LAURENT POLYNOMIAL LANDAU-GINZBURG MODELS FOR COMINUSCULE HOMOGENEOUS SPACES AND MIRROR SYMMETRY FOR THE EXCEPTIONAL FAMILY

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Declaration

The work in this thesis is based on research completed at the School of Mathematics, Statistics and Actuarial Science at the University of Kent. This thesis nor any part of it has been submitted elsewhere for any other degree or qualification. Chapter 1 contains necessary background material, for which no originality is claimed and references are given. The work in Chapter 2 is original research published open-access in [Spa21] under a Creative Commons licence agreed with *Transformation Groups*. Chapter 3 is original material resulting from own research currently unpublished or unavailable as preprint, but planned to be released on arXiv in [SW21] combined with results from a collaboration with Charles Wang at Harvard University. Credit has been given to persons and articles whenever they formed an inspiration for the results obtained in this thesis.

Abstract

This thesis considers mirror symmetry for the small quantum cohomology of cominuscule homogeneous spaces. We present two main results: Firstly, in Theorem 2.2.7, we present a type-independent Laurent polynomial expression for Rietsch's Lie-theoretic mirror model [Rie08] restricted to an algebraic torus. Secondly, in Theorems 3.1.1 and 3.1.2, we present canonical mirror models for the exceptional family in terms of projective coordinates called (generalized) Plücker coordinates, and show that these are isomorphic to Rietsch's models.

The Laurent polynomial expression resembles the potential for projective complete intersections given in [Giv96]: the sum of all the toric coordinates plus a "quantum term" consisting of a homogeneous polynomial in the toric coordinates divided by the product of all toric coordinates. This polynomial is initially enumerated by subexpressions of a given Weyl group element in another. In Corollary 2.5.12 we show that this enumeration can be replaced by diagrammatic combinatorics considering subsets of the quivers defined in [Per07, CMP08]. As we illustrate in the example of Grassmannians, these subsets can be considered as generalizations of Young diagrams.

The canonical mirror models for the exceptional family are similar to the models found for other cominuscule families in [MR20, PR13, PRW16]. One notable difference is that we find cubic and quartic homogeneous polynomials in the Plücker coordinates, whereas these polynomials were found to be at most quadratic in the cases of Lagrangian Grassmannians and quadrics, and linear for Grassmannians. Analogously to the aforementioned papers, we use a presentation of the coordinate ring of a unipotent cell by [GLS11]. However, we give a type-independent criterion for the generators of this presentation to coincide (up to a constant) with Plücker coordinates in Proposition 3.2.5, and we show the isomorphism with Rietsch's mirror model using the algebraic torus where our Laurent polynomial expression holds.

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Contents

Chapter 1

Context and Background

This thesis presents results in the context of mirror symmetry for cominuscule homogeneous spaces. The first chapter of this thesis will be devoted to introducing all concepts necessary to present our results. However, before we do so, let us start by giving a short overview of the context of this thesis and a summary of the main results. We also mention some of the implications and our intentions for future research.

An overview of the context. We should start by specifying what we mean with mirror symmetry. One could argue that in its essence, mirror symmetry strives to find relations between structures defined in different geometric theories: algebraic geometry and symplectic geometry. These relations, more commonly than not, involve finding a pair of spaces $(X, X^\vee)$ such that the symplectic structure on X is isomorphic to the algebraic structure on $X^\vee$. There are many ways to fill in the structures one wants to consider mirror symmetry for.

In this thesis, we consider the *small quantum cohomology* $qH^*(X)$ as the symplectic structure on X. One can consider this structure to be a rigorous way to count curves intersecting two given subspaces of X, generalizing the (intersection) cohomology that counts intersection points (i.e. degree-0 curves) of two subspaces. We will introduce this in Section 1.3 and illustrate it with the example of the quantum cohomology of $\mathbb{CP}^n$. For now, we note that $qH^*(X)$ is a polynomial ring over a set of "quantum parameters" that keep track of the degrees of the curves, with coefficients in $H^*(X)$.

Thus, we want to find a mirror space $X^\vee$ and an algebraic structure related to $X^\vee$ that is isomorphic to the small quantum cohomology of X. The algebraic structure we find is the coordinate ring of $X^\vee$ modulo relations generated by the derivatives of a function $\mathcal{W} : X^\vee \to \mathbb{C}$ called the *potential*. However, there is a catch: to obtain an isomorphism, one needs to localize the quantum parameters of the quantum cohomology (i.e. adjoin the inverses of the quantum parameters to the quantum cohomology). A pair $(X^\vee, \mathcal{W} : X^\vee \to \mathbb{C})$ such that $\mathbb{C}[X^\vee]/\langle \partial \mathcal{W} \rangle \cong qH^*(X)_{\mathrm{loc}}$ is called a *Landau-Ginzburg model* for X. (Note that the ideal $\langle \partial \mathcal{W} \rangle$ generated by the derivatives of $\mathcal{W}$ is known as the *Jacobi ideal* of $\mathcal{W}$ and the ring $\mathbb{C}[X^\vee]/\langle \partial \mathcal{W} \rangle$ is also known as the *Jacobi ring* associated to $(X^\vee, \mathcal{W})$. However, we will refrain from using this terminology.)

Now we have discussed mirror symmetry, let us have a look at cominuscule homogeneous spaces. We will consider homogeneous spaces as projective varieties X over $\mathbb{C}$ with a transitive action of an (almost) simple, complex linear algebraic group G. We will recall the basic definitions in Section 1.1. In particular, we will note that we can assume without loss of generality that G is simply-connected, and that there exists a parabolic subgroup $P \subset G$ such that $X \cong G/P$. After the choice of a maximal torus (diagonalizable, commutative subgroup of maximal dimension) and a Borel subgroup (subgroup of minimal dimension such that the quotient is a projective variety), we will see that this parabolic subgroup corresponds to a choice of vertices in the Dynkin diagram of G. When $P = P_k$ corresponds to a single vertex marked k such that the fundamental weight ω_k is cominuscule, the homogeneous variety $X = G/P$ is called cominuscule; see Section 1.5.

Mirror symmetry for homogeneous spaces has seen a lot of progress over the last decade and a half. Of particular relevance to this thesis is Rietsch's *Lie-theoretic Landau-Ginzburg model*, see [Rie08]. We will introduce this model in Section 1.4. This Landau-Ginzburg model holds for all homogeneous spaces, also non-cominuscule ones. It is completely defined in Lie-theoretic terms: namely it has as mirror variety a Richardson variety in the Langlands dual full flag variety, and as potential the sum of dual Chevalley generators. However, this is not the end of the story: the geometric

Satake correspondence implies a correspondence between a basis for the cohomology of a homogeneous space G/P and a set of projective coordinates on the Langlands dual homogeneous space $P^\vee \backslash G^\vee$ known as *(generalized) Plücker coordinates*, see Section 1.6. Therefore, it is convenient to find a Landau-Ginzburg model that is formulated in terms of Plücker coordinates.

For cominuscule homogeneous spaces, the correspondence between the basis of the cohomology of G/P and Plücker coordinates on $P^\vee \backslash G^\vee$ is in fact one-to-one. Thus, in the papers [MR20, PR13, PRW16] Plücker coordinate Landau-Ginzburg models were constructed and proven to be isomorphic to Rietsch's Lie-theoretic mirror models for Grassmannians, Lagrangian Grassmannians, and quadrics, respectively. (See Table 1.5.1 for the definitions of these homogeneous spaces.) Following [PRW16], these Plücker coordinate mirror models have become known as *canonical mirror models*: they consist of the complement of an anti-canonical divisor in $P^\vee \backslash G^\vee$ defined by the potential, which is a rational map in terms of the Plücker coordinates. In constructing the canonical models and proving the isomorphisms with the Lie-theoretic models, all cases relied on Laurent polynomial expressions for Rietsch's potential restricted to an algebraic torus.

Summary of the main results. This is where the results of this thesis come in. In Chapter 2 we prove a type-independent Laurent polynomial expression for Rietsch's Lie-theoretic potential restricted to an algebraic torus, see Theorem 2.2.7. In this proof, the Weyl groups associated to G and P play a large role, so we will introduce these in Section 1.2. The Laurent polynomial potential consists of the sum of the toric coordinates plus a "quantum term" consisting of a homogeneous polynomial in these coordinates divided by the product of the toric coordinates. Initially, the homogeneous polynomial is enumerated by reduced subexpressions of a given Weyl group element in another. In Corollary 2.5.12 we show that this enumeration can be replaced by subsets of a quiver defined in [Per07, CMP08]. Moreover, we give combinatorial tools to find all such subsets, see Lemma 2.5.9 and Proposition 2.5.11. The Laurent polynomial expression coincides with the known potentials for Lagrangian

Grassmannians and quadrics, and the expression offers Laurent polynomial potentials for orthogonal Grassmannians and the exceptional family, which have not been found before. These results have been published in [Spa21].

In Chapter 3, we construct canonical mirror models for the exceptional cominuscule homogeneous spaces and show that they are isomorphic to Rietsch's Lie-theoretic mirror models, see Theorems 3.1.1 and 3.1.2. These models are similar to the canonical mirror models for Grassmannians, Lagrangian Grassmannians and quadrics given in [MR20, PR13, PRW16]. However, there is one notable difference: where the terms in the potentials were at most quadratic homogeneous polynomials in Plücker coordinates for Lagrangian Grassmannians and quadrics and even linear for Grassmannians, we find cubic and quartic terms for the exceptional family. We take an analogous approach to the earlier cases by using a presentation by [GLS11] of the coordinate ring of a certain unipotent cell. However, we also give a type-independent criterion for the generators of this coordinate ring to coincide (up to a constant) with Plücker coordinates, see Proposition 3.2.5. Moreover, we simplify a part of the computations by restricting to the algebraic torus where our Laurent polynomial expression is defined. Using this torus, we also find "Plücker relations" on the mirror varieties expressing the remaining Plücker coordinates in terms of the generating ones.

Implications and future research. The first and most direct implications of these results are the advances made in developing a type-independent approach to describe mirror symmetry for cominuscule homogeneous spaces. The Laurent polynomial potential and the criterion for Plücker coordinates to be coordinate ring generators we present in this thesis have been formulated without restrictions on the type. Therefore, the largest remaining obstruction to formulating a general canonical mirror model for cominuscule homogeneous spaces is a type-independent Plücker coordinate description of a set of *generalized minors* which we introduce in Section 1.6. Obtaining such a description might turn out to be troublesome, since it requires comparisons of actions of unipotent elements on different representations. As indicated in the proof of

Lemma 3.3.2, these comparisons can be quite complicated, but restricting to the algebraic torus simplifies the calculations and might be sufficient for a type-independent consideration.

The canonical models themselves also have interesting features. For example, it is expected that they all have cluster structures. These structures have been determined for Grassmannians and quadrics in [MR20] and [PRW16]. In the second reference, a description of the cluster structures in terms of generalized minors by [GLS08] was used. Currently, I am collaborating with Charles Wang at Harvard University to see if a type-independent Plücker coordinate expression for the cluster structure based on the same reference is feasible, especially since the approach there is extremely general. We initially focused on obtaining more examples and have found the cluster structures for the mirrors of the exceptional cominuscule homogeneous spaces. In fact, to simplify a number of calculations in obtaining these, we made use of the Laurent polynomial potential and the algebraic torus on which it is defined. These simplifications work quite similarly to those we will see in Chapter 3. We are hoping to release our results on arXiv shortly [SW21], but the results were not yet at a stage to be included in this thesis.

The canonical mirrors also play a role in a stronger mirror conjecture concerning homogeneous spaces. This strengthening is obtained by replacing the quantum cohomology with the D-module associated to the Dubrovin connection, and by replacing the Landau-Ginzburg model with the D-module obtained from the Gauss-Manin system associated to the model. In the case of quadrics, an injection from the first D-module into the second is shown in [PRW16] using the description of canonical mirror. Moreover, it is known from [LT17] that the two D-modules are isomorphic for *minuscule* homogeneous spaces, i.e. G/P_k with ω_k minuscule. See Table 1.5.1 for homogeneous spaces that are either minuscule, cominuscule or both. However, note in particular that odd quadrics are cominuscule but *not* minuscule. Furthermore, the isomorphism is not formulated explicitly in [LT17]; instead, they rely on Langlands reciprocity. Thus, we believe that obtaining descriptions in terms of Plücker coordinates of the mirrors of more cominuscule cases will help to make the isomorphism more explicit and expand it to all cominuscule homogeneous spaces.

In [PRW16], the Laurent polynomial expression was subsequently used to calculate the components of a flat section of the (dual) Dubrovin connection defined in [Giv96]. This implied an expression for the *A-series* or *hypergeometric series* that forms a part of Givental's J-function. These methods could be generalized using the type-independent Laurent polynomial potential presented here.

In [Rie08], Rietsch also conjectures that the Lie-theoretic Landau-Ginzburg models give rise to oscillatory integrals that are solutions to the quantum differential equations of X, see Conjecture 8.1 there. This conjecture is verified in [MR20] for Grassmannians and [PRW16] for quadrics using the descriptions of a flat section of the Dubrovin connection obtained from the Laurent polynomial expressions. Therefore, we expect that progress can be made in resolving this conjecture for other cominuscule examples with our Laurent polynomial expression. We would like to note that the results of [LT17] imply the conjecture for all minuscule cases as well, again using different methods.

Due to the one-to-one correspondence between the Schubert classes in the (quantum) cohomology of a cominuscule homogeneous space and the (generalized) Plücker coordinates on its mirror, we can use the canonical models to investigate the quantum product (this is also known as *quantum Schubert calculus*). For example, the Plücker relations we found in Remarks 3.3.4 and 3.4.4 imply similar relations between Schubert classes through this correspondence. We have not made any investigations of this in this thesis, focusing instead on the construction and its proof. However, we would like to note that descriptions of the ring structure of the quantum cohomology of cominuscule homogeneous spaces are already available. For example, [CMP08] gives a unified expression of the quantum cohomologies of cominuscule cases and explicitly describes the ring structure for the exceptional family, achieved using different approaches. There one can also find references to the descriptions of the quantum cohomologies for the cominuscule homogeneous spaces for the classical linear algebraic groups that were already available before. Even non-cominuscule cases have been investigated: for example, for *adjoint* homogeneous spaces (these are *quasi-cominuscule* but not cominuscule) the quantum cohomologies have been described in [CP11], while the quantum cohomologies of homogeneous spaces with maximal parabolic subgroups (i.e. $X = G/P$ with $P = P_k$ associated to a single vertex k of the Dynkin diagram) for the classical linear algebraic groups have been described in [BKT09].

Another example of a future implication of the canonical models is in *homological mirror symmetry*. This is an isomorphism of triangulated categories between the Fukaya category of a variety (on the symplectic side) and the (bounded) derived category of coherent sheaves on its mirror (on the algebraic side). Recently, [Cas20] used the canonical mirror model for Grassmannians proven in [MR20] to show that homological mirror symmetry holds for Grassmannians $\mathrm{Gr}(k,n)$ for n prime. This result is a very promising step in the consideration of homological mirror symmetry for homogeneous spaces, especially now canonical models are available for (almost) all cominuscule homogenous spaces.

I will conclude with some remarks on my plans for future research, other than the investigations of cluster structures with Charles Wang mentioned in passing before. Firstly, I have been awarded an Early Career Fellowship by the London Mathematical Society to collaborate with Prof Nicolas Perrin at the Laboratoire de Mathématiques de Versailles on extending the Laurent polynomial potentials and the definition of the quivers used to enumerate their quantum terms to the adjoint homogeneous spaces we mentioned before. This would most likely be followed up by an investigation into canonical models for these spaces, as well as into the possibility of a type-independent unification of the adjoint and cominuscule cases. I am also interested in investigating the stronger mirror symmetry statements discussed before, either on the level of D-modules or triangulated categories, for cominuscule homogeneous spaces. Finally, I have recently become interested in considering the *Newton-Okounkov bodies* of cominuscule homogeneous spaces. A Newton-Okounkov body is a polytope associated to a variety parametrizing its toric deformations. In [RW19], the Laurent polynomial potential for the Grassmannian is used to describe its Newton-Okounkov body from a mirror-symmetric perspective. I would be interested in investigating these methods for the other cominuscule homogenous spaces and have contacted Prof Akihiro Higashitani at Osaka University and Dr Naoki Fujita at the University of Tokyo to start a collaboration regarding these.

A final remark regarding this chapter. We have restricted the presentation of the background material in this chapter to those concepts strictly necessary for these results. However, we have tried to give ample references for the reader interested in finding out more about them. The expert reader can most likely skip the first three or four sections. Nevertheless, we would like to encourage the expert reader to at least have a look at Examples 1.1.10, 1.2.7 and in particular Example 1.4.6, as they introduce a number of phenomena that play an important role in the narrative of Chapter 2. Sections 1.5 and 1.6 are essential for Chapters 2 and 3 respectively. Finally, Section 1.7 introduces some notation for the exceptional cominuscule homogeneous spaces that will also play a role in Chapter 3.

