Hypergeometric multiple orthogonal polynomials

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Abstract

This thesis is devoted to the analysis of multiple orthogonal polynomials for indices on the so-called step-line with respect to absolutely continuous measures on the positive real line, whose moments are given by ratios of Pochhammer symbols (also known as rising factorials). We investigate both type I and type II multiple orthogonal polynomials, though the main focus is on the type II polynomials. For the former, the characterisation includes Rodrigues-type formulas for the type I polynomials and type I functions. On the latter, the characterisation includes explicit representations as terminating generalised hypergeometric series as well as solutions of differential equations and recurrence relations, and an analysis of their asymptotic behaviour and the location of their zeros. We investigate the link of these polynomials with branched-continued-fraction representations of generalised hypergeometric series, which were introduced to solve total-positivity problems in combinatorics. The polynomials analysed here also have direct applications to the study of Painlevé equations and to random matrix theory.

We give a detailed characterisation of two new families of multiple orthogonal polynomials associated with Nikishin systems of 2 absolutely continuous measures. These measures are supported on the positive real line and on the interval $(0, 1)$ and they admit integral representations via the confluent hypergeometric function of the second kind (also known as the Tricomi function) and Gauss' hypergeometric function, respectively. The vectors of orthogonality weights satisfy matrix Pearson-type differential equations, linked to the action of the differentiation operator on the type II polynomials and type I functions as a shift in their index and parameters. As a result, the type II polynomials and type I functions satisfy Hahn's property. We further draw the links between these two families of multiple orthogonal polynomials and other known polynomial sets via limiting relations or specialisations. Examples of such connections encompass the components of the cubic decomposition of Hahn-classical threefold-symmetric 2-orthogonal polynomials as well as Jacobi-Piñeiro polynomials and multiple orthogonal polynomials with respect to Macdonald functions.

Keywords: *Multiple orthogonal polynomials, Nikishin systems, Pochhammer symbols, generalised hypergeometric series, branched continued fractions, confluent hypergeometric function, Gauss' hypergeometric function, 2-orthogonal polynomials, Hahn classical, 3-fold symmetric.*

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Chapter 1

Background and motivation

The central theme of this thesis are multiple orthogonal polynomials, which are a generalisation of orthogonal polynomials consisting of polynomials of one variable satisfying orthogonality conditions with respect to several measures. We refer to the multiple orthogonal polynomials studied in this thesis as hypergeometric because they admit explicit representations in the form of terminating generalised hypergeometric series.

Multiple orthogonal polynomials originate from Hermite-Padé approximation, a simultaneous rational approximation to several functions, and were introduced in Hermite's proof of the transcendence of e. Other applications of rational approximations using multiple orthogonal polynomials to analytic number theory include proofs of transcendence and construction of upper bounds for the measure of irrationality of π as well as demonstrations of several results on the irrationality of values of the Riemann zeta function $\zeta(s)$, including the irrationality of $\zeta(2)$, $\zeta(3)$ (the famous Apéry theorem in [3]) and infinitely many zeta values with an odd positive integer argument. See [70] for more details on the applications of Hermite-Padé rational approximation to results on the irrationality and transcendence of numbers and references for each of the specific results mentioned.

Research on multiple orthogonal polynomials has received increasing attention in the past decades, partly motivated by their applicability to different areas of Mathematics and Mathematical Physics. Adding to the aforementioned applications in analytic number theory, there are connections of multiple orthogonal polynomials to random matrix theory [9, 42], spectral theory of non-selfadjoint operators [6, 69], and the description of rational solutions to Painlevé equations [15, 73], to name a few. More recently, a surprising connection was discovered between multiple orthogonal polynomials and the branched continued fractions introduced in [61], which are a generalisation of continued fractions originating from combinatorics of lattice paths. In this document, that connection is applied in the analysis of the multiple orthogonal polynomials under study.

In this initial chapter we give an introduction to:

- measures and linear functionals;
- some relevant special functions, including hypergeometric functions;
- (standard) orthogonal polynomials;
- multiple orthogonal polynomials;
- continued fractions;
- branched continued fractions;
- totally positive matrices and production matrices.

The main aim of this introduction is to present the necessary background and give some motivation for the following chapters. Besides, we establish some terminology and notation to be used throughout the thesis. All the definitions and results presented in this introduction as well as the respective proofs can be found in the references cited throughout the chapter.

We end this introductory chapter with an outline of the thesis.

1.1 Measures and linear functionals

The standard and multiple orthogonal polynomials appearing in this thesis satisfy orthogonality conditions with respect to positive Borel measures. These measures define linear functionals on the vector space of polynomials. Therefore, we can use some known results about orthogonality and multiple orthogonality with respect to linear functionals to study the polynomials appearing here. As such, we make a brief introduction to the theories of measures and linear functionals in this section. On the former, we present the concepts of measurable spaces, positive measures, Borel measures and absolutely continuous measures as well as the Radon-Nikodym derivative and Stieltjes transform of a measure. On the latter, we describe the vector space of polynomials and its dual space, where we define the dual sequence of a polynomial sequence and two key operations. For more information on measure theory and functional theory, we refer to [16] and [68], respectively.

Measures are defined in *measurable spaces*, which are pairs (X,Σ) where X is a set and Σ is a σ-algebra on X. A *σ-algebra* on a set X is a collection Σ of subsets of X such that $\emptyset\in\Sigma$, $E\in\Sigma \implies X\backslash E\in\Sigma$, and $(E_n)_{n\in\mathbb{N}}\subset\Sigma \implies \bigcup_{n\in\mathbb{N}} E_n\in\Sigma$.

A *positive measure*, or simply *measure*, in a measurable space (X,Σ) is a function $\mu:\Sigma\to\mathbb{R}_0^+\cup\{+\infty\}$ such that $\mu(\emptyset)=0$ and $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right)=\sum_{n=0}^{\infty}\mu\left(A_n\right)$, for any countable sequence $\{A_n\}_{n\in\mathbb{N}}\subset\Sigma$ of pairwise disjoint sets.

When $\mu(X)<+\infty$, μ is a *finite measure*. We often deal with finite measures μ such that $\mu(X)=1$. In that case, we say that μ is a *probability measure*. Moreover, for $A\in\Sigma$, we refer to $\mu(A)$ as the μ-measure of A.

A *Borel measure* on a topological space X is a measure defined in the *Borel σ-algebra* of X, denoted by $\mathcal{B}(X)$, which is the smallest σ-algebra containing all open subsets of X. For instance, the Borel σ-algebra of $\mathbb{R}$ (with its usual topology), $\mathcal{B}(\mathbb{R})$, is the smallest σ-algebra that contains all the open intervals of $\mathbb{R}$ and a Borel measure in $\mathbb{R}$ is any measure defined in $\mathcal{B}(\mathbb{R})$. The Lebesgue measure in $\mathbb{R}$, which maps each real interval to its length, is a Borel measure.

Let μ and ν be two finite measures in a measurable space (X, Σ). We write $\nu \ll \mu$ and say that the measure ν is *absolutely continuous with respect to* μ if, for each set $A \in \Sigma$, $\mu(A) = 0$ implies $\nu(A) = 0$. A function $f : A \in \Sigma \to \mathbb{R}$ is Σ-*measurable* if the set $\{x \in A : f(x) \leq t\}$ belongs to Σ, for each $t \in \mathbb{R}$. This condition remains equivalent if we replace "$\leq$" by any of "$\geq$", "$<$" or "$>$" in this definition. An important result about absolutely continuous measures is the Radon-Nikodym theorem, which states that if $\nu \ll \mu$ then there exists an Σ-measurable function $w : X \to \mathbb{R}_0^+$, unique up to sets of μ-measure zero, such that $\nu(A) = \int_A w \mathrm{d}\mu$, for each set $A \in \Sigma$. The function w is called the *Radon-Nikodym derivative* of ν with respect to μ and it is sometimes denoted by $\dfrac{\mathrm{d}\nu}{\mathrm{d}\mu}$.

Our main focus will be on absolutely continuous measures with respect to the Lebesgue measure, which are simply said to be *absolutely continuous*, supported on subsets of the real line. Let μ be a measure of that type, with Radon-Nikodym derivative w with respect to the Lebesgue measure. Then we write $\mathrm{d}\mu(x) = w(x)\mathrm{d}x$ and w is called the *density* of μ. Besides, we use the term *weight function* for the density of an absolutely continuous measure.

The *support of a continuous real or complex valued function* f defined on a topological space X, denoted by $\operatorname{supp}(f)$, is the closure of the set $\{x \in X : f(x) \neq 0\}$. Similarly, the *support of a measure* μ on a topological space X, denoted by $\operatorname{supp}(\mu)$, which is defined as the complement of the union of all open subsets of X with μ-measure zero, is the smallest closed set of X whose complement has μ-measure zero. As a result, if $\nu \ll \mu$ then $\operatorname{supp}(\nu) \subset \operatorname{supp}(\mu)$ and the Radon-Nikodym derivative w of ν with respect to μ such that $\operatorname{supp}(w) = \operatorname{supp}(\nu)$ is unique.

The *Stieltjes transform* of a measure μ supported on a real interval I is given by

$$F(z) = \int_I \frac{\mathrm{d}\mu(x)}{x - z}, \quad \text{for } z \in \mathbb{C} \backslash I.$$

Now we introduce linear functionals in the vector space of polynomials with one variable and coefficients on a field $\mathbb{K}$, which here is always either $\mathbb{R}$ or $\mathbb{C}$. We denote this vector space of polynomials by $\mathcal{P}$. A *linear functional* u defined on $\mathcal{P}$ is a linear map $u : \mathcal{P} \to \mathbb{K}$. The action of u on a polynomial f is denoted by $\langle u, f\rangle$. The *moment of order* $n \in \mathbb{N}$ *of a linear functional* $u \in \mathcal{P}'$ is equal to $\langle u, f\rangle$, with $f : x \mapsto x^n$, which we denote by $\langle u, x^n\rangle$. By linearity, every linear functional u is uniquely determined by its moments or, alternatively, by the values of u at the elements of any basis of $\mathcal{P}$. The *dual space* of $\mathcal{P}$, denoted by $\mathcal{P}'$, is the vector space consisting of all linear functionals on $\mathcal{P}$.

Similarly, we define the *moment of order* $n \in \mathbb{N}$ *of a measure* μ by the integral $\int x^n \mathrm{d}\mu(x)$. If all these moments exist and are finite (which in particular implies that the measure is finite), we can define in $\mathcal{P}'$ a linear functional u such that $\langle u, p\rangle = \int p(x)\mathrm{d}\mu(x)$, for all $p \in \mathcal{P}$, and the moment sequences of the measure μ and the linear functional u defined via μ clearly coincide.

A *polynomial sequence* is a sequence $(P_n(x))_{n\in\mathbb{N}} \subset \mathcal{P}$ such that each P_n is a polynomial of degree exactly n. Note that a polynomial sequence always forms a basis of $\mathcal{P}$. The *dual sequence* of a polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is the unique sequence $(u_n)_{n\in\mathbb{N}}$ in the dual space $\mathcal{P}'$ such that, for all $n, m \in \mathbb{N}$,

$$\langle u_n, P_m\rangle = \delta_{nm} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Operations in the polynomial space $\mathcal{P}$ induce operations in the dual space $\mathcal{P}'$. For instance, the derivative of u and the left-multiplication of u by f, with $u \in \mathcal{P}'$ and $f \in \mathcal{P}$, which are the two operations on the dual space $\mathcal{P}'$ used in this thesis, namely in Chapter 2, are defined, by duality,

$$\langle u', p\rangle = -\langle u, p'\rangle \quad \text{and} \quad \langle fu, p\rangle = \langle u, fp\rangle, \quad \text{for any } p \in \mathcal{P}.$$

1.2 Special and hypergeometric functions

The multiple orthogonal polynomials under discussion here can be explicitly expressed via terminating generalised hypergeometric series. Besides, most orthogonality measures appearing in this thesis have densities involving special functions. As such, we present the definitions and some basic properties of the special functions relevant to this thesis, including the ordinary and generalised hypergeometric functions. For more details on special functions, we refer to [1, 10, 22, 51], among the many references on this topic.

The *Gamma function*, usually denoted by $\Gamma(z)$, is defined by Euler's integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t \quad \text{if } \operatorname{Re} z > 0, \tag{1.1}$$

and by analytic continuation elsewhere in the complex plane, except for the non-positive integers, via the functional relation

$$\Gamma(z+1) = z\Gamma(z). \tag{1.2}$$

The *Pochhammer symbol*, denoted by $(z)_n$, with $z \in \mathbb{C}$ and $n \in \mathbb{N}$, is defined by

$$(z)_0 = 1 \quad \text{and} \quad (z)_n = z(z+1)\cdots(z+n-1), \quad n \in \mathbb{Z}^+.$$

Due to (1.2), and because $\Gamma(z)$ is always nonzero, the Pochhammer symbol can also be defined as a ratio of Gamma function values as

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}, \quad \text{unless } z \in \mathbb{Z}_0^-.$$

In particular, $n! = (1)_n = \Gamma(n+1)$, for any $n \in \mathbb{N}$.

Furthermore, as a result of the well-known Gauss' multiplication formula, the values of the Gamma function with rational arguments satisfy

$$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{-\frac{1}{2}}, \quad \text{for any } n \geq 2.$$

The initial cases $n = 2$ and $n = 3$ of this formula imply that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \text{and} \quad \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}. \tag{1.3}$$

The former is the most widely known value of the Gamma function with a non-integer argument, while the latter identity is useful in Chapter 2.

The *Beta function* is a two-variable function, defined by Euler's Beta integral, whenever $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$, and equal to a ratio of Gamma function values:

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.4}$$

Note that $B(\alpha, \beta) = B(\beta, \alpha)$.

For parameters $\alpha, \beta, \gamma \in \mathbb{C}$ with $\gamma \notin \mathbb{Z}_0^-$, the *ordinary hypergeometric function* or *Gauss' hypergeometric function* is defined by Gauss' series

$${}_2F_1\left(\alpha, \beta; \gamma \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \tag{1.5}$$

on the disk $|z| < 1$ and by analytic continuation elsewhere. The hypergeometric function ${}_2F_1\left(\alpha, \beta; \gamma \,\middle|\, z\right)$ is a solution of the hypergeometric differential equation

$$z(1-z)F''(z) + (\gamma - (\alpha+\beta+1)z)F'(z) - \alpha\beta F(z) = 0 \tag{1.6}$$

and, if $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$, it admits the integral representation

$${}_2F_1\left(\alpha, \beta; \gamma \,\middle|\, z\right) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\mathrm{d}t. \tag{1.7}$$

The *generalised hypergeometric series* is formally defined by

$${}_pF_q\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{1.8}$$

with $p, q \in \mathbb{N}$, $z, a_1, \cdots, a_p \in \mathbb{C}$ and $b_1, \cdots, b_q \in \mathbb{C}\backslash\mathbb{Z}_0^-$. It is called the generalised hypergeometric series because it reduces to (1.5) when $(p, q) = (2, 1)$.

If $p \leq q$, (1.8) converges for all finite values of z; if $p = q+1$, it generally only converges absolutely on the disk $|z| < 1$, with convergence on the unit circle if

$$\mathrm{Re}\left(\sum_{j=0}^{p} b_j - \sum_{i=0}^{q} a_i\right) > 0;$$

and, if $p \geq q+2$, it generally diverges for all $z \neq 0$. However, if one of the parameters $a_1, \cdots, a_p$ is a non-positive integer, the series (1.8) terminates, and it defines a hypergeometric polynomial convergent everywhere in the complex plane.

The derivative of a generalised hypergeometric series is equal to a shift in the parameters, up to multiplication by a constant. To be precise,

$$\frac{\mathrm{d}}{\mathrm{d}z}\, {}_pF_q\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\middle|\, z\right) = \frac{a_1 \cdots a_p}{b_1 \cdots b_q}\, {}_pF_q\left(\begin{matrix} a_1+1, \cdots, a_p+1 \\ b_1+1, \cdots, b_q+1 \end{matrix}\middle|\, z\right). \tag{1.9}$$

Furthermore, (1.8) is a solution of the generalised hypergeometric differential equation

$$\left[\left(\prod_{j=1}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + b_j\right)\right)\frac{\mathrm{d}}{\mathrm{d}z} - \prod_{i=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + a_i\right)\right] F(z) = 0. \tag{1.10}$$

Here and always throughout this thesis, the product of differential operators is understood as their composition so we write $(\mathcal{L}_1\mathcal{L}_2)\, f := \mathcal{L}_1\,(\mathcal{L}_2 f)$. Note that if $(p, q) = (2, 1)$, (1.10) reduces to (1.6).

The generalised hypergeometric series also satisfies the confluent relations

$$\lim_{|\alpha|\to\infty} {}_{p+1}F_q\left(\begin{matrix} a_1, \cdots, a_p, \alpha \\ b_1, \cdots, b_q \end{matrix}\middle|\, \frac{z}{\alpha}\right) = {}_pF_q\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\middle|\, z\right) \tag{1.11}$$

and

$$\lim_{|\beta|\to\infty} {}_pF_{q+1}\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q, \beta \end{matrix}\middle|\, \beta z\right) = {}_pF_q\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\middle|\, z\right),$$

whenever the hypergeometric series and the limits on the left-hand side of these relations are convergent.

The Bessel functions of first and second kind, respectively denoted by $\mathrm{J}_\alpha(z)$ and $\mathrm{Y}_\alpha(z)$, are solutions to Bessel's equation

$$z^2 w'' + zw' + (z^2 - \alpha^2)w = 0.$$

The *Bessel function of the first kind* is defined by

$$\mathrm{J}_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{\left(-\frac{z^2}{4}\right)^k}{k!\Gamma(\alpha+k+1)} = \frac{\left(\frac{z}{2}\right)^\alpha}{\Gamma(\alpha+1)} {}_0F_1\left(-;\alpha+1 \,\middle|\, -\frac{z^2}{4}\right). \tag{1.12}$$

The *modified Bessel functions* of the first and second kind, respectively denoted by $\mathrm{I}_\alpha(z)$ and $\mathrm{K}_\alpha(z)$, are solutions to the modified Bessel equation

$$z^2 w'' + zw' - (z^2 + \alpha^2)w = 0.$$

When $|\arg z| \leq \frac{\pi}{4}$, the modified Bessel function of the second kind $\mathrm{K}_\alpha(z)$, which is also known as the *Macdonald function*, admits the integral representation

$$\mathrm{K}_\alpha(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\alpha \int_0^\infty \exp\left(-t - \frac{z^2}{4t}\right) \frac{\mathrm{d}t}{t^{\alpha+1}}. \tag{1.13}$$

As a result, making the change of variable $u = \frac{z^2}{4t}$, we deduce that the Macdonald function is symmetric with respect to the parameter α, that is,

$$\mathrm{K}_{-\alpha}(z) = \mathrm{K}_\alpha(z). \tag{1.14}$$

The *Airy functions* of the first and second kind, respectively denoted by $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$, are solutions to the Airy differential equation $w'' = zw$. For real variable, the Airy function of the first kind satisfies the integral representation on $\mathbb{R}^+$

$$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) \mathrm{d}t. \tag{1.15}$$

The Airy function of the first kind and its derivative can be written in terms of the Macdonald function by

$$\mathrm{Ai}(x) = \frac{2\sqrt{x}}{3\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}\mathrm{K}_{\pm\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) \text{ and } \mathrm{Ai}'(x) = -\frac{2x}{3\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}\mathrm{K}_{\pm\frac{2}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right). \tag{1.16}$$

The *confluent hypergeometric functions* of the first and second kind, respectively denoted by $\mathbf{M}(\alpha,\beta;z)$ and $\mathbf{U}(\alpha,\beta;z)$ and also known as the *Kummer* and *Tricomi function*, respectively, are solutions to Kummer's differential equation

$$z\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + (\beta - x)\frac{\mathrm{d}y}{\mathrm{d}z} - \alpha y = 0. \tag{1.17}$$

If $\mathrm{Re}\,\alpha > 0$ and $|\arg(z)| < \dfrac{\pi}{2}$, the Tricomi function admits the integral representation

$$\mathbf{U}(\alpha,\beta;z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \mathrm{e}^{-zt}t^{\alpha-1}(t+1)^{\beta-\alpha-1}\mathrm{d}t. \tag{1.18}$$

Besides that, $\mathbf{U}(0,\beta;x) = 1$ and

$$\mathbf{U}(\alpha,\beta;z) = x^{1-\beta}\,\mathbf{U}(\alpha-\beta+1,2-\beta;z). \tag{1.19}$$

1.3 Orthogonal polynomials

In this section we give a short introduction to orthogonal polynomials, to which we often refer as standard orthogonal polynomials, with the purpose of clearly distinguishing them from multiple orthogonal polynomials. The classical reference on orthogonal polynomials is [66]. We also refer to [14, 36, 37], among the many references on this topic.

We start this section by defining an orthogonal polynomial sequence. Then we recall the spectral theorem for orthogonal polynomials leading to the connection between orthogonal polynomials, recurrence relations and Jacobi matrices. Finally, we survey the well-known families of classical orthogonal polynomials, as well as the notion of symmetric polynomials and its implications in orthogonality.

Throughout this thesis, we consistently deal with monic polynomials, which means that the leading coefficient is equal to 1.

A polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is *orthogonal* with respect to a linear functional u if

$$\langle u, P_m P_n \rangle = \begin{cases} N_n \neq 0 & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

or, equivalently,

$$\left\langle u, x^k P_n \right\rangle = \begin{cases} N_n \neq 0 & \text{if } k = n, \\ 0 & \text{if } k < n. \end{cases}$$

The orthogonal polynomial sequence with respect to a linear functional u is unique, up to multiplication of each polynomial by a constant. As a result, the monic orthogonal polynomial sequence is unique.

Suppose that a polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is orthogonal with respect to a linear functional that admits an integral representation via a measure μ (or a weight function w) over its support. Then, we refer to μ (respectively, w), as the *orthogonality measure* (respectively, *orthogonality weight*) of $(P_n(x))_{n\in\mathbb{N}}$, and we say that $(P_n(x))_{n\in\mathbb{N}}$ is orthogonal with respect to μ (respectively, w).

If $(P_n(x))_{n\in\mathbb{N}}$ is orthogonal with respect to a linear functional u and $(u_n)_{n\in\mathbb{N}}$ is the dual sequence of $(P_n(x))_{n\in\mathbb{N}}$ then $u = N_0 u_0$, with $N_0 \neq 0$. Besides, there exists a unique sequence of nonzero constants $(N_n)_{n\in\mathbb{N}}$ such that $u_n = N_n^{-1} P_n u_0$, that is, $\langle u_n, f \rangle = N_n^{-1} \langle u_0, P_n f \rangle$, for any $f \in \mathcal{P}$. In fact, the choice $f = P_n$ implies that $N_n = \langle u_0, P_n^2 \rangle$. From now on, we assume that $N_0 = 1$, which means that $u = u_0$.

1.3.1 Spectral theorem and recurrence relation

The spectral theorem for orthogonal polynomials (also known as Shohat-Favard theorem) states that a polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is orthogonal if and only if, for each $n \in \mathbb{N}$, there exist coefficients β_n and $\gamma_{n+1} \neq 0$ such that $(P_n(x))_{n\in\mathbb{N}}$

satisfies the second order recurrence relation

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \tag{1.21}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$. The polynomial sequence $(R_n(x))_{n\in\mathbb{N}}$ satisfying (1.21) with initial conditions $R_0(x) = 0$ and $R_1(x) = 1$ is called the *associated polynomial sequence.*

We consider an infinite tridiagonal matrix $\mathrm{T} = (t_{i,j})_{i,j\in\mathbb{N}}$ with entries

$$t_{i,i+1} = 1, \quad t_{i,i} = \beta_i, \quad t_{i,i-1} = \gamma_i, \quad \text{and} \quad t_{i,j} = 0 \text{ if } |j - i| > 1. \tag{1.22}$$

For each $n \in \mathbb{N}$, we let T_n be the tridiagonal $n \times n$-matrix formed by the first n rows and columns of T so that

$$\mathrm{T}_n = \begin{bmatrix} \beta_0 & 1 & 0 & \cdots & 0 \\ \gamma_1 & \beta_1 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \gamma_{n-3} & \beta_{n-2} & 1 \\ 0 & \cdots & 0 & \gamma_{n-2} & \beta_{n-1} \end{bmatrix}. \tag{1.23}$$

Then, the recurrence relation (1.21) can be expressed as

$$\mathrm{T}_n \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-2}(x) \\ P_{n-1}(x) \end{bmatrix} = x \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-2}(x) \\ P_{n-1}(x) \end{bmatrix} - P_n(x) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{1.24}$$

Suppose that $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} \in \mathbb{R}^+$ for all $n \in \mathbb{N}$, and let $(p_n(x))_{n\in\mathbb{N}}$ be the orthonormal polynomial sequence defined by $p_n(x) = \lambda_n P_n(x)$, with $\lambda_n = \prod_{k=1}^{n} \left(\gamma_k^{-\frac{1}{2}}\right)$. Then $(p_n(x))_{n\in\mathbb{N}}$ satisfies the recurrence relation

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x), \tag{1.25}$$

with $a_n = \sqrt{\gamma_{n+1}}$ and $b_n = \beta_n$, for all $n \in \mathbb{N}$. The recurrence relation (1.25) can be expressed as

$$\mathrm{J}_n \begin{bmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{n-2}(x) \\ p_{n-1}(x) \end{bmatrix} = x \begin{bmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{n-2}(x) \\ p_{n-1}(x) \end{bmatrix} - p_n(x) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

where

$$\mathrm{J}_n = \begin{bmatrix} \beta_0 & \sqrt{\gamma_1} & 0 & \cdots & 0 \\ \sqrt{\gamma_1} & \beta_1 & \sqrt{\gamma_2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \sqrt{\gamma_{n-3}} & \beta_{n-2} & \sqrt{\gamma_{n-2}} \\ 0 & \cdots & 0 & \sqrt{\gamma_{n-2}} & \beta_{n-1} \end{bmatrix}.$$

is the $n \times n$-matrix formed by the first n rows and columns of the infinite *Jacobi matrix* $\mathrm{J} = (\mathrm{J}_{i,l})_{i,l \in \mathbb{N}}$ with entries

$$\mathrm{J}_{i,i} = \beta_i, \quad \mathrm{J}_{i,i+1} = \mathrm{J}_{i+1,i} = \sqrt{\gamma_{i+1}}, \quad \text{and} \quad \mathrm{J}_{i,l} = 0 \text{ if } |l - i| > 1.$$

As a result, the zeros of $p_n(x)$, which coincide with the zeros of $P_n(x)$, are the eigenvalues of the matrix J_n and $P_n(x)$ is the characteristic polynomial of J_n. Therefore, because J_n is a real symmetric matrix, the zeros of $P_n(x)$ are all real and simple. Besides, due to Cauchy's Interlacing Theorem (see [35, Th. 4.3.17]), the zeros of consecutive polynomials interlace, that is, there is always a zero of P_n between two consecutive zeros of P_{n+1}.

The infinite Jacobi matrix J acts as a symmetric linear operator defined in ℓ_2 or in a subset of ℓ_2, which is called a Jacobi operator. Under suitable extra conditions, for example if the coefficients are bounded, the Jacobi operator is self-adjoint. Therefore, there is a strong link between orthogonal polynomials and the spectral theory of symmetric and self-adjoint Jacobi operators (see [41]).

1.3.2 Classical and symmetric orthogonal polynomials

An orthogonal polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is said to be *classical* if its sequence of monic derivatives $\left(\frac{1}{n+1}\,P'_{n+1}(x)\right)_{n\in\mathbb{N}}$ is also orthogonal.

Up to a linear transformation of the variable, the classical orthogonal polynomials are the Hermite, Laguerre, Jacobi and Bessel polynomials. The Hermite polynomials, $(H_n(x))_{n\in\mathbb{N}}$, are orthogonal with respect to e^{-x^2} over $\mathbb{R}$. The Laguerre polynomials, $(L_n(x;\alpha))_{n\in\mathbb{N}}$, $\alpha > -1$, are orthogonal with respect to $e^{-x}x^{\alpha}$ over $\mathbb{R}^+$. The Jacobi polynomials, $(\mathcal{J}_n(x;\alpha,\beta))_{n\in\mathbb{N}}$, $\alpha,\beta > -1$, are orthogonal with respect to $(1-x)^{\alpha}(1+x)^{\beta}$ over the interval $(-1,1)$. As a result, the zeros of Hermite, Laguerre and Jacobi polynomials are all located on the interior of their orthogonality intervals and the zeros of consecutive polynomials interlace. The linear functional of orthogonality of the Bessel polynomials cannot be represented via any positive measure on the real line. Moreover, all their zeros are complex when the degree is even and they have only one real zero when the degree is odd.

The following are alternative defining properties of the classical orthogonal polynomials:

- They satisfy a second-order differential equation, known as Bochner's differential equation, of the type

$$\phi(x)P''_n(x) + \psi(x)P'_n(x) + n\lambda_n P_n(x) = 0, \tag{1.26}$$

  where ϕ and ψ are polynomials independent of n with degree not greater than 2 and exactly 1, respectively, and $\lambda_n = \psi'(0) - \frac{n-1}{2}\,\phi''(0) \neq 0$ for all $n \geq 1$.

- Their linear functional of orthogonality is a non-trivial solution to a first-order differential equation, commonly referred to as the Pearson equation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\,(\phi(x)u) + \psi(x)u = 0, \tag{1.27}$$

  involving the same polynomials ϕ and ψ appearing in (1.26).

- They can be generated via Rodrigues-type formulas

$$P_n(x) = \frac{a_n}{w(x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(\phi^n(x) w(x) \right), \tag{1.28}$$

where the polynomial ϕ is the same as in (1.26) and (1.27).

For the Hermite, Laguerre and Jacobi polynomials, the function $w(x)$ in the Rodrigues-type formula (1.28) is their orthogonality weight, which also satisfies the Pearson equation (1.27), while for the Bessel polynomials, $w(x) = x^{a-2} \mathrm{e}^{-\frac{b}{x}}$, for parameters a, b. The original Rodrigues formula is the one for Legendre polynomials, which are the Jacobi polynomials with parameters $\alpha = \beta = 0$.

A polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is *symmetric* if

$$P_n(-x) = (-1)^n P_n(x), \quad \text{for all } n \in \mathbb{N}. \tag{1.29}$$

or, equivalently, if there exist two polynomial sequences $\left(P_n^{[k]}(x)\right)_{n\in\mathbb{N}}$, $k \in \{0,1\}$, which we call the *quadratic components* of $(P_n(x))_{n\in\mathbb{N}}$, such that

$$P_{2n}(x) = P_n^{[0]}(x^2) \quad \text{and} \quad P_{2n+1}(x) = x P_n^{[1]}(x^2), \quad \text{for all } n \in \mathbb{N}. \tag{1.30}$$

A polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is symmetric and orthogonal if and only if it satisfies a second order recurrence relation of the form

$$P_{n+1}(x) = x\, P_n(x) - \gamma_n P_{n-1}(x), \quad n \geq 1,$$

with $\gamma_n \neq 0$, for all $n \geq 1$, and initial conditions $P_0 = 1$ and $P_1(x) = x$.

A linear functional u is *symmetric* if all its moments of odd order are equal to 0. The notions of symmetry for orthogonal polynomials and for linear functionals are naturally connected. Indeed, if $(P_n(x))_{n\in\mathbb{N}}$ is orthogonal with respect to u_0, then $(P_n(x))_{n\in\mathbb{N}}$ is symmetric if and only if u_0 is symmetric.

Up to a linear transformation of the variable, the only classical orthogonal polynomials that are also symmetric are the Hermite polynomials and the Jacobi polynomials with $\alpha = \beta$. The latter are known as the Gegenbauer or ultraspherical polynomials and include, as particular cases, the Legendre polynomials, when $\alpha = \beta = 0$, and the Chebyshev polynomials of the first and second kind, when $\alpha = \beta = -\frac{1}{2}$ and $\alpha = \beta = \frac{1}{2}$, respectively.

The quadratic components of a symmetric orthogonal polynomial sequence are also orthogonal. Let $(P_n(x))_{n\in\mathbb{N}}$ be a symmetric and orthogonal polynomial sequence with quadratic decomposition (1.30), and let $\sigma_2 : \mathcal{P}' \to \mathcal{P}'$ be the linear operator defined by $\langle \sigma_2(v), f\rangle := \left\langle v, f(x^2)\right\rangle$ for any $v \in \mathcal{P}'$ and $f \in \mathcal{P}$. Then, $\left(P_n^{[0]}(x)\right)_{n\in\mathbb{N}}$ and $\left(P_n^{[1]}(x)\right)_{n\in\mathbb{N}}$ are orthogonal with respect to $\sigma_2(u_0)$ and $\sigma_2(xu_1)$, respectively, where u_0 and u_1 are the first two elements of the dual sequence of $(P_n(x))_{n\in\mathbb{N}}$. In particular, if $(P_n(x))_{n\in\mathbb{N}}$ is orthogonal with respect to an even function $w(x)$ over the interval $(-c, c)$, where $c \in \mathbb{R}^+ \cup \{+\infty\}$, then $\left(P_n^{[0]}(x)\right)_{n\in\mathbb{N}}$ and $\left(P_n^{[1]}(x)\right)_{n\in\mathbb{N}}$ are orthogonal with respect to $x^{-\frac{1}{2}} w\left(x^{\frac{1}{2}}\right)$ and $x^{\frac{1}{2}} w\left(x^{\frac{1}{2}}\right)$, respectively, over the interval $(0, c^2)$.

1.4 Multiple orthogonal polynomials (MOPs)

In this section we present some background on multiple orthogonal polynomials. Our main references on the general theory of multiple orthogonal polynomials are [4], [36, Ch. 23], [59, Ch. 4], [49] and [74, §3]. Throughout this thesis, we often use the abbreviation MOPs for multiple orthogonal polynomials.

Other terminology used to refer to these polynomials in the bibliography include Hermite-Padé polynomials [60], polyorthogonal polynomials [59], polynomials of simultaneous orthogonality [39], vector orthogonal polynomials [76, 77] and d-orthogonal polynomials [53] (the latter terminology is only used for the type II polynomials on the so-called step-line).

The orthogonality measures of multiple orthogonal polynomials are spread across a vector of r linear functionals or measures, with $r \in \mathbb{Z}^+$, and they are polynomials of a single variable depending on a multi-index $\vec{n} = (n_0, \cdots, n_{r-1}) \in \mathbb{N}^r$ of length $|\vec{n}| = n_0 + \cdots + n_{r-1}$. In this section, we present definitions of the two types of multiple orthogonal polynomials, type I and type II, with respect to vectors of measures (these definitions can easily be rewritten replacing the orthogonality measures by linear functionals); we describe three special examples of perfect systems, Angelesco systems, AT-systems and Nikishin systems, with emphasis on the latter; we give more detail on multiple orthogonal polynomials for multi-indices on the so-called step-line and the recurrence relations satisfied by them; and we extend the concepts of classical and symmetric orthogonal polynomials to the context of multiple orthogonality.

1.4.1 Type I and type II MOPs

The *type I multiple orthogonal polynomials* for $\vec{n} = (n_0, \cdots, n_{r-1}) \in \mathbb{N}^r$ with respect to the system of r measures $(\mu_0, \cdots, \mu_{r-1})$ are given by a vector of r polynomials $(A_{\vec{n},0}, \cdots, A_{\vec{n},r-1})$, with $\deg A_{\vec{n},j} \leq n_j - 1$ for each $j \in \{0, \cdots, r-1\}$, satisfying the orthogonality and normalisation conditions

$$\sum_{j=0}^{r-1} \int x^k A_{\vec{n},j}(x) \mathrm{d}\mu_j(x) = \begin{cases} 0 & \text{if } k \in \{0, \cdots, |\vec{n}| - 2\} \\ 1 & \text{if } k = |\vec{n}| - 1. \end{cases} \tag{1.31}$$

In this thesis, we will be dealing with systems of absolutely continuous measures $\mu_j(x)$ with respect to a common measure μ. Therefore, the measures $\mu_j(x)$ can be represented via weight functions $w_j(x)$ such that $\mathrm{d}\mu_j(x) = w_j(x)\mathrm{d}\mu(x)$, for each $j \in \{0, \cdots, r-1\}$. In these cases, the *type I function* is defined as

$$Q_{\vec{n}}(x) = \sum_{j=0}^{r-1} A_{\vec{n},j}(x)\, w_j(x), \tag{1.32}$$

and the conditions in (1.31) are equivalent to

$$\int x^k Q_{\vec{n}}(x)\mathrm{d}\mu(x) = \begin{cases} 0 & \text{if } k \in \{0, \cdots, |\vec{n}|-2\}, \\ 1 & \text{if } k = |\vec{n}|-1. \end{cases} \tag{1.33}$$

The *type II multiple orthogonal polynomial* for $\vec{n} = (n_0, \cdots, n_{r-1}) \in \mathbb{N}^r$ with respect to the system of r measures $(\mu_0, \cdots, \mu_{r-1})$ consists of a monic polynomial $P_{\vec{n}}$ of degree $|\vec{n}|$ which satisfies, for each $j \in \{0, \cdots, r-1\}$, the orthogonality conditions

$$\int x^k P_{\vec{n}}(x)\mathrm{d}\mu_j(x) = 0 \quad \text{if } k \in \{0, \cdots, n_j - 1\}. \tag{1.34}$$

Observe that both types of multiple orthogonality reduce to standard orthogonality when the number of measures, r, is equal to 1.

The orthogonality conditions for multiple orthogonal polynomials give a non-homogeneous system of $|\vec{n}|$ linear equations for the $|\vec{n}|$ unknown coefficients of the vector of type I polynomials $(A_{\vec{n},0}, \cdots, A_{\vec{n},r-1})$ in (1.31) or the type II polynomials $P_{\vec{n}}(x)$ in (1.34). However, this system may not have a solution, or when a solution exists it may not be unique. The existence and uniqueness of a solution is equivalent for type I and type II polynomials. If the solution is unique, then the multi-index $\vec{n}$ is called *normal*. A system of multiple orthogonal polynomials is said to be *perfect* [52] if all the multi-indices are normal.

Furthermore, if $\vec{n} \in \mathbb{N}^r$ is a multi-index in a perfect system and $j \in \{0, \cdots, r-1\}$, then the type I polynomial $A_{\vec{n},j}$ has degree exactly $n_j - 1$ whenever $n_j \geq 1$, and the type II polynomial $P_{\vec{n}}(x)$ satisfies

$$\int x^{n_j} P_{\vec{n}}(x)\mathrm{d}\mu_j(x) \neq 0.$$

1.4.2 Special systems: Angelesco, AT- and Nikishin

A vector of measures $(\mu_0, \cdots, \mu_{r-1})$ is an *Angelesco system* (introduced in [2]) if the supports of the measures are subsets of pairwise disjoint intervals, that is, if there exist intervals $\Delta_0, \cdots, \Delta_{r-1}$ such that $\text{supp}(\mu_j) \subseteq \Delta_j$, for each $j \in \{0, \cdots, r-1\}$, and $\Delta_i \cap \Delta_j = \emptyset$, whenever $i \neq j$. Usually, the intervals are allowed to touch each other, that is, the last condition is replaced by $\mathring{\Delta}_i \cap \mathring{\Delta}_j = \emptyset$, whenever $i \neq j$. An Angelesco system is always perfect. Furthermore, for any $\vec{n} \in \mathbb{N}^r$ and each $j \in \{0, \cdots, r-1\}$, the type II and type I multiple orthogonal polynomials $P_{\vec{n}}$ and $A_{\vec{n},j}$ (with $n_j \geq 1$) have exactly n_j and $n_j - 1$, respectively, distinct zeros on $\mathring{\Delta}_j$. Therefore, $P_{\vec{n}}$ has exactly $|\vec{n}|$ zeros, all simple and located in $\bigcup_{j=0}^{r-1} \mathring{\Delta}_j$.

A vector of measures $(\mu_0, \cdots, \mu_{r-1})$ is an *AT-system* (introduced in [59, Ch. 4]), where AT stands for Algebraic Tchebyshev, on an interval Δ for a multi-index $\vec{n} = (n_0, \cdots, n_{r-1}) \in \mathbb{N}^r$ if the measures $\mu_j(x)$ are absolutely continuous with respect to a common positive measure μ on Δ, via weight functions $w_j(x)$ such that the set of functions

$$\bigcup_{j=0}^{r-1} \left\{ w_j(x), x w_j(x), \cdots, x^{n_j - 1} w_j(x) \right\}$$

forms a Chebyshev system on Δ. This latter condition is equivalent to imposing that, for any polynomials $p_0, \cdots, p_{r-1}$ not all identically equal to 0 with $\deg(p_j)$ not greater than $n_j - 1$ for each $j \in \{0, \cdots, r-1\}$, the function $\sum_{j=0}^{r-1} p_j(x) w_j(x)$ has at most $|\vec{n}| - 1$ zeros on Δ. A vector of measures $(\mu_0, \cdots, \mu_{r-1})$ is an AT-system on an interval Δ if it is an AT-system on Δ for every multi-index in $\mathbb{N}^r$.

All AT-systems are perfect. Furthermore, the type I function $Q_{\vec{n}}$ defined by (1.32) and the type II multiple orthogonal polynomial $P_{\vec{n}}$ have exactly $|\vec{n}| - 1$ and $|\vec{n}|$ simple zeros on Δ, respectively, for every multi-index $\vec{n} \neq (0, \cdots, 0) \in \mathbb{N}^r$ in an AT-system. Moreover, the zeros of the type II polynomials interlace with the zeros of their nearest neighbours (see [34, Th. 2.1]), which means that there is always a zero of $P_{\vec{n}}$ between two consecutive zeros of $P_{\vec{n}+\vec{e}_k}$, where $\vec{e}_k \in \mathbb{N}^r$, with

$k \in \{0, \cdots, r-1\}$, is the multi-index that has all entries equal to 0 except the entry of index k which is equal to 1.

A pair of measures (μ_0, μ_1) forms a *Nikishin system* of order 2, firstly introduced in [58], if μ_0 and μ_1 are both supported on an interval Δ_0 and there exists a measure σ supported on an interval Δ_1, with $\mathring{\Delta}_0 \cap \mathring{\Delta}_1 = \emptyset$, such that μ_1 is absolutely continuous with respect to μ_0, with Radon-Nikodym derivative equal to the Stieltjes transform of σ, that is,

$$\frac{\mathrm{d}\mu_1(x)}{\mathrm{d}\mu_0(x)} = \int_{\Delta_1} \frac{\mathrm{d}\sigma(t)}{x-t}. \tag{1.35}$$

A Nikishin system formed by $r > 2$ measures is defined by induction using a non-commutative product of measures $\langle \cdot, \cdot \rangle$ such that, for any two measures σ_1 and σ_2, $\langle \sigma_1, \sigma_2 \rangle$ is an absolutely continuous measure with respect to σ_1, with Radon-Nikodym derivative equal to the Stieltjes transform of σ_2, that is,

$$\mathrm{d}\langle \sigma_1, \sigma_2 \rangle (x) = \left(\int \frac{\mathrm{d}\sigma_2(t)}{x-t} \right) \mathrm{d}\sigma_1(x).$$

Observe that (1.35) is equivalent to $\mu_1 = \langle \mu_0, \sigma \rangle$. A Nikishin system of order $r > 2$ is formed by r measures $(\mu_0, \cdots, \mu_{r-1})$ supported on a common interval Δ_0 such that there exists a Nikishin system $(\sigma_1, \cdots, \sigma_{r-1})$ of order $r-1$ on an interval Δ_1, with $\mathring{\Delta}_0 \cap \mathring{\Delta}_1 = \emptyset$, such that $\mu_j = \langle \mu_0, \sigma_j \rangle$, for each $j \in \{0, \cdots, r-1\}$.

It was proved in [29] that every Nikishin system is an AT-system (see also [30] for the cases where the supporting intervals of the measures are unbounded or where consecutive intervals touch at one point). Therefore, all Nikishin systems are perfect and the zeros of their type I functions and type II multiple orthogonal polynomials satisfy the properties stated for AT-systems.

1.4.3 MOPs on the step-line and recurrence relations

A multi-index $(n_0, \cdots, n_{r-1}) \in \mathbb{N}^r$ is on the *step-line* if $n_0 \geq n_1 \geq \cdots \geq n_{r-1} \geq n_0 - 1$, or, equivalently, if there exists $m \in \mathbb{N}$ and $j \in \{0, \cdots, r-1\}$ such that

$$n_k = \begin{cases} m+1 & \text{if } 0 \leq k \leq j-1, \\ m & \text{if } j \leq k \leq r-1. \end{cases} \tag{1.36}$$

Fixing $r \in \mathbb{Z}^+$, there is a unique multi-index of length n on the step-line of $\mathbb{N}^r$, for each $n \in \mathbb{N}$. If $n = rm + j$, with $m \in \mathbb{N}$ and $j \in \{0, \cdots, r-1\}$, the multi-index of length n is $(n_0, \cdots, n_{r-1}) \in \mathbb{N}^r$ with entries as described in (1.36). Therefore, when the number of measures is fixed and we only consider multi-indices on the step-line, we can replace the multi-index of the multiple orthogonal polynomials of both type I and type II by its length without any ambiguity. Throughout the rest of this section, we assume that we are dealing with a perfect system.

The type I multiple orthogonal polynomials for the multi-index on the step-line $(n_0, \cdots, n_{r-1})$, with entries given by (1.36) and length $n = rm + j$, are given by a vector of r polynomials $(A_{n,0}, \cdots, A_{n,r-1})$, with $\deg A_{n,k} = \left\lfloor \frac{n-k-1}{r} \right\rfloor$, for each $k \in \{0, \cdots, r-1\}$, satisfying (1.31), with all "$\vec{n}$" and "$|\vec{n}|$" replaced by "n".

The type II multiple orthogonal polynomials on the step-line form a sequence with exactly one polynomial of degree n, for each $n \in \mathbb{N}$. These are often referred to as *d-orthogonal polynomials*, as introduced in [53], where d is the number of orthogonality measures. This means that a polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is *r-orthogonal* with respect to a vector of r measures $(\mu_0, \cdots, \mu_{r-1})$ if $P_n(x)$, with $n = rm + j$ for $m \in \mathbb{N}$ and $j \in \{0, \cdots, r-1\}$, is the type II multiple orthogonal polynomial for the multi-index on the step-line $(n_0, \cdots, n_{r-1})$, with entries given by (1.36). By definition, $(P_n(x))_{n\in\mathbb{N}}$ satisfies, for $j \in \{0, \cdots, r-1\}$,

$$\int x^k P_n(x) \mathrm{d}\mu_j(x) = \begin{cases} N_n \neq 0 & \text{if } n = rk + j, \\ 0 & \text{if } n \geq rk + j + 1. \end{cases} \tag{1.37}$$

Throughout most of this thesis, in particular in Chapters 2, 4 and 5, we focus on multiple orthogonal polynomials on the step-line with respect to 2 measures, with emphasis on the type II polynomials, that is, on the 2-orthogonal polynomials. For $r = 2$, the step-line as defined above corresponds to the lower step-line as illustrated in Figure 1.1. The type II polynomials and type I function for multi-indices on the upper step-line with respect to the pair of measures (μ_0, μ_1) correspond to the polynomials on the lower step-line with respect to (μ_1, μ_0), which means that moving between the lower and upper step-line corresponds to swapping the order of the measures. In Chapter 4, we investigate a family of multiple orthogonal systems which, under the action of the derivative operator, bounce from the lower to the upper step-line, and reciprocally, with shifted parameters.

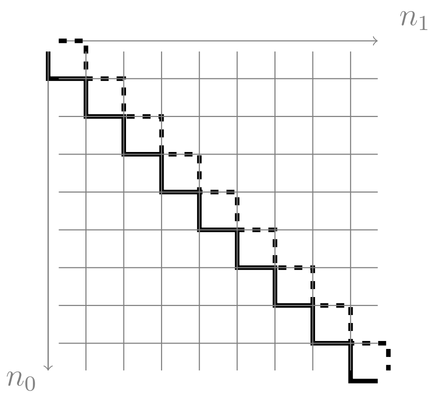

FIGURE 1.1: [47, Fig. 1] Lower and upper step-line for the multi-index $(n_0, n_1) \in \mathbb{N}^2$ in solid and dashed black line, respectively.

Like standard orthogonal polynomials, multiple orthogonal polynomials also satisfy recurrence relations of finite order. In fact, both type II and type I multiple orthogonal polynomials satisfy nearest-neighbour recurrence relations as well as recurrence relations for sequences of polynomials for indices on a path starting from the origin and with one and only one component increasing by exactly one at each step. All these recurrence relations are shown in [36, § 23.1.4] and the nearest-neighbour recurrence relations are also investigated in [71].

We are particularly interested in recurrence relations for the type II multiple orthogonal polynomials with multi-indices on the step-line. In fact, a polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is r-orthogonal if and only if it satisfies a recurrence relation of order $r+1$ of the form

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \sum_{j=1}^{r} \gamma_{n-j+1}^{[j]} P_{n-j}(x), \tag{1.38}$$

with $\gamma_n^{[r]} \neq 0$, for all $n \in \mathbb{Z}^+$, and initial conditions $P_0 = 1$ and $P_{-j} = 0$, for each $1 \leq j \leq r$. This result can be found in [53, Th. 2.1].

When $r = 2$, the relation (1.38) reduces to the third order recurrence relation

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1} P_{n-2}(x), \tag{1.39}$$

with $\gamma_n \neq 0$ for all $n \in \mathbb{Z}^+$, and initial conditions $P_0 = 1$ and $P_{-1} = P_{-2} = 0$.

When $(P_n(x))_{n\in\mathbb{N}}$ satisfies (1.38), we consider an infinite $(r+1)$-banded lower-Hessenberg matrix $\mathrm{H} = (\mathrm{h}_{i,j})_{i,j\in\mathbb{N}}$, to which we refer to as the *infinite Hessenberg matrix* associated with $(P_n(x))_{n\in\mathbb{N}}$, whose entries are

$$\mathrm{h}_{i,j} = \begin{cases} 1 & \text{if } j = i+1, \\ \beta_i & \text{if } j = i, \\ \gamma_{i-k+1}^{[k]} & \text{if } j = i-k, \text{ for } 1 \leq k \leq r, \\ 0 & \text{if } j \geq i+2 \text{ or } j \leq j-r-1. \end{cases} \tag{1.40}$$

For each $n \in \mathbb{Z}^+$, we let H_n be the lower-Hessenberg $(n \times n)$-matrix formed by the first n rows and columns of H. Then, in the same manner as the recurrence relation (1.21) satisfied by an orthogonal polynomial sequence can be expressed by (1.24) using the tridiagonal matrix (1.23), the recurrence relation (1.38) satisfied by a r-orthogonal polynomial sequence can be expressed, for each $n \in \mathbb{Z}^+$, by

$$\mathrm{H}_n \begin{bmatrix} P_0(x) \\ \vdots \\ P_{n-2}(x) \\ P_{n-1}(x) \end{bmatrix} = x \begin{bmatrix} P_0(x) \\ \vdots \\ P_{n-2}(x) \\ P_{n-1}(x) \end{bmatrix} - P_n(x) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{1.41}$$

In particular, setting $r = 2$, (1.38) can be expressed as in (1.41) with

$$\mathrm{H}_n = \begin{bmatrix} \beta_0 & 1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & \beta_1 & 1 & 0 & \cdots & 0 \\ \gamma_1 & \alpha_2 & \beta_2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \gamma_{n-3} & \alpha_{n-2} & \beta_{n-2} & 1 \\ 0 & \cdots & 0 & \gamma_{n-2} & \alpha_{n-1} & \beta_{n-1} \end{bmatrix}.$$

If $(P_n(x))_{n\in\mathbb{N}}$ is the r-orthogonal polynomial sequence satisfying (1.41), then the zeros of $P_n(x)$, $n \in \mathbb{Z}^+$, are the eigenvalues of the Hessenberg matrix H_n. So, $P_n(x)$ is the characteristic polynomial of H_n, that is, $P_n(x) = \det(x\mathrm{I}_n - \mathrm{H}_n)$.

When we are dealing with r-orthogonal polynomials, with $r \geq 2$, the corresponding $(r+1)$-banded Hessenberg matrices H_n and H, whose entries are given by (1.40), are not symmetrisable. Hence, the operator associated with H is non-selfadjoint. This leads to a connection between multiple orthogonal polynomials and the spectral theory of non-selfadjoint operators (for more information, we refer to [6] and [69]), which is much harder to explore than the spectral theory of self-adjoint Jacobi operators naturally linked to orthogonal polynomials.

1.4.4 Extensions of classical and symmetric orthogonal polynomials

As we have seen in §1.3.2, there are several equivalent defining properties of classical orthogonal polynomials. These properties give rise to different sets of multiple orthogonal polynomials, so the notion of "classical" in the context of multiple orthogonality requires further details.

Amongst these extensions of the classical orthogonal polynomials, there are the multiple orthogonal polynomials with respect to vectors of measures such that all the measures are obtained from a classical weight. For instance, by taking weights of the same type with different parameters and supported on a common interval to form AT-systems, often Nikishin systems, or by considering the same weight supported in different domains, which may be real intervals or curves on the complex plane, to form Angelesco systems. Examples of the former include the multiple Hermite, multiple Laguerre of first and second kind and Jacobi-Piñeiro polynomials, while Laguerre-Angelesco and Jacobi-Angelesco polynomials are examples of the latter. For more information on several of these families of multiple orthogonal polynomials, we refer to [5, 18] and the references therein.

Another generalisation of the classical character for orthogonal polynomials, with an emphasis on the algebraic properties of the polynomials, plays a bigger role in this thesis: a sequence of multiple orthogonal polynomials is said to be *Hahn-classical*, or to satisfy the Hahn-classical property, if the sequence of their derivatives is also multiple orthogonal. This terminology is mostly used for r-orthogonal polynomials, which, typically, satisfy orthogonality conditions with respect to a vector of weight functions satisfying a non-trivial first-order matrix differential equation, where each weight is a solution to an ordinary differential equation of order equal to the number of orthogonality measures.

The definition of a symmetric polynomial sequence can also be generalised. For $m \geq 2$, a polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is *m-fold symmetric* if

$$P_n\left(\mathrm{e}^{\frac{2\pi i}{m}}x\right) = \mathrm{e}^{\frac{2n\pi i}{m}}P_n(x), \quad \text{for all } n \in \mathbb{N}, \tag{1.42}$$

or, equivalently, if there exist m polynomial sequences $\left(P_n^{[k]}(x)\right)_{n\in\mathbb{N}}$, each supra indexed with $k \in \{0, \cdots, m-1\}$, such that

$$P_{mn+k}(x) = x^k P_n^{[k]}(x^m), \quad \text{for all } n \in \mathbb{N}. \tag{1.43}$$

Observe that a 2-fold symmetric polynomial sequence is, in fact, a symmetric polynomial sequence as defined by (1.29).

Recall that symmetric orthogonal polynomials satisfy a simpler case of the recurrence relation (1.21) with $\beta_n = 0$. Analogously, if $(P_n(x))_{n\in\mathbb{N}}$ is a $(r+1)$-fold symmetric r-orthogonal polynomial sequence, with $r \in \mathbb{Z}^+$, the coefficients in

(1.38) satisfy $\beta_n = \gamma_{n+1-j}^{[j]} = 0$, for all $n \in \mathbb{N}$ and each $j \in \{1, \cdots, r-1\}$, and $(P_n(x))_{n\in\mathbb{N}}$ satisfies a three-term recurrence relation (of order $r+1$)

$$P_{n+1}(x) = xP_n(x) - \gamma_{n-r+1}P_{n-r}(x), \tag{1.44a}$$

with

$$\gamma_n \neq 0, \text{ for all } n \in \mathbb{Z}^+, \quad \text{and} \quad P_j(x) = x^j, \text{ for each } j \in \{0, \cdots, r\}. \tag{1.44b}$$

Conversely, a polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ satisfying (1.44a)-(1.44b) is necessarily $(r+1)$-fold symmetric and r-orthogonal.

