Xing-Gang Yan, Sarah K Spurgeon and Christopher Edwards

Variable Structure Control of Complex Systems: Analysis and Design

Jan, 2017

Preface

It is well known that linear dynamical systems cannot adequately describe many phenomena commonly observed in the real world. With the advancement of science and technology, practical systems are becoming more complex in order to complete more advanced tasks. With the increasing requirements for system performance, linear system theory based study cannot satisfy the practical requirements, and the mathematical equations used to model real physical and engineering systems have become more and more complex. In reality, there are many factors which will affect system performance. To describe and explore various natural phenomena, it is necessary to consider these factors and thus to investigate complex systems as a means to model real systems more accurately. This book systemises aspects of the authors' recent achievements in the area of variable structure control alongside with some fundamental knowledge in the area.

This book focuses on the study of complex control systems in which the complexity mainly stems from nonlinearities, uncertainties, time delay, faults and/or coupling among subsystems. It provides rigorous theoretical solutions to the problem of control of complex systems but has potential application in practical systems. It should be emphasised that many theoretical studies on control systems often assume that all system states are available for control design. This assumption is not valid for real systems in many cases. To implement such control schemes, a pertinent way forward is to construct an appropriate dynamical system which is called an observer, to estimate the state variables. Unfortunately, the traditional separation principle for linear control systems usually does not hold for the nonlinear counterpart, which implies that for nonlinear systems, the properties of a state feedback control law may not be achieved when the control law is implemented with the estimated states. In connection with this, this book focuses on output feedback control design: both static output feedback and dynamical output feedback strategies, including reduced order dynamical output feedback strategies, are proposed to control complex systems such that the closed-loop systems have the desired performance.

Variable structure control techniques have been extensively studied, and widely applied to theoretical research and practical engineering systems due to their high robustness. Specifically, as one special case of variable structure controllers, sliding mode controllers are completely robust to matched uncertainties. Moreover, the sliding motion is determined by reduced order dynamics, which facilitates the reduction of the effects of mismatched uncertainties on the whole systems when compared with other methods. A key development in this book considers variable structure control for complex systems based on only output information, using mainly the Lyapunov direct method and sliding mode techniques, with the objective of enhancing the robustness against uncertainties, reduction of conservatism and enlargement of the admissible systems. Rigorous stability analysis and design methodologies are provided from a theoretical perspective for this theme. Nonlinearities appear in all the considered systems throughout the book. Both the matched and mismatched uncertainties covered in this book are nonlinear and bounded by nonlinear functions. Since the considered systems are complex and all the results are rigorous, the conditions developed for all the main results in this book are sufficient. As there is no general way to obtain the design parameters for an output feedback controller, trying to determine 'easy' test conditions with low conservatism, by separating possible known information from the system and then employing them in the design to reduce the effects of factors such as uncertainties and time delay on the system, is one of the main targets throughout this book. The book also presents novel contributions to deal with nonlinear uncertainties for time delay systems by combining the Lyapunov Razumikhin approach and variable structure techniques for different cases when delay is known and unknown respectively. It is shown that for interconnected systems, decentralised control schemes are available to cancel/reduce the effects of the interconnections on the whole system performance, under certain conditions. One of the characteristics of this monograph is that many examples and case studies with simulations are given to help readers understand the developed theoretical results and the proposed approaches.

The first two chapters present fundamental knowledge used in later developments. Chapter 1 develops some preliminary ideas regarding variable structure control. Specifically, the basic concepts and fundamental methodologies for sliding mode control and decentralised control are provided. Some of them are clarified for the first time based on the authors' understanding as a result of the authors' many years of research work in the areas. Several practical examples are given to show the potential application of complex systems. This helps readers understand the main methods used in the book intuitively from both mathematical and practical points of view. Chapter 2 presents some preliminary mathematical results and some results developed by the authors.

Chapter 3 considers static output feedback control design for both nonlinear systems and interconnected systems. For a class of fully nonlinear systems, a variable structure control based on Lyapunov methods is designed to drive and maintain the system in a 'small' region of the origin. Then, in the region, the nonlinear system is linearised and a sliding mode control is designed to stabilise the system asymptotically. Both controllers combined together stabilise the system globally. For interconnected systems, decentralised control schemes are developed and output variables embedded in the nonlinearity are separated and used in the control design to reduce conservatism. Case studies relating to a mass spring system, coupled inverted pendulums and a flight control system are provided to illustrate the developed control methodologies.

Chapter 4 considers dynamical output feedback control design for systems with mismatched uncertainties/disturbances such that the corresponding closed-loop systems are asymptotically stable. Compared with Chapter 3, all the uncertainties involved in this chapter are bounded by nonlinear functions of the system state variables instead of the output variables. The bounding functions are assumed to be known and thus it is possible to use them for control design and system analysis to reduce the effects of uncertainties. In Section 4.2, a sliding surface is designed which is independent of the designed observer, and then a sliding mode control is synthesised based on the estimated states from the designed observer and the system outputs. The controller design and the observer design are separated. The designed control can be implemented with any appropriate observer but the developed approach requires that the considered system is minimum phase. In Section 4.3, a dynamical compensator is designed first. A sliding surface is then designed for the augmented system formed by the considered system and error dynamics. It is not required that the nominal system is minimum phase. Applications to control of the High Incidence Research Model (HIRM) aircraft are given in Section 4.4. Both longitudinal and lateral aircraft dynamics based on different trim values of Mach number and height are employed in the simulation study.

Chapter 5 continues to consider dynamical output feedback controller design. It focuses on large-scale interconnected systems and uses reduced-order compensators to form the feedback loop which is particularly important for large scale systems as it may avoid 'the curse of dimensionality'. In Section 5.2, sliding mode dynamics are established and the stability is analysed using an equivalent control approach and a local coordinate transformation. A robust decentralised output feedback sliding mode control scheme is synthesized such that the interconnected system can be driven to the pre-designed sliding surface. This approach allows both the nominal isolated subsystem and the whole nominal system to be nonminimum phase. In Section 5.3, a similar structure is introduced to identify a class of nonlinear large-scale interconnected systems. By exploiting the system structure of similarity, the proposed nonlinear reduced-order control schemes allow more general forms of uncertainties. Specifically, based on a constrained Lyapunov equation, the effect of matched uncertainties is canceled completely. The study shows that a similar structure can simplify the analysis and reduce the amount of computation. Numerical simulation examples and a case study on river pollution control are provided to illustrate the results developed.

Chapters 6 and 7 consider complex systems with time delay. A Lyapunov Razumikhin approach is employed to deal with time delay throughout the two chapters. All the developed results are suitable for time varying delay and there is no limitation to the rate of change of the time varying delay as with the Lyapunov Krasovskii approach. Chapter 6 requires that the time delay is known and thus the time delay can be used in the design to reduce conservatism. Therefore the controllers are delay dependent. Chapter 7 removes the assumption that the time delay is known but the results obtained are usually conservative when compared with Chapter 6. In Chapter 6, both static and dynamical output feedback control schemes are presented for complex time delay systems; decentralised static output feedback sliding mode controllers are designed to stabilise complex interconnected time delay systems where delay exists in both the interconnections and the isolated subsystems. In Chapter 7, local stabilisation is considered for affine nonlinear control systems with uncertainties involving time-varying delay. It is not assumed that the nominal system is either linearisable or partially linearisable. Section 7.4 focuses on the stabilisation problem for a class of large scale systems with nonlinear interconnections. A decentralised static output feedback variable structure control is synthesised and a set of conditions is developed to guarantee that the considered large scale interconnected systems are stabilised uniformly asymptotically. Section 7.5 provides some examples to demonstrate the results developed in Sections 7.2–7.4. Numerical simulation examples and a case study on a mass-spring system are provided to demonstrate the theoretical results.

Chapter 8 considers fault detection and isolation (FDI) for nonlinear systems with uncertainties using particular sliding mode observers for which the design parameters can be obtained using LMI techniques. In Section 8.2, a sliding mode observer based approach is presented to estimate system faults using bounds on the uncertainties, and as a special case, a fault reconstruction scheme is available where the reconstructed signal can approximate the fault signal to any accuracy. Section 8.3 considers sensor FDI for nonlinear systems where a nonlinear diffeomorphism is introduced to explore the system structure and a simple filter is presented to ‘transform’ the sensor fault into a pseudo-actuator fault scenario. Both fault estimation and reconstruction are considered. Case studies on a robotic arm system and a mass-spring system demonstrate the effectiveness of the proposed FDI schemes.

Chapter 9 provides a decentralised strategy for the excitation control problem of multimachine power systems which are formed from an interconnected set of lower order subsystems through a network transmission. Both mismatched uncertainties in the interconnections and parametric uncertainties in the direct axis transient short circuit time constants, which affect the subsystem input distribution matrix, are considered. The proposed approach can deal with interconnection terms and parametric disturbances with large magnitude. The results obtained hold in a large region of operation if the control gain is high enough. This allows the operating point of the multimachine power system to vary to satisfy different load demands. Simulations based on a three-machine power system are presented to illustrate the proposed control scheme.

Chapter 10 makes some concluding remarks. Several specific examples are presented to show the complexity of the systems considered in this book. Some comments offer suggestions for future work. Finally, Appendixes A to D provide some results (with rigorous proofs), which are used in the book, and Appendix E presents notation and the parameters of the multimachine power system considered in Chapter 8.

The book aims to disseminate recent results in the area of variable structure control of complex systems. It is suitable for scientists and engineers in academia and industry who are interested in either variable structure techniques or complex systems including nonlinear control, decentralised control, time delay systems, robust control and fault detection and isolation. It is particularly valuable to have a combined set of references at the end of the book for ease of access to many important theoretical and practical applications. It contains many case studies and numerical examples with simulations to help readers understand and apply the developed theoretical results. The analysis and design methodologies are also useful for both undergraduate and postgraduate students in the field of nonlinear control systems design. We believe mathematicians and control engineers will find the book useful.

Last but not least, we would like to point out that this book only attempts to present part of the authors' recent achievements in the area of complex variable structure control, which is obviously built on many other previous results. Although we have tried to cover most of the recent important ideas and results in the area, the exposition is far from a complete overview of the associated subjects. The bibliography includes only the literature which has been actually used in the book. We sincerely apologies for any serious omissions, large or small.

Canterbury, United Kingdom *Xing-Gang Yan*
Canterbury, United Kingdom *Sarah K. Spurgeon*
Exeter, United Kingdom *Christopher Edwards*

April, 2016

Contents

Notation And Symbols

- $\emptyset$ —— the empty set
- $\mathbb{R}$ —— the set of real numbers
- $\mathbb{R}^+$ —— the set of nonnegative real numbers
- $\mathbb{R}^n$ —— the n dimensional Euclidean space
- $\mathbb{R}^{n\times m}$ —— the set of $n \times m$ matrices with elements in $\mathbb{R}$
- $\|\cdot\|$ —— the Euclidean norm or its induced norm
- I_n —— the unit matrix with dimension n
- $\mathrm{Im}(A)$ —— the range space of matrix A
- $A^{(j)}$ —— the j-th column vector of the matrix A
- B_r or $\mathscr{B}_r$ —— the ball $\{x \mid \|x\| < r\}$ with radius r where $r \in (0, +\infty)$
- $\overline{B}_r$ or $\overline{\mathscr{B}}_r$ —— the closure of $\mathscr{B}_r$
- $\partial\mathscr{B}_r$ or $\partial\overline{\mathscr{B}}_r$ —— the boundary of $\mathscr{B}_r$
- A^τ or A^T —— the transpose of matrix A
- $A^{-\tau}$ or A^{-T} —— the transpose of matrix A^{-1}
- $A > 0$ —— A is a symmetric positive definite matrix
- $A < 0$ —— A is a symmetric negative definite matrix
- $\overline{\sigma}(A)$ —— the maximum singular value of the matrix A
- $\underline{\sigma}(A)$ —— the minimum singular value of the matrix A
- $\lambda_{\min}(A)$ —— the minimum eigenvalue of the square matrix A
- $\lambda_{\max}(A)$ —— the maximum eigenvalue of the square matrix A
- $\mathrm{diag}\{A_1, A_2, \cdots, A_N\}$ —— a block-diagonal matrix with diagonal elements A_1, $A_2, \cdots, A_N$

- $A^{\frac{1}{2}}$ —— a symmetric positive definite matrix such that $A^{\frac{1}{2}}A^{\frac{1}{2}} = A$
- $f^{-1}(\cdot)$ —— the inverse function of the function $f(\cdot)$
- $\mathscr{L}_f$ —— the Lipschitz constant of the function $f(\cdot)$
- $L_f h$ —— derivative of the mapping $h(x) : \mathbb{R}^n \mapsto \mathbb{R}^p$, along the vector field $f(x,u) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ defined by $L_{f(x,u)}h(x) := \frac{\partial h}{\partial x} f(x,u)$
- $< d\lambda(x),\ f(x) >$ —— $L_f\lambda(x)$ where $d\lambda = \frac{\partial \lambda}{\partial x}$ is the differential of λ.
- $J_f(x)$ —— the Jacobian matrix of the function $f(x)$
- $[f,g]$ —— Lie bracket (product) of the vector fields $f(x)$ and $g(x)$, defined by $[f,\ g](x) = J_g(x)f(x) - J_f(x)g(x)$
- $ad_f^k g(x)$ —— $[f, ad_f^{k-1}g](x)$ where $ad_f^0 g(x) := g(x)$
- $\mathscr{L}_\psi(x_1,x_2)$ —— generalised Lipschitz constants about $x_1 \in \mathbb{R}^{n_1}$ uniformly for $x_2 \in \mathbb{R}^{n_2}$ where the function $\psi(x_1,x_2)$ is defined in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$
- $L_f^r h$ —— the rth order Lie derivative of the mapping $h(x) : \mathbb{R}^n \mapsto \mathbb{R}^p$, along the vector field $f(x,u) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$
- $f(x,y)$ —— $f(x_1,x_2,\ldots,x_{n_1},y_1,y_2,\ldots,y_{n_2})$ where $x = [x_1\ x_2\ \ldots\ x_{n_1}]^T \in \mathbb{R}^{n_1}$ and $y = [y_1\ y_2\ \ldots\ y_{n_2}]^T \in \mathbb{R}^{n_2}$
- $\frac{\partial f(x,y)}{\partial x}$ —— Jacobian matrix of function $f(x,y)$ relating the variable x where $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$
- $\mathrm{col}(x_1,x_2,\ldots,x_n)$ —— the coordinates $[x_1,x_2,\ldots,x_n]^T$ where $x_i \in \mathbb{R}$ for $i = 1,2,\ldots,n$
- $\mathrm{col}(x_1,x_2)$ —— the coordinates $[x_{11},x_{12},\ldots,x_{1n},x_{21},x_{22},\ldots,x_{2n}]^T$ where $x_1 = [x_{11},x_{12},\ldots,x_{1n}]^T \in \mathbb{R}^{n_1}$ and $x_2 = [x_{21},x_{22},\ldots,x_{2n}]^T \in \mathbb{R}^{n_2}$
- $\mathscr{C}_{[a,b]}$ —— represents the set of $\mathbb{R}^n$-valued continuous function on interval $[a,b]$
- x_d —— $x(t-d)$ where d (may be time varying) represents time delay
- $A := B$ —— A is defined by B
- $A \Rightarrow B$ —— A implies B
- $A \Leftrightarrow B$ —— A is equivalent to B

Chapter 1
Introduction

Control systems widely exist in the real world. Increasing requirements for system performance and reliability have resulted in increasing complexity in the dynamic systems used to model reality. Control engineers are faced with increasingly complex control systems. The development of computer science and technology coupled with developments in mathematical theory has provided the possibility for study of complex systems from both the theoretical and practical viewpoint.

