The double ratio geometric process for the analysis of recurrent events

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Abstract

Since its introduction, the geometric process (GP) has attracted extensive research attention from authors in various research communities, including probability, statistics, and reliability mathematics. However, the GP can only model a process with its gap times (i.e., times between events/failures) having a monotonic trend (either increasing or decreasing). It also implicitly assumes that the level of the modification on the hazard rate functions and that on the age after the occurrence of an event are the same, which is too restrictive and may limit its application. To overcome these drawbacks, this paper extends the GP to a new stochastic model. Probabilistic properties of the proposed model are investigated. The maximum likelihood method is used to estimate the parameters in the model. Case studies are performed to illustrate the parameter estimation process and obtain favourable performance.

This paper has online supplementary material.

Keywords: Stochastic processes, geometric processes, recurrent events, doubly geometric process, repair.

1 Introduction

The geometric process (GP), introduced by Lam (1988), defines an extension to the renewal process (RP): A sequence of random variables $\{X_k, k = 1, 2, \dots\}$ is a GP if $\{X_k, k = 1, 2, \dots\}$ are independent and the cumulative distribution function (CDF) of X_k is given by $F(a^{k-1}t)$ for $k = 1, 2, \dots$, where a is a positive constant and $F(t)$ is a cumulative distribution function. Probabilistic and statistical properties related to the GP have been investigated by Lam (2007), Aydoğdu and Karabulut (2014), Pekalp and Aydoğdu (2018), Chukova and Minkova (2020) and Biçer et al. (2021). Unlike the RP that can only fits data without a trend, the GP is able to describe a stochastically increasing or decreasing trend: $\{X_k, k = 1, 2, \dots\}$ is stochastically decreasing if $a > 1$ and it is stochastically increasing otherwise. This property has many practical applications.

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Several extensions of the GP have been proposed to overcome its drawbacks (Finkelstein, 1993; Chan et al., 2006; Braun et al., 2005; Wu and Clements-Croome, 2006; Bordes and Mercier, 2013; Wu, 2017; Wu and Wang, 2018) and find their applications, some of which include:

(a) *Scheduling maintenance policy and warranty claim analysis:* There are an enormous number of publications in applying the GP to schedule maintenance policies and optimise warranty policies, in which times between failures are assumed to be shorter and shorter and repair times becomes longer and longer over time (Gao, 2020; Wang et al., 2021; Marshall et al., 2018);

(b) *Modelling the number of infected cases of epidemic disease :* Chan et al. (2006) model the number of daily infected cases due to the severe acute respiratory syndrome (SARS) in Hong Kong and Singapore in 2002–2004, respectively; and

(c) *Modelling electricity price:* Chan et al. (2014) incorporate an extension of the GP and a time series model to propose a conditional autoregressive geometric process model with thresholds and jump and then use their proposed method to model electricity price.

In both applications (b) and (c), the authors used the threshold GP (i.e., model 3 in Table 1), which is an extension of the GP and can model a series of time-between-events data with multiple monotonically changing trends.

The reader is also referred to two recently published review papers by Arnold et al. (2020) and Wu et al. (2020) for more discussions. The existing extensions of the GP have some disadvantages, as discussed in Section 1.2. This paper therefore proposes a novel extension of the GP that will address those disadvantages. The proposed model can be applied to the real-world problems, including those above-mentioned applications of the GP. It should be noted: although this paper is developed from a reliability engineering perspective, its results can also be applied to other disciplines that need recurrent event data analysis.

1.1 Related work

Given a sequence of independent random variables $\{X_k, k = 1, 2, \dots\}$, denote $F_k(t)$ as the probability distribution of X_k. Then, the extensions/variants of the GP can be summarised in Table 1, in which column 2 lists different versions of $F_k(t)$; column 3 checks whether X_k and X_{k-1} are independent or not; column 4 shows whether the corresponding model is able to model a process with a non-monotonic trend or not; and the last column lists the corresponding references. Note: in Model 3, $1 = M_1 < M_2 < \cdots < M_i < M_{i+1} = \infty$ and $M_i \le j < M_{i+1}$. In all of the models, $a, a_k, b \in (0, +\infty)$ (i.e., non-negative numbers); $b_k, \alpha_0 \in (-\infty, +\infty)$ (i.e., real numbers); $\alpha, \beta \in [0, +\infty)$ that satisfy $\alpha^2 + \beta^2 \neq 0$; and $g(\theta, k)$ is a function of k and parameter θ.

The extensions shown in Table 1 can be broken down into the following categories.

Table 1: Extensions of the GP

Model No	$F_k(t)$	Independent?	Non-monotonic trend?	Reference
1	$F(a^{k-1}t)$	Yes	No	Lam (1988)
2	$F(a_k t)$	Yes	No	Finkelstein (1993)
3	$F(a_i^{k-M_i}t)$	Yes	Yes	Chan et al. (2006)
4	$F(k^{\alpha_0}t)$	Yes	No	Braun et al. (2005)
5	$F((\alpha a^{k-1}+\beta b^{k-1})t)$	Yes	Yes	Wu and Clements-Croome (2006)
6	$F(a^{b_k}t)$	Yes	No	Bordes and Mercier (2013)
7	$F(a^{k-1}t^{g(\theta,k)})$	Yes	Yes	Wu (2017)
8	$F(a^{k-1}t)$	No	No	Wu and Wang (2018)

- Relaxation of the infinite expected number of occurrences: The expected number of counts at an arbitrary time of the GP does not exist for the decreasing geometric process. To overcome this drawback, Braun et al. (2005) and Bordes and Mercier (2013) propose models (i.e., models 4 and 6) to extend the GP, respectively.

- Relaxation of the monotonicity assumption: The GP can only model the process with a monotonic trend. To overcome this drawback, Chan et al. (2006), Wu and Clements-Croome (2006), and Wu (2017) propose models (i.e., models 3, 5, and 7) to extend the GP and allow their models to model non-monotonic trends.

- Relaxation of the independence. The GP assumes that X_{k-1} and X_k are dependent. To overcome this drawback, Wu and Wang (2018) extends the GP by assuming that X_{k-1} and X_k are dependent (see model 8).

Model 2 aims to extend the GP by replacing a^{k-1} with a_k (Finkelstein, 1993), which cannot be classified into any of the above three categories.

Other properties of the GP variants are also succinctly summarised in Wu (2017).

1.2 Motivation

The existing variants shown in Table 1 have extended the GP, which widens their applications in practice. However, there is still room for improvement, as discussed in the following.

- The extensions for depicting the bath-tub curve include the work by Chan et al. (2006) and that by Wu and Clements-Croome (2006). However, in addition to the parameters in $F(t)$, both include more than three parameters in the process. The distributions $F(\alpha a^{k-1}+\beta b^{k-1})t)$ in the extension by Wu and Clements-Croome (2006) have four parameters (i.e., a, b, α, and β) and the distribution $F(a_i^{j-M_i}t)$ in the extension by Chan et al. (2006) have more parameters (i.e., a_i and the changing points). When sample size for model-fitting is small, a model with a large number of parameters estimated on the dataset will have large standard errors of the estimated parameters.

- The doubly geometric process (DGP) assumes that the gap times X_k follow distributions $F(a^{k-1}t^{g(k)})$, where $g(k)$ is a function of k (Wu, 2017). The DGP differs from the model proposed by Wu and Clements-Croome (2006) and the one by Chan et al. (2006) in that the shape parameters in the DGP changes over k and that the DGP has a smaller number of parameters, i.e., the DGP is more parsimonious than those two models.

There is a lack of the GP-like process that satisfies the following three requirements: (1) it can model a process with non-monotonic trends; (2) it has fewer parameters than those in models proposed in Wu and Clements-Croome (2006) and Chan et al. (2006); and (3) the shape parameter in the distributions of X_k keep fixed over k's if the lifetime distribution of the time to the first failure (i.e., X_1) is a Weibull distribution.

To meet the above three requirements, this paper proposes a new stochastic process, investigates its probabilistic properties and likelihood function properties, which presents novelty and makes contributions to the literature.

The models developed in this paper may be used in asset management, in which the number of failures needs estimation. Models with better accuracy can help asset managers to make more precise budget plans. They may also be used in other scenarios such as those applications (b) and (c) illustrated in the first paragraph under Section 1.

1.3 Overview

The remainder of this paper is structured as follows. Section 2 introduces a new stochastic process, i.e., the *double-ratio geometric process* and discusses its probabilistic properties. Section 3 introduces the maximum likelihood estimation methods for the cases when the failure data are collected from one system and multiple systems, respectively. Section 4 investigates the methods of hypothesis testing and model checking. Section 5 estimates parameters for the DRGP based on four real datasets. Section 6 concludes the paper and proposes future research.

The online supplementary material recalls the geometric process, the α-series process, and presents the real datasets used in this paper.

2 Double-ratio geometric process

Some existing definitions relating to stochastic ordering and the geometric process (GP) can be found in the online supplementary material. This section introduces a new extension of the GP process: a double-ratio geometric process (DRGP).

2.1 Double-ratio geometric process

The relationship of the hazard rate functions between the kth cycle and the $(k+1)$th cycle of the GP in Definition 3 (see the online supplementary material) is given by

$$h_k(t) = ah_{k-1}(at). \tag{1}$$

If we assume that the changes of the hazard rate functions between two consecutive gap-times are due to the effectiveness of imperfect maintenance, we can have the following interpretations:

- The first parameter a, which is at the outside of $h_{k-1}(\cdot)$ of the right hand side in equation (1), has effectiveness of decreasing (if $a < 1$) or increasing (if $a > 1$) the failure rate function after the kth repair, or we can regard this a as playing the role of modifying the hazard rate and refer to it as the hazard rate modification parameter.