1.1 Homogeneous spaces and Langlands duality

Homogeneous spaces (i.e. spaces with a transitive group action) can be considered in a number of contexts. Here, we will restrict ourselves to the context of algebraic varieties, or more specifically *projective* varieties. The transitive groups are therefore assumed to have the structure of linear algebraic groups. The theory of homogeneous spaces in this context is well established and there are many in-depth references available, e.g. [Hum75, Bor91, Spr98]. We will only present a short overview of this here, focusing on the concepts that will play a role. Moreover, we will only consider varieties over $\mathbb{C}$.

At the end of this section we illustrate all definitions for $\mathbb{CP}^2$, which is homogeneous for the simple, complex algebraic group $\mathrm{SL}_3 = \mathrm{SL}_3(\mathbb{C})$.

We will be considering transitive actions of simple, simply-connected, complex algebraic groups.

Definition 1.1.1. A *(complex) algebraic group* is a variety G (over $\mathbb{C}$) with morphisms $\mu : G \times G \to G$ and $\iota : G \to G$ that form the product and inverse maps for a group structure on G. These morphisms will henceforth be denoted by the juxtaposition ($\mu(g,h) = gh$) and reciprocal ($\iota(g) = g^{-1}$) of the elements respectively.

Example 1.1.2. Consider $\mathbb{C}^{n\times n}$ as the affine variety of complex-valued $n\times n$-matrices. Recall that $\det : \mathbb{C}^{n\times n} \to \mathbb{C}$ is polynomial in the coefficients of the matrix. Hence, one obtains $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C}) = \{M \in \mathbb{C}^{n\times n} \mid \det(M) \neq 0\}$ as a quasi-affine subvariety (open subset of an affine variety). In fact, GL_n is isomorphic to the affine variety $\{(M,t) \in \mathbb{C}^{n\times n} \times \mathbb{C} \mid t\det(M) - 1 = 0\}$ through the morphism $M \mapsto (M, \det(M)^{-1})$. In any case, GL_n is an algebraic variety. Since the inverse of a matrix is obtained using determinants, $M \mapsto M^{-1}$ is a morphism on GL_n, and it is clear from the definition of matrix multiplication that $(M_1, M_2) \mapsto M_1M_2$ is a morphism as well. It is well known that matrix multiplication and inversion induce a group structure on GL_n. Thus, GL_n is an algebraic group known as the *general linear group* of invertible $n \times n$-matrices over $\mathbb{C}$. $\diamondsuit$

Definition 1.1.3. A *linear algebraic group* is an algebraic group that is (isomorphic to) a subgroup of GL_n. A connected (nontrivial) linear algebraic group is called *semisimple* when the trivial subgroup is the only (proper) closed, connected, commutative normal subgroup. A noncommutative, semisimple algebraic group is called *simple*[1] when the trivial subgroup is the only (proper) connected, normal subgroup.

The simple complex algebraic groups can be classified using the classification of simple Lie algebras (see e.g. Sections 13.5 and 14.1 of [Hum75]). However, as they

[1]This is also called *almost simple*, see Section 14.2 of [Hum75]. This is because a simple algebraic group can have a nontrivial center (albeit discrete), which contradicts the definition of a simple (abstract) group.

allow nontrivial discrete centers, there are multiple simple complex algebraic groups of a given type. Those that have a trivial center are called *adjoint*, while those with maximal[2] discrete center are in fact *simply-connected* varieties.

type	A_n	B_n	C_n	D_n
ad	PSL_{n+1}	SO_{2n+1}	PSp_{2n}	PSO_{2n}
sc	SL_{n+1}	Spin_{2n+1}	Sp_{2n}	Spin_{2n}

Table 1.1.1: The adjoint (ad) and simply-connected (sc) simple linear algebraic groups of classical types.

Definition 1.1.4. A *homogeneous space X for a linear algebraic group G* is a projective variety with a transitive action of G.

We will only consider simple linear algebraic groups G.

Similarly to the theory of Lie groups, one can associate a Lie algebra to a linear algebraic group. Write $\mathfrak{g}$ for the Lie algebra of G and fix a set of Chevalley generators $(e_1, f_1, \dots, e_n, f_n)$, i.e. Lie algebra generators such that for each $i \in \{1, \dots, n\}$ the triple (e_i, f_i, h_i) with $h_i = [e_i, f_i]$ generates a Lie subalgebra isomorphic to $\mathfrak{sl}_2$. This gives the decomposition $\mathfrak{g} = \mathfrak{u}_+ \oplus \mathfrak{t} \oplus \mathfrak{u}_-$, where $\mathfrak{u}_+$ is generated by $\{e_i \mid 1 \le i \le n\}$, $\mathfrak{u}_-$ is generated by $\{f_i\}$, and the Cartan subalgebra $\mathfrak{t}$ is spanned by $\{h_i\}$. We denote by $\mathcal{U}_+$ and $\mathcal{U}_-$ the universal enveloping algebras of $\mathfrak{u}_+$ and $\mathfrak{u}_-$ respectively, and we write their completions as $\widehat{\mathcal{U}}_+$ and $\widehat{\mathcal{U}}_-$.

Let T be the *maximal torus*, i.e. diagonalizable, commutative subgroup of maximal dimension, of G that has $\mathfrak{t}$ as Lie algebra. Let U_+ and U_- be the subgroups of G that have $\mathfrak{u}_+$ and $\mathfrak{u}_-$ as Lie algebras, respectively. These are called *unipotent subgroups*, as they are (isomorphic to) subgroups of the (upper/lower) triangular matrices with ones on the diagonal in GL_n. We can consider U_+ and U_- as lying inside $\widehat{\mathcal{U}}_+$ and $\widehat{\mathcal{U}}_-$ respectively, generated by the one-parameter subgroups

$$x_i(a) = \exp(a\, e_i) \quad \text{and} \quad y_i(a) = \exp(a\, f_i),$$

[2]The center needs to be a subgroup of the *fundamental group* of the root system, i.e. quotient of the weight lattice by the root lattice.

for $i \in \{1, \ldots, n\}$ and $a \in \mathbb{C}$. Here $\exp(a\, e_i) = 1 + a\, e_i + \frac{1}{2}a^2\, e_i^2 + \ldots \in \widehat{\mathcal{U}}_+$ and $\exp(a\, f_i) \in \widehat{\mathcal{U}}_-$ is given analogously.

For any choice of $I_P \subset \{1, \ldots, n\}$, we can define a Lie algebra $\mathfrak{p}$ that is generated as

$$\mathfrak{p} = \langle e_i, h_i, f_j \mid i \in \{1, \ldots, n\},\ j \in I_P \rangle. \tag{1.1.1}$$

There is a unique subgroup $P \subset G$ with $\mathfrak{p}$ as Lie algebra. These subgroups have the property that G/P is a (projective) homogeneous space for G, and are called *parabolic subgroups*. The complement of I_P will play a role in the following and is denoted I^P.

Remark 1.1.5. One can define a parabolic subgroup to be a subgroup P of G such that G/P is a (projective) homogeneous space. In this way, one finds that the parabolic subgroups fall into conjugacy classes that are determined by a subset I_P of the Dynkin diagram of G (see Theorem 29.3 of [Hum75]). Fixing a maximal torus T and a unipotent subgroup U_+ then gives a unique element of each of these conjugacy classes that satisfies $TU_+ \subset P \subset G$ (see Corollary 23.1C of [Hum75]).

As noted above, the subgroup $B_+ = TU_+$ plays an important role. It is a parabolic subgroup of minimal dimension, with $I_P = \varnothing$, so that its Lie algebra is $\mathfrak{t} \oplus \mathfrak{u}_+$. Parabolic subgroups of minimal dimension are called *Borel subgroups*. There is a second Borel subgroup that will play a role, namely $B_- = TU_-$, which is called its *opposite* and has $\mathfrak{t} \oplus \mathfrak{u}_-$ as Lie algebra.

With the subgroups of G fixed as above, there is a unique parabolic subgroup P (up to symmetries of the Dynkin diagram) containing B_+ such that $G/P \cong X$ for any homogeneous space for G. Therefore, we can consider homogeneous spaces of the form $X = G/P$ without loss of generality.

Remark 1.1.6. Let $X = G/P$ be a projective space for G a simple linear algebraic group, and let $\widetilde{G}$ be the (simply-connected) universal cover of G. Let $\widetilde{P}$ be the preimage of P under $\widetilde{G} \to G$; then we have $\widetilde{G}/\widetilde{P} \cong X$. Thus, we can assume that the simple algebraic group acting transitively is simply-connected, without loss of generality.

We write $\mathcal{X}$ for the lattice of characters $\chi : T \to \mathbb{C}^*$ of the maximal torus (written additively). Within $\mathcal{X}$, we denote the set of roots by $\Phi \subset \mathcal{X}$ and a base of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is determined by the Chevalley generators. The associated sets of positive and negative roots are denoted by Π and Π_- respectively. We denote the cocharacter lattice by $\mathcal{X}^\vee$, the coroots by $\Phi^\vee$ and the simple coroots by $\alpha_i^\vee : \mathcal{X} \to \mathbb{C}$.

With a given root system Φ and character lattice $\mathcal{X}$, there exists a unique group $G^\vee$ determined by having as root system the coroots $\Phi^\vee$ and as character lattice the cocharacter lattice $\mathcal{X}^\vee$ of G. The character lattice $\mathcal{X}^\vee$ of $G^\vee$ also determines a maximal torus $T^\vee$ in $G^\vee$. The pair $(G^\vee, T^\vee)$ is called the *Langlands dual pair* associated to (G, T); we call $G^\vee$ the *Langlands dual group*.

Remark 1.1.7. As G is assumed to be simply-connected, $G^\vee$ will be adjoint.

The Langlands dual group $G^\vee$ inherits the base of simple roots $\Delta^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$, which in turn determines the decomposition of the Lie algebra $\mathfrak{g}^\vee$ of $G^\vee$ into $\mathfrak{g}^\vee = \mathfrak{u}_-^\vee \oplus \mathfrak{t}^\vee \oplus \mathfrak{u}_+^\vee$. The Langlands dual groups $U_+^\vee$, $U_-^\vee$, $B_+^\vee$, $B_-^\vee$ and $P^\vee$ are now defined analogously to above, and we write $\Pi^\vee$ and $\Pi_-^\vee$ for the sets positive and negative roots of $G^\vee$. We also obtain Chevalley generators $(e_1^\vee, f_1^\vee, \ldots, e_n^\vee, f_n^\vee)$ and define the corresponding one-parameter subgroups of $U_+^\vee$ and $U_-^\vee$

$$x_i^\vee(a) = \exp(a\, e_i^\vee) \quad \text{and} \quad y_i^\vee(a) = \exp(a\, f_i^\vee) \tag{1.1.2}$$

for $i \in \{1, \ldots, n\}$ and $a \in \mathbb{C}$. Here $\exp(a\, e_i^\vee) = 1 + a\, e_i^\vee + \frac{1}{2}a^2(e_i^\vee)^2 + \ldots \in \widehat{\mathcal{U}}_+^\vee$ in the completed universal enveloping algebra of $\mathfrak{u}_+^\vee$, and analogously for $\exp(a\, f_i^\vee)$. Note that the parabolic subgroup $P^\vee$ is associated to the same set I_P as P: that is, its Lie algebra is given by

$$\mathfrak{p}^\vee = \langle e_i^\vee, h_i^\vee, f_j^\vee \mid i \in \{1, \ldots, n\},\ j \in I_P \rangle. \tag{1.1.3}$$

Thus, the complement of the set of indices is I^P as well.

Remark 1.1.8. When the Dynkin diagram of G is simply-laced, $G^\vee$ has the same Dynkin diagram (with the same numbering of the vertices). When the Dynkin diagram of G has a double or triple edge, the Dynkin diagram of $G^\vee$ is obtained by reversing the arrows at these edges. Explicitly, if G is of type A_n, D_n or E_n, then $G^\vee$ is of the same type; if G is of type B_n, then $G^\vee$ is of type C_n, and vice versa; finally, for G of type F_4 and G_2, $G^\vee$ is of the same type but has the reverse numbering of vertices.

Apart from the Chevalley generators $(e_1^\vee, f_1^\vee, \dots, e_n^\vee, f_n^\vee)$ for $\mathfrak{g}^\vee$, we will need the corresponding dual maps $(e_i^\vee)^*, (f_i^\vee)^* \in (\mathfrak{g}^\vee)^*$ as well, satisfying

$$(e_i^\vee)^*(e_j^\vee) = \delta_{ij} = (f_i^\vee)^*(f_j^\vee) \quad \text{and} \quad (e_i^\vee)^*(f_j^\vee) = 0 = (f_i^\vee)^*(e_j^\vee) \tag{1.1.4}$$

and vanishing on $\mathfrak{t}^\vee$ and the other root spaces of $\mathfrak{g}^\vee$.

We extend these maps to be defined on arbitrary products of the Chevalley generators using the inclusions of $\mathfrak{u}_+^\vee$ and $\mathfrak{u}_-^\vee$ into their completed universal algebras $\widehat{\mathcal{U}}_+^\vee$ and $\widehat{\mathcal{U}}_-^\vee$. This in turn allows us to define $(e_i^\vee)^*$ and $(f_i^\vee)^*$ on $U_+^\vee$ and $U_-^\vee$ through the identification of the one-parameter subgroups $x_i^\vee(a) \in U_+^\vee$ and $y_i^\vee(a) \in U_-^\vee$ with $\exp(a\, e_i^\vee) \in \widehat{\mathcal{U}}_+^\vee$ and $\exp(a\, f_i^\vee) \in \widehat{\mathcal{U}}_-^\vee$ respectively. Equivalently, $(e_i^\vee)^*$ and $(f_i^\vee)^*$ are defined as the unique group homomorphisms $U_+^\vee \to \mathbb{C}$ and $U_-^\vee \to \mathbb{C}$ such that

$$(e_i^\vee)^*(x_j^\vee(a)) = a\delta_{ij} = (f_i^\vee)^*(y_j^\vee(a)). \tag{1.1.5}$$

As $G^\vee$ is in general not simply-connected, we will need to consider the universal cover $\widetilde{G}^\vee$ of $G^\vee$ in Section 2.3. As before, we define the universal covers $\widetilde{P}^\vee$, $\widetilde{T}^\vee$, $\widetilde{B}_+^\vee$ and $\widetilde{B}_-^\vee$. Note that the cover of $U_+^\vee$ is in fact isomorphic to $U_+^\vee$ and the same holds for $U_-^\vee$, so we simply identify them.

Remark 1.1.9. Considering Remark 1.1.8, we note that for a simply-connected group G with a simply-laced Dynkin diagram we have $\widetilde{G}^\vee \cong G$ as they are both simply-connected and of the same type.

Example 1.1.10. Let us illustrate the concepts of this section for $X = \mathbb{CP}^2$, which is homogeneous for $G = \mathrm{SL}_3$, the simple, simply-connected complex algebraic group of type A_2. Its Lie algebra is $\mathfrak{sl}_3$ and has the standard Chevalley generators:

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad f_i = e_i^T.$$

We find that the Cartan subalgebra elements $h_i = [e_i, f_i]$ are

$$h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Thus, $\mathfrak{t} = \mathbb{C}h_1 \oplus \mathbb{C}h_2$, $\mathfrak{u}_+ = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}[e_1, e_2]$ and $\mathfrak{u}_- = \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}[f_1, f_2]$. The subgroups of G that have these as Lie algebras are

$$T = \left\{ \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1}\mu & 0 \\ 0 & 0 & \mu^{-1} \end{bmatrix} \;\middle|\; \lambda, \mu \in \mathbb{C}^* \right\}, \quad U_+ = \left\{ \begin{bmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; z_1, z_2, z_3 \in \mathbb{C} \right\}$$

and $U_- = U_+^T$. Considering $U_+ \subset \widehat{\mathcal{U}}_+$, we find that $x_1(a)$, $a \in \mathbb{C}$, corresponds to the $u \in U_+$ with $z_1 = a$ and $z_2 = z_3 = 0$. We obtain x_2, y_1 and y_2 similarly.

The Borel subgroups $B_+ = TU_+$ and $B_- = TU_-$ can easily be deduced to be

$$B_+ = \left\{ \begin{bmatrix} \lambda & z_1 & z_3 \\ 0 & \lambda^{-1}\mu & z_2 \\ 0 & 0 & \mu^{-1} \end{bmatrix} \;\middle|\; \lambda, \mu \in \mathbb{C}^*,\ z_1, z_2, z_3 \in \mathbb{C} \right\} \quad \text{and} \quad B_- = B_+^T. \tag{1.1.6}$$

There are three parabolic subgroups containing B_+. Firstly, taking $I^P = \{1\}$, we find

$$P_1 = \left\{ \left[\begin{array}{c|cc} \lambda^{-1} & z_1 & z_2 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{λM}} \\ 0 & & \end{array} \right] \;\middle|\; \lambda \in \mathbb{C}^*,\ z_1, z_2 \in \mathbb{C},\ M \in \mathrm{SL}_2 \right\}. \tag{1.1.7}$$

The second parabolic subgroup, with $I^P = \{2\}$, is

$$P_2 = \left\{ \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{λM}} & z_1 \\ & & z_2 \\ \hline 0 & 0 & \lambda^{-1} \end{array} \right] \;\middle|\; \lambda \in \mathbb{C}^*,\ z_1, z_2 \in \mathbb{C},\ M \in \mathrm{SL}_2 \right\}.$$

Finally, the third parabolic subgroup has $I^P = \{1, 2\}$, so that $P_{1,2} = B_+$.