We end this section by making a remark that, when investigating multiple orthogonal polynomials, we are often interested in analysing their asymptotic behaviour. For that purpose, Landau's asymptotic notation is very useful. As such, throughout this thesis, we use for functions $f, g : \mathbb{N} \to \mathbb{C}$:

- Landau's little-o notation: $f = o(g)$ as $n \to \infty$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$;
- Landau's big-$\mathcal{O}$ notation: $f = \mathcal{O}(g)$ as $n \to \infty$ if there exist $M \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that $|f(n)| < M\,|g(n)|$, for all $n \geq n_0$;
- asymptotic equivalence notation: $f \sim g$ as $n \to \infty$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

1.5 Continued fractions

In this subsection, we give a brief account of the analytic theory of continued fractions, whose investigation seems to have originated from Stieltjes' memoir [65]. Our main reference for the analytic theory of continued fractions is [78]. For examples of continued fractions involving special functions, we refer to [20].

We start by giving the definition of continued fractions as well as introducing some basic concepts about these mathematical objects. Then, we show some special examples of continued fractions, namely Stieltjes and Jacobi continued fractions, also known as S-fractions and J-fractions, respectively. Next, we discuss the key role of J-fractions in the connection of continued fractions and orthogonal polynomials. Finally, we observe that S-fractions and J-fractions have representations as Stieltjes transforms, which are instrumental for us to prove that we are dealing with Nikishin systems in Chapters 4 and 5.

Following the notation in [20], a *continued fraction* is an expression of the type

$$\mathop{\mathrm{K}}_{n=0}^{\infty}\left(\frac{\alpha_n}{\beta_n}\right) := \cfrac{\alpha_0}{\beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \cdots}}}. \tag{1.45}$$

A continued fraction is a limiting case of rational fractions. For any $n \in \mathbb{Z}^+$, the *n-th-approximant* of the continued fraction (1.45) is

$$f_n = f_n \begin{pmatrix} \alpha_0, \cdots, \alpha_{n-1} \\ \beta_0, \cdots, \beta_{n-1} \end{pmatrix} = \cfrac{\alpha_0}{\beta_0 + \cfrac{\alpha_1}{\beta_1 + \ddots \cfrac{\alpha_{n-1}}{\beta_{n-1}}}}.$$

The n-th-approximant of a continued fraction can be written in the form $f_n = \dfrac{R_n}{S_n}$, where R_n and S_n denote the *n-th-numerator* and *n-th-denominator* of (1.45), which are polynomials in the indeterminates $\alpha_0, \cdots, \alpha_{n-1}$ and $\beta_0, \cdots, \beta_{n-1}$ that can be generated by the recurrence formulas

$$R_{n+1} = \beta_n R_n + \alpha_n R_{n-1} \quad \text{and} \quad S_{n+1} = \beta_n S_n + \alpha_n S_{n-1}, \quad n \in \mathbb{Z}^+, \tag{1.46}$$

with initial conditions $R_0 = 0$, $S_0 = 1$, $R_1 = \alpha_0$ and $S_0 = \beta_0$.

A continued fraction (1.45) is *convergent* if the limit of its sequence of approximants, exists and is finite, and it is *divergent* otherwise. If (1.45) is convergent then its *value* is equal to the limit of its sequence of approximants. Two continued fractions are *equivalent* if their n-th-approximants are all equal. It is straightforward that two equivalent convergent continued fractions have the same value.

A *contraction* of a continued fraction with a convergent sequence $(f_n)_{n\in\mathbb{N}}$ is a continued fraction whose convergents $(f'_n)_{n\in\mathbb{N}}$ form a subsequence of $(f_n)_{n\in\mathbb{N}}$, that is, $f'_n = f_{s(n)}$, $n \in \mathbb{N}$, where $s : \mathbb{N} \to \mathbb{N}$ is an increasing sequence. Conversely, the original continued fraction is an *extension* of its contraction. Observe that any contraction of a convergent continued fraction is also convergent and has the same value as the original continued fraction. Examples of contractions of a continued fraction with a convergent sequence $(f_n)_{n\in\mathbb{N}}$ are its even and odd part which are defined, up to equivalence, as the continued fractions with convergent sequences equal to $(f_{2n})_{n\in\mathbb{N}}$ and $(f_{2n+1})_{n\in\mathbb{N}}$, respectively.

A continued fraction (1.45) with $\alpha_n = 1$, for all $n \in \mathbb{N}$, is called a *regular or simple continued fraction*. A classical example of a regular continued fraction is obtained from (1.45) by setting $\beta_n = c_n z^{n+1 \bmod 2}$, with $c_n \in \mathbb{R}^+$ for all $n \in \mathbb{N}$ (that is, $\beta_{2k} = c_{2k} z$ and $\beta_{2k+1} = c_{2k+1}$, for each $k \in \mathbb{N}$):

$$\operatorname*{K}_{n=0}^{\infty}\left(\frac{1}{c_n z^{n+1 \bmod 2}}\right) = \cfrac{1}{c_0 z + \cfrac{1}{c_1 + \cfrac{1}{c_2 z + \cfrac{1}{c_3 + \cdots}}}}. \tag{1.47}$$

The latter continued fraction was the object of Stieltjes' work in [65]. Hence, the terms *Stieltjes continued fraction* and *S-fraction* are used for continued fractions of the form (1.47), as well as for any continued fraction which is equivalent to (1.47) or which can be obtained from (1.47) by a change of variable.

If we set $\alpha_0 = \dfrac{1}{c_0}$ and $\alpha_{n+1} = \dfrac{1}{c_n c_{n+1}}$, for each $n \in \mathbb{N}$, then (1.47) is equivalent to

$$\operatorname*{K}_{n=0}^{\infty}\left(\frac{\alpha_n}{z^{n+1 \bmod 2}}\right) = \cfrac{\alpha_0}{z + \cfrac{\alpha_1}{1 + \cdots}}. \tag{1.48}$$

Making the change of variable $w = z^{-1}$, the latter leads to

$$\underset{n=0}{\overset{\infty}{\mathrm{K}}} \left(\frac{\alpha_n w}{1} \right) = \cfrac{\alpha_0 w}{1 + \cfrac{\alpha_1 w}{1 + \cdots}}. \tag{1.49}$$

Therefore, the continued fractions (1.47), (1.48) and (1.49) are all S-fractions.

Another well-known example of continued fractions are the so-called *Jacobi continued fractions* or *J-fractions*, obtained from (1.45) by setting $\alpha_0 = a_0$, $\alpha_n = -a_n$, for any $n \geq 1$, and $\beta_n = z + b_n$, for all $n \in \mathbb{N}$, with $a_n, b_n \in \mathbb{C}$, to get

$$-\underset{n=0}{\overset{\infty}{\mathrm{K}}} \left(\frac{-a_n}{z + b_n} \right) = \cfrac{a_0}{z + b_0 - \cfrac{a_1}{z + b_1 - \cfrac{a_2}{z + b_2 - \cdots}}}. \tag{1.50}$$

If every $a_n, b_n \in \mathbb{R}^+$ then the J-fraction (1.50) can be obtained by contraction from the S-fraction (1.48). More precisely, applying to (1.48) the identity

$$z + \cfrac{\beta_0}{1 + \cfrac{\beta_1}{\lambda}} = z + \beta_0 - \frac{\beta_0 \beta_1}{\beta_1 + \lambda},$$

valid for any $\beta_0, \beta_1, \lambda \in \mathbb{C}$, we derive that (1.50), with coefficients $a_0 = \alpha_0$, $b_0 = \alpha_1$ and $a_n = \alpha_{2n-1}\alpha_{2n}$ and $b_n = \alpha_{2n} + \alpha_{2n+1}$, for $n \geq 1$, is the even part of (1.48). For more details, including how to recover α_n from a_n and b_n, see [65, Eqs. (I^a)-(I^d)] and [78, Th. 28.3].

Recalling (1.46), the n-th-approximants of the J-fraction (1.50) are rational functions $\dfrac{R_n(z)}{S_n(z)}$, where $R_n(z)$ and $S_n(z)$ are polynomial solutions of the recurrence relation

$$y_{n+1}(z) = (z + b_n)\, y_n(z) - a_n y_{n-1}(z), \quad n \in \mathbb{Z}^+,$$

with initial conditions $R_0 = 0$ and $R_1 = a_0$, $S_0 = 1$ and $S_1 = z + b_0$. Therefore, $(S_n)_{n \in \mathbb{N}}$ is the orthogonal polynomial sequence satisfying the recurrence relation

(1.21) with $\beta_n = -b_n$ and $\gamma_n = a_n$, for $n \in \mathbb{N}$, and $(R_n)_{n\in\mathbb{N}}$ are the corresponding associated polynomials.

From what we have written above, it is clear how orthogonal polynomials are intrinsically connected with continued fractions. In the context of multiple orthogonality, we will use continued fractions and, in particular, S-fractions and J-fractions to prove that certain pairs of measures form Nikishin systems, because these continued fractions can be represented via Stieltjes transforms.

In fact, Stieltjes showed in [65] that the S-fraction (1.47) can be represented as a Stieltjes transform of a measure with support in $\mathbb{R}^-$. More precisely, if every $c_n \in \mathbb{R}^+$ then there exists a probability measure in $\mathbb{R}^+$, that is, a non decreasing bounded function σ in $\mathbb{R}_0^+$ with $\sigma(0) = 0$ and $\lim\limits_{u\to\infty} \sigma(u) = 1$, such that

$$\mathop{\mathrm{K}}_{n=0}^{\infty}\left(\frac{1}{c_n z^{n+1 \bmod 2}}\right) = \frac{1}{c_0}\int_{-\infty}^{0}\frac{\mathrm{d}\sigma(-t)}{z-t} = \frac{1}{c_0}\int_{0}^{\infty}\frac{\mathrm{d}\sigma(u)}{z+u}.$$

Due to the relations between the continued fractions (1.47)-(1.50), we also have the integral representations

$$-\mathop{\mathrm{K}}_{n=0}^{\infty}\left(\frac{-a_n}{z+b_n}\right) = \mathop{\mathrm{K}}_{n=0}^{\infty}\left(\frac{\alpha_n}{z^{n+1 \bmod 2}}\right) = \alpha_0\int_{0}^{\infty}\frac{\mathrm{d}\sigma(u)}{z+u} \tag{1.51}$$

and

$$\mathop{\mathrm{K}}_{n=0}^{\infty}\left(\frac{\alpha_n w}{1}\right) = \alpha_0\int_{0}^{\infty}\frac{\mathrm{d}\sigma(u)}{w^{-1}+u} = \alpha_0\int_{0}^{\infty}\frac{w\mathrm{d}\sigma(u)}{1+uw},$$

with $\alpha_n, a_n, b_n \in \mathbb{R}^+$, for all $n \in \mathbb{N}$.

1.6 Branched continued fractions

Although various types of branched continued fractions have been introduced in the literature, the main object of this section are the branched continued fractions introduced in [61] as a fundamental tool to prove coefficientwise Hankel-total positivity of combinatorially interesting sequences of polynomials and total positivity of related matrices. As such, this section is based on [61].

Firstly, we present the connection of m-branched continued fractions with the combinatorics of m-Dyck paths as a generalisation of the connection of continued fractions with Dyck paths. In fact, m-branched continued fractions appear as a representation of the ordinary generating function of weighted m-Dyck paths. Like the authors of [61], we do not deal directly with the m-branched continued fractions, and instead we deal with the associated m-Stieltjes-Rogers polynomials, which are the generating polynomials of the same m-Dyck paths with fixed length.

A *Dyck path* is a path in the upper half-plane $\mathbb{Z} \times \mathbb{N}$, starting and ending on the horizontal axis, using steps $(1, 1)$ (called "rise" or "up step") and $(1, -1)$ (called "fall" or "down step"). More generally, a *Dyck path at level* k is a path in $\mathbb{Z} \times \mathbb{N}_{\geq k}$, starting and ending on the horizontal line at height k, using steps $(1, 1)$ and $(1, -1)$. Observe that a Dyck path always has even length, where the length is the number of the steps, because the number of rises and falls must be equal.

For an infinite set of indeterminates $\boldsymbol{\lambda} = (\lambda_i)_{i \geq 1}$, we define the *Stieltjes-Rogers polynomial* of order n (introduced in [31]), which we denote by $S_n(\boldsymbol{\lambda})$, as the generating polynomial for all Dyck paths of length $2n$, with each rise having weight 1 and each fall from height i having weight λ_i. Clearly $S_n(\boldsymbol{\lambda})$ is a homogeneous polynomial of degree n with nonnegative integer coefficients.

The *ordinary generating function* of a sequence $(a_n)_{n \in \mathbb{N}}$ is the formal series

$$f(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Let $f_0(t) = \sum_{n=0}^{\infty} S_n(\boldsymbol{\lambda})\, t^n$ be the ordinary generating function for Dyck paths with the weights specified above, considered as a formal power series in t. More generally, let $f_k(t)$ be the ordinary generating function for Dyck paths at level k with the same weights. Then, f_k is f_0 with each λ_i replaced by λ_{i+k}. Observe that we can split a Dyck path at level k of nonzero length at its last visit to level k and rewrite the path in the form $\mathcal{P}_0 U \mathcal{P}_1 D$ where $\mathcal{P}_0$ and $\mathcal{P}_1$ are arbitrary Dyck paths at level k and $k+1$, respectively, U is an up step and D is a down step. As a result, the functions $f_k(t)$ satisfy the equivalent functional equations

$$f_k(t) = 1 + \lambda_{k+1}\, t\, f_k(t) f_{k+1}(t) \quad \text{and} \quad f_k(t) = \frac{1}{1 - \lambda_{k+1}\, t\, f_{k+1}(t)}.$$

Successively iterating the former we derive the continued-fraction representation

$$f_k(t) = \frac{1}{\mathop{\mathrm{K}}_{j=1}^{\infty}\left(\frac{-\lambda_{k+j} t}{1}\right)} = \cfrac{1}{1 - \cfrac{\lambda_{k+1} t}{1 - \cfrac{\lambda_{k+2} t}{1 - \cdots}}}.$$

In particular,

$$f_0(t) = \frac{1}{\mathop{\mathrm{K}}_{j=1}^{\infty}\left(\frac{-\lambda_j t}{1}\right)} = \cfrac{1}{1 - \cfrac{\lambda_1 t}{1 - \cfrac{\lambda_2 t}{1 - \cdots}}}.$$

For any positive integer m, a *m-Dyck path* is a path in the upper half-plane $\mathbb{Z} \times \mathbb{N}$, starting and ending on the horizontal axis, using steps $(1,1)$ (called "rise") and $(1,-m)$ (called "m-fall"). More generally, a *m-Dyck path at level k* is a path in $\mathbb{Z} \times \mathbb{N}_{\geq k}$, starting and ending at height k, using steps $(1,1)$ and $(1,-m)$. Clearly the length of a m-Dyck path must be a multiple of $m+1$ and, when $m = 1$, the 1-Dyck paths are standard Dyck paths.

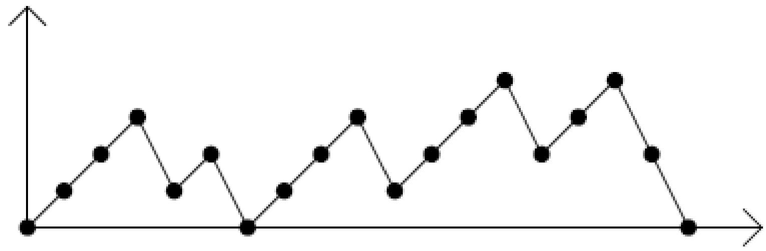

FIGURE 1.2: [61, Fig. 1] A 2-Dyck path of length 18.

For any positive integer m and any infinite set of indeterminates $\boldsymbol{\lambda} = (\lambda_i)_{i \geq m}$, we define the m-*Stieltjes-Rogers polynomial* of order n, which we denote by $S_n^{(m)}(\boldsymbol{\lambda})$, as the generating polynomial for m-Dyck paths of length $(m+1)n$, with each rise having weight 1 and each m-fall from height i having weight λ_i Clearly $S_n^{(m)}(\boldsymbol{\lambda})$ is a homogeneous polynomial of degree n with nonnegative integer coefficients.

Similarly to the generating functions defined earlier, let $f_0(t) = \sum_{n=0}^{\infty} S_n^{(m)}(\boldsymbol{\lambda})\, t^n$ be the ordinary generating function for m-Dyck paths with the weights specified above, considered as a formal power series in t and, more generally, let $f_k(t)$ be the ordinary generating function for m-Dyck paths at level k with the same weights. Then, f_k is again f_0 with each λ_i replaced by λ_{i+k}. Observe that, analogously to how we split a Dyck path at level k at its last visit to level k, we can split a m-Dyck path at level k of nonzero length at its last visit to level k and then we further split the remaining part of the path at its last return to height $k+1$, then at its last return to height $k+2$ and so on and rewrite the path in the form $\mathcal{P}_0 U \mathcal{P}_1 U \cdots \mathcal{P}_m D$ where each $\mathcal{P}_j$, $j \in \{0, \cdots, m\}$, is an arbitrary Dyck path at level $k+j$, U is a rise and D is a m-fall. Therefore, we derive the equivalent functional equations

$$f_k(t) = 1 + \lambda_{k+m}\, t \prod_{j=0}^{m} f_{k+j}(t) \quad \text{and} \quad f_k(t) = \frac{1}{1 - \lambda_{k+m}\, t \prod_{j=1}^{m} f_{k+j}(t)}.$$

Successively iterating the former we derive that

$$f_k(t) = \cfrac{1}{1 - \alpha_{k+m} t \displaystyle\prod_{i_1=1}^{m} \cfrac{1}{1 - \alpha_{k+m+i_1} t \displaystyle\prod_{i_2=1}^{m} \cfrac{1}{1 - \alpha_{k+m+i_1+i_2} t \displaystyle\prod_{i_3=1}^{m} \cfrac{1}{1 - \cdots}}}}. \tag{1.52}$$

In particular, a representation for $f_0(t)$ is obtained by taking $k = 0$ in (1.52). We call the right-hand side of (1.52) a *m-branched continued fraction*. From now on, we do not deal directly with the m-branched continued fractions but with the associated m-Stieltjes-Rogers polynomials.

The m-Stieltjes-Rogers polynomials can be generalised to form a unit lower triangular matrix whose first column entries are the m-Stieltjes-Rogers polynomials. The construction of this generalisation is connected with *partial m-Dyck paths*, which are paths in the upper half-plane $\mathbb{Z} \times \mathbb{N}$, only using steps $(1, 1)$ and $(1, -m)$, starting on the horizontal axis and allowed to end anywhere in the upper half-plane. Note that every point (x, y) of a partial m-Dyck path, and in particular the final point, satisfy $x \equiv y \mod (m + 1)$. For an infinite set of indeterminates $\boldsymbol{\lambda} = (\lambda_i)_{i \geq m}$, we define the *generalised m-Stieltjes-Rogers polynomials*, denoted by $S^{(m)}_{n,k}(\boldsymbol{\lambda})$, with $n, k \in \mathbb{N}$, as the generating polynomial for partial m-Dyck paths from $(0, 0)$ to $((m+1)n, (m+1)k)$ in which each rise gets weight 1 and each m-fall from height i gets weight λ_i. It is clear that $S^{(m)}_{n,0}(\boldsymbol{\lambda}) = S^{(m)}_{n}(\boldsymbol{\lambda})$, $S^{(m)}_{n,n}(\boldsymbol{\lambda}) = 1$ and $S^{(m)}_{n,k}(\boldsymbol{\lambda}) = 0$, if $k > n$. Therefore, the matrix $S = \left(S^{(m)}_{n,k}(\boldsymbol{\lambda})\right)_{n,k\in\mathbb{N}}$ is a unit-lower-triangular matrix, whose first column displays the ordinary m-Stieltjes-Rogers polynomials.

1.7 Total positivity and production matrices

In this section we give a brief introduction to the theory of totally positive matrices, the method of production matrices and the link between these two topics. We highlight a special class of totally positive matrices, called oscillation matrices, which has applications to the study of zeros of multiple orthogonal polynomials, as explained in Section 3.4. Our main references for totally positive matrices and oscillation matrices are [28, 32, 63], while for production matrices, we refer to [21], where they were introduced.

We call a matrix *totally positive* if all its minors are nonnegative and *strictly totally positive* if all its minors are positive. This terminology is the same used in [61] and [63]. However, in [32] and [28], the authors use the terms totally nonnegative and totally positive matrices for what we define here as totally positive and strictly totally positive matrices, respectively.

Oscillation matrices are a class of matrices intermediary between totally positive and strictly totally positive matrices. A $(n \times n)$-matrix A is an *oscillation matrix* if A is a totally positive matrix and some power of A is a strictly totally positive matrix. We are interested in oscillation matrices because they share the nice spectral properties of strictly totally positive matrices. In fact, if A is a $(n \times n)$-oscillation matrix (and, in particular, if A is strictly positive) then A has n real, positive and simple eigenvalues (see [32, Th. II-6], this result is known as the Gantmacher-Krein theorem), which interlace with the eigenvalues of the matrix obtained from A by removing either its first or last row and column (see [32, Th. II-14] and [28, Th. 5.5.2]).

Now we introduce production matrices. Let $P = (p_{i,j})_{i,j\in\mathbb{N}}$ be an infinite matrix with entries in a commutative ring. We assume that P is either row-finite or column-finite, that is, P has only finitely many nonzero entries in each row or in each column, respectively. Then, all the powers of P are well defined and we can define an infinite matrix $Q = (q_{n,k})_{n,k\in\mathbb{N}}$ by $q_{n,k} = (P^n)_{0,k}$. In particular, $q_{0,0} = 1$ and $q_{0,k} = 0$, if $k \geq 1$. We call P the *production matrix* and Q the *output matrix*.

Total positivity and production matrices are strongly connected topics because production matrices can be used to solve total-positivity problems. That is a consequence of the following result (see [61, Th. 9.4]): if P is a totally positive matrix with entries in a commutative ring and P is either row-finite or column-finite, then its output matrix is also a totally positive matrix.

For instance, the matrix $S = \left(S_{n,k}^{(m)}(\boldsymbol{\lambda})\right)_{n,k\in\mathbb{N}}$ of generalised m-Stieltjes-Rogers polynomials, for the set of indeterminates $\boldsymbol{\lambda} = (\lambda_{k+m})_{k\in\mathbb{N}}$, introduced in Section 1.6, is proved to be totally positive in the polynomial ring $\mathbb{Z}[\boldsymbol{\lambda}]$ equipped with the coefficientwise partial order (that is, a polynomial in $\mathbb{Z}[\boldsymbol{\lambda}]$ is nonnegative if all its coefficients are nonnegative), due to the total positivity of its production matrix (see [61, §9.5]). Based on [61, Prop. 8.2], this production matrix P is a $(m+1)$-banded lower-Hessenberg matrix admitting the decomposition

$$P = \prod_{i=0}^{m-1} \left(L\left(\left(\lambda_{k(m+1)+i}\right)_{k\geq 1}\right)\right) U\left(\left(\lambda_{k(m+1)-1}\right)_{k\geq 1}\right), \tag{1.53}$$

where $L\left((s_k)_{k\geq 1}\right)$ is the lower-bidiagonal infinite matrix with entries $L_{k,k} = 1$ and $L_{k+1,k} = s_{k+1}$ for all $k \in \mathbb{N}$, and $U\left((t_k)_{k\geq 1}\right)$ is the upper-bidiagonal infinite matrix with entries $U_{k,k+1} = 1$ and $U_{k,k} = t_{k+1}$, for all $k \in \mathbb{N}$. We often refer to P in (1.53) as the production matrix of the m-Stieltjes-Rogers polynomials $\left(S_n^{(m)}(\boldsymbol{\lambda})\right)_{n\in\mathbb{N}}$ or the production matrix of the corresponding branched continued fraction.

To prove that the production matrix P is totally positive in $\mathbb{Z}[\boldsymbol{\lambda}]$ equipped with the coefficientwise partial order, we combine two known results about totally positive matrices. Firstly, the product of totally positive matrices (when well-defined) is also totally positive. This is a corollary of the Cauchy-Binet formula, which gives a formula for the minors of the product of two matrices as a sum of products of minors of each matrix (see, for instance, [63, §4.6]). Secondly, a bidiagonal matrix is totally positive if and only if all its entries are nonnegative. The nonnegativity of all the entries of a matrix is naturally a necessary condition for any matrix to be totally positive. For bidiagonal matrices, it is also a sufficient condition because all their nonzero minors are simply a product of some entries. Combining these two results, we derive the total positivity of P.

1.8 Outline

The overarching theme of this thesis lies on the characterisation of multiple orthogonal polynomials with respect to absolutely continuous measures. The main object of interest is on the type II multiple orthogonal polynomials on the step-line, or r-orthogonal polynomials, which admit representations as terminating hypergeometric series. The r-orthogonal polynomials analysed here, as well as the type I functions on the step-line, satisfy the Hahn-classical property, as a consequence of the differential properties of their orthogonality weights. The main focus of the research presented here lies on multiple orthogonal polynomials with respect to 2 measures, but the techniques obtained can be extended to cases involving more orthogonality measures.

The object of study in Chapter 2 are Hahn-classical 3-fold symmetric 2-orthogonal polynomials. After presenting some known results on this collection of polynomials, we obtain explicit expressions for the orthogonality weights of the components of their cubic decomposition, and we prove that these cubic components are always Hahn-classical. This approach is instrumental to the extension of those cubic components obtained in Chapters 4 and 5. In the meantime, we show that the cubic components are always Hahn-classical, a result that is new to the theory.

In Chapter 3 we investigate multiple orthogonal polynomials with respect to measures supported on the positive real line (or on a subset of it), whose moments are ratios of Pochhammer symbols. In Chapters 4 and 5, we analyse in detail two families of multiple orthogonal polynomials with respect to vectors of two measures, which are instances of the polynomials studied in Chapter 3. In Chapter 4 the orthogonality measures are supported on the positive real line and admit integral representations involving the Tricomi confluent hypergeometric function defined by (1.18), while in Chapter 5 the orthogonality measures are supported on the interval $(0, 1)$ and admit integral representations involving Gauss' hypergeometric function defined by (1.5) and (1.7). Chapters 4 and 5 are based on joint publications with Ana Loureiro [47] and [48], respectively.

For both of these cases, we start by observing that the moments of the orthogonality measures are ratios of, respectively, two-by-one and two-by-two Pochhammer symbols. In fact, when the number of orthogonality measures is two, the multiple orthogonal polynomials originating from Chapter 3 are the two families studied in Chapters 4 and 5 and the multiple orthogonal polynomials with respect to Macdonald functions introduced in [75] and [11]. Moreover, the cubic components of any Hahn-classical 3-fold-symmetric 2-orthogonal polynomial sequence is a particular realisation of one of these three families of multiple orthogonal polynomials.

We prove that the orthogonality measures of the multiple orthogonal polynomials in Chapters 4 and 5 form Nikishin systems, which guarantees the existence and uniqueness of the entire systems of multiple orthogonal polynomials. We start our analysis of these polynomial systems by studying the differential properties of the orthogonality weights, which we use to prove that both type I functions and type II polynomials on the step-line satisfy the Hahn-classical property, and to obtain Rodrigues-type formulas generating the type I functions and polynomials on the step-line. Then, we focus on the characterisation of the type II polynomials on the step-line, via their explicit representations as terminating hypergeometric series and as solutions of third-order differential equations and recurrence relations as well as by investigating their asymptotic behaviour and the location of their zeros. Furthermore, we link these polynomials with branched-continued-fraction representations of generalised hypergeometric series, equal to the generating functions of the moment sequences of their first orthogonality measures. We also show that the polynomials studied in [75] and [11] are a limiting case of the polynomials investigated in Chapter 4, which in turn are a limiting case of the polynomials analysed in Chapter 5. At the end of Chapter 4, we present some known properties of the polynomials in [75] and [11] and we link these polynomials with branched continued fractions and production matrices.

Finally, in Chapter 6, we give a brief summary of the results in this thesis and the connection between them, leading to a discussion on possible future directions of investigation related to the research presented here.

Chapter 2

Hahn-classical 3-fold-symmetric 2-orthogonal polynomials

As explained in the introduction, a 2-orthogonal polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is Hahn-classical if its sequence of derivatives is also 2-orthogonal and is 3-*fold symmetric* if

$$P_n\left(\mathrm{e}^{\frac{2\pi i}{3}}x\right) = \mathrm{e}^{\frac{2n\pi i}{3}}P_n(x), \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

This definition is equivalent to the existence of three polynomial sequences $\left(P_n^{[k]}(x)\right)_{n\in\mathbb{N}}$ supra indexed with $k \in \{0, 1, 2\}$, the *cubic components* of $(P_n(x))_{n\in\mathbb{N}}$, such that

$$P_{3n+k}(x) = x^k P_n^{[k]}(x^3), \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

The definition (2.1) and the characterisation (2.2) of 3-fold symmetry can be obtained by taking $m = 3$ in (1.42) and (1.43), respectively.

There are four distinct families of Hahn-classical 3-fold-symmetric 2-orthogonal polynomials, up to a linear transformation of the variable, as shown in [23]. All these families were studied in detail in [50] and the four arising cases were therein denominated as cases A, B1, B2, and C. We use the same terminology.

The outline of this chapter is as follows. In Section 2.1, we introduce several known results about 3-fold-symmetric 2-orthogonal polynomials, some of them also involving the Hahn-classical property. In Section 2.2, we obtain original formulas for the orthogonality weights of the cubic components of a generic 3-fold-symmetric Hahn-classical 2-OPS (Proposition 2.8). We use these formulas in §2.2.1-2.2.4 to obtain new explicit expressions for the cubic components of each of the four aforementioned cases of Hahn-classical 3-fold-symmetric 2-orthogonal polynomials. In Section 2.3, we prove the main original result of this chapter: the cubic decomposition preserves the Hahn-classical property, that is, the cubic components of Hahn-classical 3-fold-symmetric 2-orthogonal polynomials are always Hahn-classical 2-orthogonal polynomials (Theorem 2.9). Both Proposition 2.8 and Theorem 2.9 were originally published in [47, §4].

2.1 Known results

Recalling (1.44a)-(1.44b), a polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is both 3-fold symmetric and 2-orthogonal if and only if

$$P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x), \tag{2.3}$$

with $\gamma_n \neq 0$, for all $n \geq 1$, and initial conditions $P_0 \equiv 1$, $P_1(x) = x$ and $P_2(x) = x^2$.

Note that a 3-fold-symmetric polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is never orthogonal with respect to any linear functional u. Otherwise we would have

$$0 = \langle u, P_0\, P_2\rangle = \left\langle u, x^2\right\rangle = \left\langle u, P_1^2\right\rangle \neq 0.$$

There is also a notion of 3-fold symmetry for linear functionals and for measures. A pair of measures (μ_0, μ_1), or alternatively a pair of linear functionals (u_0, u_1), is 3-fold symmetric if, for $j \in \{0, 1\}$ and $k \in \{0, 1, 2\}$ with $k \neq j$,

$$\int x^{3n+k}\mathrm{d}\mu_j(x) = 0, \quad \text{or alternatively} \quad \left\langle u_j, x^{3n+k}\right\rangle = 0.$$

Note that, if (μ_0, μ_1) is a 3-fold-symmetric pair of measures (or linear functionals), then an orthogonal polynomial sequence with respect to either μ_0 or μ_1 does not exist. The latter is a direct consequence of $\int \mathrm{d}\mu_1(x) = 0$, while the former is a result of the moments or order 1 and 2 being both equal to 0.

The notions of 3-fold symmetry for polynomials and for linear functionals are naturally connected. In fact, if $(P_n(x))_{n\in\mathbb{N}}$ is a 2-orthogonal polynomial sequence with respect to a pair of measures (μ_0, μ_1), or to a pair of linear functionals (u_0, u_1), then $(P_n(x))_{n\in\mathbb{N}}$ is 3-fold symmetric if and only if (μ_0, μ_1), or (u_0, u_1), is 3-fold symmetric as shown in [23, Th. 5.1].

Furthermore, the cubic components of a 3-fold-symmetric 2-orthogonal sequence are also 2-orthogonal. The structure of the orthogonality linear functionals of the cubic components and the expressions for their recurrence relation coefficients are obtained in [23], and they are as follows.

Lemma 2.1. *[23, §5.1 & §6.1] (cf. [50, Lemma 2.1]) Let $(P_n(x))_{n\in\mathbb{N}}$ be a 3-fold-symmetric 2-orthogonal polynomial sequence with respect to a pair of linear functionals (u_0, u_1) satisfying (2.3) and let $\{u_n\}_{n\in\mathbb{N}}$ be the corresponding dual sequence. Then, for each $k \in \{0, 1, 2\}$:*

(a) The cubic component $\left(P_n^{[k]}(x)\right)_{n\in\mathbb{N}}$ is 2-orthogonal with respect to the vector of linear functionals $\left(u_0^{[k]}, u_1^{[k]}\right)$ such that

$$u_0^{[k]} = \sigma_3(x^k u_k) \quad \text{and} \quad u_1^{[k]} = \sigma_3(x^k u_{k+3}),$$

where $\sigma_3 : \mathcal{P}' \to \mathcal{P}'$ represents the linear operator defined in $\mathcal{P}'$ by

$$\langle \sigma_3(v), f \rangle := \left\langle v, f(x^3) \right\rangle, \quad \text{for any } v \in \mathcal{P}' \text{ and } f \in \mathcal{P}.$$

(b) $\left(P_n^{[k]}(x)\right)_{n\in\mathbb{N}}$ satisfies the recurrence relation

$$P_{n+1}^{[k]}(x) = \left(x - \beta_n^{[k]}\right) P_n(x) - \alpha_n^{[k]} P_{n-1}(x) - \gamma_{n-1}^{[k]} P_{n-2}(x),$$

where, for each $n \in \mathbb{N}$,

- $\beta_n^{[k]} = \gamma_{3n-1+k} + \gamma_{3n+k} + \gamma_{3n+1+k}$,
- $\alpha_{n+1}^{[k]} = \gamma_{3n+k}\gamma_{3n+2+k} + \gamma_{3n+1+k}\gamma_{3n+2+k} + \gamma_{3n+1+k}\gamma_{3n+3+k}$,
- $\gamma_{n+1}^{[k]} = \gamma_{3n+1+k}\gamma_{3n+3+k}\gamma_{3n+5+k}$.

It was proved in [8] that the orthogonality measures of a 3-fold-symmetric 2-orthogonal polynomial sequence (with positive recurrence coefficients γ_n) are supported on subsets of the starlike set with three rays, referred to here as the 3-star and illustrated in Figure 2.1,

$$S = \bigcup_{k=0}^{2} \Gamma_k, \quad \text{with} \quad \Gamma_k = \left(0, \mathrm{e}^{\frac{2k\pi i}{3}} \infty\right) \quad \text{for each} \quad k \in \{0, 1, 2\}. \tag{2.4}$$

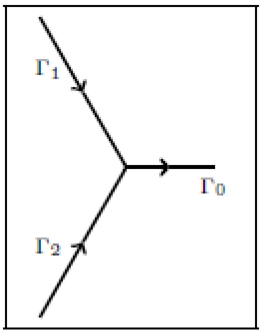

FIGURE 2.1: [50, Fig. 2] The 3-star S with rays Γ_0, Γ_1 and Γ_2.

Theorem 2.2. *[8] (cf. [50, Th. 2.1]) Let $(P_n(x))_{n\in\mathbb{N}}$ be a 2-orthogonal polynomial sequence with respect to a pair of linear functionals (u_0, u_1) satisfying (2.3) with $\gamma_n > 0$, for all $n \in \mathbb{Z}^+$. Then there exists a pair of measures (μ_0, μ_1) such that*

$$\langle u_0, f\rangle = \int_{S_\gamma} f(x)\mathrm{d}\mu_0(x) \quad \textit{and} \quad \langle u_1, f\rangle = \int_{S_\gamma} f(x)\mathrm{d}\mu_1(x),$$

where $\operatorname{supp}(\mu_0) = \operatorname{supp}(\mu_1) = S_\gamma = \bigcup_{k=0}^{2} \left(0, \gamma\, \mathrm{e}^{\frac{2k\pi i}{3}}\right)$, *for some $\gamma \in \mathbb{R}^+ \cup \{\infty\}$, and the measures μ_0 and μ_1 are invariant under rotations of angle $\dfrac{2\pi i}{3}$.*

Moreover, all the zeros of the 3-fold-symmetric 2-orthogonal polynomials (with positive recurrence coefficients γ_n) lie on the 3-star (2.4), due to the following result obtained in [12].

Theorem 2.3. *[12, Th. 2.2, d = 2] Let $(P_n(x))_{n\in\mathbb{N}}$ be a 2-orthogonal polynomial sequence with respect to a pair of linear functionals (u_0, u_1) satisfying (2.3) with $\gamma_n > 0$, for all $n \in \mathbb{Z}^+$. Then, for each $n \in \mathbb{N}$ and $j \in \{0, 1, 2\}$:*

(a) P_{3n+j} has n positive real simple zeros, which we denote by $\{x_{3n+j}^{(k)}\}_{k=1}^{n}$ with $0 < x_{3n+j}^{(1)} < \cdots < x_{3n+j}^{(n)}$, satisfying the interlacing property

$$x_{3n+3}^{(k)} < x_{3n}^{(k)} < x_{3n+1}^{(k)} < x_{3n+2}^{(k)} < x_{3n+3}^{(k+1)}.$$

for any $n \in \mathbb{Z}^+$ and $k \in \{1, \cdots, n\}$.

(b) If x is a zero of P_{3n+j}, then $e^{\frac{2\pi i}{3}}x$ and $e^{\frac{4\pi i}{3}}x$ are also zeros of P_{3n+j}.

(c) 0 is a zero of P_{3n+j} of multiplicity j when $j \in \{1, 2\}$.

Furthermore, there is a relation between the asymptotic behaviour of the zeros of 3-fold-symmetric 2-orthogonal polynomials and the corresponding recurrence coefficients γ_n, as follows.

Theorem 2.4. *[50, Th. 2.2] Let $(P_n(x))_{n\in\mathbb{N}}$ be a 2-orthogonal polynomial sequence satisfying (2.3) with $\gamma_n > 0$ and $\gamma_{2k+j} = c_j\, k^{\lambda} + o\left(k^{\lambda}\right)$, $j \in \{0, 1\}$, as $k \to \infty$, with $\lambda, c_0, c_1 \in \mathbb{R}_0^+$ and $c = \max\{c_0, c_1\} > 0$. Then, if we denote by $x_n^{(n)}$, with $n \geq 1$, the largest zero in absolute value of $P_n(x)$,*

$$\left|x_n^{(n)}\right| \leq \left(\frac{27}{4}\,c\right)^{\frac{1}{3}} n^{\frac{\lambda}{3}} + o\left(n^{\frac{\lambda}{3}}\right), \quad \text{as } n \to +\infty.$$

Now we start presenting results involving the Hahn-classical property. Firstly, we recall an alternative characterisation of the Hahn-classical property for 2-orthogonal polynomials, via a matrix Pearson-type differential equation satisfied by the orthogonality linear functionals, corresponding to [24, Th. 3.1] with $d = 2$.

Proposition 2.5. *[24, Th. 3.1, d = 2] (cf. [56, Prop. 6.2]) Let* $(P_n(x))_{n\in\mathbb{N}}$ *be a 2-orthogonal polynomial sequence with respect to a pair of linear functionals* (u_0, u_1)*, where* $(u_n)_{n\in\mathbb{N}}$ *is the dual sequence of* $(P_n(x))_{n\in\mathbb{N}}$*. Then* $(P_n(x))_{n\in\mathbb{N}}$ *is Hahn-classical if and only if* $\overline{u} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$ *satisfies a differential equation of the type*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi(x)\overline{u}\right) + \Psi(x)\overline{u} = 0, \tag{2.5}$$

with $\Phi(x) = \begin{bmatrix} \phi_{00}(x) & \phi_{01}(x) \\ \phi_{10}(x) & \phi_{11}(x) \end{bmatrix}$ *and* $\Psi(x) = \begin{bmatrix} 0 & 1 \\ \eta P_1(x) & \xi \end{bmatrix}$*, where* η *and* ξ *are nonzero constants and the* ϕ_{ij}*, with* $i, j \in \{0, 1\}$*, are polynomials such that* $\deg \phi_{ij}$ *is not greater than* 1 *when* $(i, j) \neq (1, 0)$ *and* $\deg \phi_{10}$ *is not greater than* 2*.*

The following theorem gives a necessary and sufficient condition for a 3-fold-symmetric 2-orthogonal polynomial sequence to be Hahn-classical.

Theorem 2.6. *[50, Th. 3.2] Let* $(P_n(x))_{n\in\mathbb{N}}$ *be a 2-orthogonal polynomial sequence. Then the following conditions are equivalent:*

- *The sequence* $(P_n(x))_{n\in\mathbb{N}}$ *satisfies* (2.3) *with*

$$\frac{\gamma_{n+2}}{\gamma_{n+1}} = \frac{n+3}{n+1}\,\frac{n\left(\vartheta_n - 1\right) + 1}{\left(n+4\right)\left(\vartheta_{n+1} - 1\right) + 1} \tag{2.6}$$

  where, for each $k \in \mathbb{N}$ *and* $j \in \{0, 1\}$*,*

$$\vartheta_{2k+j} = 1 + \frac{\vartheta_j - 1}{k\left(\vartheta_j - 1\right) + 1},$$

  and $\vartheta_1, \vartheta_2 \neq \dfrac{m-1}{m}$*, for any* $m \geq 1$*.*

- *The sequence* $(Q_n(x))_{n\in\mathbb{N}}$*, defined by* $Q_n(x) = \dfrac{1}{n+1}P'_{n+1}(x)$*, for all* $n \in \mathbb{N}$*, is 2-orthogonal satisfying the recurrence relation*

$$Q_{n+1}(x) = x\,Q_n(x) - \frac{n-1}{n+1}\,\vartheta_{n-1}\gamma_n Q_{n-2}(x). \tag{2.7}$$

Next we present an integral representation for the (absolutely continuous) orthogonality measures of Hahn-classical 3-fold-symmetric 2-orthogonal polynomials (with positive recurrence coefficients γ_n), via a pair of weight functions satisfying a matrix differential equation of the type in (2.5), but involving weight functions instead of linear functionals.

Theorem 2.7. *[50, Th. 3.3 & 3.1] Let $(P_n(x))_{n\in\mathbb{N}}$ be a 3-fold-symmetric Hahn-classical 2-orthogonal polynomial sequence satisfying (2.3) with $\gamma_{n+1}>0$, for all $n\in\mathbb{N}$. Then $(P_n(x))_{n\in\mathbb{N}}$ is 2-orthogonal with respect to a pair of measures (μ_0,μ_1) admitting, for both $j\in\{0,1\}$, the integral representations*

$$\int_S f(z)\mathrm{d}\mu_j(z)=\sum_{k=0}^{2}\omega^{k(j+1)}\int_0^{\gamma\omega^{3-k}} f(z)\,\mathcal{U}_j\left(\omega^k z\right)\mathrm{d}z, \tag{2.8}$$

where $\omega=\mathrm{e}^{\frac{2\pi i}{3}}$, $\gamma=\dfrac{27}{4}\lim\limits_{n\to\infty}\gamma_n$, $S=\bigcup\limits_{k=0}^{2}\left[0,\gamma\,\mathrm{e}^{\frac{2k\pi i}{3}}\right]$ and the $\mathcal{U}_j:[0,\gamma]\to\mathbb{R}$ are twice differentiable functions satisfying the matrix differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi(x)\begin{bmatrix}\mathcal{U}_0(x)\\ \mathcal{U}_1(x)\end{bmatrix}\right)+\Psi(x)\begin{bmatrix}\mathcal{U}_0(x)\\ \mathcal{U}_1(x)\end{bmatrix}=0, \tag{2.9}$$

where there exist constants $\vartheta_1,\vartheta_2\neq\dfrac{n-1}{n}$, for any $n\geq 1$, such that

$$\Phi(x)=\begin{bmatrix}\vartheta_1 & (1-\vartheta_1)\,x\\ \dfrac{2\,(1-\vartheta_2)}{\gamma_1}\,x^2 & 2\vartheta_2-1\end{bmatrix}\quad \text{and}\quad \Psi(x)=\begin{bmatrix}0 & 1\\ \dfrac{2}{\gamma_1}\,x & 0\end{bmatrix}.$$

Let $j\in\{0,1\}$ and $m\in\mathbb{N}$. Making the change of variable $x=\omega^k z$ for each $k\in\{0,1,2\}$, the integral representation in (2.8) is equivalent to

$$\int_S z^m\mathrm{d}\mu_j(z)=\sum_{k=0}^{2}\omega^{k(m-j)}\int_0^{\gamma} x^m\,\mathcal{U}_j(x)\mathrm{d}x\quad\text{for all } m\in\mathbb{N}.$$

Moreover,

$$\sum_{k=0}^{2} \omega^{k(m-j)} = \begin{cases} 3, & \text{if } m = 3n + j \\ 1 + \omega + \omega^2 = 0, & \text{if } m = 3n + k \text{ with } k \in \{0, 1, 2\} \backslash \{j\}. \end{cases}$$

Therefore, the integral representations in (2.8) imply that the pair of measures (μ_0, μ_1) is 3-fold symmetric and, for both $j \in \{0, 1\}$ and any $n \in \mathbb{N}$,

$$\int_S z^{3n+j} \mathrm{d}\mu_j(z) = 3 \int_0^{\gamma} x^{3n+j}\, \mathcal{U}_j(x) \mathrm{d}x = \int_0^{\gamma^3} t^{n+\frac{j-2}{3}}\, \mathcal{U}_j\left(t^{\frac{1}{3}}\right) \mathrm{d}t. \tag{2.10}$$

The latter integral representation is obtained via the change of variable $t = x^3$ and it will be useful in the following section.

As explained in [54], all elements of the dual sequence $(u_n)_{n\in\mathbb{N}}$ of a 2-orthogonal polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ can be written as a combination of u_0 and u_1. Namely, for each $n \in \mathbb{N}$, there exists polynomials $E_n(x)$, $a_{n-1}(x)$, $F_n(x)$ and $b_n(x)$ with $\deg E_n = \deg F_n = n$, $\deg a_{n-1} \leq n - 1$ and $\deg b_n \leq n$, such that

$$u_{2n} = E_n(x)u_0 + a_{n-1}(x)u_1 \quad \text{and} \quad u_{2n+1} = b_n(x)u_0 + F_n(x)u_1.$$

It is clear that $E_0(x) = F_0(x) = 1$ and $b_0(x) = a_{-1}(x) = 0$. Moreover, the polynomials $E_n(x)$, $a_{n-1}(x)$, $F_n(x)$ and $b_n(x)$, for $n \geq 1$, can be generated by the recursive relations given in [54, Lemma 2.2]. In particular, when $(P_n(x))_{n\in\mathbb{N}}$ is a 3-fold-symmetric 2-orthogonal polynomial sequence satisfying (2.3), the initial elements of the dual sequence (after u_0 and u_1) are

- $$u_2 = E_1 u_0 + \mathrm{a}_0 u_1 = \frac{x}{\gamma_1}\, u_0, \tag{2.11a}$$
- $$u_3 = \mathrm{b}_0 u_0 + F_1 u_1 = \frac{1}{\gamma_2}\, (x u_1 - u_0), \tag{2.11b}$$
- $$u_4 = E_2 u_0 + \mathrm{a}_1 u_1 = \frac{1}{\gamma_1 \gamma_3}\, \left(x^2 u_0 - \gamma_1 u_1\right), \tag{2.11c}$$
- $$u_5 = \mathrm{b}_2 u_0 + F_2 u_1 = \frac{1}{\gamma_2 \gamma_4} \left(x^2 u_1 - \left(1 + \frac{\gamma_2}{\gamma_1}\right) x u_0\right). \tag{2.11d}$$

2.2 Cubic components and their orthogonality weights

As explained in [23] and [50], there are four distinct families of Hahn-classical 3-fold-symmetric 2-orthogonal polynomials, up to a linear transformation of the variable. Using the terminology from [50] and the coefficients ϑ_n with the same meaning as in Theorem 2.6, the four cases to consider are

- Case A: $\vartheta_1 = \vartheta_2 = 1$. This implies that $\vartheta_n = 1$, for all $n \in \mathbb{Z}^+$.
- Case B1: $\vartheta_1 \neq 1$ and $\vartheta_2 = 1$. We set $\vartheta_1 = \dfrac{\nu+2}{\nu+1}$ and obtain

$$\vartheta_{2m+1} = \frac{m+\nu+2}{m+\nu+1} \quad \text{and} \quad \vartheta_{2m+2} = 1, \quad \text{for all } m \in \mathbb{N}. \tag{2.12a}$$

- Case B2: $\vartheta_1 = 1$ and $\vartheta_2 \neq 1$. We set $\vartheta_2 = \dfrac{\rho+2}{\rho+1}$ and obtain

$$\vartheta_{2m+1} = 1 \quad \text{and} \quad \vartheta_{2m+2} = \frac{m+\rho+2}{m+\rho+1}, \quad \text{for all } m \in \mathbb{N}. \tag{2.12b}$$

- Case C: $\vartheta_1, \vartheta_2 \neq 1$. We set $\vartheta_1 = \dfrac{\nu+2}{\nu+1}$ and $\vartheta_2 = \dfrac{\rho+2}{\rho+1}$, obtaining

$$\vartheta_{2m+1} = \frac{m+\nu+2}{m+\nu+1} \quad \text{and} \quad \vartheta_{2m+2} = \frac{m+\rho+2}{m+\rho+1}, \quad \text{for all } m \in \mathbb{N}. \tag{2.12c}$$

As observed in [23, §4], there are limiting relations connecting these 4 cases:

$$\text{case C} \xrightarrow{\rho\to\infty} \text{case B1} \xrightarrow{\nu\to\infty} \text{case A} \quad \text{and} \quad \text{case C} \xrightarrow{\nu\to\infty} \text{case B2} \xrightarrow{\rho\to\infty} \text{case A}.$$

In this section, we present explicit representations as hypergeometric polynomials for the cubic components of each of these four cases, and we obtain explicit expressions for their orthogonality weights. For the latter purpose, we firstly derive expressions for the orthogonality weights of the cubic components of a generic 3-fold-symmetric Hahn-classical 2-orthogonal polynomial sequence, as shown in the following proposition, which will be useful again in Section 2.3.

Proposition 2.8. *[47, Prop. 4.4] Suppose that $(P_n(x))_{n\in\mathbb{N}}$ is a 3-fold-symmetric 2-orthogonal polynomial sequence with respect to a pair of measures (μ_0,μ_1) admitting the integral representations given by (2.8) and that $(P_n(x))_{n\in\mathbb{N}}$ satisfies (2.3) with $\gamma_{n+1}>0$, for all $n\in\mathbb{N}$. Then the cubic components $\left(P_n^{[k]}(x)\right)_{n\in\mathbb{N}}$, $k\in\{0,1,2\}$, are 2-orthogonal with respect to the pairs of measures $\left(\mu_0^{[k]},\mu_1^{[k]}\right)$ admitting the integral representations*

$$\int f(x)\mathrm{d}\mu_j^{[k]}(x)=\int_0^{\gamma^3} f(x)\,\mathcal{U}_j^{[k]}(x)\mathrm{d}x, \tag{2.13}$$

for both $j\in\{0,1\}$, where the weight functions $\mathcal{U}_j^{[k]}(x)$ are

- $\mathcal{U}_0^{[0]}(x)=x^{-\frac{2}{3}}\mathcal{U}_0\left(x^{\frac{1}{3}}\right),$ (2.14a)
- $\mathcal{U}_0^{[1]}(x)=x^{-\frac{1}{3}}\mathcal{U}_1\left(x^{\frac{1}{3}}\right),$ (2.14b)
- $\mathcal{U}_0^{[2]}(x)=\dfrac{1}{\gamma_1}\,x^{\frac{1}{3}}\,\mathcal{U}_0\left(x^{\frac{1}{3}}\right),$ (2.14c)
- $\mathcal{U}_1^{[0]}(x)=\dfrac{1}{\gamma_2}\left(x^{-\frac{1}{3}}\mathcal{U}_1\left(x^{\frac{1}{3}}\right)-x^{-\frac{2}{3}}\mathcal{U}_0\left(x^{\frac{1}{3}}\right)\right),$ (2.14d)
- $\mathcal{U}_1^{[1]}(x)=\dfrac{1}{\gamma_1\gamma_3}\left(x^{\frac{1}{3}}\,\mathcal{U}_0\left(x^{\frac{1}{3}}\right)-\gamma_1 x^{-\frac{1}{3}}\,\mathcal{U}_1\left(x^{\frac{1}{3}}\right)\right),$ (2.14e)
- $\mathcal{U}_1^{[2]}(x)=\dfrac{1}{\gamma_2\gamma_4}\left(x^{\frac{2}{3}}\,\mathcal{U}_1\left(x^{\frac{1}{3}}\right)-\left(1+\dfrac{\gamma_2}{\gamma_1}\right)x^{\frac{1}{3}}\,\mathcal{U}_0\left(x^{\frac{1}{3}}\right)\right),$ (2.14f)

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be the dual sequence of $(P_n(x))_{n\in\mathbb{N}}$ and, for $k\in\{0,1,2\}$, let $\left(u_0^{[k]},u_1^{[k]}\right)$ be the pair of orthogonality linear functionals for the cubic component $\left(P_n^{[k]}(x)\right)_{n\in\mathbb{N}}$. Because every linear functional is uniquely determined by its moments, it suffices to prove (2.13) when $f(x)=x^n$, for any $n\in\mathbb{N}$. Recalling Lemma 2.1, the moments of $u_j^{[k]}$, with $j\in\{0,1\}$ and $k\in\{0,1,2\}$, are equal to

$$\left\langle u_j^{[k]},x^n\right\rangle=\left\langle\sigma_3\left(x^k u_{k+3j}\right),x^n\right\rangle=\left\langle x^k u_{k+3j},x^{3n}\right\rangle=\left\langle u_{k+3j},x^{3n+k}\right\rangle. \tag{2.15}$$

As a result, recalling (2.10), we deduce that

$$\left\langle u_0^{[0]},x^n\right\rangle=\left\langle u_0,x^{3n}\right\rangle=\int_0^{\gamma^3} x^{n-\frac{2}{3}}\,\mathcal{U}_0\left(x^{\frac{1}{3}}\right)\mathrm{d}x$$

and

$$\left\langle u_0^{[1]}, x^n \right\rangle = \left\langle u_1, x^{3n+1} \right\rangle = \int_0^{\gamma^3} x^{n-\frac{1}{3}} \mathcal{U}_1\left(x^{\frac{1}{3}}\right) \mathrm{d}x.$$

Therefore, (2.13) holds for $j = 0$ and $k \in \{0, 1\}$, with $\mathcal{U}_0^{[0]}(x)$ and $\mathcal{U}_0^{[1]}(x)$ given by (2.14a) and (2.14b), respectively.