- The second parameter a, which is in the parentheses of the function $h_{k-1}(\cdot)$, has an effect of modifying the age (as it multiples the time t): reducing the age for $a < 1$ or increasing the age for $a > 1$ after the kth repair, or we can regard this a as playing the role of modifying the age and refer to it as the age modification parameter.

From the above analysis, it can be seen that the underlying assumption of the GP is that the hazard rate modification parameter and the age modification parameter are the same. A similar analysis can be carried out for the α-series process. These parameter assumptions are too restrictive and may therefore limit their applications in practice. An intuitive extension of Eq. (1) is to assume that these effects are different, which leads to the following definition.

Definition 1 (Double Ratio Geometric Process) *Given a sequence of non-negative random variables* $\{Z_k^D, k = 1, 2, \dots\}$*, if they are independent and the cdf of* Z_k^D *is given by* $F_k^D(t) = 1 - \exp\{-\int_0^t b_k h(a_k u)du\}$ *for* $k = 1, 2, \dots,$ *where* a_k *and* b_k *are positive parameters (or ratios) and* $a_1 = b_1 = 1$*. We call the stochastic process the double-ratio geometric process (DRGP).*

The reason we refer to the stochastic process defined in Definition 1 as the double-ratio geometric process is that its definition is inspired by the GP and a_k and b_k may be set to α_1^{k-1} or k^{β_1}. In what follows, we set a_k and b_k to be α_1^{k-1} or k^{β_1} where needed.

It is also straightforward to see from Proposition 1 that the DRGP reduces to the GP if $a_k = b_k = a^{k-1}$ and to the α-series process if $a_k = b_k = k^{\alpha}$.

As can be seen, the first six models listed in Table 1 are special cases of the DRGP.

2.2 Probabilistic properties

In this subsection, we will investigate some probabilistic properties of the DRGP.

Proposition 1 (cdf of Z_k^D) *(i).* $F_k^D(t) = 1 - (1 - F_1^D(a_k t))^{\frac{b_k}{a_k}}$,

(ii). Assume Z_1^D follows the exponential distribution with hazard function $h(u) = \lambda$, then $F_k^D(t) = 1 - \exp\{-b_k \lambda t\}$. That is, a_k does not play a role in DRGP. Below are two special cases.

- *If $b_k = b^{k-1}$, regardless of the form of a_k, then $\{Z_k^D, k = 1, 2, \ldots\}$ is a GP with the cdf of X_k being $F_k^D(t) = 1 - \exp(-b^{k-1}\lambda t)$, and*
- *If $b_k = k^\alpha$, regardless of the form of a_k, then $\{Z_k^D, k = 1, 2, \ldots\}$ is an α-series process with the cdf of X_k being $F_k^D(t) = 1 - \exp(-k^\alpha \lambda t)$.*

The proofs of the above and the other propositions can be found in Appendices.

From (i) of Definition 1, $F_k^D(t)$ can be regarded as the cdf of the first order statistic of the $\frac{b_k}{a_k}$ random variables with their cdf's as $F_1^D(a_k t)$. Alternatively, it can also be regarded as the reliability of a series system composed of $\frac{b_k}{a_k}$ components, each of which has reliability $1 - F_1^D(a_k t)$ (although $\frac{b_k}{a_k}$ may not be an integer).

It is also of interest to see from Proposition 1 that the DRGP is exactly the same as the GP for $b_k = b^{k-1}$ and the α-series process for $b_k = k^b$, given that the time to the first event follows the exponential distribution.

From (i) of Proposition 1, one can obtain the following lemma that provides an equivalent definition of the DRGP.

Lemma 1 *Given a sequence of non-negative random variables $\{Z_k^D, k = 1, 2, \ldots\}$, if they are independent and the cdf of $a_k^{-1} Z_k^D$ is given by $F_k^D(t) = 1 - \left(1 - F_1^D(a_k t)\right)^{\frac{b_k}{a_k}}$ for $k = 1, 2, \ldots$, where a_k and b_k are positive parameters (or ratios), and $a_1 = b_1 = 1$. Then $\{Z_k^D, k = 1, 2, \ldots\}$ is a double-ratio geometric process.*

From From (i) of Proposition 1, we have

$$f_k^D(t) = b_k f_1^D(a_k t)(1 - F_1^D(a_k t))^{\frac{b_k}{a_k} - 1}. \tag{2}$$

From Eq. (2), we can calculate the expected value and the variance of Z_k^D. We can also use the following proposition to find the lower and upper bounds of the expected value of Z_k^D based on the distribution of Z_1^D.

Proposition 2 (The expected value of Z_k^D) *Suppose that both the expected value, $\mathbb{E}[Z_1^D](= \mu_1)$, and variance, $\mathbb{V}[Z_1^D](= \sigma_1^2)$, of Z_1^D exist. Then*

- *if $0 < \frac{b_k}{a_k} \le 1$, $b_k a_k^{-2} \mu_1^{\frac{b_k}{a_k} - 1} \mathbb{E}[(Z_1^D)^{2 - \frac{b_k}{a_k}}] \le \mathbb{E}[Z_k^D] \le b_k a_k^{-2} \mathbb{E}\left[Z_1^D \left(1 - \frac{\sigma_1^2}{\sigma_1^2 + \mu_1^2 - 2a_k \mu_1 Z_1^D + a_k^2 (Z_1^D)^2}\right)^{\frac{b_k}{a_k} - 1}\right]$, assuming the three expected values in the inequalities exist;*

- *if $\frac{b_k}{a_k} > 1$, the above inequalities change their directions.*

Apparently, if we simply use the rule $1-F_1^D(a_kt)\le 1$, then $\mathbb{E}[Z_k^D]\ge b_ka_k^{-2}\mu_1$ for the case $0<\frac{b_k}{a_k}\le 1$ and $\mathbb{E}[Z_k^D]<b_ka_k^{-2}\mu_1$ for the case $\frac{b_k}{a_k}>1$. On the other hand, according to Theorem 3 in Ghosh (2002), we may apply the sharpest inequalities for $\mathrm{P}(Z_1^D\ge a_kt)$ as follows: (a) If $0<a_kt<\mu_1$, then $\frac{(\mu_1-a_kt)^2}{\sigma_1^2+(\mu_1-a_kt)^2}<\mathrm{P}(Z_1^D\ge a_kt)\le 1$; (b) If $\mu_1\le a_kt\le\mu_1+\mu_1^{-1}\sigma_1^2$, then $0\le\mathrm{P}(Z_1^D\ge a_kt)\le\frac{\sigma_1^2}{\sigma_1^2+(a_kt-\mu_1)^2}$; and (c) If $a_kt>\mu_1+\mu_1^{-1}\sigma_1^2$, then $0<\mathrm{P}(Z_1^D\ge a_kt)\le\frac{\mu_1}{a_kt}$. It is easy to apply these inequalities to $b_kf_1^D(a_kt)(1-F_1^D(a_kt))^{\frac{b_k}{a_k}-1}=b_kf_1^D(a_kt)\left(\mathrm{P}(Z_1^D\ge a_kt)\right)^{\frac{b_k}{a_k}-1}$, we can then obtain the sharpest inequalities for $\mathbb{E}[Z_k^D]$:

$$\begin{aligned}&b_k\int_0^{\frac{\mu_1}{a_k}}tf_1^D(a_kt)dt+b_k\int_{\frac{\mu_1}{a_k}}^{\frac{\mu_1+\mu_1^{-1}\sigma_1^2}{a_k}}tf_1^D(a_kt)\left(\frac{\sigma_1^2}{\sigma_1^2+(a_kt-\mu_1)^2}\right)^{\frac{b_k}{a_k}-1}dt\\&+b_k\int_{\frac{\mu_1+\mu_1^{-1}\sigma_1^2}{a_k}}^{\infty}f_1^D(a_kt)\mu_1^{\frac{b_k}{a_k}-1}a_k^{1-\frac{b_k}{a_k}}t^{2-\frac{b_k}{a_k}}dt\le\mathbb{E}[Z_k^D]\le\int_0^{\frac{\mu_1}{a_k}}tb_kf_1^D(a_kt)\frac{(\mu_1-a_kt)^2}{\sigma_1^2+(\mu_1-a_kt)^2}dt.\end{aligned}\tag{3}$$

Unlike those properties shown in Remark 1 (see the online supplementary material), the stochastic monotonicity of the DRGP behaves differently from the GP, as discussed in the proposition below.

Proposition 3 (Monotonicity) *Suppose $h(t)$ is a monotonously increasing function in t, $\{Z_k^D, k=1,2,\dots\}$ is a DRGP, then*

(i). If both a_k and b_k are increasing in k, then the DRGP is stochastically decreasing;

(ii). If both a_k and b_k are decreasing in k, then the DRGP is stochastically increasing; and

(iii). If a_k (or b_k) is increasing in k and b_k (or a_k) is decreasing in k, then the DRGP may not be stochastically monotonic.

In what follows, we investigate four special scenarios, as shown in Table 2, in which $\mathbb{N}^+$ denotes the set of all positive integers.

