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Reliable Hub-and-Spoke Systems with Multiple Capacity Levels and Flow Dependent Discount Factor

Nader Azizi[a*], Said Salhi[b]

[a] *Business School, The University of Edinburgh, Edinburgh, EH89JS, UK*

[b] *Kent Business School, University of Kent, Canterbury, CT27FS, UK*

In this paper we investigate the reliable single allocation p-hub location problem with multiple capacity levels and flow dependent discount factor. We first present new and novel MIP formulations that are built upon the well-known uncapacitated FLOWLOC model proposed by O'Kelly and Bryan (1998). The proposed reliable models aim at simultaneously determining (a) the optimal location of the hubs, (b) the allocation of demand to these hubs,(c) the backup facilities for each demand point, (d) the hub capacity level to handle the normal flow, (e) the additional capacity to handle excessive rerouted flows due to possible hub disruption, (f) the values of discount factor for inter-hub links at normal and (g) the discount factor to be applied on inter-hub links should volume of flow increases because of hub disruption. The proposed mathematical models could solve small instances to optimality using a commercial optimiser such as CPLEX. To solve large instances we propose a variant of the VNS algorithm, namely, the reduced VNS. We present computational results including lower and upper bounds of the optimal solutions to problems with 15, 20 and 25 nodes and the upper bounds of the solutions to larger problems up to 170 nodes. Managerial insights for the reliable hub location problem with and without the use of flow dependent discount factors are presented and recommendations on the use of trade-off curves between the two objectives of minimising the network cost in normal and disrupted conditions are also provided.

Keywords: Location, Single allocation hub location, Reliability, Facility disruption, RVNS

[*] Corresponding author.
Email address: nader.azizi@ed.ac.uk

1. Introduction

Everyday, an enormous number of shipments deliver materials, parts and parcels to customers around the globe to support the local and global economy. Logistics networks play a major role in this competitive environment by facilitating the movement of commodities, appliances, data etc at a lower cost. Hub and spoke is one of the most popular paradigms that is used by various companies operating in air and freight transport, parcel delivery, telecommunication and computer networks. A hub system or simply hub location problem is concerned with the optimal location of hubs, the hub network design and directing the flow through this network in such a way to minimise the total transportation cost and or service level. The hub system has a number of advantages over a system with direct links between all origin-destination pairs including lower transportation and in some cases transmission cost. This is usually achieved by exploiting economy of scales in hub links and fewer direct links which ultimately leads to lower operational cost. Hub location problems are categorised into two distinctive groups namely single and multiple allocation problems. In the former, all incoming and outgoing traffic to and from every node is transferred via a single hub, while in the latter each node is allowed to receive and send flow through more than one hub.

The research on hub location problem started with the work of O'kelly (1987) who provided the first discrete model for the single allocation version of the problem. Early research on hub location concentrated on the development of efficient formulations and solution techniques. Campbell (1994) developed a linear integer formulation for the problem. Based on the idea of multicommodity flow, Ernst and Krishnamoorthy (1996) proposed a new set of formulations for both single and multiple allocation cases. A widely used formulation in the literature for single and multiple allocation p-hub median problems is given by Skorin-Kapov et al (1996) where tight linear relaxation is derived. Over the years, a number of solution techniques based on approximation and exact methods have been developed and successfully applied to a variety of hub location problems. As an example, these include Simulated Annealing (Ernst and Krishnamoorthy1999), Genetic Algorithm (Kratica et al, 2007), Hybrid GA and Tabu Search (Abdinnour-Helm 1998), Particle Swarm Optimisation (Azizi, 2019), Lagrangian Relaxation (Contreras et al, 2009), Benders Decomposition (Contreras et al. 2012) among others. The interested readers in historical development of hub location are referred to the recent surveys of Farahani et al (2013), Contreras and O'Kelly (2019), and Alumur et al (2020).

From a practical point of view, any logistic system including those with a hub and spoke topology are often negatively affected by two types of phenomena: (1) demand, cost and time uncertainty or their variation and (2) supply uncertainty. The first type of uncertainty is highly related to the level of inaccuracy in input data while the second type is associated with the risk of supply disruption. Regardless of the source, the impact of uncertainty on daily operations could be significant ranging from excessive transportation cost and low service level to customer dissatisfaction. In general, a number of approaches have been developed to deal with demand variation and supply disruption. To incorporate uncertainties in input data, researchers often recommend a stochastic programming and/ robust optimisation approach while to reduce the impact of supply disruption the common approach is usually network resilience and reliability. The focus of this study is on the latter. More specifically in this paper we propose methodologies to enhance hub network reliability. We relax two frequently made assumptions, namely, the use of unlimited hub capacity and the utilisation of a fixed discount factor in hub links.

2. Literature Review

Given the inherent similarities between network robustness and resilience or reliability, we present our literature review by looking first at network robustness in hub systems and then resilience/reliability.

2.1 Network Robustness

Variation in demand, cost (price) and time uncertainty are an inherent part of the logistics and supply chain networks which are largely unavoidable. They are often measured as the amount of deviation from the expected values of the interested data. In hub systems for instance, long term decision of hub location is made based on an estimate for the cost of unit transportation and prediction of future demand at spokes. In practice however, none of these measures are fixed and constantly fluctuate over time. For example, Yang (2009) considers demand variation in hub systems and proposes a two-stage network design model. The first stage looks for hub locations for different demand levels and the second stage attempts to determine the transport paths and flow allocation in response to demand change. Contreras et al (2011) study stochastic uncapacitated hub location problems considering uncertain demand and transportation costs. In the case of uncertain independent transportation costs, Contreras et al (2011) provide an interesting result which shows that the corresponding stochastic problem is not equivalent to its expected value problem. Alumur et al (2012) consider uncertainty in demand and hub setup cost. Another important result is given by Alumur et al (2012) where they demonstrate that the structure of the solutions of the hub location problem with and without uncertainty are likely to be different. In addition, the authors show that optimal solutions are sensitive to the inclusion of uncertainty. Ghaffari-Nasab et al (2015) explore a robust optimisation approach to single and multiple allocation hub location problem while assuming only limited features of known demand distribution. They consider the uncertainty in the capacity constraints and use the nominal demand value in the objective function. Ghaffari-Nasab et al (2018) extend their previous work and consider three variants of polyhedral uncertainty. They propose a Tabu search algorithm to solve instances of the problem. Habibzadeh Boukani et al (2016) study robust capacitated hub location problems with both multiple and single assignments under capacity and setup cost uncertainty. Merakli and Yaman (2017) work on robust uncapacitated multiple allocation hub location problem under polyhedral demand uncertainty. They used a hose uncertainty and a hybrid model to characterize demand uncertainty. Zetina et al (2017) present models and solution techniques for robust counterparts of the well-known uncapacitated hub location problem with multiple allocation considering uncertainty in demand, transportation cost and both simultaneously. The authors compare solutions obtained from deterministic, stochastic and robust models in both worst-case and risk-neutral settings and show that when transportation costs are uncertain, commodities are routed through multiple paths. De Sa et al (2018) propose a robust optimization approach for multiple assignment incomplete hub network under demand and hub arc fixed setup cost uncertainty. The authors found that a robust model performs better than the deterministic version in avoiding budget constraints violation. Zhalechian et al (2018) propose a bi-objective two-stage stochastic programming model taking into consideration both operational and disruption risks. To design hub location networks with demand uncertainty has been also studied from the hub congestion perspective. Examples of such studies are but not limited to Marianov and Serra (2003), de Camargo et al (2009), Elhedhli and Wu (2010), Rahimi et al (2016) and Azizi et al (2018).

2.2 Reliability and Resilience

The cause of hub failure or disruption vary from severe weather condition and natural disasters to labour dispute and sabotage. While a set of disruption causes like earthquake may occur less frequently, a large number of other causes are likely to strike the network at any time. Traditional approaches to hub-and-spoke network design ignores the possibility of hub disruption. These approaches are mainly concerned with the location of the facilities and the allocation of demand to these facilities in such a way to minimise the total network cost. In recent years however, a number of studies have highlighted the importance of considering hub failure at the design stage. Some of these studies are discussed here.

In the context of hub design and compare to other locational problems, the number of works dealing with reliability and hub disruption are limited. For instance, O'Kelly et al (2006) propose response strategies (e.g., delaying, cancelling, rerouting) if and when disruptions occur. These measures are important for coping with disruption but are reactive in nature and can be expensive to implement. A more robust approach to dealing with hub disruption is to consider it at the design stage when deciding where to locate hubs in the first place. Among the first studies that explore this issue is the one by Kim and O'Kelly (2009) who formulate two p-hub location problems in telecommunication network with reliability consideration. The authors investigate both single and multiple allocation cases without considering backup hubs and rerouting the affected flow. Kim (2012) extend the previous work by proposing a series of hub location models to mitigate against hub failures, including two variants in which disrupted flows can be rerouted through a single intermediate backup hub.

The use of backup hubs has also been successfully employed by An et al (2015), Tran et al (2015), Azizi et al (2016) and Rostami et al (2018). An et al (2015) study the uncapacitated reliable single allocation hub median problem considering backup facilities and propose a mixed integer nonlinear formulation to solve the problem optimally on problem instances with up to 25 nodes. The authors show that a reliable network can transport more passengers by its regular routes than a network with classical configuration. However, it is worth stressing that in their study, the flow transported in the network is assumed to be symmetrical which may not always be the case in practice. Tran et al (2015) study a similar problem where the flow transported in the network is also assumed to be symmetrical and hubs are uncapacitated. The authors propose a mixed integer nonlinear model to find the optimal location of a pre-determined number of hubs. They linearize the model and use a tabu search algorithm to solve the same problem instances up to 25 nodes. Azizi et al (2016) propose mixed integer nonlinear formulations to design uncapacitated hub-and-spoke network taking into account the probability of hub failure and re-routing cost applicable to network with sympatric and asymmetric flow. They first investigate a special case assuming disrupted flow are re-assigned to a single hub. The assumption was then relaxed to allow affected nodes to be re-allocated to any hub in the network Azizi (2019). Instances up to 81 nodes are solved using CPLEX, Genetic Algorithms and Particle Swarm Optimization.

Similar to that in Azizi et al (2016) study, Rostami et al (2018) consider an uncapacitated case where a single backup hub completely reroute the flow affected by the failed hub. The authors model the problem as a two-stage formulation and propose an interesting branch-and-cut framework based on Benders decomposition to solve large problem instances. Their computational results confirms the importance of considering hub breakdowns in the strategic planning phase of a transportation network.

Mohammadi et al (2019) study a bi-objective reliable p-hub location model considering hubs and links uncertainties to minimize both the total cost and the maximum transportation time. They solved randomly generated large instances of the problem using a hybrid algorithm based on GA and VNS algorithms. Analysing a case study, the authors report considerable cost saving when possibility of random disruptions is considered in design stage.

The performance of reliable hub networks relies heavily on the availability of adequate capacity in the network. Therefore, capacity or dimension of hubs in normal operations and more importantly at disruption situations should not be discarded and has to be decided simultaneously with other decisions like hub locations, allocations and the location of potential backup facilities among others. Nevertheless to the best of our knowledge in the large majority of previous studies, it is assumed that hub facilities have unlimited capacities and there is no fixed cost associated with hub installation. Even if capacity is considered it is assumed the capacity is known in advance and do not differentiate between the capacity in normal and disruption situations. In this paper, we will respond to these two important practical issues.

Another shortcoming of recent studies on hub systems reliability is about how economies of scale is modelled. In the classic single and multiple allocation models because of flow consolidation it is often assumed those arcs that connect all hubs carry large volume of flow. Therefore the same value of the discount factor $0 \leq \alpha \leq 1$ applied to the regular unit cost for all inter-hub traffic. It is further expected that the flow volume in all spoke to be low and therefore, no discount applied to these unit costs. However in the optimal solutions that employ this simple cost function, it has been found (e.g., see O'Kelly and Bryan,1998 and Campbell, 2013) that in many situations some of the inter-hub links carry a low level of flow while some spokes carry a relatively high level. In other words in those solutions the traffic in inter-hub links are subject to the same discount factor regardless of the volume. In practice, as noted by Alumur et al (2020) it would be uneconomical to send a partially filled large vehicle (on an inter-hub link) and multiple full small vehicles (on a spoke) when a small vehicle on the hub-link and a full large vehicle on the spoke would be sufficient. Furthermore, in general approaches to reliability in hub location problem, the objective is to find an optimal network with backup hubs to maintain network operations following an incident at a hub facility without substantially increasing the day to day operating cost. Therefore the trade-off of interest is between the expected network cost in normal and that in disrupted situations. On the other hand research have shown that assuming flow-independent costs, not only miscalculate total network cost, but also incorrectly select optimal hub locations and allocations (e.g., see O'Kelly and Bryan (1998), Campbell, 2013 and Alumur et al, 2020). In our opinion, a network with back up facilities based on wrong network costs, incorrect hub locations and allocations may not necessarily represent a "reliable" network. This study will also revisit this sensitive and important issue by proposing a reliable hub and spoke model with flow-dependent transportation costs. An illustrative example that compares three networks (a) without reliability consideration, (b) with reliability and flow dependent cost and (c) with reliability and fixed discount factor has been presented in Figure 1.

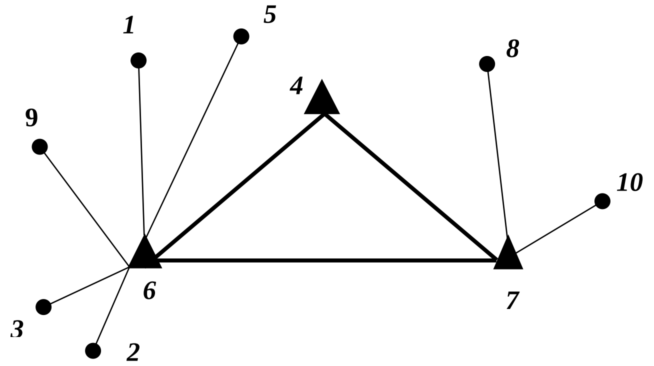


Figure 1(a). Optimal solution to the classical problem with 10 nodes and 3 hubs-CAB dataset (network cost: 491450860)

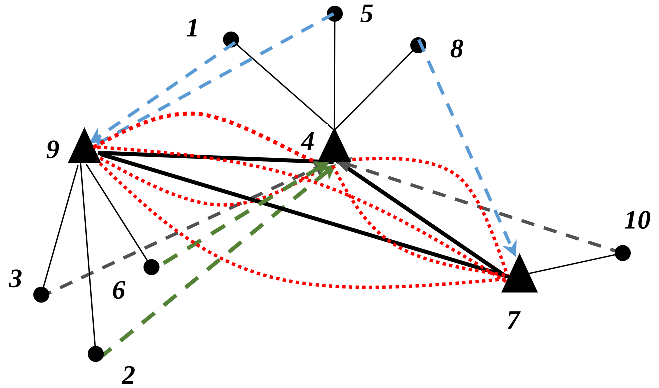


Figure 1(b). Optimal solution to the same problem with reliability consideration and flow-dependent discount factor (network cost: 904651569)

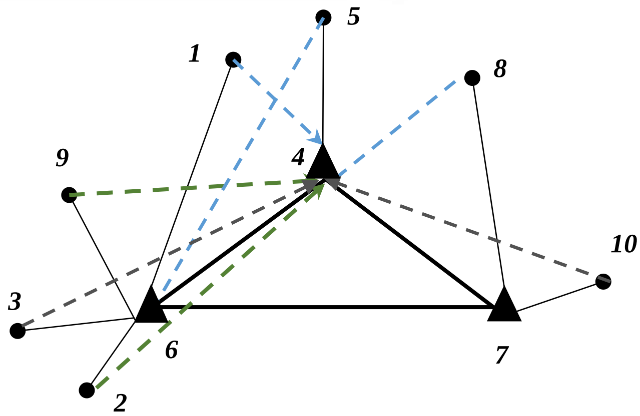


Figure 1(c). Optimal solution to the same problem with reliability consideration and fixed discount factor (network cost: 670623945)

Figure 1. Comparison between reliable with and without flow dependent cost and unreliable networks

In short, in this paper we propose models for reliable hub and spoke networks with multiple capacity levels that applies the discount based on the flow on the inter-hub connections. The proposed model aims at determining the optimal location of hubs, allocation of demand to these hubs, capacity of hub facilities both in normal and disrupted situations, the backup hubs to serve each demand point in case of disruption and the flow dependent discount factor to be applied to inter-hub connections in normal and disrupted situations. We first propose a FLOWLOC formulation with multiple capacity levels. This formulation is used as a base to develop a new mathematical model for the reliable single allocation p-hub location problem. We use the new formulation to solve small instances of the problem to optimality. A variation of this new model with fixed discount factor is then developed. This model provides upper bound whenever necessary for larger instances and hence assesses the efficiency of our proposed RVNS metaheuristic. The lower bounds are established by solving the FLOWLOC formulation with multiple capacity levels.

