nodes*

$$[\lambda \setminus \nu] = \{[2, 1, 0], [3, 1, 0], [4, 1, 1], [5, 1, 1], [6, 1, 1], [7, 1, 1]\}.$$

Now, we consider the tableau

$$\mathsf{t} = \left(\begin{array}{|c|}\hline 3 \\ \hline \end{array}, \ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} \right)$$

in $\mathrm{Std}(\lambda)$ and let $\mathsf{s} \in \mathrm{Std}(\lambda \setminus \nu)$ be the tableau with

$$\begin{array}{ll} \mathsf{s}[2,1,0] = 2 & \mathsf{s}[5,1,1] = 4 \\ \mathsf{s}[3,1,0] = 3 & \mathsf{s}[6,1,1] = 5 \\ \mathsf{s}[4,1,1] = 1 & \mathsf{s}[7,1,1] = 6. \end{array}$$

The composition $\mathsf{t} \circ \mathsf{s} \in \mathrm{Std}(\lambda)$ is the standard tableau

$$\mathsf{t} \circ \mathsf{s} = \left(\begin{array}{|c|}\hline 3 \\ \hline 6 \\ \hline 7 \\ \hline \end{array}, \ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline 8 \\ \hline 9 \\ \hline 10 \\ \hline \end{array} \right).$$

We shall now define the concept of Garnir tableaux in the setting of one-column bipartitions. As per usual the concept of defining Garnir tableaux starts by defining Garnir nodes and Garnir belts. Note that previously in this chapter we have spoken about Garnir belts in the context of quasi-hereditary quotients of the quiver Hecke algebra. In the context of the blob algebra there will be slight differences.

Recall that in Definition 1.29, the only restriction on the node $[r, c, m]$ is that $r \neq 1$. In the context of one-column bipartitions we have the following definition of a *Garnir node*. In turn this gives us the definition of a *Garnir belt* associated to a Garnir node.

Definition 1.45. Let $\lambda \in \mathrm{Bip}_1(n)$ and $A = [r, 1, m] \in [\lambda]$ be a node of the diagram of λ. The node A is called Garnir node if it is not removable.

Suppose that $A = [r, 1, m] \in [\lambda]$ is a Garnir node and let $u := \mathsf{t}^\lambda[r, 1, m]$ and $v := \mathsf{t}^\lambda[r+1, 1, m]$. It is clear from the definition of t^λ that- for $[r, 1, m]$ be Garnir node- there are two distinct cases for u and v. In particular it will either be $v = u + 1$ or $v = u + 2$ and recall that $[u, v] := \{t \in \mathbb{Z} \mid u \leq t \leq v\}$. The Garnir belt B^A is a set consisting of the nodes $(\mathsf{t}^\lambda)^{-1}(k)$ for $k \in [u, v]$.

Remark 1.46. We remark that the Garnir belts for the blob algebra are essentially the same as in Definition 1.29 up to reindexing.

Definition 1.47. For a Garnir node $A = [r, 1, m] \in [\lambda]$ we define a Garnir tableau G^A associated to A to be the λ-tableau which:

- coincides with t^λ outside the Garnir belt B^A;
- has the numbers of the set $[u, v]$ in the remaining nodes according to the following rules.
    1. If $v = u + 1$, then G^A has the numbers $u, u + 1$ from the bottom to the top in the mth column;
    2. if $v = u + 2$, then G^A has the entries $u, u + 1, u + 2$ from the bottom to the top in both components, first by filling one of the components and then by filling the other.

Note that we define *a* Garnir tableau associated to A rather that *the* Garnir tableau, as the above definition does not always give a unique tableau. In the following remark we will clarify this point and we shall give a more concrete description of the Garnir tableaux.

Remark 1.48. Let $A = [r, 1, m]$ be a Garnir node and B^A be the Garnir belt associated to A. When $v = u + 1$ there is a unique Garnir tableau G^A, since there is a unique way of placing the numbers $u, u + 1$. In particular, the tableau G^A is the tableau

$$\mathsf{G}^A = s_u \mathsf{t}^\lambda. \tag{1.26}$$

When $v = u + 2$ there are two choices of Garnir tableaux. The first choice comes from filling the first component first and then the second component, while the second choice comes from filling the second component first and then the first component. In particular, the two different Garnir tableaux are

$$\mathsf{G}^A = \begin{cases} s_u s_{u+1} \mathsf{t}^\lambda \\ s_{u+1} s_u \mathsf{t}^\lambda. \end{cases} \tag{1.27}$$

The following example aims to clear the concept of the Garnir tableaux discussed in Definition 1.47 and Remark 1.48.

Example 1.49. *Let $n = 12$, $\sigma = (0, 2)$ and $e = 4$. We consider the bipartition $\lambda = ((1^4), (1^8)) \in \mathrm{Bip}_1(12)$ and the nodes $A = [2, 1, 0]$, $B = [6, 1, 1] \in [\lambda]$ which are Garnir nodes (i.e removable). The Garnir tableaux associated to A and B are the following non-standard tableaux*

$$\mathsf{G}_1^A = \left(\begin{array}{|c|}\hline 2\\\hline 5\\\hline 4\\\hline 8\\\hline \end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline 6\\\hline 7\\\hline 9\\\hline 10\\\hline 11\\\hline 12\\\hline \end{array} \right), \mathsf{G}_2^A = \left(\begin{array}{|c|}\hline 2\\\hline 6\\\hline 5\\\hline 8\\\hline \end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline 7\\\hline 9\\\hline 10\\\hline 11\\\hline 12\\\hline \end{array} \right) \textit{ and } \mathsf{G}^B = \left(\begin{array}{|c|}\hline 2\\\hline 4\\\hline 6\\\hline 8\\\hline \end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline 5\\\hline 7\\\hline 9\\\hline 11\\\hline 10\\\hline 12\\\hline \end{array} \right)$$

where the nodes shaded in blue are the Garnir belts of each Garnir tableau. As expected, there are two distinct Garnir tableaux associated to the node A and one unique Garnir tableau associated to B. One can easily check that $\mathsf{G}_1^A = s_4 s_5 \mathsf{t}^\lambda$, $\mathsf{G}_2^A = s_5 s_4 \mathsf{t}^\lambda$ and $\mathsf{G}^B = s_{10}\mathsf{t}^\lambda$, as described in Remark 1.48.

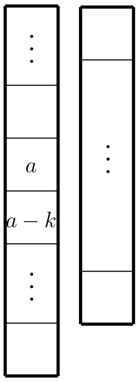
In order to make the notation simpler we introduce the notion of the left and right exposed transposition. Let $\mathsf{t} \in \mathrm{Std}(n)$ with reduced expression $w_{\mathsf{t}} = s_{i_1} \cdots s_{i_k}$. A simple transposition s_r is called left exposed (resp. right exposed) if $s_r = s_{i_j}$ for some $j \in \{1, \cdots, k\}$ and s_r commutes with s_{i_l} for all $l < j$ (resp. $l > j$). We also define the length function of the symmetric group $\mathfrak{S}_n$.

Definition 1.50. We define the length $L(\sigma)$ of a permutation $\sigma \in \mathfrak{S}_n$ to be the least number of simple permutations needed to form σ.

Lemma 1.51. *Let $\lambda \in \mathrm{Bip}_1(n)$ and $\mathsf{t} \notin \mathrm{Std}(\lambda)$ is a non-standard tableau of shape λ. Suppose that $A = [r, 1, m] \in \lambda$ is a node such that $\mathsf{t}(r, 1, m) > \mathsf{t}(r + 1, 1, m)$. Then there exists $w \in \mathfrak{S}_n$ such that $\mathsf{t} = w\mathsf{G}^{A'}$ for some Garnir node $A' \in \lambda$ and some Garnir tableau $\mathsf{G}^{A'}$ and $\mathsf{L}(w_{\mathsf{t}}) = \mathsf{L}(w) + \mathsf{L}(w_{\mathsf{G}^{A'}})$. Conversely, if $\mathsf{t} = w\mathsf{G}^{A'}$ with $\mathsf{L}(w_{\mathsf{t}}) = \mathsf{L}(w) + \mathsf{L}(w_{\mathsf{G}^{A'}})$ then $\mathsf{t} \notin \mathrm{Std}(\lambda)$.*

Proof. Let $u := \mathsf{t}^\lambda[r, 1, m]$, $v := \mathsf{t}^\lambda[r+1, 1, m]$, $a := \mathsf{t}[r, 1, m]$ and $b := \mathsf{t}[r+1, 1, m]$. First we consider the case that $v = u+1$. From our discussion in Remark 1.48 we have that $\mathsf{G}^A = s_u \mathsf{t}^\lambda$ and without loss of generality we may assume that $[r, 1, m]$ is the node with $\mathsf{t}[r, 1, m] > \mathsf{t}[r + 1, 1, m]$. If $\mathsf{t} = \mathsf{G}^A$ we have nothing to prove, so let $\mathsf{t} \neq \mathsf{G}^A$. If $a = v$ and $b = u$ the result is straightforward. Assume that $a \neq v$ and $b = a - 1$ and let $\mathsf{s} = s_{a-1}\mathsf{t}$, that is $\mathsf{t} = s_{a-1}\mathsf{s}$. Then the word $s_{a-1}s_a s_{a-1}$ appears as subword of w_{t} and by successively applying the braid Coxeter relations we end up with a subword of the form $s_u s_{u-1} s_u$ with s_u being right exposed. Note that if

$b \neq a-1$ then t will be of the form



for some $2 \leq k \leq a-1$ and we simply have the subword $s_{a-k} \cdots s_{a-2}$ on the left of $s_u s_{u-1} s_u$.

In any case we have that

$$\mathsf{t} = w' s_u s_{u-1} s_u \mathsf{t}^\lambda = w' s_u s_{u-1} \mathsf{G}^A \tag{1.28}$$

for some permutation $w' \in \mathfrak{S}_d$ and we have factorised the non-standard tableau t through the unique Garnir tableau associated to A.

Now we consider the case that $v = u+2$ and recall that the Garnir tableau associated to A are the tableaux $\mathsf{G}_1^A := s_u s_{u+1} \mathsf{t}^\lambda$ and $\mathsf{G}_2^A := s_{u+1} s_u \mathsf{t}^\lambda$. Same as in the case $v = u+1$, if $\mathsf{t} = \mathsf{G}_1^A$ or $\mathsf{t} = \mathsf{G}_2^A$ the result is straightforward. Hence we may assume that $\mathsf{t} \neq \mathsf{G}_1^A, \mathsf{G}_2^A$. If the entries $u, u+1, u+2$ occupy the nodes in B^A in t then the result is straightforward, that is $\mathsf{t} = s_{u+1} \mathsf{G}_1^A = s_u \mathsf{G}_2^A$. Now suppose that the numbers $u, u+1, u+2$ do not occupy the nodes of B^A, but those nodes contain consecutive numbers $a, a+1, a+2$. Then if $a < u$ we have that on of the subwords $s_{a+3} s_{a+2}$ or $s_{a+2} s_{a+3}$ will appear in w_t and it will be right exposed, hence

$$\mathsf{t} = w \mathsf{G}^B, \text{ where } B := (\mathsf{t}^\lambda)^{-1}(a+2) \tag{1.29}$$

for some $w \in \mathfrak{S}_n$ and some Garnir tableau G^B associated to B. If $a > u$ then either $s_{a-2} s_{a-1}$ ir $s_{a-1} s_{a-2}$ will appear as subword of w_t and it will be right exposed, hence

$$\mathsf{t} = w\mathsf{G}^C, \text{ where } C := (\mathsf{t}^\lambda)^{-1}(a-2) \tag{1.30}$$

for some $w \in \mathfrak{S}_n$ and some Garnir tableau G^C associated to C. From (1.28), (1.29) and (1.30) we have the desired result.

For the converse argument we refer to [APS19, Lemma 1.12]. □

Chapter 2

Cellularity of quasi-hereditary quotients of Hecke algebras

In this chapter we present one of the main results of this thesis concerning the quasi-hereditary quotients $\mathscr{H}_n^\sigma$, $\sigma \in \mathbb{Z}^\ell$, of cyclotomic quiver Hecke algebras. In particular, we construct a "classical-type" tableaux-theoretic cellular basis for the aforementioned quotients. These bases might look familiar to the reader and remind them the Murphy's bases for Hecke algebras. However, in this case the underlying combinatorics are different, as they rely on the non-standard order of Definition 1.24 for the cellular structure. The results of this chapter form the first section of the preprint [BCHM20] and they are personal work of the author.

The motivation for the construction of the above cellular basis comes from the representation theory of the symmetric group. The representation theory of the symmetric group lies in the intersection of two great categorical theories. The first is Khovanov–Lauda and Rouquier's categorification of quantum groups and their knot invariants [KL09, Rou] and Elias–Williamson's diagrammatic categorification in terms of endomorphisms of Bott–Samelson bimodules. Bowman–Hazi–Cox [BCH] construct explicit isomorphisms between these two diagrammatic worlds. For this construction, the "light-leaves" bases of these algebras play crucial role.

In [BCHM20] Bowman–Hazi–Cox and the author construct such bases for the quasi-hereditary quotients $\mathscr{H}_n^\sigma$, $\sigma \in \mathbb{Z}^\ell$, of cyclotomic quiver Hecke algebras.

The backbone for the construction of the light-leaves basis for the quiver Hecke algebras is the construction of the tableaux-theoretic cellular basis we mentioned above, with respect to that order relation. The first section of this chapter is devoted in defining technicalities and combinatorial analogues of the action of the dot generators of the algebras. In particular, these analogues will be considered as maps on the set of box-configurations. Prior to that we define some essential combinatorial language accompanied with examples, in order to make it easier for the reader to understand the new concepts. The second section is the most fruitful section of this chapter and it contains the technical proof that enables us to construct a chain of two-sided ideals for the quotients $\mathscr{H}_n^\sigma$, $\sigma \in \mathbb{Z}^\ell$, with respect to the order relation of Definition 1.24. The proof massively depends on the maps defined in the previous section and the order of Definition 2.2 (which is a coarsening of $\succ$). Throughout this section we use a running example which makes the technical load easier for the reader to digest. Finally, in the last section we utilise that chain of two-sided ideals and we prove that the algebras $\mathscr{H}_n^\sigma$, $\sigma \in \mathbb{Z}^\ell$, are cellular and indeed (over a field $\Bbbk$) quasi-hereditary.

2.1 Maps on box-configurations

Let $h, \ell \in \mathbb{Z}_{>0}$, $e \geq (h+1)\ell$ and $\sigma \in \mathbb{Z}^\ell$ be an (h, e)-admissible charge, as defined in Definition 1.10. Recall that

$$\mathscr{H}_n^\sigma := \mathcal{H}_n^\sigma / \mathcal{H}_n^\sigma \mathsf{e}_h \mathcal{H}_n^\sigma$$

is the quotient of the cyclotomic quiver Hecke algebra of Definition 1.11. In order to construct a tableaux-theoretic basis for the aforementioned quasi-hereditary quotients, we first need to construct a stratification of $\mathscr{H}_n^\sigma$ in which each layer is generated by an idempotent corresponding to some ℓ-partition. Hence we need to understand the effect of multiplying a generator of a given cell-stratification by a KLR dot. Towards that direction, in this section, we define combinatorial analogues of the dot generators as maps on the set of box-configurations.

Definition 2.1. Let $\lambda \in \mathcal{B}_\ell(n)$ and let $[i, j, m] \in \lambda$ be an r-box for some $r \in \mathbb{Z}/e\mathbb{Z}$. We say that $[i, j, m]$ is left-justified if either $j < e$ or there exists some $[i, j-p, m] \in \lambda$ for $1 \leq p \leq e$.

Recall the reverse lexicographic order $\succ$ on the set of box configurations as defined in Definition 1.24. The next definition is crucial for the remainder of this chapter. It defines a new order relation (which will be a coarsening of $\succ$) which will be mainly used in the proof of Proposition 2.12.

Definition 2.2. Let $\lambda \in \mathcal{B}_\ell(n)$, $r \in \mathbb{Z}/e\mathbb{Z}$. For an r-box $\alpha \in \lambda$, we define

$$Y_\alpha(\lambda) := \lambda - \alpha \cup \beta$$

where $\beta \notin \lambda$ is the box with $\beta \succ \alpha$ satisfying the following properties:

(i) has residue r;

(ii) is left-justified; and

(iii) is minimal to the order relation $\succ$ with respect to these properties.

If such box does not exist, we say that $Y_\alpha(\lambda)$ is *undefined*. We write $\lambda \succ\!\!\succ \mu$ if $\lambda = Y_\alpha(\mu)$ for some $\alpha \in \mu$ and we then extend $\succ\!\!\succ$ to a partial order on $\mathcal{B}_\ell(n)$ by taking the transitive closure. Suppose that $\{[i_k, j_k, m_k] \mid 0 < k \leq p\}$ is a set of r-boxes and that $Y_{[i_k,j_k,m_k]}(\lambda \cup [i_k, j_k, m_k]) = \lambda \cup [i_{k+1}, j_{k+1}, m_{k+1}]$ for $k \geq 1$. We define

$$Y^p_{[i_1,j_1,m_1]}(\lambda \cup [i_1, j_1, m_1]) = (\lambda \cup [i_p, j_p, m_p]).$$

Note that the only case that $Y_\alpha(\lambda)$ is undefined, is when $i = 1$ and $m = 0$.

Remark 2.3. We remark that $\lambda \succ\!\!\succ \mu$ implies that $\lambda \succ \mu$. Hence the order $\succ\!\!\succ$ of Definition 2.2 is a coarsening of $\succ$.

Example 2.4. *Let $\lambda = (3, 2^2, 1^6) \in \mathscr{P}_{3,1}(13)$ and $e = 5$ and $\sigma = 0 \in \mathbb{Z}$. The residue of the node $[3, 2, 0]$ is $\mathrm{res}[3, 2, 0] = 4$. We have that $Y_{[3,2,0]}(\lambda) = (3, 2^2, 1^6) \cup [2, 6, 0] \in \mathscr{B}_1(13)$, since the 4-node $[2, 6, 0]$ is minimal in the lexicographic order such that $[2, 6, 0] \succ [3, 2, 0]$ and $[2, 6, 0]$ is left-justified. Note that the lexicographically least 4-node satisfying the first condition of Definition 2.2, is the node $[2, 11, 0]$. However, the node $[2, 11, 0]$ is not left-justified, hence we pick the node $[2, 6, 0]$ which is also left-justified. The above are depicted in Figure 2.1.*

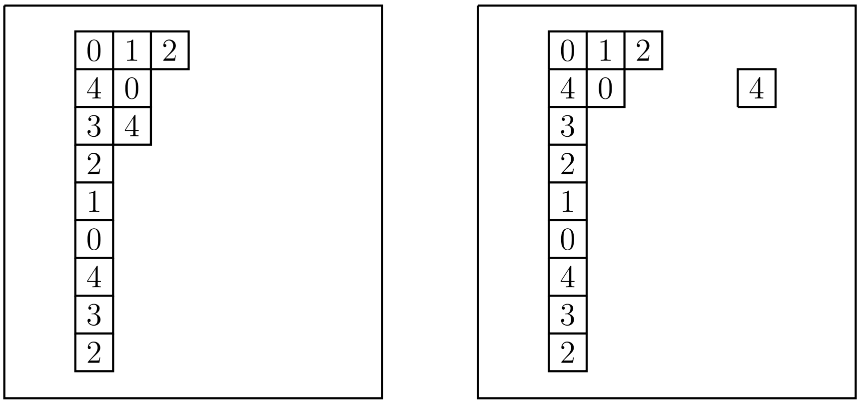


FIGURE 2.1: The pair of box-configurations is the same as in Example 1.27. Here we also record the residues of each node of the configuration. The left hand side partition is λ and the right hand side box-configuration is $Y_{[3,2,0]}(\lambda)$.

Example 2.5. *Let $h = 3$, $\ell = 1$, $e = 5$, $\sigma = 0$ and $\lambda = (3, 2^2, 1^6) \in \mathscr{P}_{3,1}(13)$ as in Example 2.4. We have that $Y^1_{[4,1,0]}(\lambda) = (3, 2^2, 1^6) \cup [3, 5, 0] - [4, 1, 0]$, $Y^2_{[4,1,0]}(\lambda) = (3, 2^2, 1^6) \cup [2, 4, 0] - [4, 1, 0]$ and $Y^3_{[4,1,0]}(\lambda) = (3, 2^2, 1^6) \cup [1, 8, 0] - [4, 1, 0]$. The above box-configurations in $\mathscr{B}_1(13)$ can be found in Figure 2.2. Note that in $Y^1_{[4,1,0]}(\lambda)$ the node $[4, 1, 0]$ does not pass through any node with adjacent residue, as it belongs in the first column of the row and $\ell = 1$.*

Given an idempotent indexed by an n-tuple $\underline{j} \in (\mathbb{Z}/e\mathbb{Z})^n$, we wish to identify in which layer of the stratification this idempotent belongs to. To this end we make the following definition.

Definition 2.6. Associated to any n-tuple $\underline{j} = (j_1, \cdots, j_n) \in (\mathbb{Z}/e\mathbb{Z})^n$ we define the tableau $\mathsf{J} \in \mathrm{Std}(n)$ to be the tableau given by placing the entry $k = 1, 2, \cdots, n$ in the lexicographically least addable j_k-node of the partition $\mathrm{Shape}(\mathsf{J}_{\leq k-1})$ or formally setting $\mathsf{J} = 0$ and $\mathrm{Shape}(\mathsf{J}) = \varnothing$ if no such node exists for some $1 \leq k \leq n$.

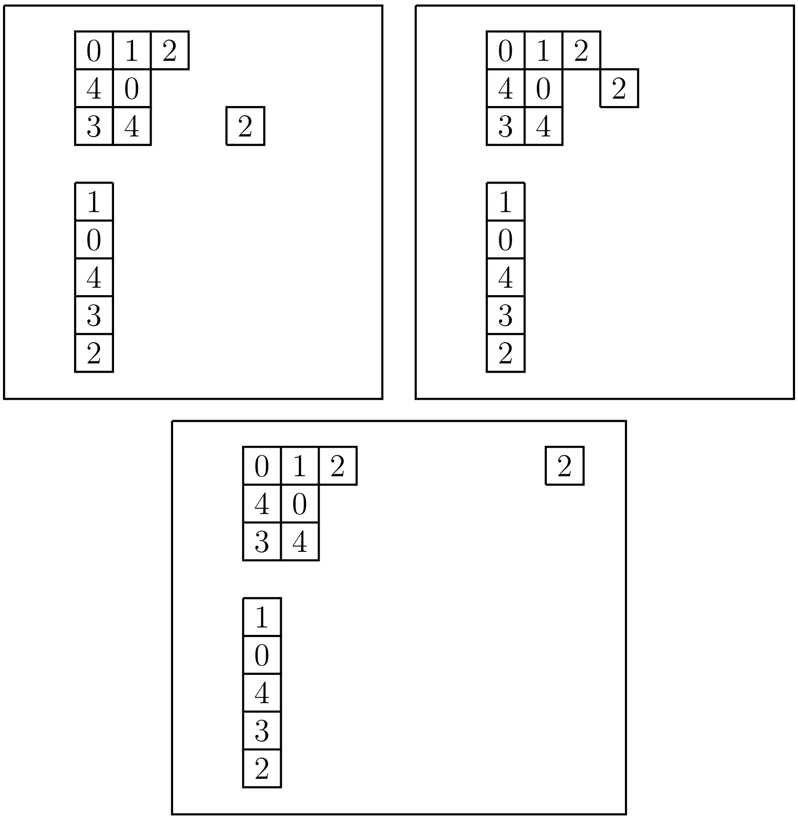


FIGURE 2.2: The box configurations $Y^1_{[4,1,0]}(\lambda)$, $Y^2_{[4,1,0]}(\lambda)$ and $Y^3_{[4,1,0]}(\lambda)$ respectively. The partition λ is depicted in Figure 2.1.

Example 2.7. *Let $h = 3$, $\ell = 1$, $e = 5$, $\sigma = 0$ and consider the tuple $\underline{j} = (0,1,4,0,3,4,2,1,0,4,3,2,2) \in (\mathbb{Z}/5\mathbb{Z})^{13}$. The tableau $\mathsf{J} \in \mathrm{Std}(13)$ described in Definition 2.6 is depicted in Figure 2.3.*

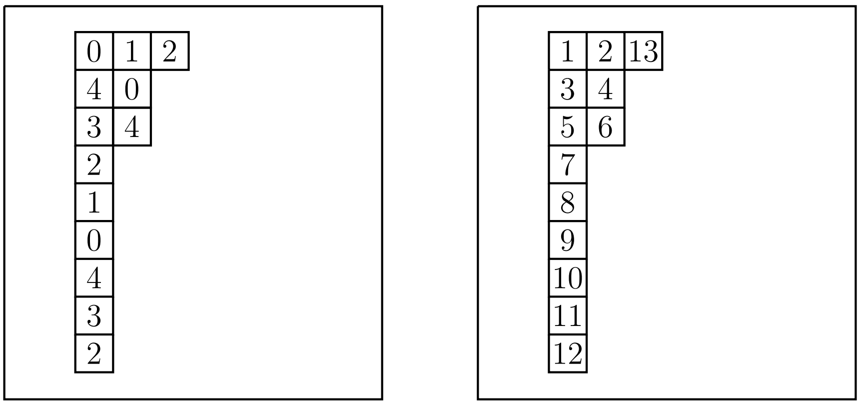


FIGURE 2.3: In the left hand side we have a residue pattern same as the vector $\underline{j}$. In the right hand side we present the tableau $\mathsf{J} \in \mathrm{Std}(13)$, constructed according to Definition 2.6.

2.2 Chain of two-sided ideals

We are now ready to construct the chain of two-sided ideals for the quotients $\mathscr{H}_n^\sigma$, $\sigma \in \mathbb{Z}^\ell$. This chain of ideals is the central technical result of this section and will be used for constructing the tableaux-theoretic cellular basis. In the section we include some further combinatorial concepts and definitions which are important in the statement and proof of the main result.

Lemma 2.8. *Let $\lambda \in \mathscr{P}_{h,\ell}(n)$ and $\sigma \in (\mathbb{Z}/e\mathbb{Z})^\ell$ be (h,e)-admissible. For any box $[i,j,m] \in \lambda$ we have that the multiset of residues of boxes in $\lambda \cap B^{[i,j,m]}$, where $B^{[i,j,m]}$ is the Garnir belt of $[i,j,m]$, is multiplicity-free (i.e. no residue appears more than once).*

Proof. This follows immediately from the definitions, since $\sigma \in (\mathbb{Z}/e\mathbb{Z})^\ell$ is (h,e)-admissible. □

Remark 2.9. Let $[i,j,m] \in \lambda$ and suppose $[i',j',m'] \in B^{[i,j,m]} \cap \lambda$ is such that

$$\mathrm{res}([i',j',m']) = \mathrm{res}([i,j,m]) - 1.$$

The most common case of such a box is $[i',j',m'] = [i,j-1,m]$. Whenever $\sigma_m - \sigma_{m-1} = h$, for $m > 0$, we also have a case where $j = 1$ and $[i',j',m'] = [i,h,m-1]$. If $\sigma_{\ell-1} - \sigma_0 = e - h$, we also have a case where $j = 1$ and $m = 0$ and $[i',j',m'] = [i-1,h,\ell-1]$. However, we note that the aforementioned cases will not appear due to restrictions on the charge σ.