To obtain the expressions for the other orthogonality weights of the cubic components from (2.15), we recall the expressions for the elements of the dual sequence u_2, u_3, u_4 and u_5 as combinations of u_0 and u_1 given by (2.11a)-(2.11d). As a result, we derive from (2.15) that

- $\left\langle u_0^{[2]}, x^n \right\rangle = \left\langle u_2, x^{3n+2} \right\rangle = \frac{1}{\gamma_1} \left\langle u_0, x^{3n+3} \right\rangle = \frac{1}{\gamma_1} \left\langle u_0^{[0]}, x^{n+1} \right\rangle = \frac{1}{\gamma_1} \left\langle x u_0^{[0]}, x^n \right\rangle,$
- $$\begin{aligned} \left\langle u_1^{[0]}, x^n \right\rangle = \left\langle u_3, x^{3n+3} \right\rangle &= \frac{1}{\gamma_2} \left(\left\langle u_1, x^{3n+1} \right\rangle - \left\langle u_0, x^{3n} \right\rangle \right) \\ &= \frac{1}{\gamma_2} \left(\left\langle u_0^{[1]}, x^n \right\rangle - \left\langle u_0^{[0]}, x^n \right\rangle \right), \end{aligned}$$
- $$\begin{aligned} \left\langle u_1^{[1]}, x^n \right\rangle = \left\langle u_4, x^{3n+1} \right\rangle &= \frac{1}{\gamma_1\gamma_3} \left(\left\langle u_0, x^{3n+3} \right\rangle - \gamma_1 \left\langle u_1, x^{3n+1} \right\rangle \right) \\ &= \frac{1}{\gamma_3} \left(\left\langle u_0^{[2]}, x^n \right\rangle - \gamma_1 \left\langle u_0^{[1]}, x^n \right\rangle \right), \end{aligned}$$
- $$\begin{aligned} \left\langle u_1^{[2]}, x^n \right\rangle = \left\langle u_5, x^{3n+2} \right\rangle &= \frac{1}{\gamma_2\gamma_4} \left(\left\langle u_1, x^{3n+4} \right\rangle - \left(1 + \frac{\gamma_2}{\gamma_1}\right) \left\langle u_0, x^{3n+3} \right\rangle \right) \\ &= \frac{1}{\gamma_2\gamma_4} \left(\left\langle x u_0^{[1]}, x^n \right\rangle - \left(1 + \frac{\gamma_2}{\gamma_1}\right) \left\langle x u_0^{[0]}, x^n \right\rangle \right). \end{aligned}$$

Therefore, (2.13) holds for all cases and the orthogonality weights satisfy

- $\mathcal{U}_0^{[2]}(x) = \frac{x}{\gamma_1} \mathcal{U}_0^{[0]}(x),$
- $\mathcal{U}_1^{[0]}(x) = \frac{1}{\gamma_2} \left(\mathcal{U}_0^{[1]}(x) - \mathcal{U}_0^{[0]}(x) \right),$
- $\mathcal{U}_1^{[1]}(x) = \frac{1}{\gamma_3} \left(\mathcal{U}_0^{[2]}(x) - \mathcal{U}_0^{[1]}(x) \right),$
- $\mathcal{U}_1^{[2]}(x) = \frac{x}{\gamma_2\gamma_4} \left(\mathcal{U}_0^{[1]}(x) - \left(1 + \frac{\gamma_2}{\gamma_1}\right) \mathcal{U}_0^{[0]}(x) \right).$

Finally, (2.14c)-(2.14f) follow directly from the latter identities combined with (2.14a) and (2.14b). □

2.2.1 Case A

Observe that multiplying the coefficients γ_n in (2.3), for all $n \in \mathbb{Z}^+$, by a common positive constant c corresponds to making a linear transformation from $P_n(x)$ to $a^{-n}P_n(ax)$, with $a = c^{\frac{1}{3}}$. Therefore, for each case (A, B1, B2, and C), we can choose any positive real value for γ_1 and keep dealing with the same 2-orthogonal polynomials, up to a linear transformation of the variable. Moreover, recalling (2.6), the values of γ_n, for $n \geq 2$, are uniquely determined by γ_1 and ϑ_n, $n \in \mathbb{Z}^+$.

Let $(P_n(x))_{n\in\mathbb{N}}$ be the Hahn-classical 3-fold-symmetric 2-orthogonal polynomial sequence corresponding to case A. Then $(P_n(x))_{n\in\mathbb{N}}$ satisfies the recurrence relation (2.3), with coefficients γ_n determined by (2.6) with $\vartheta_n = 1$, for all $n \in \mathbb{Z}^+$. Setting $\gamma_1 = 2$, we obtain $\gamma_n = n(n+1)$, for all $n \in \mathbb{Z}^+$.

Moreover, if $Q_n(x) = \dfrac{1}{n+1}P'_{n+1}(x)$, for each $n \in \mathbb{N}$, then, recalling (2.7),

$$Q_{n+1}(x) = x\,Q_n(x) - n(n-1)Q_{n-2}(x).$$

Therefore, $Q_n(x) = P_n(x)$ and $(P_n(x))_{n\in\mathbb{N}}$ is an Appell sequence.

Based on [50, Prop. 3.2] and recalling (1.16), the sequence $(P_n(x))_{n\in\mathbb{N}}$ is 2-orthogonal with respect to the pair of measures (μ_0, μ_1) with integral representations as in (2.8), involving the weight functions, defined on the positive real line,

$$\mathcal{U}_0(x) = \mathrm{Ai}(x) = \frac{2\sqrt{x}}{3\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}\,\mathrm{K}_{\pm\frac{1}{3}}\left(\frac{2}{3}\,x^{\frac{3}{2}}\right) \tag{2.17a}$$

and

$$\mathcal{U}_1(x) = -\mathrm{Ai}'(x) = \frac{2x}{3\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}\,\mathrm{K}_{\pm\frac{2}{3}}\left(\frac{2}{3}\,x^{\frac{3}{2}}\right), \tag{2.17b}$$

where $\mathrm{Ai}(x)$ is the Airy function of the first kind defined by (1.15).

Because it is a 3-fold-symmetric polynomial sequence, $(P_n(x))_{n\in\mathbb{N}}$ satisfies a cubic decomposition as in (2.2). According to [11, §5] (see also [50, §3.1.1]), the cubic components are

$$P_n^{[k]}(x) = (-9)^n \, (a_k)_n \, (b_k)_n \, {}_1F_2 \left(\begin{array}{c|c} -n & x \\ a_k, b_k & 9 \end{array} \right), \tag{2.18}$$

with $k \in \{0,1,2\}$ and

$$(a_0, b_0) = \left(\frac{1}{3}, \frac{2}{3}\right), \quad (a_1, b_1) = \left(\frac{4}{3}, \frac{2}{3}\right) \quad \text{and} \quad (a_2, b_2) = \left(\frac{4}{3}, \frac{5}{3}\right). \tag{2.19}$$

The cubic components are 2-orthogonal with respect to vectors of weight functions $\left(\mathcal{U}_0^{[k]}(x), \mathcal{U}_1^{[k]}(x)\right)$ defined on the positive real line. Inputting the weight functions (2.17a)-(2.17b) in the formulas for $\mathcal{U}_j^{[k]}(x)$, with $j \in \{0,1\}$ and $k \in \{0,1,2\}$, given in Proposition 2.8, we obtain

- $\mathcal{U}_0^{[0]}(x) = x^{-\frac{2}{3}} \mathrm{Ai}\left(x^{-\frac{1}{3}}\right) = \dfrac{2}{9\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)} \left(\dfrac{x}{9}\right)^{-\frac{1}{2}} \mathrm{K}_{\pm\frac{1}{3}}\left(\dfrac{2}{3}\sqrt{x}\right);$ (2.20a)
- $\mathcal{U}_0^{[1]}(x) = -x^{-\frac{1}{3}} \mathrm{Ai}'\left(x^{-\frac{1}{3}}\right) = \dfrac{2}{9\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)} \mathrm{K}_{\pm\frac{2}{3}}\left(\dfrac{2}{3}\sqrt{x}\right);$ (2.20b)
- $\mathcal{U}_0^{[2]}(x) = \dfrac{1}{2} x^{\frac{1}{3}} \mathrm{Ai}\left(x^{-\frac{1}{3}}\right) = \dfrac{2}{9\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)} \left(\dfrac{x}{9}\right)^{\frac{1}{2}} \mathrm{K}_{\pm\frac{1}{3}}\left(\dfrac{2}{3}\sqrt{x}\right);$ (2.20c)
- $\mathcal{U}_1^{[0]}(x) = -\dfrac{1}{6}\left(x^{-\frac{1}{3}} \mathrm{Ai}'\left(x^{\frac{1}{3}}\right) + x^{-\frac{2}{3}} \mathrm{Ai}\left(x^{\frac{1}{3}}\right)\right) = -\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{3}} \mathrm{Ai}\left(x^{\frac{1}{3}}\right)\right);$ (2.20d)
- $\mathcal{U}_1^{[1]}(x) = \dfrac{1}{24}\left(2x^{-\frac{1}{3}} \mathrm{Ai}'\left(x^{\frac{1}{3}}\right) + x^{\frac{1}{3}} \mathrm{Ai}\left(x^{\frac{1}{3}}\right)\right) = \dfrac{1}{8}\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{2}{3}} \mathrm{Ai}'\left(x^{\frac{1}{3}}\right)\right);$ (2.20e)
- $\mathcal{U}_1^{[2]}(x) = -\dfrac{1}{120}\left(x^{\frac{2}{3}} \mathrm{Ai}'\left(x^{\frac{1}{3}}\right) + 4x^{\frac{1}{3}} \mathrm{Ai}\left(x^{\frac{1}{3}}\right)\right) = -\dfrac{1}{20}\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{4}{3}} \mathrm{Ai}\left(x^{\frac{1}{3}}\right)\right).$ (2.20f)

The expressions (2.20a)-(2.20f) for the orthogonality weights can be rewritten as

$$\mathcal{U}_0^{[k]}(x) = \frac{2}{9\Gamma(a_k)\Gamma(b_k)} \left(\frac{x}{9}\right)^{\frac{a_k+b_k}{2}-1} \mathrm{K}_{a_k-b_k}\left(\frac{2}{3}\sqrt{x}\right) \tag{2.21a}$$

and

$$\mathcal{U}_1^{[k]}(x) = -\frac{1}{9a_k b_k}\frac{\mathrm{d}}{\mathrm{d}x}\left(x\,\mathcal{U}_0^{[k]}(x)\right), \tag{2.21b}$$

with (a_k, b_k) given by (2.19).

2.2.2 Case B1

Let $(P_n(x;\nu))_{n\in\mathbb{N}}$ be the Hahn-classical 3-fold-symmetric 2-orthogonal polynomial sequence, corresponding to case B1, satisfying the recurrence relation (2.3), with coefficients $\gamma_n = \gamma_n(\nu)$, $n \in \mathbb{Z}^+$, determined by (2.6), where ϑ_n is given by (2.12a). Setting $\gamma_1 = \dfrac{2}{3(\nu+2)}$, we get, for any $m \in \mathbb{N}$,

$$\gamma_{2m+1} = \frac{(2m+1)(2m+2)}{3(3m+\nu+2)} \quad \text{and} \quad \gamma_{2m+2} = \frac{(2m+2)(2m+3)(m+\nu+1)}{3(3m+\nu+2)(3m+\nu+5)}.$$

In particular,

$$\gamma_2 = \frac{2(\nu+1)}{(\nu+2)(\nu+5)}, \quad \gamma_3 = \frac{4}{\nu+5} \quad \text{and} \quad \gamma_4 = \frac{20(\nu+2)}{3(\nu+5)(\nu+8)}.$$

Based on [50, Prop. 3.3], the sequence $(P_n(x;\nu))_{n\in\mathbb{N}}$ is 2-orthogonal with respect to the pair of measures (μ_0, μ_1) with integral representations as in (2.8), involving the weight functions, defined on the positive real line,

$$\mathcal{U}_0(x;\nu) = \frac{\Gamma\left(\frac{\nu+2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)} \mathrm{e}^{-x^3}\, \mathbf{U}\left(\frac{\nu}{3}, \frac{2}{3}; x^3\right) \tag{2.22a}$$

and

$$\mathcal{U}_1(x;\nu) = \frac{\Gamma\left(\frac{\nu+5}{3}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)} x^2\mathrm{e}^{-x^3}\, \mathbf{U}\left(\frac{\nu}{3}+1, \frac{5}{3}; x^3\right). \tag{2.22b}$$

Like the polynomial sequence corresponding to case A, $(P_n(x;\nu))_{n\in\mathbb{N}}$ satisfies a cubic decomposition as in (2.2). Based on [50, §3.2.2], the cubic components are

$$P_n^{[k]}(x;\nu) = \frac{(-1)^n\,(a_k)_n\,(b_k)_n}{\left(\frac{\nu+2}{3} + \left\lfloor\frac{n+k}{2}\right\rfloor\right)_n}\, {}_2F_2\left(\left.\begin{matrix} -n,\ \frac{\nu+2}{3} + \left\lfloor\frac{n+k}{2}\right\rfloor \\ a_k, b_k \end{matrix}\ \right|\ x\right), \tag{2.23}$$

with $k \in \{0,1,2\}$ and (a_k, b_k) given by (2.19).

The cubic components are 2-orthogonal with respect to vectors of weight functions $\left(\mathcal{U}_0^{[k]}(x;\nu), \mathcal{U}_1^{[k]}(x;\nu)\right)$ defined on the positive real line. Inputting the weight functions (2.22a)-(2.22b) in the formulas for $\mathcal{U}_j^{[k]}(x)$, with $j \in \{0,1\}$ and $k \in \{0,1,2\}$, given in Proposition 2.8, we obtain

- $\mathcal{U}_0^{[0]}(x;\nu) = \dfrac{\Gamma\left(\frac{\nu+2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}\,\mathrm{e}^{-x}x^{-\frac{2}{3}}\,\mathbf{U}\left(\frac{\nu}{3},\frac{2}{3};x\right),$ (2.24a)
- $\mathcal{U}_0^{[1]}(x;\nu) = \dfrac{\Gamma\left(\frac{\nu+5}{3}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)}\,\mathrm{e}^{-x}x^{\frac{1}{3}}\,\mathbf{U}\left(\frac{\nu}{3}+1,\frac{5}{3};x\right),$ (2.24b)
- $\mathcal{U}_0^{[2]}(x;\nu) = \dfrac{\Gamma\left(\frac{\nu+5}{3}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}\,\mathrm{e}^{-x}x^{\frac{1}{3}}\,\mathbf{U}\left(\frac{\nu}{3},\frac{2}{3};x\right),$ (2.24c)

and

- $\mathcal{U}_1^{[0]}(x;\nu) = \lambda^{[0]}\,\mathrm{e}^{-x}x^{-\frac{2}{3}}\left(\dfrac{\nu+2}{3}\,x\,\mathbf{U}\left(\frac{\nu}{3}+1,\frac{5}{3};x\right)-\dfrac{1}{3}\,\mathbf{U}\left(\frac{\nu}{3},\frac{2}{3};x\right)\right),$
- $\mathcal{U}_1^{[1]}(x;\nu) = \lambda^{[1]}\,\mathrm{e}^{-x}x^{\frac{1}{3}}\left(\mathbf{U}\left(\frac{\nu}{3},\frac{2}{3};x\right)-\dfrac{2}{3}\,\mathbf{U}\left(\frac{\nu}{3}+1,\frac{5}{3};x\right)\right),$
- $\mathcal{U}_1^{[2]}(x;\nu) = \lambda^{[2]}\,\mathrm{e}^{-x}x^{\frac{1}{3}}\left(\dfrac{\nu+5}{3}\,x\,\mathbf{U}\left(\frac{\nu}{3}+1,\frac{5}{3};x\right)-\dfrac{4}{3}\,\mathbf{U}\left(\frac{\nu}{3},\frac{2}{3};x\right)\right),$

with $\lambda^{[0]} = \dfrac{\Gamma\left(\frac{\nu+8}{3}\right)}{\frac{\nu+1}{3}\,\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}$, $\lambda^{[1]} = \dfrac{\Gamma\left(\frac{\nu+8}{3}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{5}{3}\right)}$ and $\lambda^{[2]} = \dfrac{\Gamma\left(\frac{\nu+11}{3}\right)}{\frac{\nu+1}{3}\,\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{8}{3}\right)}$.

Now we rewrite the weights $\mathcal{U}_1^{[k]}(x;\nu)$ as expressions that will make the connection with the 2-orthogonal polynomials analysed in Chapter 4 more obvious. Based on [22, Eqs. 13.3.10 & 13.3.9], we have

$$x\,\mathbf{U}\left(\frac{\nu}{3}+1,\frac{5}{3};x\right) = \mathbf{U}\left(\frac{\nu}{3},\frac{2}{3};x\right)-\frac{\nu+1}{3}\,\mathbf{U}\left(\frac{\nu}{3}+1,\frac{2}{3};x\right) \tag{2.26a}$$

and

$$\mathbf{U}\left(\frac{\nu}{3},\frac{2}{3};x\right) = \mathbf{U}\left(\frac{\nu}{3},\frac{5}{3};x\right)-\frac{\nu}{3}\,\mathbf{U}\left(\frac{\nu}{3}+1,\frac{5}{3};x\right). \tag{2.26b}$$

Applying (2.26a) and (2.26b), respectively for $k\in\{0,2\}$ and $k=1$, to the expressions above for the weights $\mathcal{U}_1^{[k]}(x;\nu)$ we derive that

- $\mathcal{U}_1^{[0]}(x;\nu) = \dfrac{\Gamma\left(\frac{\nu+8}{3}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}\,\mathrm{e}^{-x}x^{-\frac{2}{3}}\left(\mathbf{U}\left(\frac{\nu}{3},\frac{2}{3};x\right)-\dfrac{\nu+2}{3}\,\mathbf{U}\left(\frac{\nu}{3}+1,\frac{2}{3};x\right)\right),$ (2.27a)
- $\mathcal{U}_1^{[1]}(x;\nu) = \dfrac{\Gamma\left(\frac{\nu+8}{3}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{5}{3}\right)}\mathrm{e}^{-x}x^{\frac{1}{3}}\left(\mathbf{U}\left(\frac{\nu}{3},\frac{5}{3};x\right)-\dfrac{\nu+5}{3}\,\mathbf{U}\left(\frac{\nu}{3}+1,\frac{5}{3};x\right)\right),$ (2.27b)
- $\mathcal{U}_1^{[2]}(x;\nu) = \dfrac{\Gamma\left(\frac{\nu+11}{3}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{8}{3}\right)}\,\mathrm{e}^{-x}x^{\frac{1}{3}}\left(\mathbf{U}\left(\frac{\nu}{3},\frac{2}{3};x\right)-\dfrac{\nu+5}{3}\,\mathbf{U}\left(\frac{\nu}{3}+1,\frac{2}{3};x\right)\right).$ (2.27c)

For each $k \in \{0,1,2\}$, we set (a_k, b_k) as in (2.19) and

$$\epsilon_k = \begin{cases} 0, & \text{if } k \in \{0,2\}, \\ 1, & \text{if } k = 1. \end{cases} \tag{2.28}$$

Then formulas (2.24a)-(2.24c) and (2.27a)-(2.27c) can be rewritten as

$$\mathcal{U}_0^{[k]}(x;\nu) = \frac{\Gamma\left(\frac{\nu}{3} + b_k + \epsilon_k\right)}{\Gamma(a_k)\Gamma(b_k)} \mathrm{e}^{-x} x^{a_k - 1} \mathbf{U}\left(\frac{\nu}{3} + \epsilon_k, \frac{2}{3} + \epsilon_k; x\right) \tag{2.29a}$$

and

$$\mathcal{U}_1^{[k]}(x;\nu) = \frac{\Gamma\left(\frac{\nu}{3} + b_k + 2\right)}{\Gamma(a_k+1)\Gamma(b_k+1)} \mathrm{e}^{-x} x^{a_k - 1} \left(\mathbf{U}\left(\frac{\nu}{3}, \frac{2}{3} + \epsilon_k; x\right) - \left(\frac{\nu}{3} + b_k + \epsilon_k\right) \mathbf{U}\left(\frac{\nu}{3} + 1, \frac{2}{3} + \epsilon_k; x\right)\right). \tag{2.29b}$$

2.2.3 Case B2

Let $(P_n(x;\rho))_{n\in\mathbb{N}}$ be the Hahn-classical 3-fold-symmetric 2-orthogonal polynomial sequence, corresponding to case B2, satisfying the recurrence relation (2.3), with coefficients $\gamma_n = \gamma_n(\rho)$, $n \in \mathbb{Z}^+$, determined by (2.6), where ϑ_n is given by (2.12b). Setting $\gamma_1 = \dfrac{2}{3(\rho+3)}$, we get, for any $m \in \mathbb{N}$,

$$\gamma_{2m+1} = \frac{(2m+1)(2m+2)(m+\rho)}{3(3m+\rho)(3m+\rho+3)} \text{ and } \gamma_{2m+2} = \frac{(2m+2)(2m+3)}{3(3m+\rho+3)}.$$

In particular,

$$\gamma_2 = \frac{2}{\rho+3}, \quad \gamma_3 = \frac{4(\rho+1)}{(\rho+3)(\rho+6)} \quad \text{and} \quad \gamma_4 = \frac{20}{3(\rho+6)}.$$

Based on [50, Prop. 3.4], the sequence $(P_n(x;\rho))_{n\in\mathbb{N}}$ is 2-orthogonal with respect to the pair of measures (μ_0, μ_1) with integral representations as in (2.8), involving the weight functions, defined on the positive real line,

$$\mathcal{U}_0(x;\rho) = \frac{\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}\,\mathrm{e}^{-x^3}\,\mathbf{U}\left(\frac{\rho+1}{3},\frac{2}{3};x^3\right) \tag{2.30a}$$

and

$$\mathcal{U}_1(x;\rho) = \frac{\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)}\,\mathrm{e}^{-x^3}x^2\,\mathbf{U}\left(\frac{\rho+1}{3},\frac{5}{3};x^3\right). \tag{2.30b}$$

Like the polynomial sequences corresponding to cases A and B1, $(P_n(x;\rho))_{n\in\mathbb{N}}$ satisfies a cubic decomposition as in (2.2). Based on [50, §3.3], the cubic components are

$$P_n^{[k]}(x;\rho) = \frac{(-1)^n\,(a_k)_n\,(b_k)_n}{\left(\frac{\rho}{3}+\left\lfloor\frac{n+k+1}{2}\right\rfloor\right)_n}\,{}_2F_2\left(\left.\begin{matrix}-n,\,\frac{\rho}{3}+\left\lfloor\frac{n+k+1}{2}\right\rfloor\\ a_k,b_k\end{matrix}\;\right|\;x\right), \tag{2.31}$$

with $k\in\{0,1,2\}$ and (a_k,b_k) given by (2.19).

The cubic components are 2-orthogonal with respect to vectors of weight functions $\left(\mathcal{U}_0^{[k]}(x;\rho),\mathcal{U}_1^{[k]}(x;\rho)\right)$ defined on the positive real line. Inputting the weight functions (2.30a)-(2.30b) in the formulas for $\mathcal{U}_j^{[k]}(x)$, with $j\in\{0,1\}$ and $k\in\{0,1,2\}$, given in Proposition 2.8, we obtain

- $$\mathcal{U}_0^{[0]}(x;\rho) = \frac{\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}\,\mathrm{e}^{-x}x^{-\frac{2}{3}}\,\mathbf{U}\left(\frac{\rho+1}{3},\frac{2}{3};x\right), \tag{2.32a}$$
- $$\mathcal{U}_0^{[1]}(x;\rho) = \frac{\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)}\,\mathrm{e}^{-x}x^{\frac{1}{3}}\,\mathbf{U}\left(\frac{\rho+1}{3},\frac{5}{3};x\right), \tag{2.32b}$$
- $$\mathcal{U}_0^{[2]}(x;\rho) = \frac{\Gamma\left(\frac{\rho}{3}+2\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}\,\mathrm{e}^{-x}x^{\frac{1}{3}}\,\mathbf{U}\left(\frac{\rho+1}{3},\frac{2}{3};x\right) \tag{2.32c}$$

as well as

- $$\mathcal{U}_1^{[0]}(x;\rho) = \lambda^{[0]}(\rho)\,\mathrm{e}^{-x}x^{-\frac{2}{3}}\left(x\,\mathbf{U}\left(\frac{\rho+1}{3},\frac{5}{3};x\right)-\frac{1}{3}\,\mathbf{U}\left(\frac{\rho+1}{3},\frac{2}{3};x\right)\right),$$
- $$\mathcal{U}_1^{[1]}(x;\rho) = \lambda^{[1]}(\rho)\mathrm{e}^{-x}x^{\frac{1}{3}}\left(\left(\frac{\rho}{3}+1\right)\mathbf{U}\left(\frac{\rho+1}{3},\frac{2}{3};x\right)-\frac{2}{3}\,\mathbf{U}\left(\frac{\rho+1}{3},\frac{5}{3};x\right)\right),$$
- $$\mathcal{U}_1^{[2]}(x;\rho) = \lambda^{[2]}(\rho)\mathrm{e}^{-x}x^{\frac{1}{3}}\left(x\,\mathbf{U}\left(\frac{\rho+1}{3},\frac{5}{3};x\right)-\frac{4}{3}\,\mathbf{U}\left(\frac{\rho+1}{3},\frac{2}{3};x\right)\right),$$

with $\lambda^{[0]}(\rho)=\dfrac{\Gamma\left(\frac{\rho}{3}+2\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}$, $\lambda^{[1]}(\rho)=\dfrac{\Gamma\left(\frac{\rho}{3}+3\right)}{\frac{\rho+1}{3}\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{5}{3}\right)}$ and $\lambda^{[2]}(\rho)=\dfrac{\Gamma\left(\frac{\rho}{3}+3\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{8}{3}\right)}$.

Now, like we did for case B1, we rewrite the weights $\mathcal{U}_1^{[k]}(x;\rho)$ as expressions that make clearer the connection with the 2-orthogonal polynomials analysed in Chapter 4. Considering (2.26a) with $\nu=\rho-2$ and (2.26b) with $\nu=\rho+1$, and applying these two formulas to the expressions above for the weights $\mathcal{U}_1^{[k]}(x;\rho)$, respectively for $k\in\{0,2\}$ and $k=1$, we derive that

- $$\mathcal{U}_1^{[0]}(x;\rho)=\frac{\Gamma\left(\frac{\rho}{3}+2\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}\,\mathrm{e}^{-x}x^{-\frac{2}{3}}\left(\mathbf{U}\left(\frac{\rho-2}{3},\frac{2}{3};x\right)-\frac{\rho}{3}\,\mathbf{U}\left(\frac{\rho+1}{3},\frac{2}{3};x\right)\right), \tag{2.33a}$$
- $$\mathcal{U}_1^{[1]}(x;\rho)=\frac{\Gamma\left(\frac{\rho}{3}+3\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{5}{3}\right)}\mathrm{e}^{-x}x^{\frac{1}{3}}\left(\mathbf{U}\left(\frac{\rho+1}{3},\frac{5}{3};x\right)-\left(\frac{\rho}{3}+1\right)\mathbf{U}\left(\frac{\rho+4}{3},\frac{5}{3};x\right)\right), \tag{2.33b}$$
- $$\mathcal{U}_1^{[2]}(x;\rho)=\frac{\Gamma\left(\frac{\rho}{3}+3\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{8}{3}\right)}\mathrm{e}^{-x}x^{\frac{1}{3}}\left(\mathbf{U}\left(\frac{\rho-2}{3},\frac{2}{3};x\right)-\left(\frac{\rho}{3}+1\right)\mathbf{U}\left(\frac{\rho+1}{3},\frac{2}{3};x\right)\right). \tag{2.33c}$$

As we did for case B1, we set, for each $k\in\{0,1,2\}$, (a_k,b_k) and ϵ_k as in (2.19) and (2.28), respectively. Then formulas (2.32a)-(2.32c) and (2.33a)-(2.33c) can be rewritten as

$$\mathcal{U}_0^{[k]}(x;\nu)=\frac{\Gamma\left(\frac{\rho+1}{3}+b_k\right)}{\Gamma(a_k)\Gamma(b_k)}\mathrm{e}^{-x}x^{a_k-1}\mathbf{U}\left(\frac{\rho+1}{3},\frac{2}{3}+\epsilon_k;x\right) \tag{2.34a}$$

and

$$\mathcal{U}_1^{[k]}(x;\nu)=\frac{\Gamma\left(\frac{\rho+4}{3}+b_k+\epsilon_k\right)}{\Gamma(a_k+1)\Gamma(b_k+1)}\mathrm{e}^{-x}x^{a_k-1} \tag{2.34b}$$
$$\left(\mathbf{U}\left(\frac{\rho-2}{3}+\epsilon_k,\frac{2}{3}+\epsilon_k;x\right)-\left(\frac{\rho-2}{3}+b_k+\epsilon_k\right)\mathbf{U}\left(\frac{\rho+1}{3}+\epsilon_k,\frac{2}{3}+\epsilon_k;x\right)\right).$$

2.2.4 Case C

Let $(P_n(x;\nu,\rho))_{n\in\mathbb{N}}$ be the Hahn-classical 3-fold-symmetric 2-orthogonal polynomial sequence, corresponding to case B2, satisfying the recurrence relation (2.3), with coefficients $\gamma_n = \gamma_n(\nu,\rho)$, $n \in \mathbb{Z}^+$, determined by (2.6), where ϑ_n is given by (2.12c). Setting $\gamma_1 = \dfrac{2}{(\nu+2)(\rho+3)}$, we get, for any $m \in \mathbb{N}$,

$$\gamma_{2m+1} = \frac{(2m+1)(2m+2)(m+\rho)}{(3m+\nu+2)(3m+\rho)(3m+\rho+3)}$$

and

$$\gamma_{2m+2} = \frac{(2m+2)(2m+3)(m+\nu+1)}{(3m+\nu+2)(3m+\nu+5)(3m+\rho+3)}.$$

In particular,

$$\gamma_2 = \frac{6(\nu+1)}{(\nu+2)(\nu+5)(\rho+3)},\ \gamma_3 = \frac{12(\rho+1)}{(\nu+5)(\rho+3)(\rho+6)} \text{ and } \gamma_4 = \frac{20(\nu+2)}{(\nu+5)(\nu+8)(\rho+6)}.$$

Based on [50, Prop. 3.5], the sequence $(P_n(x;\rho))_{n\in\mathbb{N}}$ is 2-orthogonal with respect to the pair of measures (μ_0,μ_1) with integral representations as in (2.8), involving the weight functions, defined on the interval $(0,1)$,

$$\mathcal{U}_0(x;\nu,\rho) = \frac{\Gamma\left(\frac{\nu+2}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{\nu+\rho+2}{3}\right)} (1-x^3)^{\frac{\nu+\rho-1}{3}}\, {}_2F_1\left(\left.\begin{matrix}\frac{\nu}{3},\ \frac{\rho+1}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\right|\, 1-x^3\right) \tag{2.35a}$$

and

$$\mathcal{U}_1(x;\nu,\rho) = \frac{\Gamma\left(\frac{\nu+5}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{\nu+\rho+2}{3}\right)} x^2(1-x^3)^{\frac{\nu+\rho-1}{3}}\, {}_2F_1\left(\left.\begin{matrix}\frac{\nu}{3}+1,\ \frac{\rho+1}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\right|\, 1-x^3\right). \tag{2.35b}$$

Like the other Hahn-classical 3-fold-symmetric 2-orthogonal polynomial sequences, $(P_n(x;\nu,\rho))_{n\in\mathbb{N}}$ satisfies a cubic decomposition as in (2.2). Based on [50, §3.4.1], the cubic components are

$$P_n^{[k]}(x;\nu,\rho) = \frac{(-1)^n\,(a_k)_n\,(b_k)_n}{(c_k(n;\nu))_n\,(d_k(n;\rho))_n}\, {}_3F_2\left(\left.\begin{matrix}-n, c_k(n;\nu), d_k(n;\rho)\\ a_k, b_k\end{matrix}\right|\, x\right), \tag{2.36}$$

with $k \in \{0,1,2\}$, (a_k, b_k) given by (2.19),

$$c_k(n;\nu) = \frac{\nu+2}{3} + \left\lfloor \frac{n+k}{2} \right\rfloor \quad \text{and} \quad d_k(n;\rho) = \frac{\rho}{3} + \left\lfloor \frac{n+k+1}{2} \right\rfloor .$$

Note that $c_k(n;\nu)$ and $d_k(n;\rho)$ were parameters of the cubic components of cases B1 and B2, respectively.

The cubic components are 2-orthogonal with respect to vectors of weight functions $\left(\mathcal{U}_0^{[k]}(x;\nu,\rho), \mathcal{U}_1^{[k]}(x;\nu,\rho)\right)$ defined on the interval $(0,1)$. Inputting the weight functions (2.35a)-(2.35b) in the formulas for $\mathcal{U}_j^{[k]}(x)$, with $j \in \{0,1\}$ and $k \in \{0,1,2\}$, given in Proposition 2.8, we obtain

- $$\mathcal{U}_0^{[0]}(x;\nu,\rho) = \frac{\Gamma\left(\frac{\nu+2}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{\nu+\rho+2}{3}\right)} x^{-\frac{2}{3}}(1-x)^{\frac{\nu+\rho-1}{3}} {}_2F_1\left(\left.\begin{matrix}\frac{\nu}{3}, \frac{\rho+1}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\right| 1-x\right), \quad (2.37a)$$
- $$\mathcal{U}_0^{[1]}(x;\nu,\rho) = \frac{\Gamma\left(\frac{\nu+5}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{\nu+\rho+2}{3}\right)} x^{\frac{1}{3}}(1-x)^{\frac{\nu+\rho-1}{3}} {}_2F_1\left(\left.\begin{matrix}\frac{\nu}{3}+1, \frac{\rho+1}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\right| 1-x\right), \quad (2.37b)$$
- $$\mathcal{U}_0^{[2]}(x;\nu,\rho) = \frac{\Gamma\left(\frac{\nu+5}{3}\right)\Gamma\left(\frac{\rho}{3}+2\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{\nu+\rho+2}{3}\right)} x^{\frac{1}{3}}(1-x)^{\frac{\nu+\rho-1}{3}} {}_2F_1\left(\left.\begin{matrix}\frac{\nu}{3}, \frac{\rho+1}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\right| 1-x\right) \quad (2.37c)$$

as well as

- $$\mathcal{U}_1^{[0]}(x;\nu,\rho) = \frac{\Gamma\left(\frac{\nu+8}{3}\right)\Gamma\left(\frac{\rho}{3}+2\right)}{\nu\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{\nu+\rho+2}{3}\right)} x^{-\frac{2}{3}}(1-x)^{\frac{\nu+\rho-1}{3}} \quad (2.38a)$$
$$\left((\nu+2)\,{}_2F_1\left(\left.\begin{matrix}\frac{\nu}{3}, \frac{\rho-2}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\right| 1-x\right) - 2\,{}_2F_1\left(\left.\begin{matrix}\frac{\nu}{3}, \frac{\rho+1}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\right| 1-x\right)\right),$$
- $$\mathcal{U}_1^{[1]}(x;\nu,\rho) = \frac{\Gamma\left(\frac{\nu+8}{3}\right)\Gamma\left(\frac{\rho}{3}+3\right)}{(\rho+1)\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{\nu+\rho+2}{3}\right)} x^{\frac{1}{3}}(1-x)^{\frac{\nu+\rho-1}{3}} \quad (2.38b)$$
$$\left((\rho+3)\,{}_2F_1\left(\left.\begin{matrix}\frac{\nu}{3}, \frac{\rho+1}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\right| 1-x\right) - 2\,{}_2F_1\left(\left.\begin{matrix}\frac{\nu}{3}+1, \frac{\rho+1}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\right| 1-x\right)\right),$$
- $$\mathcal{U}_1^{[2]}(x;\nu,\rho) = \frac{\Gamma\left(\frac{\nu+11}{3}\right)\Gamma\left(\frac{\rho}{3}+3\right)}{\nu\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{8}{3}\right)\Gamma\left(\frac{\nu+\rho+2}{3}\right)} x^{\frac{1}{3}}(1-x)^{\frac{\nu+\rho-1}{3}} \quad (2.38c)$$
$$\left((\nu+5)\,{}_2F_1\left(\left.\begin{matrix}\frac{\nu}{3}, \frac{\rho-2}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\right| 1-x\right) - 5\,{}_2F_1\left(\left.\begin{matrix}\frac{\nu}{3}, \frac{\rho+1}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\right| 1-x\right)\right).$$

To obtain the expressions above for $\mathcal{U}_1^{[0]}(x;\nu,\rho)$ and $\mathcal{U}_1^{[2]}(x;\nu,\rho)$, we use [22, Eq. 15.5.13] to derive that

$$\nu x\,{}_2F_1\left(\begin{matrix}\frac{\nu}{3}+1,\frac{\rho+1}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\middle|\,1-x\right)=(\nu+1)\,{}_2F_1\left(\begin{matrix}\frac{\nu}{3},\frac{\rho-2}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\middle|\,1-x\right)-{}_2F_1\left(\begin{matrix}\frac{\nu}{3},\frac{\rho+1}{3}\\ \frac{\nu+\rho+2}{3}\end{matrix}\middle|\,1-x\right).$$

For each $k\in\{0,1,2\}$, we set (a_k,b_k) as in (2.19) as well as

$$\zeta_k=\begin{cases}\frac{\nu}{3}, & \text{if } k\in\{0,2\},\\ \frac{\rho+1}{3}, & \text{if } k=1.\end{cases}\quad\text{and}\quad \xi_k=\begin{cases}\frac{\rho+1}{3}, & \text{if } k\in\{0,2\},\\ \frac{\nu}{3}+1, & \text{if } k=1.\end{cases}$$

Then formulas (2.37a)-(2.37c) and (2.38a)-(2.38c) can be rewritten as

$$\mathcal{U}_0^{[k]}(x;\nu,\rho)=\frac{\Gamma\left(\zeta_k+b_k\right)\Gamma\left(\xi_k+b_k\right)}{\Gamma(a_k)\Gamma(b_k)\Gamma\left(\frac{\nu+\rho+2}{3}\right)}\,x^{a_k-1}(1-x)^{\frac{\nu+\rho-1}{3}}\,{}_2F_1\left(\begin{matrix}\zeta_k,\xi_k\\ \frac{\nu+\rho+2}{3}\end{matrix}\middle|\,1-x\right)\tag{2.39a}$$

and

$$\begin{aligned}\mathcal{U}_1^{[k]}(x;\nu,\rho)=&\frac{\Gamma\left(\zeta_k+b_k+2\right)\Gamma\left(\xi_k+b_k+1\right)}{\zeta_k\Gamma(a_k+1)\Gamma(b_k+1)\Gamma\left(\frac{\nu+\rho+2}{3}\right)}\,x^{a_k-1}(1-x)^{\frac{\nu+\rho-1}{3}}\\ &\left((\zeta_k+b_k){}_2F_1\left(\begin{matrix}\zeta_k,\xi_k-1\\ \frac{\nu+\rho+2}{3}\end{matrix}\middle|\,1-x\right)-b_k{}_2F_1\left(\begin{matrix}\zeta_k,\xi_k\\ \frac{\nu+\rho+2}{3}\end{matrix}\middle|\,1-x\right)\right).\end{aligned}\tag{2.39b}$$

2.3 Cubic decomposition and the Hahn-classical property

The aim of this section is to prove that the cubic components of Hahn-classical 3-fold-symmetric 2-orthogonal polynomials (with positive recurrence coefficients γ_n) are always Hahn-classical, as stated in the following theorem, which is the main new result in this chapter.

Theorem 2.9. *[47, Th. 4.1] Let $(P_n(x))_{n\in\mathbb{N}}$ be a 3-fold-symmetric Hahn-classical 2-orthogonal polynomial sequence satisfying (2.3) with $\gamma_{n+1} > 0$, for all $n \in \mathbb{N}$. Then all the cubic components $\left(P_n^{[k]}(x)\right)_{n\in\mathbb{N}}$, with $k \in \{0,1,2\}$, in (2.2) are Hahn-classical 2-orthogonal polynomial sequences.*

Proof. Let $(P_n(x))_{n\in\mathbb{N}}$ be a 3-fold-symmetric Hahn-classical 2-orthogonal polynomial sequence satisfying (2.3) with $\gamma_{n+1} > 0$, for all $n \in \mathbb{N}$. Our goal is to show that, under the assumptions, the sequences of monic derivatives of the cubic components of $(P_n(x))_{n\in\mathbb{N}}$, that is, the sequences $\left(\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}P_{n+1}^{[k]}(x)\right)_{n\in\mathbb{N}}$ are 2-orthogonal, for each $k \in \{0,1,2\}$.

If $(P_n(x))_{n\in\mathbb{N}}$ is a 3-fold-symmetric Hahn-classical 2-orthogonal polynomial sequence, then the sequence of derivatives $\left(Q_n(x) := \frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(P_n(x)\right)\right)_{n\in\mathbb{N}}$ is also 3-fold symmetric and 2-orthogonal and, recalling Lemma 2.1, the same holds for the cubic components $\left(Q_n^{[k]}(x)\right)_{n\in\mathbb{N}}$, $k \in \{0,1,2\}$. As a result, it is straightforward to check that Theorem 2.9 is valid for $k = 0$, that is, $\left(\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{n+1}^{[0]}(x)\right)\right)_{n\in\mathbb{N}}$ is a 2-orthogonal polynomial sequence because, by definition of $P_{n+1}^{[0]}(x)$,

$$\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{n+1}^{[0]}(x)\right) = \frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{3n+3}\left(x^{\frac{1}{3}}\right)\right) = \frac{x^{-\frac{2}{3}}}{3n+3}P'_{3n+3}\left(x^{\frac{1}{3}}\right).$$

Therefore, by definition of $Q_n(x)$ and $Q_n^{[2]}(x)$,

$$\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{n+1}^{[0]}(x)\right) = x^{-\frac{2}{3}}Q_{3n+2}\left(x^{\frac{1}{3}}\right) = Q_n^{[2]}(x).$$

This observation had already been made by Douak and Maroni in [25].

An analogous procedure does not give an obvious way to conclude anything about the 2-orthogonality of $\left(\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}P_{n+1}^{[k]}(x)\right)_{n\in\mathbb{N}}$, for $k \in \{1,2\}$. So we take a different approach to prove these two cases: we check that the orthogonality weights satisfy a matrix differential equation of the type in (2.5). More precisely, we find matrices

$$\Phi^{[k]}(x) = \begin{bmatrix} \phi_{00} & \phi_{01} \\ \varphi(x) & \phi_{11} \end{bmatrix} \text{ and } \Psi^{[k]}(x) = \begin{bmatrix} 0 & 1 \\ \psi(x) & \xi \end{bmatrix},$$

where ξ and ϕ_{ij}, with $(i,j) \in \{(0,0),(0,1),(1,1)\}$, are constants, and φ and ψ are polynomials with $\deg\varphi \leq 1$ and $\deg\psi = 1$, such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\,\Phi^{[k]}(x)\overline{\mathcal{U}}^{[k]}(x)\right) + \Psi^{[k]}(x)\overline{\mathcal{U}}^{[k]}(x) = 0, \tag{2.40}$$

Based on Proposition 2.5, or alternatively using [47, Prop. 3.6] (which is the case $r=2$ of part (a) of Proposition 3.2), that is sufficient to prove the Hahn-classical character of $\left(P_n^{[k]}(x)\right)_{n\in\mathbb{N}}$, for $k\in\{1,2\}$.

To find the matrices $\Phi^{[k]}(x)$ and $\Psi^{[k]}(x)$, we start by rewriting formulas (2.14b) and (2.14e), (2.14c) and (2.14f) as

$$\overline{\mathcal{U}}^{[k]}(s) = T_k(s)\overline{\mathcal{U}}\left(s^{\frac{1}{3}}\right),$$

where

$$T_1(s) = \begin{bmatrix} 0 & s^{-\frac{1}{3}} \\ \dfrac{1}{\gamma_1\gamma_3}\,s^{\frac{1}{3}} & -\dfrac{1}{\gamma_3}\,s^{-\frac{1}{3}} \end{bmatrix} \text{ and } T_2(s) = \begin{bmatrix} \dfrac{1}{\gamma_1}\,s^{\frac{1}{3}} & 0 \\ -\dfrac{1}{\gamma_4}\left(\dfrac{1}{\gamma_2}+\dfrac{1}{\gamma_1}\right)s^{\frac{1}{3}} & \dfrac{1}{\gamma_2\gamma_4}\,s^{\frac{2}{3}} \end{bmatrix}.$$

These equations are naturally equivalent to

$$\overline{\mathcal{U}}\left(s^{\frac{1}{3}}\right) = T_k^{-1}(s)\overline{\mathcal{U}}^{[k]}(s),$$

with

$$T_1^{-1}(s) = \begin{bmatrix} \gamma_1\,s^{-\frac{1}{3}} & \gamma_3\,\gamma_1\,s^{-\frac{1}{3}} \\ s^{\frac{1}{3}} & 0 \end{bmatrix} \text{ and } T_2^{-1}(s) = \begin{bmatrix} \gamma_1\,s^{-\frac{1}{3}} & 0 \\ (\gamma_1+\gamma_2)\,s^{-\frac{2}{3}} & \gamma_2\,\gamma_4\,s^{-\frac{2}{3}} \end{bmatrix}.$$

If we consider the change of variable $s = x^{\frac{1}{3}}$ in the matrix differential equation (2.9) and then use the previous formula, we obtain, for both $k\in\{1,2\}$,

$$3\,x^{\frac{2}{3}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi\left(x^{\frac{1}{3}}\right)T_k^{-1}(x)\overline{\mathcal{U}}^{[k]}(x)\right) + \Psi\left(x^{\frac{1}{3}}\right)T_k^{-1}(x)\overline{\mathcal{U}}^{[k]}(x) = 0.$$

Equivalently, we can write

$$M^{[k]}(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{\mathcal{U}}^{[k]}(x)\right)+N^{[k]}(x)\overline{\mathcal{U}}^{[k]}(x)=0, \tag{2.41}$$

with

$$M^{[k]}(x)=3\,x^{\frac{2}{3}}\Phi\left(x^{\frac{1}{3}}\right)T_k^{-1}(x)$$

and

$$N^{[k]}(x)=\Psi\left(x^{\frac{1}{3}}\right)T_k^{-1}(x)+3x^{\frac{2}{3}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi\left(x^{\frac{1}{3}}\right)T_k^{-1}(x)\right).$$

Finally, we multiply (2.41) from the left by a suitable matrix to derive (2.40).

When $k=1$, we have (2.41) with

$$M^{[1]}(x)=3\begin{bmatrix}(1-\vartheta_1)\,x^{\frac{4}{3}}+\vartheta_1\gamma_1x^{\frac{1}{3}} & \vartheta_1\gamma_1\gamma_3x^{\frac{1}{3}}\\ x & 6\,(1-\vartheta_2)\,\gamma_3\,x\end{bmatrix}$$

and

$$N^{[1]}(x)=\begin{bmatrix}(3-2\vartheta_1)\,x^{\frac{1}{3}}-\vartheta_1\gamma_1x^{-\frac{2}{3}} & -\vartheta_1\gamma_1\gamma_3x^{-\frac{2}{3}}\\ 3 & -2\,(2\vartheta_2-1)\,\gamma_3\end{bmatrix},$$

which, after a multiplication by $\begin{bmatrix}0 & 1\\ x^{\frac{2}{3}} & 0\end{bmatrix}$, leads to (2.40), with

$$\Phi^{[1]}(x)=3\begin{bmatrix}\dfrac{1}{2\,(2\vartheta_2-1)\,\gamma_3} & \dfrac{1-\vartheta_2}{2\vartheta_2-1}\\ (1-\vartheta_1)\,x+\vartheta_1\gamma_1 & \vartheta_1\gamma_1\gamma_3\end{bmatrix} \text{ and } \Psi^{[1]}(x)=\begin{bmatrix}0 & 1\\ \psi^{[1]}(x) & -4\vartheta_1\gamma_1\gamma_3\end{bmatrix},$$

where $\psi^{[1]}(x)=(4\vartheta_1-3)\,x-4\vartheta_1\gamma_1$.

As a result, $\left(P_n^{[1]}(x)\right)_{n\in\mathbb{N}}$ is a Hahn-classical 2-orthogonal polynomial sequence.

To prove the result when $k=2$, we start by using (2.6), with $n=0$, to write $\gamma_1 = \frac{1}{3}\left(4\vartheta_1 - 3\right)\gamma_2$ and obtain

$$T_2^{-1}(s) = \begin{bmatrix} \frac{1}{3}\left(4\vartheta_1 - 3\right)\gamma_2\, s^{-\frac{1}{3}} & 0 \\ \frac{4}{3}\,\vartheta_1\gamma_2 s^{-\frac{2}{3}} & \gamma_2\,\gamma_4\, s^{-\frac{2}{3}} \end{bmatrix}.$$

Therefore, we derive (2.41) with

$$M^{[2]}(x) = 3\begin{bmatrix} \frac{1}{3}\vartheta_1\gamma_2\, x^{\frac{1}{3}} & \gamma_2\gamma_4\left(1-\vartheta_1\right)\, x^{\frac{1}{3}} \\ 2\left(1-\vartheta_2\right)x + \frac{4}{3}\gamma_2\left(2\vartheta_2 - 1\right) & \gamma_2\gamma_4\left(2\vartheta_2 - 1\right) \end{bmatrix}$$

and

$$N^{[2]}(x) = \begin{bmatrix} \vartheta_1\gamma_2\, x^{-\frac{2}{3}} & \vartheta_1\gamma_2\gamma_4\, x^{-\frac{2}{3}} \\ 2\left(2-\vartheta_2\right) + \frac{8}{3}\,\gamma_2\left(1-2\vartheta_2\right)x^{-1} & -2\gamma_2\gamma_4\left(2\vartheta_2 - 1\right)x^{-1} \end{bmatrix},$$

which, after a multiplication by $\begin{bmatrix} x^{\frac{2}{3}} & 0 \\ 0 & x \end{bmatrix}$, corresponds to (2.40) with

$$\Phi^{[2]}(x) = \begin{bmatrix} \dfrac{\vartheta_1\gamma_2}{3\gamma_1\gamma_4} & \left(1-\vartheta_1\right)\dfrac{\gamma_2}{\gamma_1} \\ \varphi^{[2]}(x) & 3\left(2\vartheta_2 - 1\right)\gamma_2\gamma_4 \end{bmatrix} \quad \text{and} \quad \Psi^{[2]}(x) = \begin{bmatrix} 0 & 1 \\ \psi^{[2]}(x) & -5\left(2\vartheta_2 - 1\right)\gamma_2\gamma_4 \end{bmatrix},$$

where

$$\varphi^{[2]}(x) = 6\left(1-\vartheta_2\right)x + 4\left(2\vartheta_2 - 1\right)\gamma_2$$

and

$$\psi^{[2]}(x) = 2\left(5\vartheta_2 - 4\right)x - \frac{20}{3}\,\vartheta_1\left(2\vartheta_2 - 1\right)\gamma_2.$$

Therefore, $\left(P_n^{[2]}(x)\right)_{n\in\mathbb{N}}$ is Hahn-classical. $\square$

Although we have made a remark that the case $k = 0$ of Theorem 2.9 is a consequence of an observation made by Douak and Maroni in [25], we also checked that (2.40) holds for $k = 0$, with

$$\Phi^{[0]}(x) = 3\begin{bmatrix} 1 & (1-\vartheta_1)\,\gamma_2 \\ \varphi^{[0]}(x) & (2\vartheta_2-1)\,\gamma_1\gamma_2 \end{bmatrix} \text{ and } \Psi^{[0]}(x) = \begin{bmatrix} 0 & \vartheta_1\gamma_2 \\ \psi^{[0]}(x) & -2\,(2\vartheta_2-1)\,\gamma_1\gamma_2 \end{bmatrix}$$

where $\varphi^{[0]}(x) = 2\,(1-\vartheta_2)\,x + (2\vartheta_2-1)\,\gamma_1$ and $\psi^{[0]}(x) = 2\,(2\vartheta_2-1)\,(x-\gamma_1)$.

It was already known that the cubic components for case A are Hahn-classical, because it was observed in [11, §5] that they are particular cases of 2-orthogonal polynomials with respect to Macdonald functions, which are Hahn-classical because the differentiation operator acts on them as a shift in the parameters and in the index (see Section 4.6). Similarly, we can prove that the cubic components for cases B and C are Hahn-classical by showing that they are particular realisations of the 2-orthogonal polynomials analysed in Chapters 4 and 5, on which the differentiation operator also acts as a shift in the parameters and index. However, we have proved here that the cubic decomposition preserves the Hahn-classical property for all 3-fold-symmetric Hahn-classical 2-orthogonal polynomial sequences in a much simpler way, without using the generalisations of the cubic components. A further benefit from this proof are the techniques involved, which may be adaptable to prove analogous results regarding Hahn-classical $(r+1)$-fold symmetric r-orthogonal polynomials, with $r > 2$, or Hahn-classical polynomials with respect to other lowering operators such as the q-derivative.

Chapter 3

MOPs associated with ratios of Pochhammer symbols

The object of study in this chapter are multiple orthogonal polynomials with respect to measures whose moments are ratios of Pochhammer symbols, that is,

$$\int_I x^n \mathcal{W}(x)\mathrm{d}x = \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n}, \quad \text{for } n \in \mathbb{N}, \tag{3.1}$$

with

$$r, s \in \mathbb{N} \text{ and } \{a_i\}_{i=1}^r, \{b_j\}_{j=1}^s \subset \mathbb{R}^+ \text{ such that } s \leq r \text{ and } \min_{1\leq j\leq s}\{b_j\} > \max_{1\leq i\leq r}\{a_i\}. \tag{3.2}$$

Throughout this chapter, we outline techniques to investigate hypergeometric multiple orthogonal polynomials, which we use in Chapters 4 and 5 and constitute the building blocks of an ongoing investigation of more general multiple orthogonal polynomials. The techniques introduced in Section 3.4 are new and particularly interesting. The main original results in this chapter are Theorem 3.2 (b) and Theorem 3.5. Proposition 3.1 is also new. These results and techniques will be submitted for publication as part of an ongoing joint work with Alan Sokal on the connection of branched-continued-fraction representations of hypergeometric series and multiple orthogonal polynomials with respect to measures whose moments are ratios of Pochhammer symbols.

In Section 3.1, we investigate the differential properties of the weight function which is the Radon-Nikodym derivative of an absolutely continuous measure μ with moments as in (3.1). We prove that such a weight satisfies an ordinary differential equation of order r (Proposition 3.1). In general, if $r \geq 2$, then an explicit and complete description of the (standard) orthogonal polynomials for such weights is a challenging open problem. Instead, it is more natural to investigate the multiple orthogonal polynomials with respect to a vector of r measures with moment sequences as in (3.1). For each particular case we have investigated, the vector of r orthogonality measures satisfies a matrix Pearson-type differential equation, which implies that the corresponding type I functions and type II polynomials on the step-line satisfy the Hahn-classical property (Theorem 3.2).

In Section 3.2 we discuss how to generate the type I functions on the step-line via a Rodrigues-type formula of the type (3.15) and the corresponding type I polynomials via a matrix Rodrigues-type formula of the type (3.17).

In Section 3.3 we investigate the type II polynomials on the step-line or r-orthogonal polynomials. We present auxiliary results to find representations as terminating hypergeometric series as in (3.18a) for these polynomials. Then, we explain how to use those representations to prove that the polynomials are solutions to an ordinary differential equation of order $r+1$ of the type (3.23) and to obtain explicit expressions for the coefficients of the recurrence relation of order $r+1$ (3.24) they satisfy. Next, we focus on investigating the location of the zeros of these r-orthogonal polynomials. We argue that the zeros are all simple and located in the support of the orthogonality measures, which is either the positive real line or the interval $(0, 1)$, for all the cases we have analysed. We prove a result about the asymptotic behaviour of the zeros from the respective recurrence coefficients (Theorem 3.5), with emphasis on the case where there are 2 orthogonality measures (Corollary 3.6). We also discuss how to deduce Mehler-Heine-type asymptotic formulas near the origin, which give us more information about the asymptotic behaviour of the polynomials and their zeros, and we introduce the asymptotic zero distribution of a polynomial sequence.

In Section 3.4, we focus on the connection between the type II polynomials on the step-line investigated here and branched continued fractions. The starting point is to observe that the ordinary generating function of the moment sequence given by (3.1) is a generalised hypergeometric function, which admits a branched-continued-fraction representation due to results in [61]. An important consequence of this observation is that the conditions (3.2) guarantee the existence of a measure μ on the positive real line with a moment sequence determined by (3.1). We explain how the infinite lower-Hessenberg matrix associated with the r-orthogonal polynomials is equal to the production matrix of the generalised r-Stieltjes-Rogers polynomials associated with the moment sequences of the first orthogonality measure. This identity gives alternative expressions for the recurrence coefficients, which make clear that they are all positive, and it leads to an alternative proof that the zeros of the polynomials are all simple and located in the support of the orthogonality measures with the zeros of consecutive polynomials interlacing.