Clearly, the parabolic subgroup P such that $\mathrm{SL}_3/P = \mathbb{CP}^2$ is $P = P_1$. The other homogeneous spaces are $\mathrm{SL}_3/P_2 = \mathrm{Gr}(2, 3)$ and $\mathrm{SL}_3/P_{1,2} = \mathrm{SL}_3/B_+ = \mathrm{Fl}(1, 2; 3)$ (the *full flag variety* parametrizing pairs (L, H) of a line L and a plane H in $\mathbb{C}^3$ satisfying $L \subset H$).

The Langlands dual group to $G = \mathrm{SL}_3$ is $G^\vee = \mathrm{PSL}_3 = \mathrm{SL}_3/\mathbb{Z}_3$, where $\mathbb{Z}_3 \subset \mathbb{C}$ is the set of cube roots of unity (embedded in SL_3 as multiples of the identity matrix). This is indeed the adjoint, simple, complex algebraic group of type A_2. The Langlands dual subgroups are therefore simply the quotients of the original subgroups by $\mathbb{Z}_3$; in the following, we will denote the class in PSL_3 of a matrix $M \in \mathrm{SL}_3$ by $\mathbb{Z}_3M$. Note that we have $G^\vee/P^\vee \cong \mathbb{CP}^2$, as we noted in Remark 1.1.6.

Finally, it is clear that the universal cover of $G^\vee = \mathrm{PSL}_3$ is $\widetilde{G}^\vee = \mathrm{SL}_3 = G$, in accordance with Remark 1.1.9. $\diamondsuit$

1.2 Weyl groups and Schubert classes

Lie theory is intimately related to the theory of reflection groups via the reflections on root systems. It should not be a surprise, then, that the same holds for linear algebraic groups. Indeed, the *Weyl group* associated to a linear algebraic group G is known to parametrize a cell decomposition of the group, and appropriate subsets parametrize a basis for the cohomology of homogeneous spaces for G. In Chapter 2, the Weyl group will also play an important role in deriving a Laurent polynomial expression for Rietsch's potential. Thus, we will take some time to introduce the Weyl groups associated to linear algebraic groups and their derived concepts here. We will again use $\mathbb{CP}^2 = \mathrm{SL}_3/P_1$ to illustrate this section at the end.

We keep the notation from Section 1.1. In particular, we have maximal tori $T \subset G$ and $T^\vee \subset G^\vee$. Denote by $N_G(T) \subset G$ and $N_{G^\vee}(T^\vee) \subset G^\vee$ their respective normalizers, which of course contain the tori themselves (as they are commutative subgroups). Thus, we can define the *Weyl group* of G as $W = N_G(T)/T$ and the Weyl group of

$G^\vee$ as $W^\vee = N_{G^\vee}(T^\vee)/T^\vee$. These groups are named because of the following fact:

Theorem 1.2.1 (See e.g. Theorem 27.1 of [Hum75]). The group $W = N_G(T)/T$ is isomorphic to the Weyl group of the root system of G.

Here the Weyl group of a root system is the Coxeter group generated by the reflections in the simple roots. Note that these Weyl groups only depend on the underlying Coxeter diagram of the root system, and since Remark 1.1.8 implies that G and $G^\vee$ have the same Coxeter diagram, we obtain

Corollary 1.2.2. The Weyl groups $W = N_G(T)/T$ of G and $W^\vee = N_{G^\vee}(T^\vee)/T^\vee$ of $G^\vee$ are isomorphic.

Thus, we can consider both Weyl groups at the same time without the need to introduce parallel notation. In particular, we will also refer to the Weyl group of $G^\vee$ as W.

Remark 1.2.3. Note that for type F_4 and G_2 the isomorphism in fact reverses the labeling of the simple roots, so one needs to be more careful for these types. However, as we will see in Section 1.5, these types will not play a role in our results, so we will ignore these subtleties here.

We denote the simple reflections by s_i, where each s_i corresponds to the reflection through a simple root $\alpha_i \in \Delta$ (or $\alpha_i^\vee \in \Delta^\vee$). Thus, every $w \in W$ can be expressed as $w = s_{i_1} \cdots s_{i_j}$; when j is minimal this is called a reduced expression and the integer j is called the length of w and denoted by $\ell(w) = j$. This integer coincides with the number of positive roots mapped to negative roots by w (see for example [Hum78], Lemma 10.3A). The longest element of W is denoted by w_0, and we denote $\ell(w_0) = \ell_0$. It is well known that w_0 maps Π bijectively to Π_- (and similarly $\Pi^\vee$ to $\Pi^\vee_-$), so that $\#\Pi = \#\Pi^\vee = \ell_0$.

There is also a Weyl group W_P associated to the parabolic subgroup by removing the simple reflections $\{s_i \mid i \in I^P\}$ from the generators of W, compare equation (1.1.3). Note that W_P is a Weyl group in its own right, associated to the Dynkin diagram of

$G^\vee$ with the vertices marked by I^P removed. The longest element of W_P is denoted by w_P, and we write $\ell(w_P) = \ell_P$. It maps ℓ_P positive (co-) roots to negative ones; the sets of these are denoted $\Pi_P \subset \Pi$ and $\Pi_P^\vee \subset \Pi^\vee$. Let $\Pi_{P,-} = w_P(\Pi_P) \subset \Pi_-$ and $\Pi_{P,-}^\vee = w_P(\Pi_P^\vee) \subset \Pi_-^\vee$. Note that $\Phi_P = \Pi_P \sqcup \Pi_{P,-}$ and $\Phi_P^\vee = \Pi_P^\vee \sqcup \Pi_{P,-}^\vee$ form mutually dual root systems in their own right, corresponding to the same Dynkin subdiagram as W_P.

We will need representatives in $G^\vee$ for $s_i \in W = N_{G^\vee}(T^\vee)/T^\vee$. There are two convenient representatives determined by the one-parameter subgroups:

$$\dot{s}_i = x_i^\vee(1) y_i^\vee(-1) x_i^\vee(1) \quad \text{and} \quad \bar{s}_i = x_i^\vee(-1) y_i^\vee(1) x_i^\vee(-1) = \dot{s}_i^{-1}. \tag{1.2.1}$$

Note that these elements are not their own inverses, as one would expect of reflections: they square to elements in $T^\vee$. Moreover, note that $\dot{s}_i$ and $\bar{s}_i$ only differ by a torus element and both normalize $T^\vee$.

We can extend this choice of representative to an arbitrary $w \in W$ with reduced expression $w = s_{i_1} \cdots s_{i_d}$ by setting $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_d}$ and $\bar{w} = \bar{s}_{i_1} \cdots \bar{s}_{i_d}$. Note that $\bar{w}$ is not equal to $\dot{w}^{-1}$ in general: it has the reverse product of simple reflections, i.e. $\dot{w}^{-1} = \bar{s}_{i_d} \cdots \bar{s}_{i_1}$.

We let

$$T^P = (T^\vee)^{W_P} \subset T^\vee \tag{1.2.2}$$

be the part of $T^\vee$ that is invariant under the action $W_P \times T^\vee \to T^\vee$ given by $(w, t) \mapsto \dot{w} t \dot{w}^{-1}$. Clearly, T^P has dimension $\# I^P$.

One of the classical results involving elements of the Weyl group is the following (see e.g. Theorem 28.3 of [Hum75]):

Theorem 1.2.4 (Bruhat decomposition)**.** A *Bruhat cell* $B_+ w_1 B_+$ equals another $B_+ w_2 B_+$ if and only if $w_1 = w_2 \in W$; otherwise the cells are disjoint. Moreover, we obtain a decomposition $G = \bigsqcup_{w \in W} B_+ w B_+$.

Note that since $T \subset B_+$, the choice of representative in G for the Weyl group elements is immaterial. Moreover, note that we can obtain a similar decomposition with B_+ replaced by B_-, and that $G^\vee$ allows both decompositions over the same group W (using $B_+^\vee$ or $B_-^\vee$) as well.

The Bruhat decomposition of G implies in turn Bruhat decompositions for G/B_+ (see e.g. Theorem 14.12 of [Bor91]) and for G/P (see e.g. Proposition 1.3 of [Köc91]). Namely, we have $G/B_+ = \bigsqcup_{w\in W} B_+ w B_+/B_+$. For the decomposition of G/P, we need to fix a set of minimal coset representatives of W/W_P, which we denote by $W^P \subset W$ (i.e. for every coset the representative of minimal length). With this, the Bruhat decomposition becomes $G/P = \bigsqcup_{w\in W^P} B_+ wP/P$.

The elements of W^P will play an important role throughout this thesis. In particular, we will need the minimal representative of $w_0 W_P$, which we denote by w^P. We denote its length by $\ell(w^P) = \ell$. (The length of w^P will occur frequently, so we simply write ℓ instead of ℓ^P.) Note that $w_0 = w^P w_P$, so that in particular $\ell_0 = \ell + \ell_P$. Moreover, we can now see from the Bruhat decomposition of $X = G/P$ that $\dim(X) = \ell(w^P) = \ell$. We fix reduced expressions

$$w^P = s_{r_1} \cdots s_{r_\ell} \quad \text{and} \quad w_P = s_{q_1} \cdots s_{q_{\ell_P}}, \tag{1.2.3}$$

so we obtain a reduced expression $w_0 = s_{r_1} \cdots s_{r_\ell} s_{q_1} \cdots s_{q_{\ell_P}}$.

Remark 1.2.5. We would like to note that the sequence of indices $(r_1, \ldots, r_\ell)$ of the reduced expression for w^P will play a large role throughout this thesis. The sequence $(q_1, \ldots, q_{\ell_P})$ will play less of a significant role and only appear in Chapter 2. Whenever we use (q_i) to denote the sequence of indices of the reduced expression for w_P, we will always explicitly recall this. We will use q throughout the thesis to denote the quantum parameter in the quantum cohomology of the homogeneous spaces we will consider, and in Chapter 3 we will use q_i to denote a set of homogeneous polynomials in Plücker coordinates.

There are two kinds of varieties related to the Bruhat decomposition that will play a role in the following. The first are the *open Richardson varieties*

$$\mathcal{R}_{v,w} = \big(B_+ vB_- \cap B_- wB_-\big)/B_- \ \subset \ G/B_-,$$

which are the intersections of the Bruhat cells of G/B_- of decompositions with respect to *opposite* Borel subgroups, here B_+ and B_-. These intersections are only non-empty when $v \leq w$ in the *Bruhat order*:

$$v \leq w \text{ if there is a } u \in W \text{ such that } w = uv \text{ and } \ell(w) = \ell(u) + \ell(v). \tag{1.2.4}$$

The second kind is the *Schubert variety* $\overline{B_+ wP/P} \subset G/P$ for $w \in W^P$, which is the closure of the Bruhat cell (also called *Schubert cell* in this context) $B_+ wP/P$. Note that

$$\overline{B_+ wP/P} = \bigsqcup_{v \leq w} B_+ vP/P, \tag{1.2.5}$$

where $v \leq w$ is again the Bruhat order given in (1.2.4). The dimension of this variety is $\ell(w)$ and we denote the associated cohomology class by $\sigma_w \in H^{2\ell(w^P)-2\ell(w)}(G/P)$. These classes are called *Schubert classes* and are particularly important because of the following result:

Proposition 1.2.6. The Schubert classes $\{\sigma_w \mid w \in W^P\}$ form a basis for $H^*(G/P)$.

(See for example Corollary 1.5 of [Köc91]. However, note that there the statement is formulated in terms of the Chow ring $CH^*(G/P)$, but that for homogeneous spaces this is isomorphic to the cohomology ring under doubling of degrees.)

Example 1.2.7. We continue Example 1.1.10 to illustrate the new concepts. We have $\mathbb{CP}^2 = \mathrm{SL}_3/P_1$. As G is of type A_2, we have $W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1 = s_2s_1s_2\}$. The Weyl group associated to P_1 is $W_P = \{e, s_2\}$. Note that we can represent W as a permutation group by representing s_1 and s_2 by

$$S_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

However, these matrices are not elements of SL_3. Moreover, their classes $\mathbb{Z}_3 S_i$ are not elements of PSL_3. (Recall from Example 1.1.10 that we represent the class in PSL_3 of a matrix $M \in \mathrm{SL}_3$ by $\mathbb{Z}_3 M$.) The representatives $\dot{s}_1$ and $\bar{s}_1$ of s_1 in PSL_3 defined in equation (1.2.1) are in fact

$$\dot{s}_1 = \mathbb{Z}_3 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{Z}_3 \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\bar{s}_1 = \mathbb{Z}_3 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{Z}_3 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly, we find

$$\dot{s}_2 = \mathbb{Z}_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \bar{s}_2 = \mathbb{Z}_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that $\dot{s}_1$, $\bar{s}_1$, $\dot{s}_2$ and $\bar{s}_2$ do not square to (the class of) the identity matrix:

$$\dot{s}_1^2 = \bar{s}_1^2 = \mathbb{Z}_3 \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in T^\vee, \quad \dot{s}_2^2 = \bar{s}_2^2 = \mathbb{Z}_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in T^\vee.$$

The other Weyl group elements are represented by

$$\dot{s}_1\dot{s}_2 = \mathbb{Z}_3 \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad \dot{s}_2\dot{s}_1 = \mathbb{Z}_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$
$$\bar{s}_1\bar{s}_2 = \mathbb{Z}_3 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{s}_2\bar{s}_1 = \mathbb{Z}_3 \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$
$$\dot{s}_1\dot{s}_2\dot{s}_1 = \dot{s}_2\dot{s}_1\dot{s}_2 = \bar{s}_1\bar{s}_2\bar{s}_1 = \bar{s}_2\bar{s}_1\bar{s}_2 = \mathbb{Z}_3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix};$$

Recall from Example 1.1.10 that the torus $T^\vee$ consist of classes of diagonal matrices with determinant 1. Clearly, $t \in T^P$ if it satisfies $t = \dot{s}_2 t \dot{s}_2^{-1}$, so writing

$$t = \mathbb{Z}_3 \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1}\mu & 0 \\ 0 & 0 & \mu^{-1} \end{bmatrix}$$

we find

$$\dot{s}_2 t \dot{s}_2^{-1} = \mathbb{Z}_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1}\mu & 0 \\ 0 & 0 & \mu^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \mathbb{Z}_3 \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu^{-1} & 0 \\ 0 & 0 & \lambda^{-1}\mu \end{bmatrix}.$$

From this we see that $t = \dot{s}_2 t \dot{s}_2^{-1}$ if and only if $\lambda = \mu^2$, so we find that

$$T^P = \left\{ \mathbb{Z}_3 \begin{bmatrix} \mu^2 & 0 & 0 \\ 0 & \mu^{-1} & 0 \\ 0 & 0 & \mu^{-1} \end{bmatrix} \;\middle|\; \mu \in \mathbb{C}^* \right\},$$

which is indeed of dimension $\#I^P = 1$.

Instead of giving a description of the Bruhat cells of SL_3 and Richardson varieties in SL_3/B_+, we will only describe the Schubert cells and varieties of $\mathbb{CP}^2 = \mathrm{SL}_3/P_1$. The approach is quite similar and it would take too long to discuss them all. For the Schubert cells, we first need the set W^P for $P = P_1$.

As $W_P = \{e, s_2\}$, the Weyl group W splits into the following cosets: $\{e, s_2\} = W_P$, $\{s_1, s_1 s_2\} = s_1 W_P$ and $\{s_2 s_1, s_2 s_1 s_2\} = s_2 s_1 W_P$. The longest element $w_0 = s_2 s_1 s_2$ is contained in the third coset, so that $w^P = s_2 s_1$. Indeed, the longest element of W_P is $w_P = s_2$, so we find $w^P w_P = s_2 s_1 s_2 = w_0$. Thus, we find $W^P = \{e, s_1, s_2 s_1\}$. Clearly, we have $e \leq s_1 \leq s_2 s_1$ in the Bruhat order.

According to equation (1.2.5), we find $\overline{B_+ e P/P} \subset \overline{B_+ s_1 P/P} \subset \overline{B_+ s_2 s_1 P/P}$. Let us verify this.

First, we have $\overline{B_+ e P/P} = P/P$ is a point, so $\sigma_e \in H^4(\mathbb{CP}^2)$ is the class of a point.