This book systemises some of the authors' recent research work along with fundamental concepts and methodologies in the area of variable structure control for complex systems. The complexity resulting from nonlinearities, uncertainties including modelling error, time delay, and interconnections between subsystems is considered. For various complex systems, theoretical analysis and control design using static output feedback, observer-based output feedback, and decentralised control ideas is presented based on variable structure techniques. The fault detection and isolation problem is also investigated, using sliding mode observers, where reconstruction and estimation schemes for both system faults and sensor faults will be presented. Numerous numerical examples and case studies with accompanying simulations are provided to help the reader understand and apply the developed strategies and approaches.

1.1 System Complexity

Linear dynamical systems cannot describe many commonly observed phenomena well. In the real world, nearly all systems exhibit nonlinearity. In order to reveal complex phenomena and study complex systems, it is necessary to investigate nonlinear dynamical systems as a means to model real systems more accurately.

A dynamical control system usually can be expressed by the following differential equation

$$\dot{x} = f(t, x, u) \tag{1.1}$$

where $x \in \mathbb{R}^n$ denotes the system state, $u \in \mathbb{R}^m$ represents the system input/control and $t \in \mathbb{R}^+$ is time. If a particular system output is of interest, then an algebraic equation

$$y = h(t,x) \tag{1.2}$$

or

$$y = h(t,x,u) \tag{1.3}$$

is used, where $y \in \mathbb{R}^p$ represents the system output. Equation (1.1) is called the state equation while equation (1.2) or (1.3) is said to be the output equation.

In this book, only the output equation (1.2) is considered which means that the output equation considered in this book does not involve the control variable u. The system (1.1)–(1.2) is called a *single-input single-output* (SISO) system if both u and y are scalars. It is called a *multi-input multi output* (MIMO) system if the dimensions of either u or y are bigger than one.

The complexity of a control system depends on the controlled plant and the environment. Higher requirements on the controlled system' performance usually require more advanced control techniques, which will introduce additional complexity. There are many factors which may affect control system performance and result in complex phenomena, such as nonlinearities, uncertainties/modelling errors, time delay and any interconnections existing in the system.

- **Nonlinearity:** Compared with linear systems, the study of nonlinear systems is much more difficult. Analysis and design of nonlinear control systems usually involve more advanced mathematics. Due to the existence of nonlinearities in dynamical systems, phenomena such as finite time escape, multiple isolated equilibria, limit cycles, harmonic oscillation, chaos and multiple modes of behaviour may appear [91]. These rich behaviours which exist in nonlinear dynamical systems greatly increase the complexity of the problem.
- **Uncertainty/modelling error:** Real systems unavoidably experience various uncertainties such as mechanical wear and changes in the external environment. The former may result in parametric uncertainties while the later may result in unstructured uncertainties. Moreover, it may be impossible to model a system accurately. If these modelling errors and uncertainties or disturbances are not considered, the developed strategies may not work well or may even fail to meet the design objective. Specifically, for a large-scale interconnected system, a perturbation of one subsystem can affect other subsystems and the overall performance of the network. This increases the complexity in the problem.
- **Time delay:** With the increasing expectations for the closed-loop system's dynamical performance, it is required that the established system model behaves more like the real process. Thus time delay has to be considered as many processes include after effect phenomena in their inner dynamics: for example, biology, population dynamics, economics, viscoelasticity and engineering science [141, 130]. For a time delay system, the future evolution, usually, not only depends on the present state but also on its history. Even a small delay may greatly affect the performance of a system; a stable system may become unstable, or chaotic behavior may appear due to delay in the system [126].

- **Interconnection:** In order to complete a complex task, systems have to be combined together to provide the desired performance. For example, in a manufacturing process, in order to produce the same engineering components in sufficiently large quantities, many machine tools (isolated subsystems) are interconnected together and monitored to form a large-scale system to complete the task [202]. A complex system may also be formed by interconnections between a collection of simple systems. In this case, although each subsystem may exhibit good performance in isolation, the whole system may not work well due to the interactions between the subsystems. To reduce, minimise or even employ the effects of the interconnections on the whole system is challenging. Moreover, these subsystems are usually distributed geographically in space, which results in problems such as data transfer, the reliability of the network communication channels and economic cost etc. [210, 2].

In this book, the factors mentioned above will be considered. In order to deal with the effects of uncertainties, variable structure control techniques will be employed. The Lyapunov-Razumikhin approach will be used to deal with time delay. For interconnected systems, decentralised strategies will be developed whenever possible to avoid the reliability problem caused by network links.

1.2 Variable Structure Control

Consider the control system (1.1) in the domain $D \in \mathbb{R}^n$. A corresponding *variable structure control* can be expressed as

$$u = \begin{cases} u_1(t,x), & (t,x) \in \mathbb{R}^+ \times D_1 \\ u_2(t,x), & (t,x) \in \mathbb{R}^+ \times D_2 \\ \vdots & \vdots \\ u_q(t,x), & (t,x) \in \mathbb{R}^+ \times D_q \end{cases} \tag{1.4}$$

where the functions $f_i(t,x)$ are continuous for $i = 1,2,\dots,q$. The structures of the functions $f_i(t,x)$ and $f_j(t,x)$ are different for $i \neq j$ and $i, j = 1,2,\dots,q$ ($q \geq 2$). The domains $D_i \in \mathbb{R}^n$ for $i = 1,2, \dots, q$ satisfy

i) $D_1 \cup D_2 \cup \cdots \cup D_q = D$;
ii) $D_i \cap D_j = \emptyset$ if $i \neq j$ for $i, j = 1,2,\dots,q$.

When the variable structure control in (1.4) is applied to the system (1.1), the corresponding closed-loop system becomes a *variable structure system*. Literally speaking, variable structure control is a control whose structure is changed or keeps changing in order to obtain and maintain the desired system performance during the control process.

For example, in real control design, when the response error/accuracy $e(t)$ is over the threshold, a proportional control is used to increase the response speed; when the response error/accuracy $e(t)$ is within the threshold, an integral control is employed to guarantee that the steady error requirement is satisfied. In this case, the control law may be described by

$$u = \begin{cases} k_p e(t), & \|e(t)\| > k \\ k_i \int e(t)dt, & \|e(t)\| < k \end{cases}$$

Here the positive constants k_p and k_i are called the proportional gain and integral gain respectively which are tuning parameters, and the positive constant k is called the threshold.

This example shows that sometimes it is desirable to change the control structure in order to get the desired system performance. As pointed out in [12], nonholonomic systems cannot be stabilised by continuously differentiable, time invariant state feedback control laws. However, a discontinuous control law is available to stabilise nonholonomic systems (see, e.g. [1]). This motivates the need for discontinuous control.

When the variable structure controller (1.4) is applied to the system (1.1), it usually produces a discontinuous right-hand side in the corresponding closed loop dynamical system which consists of a set of ordinary differential equations. This produces an interesting mathematical problem: the traditional definition and existence conditions for the solutions of the closed-loop system are not applicable. It is necessary to extend the classical solution. In this case the solution of the equations is defined in the Filippov sense [46] throughout the book.

In order to reject/reduce the effects of uncertainties and disturbances, different variable structure approaches have been proposed, for example, the approach based on the direct Lyaponov method in [202, 214, 210] and a discontinuous control law for nonholonomic systems in [1]. However, variable structure control which leads to a sliding motion, has underpinned the development of a systematic research methodology, which is the well known sliding mode control paradigm. Sliding mode control has dominated the literature in the area of variable structure control and thus when people talk about variable structure control, they usually mean sliding mode control. Here, it should be pointed out that not all variable structure control will lead to a sliding motion.

1.3 Sliding Mode Control

Sliding mode control, as a particular type of variable structure control, evolved from the pioneering work in Russian of Emel'yanov and Barbashin in the early 1960s. The ideas did not appear outside of Russia until the mid 1970s when a book by Itkis [81] and a survey paper by Utkin were published in English [175]. The ideas underlying the modern analysis and design of sliding mode controllers may be further dated back to publications in the early 1930s. At that time, concerns on relay systems with sliding modes for controlling the course of a ship had been proposed [55] where the terms phase plane, switching line, and even sliding mode appear [172].

Relay systems have been found in many control engineering systems. Relay control systems are a simple nonlinear system which is effective and has low cost. Sometimes they have better dynamical performance than linear systems [171]. Early rigorous studies on relay systems are found in contributions in the 1960s which were presented celebrating Filippov's achievement for differential equations with discontinuous righthand sides [47]. The study of relay systems stimulated the study of sliding mode control.

In the initial stage (before 1962), nearly all studies focused on second order linear systems. Later work was extended to higher order systems (i.e. systems with order greater than 2) but most work was still limited to linear systems with single input control. The study of nonlinear systems in state space form commenced in 1970 and muti-input control systems have been widely considered since then. The development of this state space description and mutivariable control system theory greatly promoted the development of sliding mode controllers, which also motivated the application of sliding mode techniques in practical systems [172].

In recent decades, various control approaches have been proposed and research on sliding mode control has become very active. Due to its high robustness against uncertainties/disturbances, sliding mode control has been widely combined with other approaches to provide better results in both theoretical research and practical engineering. Many interesting results have been created in adaptive sliding mode control [18, 176, 4], fuzzy sliding mode control [178, 168], backstepping based sliding mode control [162] and decentralised sliding mode control [200, 201] with applications in wide areas such as engineering systems, aircraft control, energy systems, communication networks and biology [82, 7, 77, 172, 153, 129]

1.3.1 Sliding mode control methodology

Sliding mode control changes the system dynamics by employing a discontinuous control signal. This approach has been well developed and extensively used in theoretical research and practical engineering design. It has been successfully employed to solve various control problems in combination with other control approaches.

The sliding mode control method consists of two steps:

- the design of a sliding surface such that the system considered possesses the desired performance when it is restricted to the surface;
- the design of a variable structure control which drives the system trajectory to the sliding surface in finite time and maintains a sliding motion on it thereafter.

A concise description is available in [38, 173]. In view of these two steps, the system motion can be separated into two phases: the *reaching phase* and the *sliding phase*. The former refers to the motion when the system trajectory moves towards the sliding surface and the latter concerns the motion when the system trajectory moves on the sliding surface.

1.3.1.1 Sliding phase

Consider system (1.1). In order to design a proper switching/sliding function

$$s = s(x)$$

such that the resulting sliding motion has the desired performance, one way is to find the dynamical equations which will govern the sliding motion, and then synthesize the sliding surface based on the characteristics of the sliding mode dynamics or sliding motion. It is assumed that the sliding motion exists. The following two approaches are usually employed to find the sliding mode dynamics and in this way the stability of the sliding motion is transformed to the problem of ensuring stability of an unforced system.

- **Equivalent control:** When the considered system (1.1) is limited to and moving on the sliding surface,

  $$s(x) = 0, \qquad and \qquad \dot{s}(x) = 0$$

  The time derivative of $s(x)$ along the system (1.1) is given by

  $$\dot{s} = \frac{\partial s}{\partial x}\dot{x} = \frac{\partial s}{\partial x} f(t,x,u)$$

  In the sliding motion,

  $$\frac{\partial s}{\partial x} f(t,x,u) = 0 \tag{1.5}$$

  Suppose there is a solution for u to the equation (1.5) denoted by

  $$u_{eq} = u_{eq}(t,x)$$

  which is the so-called *equivalent control* (see, page 14 in [174]). Then, the sliding mode dynamics governing the sliding motion may be obtained by

  $$\begin{cases} \dot{x} = f(t,x,u_{eq}(t,x)) \\ s(x) = 0 \end{cases} \tag{1.6}$$

  Now, assume that system (1.1) is in the following affine form,

  $$\dot{x} = F(t,x) + G(t,x)u \tag{1.7}$$

  Then, for the sliding surface $s(x) = 0$, it follows from $\dot{s}(x) = 0$ that the corresponding equivalent control is given by

$$u_{eq} = -(s(x)G(x,t))^{-1}s(x)F(t,x) \tag{1.8}$$

where $s(x)$ should be chosen such that $s(x)G(x,t)$ is nonsingular for all x in the considered domain and $t \in \mathbb{R}^+$. Substitute u_{eq} from (1.8) into the system (1.1), it follows that the corresponding sliding motion can be described by

$$\begin{cases} \dot{x} = F(t,x) - G(t,x)(s(x)G(x,t))^{-1}s(x)F(t,x) \\ s(x) = 0 \end{cases}$$

Remark 1.1. It should be noted that the equivalent control is used only to analyse the sliding motion. It is not the control signal which is actually applied to the system but it may be thought of as the control signal which must be applied "on average" to maintain the sliding motion [174, 38].