Our contribution is four folds:

- To develop a new nonlinear formulation for the reliable single allocation p-hub location problem with multiple capacity levels and flow-dependent discount factor
- To improve linear and nonlinear formulation for the above problem that could solve problem instances with and without symmetrical flow including fixed discount factor
- To design a simple but efficient VNS algorithm to tackle large instances including new randomly generated datasets up to 170 nodes which can also be used for benchmarking purposes
- To establish lower and upper bounds for instances up to 25 nodes, set upper bounds for the large instances, and to finally provide managerial insights in terms of scenario analysis.

The remainder of the paper is organised as follow. In the next section, we describe the problem followed by mathematical models in Section 4. In Section 5, a VNS-based heuristic is presented to solve instances of CAB, TR datasets and also larger randomly generated datasets up to 170 nodes. Computational results and analysis are given in Section 6. The paper ends by summarising our findings and highlighting some research avenues in the last section.

3. Problem Description

The problem we study is about determining the location of hubs, their respective capacities, the allocation of demand points to these hubs and the discount factors to inter-hub links in normal and disruption situations. The location decision select a set of nodes from a set of potential sites to establish hubs while the network design decisions concern the design of the links to connect nodes of the network. We also need to assign a backup hub to each demand point, design the hub network and find the best or ‘optimal’ routes to transport/transmit flows through the network. Our objective function minimises the sum of the expected transportation and installation costs in both normal and disrupted operations. A typical solution to such problem is illustrated in Figure 1(b).

Let N be the set of nodes that exchange traffic, K be the set of potential nodes for hubs, L be the set of capacity levels and R be the set of linear functions ($r \in R$ is one of the piecewise lines). In our case, we set $N \equiv K$. For each $k \in K$ there is a number of capacity levels e.g., $l=l_1$, l_2, or l_3 ($l \in L$) to select from to handle normal and excessive flow should any of the operating hubs become disrupted. For each $k \in K$ depending on k's capacity level there is also a number of installation costs, F^l_k . For each pair i , j (i , $j \in N| \ i \neq j$), there are $w_{ij} \geq 0$ units of traffic to be sent from i to j. The traffic must be routed through one hub k or two, hub k and hub m (k, $m \in K$). When using two hubs the traffic w_{ij} is sent on link $i{-}k$ first, then routed through the inter-hub connection $k{-}m$, to be finally delivered on link $m{-}j$ ($i{-}k{-}m{-}j| \ k \neq m$). If only one hub is used, the traffic w_{ij} is routed on link $i{-}k$ and then on link $k{-}j$ ($i{-}k{-}j$). The unit cost of traffic in links $i{-}k$ and $m{-}j$ are c_{ik} and c_{mj} respectively. Therefore, the transportation cost incurred on these links (i.e., $i{-}k$ and $m{-}j$) are $w_{ij}c_{ik}$ and $w_{ij}c_{mj}$. The cost on the inter-hub connection $k{-}m$ is a piecewise linear concave function of the total traffic on that link (O'Kelly and Bryan, 1998). The aforementioned cost function can be viewed as the lower envelope of a set R of linear functions in which each element r ($r \in R$) is a piecewise line (Klincewicz, 2002). Therefore, the total cost on the inter-hub link $k{-}m$ is given by a linear function that minimises $\left(b^r_{km} + a^r_{km} \sum_i \sum_j w_{ij} \, x_{ikmj}\right)c_{km}$. The parameters $b^r_{km} \, c_{km}$ and $a^r_{km} \, c_{km}$ are the intercept and slope of the r^{th} linear function ($r \in R$) respectively, and c_{ik} represents the cost per unit flow. Figure 2 illustrates a piecewise-linear concave cost function with three line segments for the inter-hub connection k-m. The decision variable $y^r_{km}=1$ if b^r_{km} and a^r_{km} are applied to the total flow g^r_{km} of the inter-hub k-m.

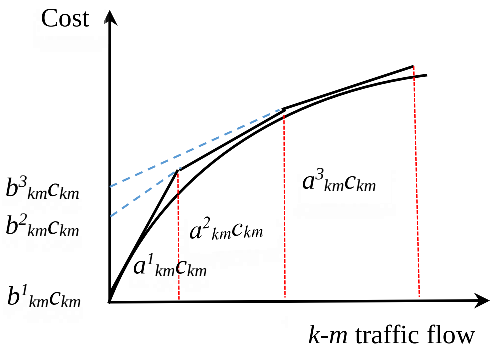


Figure 2. Piecewise-Linear Concave Cost Function on Inter-hub link k-m

4. Mathematical Formulations

In this section, we present mixed integer linear and nonlinear programming (MINLP &MILP) models for Reliable Single Allocation p Hub Location Problem with Multiple capacity levels and flow dependent discount factor (RSAPHLPM). In subsection 4.1, we first present a formulation for the Capacitated Single Allocation p-Hub Location Problem (CSAPHLPM) with multiple capacity levels and flow dependent discount factor. In subsections 4.2 and 4.3, we extend the CSAPHLPM model and introduce linear and nonlinear formulations for RSAPHLPM. For clarity, we summarise all the notation including those that have already given in the text:

Sets:	
N	the set of nodes that exchange traffic
K	the set of potential nodes for hubs
L	the set of capacity levels
R	the set of linear functions
Parameters:	
w_{ij}	Flow to be sent from node *i* to node *j* $(i, j \in N)$
c_{ij}	Transportation cost per unit flow
q_k	Probability of failure at hub *k*
F_k^l	Fixed cost for installing a hub with capacity level *l* at node *k* $(k \in N, l \in L_k)$
Γ_k^l	Capacity of a hub installed at node *k* with capacity level *l* $(k \in N, l \in L_k)$
$b^r_{km}\, c_{km}$	Intercept of piecewise *r* $(r \in R)$
$a^r_{km}\, c_{km}$	Slope of piecewise *r* $(r \in R)$
Variables:	
x_{ijkm}	1 if flow from node *i* to node *j* routed via hubs located at nodes *k* and *m* and 0 otherwise
z_{ik}	1 if node *i* is allocated to hub *k* and 0 otherwise
h_k^l	1 if capacity level *l* is decided for hub *k* and 0 otherwise
g_{km}^r	Total flow of inter-hub link *k*-*m* when segment $r(r \in R)$ is used
y_{km}^r	1 if b^r_{km} and a^r_{km} are applied to the total flow g_{km}^r of inter-hub *k*-*m*
u_{ikn}	1 if *n* is the backup hub for node *i* assigned when hub *k* is disrupted and 0 otherwise
ψ_{kn}^l	1 if node *n* is a hub with capacity level *l* when hub *k* is disrupted and 0 otherwise
f_{mnk}^r	Total flow of inter-hub link *n*-*m* when hub *k* is disrupted and segment *r* $(r \in R)$ is used
Θ_{mnk}^r	1 if b^r_{km} and a^r_{km} are applied to the total flow f_{mnk}^r of inter-hub *n*-*m* when hub *k* is disrupted and 0 otherwise
D_{ijn}^1	Auxiliary continuous variable (transformation)
D_{ijn}^2	Auxiliary continuous variable (transformation
Ω_k^l	Auxiliary continuous variable (transformation)
T_n^r	Auxiliary continuous variable (transformation)
S_n^r	Auxiliary continuous variable (transformation)
H_{nm}	Auxiliary continuous variable (transformation)
v_{iknj}	Auxiliary continuous variable (transformation)
B_n	Auxiliary continuous variable (transformation)
E_{nj}	Auxiliary continuous variable (transformation)
λ_{nmk}	Auxiliary binary variable (linearization)
γ_{ijn}	Auxiliary continuous variable (linearization)
η_{jmnk}	Auxiliary continuous variable (linearization)
ξ_n^r	Auxiliary continuous variable (linearization)
μ_n^r	Auxiliary continuous variable (linearization)
ρ_{ijn}	Auxiliary continuous variable (linearization)
π_{kn}^l	Auxiliary continuous variable (linearization)

θ_{ikmnj}	Auxiliary continuous variable (linearization)
τ_{ikmnj}	Auxiliary continuous variable (linearization)
ζ_{nmk}	Auxiliary continuous variable (linearization)
ϕ_{kn}	Auxiliary continuous variable (linearization)

4.1 CSAPHLPM Problem with Flow Dependent Discount Factor

In the hub location literature two streams of research address the capacitated single allocation hub location problem. In the first stream, the level of capacity in a potential hub node is assumed to be known. Examples within this category include Campbell (1994), Ernst and Krishnamoorthy (1999), Labbé et al (2005) and Contreras et al (2009). The other stream of research address a case where the size of a hub capacity is also a decision and has to be chosen from a range of possibilities. This group of models are often called capacitated single-allocation hub location problem with *multiple capacity levels* or CSAHLPM. The models proposed by Correia et al (2010) are the earliest works on the hub location problem with multiple capacity levels. The authors provide an interesting assessment of the efficiency of several CSAHLPM formulations and propose different sets of inequalities to enhance the models. In the following, we present an extension of their CSAHLPM model by introducing important factors, namely, the flow dependent discount factor. The proposed formulation can also be considered as an extension of the uncapacitated single allocation FLOWLOC model proposed by O'Kelly and Bryan (1998). The proposed (Classical) Formulation for Capacitated Single Allocation P-Hub Location Problem with flow dependent discount factor (CF-CSAPHLPM) is presented as follows.

CF-CSAPHLPM:

$$min\left(\sum_i\sum_k\sum_m\sum_j w_{ij}(c_{ik}+c_{mj})x_{ikmj}+\sum_r\sum_k\sum_m c_{km}\,(a^r_{km}g^r_{km}+b^r_{km}y^r_{km})+\sum_{k\in N}\sum_{l\in L_k}F^l_k\,h^l_k\right) \qquad (1)$$

S.t.

$$\sum_k z_{ik}=1 \qquad \forall i \qquad (2)$$

$$\sum_k z_{kk}=p \qquad (3)$$

$$z_{ik}\le z_{kk} \qquad \forall i,k \qquad (4)$$

$$\sum_m x_{ikmj}=z_{ik} \qquad \forall i,j,k \qquad (5)$$

$$\sum_k x_{ikmj}=z_{jm} \qquad \forall i,j,m \qquad (6)$$

$$\sum_i\sum_j\sum_m w_{ij}x_{ikmj}\le\sum_{l\in L_k}\Gamma^l_k\,h^l_k \qquad \forall k \qquad (7)$$

$$\sum_{l\in L_k} h_k^l = z_{kk} \qquad \forall k \tag{8}$$

$$\sum_r g_{km}^r = \sum_i \sum_j w_{ij}\, x_{ikmj} \qquad \forall k \neq m \tag{9}$$

$$g_{km}^r \leq y_{km}^r \sum_i \sum_j w_{ij} \qquad \forall r, k \neq m \tag{10}$$

$$\sum_r y_{km}^r \leq x_{kkmm} \qquad \forall k \neq m \tag{11}$$

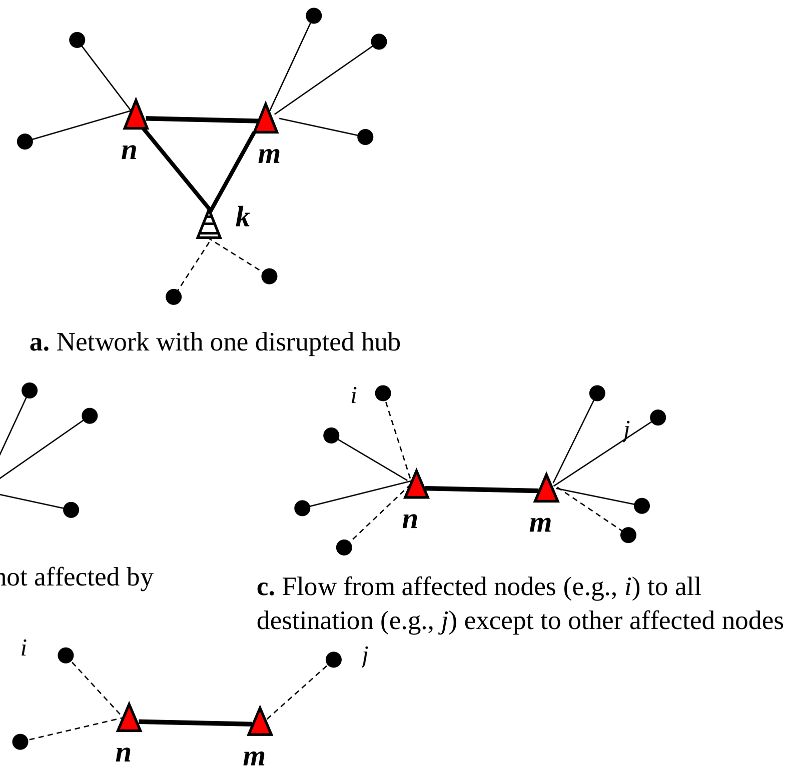
$$z_{ik}, h_k^l, y_{km}^r, x_{ikmj} \in \{0,1\} \qquad \forall i, l, r, k \neq m \tag{12}$$

$$g_{km}^r \geq 0 \qquad \forall r, k \neq m \tag{13}$$

The objective function (1) minimizes the total installation and transportation costs. Constraint set (2) ensures every node is assigned to exactly one hub. Constraint (3) limits the number of hubs to open to exact number of p facilities. Constraint (4) guarantees that a node will be assigned to an open hub. Constraint (5) and (6) ensure that all the traffic between an origin-destination pair has been routed via a hub sub-network. Constraints (7) are capacity constraints for the non-processed incoming flow at hubs. Constraints (8) are consistency constraints assuring that for each potential hub at most one capacity level can be chosen. Constraints (9) compute the amount of flow on the inter-hub connection *k-m*. Constraints (10) ensure that the flow on the inter-hub link *k-m* using piecewise *r* and the slope a_{km}^r, is associated with the right intercept b_{km}^r. Constraints (11) guarantee exactly one segment of piecewise-linear concave function is used for every inter-hub connection *k-m* (only one y_{km}^ractivated for each *k-m* when *k* and *m* are selected as hubs). In the following subsection we use this model as the base formulation and develop a reliable hub system with multiple capacity levels and flow dependent discount factor.