Definition 2.10. Let $\lambda \in \mathscr{P}_{h,\ell}(n)$. We define the Garnir adjacency set of a node $\alpha = [i,j,m]$, with $\mathrm{res}(\alpha) = r \in \mathbb{Z}/e\mathbb{Z}$, to the set of boxes $\gamma \in \lambda \cap B^\alpha$ such that $|\mathrm{res}(\gamma) - \mathrm{res}(\alpha)| \leq 1$ and denote this set by $\mathsf{Adj}-\mathsf{Gar}(\alpha)$. We also set

$$\mathrm{res}(B^\alpha) = \{\mathrm{res}(\gamma) \mid \gamma \in \mathsf{Adj}-\mathsf{Gar}(\alpha)\}.$$

Note that for any node $\alpha \notin \lambda$, the set $\mathsf{Adj}-\mathsf{Gar}(\alpha)$ has at most one node of each possible residue, by Lemma 2.8. We also define the following elements, which will be needed for the construction of the chain of two-sided ideals and the cellular basis.

Definition 2.11. Given $\mathsf{s},\mathsf{t} \in \mathrm{Std}(\lambda)$ we define the element

$$\psi^{\mathsf{s}}_{\mathsf{t}} := e_{\mathsf{s}}\psi_{\underline{w}}e_{\mathsf{t}} \in \mathscr{H}^{\sigma}_n$$

where $\underline{w} \in \mathfrak{S}_n$ any fixed reduced expression for $w^{\mathsf{s}}_{\mathsf{t}} \in \mathfrak{S}_n$.

We define $\mathscr{H}^{\geq\lambda}_n := \mathscr{H}^{\sigma}_n\langle e_{\mathsf{t}_\nu} \mid \nu \geq \lambda\rangle\mathscr{H}^{\sigma}_n$ for $\geq$ any order on $\mathscr{P}_{h,\ell}(n)$; we formally set $\mathscr{H}^{\geq 0}_n = 0$ to be the zero ideal.

The next proposition is the technical element of the main result of this chapter.

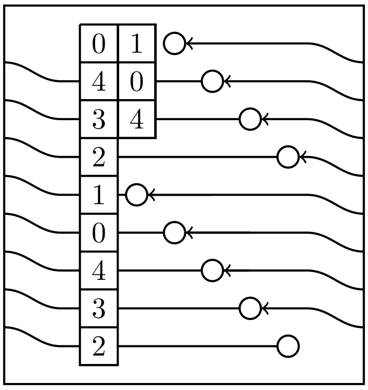


FIGURE 2.4: For $n = 13$ and $\lambda = (2^3, 1^6)$, we illustrate how the idempotent, $e_{(0,1,4,0,3,4,2,1,0,4,3,2,2)}$, labelled by J in Figure 2.3, is rewritten in the form of equation (2.4). The box moves through each row until it comes to rest at the point $\mathsf{J}^{-1}(13) = [1, 3, 0]$. This involves 8 applications of (2.1) to deduce (2.4) in this example. For the purposes of later referencing, we label the 9 boxes from bottom-to-top by $\alpha_1 = [9, 6, 0], \alpha_2 = [8, 5, 0] \ldots, \alpha_9 = [1, 3, 0]$

Proposition 2.12. *Let $\lambda \in \mathscr{P}_{h,\ell}(n-1)$ and $\alpha = [i, j, m] \notin \lambda$ with $\mathrm{res}(\alpha) = r$. We set β to be the box determined by*

$$\begin{cases} Y^{\ell+1}_{\alpha}(\lambda \cup \alpha), & \text{if } \mathsf{Adj}-\mathsf{Gar}(\alpha) = \{r-1\}; \\ Y_{\alpha}(\lambda \cup \alpha), & \text{otherwise.} \end{cases}$$

Moreover, β is undefined if $\mathsf{Adj} - \mathsf{Gar}(\alpha) = \emptyset$ *and* $r = 1$. *We set* $a = \mathsf{t}_{\lambda\cup\alpha}(\alpha)$ *and* $b = \mathsf{t}_{\lambda\cup\beta}(\beta)$. *We define*

$$\mathscr{H}_n^{(\not\succ\lambda)\cup\beta} = \sum_{\{\mu\in\mathscr{P}_{h,\ell}(n-1)\mid\mu\not\succ\lambda,\beta\notin\mu\}} \mathscr{H}_n^\sigma e_{\mathsf{t}_{\mu\cup\beta}} \mathscr{H}_n^\sigma \leq \mathscr{H}_n^{\not\succ(\lambda\cup\beta)}.$$

and we also define

$$y^{\succ}_{\mathsf{t}_{\lambda\cup\alpha}} = \begin{cases} y_a e_{\mathsf{t}_{\lambda\cup\alpha}} & \text{if there exists } \square \in \mathsf{Add}(\mathsf{t}_\lambda{\downarrow}_{\leq a-1}) \text{ with } \alpha \not\succ \square; \\ e_{\mathsf{t}_{\lambda\cup\alpha}} & \text{otherwise.} \end{cases}$$

- *If* $\lambda\cup\alpha \notin \mathscr{P}_{h,\ell}(n)$, *then we have that*

$$y^{\succ}_{\mathsf{t}_{\lambda\cup\alpha}} \in \begin{cases} \pm\psi^a_b y^{\succ}_{\mathsf{t}_{\lambda\cup\beta}}\psi^b_a + \mathscr{H}_n^{(\not\succ\lambda)\cup\beta} & \text{or} \\ \pm(y_{a-1}\psi^a_b y^{\succ}_{\mathsf{t}_{\lambda\cup\beta}}\psi^b_a - \psi^a_b y^{\succ}_{\mathsf{t}_{\lambda\cup\beta}}\psi^b_a y_a) + \mathscr{H}_n^{(\not\succ\lambda)\cup\beta} \end{cases} \tag{2.1}$$

  if β is defined (the two possible cases are detailed in the proof). If β is undefined, then $e_{\mathsf{t}_{\lambda\cup\alpha}} = 0$.

- *If* $\lambda\cup\alpha \in \mathscr{P}_{h,\ell}(n)$, *then we have that*

$$y_a y^{\succ}_{\mathsf{t}_{\lambda\cup\alpha}} \in \pm\psi^a_b y^{\succ}_{\mathsf{t}_{\lambda\cup\beta}}\psi^b_a + \mathscr{H}_n^{(\not\succ\lambda)\cup\beta} \tag{2.2}$$

  if β is defined. If β is undefined, then $y_a e_{\mathsf{t}_{\lambda\cup\alpha}} = 0$.

Remark 2.13. By Lemma 2.8, if $\lambda\cup\alpha\in\mathscr{P}_{h,\ell}(n)$ then $y^{\succ}_{\mathsf{t}_{\lambda\cup\alpha}} = e_{\mathsf{t}_{\lambda\cup\alpha}}$.

Before we proceed to the proof of Proposition 2.12 we shall try to emphasise its importance by intuitively describing the motivation behind the need of having such a technical result. The motivation is its immediate Corollary 2.15. We aim to construct a chain of two-sided ideals for the algebra $\mathscr{H}_n^\sigma$, with respect to the order $\succ$, in which each two-sided ideal is generated by an idempotent e_{t_λ} for $\lambda\in\mathscr{P}_{h,\ell}(n)$. By using equations (2.1) and (2.2) of Proposition 2.12 , we are able rewrite any element of

$$\mathcal{Y}_n := \langle e_{\underline{i}}, y_k \mid \underline{i}\in(\mathbb{Z}/e\mathbb{Z})^n, 1\leq k\leq n\rangle \tag{2.3}$$

in the required form by moving a given box α through the partition $\lambda \in \mathscr{P}_{h,\ell}(n)$, one row at a time until it comes to rest at some point β. We include a running example of this procedure for the tableau of Figure 2.3. The eight steps (indexed by nine tableaux) of this procedure are illustrated in Figure 2.4.

Remark 2.14. We remark that in the proof of Proposition 2.12 we often relate ideals and elements in algebras of smaller and larger rank by using horizontal concatenation of diagrams, in the sense of the first chapter of the thesis (see Figures 1.4, 1.5). The reason we can do that is by the definition of the reverse lexicographic order. In more detail, the order $\succ$ distinguishes between box configurations based on the first discrepancy upon reading a pair of box configurations backwards.

Proof of Proposition 2.12. We assume the equation (2.1) holds for all $\lambda \cup \alpha = \nu \notin \mathscr{P}_{h,\ell}(k)$ and equation (2.2) holds for all $\lambda \cup \alpha = \nu \in \mathscr{P}_{h,\ell}(k)$ for all $1 \leq k < n$. (The $k = 1$ base case is trivial.) By repeated applications of equations (2.1) and (2.2) we know that the algebra $\mathscr{H}_k^\sigma$ has a chain of 2-sided ideals $\mathscr{H}_k^{\succcurlyeq\nu}$ indexed by $\nu \in \mathscr{P}_{h,\ell}(k)$ for $1 \leq k < n$. In particular, for any $\underline{j} = (j_1, \ldots, j_k) \in (\mathbb{Z}/e\mathbb{Z})^k$, $1 \leq k < n$, we have that

$$\begin{cases} e_{\underline{j}} \in \pm \widehat{\psi}^{\mathsf{J}}_{\mathsf{t}_\nu} \widehat{\psi}^{\mathsf{t}_\nu}_{\mathsf{J}} + \mathscr{H}_k^{\succcurlyeq\nu} & \text{if } \mathrm{Shape}(\mathsf{J}) = \nu \in \mathscr{P}_{h,\ell}(k) \\ e_{\underline{j}} = 0 & \text{if } \mathrm{Shape}(\mathsf{J}) = \varnothing \end{cases} \tag{2.4}$$

where $\widehat{\psi}^{\mathsf{J}}_{\mathsf{t}_\nu}, \widehat{\psi}^{\mathsf{t}_\nu}_{\mathsf{J}}$ are obtained by $\psi^{\mathsf{J}}_{\mathsf{t}_\nu}, \psi^{\mathsf{t}_\nu}_{\mathsf{J}}$, respectively, by possibly adding dot decorations along the strands. Also for $\nu \in \mathscr{P}_{h,\ell}(k)$, $1 \leq k < n$, the element $y_a e_{\mathsf{t}_\nu}$ belongs to $\mathscr{H}_k^{\succcurlyeq\nu}$ for any $1 \leq a < n$.

We can now further assume that the result has been proven for all ν of the form $\mu \cup \alpha = \nu \in \mathscr{P}_{h,\ell}(n)$ for some $\mu \succcurlyeq \lambda$ in $\mathscr{P}_{h,\ell}(n-1)$, thus leaving us to prove the result for all $\nu = \lambda \cup \alpha$ for $\lambda \in \mathscr{P}_{h,\ell}(n-1)$. By Definition 2.2, $\mathrm{res}(\alpha) = \mathrm{res}(\beta)$ and we set this residue equal to $r \in \mathbb{Z}/e\mathbb{Z}$ for the remainder of the proof.

Proof of equation (2.1) for a given λ and α. We include a running example of our algorithm for $e = 5$ and $\ell = 1$ and $\lambda = (2^3, 1^6)$. There are four cases to consider, depending on the residue of the final node in the column (i.e. the residue of the strand labelled by $\alpha' := \mathsf{t}^{-1}_{\lambda\cup\alpha}(a-1)$).

(*i*) Suppose $\alpha' = \mathsf{t}^{-1}_{\lambda\cup\alpha}(a-1)$ has residue $r \in \mathbb{Z}/e\mathbb{Z}$ and so $y^{\succ}_{\mathsf{t}_{\lambda\cup\alpha}} = e_{\mathsf{t}_{\lambda\cup\alpha}}$. By application of relations R4 and R5, we have that

$$e_{\mathsf{t}_{\lambda\cup\alpha}} = \psi_{a-1}y_{a-1}e_{\mathsf{t}_{\lambda\cup\alpha}}\psi_a y_{a-1} - y_a\psi_{a-1}y_{a-1}e_{\mathsf{t}_{\lambda\cup\alpha}}\psi_{a-1}. \tag{2.5}$$

An example of the visualisation of the idempotents on the righthand-side of equation (2.5) is given in the first step of Figure 2.4; the corresponding righthand-side of equation (2.5) is depicted in Figure 2.5. Now, we have that

$$y_{a-1}e_{\mathsf{t}_{\lambda\cup\alpha}} = (y_{a-1}e_{\mathsf{t}_{\lambda\cup\alpha\downarrow_{\leq a-1}}}) \boxtimes e_r \boxtimes e_{\mathsf{t}_{\lambda\cup\alpha\downarrow_{>a}}}$$

and so by our inductive assumption for equation (2.2) for rank $a-1<n$, we have that

$$\begin{aligned} y_{a-1}e_{\mathsf{t}_{\lambda\cup\alpha}} &\in \psi_b^{a-1}y^{\succ}_{\mathsf{t}_{\lambda\cup\beta\downarrow_{\leq a-1}}}\psi^b_{a-1} \boxtimes e_{\mathsf{t}_{\lambda\cup\beta\downarrow_{\geq a}}} + \mathscr{H}_n^{(\not\succ\lambda)\cup\beta} \\ &= \psi_b^{a-1}y^{\succ}_{\mathsf{t}_{\lambda\cup\beta}}\psi^b_{a-1} + \mathscr{H}_n^{(\not\succ\lambda)\cup\beta} \end{aligned}$$

where we have implicitly used the following facts: (*i*) $Y^{d+1}_{\alpha'}(\lambda\cup\alpha) = \lambda\cup\alpha\cup\beta-\alpha'$ (*ii*) $\mathsf{t}_{\lambda\cup\alpha\downarrow_{>a}} = \mathsf{t}_{\lambda\cup\beta\downarrow_{>a}}$ and (*iii*) once we have moved α' to position β, the α box is free to move into the newly unoccupied position. Substituting this back into equation (2.5), we obtain

$$e_{\mathsf{t}_{\lambda\cup\alpha}} \in \psi^a_b y^{\succ}_{\mathsf{t}_{\lambda\cup\beta}}\psi^b_a y_{a-1} - y_a\psi^a_b y^{\succ}_{\mathsf{t}_{\lambda\cup\beta}}\psi^b_a + \mathscr{H}_n^{(\not\succ\lambda)\cup\beta} \tag{2.6}$$

as required. An example is depicted in Figure 2.5 (although we remark that the error terms belonging to $\mathscr{H}_n^{(\not\succ\lambda)\cup\beta}$ are actually all zero in this case).

(*ii*) Now suppose $\mathsf{t}^{-1}_{\lambda\cup\alpha}(a-1) = [x,y,z]$ has residue $r+1 \in \mathbb{Z}/e\mathbb{Z}$. Here we need to consider two separate cases. We first consider the case in which $[x,y,z] = [i,1,m]$, which will be the easier one. By Lemma 2.8 and relation R5 we have

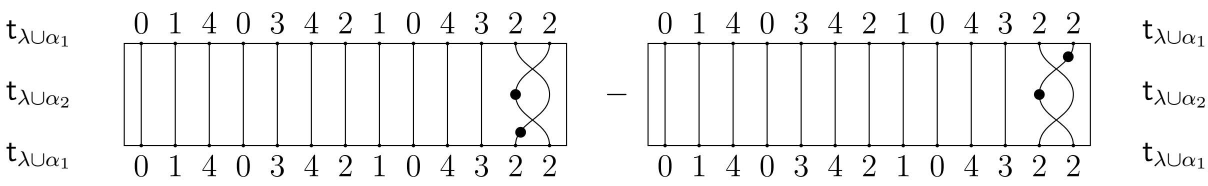


FIGURE 2.5: We continue with the example in Figure 2.4 for $\lambda = (2^6, 1^3)$. This is the righthand-side of equation (2.5) for $y_{13}e_{\mathsf{t}_{\lambda\cup\alpha_1}}$.

that

$$y^{\succ}_{\mathsf{t}_{\lambda\cup\alpha}} = y_a e_{\mathsf{t}_{\lambda\cup\alpha}} = y_{a-1}e_{\mathsf{t}_{\lambda\cup\alpha}} - e_{\mathsf{t}_{\lambda\cup\alpha}}\psi_{a-1}e_{\mathsf{t}_{\lambda\cup\beta}}\psi_{a-1}e_{\mathsf{t}_{\lambda\cup\alpha}} \tag{2.7}$$

were the former term belongs to the required ideal by equation (2.1) for rank $a-1 < n$. Note that in this case we simply have that $b = a-1$.

The second case is the one in which $y > 1$. Then the $(a-2)$th, $(a-1)$th and ath strands have residues r, $r+1$, and r respectively. We have that

$$\begin{aligned} y^{\succ}_{\mathsf{t}_{\lambda\cup\alpha}} = e_{\mathsf{t}_{\lambda\cup\alpha}} &= e_{\mathsf{t}_{\lambda\cup\alpha}}\psi_{a-2}\psi_{a-1}\psi_{a-2}e_{\mathsf{t}_{\lambda\cup\alpha}} - e_{\mathsf{t}_{\lambda\cup\alpha}}\psi_{a-1}\psi_{a-2}\psi_{a-1}e_{\mathsf{t}_{\lambda\cup\alpha}} \\ &= -e_{\mathsf{t}_{\lambda\cup\alpha}}\psi_{a-2}\psi_{a-1}y_{a-1}\psi_{a-1}\psi_{a-2}e_{\mathsf{t}_{\lambda\cup\alpha}} \\ &\quad + e_{\mathsf{t}_{\lambda\cup\alpha}}\psi_{a-1}\psi_{a-2}y_{a-2}\psi_{a-2}\psi_{a-1}e_{\mathsf{t}_{\lambda\cup\alpha}}. \end{aligned} \tag{2.8}$$

where the first equality follows from relation R6 and the second follows from relations R4 and R5. We set $\xi = \mathrm{Shape}(\mathsf{t}_\lambda{\downarrow}_{<a-2})$. The two terms in equation (2.8) factor through the elements

$$\underbrace{e_{\mathsf{t}_\lambda{\downarrow}_{<a-2}} \boxtimes e_{r+1}}_{\xi\cup[x,y,z]} \boxtimes y_1 e_{r,r} \boxtimes e_{\mathsf{t}_{\lambda\cup\alpha}{\downarrow}_{>a}} \qquad \underbrace{e_{\mathsf{t}_\lambda{\downarrow}_{<a-2}} \boxtimes y_1 e_r}_{\xi\cup[x,y-1,z]} \boxtimes e_{r,r+1} \boxtimes e_{\mathsf{t}_{\lambda\cup\alpha}{\downarrow}_{>a}} \tag{2.9}$$

respectively.

- We first consider the latter term on the righthand-side of equation (2.8) (which we will see, is the required non-zero term). We note that $[x, y-1, z]$ and α have the same residue and so $Y_{[x,y-1,z]}(\xi \cup [x, y-1, z]) = \xi \cup \beta - [x, y-1, z]$. By our inductive assumption that (2.2) holds for rank $a-2 < n$, we have that

$$e_{\mathsf{t}_{\lambda\downarrow_{<a-2}}} \boxtimes y_1 e_r = y_{a-2} e_{\mathsf{t}_{\xi\cup[x,y-1,z]}} \in \psi_b^{a-2} y^{\succ}_{\mathsf{t}_{\xi\cup\beta}} \psi^b_{a-2} + \mathscr{H}_{a-2}^{(\not\succ\xi)\cup\beta} \tag{2.10}$$

Substituting this back into the second term of equation (2.9) we obtain

$$\psi_b^{a-2} y^{\succ}_{\mathsf{t}_{\xi\cup\beta}} \psi^b_{a-2} \boxtimes e_{r,r+1} \boxtimes e_{\mathsf{t}_{\lambda\cup\alpha\downarrow_{>a}}} \in \psi_b^{a-2} y^{\succ}_{\mathsf{t}_{\lambda\cup\beta}} \psi^b_{a-2} + \mathscr{H}_n^{(\not\succ\lambda)\cup\beta}$$

and then substituting into the second term of equation (2.8) we obtain

$$\begin{aligned} e_{\mathsf{t}_{\lambda\cup\alpha}} \psi_{a-1}\psi_{a-2} y_{a-2} \psi_{a-2}\psi_{a-1} e_{\mathsf{t}_{\lambda\cup\alpha}} &\in \psi_{a-1}\psi_{a-2}\psi_b^{a-2} y^{\succ}_{\mathsf{t}_{\lambda\cup\beta}} \psi^b_{a-2}\psi_{a-2}\psi_{a-1} + \mathscr{H}_n^{(\not\succ\lambda)\cup\beta} \\ &= \psi_b^a y^{\succ}_{\mathsf{t}_{\lambda\cup\beta}} \psi^b_a + \mathscr{H}_n^{(\not\succ\lambda)\cup\beta} \end{aligned}$$

as required.

- We now consider the former term of equation (2.8) (which, we will see, is zero modulo the ideal). We have that $Y^2_{[x,y,z]}(\xi \cup [x,y,z]) = \xi \cup \gamma \not\succ \xi \cup [x,y,z]$ for γ a box of residue $r+1 \in \mathbb{Z}/e\mathbb{Z}$. We set $c = \mathsf{t}^{-1}_{\xi\cup\gamma}(\gamma)$ and We have that

$$e_{\mathsf{t}_\xi} \boxtimes e_{r+1} = e_{\mathsf{t}_{\xi\cup[x,y,z]}} = \psi_c^{a-2} e_{\mathsf{t}_{\xi\cup\gamma}} \psi^c_{a-2} \in \psi_c^{a-2} y^{\succ}_{\mathsf{t}_{\xi\cup\gamma}} \psi^c_{a-2} + \mathscr{H}_{a-2}^{(\not\succ\xi)\cup\gamma}$$

by our inductive assumption that the equation (2.1) holds for rank $a-2 < n$. We now consider the concatenation with $y_1 e_r$. We have that $Y_{[x,y-1,z]}(Y_{[x,y,z]}(\xi \cup [x,y,z]) \cup [x,y-1,z]) = Y_{[x,y-1,z]}(\xi \cup \gamma \cup [x,y-1,z]) = \xi \cup \gamma \cup \beta$ for $\gamma \succ \beta \not\succ [x,y-1,z]$ with γ a box of residue $r+1 \in \mathbb{Z}/e\mathbb{Z}$. We have that

$$e_{\mathsf{t}_{\xi\cup\gamma}} \boxtimes y_1 e_r = y_{a-1} e_{\mathsf{t}_{\xi\cup\gamma\cup[x,y-1,z]}} \in \psi_b^{a-1} y^{\succ}_{\mathsf{t}_{\xi\cup\gamma\cup\delta}} \psi^b_{a-1} + \mathscr{H}_{a-1}^{(\not\succ\xi)\cup\gamma\cup\delta}$$

by induction for rank $a-1 < n$ and (2.4). Finally, we concatenate again to obtain

$$e_{\mathsf{t}_{\lambda\downarrow_{<a-2}}} \boxtimes e_{r+1} \boxtimes y_1 e_{r,r} \boxtimes e_{\mathsf{t}_{\lambda\cup\alpha\downarrow_{>a}}} \in \psi_c^{a-2}\psi_b^{a-1} y^{\succ}_{\mathsf{t}_{\lambda\cup\gamma\cup\beta-[x,y,z]}} \psi^b_{a-1}\psi^c_{a-2} + \mathscr{H}_n^{(\not\succ\lambda)\cup\beta}$$

and we note that the idempotent on the righthand-side is labelled by $(\lambda \cup \gamma - [x,y,z]) \cup \beta$ where $\lambda \cup \gamma - [x,y,z] \not\succ \lambda$. Therefore this element belongs to the ideal $\mathscr{H}_n^{(\not\succ\lambda)\cup\beta}$ as required.

(iii) Now suppose $\alpha' = \mathsf{t}^{-1}_{\lambda\cup\alpha}(a-1)$ has residue $d \in \mathbb{Z}/e\mathbb{Z}$ such that $|d-r| > 1$. We set $\xi = \mathrm{Shape}(\mathsf{t}_\lambda{\downarrow}_{<a-1})$. By case 2 of relation R5, we have that

$$y_a^k e_{\mathsf{t}_{\lambda\cup\alpha}} = \psi_{a-1}\big(\underbrace{e_{\mathsf{t}_\lambda\downarrow_{<a-1}} \boxtimes y_1^k e_r}_{\xi\cup\alpha} \boxtimes e_d \boxtimes e_{\mathsf{t}_{\lambda\cup\alpha}\downarrow_{>a}}\big)\psi_{a-1} \tag{2.11}$$

for $k \in \{0,1\}$. By the inductive assumption for rank $a < n$ of equation (2.1), we have that

$$e_{\mathsf{t}_\lambda\downarrow_{<a-1}} \boxtimes y_1^k e_r \in \psi_b^{a-1} e_{\mathsf{t}_{\xi\cup\beta}} \psi_{a-1}^b + \mathscr{H}_{a-1}^{(\not\succ\xi)\cup\beta}$$

and so, as in the case *(ii)* above, we concatenate to deduce the result. Two examples of the visualisation of the righthand-side of equation (2.11) are given in the third and fourth steps of Figure 2.4; the corresponding elements are depicted in Figure 2.6.

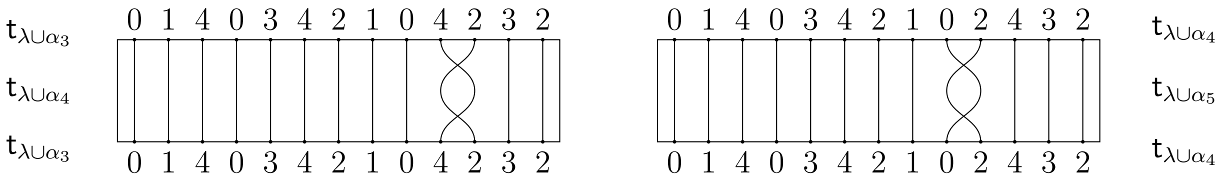


FIGURE 2.6: The righthand-side of (2.11) for $\lambda = (2^6, 1^3)$ and $\alpha = \alpha_3$ and α_4 respectively. Note that this is the case of (2.11) where $k = 0$, as there is no addable box of residue 2.

(iv) Suppose $\mathsf{t}_\lambda^{-1}(a-1) = [x,y,z]$ has residue $r-1 \in \mathbb{Z}/e\mathbb{Z}$ (thus $[x,y,x] = [i,j-1,m]$ by residue considerations). We also note that $[i-1,j,m] \notin \lambda$. To see this, if $[i-1,j,m] \in \lambda$, this implies that $\lambda \cup \alpha \in \mathscr{P}_{h,\ell}(n)$ and so the process would terminate. Let $\gamma = [i-1,j-1,m]$ and we set $c = \mathsf{t}_\lambda(\gamma)$ and let $\xi = \mathrm{Shape}(\mathsf{t}_\lambda{\downarrow}_{<a-1})$ (see

Figure 2.8 for an example). Then $y^{\succ}_{\mathsf{t}_{\lambda\cup\alpha}} = e_{\mathsf{t}_{\lambda\cup\alpha}}$ and we have that

$$\begin{aligned}
e_{\mathsf{t}_{\lambda\cup\alpha}} &= e_{\mathsf{t}_{\lambda\cup\alpha}}\psi^{c}_{a-2}\psi^{a-2}_{c}e_{\mathsf{t}_{\lambda\cup\alpha}} && (2.12)\\
&= e_{\mathsf{t}_{\lambda\cup\alpha}}\psi_{a-1}\psi^{c}_{a-1}\psi^{a-2}_{c}\psi_{a-1}e_{\mathsf{t}_{\lambda\cup\alpha}} - e_{\mathsf{t}_{\lambda\cup\alpha}}\psi^{c}_{a-1}\psi^{a}_{c}e_{\mathsf{t}_{\lambda\cup\alpha}} && (2.13)\\
&= e_{\mathsf{t}_{\lambda\cup\alpha}}\psi^{a}_{c+1}\psi^{c}_{a} - \psi^{c}_{a-1}(e_{\mathsf{t}_{\xi-\gamma}} \boxtimes e_{r-1,r,r} \boxtimes e_{\mathsf{t}_{\lambda\cup\alpha\downarrow_{>a}}})\psi^{a}_{c} && (2.14)\\
&= -e_{\mathsf{t}_{\lambda\cup\alpha}}\psi^{a}_{c}y_{c}\psi^{c}_{a} - \psi^{c}_{a-1}(e_{\mathsf{t}_{\xi-\gamma}} \boxtimes e_{r-1,r,r} \boxtimes e_{\mathsf{t}_{\lambda\cup\alpha\downarrow_{>a}}})\psi^{a}_{c} && (2.15)\\
&= -\psi^{a}_{b}y^{\succ}_{\mathsf{t}_{\lambda\cup\beta}}\psi^{b}_{a} - \psi^{c}_{a-1}(e_{\mathsf{t}_{\xi-\gamma}} \boxtimes e_{r-1,r,r} \boxtimes e_{\mathsf{t}_{\lambda\cup\alpha\downarrow_{>a}}})\psi^{a}_{c} && (2.16)
\end{aligned}$$

where the first and third equalities follow from the commuting case 2 of relation R5 and Lemma 2.8; the second equality follows from case 1 of relation R6; the fourth equality follows from relation R4; and the fifth equality is either trivial or follows from the case 3 of relation R5 (in the latter case, the error term is zero by our inductive assumption for rank $c-1<n$ of equation (2.1)). For our continuing example, the righthand-side of equation (2.16) is depicted in Figure 2.7; the box-configurations labelling the idempotents on the left and righthand-sides of (2.16) are depicted in Figure 2.8.