- **Regular form:** Another approach to find the sliding mode dynamics relating to the sliding function $s = s(x)$ for system (1.1) is to employ the well known regular form. Suppose that there exists a coordinate transformation $z = T(x)$ such that in the new coordinate system z, the sliding surface $s(x) = 0$ can be described in the form

$$z_2 = \sigma(z_1)$$

where $z_1 \in \mathbb{R}^{n-m}$, $z_2 \in \mathbb{R}^m$, $z := \mathrm{col}(z_1, z_2)$ and system (1.1) can be described by

$$\dot{z}_1 = F_1(t, z_1, z_2) \tag{1.9}$$

$$\dot{z}_2 = F_2(t, z_1, z_2, u) \tag{1.10}$$

where $u \in \mathbb{R}^m$ is the control. The Jacobian matrix $\frac{\partial F_2(t,z_1,z_2,u)}{\partial u}$ is assumed to be nonsingular in the considered domain. Note that system (1.9) is independent of the control signal and the dimension of z_2 is the same as the dimension of the control u. System (1.9)–(1.10) is the so-called *regular form*.
Based on the regular form in (1.9)–(1.10), it is clear to see that the corresponding sliding mode dynamics of system (1.1) is described by

$$\dot{z}_1 = F_1(t, z_1, \sigma(z_1)) \tag{1.11}$$

which is a reduced-order system when compared with system (1.1).
Note, if system (1.1) is in the following affine form as given in (1.7), then, the corresponding regular form can be described by

$$\dot{z}_1 = F_1(t, z_1, z_2) \tag{1.12}$$

$$\dot{z}_2 = F_2(t, z_1, z_2) + G_2(t, z_1, z_2)u \tag{1.13}$$

where the functions $F_1(\cdot)$ and $F_2(\cdot)$, and $G_2(\cdot)$ are dependent on the coordinate transformation $z = T(x)$ and the functions $F(\cdot)$ and $G(\cdot)$ respectively.

1.3.1.2 Reaching phase

In order to guarantee that the system trajectory can be driven to the sliding surface $s(x) = 0$ in finite time and a sliding motion can be maintained on it thereafter, a proper discontinuous control

$$u = u(t,x)$$

needs to be designed such that the following condition is satisfied [38, 173]

$$s^T(x)\dot{s}(x) \leq -\eta \|s(x)\| \tag{1.14}$$

for some constant $\eta > 0$. The inequality (1.14) is the so-called *reachability condition* and η is called the *reachability constant*.

From equation (1.1), it follows that

$$\dot{s} = \frac{\partial s}{\partial x}\dot{x} = \frac{\partial s}{\partial x} f(t,x,u)$$

Therefore, inequality (1.14) is equivalent to

$$s^T(x)\frac{\partial s}{\partial x} f(t,x,u) \leq -\eta \|s(x)\| \tag{1.15}$$

which explicitly contains the variable u. The sliding mode controller guaranteeing reachability can usually be synthsised from (1.15).

The following condition

$$s^T(x)\dot{s}(x) < 0$$

is also called a reachability condition but it cannot guarantee that a sliding motion takes place in finite time and thus a sliding motion may not occur in this case.

It should be emphasised that, when the designed sliding/switching function is time varying, for example,

$$s = s(t,x)$$

it is straightforward to see that the condition (1.15) used to synthesise the sliding mode control law should be updated to

$$s^T(t,x)\left(\frac{\partial s}{\partial t} + \frac{\partial s}{\partial x} f(t,x,u)\right) \leq -\eta \|s(t,x)\|$$

For this case, a design approach has been provided in [27].

1.3.2 *Sliding mode control of a mass spring damper system*

In order to illustrate the sliding mode control methodology, consider the simple mass spring damper mechanical system in Figure 1.1 where the mass M slides on a smooth surface. In Figure 1.1, X denotes the displacement from the reference

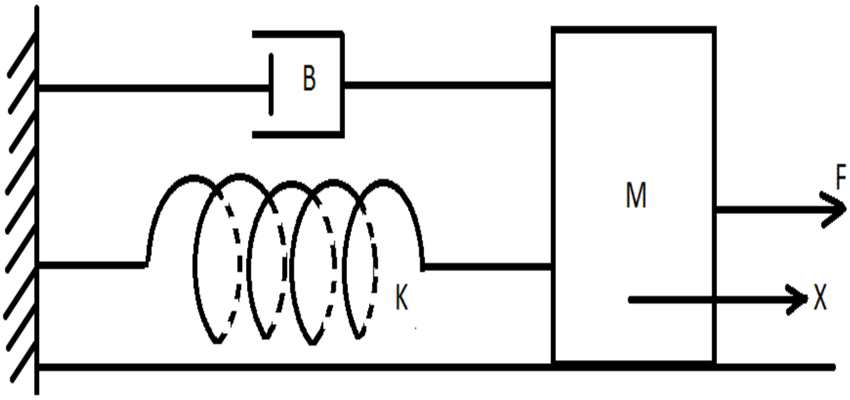

Fig. 1.1 A mass spring damper mechanical system

position, m is the mass of the object M sliding on a horizontal surface, k is the coefficient of spring K, b is the coefficient of the damper B and F is an external force which is considered as the control input u ($u = F$).

It is assumed that the mass spring damper system experiences a hardening spring which produces a restoring force described by (see [91])

$$k(1 + a^2X^2)X$$

The simple viscous damper produces a damping force described by $b\dot{X}$. From Newton's second law, the motion of the object M can be described by

$$m\ddot{X} = -b\dot{X} - k(1 + a^2X^2)X + u \tag{1.16}$$

Let $x = \text{col}(x_1, x_2) = (X, \dot{X})$. Then, $\dot{x}_1 = x_2$ and $\dot{x}_2 = \ddot{X}$. From equation (1.16), it follows that

$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 - \frac{k}{m}a^2x_1^3 + u$$

which can be rewritten by

$$\dot{x}_2 = -\Big(\frac{k}{m} + \frac{k}{m}a^2x_1^2\Big)x_1 - \frac{b}{m}x_2 + u$$

Choose $m = b = k = a = 1$ for simplicity. Then, the system (1.16) can be described in the form of (1.1) by

$$\dot{x} = \underbrace{\begin{bmatrix} x_2 \\ -(1+x_1^2)x_1 - x_2 + u \end{bmatrix}}_{f(x,u)} \tag{1.17}$$

which is a nonlinear system.

The objective is to design a sliding mode control law such that the system (1.17) is asymptotically stable.

i) **Sliding phase:** Design a linear switching function

$$s(x) = \gamma x_1 + x_2 \tag{1.18}$$

where γ is a design parameter. When system (1.17) is limited to the sliding surface, $s(x) = 0$. It follows from (1.18) that

$$x_2 = -\gamma x_1$$

Considering the structure of system (1.17), it is straightforward to see that the corresponding sliding mode dynamics are

$$\dot{x}_1 = -\gamma x_1 \tag{1.19}$$

Therefore, the sliding motion governed by the sliding mode dynamics (1.19) is asymptotically stable if the parameter γ is chosen to satisfy $\gamma > 0$.

ii) **Reaching phase:** Consider the sliding mode controller

$$u = (1+x_1^2)x_1 + x_2 - \gamma x_2 - \eta \,\mathrm{sgn}(\gamma x_1 + x_2) \tag{1.20}$$

where $\eta > 0$ is a constant. Then the closed-loop system obtained by applying the control in (1.20) to system (1.17) is given by

$$\dot{x}_1 = x_2 \tag{1.21}$$
$$\dot{x}_2 = -\gamma x_2 - \eta \,\mathrm{sgn}(\gamma x_1 + x_2) \tag{1.22}$$

By direct computation, it follows from equations (1.21)–(1.22) that

$$\begin{aligned} s(x)\dot{s}(x) &= -s(x)(\gamma \dot{x}_1 + \dot{x}_2) \\ &= -\eta s(x)\mathrm{sgn}(s(x)) \le -\eta |s| \end{aligned}$$

This guarantees that the control (1.20) can drive the trajectories of system (1.17) to the sliding surface $s(x) = 0$ with $s(\cdot)$ defined in (1.18), in finite time and maintain a sliding motion on it thereafter.

From sliding mode control theory, i) and ii) above together show that the corresponding closed-loop system is asymptotically stable. For simulation purposes, choose

$$\gamma = 0.5, \qquad \eta = 1$$

and the initial condition $x_0 = \mathrm{col}(2,1)$.

Figure 1.2 shows the phase plane portrait of the displacement x_1 and velocity x_2. From Figure 1.2, the system states (x_1,x_2) are driven to the sliding surface from the

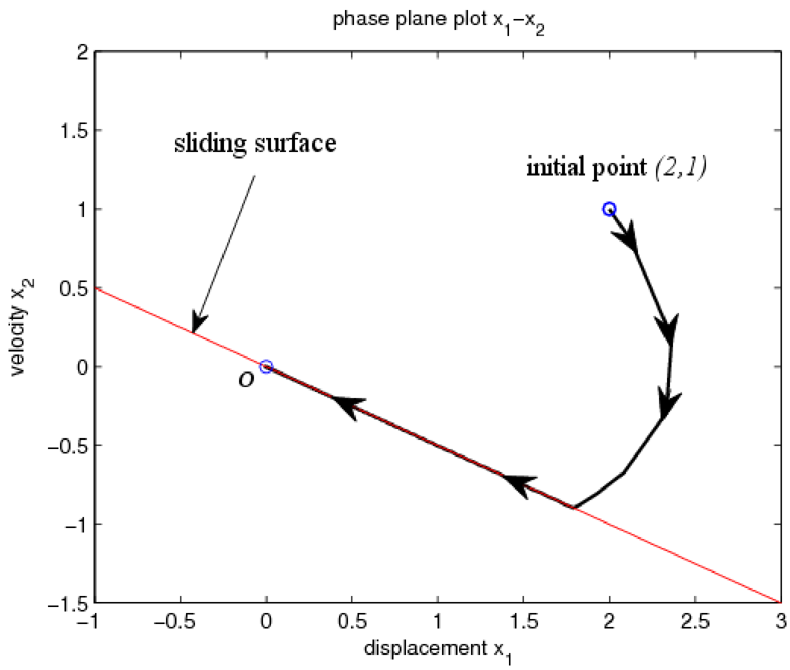

Fig. 1.2 The phase plane portrait

initial point $x_0 = (2,1)$, and then move along the sliding surface to converge to the origin.

The time responses of the displacement and velocity of the object are shown in Figure 1.3. Figure 1.4 shows the control signal imposed on the system.

It is clear to see that *chattering* appears due to the discontinuity in the control.

Chattering may be undesirable in practice because it may result in unnecessary wear and tear on the actuator components and result in unnecessary energy consumption. One way of overcoming this drawback is to introduce a boundary layer about the discontinuous surfaces (see [13]) which may affect the control accuracy. Another way is to use higher order sliding mode techniques but this requires the considered system to have a certain structure.

In this book, higher order sliding mode techniques will not be discussed. Detailed information about higher order sliding mode control can be found in [45, 5, 100, 153]) and the references therein.

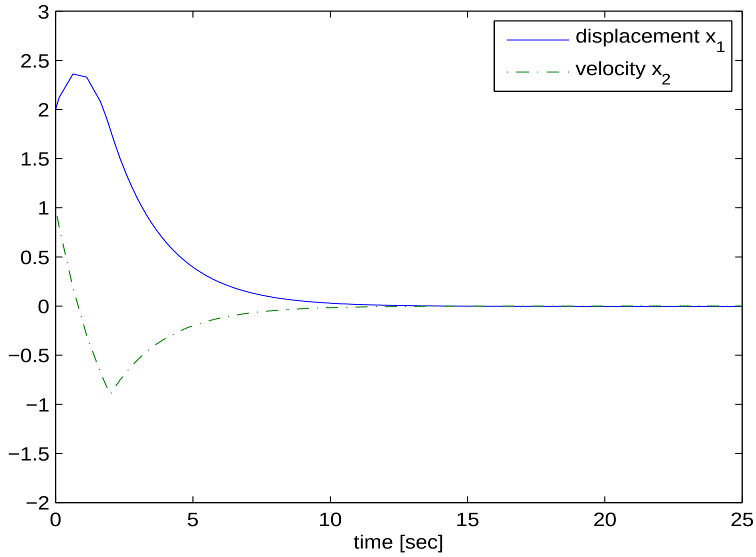

Fig. 1.3 The time responses of the displacement x_1 and velocity x_2

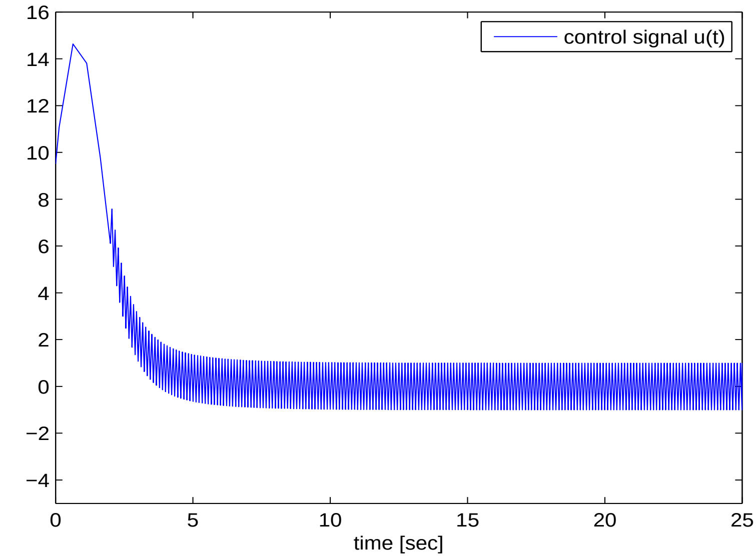

Fig. 1.4 The control signal

1.3.3 Characteristics of sliding mode control

It is observed that sliding mode control has the following characteristics:

- The sliding mode dynamics are a reduced-order system when compared with the original system dynamics.
  For system (1.1) with sliding surface $s = s(x)$, the corresponding sliding mode dynamics can be described by (1.6) or (1.11). It is clear to see that the order of the system (1.6) or (1.11) is $n - m$ where n is the dimension of the original system and m is the dimension of the control. Therefore, during the sliding motion, the system exhibits reduced-order dynamics when compared with the original system.

