MULTI-COMPONENT TODA LATTICE IN CENTRO-AFFINE $\mathbb{R}^n$

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Abstract. In this paper we use the group-based discrete moving frame method to study invariant evolutions in n-dimensional centro-affine space. We derive the induced integrable equations for invariants, which can be transformed to local and nonlocal multi-component Toda lattices under a Miura transformation, and hence establish their geometric realisations in centro-affine space.

Contents

1. Introduction

Integrable systems are closely linked with classical geometries. Geometric evolutions for curves in homogeneous spaces induce integrable flows for geometric invariants such as curvatures. The space of these invariants can be viewed as coordinates of the moduli space of curves under the group action. The moving frame approach leads to a natural description of its associated Hamiltonian structures defined on the moduli space.

A well-known example was given by Hasimoto [1]. He showed that a curve flow in Euclidean space, invariant under the Euclidean group, known as the Vortex-Filament flow, induces the nonlinear Shrödinger equation (NLS) for the curvature and torsion of the curve flow. The Vortex Filament flow is called an Euclidean realisation of NLS. Geometric realisations of other integrable systems such as the Korteweg-de Vries equation (KdV), modified KdV and sine-Gordon equation are derived in classical geometries. The method of group-based moving frame introduced by Fels and Olver[2, 3] has played a very important role in establishing the relations. There are many papers devoted to this topic. We refer to [4, 5] and the references of them.

In 2013, the method of group-based discrete moving frame was introduced by Mansfield, Marí-Beffa and Wang [6], which is essentially a sequence of moving frames with overlapping domains. It provides a powerful tool to study the link between induced completely integrable systems on discrete curvatures (or invariants) and the invariant evolutions of polygons in different geometric settings. As examples, they derived the projective $\mathbb{RP}^2$ and 2-dimensional centro-affine $\mathbb{R}^2$ discrete realisations of the modified Volterra and Toda lattices. Such study was soon extended to projective polygons in $\mathbb{RP}^n$ in [7], establishing a close relationship between the projective invariant evolutions and the Hamiltonian evolutions on the invariants of the flow.

The induced flows for invariants from geometric evolutions for curves in classical geometries can be viewed as a syzygy between differential and difference invariants [8, 9], which offers a great advantage in direct computation of the Euler-Lagrange equations in terms of invariants from given invariant Lagrangians.

This paper is devoted to the study of invariant evolutions in centro-affine $\mathbb{R}^n$ and induced integrable systems. In 3-dimensional centro-affine case, the authors of [10] studied the geometric realisations of the B-Toda lattice and C-Toda. Recently Beffa and Calini investigated the evolutions of arc length-parametrised polygons (corresponding to the case $p_s = 1$ in Section 4) in n-dimensional centro-affine space, which can be identified with the case of projective $\mathbb{RP}^{n-1}$ [11]. They proved that the Poisson brackets derived in [7] form a bi-Hamiltonian pair.

In this paper, we are going to derive the induced integrable equations from invariant evolutions in n-dimensional centro-affine space and to establish their geometric realisations. The paper is organized as follows. In section 2, we review basic facts on discrete moving frames and invariants evolutions mainly based on [6, 7]. In section 3, we give a brief introduction on multi-component Toda lattice for both local and nonlocal flows. Our main results are in section 4. We use the approach of discrete moving frame to derive the flow of invariants for a given invariant time evolution. In 3-dimensional centro-affine space, we construct a Hamiltonian pair which generate both local and nonlocal integrable differential-difference systems, which can be transformed into 3-component local and nonlocal Toda lattices under the same Miura transformation. In the general n-dimension case, although it is hard to give the Hamiltonian pair explicitly, we write down the integrable differential-difference systems, which are multi-component local and nonlocal Toda lattices.

2. Discrete moving frame and invariant evolutions

In this section we will describe basic definitions and theorems on discrete group-based moving frames and invariant evolutions. We only state results (without proofs) for the left group action and the right discrete moving frame, which are taken from [6, 7]. We refer the readers to the original papers for the details.

2.1. **Discrete moving frames.** Let M be an $n-$dimensional manifold and $G \times M \to M$ be a left action of an $r-$dimensional Lie group G on M.