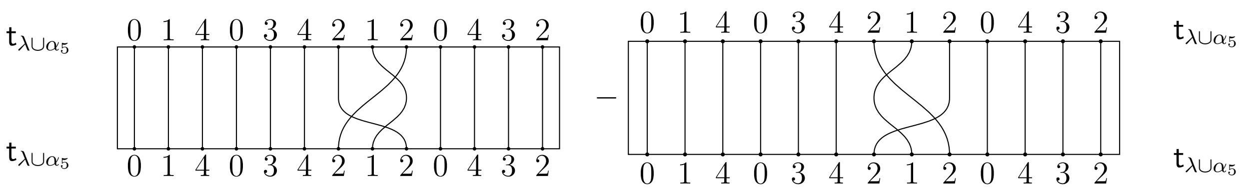


FIGURE 2.7: The righthand-side of (2.14) for $\lambda = (2^6, 1^3)$ and $\alpha = \alpha_5$ is the respective box in Figure (2.4).

We now consider the second term on the righthand-side of equation (2.16), and we have that

$$e_{\mathsf{t}_{\xi-\gamma}} \boxtimes e_{r-1,r,r} = e_{\mathsf{t}_{\xi-\gamma\cup[i,j-1,m]\cup\alpha\cup[i,j+e,m]}}.$$

We will prove that this term is zero modulo the ideal $\mathscr{H}_n^{(\not\succ\lambda)\cup\beta}$. By our inductive assumption for ranks $a-2, a-1<n$ for equation (2.1), we have that:

$$e_{\mathsf{t}_{\xi-\gamma\cup[i,j-1,m]}} \in \mathscr{H}_{a-2}^{\not\succeq\rho} \implies e_{\mathsf{t}_{\xi-\gamma\cup[i,j-1,m]\cup\alpha}} \in \mathscr{H}_{a-1}^{(\not\succeq\rho)\cup\gamma}$$

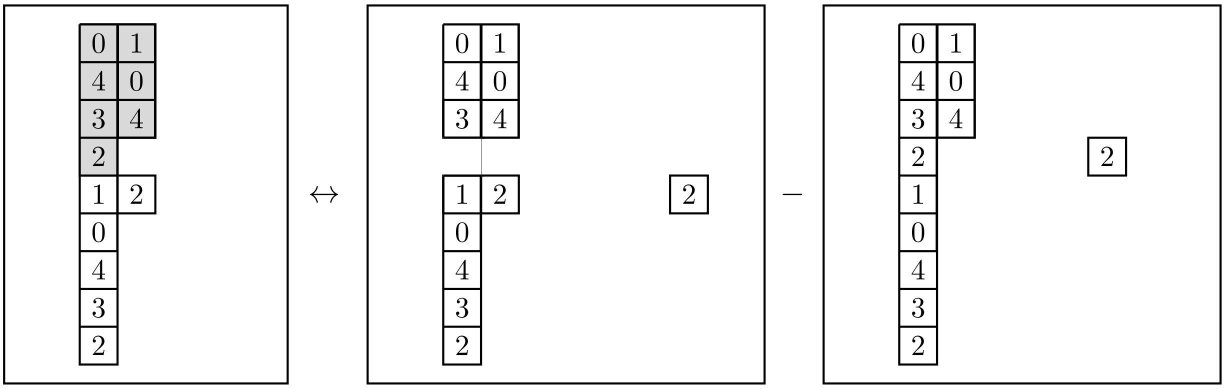


FIGURE 2.8: Let $e = 5$ and $\ell = 1$. The left hand-side is $\lambda \cup \alpha_5$ as in Figure 2.4 (with ξ shaded grey). The righthand-side labels the idempotents obtained from applying equation (2.16).

for $\rho = Y^{\ell+1}_{[i,j-1,m]}(\xi - \gamma \cup [i, j-1, m])$. Given $\pi \not\trianglerighteq (\rho \cup \gamma)$, we can left justify $\pi \cup [i, j+e, m]$ to obtain $\pi \cup \alpha$. We note that $B^\alpha \cap \pi$ contains no nodes of residue r or $r \pm 1$. Therefore

$$e_{\mathsf{t}_{\xi-\gamma\cup[i,j-1,m]\cup\alpha\cup[i,j+e,m]}} \in \mathscr{H}_a^{(\not\trianglerighteq \rho\cup\gamma)\cup\beta} \implies e_{\mathsf{t}_\xi} \boxtimes e_{r-1,r,r} \boxtimes e_{\mathsf{t}_{\lambda\cup\alpha}\downarrow_{>a}} \in \mathscr{H}_n^{(\not\succ\lambda)\cup\beta}$$

as required. (Note that $\gamma, \beta \notin \pi$ by Lemma 2.8.) See Figure 2.9 for an example.

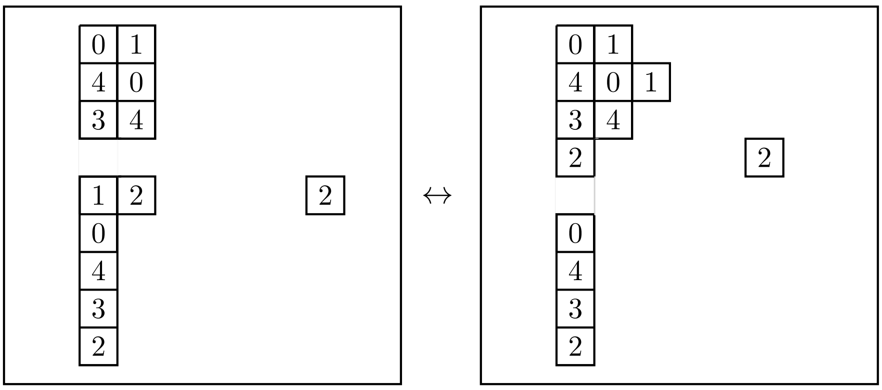


FIGURE 2.9: Rewriting the first term after the equality in equation (2.16). We have moved the 1-box using case (*iii*) and this leaves us free to move the 2-boxes up their corresponding diagonals.

Proof of equation (2.2) for a given λ and α. We assume that (2.2) holds for all $\lambda \in \mathscr{P}_{h,\ell}(n-1)$. We set $\nu = \lambda \cup \alpha$ and we have that $y^{\succ}_{\mathsf{t}_{\lambda\cup\alpha}} = e_{\mathsf{t}_{\lambda\cup\alpha}}$. Recall that

α is a box of residue $r \in \mathbb{Z}/e\mathbb{Z}$. Let $1 \leq k < a \leq n$, we know that

$$y_k y^{\succ}_{\mathsf{t}_{\lambda\cup\alpha}} \in \pm \psi^a_b y^{\succ}_{\mathsf{t}_{\lambda\cup\beta}} \psi^b_a + \mathscr{H}_n^{(\not\succ\lambda)\cup\beta} \tag{2.17}$$

by our inductive assumption on rank $k < a \leq n$ and the definition of $\not\succ$. It remains to compute the product $y_a e_{\mathsf{t}_{\lambda\cup\alpha}}$. We have that

$$y_a e_{\mathsf{t}_{\lambda\cup\alpha}} = \begin{cases} y_{a-1} e_{\mathsf{t}_{\lambda\cup\alpha}} + e_{\mathsf{t}_{\lambda\cup\alpha}} \psi^a_b e_{\mathsf{t}_{\lambda\cup\beta}} \psi^b_a e_{\mathsf{t}_{\lambda\cup\alpha}} & \text{if } r-1 \in \mathrm{res}(\mathsf{Adj}-\mathsf{Gar}(\alpha)); \\ y_b e_{\mathsf{t}_{\lambda\cup\beta}} & \text{if } r-1 \notin \mathrm{res}(\mathsf{Adj}-\mathsf{Gar}(\alpha)) \neq \emptyset; \\ 0 & \text{if } \mathrm{res}(\mathsf{Adj}-\mathsf{Gar}(\alpha)) = \emptyset. \end{cases} \tag{2.18}$$

In the first case, the statement follows from case 3 of relation R5 and the commutativity relations. To see this note that the $(a-1)$th strand has residue $r-1 \in \mathbb{Z}/e\mathbb{Z}$. In the second case, this follows from case 4 of relation R5 and the commutativity relations. To see this note that $b+1 < a$ is maximal such that the corresponding strand is of adjacent residue (namely, $r+1 \in \mathbb{Z}/e\mathbb{Z}$), by Lemma 2.8. In the third case, this follows from from the commutativity and cyclotomic relations. By equation (2.17), the dotted terms on the righthand-side of equation (2.18) belong to the required ideal. □

We let $\lambda^{[0]} \succ \lambda^{[1]} \succ \cdots \succ \lambda^{[m]}$ denote the complete set of elements of $\mathscr{P}_{h,\ell}(n)$ enumerated according to the total ordering $\succ$. Note that we use square brackets in order not to confuse the enumeration of the ℓ-partitions with the notation for components of a given ℓ-partition.

Corollary 2.15. *For $\underline{j} \in (\mathbb{Z}/e\mathbb{Z})^n$, we have that*

$$\begin{cases} e_{\underline{j}} \in \pm \widehat{\psi}^{\mathsf{J}}_{\mathsf{t}_\nu} \widehat{\psi}^{\mathsf{t}_\nu}_{\mathsf{J}} + \mathscr{H}^{\succ\nu} & \textit{if } \mathrm{Shape}(\mathsf{J}) = \nu \in \mathscr{P}_{h,\ell}(n) \\ e_{\underline{j}} = 0 & \textit{if } \mathrm{Shape}(\mathsf{J}) = \varnothing \end{cases} \tag{2.19}$$

where $\widehat{\psi}^{\mathsf{J}}_{\mathsf{t}_\nu}, \widehat{\psi}^{\mathsf{t}_\nu}_{\mathsf{J}}$ are obtained by $\psi^{\mathsf{J}}_{\mathsf{t}_\nu}, \psi^{\mathsf{t}_\nu}_{\mathsf{J}}$, respectively, by possibly adding dot decorations along the strands. For $\nu \in \mathscr{P}_{h,\ell}(n)$, the element $y_a e_{\mathsf{t}_\nu}$ belongs to $\mathscr{H}^{\succ\nu}$ for *any* $1 \leq a \leq n$*. In particular, the* $\mathbb{Z}$*-algebra* $\mathscr{H}_n^\sigma$ *has a chain of two-sided ideals*

$$0 = \mathscr{H}_n^0 \subset \mathscr{H}_n^{\succ\lambda^{[0]}} \subset \mathscr{H}_n^{\succ\lambda^{[1]}} \subset \cdots \subset \mathscr{H}_n^{\succ\lambda^{[m]}} = \mathscr{H}_n^\sigma.$$

Proof. This follows from repeated applications of Proposition 2.12 and the definition of the reverse lexicographic ordering (and the fact that $\ggcurly$ is a coarsening of $\succ$). □

2.3 A tableaux-theoretic basis

In this last section of the second chapter we construct the cellular basis for our quotients of the quiver Hecke algebras. We start the section by preparing the ground for the proof of the main theorem, which is Theorem 2.17.

Recall that the set $\mathcal{Y}_n$ defined in (2.3), contains elements in $\mathscr{H}_n^\sigma$ which are polynomials on the generators $y_1, \cdots, y_n$. The following technical result is an amalgamation of Lemma 2.4 and Proposition 2.5 of [BKW11] and it is crucial towards constructing a basis for our algebra.

Proposition 2.16. *Let* $w \in \mathfrak{S}_n$*. We let* $\underline{w}$*,* $\underline{w}'$ *be any two choices of reduced expression for* w *and let* $\underline{v}$ *be any non-reduced expression for* w*. We have that*

$$e_{\underline{i}}\psi_{\underline{w}}e_{\underline{j}} = e_{\underline{i}}\psi_{\underline{w}'}e_{\underline{j}} + \sum_{\underline{x}<\underline{w},\underline{w}'} e_{\underline{i}}\psi_{\underline{x}}f_{\underline{x}}(y)e_{\underline{j}} \tag{2.20}$$

$$e_{\underline{i}}\psi_{\underline{v}}e_{\underline{j}} = \sum_{\underline{x}<\underline{v}} e_{\underline{i}}\psi_{\underline{x}}e_{\underline{j}}g_{\underline{x}}(y) \tag{2.21}$$

$$y_k e_{\underline{i}}\psi_{\underline{w}}e_{\underline{j}} = e_{\underline{j}}\psi_{\underline{w}}e_{\underline{i}}y_{w(k)} + \sum_{\underline{x}<\underline{w}} e_{\underline{i}}\psi_{\underline{x}}e_{\underline{j}} \tag{2.22}$$

for some $f_{\underline{x}}(y), g_{\underline{x}}(y) \in \mathcal{Y}$*.*

The following theorem is the basic result of this chapter and provides a graded cellular basis for the algebra $\mathscr{H}_n^\sigma$. In order to prove this theorem we shall utilise all the technical work we did previously and in particular Proposition 2.12, hence Corollary 2.15.

Theorem 2.17. *Let $\Bbbk$ be an integral domain. The $\Bbbk$-algebra $\mathscr{H}_n^\sigma$ is a graded cellular algebra with basis*

$$\{\psi^{\mathsf{s}}_{\mathsf{t}_\lambda}\psi^{\mathsf{t}_\lambda}_{\mathsf{t}} \mid \mathsf{s},\mathsf{t} \in \mathrm{Std}(\lambda), \lambda \in \mathscr{P}_{h,\ell}(n)\}. \tag{2.23}$$

We let $\mathscr{H}_n^{\succ\lambda}$ be the $\Bbbk$-submodule of $\mathscr{H}_n^\sigma$ with basis

$$\{\psi^{\mathsf{u}}_{\mathsf{t}_\mu}\psi^{\mathsf{t}_\mu}_{\mathsf{v}} \mid \mathsf{u},\mathsf{v} \in \mathrm{Std}(\mu), \mu \in \mathscr{P}_{h,\ell}(n), \mu \succ \lambda\}.$$

Under the anti-involution $*\colon \mathscr{H}_n^\sigma \longrightarrow \mathscr{H}_n^\sigma$ we have $(\psi^{\mathsf{s}}_{\mathsf{t}_\lambda}\psi^{\mathsf{t}_\lambda}_{\mathsf{t}})^* = \psi^{\mathsf{t}}_{\mathsf{t}_\lambda}\psi^{\mathsf{t}_\lambda}_{\mathsf{s}}$. The degree function $\deg\colon \mathscr{H}_n^\sigma \longrightarrow \mathbb{Z}$ inherited from Definition 1.7. For $\Bbbk$ a field, $\mathscr{H}_n^\sigma$ is quasi-hereditary.

Proof. Let $d \in e_{\underline{i}}\mathscr{H}_n^\sigma e_{\underline{j}}$ be an arbitrary element of the algebra. By equation (2.19) we can rewrite $e_{\underline{j}}$ (or equivalently $e_{\underline{i}}$) so that

$$d = \sum_{x,y\in\mathfrak{S}_n} e_{\underline{i}} a_x e_{\mathsf{t}_\lambda} a_y e_{\underline{j}}$$

for some $a_x, a_y \in \mathscr{H}_n^\sigma$ which are linear combinations of KLR elements tracing out some bijections $x, y \in \mathfrak{S}_n$ respectively (but possibly decorated with dots and need not be reduced) and $\lambda = \mathrm{Shape}(\mathsf{J})$ (see Definition 2.6). It remains to show that $a_x, a_y \in \mathscr{H}_n^\sigma$ can be assumed to be reduced and undecorated. We establish this by induction along the Bruhat order, by working modulo the span of elements

$$\mathrm{span}_\Bbbk\{\psi_{\underline{u}} e_{\mathsf{t}_\lambda}\psi_{\underline{v}} \mid u < x \text{ or } v < y\} + \mathscr{H}_n^{\succ\lambda}. \tag{2.24}$$

If the word $\underline{x}$ is not reduced, then the element $\psi_{\underline{x}} e_{\mathsf{t}_\lambda} a_y$ is zero modulo (2.24) by equation (2.21). Given two choices $\underline{x}, \underline{x}'$ of reduced expression for $x \in \mathfrak{S}_n$ we have that $(\psi_{\underline{x}} - \psi_{\underline{x}'}) e_{\mathsf{t}_\lambda} a_{\underline{y}}$ belongs to equation (2.24) by equation (2.20). Finally, if a_x is obtained from $\psi_{\underline{x}}$ by adding a linear combination of dot decorations (at any points within the expression $\psi_{\underline{x}} = \psi_{s_{i_1}} \dots \psi_{s_{i_k}}$) then $\psi_{\underline{x}} e_{\mathsf{t}_\lambda} a_y$ is zero modulo (2.24) by equation (2.22). Thus $\mathscr{H}_n^{\succeq\lambda}/\mathscr{H}_n^{\succ\lambda}$ is spanned by elements of the form

$$\{\psi_{\underline{x}} e_{\mathsf{t}_\lambda} \psi_{\underline{y}} \mid \text{for } x, y \in \mathfrak{S}_n\} + \mathscr{H}_n^{\succ\lambda}. \tag{2.25}$$

Note that $\underline{x}, \underline{y}$ are arbitrary choices of fixed reduced expressions of $x, y \in \mathfrak{S}_n$. It remains to show that a spanning set is given by the elements $x = w^{\mathsf{s}}_{\mathsf{t}_\lambda}$, $y = w^{\mathsf{t}_\lambda}_{\mathsf{t}}$ for $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda)$. In order to prove this, we need show that when the tableaux indexing these words are non-standard, then the elements belong to the ideal $\mathscr{H}_n^{\succ\lambda}$. More precisely we will proceed by assuming that t is either column or row standard, but non-standard. We will then prove that the element e_{t} belongs to $\mathscr{H}_n^{\succ\lambda}$.

We first consider the case that t is column-standard and not row-standard, that is $\mathsf{t} \in \mathrm{CStd}(\lambda) \setminus \mathrm{Std}(\lambda)$. Then we have that $w^{\mathsf{t}_\lambda}_{\mathsf{t}}$ has a pair of crossing strands from $1 \leq i < j \leq n$ to $1 \leq w^{\mathsf{t}_\lambda}_{\mathsf{t}}(j) < w^{\mathsf{t}_\lambda}_{\mathsf{t}}(i) \leq n$ such that $\mathsf{t}_\lambda^{-1}(i) = [r, c, m]$ and $\mathsf{t}_\lambda^{-1}(j) = [r, c+1, m]$ are in the same row and in particular so that $i = j - 1$. It suffices to show that $\psi_{\underline{x}} e_{\mathsf{t}_\lambda} \psi_{\underline{y}}$ belongs to the ideal $\mathscr{H}_n^{\succ\lambda}$ for a preferred choice of $\underline{y}$; we choose $\underline{y} = s_i \underline{w}$ (for some $w \in \mathfrak{S}_n$ such that $s_i w = y$). Thus it remains to show that $e_{\mathsf{t}_\lambda} \psi_{s_i \underline{w}}$ belongs to $\mathscr{H}_n^{\succ\lambda}$. However, this immediately follows from (2.1) because $e_{\mathsf{t}_\lambda} \psi_{s_i} = \psi_{s_i} e_{s_i(\mathsf{t}_\lambda)}$ and we have that $e_{s_i(\mathsf{t}_\lambda)} \in \mathscr{H}_n^{\succeq\alpha}$ for $\alpha = Y_{\mathsf{t}_\lambda^{-1}(i+1)}(\lambda) \succ \lambda$.

We now consider the case that t is row-standard but not column-standard, that is $\mathsf{t} \in \mathrm{RStd}(\lambda) \setminus \mathrm{Std}(\lambda)$. We let k be minimal such that $\mathsf{t}{\downarrow_{<k}} \in \mathrm{Std}(\mu)$ for some $\mu \in \mathscr{P}_{h,\ell}(k-1)$ and $\mathrm{Shape}(\mathsf{t}{\downarrow_{\leq k}}) = \nu$, for some $\nu \in \mathcal{C}_{h,\ell}(k) \setminus \mathscr{P}_{h,\ell}(k)$. We have that $e_{\mathsf{t}} = e_{\mathsf{t}\downarrow_{\leq k}} \boxtimes e_{\mathsf{t}\downarrow_{>k}}$, and since $\nu \in \mathcal{C}_{h,\ell}(k) \setminus \mathscr{P}_{h,\ell}(k)$, by equation (2.1), we have that $e_{\mathsf{t}_{\leq k}} \in \mathscr{H}_k^{\succ\nu}$. Then by concatenation and the definition of the order $\succ$, we have that $e_{\mathsf{t}} \in \mathscr{H}_n^{\succ\lambda}$. This implies that $e_{\mathsf{t}_\lambda} \psi^{\mathsf{t}}_{\mathsf{t}_\lambda} = \psi^{\mathsf{t}_\lambda}_{\mathsf{t}} e_{\mathsf{t}} \in \mathscr{H}_n^{\succ\lambda}$, as required.

In addition, we notice that there is no tableau $\mathsf{s} \in \mathrm{Std}(\lambda)$, $\lambda \in \mathscr{P}_{h,\ell}(n)$, with

$$\mathrm{res}(\mathsf{s}) = (i_1, i_1+1, i_1+2, \dots, i_1+h).$$

Hence, any idempotent of the form $e_{(i_1,\cdots,i_1+h)}$ annihilates the cell modules and the algebra $\mathscr{H}_n^{\sigma}$ is of rank at least $\sum_{\lambda \in \mathscr{P}_{h,\ell}(n)} |\mathrm{Std}(\lambda)|^2$. Therefore, the spanning set is linearly independent (and hence a basis) as required.

Finally, we note that each layer of the cell chain contains an idempotent e_{t_λ}. Hence when $\Bbbk$ is a field, by [KX99a], the algebra is quasi-hereditary as required. $\square$

Remark 2.18. A nice property of the quasi-hereditary algebras is that their simple modules are generated by an idempotent hence they have the same number of cell and simple modules.

Chapter 3

Simple modules of the blob algebra

In this chapter we present the basic results of our research on the blob algebra. The main algebraic and combinatorial concepts have been defined in the first chapter and the notation will follow from there. The first section of this chapter is devoted to the alcove geometry of type $\hat{A}_1$. Alcove geometries will play central role in the proofs of this study as they provide a way of visualising our arguments. In the second section we shall briefly present basic results on the blob algebra, proven by other researchers, which form the building blocks for our proofs. The third section is devoted in the construction of homomorphisms between cell modules of the blob algebra (regardless the characteristic of the field). We also construct the images of these homomorphisms which will be crucial in the calculation of the radical of the cell modules. In section 4, by utilising the images of the homomorphisms we construct bases for the simple modules of the blob algebra over a field of characteristic zero. In order to do that we construct a B_n^σ-module with bar-invariant graded dimension and we prove it to be equal to the simple head of the corresponding simple module.

One of the main reasons we initiated our study is [dBELS18], where de Boeck, Evseev, Lyle and Speyer constructed bases for the simple modules of Temperley–Lieb algebras of type A. Our ambition was to construct such bases for simple modules of Temperley–Lieb algebras of type B over a field of characteristic zero. The blob algebra has a very rich and fruitful combinatorial theory arising from the fact that we need to consider partitions with 2 components.

The work presented in this chapter is based on the author's work in [Mic21].

3.1 Alcove geometry of type $\hat{A}_1$

Let $\{\varepsilon_1, \varepsilon_2\}$ be formal symbols. We consider the 2-dimensional Euclidean space

$$V := \bigoplus_{i=1,2} \mathbb{R}\varepsilon_i$$

with basis $\{\varepsilon_1, \varepsilon_2\}$ and let $V_{\mathbb{Z}_{\geq 0}}$ be the $\mathbb{Z}_{\geq 0}$-span of $\{\varepsilon_1, \varepsilon_2\}$. To any bipartition $\lambda = ((1^{\lambda_1}), (1^{\lambda_2})) \in \mathrm{Bip}_1(n)$ we attach a point of the Euclidean space V via the embedding $((1^{\lambda_1}), (1^{\lambda_2})) \longmapsto \sum_{i=1,2} \lambda_i \varepsilon_i$. We consider the affine Weyl group $W_{\text{aff}} \cong \hat{\mathfrak{S}}_2$ of type $\hat{A}_1$ with $\alpha_1 = \varepsilon_1 - \varepsilon_2$ the corresponding simple real root. Note that the affine Weyl group is generated by the reflection $s_{\alpha_1,-1/2}$ and the reflection $s_{\alpha_1,1/2}$, where the later corresponds to translation of the former by $e\alpha_1$. Let $(\cdot,\cdot)$ be a symmetric bilinear form on V determined by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta. For a given e-bicharge $\sigma = (\sigma_0, \sigma_1)$ we set $\rho := (\sigma_1 - \sigma_0)\varepsilon_1$.

Definition 3.1. For any $m \in \mathbb{Z}$ we define the hyperplane

$$H_{\alpha_1, m-\frac{1}{2}} := \{v \in V \mid (v + \rho, \alpha_1) = me\}. \tag{3.1}$$

and we sometimes refer such a hyperplane as a wall.

For any $m \in \mathbb{Z}$ there exists a unique reflection $s_{\alpha_1, m-1/2}$ such that

$$s_{\alpha_1, m-\frac{1}{2}} \cdot v = v - ((v + \rho, \alpha_1) - me)\alpha_1$$

for any $v \in V$. In other words $s_{\alpha_1,m-1/2}$ acts on V by reflection with respect to the hyperplane $H_{\alpha_1,m-1/2}$. From now on, since we have only one simple real root α_1, we shall write simply $H_{m-1/2}$, $s_{m-1/2}$ for the wall and the reflection corresponding to the integer $m \in \mathbb{Z}$, respectively.

For any two integers $r, s \in \mathbb{Z}$ we denote by $[r, s]$ the set $[r, s] = \{t \in \mathbb{Z} \mid r \leq t \leq s\}$. For $n \in \mathbb{Z}_{>0}$ we define $\mathrm{Path}(n)$ to be the set of maps $\pi\colon [0, n] \longrightarrow V_{\mathbb{Z}_{\geq 0}}$ such that

$$\pi(0) = 0 \text{ and } \pi(k+1) - \pi(k) \in \{\varepsilon_1, \varepsilon_2\}$$

for all $k \in [0, n-1]$ and we call its elements **paths** from 0 to n. Given a standard tableau $\mathsf{t} \in \mathrm{Std}(n)$ we define the point $\pi_{\mathsf{t}}(k)$ in the space $V_{\mathbb{Z}_{\geq 0}}$ by the formula

$$\pi_{\mathsf{t}}(k) := c_{k,1}(\mathsf{t})\varepsilon_1 + c_{k,2}(\mathsf{t})\varepsilon_2 \tag{3.2}$$

where $c_{k,i}(\mathsf{t})$ is the number of nodes of the tableau $\mathsf{t}_{\leq k}$ in the ith component. Using the aforementioned notation we shall define the path in $V_{\mathbb{Z}_{\geq 0}}$ attached to a standard tableau $\mathsf{t} \in \mathrm{Std}(\lambda)$.