Table 2: Four models

a_k	b_k	Constraint	Model name
β_1^{k-1}	β_2^{k-1}	$\beta_1,\beta_2\in(0,+\infty)$, $k\in\mathbb{N}^+$	DRGP-I
k^{α_1}	k^{α_2}	$\alpha_1,\alpha_2\in(-\infty,+\infty)$, $k\in\mathbb{N}^+$	DRGP-II
k^{α_1}	β_1^{k-1}	$\beta_1\in(0,+\infty)$, $\alpha_1\in(-\infty,+\infty)$, $k\in\mathbb{N}^+$	DRGP-III
β_1^{k-1}	k^{α_1}	$\beta_1\in(0,+\infty)$, $\alpha_1\in(-\infty,+\infty)$, $k\in\mathbb{N}^+$	DRGP-IV

Let $S_n^D=\sum_{k=1}^n Z_k^D$, $S_n^G=\sum_{k=1}^n X_k^G$, $N^D(t)=\sup\{n\colon S_n^D\le t\}$ and $N^G(t)=\sup\{n\colon S_n^G\le t\}$, where $t>0$.

Proposition 4 (Comparison) *Suppose $h(t)$ is a monotonously increasing function in t, $X_k^G \sim F_k^G(t) = 1 - \exp\{-\beta_1^{k-1}\int_0^t h(\beta_1^{k-1}u)du\}$ in the GP, $Y_k^A \sim F_k^A(t) = 1 - \exp\{-k^{\alpha_1}\int_0^t h(k^{\alpha_1}u)du\}$ in the α-series process, and $Z_k^D \sim F_k^D(t) = 1 - \exp\{-\int_0^t b_k h(a_k u)du\}$ in the DRGP.*

(i). Let $a_k = \beta_1^{k-1}$ and $b_k = \beta_2^{k-1}$, or $a_k = \beta_2^{k-1}$ and $b_k = \beta_1^{k-1}$,

- *if $\beta_2 \geq \beta_1$, then: (1) $Z_k^D \leq_{st} X_k^G$; (2) $\mathbb{E}[N^D(t)] \geq \mathbb{E}[N^G(t)]$; and (3) if $\beta_1 > 1$ and $F_1^D(\epsilon) > 0$ for all $\epsilon > 0$, $\mathbb{E}[N^D(t)] = \infty$;*
- *if $\beta_2 < \beta_1$, then: (1) $Z_k^D >_{st} X_k^G$; (2) $\mathbb{E}[N^D(t)] < \mathbb{E}[N^G(t)]$; and (3) if $0 < \beta_1 \leq 1$ and $F_1^D(0) < 1$, $\mathbb{E}[N^D(t)] < \infty$.*

(ii). Let $a_k = k^{\alpha_1}$ and $b_k = k^{\alpha_2}$, or $a_k = k^{\alpha_2}$ and $b_k = k^{\alpha_1}$,

- *If $\alpha_2 \geq \alpha_1$, then: (1) $Z_k^D \leq_{st} Y_k^A$; (2) $\mathbb{E}[N^D(t)] \geq \mathbb{E}[N^A(t)]$ (3) if $\alpha_1 \geq 1$, then $\mathbb{E}[N^D(t)] = \infty$;*
- *If $\alpha_2 < \alpha_1$, then: (1) $Z_k^D >_{st} Y_k^A$; (2) $\mathbb{E}[N^D(t)] < \mathbb{E}[N^A(t)]$ (3) if $\alpha_1 < 0$, then $\mathbb{E}[N^D(t)] < \infty$.*

(iii). Let $a_k = \beta_1^{k-1}$ and $b_k = k^{\alpha_1}$, or $a_k = k^{\alpha_1}$ and $b_k = \beta_1^{k-1}$,

- *If $\alpha_1/\ln(\beta_1) \geq (k-1)/\ln(k)$, then: (1) $Z_k^D \leq_{st} X_k^G$ and $Z_k^D \leq_{st} X_k^A$; (2) $\mathbb{E}[N^D(t)] \geq \mathbb{E}[N^G(t)]$ and $\mathbb{E}[N^D(t)] \geq \mathbb{E}[N^A(t)]$; (3) if $\beta_1 > 1$ and $F_1^D(\epsilon) > 0$ for all $\epsilon > 0$, or $\alpha_1 > 1$, then $\mathbb{E}[N^D(t)] = \infty$;*
- *If $\alpha_1/\ln(\beta_1) < (k-1)/\ln(k)$, then: (1) $Z_k^D >_{st} X_k^G$ and $Z_k^D >_{st} X_k^A$; (2) $\mathbb{E}[N^D(t)] < \mathbb{E}[N^G(t)]$ and $\mathbb{E}[N^D(t)] < \mathbb{E}[N^A(t)]$; (3) if $0 < \beta_1 \leq 1$ and $F_1^D(0) < 1$, or $\alpha_1 < 0$, then $\mathbb{E}[N^D(t)] < \infty$.*

Due to their complexity, other cases that have not been covered in Proposition 4 will be investigated in our future research.

In the special case that Z_1^D follows a Weibull distribution, we have the following discussion.

Suppose Z_1^D follows the Weibull distribution, $F_1(t) = 1 - \exp\{-\theta_1\theta_2^{-\theta_1}\int_0^t u^{\theta_1-1}du\}$, for example, then the cumulative distribution function of Z_k^D is given by

$$\begin{aligned} F_k^D(t) &= 1 - \exp\{-a_k^{\theta_1-1}b_k \int_0^t \theta_1\theta_2^{-\theta_1}u^{\theta_1-1}du\} \\ &= 1 - \exp\{-(\theta_2 a_k^{\frac{1-\theta_1}{\theta_1}} b_k^{-\frac{1}{\theta_1}})^{-\theta_1}\int_0^t \theta_1 u^{\theta_1-1}du\}. \end{aligned} \tag{4}$$

Then the shape parameter in the distribution of Z_k^D is θ_1, which is the same as that of Z_1^D. The scale parameter of Z_k^D changes from θ_2 to $\theta_2 a_k^{(1-\theta_1)/\theta_1} b_k^{-1/\theta_1}$. Hence, for this special case, the monotonicity of $\{Z_k^D, k = 1, 2, \dots\}$ can be analysed in the following four cases (a)–(d).

(a) If $a_k = \beta_1^{k-1}$ and $b_k = \beta_2^{k-1}$. Then the scale parameter of Z_k^D is $\theta_2(\beta_1^{(1-\theta_1)/\theta_1}\beta_2^{-1/\theta_1})^{k-1}$, which is positively proportional to the expectation of Z_k^D. Z_k^D increases (or decreases) stochastically in k if $(1-\theta_1)\ln\beta_1 > \ln\beta_2$ (or $(1-\theta_1)\ln\beta_1 < \ln\beta_2$), according to Proposition 3.

(b) If $a_k = k^{\alpha_1}$ and $b_k = k^{\alpha_2}$. Then the scale parameter of Z_k^D is $\theta_2 k^{((1-\theta_1)\alpha_1-\alpha_2)/\theta_1}$. Similarly, Z_k^D increases (or decreases) stochastically in k if $(1-\theta_1)\alpha_1 > \alpha_2$ (or $(1-\theta_1)\alpha_1 < \alpha_2$).

(c) If $a_k = k^{\alpha_1}$ and $b_k = \beta_1^{k-1}$. Then the scale parameter of Z_k^D is $\theta_2 k^{\alpha_1(1-\theta_1)/\theta_1}\beta_1^{(1-k)/\theta_1}$. Then Z_k^D increases stochastically in k if $\alpha_1(\theta_1-1)\ln\frac{k}{k+1} > \ln\beta_1$ and decreases stochastically otherwise.

(d) If $a_k = \beta_1^{k-1}$ and $b_k = k^{\alpha_1}$. Then the scale parameter of Z_k^D is $\theta_2\beta_1^{(1-\theta_1)(k-1)/\theta_1}k^{-\alpha_1/\theta_1}$. Then Z_k^D increases stochastically in k if $(1-\theta_1)\ln\beta_1 > \alpha_1\ln\frac{k+1}{k}$ (or $(1-\theta_1)\ln\beta_1 < \alpha_1\ln\frac{k+1}{k}$) and decreases stochastically otherwise.

The processes $\{Z_k^D, k = 1, 2, \dots\}$ in cases (a) and (b) are monotonic, while those in cases (c) and (d) are not. For cases (c) and (d), see Figures 1 and 2 for the scenario $a_k = k^\alpha$ and $b_k = \beta^{k-1}$ (i.e., the above case (c)), and Figures 3 and 4 for the scenario $a_k = \beta^{k-1}$ and $b_k = k^\alpha$ (i.e., the above case (d)), respectively.