We begin with a discrete analogue of the m^{th} order submanifold jet bundle introduced in [12]. Assume that $x : \mathbb{Z} \to M$ is a discrete function. Here we use the subscript notation $x_s = x(s)$ to denote the evaluation of x at the integer point $s \in \mathbb{Z}$. The collection of $m + 1$ points

$$x_s^{[m]} = (x_s, x_{s+1}, \cdots , x_{s+m}), \quad x_{s+i} \in M,\ i = 0 \cdots m$$

is the m^{th} order forward discrete jet at $s \in \mathbb{Z}$ denoted by $(s, x_s^{[m]})$. Then the m^{th} order forward discrete jet space $J^{[m]}$ is defined as the collection of $(s, x_s^{[m]})$, that is,

$$J^{[m]} = \bigcup_{s\in\mathbb{Z}} (s, x_s^{[m]}).$$

Let $\pi^m : J^{[m]} \to \mathbb{Z}$ denote the projection onto the discrete index

$$\pi^m(s, x_s^{[m]}) = s.$$

Then for each $s \in \mathbb{Z}$, the fiber $J^{[m]}|_s = (\pi^m)^{-1}(s) \simeq M^{m+1}$ is a smooth manifold when $m \geqslant 0$. Naturally we can extend the action of G on M to $J^{[m]}$ as follows:

$$g \cdot (s, x_s^{[m]}) = (s, g \cdot x_s, g \cdot x_{s+1}, \cdots, g \cdot x_{s+m}). \tag{1}$$

Definition 2.1. (Discrete moving frame) A discrete right (resp. left) moving frame is a $G-$equivariant map $\rho : J^{[m]} \to G$ satisfying

$$\rho(s, g \cdot x_s^{[m]}) = \rho(s, x_s^{[m]})g^{-1},$$

(resp. $g \cdot \rho(s, x_s^{[m]})$) for all $g \in G$.

For simplicity we use the notation ρ_s to denote the image of the moving frame ρ at the point $(s, x_s^{[m]})$. If ρ_s is a left moving frame, then ρ_s^{-1} is a right moving frame. As in the continuous case, the construction of a discrete moving frame is based on the choice of the cross-section. The cross section is not unique. One adequate cross section can simplify the computation. We use $\mathcal{K}_s$ to denote the cross-sections over s on $J^{[m]}$. For a discrete moving frame, its cross-section over s is replicated for all other base points $s + i$, which means the cross-section over $s + i$ is represented by $\mathcal{K}_{s+i} = \mathcal{T}^i\mathcal{K}_s$ for all $i \in \mathbb{Z}$, where $\mathcal{T}$ is the shift operator. Consequently we have that $\rho_{s+i} = \mathcal{T}^i\rho_s$.

The discrete moving frames provide a powerful approach to construct discrete invariants. We say a function $F : J^{[m]} \to \mathbb{R}$ is a discrete invariant if

$$F(g \cdot x_s^{[m]}) = F(x_s^{[m]}), \quad \text{for all } g \in G \text{ and any } x_s^{[m]} \in J^{[m]}.$$

For a right moving frame, the quantities

$$I_{s,j} := \rho_s \cdot x_j, \tag{2}$$

are invariants. The induced action on the coordinate functions also produces discrete invariants, that is, for any difference function $F : J^{[m]} \to \mathbb{R}$, the induced action on it $F(s, \rho_s \cdot x_s^{[m]})$ is a discrete invariant. We are able to describe a smaller set of generating invariants, the Maurer-Cartan invariants.

Definition 2.2. (Discrete Maurer-Cartan invariant). Let $\rho : J^{[m]} \to G$ be a right moving frame. The element of the group

$$K_s = \rho_{s+1}(\rho_s)^{-1} \tag{3}$$

is called the right Maurer-Cartan matrix for ρ.

The equivariance of ρ implies that the K_s are invariant under the group action. In addition, using (2) and (3) we have

$$K_s \cdot I_{s,j} = \rho_{s+1}\rho_s^{-1} \cdot \rho_s \cdot x_j = \rho_{s+1} \cdot x_j = I_{s+1,j} \tag{4}$$

and iterating this, we have $K_{s+1}K_s \cdot I_{s,j} = I_{s+2,j}$, and so on. Hence, the components of K_s, together with the set of all diagonal invariants, $I_{j,j} = \rho_j \cdot x_j$, generate all other invariants [6].

2.2. **Invariant evolutions.** For an evolution equation

$$(x_s)_t = F_s((x_r)), \tag{5}$$

we say it is an invariant time evolution under the action of the group G if the group action takes solutions to solutions, that is, if (x_r) is a solution, so is $(g \cdot x_r)$ for any $g \in G$. Any invariant time evolution can be explicitly expressed in terms of the invariants and the moving frame.

Here we consider homogeneous manifolds $M = G/H$ with H a closed subgroup and assume that G acts on M via left multiplication on representatives of the class. The distinguished class of H is denoted by $o \in G/H$. Let ρ_s be the discrete right moving frame satisfying $\rho_s \cdot x_s = o$ for all s. We can describe the general formula for an invariant evolution (5) in terms of the moving frame [6, 7].