Definition 3.2. Let $\mathsf{t} \in \mathrm{Std}(n)$ be a standard tableau. We define the path π_{t} corresponding to the tableau t given by the sequence of points

$$\pi_{\mathsf{t}} = (\pi_{\mathsf{t}}(0), \cdots, \pi_{\mathsf{t}}(n))$$

in the sense of relation (3.2). There is a bijection between the set $\mathrm{Std}(d)$ of standard tableaux and the set of paths $\mathrm{Path}(d)$, given by $\mathsf{t} \longmapsto \pi_{\mathsf{t}}$.

Definition 3.3. Let $\mathsf{t} \in \mathrm{Std}(d)$ and suppose that $\pi_{\mathsf{t}}(a) \in H_{m-1/2}$ is the ith intersection point of π_{t} with the hyperplane $H_{m-1/2}$. We define the path $s^i_{m-1/2} \cdot \pi_{\mathsf{t}}$ as follows

$$(s^i_{m-\frac{1}{2}} \cdot \pi_{\mathsf{t}})(k) := \begin{cases} \pi_{\mathsf{t}}(k) & \text{if } 0 \leq k \leq a \\ s_{m-1/2} \cdot \pi_{\mathsf{t}}(k) & \text{if } a < k \leq n \end{cases}.$$

We refer to the path $s^i_{m-1/2} \cdot \pi_{\mathsf{t}}$ as the **reflected path** through the ith intersection point of π_{t} with the hyperplane $H_{m-1/2}$.

Remark 3.4. Note that if the path π_{t} intersects the hyperplane $H_{m-1/2}$ at a unique point, then we shall denote the reflected path simply by $s_{m-1/2} \cdot \pi_{\mathsf{t}}$.

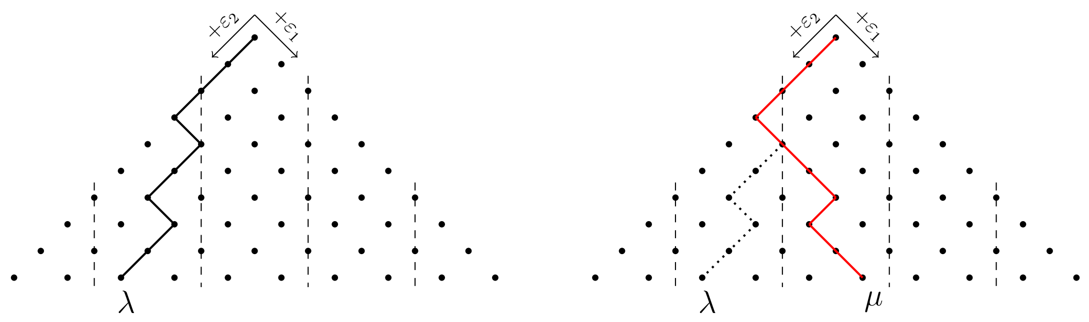


FIGURE 3.1: The path T and the reflected path $s^2_{-1/2} \cdot \mathsf{T}$.

In Figure 3.1 we visualise the last definition. We consider a Pascal triangle with points corresponding to integers and the top of the triangle corresponds to 0. We can represent the paths in V as paths in the Pascal triangle starting from the top and moving downwards. Let $\lambda = ((1^{\lambda_1}), (1^{\lambda_2})) \in \mathrm{Bip}_1(n)$ and $\mathsf{t} \in \mathrm{Std}(\lambda)$ be a standard tableau. The path π_{t} is a path starting from the top of the Pascal triangle and ending at a point corresponding to the integer $\lambda_1 - \lambda_2$ at the level n of the triangle. We draw a path with endpoint the bipartition λ and we also draw the reflected path through its second intersection point with the hyperplane $H_{-1/2}$.

Let $u, v \in V_{\mathbb{Z}_{\geq 0}}$ such that $u - v = \varepsilon_i$, $i = 1, 2$. Then we define the degree of the pair (u, v) as follows

$$\deg(u,v) := \begin{cases} 1 & \text{if } u \in H_{m-\frac{1}{2}} \text{ and } |(v+\rho,\alpha_1)| < |me| \text{ for some } m \in \mathbb{Z}; \\ -1 & \text{if } v \in H_{m-\frac{1}{2}} \text{ and } |(u+\rho,\alpha_1)| > |me| \text{ for some } m \in \mathbb{Z}; \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

By using relation (3.3) we are able to give a reinterpretation of the degree of a tableau in terms of paths.

Definition 3.5. Let $\mathsf{t} \in \mathrm{Std}(n)$ and $\pi_{\mathsf{t}} \in \mathrm{Path}(n)$ be the path corresponding to t. The integer

$$\deg(\pi_{\mathsf{t}}) := \sum_{k=0}^{n-1} \deg(\pi_{\mathsf{t}}(k), \pi_{\mathsf{t}}(k+1))$$

is the degree of the path π_{t}.

Remark 3.6. By [Pla13, Corrolary 4.6] we have that the degree of the path π_{t} coincides with the degree of the tableau t.

Using the aforementioned notions we are able to describe an alcove geometry on the Euclidean space V. We say that for any $m \in \mathbb{Z}$, the set of points

$$\mathfrak{a}_m := \{v \in V \mid me < (v + \rho, \alpha_1) < (m+1)e\}$$

forms an alcove. By the definition of the hyperplane as presented in (3.1), we can deduce that the origin, namely the point $(0, 0)$, will always lie in an alcove and not on a hyperplane.

Notation. From now on we shall not distinguish between the standard tableau and the corresponding path. Namely, we will denote the path corresponding to the tableau t by

$$\mathsf{T} = (\mathsf{T}(0), \cdots, \mathsf{T}(n)) \in \mathrm{Path}(n).$$

Moreover, let $\lambda \in \mathrm{Bip}_1(n)$ and $\mu \in \mathrm{Bip}_1(n')$ with $n' < n$. We denote by $\mathrm{Path}(\mu \to \lambda)$ the set of paths starting from the bipartition μ and ending at the bipartition λ. We also let $\mathrm{Path}(\lambda) := \mathrm{Path}(\varnothing \to \lambda)$. Namely the paths of $\mathrm{Path}(\lambda)$ are paths starting from the top of the Pascal triangle and they have the bipartition λ as endpoint. By using the above notation we have that

$$\mathrm{Path}(n) = \bigcup_{\lambda \in \mathrm{Bip}_1(n)} \mathrm{Path}(\lambda).$$

In the following example we shall summarise most of the facts we discussed above. Recall that we denote by t^t the transpose of a tableau $\mathsf{t} \in \mathrm{Std}(n)$.

Example 3.7. *Suppose* $n = 9$, $e = 4$, $\sigma = (0, 2)$ *and let* $\lambda = ((1^2), (1^7)), \mu = ((1^5), (1^4)) \in \mathrm{Bip}_1(9)$. *We consider the* λ*-tableau*

$$\mathsf{t}^t = \left(\boxed{4}\boxed{7}, \ \boxed{1}\boxed{2}\boxed{3}\boxed{5}\boxed{6}\boxed{8}\boxed{9} \right).$$

By following the description above we can construct the path corresponding to the tableau t *as in the left picture in Figure 3.1. We observe that the path* T *intersects the hyperplane* $H_{-1/2}$ *at two points which correspond to the steps* $\mathsf{t}^1_{-1/2}$ *and* $\mathsf{t}^2_{-1/2}$ *of the path. Then we obtain the reflected paths* $s^1_{-1/2} \cdot \mathsf{T}$, $s^2_{-1/2} \cdot \mathsf{T}$ *and the later is also pictured in Figure 3.1. The endpoint of the reflected paths is the bipartition* μ.

Moreover one can easily calculate the degree of the path T *to be equal to* -1. *To see this note that* $\deg(\mathsf{T}(3), \mathsf{T}(4)) = -1$ *and degree is zero otherwise. This is something we expect since* $\deg(\mathsf{t}^{-1}(4)) = -1$ *and the rest nodes of the tableau* t *are of degree* 0.

The residue sequence of the tableau t *is*

$$\mathrm{res}(\mathsf{t}) = (2, 1, 0, 0, 3, 2, 3, 1, 0)$$

and we observe that $\mathrm{res}(s^2_{-1/2} \cdot \mathsf{t}) = \mathrm{res}(\mathsf{t})$.

More generally, from [Pla13, Lemma 4.7] we have that given any two tableaux $\mathsf{t}, \mathsf{s} \in \mathrm{Std}(n)$ we have that

$$\mathrm{res}(\mathsf{t}) = \mathrm{res}(\mathsf{s}) \Longleftrightarrow \mathsf{T} = s^{j_1}_{i_1 - 1/2} \cdots s^{j_a}_{i_a - 1/2} \cdot \mathsf{S} \tag{3.4}$$

for some simple reflections $s_{i_l - 1/2}$, $1 \leq l \leq a$. Given two bipartitions $\lambda, \mu \in \mathrm{Bip}_1(n)$ and $\mathsf{T} \in \mathrm{Path}(\lambda)$, we define the set of μ-paths which can be obtained by T by a series of reflections as follows:

$$\mathrm{Path}(\mu, \mathsf{T}) := \{\mathsf{S} \in \mathrm{Path}(\mu) \mid \mathsf{S} = s^{j_1}_{i_1 - 1/2} \cdots s^{j_a}_{i_a - 1/2} \cdot \mathsf{T}, \text{ for some } s_{i_l - 1/2} \in \hat{\mathfrak{S}}_2\}.$$

Now we equip our alcove geometry with a *length function*

$$\ell\colon \mathrm{Bip}_1(n) \longrightarrow \tfrac{1}{2}\mathbb{Z},\ \lambda = ((1^{\lambda_1}),(1^{\lambda_2})) \longmapsto \begin{cases} m & \text{if } \lambda_1\varepsilon_1+\lambda_2\varepsilon_2 \in \mathfrak{a}_m \\ m-\frac{1}{2} & \text{if } \lambda_1\varepsilon_1+\lambda_2\varepsilon_2 \in H_{m-\frac{1}{2}}. \end{cases}$$

We will also give a useful geometric interpretation of the dominance order on tableaux, mentioned in Definition 1.37 in terms of the alcove geometry. Given two tableau $\mathsf{t},\mathsf{s} \in \mathrm{Std}(n)$ with $\mathrm{res}(\mathsf{t}) = \mathrm{res}(\mathsf{s})$ we say that the node $\mathsf{t}^{-1}(k)$ is less dominant that the node $\mathsf{s}^{-1}(k)$ in the sense of Definition 1.35 if and only if

$$|\ell(\mathrm{Shape}(\mathsf{t}_{\leq k})| > |\ell(\mathrm{Shape}(\mathsf{s}_{\leq k})|.$$

The tableau t is less dominant than s if and only if $\mathsf{t}^{-1}(k) \trianglelefteq \mathsf{s}^{-1}(k)$, $1 \leq k \leq n$, and there is at least one node of t strictly less dominant than the corresponding node of s.

Example 3.8. *We continue on the Example 3.7 and we have that $\ell(\lambda) = -1$ while $\ell(\mu) = 0$. The paths $\mathsf{S}_1, \mathsf{S}_2$ drawn in the following figure are the elements of $\mathsf{Path}(\mu, \mathsf{T})$.*

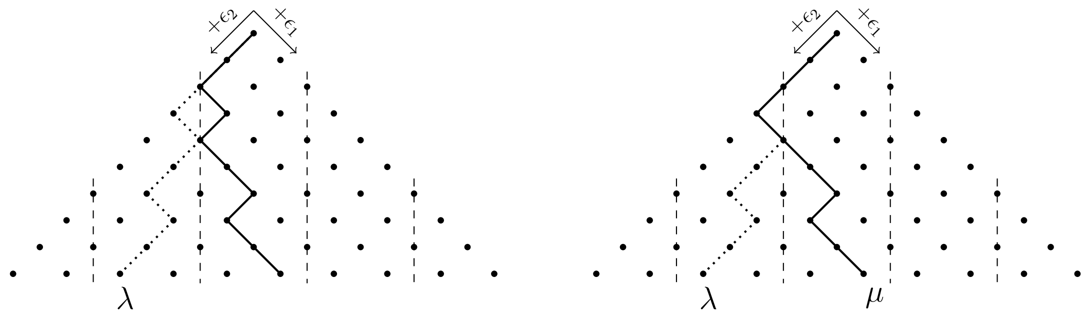


FIGURE 3.2: The paths S_1 and S_2 are solid. The path T is dotted.

In particular we have that $\mathsf{S}_1 = s^1_{-1/2} \cdot \mathsf{T}$ and $\mathsf{S}_2 = s^2_{-1/2} \cdot \mathsf{T}$. Moreover we observe that $\mathsf{S}_1 \trianglerighteq \mathsf{S}_2 \trianglerighteq \mathsf{T}$.

Recall that to each bipartition $\lambda = ((1^{\lambda_1}),(1^{\lambda_2})) \in \mathrm{Bip}_1(n)$ we can attach the integer $\lambda_1 - \lambda_2$. Hence the action of the affine Weyl group W_{aff} on the set of bipartitions of d can be described in terms of the action of W_{aff} on $\mathbb{Z}$. In particular, reflection $s_{m-1/2}$ corresponds to reflection about the integers $(\sigma_0 - \sigma_1) + me$, for all $m \in \mathbb{Z}$. Note that the integer obtained by reflecting as above corresponds to a bipartition of d. We say that two bipartitions λ, μ are *linked* with respect to the alcove geometry of type $\hat{A}_1$ and we write $\lambda \sim \mu$ if they belong to the same W_{aff}-orbit, i.e $\lambda \in W_{\text{aff}} \cdot \mu$. If $\mathsf{T} \in \mathsf{Path}(\lambda)$ then the paths linked with T are the paths of $\mathsf{Path}(\mu, \mathsf{T})$, defined above, for $\mu \sim \lambda$. The paths linked with the path T^λ will be of particular interest when we construct homomorphisms of the blob algebra. We define

$$\mathsf{Path}_{\sim}(\lambda) := \bigcup_{\mu \trianglerighteq \lambda} \mathsf{Path}(\mu, \mathsf{T}^\lambda).$$

Remark 3.9. If $\lambda, \mu \in \text{Bip}_1(n)$ are two bipartitions, we note that λ is less dominant than μ if and only if $\lambda \sim \mu$ and $|\ell(\lambda)| > |\ell(\mu)|$, i.e. λ is further away from the origin of the Pascal triangle than μ.

Let us see an example regarding the notions we discussed above.

Example 3.10. *Let* $n = 9, e = 4, \sigma = (0, 2)$ *as in Example 3.7 and* $\lambda = ((1), (1^8)) \in \text{Bip}_1(9)$. *Then*

$$(\mathsf{t}^\lambda)^t = \left(\boxed{2}, \ \boxed{1}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{7}\boxed{8}\boxed{9} \right)$$

and the paths linked with T^λ *are drawn in the following diagram*

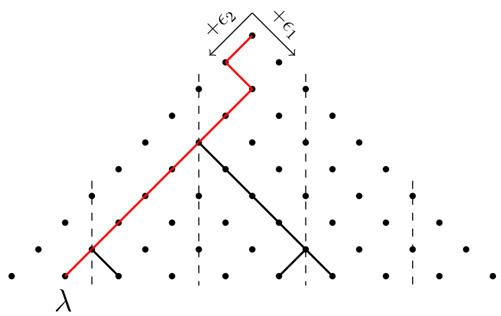


FIGURE 3.3: The red path is the path T^λ and the black ones are those linked with T^λ.

The new linked paths correspond to the tableaux

$$\mathsf{s}_1^t = \left(\boxed{2}\boxed{9}, \ \boxed{1}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{7}\boxed{8} \right)$$

$$\mathsf{s}_2^t = \left(\boxed{2|5|6|7|8}, \ \boxed{1|3|4|9} \right)$$

and

$$\mathsf{s}_3^t = \left(\boxed{2|5|6|7|8|9}, \ \boxed{1|3|4} \right).$$

Thus $\mathsf{Path}_{\sim}(\lambda) = \{\mathsf{T}^\lambda, \mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3\}$ *and one can easily see that for a given bipartition* $\mu \in \mathrm{Bip}_1(9)$, $\mathsf{Path}(\mu, \mathsf{t}^\lambda) \neq \emptyset$ *implies* $\mu \trianglerighteq \lambda$.

3.2 Algebraic structure of the blob algebra

In this section we shall briefly present some the basic properties of the blob algebra. This will include the adjustment of the general algebraic properties of Chapter 1, in the context of the blob algebra. These properties are essential for our work and have been proven by various researchers in the past. This is the reason we shall not provide detailed proofs and we shall refer to the relevant literature instead.

If $w = s_{i_1} \cdots s_{i_l} \in \mathfrak{S}_n$ is a reduced expression of an element of the symmetric group, recall that

$$\psi_w = \psi_{i_1} \cdots \psi_{i_l} \in \mathsf{B}_n^\sigma.$$

Also, recall that for any tableau $\mathsf{t} \in \mathrm{Std}(\lambda)$, $\lambda \in \mathrm{Bip}_1(n)$, we have defined the reduced expression $w_\mathsf{t} = s_{i_1} \cdots s_{i_l} \in \mathfrak{S}_n$ such that $\mathsf{t} = w_\mathsf{t} \mathsf{t}^\lambda$. Recall from Remark 1.41 that the reduced expression of t is unique up to the commuting relations of the symmetric group. We define the element

$$\psi_\mathsf{t} := \psi_{i_1} \cdots \psi_{i_l} e_{\underline{i}^\lambda},$$

and again by Remark 1.41 and (3.2) we have that the product $\psi_{i_1} \cdots \psi_{i_l}$ is unique up to the commuting KLR relation (i.e. the second relation of (R3)). Suppose that $\lambda \in \mathrm{Bip}_1(n)$ and $\mathsf{t}, \mathsf{s} \in \mathrm{Std}(\lambda)$. We set

$$\psi_{\mathsf{st}} = \psi_\mathsf{s} e_{\underline{i}^\lambda} \psi_\mathsf{t}^* \in \mathsf{B}_n^\sigma.$$

Remark 3.11. In order to connect the element $\psi_{\mathsf{st}} \in \mathsf{B}_n^\sigma$ above with the elements $\psi_{\mathsf{t}}^{\mathsf{s}}$ of Definition 2.11, we remark that

$$\psi_{\mathsf{st}} = \psi_{\mathsf{s}}^{\mathsf{t}^\lambda} \psi_{\mathsf{t}^\lambda}^{\mathsf{t}}.$$

For ease of notation, in the remainder of the thesis we shall carry on using the symbol ψ_{st}.

The following theorem summarises the fact that the blob algebra is cellular in the sense of Graham–Lehrer [GL96], as presented in the first chapter of the thesis. Moreover, it has the structure of a graded cellular algebra in the sense of [HM10]

Theorem 3.12 ([PRH14, Theorem 6.10]). *The blob algebra B_n^σ is a graded $\Bbbk$-algebra with basis*

$$\{\psi_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda) \text{ for } \lambda \in \mathrm{Bip}_1(n)\}.$$

We let $\mathsf{B}_{n, \rhd\lambda}^\sigma$ be the $\Bbbk$-submodule of B_n^σ with basis

$$\{\psi_{\mathsf{uv}} \mid \mathsf{u}, \mathsf{v} \in \mathrm{Std}(\mu) \text{ for } \mu \in \mathrm{Bip}_1(n), \mu \rhd \lambda\}.$$

Under the anti-involution $*\colon \mathsf{B}_n^\sigma \longrightarrow \mathsf{B}_n^\sigma$, we have $\psi_{\mathsf{st}}^* = \psi_{\mathsf{ts}}$. For any $\lambda \in \mathrm{Bip}_1(n)$, $\mathsf{t} \in \mathrm{Std}(\lambda)$ and $a \in \mathsf{B}_n^\sigma$ there exists $\alpha_{\mathsf{u}} \in \Bbbk$ such that for all $\mathsf{s} \in \mathrm{Std}(\lambda)$

$$a\psi_{\mathsf{st}} = \sum_{\mathsf{u} \in \mathrm{Std}(\lambda)} \alpha_{\mathsf{u}} \psi_{\mathsf{ut}} \mod \mathsf{B}_{n, \rhd\lambda}^\sigma.$$

In particular the blob algebra B_n^σ is a graded cellular algebra.

Again by the classical theory of cellular algebras as presented in the first chapter, we know that there exists a family of modules $\{\Delta(\lambda) \mid \lambda \in \mathrm{Bip}_1(\lambda)\}$ with $\Bbbk$-basis

$$\{\psi_{\mathsf{t}} \mid \mathsf{t} \in \mathrm{Std}(\lambda)\}$$

called cell or standard modules and there is a unique bilinear form $\langle\cdot,\cdot\rangle\colon \Delta(\lambda)\times\Delta(\lambda)\longrightarrow \Bbbk$ such that $\langle\psi_{\mathsf{s}},\psi_{\mathsf{t}}\rangle$ for $\mathsf{s},\mathsf{t}\in \mathrm{Std}(\lambda)$, is given by

$$\langle\psi_{\mathsf{s}},\psi_{\mathsf{t}}\rangle\psi_{\mathsf{uv}} = \psi_{\mathsf{us}}\psi_{\mathsf{tv}} \mod \mathsf{B}^{\sigma}_{n,\rhd\lambda}.$$

The radical of a cell module $\Delta(\lambda)$ is given by

$$\mathrm{rad}\ \Delta(\lambda) := \{x\in\Delta(\lambda) \mid \langle x,y\rangle = 0 \text{ for all } y\in\Delta(\lambda)\}$$

and let $L(\lambda) := \Delta(\lambda)/\mathrm{rad}\ \Delta(\lambda)$. By [MW00, Section 9], the bilinear form is non-degenerate and so B^{σ}_n is quasi-hereditary with simples $\{L(\lambda)\mid \lambda\in \mathrm{Bip}_1(n)\}$.

In this thesis we focus our interest in the graded version of the blob algebra. Many of the proofs of the known results for the graded case, use results for the ungraded case. Moreover, the results in the ungraded case are motivation for similar results when we add the graded structure.

Let M be a finite dimensional graded B^{σ}_n-module and let $M=\bigoplus_{i\in\mathbb{Z}} M_i$ be its decomposition into direct sum of homogeneous components.

Definition 3.13. We define the graded dimension of M to be the polynomial

$$\dim_t(M) := \sum_{i\in\mathbb{Z}} (\dim M_i)t^i \in \mathbb{Z}[t,t^{-1}]$$

where t is an indeterminate.

Moreover if $L(\lambda)$ is a simple graded B^{σ}_n-module, we denote by $L(\lambda)\langle k\rangle$ the graded B^{σ}_n-module obtained by shifting the grading on $L(\lambda)$ up by k, namely

$$L(\lambda)\langle k\rangle = \bigoplus_{i\in\mathbb{Z}} L(\lambda)\langle k\rangle_i = \bigoplus_{i\in\mathbb{Z}} L(\lambda)_{i-k}.$$

The following theorem summarises the work of Plaza and Ryom–Hansen using the Hu and Mathas' work on graded cellular structure of KLR algebras, [PRH14, HM10].

Theorem 3.14. *Let $L(\lambda)$, $\lambda \in \mathrm{Bip}_1(n)$ be a simple module. Then*

$$\{L(\lambda)\langle k\rangle \mid \lambda \in \mathrm{Bip}_1(n) \text{ and } k \in \mathbb{Z}\}$$

is a complete set of pairwise non-isomorphic simple graded B_n^σ-modules.

We also have the following useful proposition from [HM10] regarding the graded dimension of a simple module.

Proposition 3.15. *For any $\lambda \in \mathrm{Bip}_1(n)$ the graded dimension of the simple module $L(\lambda)$ is bar-invariant (i.e., fixed under interchanging t and t^{-1}).*

Proof. See [HM10, Proposition 1.8]. □

It is also important to know the block structure of the blob algebra. In our case of study the block structure is controlled by a *linkage property* with respect to the affine Weyl group W_{aff} of type $\hat{A}_1$.

Proposition 3.16 ([MW00, Theorem 9.3])**.** *Let $\lambda, \mu \in \mathrm{Bip}_1(n)$. Two simple modules $L(\lambda)$, $L(\mu)$ are in the same block of B_n^σ if and only if λ and μ are in the same orbit, i.e $\lambda \in W_{\mathrm{aff}} \cdot \mu$.*

For a graded B_n^σ-module M we denote by $[M\colon L(\lambda)\langle k\rangle]$ the graded multiplicity of the simple module $L(\lambda)\langle k\rangle$ as a graded composition factor of M. Then the graded decomposition number is

$$[M\colon L(\lambda)]_t := \sum_{k\in\mathbb{Z}} [M\colon L(\lambda)\langle k\rangle] t^k \in \mathbb{Z}[t, t^{-1}].$$

In particular we are interested in the decomposition matrix $\mathbf{D} = (d_{\mu\lambda})_{\mu,\lambda\in\mathrm{Bip}_1(n)}$, that is the decomposition numbers

$$d_{\mu\lambda} = [\Delta(\mu)\colon L(\lambda)]_t$$

which were computed, over a field $\Bbbk$ of characteristic zero, by Plaza [Pla13]. The closed formula for the graded decomposition number $[\Delta(\mu)\colon L(\lambda)]_t$ depends on whether the bipartition λ lies in an alcove or on a hyperplane. However using the length function we defined before we can amalgamate the two distinct formulas into one. In what follows we assume that $\mu \trianglerighteq \lambda$, since this is the only case we can have a non-zero decomposition number, by Theorem 3.12. The following theorem gives the graded decomposition numbers of the blob algebra.

Theorem 3.17 ([Pla13, Theorem 5.11, 5.15])**.** *Let $\Bbbk$ be a field of characteristic zero and let $\lambda, \mu \in \mathrm{Bip}_1(d)$ be two linked bipartitions with $\lambda \trianglelefteq \mu$. Then*

$$[\Delta(\mu) \colon L(\lambda)]_t = t^{|\ell(\lambda)|-|\ell(\mu)|}.$$

Remark 3.18. We remark that by the construction of our alcove geometry the difference $|\ell(\lambda)| - |\ell(\mu)|$ is strictly positive for $\lambda \triangleleft \mu$.

Example 3.19. *We continue with the Example 3.10 and let $\lambda_1 = ((1^2),(1^7))$, $\lambda_2 = ((1^5),(1^4))$, $\lambda_3 = ((1^6),(1^3))$. According to Theorem 3.17, the (non-zero) graded decomposition numbers are the following:*

$$[\Delta(\lambda) \colon L(\lambda)] = 1,\ [\Delta(\lambda_1) \colon L(\lambda)] = t,\ [\Delta(\lambda_2) \colon L(\lambda)] = t^2 \text{ and } [\Delta(\lambda_3) \colon L(\lambda)] = t.$$

3.3 One column homomorphisms between cell modules

In this section we shall construct homomorphisms between certain cell modules of the blob algebra. By using the fact that B_n^σ is quasi-hereditary, we know that for a given bipartition $\nu \in \mathrm{Bip}_1(n)$ we have $\mathrm{Hom}_{\mathsf{B}_n^\sigma}(\Delta(\nu'), \Delta(\nu)) \neq 0$ only if $\nu' \trianglelefteq \nu$. For the purposes of the thesis we need to construct homomorphisms between cell modules indexed by linked bipartitions which also have lengths with absolute value differing by one.