Next, we have $\overline{B_+ s_1 P/P}$. Note from (1.1.6) that $b\bar{s}_1 \in B_+ s_1$ is of the form

$$\begin{bmatrix} \lambda^{-1}\mu^{-1} & z_1 & z_3 \\ 0 & \lambda & z_2 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} z_1 & -\lambda^{-1}\mu^{-1} & z_3 \\ \lambda & 0 & z_2 \\ 0 & 0 & \mu \end{bmatrix}.$$

Here we have defined $\bar{s}_1 \in G$ analogously to the definition of $\bar{s}_i \in G^\vee$ given in (1.2.1). Recall that we described $P = P_1$ in equation (1.1.7), so that the class $b\bar{s}_1 P/P$ is uniquely determined by the value of $\lambda \in \mathbb{C}^*$, i.e. $B_+ s_1 P/P = y_1(\lambda)P/P$ for $\lambda \in \mathbb{C}^*$. Clearly, the closure of this Schubert cell is obtained by adding the point $y_1(0)P/P = P/P$. Thus, as claimed, we find that $\overline{B_+ eP/P} \subset \overline{B_+ s_1 P/P}$ and more precisely $\overline{B_+ s_1 P/P} = B_+ eP/P \sqcup B_+ s_1 P/P$. Note that $\sigma_{s_1} \in H^2(\mathbb{CP}^2)$ is the class of a line (and hyperplane).

Finally, $\overline{B_+ s_2 s_1 P/P}$. We find that elements $b\bar{s}_2\bar{s}_1 \in B_+ s_2 s_1$ are of the form

$$\begin{bmatrix} \lambda^{-1}\mu^{-1} & z_1 & z_3 \\ 0 & \lambda & z_2 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} z_3 & -\lambda^{-1}\mu^{-1} & -z_1 \\ z_2 & 0 & -\lambda \\ \mu & 0 & 0 \end{bmatrix}.$$

Thus, the class $b\bar{s}_2\bar{s}_1 P/P$ is uniquely determined by $\mu \in \mathbb{C}^*$ and $z_2 \in \mathbb{C}$. Note that this has an empty intersection with the other Schubert cells as $\mu \neq 0$. We see that $\overline{B_+ s_2 s_1 P/P} = B_+ eP/P \sqcup B_+ s_1 P/P \sqcup B_+ s_2 s_1 P/P = \mathbb{CP}^2$, verifying the inclusions $\overline{B_+ eP/P} \subset \overline{B_+ s_1 P/P} \subset \overline{B_+ s_2 s_1 P/P}$. Clearly, $\sigma_{s_2 s_1} \in H^0(\mathbb{CP}^2)$ is the fundamental class. $\diamondsuit$

1.3 Small quantum cohomology

As mentioned in the introduction, the main object of study in this thesis is the small quantum cohomology of cominuscule homogeneous spaces. This warrants an introduction to its definition, structure and the underlying motivation. However, since our approach is to investigate the quantum cohomology using mirror symmetry, almost all our considerations are done on the "algebraic side of the mirror". Therefore, we will only introduce the quantum cohomology[3] briefly.

We would like to suggest the book *An Invitation to Quantum Cohomology* by Kock and Vainsencher [KV07] as a reference (for both the small and big quantum cohomology of $\mathbb{CP}^n$).

[3]There are two concepts of quantum cohomology, which are distinguished by the names "small" and "big". We are only concerned with the "small" version, and we will drop the adjective.

As the name suggests, the quantum cohomology $qH^*(X)$ is an extension of the ordinary (intersection/de Rham) cohomology of a variety. In fact, it is a deformation in the formal sense: $qH^*(X)$ is obtained by adjoining formal "quantum parameters" and deforming the intersection product of $H^*(X)$ to a "quantum product". In particular, setting the quantum parameters to zero returns the ordinary cohomology.

This time, we will start with an example instead of ending with one:

Example 1.3.1. Consider $X = \mathbb{CP}^n$, so that $H^*(X) = \mathbb{C}[h]/\langle h^{n+1}\rangle$. Let us consider the relation $h^i \cup h^j = h^{i+j}$: we can rewrite it as $h^i \cup h^j = \langle h^i \cdot h^j \cdot h^{n-i-j}\rangle_0\, h^{i+j}$, where

$$\langle h^i \cdot h^j \cdot h^{n-i-j}\rangle_0 = \int_{\mathbb{CP}^n} h^i \cup h^j \cup h^{n-i-j} = \int_{\mathbb{CP}^n} h^n = 1,$$

denoting by $\langle \gamma_1 \cdot \gamma_2 \cdot \gamma_2\rangle_0$ the integration over $\mathbb{CP}^n$ of the intersection product of the three classes. Note that for $k \neq i+j$, we have $\langle h^i \cdot h^j \cdot h^{n-k}\rangle_0 = 0$, so we can rewrite the intersection product even further as

$$h^i \cup h^j = \sum_{k=0}^{n} \langle h^i \cdot h^j \cdot h^{n-k}\rangle_0\, h^k.$$

Now, we can interpret $\langle [Y_1] \cdot [Y_2] \cdot [Y_3]\rangle_0$ as the number of degree-0 curves (i.e. points) intersecting Y_1, Y_2 and Y_3. This leads to an obvious generalization: let $\langle [Y_1] \cdot [Y_2] \cdot [Y_3]\rangle_d$ be the number of degree-d curves intersecting Y_1, Y_2 and Y_3. Then we deform $\cup$ into the quantum product $*$ by setting

$$h^i * h^j = \sum_{d=0}^{\infty} \sum_{k=0}^{n} \langle h^i \cdot h^j \cdot h^{n-k}\rangle_d\, h^k q^d.$$

Clearly, setting $q = 0$ returns the intersection product. It turns out that for given (i, j), the number $\langle h^i \cdot h^j \cdot h^{n-k}\rangle_d$ is only non-zero for one pair (k, d), so the sum is actually finite. In fact, we find

$$h^i * h^j = \begin{cases} h^{i+j}, & i+j \leq n, \\ h^{i+j-(n+1)}q, & i+j > n. \end{cases}$$

Thus, the quantum cohomology of $X = \mathbb{CP}^n$ is $qH^*(X) = \mathbb{C}[h,q]/\langle h^{n+1} - q\rangle$. $\Diamond$

To properly define $\langle \gamma_1 \cdot \gamma_2 \cdot \gamma_2 \rangle_d$, we have to consider *stable maps* to X.

Remark 1.3.2. To preclude any complications with the definitions, we assume that X is *convex.* i.e. that $H^1(\mathbb{CP}^1, \mu^* T_X) = 0$ for all maps $\mu : \mathbb{CP}^1 \to X$. Homogeneous spaces are convex by the transitivity of the group action (see [FP97], p. 6).

Definition 1.3.3. Let C be a projective, connected, nodal curve of genus 0, and fix distinct smooth points $p_1, \ldots, p_n \in C$, which we call *marked points.* A map $\mu : C \to X$ is called *(Kontsevich) stable* if every smooth component ($\cong \mathbb{CP}^1$) mapped to a point has at least three marked or singular points. We call the $\beta \in H_2(X, \mathbb{Z})$ such that $\mu_*([C]) = \beta$ the *degree* of the map. Note that β is necessarily effective. The *moduli space of stable maps* to X of degree $\beta \in H_2(X,\mathbb{Z})_{\text{eff}}$ with n marked points is denoted $\overline{M}_{0,n}(X, \beta)$. This space has n natural *evaluation maps* given by

$$\mathrm{ev}_i : \quad \overline{M}_{0,n}(X,\beta) \to X : \quad (C, \{p_1, \ldots, p_n\}; \mu) \mapsto \mu(p_i).$$

Remark 1.3.4. Stability of a map can equivalently be characterized by the finiteness of the automorphism group of the map, see Lemma 2.3.1 of [KV07].

Remark 1.3.5. Stable maps (and their associated Gromov-Witten invariants below) can also be defined for higher genera. However, the structure of the associated moduli spaces is much less understood.

Under the assumptions of Remark 1.3.2, it is known that the moduli space is nicely behaved:

Theorem 1.3.6 ([FP97], Theorem 2)**.** For X projective, nonsingular and convex, $\overline{M}_{0,n}(X,\beta)$ is a normal projective variety of pure dimension $\dim(X) + \int_\beta c_1(T_X) + n - 3$. Locally, it is the quotient of a nonsingular variety by a finite group.

In particular, in the case of convex varieties, integration over $\overline{M}_{0,n}(X,\beta)$ is well-defined.

Definition 1.3.7. For $\gamma_1, \dots, \gamma_n \in H^*(X)$ and $\beta \in H_2(X)$, the (n-point) *Gromov-Witten invariant* $\langle \gamma_1 \cdot \ldots \cdot \gamma_n \rangle_d$ is defined as

$$\langle \gamma_1 \cdot \ldots \cdot \gamma_n \rangle_d = \int_{\overline{M}_{0,n}(X,\beta)} \mathrm{ev}_1^*(\gamma_1) \cup \ldots \cup \mathrm{ev}_n^*(\gamma_n).$$

Note that this is independent of the order of the factors.

As $\mathrm{ev}_i^*([Y])$ is (the class of) the locus of stable maps μ satisfying $\mu(p_i) \in Y$, the Gromov-Witten invariant seems to count the stable maps whose images intersect with n given subvarieties. However, for this count to work, one needs a Bertini-type transversality condition. For homogeneous spaces, the transitive group action can be used to obtain this, and the Gromov-Witten invariants indeed agree with these counts (see [FP97], p. 8). Moreover, note that the count includes more than simply the curves of a given degree, although for X a homogeneous space an open, dense subvariety of $\overline{M}_{0,n}(X,\beta)$ consists of maps whose image are smooth curves.

Only the 3-point invariants appear in the (small) quantum cohomology:

Definition 1.3.8. Let X be a smooth, projective and convex variety. Take a $\mathbb{Z}$-basis $\{\beta_1, \dots, \beta_r\} \subset H_2(X, \mathbb{Z})$, and let $\Gamma \subset H^*(X)$ be a $\mathbb{C}$-basis. For every $\gamma \in \Gamma$, denote by γ^* the the Poincaré dual element. The *quantum cohomology* $qH^*(X)$ is defined as the ring $H^*(X)[q_1, \dots, q_r]$ equipped with the *quantum product* $*$. For $\gamma_1, \gamma_2 \in H^*(X)$, this is defined as

$$\gamma_1 * \gamma_2 = \sum_{\mathbf{d} \in \mathbb{N}_0^r} \sum_{\gamma \in \Gamma} \langle \gamma_1 \cdot \gamma_2 \cdot \gamma \rangle_{\mathbf{d} \cdot \boldsymbol{\beta}} \, \mathbf{q}^{\mathbf{d}} \gamma^*, \tag{1.3.1}$$

where $\mathbf{d} \cdot \boldsymbol{\beta} = \sum_{i=1}^r d_i \beta_i$ and $\mathbf{q}^{\mathbf{d}} = \prod_{i=1}^r q_i^{d_i}$. (We denoted $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.)

The quantum cohomology of homogeneous spaces has had a lot of attention of the years since the initial conception of quantum cohomology. A complete survey of these references would lead us too far afield. Of particular relevance is the paper [CMP08], as well as the references therein. In particular, it gives a unified treatment of the quantum cohomology of ring of *cominuscule* homogeneous spaces as well as explicitly describing its ring structure for the exceptional family.

As we will discuss in Section 1.5, cominuscule homogeneous spaces are in particular homogeneous spaces G/P with P of maximal dimension, i.e. $I^P = \{k\}$ for a certain $k \in \{1, \dots, n\}$. It is easy to see that for homogeneous spaces $X = G/P$ with $P = P_k$, the second homology group $H_2(X, \mathbb{Z})$ is one-dimensional. Indeed, we can deduce this from Poincaré duality and our discussion of Schubert varieties in Section 1.2:

If $I^P = \{k\}$, then $W_P = \langle s_i \mid i \neq k \rangle$, so that $s_i W_P = W_P$ for all $i \neq k$. Thus, $s_k W_P$ is the only coset wW_P with $\ell(w) = 1$ unequal to W_P. This implies that the Schubert variety $\overline{B_+ s_k P/P}$ is the only one-dimensional Schubert variety. Now, since the Schubert classes form a basis of the cohomology, we find that $H^{2n-2}(X) = \mathbb{C}\sigma_{s_k}$, implying that $H_2(X, \mathbb{Z}) = \mathbb{Z}\beta$ for the Poincaré dual β of σ_{s_k}. (Here we are using that the Schubert classes form a basis over $\mathbb{Z}$ as well.)

Thus, in the case of homogeneous spaces G/P with P of maximal dimension, the expression in (1.3.1) simplifies significantly:

$$\gamma_1 * \gamma_2 = \sum_{d \in \mathbb{N}_0} \sum_{\gamma \in \Gamma} \langle \gamma_1 \cdot \gamma_2 \cdot \gamma \rangle_{d\beta}\, q^d \gamma^*. \tag{1.3.2}$$

Similarly, the quantum cohomology of these homogeneous spaces is a polynomial ring over a single quantum parameter q.

1.4 Rietsch's Lie-theoretic mirror model

In [Rie08], Rietsch constructs a Landau-Ginzburg model for general homogeneous spaces $X = G/P$ for P an arbitrary parabolic subgroup. The mirror variety there is a subvariety of the open Richardson variety associated to (w_P, w_0). Recall that this open Richardson variety is given by

$$X^\vee_{\mathrm{Lie}} = \mathcal{R}^\vee_{w_P, w_0} = \left(B^\vee_+ w_P B^\vee_- \cap B^\vee_- w_0 B^\vee_-\right)/B^\vee_- \ \subset\ G^\vee/B^\vee_-. \tag{1.4.1}$$

This variety turns out to be related the following subset of $G^\vee$:

$$\mathcal{Z}_P^\vee = B_-^\vee \bar{w}_0^{-1} \cap U_+^\vee T^P \bar{w}_P U_-^\vee \ \subset \ G^\vee. \tag{1.4.2}$$

Namely, there exists an isomorphism

$$\Psi : X_{\mathrm{Lie}}^\vee \times T^P \xrightarrow{\sim} \mathcal{Z}_P^\vee, \tag{1.4.3}$$

whose inverse is given by $z \mapsto (zB_-^\vee, t)$ where $z = u_+ t \bar{w}_P u_-$. This map is well-defined by Lemma 1.4.4.

Remark 1.4.1. These results are analogous to the statements in [Rie08], Section 4.1, although we have modified the definition of $\mathcal{Z}_P^\vee$ compared to [Rie08] to facilitate calculations in Section 2.4. Indeed, Ψ can be obtained as follows: For a given $t \in T^P$, a class $rB_-^\vee \in \mathcal{R}_{w_P,w_0}^\vee$ allows by definition a representative of the form $r = b_- \bar{w}_0^{-1} b'_- = b_+ \bar{w}_P t u_-$ (where $b_-, b'_- \in B_-^\vee$, $b_+ \in B_+^\vee$ and $u_- \in U_-^\vee$). Note that this representative is unique up to right-multiplication with an element of $U_-^\vee$. Writing $b'_- = t' u'_-$ for $t' \in T^\vee$ and $u'_- \in U_-^\vee$, the element $r(u'_-)^{-1}$ is independent of the choice of representative. Moreover, it is an element of $B_-^\vee \bar{w}_0^{-1}$, as $b_- \bar{w}_0^{-1} t' = b_- t'' \bar{w}_0^{-1}$ for some $t'' \in T^\vee$. Writing $b_+ = t_+ u_+$ for $t_+ \in T^\vee$ and $u_+ \in U_+^\vee$, we find that $\Psi(z) = t_+^{-1} r (u'_-)^{-1}$ is well-defined and an element of $\mathcal{Z}_P^\vee$ (using $\bar{w}_P t = t \bar{w}_P$).

Now, in [Pet97], Peterson gave a presentation of the quantum cohomology of a generalized flag variety G/P as the coordinate ring of what is subsequently called the *Peterson variety* (see e.g. [Rie08], paragraph 3.2). The coordinate ring of a well-chosen open stratum of this (non-reduced) variety gives the quantum cohomology localized at the quantum parameters (see e.g. [Rie08], equation (3.2)). In [Rie08], this open stratum is then shown to be isomorphic to the critical locus of a certain function on $\mathcal{Z}_P^\vee$. Thus, using the isomorphism Ψ of equation (1.4.3), we obtain a subvariety of $X_{\mathrm{Lie}}^\vee \times T^P$ whose coordinate ring is also isomorphic to the localized quantum cohomology of X.

Instead of presenting these statements in more detail, we will instead follow the reformulation of these results presented in [MR20], Theorem 6.5; see also Section 4.2 of [PR18]. There the critical locus is replaced with all of $X^\vee_{\mathrm{Lie}} \times T^P$, but one needs to take the quotient of the coordinate ring by an ideal generated by the derivatives of a potential.

Theorem 1.4.2 ([Rie08], Theorem 4.1, *Lie-theoretic Landau-Ginzburg model*)**.** Let $X = G/P$ be a complete homogeneous space with G a simple, simply-connected algebraic group over $\mathbb{C}$ and with P a (not necessarily maximal) parabolic subgroup. There exists a potential $\mathcal{W}_{\mathrm{Lie}} : X^\vee_{\mathrm{Lie}} \times T^P \to \mathbb{C}$ (given in Definition 1.4.5) such that'

$$qH^*(X)_{\mathrm{loc}} \cong \mathbb{C}[X^\vee_{\mathrm{Lie}} \times T^P]/\langle \partial \mathcal{W}_{\mathrm{Lie}} \rangle,$$

where $qH^*(X)_{\mathrm{loc}}$ is the (small) quantum cohomology of X with inverted quantum parameters and $\langle \partial \mathcal{W}_{\mathrm{Lie}} \rangle$ is the ideal generated by the derivatives of $\mathcal{W}_{\mathrm{Lie}}$ along $X^\vee_{\mathrm{Lie}}$.