We denote by $\Gamma_g : G/H \to G/H$ the map defined by the action of $g \in G$, that is, $\Gamma_g(x) = g \cdot x$, and by $d\Gamma_g(x)$ the tangent map of Γ_g at $x \in G/H$. Any G-invariant evolution of the form (5) can be written as

$$(x_s)_t = d\Gamma_{\rho_s^{-1}}(o)(\mathbf{v}_s), \tag{6}$$

where $\mathbf{v}_s$ is an invariant vector in the tangent space of M at x_s.

For an invariant evolution (6) there is a simple process to describe the evolution induced on the Maurer-Cartan matrices, and hence on a generating set of invariants as stated in the following theorem. Its proof for the left discrete moving frame can be found [6].

Theorem 2.1. *Let $\varsigma : G/H \to G$ be a section of G/H such that $\varsigma(o) = e \in G$, where e is the identity. Given a right moving frame ρ_s, assume that $\rho_s \cdot x_s = o$ and $\rho_s = \rho_s^H \varsigma(x_s)^{-1}$, for some $\rho_s^H \in H$. Then the invariant evolution (6) leads to the structure equation*

$$(K_s)_t = N_{s+1}K_s - K_s N_s, \tag{7}$$

where K_s is the right Maurer-Cartan matrix and $N_s = (\rho_s)_t \rho_s^{-1} \in \mathfrak{g}$. Furthermore, assume $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, where $\mathfrak{g}$ is the algebra of G, $\mathfrak{h}$ is the algebra of H and $\mathfrak{m}$ is a linear complement (which can be identified with the tangent to the image of the section ς). Then, if $N_s = N_s^{\mathfrak{h}} + N_s^{\mathfrak{m}}$ splits accordingly,

$$N_s^{\mathfrak{m}} = -d\varsigma(o)\mathbf{v}_s. \tag{8}$$

In this paper, we are going to apply the above theorem to the centro-affine space. In fact, equation (7) and condition (8) completely determine the evolution of K_s [6, 7, 10]. Note that identity (7) is similar to the zero curvature condition (without the spectral parameter) for completely integrable systems. This is a key point when we link integrable systems to invariant evolutions.

3. Multi-component Toda lattices

We are going to link the invariant evolutions in centro-affine space with multi-component Toda lattices. To be self-contained, we will recall some facts on the Toda lattices in this section.

The well-known Toda lattice [13] is given by

$$\frac{\mathrm{d}^2 u_s}{\mathrm{d}t^2} = \exp(u_{s-1} - u_s) - \exp(u_s - u_{s+1}). \tag{9}$$

Here the dependent variable u is a function of time t and discrete variable $s \in \mathbb{Z}$. It can be viewed as a discretization of the Korteweg-de Vries equation. Using the Flaschka [14, 15] coordinates

$$q_s = \frac{\mathrm{d}u_s}{\mathrm{d}t}, \qquad p_s = \exp(u_s - u_{s+1}),$$

we rewrite the Toda lattice (9) in the form

$$\frac{\mathrm{d}p_s}{\mathrm{d}t} = p_s(q_s - q_{s+1}), \qquad \frac{\mathrm{d}q_s}{\mathrm{d}t} = p_{s-1} - p_s. \tag{10}$$

Its complete integrability was first established by Flaschka and Manakov [14, 15, 16]. Its Lax representation can be written as

$$\frac{\partial \mathcal{L}}{\partial t} = [B, \mathcal{L}] = B\mathcal{L} - \mathcal{L}B, \tag{11}$$

where

$$\mathcal{L} = \mathcal{T}^{-1} + q_s + p_s\mathcal{T}, \quad B = -\mathcal{L}_{\geq 1} = -p_s\mathcal{T}.$$

The above scalar Lax representation has been generalised to higher order difference operators involving in more dependent variables [17]. From now on, we will drop the down index s without causing confusion. For instance, we simply write p for p_s and p_i for p_{s+i}. Let

$$\mathcal{L} = \mathcal{T}^{-n} + \sum_{j=1}^{n} w^j \mathcal{T}^{-n+j} + u\mathcal{T}, \quad B = \mathcal{L}_{\geq 1} = u\mathcal{T}, \tag{12}$$

where u and $w^j, j = 1, 2, \cdots n$ are dependent variables. It follows from the Lax equation (11) that the $n+1$-component Toda lattice is of the form [17]

$$\begin{cases} w_t^1 &= u - u_{-n}, \\ w_t^j &= u w_1^{j-1} - w^{j-1} u_{-n+j-1}, \quad j = 2, \cdots, n, \\ u_t &= u(w_1^n - w^n). \end{cases} \tag{13}$$

When $n = 1$, it leads to the Toda lattice (10) by letting $u = p_s$ and $w^1 = -q_s$. For any fixed n, applying an r-matrix formalism, their bi-Hamiltonian structures can be constructed. In [17], the biHamiltonian structures for two and three fields are explicitly given.