- The sliding motion is insensitive to matched uncertainty.
  Suppose system (1.1) experiences an uncertainty/disturbance. If the uncertainty or disturbance acts in the input/control channel or the effects are equivalent to an uncertainty acting in the input channel, it is called *matched uncertainty*. Otherwise it is called *mismatched uncertainty*. For example, assume that the nonlinear affine control system (1.7) experiences uncertainties $\phi(t,x)$ and $\psi(t,x)$ described by
  $$\dot{x} = F(t,x) + G(t,x)(u + \phi(t,x)) + \psi(t,x) \tag{1.23}$$
  Then, the term $\phi(t,x)$ is called matched uncertainty. In addition, if the uncertainty $\psi(t,x)$ can be modelled as
  $$\psi(t,x) = G(t,x)\chi(t,x)$$
  where $\chi(\cdot)$ represents the uncertainty, it is clear to see that the uncertainty of the term $\psi(\cdot)$ is reflected by the uncertainty $\chi(\cdot)$ which is exactly acting in the input channel. In this case $\psi(t,x)$ is also called matched uncertainty.
  From equations (1.6) or (1.11), it is straightforward to see that the dynamics governing the sliding motion are completely independent of the control and thus the system is robust to matched uncertainty.

- Uncertainties/disturbances will affect reachability.
  In order to guarantee that the trajectory of the considered system is driven to the pre-designed sliding surface, the reachability condition must be satisfied – which is interpreted as (1.15). It is clear that (1.15) involves all of the right hand side of equation (1.1). Therefore, uncertainties/disturbances may affect the reaching phase no matter whether they are matched or mismatched, but the effects of some uncertainties may be completely rejected by an appropriate control.

- The process of designing the sliding surface and sliding mode control can be 'separated'.
  The main target of sliding surface design is to ensure that the resulting sliding motion has the required performance. The main objective of the control design is that the reachability condition is satisfied so that the system can be driven to the sliding surface. In view of this, sliding surface design and sliding mode

control design can be completed separately. This property is called the design 'separation' property in this book.

The design of a sliding surface is usually not dependent on the process of the sliding mode control design. Once the sliding surface is specified, the study of the stability of the sliding motion and the reachability can be carried out separately. This has advantages when compared with other control approaches. For example, the steady state response is totally dependent on the sliding mode dynamics which is independent of the control. Therefore, in order to improve the steady-state response of the control system, it is only necessary to consider the sliding mode dynamics instead of the original system. In the reaching phase, by adjusting the parameters in the sliding mode control law, the reaching time can be reduced which may produce a fast time response, and will also maximise robustness.

1.4 Decentralised Control

In the real world, there are a number of important systems which can be modeled as dynamical equations composed of interconnections between a collection of lower-dimensional subsystems. Such classes of systems are called large-scale interconnected systems, which are often widely distributed in space [111, 117, 145]. A fundamental property of an interconnected system is that a perturbation of one subsystem can affect the other subsystems as well as the overall performance of the entire network. Decentralised control has been recognised as an effective method to control such systems.

1.4.1 Background

Large scale interconnected systems widely exist in society. A typical large scale interconnected system is the multimachine power system [182, 201]. Other examples of large scale interconnected systems that present a great challenge to both system analysts and control designers include power networks, ecological systems, biological systems and energy systems [117, 158].

For interconnected systems, the presupposition of centrality fails to hold due to either the lack of centralised information or the lack of centralised computing capability. When the number of subsystems is large, the computation time increases significantly if centralised control is employed. In the extreme case when information transfer among the subsystems is blocked, centralised control schemes simply cannot be applied. Even with engineered systems, issues such as the economic cost and reliability of communication links, particularly when systems are characterised by geographical separation, limit the appetite to develop centralised systems. From the perspective of economics and reliability, decentralised strategies are pertinent for large scale interconnected systems. This has motivated the application of decentralised control methodologies to interconnected systems [192, 87, 106]. A survey paper [2] has covered several decomposition approaches such as disjoint subsystems, overlapping subsystems, symmetric composite systems, multi-time scale systems and hierarchically structured systems to simplify the analysis and synthesis tasks for large-scale systems to reduce the computational complexity.

Decentralised control for large-scale interconnected systems has been studied extensively. Research on large-scale interconnected systems analysis and synthesis can be traced back to at least the 1970s, and the survey paper [145] clearly shows the development of this topic at that time, when almost all of the work focused on linear cases. With the advancement of technology and increasing requirements for high levels of performance, specifically in recent years, the dynamic systems used to model reality have become more complex involving nonlinearities, uncertainties, time delay and interconnection. Therefore, the study of complex interconnected systems has become increasingly important. The interest in this subject has been revived by new developments in nonlinear systems and control. The recent survey paper [216] has shown the progress made in the area of decentralised control where some of the work associated with sliding mode control, adaptive control and backstepping control has been covered.

1.4.2 Fundamental concept

From the mathematical point of view, a nonlinear large scale interconnected system composed of N n_i-th order subsystems can be described by

$$\dot{x}_i = f_i(t,x_i) + g_i(t,x_i)\big(u_i + \Delta g_i\big(t,x_i\big)\big) + \Delta f_i(t,x_i) + \sum_{\substack{j=1\\ j\neq i}}^{N} \zeta_{ij}(t,x_j) \quad (1.24)$$

$$y_i = h_i(x_i), \qquad i = 1,2,\ldots,N, \quad (1.25)$$

where $x_i \in \Omega_i \subseteq \mathbb{R}^{n_i}$ (Ω_i is a neighbourhood of the origin), $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{p_i}$ are the states, inputs and outputs of the i-th subsystem respectively for $i = 1,2,\ldots,N$. All the matrix functions $g_i(\cdot) \in \mathbb{R}^{n_i \times m_i}$ and the nonlinear vectors $f_i(\cdot) \in \mathbb{R}^{n_i}$ and $h_i(\cdot) \in \mathbb{R}^{p_i}$ with $h_i(0) = 0$ are known. The terms $\Delta g_i(\cdot)$ and $\Delta f_i(\cdot)$ represent the matched and the mismatched uncertainties respectively. The term

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \zeta_{ij}(t,x_j)$$

represents the interconnection of the i-th subsystem with the other subsystems. It is assumed that all the nonlinear functions are smooth enough such that the unforced systems have unique continuous solutions.

Definition 1.1. Consider system (1.24)–(1.25). The system

$$\dot{x}_i = f_i(t,x_i) + g_i(t,x_i)\big(u_i + \Delta g_i\big(t,x_i\big)\big) + \Delta f_i(t,x_i) \tag{1.26}$$
$$y_i = h_i(x_i), \qquad i = 1,2,\dots,N, \tag{1.27}$$

is called the i-th *isolated subsystem* of system (1.24)–(1.25), and the system

$$\dot{x}_i = f_i(t,x_i) + g_i(t,x_i)u_i \tag{1.28}$$
$$y_i = h_i(x_i), \qquad i = 1,2,\dots,N, \tag{1.29}$$

is called the i-th *nominal isolated subsystem* of system (1.24)–(1.25).

It is well known that one of the main problems for interconnected systems is to establish under what conditions the interconnected system (1.24)–(1.25) exhibits the desired performance if all the isolated subsystems (1.26)–(1.27) or all the nominal isolated subsystems (1.28)–(1.29) exhibit the required performance. Therefore, how to deal with interconnections is a key problem of interest in decentralised control.

Definition 1.2. Consider system (1.24)–(1.25). If the designed controllers u_i for the i-th subsystems depend on the time t and states x_i of the i-th subsystem only, i.e.

$$u_i = u_i(t,x_i), \qquad i = 1,2,\dots,N \tag{1.30}$$

then (1.30) is called *decentralised state feedback control*. If the controllers in (1.30) have the form

$$u_i = u_i(t,y_i), \qquad i = 1,2,\dots,N \tag{1.31}$$

that is, each local controller depends upon the time t and the outputs of the local subsystem only, then they are called *decentralised static output feedback control*. Furthermore, if the designed controllers consist of the dynamical systems

$$\dot{\hat{x}}_i = \phi_i(t,\hat{x}_i,u_i,y_i), \qquad i = 1,2,\dots,N \tag{1.32}$$

and controllers

$$u_i = u_i(t,\hat{x}_i,y_i), \qquad i = 1,2,\dots,N \tag{1.33}$$

then (1.32)–(1.33) is called *decentralised dynamical output feedback control*. Specifically, if (1.32) is an observer of the system (1.24)–(1.25), then it is called *decentralised observer-based feedback control*.

It is straightforward to see, according to Definition 1.2 above, that it is required that the dynamical systems (1.32) are decoupled in a decentralised dynamical output feedback scheme. It should be mentioned that in some of the existing work, see for example [203, 215], the designed dynamical systems (1.32) are not decoupled (in fact they are interconnected systems). In this case, the developed controllers are sometimes still called a decentralised control. However, in precise terms, such a class of controllers is not decentralised because there exists information transfer between subsystems of the designed dynamical system (see e.g. [203, 215]).

Several decades ago, most work on decentralised control focused on linear interconnected systems due to the limitation of available control paradigms that were able to deal with nonlinearity. However, the dynamics of large scale natural and engineered interconnected systems are usually highly nonlinear. It is not only the structure of the system and interconnections which produce complexity but also the nonlinearity of the dynamics themselves. It is clear that although linear dynamics may approximate the orbit of a nonlinear system locally, it does not permit the existence of the multiple states observed in real networks and does not accommodate global properties of the system. Such global properties can be crucial because they may become significant when the system is perturbed or a subsystem enters a failure state. Increasing requirements on system performance coupled with the ability to model and simulate reality by means of complex, possibly nonlinear, interconnected systems models has motivated increasing contributions in the study of such systems. This interest has been further stimulated by the simultaneous development of nonlinear systems theory and the emergence of powerful mathematical and computational tools which render the formal and constructive study of nonlinear large scale systems increasingly possible [210].

In order to help readers to understand the 'decentralised' concept, the following schematic diagram in which the interconnected system has three subsystems, is produced to show that in static decentralised output feedback control scheme, the local controller u_i of the i-th subsystem only uses the local output information y_i; no output information y_j $(j \neq i)$ is involved in the design of u_i. From Figure 1.5, it is

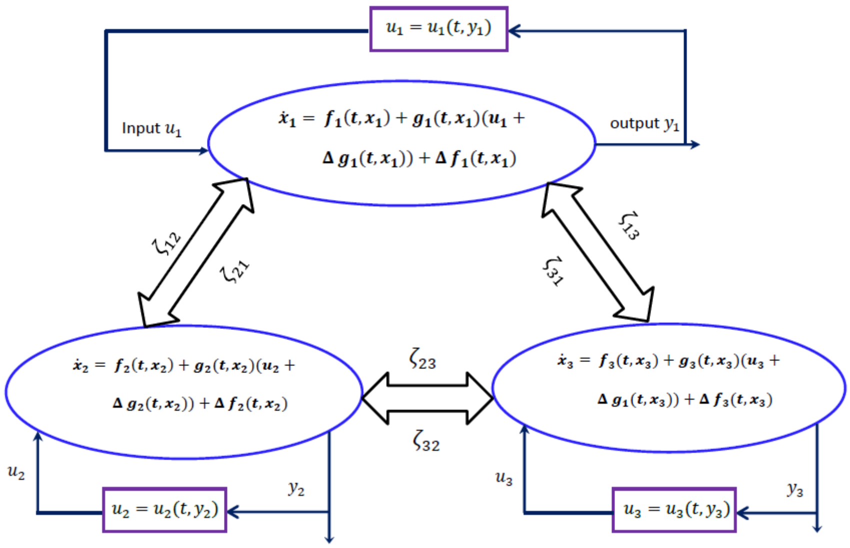

Fig. 1.5 Decentralised static output feedback schematic diagram

clear that there is no local output information transfer between the local controllers u_i and u_j $(i \neq j)$ for $i, j = 1, 2, 3$.

1.5 Examples of Complex Systems

In this section, some practical examples will be presented to show that complex systems widely exist in the real world.