4.2. Reliable Hub System with Multiple Capacity Levels and Flow-Dependent Discount Factor: The original and the improved formulation

Original formulation- When disruption occurs we assume the affected demand points will be reallocated to one of the operating hubs (i.e., backup hubs in the network). We also assume only one hub can be disrupted at a time with a certain probability which we consider to be different from one potential location (to install a hub) to another. To calculate the expected network transportation cost we distinguish the following three different types of flow: *i*) flow that need to be transported in part of the network not affected by disruption (figure 3-a), *ii*) flow initiated from the affected nodes to all unaffected nodes in the network (figure 3-b) , and *iii*) exchanged flow between all affected nodes that are re-allocated to the same and/or different backup hubs (figure 3-c). The three types of flow are used to compute the expected transportation cost of the network assuming one of the operating hubs in the network become disrupted (one at a time). The objective of the proposed reliable model minimises the sum of the expected transportation and hub installation costs in normal and disruption situations. The MINLP formulation for the Reliable Single Allocation p-Hub Location Problem with Multiple capacity levels and flow dependent discount factor, RF1-RSAPHLPM is presented as follows.



a. Network with one disrupted hub

b. Flow in part of the network not affected by disruption

c. Flow from affected nodes (e.g., *i*) to all destination (e.g., *j*) except to other affected nodes

d. Flow between all affected nodes if assigned to the same and (*i* and *i'*)/or different backup hubs (*i* and *j*)

Figure 3. Expected flow to be transported in the network following a disruption at e.g., hub *k*

RF1- RSAPHLPM:

$$
\begin{aligned}
min & \Bigg(\sum_{i}\sum_{k}\sum_{m}\sum_{j} w_{ij}(c_{ik}+c_{mj})x_{ikmj} + \sum_{r}\sum_{k}\sum_{m} c_{km}\,(a^{r}_{km}g^{r}_{km}+b^{r}_{km}y^{r}_{km}) + \\
& \sum_{k}\sum_{l\in L_k} F^{l}_{k}\,h^{l}_{k} \Bigg)\Bigg(1-\Big(\sum_{n} q_n z_{nn}\Big)\Bigg) + \Bigg(\sum_{k}\sum_{\substack{n\\ n\neq k}}\sum_{l} F^{l}_{k}\,\psi^{l}_{nk}\,q_k + \\
& \sum_{i}\sum_{k\neq n}\sum_{m\neq n}\sum_{n}\sum_{j} w_{ij}\,(c_{ik}+c_{mj})x_{ikmj}z_{nn}q_n + \sum_{\substack{m\\ m\neq n}}\sum_{\substack{n\\ n\neq k}}\sum_{\substack{k\\ k\neq m}}\sum_{r} c_{mn}\,(a^{r}_{mn}f^{r}_{mnk}+b^{r}_{mn}\Theta^{r}_{mnk})\,q_k + \\
& 2\Bigg(\sum_{\substack{i\\ i\neq k}}\sum_{\substack{k\\ k\neq m}}\sum_{m}\sum_{n}\sum_{j} w_{ij}\,(c_{in}+c_{mj})u_{ikn}z_{jm}q_k\Bigg) + \\
& \sum_{i}\sum_{k}\sum_{m}\sum_{n}\sum_{j} w_{ij}\,(c_{in}+c_{mj})u_{ikn}u_{jkm}q_k + \sum_{i}\sum_{j}\varphi_{ij}\,w_{ij}\big(q_i z_{ii}+q_j z_{jj}\big)\Bigg) \qquad (14)
\end{aligned}
$$

s.t

(2)-(11)

$$\sum_{n\neq k} u_{ikn} = z_{ik} \qquad \forall k, i\neq k \qquad (15)$$

$$u_{ikn} \leq z_{nn} \qquad \forall k, n, i\neq k \qquad (16)$$

$$\sum_{l \in L_k} \psi_{nk}^l = x_{nnkk} \qquad \forall\, k, n \tag{17}$$

$$\sum_i \sum_j w_{ij}\, z_{in} z_{kk} + \sum_{\substack{i \\ i \neq k}} \sum_{\substack{j \\ j \neq k}} w_{ij} u_{ikn} \leq \sum_{l \in L_k} \Gamma_n^l\, \psi_{nk}^l \qquad \forall n, k \tag{18}$$

$$\sum_r f_{mnk}^r = \sum_i \sum_j w_{ij}\, x_{inmj}\; z_{kk} + \sum_i \sum_j w_{ij}\, u_{ikn}\; z_{jm} + \sum_i \sum_j w_{ij}\, u_{jkm}\; z_{in} +$$
$$\sum_i \sum_j w_{ij}\, u_{ikn} u_{jkm} \quad \forall k, m, n \neq m \neq k \tag{19}$$

$$f_{mnk}^r \leq \Theta_{mnk}^r \sum_i \sum_j w_{ij} \qquad \forall k, m, r, n \neq m \neq k \tag{20}$$

$$\sum_r \Theta_{mnk}^r \leq x_{nnmm}\; z_{kk} \qquad \forall k, m, n \neq m \neq k \tag{21}$$

$$z_{ik}, h_k^l, \psi_{nk}^l, y_{km}^r, \Theta_{mnk}^r, x_{ikmj}, u_{ikn} \in \{0,1\} \quad \forall i, n, m, k, r, l \tag{22}$$

$$g_{km}^r, f_{mnk}^r \geq 0 \qquad \forall r, k, m, n \neq m \neq k \tag{23}$$

For asymmetric flow the seventh term in the objective function is replaced by the following expression

$$\left(\sum_{\substack{i \\ i \neq k}} \sum_{\substack{k \\ k \neq m}} \sum_m \sum_n \sum_j w_{ij}\, (c_{in} + c_{mj}) u_{ikn} z_{jm} q_k + \right. \tag{24}$$
$$\left. \sum_{\substack{i \\ i \neq k}} \sum_{\substack{k \\ k \neq m}} \sum_m \sum_n \sum_j w_{ij}\, (c_{im} + c_{nj}) u_{jkn} z_{im} q_k \right)$$

The objective function (14) calculates the expected sum of transportation and hub installation costs in normal operation and the expected transportation and installation cost should any of the operating hubs become disrupted. More specifically, the first and the second terms in the objective function jointly calculate the network transportation cost; the third and fourth terms computes the hub installation costs in normal and disrupted situations respectively; the fifth term calculates the expected cost of transporting flow at spoke links which are not affected by a possible hub disruption; the sixth term calculates the expected cost of the traffic at inter-hub links should any of the operating hubs become disrupted. This cost includes the cost of transporting flow from both unaffected and affected nodes that need to be transported via inter-hub links should any of the operating hubs become disrupted; the seventh term calculates the expected cost of transporting flow from all affected nodes (via backup hubs) to all destinations at spoke nodes except to other affected nodes; the eight term calculates the expected cost of transporting flow between all affected nodes at spoke nodes. The affected nodes are either assigned to the same or different backup hubs. The ninth term penalizes the loss of flow/demand in disrupted situations where the source or destination of the flow is a hub.

Constraint (15) guarantees each node *i* has only one backup facility (*n*) and it differs from the hub initially assigned to (i.e., *k*). Constraint (16) ensures the candidate backup *n* for the affected node *i* is an open "hub" facility. The constraint set (17) ensure that for each hub *n* when another hub *k* in the network is disrupted at most one capacity level can be chosen. Constraints (18) are (nonlinear) capacity constraints to ensure the sum of the unprocessed incoming flow at hub *n* and the re-routed flow through hub *n* (when hub *k* is disrupted) is not more than its capacity. The nonlinear constraints (19) compute the amount of flow on the inter-hub connection *n-m* when hub *k* is disrupted. More specifically the first term in the right hand side of the constraint calculate the exchanged flow between nodes directly connected to hubs *n* and *m* and are not affected by disruption at *k*. The second term computes the potential incoming flow originated from the affected nodes being re-allocated to *n* because of disruption at hub *k* and destined to *j* (j's are unaffected nodes assigned to another hub $m \neq n$). The third term computes the flow originated from the unaffected nodes assigned to *n* and destined to all affected nodes allocated to *m*. The fourth term computes the exchanged flow between the affected nodes whether being reallocated to the same backup hub (e.g., *m* or *n*) or to two different hubs. Constraints (20) ensure that the flow on the inter-hub *n-m* using the linear function *r* with its slope a_{nm}^{r}, is associated with the right intercept b_{nm}^{r} when hub *k* is disrupted. Constraints (21) guarantee exactly one segment of the piecewise-linear concave function is used for every inter-hub connection *n-m* when hub *k* is disrupted. In other words, there is only one Θ_{mnk}^{r} activated for each *n-m* when *n, m and k* are selected as hubs. Finally, constraints (22) and (23) enforce the binary condition and indicate the non-negativity restriction of the utilised variables.

The proposed nonlinear model contains a number of quadratic terms in both the objective function and constraints. This type of model is termed *Mixed Integer Quadratically Constrained Program (MIQCP)*. A linearized MIQCP like the above developed for the RSAPHLPM problem is computationally intensive even for small instances due to these large numbers of quadratic terms.

A Preliminary Experiment- We linearize the RF1-RSAPHLPM formulation (for simplicity, we do not present the linear model here) and conducted a limited experiment using an instance of the CAB dataset with 10 nodes and 3 hubs. We set the piecewise parameters with 4 slopes of 1.0, 0.8, 0.6 and 0.4 (see Table 3). The number of capacity levels is set to three (large, medium and small). We set the optimality gap to 0.5%. Unsurprisingly it takes about 10770 seconds (3.12 hours) of computational time for CPLEX to return the optimal solution for this relatively small problem instance. In the next subsection, we briefly discuss a procedure to transfer the RF1-RSAPHLPM into a new MINLP formulation that could be efficiently linearized. Details of the procedure could be found in Appendix A.

The improved formulation- To improve the efficiency of the RF1-RSAPHLPM formulation, wherever possible we replace the quadratic terms both in the objective function and in the constraints with nonlinear terms each defined as a product of a continuous and a binary variable (Adams and Sherali, 1990; Azizi et al, 2016). Replacing all quadratic terms will transform the model into a convex MINLP. Here, we show that substituting only part of the quadratic terms in RF1-RSAPHLPM will significantly improve the efficiency of the resulting new formulation. The improved (nonlinear) formulation is called RF2-RSAPHLPM and presented in Appendix A. The mixed-integer linear programming formulation of the RF2-RSAPHLPM will be discussed in the following section.

4.3. Linear RF2-RSAPHLPM formulation

Linearization of the improved RF2-RSAPHLPM model (see appendix A), is straightforward. The nonlinear terms in the objective function and/or in the constraints are either the product of two binary variables, or the result of multiplying a binary and a continuous variable. To linearize the model, we use the standard technique and replace the nonlinear terms with auxiliary continuous or binary variables depending on the nature of the presented nonlinearity. To enforce the relationship between the variables, additional constraints are added to the model as appropriate. For example, the first nonlinear term in the objective function of RF2-RSAPHLPM is the product of a continuous variable D^1_{ijn}, and a binary variable z_{nn}. We define the nonnegative auxiliary (continuous) variables ρ_{ijn} such that

$$\rho_{ijn} = D^1_{ijn} z_{nn} \qquad (25)$$

The relationship between these variables could be ensured by adding the following set of constraints to the model

$$\rho_{ijn} \le M_1 z_{nn} \qquad \forall i, j, n$$

$$\rho_{ijn} \le D^1_{ijn} \qquad \forall\, i, j, n$$

$$\rho_{ijn} \ge D^1_{ijn} - M_1(1 - z_{nn}) \qquad \forall i, j, n$$

The linear counterpart of the RF2-RSAPHLPM is presented as follows.

RF2-RSAPHLPM:

$$\begin{aligned}
min \Bigg(& \sum_i \sum_k \sum_m \sum_j w_{ij}(c_{ik} + c_{mj}) x_{ikmj} - \sum_i \sum_j \sum_n \rho_{ijn}\, q_n + \\
& \sum_r \sum_k \sum_m c_{km}\, (a^r_{km} g^r_{km} + b^r_{km} y^r_{km}) - \left(\sum_r \sum_n (\xi^r_n + \mu^r_n)\, q_n \right) + \\
& \sum_k \sum_{l \in L_k} F^l_k\, h^l_k - \sum_k \sum_{l \in L_k} \sum_n \pi^l_{kn}\, q_n \Bigg) + \Bigg(\sum_k \sum_{\substack{n \\ n \ne k}} \sum_l F^l_k\, \psi^l_{nk}\, q_k + \sum_i \sum_j \sum_n \gamma_{ijn}\, q_n + \\
& 2 \Bigg(\sum_{\substack{i \\ i \ne k}} \sum_{\substack{k \\ k \ne m}} \sum_m \sum_n \sum_j \left(c_{in} + c_{mj}\right) \theta_{ikmnj} q_k \Bigg) + \sum_i \sum_k \sum_m \sum_n \sum_j \left(c_{in} + c_{mj}\right) \tau_{ikmnj} q_k + \\
& \sum_{\substack{m \\ m \ne n}} \sum_{\substack{n \\ n \ne k}} \sum_{\substack{k \\ k \ne m}} \sum_r c_{mn}\, (a^r_{mn} f^r_{mnk} + b^r_{mn} \Theta^r_{mnk})\, q_k + \sum_i \sum_j \varphi_{ij}\, w_{ij} \left(q_i z_{ii} + q_j z_{jj}\right) \Bigg)
\end{aligned} \qquad (26)$$

s.t

(2)-(11);(15)-(17);(20)

$$\phi_{kn} + \sum_{\substack{i \\ i \ne k}} \sum_{\substack{j \\ j \ne k}} w_{ij} u_{ikn} \le \sum_{l \in L_k} \Gamma^l_n\, \psi^l_{nk} \qquad \forall n, k \qquad (27)$$

$$\sum_r f^r_{mnk} = \zeta_{nmk} + \sum_i \sum_j \theta_{ikmnj} + \sum_j \eta_{jmnk} + \sum_i \sum_j \tau_{ikmnj} \quad \forall k, m, n \ne m \ne k \qquad (28)$$

$$\sum_{r} \Theta_{mnk}^{r} \leq \lambda_{nmk} \qquad \forall k, m, n \neq m \neq k \tag{29}$$

$$D_{ijn}^{1} = \sum_{k} \sum_{m} w_{ij} \left(c_{ik} + c_{mj}\right) x_{ikmj} \qquad \forall\, i, j, n \tag{30}$$