From Lax operator (12), one can also derive a nonlocal multi-component Toda equation by taking its n-th root [18]. Here we write out the first two terms of this Laurent series in $\mathcal{T}$:

$$\mathcal{L}^{\frac{1}{n}} = \mathcal{T}^{-1} + \eta + \cdots, \qquad \eta = (1 + \mathcal{T}^{-1} + \cdots + \mathcal{T}^{1-n})^{-1} w^1.$$

Then the Lax flow is given by

$$\partial_t \mathcal{L} = [(\mathcal{L}^{\frac{1}{n}})_{\geq 1}, \mathcal{L}] = -[(\mathcal{L}^{\frac{1}{n}})_{\leq 0}, \mathcal{L}] = -[\mathcal{T}^{-1} + \eta, \mathcal{L}],$$

which leads to

$$\begin{cases} u_t &= u(\mathcal{T} - 1)\eta, \\ w_t^j &= w^{j+1} - w_{-1}^{j+1} - w^j(1 - \mathcal{T}^{-n+j})\eta, \quad j = 1, \cdots, n-1, \\ w_t^n &= u - u_{-1}, \quad \text{where } \eta = (1 + \mathcal{T}^{-1} + \cdots + \mathcal{T}^{1-n})^{-1} w^1. \end{cases} \tag{14}$$

In particular, if we take $n=2$, it leads to a local 3-component Toda equation[17]

$$\begin{cases} u_t &= u(w_1^2 - w^2), \\ w_t^1 &= u - u_{-2}, \\ w_t^2 &= uw_1^1 - u_{-1}w^1, \end{cases} \tag{15}$$

and a nonlocal 3-component Toda equation

$$\begin{cases} u_t &= u(\mathcal{T}-1)\mathcal{T}(1+\mathcal{T})^{-1}w^1, \\ w_t^1 &= w^2 - w_{-1}^2 - w^1(\mathcal{T}-1)(1+\mathcal{T})^{-1}w^1, \\ w_t^2 &= u - u_{-1}. \end{cases} \tag{16}$$

4. Invariant evolutions in centro-affine $\mathbb{R}^n$

Centro-affine geometry is obtained by deleting translations from the affine geometry. Let $G = SL(n,\mathcal{R})$ acts linearly on $M = \mathbb{R}^n$ as follows:

$$x \to \ g \cdot x,$$

where x is a n-vector, $g \in G$ and the product is the matrix multiplication. The n-dimensional centro-affine space $M = \mathbb{R}^n$ can be regarded as the homogeneous space $SL(n,\mathcal{R})/H$, where H is the isotropy subgroup of $e_1 = (1,0,\cdots,0)^T$, where the upper index T denotes the transpose of a matrix. We write

$$H = \begin{pmatrix} 1 & Y_{1\times(n-1)} \\ 0_{(n-1)\times 1} & A_{(n-1)\times(n-1)} \end{pmatrix}$$

and

$$G/H = \begin{pmatrix} x & 0_{1\times(n-1)} \\ y_{(n-1)\times 1} & B_{(n-1)\times(n-1)} \end{pmatrix},$$

where matrix $B_{(n-1)\times(n-1)} = \mathrm{diag}(1/x, 1, \cdots, 1)$ and (x, y^T) can be viewed as the n-coordinates on $M = \mathbb{R}^n$.

To construct the right frame ρ_s we take the normalization equation to be

$$\rho_s \cdot (x_s, x_{s+1}, x_{s+2}, \cdots, x_{s+n-1}) = (e_1, e_2, \cdots, (-1)^{n-1} p_s e_n), \tag{17}$$

where $x_{s+k} = (x_{s+k}^0, x_{s+k}^1, x_{s+k}^2, \cdots, x_{s+k}^{n-1})^T$ for $k = 0, 1, \cdots, n-1$ and

$$p_s = (-1)^{n-1}\det(x_s, x_{s+1}, x_{s+2}, \cdots, x_{s+n-1}).$$

This leads to the right Maurer-Cartan matrix

$$K_s = \rho_{s+1}\rho_s^{-1} = \begin{pmatrix} r_s^1 & 1 & 0 & \cdots & 0 & 0 \\ r_s^2 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \\ r_s^{n-2} & 0 & 0 & \cdots & 1 & 0 \\ r_s^{n-1} & 0 & 0 & \cdots & 0 & \frac{(-1)^{n-1}}{p_s} \\ p_s & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{18}$$

where $r_s^k = -\frac{\det(x_s,\cdots,x_{k-1+s},x_{s+n},x_{k+1+s}\cdots,x_{s+n-1})}{\mathcal{T}p_s}$ for $k = 1, \cdots, n-1$.