For $r = 1$, the condition $s \leq r$ in (3.2) implies that either $s = 0$ or $s = 1$. The two corresponding measures with moments as in (3.1) are closely related with classical orthogonal polynomials as they are the orthogonality measures of the Laguerre polynomials and the Jacobi polynomials on the interval $(0, 1)$, respectively. We give more detail about these two cases in Section 3.5. Besides that, we note that these two families of orthogonal polynomials include as particular cases the components of the quadratic decomposition of the Hermite and Gegenbauer polynomials, respectively, which are, up to a linear transformation of the variable, the only symmetric classical orthogonal polynomials.

The multiple orthogonal polynomials analysed in Chapters 4 and 5 satisfy orthogonality conditions with respect to measures with moments equal to the ratio of two-by-one and two-by-two Pochhammer symbols, respectively, that is, moments as in (3.1) with $(r, s) = (2, 1)$ and $(r, s) = (2, 2)$. The multiple orthogonal polynomials with respect to measures with moments equal to the products of r Pochhammer symbols, which correspond to (3.1) with $s = 0$, have already been investigated.

The multiple orthogonal polynomials introduced in [75] and [11] satisfy orthogonality conditions with respect to two measures which, after an adequate normalisation, have moments given by the product of two Pochhammer symbols. These orthogonality measures are supported on the positive real line and they admit integral representations involving the Macdonald function $\mathrm{K}_\nu(z)$ defined by (1.13). We mention some results about these multiple orthogonal polynomials in Section 4.6, where they appear as a limiting case of the polynomials characterised in Chapter 4.

More generally, the multiple orthogonal polynomials used in [44] to investigate the singular values of products of Ginibre random matrices satisfy orthogonality conditions with respect to measures whose moments, after an adequate normalisation, are equal to the products of M Pochhammer symbols, with $M \in \mathbb{Z}^+$. These orthogonality measures are supported on the positive real line and admit integral representations using weight functions expressed in terms of Meijer G-functions. As mentioned in [44, §2.3], the case $M = 2$ of the multiple orthogonal polynomials analysed therein reduces to the polynomials introduced in [75] and [11].

3.1 Differential properties

As previously mentioned, we are interested in multiple orthogonal polynomials with respect to measures whose moments are ratios of Pochhammer symbols as in (3.1), under the assumptions in (3.2). These measures admit integral representations via weight functions that are solutions of ordinary differential equations of order r. The relation between these differential equations and the moment sequences of the corresponding measures is a consequence of the following proposition.

Proposition 3.1. *Let $\mathcal{W}(x)$ be a positive r times continuously differentiable function defined on an interval $I \subset \mathbb{R}$ such that $f(x)\mathcal{W}^{(k)}(x)\Big|_I = 0$, for any polynomial $f \in \mathcal{P}$ and each $k \in \{0, \cdots, r\}$. Let μ be the measure supported on I such that $\mathrm{d}\mu(x) = \mathcal{W}(x)\mathrm{d}x$ for all $x \in I$, and suppose that all the moments of μ exist and* *are finite. Then, μ satisfies (3.1) subject to (3.2) if and only if* $\int_I \mathcal{W}(x)\mathrm{d}x = 1$ *and there are polynomials ϕ_k, with $k \in \{0, \cdots, r\}$, of degree not greater than 1 such that*

$$\int_I x^n \left(\sum_{k=0}^{r} \left(\frac{\mathrm{d}}{\mathrm{d}x} x \right)^k \Big(\phi_k(x)\mathcal{W}(x) \Big) \right) \mathrm{d}x = 0, \tag{3.3}$$

holds for all $n \in \mathbb{N}$, with ϕ_k determined by the conditions $\phi_k'(0) = 0$ if $k \geq s+1$,

$$\phi_k'(0) = (-1)^{k+1} \sum_{1 \leq j_1 < \cdots < j_{s-k} \leq s} \left(\prod_{\lambda=1}^{s-k} b_{j_\lambda} \right) \quad \textit{if} \quad k \leq s, \tag{3.4a}$$

and

$$\phi_k(0) = (-1)^k \sum_{1 \leq i_1 < \cdots < i_{r-k} \leq r} \left(\prod_{\lambda=1}^{r-k} a_{i_\lambda} \right) \quad \textit{for any} \quad 0 \leq k \leq r. \tag{3.4b}$$

Proof. Observe that the measure μ supported on I with $\mathrm{d}\mu(x) = \mathcal{W}(x)\mathrm{d}x$ for all $x \in I$ satisfies (3.1) if and only if $\int_I \mathcal{W}(x)\mathrm{d}x = 1$ and, for all $n \in \mathbb{N}$,

$$\frac{\displaystyle\int_I x^{n+1}\mathcal{W}(x)\mathrm{d}x}{\displaystyle\int_I x^n\mathcal{W}(x)\mathrm{d}x} = \frac{\displaystyle\prod_{j=1}^{s}(n+b_j)}{\displaystyle\prod_{i=1}^{r}(n+a_i)}. \tag{3.5}$$

Therefore, we simply need to prove that, for any $n \in \mathbb{N}$, (3.5) is equivalent to (3.3), where the polynomials $\phi_k(x)$, $k \in \{0, \cdots, r\}$, are the ones defined in the statement of the proposition.

For any $n \in \mathbb{N}$, (3.3) is clearly equivalent to

$$\sum_{k=0}^{r} \left(\int_I x^n \left(\frac{\mathrm{d}}{\mathrm{d}x} x \right)^k \Big(\phi_k(x)\mathcal{W}(x) \Big) \mathrm{d}x \right) = 0. \tag{3.6}$$

For any differentiable functions f, g such that $xf(x)g(x)\Big|_I = 0$, we get, via integration by parts,

$$\int_I f(x) \frac{\mathrm{d}}{\mathrm{d}x}\Big(xg(x)\Big)\mathrm{d}x = -\int_I x \frac{\mathrm{d}}{\mathrm{d}x}\Big(f(x)\Big)g(x)\mathrm{d}x.$$

Hence, using integration by parts k times,

$$\int_I x^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\,x\right)^k \big(\phi_k(x)\mathcal{W}(x)\big)\mathrm{d}x = (-1)^k \int_I \left(x\,\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (x^n)\phi_k(x)\mathcal{W}(x)\mathrm{d}x.$$

As a result, because $\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}\right)(x^n) = nx^n$,

$$\int_I x^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\,x\right)^k \big(\phi_k(x)\mathcal{W}(x)\big)\mathrm{d}x = (-1)^k n^k \int_I x^n\phi_k(x)\mathcal{W}(x)\mathrm{d}x.$$

Furthermore, $\phi_k(x) = \phi_k'(0)x + \phi_k(0)$, because $\deg(\phi_k) \leq 1$. Therefore,

$$\begin{aligned}&\int_I x^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\,x\right)^k \big(\phi_k(x)\mathcal{W}(x)\big)\mathrm{d}x\\ &= (-1)^k n^k \left(\phi_k'(0)\int_I x^{n+1}\mathcal{W}(x)\mathrm{d}x + \phi_k(0)\int_I x^n\mathcal{W}(x)\mathrm{d}x\right).\end{aligned}$$

As a result, (3.3) and (3.6) are equivalent to

$$\sum_{k=0}^{r}(-1)^k n^k \left(\phi_k'(0)\int_I x^{n+1}\mathcal{W}(x)\mathrm{d}x + \phi_k(0)\int_I x^n\mathcal{W}(x)\mathrm{d}x\right) = 0,$$

and to

$$\frac{\displaystyle\int_I x^{n+1}\mathcal{W}(x)\mathrm{d}x}{\displaystyle\int_I x^n\mathcal{W}(x)\mathrm{d}x} = \frac{\displaystyle\sum_{k=0}^{r}\left((-1)^{k+1}\phi_k'(0)n^k\right)}{\displaystyle\sum_{k=0}^{r}\left((-1)^k\phi_k(0)n^k\right)}. \tag{3.7}$$

Finally, recalling (3.4a)-(3.4b), we get

$$\sum_{k=0}^{r}\left((-1)^{k+1}\phi_k'(0)n^k\right) = \sum_{k=0}^{s}\left(\sum_{1\leq j_1<\cdots<j_{s-k}\leq s}\left(\prod_{\lambda=1}^{s-k} b_{j_\lambda}\right)n^k\right) = \prod_{j=1}^{s}(n+b_j)$$

and

$$\sum_{k=0}^{r}\left((-1)^k\phi_k(0)n^k\right) = \sum_{k=0}^{r}\left(\sum_{1\leq i_1<\cdots<i_{r-k}\leq r}\left(\prod_{\lambda=1}^{r-k} a_{i_\lambda}\right)n^k\right) = \prod_{i=1}^{r}(n+a_i).$$

Therefore, the conditions (3.5), (3.7) and (3.3) are all equivalent. □

Note that, for any differentiable functions f and ψ,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\,x\right)\big(\psi(x)f(x)\big) = x\psi(x)f'(x) + \left(\psi(x) + x\psi'(x)\right)f(x).$$

Therefore, (3.3) can also be written in the form

$$\int_I x^n \left(\sum_{k=0}^{r} x^k \tilde{\phi}_k(x)\mathcal{W}^{(k)}(x)\right) \mathrm{d}x = 0,$$

where the $\tilde{\phi}_k(x)$ are all polynomials of degree not greater than 1 and $\tilde{\phi}_r(x) = \phi_r(x)$.

For each of the cases analysed in more detail here, we find weight functions that satisfy the differential equation

$$\sum_{k=0}^{r} \left(\frac{\mathrm{d}}{\mathrm{d}x}\,x\right)^k \big(\phi_k(x)\mathcal{W}(x)\big) = \sum_{k=0}^{r} x^k \tilde{\phi}_k(x)\mathcal{W}^{(k)}(x) = 0,$$

which can be used to derive a matrix differential equation of the type in (3.8), called a matrix Pearson-type equation, satisfied by a vector of r measures that satisfy a differential equation of the type in (3.3) and are obtained from each other via shifts in the parameters.

In Theorem 3.2, we deduce differential properties for type II and type I multiple orthogonal polynomials on the step-line with respect to a vector of measures that satisfies a matrix Pearson-type differential equation. The type II result is a consequence of the characterisation of the Hahn-classical property for d-orthogonal polynomials in Proposition 2.5, corresponding to [24, Th. 3.1] with $d = 2$. Evoking similar arguments, we derive an analogous result for type I polynomials. These two results when $r = 2$ can be found in [47, §2.3].

There are two important differences distinguishing the type II result we show here from [24, Th. 3.1]. Firstly, the latter is stated and proved using linear functionals, while we restrict ourselves to the use of weight functions in both the statement and the proof of our result. Secondly, the authors of [24] consider r-orthogonal polynomials $(P_n(x))_{n\in\mathbb{N}}$ with respect to the vector of the first r elements $(u_0, \cdots, u_{r-1})$ of the dual sequence of $(P_n(x))_{n\in\mathbb{N}}$, which implies the moment of order 0 of u_j

is equal to 0 whenever $j \geq 1$. Instead we consider each orthogonality weight to be a probability density, so that all of the weights have the moment of order 0 equal to 1. This is relevant because our focus is on perfect systems and all the orthogonality weights having a nonzero moment of order 0 is a necessary condition for a system of multiple orthogonal polynomials to be perfect.

Theorem 3.2. *For $r \in \mathbb{Z}^+$, let $\overline{w}(x) = [w_0(x), \cdots, w_{r-1}(x)]^T$ be a vector of r functions supported on a common interval I such that, for each $j \in \{0, \cdots, r-1\}$, all the moments of $w_j(x)$ over I exist and are finite, and $f(x)w_j(x)\Big|_I = 0$ for any polynomial f. Suppose that there exist two $(r \times r)$-matrices $\Phi(x) = [\phi_{i,j}]_{i,j=0}^{r-1}$ and $\Psi(x) = [\psi_{i,j}]_{i,j=0}^{r-1}$, with*

- $\phi_{i,j} = \psi_{i,j} = 0$ *if $j \geq i+2$ (that is, Φ and Ψ are lower-Hessenberg matrices),*
- $\phi_{i,j}$ *and* $\psi_{i,j}$ *are constants unless $(i,j) = (r-1, 0)$,*
- $\deg \phi_{r-1,0} \leq 1$ *and* $\deg \psi_{r-1,0} = 1$,
- $\psi_{i,i+1} \neq k\phi_{i,i+1}$ *for all $k \in \mathbb{N}$ and $0 \leq i \leq r-2$,*
- $\psi'_{r-1,0}(0) \neq k\phi'_{r-1,0}(0)$, *for all $k \in \mathbb{N}$,*

such that $\overline{w}(x)$ satisfies the matrix differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x\Phi(x)\overline{w}(x)\big) + \Psi(x)\overline{w}(x) = 0. \tag{3.8}$$

If all multi-indices on the step-line are normal with respect to both $\overline{w}(x)$ and $x\Phi(x)\overline{w}(x)$, the following statements hold:

(a) If $(P_n(x))_{n\in\mathbb{N}}$ is the r-orthogonal polynomial sequence with respect to $\overline{w}(x)$, then $\left(\frac{1}{n+1} P'_{n+1}(x)\right)_{n\in\mathbb{N}}$ is r-orthogonal with respect to $x\Phi(x)\overline{w}(x)$.

(b) If $Q_n(x)$ is the type I function for the index of length $n \in \mathbb{Z}^+$ on the step-line with respect to $x\,\Phi(x)\overline{w}(x)$, then $\left(-\frac{1}{n}\,Q'_n(x)\right)$ is the type I function for the index of length $n+1$ on the step-line with respect to $\overline{w}(x)$.

The differentiable properties described in Theorem 3.2 are the main pillars for further characterisation of the multiple orthogonal polynomials under analysis and they resemble those found within the context of the very classical standard orthogonal polynomials.

Proof. Let $\overline{v}(x) = x\Phi(x)\overline{w}(x) = [v_0(x), \cdots, v_{r-1}(x)]^T$, which means that

$$v_i(x) = x \sum_{j=0}^{r-1} \phi_{i,j}(x)\, w_j(x), \quad \text{for } i \in \{0, \cdots, r-1\}. \tag{3.9a}$$

Moreover, by virtue of equation (3.8), $\frac{\mathrm{d}}{\mathrm{d}x}(\overline{v}(x)) = -\Psi(x)\overline{w}(x)$, which means that

$$v_i'(x) = -\sum_{j=0}^{r-1} \psi_{i,j}(x)\, w_j(x), \quad \text{for } i \in \{0, \cdots, r-1\}. \tag{3.9b}$$

We begin by proving statement *(a)*. For that purpose, we let $i \in \{0, \cdots, r-1\}$ and $k, n \in \mathbb{N}$. Combining (3.9a) and (3.9b), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^k v_i(x)\right) = x^k \sum_{j=0}^{r-1} \Big(\big(k\phi_{i,j}(x) - \psi_{i,j}(x)\big) w_j(x)\Big).$$

Therefore, using integration by parts, we derive

$$\int_I x^k P_{n+1}'(x) v_i(x) \mathrm{d}x = \sum_{j=0}^{r-1} \left(\int_I \big(\psi_{i,j}(x) - k\phi_{i,j}(x)\big) x^k P_{n+1}(x) w_j(x) \mathrm{d}x \right). \tag{3.10}$$

Due to the r-orthogonality of $(P_n(x))_{n\in\mathbb{N}}$ with respect to $\overline{w}(x)$, we know that, for any $k, m \in \mathbb{N}$ and $j \in \{0, \cdots, r-1\}$,

$$\int_I x^k P_m(x) w_j(x) \mathrm{d}x = \begin{cases} N_m \neq 0 & \text{if } m = rk + j, \\ 0 & \text{if } m \geq rk + j + 1. \end{cases}$$

Let $0 \leq i \leq r-2$. Then $\phi_{i,j}$ and $\psi_{i,j}$ are constants, for all $j \in \{0, \cdots, r-1\}$, and $\phi_{i,j} = \psi_{i,j} = 0$, if $j \geq i+2$. Therefore, for any $k, n \in \mathbb{N}$, (3.10) implies

$$\int_I x^k P'_{n+1}(x) v_i(x) \mathrm{d}x = \sum_{j=0}^{i+1} \left(\left(\psi_{i,j} - k\phi_{i,j}\right) \int_I x^k P_{n+1}(x) w_j(x) \mathrm{d}x \right).$$

If $n \geq rk + i + 1$, then

$$\int_I x^k P_{n+1}(x) w_j(x) \mathrm{d}x = 0, \quad \text{for each } 0 \leq j \leq i+1,$$

and, as a result,

$$\int_I x^k P'_{n+1}(x) v_i(x) \mathrm{d}x = 0.$$

If $n = rk + i$, then

$$\int_I x^k P_{rk+i+1}(x) w_j(x) \mathrm{d}x = 0, \quad \text{for each } 0 \leq j \leq i,$$

and, as a result,

$$\int_I x^k P'_{rk+i+1}(x) v_i(x) \mathrm{d}x = \left(\psi_{i,i+1} - k\phi_{i,i+1}\right) \int_I x^k P_{rk+i+1}(x) w_{i+1}(x) \mathrm{d}x \neq 0.$$

Let $i = r-1$ and $k, n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \int_I x^k P'_{n+1}(x) v_{r-1}(x) \mathrm{d}x = & \sum_{j=1}^{r-1} \left(\left(\psi_{r-1,j} - k\phi_{r-1,j}\right) \int_I x^k P_{n+1}(x) w_j(x) \mathrm{d}x \right) \\ & + \left(\psi_{r-1,0}(0) - k\phi_{r-1,0}(0)\right) \int_I x^k P_{n+1}(x) w_0(x) \mathrm{d}x \\ & + \left(\psi'_{r-1,0}(0) - k\phi'_{r-1,0}(0)\right) \int_I x^{k+1} P_{n+1}(x) w_0(x) \mathrm{d}x. \end{aligned}$$

Therefore, if $n \geq r(k+1)$,

$$\int_I x^k P'_{n+1}(x) v_{r-1}(x) \mathrm{d}x = 0,$$

and, if $n = rk + (r-1)$,

$$\int_I x^k P'_{r(k+1)}(x) v_{r-1}(x) \mathrm{d}x = \left(\psi'_{r-1,0}(0) - k\phi'_{r-1,0}(0)\right) \int_I x^{k+1} P_{r(k+1)}(x) w_0(x) \mathrm{d}x \neq 0,$$

which proves *(a)*.

To prove *(b)*, we let $\left(C_{n,0}(x), \cdots, C_{n,r-1}(x)\right)$, with $n \in \mathbb{Z}^+$, be the vector of type I multiple orthogonal polynomials for the index of length n on the step-line with respect to $\overline{v}(x)$. Then, by definition of the type I function (1.32) combined with the relation (3.9a), we have $\deg(C_{n,i}) \leq \left\lfloor \frac{n-(i+1)}{r} \right\rfloor$ for all $i \in \{0, \cdots, r-1\}$, and

$$Q_n(x) = \sum_{i=0}^{r-1} C_{n,i}(x)\, v_i(x) = \sum_{j=0}^{r-1} \left(w_j(x) \sum_{i=0}^{r-1} \left(x\phi_{i,j}(x) C_{n,i}(x) \right) \right). \tag{3.11}$$

Since $f(x)w_j(x)\Big|_I = 0$ for each $j \in \{0, \cdots, r-1\}$ and any polynomial f, it follows from the latter that

$$\int_I Q'_n(x) \mathrm{d}x = Q_n(x)\Big|_I = \sum_{j=0}^{r-1} \left(w_j(x) \sum_{i=0}^{r-1} \left(x\phi_{i,j}(x) C_{n,i}(x) \right) \right) \Bigg|_I = 0.$$

By performing integration by parts and arguing with $Q_n(x)$ satisfying orthogonality conditions as in (1.33), we have

$$\int_I x^{k+1} Q'_n(x) \mathrm{d}x = -(k+1) \int_I x^k Q_n(x) \mathrm{d}x = \begin{cases} 0, & \text{if } 0 \leq k \leq n-2, \\ -n, & \text{if } k = n-1. \end{cases}$$

Hence, we obtain

$$\int_I \left(-x^m \frac{Q'_n(x)}{n} \right) \mathrm{d}x = \begin{cases} 0, & \text{if } 0 \leq m \leq n-1, \\ 1, & \text{if } m = n. \end{cases}$$

As a result, it is sufficient to show that there exists a vector of polynomials $\left(A_{n+1,0}(x), \cdots, A_{n+1,r-1}(x)\right)$, with $\deg(A_{n+1,j}) \leq \left\lfloor \frac{n-j}{r} \right\rfloor$ for $j \in \{0, \cdots, r-1\}$, such that

$$-\frac{1}{n}Q_n'(x) = \sum_{j=0}^{r-1}\big(A_{n+1,j}(x)\,w_j(x)\big). \tag{3.12}$$

In that case, $(A_{n+1,0}(x),\cdots,A_{n+1,r-1}(x))$ is the vector of type I multiple orthogonal polynomials for the index of length $n+1$ on the step-line with respect to $\overline{w}(x)$ and, consequently, $-\frac{1}{n}\,Q_n'(x)$ is the type I function for the index of length $n+1$ on the step-line with respect to $\overline{w}(x)$.

Differentiating the first expression for $Q_n(x)$ in (3.11), we obtain

$$Q_n'(x) = \sum_{i=0}^{r-1}\left(C_{n,i}'(x)v_i(x) + C_{n,i}(x)v_i'(x)\right),$$

which, using (3.9a)-(3.9b), leads to

$$Q_n'(x) = \sum_{j=0}^{r-1}\left(w_j(x)\sum_{i=0}^{r-1}\Big(x\phi_{i,j}(x)C_{n,i}'(x) - \psi_{i,j}(x)C_{n,i}(x)\Big)\right).$$

The latter is equivalent to (3.12) if we set, for all $j\in\{0,\cdots,r-1\}$,

$$A_{n+1,j}(x) = -\frac{1}{n}\sum_{i=0}^{r-1}\Big(x\phi_{i,j}(x)C_{n,i}'(x) - \psi_{i,j}(x)C_{n,i}(x)\Big). \tag{3.13}$$

Now to prove that $(A_{n+1,0}(x),\cdots,A_{n+1,r-1}(x))$ defined by (3.13) is the vector of type I polynomials with respect to $\overline{w}(x)$ for the index of length $n+1$ on the step-line and, as a result, that $-\frac{1}{n}\,Q_n'(x)$ is the corresponding type I function, it is sufficient to check that $\deg\left(A_{n+1,j}\right) \le \left\lfloor \frac{n-j}{r}\right\rfloor$, for each $j\in\{0,\cdots,r-1\}$.

If $j=0$, then $\phi_{i,0}$ and $\psi_{i,0}$ are constants, for all $0\le i\le r-2$, $\deg\phi_{r-1,0}\le 1$ and $\deg\psi_{r-1,0}=1$. Therefore, $-nA_{n+1,0}(x)$ is equal to

$$\sum_{i=0}^{r-2}\Big(\phi_{i,0}x\,C_{n,0}'(x) - \psi_{i,0}C_{n,0}(x)\Big) + x\phi_{r-1,0}(x)C_{n,r-1}'(x) - \psi_{r-1,0}(x)C_{n,r-1}(x),$$

which has degree not greater than $\deg\left(C_{n,r-1}\right)+1 \le \left\lfloor\frac{n}{r}\right\rfloor$. Furthermore, if $n=rm$ with $m\ge 1$, then $\deg\left(A_{rm+1,0}(x)\right) = \deg\left(C_{rm,r-1}\right)+1 = m$.

If $j \in \{1, \cdots, r-1\}$, then $\phi_{i,j}$ and $\psi_{i,j}$ are constants for all $i \in \{0, \cdots, r-1\}$, and $\phi_{i,j} = \psi_{i,j} = 0$ if $j \geq i+2$. Therefore,

$$A_{n+1,j}(x) = -\frac{1}{n} \sum_{i=j-1}^{r-1} \Big(\phi_{i,j} x \, C'_{n,i}(x) - \psi_{i,j} C_{n,i}(x) \Big),$$

which has degree not greater than $\max_{j-1 \leq i \leq r-1} \deg(C_{n,i}) \leq \left\lfloor \frac{n-j}{r} \right\rfloor$. Furthermore, if $n = rm + j$ with $m \in \mathbb{N}$, then $\deg\left(A_{rm+j+1,j}(x)\right) = \deg\left(C_{rm+j,j-1}\right) = m$. □

3.2 Rodrigues-type formulas for type I MOPs

Here we explain how part (b) of Theorem 3.2 and formula (3.13) appearing in its proof can be used to find Rodrigues-type formulas generating the type I functions and the type I multiple orthogonal polynomials on the step-line.

Throughout the rest of this thesis, we consider vectors of orthogonality weights

$$\overline{\mathcal{W}}(x) = \overline{\mathcal{W}} \left(x \,\middle|\, \begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \right) = \left[\mathcal{W}_0 \left(x \,\middle|\, \begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \right), \cdots, \mathcal{W}_{r-1} \left(x \,\middle|\, \begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \right) \right]^T,$$

depending on the parameters $a_1, \cdots, a_r$ and $b_1, \cdots, b_s$ in (3.1), which satisfy differential equation of the type in (3.8), meaning that there exist matrices

$$\Phi(x) := \Phi \left(x \,\middle|\, \begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \right) \quad \text{and} \quad \Psi(x) := \Psi \left(x \,\middle|\, \begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \right), \tag{3.14}$$

satisfying the conditions on the statement of Theorem 3.2, such that

$$\frac{\mathrm{d}}{\mathrm{d}x} \Big(x \Phi(x) \overline{\mathcal{W}}(x) \Big) + \Psi(x) \overline{\mathcal{W}}(x) = 0.$$

It is instrumental to our techniques that $x \, \Phi(x) \overline{\mathcal{W}}(x)$ is equal to $\overline{\mathcal{W}}(x)$ up to a shift in the parameters. This property is common to all the cases analysed here.

Firstly, we obtain the initial type I polynomials and functions. By the conditions on the degrees of the type I polynomials, we have $A_{0,j}(x) = 0$ for all j, and $A_{1,j}(x) = 0$ whenever $j \neq 0$. In addition, we have the normalisation $A_{1,0}(x) = 1$, because we are assuming that the moment of order 0 of the orthogonality measures is always equal to 1. As a result, we have $Q_0(x) = 0$ and $Q_1(x) = \mathcal{W}_0(x)$.

Then, we use part (b) of Theorem 3.2 to generate, by induction, the type I functions on the step-line. More precisely, we derive, by concatenated differentiation of the weight function, the Rodrigues-type formulas

$$Q_{n+1}\left(x \,\middle|\, \begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}\right) = \frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathcal{W}\left(x \,\middle|\, \begin{matrix} a_1^{[n]}, \cdots, a_r^{[n]} \\ b_1^{[n]}, \cdots, b_s^{[n]} \end{matrix}\right)\right), \tag{3.15}$$

valid for each $n \in \mathbb{N}$, where $a_i^{[n]}$, $i \in \{1, \cdots, r\}$, and $b_j^{[n]}$, $j \in \{1, \cdots, s\}$, can be determined by induction on $n \in \mathbb{N}$ setting $a_i^{[0]} = a_i$, $b_j^{[0]} = b_j$ and

$$x\,\Phi\left(x \,\middle|\, \begin{matrix} a_1^{[n]}, \cdots, a_r^{[n]} \\ b_1^{[n]}, \cdots, b_s^{[n]} \end{matrix}\right)\overline{\mathcal{W}}\left(x \,\middle|\, \begin{matrix} a_1^{[n]}, \cdots, a_r^{[n]} \\ b_1^{[n]}, \cdots, b_s^{[n]} \end{matrix}\right) = \overline{\mathcal{W}}\left(x \,\middle|\, \begin{matrix} a_1^{[n+1]}, \cdots, a_r^{[n+1]} \\ b_1^{[n+1]}, \cdots, b_s^{[n+1]} \end{matrix}\right).$$

Moreover, as a consequence of (3.13), the type I multiple orthogonal polynomials on the step-line with respect to $\overline{\mathcal{W}}(x)$ and $x\,\Phi(x)\overline{\mathcal{W}}(x)$, respectively denoted by $\big(A_{n,0}(x), \cdots, A_{n,r-1}(x)\big)$ and $\big(C_{n,0}(x), \cdots, C_{n,r-1}(x)\big)$, are related, for $n \in \mathbb{N}$, by

$$n\,[A_{n+1,j}(x)]_{j=0}^{r-1} = \left(\Psi(x)^T - x\,\Phi(x)^T\frac{\mathrm{d}}{\mathrm{d}x}\right)[C_{n,j}(x)]_{j=0}^{r-1}, \tag{3.16}$$

where Φ^T and Ψ^T are the transpose of the matrices in (3.14).

In addition, recalling that $\big(A_{1,0}(x), \cdots, A_{1,r-1}(x)\big) = \big(1, 0 \cdots, 0\big)$, we obtain a matrix Rodrigues-type formula generating the type I polynomials

$$\left[A_{n+1,j}\left(x \,\middle|\, \begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}\right)\right]_{j=0}^{r-1} = \frac{1}{n!}\left(\prod_{k=0}^{n-1}\mathcal{O}\left(\begin{matrix} a_1^{[k]}, \cdots, a_r^{[k]} \\ b_1^{[k]}, \cdots, b_s^{[k]} \end{matrix}\right)\right)\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{3.17}$$

for each $n \in \mathbb{N}$, where $a_i^{[k]}$, $i \in \{1, \cdots, r\}$, and $b_j^{[k]}$, $j \in \{1, \cdots, s\}$, are the same parameters appearing in (3.15) and the raising operator $\mathcal{O}$ is defined by

$$\mathcal{O}\begin{pmatrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{pmatrix} = \Psi\left(x \,\middle|\, \begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}\right)^T - x\,\Phi\left(x \,\middle|\, \begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}\right)^T \frac{\mathrm{d}}{\mathrm{d}x}.$$

A detailed illustration of these claims for particular cases with 2 orthogonality measures is discussed in Sections 4.3 and 5.3.

3.3 Characterisation of the type II MOPs

In this section, we focus on the type II multiple orthogonal polynomials on the step-line, or r-orthogonal polynomials, with respect to vectors of measures $(\mu_0, \cdots, \mu_{r-1})$ such that μ_0 satisfies (3.1) and each μ_j, with $j \in \{1, \cdots, r-1\}$, is obtained from μ_0 by a certain shift in the parameters a_i and b_j in (3.1), which we specify for each particular case we investigate.

3.3.1 Explicit expressions as hypergeometric polynomials

The r-orthogonal polynomials under analysis here are hypergeometric polynomials, as they can be written as terminating hypergeometric series of the form

$$P_n(x) = \frac{(-1)^n \prod_{k=1}^{q} (\beta_k)_n}{\prod_{l=1}^{p} (\alpha_l)_n} \, {}_{p+1}F_q\left(\begin{matrix} -n, \alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_q \end{matrix} \,\middle|\, x\right), \quad n \in \mathbb{N}, \tag{3.18a}$$

with $p = s$, $q = r$, $\beta_i = a_i$, for $1 \leq i \leq r$, and $\alpha_j = b_j + f_j(n)$, for $1 \leq j \leq s$, where $f_j(n)$ is a non-negative function of n. By definition of the generalised hypergeometric series (1.8), this representation is equivalent to

$$P_n(x) = \sum_{k=0}^{n} \tau_{n,k} x^{n-k}, \quad \tau_{n,k} = \frac{(-1)^k}{k!} (n-k+1)_k \frac{\prod_{j=1}^{q} (n-k+\beta_j)_k}{\prod_{i=1}^{p} (n-k+\alpha_i)_k}. \tag{3.18b}$$

For specific cases, as for instance the ones in Chapters 4 and 5, we show that $(P_n(x))_{n\in\mathbb{N}}$ defined by (3.18a) satisfy the r-orthogonality conditions (1.37) with respect to $(\mu_0, \cdots, \mu_{r-1})$, as a consequence of the following Lemmas 3.3 and 3.4.

Lemma 3.3. *For $n, p, q, r, s \in \mathbb{N}$ and $\{a_i\}_{i=1}^{r}, \{b_j\}_{j=1}^{s}, \{\alpha_l\}_{l=1}^{p}, \{\beta_k\}_{k=1}^{q} \subset \mathbb{C}\backslash\mathbb{Z}_0^-$, let μ be a measure with moments given by (3.1) and $(P_n(x))_{n\in\mathbb{N}}$ a polynomial sequence defined by (3.18a). Then, for any $m, n \in \mathbb{N}$,*

$$\int x^m P_n(x) \mathrm{d}\mu(x) = \frac{(-1)^n \prod_{k=1}^{q} (\beta_k)_n \prod_{i=1}^{r} (a_i)_m}{\prod_{l=1}^{p} (\alpha_l)_n \prod_{j=1}^{s} (b_j)_m} \, {}_{p+r+1}F_{q+s} \left(\left. \begin{matrix} -n, \alpha_1, \cdots, \alpha_p, a_1+m, \cdots, a_r+m \\ \beta_1, \cdots, \beta_q, b_1+m, \cdots, b_s+m \end{matrix} \right| 1 \right). \tag{3.19}$$

Proof. Recalling (1.8), we have

$$P_n(x) = \frac{(-1)^n \prod_{k=1}^{q} (\beta_k)_n}{\prod_{l=1}^{p} (\alpha_l)_n} \sum_{\lambda=0}^{n} \left(\frac{\prod_{l=1}^{p} (\alpha_l)_\lambda}{\lambda! \prod_{k=1}^{q} (\beta_k)_\lambda} x^\lambda \right).$$

As a result,

$$\int x^m P_n(x) \mathrm{d}\mu(x) = \frac{(-1)^n \prod_{k=1}^{q} (\beta_k)_n}{\prod_{l=1}^{p} (\alpha_l)_n} \sum_{\lambda=0}^{n} \left(\frac{\prod_{l=1}^{p} (\alpha_l)_\lambda}{\lambda! \prod_{k=1}^{q} (\beta_k)_\lambda} \int x^{m+\lambda} \mathrm{d}\mu(x) \right).$$

Recalling (3.1), the latter implies that

$$\int x^m P_n(x)\mathrm{d}\mu(x) = \frac{(-1)^n \prod_{k=1}^{q}(\beta_k)_n}{\prod_{l=1}^{p}(\alpha_l)_n} \sum_{\lambda=0}^{n} \left(\frac{\prod_{l=1}^{p}(\alpha_l)_\lambda \prod_{i=1}^{r}(a_i)_{m+\lambda}}{\lambda! \prod_{k=1}^{q}(\beta_k)_\lambda \prod_{j=1}^{s}(b_j)_{m+\lambda}} \right),$$

which is equivalent to

$$\int x^m P_n(x)\mathrm{d}\mu(x) = \frac{(-1)^n \prod_{k=1}^{q}(\beta_k)_n \prod_{i=1}^{r}(a_i)_m}{\prod_{l=1}^{p}(\alpha_l)_n \prod_{j=1}^{s}(b_j)_m} \sum_{\lambda=0}^{n} \left(\frac{\prod_{l=1}^{p}(\alpha_l)_\lambda \prod_{i=1}^{r}(a_i+m)_\lambda}{\lambda! \prod_{k=1}^{q}(\beta_k)_\lambda \prod_{j=1}^{s}(b_j+m)_\lambda} \right).$$

Therefore, by definition of the generalised hypergeometric series, we derive (3.19).

□

Lemma 3.4. *[47, Lemma 3.2] Let $n, p \in \mathbb{N}$, $\{m_i\}_{i=1}^{p} \subset \mathbb{N}$ and $m := \sum_{i=1}^{p} m_i$ such that $m \leq n$ and let $\beta, \{f_i\}_{i=1}^{p}$ be complex numbers with positive real part. Then*

$$ {}_{p+1}F_p\left(\left. \begin{matrix} -n, f_1+m_1, \cdots, f_p+m_p \\ f_1, \cdots, f_p \end{matrix} \right| 1 \right) = \begin{cases} 0 & \text{if } m < n, \\ \dfrac{(-1)^n n!}{\prod_{i=1}^{p}(f_i)_{m_i}} & \text{if } m = n. \end{cases} \tag{3.20}$$

and

$$ {}_{p+2}F_{p+1}\left(\left. \begin{matrix} -n, \beta, f_1+m_1, \cdots, f_p+m_p \\ \beta+1, f_1, \cdots, f_p \end{matrix} \right| 1 \right) = \frac{n! \prod_{i=1}^{p}(f_i-\beta)_{m_i}}{(\beta+1)_n \prod_{i=1}^{p}(f_i)_{m_i}}. \tag{3.21}$$

Proof. Formula (3.21) was deduced by Minton in [57] (see also [40, Eq. 1.2]) and (3.20) can be obtained by taking the limit $\beta \to +\infty$ in (3.21). □

The representation for the r-orthogonal polynomials as terminating generalised hypergeometric series can be used to derive differential equations and recurrence relations of which they are solutions, as explained in Sections 3.3.2 and 3.3.3, respectively. We can also obtain limiting relations connecting the multiple orthogonal polynomials investigated here from their explicit representations, using the confluent relation for the generalised hypergeometric series (1.11).

3.3.2 Differential equation

Recalling the generalised hypergeometric differential equation (1.10), the polynomial defined by (3.18a) satisfies the ordinary differential equation of order equal to $\max\{p+1, q+1\}$

$$\left[\prod_{i=1}^{q}\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}+\beta_i\right)\right]\frac{\mathrm{d}}{\mathrm{d}x}\left(P_n(x)\right)=\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}-n\right)\left[\prod_{j=1}^{p}\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}+\alpha_j\right)\right]\left(P_n(x)\right). \tag{3.22}$$

Expanding both sides of (3.22), we deduce that the left-hand side is equal to

$$\sum_{k=0}^{q}\left(\eta^{[k]}x^k\,\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(P_n(x)\right)\right),$$

while the right-hand side is equal to

$$-n\zeta P_n(x)+\sum_{k=0}^{p}\left(\xi_n^{[k]}x^{k+1}\,\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(P_n(x)\right)\right),$$

where ζ, $\eta^{[k]}$ and $\xi_n^{[k]}$ are constants with $\zeta=\prod_{j=1}^{p}\alpha_j$ and $\eta^{[q]}=1=\xi_n^{[p]}$.

Therefore, (3.22) can be rewritten in the form

$$n\zeta P_n(x)+\sum_{k=0}^{\max\{p,q\}}\left(\left(\eta^{[k]}-\xi_n^{[k]}x\right)x^k\,\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(P_n(x)\right)\right)=0, \tag{3.23}$$

where $\eta^{[k]}=0$, if $p<k<q$, or $\xi_n^{[k]}=0$, if $q<k<p$.

In particular, when we consider $p=s$, $q=r$ and $s\leq r$, the order of the differential equation is $r+1$ with the coefficient of the highest order derivative equal to x^r if $s<r$ or to $x^r(1-x)$ if $s=r$.

This differential equation can be seen as a generalisation of the second-order differential equation (1.26) satisfied by the classical orthogonal polynomials.

3.3.3 Recurrence relation

The r-orthogonal polynomial sequences of the type (3.18a) naturally satisfy a $(r+1)$-order recurrence relation of the type in (1.38), that is,

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \sum_{j=1}^{r} \gamma_{n-j+1}^{[j]} P_{n-j}(x). \tag{3.24}$$

To obtain expressions for the recurrence coefficients in (3.24), we replace the polynomials $P_{n+1-k}(x)$, $k \in \{0,\cdots,r+1\}$, appearing in that recurrence relation by their corresponding expansion over the powers of x given by (3.18b), using the same notation $\tau_{n,k}$, $n \geq k$, for its coefficients. The linear independence of $(x^n)_{n\in\mathbb{N}}$ implies that we can equate the coefficients on both sides of the recurrence relation. After equating the coefficients of x^n, for $n \in \mathbb{N}$, we obtain

$$\beta_n = \tau_{n,1} - \tau_{n+1,1}.$$

In addition, a comparison of the coefficients of x^{n-k}, with $k \in \{1,\cdots,r\}$ and $n \geq k$, in (3.24) implies that

$$\gamma_{n-k+1}^{[k]} = \tau_{n,k+1} - \tau_{n+1,k+1} - \tau_{n,1}\tau_{n,k} + \tau_{n+1,1}\tau_{n,k} - \sum_{j=1}^{k-1} \tau_{n-j,k-j}\gamma_{n+1-j}^{[j]}.$$

However, for $k=r$, it is easier, from the computational point of view, to derive the expressions for $\gamma_{n+1}^{[r]}$, $n \in \mathbb{N}$, directly from the r-orthogonality conditions. To be precise, integrating both sides of (3.24) with $n = r(m+1)+j$, where $m \in \mathbb{N}$ and $j \in \{0,\cdots,r-1\}$, multiplied by x^m with respect to the measure μ_j, we obtain

$$\gamma_{rm+j+1}^{[r]} = \frac{\int x^{m+1} P_{r(m+1)+j}(x)\mathrm{d}\mu_j(x)}{\int x^m P_{rm+j}(x)\mathrm{d}\mu_j(x)}.$$

In particular, when the polynomials defined by (3.18a)-(3.18b) are 2-orthogonal with respect to a pair of measures (μ_0, μ_1), the sequence $(P_n(x))_{n\in\mathbb{N}}$ satisfies a third order recurrence relation of the type in (1.39), that is,

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1}P_{n-2}(x). \tag{3.25}$$

with coefficients

$$\beta_n = \tau_{n,1} - \tau_{n+1,1}; \tag{3.26a}$$

$$\alpha_n = \tau_{n,2} - \tau_{n+1,2} - \tau_{n,1}^2 + \tau_{n,1}\tau_{n+1,1}; \tag{3.26b}$$

$$\gamma_{2n+1} = \frac{\int x^{n+1}P_{2n+2}(x)\mathrm{d}\mu_0(x)}{\int x^n P_{2n}(x)\mathrm{d}\mu_0(x)} \quad \text{and} \quad \gamma_{2n+2} = \frac{\int x^{n+1}P_{2n+3}(x)\mathrm{d}\mu_1(x)}{\int x^n P_{2n+1}(x)\mathrm{d}\mu_1(x)}. \tag{3.26c}$$

3.3.4 Asymptotic behaviour and location of the zeros

For a (standard) orthogonal polynomial sequence with respect to a positive measure on the real line, it is known that all the zeros are simple and located in the support of the orthogonality measure, and that the zeros of consecutive polynomials interlace. These results about the location of the zeros do not necessarily hold for a r-orthogonal polynomial sequence. However, we show that they hold for all the cases we analyse in detail here, using two alternative arguments.

Firstly, we prove that, for each of these cases, the orthogonality measures form a Nikishin system. Therefore, as explained in Section 1.4, the zeros of all type II polynomials are simple and located in the support of the orthogonality measures, which here is always equal to or a subset of the positive real line, and that the zeros of nearest neighbour polynomials interlace. Alternatively, we check that, the $(r + 1)$-banded lower Hessenberg matrix H_n such that the recurrence relation satisfied by the r-orthogonal polynomials $(P_n(x))_{n\in\mathbb{N}}$ can be expressed as in (1.41) is an oscillation matrix. As a result, the zeros of $P_n(x)$, which are the eigenvalues of H_n as observed in Subsection 1.4.3, are all simple, real and positive and the zeros of consecutive polynomials interlace, as expounded in Section 3.4.

We also find an upper bound for the largest zeros of the r-orthogonal polynomials, which can be derived from the asymptotic behaviour of their recurrence coefficients, as described in the following theorem.

Theorem 3.5. *For $r \in \mathbb{Z}^+$, let $(P_n(x))_{n\in\mathbb{N}}$ be a r-orthogonal polynomial sequence satisfying (3.24), where, for any $n \in \mathbb{N}$, $\beta_n \in \mathbb{R}$, $\gamma_{n+1}^{[k]} \in \mathbb{R}$, for each $1 \leq k \leq r$, with $\gamma_{n+1}^{[r]} > 0$, and there exist constants $\beta \in \mathbb{R}_0^+$, $\gamma^{[k]} \in \mathbb{R}_0^+$, if $1 \leq k \leq r-1$, and $\gamma^{[r]} \in \mathbb{R}^+$ as well as a non-decreasing positive sequence $(M_n)_{n\in\mathbb{Z}^+}$ such that, for any $n \in \mathbb{Z}^+$:*

- $|\beta_{n-1}| \leq \beta M_n + o(M_n)$ *and* (3.27a)
- $\left|\gamma_n^{[k]}\right| \leq \gamma^{[k]} M_n^{k+1} + o\left(M_n^{k+1}\right)$, *for each $1 \leq k \leq r$.* (3.27b)

Then, if we denote by $x_n^{(n)}$, $n \in \mathbb{Z}^+$, the largest zero in absolute value of $P_n(x)$,

$$\left|x_n^{(n)}\right| \leq \min_{s\in\mathbb{R}^+} \left(s + \beta + \sum_{k=1}^{r} \frac{\gamma^{[k]}}{s^k} \right) M_n + o(M_n), \quad \text{as } n \to +\infty. \tag{3.28}$$

The particular case $r = 2$ of the latter theorem, which is presented in this text as Corollary 3.6, corresponds to [47, Th. 3.5] and the proof presented here for Theorem 3.5 is a generalisation of the proof therein.

Proof. Let $n \in \mathbb{Z}^+$ and H_n be the $(r+1)$-banded lower Hessenberg matrix such that the recurrence relation satisfied by $(P_n(x))_{n\in\mathbb{N}}$ can be expressed as in (1.41). Then, the zeroes of P_n are the eigenvalue of H_n. As a result, if $x_n^{(n)}$ is the largest zero in absolute value of $P_n(x)$, $\left|x_n^{(n)}\right|$ is equal to the spectral radius of H_n, which is, by definition, the maximum of the absolute values of the eigenvalues of H_n.

Therefore, based on [35, Cor. 6.1.8], we have

$$\left|x_n^{(n)}\right| \leq \min_{s_0,\cdots,s_{n-1}\in\mathbb{R}^+} \left(\max_{i\in\{0,\cdots,n-1\}} \left\{ \sum_{j=0}^{n-1} \left(\frac{s_j}{s_i} \left|(\mathrm{H}_n)_{i,j}\right| \right) \right\} \right),$$

which, recalling the values of the entries of H_n, implies that

$$\left|x_n^{(n)}\right| \leq \min_{s_0,\cdots,s_{n-1}\in\mathbb{R}^+}\left(\max_{i\in\{0,\cdots,n-1\}}\left\{\frac{s_{i+1}}{s_i}+|\beta_i|+\sum_{k=1}^{r}\left(\left|\gamma_{i-k+1}^{[k]}\right|\frac{s_{i-k}}{s_i}\right)\right\}\right),$$

with the convention $s_l = 0$ if $l = n$ or $l \in \mathbb{Z}^-$. In particular we can set, for any $s \in \mathbb{R}^+$,

$$s_j = \prod_{l=1}^{j}(sM_l) = s^j\prod_{l=1}^{j}M_l > 0, \quad j \in \{0,\cdots,n-1\},$$

to obtain

$$\left|x_n^{(n)}\right| \leq \min_{s\in\mathbb{R}^+}\left(\max_{i\in\{0,\cdots,n-1\}}\left\{sM_{i+1}+|\beta_i|+\sum_{k=1}^{r}\left(\left|\gamma_{i-k+1}^{[k]}\right|s^{-k}\prod_{l=0}^{k-1}M_{i-l}^{-1}\right)\right\}\right).$$

Furthermore, recalling (3.27a)-(3.27b), we derive that

$$\left|x_n^{(n)}\right| \leq \min_{s\in\mathbb{R}^+}\left(\max_{i\in\{0,\cdots,n-1\}}\left\{sM_{i+1}+\beta M_{i+1}+\sum_{k=1}^{r}\left(\frac{\gamma^{[k]}M_{i-k+1}^{k+1}}{s^k\prod_{l=0}^{k-1}M_{i-l}}\right)+o\left(M_{i+1}\right)\right\}\right).$$

Therefore, due to the sequence $(M_n)_{n\in\mathbb{Z}^+}$ being non-decreasing,

$$\left|x_n^{(n)}\right| \leq \min_{s\in\mathbb{R}^+}\left(\max_{i\in\{0,\cdots,n-1\}}\left\{\left(s+\beta+\sum_{k=1}^{r}\frac{\gamma^{[k]}}{s^k}\right)M_{i+1}+o\left(M_{i+1}\right)\right\}\right),$$

and (3.28) holds. □

Setting $r = 2$ in Theorem 3.5, we derive the following result.

Corollary 3.6. *[47, Th. 3.5] Let $(P_n(x))_{n\in\mathbb{N}}$ be a 2-orthogonal polynomial sequence satisfying (3.25), where, for any $n \in \mathbb{N}$, $\beta_n, \alpha_{n+1} \in \mathbb{R}$, $\gamma_{n+1} \in \mathbb{R}^+$ and there exist constants $\alpha, \beta \in \mathbb{R}_0^+$ and $\gamma \in \mathbb{R}^+$, with $\Delta := \gamma^2 - \dfrac{\alpha^3}{27} \geq 0$, and a non-decreasing positive sequence $(M_n)_{n\in\mathbb{Z}^+}$ such that, for any $n \in \mathbb{Z}^+$:*

- $|\beta_{n-1}| \leq \beta M_n + o\left(M_n\right)$; (3.29a)
- $|\alpha_n| \leq \alpha M_n^2 + o\left(M_n^2\right)$; (3.29b)
- $|\gamma_n| \leq \gamma M_n^3 + o\left(M_n^3\right)$. (3.29c)

Then, if we denote by $x_n^{(n)}$, $n \in \mathbb{Z}^+$, the largest zero in absolute value of $P_n(x)$,

$$\left|x_n^{(n)}\right| \leq \left(\frac{3}{2}\tau + \beta + \frac{\alpha}{2\tau}\right) M_n + o\left(M_n\right), \quad \textit{as } n \to +\infty, \tag{3.30}$$

with $\tau = \sqrt[3]{\gamma + \sqrt{\Delta}} + \sqrt[3]{\gamma - \sqrt{\Delta}}$.

Observe that Theorem 2.4 is a particular case of Corollary 3.6, obtained when we set $M_n = n^\lambda$, with $\lambda \geq 0$, and $\alpha = \beta = 0$.

Proof. Considering the case $r = 2$ of Theorem 3.5, we have

$$\left|x_n^{(n)}\right| \leq \min_{s \in \mathbb{R}^+} \left(s + \beta + \frac{\alpha}{s} + \frac{\gamma}{s^2}\right) M_n + o\left(M_n\right), \quad \text{as } n \to +\infty.$$

Therefore, to find the sharpest upper bound for $\left|x_n^{(n)}\right|$ given by this formula we need to find the minimum value on $\mathbb{R}^+$ of $f(s) = s + \beta + \frac{\alpha}{s} + \frac{\gamma}{s^2}$. With that purpose, we look for the roots of $f'(s) = 1 - \frac{\alpha}{s^2} - \frac{2\gamma}{s^3} = \frac{1}{s^3}\left(s^3 - \alpha s - 2\gamma\right)$. Due to the condition $\Delta > 0$, we know that f' has one real root and two complex roots. The real root is $\tau = \sqrt[3]{\gamma + \sqrt{\Delta}} + \sqrt[3]{\gamma - \sqrt{\Delta}} > 0$, where we are taking real and positive square and cubic roots. Furthermore, $f''(s) = \frac{2\alpha}{s^3} + \frac{6\gamma}{s^4} > 0$, for any $s \in \mathbb{R}^+$, so $f''(\tau) > 0$ and the choice $s = \tau$ gives a minimum value to $f(s)$. Finally, $f'(\tau) = 0$ implies that $\frac{2\gamma}{\tau^3} = 1 - \frac{\alpha}{\tau^2}$. As a result, $f(\tau) = \frac{3}{2}\tau + \beta + \frac{\alpha}{2\tau}$, which implies (3.30). □

Note that if we remove the terms $o\left(M_n^j\right)$ from the upper bounds for the recurrence relation coefficients in Theorem 3.5 and Corollary 3.6, then we can also remove the term $o(M_n)$ from the upper bound for the largest zeros of the polynomials.

Taking $r = 1$ in Theorem 3.5, we derive an upper bound for the zeros of orthogonal polynomials. More precisely, let $(P_n(x))_{n \in \mathbb{N}}$ be an orthogonal polynomial sequence satisfying the recurrence relation

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \tag{3.31}$$

where, for each $n \in \mathbb{N}$, $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} \in \mathbb{R}^+$, and there exist constants $\beta \geq 0$ and $\gamma > 0$ and a non-decreasing positive sequence $(M_n)_{n\in\mathbb{Z}^+}$ such that

$$|\beta_n| \leq \beta M_n + o\left(M_n\right) \quad \text{and} \quad |\gamma_{n+1}| \leq \gamma M_n^2 + o\left(M_n^2\right). \tag{3.32}$$

Then, if we denote by $x_n^{(n)}$ the largest zero in absolute value of $P_n(x)$, $n \in \mathbb{Z}^+$,

$$\left|x_n^{(n)}\right| \leq \left(\beta + 2\sqrt{\gamma}\right) M_n + o\left(M_n\right), \quad \text{as } n \to +\infty. \tag{3.33}$$

In fact, based on [43, Th. 1.4], it is known that the zeros of $P_n(x)$ are all located on intervals of the form $\left(\left(\beta - 2\sqrt{\gamma}\right) M_n + o\left(M_n\right), \left(\beta + 2\sqrt{\gamma}\right) M_n + o\left(M_n\right)\right)$.

To obtain more information on the zeros of the r-orthogonal polynomials investigated here, we also derive Mehler-Heine-type asymptotic formulas satisfied by them near the origin. The original Mehler-Heine formula gives the asymptotic behaviour of Legendre polynomials near the endpoint $x = 1$ of their interval of orthogonality $(-1, 1)$. This result was later generalised to obtain an asymptotic formula near $x = 1$ satisfied by the classical Jacobi polynomials on the interval $(-1, 1)$, involving the Bessel function of the first kind $J_\alpha(z)$ defined by (1.12). Similarly, the Laguerre polynomials satisfy a Mehler-Heine type asymptotic formula near the origin, also involving $J_\alpha(z)$. We present the Mehler-Heine-type formulas satisfied by Jacobi and Laguerre polynomials in Section 3.5 and the resulting information about the zeros of these classical orthogonal polynomials.

Examples of Mehler-Heine type formulas for multiple orthogonal polynomials can be found, for instance, in [67] and [72]. Like the Mehler-Heine-type formulas for the classical Jacobi and Laguerre polynomials, the Mehler-Heine formulas for multiple orthogonal polynomials also give important information about the location of their zeros (see [72, §4]). In later chapters, we obtain Mehler-Heine type formulas from the representations as terminating hypergeometric series for the r-orthogonal polynomials we analyse, applying the confluent relation for the generalised hypergeometric series (1.11) to polynomials of the form (3.18a). Therefore, the limits obtained are of the type ${}_0F_r\left(-; a_1, \cdots, a_r \,|\, -z\right)$, with r equal to the number of orthogonality measures, similarly to the results in [72] for the Jacobi-Piñeiro polynomials and for the multiple orthogonal polynomials with respect to the Macdonald function (for $r = 2$) and to Meijer G-functions (when $r > 2$).

When investigating the zeros of r-orthogonal polynomials, we are also interested in finding their *asymptotic zero distribution* ν, by studying the limit for the normalised zero counting measure of $P_n(x)$

$$\nu(P_n) := \frac{1}{n} \sum_{P_n(x)=0} \delta_x,$$

where δ_x is the Dirac point mass at x. If the limit exists, then we say it converges to a measure ν in the sense of the weak convergence of measures, that is,

$$\int f \mathrm{d}\nu = \lim_{n\to\infty} \int f \mathrm{d}\nu(P_n) = \lim_{n\to\infty} \left(\frac{1}{n} \sum_{P_n(x)=0} f(x) \right).$$

for all bounded and continuous functions f on I, the smallest interval containing all the zeros of $(P_n(x))_{n\in\mathbb{N}}$. Fundamental in this analysis is the uniform convergence of the ratio of two consecutive polynomials for all compact subsets in $\mathbb{C}\backslash\mathrm{I}$. To be precise, if there exists a function ρ defined in $\mathbb{C}\backslash\mathrm{I}$ such that

$$\rho(z) = \lim_{n\to\infty} \frac{P_n(z)}{P_{n+1}(z)}, \tag{3.34}$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathrm{I}$, and the zeros of $(P_n(x))_{n\in\mathbb{N}}$ are all simple with the zeros of consecutive polynomials interlacing, then, as explained in [17],

$$\lim_{n\to\infty} \frac{P_n'(z)}{nP_n(z)} = -\frac{\rho'(z)}{\rho(z)} \quad \text{and} \quad \frac{P_n'(x)}{nP_n(x)} = \int \frac{\mathrm{d}\nu(P_n)(t)}{x-t}.$$

Therefore, under suitable conditions, $-\dfrac{\rho'(z)}{\rho(z)}$ is the Stieltjes transform of ν. Thus, determining the ratio asymptotics (3.34) is crucial to find the asymptotic zero distribution ν, which may be obtainable via the Stieltjes inverse transform.