$$D_{ijn}^{2} = \sum_{k \neq n} \sum_{m \neq n} w_{ij} \left(c_{ik} + c_{mj}\right) x_{ikmj} \qquad \forall\, i, j, n \tag{31}$$

$$\Omega_{k}^{l} = F_{k}^{l} h_{k}^{l} \qquad \forall\, k, l \tag{32}$$

$$T_{n}^{r} = \sum_{k} \sum_{m} c_{km}\, a_{km}^{r} g_{km}^{r} \qquad \forall r, n \tag{33}$$

$$S_{n}^{r} = \sum_{k} \sum_{m} c_{km}\, b_{km}^{r} y_{km}^{r} \qquad \forall r, n \tag{34}$$

$$H_{nm} = \sum_{i} \sum_{j} w_{ij}\, x_{inmj} \quad \forall\, n, m \tag{35}$$

$$v_{iknj} = w_{ij} u_{ikn} \qquad \forall\, i, k, n, j \tag{36}$$

$$B_{n} = \sum_{i} \sum_{j} w_{ij}\, z_{in} \qquad \forall n \tag{37}$$

$$E_{nj} = \sum_{i} w_{ij}\, z_{in} \quad \forall j, n \neq j \tag{38}$$

$$\rho_{ijn} \leq M_{1} z_{nn} \qquad \forall i, j, n \tag{39}$$

$$\rho_{ijn} \leq D_{ijn}^{1} \qquad \forall\, i, j, n \tag{40}$$

$$\rho_{ijn} \geq D_{ijn}^{1} - M_{1}(1 - z_{nn}) \qquad \forall i, j, n \tag{41}$$

$$\xi_{n}^{r} \leq M_{2} z_{nn} \qquad \forall n, r \tag{42}$$

$$\xi_{n}^{r} \leq T_{n}^{r} \qquad \forall n, r \tag{43}$$

$$\xi_{n}^{r} \geq T_{n}^{r} - M_{2}(1 - z_{nn}) \qquad \forall n, r \tag{44}$$

$$\mu_{n}^{r} \leq M_{3} z_{nn} \qquad \forall n, r \tag{45}$$

$$\mu_{n}^{r} \leq S_{n}^{r} \qquad \forall\, n, r \tag{46}$$

$$\mu_{n}^{r} \geq S_{n}^{r} - M_{3}(1 - z_{nn}) \qquad \forall n, r \tag{47}$$

$$\pi_{kn}^{l} \leq M_{4} z_{nn} \qquad \forall k, n, l \tag{48}$$

$$\pi_{kn}^{l} \leq \Omega_{k}^{l} \qquad \forall k, n, l \tag{49}$$

$$\pi_{kn}^{l} \geq \Omega_{k}^{l} - M_{4}(1 - z_{nn}) \qquad \forall k, n, l \tag{50}$$

$$\gamma_{ijn} \leq M_{5} z_{nn} \qquad \forall i, j, n \tag{51}$$

$$\gamma_{ijn} \leq D_{ijn}^{2} \qquad \forall i, j, n \tag{52}$$

$$\gamma_{ijn} \geq D_{ijn}^{2} - M_{5}(1 - z_{nn}) \qquad \forall i, j, n \tag{53}$$

$$\zeta_{nmk} \leq M_{6} z_{kk} \qquad \forall n, m, k \tag{54}$$

$$\zeta_{nmk} \leq H_{nm} \qquad \forall n, m, k \tag{55}$$

$$\zeta_{nmk} \geq H_{nm} - M_{6}(1 - z_{kk}) \qquad \forall n, m, k \tag{56}$$

$$\lambda_{nmk} \leq z_{kk} \qquad \forall n, m, k \tag{57}$$

$$\lambda_{nmk} \leq x_{nnmm} \quad \forall n, m, k \tag{58}$$

$$\lambda_{nmk} \geq x_{nnmm} + z_{kk} - 1 \quad \forall n, m, k \tag{59}$$

$$\phi_{kn} \leq M_7 z_{kk} \quad \forall n, k \tag{60}$$

$$\phi_{kn} \leq B_n \quad \forall n, k \tag{61}$$

$$\phi_{kn} \geq B_n - M_7(1 - z_{kk}) \quad \forall n, \tag{62}$$

$$\eta_{jmnk} \leq M_8 u_{jkm} \quad \forall \tag{63}$$

$$\eta_{jmnk} \leq E_{nj} \quad \forall j, m, n, k \tag{64}$$

$$\eta_{jmnk} \geq E_{nj} - M_8\left(1 - u_{jkm}\right) \quad \forall j, m, n, k \tag{65}$$

$$\theta_{ikmnj} \leq M_9 z_{jm} \quad \forall i, k, m, n, j \tag{66}$$

$$\theta_{ikmnj} \leq v_{iknj} \quad \forall i, k, m, n, j \tag{67}$$

$$\theta_{ikmnj} \geq v_{iknj} - M_9\left(1 - z_{jm}\right) \quad \forall i, k, m, n, j \tag{68}$$

$$\tau_{ikmnj} \leq M_{10} u_{jkm} \quad \forall i, k, m, n, j \tag{69}$$

$$\tau_{ikmnj} \leq v_{iknj} \quad \forall i, k, m, n, j \tag{70}$$

$$\tau_{ikmnj} \geq v_{iknj} - M_{10}\left(1 - u_{jkm}\right) \quad \forall i, k, m, n, j \tag{71}$$

$$z_{ik}, h_k^l, \psi_{nk}^l, y_{km}^r, \Theta_{mnk}^r, x_{ikmj}, u_{ikn}, \lambda_{nmk} \in \{0,1\} \quad \forall i, j, n, m, k, r, l \tag{72}$$

$$\gamma_{ijn}, \eta_{jmnk}, D_{ijn}^1, H_{nm}, \Omega_k^l, \xi_n^r, \mu_n^r, T_n^r, S_n^r, \rho_{ijn}, \pi_{kn}^l, B_n, v_{iknj}, \theta_{ikmnj}, \tau_{ikmnj} \geq 0 \quad \forall i, j, k, n, m, r$$
$$D_{ijn}^2, g_{km}^r, f_{mnk}^r, \zeta_{nmk}, \phi_{kn} \geq 0 \quad \forall r, k, m, n \neq m \neq k \tag{73}$$

$$E_{nj} \geq 0 \quad \forall j, n \neq j \tag{74}$$

In the above formulation, the function (26) is the linearized objective function presented in (14). Constraints (27)-(29) are the linear counterparts of the constraints (18), (19) and (21) respectively. Equalities (30)-(38) define the auxiliary variables used to transform the original nonlinear model (RF1-RSAPHLPM) into a new MINLP (RF2-RSAPHLPM). (39)-(71) are auxiliary constraints to insure that the new variables representing nonlinear terms in the objective function and in the constraints will assume the appropriate values. Finally, constraints (72)-(74) enforce the binary condition and indicate the non-negativity restriction of the utilised variables.

In Table 1, we compare the number of variables in both proposed nonlinear formulations (i.e., RF1-RSAPHLPM and RF2-RSAPHLPM) and their linear counterparts. The total number of variables in the two nonlinear formulation are quite identical. However, as expected, linearizing RF2-RSAPHLPM yields to a formulation with significantly less number of binary variables. This could make the solving of those instances of the RF2-RSAPHLPM computationally less challenging. This is due to the difficulty of mixed integer programs which is usually more dependent on the number of integer variables than the number of continuous variables. Furthermore, as reported by Klincewicz (2002) and Camargo et al (2009) the FLOWLOC formulation can present large integrality gaps and may result in very long computational time (e.g., 13 hours to solve a medium size problem instance). The quality of these gaps found to be due to the use of Big-M formulation that is needed to activate the right variable with inter-hub links aggregated flow.

Table 1. Number of variables in RF1- RSAPHLPM and RF2- RSAPHLPM

Formulation	Nonlinear		Linear	
	Total# of variables	Binary Variables	Total# of variables	Binary variables
RF1-RSAPHLPM	$n^4+n^3(1+2r)+n^2(1+2r+l)+nl$	$n^4+n^3(1+r)+n^2(1+r+l)+nl$	$5n^5+n^4+n^3(2+4r)+2n^2(1+r+l)+n(l+r)$	$5n^5+n^4+n^3(4+4r)+2n^2(1.5+r+l)+nl$
RF2-RSAPHLPM	$2n^4+3n^3+2n^3r+n^2(3+2r+l)+n(2l+2r+1)$	$n^4+n^3(1+r)+n^2(1+r+l)+nl$	$3n^5+n^4+2n^3(2+r)+n^2(3+2r+l)+n(2l+4r+1)$	$n^4+2n^3+n^3r+n^2(1+r+l)+nl$

In our proposed classical formulations (i.e., CF-CSAPHLPM) this means a weak coupling between g_{km}^{r} and y_{km}^{r} in constraint (10). We identify another Big-M formulation in constraint (20). To deal with the possible instability caused by using a large number of Big-Ms, we replace all Big-M formulation in the linear RF2- RSAPHLPM with *Indicator Constraints* (Bonami et al, 2015) before solving the model by CPLEX.

RF2-RSAPHLPM vs the original formulation RF1-RSAPHLPM –we evaluate and compare the performance of the improved (linear) RF2-RSAPHLPM formulation against the original (linear) formulation of RF1-RSAPHLPM. We tested these two models on the same problem instance with 10 nodes and 3 hubs using the same parameters. Significant reduction in CPU time is observed as the improved linear formulation returns the optimal solution in about 10 minutes compared to more than 3 hours that is required for the initial formulation. The optimal solution to the problem is depicted in Figure 4. The discount factors applied to inter-hub links 4-7, 4-9 and 7-9 in normal operations are respectively piecewise linear function 2 with slope (0.8), linear function 4 with slope 0.4 and linear function 1 with slope 1.0 respectively. For disruption situations the discount factors for the same inter-hub links are linear function 3 (if hub 9 becomes disrupted), line 4 (if hub 7 is disrupted) and line 2 if 4 is disrupted instead. In summary, we argue that the observed improvement in computational efficiency of the linear RF2-RSAPHLPM formulation is a result of replacing quadratic terms with nonlinear terms defined as the product of continuous and binary variables and using indicator constraints to replace Big-M formulations. Despite the above improvement made, the formulation is still limited and cannot be used to solve instances beyond 10 nodes by CPLEX. To solve larger instances one way forward is to explore using an efficient metaheuristic. In this study, we propose a Variable Neighbourhood Search (VNS) algorithm. In order to comment on the solution quality of large instances obtained by the heuristic algorithm, we propose to use the lower bounds of the optimal solutions. In the following and prior to presenting details of the proposed heuristic, we show that optimal solutions to the CF-CSAPHLPM problems are the lower bounds for the optimal solutions to the same problems with reliability consideration i.e., RF1-RSAPHLPM.

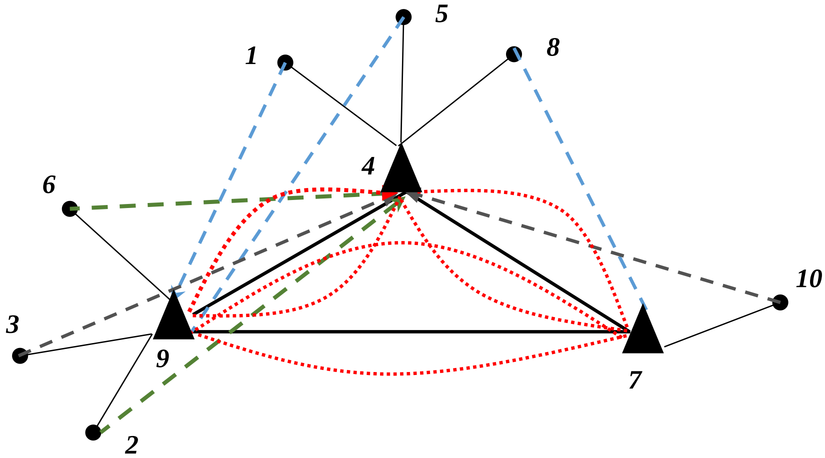


Figure 4. Graphical representation of the optimal solution to a problem with 10 nodes

4.4. Lower Bound Establishment

Preposition. An optimal solution to a CSAPHLPM problem is the lower bound for the optimal solution to the same problem with reliability consideration, RSAPHLPM.

Proof. Let $\bar{x}_C^*$ and X^*_R be the optimal solutions to CF-CSAPHLPM and RF1-RSAPHLPM respectively with the corresponding cost functions of z_C and Z_R. Assume q^k is the probability that k^{th} hub in the network will be disrupted. In the following, we will show

$$z_C(\bar{x}_C^*) \leq Z_R(X_R^*) \quad (75)$$

Beginning with the objective function of RF1-RSAPHLPM, $Z_R(X^*_R)$ could be written as

$$Z_R(X_R^*) = z_R(x_R^*)\left(1 - \sum_{k=1}^{P} q^k\right) + \sum_{k=1}^{P} ENC^k q^k \quad or$$

$$Z_R(X_R^*) = z_{RNC}^*\left(1 - \sum_{k=1}^{P} q^k\right) + \sum_{k=1}^{P} ENC^k q^k \quad (76)$$

With the same token, $z_C(\bar{x}_C^*)$ could also be written as

$$z_C(\bar{x}_C^*) = z_{CNC}^* \quad (77)$$

where z_{RNC}^* is the optimal network cost of the RF1-RSAPHLPM for a given problem (only transportation and hub installation costs) and $z_{RNC}^* = z_R(x_R^*) = z_R(z_{ik}, h_k^l, y_{km}^r, x_{ikmj})$. ENC^k is the expected network cost when k^{th} hub becomes disrupted and z_{CNC}^* is the optimal cost of the CF-CSAPHLPM for the same problem (the classical network without reliability consideration).

Similarly $z_{CNC}^* = z_C(\bar{x}_C^*) = z_C(\bar{z}_{ik}, \bar{h}_k^l, \bar{y}_{km}^r, \bar{x}_{ikmj})$. Substituting for $z_C(\bar{x}_C^*)$ and $Z_R(X_R^*)$ in (75) we obtain

$$z_{CNC}^* \leq z_{RNC}^*\left(1 - \sum_{k=1}^{p} q^k\right) + \sum_{k=1}^{p} ENC^k q^k \quad (78) \quad or$$

$$z_{CNC}^* \leq z_{RNC}^* - z_{RNC}^* \sum_{k=1}^{p} q^k + \sum_{k=1}^{p} ENC^k q^k \quad (79)$$

Now, from the RF1-RSAPHLPM formulation, it can be easily verified that $\sum_{k=1}^{p} ENC^k q^k and\ Z_{RNC}^* \sum_{k=1}^{p} q^k$ are quite identical as both terms account for the total hub installation and transportation costs therefore they will largely cancel each other out. Also given the low probability of hub failure, the result of both expressions $\sum_{k=1}^{p} ENC^k q^k$ and $z_{RNC}^* \sum_{k=1}^{p} q^k$ are negligible compare to z_{RNC}^* (also shown in the following numerical example) and they could be ignored. As a result, (79) will reduce to (80) and one just need to show that

$$z_{CNC}^* \leq z_{RNC}^* \qquad (80)$$

As CF-CSAPHLPM is the base formulation for the RF1-RSAPHLPM (it could be directly obtained from RF1-RSAPHLPM by setting $q^k = 0, \forall k$), the two cost functions z_C and z_R are identical i.e.,

$$z_C \equiv z_R \qquad (81)$$

Also an optimal network configuration obtained for the RF1-RSAPHLPM, $x_R^*(z_{ik}, h_k^l, y_{km}^r, x_{ikmj})$ ll satisfy all the CF-CSAPHLPM constraints therefore, it will be a feasible solution to CF-CSAPHLPM. This will result in

$$z_C(\bar{x}_C^*) \leq z_C(x_R^*) \qquad (82)$$

With (81)

$$z_C(\bar{x}_C^*) \leq z_R(x_R^*) \quad or \quad z_{CNC}^* \leq z_{RNC}^* \qquad (83)$$

Otherwise, it will be a contradiction because $\bar{x}_C^*(\bar{z}_{ik}, \bar{h}_k^l, \bar{y}_{km}^r, \bar{x}_{ikmj})$ is the optimal solution to the CF-CSAPHLPM. In conclusion, an optimal solution to CF-CSAPHLPM is a lower bound of the optimal solution to RF1-RSAPHLPM i.e., $z_C(\bar{x}_C^*) \leq Z_R(X_R^*)$.