The potential $\mathcal{W}_{\mathrm{Lie}}$ is presented in [PR18] as the pull-back of a potential defined on $\mathcal{Z}^\vee_P$ along the isomorphism $\Psi : X^\vee_{\mathrm{Lie}} \times T^P \xrightarrow{\sim} \mathcal{Z}^\vee_P$ from equation (1.4.3). To state this potential, we introduce to the following subset of $U^\vee_-$, which is an intersection of $U^\vee_-$ with a Bruhat cell (see Section 1.2):

$$U^P_- = U^\vee_- \cap B^\vee_+ \bar{w}_P \bar{w}_0 B^\vee_+ = U^\vee_- \cap B^\vee_+ (\dot{w}^P)^{-1} B^\vee_+ \ \subset \ U^\vee_-. \tag{1.4.4}$$

Remark 1.4.3. Note that the two definitions of U^P_- coincide: $(w^P)^{-1} = w_P w_0$ and the chosen representatives only differ by a torus element, so that $B^\vee_+ \bar{w}_P \bar{w}_0 B^\vee_+ = B^\vee_+ (\dot{w}^P)^{-1} B^\vee_+$. We will use both expressions; the first will be more convenient in Chapter 2, while the second will be used in Chapter 3.

This set has the following property, which will also be important in Section 2.2:

Lemma 1.4.4 ([PR18], Proposition 5.1)**.** Every $z \in \mathcal{Z}^\vee_P$ has a *unique* decomposition $z = u_+ t \bar{w}_P u_-$ with $u_+ \in U^\vee_+$, $t \in T^P$ and $u_- \in U^P_-$. Moreover, the map $\mathcal{Z}^\vee_P \to U^P_- \times T^P$ given by $z \mapsto (u_-, t)$ is an isomorphism. In particular, fixing (u_-, t) determines u_+.

The proof of this result in [PR18] can be carried over to the general case without any modification, so it will be omitted here.

The potential on $\mathcal{Z}_P^\vee$ is now defined as follows:

Definition 1.4.5. Define the potential $\mathcal{W}_{\mathcal{Z}_P^\vee} : \mathcal{Z}_P^\vee \to \mathbb{C}$ as the map

$$\mathcal{W}_{\mathcal{Z}_P^\vee} : \quad z = u_+ t \bar{w}_P u_- \longmapsto \mathcal{E}^*(u_+^{-1}) + \mathcal{F}^*(u_-),$$

where $\mathcal{E}^* = \sum_{i=1}^n (e_i^\vee)^*$ and $\mathcal{F}^* = \sum_{i=1}^n (f_i^\vee)^*$, and where the decomposition of $z = u_+ t \bar{w}_P u_-$ is the unique decomposition with $u_- \in U_-^P$ as stated in Lemma 1.4.4. Moreover, the potential $\mathcal{W}_{\text{Lie}} : X_{\text{Lie}}^\vee \times T^P \to \mathbb{C}$ mentioned in Theorem 1.4.2 is given by $\mathcal{W}_{\text{Lie}} = \mathcal{W}_{\mathcal{Z}_P^\vee} \circ \Psi$ with Ψ given in equation (1.4.3).

Example 1.4.6. We will illustrate Rietsch's mirror model for $\mathbb{CP}^2$, continuing Examples 1.1.10 and 1.2.7. We will need to consider the subvariety $\mathcal{Z}_P^\vee = B_-^\vee \bar{w}_0^{-1} \cap U_+^\vee T^P \bar{w}_P U_-^P$ of $G^\vee = \text{PSL}_3 = \text{SL}_3/\mathbb{Z}_3$. Let us start with the elements of $U_+^\vee T^P \bar{w}_P U_-^P$. Recall that $U_-^P = U_-^\vee \cap B_+^\vee (\dot{w}^P)^{-1} B_+^\vee$ is the intersection of $U_-^\vee$ with a Bruhat cell. This cell contains elements of the form (note that $(\dot{w}^P)^{-1} = (\dot{s}_2 \dot{s}_1)^{-1} = \bar{s}_1 \bar{s}_2$)

$$\begin{aligned}
&\mathbb{Z}_3 \begin{bmatrix} \lambda_1 & z_1 & z_3 \\ 0 & \lambda_1^{-1}\lambda_2 & z_2 \\ 0 & 0 & \lambda_2^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_3 & z_4 & z_6 \\ 0 & \lambda_3^{-1}\lambda_4 & z_5 \\ 0 & 0 & \lambda_4^{-1} \end{bmatrix} \\
&= \mathbb{Z}_3 \begin{bmatrix} \lambda_3 z_1 & z_1 z_4 + \lambda_3^{-1}\lambda_4 z_3 & z_1 z_6 + z_3 z_5 + \lambda_1 \lambda_4^{-1} \\ \lambda_1^{-1}\lambda_2\lambda_3 & \lambda_1^{-1}\lambda_2 z_4 + \lambda_3^{-1}\lambda_4 z_2 & \lambda_1^{-1}\lambda_2 z_6 + z_2 z_5 \\ 0 & \lambda_2^{-1}\lambda_3^{-1}\lambda_4 & \lambda_2^{-1} z_5 \end{bmatrix}
\end{aligned}$$

where $\lambda_i \in \mathbb{C}^*$ and $z_i \in \mathbb{C}$. The elements in the intersection of this cell with $U_-^\vee$ are thus of the form

$$\mathbb{Z}_3 \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ 0 & a_2 & 1 \end{bmatrix} = y_1^\vee(a_1) y_2^\vee(a_2)$$

for $a_1, a_2 \in \mathbb{C}^*$. (One can check that $z_i \in \mathbb{C}$ give enough degrees of freedom to make the upper triangle of the matrix the required form.)

Note that this is an algebraic torus isomorphic to $(\mathbb{C}^*)^2$ and that the indices $(1,2)$ are exactly the indices of the reduced expression of $(w^P)^{-1} = s_1 s_2$, the inverse expression of w^P. The algebraic torus U_-° consisting of elements of the form $y_{r_\ell}^\vee(a_\ell)\cdots y_{r_1}^\vee(a_1)$ where $a_i \in \mathbb{C}^*$ and $(r_1,\ldots,r_\ell)$ is such that $w^P = s_{r_1}\cdots s_{r_\ell}$ will play an important role in Chapters 2 and 3, see Definition 2.2.1. However, in general we have that $U_-^\circ \subset U_-^P$ is a *strict* dense subset. In fact, it is the restriction to the torus $\mathcal{Z}_P^\circ = B_-^\vee \bar{w}_0^{-1} \cap U_+^\vee T^P \bar{w}_P U_-^\circ$ inside $\mathcal{Z}_P^\vee$ (see Definition 2.2.3) where the Laurent polynomial expression is defined, see Theorem 2.2.7. Moreover, by restricting to U_-° we will be able to prove the isomorphism between Rietsch's Lie-theoretic model and the canonical models for the exceptional cominuscule homogeneous spaces in Theorems 3.1.1 and 3.1.2.

Now we have a description of U_-^P, we can work out $U_+^\vee T^P \bar{w}_P U_-^P$: its elements are of the form (note that $\bar{w}_P = \bar{s}_2$)

$$\begin{aligned}
&\mathbb{Z}_3 \begin{bmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu^2 & 0 & 0 \\ 0 & \mu^{-1} & 0 \\ 0 & 0 & \mu^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ 0 & a_2 & 1 \end{bmatrix} \\
&= \mathbb{Z}_3 \begin{bmatrix} \mu^2 + \mu^{-1}a_1 z_3 & \mu^{-1}z_3 - \mu^{-1}a_2 z_1 & -\mu^{-1}z_3 \\ \mu^{-1}a_1 z_2 & \mu^{-1}z_2 - \mu^{-1}a_2 & -\mu^{-1} \\ \mu^{-1}a_1 & \mu^{-1} & 0 \end{bmatrix}
\end{aligned}$$

where $\mu, a_1, a_2 \in \mathbb{C}^*$ and $z_1, z_2, z_3 \in \mathbb{C}$.

On the other hand, the elements of $B_-^\vee \bar{w}_0^{-1}$ are of the form (note that $\bar{w}_0^{-1} = (\bar{s}_2\bar{s}_1\bar{s}_2)^{-1} = \dot{s}_2\dot{s}_1\dot{s}_2$)

$$\mathbb{Z}_3 \begin{bmatrix} \mu_1 & 0 & 0 \\ w_1 & \mu_1^{-1}\mu_2 & 0 \\ w_3 & w_2 & \mu_2^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \mathbb{Z}_3 \begin{bmatrix} 0 & 0 & \mu_1 \\ 0 & -\mu_1^{-1}\mu_2 & w_1 \\ \mu_2^{-1} & -w_2 & w_3 \end{bmatrix}$$

where $\mu_1, \mu_2 \in \mathbb{C}^*$ and $w_1, w_2, w_3 \in \mathbb{C}$. Comparing these elements with the elements of $U_+^\vee T^P \bar{w}_P U_-^P$, we find that the three zero entries in the top-left of this expression imply that $z_2 = 0$, $z_3 = -\mu^3/a_1$ and in turn $z_1 = -\mu^3/a_1 a_2$. (In particular, we find that $z_1 \neq 0$ and $z_3 \neq 0$.) There are no restrictions on $\mu, a_1, a_2 \in \mathbb{C}^*$. Thus, elements

$\mathcal{Z}_P^\vee$ can be decomposed in the form $u_+ t\bar{w}_P u_-$ as

$$\mathbb{Z}_3\begin{bmatrix} 1 & -\frac{\mu^3}{a_1a_2} & -\frac{\mu^3}{a_1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mu^2 & 0 & 0 \\ 0 & \mu^{-1} & 0 \\ 0 & 0 & \mu^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ 0 & a_2 & 1 \end{bmatrix}.$$

Now, note that $(e_i^\vee)^* : U_+^\vee \to \mathbb{C}$ and $(f_i^\vee)^* : U_-^\vee \to \mathbb{C}$ ($i \in \{1,2\}$) are given by

$$(e_i^\vee)^* : \mathbb{Z}_3\begin{bmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix} \mapsto z_i, \quad \text{and} \quad (f_i^\vee)^* : \mathbb{Z}_3\begin{bmatrix} 1 & 0 & 0 \\ w_1 & 1 & 0 \\ w_3 & w_2 & 1 \end{bmatrix} \mapsto w_i,$$

and that $(e_i^\vee)^*(u^{-1}) = -(e_i^\vee)^*(u)$ for $u \in U_+^\vee$. Thus, $\mathcal{W}_{\mathcal{Z}_P^\vee} : \mathcal{Z}_P^\vee \to \mathbb{C}$ is given by

$$\begin{aligned} \mathcal{W}_{\mathcal{Z}_P^\vee}(z) &= (e_1^\vee)^*(u_+^{-1}) + (e_2^\vee)^*(u_+^{-1}) + (f_1^\vee)^*(u_-) + (f_2^\vee)^*(u_-) \\ &= -(-\tfrac{\mu^3}{a_1a_2}) + 0 + a_1 + a_2 = a_1 + a_2 + q\frac{1}{a_1a_2}, \end{aligned}$$

writing $q = \mu^3 = \alpha_1^\vee(t)$. Note that the choice of representative in SL_3 of $t \in \mathrm{PSL}_3$ is ambiguous: we can replace μ by multiplying it with a cube root of unity without changing its class in PSL_3, but all choices have $\alpha^\vee(t) = \mu^3$, so that $\alpha_1^\vee : T^P \to \mathbb{C}^*$ given by $t \mapsto \mu^3$ is in fact an isomorphism. This holds because $G^\vee$ is adjoint, which is in turn implied by our initial assumption that G is simply-connected.

Before comparing the ring $\mathbb{C}[\mathcal{Z}_P^\vee]/\langle \partial\mathcal{W}_{\mathcal{Z}_P^\vee}\rangle$ with the quantum cohomology of $\mathbb{CP}^2$, we would like to make a number of comments on the (toric, as $U_-^P = U_-^\circ$ in this case) expression for the potential: Firstly, note that we found that $\mathcal{F}^*(u_-) = \sum_i a_i$ is the sum of the toric coordinates on U_-°. In Lemma 2.3.4, we will show that this always holds on U_-°, even when $U_-^\circ \subsetneq U_-^P$. Secondly, we found that $(e_i^\vee)^*(u_+^{-1}) = 0$ for $i \neq 1$ (i.e. $i = 2$) where $P = P_1$. This also holds in general on U_-°, see Lemma 2.3.6. Thirdly, the fact that the denominator of the quantum term always equals the product of the toric coordinates on U_-° will be shown in general in Lemma 2.3.7. Finally, we will see in Lemma 2.3.8 that the numerator of the quantum term on U_-° corresponds to the subexpressions of $w' = w^P(w'')^{-1}$ inside w^P, where w'' is the minimal coset representative of $w_P s_k W_P$ for k such that $P = P_k$. However, for $\mathbb{CP}^2$ (and indeed all

$\mathbb{CP}^n$), we find that $w_P s_1 = w^P$, so that $w' = e$ is the identity element and we find that the numerator is simply 1. In fact, these four lemmas will be the majority of the proof of the Laurent polynomial expression we give in Theorem 2.2.7; the lemmas will be proven in Section 2.4.

Now, let us finally verify Theorem 1.4.2 for $\mathbb{CP}^2$ by calculating $\mathbb{C}[\mathcal{Z}_P^\vee]/\langle \partial \mathcal{W}_{\mathcal{Z}_P^\vee}\rangle$. Recall that in Theorem 1.4.2 the ideal $\langle \partial \mathcal{W}_{\text{Lie}}\rangle$ is generated by the derivatives *along* $X_{\text{Lie}}^\vee$. For $\mathbb{CP}^2$, these correspond on $\mathcal{Z}_P^\vee$ to the derivatives with respect to a_1 and a_2. Note that

$$\langle \partial \mathcal{W}_{\mathcal{Z}_P^\vee}\rangle = \left\langle 1 - q\tfrac{1}{a_1^2 a_2}, 1 - q\tfrac{1}{a_1 a_2^2}\right\rangle \supset \langle a_1 - a_2\rangle.$$

(Multiply the first generator with a_1 and the second with a_2 and take the difference.) Thus, we find that

$$\mathbb{C}[\mathcal{Z}_P^\vee]/\langle \partial \mathcal{W}_{\mathcal{Z}_P^\vee}\rangle = \mathbb{C}[a_1^{\pm 1}, a_2^{\pm 1}, q^{\pm 1}]/\big\langle 1 - q\tfrac{1}{a_1^2 a_2}, 1 - q\tfrac{1}{a_1 a_2^2}\big\rangle \cong \mathbb{C}[a^{\pm 1}, q^{\pm}]/\langle a^3 - q\rangle.$$

(Note that the inverse generators a_i^{-1} and q^{-1} come from the fact that $a_i \in \mathbb{C}^*$ and $q = \mu^3 \in \mathbb{C}^*$.) Recall that $qH^*(\mathbb{CP}^2) \cong \mathbb{C}[h, q]/\langle h^3 - q\rangle$ from Example 1.3.1, then we clearly have

$$\mathbb{C}[\mathcal{Z}_P^\vee]/\langle \partial \mathcal{W}\rangle \cong \mathbb{C}[a^{\pm 1}, q^{\pm 1}]/\langle a^3 - q\rangle \cong \mathbb{C}[h, q^{\pm 1}]/\langle h^3 - q\rangle \cong qH^*(\mathbb{CP}^2)_{\text{loc}},$$

as was claimed in Theorem 1.4.2. $\Diamond$

1.5 Cominuscule homogeneous spaces and minuscule representations

In section 2.2 we will give a Laurent polynomial expression for the potential $\mathcal{W}_{\mathcal{Z}_P^\vee}$ restricted to an open algebraic torus inside $\mathcal{Z}_P^\vee$, after assuming the homogeneous space is *cominuscule*. In this section we will discuss this property, fix further notation and finally consider *minuscule representations* of a given Lie algebra.

We will maintain all the assumptions and conventions of Sections 1.1 and 1.2. In particular, we assume that $X = G/P$ is a homogeneous space for a (complex) linear algebraic group G of rank n that is both simple and simply-connected. Now, we assume in addition that X is *minimal*. This is equivalent to the assumption that P is a *maximal* parabolic subgroup; that is, $I^P = \{k\}$ for a single index $k \in \{1, \ldots, n\}$, see equation (1.1.1). Thus, P is associated to a single vertex k of the Dynkin diagram of G. This fact is denoted by $P = P_k$. Note that we can also write $P^\vee = P_k^\vee$ as $P^\vee$ is associated to the kth vertex of the Dynkin diagram of $G^\vee$ (see equation (1.1.3)).