Example 1 *Suppose $h(t) = \theta_1\theta_2^{-\theta_1}t^{\theta_1-1}$. The dots in Figures 1 and 2 represent the mean of the random variables following $F_k(t)$ for model DRGP-III with different parameters α_1 and β_1, respectively. The dots in Figures 3 and 4 represent the means of the random variables following $F_k(t)$ for model DRGP-IV with different parameters α_1 and β_1, respectively. The X-axis and the Y-axis in each figure represents k and the means of the random variables following $F_k(t)$, respectively.*

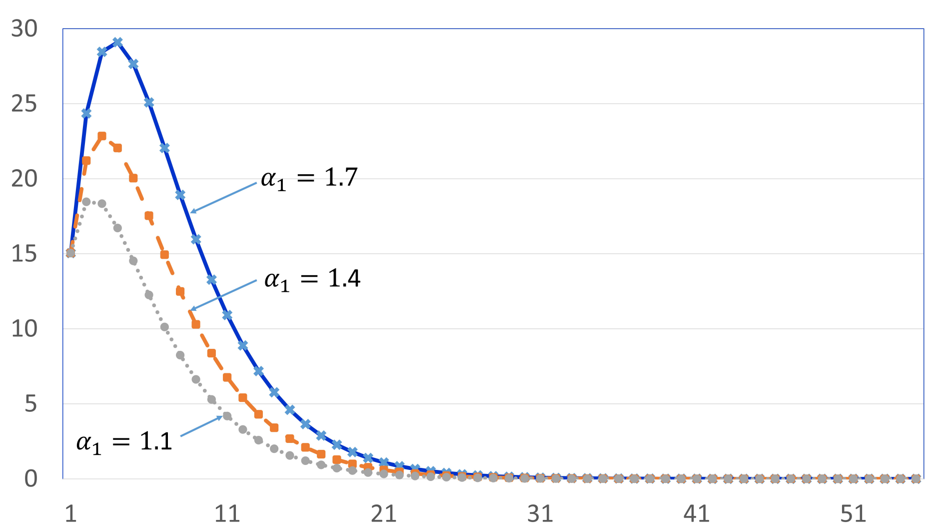

Figure 1: DRGP-III: $\theta_1 = 0.6, \theta_2 = 10, \beta_1 = 1.2$.

Figures 1, 2, 3, and 4 portray non-monotonic trends in the expected values of DRGP against k.

It is noted that, for a given set of parameters α_1 and β_1, the trend of a stochastic process that DRGP III and DRGP IV can describe can be increasing (e.g., with condition (ii) in Proposition 3), decreasing (e.g., with condition (i) in Proposition 3), or both (e.g., Figures 1–4), or constant (only if $a_k = b_k = 1$). Nevertheless, DRGP III and DRGP IV are unable to model the failure process with a constant trend

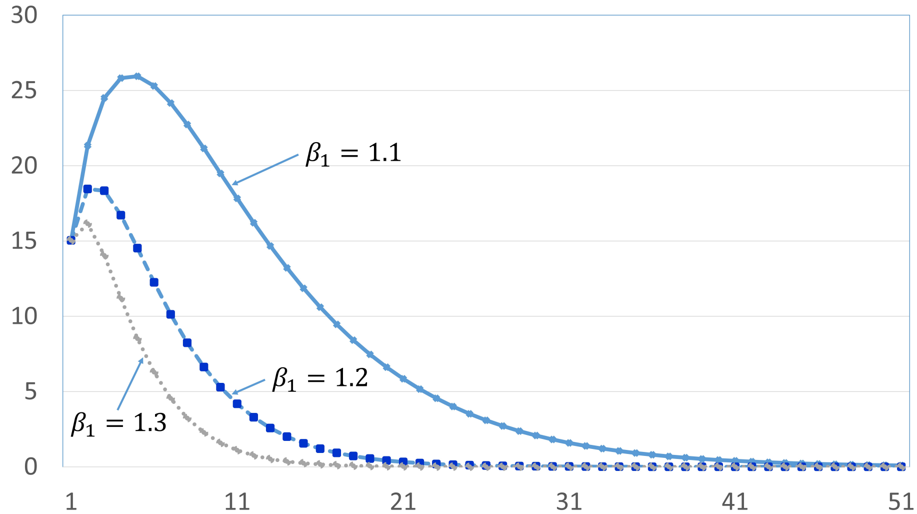

Figure 2: DRGP-III: $\theta_1 = 0.6, \theta_2 = 10, \alpha_1 = 1.1$.

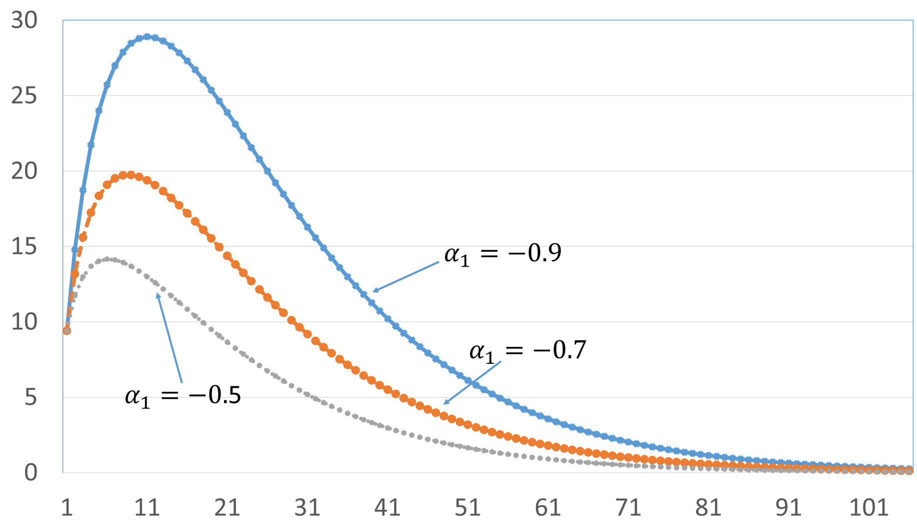

Figure 3: DRGP-IV: $\theta_1 = 1.2, \theta_2 = 10, \beta_1 = 1.5$.

followed by an increasing/decreasing trend since the condition $a_k = b_k = 1$ and $a_k \neq 1$ (or $b_k \neq 1$) are mutually exclusive events.

3 Parameter estimation

This section presents the likelihood functions for two scenarios: the failure data are collected from a single system scenario and from a multiple system scenario, respectively.

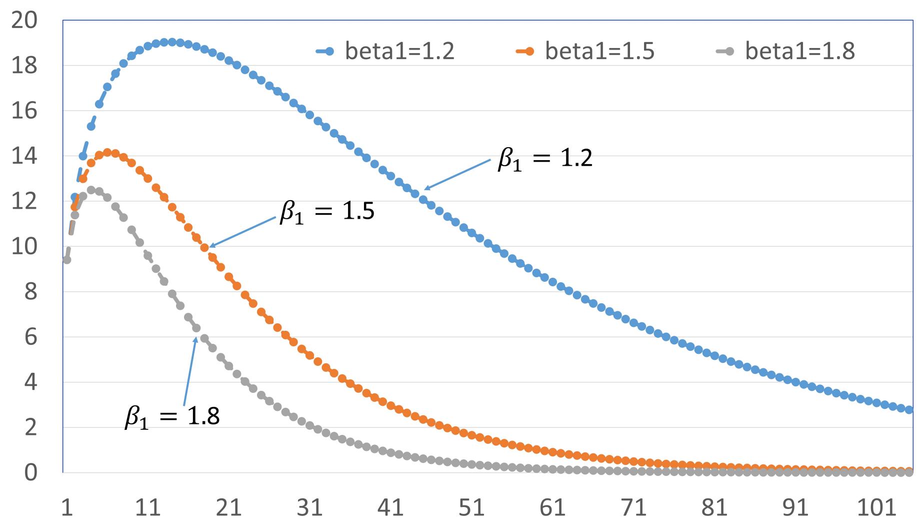

Figure 4: DRGP-IV: $\theta_1 = 1.2, \theta_2 = 10, \alpha_1 = -0.5$.

3.1 Likelihood for a single system

Suppose n successive failures have been observed on a system. The gap times are $\{t_1, t_2, \ldots, t_n\}$, that is, the failures occurred at times $t_1, t_1 + t_2, \ldots, \sum_{k=1}^{n} t_k$. The likelihood of the DRGP is given by

$$L(\boldsymbol{\theta}; \boldsymbol{t}, n) = \prod_{k=1}^{n} \left(b_k f_1^D(a_k t_k)(1 - F_1^D(a_k t_k))^{\frac{b_k}{a_k} - 1} \right), \tag{5}$$

where $\boldsymbol{\theta}$ is the vector of parameters containing a_k and b_k, and the parameters in $F_1^D(t)$; $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$. Then the log likelihood of the DRGP is given by

$$\ln L(\boldsymbol{\theta}; \boldsymbol{t}, n) = \sum_{k=1}^{n} \left(\ln(b_k) + \ln(f_1^D(a_k t_k)) + \left(\frac{b_k}{a_k} - 1 \right) \ln(1 - F_1^D(a_k t_k)) \right). \tag{6}$$

Let $l(\boldsymbol{\theta}; \boldsymbol{t}, n) = \ln L(\boldsymbol{\theta}; \boldsymbol{t}, n)$. Then the Fisher's score function can be obtained by $\boldsymbol{u}(\boldsymbol{\theta}) = \frac{\partial l(\boldsymbol{\theta}; \boldsymbol{t}, n)}{\partial \boldsymbol{\theta}}$. By setting $\boldsymbol{u}(\boldsymbol{\theta}) = 0$, we can obtain the maximum likelihood estimate of $\boldsymbol{\theta}$: $\widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} l(\boldsymbol{\theta}; \boldsymbol{t}, n)$. Define the observed information matrix $I(\boldsymbol{\theta})$ of the log-likelihood function as $I(\boldsymbol{\theta}) = E\left[-\frac{\partial^2 l(\boldsymbol{\theta}; \boldsymbol{t}, N)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} \right]$. Then, using the Newton-Raphson method, we have $\widehat{\boldsymbol{\theta}} = \boldsymbol{\theta}_0 + \boldsymbol{I}^{-1}(\boldsymbol{\theta}_0) \boldsymbol{u}(\boldsymbol{\theta}_0)$.