Let $\mu \in \mathrm{Bip}_1(n)$ be a bipartition with $\ell(\mu) = m$ or $\ell(\mu) = m - 1/2$ for some integer $m \in \mathbb{Z}$. Equivalently μ lies in the alcove $\mathfrak{a}_m$ or in the hyperplane $H_{m-1/2}$.

1. Suppose that $\ell(\mu) = m$, $m \in \mathbb{Z}$. There exist at most two bipartitions λ, λ' with $\lambda, \lambda' \sim \mu$ satisfying $|\ell(\lambda)| = |\ell(\lambda')| = |\ell(\mu)| + 1$. We wish to distinguish between these bipartitions (when they exist).

   - If $m \leq 0$ is a non-positive integer then we let $\ell(\lambda) < 0$ and $\ell(\lambda') > 0$;
   - if $m > 0$ is a positive integer then we let $\ell(\lambda) > 0$ and $\ell(\lambda') < 0$.

   In this case we shall construct maps in the sets $\mathrm{Hom}_{\mathsf{B}_n^\sigma}(\Delta(\lambda'), \Delta(\mu))$ and $\mathrm{Hom}_{\mathsf{B}_n^\sigma}(\Delta(\lambda), \Delta(\mu))$.

2. Suppose that $\ell(\mu) = m - 1/2$, $m \in \mathbb{Z}$. In this case we fix the unique bipartition λ' with $\lambda' \sim \mu$ and $|\ell(\lambda')| = |\ell(\mu)| + 1$, such that

   - if $m \leq 0$ is a non-positive integer then $\ell(\lambda') > 0$;
   - if $m > 0$ is a positive integer then $\ell(\lambda') < 0$.

   In this case we shall construct map in the set $\mathrm{Hom}_{\mathsf{B}_n^\sigma}(\Delta(\lambda'), \Delta(\mu))$.

Notation. From now on we will make the following abuse of notation. We shall not distinguish between the tableau t and the attached path T and both will be denoted by T. Moreover we shall denote by $\mathsf{t}^i_{m-1/2}$ the ith intersection point of the path T with the hyperplane $H_{m-1/2}$.

In what follows we shall restrict ourselves in the case that $m \leq 0$ and we shall construct the maps we discussed above. Note that the results are not affected by whether $m \leq 0$ or $m > 0$. This is just a convention in order to save space since everything is analogous for $m > 0$.

Definition 3.20. Let $\mu \in \mathrm{Bip}_1(n)$ be a bipartition.

1. If $\ell(\mu) = m$, $m \leq 0$, we define the maps $\varphi^\mu_\lambda \colon \Delta(\lambda) \longrightarrow \Delta(\mu)$ and $\varphi^\mu_{\lambda'} \colon \Delta(\lambda') \longrightarrow \Delta(\mu)$ as follows:

$$\varphi^\mu_\lambda(\psi_{\mathsf{T}^\lambda}) := \psi_{s_{m-1/2} \cdot \mathsf{T}^\lambda} \tag{3.5}$$

   and

$$\varphi^\mu_{\lambda'}(\psi_{\mathsf{T}^{\lambda'}}) := \psi_{s_{1/2} \cdot \mathsf{T}^{\lambda'}} \tag{3.6}$$

where the paths $s_{m-1/2} \cdot \mathsf{T}^{\lambda}$ and $s_{1/2} \cdot \mathsf{T}^{\lambda'}$ are the reflections of the paths T^{λ}, $\mathsf{T}^{\lambda'}$ through the hyperplanes $H_{m-1/2}$, $H_{1/2}$ respectively.

2. If $\ell(\mu) = m - 1/2$, $m \leq 0$, then we define the map $\varphi^{\mu}_{\lambda'} \colon \Delta(\lambda') \longrightarrow \Delta(\mu)$ on the same way as in equation (3.6).

Remark 3.21. Note that each of the paths T^{λ} and $\mathsf{T}^{\lambda'}$ intersects the hyperplanes $H_{m-1/2}$ and $H_{1/2}$ at precisely one point and we have dropped the superscripts.

Remark 3.22. In the case that $m > 0$, we can define the maps φ^{μ}_{λ} and $\varphi^{\mu}_{\lambda'}$ in an analogous way. In particular, if $\ell(\mu) = m > 0$, then

$$\varphi^{\mu}_{\lambda}(\psi_{\mathsf{T}^{\lambda}}) := \psi_{s_{m+1/2} \cdot \mathsf{T}^{\lambda}} \text{ and } \varphi^{\mu}_{\lambda'}(\psi_{\mathsf{T}^{\lambda'}}) := \psi_{s_{-1/2} \cdot \mathsf{T}^{\lambda'}}$$

where $s_{m+1/2} \cdot \mathsf{T}^{\lambda}$ and $s_{-1/2} \cdot \mathsf{T}^{\lambda'}$ are the reflections of the paths T^{λ}, $\mathsf{T}^{\lambda'}$ through the hyperplanes $H_{m+1/2}$, $H_{-1/2}$ respectively. If $\ell(\mu) = m - 1/2$ then the desired map is $\varphi^{\mu}_{\lambda}(\psi_{\mathsf{T}^{\lambda'}}) = \psi_{s_{-1/2} \cdot \mathsf{T}^{\lambda'}}$.

In the next proposition we shall prove that the maps of Definition 3.20 are indeed B^{σ}_{n}-module homomorphisms. We cover the $m \leq 0$ case, since the other case works analogously.

At this point we shall prove that the relations of Proposition 3.23 hold in a cell module. In addition, we shall prove that the relations below form a presentation for the cell modules of the blob algebra. Both results are essential for proving that the maps of Definition 3.20 are module homomorphisms.

Proposition 3.23 (Relations for cell modules)**.** *Let* $\lambda = ((1^{\lambda_1}), (1^{\lambda_2})) \in \mathrm{Bip}_1(n)$. *Then*

$$
\begin{aligned}
e(\underline{i})\psi_{\mathsf{T}^{\lambda}} &= \delta_{\underline{i},\underline{i}^{\lambda}}\psi_{\mathsf{T}^{\lambda}}, \ \ \delta_{\underline{i},\underline{i}^{\lambda}} \text{ the Kronecker delta,} && (3.7)\\
y_s\psi_{\mathsf{T}^{\lambda}} &= 0 && (3.8)\\
\psi_r\psi_{\mathsf{T}^{\lambda}} &= \begin{cases} \psi_{\mathsf{T}^{\lambda}_{r\leftrightarrow r+1}} & \text{if } r, r+1 \text{ are in different components} \\ 0 & \text{otherwise} \end{cases} && (3.9)\\
\psi_{t+1}\psi_t\psi_{\mathsf{T}^{\lambda}} &= 0 && (3.10)\\
\psi_t\psi_{t+1}\psi_{\mathsf{T}^{\lambda}} &= 0 && (3.11)
\end{aligned}
$$

for all $1 \leq r \leq n-1$, $1 \leq s \leq n$ *and* $1 \leq t \leq 2\min\{\lambda_1, \lambda_2\} - 2$. *We refer to the relations (3.9)-(3.11) as Garnir relations.*

Proof. Let $e_{\underline{i}}$, $\underline{i} \in I^n$ be a KLR idempotent of B_n^σ. By the orthogonality relation we have that

$$e_{\underline{i}}\psi_{\mathsf{T}^\lambda} = e_{\underline{i}}e_{\underline{i}^\lambda} = \begin{cases} \psi_{\mathsf{T}^\lambda} & \text{if } \underline{i} = \underline{i}^\lambda \\ 0 & \text{otherwise.} \end{cases}$$

The element $\psi_r e_{\underline{i}^\lambda}$ corresponds to a tableau with residue sequence

$$(i_1^\lambda, \cdots, i_{r+1}^\lambda, i_r^\lambda, \cdots, i_n^\lambda).$$

We use the fact that for any standard tableau T the element ψ_{w_T} is unique up to the second KLR relation of (R3) by Remark 1.41. If the nodes of T^λ occupied by the entries r, $r+1$ are in the same component, then any tableau with such residue sequence indexes elements in the ideal $\mathsf{B}_{n,\rhd\lambda}^\kappa$, hence $\psi_r e_{\underline{i}^\lambda} = 0$ modulo more dominant terms. If they are in different components then the only choice for a tableau with the above residue sequence and corresponding permutation consisting of the generator ψ_r is the tableau $\mathsf{T}_{r\leftrightarrow r+1}^\lambda$, hence $\psi_r e_{\underline{i}^\lambda} = \psi_{\mathsf{T}_{r\leftrightarrow r+1}^\lambda}$. The element $y_s e_{\underline{i}^\lambda}$ corresponds to a tableau with residue sequence $\underline{i}^\lambda \in I^n$. The unique tableau with that residue sequence is T^λ. However

$$\deg(y_s e_{\underline{i}^\lambda}) = 2 \neq 0 = \deg(e_{\underline{i}^\lambda})$$

thus $y_s e_{\underline{i}^\lambda}$. Regarding relation (3.10), if $t, t+1$ are in the same component then the result follows from (3.9). If t, $t+1$ are in different components then the element $\psi_{t+1}\psi_t e_{\underline{i}^\lambda}$ corresponds to a tableau with residue sequence

$$(i_1^\lambda, \cdots, i_{t+1}^\lambda, i_{t+2}^\lambda, i_t^\lambda, \cdots, i_n^\lambda).$$

But such standard λ-tableau does not exist hence $\psi_{t+1}\psi_t e_{\underline{i}^\lambda} = 0$ modulo terms in the ideal $\mathsf{B}_{n,\rhd\lambda}^\kappa$ (we note that there are μ-tableaux of this residue sequence, for $\mu \rhd \lambda$). Similarly we prove relation (3.11). $\square$

Proposition 3.24. *Let $\lambda \in \mathrm{Bip}_1(n)$. The generator $\psi_{\mathsf{T}^\lambda}$ and relations of Proposition 3.23 form a presentation for the cell module $\Delta(\lambda)$.*

Proof. By Proposition 3.23 we have that the desired relations are satisfied. By Lemma 1.51 and the fact that we know a basis for the cell modules, we deduce that this list of relations is complete. Hence this is enough for proving that the relations, together with the generator $\psi_{\mathsf{T}^\lambda}$, form a presentation for the cell module $\Delta(\lambda)$. □

Proposition 3.25. *Let $\mu \in \mathrm{Bip}_1(n)$ with $\ell(\mu) = m$, $m \leq 0$. The maps $\varphi^\mu_\lambda \colon \Delta(\lambda) \longrightarrow \Delta(\mu)$ and $\varphi^\mu_{\lambda'} \colon \Delta(\lambda') \longrightarrow \Delta(\mu)$ of Definition 3.20 are homomorphisms of B^σ_n-modules.*

Proof. We shall prove the result for the map φ^μ_λ and then similar arguments apply for the map $\varphi^\mu_{\lambda'}$. By Proposition 3.24 we need to show that the relations (3.9)-(3.11) of Theorem 3.23 are satisfied. Recall that $\varphi^\mu_\lambda(\psi_{\mathsf{T}^\lambda}) = \psi_{s_{m-1/2}\cdot \mathsf{T}^\lambda}$ and let us denote $\mathsf{S} := s_{m-1/2} \cdot \mathsf{T}^\lambda$. Also let $\mathsf{S}(q) = \mathsf{T}^\lambda(q) \in H_{m-1/2}$, for some $1 \leq q \leq n-1$, be the unique reflection point of the path T^λ through the hyperplane $H_{m-1/2}$. Then

$$
\begin{aligned}
e_{\underline{i}}\varphi^\mu_\lambda(\psi_{\mathsf{T}^\lambda}) =& e_{\underline{i}}\psi_{\mathsf{S}} \\
=& \psi_{w_{\mathsf{S}}} e_{(w_{\mathsf{S}}^{-1}\underline{i})} e_{\underline{i}^\mu} \\
=& \delta_{\underline{i},\mathrm{res}(\mathsf{S})} \psi_{w_{\mathsf{S}}} e_{\underline{i}^\mu} \\
=& \delta_{\underline{i},\mathrm{res}(\varphi^\mu_\lambda(\psi_{\mathsf{T}^\lambda}))} \varphi^\mu_\lambda(\psi_{\mathsf{T}^\lambda})
\end{aligned}
\tag{3.12}
$$

for any idempotent $e_{\underline{i}}$ and so relation (3.7) holds. Now consider the generator y_s for some $1 \leq s \leq n$. We claim

$$y_s\varphi^\mu_\lambda(\psi_{\mathsf{T}^\lambda}) = y_s\psi_{\mathsf{S}} = 0. \tag{3.13}$$

To see the claim we note that $\deg(\mathsf{T}^\lambda) = 0$, $\deg(\mathsf{S}) = 1$ and the degree of the element $y_s\psi_{\mathsf{S}}$ is equal to 3. By residue considerations we can see that there does not exist a path of degree 3 terminating at μ with residue sequence $\mathrm{res}(\mathsf{S})$. Hence relation (3.8) holds.

Consider the element $\psi_r\varphi^{\mu}_{\lambda}(\psi_{\mathsf{T}^\lambda})$, for some $1 \leq r \neq q < n$. then

$$\begin{aligned}\psi_r\varphi^{\mu}_{\lambda}(\psi_{\mathsf{T}^\lambda}) &= \psi_r\psi_{\mathsf{S}} \\ &= \begin{cases}\psi_{\mathsf{S}_{r\leftrightarrow r+1}} & \text{if } r, r+1 \text{ are in different components} \\ 0 & \text{otherwise}\end{cases} \\ &= \begin{cases}\varphi^{\mu}_{\lambda}(\psi_{\mathsf{T}^\lambda})_{r\leftrightarrow r+1} & \text{if } r, r+1 \text{ are in different components} \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

In order to prove that relation (3.9) holds we need to consider the case $r = q$. By construction, the simple transposition s_q exists in w_{S} and it is left exposed. Hence $\psi_{\mathsf{S}} = \psi_q\psi_{i_1}\cdots\psi_{i_l}e_{\underline{i}^\mu}$. Since $\mathrm{res}(\mathsf{S}^{-1}(q)) = \mathrm{res}(\mathsf{S}^{-1}(q+1)) + 1$ (since we reflected through a hyperplane at this point) we have that

$$\begin{aligned}\psi_q\psi_{\mathsf{S}} &= \psi_q^2\psi_{s_q\mathsf{S}} \\ &= (y_{q+1} - y_q)\psi_{s_q\mathsf{S}}.\end{aligned}$$

But both summands are zero and we can see that by using the same arguments as in the proof of relation (3.13). Thus $\psi_q\psi_{\mathsf{S}} = 0$ and so relation (3.9) holds.

Consider the product $\psi_{r+1}\psi_r\varphi^{\mu}_{\lambda}(\psi_{\mathsf{T}^\lambda})$, for $r = \mathsf{T}^\lambda(A)$ where A is a Garnir node as in the statement of Theorem 3.23. Then

$$\psi_{r+1}\psi_r\varphi^{\mu}_{\lambda}(\psi_{\mathsf{T}^\lambda}) = \psi_{r+1}\psi_r\psi_{\mathsf{S}} = \psi_{r+1}\psi_{s_r(\mathsf{S})}$$

and we deduce that the product is zero (modulo terms in the ideal $\mathsf{B}^{\sigma}_{n,\rhd\mu}$), since there does not exist standard μ-tableau with residue sequence $\mathrm{res}(s_{r+1}s_r\mathsf{S})$; this we use an argument identical to the proof of relation (3.12). Hence relation (3.10) is satisfied. Similarly we prove that relation (3.11) is also satisfied. $\square$

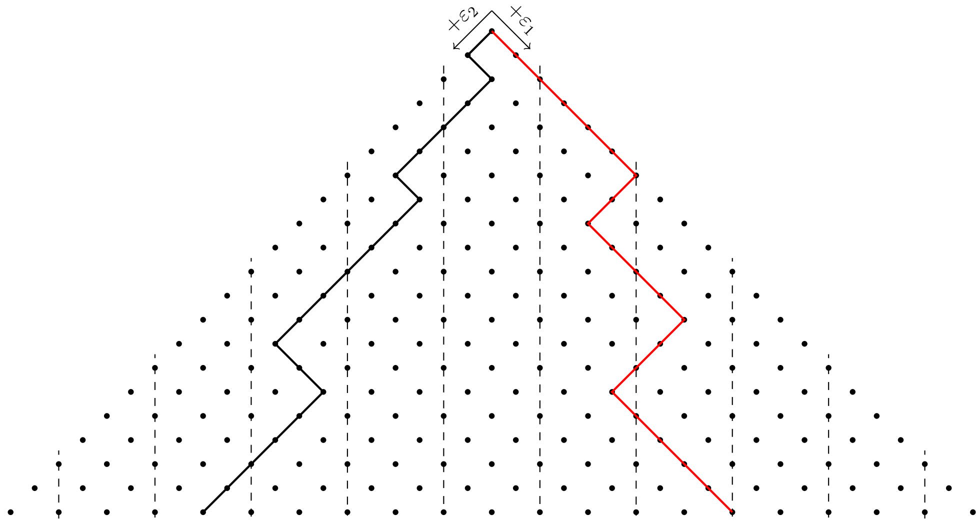


FIGURE 3.4: The black path is a length increasing path whereas the red path is non-length-increasing.

The proof is identical in the case that $\mu \in \mathrm{Bip}_1(n)$ lies on a hyperplane. We present the result without repeating the proof.

Proposition 3.26. *Let $\mu \in \mathrm{Bip}_1(n)$ with $\ell(\mu) = m - 1/2$, $m \leq 0$. The map $\varphi^{\mu}_{\lambda'} \colon \Delta(\lambda') \longrightarrow \Delta(\mu)$ of Definition 3.20 is a homomorphism of B^{σ}_n-modules.*

Proof. The proof is identical to the proof of Proposition 3.25. □

In the last part of this section, we shall discuss a specific type of paths which will be the building blocks of the main proofs of this chapter.

Definition 3.27. Let $\mathsf{T} \in \mathrm{Std}(n)$ be a standard tableau. The path T is called length increasing if

$$|\ell(\mathrm{Shape}(\mathsf{T}_{\leq k}))| \leq |\ell(\mathrm{Shape}(\mathsf{T}_{\leq k+1}))|$$

for all $1 \leq k < n$.

An example of a length increasing path along with a non-length-increasing path can be seen in Figure 3.4.

An alternative criterion to be a length increasing path of shape μ is that for a given bipartition $\mu \in \mathrm{Bip}_1(n)$ with $\ell(\mu) = m < 0$ (resp. $\ell(\mu) = m > 0$) every length increasing path in $\mathrm{Path}(\mu)$ intersects with the hyperplanes $H_{-1/2}, \cdots, H_{m+1/2}$ (resp. $H_{1/2}, \cdots, H_{m-1/2}$) at exactly one point and it does not intersect $H_{m+3/2}, H_{m+5/2}, \cdots$ (resp. $H_{m-3/2}, H_{m-5/2}, \cdots$). In the exceptional case that $\ell(\mu) = 0$ we have that a length increasing path is a path that does not leave the fundamental alcove at any point. This criterion can be also visualised in Figure 3.4.

The following lemma shows that the generators of the cell modules indexed by length increasing paths belong to the simple head.

Lemma 3.28. *Let $\nu \in \mathrm{Bip}_1(n)$ and $\mathsf{T} \in \mathrm{Path}(\nu)$ be a length increasing path. Then the element ψ_{T} belongs to the simple module $L(\nu)$.*

Proof. Let $\mathrm{res}(\mathsf{T}) \in I^n$ be the residue sequence of T. Since T is length increasing, the set $\mathrm{Path}(\nu', \mathsf{T}^{\nu})$ is non-empty only if $\nu' \trianglerighteq \nu$. Hence we have that $e_{\mathrm{res}(\mathsf{T})}\Delta(\nu') = 0$, for any bipartition $\nu' \triangleleft \nu$. Thus $e_{\mathrm{res}(\mathsf{T})}L(\nu') = 0$, for any bipartition $\nu' \triangleleft \nu$. This shows that the element ψ_{T} belongs to a composition factor of $\Delta(\mu)$ not of the form $L(\nu')$, $\nu' \triangleleft \nu$, so it belongs to the simple head $L(\nu)$. $\square$

3.4 Image of the homomorphisms

In this section we shall construct the image of the homomorphisms $\varphi^{\mu}_{\lambda'}$ and φ^{μ}_{λ} of Definition 3.20. This will be the decisive step for constructing the radical of a cell module $\Delta(\mu)$ since the images of the homomorphisms are closely related with that, as we will see in the next section of this chapter.

Same as in last section we cover the case that $m \leq 0$, since all the arguments work equally in the case $m > 0$ up to relabelling hyperplanes. In the alcove case we compute the image of both φ^{μ}_{λ}, $\varphi^{\mu}_{\lambda'}$ whereas in the hyperplane case it is only necessary to consider the homomorphism φ^{μ}_{λ}.

Suppose that $\mathsf{T}_1 \in \mathrm{Path}(\lambda')$ is a length increasing path. The image of the element ψ_{T_1} under the homomorphism $\varphi^{\mu}_{\lambda'}$ is

$$\varphi^{\mu}_{\lambda'}(\psi_{\mathsf{T}_1}) = \psi_{s_{1/2}\cdot\mathsf{T}_1}$$

since the path $s_{1/2} \cdot \mathsf{T}_1$ is the unique path with residue sequence equal to $\mathrm{res}(\mathsf{T})$ terminating at the bipartition μ. For the same reason, if $\ell(\mu) = m$, $m \leq 0$ and $\mathsf{T}_2 \in \mathrm{Path}(\lambda)$ is a length increasing path then the image of the element ψ_{T_2} under the homomorphism φ^{μ}_{λ} is

$$\varphi^{\mu}_{\lambda}(\psi_{\mathsf{T}_2}) = \psi_{s_{m-1/2}\cdot\mathsf{T}_2}.$$

The following proposition is one of the main results of the section and describes a spanning set for the image of the homomorphism $\varphi^{\mu}_{\lambda'}$. Note that the result holds for both $\ell(\mu) = m$ and $\ell(\mu) = m - 1/2$, $m \leq 0$.

Proposition 3.29. *The homomorphism* $\varphi^{\mu}_{\lambda'} \colon \Delta(\lambda') \longrightarrow \Delta(\mu)$ *of Definition 3.20 is an injective homomorphism. Moreover*

1. *if* $m \leq 0$

$$\mathrm{Im}\varphi^{\mu}_{\lambda'} = \mathrm{span}_{\Bbbk}\{\psi_{\mathsf{U}} \mid \mathsf{U} \in \mathrm{Path}(\mu),\ \mathsf{U} \textit{ intersects } H_{1/2}\},$$

2. *if* $m > 0$

$$\mathrm{Im}\varphi^{\mu}_{\lambda'} = \mathrm{span}_{\Bbbk}\{\psi_{\mathsf{U}} \mid \mathsf{U} \in \mathrm{Path}(\mu),\ \mathsf{U} \textit{ intersects } H_{-1/2}\}.$$

Proof. We cover the case $m \leq 0$ as the other one works similarly. Take any path $\mathsf{U} \in \mathrm{Path}(\mu)$ and suppose that it intersects the hyperplane $H_{1/2}$ at n-many points and let $\mathsf{u}^n_{1/2}$ be the final one. Then we notice that the reflection $s^n_{1/2} \cdot \mathsf{U}$ through the final point that U intersects the hyperplane $H_{1/2}$ gives a path terminating at λ'. This shows that there is a bijection between the paths in $\mathrm{Path}(\mu)$ intersecting $H_{1/2}$ and the paths in $\mathrm{Path}(\lambda')$. We will prove that any path intersecting the hyperplane $H_{1/2}$ belongs indeed to the image of $\varphi^{\mu}_{\lambda'}$ and thus the result will follow from the fact that the dimension of $\Delta(\lambda')$ is equal to the dimension of the image of $\varphi^{\mu}_{\lambda'}$.

We consider the path T^{μ} and we fix integer $a \in \mathbb{Z}$, $1 \leq a \leq n-1$. Let $\mathsf{T} \in \mathrm{Path}(\mu)$ be the maximal path, under the lexicographic order, with the property $\mathsf{T}(a) \in H_{1/2}$ (see Figures 3.5, 3.6). We proceed by considering each value, a, one at a time. Since T must intersect $H_{1/2}$ at some point by assumption, this allows us to consider all such paths.

Note that the reflection $s_{1/2} \cdot \mathsf{T}$ of the path T through the hyperplane $H_{1/2}$ is a length increasing path in $\mathrm{Path}(\lambda')$. Hence the element ψ_{T} belongs to the image of the homomorphism $\varphi^{\mu}_{\lambda'}$. Let $\mathsf{U} \in \mathrm{Path}(\mu)$ be any path which intersects the hyperplane $H_{1/2}$ at the point $\mathsf{U}(a) = \mathsf{T}(a)$, with $w_{\mathsf{U}} = s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_n$ its reduced expression. Since T is the maximal path, under the lexicographic order, with $\mathsf{T}(a) \in H_{1/2}$ we have that w_{T} is a subword of w_{U}.

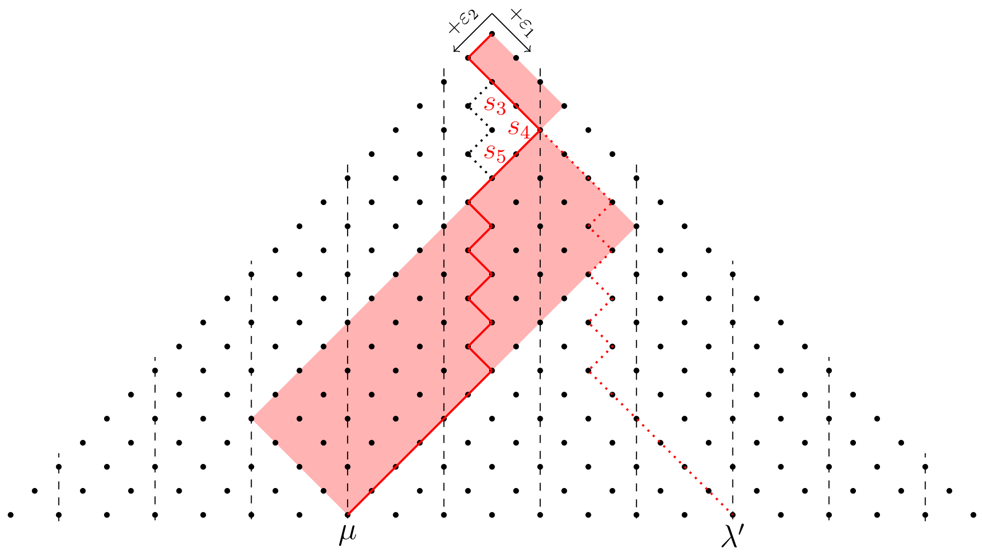


FIGURE 3.5: For $n = 20$, $\mu = ((1^7), (1^{13}))$, $\ell(\mu) = -3/2$, $e = 4$, $\sigma = (0, 2)$ and $a = 4$ the red path is the path T. The shaded area corresponds to all the paths that can be obtained from T and the elements corresponding to them belong to the image of φ^{μ}_{λ}. Any path within the shaded region has $s_4 s_3 s_5$ as a subword.

Note that the subword w_{T} will be right exposed, as otherwise the condition $\mathsf{U}(a) \in H_{1/2}$ would not hold. We can rewrite the reduced expression w_{U} as

$$w_{\mathsf{U}} = s_{i_1} \cdots s_{i_l} w_{\mathsf{T}}, \; 1 \leq l \leq k$$

hence $\mathsf{U} = s_{i_1} \cdots s_{i_l}\mathsf{T}$, $1 \le l \le k$.