Because of the maximality of $P^\vee = P_k^\vee$, we know that $W_P = \langle s_i \mid i \neq k \rangle$. In particular, the invariant torus $T^P \subset T^\vee$ is one-dimensional and $\alpha_k^\vee : T^P \to \mathbb{C}^*$ gives an isomorphism ($G^\vee$ being adjoint; we saw this in Example 1.4.6). The element $w_P s_k \in W$ will turn out to be of particular interest; we will write $w'' \in W^P$ for the minimal coset representative of $w_P s_k W_P$ and denote its length by $\ell(w'') = \ell'' \leq \ell = \ell(w^P)$. Note that $w_P s_k w_P = s_0$, where s_0 is the reflection through the highest (co-) root, so that we can equivalently define w'' as the minimal coset representative of $s_0 W_P$. Define $w' \in W$ by

$$w' = w^P (w'')^{-1} \tag{1.5.1}$$

and write $\ell(w') = \ell'$; clearly $\ell = \ell' + \ell''$.

The second assumption we will impose on $X = G/P_k$ is that it is a *cominuscule* homogeneous space. A minimal homogeneous space $X = G/P_k$ is called (co-) minuscule if the fundamental weight ω_k is (co-) minuscule. Recall that the fundamental weights $\{\omega_1, \ldots, \omega_n\}$ form a basis of the character lattice $\mathcal{X}$ dual to the simple coroot basis $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ of the cocharacter lattice $\mathcal{X}^\vee$.

Definition 1.5.1. A fundamental weight ω_i is called *minuscule* if it satisfies one of the following equivalent conditions (see also [Bou68], Section VI.1, exercise 24):

(i) For every $\alpha^\vee \in \Phi^\vee$, $\langle\!\langle \omega_i, \alpha^\vee \rangle\!\rangle \in \{-1, 0, +1\}$, where $\langle\!\langle \cdot, \cdot \rangle\!\rangle : \mathcal{X} \times \mathcal{X}^\vee \to \mathbb{C}$ denotes the dual pairing.

(ii) For $\alpha_0^\vee$ the longest root of the root system $\Phi^\vee$, $\langle\!\langle \omega_i, \alpha_0^\vee \rangle\!\rangle = 1$.

(iii) The coefficient of $\alpha_i^\vee$ in $\alpha_0^\vee$ is 1.

A fundamental weight ω_i is called *cominuscule* if the corresponding coweight $\omega_i^\vee$ is minuscule.

(Recall that the coweights $\{\omega_1^\vee, \dots, \omega_n^\vee\}$ form a basis of $\mathcal{X}^\vee$ dual to the basis of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$ of $\mathcal{X}$.) The list of minuscule and cominuscule fundamental weights is well-known; we have included it in Table 1.5.1 together with the associated minimal homogeneous spaces $X = G/P_k$.

Thus, assuming that $X = G/P_k$ is cominuscule means that $\omega_k^\vee$ is minuscule, which in turn implies that the fundamental weight representation $V(\omega_k^\vee)$ is minuscule; that is, the Weyl group acts transitively on the weight spaces of $V(\omega_k^\vee)$. Here the simple reflection $s_i \in W$ acts on a vector $v_{\mu^\vee}$ of weight $\mu^\vee$ by mapping it to the vector $\bar{s}_i \cdot v_{\mu^\vee}$ of weight $s_i(\mu^\vee)$.

Recall that the fundamental weight representation $V(\omega_k^\vee)$ is the highest weight representation of $\mathfrak{g}^\vee$ with $\omega_k^\vee$ as highest weight. For any choice of a highest-weight vector $v^+_{\omega_k^\vee}$, we obtain the representation as $V(\omega_k^\vee) = \mathcal{U}_-^\vee \cdot \mathbb{C}v^+_{\omega_k^\vee}$, where $\mathcal{U}_-^\vee$ is the universal enveloping algebra of $\mathfrak{u}_-^\vee$ (see for example [Hum78], Theorem 20.2). Thus, $\{f_{i_1}^\vee \cdots f_{i_j}^\vee \cdot v^+_{\omega_k^\vee} \mid j \geq 0\}$ spans $V(\omega_k^\vee)$. We want to compare the actions of $\mathcal{U}_-^\vee$ and W, so we will need some results on the structure of minuscule representations.

Remark 1.5.2. To be able to apply the results directly to our case, we will change the notation of the following theorem to conform to our situation; for example, we write $\mathfrak{g}^\vee$ for a general Lie algebra, as we want to apply the theorem to the minuscule highest weight representation of the Lie algebra $\mathfrak{g}^\vee$ of the (adjoint) Langlands dual group $G^\vee$ associated to the (simply-connected) group G such that $X = G/P_k$.

We denote the Cartan integers of a root system $\Phi^\vee$ by $a_{ij} \in \mathbb{Z}$, $i, j \in \{1, \ldots, n\}$. They are given by $a_{ij} = 2\frac{(\alpha_j^\vee, \alpha_i^\vee)}{(\alpha_i^\vee, \alpha_i^\vee)}$ for any choice of non-degenerate, symmetric, bilinear form $(\cdot, \cdot)$ (e.g. the Killing form). We also use the notation $c(\mu^\vee, \alpha_i^\vee) = 2\frac{(\mu^\vee, \alpha_i^\vee)}{(\alpha_i^\vee, \alpha_i^\vee)}$ for $\mu^\vee$ a general weight.

Theorem 1.5.3 (Green [Gre08])**.** Let $\mathfrak{g}^\vee$ be a simple Lie algebra and fix a set of simple roots $\Delta^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ and Chevalley generators $(e_i^\vee, f_i^\vee)$. Suppose $V(\lambda^\vee)$ is a minuscule representation of $\mathfrak{g}^\vee$ with highest weight $\lambda^\vee$. Denote by $M(\lambda^\vee)$ the weights of $V(\lambda^\vee)$ and let $\mu^\vee \in M(\lambda^\vee)$ be an arbitrary weight. The following statements hold:

(i) $c(\mu^\vee, \alpha_i^\vee) \in \{-1, 0, 1\}$, and $\mu^\vee - c\alpha_i^\vee \in M(\lambda^\vee)$ if and only if $c = c(\mu^\vee, \alpha_i^\vee)$.

(ii) Each of the weight spaces is one-dimensional.

(iii) Given a highest weight vector $v^+_{\lambda^\vee}$, we can find a basis $\{v_{\mu^\vee} \mid \mu^\vee \in M(\lambda^\vee)\}$ with the following properties: $v_{\mu^\vee}$ has weight $\mu^\vee$ and the basis vector of the highest weight $v_{\lambda^\vee}$ coincides with $v^+_{\lambda^\vee}$; the Chevalley generators act on $v_{\mu^\vee}$ as

$$e_i^\vee \cdot v_{\mu^\vee} = \begin{cases} v_{\mu^\vee + \alpha_i^\vee}, & \text{if } c(\mu^\vee, \alpha_i^\vee) = -1, \\ 0, & \text{otherwise,} \end{cases}$$

$$f_i^\vee \cdot v_{\mu^\vee} = \begin{cases} v_{\mu^\vee - \alpha_i^\vee}, & \text{if } c(\mu^\vee, \alpha_i^\vee) = +1, \\ 0, & \text{otherwise,} \end{cases}$$

and $h_i^\vee \cdot v_{\mu^\vee} = c(\mu^\vee, \alpha_i)\, v_{\mu^\vee}$.

(iv) For any $v \in V(\lambda^\vee)$ and any $i \in \{1, \ldots, n\}$, we have $(e_i^\vee)^2 \cdot v = 0$ and $(f_i^\vee)^2 \cdot v = 0$. Moreover, if $j \in \{1, \ldots, n\}$ is such that the Cartan integer $a_{ij} = -1$ (i.e. when in the Dynkin diagram we have $i \;\bullet\!\!-\!\!\bullet\; j$ or $i \;\bullet\!\!\Rightarrow\!\!\bullet\; j$)[4] we have both $e_i^\vee e_j^\vee e_i^\vee \cdot v = 0$ and $f_i^\vee f_j^\vee f_i^\vee \cdot v = 0$. Finally, if $a_{ij} < 0$ we have $e_i^\vee f_j^\vee \cdot v = 0 = f_j^\vee e_i^\vee \cdot v$.

[4]Strictly speaking, we also have $a_{ij} = -1$ when we have $i \;\bullet\!\!\Rrightarrow\!\!\bullet\; j$, but these edges only appear in the Dynkin diagram of type G_2, and the corresponding Lie algebra does not have any minuscule representations.

Applying Theorem 1.5.3 to the definition of $\dot{s}_i$ and $\bar{s}_i$ given in equation (1.2.1) we obtain the following corollary directly:

Corollary 1.5.4. With the assumptions of Theorem 1.5.3, write $c = c(\mu^\vee, \alpha_i^\vee)$, then we have $s_i(\mu^\vee) = \mu^\vee - c\alpha_i^\vee$ and

$$\dot{s}_i \cdot v_{\mu^\vee} = \begin{cases} v_{\mu^\vee+\alpha_i^\vee} = e_i^\vee \cdot v_{\mu^\vee}, & \text{if } c = -1, \\ v_{\mu^\vee}, & \text{if } c = 0, \\ -v_{\mu^\vee-\alpha_i^\vee} = -f_i^\vee \cdot v_{\mu^\vee}, & \text{if } c = +1, \end{cases}$$

$$\bar{s}_i \cdot v_{\mu^\vee} = \begin{cases} -v_{\mu^\vee+\alpha_i^\vee} = -e_i^\vee \cdot v_{\mu^\vee}, & \text{if } c = -1, \\ v_{\mu^\vee}, & \text{if } c = 0, \\ v_{\mu^\vee-\alpha_i^\vee} = f_i^\vee \cdot v_{\mu^\vee}, & \text{if } c = +1. \end{cases}$$

Conversely, we have

$$e_i^\vee \cdot v_{\mu^\vee} = \begin{cases} v_{\mu^\vee+\alpha_i^\vee} = \dot{s}_i \cdot v_{\mu^\vee} = -\bar{s}_i \cdot v_{\mu^\vee}, & \text{if } c = -1, \\ 0, & \text{otherwise}, \end{cases}$$

$$f_i^\vee \cdot v_{\mu^\vee} = \begin{cases} v_{\mu^\vee-\alpha_i^\vee} = \bar{s}_i \cdot v_{\mu^\vee} = -\dot{s}_i \cdot v_{\mu^\vee}, & \text{if } c = +1, \\ 0, & \text{otherwise}. \end{cases}$$

The explicit description of the action of a Lie algebra on a minuscule representation in Theorem 1.5.3 and the comparison of the actions of Chevalley generators and Weyl group elements in Corollary 1.5.4 allow us to easily compare the actions of $\mathcal{U}_-^\vee$ and W. We will do so in Section 2.1. The coset representatives W^P will turn out to play an important role in this comparison.

1.6 Generalized Plücker coordinates and the coordinate ring of $(U_-^P)^T$

For cominuscule homogeneous spaces $X = G/P_k$, a consequence of the geometric Satake correspondence is that the Schubert classes that form a basis of the cohomology are in one-to-one correspondence with a set of projective coordinates called *(generalized) Plücker coordinates* on the *Langlands dual homogeneous space*

$$\mathbb{X}^\vee = P_k^\vee \backslash G^\vee = \widetilde{P}_k^\vee \backslash \widetilde{G}^\vee. \tag{1.6.1}$$

(Following a similar reasoning to Remark 1.1.6, we note that both expressions give the same homogeneous space. We will most frequently use the second expression.) See [MR20], p. 3, for more references regarding the geometric Satake correspondence. Thus, if we reformulate the potential in Definition 1.4.5 in terms of Plücker coordinates, we obtain a description of the localized quantum cohomology in terms of Schubert classes and relations between them. Plücker coordinate expressions for Rietsch's potential have been given for Grassmannians [MR20], quadrics [PRW16], and Lagrangian Grassmannians [PR13]. In [PRW16], this was obtained using a description of the coordinate ring of U_-^P by [GLS11].

In Chapter 3, we will present Plücker coordinate expressions for the potentials of the exceptional cominuscule homogeneous spaces, the Cayley plane and the Freudenthal variety. Our approach will be analogous to [PRW16] (with some deviations), so we will introduce Plücker coordinates and the description of the coordinate ring of the transpose of U_-^P in this section. We will discuss the Cayley plane and the Freudenthal variety in Section 1.7.

Generalized Plücker coordinates. As we discussed in Section 1.5, the assumption that $X = G/P_k$ is cominuscule implies that $V(\omega_k^\vee)$ is minuscule. Now, note that $V(\omega_k^\vee)$ has $w_0 \cdot \omega_k^\vee$ as lowest weight, which equals either $-\omega_k^\vee$ (in the cases of type B_n, C_n, D_{2n} and E_7) or $-\omega_{\sigma_0(k)}^\vee$ for $\sigma_0 : \{1, \ldots, n\} \to \{1, \ldots, n\}$ the symmetry of the

Dynkin diagram of G (in the cases of A_n, D_{2n+1} and E_6). In either case, we see from

$$\langle\!\langle -w_0 \cdot \omega_k^\vee, \alpha \rangle\!\rangle = \langle\!\langle -w_0 \cdot \omega_k^\vee, -w_0 \cdot \alpha' \rangle\!\rangle = \langle\!\langle \omega_k^\vee, \alpha' \rangle\!\rangle \in \{-1, 0, +1\}$$

(where $\alpha \in \Phi$ and $\alpha' = -w_0 \cdot \alpha$), that $-w_0 \cdot \omega_k^\vee$ is minuscule as well, see Definition 1.5.1. Thus, we can consider the minuscule fundamental representation $V(-w_0 \cdot \omega_k^\vee)$, which now has $-\omega_k^\vee$ as lowest weight. Taking the dual, we find that $V(-w_0 \cdot \omega_k^\vee)^*$ has highest weight $\omega_k^\vee$ and a *right* action of $\widetilde{G}^\vee$.

As $V(-w_0 \cdot \omega_k^\vee)$ is minuscule, Theorem 1.5.3 tells us that it has a natural basis $\{v_i\}$ after fixing a lowest weight vector, which we denote by v_0. Taking the dual basis $\{v_i^*\}$ for $V(-w_0 \cdot \omega_k^\vee)^*$, we note that v_0^* is the highest weight vector of weight $\omega_k^\vee$. Thus, we have $[v_0^*] \cdot \widetilde{P}^\vee = [v_0^*] \in \mathbb{P}\big(V(-w_0 \cdot \omega_k^\vee)^*\big)$ and we obtain a natural embedding $\mathbb{X}^\vee \hookrightarrow \mathbb{P}\big(V(-w_0 \cdot \omega_k^\vee)^*\big)$ given by $\widetilde{P}^\vee g \mapsto [v_0^* \cdot g]$, using the fact that $\widetilde{P}^\vee$ and $\{y_k^\vee(a) \mid a \in \mathbb{C}\}$ generate $\widetilde{G}^\vee$. Using this embedding, we can define the coordinates as follows:

Definition 1.6.1. For any basis element v_i as fixed above, and $g \in \widetilde{G}^\vee$, define the map $p_i : \widetilde{G}^\vee \to \mathbb{C}$ by

$$p_i(g) = (v_0^* \cdot g)(v_i) = v_0^*(g \cdot v_i). \tag{1.6.2}$$

These maps induce projective coordinates on $\mathbb{X}^\vee$ called *(generalized) Plücker coordinates.*

We will abuse notation and call the maps $p_i : \widetilde{G}^\vee \to \mathbb{C}$ Plücker coordinates as well.

The coordinate ring of $U_+^P = (U_-^P)^T \subset \widetilde{G}^\vee$. Recall from equation (1.4.4) that we defined $U_-^P = U_-^\vee \cap B_-^\vee(\dot{w}^P)^{-1}B_-^\vee$. In Proposition 8.5 of [GLS11], the coordinate rings of these *unipotent cells* have been described using a dual PBW basis compatible with a reduced expression. However, the result is formulated using cells of $U_+^\vee$ instead of $U_-^\vee$: we will have to use the *transposition anti-automorphism* $\cdot^T$ to translate their results to cells of $U_-^\vee$. Recall that this anti-automorphism is given by mapping $x_i^\vee(a)$ to $y_i^\vee(a)$ and vice versa, while acting trivially on $\widetilde{T}^\vee$. Note that $\bar{s}_i^T = \dot{s}_i = \bar{s}_i^{-1}$.

We would like to emphasize that the results in [GLS11] hold for general unipotent cells, but we will only consider the case of U_-^P for $P = P_k$ with ω_k cominuscule here.

Writing U_+^P for the transpose of U_-^P, we clearly have

$$U_+^P = (U_-^P)^T = U_+^\vee \cap B_+^\vee \dot{w}^P B_+^\vee. \tag{1.6.3}$$

The coordinates are defined on U_+^P using the embedding of $U_+^\vee$ in the completed universal enveloping algebra $\widehat{\mathcal{U}}_+^\vee$ of $\mathfrak{u}_+^\vee$. Namely, they form a subset of a basis for the *graded dual* of $\widehat{\mathcal{U}}_+^\vee$. As the universal enveloping algebra is in particular a infinite-dimensional vector space, we need to be more careful in defining the dual. We will make use of the Chevalley generators $(e_1^\vee, f_1^\vee, \ldots, e_n^\vee, f_n^\vee)$.