1.5.1 One-machine infinite-bus system

Consider a simple power system where a large-turbine generator set connects with an infinite bus. The motion equation of the machine's rotor can be described by (see, for example, [107])

$$H\frac{d^2\delta}{dt^2} = M_m(t) - \frac{E_q V_s}{X_\delta}\sin\delta(t) \tag{1.34}$$

where $\delta(t)$ is the generator's rotor angle, M_m is the mechanical input torque, H is the moment of inertia of the machine, E_q is the transient potential of the q-axis of the generator, V_s is the voltage of the infinite bus which is constant, X_δ is the sum of the transient inductance of the shaft of generator, the inductance of the transformer and the inductance of the transmission line.

For simplicity, assume that E_q is constant. Let

$$x_1 = \delta \qquad \text{and} \qquad x_2 = \dot{\delta}$$

where x_2 represents the angular velocity. The letter M_m denotes the control input u. Then the system (1.34) modelling the one-machine infinite-bus is described by

$$\dot{x} = \begin{bmatrix} x_2 \\ -a_1 \sin x_1 + a_2 u \end{bmatrix} \tag{1.35}$$

where $x := \mathrm{col}(x_1, x_2)$, and

$$a_1 := \tfrac{E_q V_s}{H X_\delta}$$
$$a_2 := \tfrac{1}{H}$$

System (1.35) is a nonlinear affine system as it can be described by

$$\dot{x} = \begin{bmatrix} x_2 \\ -a_1 \sin x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ a_2 \end{bmatrix} u$$

where the input distribution is a constant matrix.

1.5.2 PVTOL aircraft

The well-known Planar Vertical Take-Off and Landing (PVTOL) represents a challenging nonlinear control problem. It is motivated by the need to stabilise an aircraft which is able to take-off vertically such as helicopters and some special aircraft.

The mathematical model describing an aircraft that evolves in a vertical plane usually has three degrees of freedom (X,Y,ϕ) corresponding to its position (X,Y) and orientation in the plane ϕ. The PVTOL is composed of two independent thrusters that produce a force and a moment on the aircraft. The dynamical model of the PVTOL aircraft can be obtained using the Lagrangian approach or Newtons laws, which are given in [191], as follows

$$\begin{aligned} m\ddot{X} &= -(\sin\phi)U_1 + \varepsilon_0(\cos\phi)U_2 \\ m\ddot{Y} &= (\cos\phi)U_1 + \varepsilon_0(\cos\phi)U_2 - mg \\ J\ddot{\phi} &= U_2 \end{aligned}$$

where (X,Y) is the center of mass of the aircraft, θ is the roll angle, mg is the gravity force imposed at the aircraft center of mass and J is the mass moment of inertia around the axis through the aircraft center of the mass and along the fuselage, the control U_1 is the thrust directed to the bottom of aircraft and the control U_2 the moment around the aircraft center of the mass, ε_0 is the quantity of lateral force induced by the rolling moment which characterizes the coupling between the rolling moment and the lateral acceleration of the aircraft.

Let

$$\bar{x} = -X/g, \quad \bar{y} = -Y/g, \quad u_1 = U_1/mg$$

$$U_2 = U_2/mg, \quad \varepsilon = \varepsilon_0 J/mg$$

Then the normalised PVTOL aircraft dynamics can be described by [16, 191]

$$\ddot{\bar{x}} = -(\sin\phi)u_1 + \varepsilon(\cos\phi)u_2 \tag{1.36}$$

$$\ddot{\bar{y}} = (\cos\phi)u_1 + \varepsilon(\cos\phi)u_2 - 1 \tag{1.37}$$

$$\ddot{\phi} = u_2 \tag{1.38}$$

The dynamical equations (1.36)–(1.38) can be described in (1.1) as follows

$$\dot{x} = \begin{bmatrix} x_2 \\ -(\sin x_5)u_1 + \varepsilon(\cos x_5)u_2 \\ x_4 \\ (\cos x_5)u_1 + \varepsilon(\cos x_5)u_2 - 1 \\ x_6 \\ u_2 \end{bmatrix} \tag{1.39}$$

where

$$x_1 := \bar{x}, \quad x_2 := \dot{\bar{x}}, \quad x_3 := \bar{y}$$

$$x_4 := \dot{y}, \quad x_5 := \phi, \quad x_6 := \dot{\phi}$$

System (1.39) can be rewritten by

$$\dot{x} = \begin{bmatrix} x_2 \\ 0 \\ x_4 \\ -1 \\ x_6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\sin x_5 & \varepsilon \cos x_5 \\ 0 & 0 \\ \cos x_5 & \varepsilon \cos x_5 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $x := \mathrm{col}(x_1, x_2, \ldots, x_6)$, and thus it represents an affine nonlinear control system. In general, ε is unknown but it is very small and can be neglected. In this case, the model can be simplified as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(\sin x_5)u_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= (\cos x_5)u_1 - 1 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= u_2 \end{aligned}$$

This is a nonlinear system.

1.5.3 Stirred tank reactor

Consider an industrial jacketed continuous stirred tank reactor (JCSTR) with a delayed recycle stream [116]. The reactions within the JCSTR are assumed unimolecular and irreversible (exothermic). Perfect mixing is assumed and the heat losses are neglected. The reactor accepts a feed of reactant which contains a substance A with initial concentration C_{A_0}. Cooling of the tank is achieved by a flow of water around the jacket and the water flow in the jacket F_J is controlled by actuating a valve.

Suppose that a fresh feed of pure substance A is to be mixed with a recycled stream of unreacted substance A with a recycle flow rate

$$1 - c, \qquad (0 \leq c \leq 1)$$

where c is the coefficient of recirculation.

The change of concentration arises from three terms: the amount of substance A that is added with feed under recycling, the amount of substance A that leaves with the product flow, and the amount of the substance A that is used up in the reaction. The change in the temperature of the fluid comes from the following four factors: the heat that enters with the feed flow under recycling, the heat that leaves with the product flow, the heat created by the reaction and the heat that is transferred to the cooling jacket. There are three terms associated with the changes of the temperature of the fluid in the jacket: one term representing the heat entering the jacket with the cooling fluid flow, another term accounting for the heat leaving the jacket with the outflow of cooling liquid and a third term representing the heat transferred from the fluid in the reaction tank to the fluid in the jacket.

Under conditions of constant hold-up, constant densities and perfect mixing, the energy and material balances can be expressed mathematically as [116]:

$$
\begin{aligned}
\dot{C}_A &= (FV)^{-1}\left(cC_{A_0} - cC_A - cC_A(t-d)\right) - k_1 C_A e^{-\frac{k_2}{T}} \\
\dot{T} &= (FV)^{-1}\left(cT_0 - cT - cT(t-d)\right) - k_1 k_3 C_A e^{-\frac{k_2}{T}} - k_4(T - T_J(t)) \\
\dot{T}_J &= (F_J V_J)^{-1}\left(T_{J_0} - T_J\right) - k_5(T - T_J)
\end{aligned}
$$

where C_A is the concentration of the substance A, T is the temperature of the fluid in the tank, T_J is the temperature of the jacket, V is the volume of the tank (gallons), F is the feed entry rate, the initial temperature is T_0, and d represents the transport delay in the recycled stream.

It is straightforward to see that system (1.40)–(1.40) is a nonlinear time-delay control system and can be described in the form of (1.1) as

$$
\dot{x} = \begin{pmatrix} (FV)^{-1}\left(cC_{A_0} - cx_1 - cx_1(t-d)\right) - k_1 x_1 e^{-\frac{k_2}{x_2}} \\ (FV)^{-1}\left(cT_0 - cx_2 - cx_2(t-d)\right) - k_1 k_3 x_1 e^{-\frac{k_2}{x_2}} - k_4(x_2 - x_3(t)) \\ (uV_J)^{-1}\left(T_{J_0} - x_3\right) - k_5(x_2 - x_3) \end{pmatrix} \tag{1.40}
$$

where $x_1 = C_A$, $x_2 = T$, $x_3 = T_J$, $x = \text{col}(x_1, x_2, x_3)$ is the system states and $u = F_J$ is the system input. The letter d represents the time delay.

1.5.4 *Coupled inverted pendula on carts*

Consider a coupled inverted pendulum connected by a moving spring mounted on two carts as shown in Figure 1.6. It is assumed that the pivot position of the moving spring is a function of time which can change along the full length l of the pendula. The input to each pendulum is the torque u_i applied at the pivot which is produced by the external forces F_1 and F_2 applied to the carts.

Let

$$
z_1 = \text{col}(\theta_1, \dot{\theta}_1)^T, \qquad \text{and} \qquad z_2 = \text{col}(\theta_2, \dot{\theta}_2)^T
$$

Then the dynamical model for the two coupled inverted pendulum system is given by (see [149]):

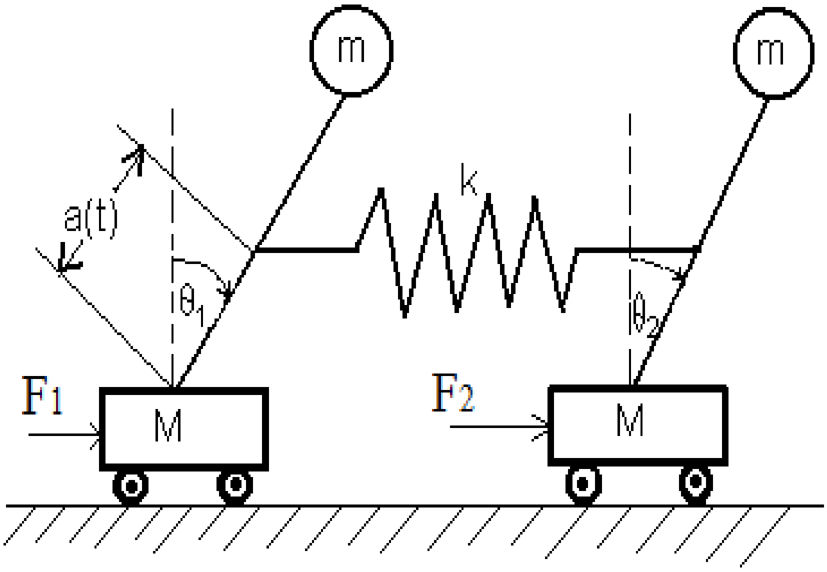

Fig. 1.6 Two coupled inverted pendula on carts

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ \frac{g}{cl} - \frac{ka(t)(a(t)-cl)}{cml^2} & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ \frac{1}{cml^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 & 0 \\ \frac{ka(t)(a(t)-cl)}{cml^2} & 0 \end{bmatrix} x_2 - \begin{bmatrix} 0 \\ \frac{m}{M}(\sin\theta_1)\dot{\theta}_1^2 + \frac{ka(t)(a(t)-cl)}{cml^2}(s_1 - s_2) \end{bmatrix} \tag{1.41}$$

$$\dot{x}_2 = \begin{bmatrix} 0 & 1 \\ \frac{g}{cl} - \frac{ka(t)(a(t)-cl)}{cml^2} & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ \frac{1}{cml^2} \end{bmatrix} u_2 + \begin{bmatrix} 0 & 0 \\ \frac{ka(t)(a(t)-cl)}{cml^2} & 0 \end{bmatrix} x_1 - \begin{bmatrix} 0 \\ \frac{m}{M}(\sin\theta_2)\dot{\theta}_2^2 + \frac{ka(t)(a(t)-cl)}{cml^2}(s_2 - s_1) \end{bmatrix} \tag{1.42}$$

where s_1 and s_2 are the positions of the two carts,

$$c = M/(M+m)$$

and k and g are the spring and gravity constants, respectively. The system (1.41)–(1.42) is a nonlinear interconnected control system.

1.5.5 Multimachine power systems

Power systems play an important role in the practical world. The classical model of power systems was given by Bergen [8], and based on this, a multimachine power system consisting of N synchronous generators interconnected through a transmission network is described by the following equations [67]:

- Mechanical equations

$$\dot{\delta}_i = \omega_i, \tag{1.43}$$

$$\dot{\omega}_i = -\frac{D_i}{2H_i}\omega_i + \frac{\omega_0}{2H_i}(P_{mi0} - P_{ei}) \tag{1.44}$$

- Generator electrical dynamics:

$$\dot{E}'_{qi} = \frac{1}{T'_{doi}}(E_{fi} - E_{qi}). \tag{1.45}$$

- Electrical equations

$$E_{qi} = E'_{qi} - (x_{di} - x'_{di})I_{di}, \tag{1.46}$$

$$E_{fi} = K_{ci}u_{fi} \tag{1.47}$$

$$P_{ei} = \sum_{j=1}^{N} E'_{qi}E'_{qj}B_{ij}\sin(\delta_i - \delta_j) \tag{1.48}$$

$$Q_{ei} = -\sum_{j=1}^{N} E'_{qi}E'_{qj}B_{ij}\cos(\delta_i - \delta_j) \tag{1.49}$$

$$I_{qi} = \sum_{j=1}^{N} E'_{qj}B_{ij}\sin(\delta_i - \delta_j) \tag{1.50}$$

$$I_{di} = \sum_{j=1}^{N} E'_{qj}B_{ij}\cos(\delta_i - \delta_j) \tag{1.51}$$

$$E_{qi} = x_{adi}I_{fi} \tag{1.52}$$

$$V_{ti} = \sqrt{(E'_{qi} - x'_{di}I_{di})^2 + (x'_{di}I_{qi})^2} \tag{1.53}$$

where δ_i is the i-th generator power angle [rad], and ω_i is the relative speed [rad/s], E'_{qi} represents the transient EMF in the quadrature axis [p.u.], and u_{fi} is the input of the amplifier of the i-th generator for $i = 1, 2, \ldots, N$. The physical meanings of all the other symbols/notation used above are shown in Appendix E.1.

This model has been used by many authors to study multimachine power systems [67, 193, 108, 182]. The multimachine power system shown above can be expressed in the form of (1.24) (see, for example, Chapter 9).