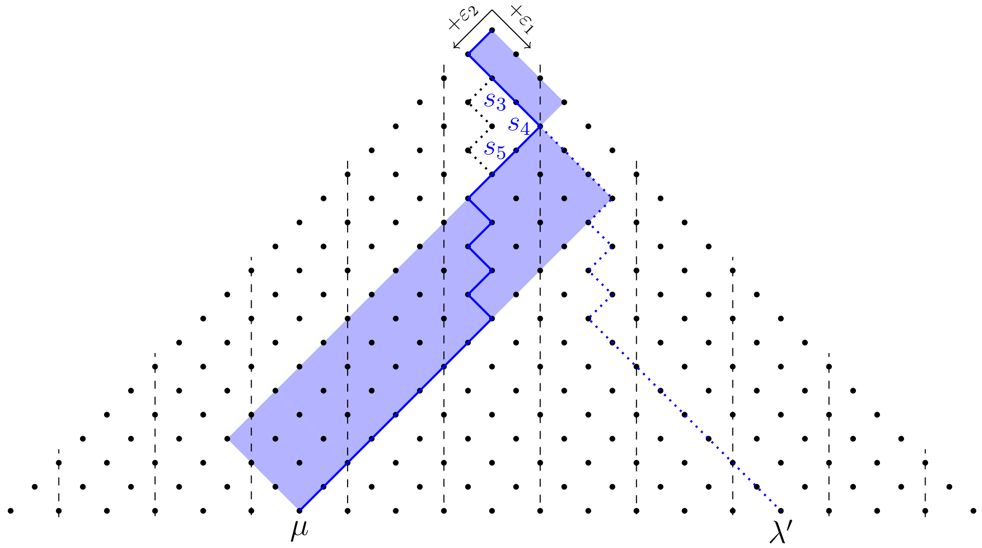


FIGURE 3.6: For $n = 20$, $\mu = ((1^6),(1^{14}))$, $\ell(\mu) = -2$, $e = 4$, $\sigma = (0,2)$ and $a = 4$ the blue path is the path T. The shaded area corresponds to all the paths that can be obtained from T and the elements corresponding to them belong to the image of φ^{μ}_{λ}. Any path within the shaded region has $s_4s_3s_5$ as a subword.

Then the basis element ψ_{U} corresponding to the path U can be written as

$$\psi_{\mathsf{U}} = \psi_{i_1} \cdots \psi_{i_l}\psi_{\mathsf{T}}$$

and it belongs to the image of φ^{μ}_{λ}, since ψ_{T} does. By repeating the same procedure for all admissible integers $a \in \mathbb{Z}$, $1 \le a \le n$, we prove that all paths in $\mathrm{Path}(\mu)$ which intersect the hyperplane $H_{1/2}$ correspond to elements in the image of the homomorphism φ^{μ}_{λ}. $\square$

Example 3.30. *Let $n = 20$, $e = 4$, $\sigma = (0,2)$ and $\mu = ((1^7),(1^{13})) \in \mathrm{Bip}_1(20)$. If $\mathsf{T} = s_4s_3s_5\mathsf{T}^{\mu}$ then we observe that the path*

$$\mathsf{S} = s_{15}s_{14}s_{16}s_{18}s_{13}s_{15}s_{17}s_3s_5s_{12}s_{14}s_{16}s_2s_4s_6s_{11}s_{13}s_{15}s_1s_3s_5s_7s_{10}s_{12}s_{14}\mathsf{T}^{\mu}$$

can be written as

$$\begin{aligned}\mathsf{S} &= s_{15}s_{14}s_{16}s_{18}s_{13}s_{15}s_{17}s_3s_5s_{12}s_{14}s_{16}s_2s_6s_{11}s_{13}s_{15}s_1s_7s_{10}s_{12}s_{14}(s_4s_3s_5\mathsf{T}^{\mu}) \\ &= s_{15}s_{14}s_{16}s_{18}s_{13}s_{15}s_{17}s_3s_5s_{12}s_{14}s_{16}s_2s_6s_{11}s_{13}s_{15}s_1s_7s_{10}s_{12}s_{14}\mathsf{T}\end{aligned}$$

by using the Coxeter relations of the symmetric group (see Figure 3.7).

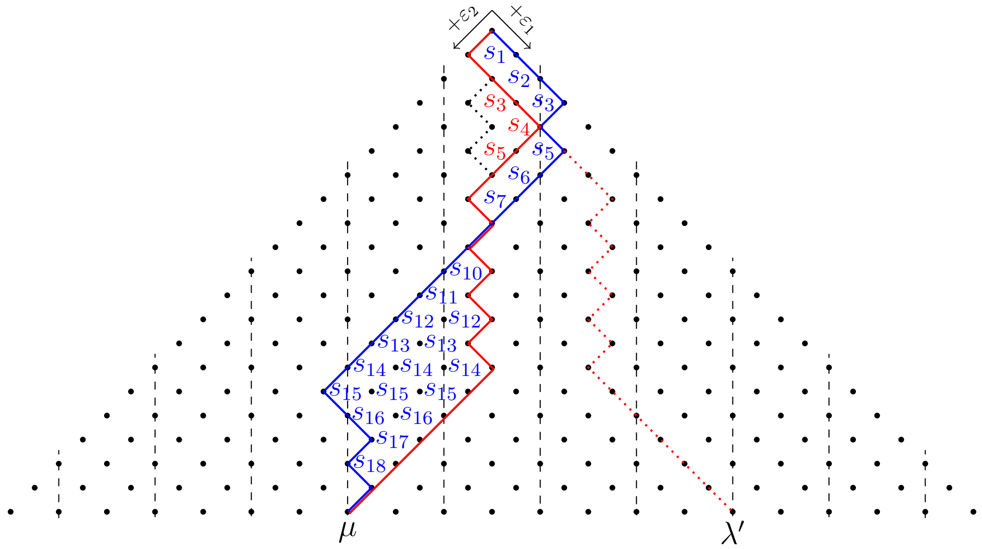


FIGURE 3.7: The red path is the path T intersecting the hyperplane $H_{1/2}$ at the point $\mathsf{T}(4)$. The blue path is the path S which intersects $H_{1/2}$ at $\mathsf{S}(4) = \mathsf{T}(4)$ and it belongs to the image of the homomorphism $\varphi^{\mu}_{\lambda'}$.

Hence the basis element ψ_{S} can be written as

$$\psi_{\mathsf{S}} = \psi_{15}\psi_{14}\psi_{16}\psi_{18}\psi_{13}\psi_{15}\psi_{17}\psi_{3}\psi_{5}\psi_{12}\psi_{14}\psi_{16}\psi_{2}\psi_{6}\psi_{11}\psi_{13}\psi_{15}\psi_{1}\psi_{7}\psi_{10}\psi_{12}\psi_{14}\psi_{\mathsf{T}} \in \Delta(\mu)$$

and since the element ψ_{T} belongs to the image of $\varphi^{\mu}_{\lambda'}$, we have that ψ_{S} also belongs to the image of $\varphi^{\mu}_{\lambda'}$.

Recall that the homomorphism $\varphi^{\mu}_{\lambda} \colon \Delta(\lambda) \longrightarrow \Delta(\mu)$ only exists when $\ell(\mu) = m$, that is the bipartition μ lies in the alcove $\mathfrak{a}_m$. The construction of the spanning set for the image of the homomorphism φ^{μ}_{λ} is the next important result of the thesis towards our aim to construct bases for the irreducible representations of B^{σ}_{n}. For completeness we give the spanning sets for both $m \leq 0$ and $m > 0$.

Proposition 3.31. *The homomorphism $\varphi^{\mu}_{\lambda} \colon \Delta(\lambda) \longrightarrow \Delta(\mu)$ of Definition 3.20 is an injective homomorphism. Moreover*

1. *if $m \leq 0$*

$$\mathrm{Im}\varphi^{\mu}_{\lambda} = \mathrm{span}_{\Bbbk}\left\{ \psi_{\mathsf{U}} \;\middle|\; \begin{array}{c} \mathsf{U}{\in}\mathrm{Path}(\mu),\ \mathsf{U}\ \textit{last intersects}\ H_{m-1/2}\ \textit{or} \\ \textit{intersects}\ H_{1/2}\ \textit{after intersecting}\ H_{-1/2} \end{array} \right\},$$

2. if $m > 0$

$$\mathrm{Im}\varphi_\lambda^\mu = \mathrm{span}_{\Bbbk}\left\{\psi_{\mathsf{U}} \mid \begin{array}{c}\mathsf{U}\in\mathrm{Path}(\mu),\ \mathsf{U}\ \textit{last intersects } H_{m+1/2}\ \textit{or}\\ \textit{intersects } H_{-1/2}\ \textit{after intersecting } H_{+1/2}\end{array}\right\}.$$

Before presenting the proof, we shall give an example which illustrates which paths we are referring to in the statement of Proposition 3.31.

Example 3.32. *Let $n = 20$, $e = 4$, $\sigma = (0, 2)$ and consider the bipartition $\mu = ((1^6), (1^{14}))$ with $\ell(\mu) = m = -2$. Then $\lambda = ((1^4), (1^{16}))$ is the bipartition linked with μ with $\ell(\lambda) = -3$ (see Figure 3.8). The hyperplanes that we shall be interested in are $H_{-1/2}$, $H_{1/2}$ which are the hyperplanes of the fundamental alcove and the hyperplane $H_{m-1/2} = H_{-5/2}$ which is the left hyperplane of the alcove $\mathfrak{a}_{-1}$.*

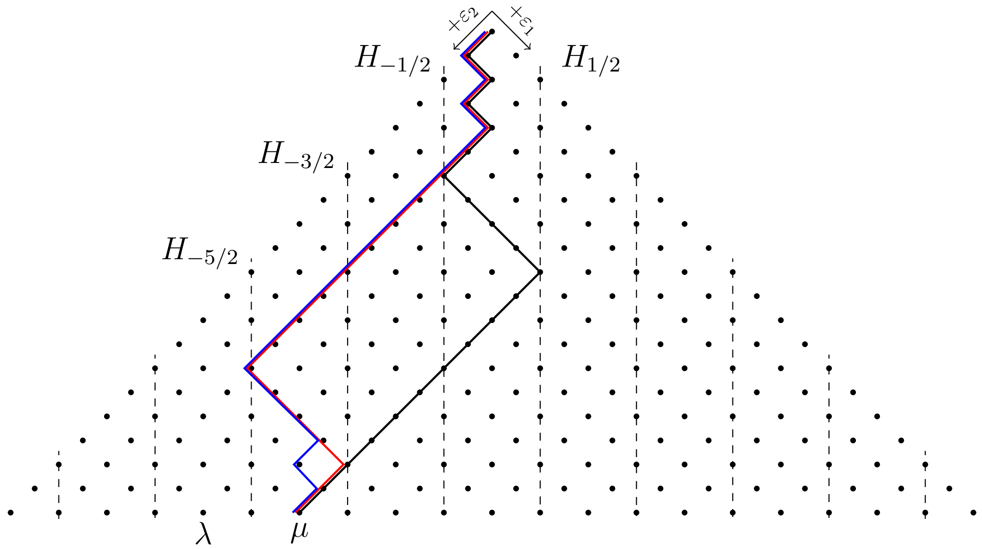


FIGURE 3.8: The blue and the black path label elements which belong in the image of φ_λ^μ whereas the red path labels an element not in the image of φ_λ^μ.

The black path is a path which intersects the hyperplane $H_{1/2}$ after intersecting the hyperplane $H_{-1/2}$. The blue path is a path last intersecting the hyperplane $H_{-5/2}$. Both paths belong to the image of the homomorphism φ_λ^μ. On the other hand the red path does intersect the hyperplane $H_{-5/2}$, but it last intersects $H_{-3/2}$ and it does not belong to the image of φ_λ^μ.

Let $\mu = ((1^{\mu_1}), (1^{\mu_2})) \in \mathrm{Bip}_1(n)$ with $\ell(\mu) = m < 0$, i.e. $\mu_1 < \mu_2$. We shall construct a path $\mathsf{T} \in \mathrm{Path}(\mu)$ which intersects hyperplanes $H_{-1/2}, \cdots, H_{m+3/2}, H_{m-1/2}$

at exactly one point, hyperplane $H_{m+1/2}$ at exactly two points and it does not intersect hyperplane $H_{1/2}$, as follows. Let $j \in \mathbb{Z}$, $1 \leq j \leq n-1$ be such that $\mathsf{T}(i) = \mathsf{T}^{\mu}(i)$, for any $1 \leq i \leq j+1$, and recall from relation (3.2) that $\mathsf{T}(j) = c_{j,1}(\mathsf{T})\varepsilon_1 + c_{j,2}(\mathsf{T})\varepsilon_2$. Also let $q > j$ such that $\mathsf{T}(q) \in H_{-1/2}$ and $\mathsf{T}(q) = c_{q,1}(\mathsf{T})\varepsilon_1 + c_{q,2}(\mathsf{T})\varepsilon_2$ with $c_{q,1}(\mathsf{T}) = c_{j,1}(\mathsf{T})$ and $c_{q,2}(\mathsf{T}) = c_{j,2}(\mathsf{T}) + n - j$. We denote by $a \in \mathbb{Z}$ the integer with the property $\mathsf{T}(a) \in H_{m-1/2}$ and $c_{a,1}(\mathsf{T}) = c_{j,1}(\mathsf{T})$, $c_{a,2}(\mathsf{T}) = c_{n,2}(\mathsf{T}) + |\ell(\mu)|e$. Finally, let $\mathsf{T}(b) \in H_{m+1/2}$ be the second intersection point of T with $H_{m+1/2}$. Note that the integers j, q, a, b determine the path T. The diagram corresponding to the basis element ψ_{T} is presented in Figure 3.9.

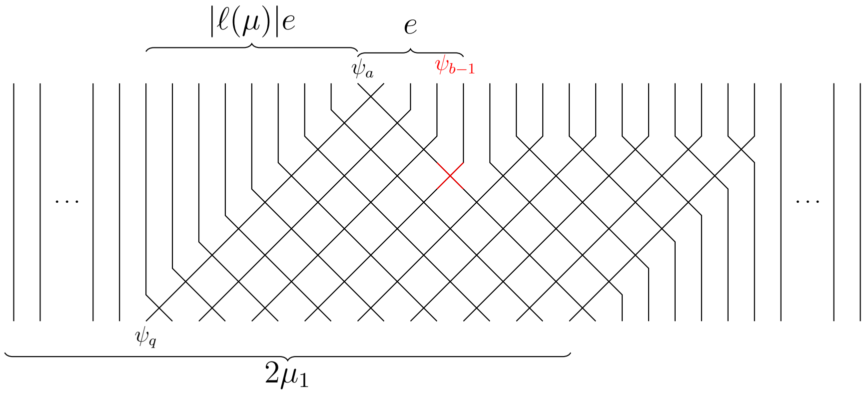


FIGURE 3.9: The general form of the diagram corresponding to the element ψ_{T}. Here the crossing marked with red is the crossing ψ_{b-1} and the colouring has nothing to do with the residues adjacent to it.

Notation. Let $\nu \in \mathrm{Bip}_1(n)$ and $\mathsf{T} \in \mathrm{Path}(\nu)$ be a path. We denote by $\mathsf{t}^{\mathrm{last}}_{q-1/2}$ the last intersection point of the path T with the hyperplane $H_{q-1/2}$, for some $q \in \mathbb{Z}$. Also we denote by $s^{\mathrm{last}}_{q-1/2}$ the reflection through that point with respect to the hyperplane $H_{q-1/2}$.

Proof of Proposition 3.31. Same as in the proof of Proposition 3.29 we cover the case $m \leq 0$. Let $a \in \mathbb{Z}$, $1 \leq a < n$ be a fixed integer such that if $\alpha = ((1^{\alpha_1}), (1^{\alpha_2}))$ is a bipartition of a, then $\alpha_1 - \alpha_2 \in H_{m-1/2}$. Also let $\mathsf{T}^{\alpha} \in \mathrm{Std}(\alpha)$ be the α-tableau which is maximal with respect to the order of Definition 1.37. Consider the skew bipartition $\lambda \backslash \alpha$ and let $\mathsf{S}, \mathsf{S}' \in \mathrm{Path}(\alpha \to \lambda)$ be length increasing paths which are highest and lowest in the lexicographic order respectively (see Figure 3.10). Recall that the elements of the set $\mathrm{Path}(\alpha \to \lambda)$ are paths starting from the bipartition

$\alpha \in H_{m-1/2}$ and endpoint the bipartition λ. All the remaining length increasing $\lambda\backslash\alpha$-paths placed between S and S' can be obtained by multiplying with a product of transpositions on the tableau S and we focus on S, S' for the ease of notation. We define the standard λ-tableaux $\mathsf{T} := \mathsf{T}^\alpha \circ \mathsf{S}$ (in the sense of Definition 1.43) and $\mathsf{T}' := \mathsf{T}^\alpha \circ \mathsf{S}'$ and let $\hat{\mathsf{T}} := s_{m-1/2} \cdot \mathsf{T}$ and $\hat{\mathsf{T}}' := s_{m-1/2} \cdot \mathsf{T}'$ be the reflection of those paths through the unique point they intersect the hyperplane $H_{m-1/2}$. Note that since the paths T, T' are length increasing paths, the basis elements $\psi_{\hat{\mathsf{T}}}$, $\psi_{\hat{\mathsf{T}}'}$ corresponding to the paths $\hat{\mathsf{T}}$, $\hat{\mathsf{T}}'$ belong the image of the homomorphism φ^μ_λ.

We shall prove that if the generators ψ_r, $a < r < n$ act on $\psi_{\hat{\mathsf{T}}}$ then $\psi_r\psi_{\hat{\mathsf{T}}}$ is a non-zero element and it corresponds to a path which either last intersects $H_{m-1/2}$ or intersects $H_{1/2}$ after intersecting $H_{-1/2}$. Since $\psi_r\psi_{\hat{\mathsf{T}}}$ belongs to the image of φ^μ_λ, the new element will also belong to the image of φ^μ_λ. For any $a < r < d$ such that $s_r\mathsf{T}$ does not intersect $H_{m-1/2}$, $H_{m+1/2}$, it is straightforward that $\psi_r\psi_{\hat{\mathsf{T}}} = \psi_{s_r\hat{\mathsf{T}}}$ because $s_r\mathsf{T}$ is the unique tableau with the desired residue sequence. Let $b \in \mathbb{Z}$, $a < b < n$, such that $(s_b\mathsf{T})(b) \in H_{m+1/2}$. Since $s_b\hat{\mathsf{T}} \preceq \hat{\mathsf{T}}$ we also have that

$$\psi_b\psi_{\hat{\mathsf{T}}} = \psi_{s_b\hat{\mathsf{T}}}$$

and the element $\psi_b\psi_{\hat{\mathsf{T}}}$ is a non zero element which belongs to the image of the homomorphism φ^μ_λ. We also need to prove that $\psi_r\psi_{\hat{\mathsf{T}}'}$, $1 < r < n$ is a non zero element which belongs to the radical. Consider the element $\psi_{\hat{\mathsf{T}}'}$ and let $b \in \mathbb{Z}$ be such that $(s_b\mathsf{T}')(b) \in H_{m-3/2}$. This is the only interesting case as for the rest cases the result is straightforward. The transposition s_b will appear in the reduced expression of $\hat{\mathsf{T}}'$ and it will be left exposed. Hence

$$\psi_b\psi_{\hat{\mathsf{T}}'} = \psi_b^2\psi_{i_1}\cdots\hat{\psi}_b\cdots\psi_{i_k}e_{\underline{i}^\mu}$$

with $\psi_{s_b\hat{\mathsf{T}}'} = \psi_{i_1}\cdots\hat{\psi}_b\cdots\psi_{i_k}e_{\underline{i}^\mu}$, where by $\hat{\psi}_b$ we mean that the generator ψ_b does not appear in the product. Since $\mathrm{res}((s_b\hat{\mathsf{T}}')^{-1}(b)) = \mathrm{res}((s_b\hat{\mathsf{T}}')^{-1}(b+1)) + 1$, by applying the KLR relation (R5) we have that

$$\psi_b\psi_{\hat{\mathsf{T}}'} = (y_{b+1} - y_b)\psi_{s_b\hat{\mathsf{T}}'}.$$

Step 1: We shall prove that $y_{b+1}\psi_{s_b\hat{\mathsf{T}}'} = 0$. Let $(s_b\hat{\mathsf{T}}')(n) \in H_{-1/2}$, for some $q \in \mathbb{Z}$, be the unique intersection point of the path $s_b\hat{\mathsf{T}}'$ with the hyperplane $H_{-1/2}$.

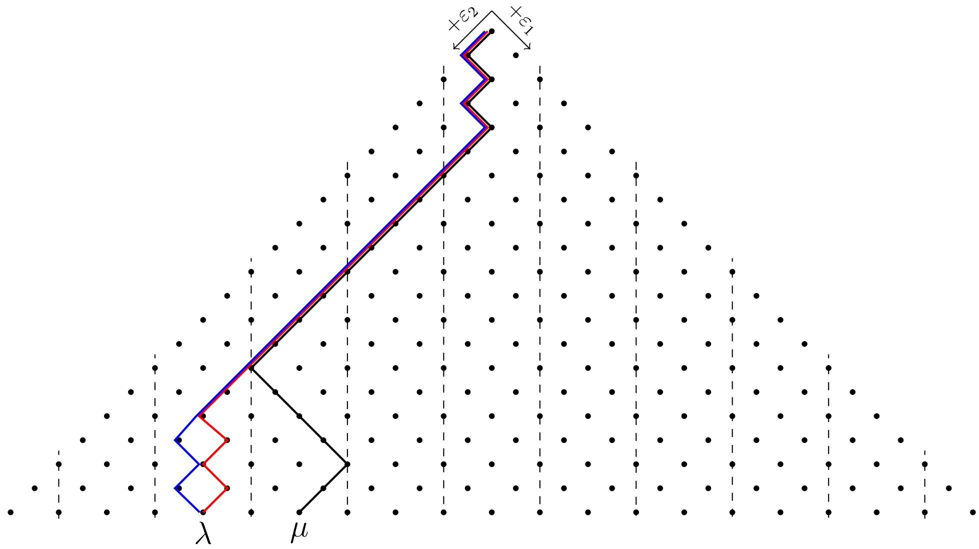
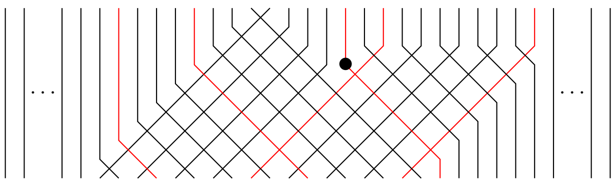


Figure 3.10: Let $n = 20$, $e = 4$ and $\sigma = (0, 2)$. For $a = 14$ and $b = 18$ the paths T (red), T' (blue) and $s_{18}\hat{\mathsf{T}}'$ (black) are depicted above. The paths S and S' are the bits of the T and T starting from the hyperplane $H_{-5/2}$ all the way down to λ. In this case $q = 6$.

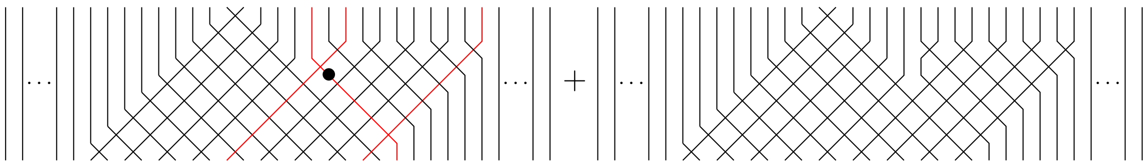
In order to compute the product $y_{b+1}\psi_{s_b\hat{\mathsf{T}}}$ it is easier to consider the diagrammatic presentation of our algebra. In particular the diagram of the element $s_b\psi_{\hat{\mathsf{T}}'}$ is of the form of Figure 3.9.

Note that the diagram consists of strands moving towards up to the right (UR-strands) and strands moving towards up to the left (UL-strands). If the l^{th} UR-strand (resp. UL-strand) carries the residue $i \in \mathbb{Z}/e\mathbb{Z}$ then the $(l+xe)^{\text{th}}$, $x \in \mathbb{Z}_{>0}$, UR-strand (resp. UL-strand) also carries the residue $i \in \mathbb{Z}/e\mathbb{Z}$. We colour strands carrying the same residue with the same colour.

We apply the generator y_{b+1} on the element $\psi_{s_b\hat{\mathsf{T}}'}$ and we obtain the element corresponding to the following diagram.

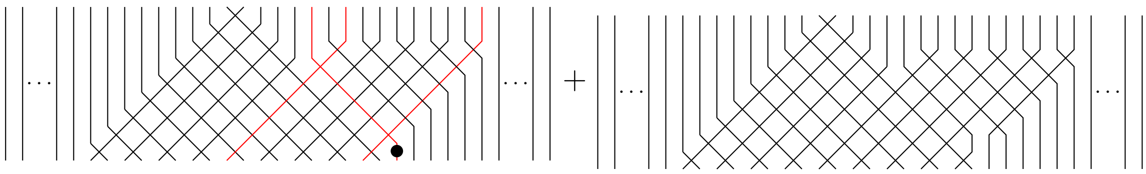


By applying the KLR-relation (R4) in the case that the residues coincide we get the following combination of diagrams:

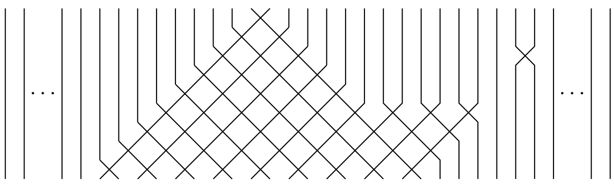


(3.14)

We take the first summand of (3.14) and by reapplying the KLR relation (R4) we obtain two more summands.



Those new summands are both equal to zero. The first one by the Garnir relation (3.8) and the second one because it corresponds to a non standard tableau. We now consider the second summand of (3.14) and we apply the KLR relation (R6). We obtain the element



which is equal to zero because of the Garnir relations. Finally we get that $y_{b+1}\psi_{s_b\hat{\mathsf{T}}'} = 0$.