Consider the Cartan (or root space) decomposition

$$\mathfrak{g}^\vee = \mathfrak{t}^\vee \oplus \bigoplus_{\alpha^\vee \in \Phi^\vee} \mathfrak{u}_{\alpha^\vee}^\vee,$$

where for $\alpha^\vee \in \Pi^\vee$ a positive root we have that $\mathfrak{u}_{+\alpha^\vee}^\vee \subset \mathfrak{u}_+^\vee$ and $\mathfrak{u}_{-\alpha^\vee}^\vee \subset \mathfrak{u}_-^\vee$ are one-dimensional subspaces (see for example [Hum78], Section 8.5). Clearly, $\mathfrak{u}_{+\alpha_i^\vee}^\vee = \mathbb{C}e_i^\vee$ is spanned by the corresponding Chevalley generator, $\mathfrak{u}_{+s_i(\alpha_j^\vee)}^\vee = \mathbb{C}[e_i^\vee, e_j^\vee]$ is spanned by the commutator of the generators if $(s_is_j)^2 \neq 1$, and so on.

Now, consider the *positive root lattice* $\mathbb{N}_0\Delta^\vee \cong \mathbb{N}_0^n$, i.e. all non-negative integral linear combinations of the simple roots $\alpha_i^\vee$. For $\alpha^\vee = \sum_{i=1}^n d_i\alpha_i^\vee \in \mathbb{N}_0\Delta^\vee$, denote by $\widehat{\mathcal{U}}_+^\vee(\alpha^\vee)$ the subspace spanned by those pure tensors $e_{i_1}^\vee \cdots e_{i_j}^\vee$ such that exactly d_i of the factors have index i. For example, $\widehat{\mathcal{U}}_+^\vee(\alpha_i^\vee) = \mathbb{C}e_i^\vee = \mathfrak{u}_{+\alpha_i^\vee}^\vee$ for $\alpha_i^\vee \in \Delta^\vee$, and $\widehat{\mathcal{U}}_+^\vee(2\alpha_1^\vee + \alpha_2^\vee) = \mathbb{C}e_1^\vee e_1^\vee e_2^\vee \oplus \mathbb{C}e_1^\vee e_2^\vee e_1^\vee \oplus \mathbb{C}e_2^\vee e_1^\vee e_1^\vee$. (If $2\alpha_1^\vee + \alpha_2^\vee \in \Pi^\vee$, note that $\mathfrak{u}_{2\alpha_1^\vee+\alpha_2^\vee}^\vee = \mathbb{C}[e_1^\vee, [e_1^\vee, e_2^\vee]] \subset \widehat{\mathcal{U}}_+^\vee(2\alpha_1^\vee + \alpha_2^\vee)$.) This endows $\widehat{\mathcal{U}}_+^\vee$ with an $\mathbb{N}_0\Delta^\vee$-grading that naturally extends the root space decomposition of $\mathfrak{u}_+^\vee$ (i.e. $\mathfrak{u}_{+\alpha^\vee}^\vee \subset \widehat{\mathcal{U}}_+^\vee(\alpha^\vee)$ for $\alpha^\vee \in \Pi^\vee \subset \mathbb{N}_0\Delta^\vee$). The graded dual is defined as the direct sum of the duals of these finite-dimensional subspaces,

$$\left(\widehat{\mathcal{U}}_+^\vee\right)_{\mathrm{gr}}^* = \bigoplus_{\alpha^\vee \in \mathbb{N}_0\Delta^\vee} \left(\widehat{\mathcal{U}}_+^\vee(\alpha^\vee)\right)^*.$$

To define a basis on this dual, we start by recalling that we fixed a reduced expression for the minimal representative $w^P \in W^P$ of the coset $w_0 W_P$ and denoted its sequence of indices by $(r_i)_{i=1}^{\ell}$ where $\ell = \ell(w^P)$, see equation (1.2.3). As a Weyl group element, w^P maps a set $\Pi^\vee_{w^P} = \{\beta^\vee_{w^P}(j) \mid j = 1, \ldots, \ell\} \subset \Pi^\vee$ of positive roots to negative roots. It is well known that these are given by

$$\beta^\vee_{w^P}(1) = \alpha^\vee_{r_\ell} \quad \text{and} \quad \beta^\vee_{w^P}(j) = s_{r_\ell} \cdots s_{r_{\ell-j+2}}(\alpha^\vee_{r_{\ell-j+1}}) \quad \text{for } j \in \{2, \ldots, \ell\}, \tag{1.6.4}$$

(for example, this follows from [Hum78], Section 10.2). Note that $s_{r_\ell} \cdots s_{r_{\ell-j+2}}$ are the first $j-1$ factors of the reduced expression for $(w^P)^{-1}$ obtained by reversing the simple reflections in (1.2.3). Since $w_0 = w^P w_P$, it is easy to see that $\Pi^\vee = \Pi^\vee_{w^P} \sqcup \Pi^\vee_P$, where one recalls from Section 1.2 that $\Pi^\vee_P$ is the set of positive (co-) roots mapped to negative ones by w_P. We also remark that both $\Pi^\vee_{w^P}$ and $\Pi^\vee_P$ are *bracket closed* subsets: if $\alpha^\vee, \beta^\vee \in \Pi^\vee_{w^P}$ and $\alpha^\vee + \beta^\vee \in \Pi^\vee$ then $\alpha^\vee + \beta^\vee \in \Pi^\vee_{w^P}$, and the same holds for roots in $\Pi^\vee_P$. (This follows from the linearity of action of the Weyl group elements on roots.) See also [GLS11], Section 4.3, and the references therein.

For each of the positive roots $\beta^\vee_{w^P}(m)$, $m \in \{1, \ldots, \ell\}$, we choose a basis element $e^\vee_{w^P}(m)$ for $\mathfrak{u}^\vee_{+\beta^\vee_{w^P}(m)}$. In particular, we take $e^\vee_{w^P}(1) = e^\vee_k$, as $\beta^\vee_{w^P}(1) = \alpha^\vee_k$.

Remark 1.6.2. In [GLS11], the basis elements for the one-dimensional subspaces $\mathfrak{u}^\vee_{+\beta^\vee_{w^P}(m)}$ are not defined more explicitly, nor will we need an explicit choice of basis vector in the following. However, it is easy to define a set of basis elements $e^\vee_{w^P}(m) \in \mathfrak{u}^\vee_{+\beta^\vee_{w^P}(m)}$ for $m \in \{1, \ldots, \ell\}$ using the following algorithm: consider the defining expression of $\beta^\vee_{w^P}(m)$ in (1.6.4); remove all simple reflections acting trivially; replace each remaining simple reflection s_i by the corresponding $\mathrm{ad}(e^\vee_i)$; and finally replace $\alpha^\vee_i$ with the corresponding $e^\vee_i$.

Now, we complete the elements $\{e^\vee_{w^P}(m) \mid m = 1, \ldots, \ell\}$ to a basis of $\mathfrak{u}^\vee_+$ by choosing generating elements in $\mathfrak{u}^\vee_{+\alpha^\vee}$ for $\alpha^\vee \in \Pi^\vee \setminus \Pi^\vee_{w^P} = \Pi^\vee_P$. To simplify notation, we will denote these additional elements by $e^\vee_{w^P}(m)$ for $m \in \{\ell+1, \ldots, \ell+\ell_P\}$, where $\ell + \ell_P = \ell_0 = \ell(w_0)$ is the number of positive roots. Here we will also assume that the Chevalley generators $e_i^\vee$ are part of this basis. With a basis for $\mathfrak{u}_+^\vee$, we can in turn define a PBW basis for $\widehat{\mathcal{U}}_+^\vee$, the elements of which we denote by

$$e_{w^P}^\vee(\mathbf{m}) = \big(e_{w^P}^\vee(1)\big)^{m_1}\big(e_{w^P}^\vee(2)\big)^{m_2}\cdots\big(e_{w^P}^\vee(\ell_0)\big)^{m_{\ell_0}}, \tag{1.6.5}$$

for $\mathbf{m} = (m_1, \ldots, m_{\ell_0}) \in \mathbb{N}_0^{\ell_0}$. As each $e_{w^P}^\vee(m)$ is homogeneous with respect to the grading on $\widehat{\mathcal{U}}_+^\vee$ (this follows from Remark 1.6.2), the PBW basis elements are homogeneous as well.

We can now define the dual PBW basis for $\big(\widehat{\mathcal{U}}_+^\vee\big)^*_{\mathrm{gr}}$ by taking the finite-dimensional vector space dual bases and extending the functionals trivially on the complement. We denote these dual basis elements $p_{\mathbf{m}}^{\mathrm{GLS}}$, so that $p_{\mathbf{m}}^{\mathrm{GLS}}\big(e_{w^P}^\vee(\mathbf{n})\big)$ equals 1 if $\mathbf{m} = \mathbf{n} \in \mathbb{N}_0^{\ell_0}$ and 0 otherwise. Whenever $\mathbf{m} = (0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 at the mth position, we simply write $p_m^{\mathrm{GLS}} = p_{\mathbf{m}}^{\mathrm{GLS}}$; indeed, in this case we have $e_{w^P}^\vee(\mathbf{m}) = e_{w^P}^\vee(m)$. This basis is called the *dual PBW basis compatible with the reduced expression for* w^P. Using the natural embedding of $U_+^\vee$ into $\widehat{\mathcal{U}}_+^\vee$, we can consider the maps p_m^{GLS} on $U_+^\vee$ as well.

From the results in [GLS11], it follows that the coordinate ring of U_+^P is generated by the dual basis elements p_m^{GLS} for $m \in \{1, \ldots, \ell\}$ where $\ell = \ell(w^P)$. However, to fully express the coordinate ring, we also need to introduce a number of *generalized minors* in the sense of Fomin and Zelevinsky [FZ99]. Recall that the highest weight space of any highest weight representation is one-dimensional. Thus, taking a highest weight vector $v_{\omega_i^\vee}^+$, we can define the projection $V(\omega_i^\vee) \to \mathbb{C}v_{\omega_i^\vee}^+$ (parallel to a basis of weight vectors). We denote the coefficient c such that a vector $v \in V(\omega_i^\vee)$ projects to $cv_{\omega_i^\vee}^+$ by

$$\big\langle v, v_{\omega_i^\vee}^+ \big\rangle = c. \tag{1.6.6}$$

Note that the highest weight representation $V(\omega_i^\vee)$ is implicit in the notation. Using the action of $\widetilde{G}^\vee$ on these representations, we obtain maps $\widetilde{G}^\vee \to \mathbb{C}$ given by $g \mapsto \langle g \cdot v_{\omega_i^\vee}^+, v_{\omega_i^\vee}^+\rangle$. For $w_1, w_2 \in W$, the generalized minors are given in Definition 1.4 of [FZ99] as

$$\Delta_{w_1(\omega_i^\vee), w_2(\omega_i^\vee)}(g) = \Big\langle (\bar{w}_1)^{-1} g \bar{w}_2 \cdot v_{\omega_i^\vee}^+, v_{\omega_i^\vee}^+ \Big\rangle.$$

(The definition in [FZ99] uses decompositions of g instead, but can easily be seen to agree with this expression. See for example Proposition 7.2 of [GLS11].) In particular, we will be concerned with the following minors:

$$\Delta_{\omega_i^\vee,(w^P)^{-1}(\omega_i^\vee)}(g) = \left\langle g(\dot{w}^P)^{-1} \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee} \right\rangle. \tag{1.6.7}$$

Note that we wrote $(\dot{w}^P)^{-1} = \overline{(w^P)^{-1}}$ to simplify the expression; it has the reverse reduced expression compared to w^P. We can now state the result:

Proposition 1.6.3 (Proposition 8.5 of [GLS11]). The cell $U_+^P = (U_-^P)^T \subset \widetilde{G}^\vee$, see equation (1.6.3), is an affine variety with coordinate ring isomorphic to

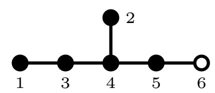
$$\mathbb{C}[p_1^{\mathrm{GLS}}, \ldots, p_\ell^{\mathrm{GLS}}][\Delta^{-1}_{\omega_1^\vee,(w^P)^{-1}(\omega_1^\vee)}, \ldots, \Delta^{-1}_{\omega_n^\vee,(w^P)^{-1}(\omega_n^\vee)}]. \tag{1.6.8}$$

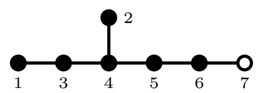
In Proposition 3.2.5, we will give a (type-independent) criterion for a generator p_m^{GLS} to correspond to a Plücker coordinate under the translation isomorphism $\cdot^T$: $U_-^P \to U_+^P$ (and up to a constant). Unfortunately, we have not yet found a type-independent approach to express the localized generalized minors in terms of Plücker coordinates. However, using the open, dense torus $U_-^\circ \subset U_-^P$ (or more precisely: its transpose) we can directly verify Plücker coordinate expressions for the minors in the case of the Cayley plane E_6^{sc}/P_6 (see Section 3.3) and in the case of the Freudenthal variety E_7^{sc}/P_7 (see Section 3.4). The remainder of Chapter 3 will be devoted to show an isomorphism between Rietsch's Lie-theoretic Landau-Ginzburg model (as discussed in Section 1.4) and the so-called *canonical Landau-Ginzburg model*, which we will define in Section 3.1 for these *exceptional cominuscule homogeneous spaces* E_n^{sc}/P_n. Therefore, the final section of this introductory chapter will be devoted to introducing these.

1.7 The exceptional cominuscule family

As we can see from Table 1.5.1, of the exceptional types, only E_6 and E_7 allow cominuscule weights. In type E_6, there are two cominuscule weights, ω_1 and ω_6. However, the homogeneous spaces E_6^{sc}/P_1 and E_6^{sc}/P_6 are isomorphic. There is only one cominuscule weight in type E_7, namely ω_7. Thus, in practice, we only need to consider two exceptional cominuscule homogeneous spaces: the Cayley plane $\mathbb{OP}^2 = E_6^{\mathrm{sc}}/P_6$ and the Freudenthal variety E_7^{sc}/P_7. Whenever we consider both varieties at the same time, we will call them "cominuscule E_n^{sc}/P_n" (we are adding the adjective "cominuscule" to exclude the *quasi-cominuscule* E_8^{sc}/P_8).

To avoid any confusion: we follow the labeling on the Dynkin diagrams used by [Bou68]:

E_6 : 

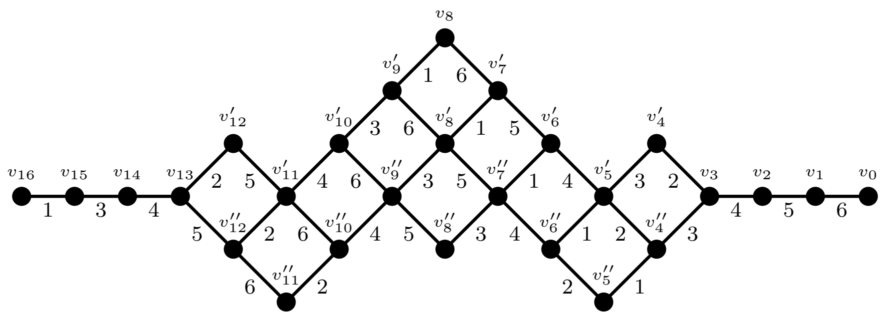
E_7 : 

The Cayley plane and the Freudenthal variety are similar enough to allow for the same approach, but they do have some notable differences. For example, for type E_7 the longest Weyl group element w_0 maps the fundamental weights ω_i to $-\omega_i$, whereas the longest Weyl group element w_0 in type E_6 maps them to $-\omega_{\sigma_0(i)}$, where σ_0 is the obvious Dynkin diagram symmetry. Note that this implies that $-w_0 \cdot \omega_6^\vee = \omega_1^\vee$ for E_6 and $-w_0 \cdot \omega_7^\vee = \omega_7^\vee$ for E_7.

In fact, we will need very few geometric properties of the exceptional cominuscule homogeneous spaces other than the structure of their cohomologies. The cohomology ring, and indeed the (small) quantum cohomology ring, of both varieties are already known; see [CMP08] for presentations of both as well as for further references. Rietsch's Lie-theoretic mirror model (discussed in Section 1.4) holds for these exceptional cominuscule homogeneous spaces as well. However, as we remarked at the start of Section 1.6, our goal is a mirror model in terms of Plücker coordinates. In Chapter 3, we will construct such mirror models for the exceptional cominuscule homogeneous spaces.

The Cayley plane. The exceptional cominuscule variety E_6^{sc}/P_6 is often denoted by $\mathbb{OP}^2$ as it can be interpreted as the variety of (complex) octonion lines in three-dimensional (complex) octonion space. However, we will not use this description here and simply consider the Cayley plane as a homogeneous space of type E_6. The geometry and the Chow ring (and thus the cohomology) of $\mathbb{OP}^2$ is discussed in [IM05][5].