If the total number n of observations is large and appropriate conditions on the model hold, $\widehat{\boldsymbol{\theta}}$ can be treated as approximately normally distributed, based on which the confidence limits for parameters can be obtained.

3.2 A special case: Z_1^D following the Weibull distribution

In this section, we investigate the likelihood function on a special case.

Proposition 5 (Equivalence) *Let Z_1^D follow the Weibull distribution $F_1^D(t) = 1 - \exp\left\{-\left(\frac{a_k t}{\theta_2}\right)^{\theta_1}\right\}$, where $a_k = k^{\alpha_1}$ and $b_k = \beta_1^{k-1}$ for model DRGP-III, and $a_k = \beta_2^{k-1}$ and $b_k = k^{\alpha_2}$ for model DRGP-IV. Denote the maximum log-likelihood estimates from Eq (22) for model DRGP-III and model DRGP-IV by $\widehat{l}_1$ and $\widehat{l}_2$, respectively. Then the two models DRGP-III and DRGP-IV are equivalent with respect of modelling a given dataset based on the maximum likelihood estimation from the following perspectives.*

(i). $\widehat{l}_1 = \widehat{l}_2$;

(ii). $\widehat{\theta}_1$ and $\widehat{\theta}_2$ from model DRGP-III equal $\widehat{\theta}_1$ and $\widehat{\theta}_2$ from model DRGP-IV, respectively; and

(iii). $\widehat{\beta}_1 = \widehat{\beta}_2^{\widehat{\theta}_1 - 1}$ and $\widehat{\alpha}_1 = \frac{\widehat{\alpha}_2}{\widehat{\theta}_1 - 1}$.

According to Proposition 5, due to the equivalence between DRGP-III and DRGP-IV, we investigate the maximum likelihood estimates for model DRGP-III in the following.

Let $a_k = k^{\alpha_1}$ and $b_k = \beta_1^{k-1}$, then

$$\ln L = n \ln \theta_1 - n\theta_1 \ln \theta_2 + \frac{n(n-1)\ln \beta_1}{2} + \sum_{k=1}^{n} \left((\theta_1 - 1)(\alpha_1 \ln k + \ln t_k) - k^{\alpha_1(\theta_1-1)} \beta_1^{k-1} \left(\frac{t_k}{\theta_2}\right)^{\theta_1} \right). \quad (7)$$

Let $\frac{\partial \ln L}{\partial \alpha_1} = 0$, $\frac{\partial \ln L}{\partial \beta_1} = 0$, $\frac{\partial \ln L}{\partial \theta_1} = 0$ and $\frac{\partial \ln L}{\partial \theta_2} = 0$, after some simplifications, we obtain the following equations. That is, the estimated parameters should satisfy the conditions listed in Eqs. (8), (9), (10), and (11).

$$\sum_{k=1}^{n} (k-1) k^{\alpha_1(\theta_1-1)} \beta_1^{k-1} \left(\frac{t_k}{\theta_2}\right)^{\theta_1} = \frac{n(n-1)}{2}, \quad (8)$$

$$\sum_{k=1}^{n} (\ln k) k^{\alpha_1(\theta_1-1)} \beta_1^{k-1} \left(\frac{t_k}{\theta_2}\right)^{\theta_1} = \sum_{k=1}^{n} \ln k, \quad (9)$$

$$\sum_{k=1}^{n} (\ln t_k) k^{\alpha_1(\theta_1-1)} \beta_1^{k-1} \left(\frac{t_k}{\theta_2}\right)^{\theta_1} = \frac{n}{\theta_1} + \sum_{k=1}^{n} \ln t_k, \quad (10)$$

and

$$\sum_{k=1}^{n} k^{\alpha_1(\theta_1-1)} \beta_1^{k-1} \left(\frac{t_k}{\theta_2}\right)^{\theta_1} = n. \quad (11)$$

From Eq. (11), we have $\theta_2 = \left(\frac{1}{n} \sum_{k=1}^{n} k^{\alpha_1(\theta_1-1)} \beta_1^{k-1} t_k^{\theta_1} \right)^{1/\theta_1}$.

3.3 Likelihood for several systems with random effect

In this subsection, we assume m systems were working in different operating conditions and/or in different working load. For simplicity, we restrict our attention to the case with no observed covariates. Suppose

n_j successive failures have been observed on the jth system, which has gap times $\{t_{j,1}, t_{j,2}, \ldots, t_{j,n_j}\}$. We assume the hazard rate function of Z_k^D is given by

$$h_k(t) = \gamma_j b_k h_1(a_k t), \tag{12}$$

where γ_j are unobserved and represent heterogeneity between systems. We further assume γ_j are independently distributed according to a probability distribution $G(.)$ and the expected value of γ_j equals 1. Then the likelihood of the jth system is given by

$$L_j = \prod_{k=1}^{n} \left(\gamma_j b_k f_1^D(a_k t_k)(1 - F_1^D(a_k t_k))^{\frac{\gamma_j b_k}{a_k} - 1} \right), \tag{13}$$

and the likelihood of the m systems is given by

$$L = \prod_{j=1}^{m} \int \left(\prod_{k=1}^{n} \gamma_j b_k f_1^D(a_k t_k)(1 - F_1^D(a_k t_k))^{\frac{\gamma_j b_k}{a_k} - 1} \right) dG(\gamma_j). \tag{14}$$

Similar to Proposition 5, we can analogously prove that models DRGP III and DRGP IV are equivalent with respect to the likelihood function shown in Eq. (14) when the probability distribution of time to first failure is the Weibull distribution. $G(.)$ is usually assumed to be the gamma distribution.

4 Hypothesis testing and model checking

Given a set of gap-time observations $\{X_1, X_2, \ldots\}$. Denote $U_k = X_{2k}^D / X_{2k-1}^D$ and $U_k' = X_{2k+1}^D / X_{2k}^D$. The process for selecting the GP-like models is illustrated in Figure 5, where the models in the box "use other models such as NHPP" can be other imperfect repair processes, including virtual age models or superposition of imperfect repair processes. More elaboration is given below.

Step 1. Checking the independence: This step aims to check where $\{X_1, X_2, \ldots,\}$ are an independent series.

- If the test shows that they are independently and identically distributed, the renewal process may be fitted to the series of observations;
- if the test shows that they are independently but not identically distributed, go to Step 2;
- if the test shows that they are neither independently nor identically distributed, other models such as the virtual age models or superposition of imperfect repair models (Wu, 2019) can be used.

Step 2. Checking U_k and U_k': According to Theorem 4.2.1 in Lam (2007), if $\{X_1, X_2, \ldots\}$ is a GP, then $\{U_k, k = 1, 2, \ldots\}$ and $\{U_k', k = 1, 2, \ldots\}$ are two sequences of i.i.d. random variables, respectively.

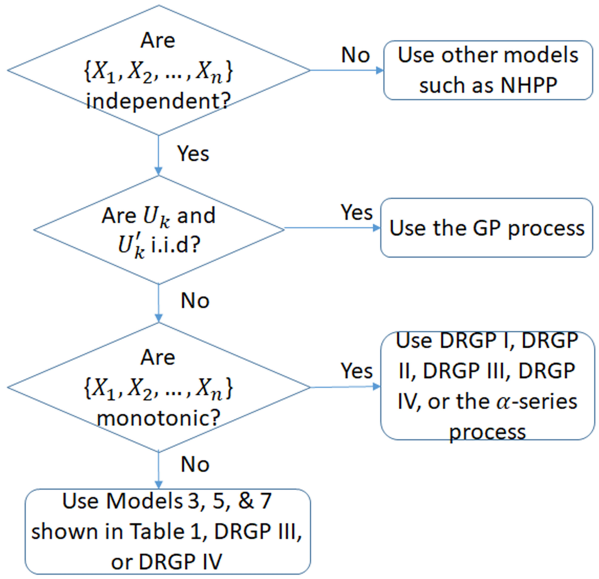

Figure 5: Hypothesis testing

Step 3. Checking the non-monotonicity: Since DRGP III and DRGP IV are able to model the non-monotonic trend in the data whereas models such as DRGP I, DRGP II and the α-series process are not, one can check whether $\{X_1, X_2, \dots,\}$ show a monotonic trend or not. If not, DRGP I, DRGP II and the α-series process may be used; otherwise, DRGP III or DRGP IV can be used. Apparently, models 3, 5, and 7 listed in Table 1 can also be used for such a set of observations with a non-monotonic trend. References on testing for non-monotonicity include Viertävä and Vaurio (2009).

As long as $\{X_1, X_2, \dots\}$ are tested independent, the DRGP models can be applied to fit the dataset as the GP is a special case of the DRGP.