1.5.6 A biochemical system – peroxidase-oxidase reaction

As a biochemical system, the peroxidase-oxidase (PO) reaction exhibits many complex dynamical behaviors. A great deal of experimental and theoretical work has been devoted to determining the mechanism by which oscillations and chaos arise in the PO reaction.

In addition to oscillatory and chaotic behavior, the PO reaction exhibits bistability. Due to its suspected kinetic source: the inhibition of the enzyme by molecular oxygen, both autocatalysis and inhibition, i.e. positive and negative feedback are needed in the reaction mechanism for this system. A simple model for the PO reaction is described in [30, 181] as follows

$$\begin{aligned}
\dot{A} &= -k_1 ABX - k_3 ABY + k_7 - k_9 A \\
\dot{B} &= -k_1 ABX - k_3 ABY + k_8 \\
\dot{X} &= k_1 ABX - 2k_2 X^2 + 2k_3 ABY - k_4 X + k_6 \\
\dot{Y} &= -k_3 ABY + 2k_2 X^2 - k_5 Y
\end{aligned}$$

where A is the concentration of dissolved O_2, B is the concentration of Nicotinamide adenine dinucleotide, and X and Y are concentrations of two critical intermediates, X and Y.

Typically all parameters except k_1 are constant. The parameter k_1 can be considered as a bifurcation parameter. Chaos is found only within a certain range of parameter values. Variations in k_1 reproduce the experimental behaviour observed when the enzyme concentration is changed. Thus k_1 can be considered as being related to the enzyme catalyst concentration [30, 181].

This section has provided practical examples of complex systems. Some will be used to demonstrate the developed results later in the text and additional examples will be given in the subsequent chapters.

1.6 Outline of this Book

This monograph systematically summarises the authors' recent results in the area of variable structure systems. It will focus on the analysis and design of complex systems where sliding mode techniques and the Lyapunov approach are the two main methods used throughout the monograph. Simulation examples and/or case studies are presented in each chapter to help readers understand the obtained theoretical results and utilise the proposed design approaches.

The book is organised as follows. Firstly, the fundamental mathematical knowledge and basic control theory employed in the subsequent analysis and design in this monograph will be presented in Chapter 2. Considering that static output feedback control design is more convenient for real implementation when compared with state feedback control, in Chapter 3, robust static output controllers are designed to globally asymptotically stabilise the system, and then a decentralised static output feedback sliding mode control scheme follows for a class of nonlinear interconnected systems.

As static output feedback control imposes strong limitations on the considered system, dynamical feedback control is investigated in Chapter 4 where both minimum phase and non-minimum phase systems are considered. Chapter 4 studies dynamical output feedback control for nonlinear interconnected systems. Since large scale interconnected systems have higher dimension, and dynamical output feedback will greatly increase the dimension of the closed-loop system, reduced-order observer based feedback controllers are considered in Chapter 5.

Time delay is a factor which increases system complexity. Chapters 6 and 7 concentrate on the study of nonlinear time delay systems where the Lyapunov-Razumikhin approach is used to deal with the time delay. Under the assumption that the time delays are known, control schemes for nonlinear time delay systems, and a decentralised control strategy for interconnected systems are proposed in Chapter 6. In practice, knowledge of the time delay is not always available for design. In connection with this, memoryless variable structure controllers are presented in Chapter 7.

Chapter 8 discusses model based fault detection and isolation for nonlinear systems with uncertainties. The reconstruction and/or estimation of both system faults and sensor faults are considered based on a sliding mode observer scheme. LMI techniques are employed to facilitate the design of the parameters. A coordinate transformation is employed to explore the system structure when the considered system is fully nonlinear.

Applications of decentralised sliding mode control schemes to multimachine power systems are presented in Chapter 9. Simulation studies on three machine power systems confirm the theoretical results.

Finally, Chapter 10 concludes the book by providing some comments on the developed methods, some specific examples to show the complexity of control systems, and some suggestions for future developments in the area of variable structure control.

Chapter 2
Mathematical Background

This chapter presents some fundamental mathematical knowledge and basic results which facilitate the analysis and design in the subsequent chapters. The motivation is to help readers understand the theoretical work presented in this book.

2.1 Lipschitz Function

This section will present the well known Lipschitz condition and the generalised Lipschitz condition.

2.1.1 Lipschitz Condition

Definition 2.1. A function $f(x):\ \mathbb{R}^n \mapsto \mathbb{R}^m$ is said to satisfy the *Lipschitz condition* in the domain $\Omega \subset \mathbb{R}^n$ if there exists a nonnegative constant L such that the inequality

$$f(x) - f(\hat{x}) \leq L\|x - \hat{x}\| \tag{2.1}$$

holds for any $x \in \Omega$ and $\hat{x} \in \Omega$. Then L is called the *Lipschitz constant* and $f(x)$ is called a *Lipschitz function* in Ω. If $\Omega = \mathbb{R}^n$, then $f(x)$ is said to satisfy the *global Lipschitz condition.*

From Definition 2.1, it is clear that a Lipschitz function must be continuous. However the converse is not true and a typical example is the scalar function

$$f(x) = x^{1/3}$$

in a neighbourhood of the origin $x = 0$. A Lipschitz function may not be differentiable and a simple example is the scalar function

$$f(x) = |x|$$

at the origin $x = 0$ in $x \in \mathbb{R}$. Moreover, a differentiable function may not be Lipschitz on a compact set, for example the function

$$f(x) = \begin{cases} x^{\alpha} \sin \frac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases} \tag{2.2}$$

is not Lipschitz in the compact set $x \in [0,\ 1]$ for any constant α satisfying $1 < \alpha < 2$. The reason is that the derivative of the function $f(x)$ defined in (2.2) is not bounded in the interval $[0,\ 1]$.

Lemma 2.1. *[91] Consider a function $f(x):\ \mathbb{R}^n \mapsto \mathbb{R}^m$ which is differentiable in the domain Ω. If its Jacobian matrix is bounded in Ω, that is, there exists a constant L such that*

$$\|J_f\| \leq L$$

for any $x \in \Omega$, then $f(x)$ satisfies the Lipshitz condition, and the inequality (2.1) holds.

2.1.2 Generalised Lipschitz condition

The well known Lipschitz condition in Section 2.1.1 will be extended to a more general case which will be used later in the analysis.

Definition 2.2. A function $f(x_1, x_2, x_3) : \Omega_1 \times \Omega_2 \times \Omega_3 \mapsto \mathbb{R}^n$ is said to satisfy a generalised Lipschitz condition with respect to (w.r.t.) the variables $x_1 \in \Omega_1 \subset \mathbb{R}^{n_1}$ and $x_2 \in \Omega_2 \subset \mathbb{R}^{n_2}$ uniformly for x_3 in $\Omega_3 \subset \mathbb{R}^{n_3}$ if there exist nonnegative continuous functions $\mathscr{L}_{f1}(\cdot)$ and $\mathscr{L}_{f2}(\cdot)$ defined in Ω_3 such that for any $\hat{x}_1, x_1 \in \Omega_1$ and $\hat{x}_2, x_2 \in \Omega_2$, the inequality

$$\|f(x_1, x_2, x_3) - f(\hat{x}_1, \hat{x}_2, x_3)\| \leq \mathscr{L}_{f1}(x_3)\,\|x_1 - \hat{x}_1\| + \mathscr{L}_{f2}(x_3)\,\|x_2 - \hat{x}_2\|$$

holds for any $x_3 \in \Omega_3$. Then, $f(\cdot)$ is called a generalised Lipschitz function, and $\mathscr{L}_{f1}(\cdot)$ and $\mathscr{L}_{f2}(\cdot)$ are called generalised Lipschitz bounds. Further, if $\Omega_1 = \mathbb{R}^{n_1}$ and $\Omega_2 = \mathbb{R}^{n_2}$, then, it is said that $f(\cdot)$ satisfies a global generalised Lipschitz condition w.r.t. x_1 and x_2 uniformly for x_3 in Ω_3.

Remark 2.1. The symbols $\mathscr{L}_{f1}(\cdot)$ and $\mathscr{L}_{f2}(\cdot)$ introduced above are usually nonnegative functions instead of constants. This is different from the Lipschitz condition. Thus, the nonnegative continuous functions $\mathscr{L}_{f1}(x_3)$ and $\mathscr{L}_{f2}(x_3)$ are called generalised Lipschitz bounds which correspond to the Lipschitz constant for the Lipschitz condition.

Clearly, the generalised Lipschitz condition is more relaxed than the Lipschitz condition. For example, the function

$$f(x_1, x_2, x_3) := x_1 x_3^2 + x_2 x_3$$

with $x_1, x_2, x_3 \in \mathbb{R}$ does not satisfy the global Lipschitz condition. However, from the inequality that for any $\mathrm{col}(x_1, x_2, x_3) \in \mathbb{R}^3$ and $\mathrm{col}(\hat{x}_1, \hat{x}_2, x_3) \in \mathbb{R}^3$

$$|f(x_1, x_2, x_3) - f(\hat{x}_1, \hat{x}_2, x_3)| \leq |x_1 - \hat{x}_1| x_3^2 + |x_2 - \hat{x}_2|\ |x_3|$$

it is clear to see that $f(\cdot)$ satisfies the global generalised Lipschitz condition w.r.t. x_1 and x_2, uniformly for $x_3 \in \mathbb{R}$.

2.2 Comparison Functions

This section will present the definitions and properties of the class $\mathscr{K}$ function and related functions.

Definition 2.3. (see [91]) A continuous function $\alpha:\ [0,a) \mapsto \mathbb{R}^+$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathscr{K}_\infty$ if $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$.

Definition 2.4. (see [91]) A continuous function $\beta:\ [0,\ a) \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to belong to class $\mathscr{K}\mathscr{L}$ if, for any given $s \in \mathbb{R}^+$, the mapping $\beta(r,s)$ belongs to class $\mathscr{K}$ with respect to the variable r, and for any given $r \in [0,\ a)$, the mapping $\beta(r,s)$ is decreasing with respect to the variable s and $\lim_{s\to\infty} \beta(r,s) = 0$.

Definition 2.5. If a class $\mathscr{K}$ function is a C^1 function, then it is said to belong to class $\mathscr{K}C^1$. A continuous function $\beta: \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to be a class $\mathscr{K}I$ function if for any given $x \in \mathbb{R}^n$ the function $\beta(x,s)$ is increasing with respect to the variable s in $\mathbb{R}^+$, that is, $\beta(x,s_1) \leq \beta(x,s_2)$ for any $0 \leq s_1 \leq s_2$.

The functions defined in Definitions 2.3 and 2.4 above are directly from [91]. The new concepts of class $\mathscr{K}C^1$ functions and class $\mathscr{K}I$ functions are introduced in Definition 2.5 will be used for later analysis.

The following new concept is introduced, which will be termed a class $\mathscr{W}\mathscr{S}$ function and will be used in Section 7.3.

Definition 2.6. A continuous function $\beta(t,x_1,x_2): \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\beta(t,0,0) = 0$ is said to be weak w.r.t the variable x_1 and strong w.r.t. the variable x_2 if there exist functions $\chi_1(t,x_1,x_2)$ and $\chi_2(t,x_1,x_2)$ such that

$$\beta(t,x_1,x_2) = \chi_1(t,x_1,x_2)x_1 + \chi_2(t,x_1,x_2)x_2 \tag{2.3}$$

where both $\chi_1(\cdot,\cdot,x_2)$ and $\chi_2(\cdot,\cdot,x_2)$ are continuous and nondecreasing w.r.t. the variable x_2. Further, the function $\beta(t,x_1,x_2)$ is said to be a class $\mathscr{W}\mathscr{S}$ function w.r.t. the variables x_1 and x_2.

Remark 2.2. It should be noted that if a function $\beta(t,x_1,x_2): \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\beta(t,0,0) = 0$ is smooth enough, then it follows from [3] that there exist continuous functions $\beta_1(\cdot)$ and $\beta_2(\cdot)$ such that the expression

$$\beta(t,x_1,x_2) = \beta_1(t,x_1,x_2)x_1 + \beta_2(t,x_1,x_2)x_2$$

holds. Moreover, if $\beta_1(t,x_1,x_2)$ and $\beta_2(t,x_1,x_2)$ are nondecreasing w.r.t. x_2, then $\beta(t,x_1,x_2)$ is a class $\mathscr{WS}$ function w.r.t. x_1 and x_2.