We will drop the down index s without causing confusion. For example, when $n=2$, the normalization equation (17) becomes

$$\rho \cdot (x, x_1) = (e_1, -pe_2), \quad p = -\det(x, x_1) = -(x^0 x_1^1 - x_1^0 x^1).$$

Solving it, we get the left moving frame

$$\rho^{-1} = (x, -\frac{x_1}{p}),$$

and thus the corresponding right Maurer-Cartan matrix is

$$K = \rho_1 \rho^{-1} = \begin{pmatrix} r^1 & -\frac{1}{p} \\ p & 0 \end{pmatrix}, \qquad r^1 = -\frac{\det(x, x_2)}{p_1}. \tag{19}$$

In this case, the link between invariant evolutions and integrable systems has been discussed in [6]. Next we will focus on the invariant evolutions and the related integrable systems when $n = 3$.

4.1. **The case of centro-affine $\mathbb{R}^3$.** It follows from (18) that the right Maurer-Cartan matrix in this case (after dropping the down index s) is

$$K = \rho_1 \rho^{-1} = \begin{pmatrix} r^1 & 1 & 0 \\ r^2 & 0 & \frac{1}{p} \\ p & 0 & 0 \end{pmatrix}, \tag{20}$$

where $p = \det(x, x_1, x_2), r^1 = -\frac{1}{p_1}\det(x, x_3, x_2), r^2 = -\frac{1}{p_1}\det(x, x_1, x_3)$ and

$$\rho^{-1} = (x, x_1, \frac{x_2}{p}). \tag{21}$$

From (6), the general invariant evolution is given by

$$(x)_t = \rho^{-1}\mathbf{v}, \qquad \mathbf{v} = \left(v^1, v^2, v^3\right)^T, \tag{22}$$

where v^1, v^2, v^3 are arbitrary functions of the invariants r^1, r^2 and p and their shifts. Using (21) and (22), we can easily see that the first column of N is $-\mathbf{v}$. From the structure equation (7) we obtain

$$N \quad = \quad \begin{pmatrix} -v^1 & \frac{-\mathcal{T}v^3}{p} & \frac{-p_1\mathcal{T}^2v^2 + r_1^2\mathcal{T}^2v^3}{p^2} \\ -v^2 & -\mathcal{T}v^1 + \frac{r^1}{p}\mathcal{T}v^3 & \frac{r^1}{p^2}(p_1\mathcal{T}^2v^2 - r_1^2\mathcal{T}^2v^3) - \frac{\mathcal{T}^2v^3}{pp_1} \\ -v^3 & -p\mathcal{T}v^2 + r^2\mathcal{T}v^3 & v^1 + \mathcal{T}v^1 - \frac{r^1}{p}\mathcal{T}v^3 \end{pmatrix},$$

and the evolution equation

$$\frac{d}{dt}\left(p, r^1, r^2\right)^T = \mathrm{P}\mathbf{v}, \tag{23}$$

where

$$\mathrm{P} = \begin{pmatrix} p(1 + \mathcal{T} + \mathcal{T}^2) & -r^2p_1\mathcal{T}^2 & r^2r_1^2\mathcal{T}^2 - r^1\mathcal{T} - \frac{r_1^1p}{p_1}\mathcal{T}^2 \\ r^1(1 - \mathcal{T}) & 1 - \frac{pp_2}{p_1^2}\mathcal{T}^3 & \frac{pr_2^2\mathcal{T}^3}{p_1^2} - \frac{r^2\mathcal{T}^2}{p_1} \\ r^2(1 - \mathcal{T}^2) & \frac{r_1^1pp_2}{p_1^2}\mathcal{T}^3 - r^1\mathcal{T} & \frac{1}{p} + \frac{r_1^1r^2}{p_1}\mathcal{T}^2 - \frac{p\mathcal{T}^3}{p_1p_2} - \frac{pr_1^1r_2^2\mathcal{T}^3}{p_1^2} \end{pmatrix}. \tag{24}$$

Remark 4.1. Under the transform $p = p_k, r^1 = q_kp_k, r^2 = -r_k$, we can get the corresponding evolution equations in [10].

Let us define a matrix

$$C = \begin{pmatrix} (\mathcal{T}^2 - 1)^{-1}p & -(\mathcal{T} + 1)^{-1}r^1 & 0 \\ 0 & -\mathcal{T}^{-1}r^2 & -\mathcal{T}^{-2}\frac{p}{p_1} \\ 0 & -\mathcal{T}^{-1}p & 0 \end{pmatrix}. \tag{25}$$