5 Numerical example

Chapter 5 in Lam (2007) compares the performance of the GP with three other models on a number of real world datasets and shows the superiority of the GP. In the following, we show four cases with the best model performance in terms of the AICc (corrected Akaike Information Criterion) for the DRGP III and DRGP IV, respectively. It should be noted that the AICc penalises more on the number of parameters in a model than AIC and it is therefore recommended for measuring model performance on small datasets. Let $\hat{L}$ be the maximum value of the likelihood function for a model and q be the number of parameters in the model, the definition of the AICc is given by

$$\text{AIC}_c = -2\ln(\hat{L}) + 2q + \frac{2q^2 + 2q}{n - q - 1}. \tag{15}$$

We estimate the DRGP models on four real datasets and compare their performance with that of three other models that can model non-monotonic trends: the threshold geometric process (TGP), the extended geometric process (EGP) and the doubly geometric process (DGP), which are models 3, 5 and 7 in Table 1, respectively. The four datasets are listed in Table 3, in which column 1 includes the dataset number, column 2 includes the name of a dataset, column 3 shows the sample size, column 4 is a one sentence description of the dataset, and the last column is the source of the dataset. Dataset 4 is a time-between-failure dataset including both failure times and preventive maintenance times. For convenience, the data are also reproduced in Table 1 in Online Appendix.

We first use the Ljung-Box test to test the autocorrelations in the data in each dataset. The null hypothesis of the Ljung-Box test assumes that there is no serial correlation in the series under testing. From the test, we conclude that there is no serial correlation in each dataset as the p-value of the Ljung-Box test is greater than 0.05 (which is the significance level we use in this paper), as shown in column 2 in Table 4. We then test the non-monotonicity of each dataset by applying the testing methods, V_1 and V_4, proposed by Viertävä and Vaurio (2009), where the null hypothesis assumes that the data under testing is monotonic and both V_1 and V_4 therefore follows the standard normal distribution. The tests show that the p-values of V_1 and V_4 on datasets 1 and 2 are smaller than 0.025, the p-value of V_1 on dataset 3 is smaller than 0.025, and the p-value of V_4 on dataset 3 is slightly larger than 0.025, respectively. We conclude that the four datasets show non-monotonic trends, which justifies the application of the three models, i.e., TGP, EGP, DGP, and DRGP in our comparison.

Assume that $F_1(t) = 1 - \exp\{-\theta_2^{-\theta_1} t^{\theta_1}\}$ for all of the models. Table 5 shows the AICc values of the models, from which one can see that the AICc values of the models DRGP-III and DRGP-IV are the smallest on the respective dataset. This suggests that models DRGP-III and DRGP-IV outperforms the three other models on these four datasets in terms of AICc.

The parameters of these two models are shown in Table 6, where the values in the parentheses are standard errors of the corresponding estimates, obtained by computing the inverse of the square root of the diagonal elements of the observed Fisher information matrix. Figure 6 illustrates the cumulative failure intensity (CFI) of the observations (empirical CFI) and the CFI from model DRGP III on the first dataset (i.e., dataset LHD3). Each gap-time of the CFI from model DRGP III is generated based on the averages of the gap-times of 5,000 iterations from the model with parameters shown in Table 6.

6 Conclusions

This paper proposed a new stochastic process, which is referred to as the double-ratio geometric process (DRGP). It considered four special scenarios with different parameter settings and referred them to as DRGPs I, II, III and IV, respectively. The paper then investigated some probabilistic properties of the DRGPs, proposed to use the maximum likelihood method to estimate the parameters in the DRGPs, and illustrated their applications in a numerical example.

Table 3: The datasets.

No.	Dataset	n	Description	Reference
1	LHD3	25	failures of a load-haul-dump (LHD) machine deployed at Kiruna mine, Sweden	Kumar and Klefsjö (1992)
2	LHD11	28	failures of a load-haul-dump (LHD) machine deployed at Kiruna mine, Sweden	Kumar and Klefsjö (1992)
3	Calvert Cliffs	23	diesel generator failure data from power plant "Calvert Cliffs"	Kvam et al. (2002)
4	Pump D	30	reliability data collected from a main pump (A) at an oil refinery	Percy and Alkali (2007)

Table 4: Test for non-monotonic trend.

No.	p-value from the Ljung-Box test	p-value from V_1	p-value from V_4
1	0.886	0.00866	0.0216
2	0.899	0.0159	0.0114
3	0.509	0.0194	0.0276
4	0.107	0.00874	0.00180

Table 5: AICc values of the models

No.	EGP	DGP	TGP	DRGP-I	DRGP-II	DRGP-III	DRGP-IV
1	310.198	301.345	309.431	307.040	305.281	300.228	300.228
2	328.790	323.711	324.103	325.802	324.983	321.312	321.312
3	224.545	221.461	222.055	221.490	222.299	219.632	219.632
4	297.200	300.939	298.127	306.622	308.924	295.031	295.031

Table 6: Parameters

No.	DRGP-III ($\hat{a}_k = k^{\hat{\alpha}_1}$ and $\hat{b}_k = \hat{\beta}_1^{k-1}$)				DRGP-IV ($\hat{a}_k = \hat{\beta}_2^{k-1}$ and $\hat{b}_k = k^{\hat{\alpha}_2}$)			
	$\widehat{\alpha}_1$	$\widehat{\beta_1}$	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\alpha}_2$	$\widehat{\beta}_2$	$\widehat{\theta}_1$	$\widehat{\theta}_2$
1	6.942	0.848	1.231	553.510	1.603	0.490	1.231	553.510
	(5.502)	(0.0621)	(0.200)	(308.283)	(0.671)	(0.287)	(0.200)	(340.652)
2	14.233	0.891	1.085	347.046	1.205	0.254	1.085	347.046
	(27.230)	(0.0555)	(0.166)	(251.771)	(0.617)	(0.660)	(0.165)	(251.777)
3	-29.898	0.855	0.960	92.937	1.192	50.442	0.960	92.937
	(144.543)	(0.0861)	(0.182)	(106.597)	(0.935)	(920.240)	(0.176)	(106.516)
4	-8.996	1.281	1.297	4.629	-2.673	2.304	1.297	4.629
	(4.827)	(0.0787)	(0.185)	(2.985)	(0.689)	(1.0352)	(0.186)	(2.985)

The findings include: (1) the DRGP is an extension of many existing extensions of the geometric process (GP); (2) if the distribution of the time to first event follows the exponential distribution, then the DRGP reduces to the geometric process or the α-series process, depending on the parameter setting in the DRGP; (3) for the four DRGPs, two of the DRGP models can model observations with the non-

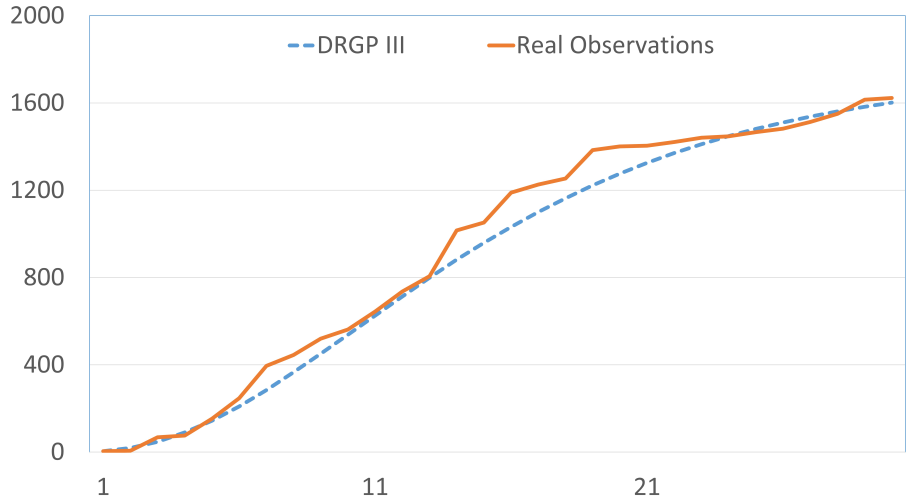

Figure 6: LD3: Empirical CFI and estimated CFI.

monotonic trend and the others cannot; and (4) if the distribution of the time to first event follows the Weibull distribution, two of the DRGP models estimated from the maximum likelihood estimation methods are equivalent in the sense that the two models have the same maximum likelihood values and the same estimated Weibull distribution.

As it can be seen, the DRGP retains the shape parameter in the process if the distribution of the time to first event is the Weibull distribution, which distinguishes the DRGP from other models such as the doubly geometric process and the threshold geometric process. Our future work aims to develop a hypothesis testing method to test whether a series of gap times may remain the same shape parameter or not.

Acknowledgement

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Appendices

Proofs

Proof of Proposition 1.

(i). We can obtain the distribution of Z_k^D as following.

$$\begin{aligned}F_k^D(t) &= 1 - \exp\{-\int_0^t b_k h(a_k u)du\} \\ &= 1 - \exp\{-\frac{b_k}{a_k}\int_0^{a_k t} h(u)du\} \\ &= 1 - \left(\exp\{-\int_0^{a_k t} h(u)du\}\right)^{\frac{b_k}{a_k}} \\ &= 1 - (1 - F_1^D(a_k t))^{\frac{b_k}{a_k}}.\end{aligned} \tag{16}$$

(ii). Let $F_1^D(t) = 1 - \exp\{-\int_0^t \frac{1}{\theta_1}du\}$. Then from Eq. (16), $F_k^D(t) = 1 - (1 - F_1^D(a_k t))^{\frac{b_k}{a_k}} = 1 - \exp\{-\frac{b^{k-1}}{\theta_1}t\} = F_1^D(b^{k-1}t)$. Then $\{Z_k^D, k = 1, 2, \dots\}$ is a GP with ratio b.

Similarly, the proof of the second bullet in (ii) can be established. □

Proof of Proposition 2.