We apply this analysis in a particular case in Section 5.4.4.

3.4 Link to branched continued fractions

By definition of the generalised hypergeometric series, the ordinary generating function of the ratio of Pochhammer symbols appearing in (3.1) is

$$\sum_{n=0}^{\infty}\left(\frac{\prod_{i=1}^{r}(a_i)_n}{\prod_{j=1}^{s}(b_j)_n}t^n\right)={}_{r+1}F_s\left(\left.\begin{matrix}a_1,\cdots,a_r,1\\ b_1,\cdots,b_s\end{matrix}\right|t\right)=\frac{{}_{r+1}F_s\left(\left.\begin{matrix}a_1,\cdots,a_r,1\\ b_1,\cdots,b_s\end{matrix}\right|t\right)}{{}_{r+1}F_s\left(\left.\begin{matrix}a_1,\cdots,a_r,0\\ b_1,\cdots,b_s\end{matrix}\right|t\right)}. \tag{3.35}$$

In this section, we explain how branched-continued-fraction representations of the generating function (3.35) can be used to obtain information about the r-orthogonal polynomials under analysis here. Firstly, we observe that the generating function (3.35) can be expressed as a ratio of contiguous generalised hypergeometric series of the type in [61, Eq. 14.2], with $a_{r+1} = 1$. Then, we can use [61, Ths. 13.1, 14.5, 14.6] to express the moments in (3.1) as m-Stieltjes-Rogers polynomials of order n, where $m = \max\{r, s\}$, denoted by $S_n^{(m)}(\boldsymbol{\lambda})$ as in Section 1.6. Furthermore, based on [61, Cor. 14.4], a corollary of these results is that, if we impose the conditions (3.2), then the moments in (3.1) form a Stieltjes moment sequence, i.e. there exists a measure μ on the positive real line satisfying (3.1).

We want to check that, for each r-orthogonal polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ under study with a first orthogonality weight satisfying (3.1), the infinite Hessenberg matrix H associated with $(P_n(x))_{n\in\mathbb{N}}$ is equal to the production matrix P of the generalised m-Stieltjes-Rogers polynomials $S = \left(S_{n,k}^{(m)}(\boldsymbol{\lambda})\right)_{n,k\in\mathbb{N}}$. Recall that this production matrix P admits the decomposition in bidiagonal matrices given by (1.53). Therefore, if we check that H is equal to the production matrix P with $\lambda_{k+m} > 0$ for all $k \in \mathbb{N}$, then H is a totally-positive matrix, because it is the product of bidiagonal matrices with positive entries in the nonzero diagonals.

Furthermore, the finite Hessenberg $(n\times n)$-matrices $(\mathrm{H}_n)_{n\in\mathbb{Z}^+}$, obtained from H by truncation, such that $(P_n(x))_{n\in\mathbb{N}}$ satisfies (1.41) are oscillation matrices, because a

$(n \times n)$-matrix is an oscillation matrix if and only if it is totally positive, nonsingular, and all the entries lying in its subdiagonal and its superdiagonal are positive (see [63, Th. 5.2]). As a result, the zeros of $P_n(x)$, which are the eigenvalues of H_n, are simple, real, and positive, and the zeros of consecutive polynomials interlace (similar to the main result in [64, §9.2]). Another consequence of the decomposition (1.53) for the infinite Hessenberg matrix H associated with a r-orthogonal polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ is that all its recurrence coefficients are positive.

As we mostly deal with multiple orthogonal polynomials with respect to 2 measures, we now set $m = 2$ in (1.53). Then the production matrix of the 2-Stieltjes-Rogers polynomials $\left(S_n^{(2)}(\boldsymbol{\lambda})\right)_{n\in\mathbb{N}}$ for the set of indeterminates $\boldsymbol{\lambda} = (\lambda_{k+2})_{k\in\mathbb{N}}$ is

$$\begin{bmatrix} 1 & & & \\ \lambda_3 & 1 & & \\ & \lambda_6 & 1 & \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 1 & & & \\ \lambda_4 & 1 & & \\ & \lambda_7 & 1 & \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \lambda_2 & 1 & & & \\ & \lambda_5 & 1 & & \\ & & \lambda_8 & 1 & \\ & & & \ddots & \ddots \end{bmatrix}, \tag{3.36}$$

which is equal to the $(2,1)$-banded lower Hessenberg matrix

$$\begin{bmatrix} \lambda_2 & 1 & & & \\ \lambda_2\lambda_3 + \lambda_2\lambda_4 & \lambda_3 + \lambda_4 + \lambda_5 & 1 & & \\ \lambda_2\lambda_4\lambda_6 & \lambda_4\lambda_6 + \lambda_5\lambda_6 + \lambda_5\lambda_7 & \lambda_6 + \lambda_7 + \lambda_8 & 1 & \\ & \lambda_5\lambda_7\lambda_9 & \lambda_7\lambda_9 + \lambda_8\lambda_9 + \lambda_8\lambda_{10} & \lambda_9 + \lambda_{10} + \lambda_{11} & 1 \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Therefore, the production matrix of the 2-Stieltjes-Rogers polynomials $S_n^{(2)}(\boldsymbol{\lambda})$ for the set of indeterminates $\boldsymbol{\lambda} = (\lambda_{k+2})_{k\in\mathbb{N}}$, is equal to the Hessenberg matrix associated with the 2-orthogonal polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ satisfying the recurrence relation (3.25) if and only if, setting $\lambda_0 = \lambda_1 = 0$ then, for any $n \in \mathbb{N}$,

- $\beta_n = \lambda_{3n} + \lambda_{3n+1} + \lambda_{3n+2}$,
- $\alpha_{n+1} = \lambda_{3n+1}\lambda_{3n+3} + \lambda_{3n+2}\lambda_{3n+3} + \lambda_{3n+2}\lambda_{3n+4}$,
- $\gamma_{n+1} = \lambda_{3n+2}\lambda_{3n+4}\lambda_{3n+6}$.

3.5 Some classical orthogonal polynomials

We end this chapter with a brief analysis of the results obtained throughout the chapter specialised to the case of (1-)orthogonal classical polynomials. In Chapters 4 and 5 we give a detailed account of these general results applied to multiple orthogonal polynomials with respect to 2 measures.

As mentioned in the beginning of this chapter, the standard orthogonal polynomials with respect to a single measure μ satisfying (3.1), with $r = 1$ and $s \in \{0, 1\}$, are the Laguerre polynomials and the Jacobi polynomials on the interval $(0, 1)$. In this section we show that several well-known properties of these classical orthogonal polynomials can be derived using results presented in the previous sections. These properties can be found in several references including [66]. However, the polynomials therein are not monic and the orthogonality interval of the Jacobi polynomials is, as usual, $(-1, 1)$; hence, the results presented here and therein are equivalent but not equal.

3.5.1 Laguerre polynomials

For $a \in \mathbb{R}^+$, we define the gamma probability density function on the positive real line as

$$\mathcal{W}(x; a) = \frac{\mathrm{e}^{-x} x^{a-1}}{\Gamma(a)}, \tag{3.37}$$

where $\Gamma(a)$ is the Gamma function defined by (1.1).

The moments of the measure μ supported on $\mathbb{R}^+$, with $\mathrm{d}\mu(x) = \mathcal{W}(x; a)\mathrm{d}x$, are given, for $n \in \mathbb{N}$, by

$$\int_0^\infty x^n \mathcal{W}(x; a)\mathrm{d}x = \int_0^\infty \frac{\mathrm{e}^{-x} x^{a+n-1}}{\Gamma(a)}\mathrm{d}x = \frac{\Gamma(a+n)}{\Gamma(a)} = (a)_n\,. \tag{3.38}$$

Therefore, the measure μ satisfies (3.1) with $(r, s) = (1, 0)$ and $a_1 = a$.

The weight function defined by (3.37) satisfies the first-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\mathcal{W}(x;a)\right)+(x-a)\mathcal{W}(x;a)=0. \tag{3.39}$$

Note that, by Proposition 3.1, (3.38) is equivalent to all the moments of the left-hand side of (3.39) integrated over the positive real line vanishing.

Let $\left(P_n(x;a)\right)_{n\in\mathbb{N}}$ and $\left(Q_n(x;a)\right)_{n\in\mathbb{N}}$ be, respectively, the sequences of monic orthogonal polynomials and type I functions on the step-line with respect to $\mathcal{W}(x;a)$. Then, by definition of the type I function, $Q_0(x;a)=0$ and

$$Q_{n+1}(x;a)=\frac{1}{n!\,(a)_n}\,P_n(x;a)\mathcal{W}(x;a),\quad n\in\mathbb{N},$$

where the normalisation constant is derived from (1.33) and (3.41).

The differential equation (3.39) is a scalar particular case of (3.8) with

$$\overline{w}(x)=\mathcal{W}(x;a),\quad \Phi(x)=1\quad\text{and}\quad \Psi(x)=x-a=P_1(x;a).$$

Then, $\Phi(x)\mathcal{W}(x;a)=a\mathcal{W}(x;a+1)$ and, using Theorem 3.2, we have

$$P'_{n+1}(x;a)=(n+1)P_n(x;a+1)\quad\text{and}\quad Q_{n+1}(x;a)=-\frac{1}{n}\,Q'_n(x;a+1).$$

Therefore, we deduce, by induction on $n\in\mathbb{N}$, that

$$Q_{n+1}(x;a)=\frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(\mathcal{W}(x;a+n)\big)=\frac{(-1)^n}{n!\,(a)_n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(x^n\,\mathcal{W}(x;a)\big),$$

As a result, we obtain a Rodrigues-type formula generating the Laguerre polynomials:

$$P_n(x;a)=\frac{(-1)^n}{\mathcal{W}(x;a)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(x^n\,\mathcal{W}(x;a)\big)=(-1)^n\,\mathrm{e}^x\,x^{1-a}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(\mathrm{e}^{-x}\,x^{a+n-1}\big).$$

Furthermore, we can express $P_n(x;a)$ as a terminating hypergeometric series:

$$\begin{aligned} P_n(x;a) &= (-1)^n \, (a)_n \, {}_1F_1\left(-n; a \,|\, x\right) \\ &= \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \, (n-k+a)_k \, (n-k+1)_k \; x^{n-k} \right). \end{aligned} \tag{3.40}$$

To check the orthogonality conditions of $P_n(x;a)$ defined by (3.40) with respect to $\mathcal{W}(x;a)$, we use Lemma 3.3 and compute, for $n, m \in \mathbb{N}$,

$$\int_0^\infty x^m P_n(x;a)\mathcal{W}(x;a)\mathrm{d}x = (-1)^n \, (a)_n \, (a)_m \, {}_2F_1 \left(\begin{matrix} -n, a+m \\ a \end{matrix} \middle| \, 1 \right).$$

Then, applying (3.20) to the latter hypergeometric function, we deduce that

$$\int_0^\infty x^m P_n(x;a)\mathcal{W}(x;a)\mathrm{d}x = \begin{cases} 0 & \text{if } m < n, \\ n! \, (a)_n > 0 & \text{if } m = n. \end{cases} \tag{3.41}$$

Recalling the differential equation for hypergeometric polynomials (3.22), the Laguerre polynomials defined by (3.40) satisfy the differential equation

$$\left(x \, \frac{\mathrm{d}}{\mathrm{d}x} + a \right) \frac{\mathrm{d}}{\mathrm{d}x} \left(P_n(x;a) \right) = \left(x \, \frac{\mathrm{d}}{\mathrm{d}x} - n \right) \left(P_n(x;a) \right),$$

which can be expanded to find

$$x P_n''(x;a) + (a-x) P_n'(x;a) + n P_n(x;a) = 0.$$

Applying (3.26a)-(3.26c) to (3.40)-(3.41), we deduce that $(P_n(x;a))_{n\in\mathbb{N}}$ satisfies the recurrence relation (3.31) with

$$\beta_n = 2n + a \quad \text{and} \quad \gamma_{n+1} = (n+1)(n+a) \quad \text{for all } n \in \mathbb{N}. \tag{3.42}$$

These expressions for the recurrence coefficients can be rewritten as

$$\beta_n = \lambda_{2n} + \lambda_{2n+1} \quad \text{and} \quad \gamma_{n+1} = \lambda_{2n+1}\lambda_{2n+2} \quad \text{for } n \in \mathbb{N}, \tag{3.43}$$

with

$$\lambda_{2n} = n \quad \text{and} \quad \lambda_{2n+1} = n + a \quad \text{for } n \in \mathbb{N}. \tag{3.44}$$

The coefficients $(\lambda_{n+1})_{n\in\mathbb{N}}$ appear in the continued fraction representation of the generating function of the moment sequence (3.38), which is equal to ${}_2F_0\left(a, 1; - \,\middle|\, t\right)$. In fact, setting $\alpha_0 = 1$ and $\alpha_n = -\lambda_n t$ for all $n \in \mathbb{Z}^+$, we have (see [78, Eq. 92.2])

$$\sum_{n=0}^{\infty} (a)_n\, t^n = {}_2F_0\left(a, 1; - \,\middle|\, t\right) = \underset{n=0}{\overset{\infty}{\mathrm{K}}} \left(\frac{\alpha_n}{1}\right).$$

Furthermore, (3.43) gives the decomposition, involving the coefficients in (3.44),

$$\begin{bmatrix} \beta_0 & 1 & & & \\ \gamma_1 & \beta_1 & 1 & & \\ & \gamma_2 & \beta_2 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \lambda_2 & 1 & & \\ & \lambda_4 & 1 & \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & & & \\ & \lambda_3 & 1 & & \\ & & \lambda_5 & 1 & \\ & & & \ddots & \ddots \end{bmatrix}, \tag{3.45}$$

for the infinite tridiagonal matrix whose entries are determined by (1.22) and (3.42). Moreover, recalling (1.53), the matrix (3.45) is equal to the production matrix of the Stieltjes-Rogers polynomials $(S_n(\boldsymbol{\lambda}))_{n\in\mathbb{N}}$, with $S_n(\boldsymbol{\lambda}) = (a)_n$.

Recalling the confluent relation (1.11), we derive a Mehler-Heine type formula near the origin for the Laguerre polynomials defined by (3.40):

$$\lim_{n\to\infty} \left(\frac{(-1)^n}{(a)_n} P_n\left(\frac{z}{n}; a\right) \right) = \lim_{n\to\infty} \left({}_1F_1\left(-n; a \,\middle|\, \frac{z}{n}\right) \right) = {}_0F_1\left(-; a \,\middle|\, -z\right), \tag{3.46}$$

which, by definition of the Bessel function of first kind (1.12), is equivalent to

$$\lim_{n\to\infty} \left(\frac{(-1)^n}{\Gamma(a+n)} P_n\left(\frac{z}{n}; a\right) \right) = z^{\frac{1-a}{2}} \mathrm{J}_{a-1}\left(2\sqrt{z}\right). \tag{3.47}$$

The Mehler-Heine-type asymptotic formulas (3.46)-(3.47) converge uniformly in every compact region of the complex plane and they give important information about the location of the zeros of the Laguerre polynomials near the origin. In fact, if we fix $a \in \mathbb{R}^+$ and denote the zeros of $P_n(x;a)$, which are all simple and positive, by $\left(x_k^{(n)}\right)_{1\le k\le n}$ and the infinite positive zeros of $\mathrm{J}_{a-1}(z)$, which are also all simple, by $(j_k)_{k\in\mathbb{Z}^+}$, with the zeros written in increasing order for both cases, then we have $\lim_{n\to\infty} x_k^{(n)} = j_k$, for any fixed $k \in \mathbb{Z}^+$.

Moreover, note that the recurrence coefficients (3.42) satisfy the upper bound in (3.32) with $M_n = n$, $\beta = 2$ and $\gamma = 1$. As a result, using (3.33), we derive an upper bound for the largest zero of $P_n(x;a)$: $x_n^{(n)} \le 4n + o(n)$, as $n \to +\infty$.

3.5.2 Jacobi polynomials

For $a, b \in \mathbb{R}^+$ such that $a < b$, we define the beta probability density function on the interval $(0,1)$ as

$$\mathcal{W}(x;a;b) = \frac{x^{a-1}(1-x)^{b-a-1}}{B(a,b-a)}, \tag{3.48}$$

where $B(\alpha,\beta)$ is the Beta function defined by (1.4). It is straightforward to deduce from the definition that $\mathcal{W}(1-x;a;b) = \mathcal{W}(x;b-a;b)$.

The moments of the measure μ supported on the interval $(0,1)$, with $\mathrm{d}\mu(x) = \mathcal{W}(x;a;b)\mathrm{d}x$, are given, for $n \in \mathbb{N}$, by

$$\int_0^1 x^n \mathcal{W}(x;a;b)\mathrm{d}x = \int_0^1 \frac{x^{a+n-1}(1-x)^{b-a-1}}{B(a,b-a)}\mathrm{d}x = \frac{B(a+n,b-a)}{B(a,b-a)} = \frac{(a)_n}{(b)_n}. \tag{3.49}$$

Therefore, the measure μ satisfies (3.1) with $(r,s) = (1,1)$ and $(a_1;b_1) = (a;b)$.

The weight function defined by (3.48) satisfies the first-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x(1-x)\mathcal{W}(x;a;b)\big) + (bx-a)\mathcal{W}(x;a;b) = 0. \tag{3.50}$$

Note that, by Proposition 3.1, (3.49) is equivalent to all the moments of the left-hand side of (3.50) integrated over the interval $(0,1)$ vanishing.

Let $(P_n(x;a;b))_{n\in\mathbb{N}}$ and $(Q_n(x;a;b))_{n\in\mathbb{N}}$ be, respectively, the sequences of monic orthogonal polynomials and type I functions on the step-line with respect to $\mathcal{W}(x;a;b)$. Then, by definition of the type I function, $Q_0(x;a;b)=0$ and

$$Q_{n+1}(x;a;b)=\frac{(b)_{2n}\,(b+n-1)_n}{n!\,(a)_n\,(b-a)_n}\,P_n(x;a;b)\mathcal{W}(x;a;b),\quad n\in\mathbb{N},$$

where the normalisation constant is derived from (1.33) and (3.52).

The differential equation (3.50) is a scalar particular case of (3.8) with

$$\overline{w}(x)=\mathcal{W}(x;a;b),\quad \Phi(x)=1-x\quad\text{and}\quad \Psi(x)=bx-a=b\,P_1(x;a;b).$$

Then, $\Phi(x)\mathcal{W}(x;a)=\dfrac{a(b-a)}{b(b+1)}\,\mathcal{W}(x;a+1;b+2)$ and, using Theorem 3.2, we have

$$P'_{n+1}(x;a;b)=(n+1)P_n(x;a+1;b+2)\ \text{ and }\ Q_{n+1}(x;a;b)=-\frac{1}{n}\,Q'_n(x;a+1;b+2).$$

Therefore, we deduce, by induction on $n\in\mathbb{N}$, that

$$\begin{aligned}Q_{n+1}(x;a;b)&=\frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(\mathcal{W}(x;a+n;b+2n)\big)\\&=\frac{(-1)^n\,(b)_{2n}}{n!\,(a)_n\,(b-a)_n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(x^n(1-x)^n\,\mathcal{W}(x;a,b)\big).\end{aligned}$$

As a result, we obtain a Rodrigues-type formula generating the Jacobi polynomials on the interval $(0,1)$:

$$P_n(x;a;b)=\frac{(-1)^n}{(b+n-1)_n}\,x^{1-a}(1-x)^{1+a-b}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(x^{a+n-1}(1-x)^{b-a+n-1}\big).$$

Furthermore, we can express $P_n(x;a;b)$ as a terminating hypergeometric series:

$$\begin{aligned}P_n(x;a;b)&=\frac{(-1)^n\,(a)_n}{(b+n-1)_n}\,{}_2F_1\left(\begin{matrix}-n,b+n-1\\a\end{matrix}\,\middle|\,x\right)\\&=\sum_{k=0}^{n}\left(\frac{(-1)^k\,(n-k+1)_k\,(n-k+a)_k}{k!\,(2n-1-k+b)_k}\,x^{n-k}\right).\end{aligned}\tag{3.51}$$

In fact, we can check the orthogonality conditions of $(P_n(x;a;b))_{n\in\mathbb{N}}$ with respect to $\mathcal{W}(x;a;b)$, using Lemma 3.3 to compute, for any $n,m\in\mathbb{N}$,

$$\int_0^1 x^m P_n(x;a;b)\mathcal{W}(x;a;b)\mathrm{d}x = \frac{(-1)^n (a)_n (a)_m}{(b+n-1)_n (b)_m}\, {}_3F_2\left(\left.\begin{matrix}-n, a+m, b+n-1\\ a, b+m\end{matrix}\,\right|\, 1\right).$$

Recalling (3.20) and (3.21), we have

$${}_3F_2\left(\left.\begin{matrix}-n, a+m, b+n-1\\ a, b+m\end{matrix}\,\right|\, 1\right) = \begin{cases} 0 & \text{if } m\leq n-1,\\ \dfrac{(-1)^n n!\,(b-a)_n}{(a)_n (b+n)_n} & \text{if } m=n.\end{cases}$$

Therefore,

$$\int_0^1 x^m P_n(x;a;b)\mathcal{W}(x;a;b)\mathrm{d}x = \begin{cases} 0 & \text{if } m\leq n-1,\\ \dfrac{n!\,(a)_n (b-a)_n}{(b)_{2n} (b+n-1)_n} > 0 & \text{if } m=n.\end{cases} \tag{3.52}$$

Recalling the differential equation for hypergeometric polynomials (3.22), the Jacobi polynomials defined by (3.51) satisfy the differential equation

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}+a\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(P_n(x;a;b)\right) = \left(x\frac{\mathrm{d}}{\mathrm{d}x}-n\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}+(b+n-1)\right)\left(P_n(x;a;b)\right),$$

which can be expanded to find

$$x(1-x)P_n''(x;a;b)+(a-bx)P_n'(x;a;b)+n(b+n-1)P_n(x;a;b)=0.$$

Applying (3.26a)-(3.26c) to (3.51)-(3.52), we deduce that $(P_n(x;a;b))_{n\in\mathbb{N}}$ satisfies the recurrence relation (3.31) with

$$\beta_n = \frac{2n(n+b-1)+a(b-2)}{(2n+b-2)(2n+b)} \quad\text{and}\quad \gamma_{n+1} = \frac{(n+1)(n+a)(n+b-a)(n+b-1)}{(2n+b-1)(2n+b)^2(2n+b+1)} \tag{3.53}$$

for all $n\in\mathbb{N}$.

These expressions for the recurrence coefficients can be rewritten as in (3.43) with

$$\lambda_{2n} = \frac{n(n+b-a-1)}{(2n+b-2)(2n+b-1)} \quad \text{and} \quad \lambda_{2n+1} = \frac{(n+a)(n+b-1)}{(2n+b-1)(2n+b)} \quad \text{for } n \in \mathbb{N}. \tag{3.54}$$

The coefficients $(\lambda_{n+1})_{n\in\mathbb{N}}$ appear in the continued fraction representation of the generating function of the moment sequence (3.49), which is equal to ${}_2F_1\left(a,1;b\,|\,t\right)$, To be precise, setting $\alpha_0 = 1$ and $\alpha_n = -\lambda_n t$, for all $n \in \mathbb{Z}^+$, we have, based on a degenerate case of Gauss' continued fraction (see [78, Eq. 89.16]),

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} t^n = {}_2F_1\left(a,1;b\,|\,t\right) = \mathop{\mathrm{K}}_{n=0}^{\infty} \left(\frac{\alpha_n}{1}\right).$$

Furthermore, analogously to what happened for the Laguerre polynomials, the infinite tridiagonal matrix whose entries are determined by (1.22) and the recurrence coefficients (3.53) have a decomposition of the form (3.45), involving the coefficients in (3.54), which is equal to the production matrix of the Stieltjes-Rogers polynomials $(S_n(\boldsymbol{\lambda}))_{n\in\mathbb{N}}$, with $S_n(\boldsymbol{\lambda}) = \dfrac{(a)_n}{(b)_n}$.

Recalling the confluent relation (1.11), we have

$$\lim_{b\to\infty} \left({}_2F_1\left(-n, b+n-1; a \,\middle|\, \frac{x}{b}\right)\right) = {}_1F_1\left(-n; a\,|\,x\right).$$

Therefore, we derive the following limiting relation between the Jacobi and Laguerre polynomials defined by (3.51) and (3.40), equivalent to [66, Eq. 5.3.4]:

$$\lim_{b\to\infty} \left(b^n P_n\left(\frac{x}{b}; a; b\right)\right) = P_n(x;a).$$

Applying again (1.11) to (3.51), and now taking $n \to \infty$ instead of $b \to \infty$, we obtain a Mehler-Heine type formula near the origin for the Jacobi polynomials on the interval $(0,1)$:

$$\lim_{n\to\infty} \left(\frac{(-1)^n (b+n-1)_n}{(a)_n} P_n\left(\frac{z}{n^2}; a; b\right)\right) = {}_0F_1\left(-; a\,|\,-z\right). \tag{3.55}$$

Let $\left(\mathcal{J}_n(y;\alpha;\beta)\right)_{n\in\mathbb{N}}$ be the monic classical Jacobi polynomials, which are orthogonal with respect to $(1-y)^{\alpha}(1+y)^{\beta}$, with $\alpha,\beta>-1$, on the interval $(-1,1)$. Then, $P_n(x;a;b)=\mathcal{J}_n(1-2x;a-1;b-a-1)$. Recalling the definition of the Bessel function of the first kind (1.12) and taking $w=2\sqrt{z}$, (3.55) is equivalent to the Mehler-Heine-type formula for the classical Jacobi polynomials

$$\lim_{n\to\infty} n^{-\alpha}\mathcal{J}_n\left(\cos\left(\frac{w}{n}\right);\alpha,\beta\right)=\lim_{n\to\infty} n^{-\alpha}\mathcal{J}_n\left(1-\frac{w^2}{2n^2};\alpha,\beta\right)=\left(\frac{w}{2}\right)^{-\alpha}\mathrm{J}_{\alpha}\left(w\right). \tag{3.56}$$

The case $\alpha=\beta=0$ of (3.56) is the original Mehler-Heine formula for the Legendre polynomials. The Mehler-Heine-type asymptotic formulas (3.55)-(3.56) converge uniformly in every compact region of the complex plane. Therefore, they give important information about the location of the zeros of the classical Jacobi polynomials near the endpoint $x=1$, or alternatively the zeros of the Jacobi polynomials on the interval $(0,1)$ near the origin. If we denote the zeros of the classical Jacobi polynomials $\mathcal{J}_n(y;\alpha,\beta)$, $n\in\mathbb{Z}^+$ and $\alpha,\beta>-1$, in decreasing order by $\left(y_k^{(n)}\right)_{1\leq k\leq n}$, and we write $y_k^{(n)}=\cos\left(\theta_k^{(n)}\right)$ with $0<\theta_k^{(n)}<\pi$, for each $1\leq k\leq n$, and the (infinite and simple) positive zeros of $\mathrm{J}_{a-1}\left(z\right)$ in increasing order by $(j_k)_{k\in\mathbb{Z}^+}$, then $\lim\limits_{n\to\infty} n\theta_k^{(n)}=j_k$ for a fixed $k\in\mathbb{Z}^+$.

Furthermore, the asymptotic behaviour of the classical Jacobi polynomials and the location of their zeros near the endpoint $y=-1$ can be obtained from (3.56), using the relation $\mathcal{J}_n(-y;\alpha,\beta)=(-1)^n\mathcal{J}_n(y;\beta,\alpha)$ for $n\in\mathbb{N}$. Similarly, we can obtain the asymptotic behaviour of the Jacobi polynomials on the interval $(0,1)$ and the location of their zeros near the origin from the Mehler-Heine-type formula (3.55) and near $x=1$ via the relation

$$P_n(1-x;a;b)=(-1)^nP_n(x;b-a;b)\quad \text{for } n\in\mathbb{N},$$

which is a consequence of the property $\mathcal{W}(1-x;a;b)=\mathcal{W}(x;b-a;b)$ of the weight function.

3.5.3 Quadratic decompositions

As mentioned in Subsection 1.3.2, the only symmetric classical orthogonal polynomials, up to a linear transformation of the variable, are the Hermite and Gegenbauer polynomials. The Laguerre polynomials and the Jacobi polynomials on the interval $(0, 1)$ generalise the quadratic components of these two families of orthogonal polynomials.

Based on [66, Eq. 5.6.1], the Hermite polynomials $(H_n(x))_{n\in\mathbb{N}}$, which are orthogonal with respect to the symmetric weight function e^{-x^2} over the real line, satisfy the quadratic decomposition

$$H_{2n}(x) = H_n^{[0]}(x^2) \quad \text{and} \quad H_{2n+1}(x) = xH_n^{[1]}(x^2) \quad \text{for all } n \in \mathbb{N},$$

with

$$H_n^{[0]}(x) = P_n\left(x; \frac{1}{2}\right) \quad \text{and} \quad H_n^{[1]}(x) = P_n\left(x; \frac{3}{2}\right),$$

where $P_n(x; a)$ are the Laguerre polynomials defined by (3.40).

The Gegenbauer or ultraspherical polynomials, $(C_n(x; \lambda))_{n\in\mathbb{N}}$ with $\lambda > -\dfrac{1}{2}$, are orthogonal with respect to the symmetric weight function $(1-x^2)^{\lambda-\frac{1}{2}}$ over the interval $(-1, 1)$. This means that $C_n(x; \lambda) = \mathcal{J}_n\left(x; \lambda - \frac{1}{2}, \lambda - \frac{1}{2}\right)$, where $(\mathcal{J}_n(y; \alpha; \beta))_{n\in\mathbb{N}}$ are again the classical Jacobi polynomials on the interval $(-1, 1)$.

Based on [66, Eq. 4.7.30], the Gegenbauer polynomials satisfy the quadratic decomposition

$$C_{2n}(x; \lambda) = C_n^{[0]}(x^2; \lambda) \quad \text{and} \quad C_{2n+1}(x; \lambda) = xC_n^{[1]}(x^2; \lambda) \quad \text{for all } n \in \mathbb{N},$$

with

$$C_n^{[0]}(x; \lambda) = P_n\left(x; \frac{1}{2}; \lambda + 1\right) \quad \text{and} \quad C_n^{[1]}(x; \lambda) = P_n\left(x; \frac{3}{2}; \lambda + 2\right),$$

where $P_n(x; a; b)$ are the Jacobi polynomials on the interval $(0, 1)$ defined by (3.51).

Chapter 4

MOPs with respect to the modified Tricomi weights

In this chapter we investigate the multiple orthogonal polynomials with respect to two absolutely continuous measures supported on the positive real line and admitting integral representations via weight functions $\mathcal{W}(x;a,b;c)$ and $\mathcal{W}(x;a,b;c+1)$, involving the confluent hypergeometric function of the second kind (1.18), aka the Tricomi function, and defined, for $a,b,c \in \mathbb{R}^+$ with $c > \max\{a,b\}$, by

$$\mathcal{W}(x;a,b;c) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \, \mathrm{e}^{-x} x^{a-1} \, \mathbf{U}\left(c-b, a-b+1; x\right). \tag{4.1}$$

This chapter consists entirely of original work, except for Section 4.6. Most of this work was published in [47]. The exceptions are Theorem 4.8, Proposition 4.12, and the results in Section 4.5, which are new unpublished work obtained after the publication of [47]. We hereby explore the results obtained in [47], bringing in a new broader perspective based on the investigation expounded in Chapter 3.

Note that the parameters a and b are interchangeable because, using (1.19),

$$\begin{aligned}\mathcal{W}(x;a,b;c) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \, \mathrm{e}^{-x} x^{a-1} \, \mathbf{U}\left(c-b, a-b+1; x\right) \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \, \mathrm{e}^{-x} x^{b-1} \, \mathbf{U}\left(c-a, b-a+1; x\right) = \mathcal{W}(x;b,a;c).\end{aligned}$$

The conditions $a, b, c \in \mathbb{R}^+$ and $c > \max\{a, b\}$ guarantee that $\mathcal{W}(x; a, b; c)$ is integrable over the positive real line. In fact (see [22, Eq. 13.10.7]),

$$\int_0^\infty \mathrm{e}^{-x} x^{a-1}\, \mathbf{U}\left(c, a+1-b; x\right) \mathrm{d}x = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}.$$

Therefore, $\mathcal{W}(x; a, b; c)$ is a probability density function on $\mathbb{R}^+$ with moments

$$\int_0^\infty x^n\, \mathcal{W}(x; a, b; c)\, \mathrm{d}x = \frac{(a)_n\,(b)_n}{(c)_n}, \quad n \in \mathbb{N}. \tag{4.2}$$

Hence, the measure μ supported on the positive real line with $\mathrm{d}\mu(x) = \mathcal{W}(x; a, b; c)\, \mathrm{d}x$ satisfies (3.1) with $(r, s) = (2, 1)$ and $(a_1, a_2; b_1) = (a, b; c)$.

In Section 4.1, we prove that the weight functions $\mathcal{W}(x; a, b; c)$ and $\mathcal{W}(x; a, b; c+1)$ form a Nikishin system on the positive real line (Theorem 4.1). As a result, all multiple orthogonal polynomials of both type I and type II with respect to these weight functions exist and are unique for every multi-index and their zeros satisfy the properties common to all Nikishin systems. We also obtain an explicit formula for the generating measure of the ratio of the weight functions (Proposition 4.2).

In Section 4.2, we describe the weight function $\mathcal{W}(x; a, b; c)$ and the two vectors of weights in (4.7) as solutions to a second-order ordinary differential equation (Proposition 4.3) and a matrix first-order differential equation (Theorem 4.5), respectively. The latter implies that both type II polynomials and type I functions on the step-line satisfy the Hahn property because the differentiation operator acts on them as a shift in the parameters and in the index (Theorem 4.6).

In Section 4.3, we focus on the type I polynomials on the step-line: we obtain a Rodrigues-type formula generating the type I functions (Theorem 4.7) and a matrix Rodrigues-type formula generating the type I polynomials (Theorem 4.8).

Section 4.4 is devoted to a detailed characterisation of the type II multiple orthogonal polynomials on the step-line, that is, the 2-orthogonal polynomials. This characterisation includes finding an explicit representation for these polynomials as terminating generalised hypergeometric series ${}_2F_2$ (Theorem 4.9), and obtaining a third order differential equation (Theorem 4.10) and recurrence relation (Theorem 4.11) of which these 2-orthogonal polynomials are a solution. It turns out that the recurrence coefficients are unbounded and asymptotically periodic of period 2, and we believe this to be the first explicit example of a Nikishin system associated with asymptotically periodic unbounded recurrence coefficients. We also derive a Mehler-Heine-type asymptotic formula satisfied by the 2-orthogonal polynomials (Proposition 4.12), which gives us information about their zeros near the origin, and we find an upper bound for their largest zero (Theorem 4.13). Finally, we show that the cubic components of cases B1 and B2 of the Hahn-classical 3-fold-symmetric 2-orthogonal polynomials analysed in Chapter 2 are particular cases of the polynomials characterised here.

The starting point of Section 4.5 is a branched-continued-fraction representation for the ordinary generating function of the moment sequence given by (4.2), which is a generalised hypergeometric series ${}_3F_1$. We obtain explicit formulas for the coefficients of this branched continued fraction, which we use to obtain alternative expressions for the recurrence coefficients of the 2-orthogonal polynomials characterised in Section 4.4. As a result, the recurrence coefficients are all positive and we derive a decomposition of the infinite lower-Hessenberg matrix associated with these 2-orthogonal polynomials as a product of bidiagonal matrices. This shows that this lower-Hessenberg matrix is the production matrix of a sequence of 2-Stieltjes-Rogers polynomials: the moment sequence given by (4.2).

Finally, in Section 4.6, we prove that the multiple orthogonal polynomials with respect to a pair of weights supported on the positive real line and involving the Macdonald function, introduced in [75] and [11], are a limiting case of the polynomials characterised in Section 4.4. Furthermore, we show that some properties of the multiple orthogonal polynomials with respect to Macdonald functions can be derived from results presented in Chapter 3. These properties include a decomposition of the recurrence coefficients via the indeterminates of a branched-continued-fraction representation of the generating function of the moment sequence of the Macdonald weight: a generalised hypergeometric series ${}_3F_0$.

4.1 Nikishin system

The first question to address when investigating the system of multiple orthogonal polynomials with respect to $\mathcal{W}(x;a,b;c)$ and $\mathcal{W}(x;a,b;c+1)$ is on whether such a system exists and it is unique. We are able to answer affirmatively to these questions by using the connection between continued fractions (in this case, J-fractions) and Stieltjes transforms introduced in Section 1.5 to prove that these weight functions form a Nikishin system, as explained in Theorem 4.1.

As a consequence of the system being Nikishin, we guarantee the existence and uniqueness of the multiple orthogonal polynomials of type I, $(A_{\vec{n}}, B_{\vec{n}})$, and type II, $P_{\vec{n}}$, with respect to the vector of weight functions $\big(\mathcal{W}(x;a,b;c), \mathcal{W}(x;a,b;c+1)\big)$, for any $\vec{n} = (n_0, n_1) \in \mathbb{N}^2$. Furthermore, the type I polynomials $A_{\vec{n}}$ and $B_{\vec{n}}$ have exactly degree $n_0 - 1$ and $n_1 - 1$, respectively, the type I function $Q_{\vec{n}}$ and the type II multiple orthogonal polynomial $P_{\vec{n}}$ have exactly $|\vec{n}| - 1$ and $|\vec{n}|$ simple zeros on $\mathbb{R}^+$, respectively, and the zeros of the type II polynomials interlace with the zeros of their nearest neighbours, as explained in the Subsection 1.4.2.

Theorem 4.1. *[47, Th. 2.1] For $a, b, c \in \mathbb{R}^+$ such that $c > \max\{a, b\}$, let $\mathcal{W}(x;a,b;c)$ be defined by (4.1). Then, setting $\alpha_0 = c$, $\alpha_n = -(c-a+n)(c-b+n)$ for $n \geq 1$, and $\beta_n = 2c - a - b + 2n + 1$ for all $n \in \mathbb{N}$, we have*

$$\frac{\mathcal{W}(x;a,b;c+1)}{\mathcal{W}(x;a,b;c)} = \mathop{\mathrm{K}}_{n=0}^{\infty} \left(\frac{\alpha_n}{x + \beta_n} \right), \tag{4.3}$$

Therefore, there exists a probability density measure σ in $\mathbb{R}^+$ such that

$$\frac{\mathcal{W}(x;a,b;c+1)}{\mathcal{W}(x;a,b;c)} = c \int_{-\infty}^{0} \frac{\mathrm{d}\sigma(-t)}{x - t}, \tag{4.4}$$

and the vector of weight functions $\big(\mathcal{W}(x;a,b;c), \mathcal{W}(x;a,b;c+1)\big)$ forms a Nikishin system on the positive real line.

Proof. Following the definition of $\mathcal{W}(x;a,b;c)$,

$$\frac{\mathcal{W}(x;a,b;c+1)}{\mathcal{W}(x;a,b;c)} = \frac{c\,\mathbf{U}\left(c-b+1,a-b+1;x\right)}{\mathbf{U}\left(c-b,a-b+1;x\right)}.$$

According to [22, Eq. 13.3.7], we have

$$\begin{aligned}&(c-a)(c-b)\mathbf{U}\left(c-b+1,a-b+1;x\right)\\ &=(x+2c-a-b-1)\mathbf{U}\left(c-b,a-b+1;x\right)-\mathbf{U}\left(c-b-1,a-b+1;x\right),\end{aligned}$$

which implies that

$$\frac{\mathcal{W}(x;a,b;c+1)}{\mathcal{W}(x;a,b;c)} = \frac{c}{(c-a)(c-b)}\left((x+2c-a-b-1)-\frac{\mathbf{U}\left(c-b-1,a-b+1;x\right)}{\mathbf{U}\left(c-b,a-b+1;x\right)}\right).$$

Furthermore, based on [20, Eq. 16.1.20], we derive

$$\frac{\mathbf{U}\left(c-b-1,a-b+1;x\right)}{\mathbf{U}\left(c-b,a-b+1;x\right)} = (x+2c-a-b-1)+\mathop{\mathrm{K}}_{n=0}^{\infty}\left(\frac{\tilde{\alpha}_n}{x+\beta_n}\right),$$

with $\tilde{\alpha}_n = -(c-a+n)(c-b+n)$ and $\beta_n = 2c-a-b+2n+1$ for all $n\in\mathbb{N}$.

Combining the two latter equations, we deduce that (4.3) holds. Moreover, because $c>\max\{a,b\}$, we have $\alpha_n,\beta_n>0$ for any $n\in\mathbb{N}$. Therefore, recalling (1.51), (4.3) implies (4.4), and we have proved that $\big(\mathcal{W}(x;a,b;c),\mathcal{W}(x;a,b;c+1)\big)$ forms a Nikishin system on the positive real line. □

The generating measure σ in (4.4) can be found via the Stieltjes-Perron inversion formula. As such, we have

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = \lim_{\epsilon\to0^+}\frac{G(-t-i\epsilon)-G(-t+i\epsilon)}{2\pi i} = \lim_{\epsilon\to0^+}\frac{G(\mathrm{e}^{-i\pi}(t+i\epsilon))-G(\mathrm{e}^{i\pi}(t-i\epsilon))}{2\pi i}, \tag{4.5}$$

where

$$G(x) = \frac{\mathcal{W}(x;a,b;c+1)}{c\,\mathcal{W}(x;a,b;c)} = \frac{\mathbf{U}\left(c-b+1,a-b+1;x\right)}{\mathbf{U}\left(c-b,a-b+1;x\right)},$$

which is well defined because $c>b$ implies that $\mathbf{U}\left(c-b,a-b+1;x\right)$ has no zeros in the sector $|\arg(x)|<\pi$ (see [22, §13.9(ii)]).

During the remainder of this section, we set $\alpha = c-b$ and $\beta = a-b+1$ to shorten the notation. Based on [22, Eqs. 13.3.22 & 13.3.10], we have

$$x\,\mathbf{U}'\left(\alpha,\beta;x\right) = \alpha(\alpha-\beta+1)\mathbf{U}\left(\alpha+1,\beta;x\right) - \alpha\,\mathbf{U}\left(\alpha,\beta;x\right). \tag{4.6}$$

Therefore, we can rewrite

$$G(x) = \frac{1}{\alpha(\alpha-\beta+1)}\left(x\,\frac{\mathbf{U}'(\alpha,\beta;x)}{\mathbf{U}\left(\alpha,\beta;x\right)} + \alpha\right)$$

and (4.5) reads as

$$\begin{aligned}\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = \lim_{\epsilon\to0^+}\Bigg(&\frac{\mathrm{e}^{-\pi i}(t+i\epsilon)\mathbf{U}'(\alpha,\beta;\mathrm{e}^{-\pi i}(t+i\epsilon))\mathbf{U}\left(\alpha,\beta;\mathrm{e}^{\pi i}(t-i\epsilon)\right)}{2\pi i\alpha(\alpha-\beta+1)\left|\mathbf{U}\left(\alpha,\beta;\mathrm{e}^{\pi i}(t-i\epsilon)\right)\right|^2}\\ &-\frac{\mathrm{e}^{\pi i}(t-i\epsilon)\mathbf{U}\left(\alpha,\beta;\mathrm{e}^{-\pi i}(t+i\epsilon)\right)\mathbf{U}'(\alpha,\beta;\mathrm{e}^{\pi i}(t-i\epsilon))}{2\pi i\alpha(\alpha-\beta+1)\left|\mathbf{U}\left(\alpha,\beta;\mathrm{e}^{\pi i}(t-i\epsilon)\right)\right|^2}\Bigg).\end{aligned}$$

In [38, Eqs. 3.4-3.5], it was shown that, for non-integer values of β, we have

$$\begin{aligned}\lim_{\epsilon\to0^+}\Bigg(&\frac{\mathbf{U}'(\alpha,\beta+1;\mathrm{e}^{-\pi i}(t+i\epsilon))\mathbf{U}\left(\alpha,\beta+1;\mathrm{e}^{\pi i}(t-i\epsilon)\right)}{2\pi i\left|\mathbf{U}\left(\alpha,\beta;\mathrm{e}^{\pi i}(t-i\epsilon)\right)\right|^2}\\ &-\frac{\mathbf{U}\left(\alpha,\beta+1;\mathrm{e}^{-\pi i}(t+i\epsilon)\right)\mathbf{U}'(\alpha,\beta+1;\mathrm{e}^{\pi i}(t-i\epsilon))}{2\pi i\left|\mathbf{U}\left(\alpha,\beta;\mathrm{e}^{\pi i}(t-i\epsilon)\right)\right|^2}\Bigg)\\ &=\frac{-t^{-\beta}\,\mathrm{e}^{-t}}{\Gamma(\alpha)\Gamma(\alpha-\beta+1)\left|\mathbf{U}\left(\alpha,\beta;\mathrm{e}^{\pi i}t\right)\right|^2}.\end{aligned}$$

The latter was obtained by expressing the function $\mathbf{U}$ as a linear combination of two independent solutions to the confluent differential equation as in [22, Eq. 13.2.42] to then use the expression for the Wronskian of those two functions given in [22, Eq. 13.2.34]. Therefore, we deduce that

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = \frac{t^{1-\beta}\,\mathrm{e}^{-t}}{\Gamma(\alpha+1)\Gamma(\alpha-\beta+2)\left|\mathbf{U}\left(\alpha,\beta;\mathrm{e}^{\pi i}t\right)\right|^2},$$

and we obtain the following result.

Proposition 4.2. *[47, Prop. 2.2] For $a, b, c \in \mathbb{R}^+$ such that $c > \max\{a, b\}$, let $\mathcal{W}(x; a, b; c)$ be defined by (4.1). If $a - b \notin \mathbb{Z}$, then relation (4.4) can be written as*

$$\frac{\mathcal{W}(x; a, b; c+1)}{\mathcal{W}(x; a, b; c)} = c \int_0^\infty \frac{t^{b-a}\,\mathrm{e}^{-t}\,|\,\boldsymbol{U}(c-b, a-b+1; -t)|^{-2}\,\mathrm{d}t}{(x+t)\Gamma(c-b+1)\Gamma(c-a+1)}.$$

4.2 Differential properties

From this point forth, we index the vector $\left[\mathcal{W}(x; a, b; c), \mathcal{W}(x; a, b; c+1)\right]^T$ of functions defined by (4.1), with an extra parameter $\epsilon \in \{0, 1\}$, by considering

$$\overline{\mathcal{W}}^{[\epsilon]}(x; a, b; c) := \begin{bmatrix} \mathcal{W}(x; a, b; c+\epsilon) \\ \mathcal{W}(x; a, b; c+1-\epsilon) \end{bmatrix}. \tag{4.7}$$

The parameter $\epsilon \in \{0, 1\}$ swaps the roles of the weight functions $\mathcal{W}(x; a, b; c)$ and $\mathcal{W}(x; a, b; c+1)$ causing a reflection of the multiple orthogonal polynomials of both types with respect to the diagonal polynomials, which remain the same. When focusing on the step-line, changing the parameter $\epsilon \in \{0, 1\}$ corresponds to a flip between the multiple orthogonal polynomials for the lower and the upper step-line multi-indexes. In particular, both type II and type I polynomials (and the type I function) for the multi-indices of even length on the step-line remain unchanged when changing the parameter $\epsilon \in \{0, 1\}$.

There are further motivations for the introduction of this parameter ϵ. Under the action of the derivative operator, the multiple orthogonal system for $\overline{\mathcal{W}}^{[\epsilon]}(x; a, b; c)$ bounces from the lower to the upper step-line and reciprocally with shifted parameters, as expounded in Theorem 4.6, which comes as a consequence of the description of the vector of weights (4.7) as a solution to a matrix first order differential equation in Theorem 4.5. Beforehand, in Proposition 4.3, we describe the weight function $\mathcal{W}(x; a, b; c)$ as a solution to a second-order differential equation.

Proposition 4.3. *[47, Prop. 2.3] For* $a, b, c \in \mathbb{R}^+$ *such that* $c > \max\{a, b\}$*, let* $\mathcal{W}(x) = \mathcal{W}(x; a, b; c)$ *be the weight function defined by (4.1). Then*

$$x^2\mathcal{W}''(x) + (x - (a + b - 3))\, x\mathcal{W}'(x) + ((a - 1)(b - 1) - (c - 2)x)\, \mathcal{W}(x) = 0 \tag{4.8a}$$

which, defining the operator $\Theta : \mathcal{P} \to \mathcal{P}$ *by* $\Theta\left(f(x)\right) = \dfrac{\mathrm{d}}{\mathrm{d}x}\big(xf(x)\big)$*, is equivalent to*

$$\Theta^2\big(\mathcal{W}(x)\big) + \Theta\big(\,(x - (a + b))\, \mathcal{W}(x)\big) + (ab - cx)\mathcal{W}(x) = 0. \tag{4.8b}$$

Note that, recalling Proposition 3.1, the formula for the moments of $\mathcal{W}(x; a, b; c)$ given by (4.2) is equivalent to all the moments of the left-hand side of (4.8b) integrated over the positive real line vanishing.

Proof. It is easy to check that the differential equations (4.8a) and (4.8b) are equivalent because expanding the latter we obtain the former. Hence, it is sufficient to prove that (4.8a) holds. To simplify the notation, we set $\lambda = \dfrac{\Gamma(c)}{\Gamma(a)\Gamma(b)}$ and we write $\mathbf{U}(x) = \mathbf{U}\left(c - b, a + 1 - b; x\right)$, so that

$$\mathcal{W}(x) = \mathcal{W}(x; a, b; c) = \lambda\, \mathrm{e}^{-x}x^{a-1}\, \mathbf{U}(x).$$

Differentiating the latter with respect to x, we obtain

$$\mathcal{W}'(x) = \lambda\, \mathrm{e}^{-x}x^{a-2}\left(x\, \mathbf{U}'(x) + (a - 1 - x)\mathbf{U}(x)\right). \tag{4.9}$$

Another differentiation brings

$$\begin{aligned}\mathcal{W}''(x) = \lambda\, \mathrm{e}^{-x}x^{a-3}\Big(&x^2\mathbf{U}''(x) + 2(a - 1 - x)x\mathbf{U}'(x)\\ &+ \big(x^2 - 2(a - 1)x + (a - 1)(a - 2)\big)\, \mathbf{U}(x)\Big).\end{aligned}$$

Based on (1.17), $x\mathbf{U}''(x) = (x - a - 1 + b)\mathbf{U}'(x) + (c - b)\mathbf{U}(x)$, which implies that

$$\begin{aligned}\mathcal{W}''(x) = \lambda\, \mathrm{e}^{-x}x^{a-3}\Big(&(a + b - 3 - x)x\, \mathbf{U}'(x)\\ &+ \big(x^2 + (c - b - 2a - 2)x + (a - 1)(a - 2)\big)\, \mathbf{U}(x)\Big).\end{aligned}$$

Finally, combining the latter expression with the definition of $\mathcal{W}(x;a,b;c)$ and (4.9), we deduce (4.8a). $\square$

In the following result, we obtain expressions for $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\,\mathcal{W}(x;a,b;c+\epsilon)\right)$, $\epsilon\in\{0,1\}$, as combinations of $\mathcal{W}(x;a,b;c)$ and $\mathcal{W}(x;a,b;c+1)$.

Lemma 4.4. *[47, Lemma. 2.4] For $a,b,c\in\mathbb{R}^+$ such that $c>\max\{a,b\}$, let $\mathcal{W}(x;a,b;c)$ be the weight function defined by (4.1). Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\mathcal{W}(x;a,b;c)\right)=-(x+c-a-b)\mathcal{W}(x;a,b;c)+\frac{(c-a)(c-b)}{c}\,\mathcal{W}(x;a,b;c+1) \tag{4.10a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\mathcal{W}(x;a,b;c+1)\right)=c\,\mathcal{W}(x;a,b;c+1)-c\,\mathcal{W}(x;a,b;c). \tag{4.10b}$$

Proof. Combining (4.9) and (4.6), we can write

$$\mathcal{W}'(x;a,b;c)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,\mathrm{e}^{-x}x^{a-2}\,g(x)$$

with

$$g(x)=(c-a)(c-b)\mathbf{U}\left(c-b+1,a-b+1;x\right)-(x+c-a-b+1)\mathbf{U}\left(c-b,a-b+1;x\right).$$

As a result,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\mathcal{W}(x;a,b;c)\right)=x\,\mathcal{W}'(x;a,b;c)+\mathcal{W}(x;a,b;c)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,\mathrm{e}^{-x}x^{a-1}\tilde{g}(x),$$

with

$$\tilde{g}(x)=(c-a)(c-b)\mathbf{U}\left(c-b+1,a-b+1;x\right)-(x+c-a-b)\mathbf{U}\left(c-b,a-b+1;x\right),$$

which implies that (4.10a) holds.

Making the shift $c \to c+1$ in (4.10a) and (4.1), we obtain, respectively,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left(x\mathcal{W}(x;a,b;c+1)\right)\\ &= -\left(x+c-a-b+1\right)\mathcal{W}(x;a,b;c+1)+\frac{(c-a+1)(c-b+1)}{c+1}\mathcal{W}(x;a,b;c+2).\end{aligned}$$

and

$$\begin{aligned}&\frac{(c-a+1)(c-b+1)}{c+1}\mathcal{W}(x;a,b;c+2)\\ &=(x+2c-a-b+1)\mathcal{W}(x;a,b;c+1)-c\,\mathcal{W}(x;a,b;c).\end{aligned}$$

As a result, combining the two latter equations, we deduce that (4.10b) holds. □

Based on the previous lemma, we can write the vector of weights $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$ as a solution to a matrix first order equation of Pearson type as follows.