Numerical Example. Through the following example, we also show for illustration purposes the validation of inequality (79). We tested six problems with 10 and 25 nodes from CAB dataset. The lower bounds are obtained by solving the CF-CSAPHLPM formulation. As shown in Table 2, in all cases, z_{CNC}^* is lower than the total *Obj.* cost and both $\sum_{k=1}^{P} ENC^k q^k$ and $z_{RNC}^* \sum_{k=1}^{p} q^k$ are significantly less than z_{RNC}^*.

Table 2. Numerical example for the lower bound establishment

Problem		RSAPHLPM				Lower Bound z_{CNC}^*
		Total Obj.[1]	Network cost (z_{RNC}^*)	$\sum_{k=1}^{P} ENC^k q^k$	$z_{RNC}^* \sum_{k=1}^{p} q^k$	
cab10	*f1*	921381261	908463496	92862553	79944788	908463496
	f2	904651569	890999439	92060080	78407951	890999439
	f3	845120640	829755726	98340487	82975573	829246258
cab25	*f1*	10104901940	9873082506	1288239263	1056419828	9868585775
	f2	8857144502	8524475034	1304459622	971790154	8524475034
	f3	7780167054	7394637449	1228518274	842988669	7391365540

1) $Total\ Obj = z_{RNC}^* - z_{RNC}^* \sum_{k=1}^{p} q^k + \sum_{k=1}^{P} ENC^k q^k$

5. A Variable Neighbourhood Search Implementation

Variable neighbourhood search is originally developed by Mladenović and Hansen (1997) and has been successfully applied to many combinatorial optimization problems including hub location. The interested reader may refer to Hansen et al. (2010) for an excellent review of the VNS algorithm and its applications, and for an overview on heuristic in general including VNS in Salhi (2017).

In this study, we developed a VNS-based search algorithm to solve more realistic instances of the model, as CPLEX is found to be inappropriate for solving larger instances. The proposed simple algorithm could be regarded as a variation of the *Reduced VNS* (RVNS) which is a variant of VNS that does not require a local search. This variant is found to be useful for instances that require relatively an excessive amount of computational effort. See Hansen et al. (2010) and Salhi (2017) regarding more details on this variant and other related ones. We would like to stress that we are not proposing novel attributes to this VNS implementation but carried out to generate good feasible solutions to such complex problem where the exact methods fails to do so.

Our proposed reduced VNS is similar to the classical RVNS except of the following attributes, namely, the way the new solution is accepted, the number of solutions to be examined from each neighbourhood structure and the generation of the initial solution that is linked to the mathematical model. The main steps of our RVNS implementation are provided as follows. The initial solution is generated using the following scheme. First *n* number of solutions (*n* is the number of nodes) are constructed and the best of these is then selected as our initial solution. To construct these solutions, first a number of nodes e.g., 3 are randomly selected as hub facilities then the remaining nodes are assigned to these hubs according to their proximity to each of these facilities. This process is repeated for *n* times and the solution with the lowest cost is then selected as the RVNS algorithm initial solution.

Neighbourhood structures. In the proposed RVNS, four neighbourhood structures are considered (see Figure 6). In other words we have $N_k(.)$; $k=1,\ldots,k_{max}=4$. In the first structure (type 1), in $N_1(.)$, a neighbouring solution is generated by replacing a hub facility with one of the demand points (i.e., node) allocated to this hub. In the second structure (type 2), $N_2(.)$, the allocation of two non-hub nodes is exchanged to create another solution. In the third structure (type 3), $N_3(.)$, the allocation of a randomly selected node is changed to generate a neighbouring solution. In the fourth structure (type 4), $N_4(.)$, , a hub is selected randomly and all of the nodes allocated to this facility are reallocated to another (randomly selected) facility.

The search begins with an initial solution (S) produced through the above procedure. The four neighbourhood structures described above are then used sequentially to generate a number of solutions within each neighbourhood. For types 1, 2, 3 and 4 structures, a total of NI_1 (NI_1=Number of nodes-Number of hubs), NI_2(NI_2 =Number of nodes), NI_3(NI_3 =Number of nodes), and NI_4 (NI_4 =Number of hubs) solutions are generated respectively. The overall best solution found in each iteration of the algorithm (i.e., when all four neighbourhoods are tested) is recorded as the iteration best solution (S_{itr}*).

Input: NI_i (The number of solutions evaluated within the neighbourhood structure *i*), Tmax (search time) and all other parameters used in the model

$t \leftarrow 0$; $S \leftarrow S^{ini}$;

$S^* \leftarrow S$; $f(S^*) \leftarrow f(S)$

$S_{itr}^* \leftarrow S$; $f(S_{itr}^*) \leftarrow f(S)$

Repeat

 For *i* =1 to 4

  For *j* =1 to NI_i (NI_1=Number of nodes-Number of hubs; NI_2 = NI_3 = Number of nodes; NI_4 = Number of Hubs)

   Generate a new solution, S^{cur} *, from* $N_i(S)$*,*

   Calculate the cost of S^{cur}*,* $f(S^{cur})$

   Update S_{itr}^* *and* $f(S_{itr}^*)$ *using* S^{cur} and $f(S^{cur})$

  Next *j*

 Next *i*

  Update S^* *and* $f(S^*)$ *using* S_{itr}^* and $f(S_{itr}^*)$

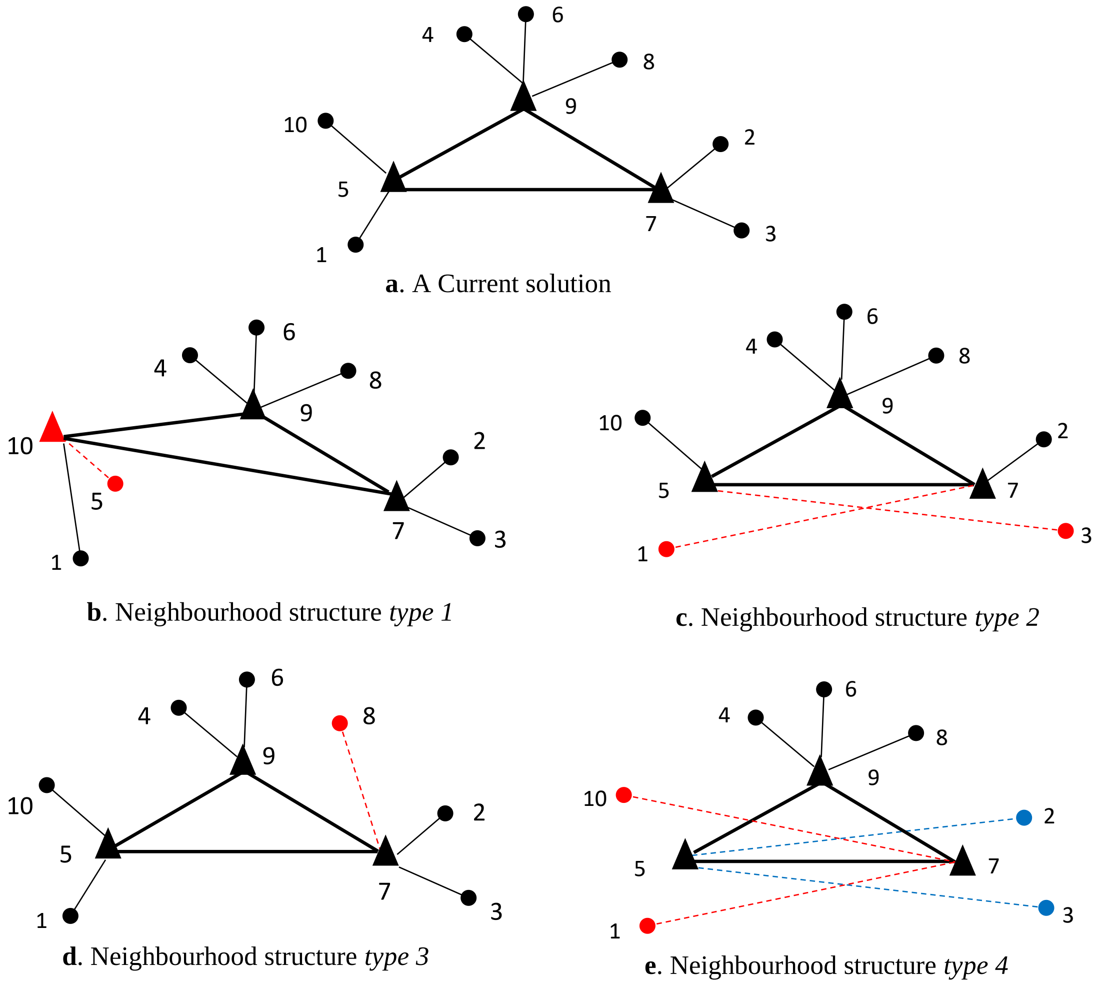
   $S \leftarrow S_{itr}^*$

  Update t

While $t <$ Tmax

Report the best solution found

Figure 5. The pseudocode of the proposed RVNS



a. A Current solution

b. Neighbourhood structure *type 1*

c. Neighbourhood structure *type 2*

d. Neighbourhood structure *type 3*

e. Neighbourhood structure *type 4*

Figure 6. The four neighbourhood structures of the proposed RVNS

This solution is used as the initial solution for the next iteration of the algorithm. This process is repeated until the stopping criterion is satisfied. In this study, the proposed algorithm is run for a fixed time depending on the size of the benchmark problems. The pseudoco de of the proposed VNS is presented in Figure 5.

6. Computational Results and Analysis

In this section, we examine three problems and models namely, the CSAPHLPM and our proposed formulation for this classical problem CF-CSAPHLPM, the RSAPHLPM and the proposed formulation RF2-RSAPHLPM, and the RSAPHLPM problem with fixed discount factor and our proposed formulation RF3-RSAPHLPM. As the focus of this study is on RSAPHLPM with flow dependent cost, we present RF3-RSAPHLPM in Appendix B. We solve instances of the CF-CSAPHLPM by CPLEX and by our proposed RVNS to: (1) provide some results for this new problem (2) demonstrate the performance of the RVNS and (3) provide lower bounds for solutions to RF2-RSAPHLPM that CPLEX is unable to solve. We solve small instances of RF2-RSAPHLPM by CPLEX while solutions to medium and large problem instances are obtained by the proposed RVNS. Finally, small instances of RSAPHLPM with fixed discount factor (RF3-RSAPHLPM) are solved by CPLEX to compare solutions to reliable hub location problem with and without flow dependent cost. While medium and large instances have been solved by the RVNS to provide upper bounds to the optimal solutions. The mathematical models are run in CPLEX 12.9 and the experiments are carried out on a PC–Intel Core i5 CPU@2.6 GHZ with 16.0 GB RAM.

6.1. The CF-CSAPHLPM Model

We begin the computational analysis by solving eight problem instances from the CAB dataset with 10, 15, 20 and 25 nodes and with 3 and 5 hubs using the CF-CSAPHLPM formulation. For every potential hub, we generate three capacity levels: Small (S), Medium (M) and Large (L). The capacities are equivalent to 30% (small capacity), 60% (medium capacity) and 90% (large capacity) of the total flow in the network (i.e., $\sum_{i,j} w_{i,j}$). The associated fixed costs are set to $50(10^6)$, $100(10^6)$ and $150(10^6)$ respectively. We set the piecewise parameters with 4 slopes of 1.0, 0.8, 0.6 and 0.4 (see Table 3).

Table 3. Piecewise-linear concave cost functions

Flow (k-m) (×1000)	Function 1 Slopes (*f1*)	Function 2 Slopes (*f2*)	Function 3 Slopes (*f3*)
$0 \le f'_{km} < 50$	1.0	1.0	0.8
$50 \le f'_{km} < 100$	0.9	0.8	0.6
$100 \le f'_{km} < 200$	0.8	0.6	0.4
$200 \le f'_{km}$	0.7	0.4	0.2

The problem instances are solved by the proposed RVNS and CPLEX 12.9. The computational results are presented in Table 4. In this table, the optimal hub location and demand allocations are presented in the fifth column. For instance, for the first problem instance, 6-6-6-4-6-6-7-7-6-7 means the three selected hubs are 6, 4 and 7; nodes 1, 2, 3, 5,6 and 9 are allocated to hub 6; 4 allocated to itself; and 8 and 10 are allocated to hub 7. In column 6, the selected capacity levels for hub 6, 4 and 7 are presented. In the last column we report the recommended inter-hub discount factors for each pair of hubs in the same order of hubs in column 6 (i.e., between hubs 6 and 4; 6 and 7 and 4 and 7 respectively). The computational time for CPLEX ranges from 10 seconds for the smallest instance with 10 nodes to 17 hours for the large problem instance with 25 nodes and 5 hubs.

As shown in Table 4, the proposed RVNS solves all problem instances to optimality in less than 1 second. This result clearly show the efficiency of the proposed VNS and its capability of tackling relatively large instances of the problem within very short computing times.

6.2. The RF2-RSAPHLPM Model

We conducted a series of computational analysis using the two standard hub location datasets (i.e., CAB and TR) and a number of larger Randomly Generated Problems (RGPs) which we constructed. The computational results pertain to the benchmark problems are discussed in the following subsections.

6.2.1 The small and medium size problems (the CAB and TR datasets)

The CAB dataset include problem instances of 10, 15, 20 and 25 nodes. The selected small and medium size problem instances from the TR dataset include problems with 10 and 25 nodes. The three capacity levels, small, medium and large, are generated by calculating 50%, 70% and 99% of total flow to be transported in the network (i.e., $\sum_{i,j} w_{i,j}$) respectively. The fixed cost to stablish small, medium and large hubs are set to $50(10^6)$, $100(10^6)$ and $150(10^6)$ respectively. In our computational experiments, we used three different piecewise-linear concave cost functions with four slopes proposed by Klincewicz (2002). The three functions represent moderate (*f1*), intermediate (*f2*) and aggressive (*f3*) scale economies and are presented in Table 3. The probabilities of hub failure have been generated from *Uniform distribution* (0.01, 0.05). A total of 18 small and medium size problems are generated based on the CAB and TR datasets and the three piecewise linear concave cost functions. The computational results are presented in Table 5. In this table, the values under the columns “best solution” and “time” are the best solution obtained over 20 runs and the time when this solution has been found within a limited computational time. The computational times for problems with 10, 15, 20 and 25 are set to 3, 10, 20, and 40 seconds respectively. Note that the solutions found for all small instances with 10 nodes are optimal solutions as optimality is guaranteed by CPLEX and solutions are confirmed by the proposed RVNS.