Lemma 2.2. *(see [91]) Assume that $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are class $\mathscr{K}$ functions in $[0,\ a)$, $\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ are class $\mathscr{K}_\infty$ functions, and $\beta(\cdot)$ is a class $\mathscr{KL}$ function defined in $[0,\ a)\times\mathbb{R}^+$. Then, the following results hold:*

- *the inverse function $\alpha_1^{-1}(\cdot)$ is a class $\mathscr{K}$ function defined in $[0,\ \alpha_1(a))$.*
- *the inverse function $\alpha_3^{-1}(\cdot)$ is a class $\mathscr{K}_\infty$ function defined in $[0,\ \infty)$.*
- *the composite function $\alpha_1 \circ \alpha_2$ is a class $\mathscr{K}$ function.*
- *the composite function $\alpha_3 \circ \alpha_4$ is a class $\mathscr{K}_\infty$ function.*
- *the function $\sigma(r,s) = \alpha_1(\beta(\alpha_2(r),s))$ is a class $\mathscr{KL}$ function.*

Lemma 2.3. *The following results hold:*

i) If $\beta(x,s) : \mathbb{R}^n \times R^+ \mapsto \mathbb{R}^+$ is a class $\mathscr{K}I$ function, then $\beta^2(x,s)$ is a class $\mathscr{K}I$ function.

ii) Suppose that a function $\phi_1 :\ [0,a) \mapsto \mathbb{R}^+$ is a C^1 function with $\phi_1(0) = 0$. Then there exists a continuous function $\phi_2(\cdot)$ in $[0,a)$ such that

$$\phi_1(s) = \phi_2(s)s, \qquad s \in [0,a)$$

Proof: i) Suppose that $\beta(x,s) : \mathbb{R}^n \times R^+ \mapsto \mathbb{R}^+$ is a class $\mathscr{K}I$ function. Then for any $0 \le s_1 \le s_2$ and $x \in \mathbb{R}^n$,

$$\beta(x,s_1) \le \beta(x,s_2)$$

Since $\beta(x,s) \ge 0$ for any $(x,s) \in \mathbb{R}^n \times \mathbb{R}^+$

$$\begin{aligned} & \beta^2(x,s_1) - \beta^2(x,s_2) \\ = & \ (\beta(x,s_1) + \beta(x,s_2))(\beta(x,s_1) - \beta(x,s_2)) \\ \le & \ 0 \end{aligned}$$

This shows that $\beta^2(x,s)$ is a class $\mathscr{K}I$ function

ii) Since the function $\phi_1(\cdot)$ is a C^1 function in $[0,a)$, its derivative $\frac{d\phi_1(s)}{ds}$ is continuous in $[0,a)$. For any $s \in [0,a)$, construct a function

$$\phi_2(s) := \begin{cases} \dfrac{\phi_1(s)}{s}, & s \neq 0 \\ \dfrac{d\phi_1(s)}{ds}\Big|_{s=0}, & s = 0 \end{cases} \tag{2.4}$$

From the definition of $\phi_2(\cdot)$, it is clear to see that

1) if $s \neq 0$, then $\phi_1(s) = \phi_2(s)s$;
2) if $s = 0$, then from $\phi_1(0) = 0$, $\phi_1(s) = \phi_2(s)s$.

Therefore, the expression

$$\phi_1(s) = \phi_2(s)s$$

holds for $s \in [0,a)$. It remains to prove that the function $\phi_2(\cdot)$ defined in (2.4) is continuous in $[0,a)$.

It is clear that $\phi_2(s)$ is continuous in $(0,a)$. Since ϕ_1 is a C^1 function in $[0,a)$, from the continuity of $\frac{d\phi_1(s)}{ds}$ at $s=0$,

$$\lim_{s\to 0^+} \phi_2(s) = \lim_{s\to 0^+} \frac{\phi_1(s)}{s} = \frac{d\phi_1(s)}{ds} \mid_{s=0} = \phi_2(0)$$

which implies that $\phi_2(\cdot)$ is continuous at $s=0$. Therefore $\phi_2(\cdot)$ is continuous in $[0,a)$.

Hence the conclusion follows. ∇

2.3 Lyapunov Stability Theorems

The results given in this section are available in [91].

Consider the nonlinear system

$$\dot{x}(t) = f(t, x(t)) \tag{2.5}$$

where the function $f : \mathbb{R}^+ \times D \mapsto \mathbb{R}^n$ is continuous and $D \subset \mathbb{R}^n$ is a domain which contains the origin $x=0$. It is assumed that

$$f(t,0) = 0, \qquad t \in \mathbb{R}^+$$

which implies that the origin is an equilibrium point of the system.

Definition 2.7. The equilibrium point $x=0$ of system (2.5) is called exponential stable if there exist positive constants c_i for $i=1,2,3$ such that for any $x(t_0)$ satisfying $\|x(t_0)\| \le c_1$,

$$\|x(t)\| \le c_2 \|x(t_0)\| e^{-c_3(t-t_0)} \tag{2.6}$$

If inequality (2.6) holds for any $x(t_0) \in \mathbb{R}^n$, then, the equilibrium point $x=0$ of system (2.5) is called globally exponentially stable.

2.3.1 Asymptotic stability

Theorem 2.1. *Consider system (2.5). Let* $V : \mathbb{R}^+ \times D \mapsto \mathbb{R}^+$ *be a continuously differentiable function such that*

$$W_1(x) \le V(t,x) \le W_2(x)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x)$$

for any $t \in \mathbb{R}^+$ and $x \in D$, where $W_i(x)$ for $i = 1,2,3$ are continuous positive definite functions in D. Then $x = 0$ is uniformly asymptotically stable. Further if $D = \mathbb{R}^n$, and $w(x)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable.

2.3.2 Exponential stability

Theorem 2.2. *Consider system (2.5). Let $V : \mathbb{R}^+ \times D \mapsto \mathbb{R}^+$ be a continuously differentiable function such that for $t \in \mathbb{R}^+$ and $x \in D$,*

$$k_1\|x\|^a \le V(t,x) \le k_2\|x\|^a$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -k_3\|x\|^a$$

where k_i for $i = 1,2,3$ and a are positive constants. Then $x = 0$ is exponentially stable. Further if $D = \mathbb{R}^n$, then $x = 0$ is global exponentially stable.

Comparing Theorems 2.1 and 2.2 above, it is straightforward to see that exponential stability implies uniform asymptotic stability.

2.3.3 Converse Lyapunov theorem

The following result is the well known Converse Lyapunov Theorem.

Theorem 2.3. *Consider system (2.5) in domain $D := \mathscr{B}_r = \{x \in \mathbb{R}^n \mid \|x\| < r\}$. Let $\beta(\cdot)$ be a class $\mathscr{K}\mathscr{L}$ function and r_0 be a positive constant such that*

$$\beta(r_0, 0) < r \qquad \text{and} \qquad \mathscr{B}_{r_0} := \{x \mid \|x\| < r_0\}$$

Assume that the Jacobian matrix $\frac{\partial f}{\partial x}$ is bounded[1] in domain D uniformly for $t \in \mathbb{R}^+$, and that the trajectory of system (2.5) satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \qquad x(t_0) \in \mathscr{B}_{r_0}, \quad t \ge t_0 \ge 0$$

Then, there exists a continuously differentiable function $V : \mathbb{R}^+ \times \mathscr{B}_{r_0} \mapsto \mathbb{R}^+$ such that

[1] If the function $f(\cdot)$ in (2.5) is continuously differentiable in the ball $\overline{\mathscr{B}}_r$, then $\frac{\partial f}{\partial x}$ is bounded in the domain $D = \mathscr{B}_r$.

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -\alpha_3(\|x\|)$$
$$\left\|\frac{\partial V}{\partial x}\right\| \le \alpha_4(\|x\|)$$

where α_i for $i = 1,2,3,4$ are class $\mathscr{K}$ functions defined on the interval $[0, r_0]$. The function $V(\cdot)$ can be chosen independent of time t if $f(\cdot)$ in system (2.5) is independent of the time t.

2.4 Uniformly Ultimate Boundedness

For a given system (2.5), if asymptotic stability is not possible, uniform ultimate bounded stability can be considered. This is very useful in practical cases.

Theorem 2.4. *Consider system (2.5). Let $V : \mathbb{R}^+ \times D \mapsto \mathbb{R}^+$ be a continuously differentiable function such that in $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$,*

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad \text{for any} \quad \|x\| \ge \mu > 0$$

where $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are class $\mathscr{K}$ functions and $W_3(\cdot)$ is a continuous positive definite function in domain D. Then $x = 0$ is uniformly ultimately bounded[2]*. Further if $D = \mathbb{R}^n$, and $\alpha_1(\cdot)$ belongs to class $\mathscr{K}_\infty$, then $x = 0$ is globally uniformly ultimately bounded.*

Proof: See the reference [91] (Theorem 4.18, p. 172). #

From Theorem 2.4, the following result is ready to be presented:

Lemma 2.4. *Consider the nonlinear system*

$$\dot{x} = \omega(x) \tag{2.7}$$

where $x \in \mathbb{R}^n$ is the system state, and the function $\omega(\cdot)$ is continuous in $\mathbb{R}^n$. Let $\mathscr{V} : \mathbb{R}^n \mapsto \mathbb{R}^+$ be a continuously differentiable class $\mathscr{K}_\infty$ function of $\|x\|$ such that the inequality

$$\frac{\partial \mathscr{V}}{\partial x}\omega(x) \le -\vartheta(\|x\|), \qquad x \in \mathbb{R}^n \setminus \mathscr{B}_\mu \tag{2.8}$$

[2] The ultimate bound depends on the parameters μ, which can be estimated using the result given in Theorem 4.18 in [91].

holds for some domain $\mathscr{B}_\mu$, where ϑ is a class $\mathscr{K}$ function, and μ is a positive constant. Then, the trajectory of system (2.7) enters into the domain $\mathscr{B}_\mu$ in finite time.

Proof: From the condition of Lemma 2.4, there exists a class $\mathscr{K}_\infty$ function $\vartheta_1(\cdot)$ such that

$$\mathscr{V}(x) = \vartheta_1(\|x\|). \tag{2.9}$$

Then, from (2.8), (2.9) and using Theorem 2.4, the trajectory of system (2.7) is driven to the domain $\overline{\mathscr{B}}_\mu$ in a finite time, and remains there. That means there exists t_1 such that $x \in \overline{\mathscr{B}}_\mu$ for $t \geq t_1$.

The aim now is to prove that the trajectory of system (2.7) enters into $\mathscr{B}_\mu$ in a finite time. Suppose for a contradiction that this is not the case, then there exists some time t_2 such that the solution $x(x_0,t)$ of system (2.7) starting from some point x_0 satisfies $x(x_0,t) \in \partial\overline{\mathscr{B}}_\mu$ after t_2. This is equivalent to

$$\|x(x_0,t)\| = \mu, \qquad t \geq t_2. \tag{2.10}$$

By (2.9) and (2.10), it follows that

$$\mathscr{V}(x(x_0,t)) = \vartheta_1(\|x(x_0,t)\|) = \vartheta_1(\mu), \qquad t \geq t_2. \tag{2.11}$$

where μ is a positive constant. This shows that $\dot{\mathscr{V}}\,|_{(2.7)} \equiv 0$ after t_2, and it contradicts (2.8). Hence, the conclusion follows. #

Remark 2.3. Lemma 2.4 demonstrates that the solution enters the open set $\mathscr{B}_\mu$ in finite time and remains on $\overline{\mathscr{B}}_\mu$. It does not claim that the solution subsequently remains in $\mathscr{B}_\mu$.

2.5 Razumikhin Theorem

Consider a time-delay system

$$\dot{x}(t) = f(t, x(t - d(t)) \tag{2.12}$$

with an initial condition

$$x(t) = \phi(t), \qquad t \in [-\overline{d}, 0]$$

where the function vector $f : \mathbb{R}^+ \times \mathscr{C}_{[-\overline{d},0]} \mapsto \mathbb{R}^n$ takes $\mathbb{R} \times$ (bounded sets of $\mathscr{C}_{[-\overline{d},0]}$) into bounded sets in $\mathbb{R}^n$; $d(t) > 0$ is the time delay and

$$\overline{d} := \sup_{t \in \mathbb{R}^+} \{d(t)\} < \infty$$

which implies that the time delay $d(t)$ has a finite upper bound in $t \in \mathbb{R}^+$.

Theorem 2.5. *(***Razumikhin Theorem***) If there exist class $\mathscr{K}_\infty$ functions $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$, a class $\mathscr{K}$ function $\zeta_3(\cdot)$ and a continuous function $V_1(\cdot) : [-\overline{d},\infty] \times \mathbb{R}^n \mapsto \mathbb{R}^+$ satisfying*

$$\zeta_1(\|x\|) \leq V_1(t,x) \leq \zeta_2(\|x\|), \qquad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^n$$

such that the time derivative of V_1 along the solution of system (2.12) satisfies

$$\dot{V}_1(t,x) \leq -\zeta_3(\|x\|) \quad \text{if} \quad V_1(t-d,x(t-d)) \leq V_1(t,x(t)) \tag{2.13}$$

for any $d \in [0,\ \overline{d}]$, then the system (2.12) is uniformly stable. If in addition, $\zeta_3(\tau) > 0$ for $\tau > 0$ and there exists a continuous non-decreasing function $\xi(\tau) > \tau$ for $\tau > 0$ such that (2.13) is strengthened to

$$\dot{V}_1(t,x) \leq -\zeta_3(\|x\|) \quad \text{if} \quad V_1(t-d,x(t-d)) \leq \xi(V_1(t,x(t))) \tag{2.14}$$

for $d \in [0,\ \overline{d}]$, then the system (2.12) is uniformly asymptotic stable.

Proof: See pages 14-15 in [65]. ∇

From the Razumikhin Theorem 2.5. the following conclusion can be obtained directly:

Lemma 2.5. *Consider the time delay system (2.12). If there exist constants $\gamma > 0$ and $\zeta > 1$ and a function*

$$V_2(x(t)) = x^T \tilde{P} x$$

with $\tilde{P} > 0$ such that the time derivative of $V_2(\cdot)$ along the solution of system (2.12) satisfies

$$\dot{V}_2 \mid_{(2.12)} \leq -\gamma \left\| \tilde{P}^{\frac{1}{2}} x(t) \right\|^2 \tag{2.15}$$

whenever

$$\|\tilde{P}^{\frac{1}{2}} x(t+\theta)\| \leq \zeta \|\tilde{P}^{\frac{1}{2}} x(t)\|$$

for any $\theta \in [-\overline{d}, 0]$, then, system (2.12) is uniformly asymptotic stable.