Theorem 4.5. *[47, Th. 2.5] For $a,b,c \in \mathbb{R}^+$ such that $c > \max\{a,b\}$ and $\epsilon \in \{0,1\}$, let $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$ be defined as in (4.7), and define the matrices*

$$\Phi^{[0]}(x)=\begin{bmatrix}0 & \dfrac{c+1}{ab}\\ \dfrac{c}{ab} & 0\end{bmatrix}\ \text{and}\ \Psi^{[0]}(x)=\begin{bmatrix}\dfrac{c(c+1)}{ab} & -\dfrac{c(c+1)}{ab}\\ \dfrac{c}{ab}\left(x+c-a-b\right) & -\dfrac{(c-a)(c-b)}{ab}\end{bmatrix};$$

$$\Phi^{[1]}(x)=\begin{bmatrix}\dfrac{c+1}{ab} & 0\\ \dfrac{(c+1)(c+2)(x+2c-a-b+1)}{ab(c-a+1)(c-b+1)} & -\dfrac{c(c+1)(c+2)}{ab(c-a+1)(c-b+1)}\end{bmatrix}\ \text{and}$$

$$\Psi^{[1]}(x)=\begin{bmatrix}-\dfrac{c(c+1)}{ab} & \dfrac{c(c+1)}{ab}\\ -\dfrac{(c+1)^2(c+2)}{ab(c-a+1)(c-b+1)}\left(x+c-\dfrac{ab}{c+1}\right) & \dfrac{c(c+1)^2(c+2)}{ab(c-a+1)(c-b+1)}\end{bmatrix}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\,\Phi^{[\epsilon]}(x)\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)\right)+\Psi^{[\epsilon]}(x)\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)=0, \tag{4.11}$$

and

$$x\,\Phi^{[\epsilon]}(x)\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)=\overline{\mathcal{W}}^{[1-\epsilon]}(x;a+1,b+1;c+1+\epsilon). \tag{4.12}$$

Proof. To simplify the notation, we let $\overline{\mathcal{W}}^{[\epsilon]}(x) = \overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$, $\epsilon \in \{0,1\}$. The equations (4.10a)-(4.10b) in Lemma 4.4 can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\overline{\mathcal{W}}^{[\epsilon]}(x)\right) = \Omega^{[\epsilon]}(x)\overline{\mathcal{W}}^{[\epsilon]}(x),$$

with

$$\Omega^{[0]}(x) = \begin{bmatrix} -(x+c-a-b) & \dfrac{(c-a)(c-b)}{c} \\ -c & c \end{bmatrix}$$

and

$$\Omega^{[1]}(x) = \begin{bmatrix} c & -c \\ \dfrac{(c-a)(c-b)}{c} & -(x+c-a-b) \end{bmatrix}.$$

As a result, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\,\Phi^{[\epsilon]}(x)\,\overline{\mathcal{W}}^{[\epsilon]}(x)\right) = \left(x\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi^{[\epsilon]}(x)\right) + \Phi^{[\epsilon]}(x)\Omega^{[\epsilon]}(x)\right)\overline{\mathcal{W}}^{[\epsilon]}(x),$$

which corresponds to (4.11), after checking that

$$\Psi^{[0]}(x) = -\Phi^{[0]}\Omega^{[0]}(x) \quad \text{and} \quad \Psi^{[1]}(x) = -\Phi^{[1]}(x)\Omega^{[1]}(x) - x\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi^{[1]}(x)\right).$$

Now, we let

$$\begin{bmatrix} \mathcal{V}_0^{[\epsilon]}(x) \\ \mathcal{V}_1^{[\epsilon]}(x) \end{bmatrix} = x\,\Phi^{[\epsilon]}(x)\overline{\mathcal{W}}^{[\epsilon]}(x), \quad \epsilon \in \{0,1\}.$$

In order to prove (4.12), we need to check that

$$\begin{bmatrix} \mathcal{V}_0^{[\epsilon]}(x) \\ \mathcal{V}_1^{[\epsilon]}(x) \end{bmatrix} = \begin{bmatrix} \mathcal{W}(x;a+1,b+1;c+2) \\ \mathcal{W}(x;a+1,b+1;c+1+2\epsilon) \end{bmatrix}.$$

Firstly, observe that both $\mathcal{V}_0^{[0]}(x)$ and $\mathcal{V}_0^{[1]}(x)$ are equal to

$$\frac{c+1}{ab}\,x\,\mathcal{W}(x;a,b;c+1) = \frac{\Gamma(c+2)\mathrm{e}^{-x}x^a}{\Gamma(a+1)\Gamma(b+1)}\,\mathbf{U}\left(c-b+1,a-b+1;x\right).$$

Hence,

$$\mathcal{V}_0^{[0]}(x) = \mathcal{V}_0^{[1]}(x) = \mathcal{W}(x; a+1, b+1; c+2).$$

Moreover,

$$\mathcal{V}_1^{[0]}(x) = \frac{c}{ab}\, x\, \mathcal{W}(x; a, b; c) = \frac{\Gamma(c+1)\mathrm{e}^{-x}x^a}{\Gamma(a+1)\Gamma(b+1)}\, \mathbf{U}\,(c+1, a-b+1; x)\,,$$

thus

$$\mathcal{V}_1^{[0]}(x) = \mathcal{W}(x; a+1, b+1; c+1).$$

Finally,

$$\mathcal{V}_1^{[1]}(x) = \frac{(c+1)(c+2)x}{ab(c-a+1)(c-b+1)}\Big((x+2c-a-b+1)\mathcal{W}(x; a, b; c+1) - c\,\mathcal{W}(x; a, b; c)\Big),$$

which, recalling (4.1) (with the shift $c \to c+1$), can be rewritten as

$$\mathcal{V}_1^{[1]}(x) = \frac{\Gamma(c+3)\mathrm{e}^{-x}x^a}{\Gamma(a+1)\Gamma(b+1)}\, \mathbf{U}\,(c-b+2, a-b+1; x) = \mathcal{W}(x; a+1, b+1; c+3).$$

□

Combining the latter result with Theorem 3.2, we show that the type II multiple orthogonal polynomials and the type I functions on the step-line satisfy the Hahn-classical property, because the differentiation with respect to the variable gives a shift in the parameters as well as in the index, as described in the following result.

Theorem 4.6. *[47, Th. 2.9] For $a, b, c \in \mathbb{R}^+$ such that $c > \max\{a, b\}$, $\epsilon \in \{0, 1\}$ and $n \in \mathbb{N}$, let $P_n^{[\epsilon]}(x; a, b; c)$ and $Q_n^{[\epsilon]}(x; a, b; c)$ be, respectively, the type II multiple orthogonal polynomial and the type I function for the index of length n on the step-line with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x; a, b; c)$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(P_{n+1}^{[\epsilon]}(x; a, b; c)\Big) = (n+1)P_n^{[1-\epsilon]}(x; a+1, b+1; c+1+\epsilon) \tag{4.13}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(Q_n^{[1-\epsilon]}(x; a+1, b+1; c+1+\epsilon)\big) = -n\, Q_{n+1}^{[\epsilon]}(x; a, b; c). \tag{4.14}$$

Proof. Let $\Phi^{[\epsilon]}(x;a,b;c)$ be defined as in Theorem 4.5.

Part (a) of Theorem (3.2) ensures that $\left(\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(P^{[\epsilon]}_{n+1}(x;a,b;c)\right)\right)_{n\in\mathbb{N}}$ is 2-orthogonal with respect to the vector of weights $x\Phi^{[\epsilon]}(x;a,b;c)\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$. Similarly, part (b) of Theorem (3.2) implies that, if $R^{[\epsilon]}_n(x;a,b;c)$ is the type I function for the index of length n on the step-line with respect to $x\Phi^{[\epsilon]}(x;a,b;c)\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$ then $-\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}x}\left(R^{[\epsilon]}_n(x;a,b;c)\right)$ is the type I function for the index of length $n+1$ on the step-line with respect to the vector of weights $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$.

Therefore, by virtue of (4.12), we conclude that both (4.13) and (4.14) hold. □

4.3 Rodrigues-type formulas for type I MOPs

We denote by $\left(A^{[\epsilon]}_n(x;a,b;c), B^{[\epsilon]}_n(x;a,b;c)\right)$ the vector of type I multiple orthogonal polynomials with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$, $\epsilon\in\{0,1\}$, for the multi-index of length n on the step-line. Then, the type I functions on the step-line are

$$Q^{[\epsilon]}_n(x;a,b;c) = A^{[\epsilon]}_n(x;a,b;c)\mathcal{W}(x;a,b;c+\epsilon) + B^{[\epsilon]}_n(x;a,b;c)\mathcal{W}(x;a,b;c+1-\epsilon),$$

and they can be generated via a Rodrigues-type formula of the type in (3.15) as described in the following result.

Theorem 4.7. *[47, Th. 2.10] For $a,b,c\in\mathbb{R}^+$ such that $c>\max\{a,b\}$, $\epsilon\in\{0,1\}$ and $n\in\mathbb{N}$, let $Q^{[\epsilon]}_n(x;a,b;c)$ be the type I function with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$ for the index of length n on the step-line. Then, for any $n\in\mathbb{N}$,*

$$Q^{[\epsilon]}_{n+1}(x;a,b;c) = \frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathcal{W}\left(x;a+n,b+n;c+n+\left\lfloor\frac{n+1+\epsilon}{2}\right\rfloor\right)\right). \tag{4.15}$$

Proof. We proceed by induction on $n\in\mathbb{N}$. For $n=0$, the relation (4.15) trivially holds, because it reads as $Q^{[\epsilon]}_1(x;a,b;c)=\mathcal{W}(x;a,b;c+\epsilon)$.

Using the differential formula (4.14) and then evoking the assumption that (4.15) holds for a fixed $n \in \mathbb{N}$, we obtain

$$\begin{aligned} Q_{n+2}^{[\epsilon]}(x;a,b;c) &= -\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(Q_{n+1}^{[1-\epsilon]}(x;a+1,b+1;c+1+\epsilon)\right) \\ &= \frac{(-1)^{n+1}}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(\mathcal{W}\left(x;a+n+1,b+n+1;c+n+1+\left\lfloor\frac{n+2+\epsilon}{2}\right\rfloor\right)\right). \end{aligned}$$

If we equate the first and latter members, we readily see that (4.15) holds for $n+1$ and, as a result, we can state that it holds for all $n \in \mathbb{N}$ by induction. □

Moreover, applying formula (3.16) to $\left(A_n^{[\epsilon]}(x;a,b;c), B_n^{[\epsilon]}(x;a,b;c)\right)$, we deduce that, for any $n \in \mathbb{N}$,

$$n\begin{bmatrix} A_{n+1}^{[\epsilon]}(x;a,b;c) \\ B_{n+1}^{[\epsilon]}(x;a,b;c) \end{bmatrix} = \mathcal{O}^{[\epsilon]}(a,b;c)\begin{bmatrix} A_n^{[1-\epsilon]}(x;a+1,b+1;c+\epsilon) \\ B_n^{[1-\epsilon]}(x;a+1,b+1;c+\epsilon) \end{bmatrix},$$

where the raising operator $\mathcal{O}^{[\epsilon]}(a,b;c)$, involving the transpose of the matrices in Theorem 4.5, is defined by

$$\mathcal{O}^{[\epsilon]}(a,b;c) = \left(\Psi^{[\epsilon]}(x)\right)^T - x\left(\Phi^{[\epsilon]}(x)\right)^T\frac{\mathrm{d}}{\mathrm{d}x}. \tag{4.16}$$

Therefore, recalling that $\left(A_1^{[\epsilon]}(x;a,b;c), B_1^{[\epsilon]}(x;a,b;c)\right) = (1,0)$, we obtain a matrix Rodrigues-type formula of the type (3.17) generating the type I polynomials on the step-line. Precisely, we have the following result.

Theorem 4.8. *For $a,b,c \in \mathbb{R}^+$ such that $c > \max\{a,b\}$, $\epsilon \in \{0,1\}$ and $n \in \mathbb{N}$, let $\left(A_n^{[\epsilon]}(x;a,b;c), B_n^{[\epsilon]}(x;a,b;c)\right)$ be the vector of type I multiple orthogonal polynomials with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$ for the index of length n on the step-line. Then, for any $n \in \mathbb{N}$,*

$$\begin{bmatrix} A_{n+1}^{[\epsilon]}(x;a,b;c) \\ B_{n+1}^{[\epsilon]}(x;a,b;c) \end{bmatrix} = \frac{1}{n!}\left(\prod_{k=0}^{n-1}\mathcal{O}^{[\epsilon]}\left(a+k,b+k;c+k+\left\lfloor\frac{k+1+\epsilon}{2}\right\rfloor\right)\right)\begin{bmatrix}1\\0\end{bmatrix},$$

where the raising operator $\mathcal{O}^{[\epsilon]}(a,b;c)$ is defined by (4.16).

4.4 Characterisation of the type II MOPs

We start this section by giving an explicit representation as terminating hypergeometric series ${}_2F_2$ for the type II multiple orthogonal polynomials on the step-line with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$. We use this representation to obtain the third-order differential equation and recurrence relation satisfied by these polynomials as well as a Mehler-Heine-type asymptotic formula near the origin satisfied by them. The latter and the asymptotic behaviour of the recurrence coefficients give us information about the zeros of these 2-orthogonal polynomials. Finally, we highlight the particular cases connected to Hahn-classical 3-fold-symmetric 2-orthogonal polynomials.

4.4.1 Explicit expressions as hypergeometric polynomials

Based on the moments expression (4.2), the explicit formula for the 2-orthogonal polynomials with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$ is presented in the following theorem.

Theorem 4.9. *[47, Th. 3.1] For $a,b,c\in\mathbb{R}^+$ such that $c>\max\{a,b\}$ and $\epsilon\in\{0,1\}$, let $\left(P_n^{[\epsilon]}(x;a,b;c)\right)_{n\in\mathbb{N}}$ be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$. Then*

$$P_n^{[\epsilon]}(x;a,b;c)=\frac{(-1)^n\,(a)_n\,(b)_n}{\left(c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor\right)_n}\,{}_2F_2\left(\left.\begin{matrix}-n,c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor\\ a,b\end{matrix}\,\right|\,x\right),\tag{4.17}$$

or, equivalently,

$$P_n^{[\epsilon]}(x;a,b;c)=\sum_{k=0}^{n}\left(\frac{(-1)^k\,(1+n-k)_k\,(a+n-k)_k\,(b+n-k)_k}{k!\,\left(c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor+n-k\right)_k}\,x^{n-k}\right).\tag{4.18}$$

To prove this theorem we check that $\left(P_n^{[\epsilon]}(x)\right)_{n\in\mathbb{N}}$ in (4.17) satisfies the 2-orthogonality conditions with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$, which are:

$$\int_0^\infty x^k P_n^{[\epsilon]}(x;a,b;c)\mathcal{W}(x;a,b;c+\epsilon)\mathrm{d}x = \begin{cases} 0 & \text{if } n \geq 2k+1, \\ N_{2k}^{[\epsilon]}(a,b;c) \neq 0 & \text{if } n = 2k, \end{cases} \tag{4.19a}$$

and

$$\int_0^\infty x^k P_n^{[\epsilon]}(x;a,b;c)\mathcal{W}(x;a,b;c+1-\epsilon)\mathrm{d}x = \begin{cases} 0 & \text{if } n \geq 2k+2, \\ N_{2k+1}^{[\epsilon]}(a,b;c) \neq 0 & \text{if } n = 2k+1. \end{cases} \tag{4.19b}$$

Actually, as we are dealing with a Nikishin system, the existence of a 2-orthogonal polynomial sequence with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$ is guaranteed. By virtue of formula (1.9) for the derivative of a generalised hypergeometric series, it is rather straightforward to show that the polynomials given by (4.17) satisfy the differential property (4.13), which the 2-orthogonal polynomial sequence with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$ must satisfy, as stated in Theorem 4.6. Therefore, it would be sufficient to check the orthogonality conditions (4.19a)-(4.19b) when $k = 0$ to then prove the result by induction on $n \in \mathbb{N}$ (the degree of the polynomials). However, we opt for checking that the polynomials in (4.17) satisfy all the orthogonality conditions in (4.19a)-(4.19b). On the one hand, this process enables us to show directly that these polynomials are indeed 2-orthogonal with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$ without using the Nikishin property. On the other hand, this proof provides a method to derive explicit expressions for the nonzero coefficients $N_n^{[\epsilon]}(a,b;c)$ in (4.19a)-(4.19b), which are used in Subsection 4.4.3 to obtain explicit expressions for the positive γ_n-coefficients in the third order recurrence relation (4.24) satisfied by these polynomials.

Proof of Theorem 4.9. Recalling the expression for the moments given by (4.2), we can use Lemma 3.3 to derive that, for $k, n \in \mathbb{N}$ and $\epsilon \in \{0, 1\}$,

$$\int_0^\infty x^k P_n^{[\epsilon]}(x; a, b; c)\mathcal{W}(x; a, b; c+\epsilon)\mathrm{d}x$$
$$= \frac{(-1)^n (a)_n (b)_n (a)_k (b)_k}{\left(c + \left\lfloor \frac{n+\epsilon}{2} \right\rfloor\right)_n (c+\epsilon)_k} \, {}_4F_3\left(\left.\begin{matrix} -n, a+k, b+k, c+\left\lfloor \frac{n+\epsilon}{2} \right\rfloor \\ a, b, c+\epsilon+k \end{matrix}\right| 1\right).$$

If $n \geq 2k+1$, then $\left\lfloor \dfrac{n+\epsilon}{2} \right\rfloor - \epsilon + k \leq \dfrac{n}{2} + k < n$. Hence, using (3.20) in Lemma 3.4, we deduce that

$${}_4F_3\left(\left.\begin{matrix} -n, a+k, b+k, c+\left\lfloor \frac{n+\epsilon}{2} \right\rfloor \\ a, b, c+\epsilon+k \end{matrix}\right| 1\right) = 0.$$

Therefore,

$$\int_0^\infty x^k P_n^{[\epsilon]}(x; a, b; c)\mathcal{W}(x; a, b; c+\epsilon)\mathrm{d}x = 0, \quad \text{for any } n \geq 2k+1. \tag{4.20a}$$

Moreover,

$$\int_0^\infty x^k P_{2k}^{[\epsilon]}(x; a, b; c)\mathcal{W}(x; a, b; c+\epsilon)\mathrm{d}x$$
$$= \frac{(a)_{2k} (b)_{2k} (a)_k (b)_k}{(c+k)_{2k} (c+\epsilon)_k} \, {}_4F_3\left(\left.\begin{matrix} -2k, a+k, b+k, c+k \\ a, b, c+\epsilon+k \end{matrix}\right| 1\right).$$

For $\epsilon = 0$ the latter hypergeometric series simplifies to a ${}_3F_2$, which, on account of the identity (3.20), can be evaluated to

$${}_3F_2\left(\left.\begin{matrix} -2k, a+k, b+k \\ a, b \end{matrix}\right| 1\right) = \frac{(2k)!}{(a)_k (b)_k},$$

whilst for $\epsilon = 1$ we use (3.21) to get

$${}_4F_3\left(\left.\begin{matrix} -2k, a+k, b+k, c+k \\ a, b, c+k+1 \end{matrix}\right| 1\right) = \frac{(2k)! \, (c-a+1)_k (c-b+1)_k}{(a)_k (b)_k (c+k+1)_{2k}}.$$

As a result, we have

$$\int_0^\infty x^k P_{2k}^{[\epsilon]}(x;a,b;c)\mathcal{W}(x;a,b;c+\epsilon)\mathrm{d}x \tag{4.20b}$$
$$= \begin{cases} \dfrac{(2k)!\,(a)_{2k}\,(b)_{2k}}{(c)_{3k}} > 0, & \text{if } \epsilon = 0, \\[2ex] \dfrac{(2k)!\,(a)_{2k}\,(b)_{2k}\,(c-a+1)_k\,(c-b+1)_k}{(c+1)_{3k}\,(c+k)_{2k}} > 0, & \text{if } \epsilon = 1. \end{cases}$$

Hence, recalling (4.20a), we conclude that (4.19a) holds for all $k, n \in \mathbb{N}$, with the nonzero coefficients $N_{2k}^{[\epsilon]}(a,b;c)$ being given by (4.20b).

Analogously, we recall (4.2) and use Lemma 3.3 to derive that

$$\int_0^\infty x^k P_n^{[\epsilon]}(x;a,b;c)\mathcal{W}(x;a,b;c+1-\epsilon)\mathrm{d}x$$
$$= \frac{(-1)^n\,(a)_n\,(b)_n\,(a)_k\,(b)_k}{\left(c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor\right)_n (c+1-\epsilon)_k}\,{}_4F_3\left(\left.\begin{matrix} -n, a+k, b+k, c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor \\ a, b, c+1-\epsilon+k \end{matrix}\,\right|\, 1\right).$$

If $n \geq 2k+2$, then $\left\lfloor\dfrac{n+\epsilon}{2}\right\rfloor - 1 + \epsilon + k < n$ and identity (3.20) leads to

$${}_4F_3\left(\left.\begin{matrix} -n, a+k, b+k, c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor \\ a, b, c+1-\epsilon+k \end{matrix}\,\right|\, 1\right) = 0.$$

As a result,

$$\int_0^\infty x^k P_n^{[\epsilon]}(x;a,b;c)\mathcal{W}(x;a,b;c+1-\epsilon)\mathrm{d}x = 0, \quad \text{for any } n \geq 2k+2. \tag{4.21a}$$

When $n = 2k+1$,

$$\int_0^\infty x^k P_{2k+1}^{[\epsilon]}(x;a,b;c)\mathcal{W}(x;a,b;c+1-\epsilon)\mathrm{d}x$$
$$= -\frac{(a)_{2k+1}\,(b)_{2k+1}\,(a)_k\,(b)_k}{(c+k+\epsilon)_{2k+1}\,(c+1-\epsilon)_k}\,{}_4F_3\left(\left.\begin{matrix} -(2k+1), a+k, b+k, c+k+\epsilon \\ a, b, c+1-\epsilon+k \end{matrix}\,\right|\, 1\right).$$

In order to evaluate the terminating hypergeometric series in the latter expression, we use (3.21) when $\epsilon = 0$ and (3.20) when $\epsilon = 1$ to deduce that

$$ {}_4F_3\left(\left. \begin{array}{c} -(2k+1), a+k, b+k, c+k+\epsilon \\ a, b, c+1-\epsilon+k \end{array} \right| 1 \right) = \begin{cases} \frac{(2k+1)!(c-a+1)_k(c-b+1)_k}{(c+1+k)_{2k+1}(a)_k(b)_k}, & \text{for } \epsilon = 0, \\ -\frac{(2k+1)!}{(a)_k(b)_k(c+k)}, & \text{for } \epsilon = 1. \end{cases} $$

Therefore,

$$ \int_0^\infty x^k P^{[\epsilon]}_{2k+1}(x;a,b;c)\mathcal{W}(x;a,b;c+1-\epsilon)\mathrm{d}x \tag{4.21b} $$
$$ = \begin{cases} -\dfrac{(2k+1)!\,(a)_{2k+1}\,(b)_{2k+1}\,(c)_k\,(c-a+1)_k\,(c-b+1)_k}{(c)_{3k+1}\,(c+1)_{3k+1}} < 0, & \text{for } \epsilon = 0, \\ \dfrac{(2k+1)!\,(a)_{2k+1}\,(b)_{2k+1}}{(c)_{3k+2}} > 0, & \text{for } \epsilon = 1. \end{cases} $$

The latter identity and (4.21a) ensure that (4.19b) holds for any $k, n \in \mathbb{N}$, with the nonzero coefficients $N^{[\epsilon]}_{2k+1}(a,b;c)$ being given by (4.21b). □

4.4.2 Differential equation

The hypergeometric 2-orthogonal polynomials defined by (4.17)-(4.18) are solutions to a third order differential equation of the type in (3.23), as described in the following result.

Theorem 4.10. *[47, Th. 3.3] For $a, b, c \in \mathbb{R}^+$ such that $c > \max\{a, b\}$ and $\epsilon \in \{0, 1\}$, let $\left(P^{[\epsilon]}_n(x) = P^{[\epsilon]}_n(x;a,b;c)\right)_{n\in\mathbb{N}}$ be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$. Then*

$$ x^2\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(P^{[\epsilon]}_n(x)\right) - x\varphi(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(P^{[\epsilon]}_n(x)\right) + \psi^{[\epsilon]}_n(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(P^{[\epsilon]}_n(x)\right) + n\zeta^{[\epsilon]}_n P^{[\epsilon]}_n(x) = 0, \tag{4.22} $$

where $\varphi(x) = x - (a+b+1)$, $\psi^{[\epsilon]}_n(x) = \left(\left\lfloor \dfrac{n+1-\epsilon}{2} \right\rfloor - (c+1)\right)x + ab$ and $\zeta^{[\epsilon]}_n = c + \left\lfloor \dfrac{n+\epsilon}{2} \right\rfloor$.

Proof. Combining the explicit formula for the polynomials given by (4.17) with the differential equation (3.22) satisfied by hypergeometric polynomials, we obtain

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}+a\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}+b\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(P_n^{[\epsilon]}(x)\right)=\left(x\frac{\mathrm{d}}{\mathrm{d}x}+\zeta_n^{[\epsilon]}\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}-n\right)\left(P_n^{[\epsilon]}(x)\right). \tag{4.23}$$

The left-hand side of (4.23) is equal to

$$x^2\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(P_n^{[\epsilon]}(x)\right)+(a+b+1)x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(P_n^{[\epsilon]}(x)\right)+ab\frac{\mathrm{d}}{\mathrm{d}x}\left(P_n^{[\epsilon]}(x)\right),$$

and, observing that $\zeta_n^{[\epsilon]}-n+1=c+1-\left\lfloor\dfrac{n+1-\epsilon}{2}\right\rfloor$, the right-hand side is

$$x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(P_n^{[\epsilon]}(x)\right)+\left(c+1-\left\lfloor\frac{n+1-\epsilon}{2}\right\rfloor\right)x\frac{\mathrm{d}}{\mathrm{d}x}\left(P_n^{[\epsilon]}(x)\right)-n\zeta_n^{[\epsilon]}P_n(x).$$

Combining the expressions above for both sides of (4.23), we obtain (4.22). □

4.4.3 Recurrence relation

As a 2-orthogonal sequence, the hypergeometric polynomials $P_n^{[\epsilon]}(x)=P_n^{[\epsilon]}(x;a,b;c)$ defined by (4.17) necessarily satisfy a third order recurrence relation of the form

$$P_{n+1}^{[\epsilon]}(x)=\left(x-\beta_n^{[\epsilon]}\right)P_n^{[\epsilon]}(x)-\alpha_n^{[\epsilon]}P_{n-1}^{[\epsilon]}(x)-\gamma_{n-1}^{[\epsilon]}P_{n-2}^{[\epsilon]}(x), \tag{4.24}$$

and we obtain here explicit expressions for the recurrence coefficients.

Recalling formulas (3.26a)-(3.26b), and the expansion of $P_n^{[\epsilon]}(x)$ over the powers of x given by (4.18), we derive that

$$\beta_n^{[\epsilon]}=\tau_{n,1}^{[\epsilon]}-\tau_{n+1,1}^{[\epsilon]}\quad\text{and}\quad\alpha_n^{[\epsilon]}=\tau_{n,2}^{[\epsilon]}-\tau_{n+1,2}^{[\epsilon]}-\left(\tau_{n,1}^{[\epsilon]}\right)^2+\tau_{n,1}^{[\epsilon]}\tau_{n+1,1}^{[\epsilon]}, \tag{4.25}$$

with

$$\tau_{n,1}^{[\epsilon]}=-\frac{n(n-1+a)(n-1+b)}{n+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor-1+c}\quad\text{and}\quad\tau_{n,2}^{[\epsilon]}=\frac{(n-1)_2\,(n-2+a)_2\,(n-2+b)_2}{2\left(n+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor-2+c\right)_2}.$$

Throughout this subsection we use the notation $n \equiv_2 \epsilon$, with $n \in \mathbb{N}$ and $\epsilon \in \{0,1\}$, meaning that $n \equiv \epsilon \pmod 2$. The formulas in (4.25) lead, for each $n \in \mathbb{N}$, to

$$\beta_n^{[\epsilon]}(a,b;c) = \frac{2(n+1)(n+a)(n+b)}{3n+\epsilon+2c} - \frac{2n(n+a-1)(n+b-1)}{3n+\epsilon+2c-2} \quad \text{if } n \equiv_2 \epsilon, \tag{4.26a}$$

and

$$\beta_n^{[\epsilon]}(a,b;c) = \frac{2(n+1)(n+a)(n+b)}{3n+\epsilon+2c+1} - \frac{2n(n+a-1)(n+b-1)}{3n+\epsilon+2c-3} \quad \text{if } n \not\equiv_2 \epsilon; \tag{4.26b}$$

as well as to

$$\begin{aligned}\alpha_{n+1}^{[\epsilon]}(a,b;c) \;&= \frac{2(n+1)(n+a)(n+b)}{3n+2c+\epsilon}\left(\frac{(n+2)(n+a+1)(n+b+1)}{3n+2c+\epsilon+2}\right. \\ &\left. - \frac{2(n+1)(n+a)(n+b)}{3n+2c+\epsilon} + \frac{n(n+a-1)(n+b-1)}{3n+2c+3\epsilon-2}\right) \quad \text{if } n \equiv_2 \epsilon,\end{aligned} \tag{4.26c}$$

and

$$\alpha_{n+1}^{[\epsilon]}(a,b;c) = \alpha_{n+1}^{[1-\epsilon]}(a,b;c+\epsilon) \quad \text{if } n \not\equiv_2 \epsilon. \tag{4.26d}$$

Expanding these formulas we obtain

$$\beta_n^{[\epsilon]}(a,b;c) = \begin{cases} \dfrac{14n^3 + \mathcal{O}(n^2)}{(3n+\epsilon+2c-2)(3n+\epsilon+2c)} & \text{if } n \equiv_2 \epsilon, \\ \dfrac{10n^3 + \mathcal{O}(n^2)}{(3n+\epsilon+2c-3)(3n+\epsilon+2c+1)} & \text{if } n \not\equiv_2 \epsilon; \end{cases}$$

and

$$\alpha_{n+1}^{[\epsilon]}(a,b;c) = \begin{cases} \dfrac{2(n+1)(n+a)(n+b)\left(26n^3 + \mathcal{O}(n^2)\right)}{(3n+2c+\epsilon-2)(3n+2c+\epsilon)^2(3n+2c+\epsilon)} & \text{if } n \equiv_2 \epsilon, \\ \dfrac{2(n+1)(n+a)(n+b)\left(26n^3 + \mathcal{O}(n^2)\right)}{(3n+2c+\epsilon-1)(3n+2c+\epsilon+1)^2(3n+2c+\epsilon+3)} & \text{if } n \not\equiv_2 \epsilon. \end{cases}$$

Furthermore, recalling (3.26c), we have, for each $m \in \mathbb{N}$,

$$\gamma_{2m+1}^{[\epsilon]}(a,b;c) = \frac{\int_0^\infty x^{m+1} P_{2m+2}^{[\epsilon]}(x;a,b;c)\mathcal{W}(x;a,b;c+\epsilon)\mathrm{d}x}{\int_0^\infty x^{m} P_{2m}^{[\epsilon]}(x;a,b;c)\mathcal{W}(x;a,b;c+\epsilon)\mathrm{d}x}$$

and

$$\gamma_{2m+2}^{[\epsilon]}(a,b;c) = \frac{\int_0^\infty x^{m+1} P_{2m+3}^{[\epsilon]}(x;a,b;c)\mathcal{W}(x;a,b;c+1-\epsilon)\mathrm{d}x}{\int_0^\infty x^{m} P_{2m+1}^{[\epsilon]}(x;a,b;c)\mathcal{W}(x;a,b;c+1-\epsilon)\mathrm{d}x}.$$

Hence, due to (4.20b) and (4.21b),

$$\gamma_{n+1}^{[\epsilon]}(a,b;c) = \frac{(n+1)_2\,(n+a)_2\,(n+b)_2}{\left(\frac{3n+\epsilon}{2}\right)_3} \quad \text{if } n \equiv_2 \epsilon; \tag{4.26e}$$

and

$$\gamma_{n+1}^{[\epsilon]}(a,b;c) = \frac{(n+1)_2\,(n+a)_2\,(n+b)_2\left(\frac{n+1+\epsilon}{2}+c-1\right)\left(\frac{n+1+\epsilon}{2}+c-a\right)\left(\frac{n+1+\epsilon}{2}+c-b\right)}{\left(\frac{3n+\epsilon-1}{2}\right)_3\left(\frac{3n+\epsilon+1}{2}\right)_3}$$

if $n \not\equiv_2 \epsilon$. (4.26f)

As a consequence, we proved the following result.

Theorem 4.11. *[47, Th. 3.4] For $a,b,c \in \mathbb{R}^+$ such that $c > \max\{a,b\}$ and $\epsilon \in \{0,1\}$, let $\left(P_n^{[\epsilon]}(x) := P_n^{[\epsilon]}(x;a,b;c)\right)_{n\in\mathbb{N}}$ be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$. Then $\left(P_n^{[\epsilon]}(x)\right)_{n\in\mathbb{N}}$ satisfies the recurrence relation (4.24), with coefficients given by (4.26a)-(4.26f). Furthermore, $\gamma_{n+1}^{[\epsilon]} > 0$, for all $n \in \mathbb{N}$, and the recurrence coefficients have the following asymptotic behaviour of period 2, as $n \to \infty$:*

$$\beta_n^{[\epsilon]} \sim \begin{cases} \frac{14}{9}n & \text{if } n \equiv_2 \epsilon, \\ \frac{10}{9}n & \text{if } n \not\equiv_2 \epsilon; \end{cases} \quad \alpha_n^{[\epsilon]} \sim \frac{52}{81}n^2; \text{ and } \gamma_n^{[\epsilon]} \sim \begin{cases} \frac{8}{729}n^3 & \text{if } n \equiv_2 \epsilon, \\ \frac{8}{27}n^3 & \text{if } n \not\equiv_2 \epsilon. \end{cases} \tag{4.27}$$

4.4.4 Asymptotic behaviour and location of the zeros

Let $\left(P_n^{[\epsilon]}(x) := P_n^{[\epsilon]}(x;a,b;c)\right)_{n\in\mathbb{N}}$ be again the monic 2-orthogonal polynomial sequence with respect to the vector of weight functions $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$, $\epsilon \in \{0,1\}$. As we have already stated in Section 4.1, due to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$ forming a Nikishin system, $P_n^{[\epsilon]}(x)$ has n simple zeros, all located on the positive real line, and the zeros of consecutive polynomials interlace as there is always a zero of $P_n^{[\epsilon]}(x)$ between two consecutive zeros of $P_{n+1}^{[\epsilon]}(x)$.

An upper bound for the largest zero of each $P_n^{[\epsilon]}(x)$, $n \in \mathbb{Z}^+$, is derived immediately from Corollary 3.6, using the asymptotic behaviour of the recurrence coefficients obtained in Theorem 4.11. Furthermore, the asymptotic behaviour of the zeros near the origin is related to the location of the zeros of the generalised hypergeometric function ${}_0F_2\left(-;a,b\,|\,-z\right)$, which are all real and positive, as a consequence of the following Mehler-Heine-type asymptotic formula.

Proposition 4.12. *For $a,b,c \in \mathbb{R}^+$ such that $c > \max\{a,b\}$ and $\epsilon \in \{0,1\}$, let $\left(P_n^{[\epsilon]}(x) = P_n^{[\epsilon]}(x;a,b;c)\right)_{n\in\mathbb{N}}$ be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathcal{W}}^{[\epsilon]}(x;a,b;c)$. Then,*

$$\lim_{n\to\infty}\left(\frac{(-1)^n\left(c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor\right)_n}{(a)_n\,(b)_n}\,P_n^{[\epsilon]}\left(\frac{z}{n^2};a,b;c\right)\right) = {}_0F_2\left(\begin{matrix}-\\a,b\end{matrix}\,\middle|\,-\frac{z}{2}\right), \tag{4.28}$$

uniformly on compact subsets of $\mathbb{C}$.

Proof. Recalling (4.17), we get

$$\frac{(-1)^n\left(c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor\right)_n}{(a)_n\,(b)_n}P_n^{[\epsilon]}\left(\frac{z}{n^2};a,b;c\right) = {}_2F_2\left(\begin{matrix}-n,c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor\\a,b\end{matrix}\,\middle|\,\frac{z}{n^2}\right).$$

Applying the confluent relation for the generalised hypergeometric series (1.11) twice to the identity above, we obtain (4.28). □

Our results on the location of the zeros of the 2-orthogonal polynomials $P_n^{[\epsilon]}(x)$ are summarised in the following theorem.

Theorem 4.13. *(cf. [47, Cor. 3.6]) For* $a, b, c \in \mathbb{R}^+$ *such that* $c > \max\{a, b\}$ *and* $\epsilon \in \{0, 1\}$*, let* $\left(P_n^{[\epsilon]}(x) = P_n^{[\epsilon]}(x; a, b; c)\right)_{n\in\mathbb{N}}$ *be the monic 2-orthogonal polynomial sequence with respect to* $\overline{\mathcal{W}}^{[\epsilon]}(x; a, b; c)$*. Then* $P_n^{[\epsilon]}(x)$ *has* n *zeros* $\left(x_k^{(n)}\right)_{k=1}^{n}$ *such that*

$$0 < x_1^{(n)} < \cdots < x_n^{(n)} < M \cdot n + o(n), \quad \text{as} \quad n \to +\infty,$$

where $M = \dfrac{3}{2}\tau + \beta + \dfrac{\alpha}{2\tau} \approx 3.484$*, with* $\alpha = \dfrac{52}{81}$, $\beta = \dfrac{14}{9}$, $\gamma = \dfrac{8}{27}$,
$\Delta = \gamma^2 - \dfrac{\alpha^3}{27} = \dfrac{1119104}{14348907} > 0$ *and* $\tau = \sqrt[3]{\gamma + \sqrt{\Delta}} + \sqrt[3]{\gamma - \sqrt{\Delta}}$.

Furthermore, if we denote the zeros of ${}_0F_2\left(-; a, b \,|\, -z\right)$*, in increasing order, by* $(f_k)_{k\in\mathbb{Z}^+}$*, we have*

$$\lim_{n\to\infty} \left(n^2 x_k^{(n)}\right) = 2 f_k.$$

We illustrate the upper bound for the zeros given above with $(a, b; c) = (3, 2.5; 7.5)$ in Figure 4.1, produced in *Maple*. The curve $y = 3.484n$ clearly gives an upper bound for the largest zero of $P_n^{[\epsilon]}(x)$, $\epsilon \in \{0, 1\}$.

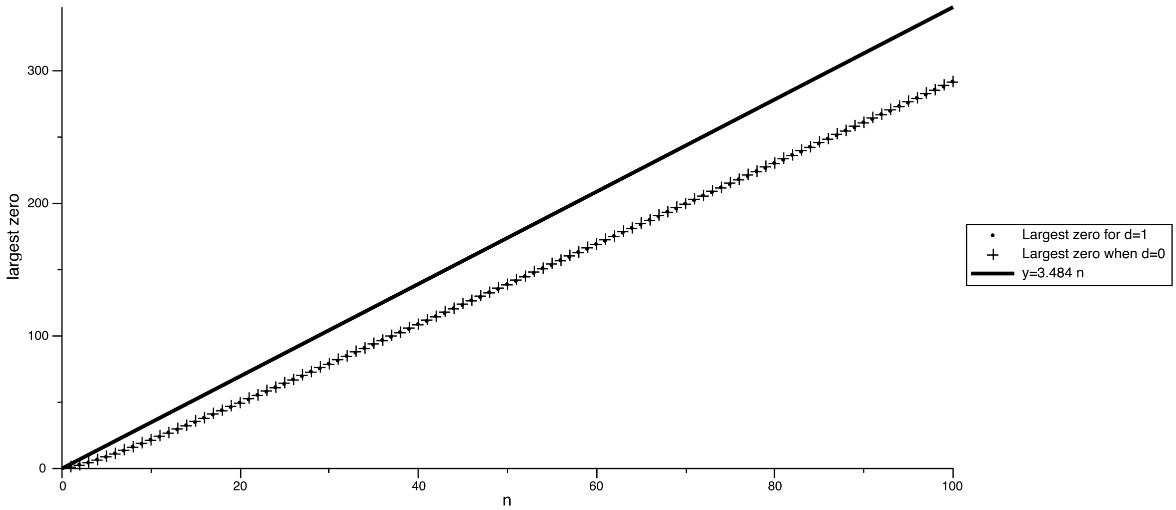

FIGURE 4.1: [47, Fig. 2] Joint plots of the largest zeros of $P_n^{[\epsilon]}(x; 3, 2.5, 7.5)$ for $\epsilon = 0$ (crosses) and $\epsilon = 1$ (dots) for each $n = 1, \ldots, 100$ with the upper bound curve $y = 3.484x$ in solid line.

4.4.5 The cubic components of cases B1 and B2

The aim of this subsection is to show that the cubic components of the Hahn-classical 3-fold-symmetric 2-orthogonal polynomial sequences corresponding to cases B1 and B2 (using the same terminology as in Chapter 2) are particular cases of the 2-orthogonal polynomials we have characterised in this section. We check this relation by comparing both the explicit expressions for the polynomials, as we have done in [47, §4], and, alternatively, by examining the expressions for their orthogonality weights.

Let $(S_n(x;\nu))_{n\in\mathbb{N}}$ and $(T_n(x;\rho))_{n\in\mathbb{N}}$ be the Hahn-classical 3-fold-symmetric 2-orthogonal polynomial sequences in Subsections 2.2.2 and 2.2.3 corresponding to cases B1 and B2, respectively, whose cubic components $\left(S_n^{[k]}(x;\nu)\right)_{n\in\mathbb{N}}$ and $\left(T_n^{[k]}(x;\rho)\right)_{n\in\mathbb{N}}$, $k\in\{0,1,2\}$, are the polynomials explicitly represented as terminating ${}_2F_2$ hypergeometric series by (2.23) and (2.31).

These representations can be obtained taking particular choices on the parameters $(a,b;c;\epsilon)$ in (4.17). For that purpose, we set (a_k,b_k) and ϵ_k as in (2.19) and (2.28), that is,

$$a_k = \begin{cases} \frac{1}{3} & \text{if } k=0, \\ \frac{4}{3} & \text{if } k\in\{1,2\}; \end{cases} \quad b_k = \begin{cases} \frac{2}{3} & \text{if } k\in\{0,1\}, \\ \frac{5}{3} & \text{if } k=2; \end{cases} \quad \text{and } \epsilon_k = \begin{cases} 0 & \text{if } k\in\{0,2\}, \\ 1 & \text{if } k=1; \end{cases} \tag{4.29}$$

and, comparing (4.17) with (2.23) and (2.31), we deduce that

$$S_n^{[k]}(x;\nu) = P_n^{[\epsilon_k]}\left(x;a_k,b_k;\frac{\nu}{3}+b_k\right) \tag{4.30a}$$

and

$$T_n^{[k]}(x;\rho) = P_n^{[1-\epsilon_k]}\left(x;a_k,b_k;\frac{\rho-1}{3}+a_k\right), \tag{4.30b}$$

for each $k\in\{0,1,2\}$.

Alternatively, formulas (4.30a)-(4.30b) can be deduced from the expressions for the orthogonality weights of the cubic components in Subsections 2.2.2 and 2.2.3.

In fact, formulas (2.29a)-(2.29b) and (2.34a)-(2.34b) are equivalent to state that, setting $c_k = \frac{\nu}{3} + b_k$ and $c_k = \frac{\rho+1}{3} - \epsilon_k$, for cases B1 and B2, respectively,

$$\mathcal{U}_0^{[k]}(x) = \mathcal{W}\left(x; a_k, b_k; c_k + \epsilon_k\right)$$

and

$$\mathcal{U}_1^{[k]}(x) = \frac{c_k(c_k+1)}{a_k b_k}\Big(\mathcal{W}\left(x; a_k, b_k; c_k\right) - \mathcal{W}\left(x; a_k, b_k; c_k + 1\right)\Big).$$

Therefore, the 2-orthogonal polynomials with respect to $\left(\mathcal{U}_0^{[k]}(x), \mathcal{U}_1^{[k]}(x)\right)$ and to $\Big(\mathcal{W}(x; a_k, b_k; c_k + \epsilon_k), \mathcal{W}(x; a_k, b_k; c_k + 1 - \epsilon_k)\Big)$ are the same, which means that formulas (4.30a)-(4.30b) hold.

4.5 Link to branched continued fractions

Recalling (3.35), with $(r, s) = (2, 1)$, the ordinary generating function of the moment sequence given by (4.2) is

$$\sum_{n=0}^{\infty}\left(\frac{(a)_n\,(b)_n}{(c)_n}\,t^n\right) = {}_3F_1\left(a, b, 1; c \,|\, t\right) = \frac{{}_3F_1\left(a, b, 1; c \,|\, t\right)}{{}_3F_1\left(a, b, 0; c \,|\, t\right)}.$$

Therefore, using [61, Th. 14.6], the ratios of Pochhammer symbols appearing in the latter formula, which are the moments in (4.2), are equal to the 2-Stieltjes-Rogers polynomials $S_n^{(2)}(\boldsymbol{\lambda}^{[1]})$, with coefficients $\boldsymbol{\lambda}^{[1]} = \left(\lambda_{k+2}^{[1]}\right)_{k\in\mathbb{N}}$, given by

$$\lambda_{k+2}^{[1]} = \begin{cases} \dfrac{\left(c_k^{[1]} - a_k'\right)\displaystyle\prod_{i\in\{1,2,3\},\, i\not\equiv k \bmod 3} a_{i,k}}{\left(c_k^{[1]} - 1\right)c_k^{[1]}} & \text{if } k \text{ is even,} \\[4ex] \dfrac{\displaystyle\prod_{i\in\{1,2,3\},\, i\not\equiv k \bmod 3} a_{i,k}}{c_k^{[1]}} & \text{if } k \text{ is odd,} \end{cases} \tag{4.32}$$

with $a_1 = a$, $a_2 = b$, $a_3 = 1$; $c_k^{[1]} = c + \left\lfloor \frac{k}{2} \right\rfloor$;

$$a_{i,k} = a_i + 1 + \left\lfloor \frac{k-i}{3} \right\rfloor = \begin{cases} a+m & \text{if } i=1 \text{ and } k \in \{3m-2, 3m-1, 3m\}, \\ b+m & \text{if } i=2 \text{ and } k \in \{3m-1, 3m, 3m+1\}, \\ m+1 & \text{if } i=3 \text{ and } k \in \{3m, 3m+1, 3m+2\}; \end{cases}$$

and

$$a'_k = a_{[k-1 \bmod 3]+1,k} = \begin{cases} a_{3,3m} = m+1 & \text{if } k = 3m, \\ a_{1,3m+1} = a+m+1 & \text{if } k = 3m+1, \\ a_{2,3m+2} = b+m+1 & \text{if } k = 3m+2. \end{cases}$$

Note that (4.32) with $k = 0$ gives $\lambda_2^{[1]} = \frac{ab}{c}$, because $a'_0 = 1$. The coefficients in (4.32) can be rewritten, for each $n \in \mathbb{N}$, as

- $\lambda_{6n+2}^{[1]} = \dfrac{a_{1,6n}a_{2,6n}\left(c_{6n}^{[1]} - a'_{6n}\right)}{\left(c_{6n}^{[1]} - 1\right)c_{6n}^{[1]}} = \dfrac{(2n+a)(2n+b)(n+c-1)}{(3n+c-1)(3n+c)}$;
- $\lambda_{6n+3}^{[1]} = \dfrac{a_{2,6n+1}a_{3,6n+1}}{c_{6n+1}^{[1]}} = \dfrac{(2n+1)(2n+b)}{3n+c}$;
- $\lambda_{6n+4}^{[1]} = \dfrac{a_{1,6n+2}a_{3,6n+2}\left(c_{6n+2}^{[1]} - a'_{6n+2}\right)}{\left(c_{6n+2}^{[1]} - 1\right)c_{6n+2}^{[1]}} = \dfrac{(2n+1)(2n+a+1)(n+c-b)}{(3n+c)(3n+c+1)}$;
- $\lambda_{6n+5}^{[1]} = \dfrac{a_{1,6n+3}a_{2,6n+3}}{c_{6n+3}^{[1]}} = \dfrac{(2n+a+1)(2n+b+1)}{3n+c+1}$;
- $\lambda_{6n+6}^{[1]} = \dfrac{a_{2,6n+4}a_{3,6n+4}\left(c_{6n+4}^{[1]} - a'_{6n+4}\right)}{\left(c_{6n+4}^{[1]} - 1\right)c_{6n+4}^{[1]}} = \dfrac{2(n+1)(2n+b+1)(n+c-a)}{(3n+c+1)(3n+c+2)}$;
- $\lambda_{6n+7}^{[1]} = \dfrac{a_{1,6n+5}a_{3,6n+5}}{c_{6n+5}^{[1]}} = \dfrac{2(n+1)(2n+a+2)}{3n+c+2}$.

Similarly to (4.32), we define

$$\lambda_{k+2}^{[0]} = \begin{cases} \dfrac{\displaystyle\prod_{i\in\{1,2,3\},\, i\not\equiv k \bmod 3} a_{i,k}}{c_k^{[0]}} & \text{if } k \text{ is even,} \\ \dfrac{\left(c_k^{[0]} - a'_k\right) \displaystyle\prod_{i\in\{1,2,3\},\, i\not\equiv k \bmod 3} a_{i,k}}{\left(c_k^{[0]} - 1\right) c_k^{[0]}} & \text{if } k \text{ is odd,} \end{cases}$$

where, for $i \in \{1,2,3\}$ and $k \in \mathbb{N}$, a_i, $a_{i,k}$ and a'_k are defined as before and

$$c_k^{[0]} = c + \left\lfloor \frac{k+1}{2} \right\rfloor .$$

Then, we have, for any $n \in \mathbb{N}$,

- $\lambda_{6n+2}^{[0]} = \dfrac{a_{1,6n} a_{2,6n}}{c_{6n}^{[0]}} = \dfrac{(2n+a)(2n+b)}{3n+c}$;
- $\lambda_{6n+3}^{[0]} = \dfrac{a_{2,6n+1} a_{3,6n+1} \left(c_{6n+1}^{[0]} - a'_{6n+1}\right)}{\left(c_{6n+1}^{[0]} - 1\right) c_{6n+1}^{[0]}} = \dfrac{(2n+1)(2n+b)(n+c-a)}{(3n+c)(3n+c+1)}$;
- $\lambda_{6n+4}^{[0]} = \dfrac{a_{1,6n+2} a_{3,6n+2}}{c_{6n+2}^{[0]}} = \dfrac{(2n+1)(2n+a+1)}{3n+c+1}$;
- $\lambda_{6n+5}^{[0]} = \dfrac{a_{1,6n+3} a_{2,6n+3} \left(c_{6n+3}^{[0]} - a'_{6n+3}\right)}{\left(c_{6n+3}^{[0]} - 1\right) c_{6n+3}^{[0]}} = \dfrac{(2n+a+1)(2n+b+1)(n+c)}{(3n+c+1)(3n+c+2)}$;
- $\lambda_{6n+6}^{[0]} = \dfrac{a_{2,6n+4} a_{3,6n+4}}{\left(c_{6n+4}^{[0]} - 1\right) c_{6n+4}^{[0]}} = \dfrac{2(n+1)(2n+b+1)}{3n+c+2}$;
- $\lambda_{6n+7}^{[0]} = \dfrac{a_{1,6n+5} a_{3,6n+5} \left(c_{6n+5}^{[0]} - a'_{6n+5}\right)}{\left(c_{6n+5}^{[0]} - 1\right) c_{6n+5}^{[0]}} = \dfrac{2(n+1)(2n+a+2)(n+c-b+1)}{(3n+c+2)(3n+c+3)}$.

Let $\epsilon \in \{0,1\}$ and $a, b, c \in \mathbb{R}^+$ such that $c > \max\{a, b, \epsilon\}$. Using the expressions above for the coefficients $\left(\lambda_{k+2}^{[\epsilon]}\right)_{k\in\mathbb{N}}$ and setting $\lambda_0^{[\epsilon]} = \lambda_1^{[\epsilon]} = 0$ (which is consistent with the formulas for $\lambda_{6n+6}^{[\epsilon]}$ and $\lambda_{6n+7}^{[\epsilon]}$, with $n = -1$), we checked, using Maple, that the recurrence coefficients given by (4.26a)-(4.26f) can be rewritten as follows, for any $n \in \mathbb{N}$:

- $\beta_{2n}^{[\epsilon]}(a,b;c-\epsilon) = \lambda_{6n}^{[\epsilon]} + \lambda_{6n+1}^{[\epsilon]} + \lambda_{6n+2}^{[\epsilon]};$ (4.33a)

- $\alpha_{2n+1}^{[\epsilon]}(a,b;c-\epsilon) = \lambda_{6n+1}^{[\epsilon]}\lambda_{6n+3}^{[\epsilon]} + \lambda_{6n+2}^{[\epsilon]}\lambda_{6n+3}^{[\epsilon]} + \lambda_{6n+2}^{[\epsilon]}\lambda_{6n+4}^{[\epsilon]};$ (4.33b)

- $\gamma_{2n+1}^{[\epsilon]}(a,b;c-\epsilon) = \lambda_{6n+2}^{[\epsilon]}\lambda_{6n+4}^{[\epsilon]}\lambda_{6n+6}^{[\epsilon]};$ (4.33c)

- $\beta_{2n+1}^{[\epsilon]}(a,b;c-\epsilon) = \lambda_{6n+3}^{[\epsilon]} + \lambda_{6n+4}^{[\epsilon]} + \lambda_{6n+5}^{[\epsilon]};$ (4.33d)

- $\alpha_{2n+2}^{[\epsilon]}(a,b;c-\epsilon) = \lambda_{6n+4}^{[\epsilon]}\lambda_{6n+6}^{[\epsilon]} + \lambda_{6n+5}^{[\epsilon]}\lambda_{6n+6}^{[\epsilon]} + \lambda_{6n+5}^{[\epsilon]}\lambda_{6n+7}^{[\epsilon]};$ (4.33e)

- $\gamma_{2n+2}^{[\epsilon]}(a,b;c-\epsilon) = \lambda_{6n+5}^{[\epsilon]}\lambda_{6n+7}^{[\epsilon]}\lambda_{6n+9}^{[\epsilon]}.$ (4.33f)

Therefore, the infinite lower-Hessenberg matrix associated with the recurrence relation satisfied by $P_n^{[\epsilon]}(x;a,b;c-\epsilon)$, the 2-orthogonal polynomials with respect to $\big(\mathcal{W}(x;a,b;c),\mathcal{W}(x;a,b;c+1-2\epsilon)\big)$, presented in Theorem 4.11, can be decomposed as in (3.36), replacing $(\lambda_{k+2})_{k\in\mathbb{N}}$ by the coefficients $\boldsymbol{\lambda}^{[\boldsymbol{\epsilon}]} = \left(\lambda_{k+2}^{[\epsilon]}\right)_{k\in\mathbb{N}}$. Moreover, this is the production matrix of the sequence of 2-Stieltjes-Rogers polynomials $\big(S_n^{(2)}\left(\boldsymbol{\lambda}^{[\boldsymbol{\epsilon}]}\right)\big)_{n\in\mathbb{N}}$, and $S_n^{(2)}\left(\boldsymbol{\lambda}^{[\mathbf{1}]}\right) = \dfrac{(a)_n\,(b)_n}{(c)_n}$ for all $n\in\mathbb{N}$.

The recurrence coefficients determined by (4.33a)-(4.33f) are all positive and it is clear that, for both $\epsilon\in\{0,1\}$ and any fixed $j\in\mathbb{N}$,

$$\lambda_{6n+j}^{[\epsilon]} \sim \begin{cases} \dfrac{4}{3}\,n & \text{if } j\equiv\epsilon \pmod 2, \\[2mm] \dfrac{4}{9}\,n & \text{if } j\not\equiv\epsilon \pmod 2, \end{cases} \qquad \text{as } n\to\infty.$$

As a result, the asymptotic behaviour (4.27) of the recurrence coefficients in Theorem 4.11 can be derived directly from formulas (4.33a)-(4.33f). Furthermore, recalling the observations made in Section 3.4, we have an alternative proof that the zeros of $P_n^{[\epsilon]}(x)$ are all simple, real and positive with the zeros of consecutive polynomials interlacing, without using the Nikishin property of the system.

4.6 A limiting case: MOPs with respect to Macdonald functions

Throughout this section, we always assume that $a, b \in \mathbb{R}^+$. Under this assumption, we define the weight function supported on the positive real line

$$\mathcal{V}(x; a, b) = \frac{2}{\Gamma(a)\Gamma(b)} x^{\frac{a+b}{2}-1} \mathrm{K}_{a-b}\left(2\sqrt{x}\right), \tag{4.34}$$

where $\mathrm{K}_\nu(z)$ is the Macdonald function defined by (1.13). Observe that, as a consequence of (1.14), we have $\mathcal{V}(x; b, a) = \mathcal{V}(x; a, b)$.

Furthermore, we denote the vector $\left[\mathcal{V}(x; a, b),\, \mathcal{V}(x; a, b+1)\right]^T$ by $\overline{\mathcal{V}}(x; a, b)$. As shown in [75], the vector of weight functions $\overline{\mathcal{V}}(x; a, b)$ forms a Nikishin system. Hence all the multiple orthogonal polynomials with respect to $\overline{\mathcal{V}}(x; a, b)$ exist and are unique, and the polynomials satisfy all the previously mentioned properties common to all Nikishin systems. The multiple orthogonal polynomials on the step-line with respect to this system were introduced in [75] (for both type I and type II) and [11] (only for type II) and were also investigated in [19].