RVNS vs HPSO- The performance of the proposed RVNS has been also compared with a hybrid Particle Swarm Optimisation (PSO) algorithm (Azizi, 2019). The hybrid algorithm (HPSO) is made of three main components: (1) a classical PSO search engine (2) an elite list made of global best solutions and (3) a crossover operator. We implemented the HPSO and solved instances of the proposed model for reliable hub systems. The results are presented in Table 6. We use the same 18 small and medium size

Table 4. Computational results for CPLEX & RVNS: CSAPHLPM problem

Problem instance	p	Optimal solution*	Time (RVNS/ sec.)	Hubs & allocations	Capacity	Inter-hub discount factors
cab10	3	952124311*	<1	6-6-6-4-6-6-7-7-6-7	6(M)4(S)7(S)	0.4-0.8-0.8
	5	962703933*	<1	1-6-6-4-4-6-7-7-9-7	1(S)6(S)4(S) 7(S)9(S)	1-1-1-1-0.6-1-1-0.8-0.8-1
cab15	3	2740126717*	<1	13-4-4-4-4-4-13-8-4 13-4-8-13-13-4	13(S)4(M)8(S)	0.4-0.6-0.4
	5	2656972877*	<1	1-6-6-4-6-6-7-8-6-7-4-8-1-1-4	1(S)6(S)4(S) 7(S)8(S)	0.6-0.8-1-1-0.6-1-0.8-0.8-0.6-0.8
cab20	3	5741145734*	<1	4-17-17-4-4-17-7-7- 4-7-4-7-4-17-4-7-17- 17-7-17	4(S)17(M)7(S)	0.4-0.4-0.4
	5	5350957256*	<1	13-17-17-4-4-4-13-8- 4-13-4-8-13-14-4-13- 17-17-8-17	13(S)17(M)4 (S)8(S)14(S)	0.4-0.6-0.6-0.8-0.4-0.4-0.4-0.4-0.6-1
cab25	3	8624475034*	<1	4-18-18-4-4-4-4-4-4- 4-4-12-4-18-4-4-18- 18-12-18-4-12-12-18-18	4(M)18(M) 12(S)	0.4-0.4-0.4
	5	7975216282*	<1	4-17-17-4-4-4-7-7-4- 7-4-12-7-14-4-7-17- 17-12-17-4-12-12-14- 17	4(S)17(M)7(S) 12(S)14(S)	0.4-0.4-0.4-0.6-0.4-0.4-0.4-0.6-1-1

problems generated from CAB and TR datasets and the three piecewise linear concave cost functions (see Table 3). The swarm size is set to 25 for all test problems. C1 and C2, the acceleration coefficients are set to 2 and the inertia factor, ω, to 1. The HPSO algorithm is also run 20 times for each test problem and the computational times are the same used to run the RVNS earlier. The best solutions found by each algorithm have been reported in Table 6. The results show that both algorithms performs well in solving small instances with 10 nodes. However, for the rest of the problem with 15, 20 and 25 nodes, the RVNS clearly outperforms the HPSO. The gaps between the cost of the best solutions found by the RVNS and the HPSO increases as the size of the problem increases (see also Figure 7).

6.2.2 The large problem instances (TR and randomly generated datasets)

To provide solutions to large instances of the reliable hub location problem with flow dependent cost RSAPHLPM, and to further evaluate the performance of the proposed RVNS, we have used the two largest instances of the TR dataset with 55 and 81 nodes and also randomly generated larger problems with 100, 130, 150 and 170 nodes. The generation is carried out as follows. The flow and unit transportation cost between each origin-destination pair have been generated from the *Poisson* (λ=10,000) and *Uniform* (500, 1000) distributions respectively. The probabilities of hub failure have been generated also from *Uniform distribution* (0.01, 0.09). Similar to that for the standard datasets, the three capacity levels, small, medium and large, are generated by calculating 50%, 70% and 99% of total flow to be transported in the network (i.e., $\sum_{i,j} w_{i,j}$) respectively. We use the same three different piecewise-linear concave cost functions with four slopes presented in Table 3. The computational time for problems with 55, 81, 100, 130, 150 and 170 are 90, 120, 900, 1300, 2400 and 4200 seconds respectively. To solve these problems the RVNS is run 10 times and the best solution found is recorded. Due to space limitation, we only report the value of the objective function and the location of hubs for large instances (see Table 7). Details of the solutions could be found in the supplementary file.

6.2.3. The lower bounds: Solving instances of CF-CSAPHLPM by CPLEX

In order to comment on the solution quality of the problem instances with more than 10 nodes which are obtained by the RVNS algorithm, we have compared our results with the optimal solutions' lower bounds. In section 4.2, we showed that an optimal solution to a *CSAPHLPM* problem provides a lower bound to the optimal solution for the same problem with reliability consideration. The lower Bounds (LBs) for the problems up to 25 nodes are found by solving instances of CF-CSAPHLPM formulation using CPLEX. To solve instances of CSAPHLPM and RSAPHLPM, the same values for all the parameters (e.g., hub capacities) have been used.

The computational results for the LBs are summarised in Table 8. In this table, for problem instances with 10 nodes the *"% Gap"* represents the gap between the objective function of the solutions returned by the RVNS and the CPLEX. For larger instances with 15, 20, and 25 nodes, it represents the gap between the objective function of the RVNS solutions and the lower bound of their optimal solutions obtained by solving CF-CSAPHLPM formulation. The reported percentage gaps vary from 0 for small instances with 10 nodes to a maximum of 4.9% for a problem instance with 25 nodes. In Table 8, we also report the percentage of UB improvements. This will be discussed in detail in the following section.

6.2.4. The upper bounds: Solving Reliable Model with Fixed Discount Factor

The two proposed models, RF1-RSAPHLPM and RF2-RSAPHLPM**,** can be easily modified to develop a reliable capacitated single allocation hub location problem with multiple capacity levels and *fixed cost*. As the focus of this paper is on models with flow dependent discount factor, we present only a nonlinear model (RF3-RSAPHLPM) for this problem in Appendix B. The purpose of developing and presenting RF3-RSAPHLPM is two-fold. First, to estimate the Upper Bounds (UBs) for the optimal solutions to all problem instances of reliable model with flow dependent cost. Second to compare solutions to the problem with and without a flow dependent discount factor. We proceed by solving instances of RF3-RSAPHLPM for extreme or near extreme values of α (e.g., α = 0.99 in *f1* and *f2;* α = 0.80 in *f3*). This enables us to estimate upper bounds for all problem instances with 10, 15, 20, 25, 55, 81, 100, 130, 150 and 170 nodes. The UBs for problem instances with 10 and 15 nodes in Table 8 are obtained by CPLEX and RVNS (i.e., they are optimal solutions). For the rest of the problems, the upper bounds are the solutions to the RSAPHLPM problems with fixed discount factor provided by the RVNS.

The %UB improvement compares the UBs (6th column) and the solutions to the RSAPHLPM with flow dependent cost also obtained by the RVNS (3nd column-RVNS).

Reliable models with and without flow dependent cost. In Figure 8, we depict solutions obtained using linear RF2-RSAPHLPM (the model with flow dependent cost) and RF3-RSAPHLPM (the model with fixed discount factor). We solved RF3-RSAPHLPM for three values of α (0.8, 0.5 and 0.2) and RF2-RSAPHLPM for three concave piecewise linear functions (see Table 3) representing moderate, intermediate and aggressive scale economies. Analysing the solutions presented in Figure 8, as expected, the value of the objective function decreases as smaller values of α and functions with more aggressive slopes are employed. The proposed network topologies in solutions with fixed α appears to be sensitive to lower discount factors i.e., higher value of α. In those networks where the value of α decreases from 0.5 to 0.2 the hub locations, demand allocations and backups remain unchanged. In contrast, it is worth noting that in the solutions with flow dependent discount factors, the network topology is more sensitive to the use of functions with intermediate to aggressive scale economy (i.e., lower values of slopes). Comparing the values of the objective function returned by the model with fixed α and the one with the model with flow dependent α, it is clear that the total transportation costs are underestimated in the solutions obtained by the model that utilises a fixed α. For example, to calculate the total transportation cost of the network (Figure 8-e) in normal operation, the proposed model recommends the use of three different slopes in inter-hub links 4-7, 4-9 and 9-7. The utilised slopes for the three inter-hub links are 0.8, 0.4 and 1.0 respectively.

As the amount of flow transported in some inter-hub links increases should any of the operating hubs in the same network become disrupted, the model adopts new slopes of 0.6, 0.4 and 0.8 to respond to the increased flow being transported in those links. Whereas in all solutions provided by the model with fixed α, the inter-hub links are treated the same way regardless of the situation (normal or disruption) and irrespective of the amount of flow being transported. This is an important observation that needs to be considered at the strategic stage of locating the chosen hubs.

6.2.5. Trade-off Curves

The objective function of the proposed reliable model RF2-RSAPHLPM minimises the total transportation and hub installation cost in normal (*Obj.1*) and disrupted (*Obj.2*) conditions. The model could be presented as a bi-objective program that minimises a weighted sum $\omega \times Obj.1 + (1-\omega) \times Obj.2$ of the two objectives where $0 \leq \omega \leq 1$. Solving the model for different values of ω and a given piecewise linear function, a trade-off curve between the network cost in normal and disrupted conditions could be generated. For illustration, we constructed three trade-off curves for the RSAPHLPM using the 20 nodes problem instance from CAB dataset and the three piecewise linear functions of *f1*, *f2* and *f3*. The results are depicted in Figure 9. The horizontal and vertical axes outline the (expected) network cost in normal and disrupted conditions respectively. The first point (from left) on each curve represent a solution to RSAPHLPM with ω=1 and the last point represent a solution with ω=0.05. The three trade-off curves

Table 5. Computational results for RVNS: RSAPHLPM (small and medium instances)

Problem instance		Best/optimal solution*	Time (sec)	Hubs & assignments *backup hubs*	Capacity	
					Normal [hub(size)]	**Disruption** [hub(size)- if disrupted]
cab10	*f1*	921381261*	<1	4 9 9 4 4 9 7 4 9 7 *9 4 4 0 9 4 0 7 0 4*	4(S)9(S)7(S)	4(M)-9;4(S)-7;7(S)-4;7(S)-9;9(S)-4;9(S)-7
	f2	904651569*	<1	4 9 9 4 4 9 7 4 9 7 *9 4 4 0 9 4 0 7 0 4*	4(S)9(S)7(S)	4(M)-9;4(S)-7;7(S)-4;7(S)-9;9(S)-4;9(S)-7
	f3	845120640*	<1	4 6 6 4 6 6 7 7 6 7 *6 4 4 0 4 0 0 4 4 4*	4(S)6(S)7(S)	4(M)-6;4(S)-7;7(S)-4;7(S)-6;6(S)-4;6(S)-7
cab15	*f1*	2901237982	5.8	5 5 5 4 5 5 11 11 4 11 11 11 5 5 4 *4 4 4 0 0 4 5 4 5 5 0* *4 11 4 11*	5(S)4(S)11(S)	5(S)-4;5(S)-11;4(M)-5;4(S)-11(S);11(S)-5;11(S)-4
	f2	2779810667	3.2	13 4 4 4 13 4 13 8 4 13 13 8 13 13 4 *4 13 13 0 4 13 8 0 13* *8 4 13 0 4 13*	13(S)4(S)8(S)	13(M)-4;13(S)-8;4(M)-13;4(S)-8;8(S)-13;8(S)-4
	f3	2592554585	5.0	13 4 4 4 13 4 13 8 4 13 13 8 13 13 4 *4 13 13 0 4 13 8 0 13* *8 4 13 0 4 13*	13(S)4(S)8(S)	13(M)-4;13(S)-8;4(M)-13;4(S)-8;8(S)-13;8(S)-4
cab20	*f1*	6451879841	12.3	4 18 18 4 4 4 7 7 4 7 4 7 4 18 4 7 18 18 7 18 *18 4 4 0 18 18 0 4 18* *4 7 4 18 4 18 4 4 0 4* *4*	4(S)18(S)7(S)	4(L)-18;4(S)-7;18(M)-4;18(M)-7;7(S)-4;7(S)-18
	f2	5809183976	11.8	4 18 18 4 4 4 7 7 4 7 4 7 4 18 4 7 18 18 7 18 *18 4 4 0 18 18 0 4 18* *18 7 4 7 4 18 4 4 0 4* *4*	4(S)18(S)7(S)	4(L)-18;4(S)-7;18(M)-4;18(S)-7;7(S)-4;7(S)-18
	f3	5253730319	23.5	4 18 18 4 4 4 7 7 4 7 4 7 7 18 4 7 18 18 7 18 *7 4 4 0 18 18 0 4 18* *4 7 4 4 7 7 4 4 0 4 4*	4(S)18(S)7(S)	4(M)-18;4(S)-7;18(M)-4;18(S)-7;7(S)-4;7(S)-18
cab25	*f1*	10104901940	10.0	4 2 2 4 4 2 4 4 4 4 4 12 4 2 4 4 2 2 12 2 4 12 12 2 2 *2 0 4 0 2 4 2 12 2 2* *2 0 12 4 2 2 4 4 4 4* *2 4 4 4 4*	4(S)2(S)12(S)	4(L)-2;4(S)-12;2(M)-4;2(S)-12;12(S)-4;12(S)-2
	f2	8857144503	26.12	4 18 18 4 4 4 4 4 4 4 4 12 4 18 4 4 18 18 12 18 4 12 12 18 18 *18 4 4 0 18 18 18 18* *18 18 18 0 18 4 18 18* *4 0 18 4 18 4 18 4 4*	4(S)18(S) 12(S)	4(L)-18;4(S)-12;18(L)-4;18(S)-12;12(S)-4;12(S)-18
	f3	7780167054	19.8	4 18 18 4 4 4 4 4 4 4 4 12 4 18 4 4 18 18 12 18 4 12 12 18 18 *18 4 4 0 18 18 18 18* *18 18 18 0 18 4 12 18* *4 0 4 4 18 4 4 4 4*	4(S)18(S) 12(S)	4(L)-18;4(S)-12;18(M)-4;18(S)-12;12(S)-4;12(S)-18