- If $0 < \frac{b_k}{a_k} \leq 1$, according to Markov's inequality, $\mathrm{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$, we have

$$\begin{aligned}f_k^D(t) &= b_k f_1^D(a_k t)(1 - F_1^D(a_k t))^{\frac{b_k}{a_k}-1} \\ &\geq b_k f_1^D(a_k t)(\frac{\mathbb{E}[Z_1^D]}{a_k t})^{\frac{b_k}{a_k}-1},\end{aligned} \tag{17}$$

then

$$\begin{aligned}E[Z_k^D] &= \int_0^\infty t f_k^D(t)dt \\ &\geq \int_0^\infty b_k f_1^D(a_k t)(\frac{\mathbb{E}[Z_1^D]}{a_k})^{\frac{b_k}{a_k}-1} t^{2-\frac{b_k}{a_k}} dt \\ &= b_k a_k^{-2}\mathbb{E}[(Z_1^D)^{2-\frac{b_k}{a_k}}](\mathbb{E}[Z_1^D])^{\frac{b_k}{a_k}-1} \\ &= b_k a_k^{-2}\mu_1^{\frac{b_k}{a_k}-1}\mathbb{E}[(Z_1^D)^{2-\frac{b_k}{a_k}}].\end{aligned} \tag{18}$$

On the other hand, using the Cauchy–Schwarz inequality, we have $\mathbb{E}[X] = \mathbb{E}[\mathbf{1}_{\{X>0\}}X] \leq (\mathrm{P}(X > 0))^{1/2}\mathbb{E}[X^2]^{1/2}$, where the indicator function $\mathbf{1}_{\{X>0\}} = 1$ if $X > 0$ and $\mathbf{1}_{\{X\leq 0\}} = 0$. Hence,

$$\mathrm{P}(X > 0) \geq \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}, \tag{19}$$

then

$$
\begin{aligned}
1 - F_1^D(a_k t) &= \mathrm{P}(Z_1^D - a_k t > 0) \\
&\geq \frac{(\mathbb{E}[Z_1^D - a_k t])^2}{\mathbb{E}[(Z_1^D - a_k t)^2]} \\
&= \frac{(\mathbb{E}[Z_1^D])^2 - 2a_k t\mathbb{E}[Z_1^D] + a_k^2 t^2}{\mathbb{E}[(Z_1^D)^2] - 2a_k t\mathbb{E}[Z_1^D] + a_k^2 t^2} \\
&= 1 + \frac{(\mathbb{E}[Z_1^D])^2 - \mathbb{E}[(Z_1^D)^2]}{\mathbb{E}[(Z_1^D)^2] - 2a_k t\mathbb{E}[Z_1^D] + a_k^2 t^2} \\
&= 1 - \frac{\mathbb{V}[Z_1^D]}{\mathbb{E}[(Z_1^D)^2] - 2a_k t\mathbb{E}[Z_1^D] + a_k^2 t^2},
\end{aligned} \tag{20}
$$

then

$$
\begin{aligned}
\mathbb{E}[Z_k^D] &= \int_0^\infty t f_k^D(t) dt \\
&= \int_0^\infty t b_k f_1^D(a_k t)(1 - F_1^D(a_k t))^{\frac{b_k}{a_k}-1} dt \\
&\leq \int_0^\infty t b_k f_1^D(a_k t) \left(1 - \frac{\mathbb{V}[Z_1^D]}{\mathbb{E}[(Z_1^D)^2] - 2a_k t\mathbb{E}[Z_1^D] + a_k^2 t^2}\right)^{\frac{b_k}{a_k}-1} dt \\
&= b_k a_k^{-2} \mathbb{E}\left[Z_1^D \left(1 - \frac{\mathbb{V}[Z_1^D]}{\mathbb{E}[(Z_1^D)^2] - 2a_k Z_1^D \mathbb{E}[Z_1^D] + a_k^2 (Z_1^D)^2}\right)^{\frac{b_k}{a_k}-1} \right] \\
&= b_k a_k^{-2} \mathbb{E}\left[Z_1^D \left(1 - \frac{\sigma_1^2}{\sigma_1^2 + \mu_1^2 - 2a_k \mu_1 Z_1^D + a_k^2 (Z_1^D)^2}\right)^{\frac{b_k}{a_k}-1} \right].
\end{aligned} \tag{21}
$$

This establishes the inequalities for the case $0 < \frac{b_k}{a_k} \leq 1$.

- If $\frac{b_k}{a_k} > 1$, a similar proof can be established. $\square$

Proof of Proposition 3.

For (i) and (ii), it is easy to prove that $\{Z_k^D, k = 1, 2, \dots\}$ are decreasing (or increasing) stochastically and monotonically because $\frac{b_{k+1}h(a_{k+1}t)}{b_k h(a_k t)} > 1$ (or < 1) if both a_k and b_k are increasing (or decreasing) in k.

On (iii), if $\frac{b_{k+1}h(a_{k+1}t)}{b_k h(a_k t)}$ is independent of k, $\{Z_k^D, k = 1, 2, \dots\}$ are decreasing (or increasing) stochastically and monotonically. Otherwise, $\{Z_k^D, k = 1, 2, \dots\}$ may not be stochastically monotonic. $\square$

Proof of Proposition 4.

With Lemma 2 (see the online supplementary material), it is easy to prove the relationship between Z_k^D and X_k^G and that between Z_k^D and X_k^A, i.e., Part (1) in each bullet), by comparing the corresponding hazard rate functions of X_k^G, Y_k^A, and Z_k^D, respectively.

On the comparisons between the expected counts between $N^A(t)$, $N^G(t)$, and $N^A(t)$ (i.e., Part (2) in each bullet), we can prove as follows. If $Z_k^D \leq_{\mathrm{st}} X_k^G$, then $S_n^D \leq_{\mathrm{st}} S_n^G$, and thus $N^D(t) \geq_{\mathrm{st}} N^G(t)$. Then,

$\mathbb{E}[N^D(t)] \geq \mathbb{E}[N^G(t)]$. Similar proofs can be established on the other relationship between $\mathbb{E}[N^D(t)]$ and $\mathbb{E}[N^A(t)]$.

On the bound of $\mathbb{E}[N^D(t)]$, i.e., Part (3) in each bullet, we can prove them by using Theorem 1 and Theorem 2 in Braun et al. (2005). $\square$

Proof of Proposition 5.

Since $F_1^D(t) = \exp\left\{-\left(\frac{a_k t}{\theta_2}\right)^{\theta_1}\right\}$, then, $f_1^D(a_k t) = \frac{\theta_1}{\theta_2}\left(\frac{a_k t}{\theta_2}\right)^{\theta_1 - 1}\exp\left\{-\left(\frac{a_k t}{\theta_2}\right)^{\theta_1}\right\}$ and $\int_0^{a_k t} h(u)du = \left(\frac{a_k t}{\theta_2}\right)^{\theta_1}$. Plugging them into Eq. (6), we obtain

$$\ln L = n\ln\theta_1 - n\theta_1\ln\theta_2 + \sum_{k=1}^{n}\left(\ln(b_k) + (\theta_1 - 1)(\ln a_k + \ln t_k) - a_k^{\theta_1 - 1} b_k \left(\frac{t_k}{\theta_2}\right)^{\theta_1}\right). \tag{22}$$

If $a_k = k^{\alpha_1}$ and $b_k = \beta_1^{k-1}$, then

$$\ln L_1 = n\ln\theta_1 - n\theta_1\ln\theta_2 + \frac{n(n-1)\ln\beta_1}{2} + \sum_{k=1}^{n}\left((\theta_1 - 1)(\alpha_1\ln k + \ln t_k) - k^{\alpha_1(\theta_1 - 1)}\beta_1^{k-1}\left(\frac{t_k}{\theta_2}\right)^{\theta_1}\right). \tag{23}$$

If $a_k = \beta_2^{k-1}$ and $b_k = k^{\alpha_2}$, then

$$\ln L_2 = n\ln\theta_1 - n\theta_1\ln\theta_2 + \frac{n(n-1)(\theta_1 - 1)\ln\beta_2}{2} + \sum_{k=1}^{n}\left(\alpha_2\ln k + (\theta_1 - 1)\ln t_k - k^{\alpha_2}\beta_2^{(k-1)(\theta_1 - 1)}\left(\frac{t_k}{\theta_2}\right)^{\theta_1}\right). \tag{24}$$

Let $\beta_1 = \beta_2^{\theta_1 - 1}$ and $\alpha_1 = \frac{\alpha_2}{\theta_1 - 1}$, then $\widehat{l}_1 = \widehat{l}_2$. As such, the maximum likelihood obtained from Eq. (22) for model DRGP-III ($a_k = k^{\alpha_1}$ and $b_k = \beta_1^{k-1}$) and model DRGP-IV ($a_k = \beta_2^{k-1}$ and $b_k = k^{\alpha_2}$) are the same, with the same values of θ_1 and θ_2 for the two models, respectively. This establishes the proposition. $\square$

The geometric process and related work

Let X and Y be two random variables with cumulative distribution functions F and G, survival functions $\overline{F}$ and $\overline{G}$, probability density functions f and g, and hazard rate functions $r_F = f/\overline{F}$ and $r_G = g/\overline{G}$, respectively.

Definition 2 (Stochastic order) *(see p. 404 in Ross (1996)) If for every real number t, the inequality*

$$\overline{F}(t) \geq \overline{G}(t)$$

holds, then X is stochastically greater than or equal to Y, or $X \geq_{st} Y$. Equivalently, Y is stochastically less than or equal to X, or $Y \leq_{st} X$.