Then we compute operator multiplication PC and obtain the following pseudo-difference anti-symmetric operator

$$\begin{pmatrix} p(1+\mathcal{T}+\mathcal{T}^2)(\mathcal{T}^2-1)^{-1}p & p\mathcal{T}(\mathcal{T}+1)^{-1}r^1 & pr^2 \\ -r^1(\mathcal{T}+1)^{-1}p & \begin{matrix} r^1(\mathcal{T}-1)(\mathcal{T}+1)^{-1}r^1 \\ -\mathcal{T}^{-1}r^2+r^2\mathcal{T} \end{matrix} & \frac{p}{p_1}\mathcal{T}-\mathcal{T}^{-2}\frac{p}{p_1} \\ -pr^2 & \frac{p}{p_1}\mathcal{T}^2-\mathcal{T}^{-1}\frac{p}{p_1} & \frac{r^1}{p}\mathcal{T}^{-1}p-p\mathcal{T}\frac{r^1}{p} \end{pmatrix}. \tag{26}$$

In fact, this operator is a Hamiltonian operator as stated in the following theorem.

Theorem 4.1. *The operator $\mathcal{H}=PC$ given by (26) is a Hamiltonian operator and it forms a Hamiltonian pair with*

$$\mathcal{H}^0=\begin{pmatrix} 0 & 0 & p \\ 0 & \mathcal{T}-\mathcal{T}^{-1} & 0 \\ -p & 0 & 0 \end{pmatrix}. \tag{27}$$

Proof. Let us introduce the transformation

$$u=\frac{p}{p_1}, v=-r^2, w=r^1, \tag{28}$$

whose Frechet derivative is

$$D_{(u,v,w)}=\begin{pmatrix} \frac{1}{p_1}-\frac{p}{p_1^2}\mathcal{T} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}=\begin{pmatrix} \frac{1}{p_1}-\frac{u}{p_1}\mathcal{T} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{29}$$

Under transformation (28), $\mathcal{H}^0$ and $\mathcal{H}$ become

$$D_{(u,v,w)}\mathcal{H}^0D^{\dagger}_{(u,v,w)}=\begin{pmatrix} 0 & -u(1-\mathcal{T}) & 0 \\ (1-\mathcal{T}^{-1})u & 0 & 0 \\ 0 & 0 & \mathcal{T}-\mathcal{T}^{-1} \end{pmatrix} \tag{30}$$

and

$$\begin{aligned} &D_{(u,v,w)}\mathcal{H}D^{\dagger}_{(u,v,w)} \\ &=\begin{pmatrix} u(\mathcal{T}^{-1}-\mathcal{T}^2)(\mathcal{T}+1)^{-1}u & -u(\mathcal{T}-1)v & u(1-\mathcal{T})(\mathcal{T}^{-1}+1)^{-1}w \\ v(\mathcal{T}^{-1}-1)u & w\mathcal{T}^{-1}u-u\mathcal{T}w & -u\mathcal{T}^2+\mathcal{T}^{-1}u \\ w(\mathcal{T}+1)^{-1}(\mathcal{T}^{-1}-1)u & \mathcal{T}^{-2}u-u\mathcal{T} & \begin{matrix} w(\mathcal{T}-1)(\mathcal{T}+1)^{-1}w \\ +\mathcal{T}^{-1}v-v\mathcal{T} \end{matrix} \end{pmatrix}, \end{aligned}$$

which form a Hamiltonian pair for three component Toda system with Lax operator

$$L=\mathcal{T}^{-2}+w\mathcal{T}^{-1}+v+u\mathcal{T}\ .$$

Here we used the same notation as in [17], where the Hamiltonian pair is explicitly given. Thus we proved the statement. □

Remark 4.2. (1) This theorem can also be proved by verifying operator $\mathcal{H}$ defining a Poisson bivector as used in [7].

(2) Another method to prove this statement is using recent results on preHamiltonian operators [19, 20]. We call a difference operator preHamiltonian if its image is a Lie subalgebra with respect to the Lie bracket of evolutionary vector fields. Direct computation shows that operator P is indeed a preHamiltonian operator.

Theorem 4.1 induces the following result on the invariant evolutions (22).

Theorem 4.2. *The invariant evolution in the centro-affine space* $\mathbb{R}^3$ *described by*

$$(x)_t = (x, x_1, \frac{x_2}{p}) \begin{pmatrix} 0 \\ -\frac{p_{-2}}{p_{-1}} \\ 0 \end{pmatrix} \tag{31}$$

induces the integrable system

$$\begin{pmatrix} p_t \\ r^1_t \\ r^2_t \end{pmatrix} = \begin{pmatrix} pr^2 \\ \frac{p}{p_1} - \frac{p_{-2}}{p_{-1}} \\ r^1 \frac{p_{-1}}{p} - \frac{p}{p_1} r^1_1 \end{pmatrix} = \mathcal{H}\delta r^2 = \mathcal{H}_0 \delta \left(\frac{p_{-1}}{p} r^1 + \frac{(r^2)^2}{2} \right), \tag{32}$$

where a Hamiltonian pair $\mathcal{H}$ *and* $\mathcal{H}^0$ *is given in Theorem 4.1, and it becomes* 3*-component Toda lattice* (15) *under the transformation*

$$u = \frac{p}{p_1}, w^1 = r^1, w^2 = -r^2. \tag{33}$$

Proof. Taking $\mathbf{v} = C\left(0, 0, 1\right)^T = \left(0, -\frac{p_{-2}}{p_{-1}}, 0\right)^T$, we get (31) from (22) and (32) from (23). Moreover, equation (32) is a biHamiltonian systems since $\mathcal{H}$ and $\mathcal{H}^0$ form a Hamiltonian pair following from Theorem 4.1.