Step 2: Now we shall consider the product $y_b\psi_{s_b\hat{\mathsf{T}}'}$ and we shall distinguish between two cases according to the length of μ. If $|\ell(\mu)| > 1$, then the unique element of $\mathrm{Path}(\mu, s_b\hat{\mathsf{T}}')$ with degree equal to $\deg(s_b\hat{\mathsf{T}}') + 2$ is the path $\mathsf{V}_1 :=$

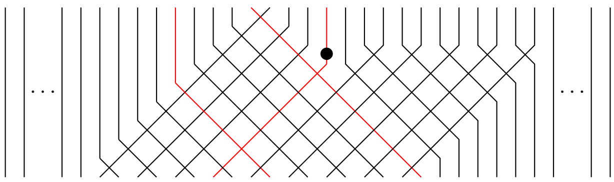
$s_{m+1/2}s_{m+3/2}s_{1/2}s_{-1/2} \cdot (s_b\hat{\mathsf{T}}')$, hence

$$\psi_b\psi_{\hat{\mathsf{T}}'} = \alpha_{\mathsf{V}_1}\psi_{\mathsf{V}_1}$$

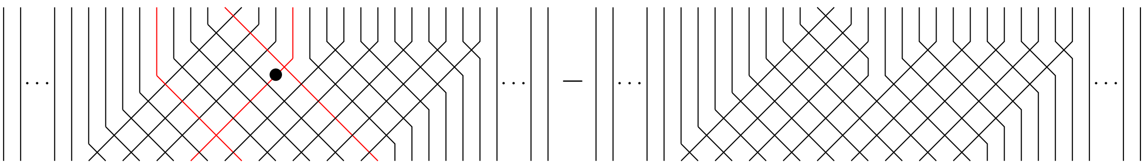
with $\alpha_{\mathsf{V}_1} \in \Bbbk$. If $|\ell(\mu)| = 1$, then the unique path in $\mathrm{Path}(\mu, s_b\hat{\mathsf{T}}')$ with the desired property is the path $\mathsf{V}_2 := s^2_{1/2}s^1_{1/2} \cdot (s_b\hat{\mathsf{T}}')$, hence

$$\psi_b\psi_{s_b\hat{\mathsf{T}}'} = \alpha_{\mathsf{V}_2}\psi_{\mathsf{V}_2}$$

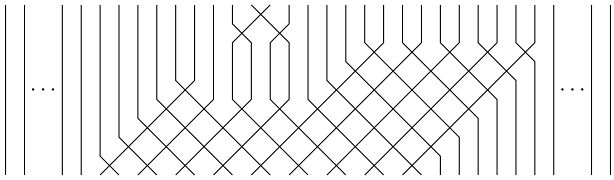
with $\alpha_{\mathsf{V}_2} \in \Bbbk$. In order to prove that the homomorphism φ^μ_λ is injective it suffices to prove that the scalars α_{V_1}, $\alpha_{\mathsf{V}_2} \in \Bbbk$ are non-zero. We prove it for the scalar $\alpha_{\mathsf{V}_1} \in \Bbbk$ since the proof for $\alpha_{\mathsf{V}_2} \in \Bbbk$ will be a subcase. The element $y_b\psi_{s_b\hat{\mathsf{T}}'}$ corresponds to the diagram



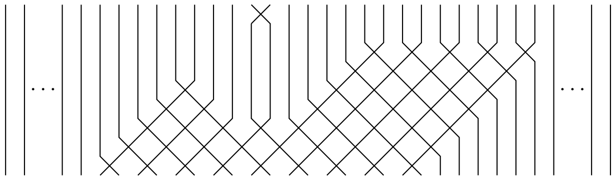
By the KLR relation (R4) we have that the above element is equal to the following combination of diagrams:



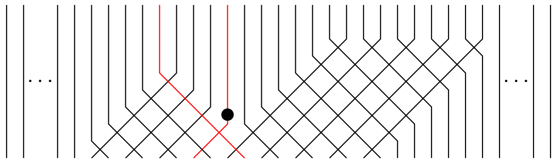
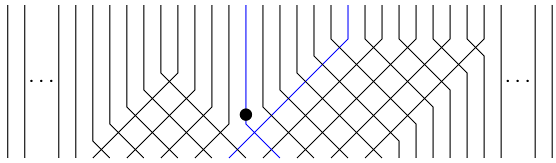
By using similar arguments as above, the first summand is zero. We now consider the second summand and we apply the KLR-relation (R6), hence we obtain the diagram

$-$ 

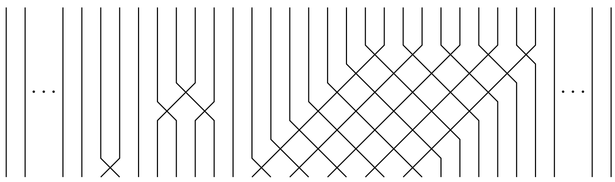
We apply the KLR-relation (R5) for the case that the residues are not equal and the do not differ by one. The we obtain the diagram

$-$ 

in which we can apply the KLR relation (R5) for the case that the residues differ by one. Then we get the following sum of diagrams.

$-$  $+$ 

where strands with different colours carry different residues which differ by one. We apply the KLR relations (R4) and (R6) appropriately until we obtain reduced diagrams. Then the only non-zero summand is of the form

(± 1) 

Hence we have proven that the scalar $\alpha_{\mathsf{V}_1} \in \Bbbk$ is equal to ± 1 and we shall not be interested in keeping track of its value. As a result the homomorphism φ_{λ}^{μ} is injective homomorphism.

In any case the element $\psi_b \psi_{\hat{\mathsf{T}}'}$ corresponds to the path $\mathsf{V}_1 := s_{m+1/2} s_{m+3/2} s_{1/2} s_{-1/2} \cdot (s_b \hat{\mathsf{T}}')$ which intersects the hyperplane $H_{1/2}$ after intersecting the hyperplane $H_{-1/2}$.

By repeating the same procedure for all admissible integers $a \in \mathbb{Z}$ we prove that the paths which either last intersect $H_{m-1/2}$ or intersect $H_{1/2}$ after intersecting $H_{-1/2}$, correspond to elements in the image of φ_λ^μ.

Dimension count: In order to complete the proof we need to prove that any element in the image of φ_λ^μ either last intersects $H_{m-1/2}$ or intersects $H_{1/2}$ after intersecting $H_{-1/2}$. For that purpose it suffices to show that the map

$$\Phi\colon \mathrm{Path}(\lambda) \longrightarrow \mathrm{Path}(\mu)$$

defined by

$$\Phi(\mathsf{U}) := \begin{cases} s^{\text{last}}_{m-1/2} \cdot \mathsf{U}, & \text{if } \mathsf{U} \text{ last intersects } H_{m-1/2} \\ s^{\text{last}}_{m+1/2} s^{\text{last}}_{1/2} s^{\text{last}}_{-1/2} \cdot \mathsf{U}, & \text{otherwise} \end{cases}$$

is an injective map of degree one, with image containing the paths in $\mathrm{Path}(\mu)$ which either last intersect $H_{m-1/2}$ or intersect $H_{1/2}$ after intersecting $H_{-1/2}$. Let $\mathsf{U} \in \mathrm{Path}(\mu)$ be a path which last intersects the hyperplane $H_{m-1/2}$ at the point $\mathsf{u}^{\text{last}}_{m-1/2}$. Then we have that

$$\Phi(s^{\text{last}}_{m-1/2} \cdot \mathsf{U}) = \mathsf{U}$$

hence U belongs to the image of the map Φ. Consider an arbitrary path $\mathsf{V} \in \mathrm{Path}(\mu)$ which intersects both hyperplanes $H_{-1/2}$ and $H_{1/2}$. Suppose that if $\mathsf{v}^{\text{last}}_{1/2} = \mathsf{V}(n_2)$ is the last intersection point with the hyperplane $H_{1/2}$, then there exists an intersection point $\mathsf{v}^{l}_{-1/2} = \mathsf{V}(n_1)$ with $n_1 < n_2$ and assume that n_1 is the greatest integer with that property. Moreover let $\mathsf{v}^{\text{last}}_{m+1/2} = \mathsf{V}(n_3)$ be the last intersection point of V with $H_{m+1/2}$. Then

$$\Phi(s^{l}_{-1/2} s^{\text{last}}_{1/2} s^{\text{last}}_{m+1/2} \cdot \mathsf{V}) = \mathsf{V}$$

hence V belongs to the image of Φ. Since both those types of paths belong to the image of φ_λ^μ we have proven that any element in the image corresponds to a path of that form. The fact that Φ is of degree 1 is straightforward by its construction. $\square$

3.5 Bases of simple modules

This is the last section of this chapter. We shall use all the work done in the previous sections and we shall construct the bases for the simple modules of the blob algebra over a field of characteristic zero.

From now on let $\Bbbk$ be a field of characteristic 0. Recall from Section 3.2 that for a given bipartition $\mu \in \mathrm{Bip}_1(n)$ with $\ell(\mu) \leq 0$ we fix two bipartitions λ, λ' and consider the homomorphisms φ^{μ}_{λ}, $\varphi^{\mu}_{\lambda'}$ of Definition 3.20. Note that everything works on the same way if $\ell(\mu) > 0$, so we restrict ourselves to the previous case. Let us denote by $\mathrm{Im}\varphi^{\mu}_{\lambda'}$ and $\mathrm{Im}\varphi^{\mu}_{\lambda}$ the images of the above homomorphisms, constructed in Propositions 3.29 and 3.31 respectively. We denote by $E(\mu)$ the quotient module

$$E(\mu) := \Delta(\mu)/(\mathrm{Im}\ \varphi^{\mu}_{\lambda'} + \mathrm{Im}\ \varphi^{\mu}_{\lambda}),$$

i.e. the cell module $\Delta(\mu)$ modulo the sum of the images of the homomorphisms. From the results of the previous section we have that when μ belongs to an alcove, $E(\mu)$ is spanned by elements corresponding to paths which do not intersect the hyperplane $H_{1/2}$ and they do not last intersect the hyperplane $H_{m-1/2}$. In the hyperplane case we have that the module $E(\mu)$ is spanned by elements ψ_{T} where T is a path which does not intersect the hyperplane $H_{1/2}$.

A key step towards constructing our bases for the simple modules is the fact that the graded dimension of $E(\mu)$ is bar invariant. For this reason the following construction is of particular importance.

Construction 3.33. *For any path $\mathsf{T} \in \mathrm{Path}(\mu)$ with $\psi_{\mathsf{T}} \in E(\mu)$ we shall construct a path $\bar{\mathsf{T}} \in \mathrm{Path}(\mu)$ with $\psi_{\bar{\mathsf{T}}} \in E(\mu)$ and $\deg(\bar{\mathsf{T}}) = -\deg(\mathsf{T})$. We denote by $\mathsf{t}^1_{q-1/2}, \mathsf{t}^2_{q-1/2}, \cdots$ the intersection points of T with the hyperplane $H_{q-1/2}$ for some $q \leq 0$. For the construction of $\bar{\mathsf{T}}$ we focus our attention on the intersection points of T with the hyperplanes. Let $\mathsf{T}^i_{q-1/2}$ be an intersection point of T with the hyperplane $H_{q-1/2}$. If the next point that T intersects any hyperplane is the point $\mathsf{t}^{i+1}_{q-1/2}$ then for all point between $\mathsf{T}^i_{q-1/2}$ and $\mathsf{T}^{i+1}_{n-1/2}$ (which belong to an alcove) we have that $\bar{\mathsf{T}}(a) := (s^i_{n-1/2} \cdot \mathsf{T})(a)$. We need to consider the case that the next intersection*

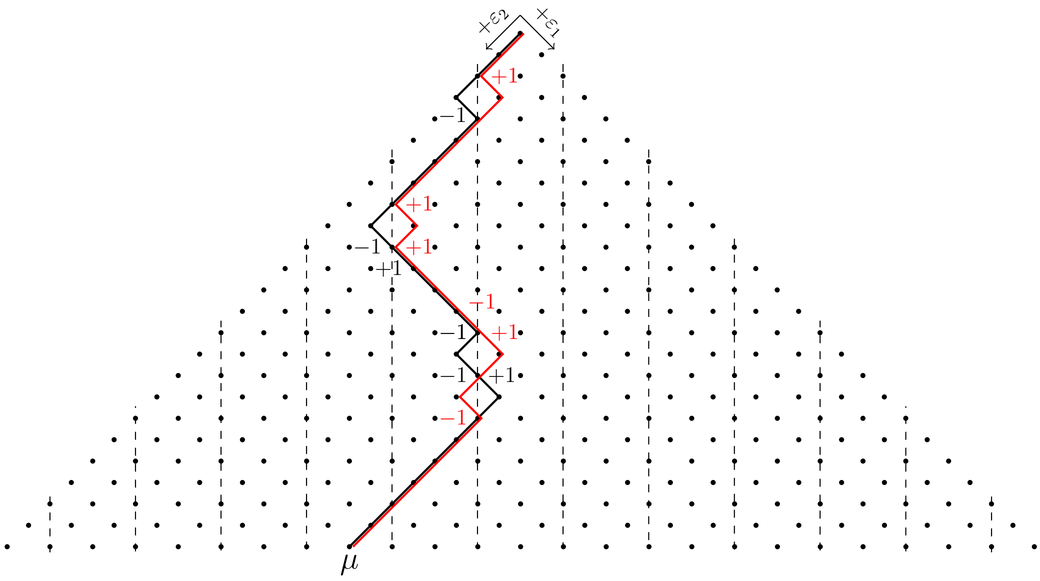


Figure 3.11: The black path is the path T and the red path is the path $\bar{\mathsf{T}}$. The numbers in black and red are the integers contributing to the degree of T and $\bar{\mathsf{T}}$ respectively.

point is $\mathsf{t}^j_{q-3/2}$*, for some* j*, or* $\mathsf{t}^i_{q-1/2}$ *is the last intersection point of* T *with any hyperplane. In these cases, for the points between* $\mathsf{t}^i_{q-1/2}$ *and* $\mathsf{t}^j_{q-3/2}$ *or the points from* $\mathsf{t}^i_{q-1/2}$ *until the end of the path respectively, we have that* $\bar{\mathsf{T}}(a) := \mathsf{T}(a)$*.*

Note that the above construction does not depend on whether the bipartition μ lies in an alcove or on a hyperplane.

Example 3.34. *Suppose that* $n = 24$*,* $e = 4$ *and* $\sigma = (0, 2)$*. We consider the bipartition* $\mu = ((1^8), (1^{16})) \in \mathrm{Bip}_1(24)$ *and let* $\mathsf{T} \in \mathrm{Path}(\mu)$ *be the black path in Figure 3.11 which corresponds to the basis element* $\psi_{\mathsf{T}} \in E(((1^8), (1^{16})))$*. The path* T *has degree* $\deg(\mathsf{T}) = -2$*. The path* $\bar{\mathsf{T}} \in \mathrm{Path}(\mu)$ *obtained by the procedure we described before, is the red path in Figure 3.11. One can readily check that* $\deg(\bar{\mathsf{T}}) = 2 = -\deg(\mathsf{T})$*.*

Remark 3.35. Suppose that $\mu \in \mathrm{Bip}_1(n)$ with $\ell(\mu) = m > 0$. Then the elements spanning the module $E(\mu)$ are of the form ψ_{T} where T is a path which does not intersect $H_{-1/2}$ and does not last intersect the hyperplane $H_{m+1/2}$.

Using the notions we defined above we can state and prove the following theorems. Those theorems are two of the main results of our paper and gives a precise description of the basis of an irreducible representation of the blob algebra over a field of characteristic 0, in the alcove and hyperplane cases.

Theorem 3.36. *Let $\mu \in \mathrm{Bip}_1(n)$ with $\ell(\mu) = m$. The module $E(\mu)$ is equal to the simple head $L(\mu)$, hence*

1. *if $m \leq 0$*

$$L(\mu) = \mathrm{span}_{\Bbbk} \left\{ \psi_{\mathsf{T}} \mid \begin{array}{c} \mathsf{T} \in \mathrm{Path}(\mu),\ \mathsf{T} \textit{ does not intersect } H_{1/2} \\ \textit{and does not last intersect } H_{m-1/2} \end{array} \right\},$$

2. *if $m > 0$*

$$L(\mu) = \mathrm{span}_{\Bbbk} \left\{ \psi_{\mathsf{T}} \mid \begin{array}{c} \mathsf{T} \in \mathrm{Path}(\mu),\ \mathsf{T} \textit{ does not intersect } H_{-1/2} \\ \textit{and does not last intersect } H_{m+1/2} \end{array} \right\}.$$

Proof. We consider the quotient module $E(\mu)$. For any path $\mathsf{T} \in \mathrm{Path}(\mu)$ with $\psi_{\mathsf{T}} \in E(\mu)$ we have already shown (see Construction 3.33) that there exists a path $\bar{\mathsf{T}} \in \mathrm{Path}(\mu)$ with $\deg(\bar{\mathsf{T}}) = -\deg(\mathsf{T})$ and therefore $\dim_t(E(\mu))$ is bar-invariant. Suppose that $E(\mu)$ is not simple. Then it will have a simple constituent $L(\alpha)$, with $\alpha \lhd \mu$, with multiplicity equal to the decomposition number $d_{\mu\alpha}(t) = t^i \in t\mathbb{N}_0(t)$, by Theorem 3.17 (note that this holds because we are working over a field of characteristic zero). Moreover, by Theorem 3.15 we know that the simple modules have bar-invariant characters. Considering the above, the graded dimension of $E(\mu)$ is not bar-invariant, as strictly positive shifts of bar-invariant polynomials are not bar-invariant polynomials, which is a contradiction. Hence, we have proven that $E(\mu) = L(\mu)$. $\square$

The following theorem is the analogous of Theorem 3.36 in the hyperplane case.

Theorem 3.37. *Let $\mu \in \mathrm{Bip}_1(n)$ with $\ell(\mu) = m - 1/2$. The module $E(\mu)$ is equal to the simple head $L(\mu)$, hence*

1. *if* $m \leq 0$

$$L(\mu) = \mathrm{span}_{\Bbbk}\{\psi_{\mathsf{T}} \mid \mathsf{T} \in \mathrm{Path}(\mu),\ \mathsf{T} \textit{ does not intersect } H_{1/2}\},$$

2. *if* $m > 0$

$$L(\mu) = \mathrm{span}_{\Bbbk}\{\psi_{\mathsf{T}} \mid \mathsf{T} \in \mathrm{Path}(\mu),\ \mathsf{T} \textit{ does not intersect } H_{-1/2}\}.$$

Proof. The proof is identical to the proof of Theorem 3.36. $\square$

Chapter 4

BGG resolutions

In 1975 Bernstein, Gelfand and Gelfand [BGG75] constructed resolutions for finite Weyl groups in the context of finite dimensional Lie algebras. In more detail they constructed resolutions of simple modules by Verma modules. These resolutions are known after their names as Bernstein–Gelfand–Gelfand (BGG) resolutions. Parabolic BGG resolutions were constructed by Lepowsky [Lep77] and have gone on to have applications in the study of the Laplacian space [Eas05]. Kac–Kazhdan conjectured that these BGG resolutions would generalise to all Kac–Moody Lie algebras, [KK79]. Their conjecture was proven in several cases, such as for the affine Weyl group $\hat{\mathfrak{S}}_2$ by Wakimoto [Wak86], in the classical type by Hayashi [Hay88] and in the general case by Feigin, Frenkel, and Ku [FF92, Ku89]. The result was extended arbitrary fields by Mathieu [Mat96]. Bowman–Hazi–Norton [BHN20] constructed BGG resolutions in the context of affine symmetric groups and a maximal finite parabolic subgroup over $\mathbb{C}$. In the context of the modular representation theory of the symmetric group and Hecke algebras, BGG resolutions were first used by Bowman–Norton–Simental [BNS] with applications in the calculation of Betti numbers and Castelnuovo–Mumford regularity.

In this chapter we generalise the findings of [BNS] in the case of $\widehat{\mathfrak{S}}_2$. Namely, we prove that *all* simple modules of the blob algebra admit BGG resolutions over a field of characteristic zero. More precisely we shall construct resolutions of cell modules for each simple B_n^σ-module indexed by a bipartition which belongs to an alcove. BGG resolutions of simple modules indexed by bipartitions on a hyperplane are simpler to construct and they are used in the proof of the more general case. In the first section we calculate the composition of one-column homomorphisms between cell modules of the blob algebra. Knowing these compositions will allow us to ensure that the diagrams of modules that we construct are indeed chain complexes. In the second section we construct the BGG resolutions in the hyperplane case. Finally, in the third section we present the main result of this chapter which is the construction of BGG resolutions for all simple modules of the blob algebra in the alcove case.

The results of this chapter are the author's work in the last section of [Mic21].

4.1 Composition of one-column homomorphisms

In this section we shall compute the composition of certain one-column homomorphisms. We consider two bipartitions $\alpha, \gamma \in \mathrm{Bip}_1(n)$ such that $|\ell(\alpha)| = |\ell(\gamma)| + 2$ and without loss of generality we may assume that $\ell(\gamma) < 0$. Then we can either have $\ell(\alpha) < 0$ or $\ell(\alpha) > 0$ and let $\beta, \beta' \in \mathrm{Bip}_1(d)$ be the bipartitions with $|\ell(\beta)| = |\ell(\beta')| = |\ell(\gamma)| + 1$ for which we have constructed the homomorphisms $\varphi^{\gamma}_{\beta}, \varphi^{\gamma}_{\beta'}$ of Chapter 3. In a case as above we can consider the following "diamond" diagram:

$$\begin{tikzcd}
 & \Delta(\gamma) & \\
\Delta(\beta) \arrow[ur, "\varphi^{\gamma}_{\beta}"] & & \Delta(\beta') \arrow[ul, "\varphi^{\gamma}_{\beta'}"'] \\
 & \Delta(\alpha) \arrow[ul, "\varphi^{\beta}_{\alpha}"] \arrow[ur, "\varphi^{\beta'}_{\alpha}"'] &
\end{tikzcd}$$

The aim of this section is to compute the compositions of the homomorphisms in such diamonds and prove that those are commutative or anti-commutative.

In the next proposition we shall assume that $\ell(\alpha) < 0$, as everything works similarly when $\ell(\alpha) > 0$.

Proposition 4.1. *Let $\alpha, \gamma \in \mathrm{Bip}_1(n)$ with $|\ell(\alpha)| = |\ell(\gamma)| + 2$. Then*

$$(\varphi_\beta^\gamma \circ \varphi_\alpha^\beta)(\psi_{\mathsf{T}^\alpha}) = (-1)^{|\ell(\gamma)|}\psi_{s_{\ell(\gamma)+1/2}s_{1/2}s_{-1/2}s_{\ell(\beta)-1/2}\cdot\mathsf{T}^\alpha}$$

and

$$(\varphi_{\beta'}^\gamma \circ \varphi_\alpha^{\beta'})(\psi_{\mathsf{T}^\alpha}) = \psi_{s_{1/2}s_{-1/2}\cdot\mathsf{T}^\alpha}.$$

In particular the diamond will either be commutative or anti-commutative, depending on the number $|\ell(\gamma)|$.

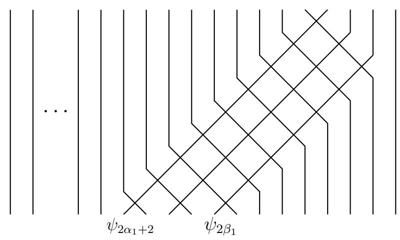
Proof. Let $\alpha = ((1^{\alpha_1}), (1^{\alpha_2}))$, $\beta = ((1^{\beta_1}), (1^{\beta_2}))$ and $\gamma = ((1^{\gamma_1}), (1^{\gamma_2}))$. The composition $\Delta(\alpha) \longrightarrow \Delta(\beta) \longrightarrow \Delta(\gamma)$ is harder to compute than the composition $\Delta(\alpha) \longrightarrow \Delta(\beta') \longrightarrow \Delta(\gamma)$ and we shall start by computing it. Consider the generator ψ_{T^α} of the cell module $\Delta(\alpha)$. Then

$$\begin{aligned}(\varphi_\beta^\gamma \circ \varphi_\alpha^\beta)(\psi_{\mathsf{T}^\alpha}) &= \varphi_\beta^\gamma(\varphi_\alpha^\beta(\psi_{\mathsf{T}^\alpha})) \\ &= \varphi_\beta^\gamma(\psi_{s_{\ell(\beta)-1/2}\cdot\mathsf{T}^\alpha}).\end{aligned}$$

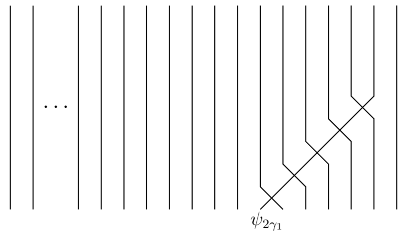
Let $\mathsf{T} := s_{\ell(\beta)-1/2}\cdot\mathsf{T}^\alpha \in \mathsf{Path}(\beta)$ and $\mathsf{T} = w_\mathsf{T}\mathsf{T}^\beta$. Then we have that

$$\begin{aligned}\varphi_\beta^\gamma(\psi_{s_{\ell(\beta)-1/2}\cdot\mathsf{T}^\alpha}) &= \varphi_\beta^\gamma(\psi_{w_\mathsf{T}}\psi_{\mathsf{T}^\beta}) \\ &= \psi_{w_\mathsf{T}}\varphi_\beta^\gamma(\psi_{\mathsf{T}^\beta}) \\ &= \psi_{w_\mathsf{T}}\psi_{s_{\ell(\beta)+1/2}\cdot\mathsf{T}^\beta}.\end{aligned}$$

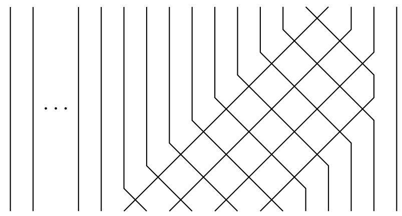
The element $\psi_\mathsf{T} \in \Delta(\beta)$ corresponds to a diagram of the form



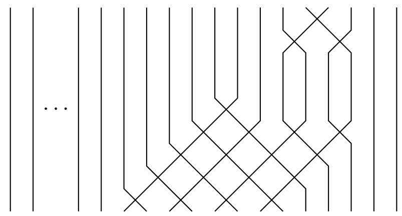
while the element $\psi_{s_{\ell(\beta)+1/2}\cdot \mathsf{T}^\beta} \in \Delta(\gamma)$ corresponds to a diagram of the form



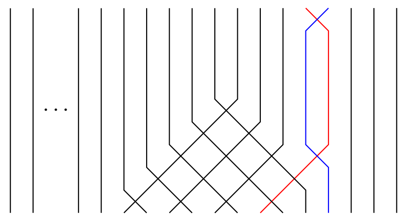
For the multiplication $\psi_{w_\mathsf{T}}\psi_{s_{\ell(\beta)+1/2}\cdot \mathsf{T}^\beta}$ we concatenate the above diagrams. Hence we obtain the diagram



and by applying the KLR-relation (R6) in the case the middle residue is less by one that the adjacent residues we get the diagram



Then we apply the KLR relation (R5) in the case that the residues are not equal and they do not differ by one and we get the diagram



Since the strands we have marked in red and blue carry the different but adjacent residues, we apply the KLR-relation (R5) and we reduce our computation to a computation similar to the one in the proof of Proposition 3.31 where we compute the effect of y-generators. Hence, we have that

$$\begin{aligned}(\varphi_{\beta}^{\gamma} \circ \varphi_{\alpha}^{\beta})(\psi_{\mathsf{T}^{\alpha}}) &= \psi_{w_{\mathsf{T}}}\psi_{s_{\ell(\beta)+1/2}\cdot\mathsf{T}^{\beta}} \\ &= (-1)^{|\ell(\gamma)|}\psi_{s_{\ell(\gamma)+1/2}s_{1/2}s_{-1/2}s_{\ell(\beta)-1/2}\cdot\mathsf{T}^{\alpha}}.\end{aligned}$$

Now we shall compute the composition $\Delta(\alpha) \longrightarrow \Delta(\beta') \longrightarrow \Delta(\gamma)$ on the generator $\psi_{\mathsf{T}^{\alpha}}$ of the cell module $\Delta(\alpha)$. We have that

$$\begin{aligned}(\varphi_{\beta'}^{\gamma} \circ \varphi_{\alpha}^{\beta'})(\psi_{\mathsf{T}^{\alpha}}) &= \varphi_{\beta'}^{\gamma}(\varphi_{\alpha}^{\beta'}(\psi_{\mathsf{T}^{\alpha}})) \\ &= \varphi_{\beta'}^{\gamma}(\psi_{s_{-1/2}\cdot\mathsf{T}^{\alpha}})\end{aligned}$$

Let $\mathsf{S} := s_{-1/2} \cdot \mathsf{T}^{\alpha} \in \mathsf{Path}(\beta')$ and $\mathsf{S} = w_{\mathsf{S}}\mathsf{T}^{\beta'}$. Then we have that

$$\varphi_{\beta'}^{\gamma}(\psi_{s_{-1/2}\cdot\mathsf{T}^{\alpha}}) = \varphi_{\beta'}^{\gamma}(\psi_{\mathsf{S}}) = \varphi_{\beta'}^{\gamma}(\psi_{w_{\mathsf{S}}}\psi_{\mathsf{T}^{\beta'}}) = \psi_{w_{\mathsf{S}}}\varphi_{\beta'}^{\gamma}(\psi_{\mathsf{T}^{\beta'}}).$$

By Definition 3.20 we have that $\varphi_{\beta'}^{\gamma}(\psi_{\mathsf{T}^{\beta'}}) = \psi_{s_{1/2}\cdot\mathsf{T}^{\beta'}}$ and let $\mathsf{U} := s_{1/2} \cdot \mathsf{T}^{\beta'} \in \mathsf{Path}(\gamma)$. Then

$$\varphi_{\beta'}^{\gamma}(\psi_{s_{-1/2}\cdot\mathsf{T}^{\alpha}}) = \psi_{w_{\mathsf{S}}}\psi_{s_{1/2}\cdot\mathsf{T}^{\beta'}} = \psi_{w_{\mathsf{S}}}\psi_{w_{\mathsf{U}}}\psi_{\mathsf{T}^{\gamma}}.$$

The final equality gives the desired result because the product of generators $\psi_{w_{\mathsf{S}}}\psi_{w_{\mathsf{U}}}$ corresponds to reduced transposition, hence $\psi_{w_{\mathsf{S}}}\psi_{w_{\mathsf{U}}}\psi_{\mathsf{T}^{\gamma}} \in \mathsf{Path}(\gamma)$ is equal to the element $\psi_{s_{-1/2}s_{1/2}\cdot\mathsf{T}^{\alpha}}$ as required. $\square$

4.2 BGG resolution for the hyperplane case

Let $\Bbbk$ be a field of characteristic zero. In this section we attach to any bipartition $\lambda \in \mathrm{Bip}_1(n)$ a complex $C_{\bullet}(\lambda)$ called the BGG resolution for the irreducible representation $L(\lambda)$.