From Definition 2, one can define the monotonicity of a stochastic process: Given a stochastic process $\{X_k, k = 1, 2, ...\}$, if $X_k \leq_{\text{st}} X_{k+1}$ ($X_k \geq_{\text{st}} X_{k+1}$) for $k = 1, 2, ...$, then $\{X_k, k = 1, 2, ...\}$ is said stochastically to be increasing (decreasing).

Lemma 2 *(p. 405, Ross (1996)) Assume that X and Y are two random variables, then*

$$X \geq_{st} Y \; if \; only \; if \; \mathbb{E}[\nu(X)] \geq \mathbb{E}[\nu(Y)],$$

for all increasing functions $\nu(.)$.

Definition 3 *(p. 4, Shaked and Shanthikumar (2007)) X is said to be smaller than Y in hazard rate ordering (denoted by $X \leq_{hr} Y$) if $\overline{G}(x)/\overline{F}(x)$ is increasing in x.*

From Definition 3, Lemma 3 can be obtained.

Lemma 3 *(p. 18, Shaked and Shanthikumar (2007)) Suppose the hazard rates $r_G(t)$ and $r_F(t)$ exist.*

(i) $X \geq_{hr} Y$ if and only if $r_G(t) \geq r_F(t)$ for every t; and

(ii) $X \geq_{hr} Y$ entails $X \geq_{st} Y$.

Lam proposes the definition of the GP, as shown below (Lam, 1988).

Definition 4 (Geometric Process) *(Lam, 1988) Given a sequence of non-negative random variables $\{X_k^G, k = 1, 2, \dots\}$, if they are independent and the cdf of X_k^G is given by $F^G(a^{k-1}x)$ for $k = 1, 2, \dots$, where $a(> 0)$ is a positive constant, then $\{X_k^G, k = 1, 2, \cdots\}$ is called a geometric process (GP) and a is called the ratio of the GP.*

We refer to the kth inter-arrival time as the random variable X_k^G in what follows.

Remark 1 *From Definition 2 and Lemma 4, we have the following results.*

- *If $a > 1$, then $\{X_k^G, k = 1, 2, \cdots\}$ is stochastically decreasing.*
- *If $a < 1$, then $\{X_k^G, k = 1, 2, \cdots\}$ is stochastically increasing.*
- *If $a = 1$, then $\{X_k^G, k = 1, 2, \cdots\}$ is a renewal process (RP).*
- *If $\{X_k^G, k = 1, 2, \dots\}$ is a GP and X_1^G follows the Weibull distribution, then the shape parameters in the distributions of X_k^G for $k = 2, 3, \dots$ remain the same as that of X_1^G. This observation is not specific to the Weibull distribution and holds for many other distributions with a scale and shape parameter such as the Gamma distribution.*
- *Assume that $\{X_k^G, k = 1, 2, \dots\}$ follows the GP. Suppose that both the expected value and the variance of X_1^G exist. Denote $\mu_1 = \mathbb{E}[X_1^G]$ and $\sigma_1^2 = \mathbb{V}[X_1^G]$. Then the expected value and the variance of X_k^G are $\mathbb{E}[X_k^G] = a^{(1-k)}\mu_1$ and $\mathbb{V}[X_k^G] = a^{(2-2k)}\sigma_1^2$, respectively.*

Considering the GP only allows for logarithmic growth or explosive growth, but nothing in between, Braun et al. (2005) proposes a variant, which assumes that the distributions of the gap times are different from that of the GP, as shown in the following definition.

Definition 5 (α-series Process) *(Braun et al., 2005) Given a sequence of non-negative random variables $\{Y_k^A, k = 1, 2, \dots\}$, if they are independent and the cdf of Y_k^A is given by $F^A(k^{-\alpha}y)$ for $k = 1, 2, \dots,$ where α is a positive constant, then $\{Y_k^A, k = 1, 2, \cdots\}$ is called an α-series process with ratio α.*

Braun et al. (2005) also prove that the expected number of event counts before a given time, or analogously, the Mean Cumulative Function (MCF) (or, the renewal function), tends to be infinite for the decreasing GP (Braun et al., 2005). As such, they propose Definition 5 as a complement.

7 The four datasets used in the paper

The four datasets used in the paper are shown in Table 7, in which each observation represents time between two consecutive events, i.e., time between failures.

References

Arnold, R., Chukova, S., Hayakawa, Y., and Marshall, S. (2020). Geometric-like processes: An overview and some reliability applications. *Reliability Engineering & System Safety*, page 106990.

Aydoğdu, H. and Karabulut, İ. (2014). Power series expansions for the distribution and mean value function of a geometric process with Weibull interarrival times. *Naval Research Logistics (NRL)*, 61(8):599–603.

Biçer, C., Bakouch, H. S., and Biçer, H. D. (2021). Inference on parameters of a geometric process with scaled muth distribution. *Fluctuation and Noise Letters*, 20(01):2150006.

Bordes, L. and Mercier, S. (2013). Extended geometric processes: Semiparametric estimation and application to reliability. *Journal of the Iranian Statistical Society*, 12(1):1–34.

Braun, W. J., Li, W., and Zhao, Y. Q. (2005). Properties of the geometric and related processes. *Naval Research Logistics (NRL)*, 52(7):607–616.

Chan, J. S., Choy, S. B., and Lam, C. P. (2014). Modeling electricity price using a threshold conditional autoregressive geometric process jump model. *Communications in Statistics-Theory and Methods*, 43(10-12):2505–2515.

Chan, J. S., Yu, P. L., Lam, Y., and Ho, A. P. (2006). Modelling sars data using threshold geometric process. *Statistics in Medicine*, 25(11):1826–1839.

Table 7: The datasets.

LHD 3–Dataset 1 (Kumar and Klefsjö, 1992)	LHD 11–Dataset 2 (Kumar and Klefsjö, 1992)	Calvert Cliffs–Dataset 3 (Kvam et al., 2002)	Pump D –Dataset 4 (Percy and Alkali, 2007)
637	353	9	4
40	96	79	2
397	49	24	62
36	211	27	8
54	82	69	77
53	175	23	94
97	79	59	148
63	117	17	50
216	26	2	75
118	4	14	42
125	5	4	82
25	60	20	92
4	39	2	70
101	35	5	210
184	258	84	36
167	97	22	137
81	59	14	37
46	3	1	28
18	37	55	130
32	8	90	17
219	245	3	3
405	79	110	17
20	49	139	20
248	31		6
140	259		19
	283		16
	150		31
	24		37
			65
			8

Chukova, S. and Minkova, L. (2020). Geometric pólya-aeppli process. *Stochastics*, 92(8):1261–1275.

Finkelstein, M. S. (1993). A scale model of general repair. *Microelectronics Reliability*, 33(1):41–44.

Gao, W. (2020). Optimal sequential preventive maintenance policy for a repairable system with maintenance windows. *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, 234(4):963–977.

Ghosh, B. (2002). Probability inequalities related to markov's theorem. *The American Statistician*, 56(3):186–190.

Kumar, U. and Klefsjö, B. (1992). Reliability analysis of hydraulic systems of lhd machines using the power law process model. *Reliability Engineering & System Safety*, 35(3):217–224.

Kvam, P. H., Singh, H., and Whitaker, L. R. (2002). Estimating distributions with increasing failure rate in an imperfect repair model. *Lifetime Data Analysis*, 8(1):53–67.

Lam, Y. (1988). A note on the optimal replacement problem. *Advances in Applied Probability*, 20(2):479–482.

Lam, Y. (2007). *The geometric process and its applications.* World Scientific, London, UK.

Marshall, S., Arnold, R., Chukova, S., and Hayakawa, Y. (2018). Warranty cost analysis: Increasing warranty repair times. *Applied Stochastic Models in Business and Industry*, 34(4):544–561.

Pekalp, M. H. and Aydoğdu, H. (2018). An integral equation for the second moment function of a geometric process and its numerical solution. *Naval Research Logistics (NRL)*, 65(2):176–184.

Percy, D. F. and Alkali, B. M. (2007). Scheduling preventive maintenance for oil pumps using generalized proportional intensities models. *International Transactions in Operational Research*, 14(6):547–563.

Ross, S. M. (1996). *Stochastic processes.* John Wiley & Sons New York, 2 edition.

Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic orders.* Springer Science & Business Media.

Viertävä, J. and Vaurio, J. K. (2009). Testing statistical significance of trends in learning, ageing and safety indicators. *Reliability Engineering & System Safety*, 94(6):1128–1132.

Wang, J., Zhu, S., and Du, S. (2021). Analysis of a two-dimensional stair-case warranty policy with preventive maintenance. *IMA Journal of Management Mathematics*, 32(1):51–67.

Wu, D., Peng, R., and Wu, S. (2020). A review of the extensions of the geometric process, applications, and challenges. *Quality and Reliability Engineering International*, 36(2):436–446.

Wu, S. (2017). Doubly geometric processes and applications. *Journal of the Operational Research Society*, pages 1–13.

Wu, S. (2019). A failure process model with the exponential smoothing of intensity functions. *European Journal of Operational Research*, 275(2):502–513.

Wu, S. and Clements-Croome, D. (2006). A novel repair model for imperfect maintenance. *IMA Journal of Management Mathematics*, 17(3):235–243.

Wu, S. and Wang, G. (2018). The semi-geometric process and some properties. *IMA Journal of Management Mathematics*, 29(2):229–245.