The measure μ supported on $\mathbb{R}^+$, with $\mathrm{d}\mu(x) = \mathcal{V}(x; a, b)\mathrm{d}x$, satisfies (3.1), with $(r, s) = (2, 0)$ and $(a_1, a_2) = (a, b)$, because, based on [22, Eq. 10.43.19],

$$\int_0^\infty x^n \mathcal{V}(x; a, b)\mathrm{d}x = (a)_n\, (b)_n\,, \quad \text{for all } n \in \mathbb{N}. \tag{4.35}$$

Therefore, the multiple orthogonal polynomials with respect to $\overline{\mathcal{V}}(x; a, b)$ are another instance of the multiple orthogonal polynomials studied in Chapter 3, and we can derive some of their properties using results from Chapter 3.

Based on [19, Th. 2] or [11, Th. 4.1], the 2-orthogonal polynomials with respect to $\overline{\mathcal{V}}(x; a, b)$, for $a, b \in \mathbb{R}^+$, are the hypergeometric polynomials

$$R_n(x; a, b) = (-1)^n\, (a)_n\, (b)_n\; {}_1F_2\left(\begin{matrix} -n \\ a, b \end{matrix}\,\middle|\, x\right), \tag{4.36}$$

or, equivalently,

$$R_n(x;a,b)=\sum_{k=0}^{n}\left(\frac{(-1)^k}{k!}\,(n-k+1)_k\,(n-k+a)_k\,(n-k+b)_k\;x^{n-k}\right).$$

We can use Lemmas 3.3 and 3.4 to check that $R_n(x;a;b)$ satisfies the 2-orthogonality conditions with respect to $\overline{\mathcal{V}}(x;a,b)$. Recalling (4.35), we can use Lemma 3.3 to compute, for $k,n\in\mathbb{N}$ and $j\in\{0,1\}$,

$$\begin{aligned}&\int_0^\infty x^k R_n(x;a,b)\mathcal{V}(x;a,b+j)\mathrm{d}x\\ =&(-1)^n\,(a)_n\,(b)_n\,(a)_k\,(b+j)_k\,{}_3F_2\left(\left.\begin{matrix}-n,a+k,b+k+j\\ a,b\end{matrix}\,\right|\,1\right).\end{aligned}$$

Furthermore, using Equation (3.20) from Lemma 3.4, we have

$${}_3F_2\left(\left.\begin{matrix}-n,a+k,b+k+j\\ a,b\end{matrix}\,\right|\,1\right)=\begin{cases}\dfrac{(-1)^j(2k+j)!}{(a)_k\,(b)_{k+j}} & \text{if } n=2k+j,\\ 0 & \text{if } n\geq 2k+j+1.\end{cases}$$

Therefore, we derive that

$$\int_0^\infty x^k R_n(x;a,b)\mathcal{V}(x;a,b+j)\mathrm{d}x=\begin{cases}(2k+j)!\,(a)_{2k+j}\,(b+j)_{2k} & \text{if } n=2k+j,\\ 0 & \text{if } n\geq 2k+1+j.\end{cases}$$

Hence, the sequence $(R_n(x;a,b))_{n\in\mathbb{N}}$ is indeed 2-orthogonal with respect to $\overline{\mathcal{V}}(x;a,b)$.

The polynomials $R_n(x;a,b)$ are a limiting case of the 2-orthogonal polynomials $P_n^{[\epsilon]}(x;a,b;c)$ with respect to $\mathcal{W}^{[\epsilon]}(x;a,b;c)$ defined by (4.7), as

$$\lim_{c\to\infty}\left(c^n P_n^{[\epsilon]}\left(\frac{x}{c};a,b;c\right)\right)=R_n\left(x;a,b\right),\quad \text{for both } \epsilon\in\{0,1\}. \tag{4.37}$$

Furthermore, the weight functions $\mathcal{V}(x;a,b)$ and $\mathcal{W}(x;a,b;c)$, defined, respectively, by (4.34) and (4.1) also satisfy a corresponding confluent relation

$$\lim_{c\to\infty}\left(\frac{1}{c}\,\mathcal{W}\left(\frac{x}{c};a,b;c\right)\right)=\mathcal{V}\left(x;a,b\right). \tag{4.38}$$

To prove the confluent relation (4.37) satisfied by the 2-orthogonal polynomials $P_n^{[\epsilon]}(x;a,b;c)$ and $R_n\left(x;a,b\right)$, we recall (4.17) and get

$$c^n P_n^{[\epsilon]}\left(\frac{x}{c};a,b;c\right) = \frac{(-1)^n\left(a\right)_n\left(b\right)_n c^n}{\left(c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor\right)_n}\,{}_2F_2\left(\left.\begin{matrix}-n,c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor\\ a,b\end{matrix}\right|\frac{x}{c}\right). \tag{4.39}$$

Therefore, observing that

$$c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor \sim c \text{ and } \left(c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor\right)_n \sim c^n, \text{ as } c\to\infty,$$

and applying (1.11) to the polynomials in (4.39), we deduce that

$$\lim_{c\to\infty}\left(\frac{c^n}{\left(c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor\right)_n}{}_2F_2\left(\left.\begin{matrix}-n,c+\left\lfloor\frac{n+\epsilon}{2}\right\rfloor\\ a,b\end{matrix}\right|\frac{x}{c}\right)\right) = {}_1F_2\left(\left.\begin{matrix}-n\\ a,b\end{matrix}\right|x\right),$$

which, by definition of $P_n^{[\epsilon]}(x;a,b;c)$ and $R_n(x;a,b)$, is equivalent to (4.37).

Now we prove the confluent relation (4.38) satisfied by the weight functions $\mathcal{W}(x;a,b;c)$ and $\mathcal{V}(x;a,b)$. Based on [27, Eq. 6.9.19], we have

$$\lim_{c\to\infty}\left(\Gamma(c-a)\mathbf{U}\left(c-b,a-b+1;\frac{x}{c}\right)\right) = 2x^{\frac{b-a}{2}}\mathrm{K}_{a-b}\left(2\sqrt{x}\right), \tag{4.40}$$

Moreover, because $\Gamma(c-a)\sim c^{-a}\Gamma(c)$ as $c\to\infty$ (see [22, Eq. 5.11.12]), and $\mathrm{e}^{-\frac{x}{c}}\xrightarrow{c\to\infty}1$, the limiting relation (4.40) implies that

$$\lim_{c\to\infty}\left(\frac{\Gamma(c)}{c}\,\mathrm{e}^{-\frac{x}{c}}\left(\frac{x}{c}\right)^{a-1}\mathbf{U}\left(c-b,a-b+1;\frac{x}{c}\right)\right) = 2x^{\frac{a+b}{2}-1}\mathrm{K}_{a-b}\left(2\sqrt{x}\right),$$

which, by definition of $\mathcal{W}(x;a,b;c)$ and $\mathcal{V}(x;a,b)$, is equivalent to (4.38).

Let $\left(\mathcal{S}_n(x)\right)_{n\in\mathbb{N}}$ be the Hahn-classical 3-fold symmetric 2-orthogonal polynomial sequence corresponding to case A (using the same terminology as in Chapter 2), whose cubic components $\left(\mathcal{S}_n^{[k]}(x)\right)_{n\in\mathbb{N}}$, $k\in\{0,1,2\}$, are the polynomials explicitly represented as terminating ${}_1F_2$ hypergeometric series by (2.18). As observed in [11, Th. 5.1], the cubic components $\mathcal{S}_n^{[k]}(x)$ are particular cases of the 2-orthogonal polynomials $\left(R_n(x;a,b)\right)_{n\in\mathbb{N}}$. In fact, if we set again (a_k,b_k) as in (2.19) and (4.29), then comparing (2.18) with (4.36), we get, for each $k \in \{0,1,2\}$,

$$\mathcal{S}_n^{[k]}(x) = 9^n R_n\left(\frac{x}{9}; a_k, b_k\right). \tag{4.41}$$

Alternatively, this result can be derived from the expressions (2.21a)-(2.21b) for the orthogonality weights of the cubic components for case A, because these can be rewritten as

$$\mathcal{U}_0^{[k]}(x) = \frac{1}{9}\,\mathcal{V}\left(\frac{x}{9}; a_k, b_k\right)$$

and

$$\mathcal{U}_1^{[k]}(x) = \frac{1}{81a_k}\left(\mathcal{V}\left(\frac{x}{9}; a_k, b_k+1\right) - \mathcal{V}\left(\frac{x}{9}; a_k, b_k\right)\right).$$

Considering (4.41) and (4.30a)-(4.30b), the confluent relation (4.37) generalises the limiting relation observed in [23] for Hahn-classical 3-fold symmetric 2-orthogonal polynomials: taking $\nu, \rho \to \infty$ in cases B1 and B2, respectively, leads to case A.

Now, we use results from Chapter 3 to derive some properties of the 2-orthogonal polynomials $R_n(x;a,b)$ and their orthogonality weights $\mathcal{V}(x;a,b)$. These properties were obtained in [75], [11] and [19].

The weight function defined by (4.34) satisfies the differential equation (cf. [11, Eq. 4.14])

$$x^2\,\mathcal{V}''(x;a,b) - (a+b-3)x\,\mathcal{V}'(x;a,b) - \big(x-(a-1)(b-1)\big)\mathcal{V}(x;a,b) = 0, \tag{4.43a}$$

which, defining the operator $\Theta : \mathcal{P} \to \mathcal{P}$ by $\Theta\left(f(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\big(xf(x)\big)$ as in Proposition (4.3), is equivalent to

$$\Theta^2\big(\mathcal{V}(x;a,b)\big) - (a+b)\Theta\big(\mathcal{V}(x;a,b)\big) - (x-ab)\mathcal{V}(x;a,b) = 0. \tag{4.43b}$$

Note that, recalling Proposition 3.1, the formula for the moments given by (4.35) is equivalent to all the moments of the left-hand side of (4.43b) integrated over the positive real line vanishing.

The differential equations (4.43a)-(4.43b) can be used to derive a matrix differential equation satisfied by the vector of weight functions $\overline{\mathcal{V}}(x;a,b)$. To be precise, if we define the matrices

$$\Phi(a,b) = \begin{bmatrix} \dfrac{1}{ab} & 0 \\ 0 & \dfrac{1}{a(b+1)} \end{bmatrix} \quad \text{and} \quad \Psi(x;a,b) = \begin{bmatrix} -\dfrac{1}{a} & \dfrac{1}{a} \\ \dfrac{x}{ab(b+1)} & -\dfrac{1}{b+1} \end{bmatrix},$$

then we have

$$x\,\Phi(a,b)\overline{\mathcal{V}}(x;a,b) = \overline{\mathcal{V}}(x;a+1,b+1) \tag{4.44}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x\,\Phi(a,b)\overline{\mathcal{V}}(x;a,b)\big) + \Psi(x;a,b)\overline{\mathcal{V}}(x;a,b) = 0. \tag{4.45}$$

For each $n \in \mathbb{N}$, let $Q_n(x;a,b)$ and $R_n(x;a,b)$ be, respectively, the type I function and the 2-orthogonal polynomial with respect to $\overline{\mathcal{V}}(x;a,b)$ for the multi-index of length n on the step-line. Based on Theorem 3.2, and as a consequence of (4.44) and (4.45), we obtain the differential formulas (cf. [75, Ths. 1 & 2])

$$R'_{n+1}(x;a,b) = (n+1)R_n(x;a+1,b+1), \tag{4.46}$$

and

$$Q'_n(x;a+1,b+1) = -n\,Q_{n+1}(x;a,b). \tag{4.47}$$

Observe that (4.46) could alternatively be obtained by applying the formula (1.9) for the derivative of a generalised hypergeometric series to the representation as a ${}_1F_2$ for $R_n(x;a,b)$ given by (4.36).

Furthermore, (4.47) can be used to obtain a Rodrigues-type formula generating the type I functions $Q_{n+1}(x;a,b)$, for $n \in \mathbb{N}$ (cf. [75, Th. 3]):

$$Q_{n+1}(x;a,b) = \frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(\mathcal{W}(x;a+n,b+n)\Big) = \frac{(-1)^n}{n!\,(a)_n\,(b)_n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(x^n\mathcal{W}(x;a,b)\Big).$$

Now we focus on the characterisation of the 2-orthogonal polynomials $R_n(x;a,b)$ defined by (4.36). Recalling (3.22), they satisfy the third-order differential equation (cf. [11, Eq. 2.25])

$$x^2 R_n'''(x;a,b) + (a+b+1)xR_n''(x;a,b) - (x-ab)R_n'(x;a,b) + nR_n(x;a,b) = 0.$$

Furthermore, as a 2-orthogonal polynomial sequence, $(R_n(x;a,b))_{n\in\mathbb{N}}$ necessarily satisfies a third order recurrence relation of the form

$$P_{n+1}(x;a,b) = (x-\beta_n)\,P_n(x;a,b) - \alpha_n P_{n-1}(x;a,b) - \gamma_{n-1}P_{n-2}(x;a,b), \quad (4.48)$$

and, recalling (3.26a)-(3.26c), the recurrence coefficients are given, for each $n \in \mathbb{N}$, by (cf. [75, Th. 4])

- $\beta_n = 3n^2 + (2a+2b-1)n + ab;$ (4.49a)
- $\alpha_{n+1} = (n+1)(n+a)(n+b)(3n+a+b+1);$ (4.49b)
- $\gamma_{n+1} = (n+1)(n+2)(n+a)(n+a+1)(n+b)(n+b+1).$ (4.49c)

As it was mentioned in the beginning of this section, $\overline{\mathcal{V}}(x;a,b)$ forms a Nikishin system. Therefore, the 2-orthogonal polynomials $R_n(x;a,b)$, have n real, positive and simple zeros, which we denote by $\left(x_k^{(n)}\right)_{k=1}^{n}$, and the zeros of consecutive polynomials interlace.

It is clear from (4.49a)-(4.49c) that the asymptotic behaviour of the recurrence coefficients is

$$\beta_n \sim 3n^2, \quad \alpha_n \sim 3n^4, \quad \text{and} \quad \gamma_n \sim n^6, \quad \text{as } n \to \infty.$$

As a result, using Corollary 3.6 with $M_n = n^2$, $\alpha = \beta = 3$ and $\gamma = 1$, we obtain an upper bound for the largest zero of $R_n(x;a,b)$ and we have

$$0 < x_1^{(n)} < \cdots < x_n^{(n)} < \frac{27}{4}n^2 + o(n^2), \quad n \to +\infty. \quad (4.50)$$

Applying the confluent relation (1.11) to (4.36), we deduce that the 2-orthogonal polynomials $R_n(x;a,b)$ satisfy a Mehler-Heine-type asymptotic formula near the origin (cf. [72, Th. 6])

$$\lim_{n\to\infty} \frac{(-1)^n}{(a)_n\,(b)_n} P_n\left(\frac{x}{n};a,b\right) = \lim_{n\to\infty} {}_1F_2\left(\begin{matrix}-n\\ a,b\end{matrix}\,\middle|\, \frac{x}{n}\right) = {}_0F_2\left(\begin{matrix}-\\ a,b\end{matrix}\,\middle|\, -x\right). \tag{4.51}$$

which converges uniformly on compact subsets of $\mathbb{C}$.

As a consequence of this asymptotic behaviour of the 2-orthogonal polynomials $R_n(x;a,b)$ near 0, we can relate their zeros near 0 with the zeros of ${}_0F_2\left(-;a,b\,|\,-z\right)$, which are all real and positive. In fact, if we denote the zeros of $R_n(x;a,b)$ as in (4.50) and the zeros of ${}_0F_2\left(-;a,b\,|\,-z\right)$, in increasing order, by $(f_k)_{k\in\mathbb{Z}^+}$, then, as a consequence of (4.51) and as explained in [72, §4],

$$\lim_{n\to\infty} n x_k^{(n)} = f_k.$$

Recalling (3.35), with $(r,s)=(2,0)$, the ordinary generating function of the moment sequence given by (4.35) is

$$\sum_{n=0}^{\infty} (a)_n\,(b)_n\,t^n = {}_3F_0\left(a,b,1;-\,|\,t\right) = \frac{{}_3F_0\left(a,b,1;-\,|\,t\right)}{{}_3F_0\left(a,b,0;-\,|\,t\right)}.$$

As a result, using [61, Th. 13.1], the product of Pochhammer symbols $(a)_n\,(b)_n$ is equal to the 2-Stieltjes-Rogers polynomial $S_n^{(2)}(\boldsymbol{\lambda})$, with coefficients $\boldsymbol{\lambda}=(\lambda_k)_{k\geq 2}$ given, for any $n\in\mathbb{N}$, by

- $\lambda_{3n+2} = (n+a)(n+b)$;
- $\lambda_{3n+3} = (n+1)(n+b)$;
- $\lambda_{3n+4} = (n+1)(n+a+1)$.

Therefore, setting $\lambda_0 = \lambda_1 = 0$, which is consistent with the formulas above, the recurrence coefficients (4.49a)-(4.49c) can be rewritten, for each $n \in \mathbb{N}$, as

- $\beta_n = \lambda_{3n} + \lambda_{3n+1} + \lambda_{3n+2}$,
- $\alpha_{n+1} = \lambda_{3n+1}\lambda_{3n+3} + \lambda_{3n+2}\lambda_{3n+3} + \lambda_{3n+2}\lambda_{3n+4}$,
- $\gamma_{n+1} = \lambda_{3n+2}\lambda_{3n+4}\lambda_{3n+6}$.

Therefore, the infinite lower-Hessenberg matrix associated with the recurrence relation (4.48) can be decomposed as in (3.36), via the coefficients $(\lambda_{k+2})_{k\in\mathbb{N}}$ given by (4.6). This is the production matrix of the sequence of 2-Stieltjes-Rogers polynomials $S_n^{(2)}(\boldsymbol{\lambda})$, $n \in \mathbb{N}$. Therefore, recalling the observations made in Section 3.4, we have an alternative proof, without using the Nikishin property of the system, that the zeros of $R_n(x; a, b)$ are all simple, and located on the positive real line, with the zeros of consecutive polynomials interlacing.

Chapter 5

MOPs with respect to Gauss' hypergeometric function

In this chapter we investigate the multiple orthogonal polynomials with respect to two absolutely continuous measures supported on the interval $(0,1)$ and admitting integral representations via weight functions $\mathcal{W}(x;a,b;c,d)$ and $\mathcal{W}(x;a,b+1;c+1,d)$, involving Gauss' hypergeometric function (1.5) and defined by

$$\mathcal{W}(x;a,b;c,d) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(\delta)}\, x^{a-1}(1-x)^{\delta-1}\, {}_2F_1\left(\left.\begin{matrix} c-b, d-b \\ \delta \end{matrix}\right| 1-x\right), \quad (5.1)$$

$$\text{with } a,b,c,d \in \mathbb{R}^+ \text{ such that } \min\{c,d\} > \max\{a,b\} \text{ and } \delta = c+d-a-b. \quad (5.2)$$

This chapter is based on the work presented in [48]. Similarly to Chapter 4, most of the underlying techniques in this chapter are specialisations of those introduced in Chapter 3. All the results obtained are original and new to the literature.

Observe that the parameters a and b are interchangeable and the same happens with c and d. The latter is a straightforward consequence of combining (5.1) with (1.5), while the former holds because, using [22, Eq. 15.8.1], we have

$${}_2F_1\left(\left.\begin{matrix} c-b, d-b \\ \delta \end{matrix}\right| 1-x\right) = x^{b-a}\, {}_2F_1\left(\left.\begin{matrix} d-a, c-a \\ \delta \end{matrix}\right| 1-x\right).$$

Under the assumptions (5.2), $\mathcal{W}(x;a,b;c,d)$ is integrable over the positive real line. In fact (see [13, Eq. 2.21.1.11] or [33, Eq. 7.512.4])

$$\int_0^1 x^{a-1}(1-x)^{\delta-1}\,{}_2F_1\left(\begin{matrix} c-b, d-b \\ \delta \end{matrix}\,\middle|\, 1-x\right)\mathrm{d}x = \frac{\Gamma(a)\Gamma(b)\Gamma(\delta)}{\Gamma(c)\Gamma(d)}.$$

Therefore, $\mathcal{W}(x;a,b;c,d)$ is a probability density function on $(0,1)$ with moments

$$\int_0^1 x^n \mathcal{W}(x;a,b;c,d)\mathrm{d}x = \frac{(a)_n\,(b)_n}{(c)_n\,(d)_n}, \quad n \in \mathbb{N}, \tag{5.3}$$

Hence, the measure μ supported on the interval $(0,1)$, with $\mathrm{d}\mu(x) = \mathcal{W}(x;a,b;c,d)\,\mathrm{d}x$, satisfies (3.1) with $(r,s) = (2,2)$ and $(a_1,a_2;b_1;b_2) = (a,b;c,d)$.

In Section 5.1, we prove that the weight functions $\mathcal{W}(x;a,b;c,d)$ and $\mathcal{W}(x;a,b+1;c+1,d)$ form a Nikishin system on the interval $(0,1)$ (Theorem 5.1). This readily implies that the multiple orthogonal polynomials of both type I and type II with respect to these weight functions exist and are unique for every multi-index and their zeros satisfy the properties common to all Nikishin systems. We also obtain explicit formulas for the generating measure of the ratio of the weight functions.

In Section 5.2, we describe the weight function $\mathcal{W}(x;a,b;c,d)$ and the vector of weights $\left[\mathcal{W}(x;a,b;c,d),\, \mathcal{W}(x;a,b+1;c+1,d)\right]^T$ as solutions to a second-order ordinary differential equation (Proposition 5.2) and a matrix first-order differential equation (Theorem 5.3), respectively. The latter implies that both type II polynomials and type I functions on the step-line satisfy the Hahn property because the differentiation operator acts on them as a shift in the parameters and in the index (see Theorem 5.4).

In Section 5.3, we focus on the type I polynomials on the step-line: we obtain a Rodrigues-type formula generating the type I functions (Theorem 5.5) and a matrix Rodrigues-type formula generating the type I polynomials (Theorem 5.6).

Section 5.4 is devoted to a detailed characterisation of the type II multiple orthogonal polynomials on the step-line, that is, the 2-orthogonal polynomials. This characterisation includes finding an explicit representation for these polynomials as terminating generalised hypergeometric series ${}_3F_2$ (Theorem 5.7), and obtaining a third order differential equation (Theorem 5.8) and recurrence relation (Theorem 5.9) of which these 2-orthogonal polynomials are a solution. The recurrence coefficients are bounded and they have the same asymptotic behaviour as the recurrence coefficients of the Jacobi-Piñeiro 2-orthogonal polynomials. As a result, the ratio asymptotics of two consecutive polynomials and the asymptotic zero distribution (Theorem 5.10) as well as a Mehler-Heine-type formula for the asymptotic behaviour near the origin (Proposition 5.11) and the consequent information about the zeros near the origin all coincide with the corresponding results observed for the Jacobi-Piñeiro polynomials in [17, 72]. Next, we observe that if we impose the condition $d = c + \frac{1}{2}$, then the polynomials characterised in Section 5.4 reduce to the Jacobi-type 2-orthogonal polynomials investigated in [46]. A particular case of these polynomials corresponds to a 2-orthogonal polynomial sequence with constant recurrence coefficients. We end Section 5.4 by showing that the cubic components of case C of the Hahn-classical 3-fold-symmetric 2-orthogonal polynomials analysed in Chapter 2 are particular cases of the 2-orthogonal polynomials studied here and that the polynomials characterised in Chapter 4 are a limiting case of the polynomials analysed in this chapter.

The starting point of Section 5.5 is a branched-continued-fraction representation for the ordinary generating function of the moment sequence given by (5.3), which is a generalised hypergeometric series ${}_3F_2$. We obtain explicit formulas for the coefficients of this branched continued fraction, which we use to obtain alternative expressions for the recurrence coefficients of the 2-orthogonal polynomials characterised in Section 5.4. As a result, the recurrence coefficients are all positive and we derive a decomposition of the infinite lower-Hessenberg matrix associated with these 2-orthogonal polynomials as a product of bidiagonal matrices, which is equal to the production matrix of a sequence of 2-Stieltjes-Rogers polynomials: the moment sequence given by (5.3).

5.1 Nikishin system

In this section, we use the connection between continued fractions (in this case, S-fractions) and Stieltjes transforms introduced in Section 1.5 to prove that the pair of weight functions $\big(\mathcal{W}(x;a,b+1;c+1,d),\mathcal{W}(x;a,b;c,d)\big)$ forms a Nikishin system, as explained in Theorem 5.1.

As a result, the type I and II multiple orthogonal polynomials with respect to these weight functions exist and are unique for every multi-index $(n_0,n_1)\in\mathbb{N}^2$; the type I multiple orthogonal polynomials $A_{(n_0,n_1)}$ and $B_{(n_0,n_1)}$ have degree exactly n_0-1 and n_1-1, respectively; and the type II multiple orthogonal polynomial $P_{(n_0,n_1)}$ has n_0+n_1 simple zeros located on the interval $(0,1)$ such that there is always a zero of $P_{(n_0,n_1)}$ between two consecutive zeros of $P_{(n_0+1,n_1)}$ or $P_{(n_0,n_1+1)}$.

Theorem 5.1. *[48, Th. 2.1] Let $\mathcal{W}(x):=\mathcal{W}(x;a,b;c,d)$ be given by (5.1), under the assumptions (5.2). Then,*

$$\frac{\mathcal{W}(x;a,b;c,d)}{\mathcal{W}(x;a,b+1;c+1,d)}=\mathop{\mathrm{K}}_{n=0}^{\infty}\left(\frac{\alpha_n}{1}\right), \tag{5.4}$$

where $\alpha_0=\dfrac{b}{c}$ and, for $n\geq 1$, $\alpha_n=(1-g_{n-1})\,g_n(x-1)$, with $g_0=0$ and

$$g_0=0;\quad g_{2k+1}=\frac{c-b+k}{\delta+2k}\quad \text{and}\quad g_{2k+2}=\frac{d-b+k}{\delta+2k+1}\quad \text{for all } k\in\mathbb{N}. \tag{5.5}$$

Moreover, there exist probability density measures σ in $(0,1)$ and θ in $(1,+\infty)$ such that

$$\frac{\mathcal{W}(x;a,b;c,d)}{\mathcal{W}(x;a,b+1;c+1,d)}=\frac{b}{c}\int_0^1\frac{\mathrm{d}\sigma(t)}{1+t(x-1)}=\frac{b}{c}\int_{-\infty}^0\frac{\mathrm{d}\theta(1-s)}{x-s}. \tag{5.6}$$

Therefore, the pair of weight functions $\big(\mathcal{W}(x;a,b+1;c+1,d),\mathcal{W}(x;a,b;c,d)\big)$ forms a Nikishin system on the interval $(0,1)$.

Proof. Recalling (5.1),

$$\frac{\mathcal{W}(x;a,b;c,d)}{\mathcal{W}(x;a,b+1;c+1,d)} = \frac{b}{c}\,\frac{{}_2F_1\left(c-b,d-b;\delta\,|\,1-x\right)}{{}_2F_1\left(c-b,d-b-1;\delta\,|\,1-x\right)}. \tag{5.7}$$

Therefore, the ratio of weight functions above admits a representation similar to Gauss' continued fraction. In fact, accordingly to [45, Eqs. 2.7-2.8], the ratio of weights in (5.7) can be represented by a continued fraction of the type on the right-hand side of (5.4), with $\alpha_0 = \dfrac{b}{c}$ and, for $n \geq 1$, $\alpha_n = \tilde{\alpha}_n(x-1)$, where $(\tilde{\alpha}_n)_{n\in\mathbb{Z}^+}$ is the chain sequence (see [78, Ch. IV, §19] for more information on chain sequences) of the form $\tilde{\alpha}_n = (1-g_{n-1})\,g_n$, with $(g_n)_{n\in\mathbb{N}}$ given by (5.5).

Observe that $\min\{c,d\} > \max\{a,b\}$ implies that $0 < g_n < 1$ for all $n \geq 1$. As a result, the continued fraction described above is of the type in [78, Eq. 27.8]. Therefore, based on [78, Eq. 67.5], there exists a probability density function σ in $(0,1)$ such that the first integral representation in (5.6) holds. Alternatively, combining [78, Ths. 66.1 & 27.5], we deduce that there exists a probability density function θ in $(1,+\infty)$ such that

$$\frac{\mathcal{W}(x;a,b;c,d)}{\mathcal{W}(x;a,b+1;c+1,d)} = \frac{b}{c}\int_{-\infty}^{-1}\frac{\mathrm{d}\theta(-u)}{x-1-u}.$$

The second integral representation in (5.6) is obtained from the one above via the change of variable $s = u+1$. □

Under the additional assumption $b > a-1$ and using a recent result in [26, §4, Ex. 2] (which Alex Dyachenko kindly shared with us whilst in draft form), the generating measure σ in (5.6) admits the integral representation

$$\frac{\mathcal{W}(x;a,b;c,d)}{\mathcal{W}(x;a,b+1;c+1,d)} = \frac{b}{c}\left(\int_0^1 \frac{\lambda\, t^{c+d-2b-2}(1-t)^{b-a}\mathrm{d}t}{(1+t(x-1))\left|{}_2F_1\left(c-b,d-b-1;\delta\,|\,t^{-1}\right)\right|^2} + K\right),$$

with

$$\lambda = \frac{(\Gamma(\delta))^2}{\Gamma(c-b)\Gamma(d-b)\Gamma(d-a)\Gamma(c-a+1)} \text{ and } K = \begin{cases} 0 & \text{if } d \leq c+1,\\[4pt] \dfrac{d-c-1}{d-b-1} & \text{if } d \geq c+1.\end{cases}$$

Note that K represents a mass point at the origin when $d > c+1$. The change of variable $t = (1-s)^{-1}$ in the latter integral representation gives an explicit canonical representation as a Stieltjes transform for the ratio of the weight functions:

$$\frac{\mathcal{W}(x;a,b;c,d)}{\mathcal{W}(x;a,b+1;c+1,d)} = \frac{b}{c}\left(\int_{-\infty}^{0} \frac{\lambda(-s)^{b-a}(1-s)^{1-\delta}\,\mathrm{d}s}{(x-s)\left|{}_2F_1\left(c-b, d-b-1; \delta \,\middle|\, 1-s\right)\right|^2} + K\right).$$

5.2 Differential properties

We describe the weight function $\mathcal{W}(x;a,b;c,d)$ in (5.1) and the vector

$$\overline{\mathcal{W}}(x;a,b;c,d) := \begin{bmatrix} \mathcal{W}(x;a,b;c,d) \\ \mathcal{W}(x;a,b+1;c+1,d) \end{bmatrix}, \tag{5.9}$$

as solutions to a second-order ordinary differential equation and to a first order matrix differential equation, respectively. The latter is crucial to prove that the derivative operator acts on the multiple orthogonal polynomials of type II and on the functions of type I on the step-line as a shift in their index and parameters, revealing that they satisfy the Hahn-classical property.

Proposition 5.2. *(cf. [48, Prop. 2.2]) Let $\mathcal{W}(x) := \mathcal{W}(x;a,b;c,d)$ be given by (5.1), under the assumptions (5.2). Then*

$$(1-x)x^2\mathcal{W}''(x) + \eta(x)x\,\mathcal{W}'(x) + \psi(x)\mathcal{W}(x) = 0, \tag{5.10a}$$

with $\eta(x) = (c+d-5)x - (a+b-3)$ and $\psi(x) = (a-1)(b-1) - (c-2)(d-2)x$. Equivalently, defining the operator $\Theta : \mathcal{P} \to \mathcal{P}$ by $\Theta\left(f(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\big(xf(x)\big)$,

$$\Theta^2\big((1-x)\mathcal{W}(x)\big) + \Theta\big(\left((c+d)x - (a+b)\right)\mathcal{W}(x)\big) + (ab - cdx)\mathcal{W}(x) = 0. \tag{5.10b}$$

Based on Proposition 3.1, the formula for the moments of $\mathcal{W}(x;a,b;c,d)$ given by (5.3) is equivalent to all the moments of the left-hand side of (5.10b) integrated over the interval $(0,1)$ vanishing.

Proof. The equivalence between (5.10a) and (5.10b) is easily checked by expanding the latter to obtain the former. As such, it is sufficient to prove (5.10a). To simplify the notation, we denote ${}_2F_1\left(c-b,d-b;\delta\,|\,z\right)$ by $F(z)$. Then, after differentiating (5.1) twice, we have

$$\mathcal{W}^{(j)}(x) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(\delta)}\, x^{a-1-j}(1-x)^{\delta-1-j}F_j(x), \quad \text{for } j \in \{0,1,2\}, \tag{5.11}$$

with $F_0(x) = F(1-x)$,

$$\begin{aligned} F_1(x) &= \big((2-a-\delta)x + (a-1)\big)F_0(x) + x(1-x)F_0'(x) \\ &= \big((2+b-c-d)x + (a-1)\big)F(1-x) - x(1-x)F'(1-x) \end{aligned}$$

and

$$\begin{aligned} F_2(x) &= \big((4+b-c-d)x + (a-2)\big)F_1(x) + x(1-x)F_1'(x) \\ &= x^2(1-x)^2F''(1-x) + 2x(1-x)g_1(x)F'(1-x) + g_2(x)F(1-x), \end{aligned}$$

where $g_1(x) = (c+d-b-2)x + (1-a)$ and

$$g_2(x) = (c+d-b-2)(c+d-b-3)x^2 - 2(a-1)(c+d-b-3)x + (a-1)(a-2).$$

Recalling (1.6), we derive that $F(1-x) = {}_2F_1\left(c-b,d-b;\delta\,|\,1-x\right)$ satisfies

$$x(1-x)F''(1-x) = (c-b)(d-b)F(1-x) - \big((c+d-2b+1)x + (b-a-1)\big)F'(1-x).$$

As a result, we can rewrite $F_2(x)$ as

$$F_2(x) = -\eta(x)F_1(x) - (1-x)\psi(x)F_0(x),$$

and, combining the latter relation with (5.11), we find (5.10a). ☐

Next, we present a system of first order differential equations for which the vector (5.9) is a solution, as described in the following theorem.

Theorem 5.3. *[48, Th. 2.3] Let* $\overline{\mathcal{W}}(x;a,b;c,d)$ *as defined in* (5.9)*, subject to* (5.2)*. Then, the following identities hold*

$$x\,\Phi(x)\overline{\mathcal{W}}(x;a,b;c,d) = \overline{\mathcal{W}}(x;a+1,b+1;d+1,c+2) \tag{5.12}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(x\,\Phi(x)\overline{\mathcal{W}}(x;a,b;c,d)\Big) + \Psi(x)\overline{\mathcal{W}}(x;a,b;c,d) = 0, \tag{5.13}$$

where

$$\Phi(x) := \Phi(x;a,b;c,d) = \begin{bmatrix} \dfrac{c(c+1)d}{ab(c-b)} & -\dfrac{(c+1)d}{a(c-b)} \\ -\dfrac{c(c+1)d(d+1)}{ab(b+1)(d-a)}\,x & \dfrac{(c+1)d(d+1)}{a(b+1)(d-a)} \end{bmatrix} \tag{5.14}$$

and

$$\Psi(x) := \Psi(x;a,b;c,d) = \begin{bmatrix} -\dfrac{c(c+1)d}{a(c-b)} & \dfrac{c(c+1)d}{a(c-b)} \\ \dfrac{c(c+1)d^2(d+1)}{ab(b+1)(d-a)}\,x & -\dfrac{(c+1)d(d+1)}{(b+1)(d-a)} \end{bmatrix}. \tag{5.15}$$

Proof. In order to prove (5.12), we need to check that

$$\begin{bmatrix} \mathcal{V}_0(x) \\ \mathcal{V}_1(x) \end{bmatrix} := x\,\Phi(x) \begin{bmatrix} \mathcal{W}(x;a,b;c,d) \\ \mathcal{W}(x;a,b+1;c+1,d) \end{bmatrix} = \begin{bmatrix} \mathcal{W}(x;a+1,b+1;d+1,c+2) \\ \mathcal{W}(x;a+1,b+2;d+2,c+2) \end{bmatrix}.$$

To shorten the notation, we write $\alpha = c-b$ and $\beta = d-b$. Then,

$$\begin{aligned} \mathcal{V}_0(x) &= x\,\frac{(c+1)d}{a\alpha}\Big(\frac{c}{b}\,\mathcal{W}(x;a,b;c,d) - \mathcal{W}(x;a,b+1;c+1,d)\Big) \\ &= \frac{\Gamma(c+2)\Gamma(d+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(\delta)}\,x^a(1-x)^{\delta-1}\,\frac{\tilde{V_0}(x)}{\alpha}, \end{aligned}$$

with

$$\tilde{V_0}(x) = {}_2F_1\left(\left.\begin{matrix}\alpha,\beta\\ \delta\end{matrix}\,\right|\,1-x\right) - {}_2F_1\left(\left.\begin{matrix}\alpha,\beta-1\\ \delta\end{matrix}\,\right|\,1-x\right).$$

Based on [22, Eqs. 15.5.15, 15.5.16], respectively, we have

$$\alpha\,{}_2F_1\left(\begin{matrix}\alpha+1,\beta\\ \delta+1\end{matrix}\,\middle|\,1-x\right)=\delta\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \delta\end{matrix}\,\middle|\,1-x\right)-(\delta-\alpha)\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \delta+1\end{matrix}\,\middle|\,1-x\right)$$

and

$$(\delta-\alpha)(1-x){}_2F_1\left(\begin{matrix}\alpha,\beta\\ \delta+1\end{matrix}\,\middle|\,1-x\right)=\delta\,{}_2F_1\left(\begin{matrix}\alpha,\beta-1\\ \delta\end{matrix}\,\middle|\,1-x\right)-\delta x\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \delta\end{matrix}\,\middle|\,1-x\right). \tag{5.16}$$

Hence, we deduce that

$$\alpha(1-x)\,{}_2F_1\left(\begin{matrix}\alpha+1,\beta\\ \delta+1\end{matrix}\,\middle|\,1-x\right)=\delta\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \delta\end{matrix}\,\middle|\,1-x\right)-\delta\,{}_2F_1\left(\begin{matrix}\alpha,\beta-1\\ \delta\end{matrix}\,\middle|\,1-x\right), \tag{5.17}$$

and we obtain

$$\tilde{V}_0(x)=\frac{\alpha}{\delta}(1-x)\,{}_2F_1\left(\begin{matrix}\alpha+1,\beta\\ \delta+1\end{matrix}\,\middle|\,1-x\right).$$

Therefore,

$$\begin{aligned}\mathcal{V}_0(x)&=\frac{\Gamma(c+2)\Gamma(d+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(\delta+1)}\,x^a(1-x)^\delta\,{}_2F_1\left(\begin{matrix}\alpha+1,\beta\\ \delta+1\end{matrix}\,\middle|\,1-x\right)\\&=\mathcal{W}(x;a+1,b+1;d+1,c+2).\end{aligned} \tag{5.18}$$

Moreover,

$$\begin{aligned}\mathcal{V}_1(x)&=x\,\frac{(c+1)d(d+1)}{a(b+1)(\delta-\alpha)}\Big(\mathcal{W}(x;a,b+1;c+1,d)-\frac{c}{b}\,x\,\mathcal{W}(x;a,b;c,d)\Big)\\&=\frac{\Gamma(c+2)\Gamma(d+2)}{\Gamma(a+1)\Gamma(b+2)\Gamma(\delta)}\,x^a(1-x)^{\delta-1}\,\frac{\tilde{V}_1(x)}{\delta-\alpha},\end{aligned}$$

with

$$\tilde{V}_1(x)={}_2F_1\left(\begin{matrix}\alpha,\beta-1\\ \delta\end{matrix}\,\middle|\,1-x\right)-x\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \delta\end{matrix}\,\middle|\,1-x\right).$$

Applying (5.16) to the latter, we obtain

$$\tilde{V}_1(x) = \frac{\delta-\alpha}{\delta}\,(1-x)\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \delta+1\end{matrix}\,\middle|\,1-x\right).$$

Therefore,

$$\begin{aligned}\mathcal{V}_1(x) &= \frac{\Gamma(c+2)\Gamma(d+2)x^a(1-x)^\delta}{\Gamma(a+1)\Gamma(b+2)\Gamma(\delta+1)}\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \delta+1\end{matrix}\,\middle|\,1-x\right)\\ &= \mathcal{W}(x;a+1,b+2;d+2,c+2).\end{aligned} \tag{5.19}$$

In order to prove (5.13), we need to check that

$$\begin{bmatrix}\mathcal{V}_0'(x)\\ \mathcal{V}_1'(x)\end{bmatrix} = \begin{bmatrix}\dfrac{c(c+1)d}{a\alpha}\Big(\mathcal{W}(x;a,b;c,d)-\mathcal{W}(x;a,b+1;c+1,d)\Big)\\ \dfrac{(c+1)d(d+1)}{(b+1)(\delta-\alpha)}\Big(\mathcal{W}(x;a,b+1;c+1,d)-\dfrac{cd}{ab}\,x\,\mathcal{W}(x;a,b;c,d)\Big)\end{bmatrix}.$$

Recalling (5.18),

$$\begin{aligned}\mathcal{V}_0'(x) &= \frac{\Gamma(c+2)\Gamma(d+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(\delta+1)}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^a(1-x)^\delta\,{}_2F_1\left(\begin{matrix}\alpha+1,\beta\\ \delta+1\end{matrix}\,\middle|\,1-x\right)\right]\\ &= \frac{\Gamma(c+2)\Gamma(d+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(\delta+1)}\,x^{a-1}(1-x)^{\delta-1}\,G_0(x),\end{aligned}$$

with $G_0(x)$ equal to

$$(a(1-x)-\delta x)\,{}_2F_1\left(\begin{matrix}\alpha+1,\beta\\ \delta+1\end{matrix}\,\middle|\,1-x\right)-\frac{(\alpha+1)\beta}{\delta+1}\,x(1-x)\,{}_2F_1\left(\begin{matrix}\alpha+2,\beta+1\\ \delta+2\end{matrix}\,\middle|\,1-x\right).$$

Using [22, Eq. 15.5.19],

$$\begin{aligned}&\frac{(\alpha+1)\beta}{\delta+1}\,x(1-x)\,{}_2F_1\left(\begin{matrix}\alpha+2,\beta+1\\ \delta+2\end{matrix}\,\middle|\,1-x\right) =\\ &\delta\,{}_2F_1\left(\begin{matrix}\alpha,\beta-1\\ \delta\end{matrix}\,\middle|\,1-x\right) - \big(\delta-(\alpha+\beta)(1-x)\big)\,{}_2F_1\left(\begin{matrix}\alpha+1,\beta\\ \delta+1\end{matrix}\,\middle|\,1-x\right).\end{aligned}$$

Hence,

$$G_0(x) = -\delta\, {}_2F_1\left(\begin{matrix}\alpha, \beta-1\\ \delta\end{matrix}\,\middle|\, 1-x\right) + b(1-x)\, {}_2F_1\left(\begin{matrix}\alpha+1, \beta\\ \delta+1\end{matrix}\,\middle|\, 1-x\right),$$

and, recalling (5.17), we have

$$G_0(x) = \frac{\delta}{\alpha}\left[b\, {}_2F_1\left(\begin{matrix}\alpha, \beta\\ \delta\end{matrix}\,\middle|\, 1-x\right) - c\, {}_2F_1\left(\begin{matrix}\alpha, \beta-1\\ \delta\end{matrix}\,\middle|\, 1-x\right)\right],$$

Therefore, setting $K_0 = \dfrac{\Gamma(c+2)\Gamma(d+1)}{\Gamma(a+1)\Gamma(b)\Gamma(\delta)\alpha}$, we deduce that

$$\begin{aligned}\mathcal{V}_0'(x) &= K_0\, x^{a-1}(1-x)^{\delta-1}\left[{}_2F_1\left(\begin{matrix}\alpha, \beta\\ \delta\end{matrix}\,\middle|\, 1-x\right) - \frac{c}{b}\, {}_2F_1\left(\begin{matrix}\alpha, \beta-1\\ \delta\end{matrix}\,\middle|\, 1-x\right)\right]\\ &= \frac{c(c+1)d}{a\alpha}\Big(\mathcal{W}(x; a, b; c, d) - \mathcal{W}(x; a, b+1; c+1, d)\Big).\end{aligned}$$

Recalling (5.19),

$$\begin{aligned}\mathcal{V}_1'(x) &= \frac{\Gamma(c+2)\Gamma(d+2)}{\Gamma(a+1)\Gamma(b+2)\Gamma(\delta+1)}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^a(1-x)^\delta\, {}_2F_1\left(\begin{matrix}\alpha, \beta\\ \delta+1\end{matrix}\,\middle|\, 1-x\right)\right]\\ &= \frac{\Gamma(c+2)\Gamma(d+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(\delta+1)}\, x^{a-1}(1-x)^{\delta-1}\, G_1(x),\end{aligned}$$

with $G_1(x)$ equal to

$$(a(1-x)-\delta x)\, {}_2F_1\left(\begin{matrix}\alpha, \beta\\ \delta+1\end{matrix}\,\middle|\, 1-x\right) - \frac{\alpha\beta}{\delta+1}\, x(1-x)\, {}_2F_1\left(\begin{matrix}\alpha+1, \beta+1\\ \delta+2\end{matrix}\,\middle|\, 1-x\right).$$

Using [22, Eq. 15.5.19],

$$\begin{aligned}&\frac{\alpha\beta}{\delta+1}\, x(1-x)\, {}_2F_1\left(\begin{matrix}\alpha+1, \beta+1\\ \delta+2\end{matrix}\,\middle|\, 1-x\right) =\\ &\delta\, {}_2F_1\left(\begin{matrix}\alpha-1, \beta-1\\ \delta\end{matrix}\,\middle|\, 1-x\right) + \big((\alpha+\beta-1)(1-x)-\delta\big)\, {}_2F_1\left(\begin{matrix}\alpha, \beta\\ \delta+1\end{matrix}\,\middle|\, 1-x\right).\end{aligned}$$

So,

$$G_1(x) = -\delta\,{}_2F_1\left(\left.\begin{matrix}\alpha-1,\beta-1\\ \delta\end{matrix}\,\right|\,1-x\right) + (b+1)(1-x)\,{}_2F_1\left(\left.\begin{matrix}\alpha,\beta\\ \delta+1\end{matrix}\,\right|\,1-x\right). \tag{5.20}$$

Based on [22, Eq. 15.5.13], we have

$$\begin{gathered}(\delta-\alpha)(1-x)\,{}_2F_1\left(\left.\begin{matrix}\alpha-1,\beta-1\\ \delta\end{matrix}\,\right|\,1-x\right) = \\ (a-b+1)\,{}_2F_1\left(\left.\begin{matrix}\alpha,\beta-1\\ \delta\end{matrix}\,\right|\,1-x\right) + (\beta-1)\,x\,{}_2F_1\left(\left.\begin{matrix}\alpha,\beta\\ \delta\end{matrix}\,\right|\,1-x\right).\end{gathered}$$

Applying the latter and (5.16) to (5.20), we deduce that

$$G_1(x) = \frac{\delta}{\delta-\alpha}\left[a\,{}_2F_1\left(\left.\begin{matrix}\alpha,\beta-1\\ \delta\end{matrix}\,\right|\,1-x\right) - dx\,{}_2F_1\left(\left.\begin{matrix}\alpha,\beta\\ \delta\end{matrix}\,\right|\,1-x\right)\right].$$

Therefore, setting $K_1 = \dfrac{\Gamma(c+2)\Gamma(d+2)}{\Gamma(a)\Gamma(b+2)\Gamma(\delta)(\delta-\alpha)}$,

$$\begin{aligned}\mathcal{V}_1'(x) &= K_1\,x^{a-1}(1-x)^{\delta-1}\left[{}_2F_1\left(\left.\begin{matrix}\alpha,\beta\\ \delta\end{matrix}\,\right|\,1-x\right) - \frac{d}{a}\,x\,{}_2F_1\left(\left.\begin{matrix}\alpha,\beta-1\\ \delta\end{matrix}\,\right|\,1-x\right)\right] \\ &= \frac{(c+1)d(d+1)}{(b+1)(\delta-\alpha)}\Big(\mathcal{W}(x;a,b+1;c+1,d) - \frac{cd}{ab}\,x\,\mathcal{W}(x;a,b;c,d)\Big).\end{aligned}$$

□

Combining the latter result with Theorem 3.2, we show that the type II multiple orthogonal polynomials and the type I functions on the step-line satisfy the Hahn-classical property, because the differentiation with respect to the variable gives a shift in the parameters as well as in the index, as described in the following result.

Theorem 5.4. *[48, Th. 2.4] Let $P_n(x;a,b;c,d)$ and $Q_n(x;a,b;c,d)$, with $n\in\mathbb{N}$, be, respectively, the type II multiple orthogonal polynomial and the type I function for the index of length n on the step-line with respect to $\overline{\mathcal{W}}(x;a,b;c,d)$, defined by (5.9), under the assumptions in (5.2). Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{n+1}(x;a,b;c,d)\right)=(n+1)P_n(x;a+1,b+1;d+1,c+2) \tag{5.21}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(Q_n(x;a+1,b+1;d+1,c+2)\big)=-n\,Q_{n+1}(x;a,b;c,d). \tag{5.22}$$

Proof. Let $\Phi(x)$ be defined by (5.14) and denote $\overline{\mathcal{W}}(x;a,b;c,d)$ by $\overline{\mathcal{W}}(x)$.

Since $\overline{\mathcal{W}}(x)$ satisfies the equation (5.13) and on account of the degrees of the polynomial entries in the matrices $\Phi(x)$ and $\Psi(x)$, then part (a) of Theorem 3.2 ensures that the sequence $\big((n+1)^{-1}\,P'_{n+1}(x;a,b;c,d)\big)_{n\in\mathbb{N}}$ is 2-orthogonal with respect to the vector of weights $x\Phi(x)\overline{\mathcal{W}}(x)$. Similarly, part (b) of Theorem 3.2 implies that, if $R_n(x)$ is the type I function for the index of length n on the step-line with respect to $x\,\Phi(x)\overline{\mathcal{W}}(x)$, then $-n^{-1}\,R'_n(x)$ is the type I function for the index of length $n+1$ on the step-line with respect to the vector of weights $\overline{\mathcal{W}}(x)$.

Therefore, by virtue of (5.12), we conclude that both (5.21) and (5.22) hold. □

5.3 Rodrigues-type formulas for type I MOPs

Due to the differential relation (5.22), the type I functions on the step-line can be generated by a Rodrigues-type formula of the type in (3.15) as described in the following result.

Theorem 5.5. *[48, Th. 2.5] Let $Q_n(x) := Q_n(x; a, b; c, d)$, $n \in \mathbb{N}$, be the type I function for the index of length n on the step-line with respect to $\overline{\mathcal{W}}(x; a, b; c, d)$ defined by (5.9), under the assumptions in (5.2). Then, for any $n \in \mathbb{N}$,*

$$Q_{n+1}(x) = \frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathcal{W}\left(x; a+n, b+n; c+\left\lfloor\frac{n+1}{2}\right\rfloor+n, d+\left\lfloor\frac{n}{2}\right\rfloor+n\right)\right). \tag{5.23}$$

Proof. We proceed by induction on $n \in \mathbb{N}$. For $n = 0$, (5.23) trivially holds, because it reads as $Q_1(x; a, b; c, d) = \mathcal{W}(x; a, b; c, d)$.

Using (5.22), we have

$$Q_{n+2}(x; a, b; c, d) = -\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(Q_{n+1}(x; a+1, b+1; d+1, c+2)\right).$$

Then, evoking the assumption that (5.23) holds for a fixed $n \in \mathbb{N}$, we obtain

$$Q_{n+2}(x; a, b; c, d) = \frac{(-1)^{n+1}}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(\mathcal{W}\left(x; a+n+1, b+n+1; d_{n+1}, c_{n+1}\right)\right),$$

with

$$d_{n+1} = d+\left\lfloor\frac{n+1}{2}\right\rfloor+n+1 \quad \text{and} \quad c_{n+1} = c+\left\lfloor\frac{n+2}{2}\right\rfloor+n+1. \tag{5.24}$$

Therefore, using the interchangeability of the parameters, (5.23) holds for $n+1$ and the result follows by induction. □

Moreover, applying formula (3.16) to $\big(A_n(x;a,b;c,d), B_n(x;a,b;c,d)\big)$, we deduce that, for any $n \in \mathbb{N}$,

$$n \begin{bmatrix} A_{n+1}(x;a,b;c,d) \\ B_{n+1}(x;a,b;c,d) \end{bmatrix} = \mathcal{O}(a,b;c,d) \begin{bmatrix} A_n(x;a+1,b+1;d+1,c+2) \\ B_n(x;a+1,b+1;d+1,c+2) \end{bmatrix},$$

where the raising operator $\mathcal{O}(a,b;c,d)$, involving the transpose of the matrices (5.14) and (5.15) introduced in Theorem 5.3, is defined by

$$\mathcal{O}(a,b;c,d) = \left(\Psi(x;a,b;c,d)\right)^T - x \left(\Phi(x;a,b;c,d)\right)^T \frac{\mathrm{d}}{\mathrm{d}x}. \tag{5.25}$$

Therefore, recalling that $\big(A_1(x;a,b;c,d), B_1(x;a,b;c,d)\big) = (1,0)$, we obtain a matrix Rodrigues-type formula of the type (3.17) generating the type I polynomials on the step-line as follows.

Theorem 5.6. *Let $\big(A_n(x;a,b;c,d), B_n(x;a,b;c,d)\big)$, with $n \in \mathbb{N}$, be the vector of type I multiple orthogonal polynomials for the index of length n on the step-line with respect to $\overline{\mathcal{W}}(x;a,b;c,d)$ defined by (5.9), under the assumptions in (5.2). Then, defining the raising operator $\mathcal{O}(a,b;c,d)$ by (5.25), we have, for any $n \in \mathbb{N}$,*

$$\begin{bmatrix} A_{n+1}(x;a,b;c,d) \\ B_{n+1}(x;a,b;c,d) \end{bmatrix} = \frac{1}{n!} \left(\prod_{k=0}^{n-1} \mathcal{O}\left(a+k, b+k; \tilde{c}_k, \tilde{d}_k\right) \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

with

$$\tilde{c}_k = \begin{cases} c+3j & \text{if } k=2j, \\ d+3j+2 & \text{if } k=2j+1; \end{cases} \quad \text{and } \tilde{d}_k = \begin{cases} d+3j & \text{if } k=2j, \\ c+3j+1 & \text{if } k=2j+1. \end{cases} \tag{5.26}$$

Note that, comparing the parameters defined by (5.24) and (5.26), we have $\{c_{n+1}, d_{n+1}\} = \{\tilde{c}_{n+1}, \tilde{d}_{n+1}\}$.

5.4 Characterisation of the type II MOPs

We start this section by giving an explicit representation as terminating hypergeometric series ${}_3F_2$ for the type II multiple orthogonal polynomials on the step-line with respect to $\overline{\mathcal{W}}(x; a, b; c, d)$. Then, we use this representation to obtain the third-order differential equation and recurrence relation satisfied by these polynomials, which share the asymptotic behaviour of the recurrence coefficients with the Jacobi-Piñeiro polynomials. As a result, we show that they also share the ratio asymptotics and the asymptotic zero distribution. We also derive a Mehler-Heine-type asymptotic formula near the origin satisfied by the polynomials investigated here, which give us more information about the asymptotic behaviour of their zeros. Finally, we highlight the particular cases connected to Jacobi-type 2-orthogonal polynomials and to the cubic components of Hahn-classical 3-fold-symmetric 2-orthogonal polynomials, and we show that the polynomials characterised in Chapter 4 are a limiting case of the polynomials analysed in this chapter.

5.4.1 Explicit expressions as hypergeometric polynomials

Based on the moments expression (5.3), we deduce an explicit representation for the type II multiple orthogonal polynomials on the step-line with respect to $\overline{\mathcal{W}}(x; a, b; c, d)$ as generalised hypergeometric series.