In the case that the simple representation is indexed by a bipartition $\lambda \in \mathrm{Bip}_1(n)$ with $\lambda \in H_{q-1/2}$, $q \in \mathbb{Z}$, the BGG resolution has an easy form. In the next proposition we construct a BGG resolution for the irreducible representation $L(\lambda)$, $\lambda \in H_{q-1/2}$, for some $q \in \mathbb{Z}$.

Proposition 4.2. *Let $\lambda \in \mathrm{Bip}_1(n)$ with $\lambda \in H_{q-1/2}$, for some $q \in \mathbb{Z}$. We have a short exact sequence*

$$C_\bullet(\lambda)\colon \qquad 0 \longrightarrow \Delta(\mu)\langle |\ell(\mu)| - |\ell(\lambda)| \rangle \xrightarrow{\varphi_\mu^\lambda} \Delta(\lambda) \longrightarrow L(\lambda) \longrightarrow 0$$

where $\mu \in \mathrm{Bip}_1(n)$ with $|\ell(\mu)| = |\ell(\lambda)| + 1$ and $\ell(\mu) = -(\ell(\lambda) + 1)$.

Proof. The result is straightforward by using the fact that $\mathrm{Coker}(\varphi_\mu^\lambda) = L(\lambda)$. □

4.3 BGG resolutions for the alcove case

In this section we construct BGG resolutions for the simple modules indexed by bipartitions which belong to an alcove. Let $\lambda \in \mathrm{Bip}_1(n)$ be a bipartition such that $\lambda \in \mathfrak{a}_q$, $q \in \mathbb{Z}$ and let us denote by ν_i, ν_i' the bipartitions -in the same linkage class as λ- such that $|\ell(\nu_i)| = |\ell(\nu_i')| = |\ell(\lambda)| + i$. We set

$$C_\bullet(\lambda) := (C_i(\lambda))_{i \geq 0}$$

where

$$C_0(\lambda) := \Delta(\lambda)$$

and

$$C_i(\lambda) := \bigoplus_{\nu = \nu_i, \nu_i'} \Delta(\nu)\langle |\ell(\nu)| - |\ell(\lambda)| \rangle$$

for $i > 0$. We define the maps

$$\delta_i\colon C_{i+1}(\lambda) \longrightarrow C_i(\lambda)$$

between those components. For $i = 0$ we have that

$$\delta_0 := \begin{pmatrix} \varphi^{\lambda}_{\nu_1} & \varphi^{\lambda}_{\nu'_1} \end{pmatrix}. \tag{4.1}$$

For $i > 0$ we shall distinguish between two cases on the number $|\ell(\lambda)| + i$. In particular if $|\ell(\lambda)| + i$ is even, we set

$$\delta_i := \begin{pmatrix} -\varphi^{\nu_i}_{\nu_{i+1}} & \varphi^{\nu_i}_{\nu'_{i+1}} \\ \varphi^{\nu'_i}_{\nu_{i+1}} & -\varphi^{\nu'_i}_{\nu'_{i+1}} \end{pmatrix} \tag{4.2}$$

whereas if it is odd, we set

$$\delta_i := \begin{pmatrix} \varphi^{\nu_i}_{\nu_{i+1}} & -\varphi^{\nu_i}_{\nu'_{i+1}} \\ -\varphi^{\nu'_i}_{\nu_{i+1}} & \varphi^{\nu'_i}_{\nu'_{i+1}} \end{pmatrix}. \tag{4.3}$$

Note that there is the possibility that not both the rightmost and leftmost alcove contain bipartitions linked with λ. In that case let $\nu_1, \nu'_1, \cdots, \nu_k \in \mathrm{Bip}_1(d)$ be the bipartitions linked with λ. Then we define the maps $\delta_i \colon C_{i+1}(\lambda) \longrightarrow C_i(\lambda)$ are defined exactly like the maps (4.1), (4.2) and (4.3) for $0 \leq i \leq k-1$. For $i = k$ we define

$$\delta_k := \begin{pmatrix} \varphi^{\nu_{k-1}}_{\nu_k} \\ \varphi^{\nu'_{k-1}}_{\nu_k} \end{pmatrix}. \tag{4.4}$$

Proposition 4.3. *Let $\lambda \in \mathrm{Bip}_1(n)$ be a bipartition such that $\lambda \in \mathfrak{a}_q$, $q \in \mathbb{Z}$. For the pair $(C_\bullet(\lambda), (\delta_i)_{i\geq 0})$ we have that*

$$\mathrm{Im}(\delta_{i+1}) \subset \mathrm{Ker}(\delta_i).$$

for any $i \geq 0$, in other words the pair $(C_\bullet(\lambda), (\delta_i)_{i\in\mathbb{Z}})$ is a (chain) complex.

Proof. The result is straightforward from Proposition 4.1. $\square$

Definition 4.4. Recall that $I = \mathbb{Z}/e\mathbb{Z}$ and let $r \in I$ be a given residue. The r-restriction functor

$$r - \mathsf{res}^n_{n-1} \colon \mathrm{mod} - \mathsf{B}^{\sigma}_n \longrightarrow \mathrm{mod} - \mathsf{B}^{\kappa}_{n-1}$$

is defined by

$$M \longmapsto \sum_{\underline{i}=(i_1,i_2,\cdots,i_{n-1},r)\in I^{n-1}\times\{r\}} \mathsf{e}_{\underline{i}} M$$

and we have that

$$\mathsf{res}^n_{n-1} = \sum_{r\in I} r - \mathsf{res}^n_{n-1}.$$

The following remark discusses the way we can restrict from modules in an algebra of a given rank n, to modules in algebras of lower rank. It also introduces for the first time the notation E_r, for $r \in \mathbb{Z}/e\mathbb{Z}$ which will be used later in the proofs of this section.

Remark 4.5. Suppose that $\lambda \in \mathrm{Bip}_1(n)$ and $\lambda \in \mathfrak{a}_q$, $q \in \mathbb{Z}$. If $r \in I$ then we have that λ has either 0 or 1 removable r-nodes. We shall denote by $\mathsf{E}_r(\lambda)$ the unique bipartition which differs from λ by removing an r-node. Consider the cell module $\Delta_n(\lambda) \in \mathrm{mod} - \mathsf{B}^\sigma_n$. We have that

$$r - \mathsf{res}^n_{n-1}(\Delta_n(\lambda)) = \begin{cases} \Delta_{n-1}(\mathsf{E}_r(\lambda)), & \text{if } \mathsf{Rem}_r(\lambda) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

where $\Delta_{n-1}(\mathsf{E}_r(\lambda))$ is a cell module in $\mathrm{mod} - \mathsf{B}^\kappa_{n-1}$.

Definition 4.6. Let $\lambda \in \mathrm{Bip}_1(n)$. The complex $(C_\bullet(\lambda), (\delta_i)_{i\geq 0})$ is called BGG $\mathsf{resolution}$ of $L(\lambda)$ if

$$H_i(C_\bullet(\lambda)) = \begin{cases} L(\lambda), & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.7 is the basic result of the thesis on BGG resolutions associated to simple modules of the blob algebra. We follow the same tactics as in the previous chapters of the thesis and we include an example prior to the main proof. This will help the reader to absorb the technicalities and will make the flow of the proof easier. We first state Theorem 4.7 and the example follows.

Theorem 4.7. *Let $\lambda \in \mathrm{Bip}_1(n)$ be a bipartition such that $\lambda \in \mathfrak{a}_q$, $q \in \mathbb{Z}$. The B^σ_n-complex*

$$C_\bullet(\lambda) := \bigoplus_{\nu \trianglelefteq \lambda} \Delta(\nu)\langle |\ell(\nu)| - |\ell(\lambda)| \rangle.$$

with differentials $\delta_i \colon C_{i+1}(\lambda) \longrightarrow C_i(\lambda)$ *the maps defined above is a BGG resolution for the simple representation* $L(\lambda)$*. Moreover we have that*

$$\mathsf{res}^n_{n-1}(L_n(\lambda)) = \bigoplus_{\square \in \mathsf{Rem}(\lambda)} L_{n-1}(\lambda - \square).$$

Remark 4.8. Note that the subscript in $L_n(\lambda)$ is used to emphasise that $L(\lambda)$ is a module of the algebra B^σ_n. This is used when we need to distinguish from modules in algebras in other ranks, such as B^σ_{n-1}.

Example 4.9. *Let* $n = 9$*,* $e = 4$ *and* $\sigma = (0, 2) \in \mathbb{Z}^2$*. We consider the bipartitions* $\lambda = ((1^2), (1^7)), \mu = ((1^3), (1^6)) \in \mathrm{Bip}_1(9)$ *and let* $r \in \{0, 1, 2, 3\} = \mathbb{Z}/4\mathbb{Z}$*. We shall calculate the images of the bipartitions* λ, μ *under the restriction* E_r*,* $r \in \mathbb{Z}/4\mathbb{Z}$*, which will belong to* $\mathrm{Bip}_1(8)$*. For the bipartition* λ *we have that*

$$\mathsf{E}_0(\lambda) = ((1^2), (1^6))$$
$$\mathsf{E}_3(\lambda) = ((1), (1^7))$$

while for μ *we have that*

$$\mathsf{E}_1(\mu) = ((1^3), (1^5))$$
$$\mathsf{E}_2(\mu) = ((1^2), (1^6)).$$

The bipartitions λ, μ *as well as their restrictions are depicted in Figure 4.1. We remark that the restrictions can either keep belonging to an alcove or a hyperplane (in lower rank).*

Proof of Theorem 4.7. Let $\lambda \in \mathrm{Bip}_1(n)$ with $\lambda \in \mathfrak{a}_q$, for some $q \leq 0$. Note that everything works analogously when $q > 0$. In order to prove that our B^σ_n-complex is a BGG resolution for the simple representation $L(\lambda)$ we need to show that

$$H_i(C_\bullet(\lambda)) = \begin{cases} L_n(\lambda), & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

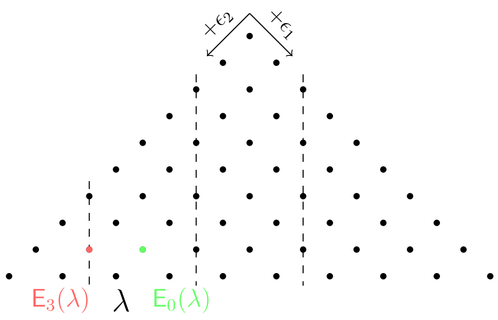


FIGURE 4.1: The bipartitions $\lambda = ((1^2), (1^7)), \mu = ((1^3), (1^6))$ and their restrictions under E_r, $r \in \mathbb{Z}/4\mathbb{Z}$. The points in the alcove geometry corresponding to the restrictions can be distinguished by different colours.

Recall that BGG resolutions and bases for the hyperplane case are already constructed in Proposition 4.2. We shall proceed by induction on the rank n of the blob algebra. We assume that the theorem holds for any bipartition $\lambda \in \mathrm{Bip}_1(n-1)$ where λ belongs to an alcove. We also have that

$$\mathsf{res}^n_{n-1}(C_\bullet(\lambda)) = \bigoplus_{r \in I} C_\bullet(\mathsf{E}_r(\lambda)).$$

We shall consider one residue at a time. The bipartition λ belongs to an alcove, hence as we mentioned in Remark 4.5 there will be either 0 or 1 removable r-nodes. For each residue we have 3 different cases.

- Suppose that the bipartition $\mathsf{E}_r(\lambda)$ belongs to the hyperplane $H_{q-1/2}$. In terms of the alcove geometry, one can think of it as the hyperplane of the alcove $\mathfrak{a}_q$ which is further away from the origin than λ. As an example we refer to the restriction $E_3(\lambda)$ in Figure 4.1. Note that since the r-restriction functor is exact, we have that $r - \mathsf{res}^n_{n-1}(C_\bullet(\lambda))$ is a complex. Consider a bipartition $\mu \in \mathrm{Bip}_1(n)$ such that μ is less dominant that λ and $\mathsf{E}_r(\mu)$ is a bipartition. In the case we examine, all the bipartitions of n with the aforementioned property come into pairs (ν^+, ν^-) with $\nu^+ \triangleleft \nu^-$ and $|\ell(\nu^+)| = |\ell(\nu^-)| + 1$. They also have the additional property

$$\mathsf{E}_r(\nu^+) = \mathsf{E}_r(\nu^-) = \nu$$

where ν is linked with $\mathsf{E}_r(\lambda)$. Then we have that

$$r - \mathsf{res}^n_{n-1}(\Delta_n(\nu^+)) = r - \mathsf{res}^n_{n-1}(\Delta_n(\nu^-)) = \Delta_{n-1}(\nu).$$

Now consider the homomorphism $\varphi^{\nu^-}_{\nu^+} \in \mathrm{Hom}_{\mathsf{B}^\sigma_n}(\Delta_n(\nu^+), \Delta_n(\nu^-))$ for some bipartition $\nu \triangleleft \mathsf{E}_r(\lambda)$. Under the r-restriction functor we have that

$$r - \mathsf{res}^n_{n-1}(\varphi^{\nu^-}_{\nu^+}) = 1_\nu \in \mathrm{End}_{\mathsf{B}^\sigma_n}(\Delta_{n-1}(\nu)).$$

In other words the identity morphism appears into all the differentials of the r-restricted complex $r - \mathsf{res}^n_{n-1}(C_\bullet(\lambda))$, hence the complex is exact. In particular the homology

$$H_i(r - \mathsf{res}^n_{n-1}(C_\bullet(\lambda))) = 0$$

for any $i \geq 0$.

- Suppose that the bipartition $\mathsf{E}_r(\lambda)$ belongs to the hyperplane $H_{q+1/2}$, that is the hyperplane closer to the origin (see $\mathsf{E}_1(\mu)$ in Figure 4.1). Recall that we denote by $\nu_1' \in \mathrm{Bip}_1(n)$ the bipartition such that $|\ell(\nu_1')| = |\ell(\lambda)| + 1$ with ν_1' belonging to the positive alcoves. The pair of bipartitions $\mathsf{E}_r(\lambda)$, $\mathsf{E}_r(\nu_1) \in \mathrm{Bip}_1(n-1)$ form a BGG resolution of the simple $L_{n-1}(\mathsf{E}_r(\lambda))$ as in Proposition 4.2. Apart form those bipartitions, all the rest bipartitions $\mu \in \mathrm{Bip}_1(n)$ which are strictly less dominant than λ and $\mathsf{E}_r(\mu) \in \mathrm{Bip}_1(n)$, pair up in the exact same way as in the previous case when restricted under the r-restriction functor. Hence

$$H_i(r - \mathsf{res}^n_{n-1}(C_\bullet(\lambda))) = 0$$

for $i > 0$. From Proposition 4.2 we have that

$$H_0(r - \mathsf{res}^n_{n-1}(C_\bullet(\lambda))) = L(\lambda).$$

- Suppose that the bipartition $\mathsf{E}_r(\lambda)$ remains to the alcove $\mathfrak{a}_q$. Examples of such case are the restrictions $\mathsf{E}_0(\lambda)$, $\mathsf{E}_2(\mu)$ in Figure 4.1. Then the complex $r - \mathsf{res}^n_{n-1}(C_\bullet(\lambda))$ is given by

$$r - \mathsf{res}^n_{n-1}\left(\bigoplus_{\nu \trianglelefteq \lambda} \Delta_n(\nu)\langle \ell(\nu)\rangle\right)$$

with differentials given by

$$r - \mathsf{res}^n_{n-1}(\delta_i)\colon r - \mathsf{res}^n_{n-1}(C_{i+1}(\lambda)) \longrightarrow r - \mathsf{res}^n_{n-1}(C_i(\lambda)).$$

Note that if $\mathsf{Rem}_r(\nu) \neq \emptyset$, we have that

$$r - \mathsf{res}^n_{n-1}(\Delta_n(\nu)\langle \ell(\nu)\rangle) = \Delta_{n-1}(\mathsf{E}_r(\nu))\langle \ell(\nu)\rangle$$

since $\ell(\nu) = \ell(\mathsf{E}_r(\nu))$, otherwise we have that

$$r - \mathsf{res}^n_{n-1}(\Delta_n(\nu)) = 0.$$

Let $\nu, \nu' \in \mathrm{Bip}_1(n)$ be bipartitions such that $\mathsf{Rem}_r(\nu), \mathsf{Rem}_r(\nu') \neq \emptyset$. Then

$$r - \mathsf{res}^n_{n-1}(\varphi^{\nu'}_{\nu}) = \varphi^{\mathsf{E}_r(\nu')}_{\mathsf{E}_r(\nu)}$$

Hence we get that

$$r - \mathsf{res}^n_{n-1}(C_\bullet(\lambda)) = C_\bullet(\mathsf{E}_r(\lambda))$$

and by the induction hypothesis we have that $H_0(C_\bullet(\mathsf{E}_r(\lambda))) = L_{n-1}(\mathsf{E}_r(\lambda))$, while $H_i(C_\bullet(\mathsf{E}_r(\lambda))) = 0$, for all $i > 0$. Thus $r - \mathsf{res}^n_{n-1}(H_0(C_\bullet(\lambda))) = H_0(C_\bullet(\mathsf{E}_r(\lambda))) = L_{n-1}(\mathsf{E}_r(\lambda))$ and $r - \mathsf{res}^n_{n-1}(H_i(C_\bullet(\lambda))) = 0$, for all $i > 0$.

Using the work we have done above we have proven that

$$\mathsf{res}^n_{n-1}(H_i(C_\bullet(\lambda))) = \begin{cases} \bigoplus_{r\in I} L_{n-1}(\mathsf{E}_r(\lambda)), & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Moreover we have that the cokernel of the differential δ_0 projects onto the simple representation $L(\lambda)$. The above argument gives us

$$\mathsf{res}^n_{n-1}(L_n(\lambda)) \subset \bigoplus_{r\in I} L_{n-1}(\mathsf{E}_r(\lambda)).$$

In addition, by Theorem 3.36, we have that the cardinality of the basis of the simple representation $L_n(\lambda)$ is equal to the sum of the cardinalities of the bases for the simple representations $L_{n-1}(\mathsf{E}_r(\lambda))$, for all $r \in I$. Thus

$$\mathsf{res}^n_{n-1}(L_n(\lambda)) = \bigoplus_{r\in I} L_{n-1}(\mathsf{E}_r(\lambda)).$$

and we conclude that

$$\mathsf{res}^n_{n-1}(H_i(C_\bullet(\lambda))) = \begin{cases} \mathsf{res}^n_{n-1}(L_n(\lambda)), & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Since $\mathsf{res}^n_{n-1}(L_n(\nu)) \neq 0$, for any $\nu \trianglelefteq \lambda$, despite the fact that $r - \mathsf{res}^n_{n-1}(L_n(\nu)) = 0$, for some $r \in I$, we have that

$$H_i(C_\bullet(\lambda)) = \begin{cases} L_n(\lambda), & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

and the proof is complete. □

Bibliography

[AK94] S. Ariki and K. Koike. A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations. *Adv. Math.*, 106(2):216–243, 1994.

[APS19] S. Ariki, E. Park, and L. Speyer. Specht modules for quiver Hecke algebras of type C. *Publications of the Research Institute for Mathematical Sciences*, 55(3):565–626, 2019.

[BGG75] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand. Differential operators on the base affine space and a study of $\mathfrak{g}$-modules. *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 21–64, 1975.

[BC18] C. Bowman and A. G. Cox. Modular decomposition numbers of cyclotomic Hecke and diagrammatic Cherednik algebras: A path theoretic approach. *Forum Math. Sigma*, 6(e11), 2018.

[BCH] C. Bowman, A. Cox, and A. Hazi. Path isomorphisms between quiver Hecke and diagrammatic Bott–Samelson endomorphism algebras. arXiv:2005.02825. preprint.

[BCHM20] C. Bowman, A. Cox, A. Hazi, and D. Michailidis. Path combinatorics and light leaves for quiver Hecke algebras. *Math. Z.*, 2021.

[BHN20] C. Bowman, A. Hazi, and E. Norton. The modular Weyl-Kac character formula. arXiv:2004.13082. preprint.

[BNS] C. Bowman, E. Norton, and J. Simental. Characteristic-free bases and BGG resolutions of unitary simple modules for quiver Hecke and Cherednik algebras. arXiv:1803.08736 . preprint.

[Bow] C. Bowman. The many graded cellular bases of Hecke algebras. *To appear in Amer. J. Math.*, 2021.

[BK09] J. Brundan and A. Kleshchev. Blocks of cyclotomic Hecke algebras and Khovanov–Lauda algebras. *Invent. Math.*, 178:451–484, 2009.

[BKW11] J. Brundan, A. Kleshchev, and W. Wang. Graded Specht modules. *J. Reine Angew. Math.*, 655:61–87, 2011.

[CL74] R. W. Carter and G. Lusztig. On the modular representations of the general linear and symmetric groups. *Math. Z.*, 136:193–242, 1974.

[dBELS18] M. de Boeck, A. Evseev, S. Lyle, and L. Speyer. On bases of some simple modules of symmetric groups and Hecke algebras. *Transform. Groups*, 23(3):631–669, 2018.

[DJ92] R. Dipper and G. James. Representations of Hecke algebras of type B_n. *Journal of algebra*, 146(2):454–481, 1992.

[DJM98] R. Dipper, G. James, and A. Mathas. Cyclotomic q-Schur algebras. *Math. Z.*, 229(3):385–416, 1998.

[Dot04] S. Doty. New versions of Schur-Weyl duality. *Finite groups 2003:59–72, De Gruyter*, 2008.

[Eas05] M. Eastwood. Higher symmetries of the Laplacian. *Ann. of Math. (2)*, 161(3):1645–1665, 2005.

[FF92] B. Feigin and E. Frenkel. Affine Kac-Moody algebras at the critical level and Gelfand-Dikiui algebras. In *Infinite analysis, Part A, B (Kyoto, 1991)*, volume 16 of *Adv. Ser. Math. Phys.*, pages 197–215. World Sci. Publ., River Edge, NJ, 1992.

[GL96] J. Graham and G. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996.

[Gre80] J. A. Green. *Polynomial representations of* GL_n, volume 830 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1980.

[Hay88] T. Hayashi. Sugawara operators and Kac-Kazhdan conjecture. *Invent. Math.*, 94(1):13–52, 1988.

[HMP18] A. Hazi, P. Martin, and A. Parker. Indecomposable tilting modules for the blob algebra. *J. Algebra*, 568:273–313, 2021.

[HM10] J. Hu and A. Mathas. Graded cellular bases for the cyclotomic Khovanov–Lauda–Rouquier algebras of type A. *Adv. Math.*, 225(2):598–642, 2010.

[Jam77] G. D. James. A characteristic-free approach to the representation theory of $\mathfrak{S}_n$. *J. Algebra*, 46(2):430–450, 1977.

[Jon97] V. F. R. Jones. A polynomial invariant for knots via von Neumann algebras. In *Fields Medallists' Lectures*, pages 448–458. World Scientific, 1997.

[Kau90] L. H. Kauffman. An invariant of regular isotopy. *Transactions of the American Mathematical Society*, 318(2):417–471, 1990.

[KK79] V. G. Kac and D. A. Kazhdan. Structure of representations with highest weight of infinite-dimensional Lie algebras. *Adv. in Math.*, 34(1):97–108, 1979.

[KL09] M. Khovanov and A. Lauda. A diagrammatic approach to categorification of quantum groups. I. *Represent. Theory*, 13:309–347, 2009.

[Ku89] J. Ku. Structure of the Verma module $M(-\rho)$ over Euclidean Lie algebras. *J. Algebra*, 124(2):367–387, 1989.

[KX99a] S. Koenig and C. Xi. When is a cellular algebra quasi-hereditary? *Math. Ann.*, 315(2):281–293, 1999.

[KX99b] S. König and C. Xi. On the number of cells of a cellular algebra. *Communications in Algebra*, 27(11):5463–5470, 1999.

[Lep77] J. Lepowsky. A generalization of the Bernstein-Gelfand-Gelfand resolution. *J. Algebra*, 49(2):496–511, 1977.

[LP] N. Libedinsky and D. Plaza. Blob algebra approach to modular representation theory. arXiv:1801.07200. preprint.

[LPRH] D. Lobos, D. Plaza, and S. Ryom-Hansen. The nil-blob algebra: An incarnation of type $\tilde{A}_1$ soergel calculus and of the truncated blob algebra. *arXiv:2001.00073*.

[MS94] P. Martin and H. Saleur. The blob algebra and the periodic Temperley–Lieb algebra. *Lett. Math. Phys.*, 30(3):189–206, 1994.

[MW00] P. Martin and D. Woodcock. On the structure of the blob algebra. *J. Algebra*, 225(2):957–988, 2000.

[MW03] P. Martin and D. Woodcock. Generalized blob algebras and alcove geometry. *LMS J. Comput. Math.*, 6:249–296, 2003.

[Mat99] A. Mathas. *Iwahori–Hecke Algebras and Schur Algebras of the Symmetric Group*, volume 15 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.

[Mat96] O. Mathieu. On some modular representations of affine Kac-Moody algebras at the critical level. *Compositio Math.*, 102(3):305–312, 1996.

[Mic21] D. Michailidis. Bases and BGG resolutions of simple modules of Temperley-Lieb algebras of type B. *To appear in International Mathematics Research Notices.*

[Pla13] D. Plaza. Graded decomposition numbers for the blob algebra. *J. Algebra*, 394:182–206, 2013.

[PRH14] D. Plaza and S. Ryom-Hansen. Graded cellular bases for Temperley–Lieb algebras of type A and B. *J. Algebraic Combin.*, 40(1):137–177, 2014.

[Rou] R. Rouquier. 2-Kac–Moody algebras. arXiv:0812.5023. preprint.

[Sch27] I. Schur. Über die rationalen Darstellungen der allgemeinen linearen Gruppe. *Sitzber. Königl Preuß. Ak. Wiss., Physikal.-Math. Klasse*, pages 58–75, 1927.

[ST54] G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canad. J. Math.*, 6:274–304, 1954.

[Wak86] M. Wakimoto. Fock representations of the affine Lie algebra $A_1^{(1)}$. *Comm. Math. Phys.*, 104(4):605–609, 1986.

[Wil17] G. Williamson. Schubert calculus and torsion explosion. *J. Amer. Math. Soc.*, 30(4):1023–1046, 2017. With a joint appendix with A. Kontorovich and P. J. McNamara.