Under the transformation we have $u_t = \frac{p_t}{p_1} - \frac{p}{p_1^2} p_{1,t} = \frac{p}{p_1} r^2 - \frac{p}{p_1} r^2_1 = u(w^2_1 - w^2)$. After direct calculation for $w^i_t, i = 1, 2$, we obtain the 3-component Toda lattice as stated. □

Notice that $e_2 = \left(0, 1, 0\right)^T$ is in the kernel of $\mathcal{H}^0$. if we take

$$\mathbf{v} = -Ce_2 = \left((\mathcal{T} + 1)^{-1} r^1, \mathcal{T}^{-1} r^2, \mathcal{T}^{-1} p\right)^T$$

in (23), we get the following integrable nonlocal equation

$$\begin{cases} p_t &= -p(1 + \mathcal{T})^{-1} r^1_1, \\ r^1_t &= r^2_{-1} - r^2 - r^1(\mathcal{T} - 1)(1 + \mathcal{T})^{-1} r^1, \\ r^2_t &= \frac{p_{-1}}{p} - \frac{p}{p_1}, \end{cases} \tag{34}$$

which becomes the nonlocal 3-component Toda lattice (16) under the transformation (33).

4.2. **Towards the general centro-affine case.** For n-dimensional centro-affine space, we can, in principle, carry out the same study as we did for the case of centro-affine $\mathbb{R}^3$. However, the explicit formula of operator P (in (24)) in this case is rather large and thus we won't write down here. We simply present the expression for matrix N and some results.

Assume that the general invariant evolution is given by

$$(x)_t = \rho^{-1} \mathbf{v}, \qquad \mathbf{v} = \left(v^1, v^2, \cdots, v^n\right)^T, \tag{35}$$

where $v^1, v^2, \cdots, v^n$ are arbitrary functions of the invariants $r^k, k = 1, \cdots, n-1$ and p and their shifts. We know that $N = \rho_t \rho^{-1}$. Thus the first column of N is

$-\mathbf{v}$, the $k-$th column of N is $M^{k-1}\mathbf{v}$ for $k = 2, \cdots, n-1$ with

$$M = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{p}\mathcal{T} \\ \mathcal{T} & 0 & 0 & \cdots & 0 & 0 & \frac{-r^1}{p}\mathcal{T} \\ \vdots & \vdots & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{T} & 0 & \frac{-r^{n-2}}{p}\mathcal{T} \\ 0 & 0 & 0 & \cdots & 0 & (-1)^{n-1}p\mathcal{T} & (-1)^n r^{n-1}\mathcal{T} \end{pmatrix}, \tag{36}$$

and the last column of N can be obtained from the structure equation (7), namely,

$$\begin{aligned} N_{1,n} &= \frac{(-1)^{n-1}}{p^2}\mathcal{T}N_{n,n-1}, \\ N_{j+1,n} &= \frac{(-1)^{n-1}\mathcal{T}N_{j,n-1}}{p} - r^j N_{1,n}, \quad j = 1, \cdots, n-2, \\ N_{n,n} &= -(N_{1,1} + \cdots + N_{n-1,n-1}), \end{aligned}$$

reflecting the fact that matrix N is traceless.

Theorem 4.3. *The invariant evolution in the centro-affine space $\mathbb{R}^{n+1}$ given by*

$$(x)_t = (x, x_1, \cdots, \frac{(-1)^n}{p}x_n)\left(0, -\tfrac{p_{-n}}{p_{1-n}}, 0, \cdots, 0\right)^T \tag{37}$$

induces the integrable system

$$\begin{cases} p_t &= (-1)^n r^n p, \\ r^1_t &= -\frac{p_{-n}}{p_{1-n}} + \frac{p}{p_1}, \\ r^j_t &= r^{j-1}\frac{p_{j-1-n}}{p_{j-n}} - r^{j-1}_1\frac{p}{p_1}, \quad j = 2, \cdots, n, \end{cases} \tag{38}$$