Table 5. continue

Problem instance		Best/optimal solution*	Time (sec)	Hubs & assignments *backup hubs*	Capacity	
					Normal [hub(size)]	**Disruption** [hub(size)- if disrupted]
tr10	*f1*	811736069*	2.7	1 2 2 1 5 2 1 5 1 5 *0 0 5 5 0 5 5 1 5 2*	1(S)2(S)5(S)	1(S)-2;1(S)-5;2(S)-1;2(S)-5;5(S)-1;5(S)-2
	f2	784598185*	<1	1 2 2 1 5 2 1 1 1 2 *0 0 5 5 0 5 5 5 5 5*	1(S)2(S)5(S)	1(S)-2;1(S)-5;2(S)-1;2(S)-5;5(S)-1;5(S)-2
	f3	715911193*	<1	1 2 2 1 1 2 7 1 7 2 *0 0 1 7 2 1 0 2 2 1*	1(S)2(S)7(S)	1(M)-2;1(S)-7;2(M)-1;2(S)-7;7(S)-1;7(S)-2
tr25	*f1*	20009637631	17.7	1 1 3 3 15 15 1 15 1 1 1 15 15 3 15 3 15 15 1 1 3 15 1 3 15 *0 3 0 15 1 3 3 3 3 3 3 3 3 1 0 1 1 3 3 3 15 3 3 15 3*	1(S)3(S)15(S)	1(S)-3;1(S)-15;3(S)-1;3(M)-15;15(S)-1;15(M)-3
	f2	17011148770	22.5	1 1 3 3 15 15 1 15 1 1 1 15 15 3 15 3 15 15 1 1 3 15 1 3 15 *0 15 0 15 3 3 3 3 3 3 3 3 3 1 0 1 3 3 3 3 15 3 3 15 3*	1(S)3(S)15(S)	1(S)-3;1(S)-15;3(S)-1;3(M)-15;15(M)-1;15(M)-3
	f3	14885007407	22.1	9 9 3 3 15 15 9 15 9 9 9 15 15 3 15 3 15 15 9 9 3 15 9 3 15 *3 3 0 15 3 3 3 3 0 3 3 3 3 9 0 15 3 3 3 15 15 9 3 9 3*	9(S)3(S)15(S)	9(S)-3;9(S)-15;3(S)-9;3(M)-15;15(M)-3;15(M)-9

Table 6. Computational results RVNS vs HPSO: RSAPHLPM problem

Problem instance		RVNS		HPSO		%Cost difference
		Best/optimal solution	Time (sec)	Best/optimal solution	Time (sec)	
cab10	*f1*	921381261	<1	921381261	<1	0
	f2	904651569	<1	*904651569*	<1	0
	f3	845120640	<1	*845120640*	<1	0
tr10	*f1*	811736069	2.7	811736069	1.7	0
	f2	784598185	<1	*784598185*	<1	0
	f3	715911193	<1	715911193	<1	0
cab15	*f1*	2901237982	5.8	2903024099	7.4	0.06
	f2	2779810667	3.2	*2819029214*	7.8	1.40
	f3	2592554585	5.0	*2650218030*	2.4	2.18
cab20	*f1*	6451879841	12.3	6586574486	15.8	2.04
	f2	5809183976	11.8	6028691753	16.6	3.60
	f3	5253730319	23.5	5506677408	20.0	4.59
cab25	*f1*	10112461057	9.7	11038422686	37.8	8.39
	f2	8859484032	27.4	9485522693	31.37	6.60
	f3	7775056414	23.1	*8480890437*	40	8.30
tr25	*f1*	20009637631	17.7	21740871911	40	7.96
	f2	17011148770	22.5	*19111442931*	37.6	10.99
	f3	14885007407	22.1	*17449319705*	40.0	14.70

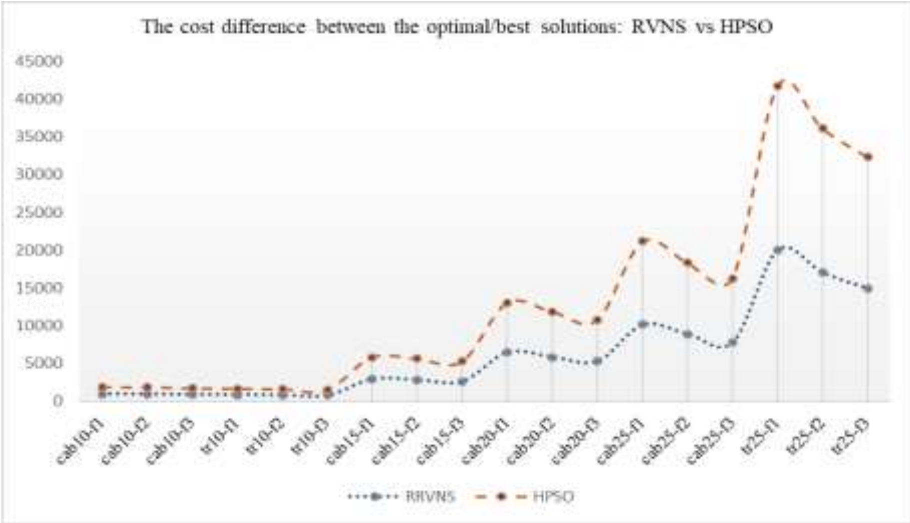
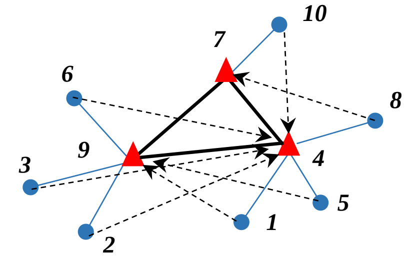


Figure 7. Comparison of the solution costs obtained by the RVNS and HPSO

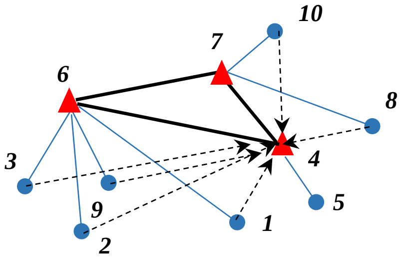
Table 7. Summary of the computational results for the RVNS: RSAPHLPM (large instances)

Problem instance		**Best solution found**	**Time (sec)**	**Hubs**
tr55	*f1*	34177996143	70	34 4 30
	f2	29198978757	52	34 4 30
	f3	25605287546	69	34 4 26
tr81	*f1*	60658500004	118	38 3 41
	f2	53032302897	54	46 6 41
	f3	47167485137	21	44 6 41
rgp100	*f1*	142305033291	866	5 3 100 52 63
	f2	136160884921	883	20 4 8 76 80
	f3	127844345945	836	1 3 79 53 52
rgp130	*f1*	248860360811	1299	79 94 112 12 37 50 20
	f2	228386662030	1300	26 71 37 106 56 20 11
	f3	215060898295	1300	10 94 5 118 127 13 74
rgp150	*f1*	316905136914	4200	119 105 122 106 11 111 69 95 66
	f2	296082212247	2075	11 13 122 89 83 86 119 116 87
	f3	277993079733	2400	57 92 68 32 27 86 82 149 30
rgp170	*f1*	390287053301	4200	77 131 108 7 156 113 64 123 22 33 146
	f2	369811366515	4200	94 155 136 71 27 108 163 151 32 61 164
	f3	349808328948	4200	94 92 124 19 5 123 93 88 126 47 146

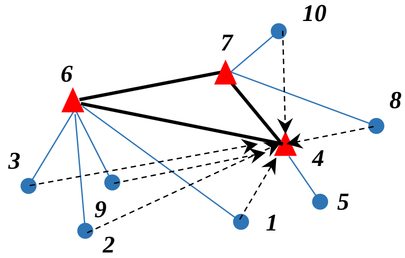
illustrate remarkable results. The curves are clearly "steep" indicating a large reduction in *Obj.2* (i.e., expected network cost) could be achieved with only small increase in *Obj.1* (i.e., the normal network cost). The first 10 solutions in the curve representing solutions with *f1* piecewise linear function (starting with the solution obtained with ω=1) are listed in Table 9. Managers are unlikely to accept large increases in day-to-day operating cost (solutions in right of the trade-off curves) to improve network reliability but might be willing to spend 2% more to reduce the expected network cost by 32% (9th solution in Table 9).



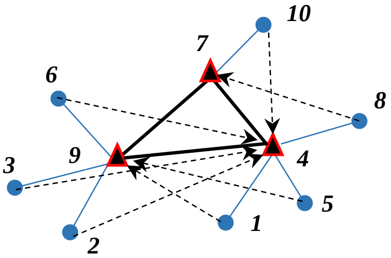
a) α = 0.8; obj: 880935159



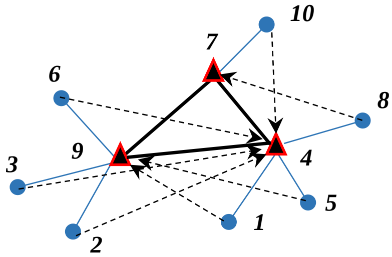
b) α = 0.5; obj: 779505164



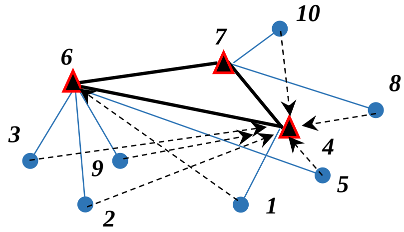
c) α = 0.2; obj: 670623945



d) f1; obj: 921381261



e) f2 ; obj: 904651569



f) f3 ; obj: 845120640

Figure 8. Comparison of networks made by models with and without flow dependent discount factors

Table 8. Estimated lower and upper bounds for all benchmark problems: RSAPHLPM problem

Problem instance		RVNS	CPLEX	Lower Bound (CF-CSAHPLPM)	Upper Bound (RF3-RSAPHLPM)	%GAP(RVNS vs CPLEX/ LB)	%UB Improvement
cab10	*f1*	921381261*	921381261*	908463496	923817005	0	-
	f2	904651569*	904651569*	890999439	923817005	0	-
	f3	845120640*	845120640*	829246258	866641210	0	-
tr10	*f1*	811736069*	811736069*	802573860	823026752	0	-
	f2	784598185*	784598185*	774825101	823026752	0	-
	f3	724133151*	724133151*	707446344	765220126	0	-
cab15	*f1*	2901237982	-	2868296327	2929587177	1.1	1.0
	f2	2779810667	-	2736792284	2929587177	1.5	5.1
	f3	2592554585	-	2528995100	2780188705	2.5	6.7
cab20	*f1*	6451879841	-	6358396687	6782480233	1.4	4.9
	f2	5809183976	-	5662188829	6782480233	2.5	14.4
	f3	5253730319	-	5007619054	6266798949	4.7	16.2
cab25	*f1*	10104901940	-	9868585775	10723982767	2.4	5.7
	f2	8857144502	-	8524475034	10723982767	3.8	17.4
	f3	7780167054	-	7391365540	9807200045	4.9	20.7
tr25	*f1*	20009637631	-	19690928564	21954170964	1.6	8.9
	f2	17011148770	-	16627593981	21954170964	2.3	22.5
	f3	14885007407	-	14343357648	19899361591	3.6	25.2
tr55	*f1*	34177996143	-	-	37100361827	-	7.9
	f2	29198978757	-	-	37100361827	-	21.3
	f3	25605287546	-	-	34117994191	-	25.0
tr81	*f1*	60658500004	-	-	64850309011	-	6.5
	f2	53032302897	-	-	64850309011	-	18.2
	f3	47167485137	-	-	60569224181	-	22.1
rgp100	*f1*	142305033291	-	-	143675567040	-	1.0
	f2	136160884921	-	-	143675567040	-	5.2
	f3	127844345945	-	-	142333944538	-	10.2
rgp130	*f1*	248860360811	-	-	262035124539	-	5.0
	f2	228386662030	-	-	262035124539	-	12.8
	f3	215060898295	-	-	256174176743	-	16.0

Table 8. Continued

Problem instance		RVNS	CPLEX	Lower Bound (CF-CSAHPLPM)	Upper Bound (RF3-RSAPHLPM)	%GAP(RVNS vs CPLEX/ LB)	%UB Improvement
rgp150	*f1*	316905136914	-	-	363026747683	-	12.7
	f2	296082212247	-	-	363026747683	-	18.4
	f3	277993079733	-	-	355059496381	-	21.7
rgp170	*f1*	390287053301	-	-	477780897813	-	18.3
	f2	369811366515	-	-	477780897813	-	22.6
	f3	349808328948	-	-	466130632708	-	25.0

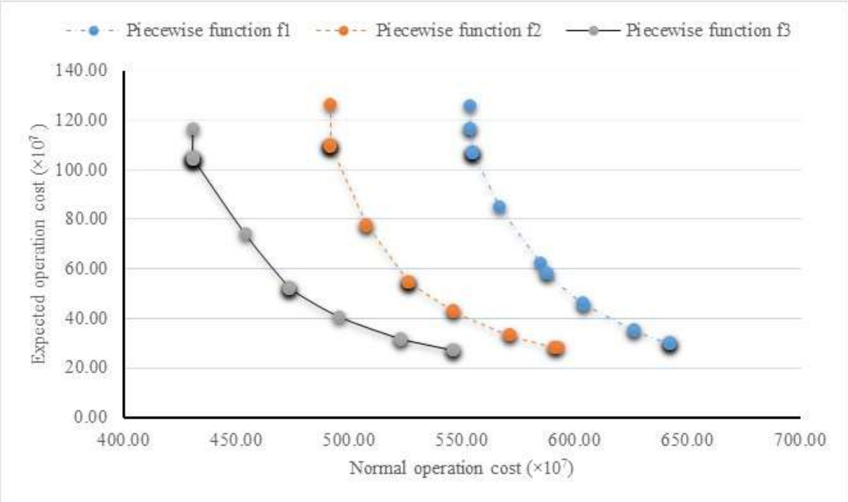


Figure 9. Trade-off curves for 20 nodes problem from CAB dataset

Table 9. First 10 solutions in trade-off curve with "piecewise function *f1*"

Solution	Obj.1 ($\times 10^7$)	Obj.2 ($\times 10^7$)	%Increase *Obj.1*	%Decrease *Obj.2*
1	553.53	125.58	-	-
2	553.53	116.32	0.00	7.37
3	553.53	116.37	0.00	7.33
4	554.94	106.62	0.25	15.09
5	554.94	106.57	0.25	15.13
6	554.94	106.62	0.25	15.09
7	554.94	106.89	0.25	14.88
8	554.94	106.75	0.25	14.99
9	566.84	84.96	2.35	32.34
10	584.87	62.18	5.36	50.49

7. Conclusions and Suggestions

In this research, we investigate strategies for the design of reliable and resilient hub and spoke systems under random hub failure. In our approach, every demand point in the network will have a backup hub and backup route designed to maintain network operations if a hub failure occurs. In case of network disruption, the existing hubs may need to cope with a significant amount of additional flow that needed to be rerouted through these hubs. This fact highlights the importance of available hub capacities in disruption situation especially in cases where backup facilities and rerouting the traffic are considered. In our proposed approach we model reliable hub and spoke systems with multiple capacity levels. Our proposed models differentiate between the required hub capacities in normal and in disruption cases. This will allow managers to estimate the lowest and the highest level of capacities needed to satisfy both existing and the expected demand should any hub failure occur in future.