Proof: From the definition of $V_2(\cdot)$ it follows that

$$\lambda_{\min}(\tilde{P})\|x\|^2 \leq V_2(t,x(t)) \leq \lambda_{\max}(\tilde{P})\|x\|^2$$

and from (2.15)

$$\dot{V}_2 \mid_{(2.12)} \leq -\gamma x(t)^T \tilde{P} x(t) \leq -\gamma \lambda_{\max}(\tilde{P})\|x\|^2.$$

It is clear that the inequality

$$V_2(x(t+\theta)) \leq \zeta^2 V_2(x(t))$$

is equivalent to the inequality

$$\|\tilde{P}^{\frac{1}{2}} x(t+\theta)\| \leq \zeta \|\tilde{P}^{\frac{1}{2}} x(t)\|$$

Then, from Razumikhin Theorem 2.5 and $\tilde{P} > 0$, the conclusion follows by letting

$$\gamma_1(\tau) = \lambda_{\min}(\tilde{P})\tau^2, \qquad \gamma_2(\tau) = \lambda_{\max}(\tilde{P})\tau^2$$
$$\gamma_3(\tau) = \gamma\lambda_{\min}(\tilde{P})\tau^2, \qquad \gamma_4(\tau) = \zeta^2\tau$$

in Theorem 2.5. #

2.6 Output Sliding Surface Design

In order to form an output feedback sliding mode control scheme, it is usually required that the designed switching function is a function of the system outputs. The corresponding sliding surface is called *an output sliding surface* in this book. The output sliding surface algorithm proposed in [37, 38] is outlined here, and this will be frequently used in the sequel.

Consider initially a linear system

$$\dot{x} = Ax + Bu \tag{2.16}$$
$$y = Cx, \tag{2.17}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are the states, inputs and outputs respectively and assume $m \leq p < n$. The triple (A, B, C) comprises constant matrices of appropriate dimensions with B and C both being of full rank.

For system (2.16)-(2.17), it is assumed that

$$\text{rank}(CB) = m$$

Then, from [37] it can be shown that a coordinate transformation $\widetilde{x} = \widetilde{T}x$ exists such that the system triple (A, B, C) with respect to the new coordinate $\widetilde{x}$ has the following structure

$$\begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad \begin{bmatrix} 0 & \check{T} \end{bmatrix} \tag{2.18}$$

where $\widetilde{A}_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $B_2 \in \mathbb{R}^{m\times m}$ and $\check{T} \in \mathbb{R}^{p\times p}$ is orthogonal. Further, it is assumed that the system $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ with $\widetilde{C}_1$ defined by

$$\widetilde{C}_1 = \begin{bmatrix} 0_{(p-m)\times(n-m)} & I_{p-m} \end{bmatrix} \tag{2.19}$$

is output feedback stabilisable i.e. there exists a matrix $K \in \mathbb{R}^{m\times(p-m)}$ such that

$$\widetilde{A}_{11} - \widetilde{A}_{12}K\widetilde{C}_1$$

is stable. It is shown in [37, 38] that a necessary condition for $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ to be stabilisable is that the invariant zeros of (A, B, C) lie in the open left half-plane. In [37, 38] a sliding surface of the form

$$FCx = 0 \tag{2.20}$$

is proposed where

$$F = F_2 \begin{bmatrix} K & I_m \end{bmatrix} \breve{T}^{\tau} \tag{2.21}$$

and $F_2 \in \mathbb{R}^{m \times m}$ is any nonsingular matrix.

If a further coordinate change is introduced based on the nonsingular transformation $z = \widehat{T}\widetilde{x}$ with $\widehat{T}$ defined by

$$\widehat{T} = \begin{bmatrix} I_{n-m} & 0 \\ K\widetilde{C}_1 & I_m \end{bmatrix}$$

then in the new coordinates z, system (2.16)–(2.17) has the following form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \qquad \widehat{C}$$

where $A_{11} = \widetilde{A}_{11} - \widetilde{A}_{12}K\widetilde{C}_1$ is stable and $\widehat{C}$ satisfies

$$F\widehat{C} = \begin{bmatrix} 0 & F_2 \end{bmatrix}$$

with F_2 nonsingular. From the analysis above, the following conclusion is obtained directly:

Lemma 2.6. *Consider system (2.16)-(2.17). Suppose that*

i) $\text{rank}(CB) = m$*;*
ii) the invariant zeros of (A, B, C) *lie in the open left half-plane;*
iii)the matrix triple $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ *is output feedback stabilisable where* $(\widetilde{A}_{11}, \widetilde{A}_{12})$ *and* $\widetilde{C}_1$ *are defined respectively by (2.18) and (2.19).*

Then,

i) there exists a transformation $z = Tx$ *such that in the new coordinate* z *system (2.16)–(2.17) has the following form*

$$\dot{z} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} z + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \tag{2.22}$$

$$y = \begin{bmatrix} 0 & C_2 \end{bmatrix} z, \tag{2.23}$$

where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ *is stable. Both matrices* $B_2 \in \mathbb{R}^{m\times m}$ *and* $C_2 \in \mathbb{R}^{p\times p}$ *are nonsingular;*
ii) there exists a matrix F *such that* $FCx = 0$ *provides a stable sliding motion for system (2.16)-(2.17) and* $F \begin{bmatrix} 0 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & F_2 \end{bmatrix}$ *where* $F_2 \in \mathbb{R}^{m\times m}$ *is nonsingular.*

Proof: All that remains to be shown is that the output distribution matrix has the form given in (2.23) and that C_2 is nonsingular. The output distribution matrix in the new coordinates is given by

$$\begin{bmatrix} 0 & \check{T} \end{bmatrix} \widehat{T}^{-1} = \begin{bmatrix} 0 & \check{T} \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 \\ -K\widetilde{C}_1 & I_m \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \check{T} \end{bmatrix} \begin{bmatrix} I_{n-p} & 0 & 0 \\ 0 & I_{p-m} & 0 \\ 0 & -K & I_m \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \check{T} \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 \\ 0 & \begin{bmatrix} I_{p-m} & 0 \\ -K & I_m \end{bmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \check{T} \begin{bmatrix} I_{p-m} & 0 \\ -K & I_m \end{bmatrix} \end{bmatrix}.$$

and so by inspection,

$$C_2 = \check{T} \begin{bmatrix} I_{p-m} & 0 \\ -K & I_m \end{bmatrix}$$

which is nonsingular. Hence the result follows. #

From the analysis above, it is clear to see that the coordinate transformation

$$z = Tx$$

where $T := \widehat{T}\widetilde{T}$, transfers the system (2.16)-(2.17) to the regular form (2.22)–(2.23). Choose the sliding surface

$$\mathscr{S} = \{x \mid FCx = 0,\ x \in \mathbb{R}^n\} \tag{2.24}$$

Then, the analysis above shows that the sliding motion of system (2.16)-(2.17) corresponding to the sliding surface (2.24) is asymptotically stable. The sliding surface (2.24) can be described by

$$\mathscr{S} = \{y \mid Fy = 0,\ y \in \mathbb{R}^p\} \tag{2.25}$$

which is a subspace of the *output space*. Therefore $\mathscr{S}$ in (2.24) or (2.25) denote *output sliding surfaces*.

Remark 2.4. Lemma 2.6 gives a condition for the existence of the output switching surface (2.20) on which system (2.16) is stable. It should be emphasized that the sliding surface given by Lemma 2.6 can be obtained from a systematic algorithm together with any output feedback pole placement algorithm of choice. Details of appropriate algorithms and how to determine the switching surface (2.20) is described in [37, 38] where the necessary and sufficient condition to guarantee the existence of the matrix F is available in Proposition 5.2 of [38]. If $p = m$ then there is no design freedom and the sliding motion is governed by the invariant zeros of (A,B,C).

2.7 Geometric Structure of Nonlinear System

Consider the nonlinear system

$$\dot{x}(t) = F(x(t), u(t)) \tag{2.26}$$

$$y(t) = h(x(t)), \qquad x_0 = x(0) \tag{2.27}$$

where $x \in \Omega \subset \mathscr{R}^n$ (and Ω is a neighbourhood of x_0), $u = \mathrm{col}(u_1, u_2, \ldots, u_m) \in \mathscr{U} \in \mathscr{R}^m$, and $y = \mathrm{col}(y_1, y_2, \ldots, y_p) \in \mathscr{R}^p$ are the state variables, inputs and outputs respectively where $\mathscr{U}$ is an admissible control set. $F(x,u)$ is a known smooth vector field in $\Omega \times \mathscr{U}$ and the known function $h : \Omega \mapsto \mathscr{R}^p$ is smooth. For convenience, the system (2.26)–(2.27) is also denoted by the pair $(F(x,u), h(x))$.

Definition 2.8. (See, e.g. [58]) System (2.26)–(2.27) is said to be *observable* at $(x_0, u_0) \in \Omega \times \mathscr{U}$ if there exists a neighbourhood $\mathscr{N}$ of (x_0, u_0) in $\Omega \times \mathscr{U}$ and a set of nonnegative integer numbers $\{r_1, r_2, \cdots, r_p\}$ with $\sum_{i=1}^{p} r_i = n$ such that

1) for all $(x,u) \in \mathscr{N}$

$$\frac{\partial}{\partial u_j} L_{F(x,u)}^k h_i(x) = 0 \tag{2.28}$$

for indices $i = 1, 2, \ldots, p$, $k = 0, 1, 2, \ldots, r_i - 1$ and $j = 1, 2, \ldots, m$;

2) the $p \times m$ matrix $M(x,u) := \{\frac{\partial}{\partial u_j} L_{F(x,u)}^{r_i} h_i(x)\}$ has rank p in (x_0, u_0)

Then, $\{r_1, r_2, \cdots, r_p\}$ is called the *observability index* of system (2.26)–(2.27) at (x_0, u_0). Further, system (2.26)–(2.27) is said to be *uniformly observable* in $\Omega \times \mathscr{U}$ if for any $(x_0, u_0) \in \Omega \times \mathscr{U}$, the system is observable and the observability indices are fixed.

Assume the pair $(F(x,u), h(x))$ has uniform observability index $\{r_1, r_2, \cdots, r_p\}$ with $\sum_{i=1}^{p} r_i = n$ in the domain $\Omega \times \mathscr{U}$. Construct a nonlinear transformation $T : x \mapsto z$ as follows:

$$z_{i1} = h_i(x) \tag{2.29}$$

$$z_{i2} = L_{F(x,u)} h_i(x) \tag{2.30}$$

$$\vdots$$

$$z_{ir_i} = L_{F(x,u)}^{r_i - 1} h_i(x) \tag{2.31}$$

where $z_i := \mathrm{col}(z_{i1}, z_{i2}, \cdots, z_{ir_i})$ for $i = 1, 2, \ldots, p$ and $z := \mathrm{col}(z_1, z_2, \cdots, z_p)$.

It follows from Definition 2.8 that $M(x,u)$ has rank p in $\Omega \times \mathscr{U}$, implying that all z_i are independent of the control u, which combined with the restriction $\sum_{i=1}^{p} r_i = n$ means that the corresponding Jacobian matrix of $T(x)$, $\frac{\partial T}{\partial x}$, is nonsingular. Therefore, (2.29)–(2.31) is a diffeomorphism in the domain Ω, and $z = \mathrm{col}(z_1, z_2, \ldots, z_p)$ forms a new coordinate system which can be obtained by direct computation from (2.29)–(2.31).

Since $L^j_{F(x,u)}h_i(x)$ is independent of u for all $i = 1,2,\ldots,p$ and $j = 1,2,\ldots,r_i-1$, it follows by direct computation that for $i = 1,2,\ldots,p$

$$\begin{aligned}
\dot z_{i1} &= \frac{\partial h_i}{\partial x}F(x,u) = L_{F(x,u)}h_i(x) = z_{i2} \\
\dot z_{i2} &= \frac{\partial\left(L_{F(x,u)}h_i(x)\right)}{\partial x}F(x,u) = L^2_{F(x,u)}h_i(x) = z_{i3} \\
&\vdots \\
\dot z_{ir_{i-1}} &= L^{r_i-1}_{F(x,u)}h_i(x) = z_{ir_i} \\
\dot z_{ir_i} &= L^{r_i}_{F(x,u)}h_i(x)
\end{aligned}$$

Therefore, in the new coordinates z defined by (2.29)–(2.31), system (2.26)–(2.27) has the following form

$$\begin{aligned}
\dot z &= Az + B\Phi(z,u) \\
y &= Cz
\end{aligned}$$

where

$$A = \operatorname{diag}\{A_1,\ldots,A_p\}, \quad B = \operatorname{diag}\{B_1,\ldots,B_p\} \quad \text{and} \quad C = \operatorname{diag}\{C_1,\ldots,C_p\}$$

where $A_i \in \mathscr{R}^{r_i\times r_i}$, $B_i \in \mathscr{R}^{r_i\times 1}$ and $C_i \in \mathscr{R}^{1\times r_i}$ for $i = 1,2,\ldots,p$ are defined by

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad C_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{2.32}$$

and

$$\Phi(z,u) := \begin{bmatrix} \phi_1(z,u) \\ \phi_2(z,u) \\ \vdots \\ \phi_p(z,u) \end{bmatrix} := \begin{bmatrix} L^{r_1}_{F(x,u)}h_1(x) \\ L^{r_2}_{F(x,u)}h_2(x) \\ \vdots \\ L^{r_p}_{F(x,u)}h_p(x) \end{bmatrix}_{x=T^{-1}(z)} \tag{2.33}$$

where $\phi_i : T(\Omega)\times\mathscr{U} \mapsto \mathscr{R}$ for $i = 1,2,\ldots,p$.

2.8 Summary

This chapter has presented the fundamental concepts and results which underpin the theoretical analysis in this book. Some of the results are taken from the existing literature and others are developed by the authors, but with rigorous proofs provided. The content covers Lipschitz conditions, comparison functions, stability of nonlinear systems, the converse Lyapunov theorem and uniform ultimate boundedness. The well known Razumikhin Theorem has been presented, for the readers' convenience, and will be employed to deal with time delay systems throughout the book. Section 2.5 summarises the output sliding surface design approach proposed in [38] which will be frequently used in the sequel. Finally the geometric structure of nonlinear systems with uniform observability index has been provided.