which becomes the $n+1$-component Toda lattice (13) under the transformation

$$u = \frac{p}{p_1}, \; w^k = (-1)^{k+1}r^k, \; k = 1, \cdots, n. \tag{39}$$

Proof. From the given invariant evolution (37), we know $\mathbf{v} = \left(0, -\frac{p_{-n}}{p_{1-n}}, 0, \cdots, 0\right)^T$. Thus we get the expression of N using the formula above. Alternatively we determine N using the structure equation (7): the nonzero entries are

$$\begin{aligned} &N_{i+1,i} = \mathcal{T}^{i-1}\frac{p_{-n}}{p_{1-n}}, \; i = 1, 2, \cdots, n-1; \quad N_{n+1,n} = (-1)^n p_{-1}; \\ &N_{1,n+1} = \frac{1}{p}; \quad N_{i+1,n+1} = -\frac{r^i}{p}, \; i = 1, 2, \cdots, n-1 \end{aligned}$$

and further obtain the corresponding flow of invariants as given by (38). The second part of the statement can be proved by direct calculation of changing variables. □

Similarly, we can get the result of the invariant evolution linking with nonlocal multi-component Toda lattices as in the centro-affine $\mathbb{R}^3$.

Theorem 4.4. *The invariant evolution in the centro-affine space $\mathbb{R}^{n+1}$ given by*

$$(x)_t = (x, x_1, \cdots, (-1)^n\frac{x_n}{p})\mathcal{T}^{-1}\left(\textstyle\sum_{k=0}^{n-2}\mathcal{T}^{-k}\eta, r^2, r^3, \cdots, r^n, p\right)^T, \tag{40}$$

where $\eta = (1+\mathcal{T}^{-1}+\cdots+\mathcal{T}^{1-n})^{-1}r^1$*, induces the integrable system*

$$\begin{cases} p_t &= -p\eta, \\ r_t^j &= r_{-1}^{j+1} - r^{j+1} - r^j(1-\mathcal{T}^{-n+j})\eta, \quad j=1,\cdots,n-1, \\ r_t^n &= (-1)^n\left(\frac{p_{-1}}{p} - \frac{p}{p_1}\right) \end{cases} \tag{41}$$

which becomes the nonlocal $n+1$*-component Toda lattice (14) under the transformation* (39).

Proof. The second part of the statement can be proved by direct calculation of changing variables. For the proof of the first part of statement, we denote

$$\overline{\eta} = (1+\mathcal{T}^{-1}+\cdots+\mathcal{T}^{2-n})\eta. \tag{42}$$

From the given invariant evolution (40), it leads to the first column of matrix N as $-\left(\mathcal{T}^{-1}\overline{\eta}, \mathcal{T}^{-1}r^2, \mathcal{T}^{-1}r^3, \cdots, \mathcal{T}^{-1}r^n, \mathcal{T}^{-1}p\right)^T$. Using the structure equation (7), we can determine N as follows

$$N = -\begin{pmatrix} \mathcal{T}^{-1}\overline{\eta} & 1 & 0 & \cdots & 0 & 0 \\ \mathcal{T}^{-1}r^2 & \overline{\eta}-r^1 & 1 & \cdots & 0 & 0 \\ \mathcal{T}^{-1}r^3 & 0 & \mathcal{T}(\overline{\eta}-r^1) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \mathcal{T}^{-1}r^{n-2} & 0 & 0 & \cdots & 1 & 0 \\ \mathcal{T}^{-1}r^n & 0 & 0 & \cdots & \mathcal{T}^{n-2}(\overline{\eta}-r^1) & \frac{(-1)^n}{p} \\ \mathcal{T}^{-1}p & 0 & 0 & \cdots & 0 & \xi \end{pmatrix},$$

where $\xi = (\mathcal{T}^{n-1}–\mathcal{T}^{-1})\overline{\eta} + (1–\mathcal{T}^{n-1})r^1$. Moreover, we get the flow for invariants

$$\begin{cases} p_t = p(\mathcal{T}^{-1}\overline{\eta} - r^1), \\ r_t^j = r^j(\mathcal{T}^{-1}-\mathcal{T}^{j-1})\overline{\eta} + (\mathcal{T}^{-1}-1)r^{j+1} - r^j(1-\mathcal{T}^{j-1})r^1, \ j=1,\cdots,n-1, \\ r_t^n = r^n(\mathcal{T}^{-1}-\mathcal{T}^{n-1})\overline{\eta} - r^n(1-\mathcal{T}^{n-1})r^1 + (-1)^n\left(\frac{p_{-1}}{p} - \frac{p}{p_1}\right). \end{cases}$$

Note that N is traceless under the given relation between η and r^1, that is,

$$r^1 = (1+\mathcal{T}^{-1}+\cdots+\mathcal{T}^{1-n})\eta.$$

Substituting it and (42) into the above flow, we obtain the system (41). □

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