Unlike other studies that concern reliability and resilience within the context of hub and spoke, we propose models that measure the inter-hub transportation cost using flow dependent discount factors. As the volume of the flow being transported/transmitted on inter-hub links may differ in normal operation and under disruption, our approach adds an invaluable attribute by considering situation-dependent discount factors. This is achieved by having one for each link under normal operations and another for the case of disruption. The proposed linear model could solve small instances using CPLEX. For larger instances, we build a simple but efficient RVNS based algorithm that considers the insight of the problem. We present computational results for existing datasets (up to 81 nodes) and newly generated problem instances up to 170 nodes. We also highlight the efficiency and effectiveness of our proposed algorithm by

(1) solving instances of F1-CSAPHLPM (i.e., single allocation p-hub location problem with multiple capacity levels and flow discount factors) all to optimality in a very short computing times (2) comparing its performance against a hybrid PSO (HPSO) and (3) reporting tight (i.e., relatively very low) gaps between our RVNS results and the lower bounds for medium instances with 25 nodes, as well as gaps based on lower and upper bounds for the larger instances.

We provide an interesting comparison of the networks obtained from formulations with and without flow dependent discount factors. The result of our analysis reinforces previous research finding in that in solutions with fixed discount factors the actual network cost is underestimated. We also highlight an important observation that the optimal solution of a reliable hub and spoke problem with and without flow dependent discount factor may differ significantly in terms of hub location, demand allocation and/ backup selections and all together. This is an important ad strategic fact that senior management could not ignore. We also conduct a scenario analysis by further modifying the objective of the proposed reliable model to minimise the weighted sum of two objectives (i.e., the network cost in normal and in disrupted conditions). A plot of the trade-off curves for a problem instance with 20 nodes is used as an example to empirically demonstrate that significant improvements in reliability could be achieved with small increase in the network cost in normal condition.

The following research avenues would be worthwhile exploring. One way would be to develop efficient exact methods that could be used to solve large instances of the problem. This could be achieved by introducing efficient valid inequalities. Furthermore, the way the discount factor is expressed in our formulations is a good and correct representation of economies of scale though it leads to intractable model especially for large instances. The use of e.g., linear expressions that could properly characterise and model economies of scale would reduce model's complexity and increase its tractability. In term of heuristic search we adopted a variation of reduced VNS in a simplest form but this metaheuristic could be made stronger by generating for each neighbourhood p_k neighbours instead (k=1,…,k_{max}) where p_k needs to be identified adaptively. The proposed RVNS could be also hybridised even more with large neighbourhood search resulting in a more powerful adaptive search engine.

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Appendix A. Model transformation and RF2-RSAPHLPM nonlinear formulation

Transformation of the quadratics terms - Beginning with the first quadratic term in the objective function. This expression calculates the transportation cost of the flow in spoke links in normal operations. It is written as

$$\sum_i \sum_k \sum_m \sum_n \sum_j w_{ij} \left(c_{ik} + c_{mj}\right) x_{ikmj} z_{nn} q_n$$

Now let define the continuous axillary variable D^1_{ijn} as

$$D^1_{ijn} = \sum_k \sum_m w_{ij} \left(c_{ik} + c_{mj}\right) x_{ikmj} \qquad \forall\, i, j, n \qquad (1 - A)$$

Then the above expression in the objective function could be re-written as

$$\sum_i \sum_j \sum_n D^1_{ijn}\, z_{nn} q_n$$

Similarly, each of the following quadratic terms in the objective function and those in constraints could be replaced by a product of a continuous and binary variables as shown below in order of their appearance in the model

$$\sum_r \sum_k \sum_m \sum_n c_{km} \left(a^r_{km} g^r_{km} + b^r_{km} y^r_{km}\right) z_{nn} q_n \equiv \sum_r \sum_n \left(T^r_n + S^r_n\right) z_{nn} q_n$$

$$\sum_k \sum_{l \in L_k} \sum_n F^l_k\, h^l_k z_{nn} q_n \equiv \sum_k \sum_{l \in L_k} \sum_n \Omega^l_k\, z_{nn} q_n$$

$$\sum_i \sum_{k \neq n} \sum_{m \neq n} \sum_n \sum_j w_{ij} \left(c_{ik} + c_{mj}\right) x_{ikmj} z_{nn} q_n \equiv \sum_i \sum_j \sum_n D^2_{ijn}\, z_{nn} q_n$$

$$\sum_{\substack{i \\ i \neq k}} \sum_{\substack{k \\ k \neq m}} \sum_m \sum_n \sum_j w_{ij} \left(c_{in} + c_{mj}\right) u_{ikn} z_{jm} q_k$$

$$\equiv \sum_{\substack{i \\ i \neq k}} \sum_{\substack{k \\ k \neq m}} \sum_m \sum_n \sum_j \left(c_{in} + c_{mj}\right) v_{iknj} z_{jm} q_k$$

$$\sum_i \sum_k \sum_m \sum_n \sum_j w_{ij} \left(c_{in} + c_{mj}\right) u_{ikn} u_{jkm} q_k$$

$$\equiv \sum_i \sum_k \sum_m \sum_n \sum_j \left(c_{in} + c_{mj}\right) v_{iknj} u_{jkm} q_k$$

$$\sum_i \sum_j w_{ij}\, x_{inmj}\; z_{kk} \equiv H_{nm} z_{kk}$$

$$\sum_i \sum_j w_{ij}\, z_{in} z_{kk} \equiv B_n z_{kk}$$

$$\sum_i \sum_j w_{ij}\, u_{jkm}\; z_{in} \equiv \sum_j E_{nj}\, u_{jkm}$$

Where

$$D_{ijn}^2 = \sum_{k \neq n} \sum_{m \neq n} w_{ij}\left(c_{ik} + c_{mj}\right) x_{ikmj} \qquad \forall\, i, j, n \tag{2-A}$$

$$\Omega_k^l = F_k^l h_k^l \qquad \forall\, k, l \tag{3-A}$$

$$T_n^r = \sum_k \sum_m c_{km}\, a_{km}^r g_{km}^r \qquad \forall r, n \tag{4-A}$$

$$S_n^r = \sum_k \sum_m c_{km}\, b_{km}^r y_{km}^r \qquad \forall r, n \tag{5-A}$$

$$H_{nm} = \sum_i \sum_j w_{ij}\, x_{inmj} \quad \forall\, n, m \tag{6-A}$$

$$v_{iknj} = w_{ij} u_{ikn} \qquad \forall\, i, k, n, j \tag{7-A}$$

$$B_n = \sum_i \sum_j w_{ij}\, z_{in} \qquad \forall n \tag{8-A}$$

$$E_{nj} = \sum_i w_{ij}\, z_{in} \quad \forall j, n \neq j \tag{9-A}$$

The improved nonlinear RF2-RSAPHLPM formulation

$$min\left(\sum_i \sum_k \sum_m \sum_j w_{ij}(c_{ik} + c_{mj}) x_{ikmj} - \sum_i \sum_j \sum_n D_{ijn}^1\, z_{nn} q_n \;\; + \right.$$

$$\sum_r \sum_k \sum_m c_{km}\,(a_{km}^r g_{km}^r + b_{km}^r y_{km}^r) - \left(\sum_r \sum_n (T_n^r + S_n^r)\, z_{nn} q_n\right) +$$

$$\left.\sum_k \sum_{l \in L_k} F_k^l\, h_k^l - \sum_k \sum_{l \in L_k} \sum_n \Omega_k^l\, z_{nn} q_n\right) +$$

$$\left(\sum_k \sum_{\substack{n \\ n \neq k}} \sum_l F_k^l\, \psi_{nk}^l\, q_k + \sum_i \sum_j \sum_n D_{ijn}^2\, z_{nn} q_n \right. +$$

$$2\left(\sum_{\substack{i \\ i \neq k}} \sum_{\substack{k \\ k \neq m}} \sum_m \sum_n \sum_j \left(c_{in} + c_{mj}\right) v_{iknj} z_{jm} q_k\right) +$$

$$\sum_i \sum_k \sum_m \sum_n \sum_j \left(c_{in} + c_{mj}\right) v_{iknj} u_{jkm} q_k +$$

$$\sum_{\substack{m \\ m\neq n}} \sum_{\substack{n \\ n\neq k}} \sum_{\substack{k \\ k\neq m}} \sum_{r} c_{mn} \left(a^r_{mn} f^r_{mnk} + b^r_{mn} \Theta^r_{mnk} \right) q_k +$$

$$\left. \sum_i \sum_j \varphi_{ij} \, w_{ij} \left(q_i z_{ii} + q_j z_{jj} \right) \right) \qquad (10-A)$$

s.t

(2)-(11); (2-A); (1-A)-(9-A)

$$\sum_{n\neq k} u_{ikn} = z_{ik} \qquad \forall k, i \neq k \qquad (11-A)$$

$$u_{ikn} \leq z_{nn} \qquad \forall k, n, i \neq k \qquad (12-A)$$

$$\sum_{l \in L_k} \psi^l_{nk} = x_{nnkk} \qquad \forall\, k, n \qquad (13-A)$$

$$f^r_{mnk} \leq \Theta^r_{mnk} \sum_i \sum_j w_{ij} \qquad \forall k, m, r, n \neq m \neq k \qquad (14-A)$$

$$B_n z_{kk} + \sum_{\substack{i \\ i\neq k}} \sum_{\substack{j \\ j\neq k}} w_{ij} u_{ikn} \leq \sum_{l \in L_k} \Gamma^l_n \, \psi^l_{nk} \qquad \forall n, k \qquad (15-A)$$

$$\sum_r f^r_{mnk} = H_{nm} z_{kk} + \sum_j \sum_j v_{iknj} \, z_{jm} + \sum_j E_{nj} \, u_{jkm} \; +$$

$$\sum_j \sum_j v_{iknj} \, u_{jkm} \qquad \forall k, m, n \neq m \neq k \in N \qquad (16-A)$$

$$\sum_r \Theta^r_{mnk} \leq x_{nnmm} \; z_{kk} \qquad \forall k, m, n \neq m \neq k \qquad (17-A)$$

$$z_{ik}, h^l_k\,, \psi^l_{nk}, y^r_{km}, \Theta^r_{mnk}, x_{ikmj}, u_{ikn} \in \{0,1\} \quad \forall i, n, m, k, r, l \qquad (18-A)$$

$$D^2_{ijn}, g^r_{km}, f^r_{mnk} \geq 0 \qquad \forall rk, m, n \neq m \neq k \qquad (19-A)$$

$$D^1_{ijn}, H_{nm}, \Omega^l_k, T^r_n, S^r_n, B_n, v_{iknj} \geq 0 \qquad \forall i, j, k, n, r, l \qquad (20-A)$$

$$E_{nj} \geq 0 \quad \forall j, n \neq j \qquad (21-A)$$

Equalities (1-A)-(9-A) defines the auxiliary variables used to transfer the original model (Appendix I) into a convex MINLP. Constraints (18A)-(21-A) enforce the binary condition and indicate the non-negativity restriction of the utilised variables.

Appendix B. Reliable single allocation p-hub location problem with multiple capacity levels and fixed discount factor: RF3-RSAPHLPM nonlinear formulation.

$$min\left(\sum_{i}\sum_{k}\sum_{m}\sum_{j} w_{ij}(c_{ik}+c_{mj})x_{ikmj}+\sum_{k}\sum_{m}\alpha c_{km}\, g_{km}+\right.$$

$$\left.\sum_{k}\sum_{l\in L_k} F_k^l\, h_k^l\right)\left(1-\left(\sum_{n} q_n z_{nn}\right)\right)+$$

$$\left(\sum_{k}\sum_{\substack{n\\n\neq k}}\sum_{l} F_k^l\, \psi_{nk}^l\, q_k+\sum_{i}\sum_{k\neq n}\sum_{m\neq n}\sum_{n}\sum_{j} w_{ij}\left(c_{ik}+c_{mj}\right)x_{ikmj}z_{nn}q_n+\right.$$

$$2\left(\sum_{\substack{i\\i\neq k}}\sum_{\substack{k\\k\neq m}}\sum_{m}\sum_{n}\sum_{j} w_{ij}\left(c_{in}+c_{mj}\right)u_{ikn}z_{jm}q_k\right)+$$

$$\sum_{i}\sum_{k}\sum_{m}\sum_{n}\sum_{j} w_{ij}\left(c_{in}+c_{mj}\right)u_{ikn}u_{jkm}q_k+$$

$$\left.\sum_{\substack{k\\k\neq m}}\sum_{\substack{m\\m\neq n}}\sum_{\substack{n\\n\neq k}}\alpha c_{nm}\left(f_{mnk}\right)q_k+\sum_{i}\sum_{j}\varphi_{ij}\, w_{ij}\left(q_i z_{ii}+q_j z_{jj}\right)\right) \qquad (1-B)$$

s.t

$$\sum_{k} z_{ik}=1 \qquad i\in N \qquad (2-B)$$

$$\sum_{k} z_{kk}=p \qquad (3-B)$$

$$z_{ik}\leq z_{kk} \qquad i,k\ \in N \qquad (4-B)$$

$$\sum_{m} x_{ikmj}=z_{ik} \qquad i,j,k\in N \qquad (5-B)$$

$$\sum_{k} x_{ikmj}=z_{jm} \qquad i,j,m\in N \qquad (6-B)$$

$$\sum_{l} h_k^l=z_{kk} \qquad k\in N \qquad (7-B)$$

$$\sum_{n\neq k} u_{ikn}=z_{ik} \qquad i,i\neq k\ ,k\in N \qquad (8-B)$$

$$u_{ikn}\leq z_{nn} \qquad i,i\neq k,k,n\ \in N \qquad (9-B)$$

$$\sum_{l \in L_k} \psi_{nk}^{l} = x_{nnkk} \qquad k, n \in N \qquad (10-B)$$

$$\left(\sum_{i} \sum_{j} \sum_{m} w_{ij}\, x_{ikmj} \right) \leq \sum_{l \in L_k} \Gamma_k^l\, h_k^l \qquad k \in N \qquad (11-B)$$

$$\sum_{i} \sum_{j} w_{ij}\, x_{inkk} + \sum_{\substack{i \\ i \neq k}} \sum_{\substack{j \\ j \neq k}} w_{ij} u_{ikn} \leq \sum_{l \in L_k} \Gamma_n^l\, \psi_{nk}^l \qquad n, k \in N \qquad (12-B)$$

$$g_{km} = \sum_{i} \sum_{j} w_{ij}\, x_{ikmj} \qquad k \neq m \in N \qquad (13-B)$$

$$f_{mnk} = \sum_{i} \sum_{j} w_{ij}\, x_{inmj}\; z_{kk} + \sum_{i} \sum_{j} w_{ij}\, u_{ikn}\; z_{jm} + \sum_{i} \sum_{j} w_{ij}\, u_{jkm}\; z_{in} +$$

$$\sum_{j} \sum_{j} w_{ij}\, u_{ikn} u_{jkm} \qquad n \neq m \neq k \in N \qquad (14-B)$$

$$z_{ik}, h_k^l, \psi_{nk}^l, x_{ikmj}, u_{ikn} \in \{0,1\} \quad \forall i, n, m, k, r, l \qquad (15-B)$$

$$g_{km}, f_{mnk} \geq 0 \qquad \forall r, k, m, n \neq m \neq k \qquad (16-B)$$