Theorem 5.7. *[48, Th. 3.1] Let $\left(P_n(x) := P_n(x; a, b; c, d)\right)_{n\in\mathbb{N}}$ be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathcal{W}}(x; a, b; c, d)$ defined by (5.9), under the assumptions in (5.2). Then*

$$P_n(x) = \frac{(-1)^n (a)_n (b)_n}{\left(c + \left\lfloor \frac{n}{2} \right\rfloor\right)_n \left(d + \left\lfloor \frac{n-1}{2} \right\rfloor\right)_n} {}_3F_2 \left(\left. \begin{matrix} -n, c + \left\lfloor \frac{n}{2} \right\rfloor, d + \left\lfloor \frac{n-1}{2} \right\rfloor \\ a, b \end{matrix} \right| x \right). \tag{5.27}$$

or, equivalently,

$$P_n(x) = \sum_{k=0}^{n} \left(\frac{(-1)^k (1+n-k)_k (a+n-k)_k (b+n-k)_k}{k! \left(c + \left\lfloor \frac{n}{2} \right\rfloor + n - k\right)_k \left(d + \left\lfloor \frac{n-1}{2} \right\rfloor + n - k\right)_k} x^{n-k} \right). \tag{5.28}$$

To prove Theorem 5.7 we need to show that the sequence $(P_n(x))_{n\in\mathbb{N}}$ defined by (5.27) satisfies the 2-orthogonality conditions with respect to $\overline{\mathcal{W}}(x;a,b;c,d)$, that is, we need to check that, for each $j\in\{0,1\}$,

$$\int_0^1 x^k P_n(x)\mathcal{W}(x;a,b+j;c+j,d)\mathrm{d}x = \begin{cases} 0, & \text{if } n\geq 2k+j+1,\\ N_n(a,b;c,d)\neq 0, & \text{if } n=2k+j. \end{cases} \tag{5.29}$$

Similarly to how we argued in Section 4.4.1, the existence of a 2-orthogonal polynomial sequence with respect to $\overline{\mathcal{W}}(x;a,b;c,d)$ is guaranteed, because we are dealing with a Nikishin system. As a result, we can use the differential property (5.21) stated in Theorem 5.4 to show that it is sufficient to check the orthogonality conditions (5.29) when $k=0$ to then prove the result by induction on the degree of the polynomials. However, as in Section 4.4.1, we opt for checking that the polynomials in (5.27) satisfy all the orthogonality conditions (5.29), because this process enables us to show directly that these polynomials are indeed 2-orthogonal with respect to $\overline{\mathcal{W}}(x;a,b;c,d)$, without using with the Nikishin property, as well as it provides a method to derive explicit expressions for the nonzero coefficients $N_n(a,b;c,d)$ in (5.29), which are used in Subsection 5.4.3 to obtain explicit expressions for the positive γ_n-coefficients in the third order recurrence relation (5.35) satisfied by these polynomials.

Proof of Theorem 5.7. Recalling the expression for the moments given by (5.3), we can use Lemma 3.3 to derive that, for $k,n\in\mathbb{N}$ and $j\in\{0,1\}$,

$$\begin{aligned} &\int_0^1 x^k P_n(x)\mathcal{W}(x;a,b+j;c+j,d)\mathrm{d}x \\ &= \lambda_{n,k}^{(j)}\,{}_5F_4\left(\left.\begin{matrix} -n, a+k, b+k+j, c+\left\lfloor\frac{n}{2}\right\rfloor, d+\left\lfloor\frac{n-1}{2}\right\rfloor \\ a, b, c+k+j, d+k \end{matrix}\,\right|\, 1\right), \end{aligned} \tag{5.30}$$

with

$$\lambda_{n,k}^{(j)} = \frac{(-1)^n\,(a)_n\,(b)_n\,(a)_k\,(b+j)_k}{\left(c+\left\lfloor\frac{n}{2}\right\rfloor\right)_n\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)_n (c+j)_k\,(d)_k}.$$

For any $n \in \mathbb{N}$, $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor = n-1$ and, if $n \geq 2k+j+1$ with $j \in \{0,1\}$, then $\left\lfloor \frac{n}{2} \right\rfloor \geq k+j$ and $\left\lfloor \frac{n-1}{2} \right\rfloor \geq k$. Therefore, using (3.20) in Lemma 3.4, we deduce that, for both $j \in \{0,1\}$,

$$ {}_5F_4\left(\begin{matrix} -n, a+k, b+k+j, c+\left\lfloor \frac{n}{2} \right\rfloor, d+\left\lfloor \frac{n-1}{2} \right\rfloor \\ a, b, c+k+j, d+k \end{matrix} \middle| \, 1 \right) = 0, \quad \text{for any } n \geq 2k+j+1. $$

As a result,

$$ \int_0^1 x^k P_n(x) \mathcal{W}(x; a, b+j; c+j, d) \mathrm{d}x = 0, \quad \text{for any } n \geq 2k+j+1. $$

Taking $j=0$ and $n=2k$ in (5.30), we get

$$ \begin{aligned} &\int_0^1 x^k P_{2k}(x) \mathcal{W}(x; a, b; c, d) \mathrm{d}x \\ &= \frac{(a)_{2k}\,(b)_{2k}\,(a)_k\,(b)_k}{(c)_{3k}\,(d)_{3k-1}\,(d+k-1)}\, {}_4F_3\left(\begin{matrix} -2k, a+k, b+k, d+k-1 \\ a, b, d+k \end{matrix} \middle| \, 1 \right). \end{aligned} $$

Using (3.21), we have

$$ {}_4F_3\left(\begin{matrix} -2k, a+k, b+k, d+k-1 \\ a, b, d+k \end{matrix} \middle| \, 1 \right) = \frac{(2k)!\,(d-a)_k\,(d-b)_k}{(a)_k\,(b)_k\,(d+k)_{2k}}. $$

Therefore,

$$ \int_0^1 x^k P_{2k}(x) \mathcal{W}(x; a, b; c, d) \mathrm{d}x = \frac{(2k)!\,(a)_{2k}\,(b)_{2k}\,(d-a)_k\,(d-b)_k}{(c)_{3k}\,(d)_{3k}\,(d+k-1)_{2k}} > 0, \tag{5.31} $$

and (5.29) holds for any $k, n \in \mathbb{N}$ when $j=0$.

Similarly, taking $j=1$ and $n=2k+1$ in (5.30),

$$ \begin{aligned} &\int_0^1 x^k P_{2k+1}(x) \mathcal{W}(x; a, b+1; c+1, d) \mathrm{d}x \\ &= -\frac{(a)_{2k+1}\,(b)_{2k+1}\,(a)_k\,(b+1)_k}{(c+1)_{3k}\,(c+k)\,(d)_{3k+1}}\, {}_4F_3\left(\begin{matrix} -2k-1, a+k, b+k+1, c+k \\ a, b, c+k+1 \end{matrix} \middle| \, 1 \right). \end{aligned} $$

Using again (3.21), we get

$$
{}_4F_3\left(\begin{matrix}-2k-1, a+k, b+k+1, c+k\\ a, b, c+k+1\end{matrix}\,\middle|\, 1\right) = -\frac{(2k+1)!\,(c-a+1)_k\,(c-b)_{k+1}}{(a)_k\,(b)_{k+1}\,(c+k+1)_{2k+1}},
$$

so that

$$
\begin{aligned}
&\int_0^1 x^k P_{2k+1}(x)\mathcal{W}(x;a,b+1;c+1,d)\mathrm{d}x\\
&=\frac{(2k+1)!\,(a)_{2k+1}\,(b+1)_{2k}\,(c-a+1)_k\,(c-b)_{k+1}}{(c+1)_{3k+1}\,(c+k)_{2k+1}\,(d)_{3k+1}} > 0,
\end{aligned}
\tag{5.32}
$$

ensuring that (5.29) also holds for any $k, n \in \mathbb{N}$ when $j = 1$. □

5.4.2 Differential equation

The hypergeometric 2-orthogonal polynomials defined by (5.27)-(5.28) are solutions to a third order differential equation of the type in (3.23), as described in the following result.

Theorem 5.8. *[48, Th. 3.3] Let $(P_n(x) := P_n(x;a,b;c,d))_{n\in\mathbb{N}}$ be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathcal{W}}(x;a,b;c,d)$ defined by (5.9), under the assumptions in (5.2). Then*

$$
x^2(1-x)P_n'''(x) - x\varphi(x)P_n''(x) + \psi_n(x)P_n'(x) + n\lambda_n P_n(x) = 0, \tag{5.33}
$$

with

$$
\begin{aligned}
\varphi(x) &= (c+d+2)x - (a+b+1),\\
\psi_n(x) &= \big((n-1)(c+d+n) - \lambda_n\big)x + ab,\\
\lambda_n &= \left(c + \left\lfloor \frac{n}{2} \right\rfloor\right)\left(d + \left\lfloor \frac{n-1}{2} \right\rfloor\right).
\end{aligned}
$$

Proof. Combining the explicit formula for the 2-orthogonal polynomials as terminating hypergeometric series (5.27) and the generalised hypergeometric differential equation (1.10), we obtain

$$\begin{aligned}&\left[\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}+a\right)\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}+b\right)\frac{\mathrm{d}}{\mathrm{d}x}\right]P_n(x)=\\ &=\left[\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}-n\right)\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}+c+\left\lfloor\frac{n}{2}\right\rfloor\right)\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}+d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)\right]P_n(x).\end{aligned} \tag{5.34}$$

Expanding the left-hand side of (5.34), we get

$$\left[\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}+a\right)\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}+b\right)\frac{\mathrm{d}}{\mathrm{d}x}\right]P_n(x)=x^2P_n'''(x)+(a+b+1)x\,P_n''(x)+abP_n'(x).$$

Similarly, recalling that $\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{n-1}{2}\right\rfloor=n-1$, for any $n\in\mathbb{N}$, we derive

$$\begin{aligned}&\left[\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}+c+\left\lfloor\frac{n}{2}\right\rfloor\right)\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}+d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)\right]P_n(x)\\ &=x^2P_n''(x)+(c+d+n)x\,P_n'(x)+\lambda_nP_n(x).\end{aligned}$$

Therefore, the right-hand side of (5.34) is

$$\begin{aligned}&\left[\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}-n\right)\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}+c+\left\lfloor\frac{n}{2}\right\rfloor\right)\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}+d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)\right]P_n(x)\\ &=x^3\,P_n'''(x)+(c+d+2)x^2\,P_n''(x)+(\lambda_n-(n-1)(c+d+n))\,xP_n'(x)-n\lambda_nP_n(x).\end{aligned}$$

Finally, combining the expressions for both sides of (5.34), we obtain the differential equation (5.33). □

5.4.3 Recurrence relation

As a 2-orthogonal sequence, the hypergeometric polynomials $P_n(x)$ defined by (5.27) necessarily satisfy a third order recurrence relation of the form

$$P_{n+1}(x)=(x-\beta_n)\,P_n(x)-\alpha_nP_{n-1}(x)-\gamma_{n-1}P_{n-2}(x). \tag{5.35}$$

Our purpose here is to obtain explicit expressions for the recurrence coefficients involved.

Recalling formulas (3.26a)-(3.26b), and the expansion of $P_n(x)$ over the powers of x given by (5.28), we find that

$$\beta_n = \tau_{n,1} - \tau_{n+1,1} \quad \text{and} \quad \alpha_n = \tau_{n,2} - \tau_{n+1,2} - (\tau_{n,1})^2 + \tau_{n,1}\tau_{n+1,1},$$

with

$$\tau_{n,1} = -\frac{n(n+a-1)(n+b-1)}{\left(n+\lfloor\frac{n}{2}\rfloor+c-1\right)\left(n+\lfloor\frac{n-1}{2}\rfloor+d-1\right)}$$

and

$$\tau_{n,2} = \frac{(n-1)_2\,(n+a-2)_2\,(n+b-2)_2}{2\left(n+\lfloor\frac{n}{2}\rfloor+c-2\right)_2\left(n+\lfloor\frac{n-1}{2}\rfloor+d-2\right)_2}.$$

Hence, we obtain, for each $k \in \mathbb{N}$,

$$\beta_{2k}(a,b;c,d) = \frac{(2k+1)(2k+a)(2k+b)}{(3k+c)(3k+d)} - \frac{2k(2k+a-1)(2k+b-1)}{(3k+c-1)(3k+d-2)}$$

and

$$\beta_{2k+1}(a,b;c,d) = \frac{(2k+2)(2k+a+1)(2k+b+1)}{(3k+c+2)(3k+d+1)} - \frac{(2k+1)(2k+a)(2k+b)}{(3k+c)(3k+d)};$$

as well as

$$\alpha_{2k+1}(a,b;c,d) = \frac{(2k+1)(2k+a)(2k+b)}{(3k+c)(3k+d)}\left(\frac{k(2k+a-1)(2k+b-1)}{(3k+c-1)(3k+d-1)} - \frac{(2k+1)(2k+a)(2k+b)}{(3k+c)(3k+d)} + \frac{(k+1)(2k+a+1)(2k+b+1)}{(3k+c+1)(3k+d+1)}\right),$$

and

$$\alpha_{2k+2}(a,b;c,d) = \frac{2(k+1)(2k+a+1)(2k+b+1)}{(3k+c+2)(3k+d+1)}\left(\frac{(2k+1)(2k+a)(2k+b)}{2(3k+c+1)(3k+d)} - \frac{2(k+1)(2k+a+1)(2k+b+1)}{(3k+c+2)(3k+d+1)} + \frac{(2k+3)(2k+a+2)(2k+b+2)}{2(3k+c+3)(3k+d+2)}\right).$$

Expanding these formulas, we derive that

- $\beta_{2k}(a,b;c,d) = \dfrac{36k^4 + \mathcal{O}(k^3)}{(3k+c-1)(3k+c)(3k+d-2)(3k+d)}$;
- $\beta_{2k+1}(a,b;c,d) = \dfrac{36k^4 + \mathcal{O}(k^3)}{(3k+c)(3k+c+2)(3k+d)(3k+d+1)}$;
- $\alpha_{2k+1}(a,b;c,d) = \dfrac{(2k+1)(2k+a)(2k+b)\left(54k^5 + \mathcal{O}(k^4)\right)}{(3k+c-1)_2(3k+c)_2(3k+d-1)_2(3k+d)_2}$;
- $\alpha_{2k+2}(a,b;c,d) = \dfrac{(2k+1)(2k+a)(2k+b)\left(54k^5 + \mathcal{O}(k^4)\right)}{(3k+c+2)_2(3k+c+3)_2(3k+d+1)_2(3k+d+2)_2}$.

Furthermore, recalling (3.26c), we have, for each $k \in \mathbb{N}$ and $j \in \{0,1\}$,

$$\gamma_{2k+1+j}(a,b;c,d) = \frac{\displaystyle\int_0^1 x^{k+1} P_{2k+2+j}(x;a,b;c,d)\mathcal{W}(x;a,b+j;c+j,d)\mathrm{d}x}{\displaystyle\int_0^1 x^{k} P_{2k+j}(x;a,b;c,d)\mathcal{W}(x;a,b+j;c+j,d)\mathrm{d}x}.$$

Based on the latter alongside with (5.31) and (5.32), we deduce that, for all $k \in \mathbb{N}$,

$$\gamma_{2k+1}(a,b;c,d) = \frac{(2k+1)_2(2k+a)_2(2k+b)_2(d-1+k)(d-a+k)(d-b+k)}{(3k+c)_3(3k+d-1)_3(3k+d)_3}$$

and

$$\gamma_{2k+2}(a,b;c,d) = \frac{(2k+2)_2(2k+a+1)_2(2k+b+1)_2(c+k)(c-a+k+1)(c-b+k+1)}{(3k+c+1)_3(3k+c+2)_3(3k+d+1)_3}.$$

As a consequence, we have just proved the following result.

Theorem 5.9. *[48, Th. 3.4] Let $\left(P_n(x) := P_n(x;a,b;c,d)\right)_{n\in\mathbb{N}}$ be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathcal{W}}(x;a,b;c,d)$ defined by (5.9), under the assumptions in (5.2). Then $\left(P_n(x)\right)_{n\in\mathbb{N}}$ satisfies the recurrence relation*

$$P_{n+1}(x) = (x-\beta_n)\,P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1}P_{n-2}(x),$$

where, for each $n \in \mathbb{N}$,

$$\beta_n = \frac{(n+1)(n+a)(n+b)}{\left(n+c'_{n-1}\right)\left(n+c'_n\right)} - \frac{n(n+a-1)(n+b-1)}{\left(n+c'_{n-1}-1\right)\left(n+c'_n-2\right)}, \tag{5.36a}$$

$$\begin{aligned}\alpha_{n+1} = \frac{(n+1)(n+a)(n+b)}{(n+c'_{n-1})(n+c'_n)}\Bigg(&\frac{n(n+a-1)(n+b-1)}{2(n+c'_{n-1}-1)(n+c'_n-1)} \\ &- \frac{(n+1)(n+a)(n+b)}{(n+c'_{n-1})(n+c'_n)} + \frac{(n+2)(n+a+1)(n+b+1)}{2(n+c'_{n-1}+1)(n+c'_n+1)}\Bigg),\end{aligned} \tag{5.36b}$$

and

$$\gamma_{n+1} = \frac{(n+1)_2\,(n+a)_2\,(n+b)_2\,(c'_n-1)(c'_n-a)(c'_n-b)}{\left(n+c'_{n-1}\right)_3(n+c'_n)_3\,(n+c'_n-1)_3}, \tag{5.36c}$$

with

$$c'_n = \begin{cases} c+m & \text{if } n = 2m-1, \\ d+m & \text{if } n = 2m. \end{cases} \tag{5.37}$$

Furthermore, $\gamma_{n+1} > 0$, for all $n \in \mathbb{N}$, and the recurrence coefficients are real, bounded and, as $n \to \infty$,

$$\beta_n \to 3\left(\frac{4}{27}\right) = \frac{4}{9}, \quad \alpha_n \to 3\left(\frac{4}{27}\right)^2 = \frac{16}{243} \text{ and } \gamma_n \to \left(\frac{4}{27}\right)^3 = \frac{64}{19683}. \tag{5.38}$$

5.4.4 Asymptotic behaviour and connection with Jacobi-Piñeiro polynomials

The Jacobi-Piñeiro polynomials are multiple orthogonal polynomials with respect to several classical Jacobi weights on the same interval. They are usually defined as the multiple orthogonal polynomials with respect to a vector of measures $(\mu_0, \cdots, \mu_{r-1})$ supported on the interval $(0,1)$, where $d\mu_i(x) = x^{\alpha_i}(1-x)^{\beta}dx$, with $\beta, \alpha_0, \cdots, \alpha_{r-1} > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$. The Jacobi-Piñeiro polynomials were introduced by Piñeiro in [62], with $\beta = 0$. The polynomials in [62] are a limiting case of the polynomials investigated here, because the choice of $c = a$ and $d = b+1$ gives

$$\mathcal{W}(x; a, b; c, d) = bx^{b-1} \quad \text{and} \quad \mathcal{W}(x; a, b+1; c+1, d) = ax^{a-1}.$$

See [18, §3.1] for explicit formulas for the type II Jacobi-Piñeiro polynomials with respect to 2 measures as well as explicit expressions for the recurrence coefficients of the polynomials on the step-line. The asymptotic behaviour (5.38) of the recurrence coefficients in Theorem 5.9 coincides with the asymptotic behaviour obtained in [18] for the coefficients of the recurrence relation satisfied by the Jacobi-Piñeiro 2-orthogonal polynomials. Based on this relation, the polynomials investigated here share the ratio asymptotics and the asymptotic zero distribution with the Jacobi-Piñeiro polynomials, which are expressed in the following result.

Theorem 5.10. *Let* $(P_n(x) := P_n(x;a,b;c,d))_{n\in\mathbb{N}}$ *be the monic* 2*-orthogonal polynomial sequence with respect to* $\overline{\mathcal{W}}(x;a,b;c,d)$ *defined by* (5.9)*, under the assumptions in* (5.2)*. Then:*

(a) The ratios of consecutive polynomials satisfy the asymptotic behaviour:

$$\lim_{n\to\infty}\frac{P_n(z)}{P_{n+1}(z)}=\frac{27}{4}\left(\frac{3}{2}\,z^{\frac{1}{3}}\left(\mathrm{e}^{\frac{4\pi i}{3}}\left(-1+\sqrt{1-z}\right)^{\frac{1}{3}}+\mathrm{e}^{\frac{2\pi i}{3}}\left(-1-\sqrt{1-z}\right)^{\frac{1}{3}}\right)-1\right), \tag{5.39}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,1]$.

(b) The asymptotic zero distribution of $(P_n(x))_{n\in\mathbb{N}}$*, which we denote by* ν*, is supported on the interval* $(0,1)$*, with density*

$$\frac{\mathrm{d}\nu}{\mathrm{d}x}=\frac{\sqrt{3}}{4\pi}\,\frac{\left(1+\sqrt{1-x}\right)^{\frac{1}{3}}+\left(1-\sqrt{1-x}\right)^{\frac{1}{3}}}{x^{\frac{2}{3}}\sqrt{1-x}},\quad \text{for } x\in(0,1). \tag{5.40}$$

Proof. We have shown in Section 5.1 that $\overline{\mathcal{W}}(x;a,b;c,d)$ forms a Nikishin system, and, consequently, $P_n(x)$ has n simple zeros, all located on the interval $(0,1)$, with the zeros of consecutive polynomials interlacing as there is always a zero of $P_n(x)$ between two consecutive zeros of $P_{n+1}(x)$. Therefore, using the asymptotic behaviour (5.38) of the recurrence coefficients in Theorem 5.9, which are all real, formulas (5.39) and (5.40) can be obtained directly from [17, Th. 3.1] and [17, Th. 2.1], respectively. Alternatively, the ratio asymptotics formula (5.39) can also be derived from [7, Lemma 3.2 & Remark 3.1]. □

Next, we derive a Mehler-Heine asymptotic formula near the origin satisfied by the 2-orthogonal polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$.

Proposition 5.11. *Let $(P_n(x) := P_n(x;a,b;c,d))_{n\in\mathbb{N}}$ be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathcal{W}}(x;a,b;c,d)$ defined by (5.9), under the assumptions in (5.2). Then,*

$$\lim_{n\to\infty}\left(\frac{(-1)^n\left(c+\left\lfloor\frac{n}{2}\right\rfloor\right)_n\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)_n}{(a)_n\,(b)_n}\,P_n\left(\frac{z}{n^3}\right)\right) = {}_0F_2\left(\left.\begin{matrix}-\\ a,b\end{matrix}\,\right|\,-\frac{z}{4}\right), \tag{5.41}$$

uniformly on compact subsets of $\mathbb{C}$.

Note that the limit (5.41) in the Mehler-Heine formula coincides with the limit in the corresponding result for the Jacobi-Piñeiro polynomials [72, Th. 2], with $r=2$ and $q_1=q_2=1/2$.

Proof. Recalling Theorem 5.7, we have

$$\frac{(-1)^n\left(c+\left\lfloor\frac{n}{2}\right\rfloor\right)_n\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)_n}{(a)_n\,(b)_n}\,P_n\left(\frac{z}{n^3}\right) = {}_3F_2\left(\left.\begin{matrix}-n,c+\left\lfloor\frac{n}{2}\right\rfloor,d+\left\lfloor\frac{n-1}{2}\right\rfloor\\ a,b\end{matrix}\,\right|\,\frac{z}{n^3}\right).$$

Furthermore, $c+\left\lfloor\frac{n}{2}\right\rfloor,\ d+\left\lfloor\frac{n-1}{2}\right\rfloor \sim \frac{n}{2}$ as $n\to\infty$. Therefore, applying the confluent relation (1.11) to the generalised hypergeometric series on the right-hand side of the latter equation, we obtain (5.41). □

Based on the Mehler-Heine-type asymptotic formula (5.41), we obtain a result about the asymptotic behaviour of the zeroes of $P_n(x;a,b;c,d)$ near the origin, coinciding with the corresponding result for the 2-orthogonal Jacobi-Piñeiro polynomials obtained in [72, §4]. If we denote the zeros of $P_n(x;a,b;c,d)$ by $\left(x_k^{(n)}\right)_{1\le k\le n}$, which are all located on the interval $(0,1)$, and the zeros of the generalised hypergeometric series ${}_0F_2\left(-;a,b\,|\,-z\right)$, which are all real and positive, by $(f_k)_{k\in\mathbb{Z}^+}$, with the zeros written in increasing order for both cases, then we have

$$\lim_{n\to\infty}\left(n^3 x_k^{(n)}\right) = 4f_k.$$

5.4.5 Particular cases

5.4.5.1 Jacobi-type 2-orthogonal polynomials

Using the coefficients c'_n introduced in (5.37), the explicit expression for the 2-orthogonal polynomials given by (5.27) can be rewritten as

$$P_n(x;a,b;c,d) = \frac{(-1)^n\,(a)_n\,(b)_n}{\left(c'_{n-2}\right)_n\left(c'_{n-1}\right)_n}\,{}_3F_2\left(\left.\begin{matrix}-n, c'_{n-2}, c'_{n-1}\\ a, b\end{matrix}\,\right|\, x\right)$$

Furthermore, if $d = c + \frac{1}{2}$, then $c'_n = c + \frac{n+1}{2}$, for any $n \in \mathbb{N}$, and the expression above becomes

$$P_n\left(x;a,b;c,c+\frac{1}{2}\right) = \frac{(-4)^n\,(a)_n\,(b)_n}{(2c-1+n)_{2n}}\,{}_3F_2\left(\left.\begin{matrix}-n, c+\frac{n-1}{2}, c+\frac{n}{2}\\ a, b\end{matrix}\,\right|\, x\right). \quad (5.42)$$

The latter polynomials coincide, up to a linear transformation of the variable, with the Jacobi-type 2-orthogonal polynomials investigated in [46], with $c = \frac{\nu+1}{2}$.

A particular case of (5.42) of special interest arises when $(a,b;c,d) = \left(\frac{4}{3},\frac{5}{3};2,\frac{5}{2}\right)$. In this case, we have $c'_n = \frac{n+5}{2}$ for all $n \in \mathbb{N}$, and the recurrence coefficients given by (5.36a)-(5.36c) are all constant and equal to the limits in (5.38), that is,

$$\beta_n\left(\frac{4}{3},\frac{5}{3};2,\frac{5}{2}\right) = \frac{4}{9}, \quad \alpha_{n+1}\left(\frac{4}{3},\frac{5}{3};2,\frac{5}{2}\right) = \frac{16}{243} \quad \text{and} \quad \gamma_{n+1}\left(\frac{4}{3},\frac{5}{3};2,\frac{5}{2}\right) = \frac{64}{19683}.$$

Therefore, the sequence $\left(P_n\left(x;\frac{4}{3},\frac{5}{3};2,\frac{5}{2}\right)\right)_{n\in\mathbb{N}}$ satisfies the third-order recurrence relation with constant coefficients

$$P_{n+1}(x) = \left(x - \frac{4}{9}\right)P_n(x) - \frac{16}{243}\,P_{n-1}(x) - \frac{64}{19683}\,P_{n-2}(x).$$

Moreover, observing that $\left(\frac{4}{3}\right)_n \left(\frac{5}{3}\right)_n = \frac{(n+1)_{2n+2}}{2\cdot 27^n}$, for any $n \in \mathbb{N}$, we can derive from (5.42) that

$$P_n\left(x; \frac{4}{3}, \frac{5}{3}; 2, \frac{5}{2}\right) = \frac{(n+1)(n+2)}{2}\left(\frac{-4}{27}\right)^n {}_3F_2\left(\left.\begin{matrix} -n, \frac{n+3}{2}, \frac{n}{2}+2 \\ \frac{4}{3}, \frac{5}{3} \end{matrix}\right| x\right). \tag{5.43}$$

By definition, the polynomials in (5.43) are 2-orthogonal with respect to

$$\overline{\mathcal{W}}\left(x; \frac{4}{3}, \frac{5}{3}; 2, \frac{5}{2}\right) = \begin{bmatrix} \frac{\Gamma(2)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{3}{2}\right)} x^{\frac{1}{3}}(1-x)^{\frac{1}{2}}\, {}_2F_1\left(\frac{1}{3}, \frac{5}{6}; \frac{3}{2} \middle| 1-x\right) \\ \frac{\Gamma(3)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{8}{3}\right)\Gamma\left(\frac{3}{2}\right)} x^{\frac{1}{3}}(1-x)^{\frac{1}{2}}\, {}_2F_1\left(\frac{1}{3}, -\frac{1}{6}; \frac{3}{2} \middle| 1-x\right) \end{bmatrix}.$$

Based on [22, Eq. 15.4.9], we have

$$(1-x)^{\frac{1}{2}}\, {}_2F_1\left(\frac{1}{3}, \frac{5}{6}; \frac{3}{2} \middle| 1-x\right) = \frac{3}{2}\left(\left(1+\sqrt{1-x}\right)^{\frac{1}{3}} - \left(1-\sqrt{1-x}\right)^{\frac{1}{3}}\right)$$

and

$$(1-x)^{\frac{1}{2}}\, {}_2F_1\left(\frac{1}{3}, -\frac{1}{6}; \frac{3}{2} \middle| 1-x\right) = \frac{3}{8}\left(\left(1+\sqrt{1-x}\right)^{\frac{4}{3}} - \left(1-\sqrt{1-x}\right)^{\frac{4}{3}}\right).$$

Furthermore, recalling formulas (1.2) and (1.3), we get

$$\frac{\Gamma(2)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{3}{2}\right)} = \frac{27\sqrt{3}}{8\pi} \quad \text{and} \quad \frac{\Gamma(3)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{8}{3}\right)\Gamma\left(\frac{3}{2}\right)} = \frac{81\sqrt{3}}{20\pi}.$$

As a result, we have shown that the polynomials in (5.43) are 2-orthogonal with respect to the vector of weights

$$\overline{\mathcal{W}}\left(x; \frac{4}{3}, \frac{5}{3}; 2, \frac{5}{2}\right) = \begin{bmatrix} \frac{81\sqrt{3}}{16\pi} x^{\frac{1}{3}}\left(\left(1+\sqrt{1-x}\right)^{\frac{1}{3}} - \left(1-\sqrt{1-x}\right)^{\frac{1}{3}}\right) \\ \frac{243\sqrt{3}}{160\pi} x^{\frac{1}{3}}\left(\left(1+\sqrt{1-x}\right)^{\frac{4}{3}} - \left(1-\sqrt{1-x}\right)^{\frac{4}{3}}\right) \end{bmatrix}. \tag{5.44}$$

Observe the similarities between the orthogonality weights in (5.44) and the density (5.40) of the asymptotic zero distribution.

5.4.5.2 The cubic components of case C

The aim of this subsection is to show that the cubic components of the Hahn-classical 3-fold-symmetric 2-orthogonal polynomial sequences corresponding to case C (using the same terminology as in Chapter 2) are particular cases of the 2-orthogonal polynomials which we have characterised in this section. We check this relation by comparing both the explicit expressions for the polynomials, as we have done in [48, §3.6], and, alternatively, by examining the expressions for their orthogonality weights.

Let $(S_n(x;\nu;\rho))_{n\in\mathbb{N}}$ be the Hahn-classical 3-fold-symmetric 2-orthogonal polynomial sequence in Subsection 2.2.4 corresponding to case C, whose cubic components $\left(S_n^{[k]}(x;\nu,\rho)\right)_{n\in\mathbb{N}}$, $k\in\{0,1,2\}$, are the polynomials explicitly represented as terminating ${}_3F_2$ hypergeometric series by (2.36). These representations can be obtained taking particular choices on the parameters $(a,b;c,d)$ in (5.27). In fact, comparing (5.27) with (2.36), we deduce that

$$S_n^{[k]}(x;\nu,\rho) = P_n\left(x;a_k,b_k;c_k,d_k\right), \quad \text{for } k\in\{0,1,2\}, \tag{5.45a}$$

with $(a_k,b_k;c_k,d_k)$ for $k=0,1,2$, respectively equal to

$$\left(\frac{1}{3},\frac{2}{3};\frac{\nu+2}{3},\frac{\rho}{3}+1\right), \ \left(\frac{4}{3},\frac{2}{3};\frac{\rho}{3}+1,\frac{\nu+5}{3}\right) \text{ and } \left(\frac{4}{3},\frac{5}{3};\frac{\nu+5}{3},\frac{\rho}{3}+2\right). \tag{5.45b}$$

Alternatively, formulas (5.45a)-(5.45b) can be deduced from the expressions for the orthogonality weights of the cubic components in Subsections 2.2.4. In fact, setting $(a_k,b_k;c_k,d_k)$ by (5.45b), formulas (2.39a)-(2.39b) are equivalent to

$$\mathcal{U}_0^{[k]}(x) = \mathcal{W}\left(x;a_k,b_k;c_k,d_k\right)$$

and

$$\mathcal{U}_1^{[k]}(x) = \frac{c_k(c_k+1)d}{a_k(c_k-b_k)}\Big(\mathcal{W}\left(x;a_k,b_k+1;c_k+1,d_k\right)-\mathcal{W}\left(x;a_k,b_k;c_k,d_k\right)\Big).$$

Hence, the 2-orthogonal polynomials with respect to $\left(\mathcal{U}_0^{[k]}(x), \mathcal{U}_1^{[k]}(x)\right)$ and to $\left(\mathcal{W}(x; a_k, b_k; c_k, d_k), \mathcal{W}(x; a_k, b_k+1; c_k+1, d_k)\right)$ are the same, which means that the formula (5.45a) holds.

5.4.6 Confluent relation with the MOPs in Chapter 4

There are confluent relations between the 2-orthogonal polynomials analysed in this chapter and the ones investigated in Chapter 4. These relations generalise the limiting relations between case C and cases B1 and B2 in [50], similarly to how the confluent relations shown in Section 4.6 generalise the ones between cases B1 and B2 and case A.

Throughout this section, we denote by $\mathcal{V}(x; a, b; c)$ the weight function defined by (4.1), with $a, b, c \in \mathbb{R}^+$ and $c > \max\{a, b\}$, and by $R_n^{[\epsilon]}(x; a, b; c)$ the 2-orthogonal polynomials with respect to $\left(\mathcal{V}(x; a, b; c+\epsilon), \mathcal{V}(x; a, b; c+1-\epsilon)\right)$, with $\epsilon \in \{0, 1\}$, analysed in Chapter 4.

The polynomials $R_n^{[\epsilon]}(x; a, b; c)$ are a limiting case of the 2-orthogonal polynomials $P_n(x; a, b; c, d)$ with respect to $\left(\mathcal{W}(x; a, b; c, d), \mathcal{W}(x; a, b+1; c+1, d)\right)$, because they satisfy the limiting relations

$$\lim_{d\to\infty} \left(d^n P_n\left(\frac{x}{d}; a, b; c, d\right)\right) = R_n^{[0]}\left(x; a, b; c\right) \tag{5.47a}$$

and

$$\lim_{c\to\infty} \left(c^n P_n\left(\frac{x}{c}; a, b; c, d\right)\right) = R_n^{[1]}\left(x; a, b; d-1\right). \tag{5.47b}$$

Furthermore, the weight functions $\mathcal{W}(x; a, b; c, d)$, defined by (5.1), and $\mathcal{V}(x; a, b; c)$, defined as in (4.1), also satisfy similar confluent relations:

$$\lim_{d\to\infty} \left(\frac{1}{d}\mathcal{W}\left(\frac{x}{d}; a, b; c, d\right)\right) = \mathcal{V}\left(x; a, b; c\right) \tag{5.48a}$$

and

$$\lim_{c\to\infty} \left(\frac{1}{c}\mathcal{W}\left(\frac{x}{c}; a, b; c, d\right)\right) = \mathcal{V}\left(x; a, b; d\right). \tag{5.48b}$$

To derive the confluent relations (5.47a)-(5.47b) satisfied by the 2-orthogonal polynomials $P_n(x;a,b;c,d)$ and $R_n^{[\epsilon]}(x;a,b;c)$, we use the confluent relation for the generalised hypergeometric series (1.11) to find that

$$\lim_{d\to\infty}\left({}_3F_2\left(\left.\begin{matrix}-n, c+\left\lfloor\frac{n}{2}\right\rfloor, d+\left\lfloor\frac{n-1}{2}\right\rfloor\\ a,b\end{matrix}\right|\frac{x}{d}\right)\right) = {}_2F_2\left(\left.\begin{matrix}-n, c+\left\lfloor\frac{n}{2}\right\rfloor\\ a,b\end{matrix}\right|x\right),$$

and

$$\lim_{c\to\infty}\left({}_3F_2\left(\left.\begin{matrix}-n, c+\left\lfloor\frac{n}{2}\right\rfloor, d+\left\lfloor\frac{n-1}{2}\right\rfloor\\ a,b\end{matrix}\right|\frac{x}{c}\right)\right) = {}_2F_2\left(\left.\begin{matrix}-n, d+\left\lfloor\frac{n-1}{2}\right\rfloor\\ a,b\end{matrix}\right|x\right).$$

As a result, recalling the explicit representations (5.27) and (4.17) for the polynomials $P_n(x;a,b;c,d)$ and $R_n^{[\epsilon]}(x;a,b;c)$, respectively, we obtain (5.47a) and (5.47b).

Now we prove the confluent relations (5.48a)-(5.48b) connecting the weight functions $\mathcal{W}(x;a,b;c,d)$ and $\mathcal{V}(x;a,b;c)$. These two relations are clearly equivalent, due to the interchangeability of the parameters c and d in (5.1), so it is sufficient to prove (5.48a). By definition,

$$\frac{1}{d}\mathcal{W}\left(\frac{x}{d};a,b;c,d\right) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(\delta)}\frac{x^{a-1}}{d^a}\left(1-\frac{x}{d}\right)^{\delta-1}{}_2F_1\left(\left.\begin{matrix}c-b, d-b\\ \delta\end{matrix}\right|1-\frac{x}{d}\right).$$

Based on [22, Eq. 15.8.1], we have

$${}_2F_1\left(\left.\begin{matrix}c-b, d-b\\ \delta\end{matrix}\right|1-\frac{x}{d}\right) = \left(\frac{x}{d}\right)^{b-c}{}_2F_1\left(\left.\begin{matrix}c-b, c-a\\ \delta\end{matrix}\right|1-\frac{d}{x}\right)$$

The limiting relation in [27, Eq. 6.8.1], connecting the hypergeometric and Tricomi functions, implies that

$$\lim_{d\to\infty}{}_2F_1\left(\left.\begin{matrix}c-b, c-a\\ \delta\end{matrix}\right|1-\frac{d}{x}\right) = x^{c-b}\,\mathbf{U}\left(c-b, a-b+1; x\right).$$

Furthermore, because $\delta = d + c - a - b \sim d$ as $d \to \infty$, we have

$$\lim_{d\to\infty} \left(1 - \frac{x}{d}\right)^{\delta-1} = \mathrm{e}^{-x} \quad \text{and} \quad \lim_{d\to\infty} \frac{\Gamma(d)}{\Gamma(\delta)} = d^{a+b-c}.$$

Therefore, we deduce that

$$\lim_{d\to\infty} \left(\frac{1}{d}\mathcal{W}\left(\frac{x}{d}; a, b; c, d\right)\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\, x^{a-1}\, \mathbf{U}\,(c - b, a - b + 1; x) = \mathcal{V}(x; a, b; c),$$

which means that (5.48a) holds. Analogously, (5.48b) also holds.

5.5 Link to branched continued fractions

Recalling (3.35), with $(r, s) = (2, 2)$, the ordinary generating function of the moment sequence given by (5.3) is

$$\sum_{n=0}^{\infty} \left(\frac{(a)_n\,(b)_n}{(c)_n\,(d)_n}\, t^n\right) = {}_3F_2\,(a, b, 1; c, d\,|\,t\,) = \frac{{}_3F_2\,(a, b, 1; c, d\,|\,t\,)}{{}_3F_2\,(a, b, 0; c, d\,|\,t\,)}.$$

Therefore, using [61, Th. 14.5], the ratios of Pochhammer symbols appearing in the latter formula, which are the moments in (5.3), are equal to the 2-Stieltjes-Rogers polynomials $S_n^{(2)}(\boldsymbol{\lambda})$, with coefficients $\boldsymbol{\lambda} = (\lambda_{k+2})_{k\in\mathbb{N}}$, given by

$$\lambda_{k+2} = \frac{(c'_k - a'_k) \displaystyle\prod_{i\in\{1,2,3\},\, i\not\equiv k \bmod 3} a_{i,k}}{(c'_k - 1)\, c_{1,k}\, c_{2,k}} \quad \text{for all } k \in \mathbb{N}, \tag{5.49}$$

with $a_1 = a$, $a_2 = b$, $a_3 = 1$, $c_1 = c$, $c_2 = d$;

$$a_{i,k} = a_i + 1 + \left\lfloor \frac{k - i}{3} \right\rfloor = \begin{cases} a + m & \text{if } i = 1 \text{ and } k \in \{3m-2, 3m-1, 3m\}, \\ b + m & \text{if } i = 2 \text{ and } k \in \{3m-1, 3m, 3m+1\}, \\ 1 + m & \text{if } i = 3 \text{ and } k \in \{3m, 3m+1, 3m+2\}; \end{cases}$$

$$a'_k = a_{[k-1 \bmod 3]+1,k} = \begin{cases} a_{3,3m} = m+1, & \text{if } k = 3m, \\ a_{1,3m+1} = a+m+1, & \text{if } k = 3m+1, \\ a_{2,3m+2} = b+m+1, & \text{if } k = 3m+2; \end{cases}$$

$$c_{i,k} = c_i + 1 + \left\lfloor \frac{k-i}{2} \right\rfloor = \begin{cases} c+m, & \text{if } i = 1 \text{ and } k \in \{2m-1, 2m\}, \\ d+m, & \text{if } i = 2 \text{ and } k \in \{2m, 2m+1\}; \end{cases}$$

and $c'_k = c_{[k-1 \mod 2]+1,k}$ is the same as in (5.37).

Note that, for any $k \in \mathbb{N}$, $\{c_{1,k}, c_{2,k}\} = \{c'_k, c'_{k-1}\}$. As a result, the coefficients in (5.49) can be expressed as follows:

- $\lambda_{3n+2} = \dfrac{a_{1,3n} a_{2,3n} \left(c'_{3n} - a'_{3n}\right)}{\left(c'_{3n} - 1\right) c'_{3n} c'_{3n-1}} = \dfrac{(a+n)(b+n)(c'_n - 1)}{(c'_n + n - 1)(c'_n + n)(c'_{n-1} + n)}$;
- $\lambda_{3n+3} = \dfrac{a_{2,3n+1} a_{3,3n+1} \left(c'_{3n+1} - a'_{3n+1}\right)}{\left(c'_{3n+1} - 1\right) c'_{3n+1} c'_{3n}} = \dfrac{(n+1)(b+n)(c'_{n-1} - a)}{(c'_{n-1} + n)(c'_{n-1} + n + 1)(c'_n + n)}$;
- $\lambda_{3n+4} = \dfrac{a_{1,3n+2} a_{3,3n+2} \left(c'_{3n+2} - a'_{3n+2}\right)}{\left(c'_{3n+2} - 1\right) c'_{3n+2} c'_{3n+1}} = \dfrac{(n+1)(a+n+1)(c'_n - b)}{(c'_{n-1} + n + 1)(c'_n + n)(c'_n + n + 1)}$,

for any $n \in \mathbb{N}$.

Let $a, b, c, d \in \mathbb{R}^+$ such that $\min\{c, d\} > \max\{a, b\}$. Using the expressions above for the coefficients $(\lambda_{k+2})_{k \in \mathbb{N}}$ and setting $\lambda_0 = \lambda_1 = 0$, we checked, using Maple, that the recurrence coefficients given by (5.36a)-(5.36c) can be rewritten, for any $n \in \mathbb{N}$, as follows:

- $\beta_n = \lambda_{3n} + \lambda_{3n+1} + \lambda_{3n+2}$; (5.50)
- $\alpha_{n+1} = \lambda_{3n+1}\lambda_{3n+3} + \lambda_{3n+2}\lambda_{3n+3} + \lambda_{3n+2}\lambda_{3n+4}$; (5.51)
- $\gamma_{n+1} = \lambda_{3n+2}\lambda_{3n+4}\lambda_{3n+6}$. (5.52)

Therefore, the infinite lower-Hessenberg matrix associated with the recurrence relation satisfied by $P_n(x;a,b;c,d)$, the 2-orthogonal polynomials with respect to $\big(\mathcal{W}(x;a,b;c,d),\mathcal{W}(x;a,b+1;c+1,d)\big)$, presented in Theorem 5.9, can be decomposed as in (3.36), with the coefficients $\boldsymbol{\lambda}=(\lambda_{k+2})_{k\in\mathbb{N}}$ given by (5.5). Moreover, this is the production matrix of the sequence of 2-Stieltjes-Rogers polynomials

$$S_n^{(2)}(\boldsymbol{\lambda})=\frac{(a)_n\,(b)_n}{(c)_n\,(d)_n},\quad n\in\mathbb{N}.$$

As a result, the recurrence coefficients determined by (4.33a)-(4.33f) are all positive. In addition, it is clear from (5.5) that $\lambda_k\to\dfrac{4}{27}$, as $k\to\infty$. Hence, the asymptotic behaviour (5.38) of the recurrence coefficients in Theorem 5.9 can be derived directly from formulas (5.50)-(5.52). Furthermore, applying Corollary 3.6 to these 2-orthogonal polynomials, their zeros have absolute value smaller than 1. Therefore, recalling the observations made in Section 3.4, we have an alternative proof, independent of the system being Nikishin, that the zeros of $P_n(x;a,b;c,d)$ are all simple and located in the interval $(0,1)$, with the zeros of consecutive polynomials interlacing.

Chapter 6

Final remarks

The central topic of this thesis are the multiple orthogonal polynomials with respect to r absolutely continuous measures on the positive real line, whose moments are ratios of Pochhammer symbols as in (3.1), under the assumptions (3.2). In Chapter 3, we outlined, developed, and detailed an array of techniques and results, useful in the subsequent research presented here and in a future general investigation of other multiple orthogonal polynomials. In Chapters 4 and 5, we gave detailed characterisations of two families of multiple orthogonal polynomials, with emphasis on the type II polynomials on the step-line, with respect to pairs of measures whose moments are ratios of two-by-one (4.2) and two-by-two (5.3) Pochhammer symbols, respectively. Therefore, these families are particular cases of the polynomials studied in Chapter 3. In fact, when the number of orthogonality measures is equal to 2, these are two of the three cases to consider, with the remaining one being the multiple orthogonal polynomials with respect to Macdonald functions. The latter polynomials have appeared in Section 4.6 as a limiting case of the polynomials analysed in Chapter 4, which are themselves a limiting case of the polynomials characterised in Chapter 5. All these three families of multiple orthogonal polynomials are Nikishin systems.

Moreover, we have shown that the cubic components of Hahn-classical 3-fold-symmetric 2-orthogonal polynomials, the object of study in Chapter 2, are particular realisations of one of these three families of multiple orthogonal polynomials. We proved in Section 2.3 that these cubic components of Hahn-classical 3-fold-symmetric 2-orthogonal polynomials are also Hahn-classical. We believe that the proof of this result may be extended to prove that the components of Hahn-classical $(r+1)$-fold symmetric r-orthogonal polynomials, with $r>2$, are also Hahn-classical, as well as to prove analogous results regarding Hahn-classical polynomials with respect to other lowering operators, such as the q-derivative.

The most natural future direction of research following the work presented here is the investigation of the multiple orthogonal polynomials with respect to vectors of $r>2$ absolutely continuous measures on the positive real line, whose moments are ratios of Pochhammer symbols as in (3.1), subject to (3.2). The investigation of these multiple orthogonal polynomials will build on the techniques and results developed in Chapter 3, using the analysis of the multiple orthogonal polynomials in Chapters 4 and 5 as a source of inspiration. An interesting question about these families of multiple orthogonal polynomials is whether they all are Nikishin systems, as it happens in the cases with 2 orthogonality measures. The newly found connection of these polynomials with branched continued fractions will play a fundamental role in the future analysis of the polynomials, and reciprocally the study of this connection will lead to new results on branched continued fractions and, consequently, to solve total-positivity problems of combinatorial interest.

The decompositions of infinite Hessenberg matrices associated with 2-orthogonal polynomials in Sections 4.5 and 5.5 are explicit examples of the applications of branched continued fractions in the analysis of multiple orthogonal polynomials. The results in Section 4.5 involve sequences of coefficients $\left(\lambda_{k+2}^{[\epsilon]}\right)_{k\in\mathbb{N}}$, with $\epsilon\in\{0,1\}$. The coefficients $\left(\lambda_{k+2}^{[1]}\right)_{k\in\mathbb{N}}$ were obtained from a 2-branched-continued-fraction representation of the generalised hypergeometric series ${}_3F_1\left(a,b,1;c\,\middle|\,t\right)$, the generating function of the 2-Stieltjes-Rogers polynomials $S_n^{(2)}\left(\left(\lambda_{k+2}^{[1]}\right)_{k\in\mathbb{N}}\right)$, equal to $\frac{(a)_n\,(b)_n}{(c)_n}$ for all $n \in \mathbb{N}$. Very recently, we proved that the 2-Stieltjes-Rogers polynomials $S_n^{(2)}\left(\left(\lambda_{k+2}^{[0]}\right)_{k\in\mathbb{N}}\right)$ are also equal to $\frac{(a)_n\,(b)_n}{(c)_n}$ for all $n \in \mathbb{N}$, and, consequently, that their ordinary generating function, the 2-branched continued fraction with coefficients $\left(\lambda_{k+2}^{[0]}\right)_{k\in\mathbb{N}}$, is ${}_3F_1\left(a, b, 1; c \,|\, t\right)$. This result is a particular case of a generalisation of the branched-continued-fraction representations for ratios of hypergeometric series described in [61, §14], to be submitted for publication in a joint work with Alan Sokal. This is an example of an original result on branched continued fractions obtained as a fruit of the study of their connection with multiple orthogonal polynomials.

Throughout Chapter 3, we imposed the conditions (3.2) to guarantee the existence of an absolutely continuous measure on the positive real line with moments as in (3.1). However, most results in Chapter 3 (in fact, all except the ones on the asymptotic behaviour and the location of the zeros) can be adapted to remain valid without the existence of a measure with moments as in (3.1), whose role is replaced by a linear functional, in those cases. Therefore, the investigation outlined in Chapter 3 can be extended by weakening the conditions (3.2).

For instance, we can remove the assumption that $\min_{1\leq j\leq s}\{b_j\} > \max_{1\leq i\leq r}\{a_i\}$ and replace the condition of the parameters being all positive by the less restrictive condition of no parameters being non-positive integers, while keeping $s \leq r$. This is an extension of the modification of the classical Laguerre and Jacobi orthogonal polynomials obtained by weakening the conditions $\alpha > -1$ or $\alpha, \beta > -1$ and simply assuming that the parameters cannot be negative integers. These modifications preserve most formal properties of the classical polynomials, but the existence of an orthogonality measure on the real line is not available (see [55]).

Another lifting of restrictions in (3.1), perhaps of further interest, is to remove the condition $s \leq r$, leading to an investigation of multiple orthogonal polynomials with respect to vectors of s linear functionals with moments given by (3.1) with $r \leq s$. The simplest case, with $(r, s) = (0, 1)$, corresponds to the classical Bessel orthogonal polynomials. Therefore, this investigation would lead to an extension of the Bessel polynomials to the context of multiple orthogonality.

We showed that the cubic components of Hahn-classical 3-fold-symmetric 2-orthogonal polynomials are particular cases of the 2-orthogonal polynomials with respect to the Macdonald, Tricomi or hypergeometric weights. This is analogous to how the components of the quadratic decomposition of the Hermite and Gegenbauer polynomials, the only symmetric and classical orthogonal polynomials up to a linear transformation of the variable, are particular realisations of the Laguerre and Jacobi polynomials (on the interval $(0,1)$), respectively, as seen in Subsection 3.5.3. As such, we conjecture that, for any $r \in \mathbb{Z}^+$, the components of the decomposition (1.43) of a $(r+1)$-fold symmetric r-orthogonal polynomial sequence are always particular realisations of a family of multiple orthogonal polynomials with respect to a vector of absolutely continuous measures on the positive real line, whose moments are ratios of Pochhammer symbols as in (3.1), subject to (3.2).

A related possible direction of future research is to consider the 2-orthogonal polynomials with respect to the Macdonald, Tricomi and hypergeometric weights as the cubic components of new 3-fold-symmetric 2-orthogonal polynomial sequences, which are not Hahn-classical. This idea is inspired by the generalised Hermite and Gegenbauer semi-classical orthogonal polynomials defined as the symmetric polynomials with quadratic components equal to Laguerre polynomials and Jacobi polynomials on the interval $(0,1)$, respectively (see [14, Ch.I.8 & Ch.V.2]). More generally, one may construct $(r+1)$-fold-symmetric r-orthogonal polynomial sequences, whose components of the decomposition (1.43) are r-orthogonal polynomials with respect to a vector of absolutely continuous measures on the positive real line, whose moments are ratios of Pochhammer symbols as in (3.1).

Our main focus throughout this thesis was on the type II polynomials on the step-line, the 2-orthogonal polynomials. However, in Sections 4.3 and 5.3 we derived Rodrigues-type formulas generating the type I polynomials and functions on the step-line with respect to the Tricomi and hypergeometric weights, respectively. A relevant future development in the characterisation of these type I polynomials would be to find explicit formulas for them, analogous to the ones obtained in [19, §2] for the type I polynomials on the step-line with respect to the Macdonald weights. Furthermore, the study of the systems of multiple orthogonal polynomials investigated in Chapters 4 and 5 for multi-indices away from the step-line and, in particular, the study of the (standard) orthogonal polynomials with respect to the Tricomi and hypergeometric weights defined by (4.1) and (5.1) remains an open (and challenging) problem. Recently, the orthogonal polynomials with respect to the Macdonald weight (4.34) were analysed in [79].

An important problem that remains open about the 2-orthogonal polynomials with respect to the Tricomi weights analysed in Chapter 4 is to find their asymptotic zero distribution. We defer this investigation where, amongst other results, a sharper upper bound for their largest zeros, than the one in Theorem 4.13, could likely be obtained by adapting Theorem 3.5 to incorporate the asymptotic periodicity of the recurrence coefficients.

The analysis of multiple orthogonal polynomials with respect to weights involving the confluent hypergeometric function of the first kind is another possible direction for future work closely related to the research in Chapter 4. Such an investigation may lead to the characterisation of a family of multiple orthogonal polynomials with respect to a pair of orthogonality measures whose moments have a different structure to the ones in (3.1). Equally compelling is to unfold the connection of those multiple orthogonal polynomials with branched continued fractions.

The Mehler-Heine asymptotic behaviour near the origin of the 2-orthogonal polynomials under analysis in Chapter 5 is given in Proposition 5.11 and it coincides with the corresponding asymptotic behaviour of the Jacobi-Piñeiro polynomials. Similarly to the Jacobi-Piñeiro polynomials, the Mehler-Heine asymptotic behaviour of the 2-orthogonal polynomials investigated in Chapter 5 near the endpoint 1 remains unknown, and its analysis is a possible direction for future work related to the research presented in this thesis. We believe that this asymptotic behaviour involves the Bessel function $\mathrm{J}_{\delta-1}$ because the orthogonality weights behave as $\mathcal{O}\left((1-x)^{\delta-1}\right)$ near $x=1$. A similar remark is made for Jacobi-Piñeiro polynomials in [72, §5].

To sum up, we have presented, in Chapters 4 and 5, detailed characterisations of two new families of Nikishin systems of multiple orthogonal polynomials. These polynomial systems are connected with Hahn-classical 3-fold-symmetric 2-orthogonal polynomials, which were studied in Chapter 2. Both are part of a bigger picture: the multiple orthogonal polynomials with respect to absolutely continuous measures on the positive real line with moments equal to ratios of Pochhammer symbols, an investigation we outlined in Chapter 3. This research is still in an initial stage with a lot of questions to be addressed, particularly on their connection with branched continued fractions. Furthermore, as expounded here, there are several possible future directions of research related to the work presented in this thesis and applications to related fields.

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