For the Cayley plane $X = E_6^{\mathrm{sc}}/P_6$, the embedding of the Langlands dual homogeneous space is $\mathbb{X}^\vee \hookrightarrow \mathbb{P}\big(V(\omega_1^\vee)^*\big)$. Since the fundamental weight representation $V(\omega_1^\vee)$ is minuscule, the Weyl group acts transitively on the weights. Thus, we can define a graph that the weights of $V(\omega_1^\vee)$ as vertices and has an edge labeled i between two weights if s_i maps one weight to the other (and vice versa). The graph we obtain is the following:



(1.7.1)

Here, we have set the highest weight space at the left and denoted a choice of highest weight vector by v_{16}. Applying equation (1.6.2) to each of the basis vectors $v_i^{(j)}$ gives the full set of Plücker coordinates.

Using Corollary 1.5.4, we see that the action of the representatives $\dot{s}_i, \bar{s}_i \in \widetilde{G}^\vee$ can be directly related to the action of $e_i^\vee, f_i^\vee \in \mathfrak{g}^\vee$. Here, applied to $V(\omega_1^\vee)$, this translates as follows: an edge marked i between two vectors v (on the left) and w (on the right) means that $\bar{s}_i \cdot v = f_i^\vee \cdot v = w$ and $\dot{s}_i \cdot w = e_i^\vee \cdot w = v$.

The diagram for $V(\omega_6^\vee)$ is in fact the same, however, now the highest weight space is on the right, and the actions of $\bar{s}_i$ and $\dot{s}_i$ go in the opposite direction. (In other words, exchange the position of v and w in the above.)

[5]Note that a different labeling of the Dynkin diagram is used in [IM05].

More importantly, this graph is identical to the *Hasse diagram* for the Cayley plane: the Hasse diagram has as vertices the Schubert classes, and edges mark inclusions. The identification between the graph above and the Hasse diagram is obtained by replacing the weight vector $v_i^{(j)}$ with the Schubert class $\sigma_i^{(j)}$, where we use the labeling of Schubert classes given in [CMP08], Section 2.3.[6] This correspondence is implied by the geometric Satake correspondence, which we mentioned at the start of this section. Indeed, we have chosen the labeling for the basis of the representation in such a way that these match up with the labeling for the Schubert classes of [CMP08]. Note that the fundamental class $[X] = \sigma_0 = 1 \in H^0(X)$ of the Cayley plane is on the right, and σ_1 is the class of a hyperplane. Considering the diagram in this way, an edge i from σ_{w_1} (on the left) to σ_{w_2} (on the right) denotes that $w_2 = s_i w_1$. (Here σ_w is the class of the Schubert variety $\overline{B_+ wP/P}$ as in Section 1.2.)

Now, the interpretation of (1.7.1) as the Hasse diagram indicates a final interpretation: we can interpret the diagram as the partial ordering of the elements of W^P under the Bruhat order given in (1.2.4). Thus, any path from the left to the right in the diagram in (1.7.1) gives a reduced expression for the minimal coset representative w^P of $w_0 W_P$; we will fix the following:

$$w^P = s_1 s_3 s_4 s_2 s_5 s_4 s_3 s_1 s_6 s_5 s_4 s_3 s_2 s_4 s_5 s_6. \tag{1.7.2}$$

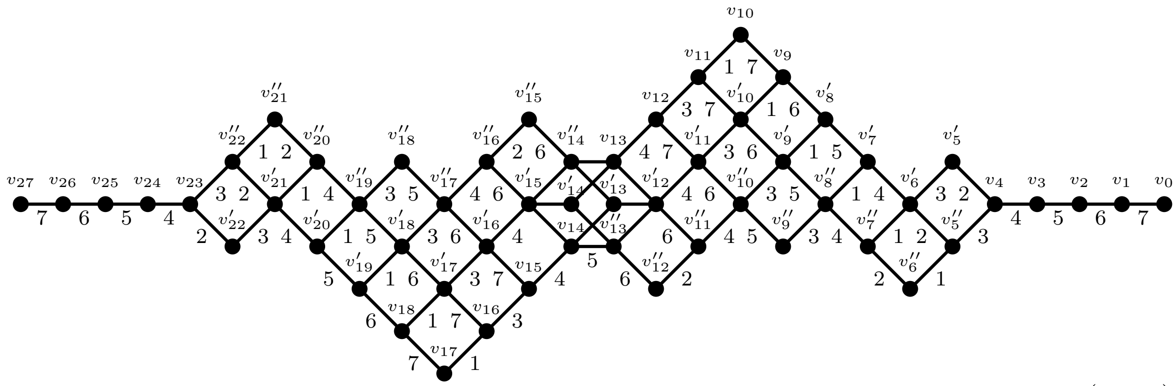
Whenever we consider the Cayley plane, the sequence $(r_i)_{i=1}^{16}$ will denote the sequence of indices of this reduced expression by, i.e. $r_1 = 1$, $r_2 = 3$ and so on.

The Freudenthal variety. The Freudenthal variety is also related to (complex) octonions, namely it is the closed orbit of E_7 on the *Zorn algebra*, see [CMP08] Section 2.3. The most extensive analysis of the Freudenthal variety can be found in [Fre64].

For the Freudenthal variety, the Langlands dual homogeneous space is embedded $\mathbb{X}^\vee \hookrightarrow \mathbb{P}\big(V(\omega_7^\vee)^*\big)$. The minuscule fundamental weight representation $V(\omega_7^\vee)$ has the

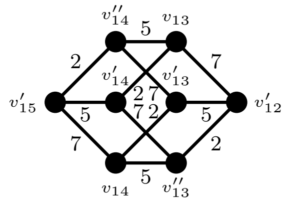
[6] Note that [CMP08] consider $\mathbb{OP}^2$ as E_6^{sc}/P_1, so the diagram would have the reverse labeling of edges.

following weight structure:



(1.7.3)

Here we used the same conventions as for the diagram in (1.7.1). Note that the center consists of the following commutative cube:



with all southwest-northeast edges marked 2, all west-east edges marked 5 and all northwest-southeast edges marked 7. We have again used the same labeling for the basis vectors as used for the Schubert classes of the cohomology of E_7^{sc}/P_7 in [CMP08] in accordance with the geometric Satake correspondence.

We have the same interpretations of diagram (1.7.3) as the actions of $e_i^\vee$ and $f_i^\vee$ on $V(\omega_1^\vee)$, as the Hasse diagram for the cohomology, and as the partial order on the elements of W^P under the Bruhat order.

Although any path from left to right in (1.7.3) gives a reduced expression for w^P, we fix the following:

$$w^P = s_7 s_6 s_5 s_4 s_3 s_2 s_4 s_5 s_6 s_1 s_3 s_4 s_2 s_5 s_7 s_4 s_3 s_1 s_6 s_5 s_4 s_2 s_3 s_4 s_5 s_6 s_7. \tag{1.7.4}$$

Note that we have $(w^P)^{-1} = w^P$, which did not hold in the case of the Cayley Plane $\mathbb{OP}^2 = E_6^{\mathrm{sc}}/P_6$.

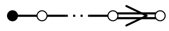
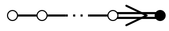
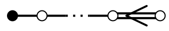
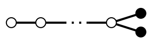
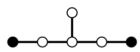
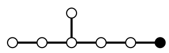
type and weight			(co-) minuscule	variety	dim	index
A_{n-1}		any k	both	$\mathrm{Gr}(k,n)$	$k(n-k)$	n
B_n		1	cominuscule	Q_{2n-1}	$2n-1$	$2n-1$
B_n		n	minuscule	$\mathrm{OG}(n,2n+1)$	$\frac{1}{2}n(n+1)$	$2n$
C_n		1	minuscule	$\mathbb{CP}^{2n-1}$	$2n-1$	$2n$
C_n		n	cominuscule	$\mathrm{LG}(n,2n)$	$\frac{1}{2}n(n+1)$	$n+1$
D_n		1	both	Q_{2n-2}	$2n-2$	$2n-2$
D_n		$n-1$ or n	both	$\mathrm{OG}(n,2n)$	$\frac{1}{2}n(n-1)$	$2n-2$
E_6		1 or 6	both	$\mathbb{OP}^2 = E_6^{\mathrm{sc}}/P_6$	16	12
E_7		7	both	E_7^{sc}/P_7	27	18

Table 1.5.1: Table listing for each type the fundamental weights that are minuscule, cominuscule or both, the associated homogeneous spaces and their dimensions and indices. In this table, Q_n denotes a quadric of dimension n; $\mathrm{Gr}(k,n)$ denotes the Grassmannian of k-dimensional subspaces in $\mathbb{C}^n$; $\mathrm{LG}(n,2n)$ denotes the Lagrangian Grassmannian of maximal isotropic subspaces with respect to the standard symplectic form; $\mathrm{OG}(n,2n)$ and $\mathrm{OG}(n,2n+1)$ denote (one of the two isomorphic connected components of) the orthogonal Grassmannians of maximal isotropic subspaces with respect to the standard quadratic form; $\mathbb{OP}^2 = E_6^{\mathrm{sc}}/P_6$ denotes the Cayley plane which is a homogeneous space for E_6^{sc}, the simply-connected Lie group of type E_6; and finally E_7^{sc}/P_7 is called the Freudenthal variety and is homogeneous for E_7^{sc}, the simply-connected Lie group of type E_7. Note that the two varieties that are only minuscule are redundant: the type-B_n minuscule variety $\mathrm{OG}(n,2n+1)$ is isomorphic to the variety $\mathrm{OG}(n+1,2n+2)$ which is both minuscule and cominuscule as a type-D_{n+1} homogeneous space; similarly, the type-C_n minuscule variety $\mathbb{CP}^{2n-1}$ is of course the same as $\mathrm{Gr}(1,2n)$, which is both minuscule and cominuscule as a type-A_{2n-1} homogeneous space. Adapted from [CMP08].

Chapter 2

Laurent polynomial LG-models for cominuscule homogeneous spaces

In this chapter, we present the results published in [Spa21], namely a Laurent polynomial expression in the case of cominuscule homogeneous spaces for Rietsch's Lie-theoretic Landau-Ginzburg models for the small quantum cohomology.

In Section 1.4, we introduced Rietsch's Lie-theoretic mirror models for homogeneous spaces [Rie08]. These mirrors model the small quantum cohomology in terms of a potential defined Lie-theoretically on a specific Langlands dual Richardson variety, see equation (1.4.1). These models are constructed completely type-independently and hold on all homogeneous spaces.

However, as we discussed in Section 1.6, especially in the case of cominuscule homogeneous spaces there are advantages to obtaining a Landau-Ginzburg model expressed in terms of (generalized) Plücker coordinates. In particular, there is a direct correspondence between the Schubert basis of the cohomology of the homogeneous space and the Plücker coordinates on its mirror, see for example Section 1.7. This is a consequence of the geometric Satake correspondence, see [MR20] p. 3 for references.

Consequently, in the articles [MR20, PR13, PR18, PRW16], Rietsch's Lie-theoretic mirror model is reformulated in terms of these Plücker coordinates in a case-by-case approach. The cases in question were Grassmannians, Lagrangian Grassmannians, and (odd and even) quadrics, respectively. Following [PRW16], these reformulated models have become known as *canonical mirror models.*

In obtaining these canonical models, each of these articles relied on a local Laurent polynomial expression for Rietsch's potential. For Grassmannians, such an expression was already available, but the articles [PR18, PRW16, PR13] had to develop new expressions. We have compared the approaches in these articles and have been able to generalize the expressions using knowledge of the general structure of the involved minuscule representations obtained from [Gre08], which we discussed in Section 1.5.

The Laurent polynomial expression is defined on an algebraic torus $\mathcal{Z}_P^\circ \subset \mathcal{Z}_P^\vee$ given in Definition 2.2.3. It is known that this is an open, dense torus inside $\mathcal{Z}_P^\vee$. The Laurent polynomial expression we find is as follows:

$$\mathcal{W}_{\mathcal{Z}_P^\circ}(z) = \sum_{i=1}^{\ell} a_i + q\frac{\sum_{(i_j)\in\mathcal{I}} a_{i_1}\cdots a_{i_{\ell'}}}{\prod_{i=1}^{\ell} a_i},$$

where $\ell = \dim(X) = \dim(X^\vee_{\mathrm{Lie}})$, the a_i are the toric coordinates for $i \in \{1,\ldots,\ell\}$, and q is the quantum parameter. The set $\mathcal{I}$ is the set of subexpressions of a certain Weyl group element w' in a fixed reduced expression of the minimal coset representative w^P of the longest Weyl group element, see equation (1.2.3) for w^P, equation (1.5.1) for w' and Definition 2.2.5 for $\mathcal{I}$. For the full statement, see Theorem 2.2.7.

Notice that this expression is reminiscent of the Laurent polynomial potential for projective complete intersections given in [Giv96]. Indeed, it is given as the sum of the toric coordinates a_i plus a quantum term consisting of a homogeneous polynomial divided by the product of all the toric coordinates. We give a second type-independent description for this homogeneous polynomial: we replace the summation over $\mathcal{I}$ by a summation over the set $\mathcal{S}$ of special subsets of the quiver $Q_{\mathbb{X}^\vee}$ associated[1] to w^P by [Per07, CMP08], see Definitions 2.5.3 and 2.5.5 as well as Corollary 2.5.12. These subsets of $Q_{\mathbb{X}^\vee}$ can be considered as generalizations of Young tableaux used in a similar way.

[1]Note that we are associating the quiver to $\mathbb{X}^\vee$ in this thesis instead of to X as in [Spa21]. We will explain this change of notation in Remark 2.5.4.

We use the second type-independent expression to obtain Laurent polynomial potentials for all cominuscule homogeneous spaces in Section 2.6: Grassmannians $\mathrm{Gr}(k,n)$, quadrics Q_n, Lagrangian Grassmannians $\mathrm{LG}(n,2n)$, orthogonal Grassmannians $\mathrm{OG}(n,2n)$, the Cayley plane E_6^{sc}/P_6, and the Freudenthal variety E_7^{sc}/P_7. The obtained expressions for quadrics and Lagrangian Grassmannians coincide with those given earlier in [PRW16] and [PR13]. This is to be expected from the fact that the type-independent expression is a generalization of these cases. However, to the best of our knowledge, the expressions for orthogonal Grassmannians (for general n), the Cayley plane and the Freudenthal variety are new.

We have already mentioned the implications of this expression in the introduction to Chapter 1, so we will not repeat them here.

The outline of this chapter is as follows. We begin in Section 2.1 with using the results from Section 1.5 to get a more elaborate description of the action of W on the minuscule representation $V(\omega_k^\vee)$. Next, in Section 2.2 we state our Laurent polynomial expression (Theorem 2.2.7) for the potential restricted to an open dense subset. We prove this expression in Section 2.3, postponing the proof of a number of intermediate results to Section 2.4. We deduce an alternative description of the quantum term (Corollary 2.5.12) using subsets of a specific quiver in Section 2.5, which simplifies the calculation of the Laurent polynomials. Finally, we apply the expression in Corollary 2.5.12 to all the cominuscule homogeneous spaces in Section 2.6, verifying that the expression coincide with [PRW16] and [PR13] for quadrics and Lagrangian Grassmannians, and obtaining new Laurent polynomial potentials for orthogonal Grassmannians (Subsection 2.6.4), the Cayley plane (Subsection 2.6.5) and the Freudenthal variety (Subsection 2.6.6).

2.1 A comparison of the actions of W and $\mathcal{U}_-^\vee$ on minuscule representations

Let G be a simply-connected (complex) linear algebraic group such that $X = G/P_k$ is cominuscule, so that the Langlands dual group $G^\vee$ is adjoint. We consider the highest weight representation $V(\omega_k^\vee)$ of the Lie algebra $\mathfrak{g}^\vee$ of $G^\vee$. As we mentioned in Section 1.5, we have two actions on $V(\omega_k^\vee)$: the actions of W and $\mathcal{U}_-^\vee$. We will need to compare these two actions to find the Laurent polynomial expression for Rietsch's Lie-theoretic potential.

Now, since the representation $V(\omega_k^\vee)$ is minuscule, Theorem 1.5.3 and Corollary 1.5.4 apply to $V(\omega_k^\vee)$. We can use these results to obtain the following facts regarding the action of the Weyl group W on the highest weight vector $v^+_{\omega_k^\vee}$:

Lemma 2.1.1. Consider the highest weight vector $v^+_{\omega_k^\vee}$ of the minuscule fundamental weight representation $V(\omega_k^\vee)$, where k is such that $X = G/P_k$.

(i) Given an arbitrary $w \in W$ with minimal coset representative $w_c \in W^P$, then we have $\bar{w} \cdot v^+_{\omega_k^\vee} = \bar{w}_c \cdot v^+_{\omega_k^\vee}$ and $\dot{w} \cdot v^+_{\omega_k^\vee} = \dot{w}_c \cdot v^+_{\omega_k^\vee}$.

(ii) An element $w_c \in W^P$ with reduced expression $w_c = s_{i_1} \cdots s_{i_c}$ acts on the vector $v^+_{\omega_k^\vee}$ by $\bar{w}_c \cdot v^+_{\omega_k^\vee} = f^\vee_{i_1} \cdots f^\vee_{i_c} \cdot v^+_{\omega_k^\vee}$ and by $\dot{w}_c \cdot v^+_{\omega_k^\vee} = (-1)^c f^\vee_{i_1} \cdots f^\vee_{i_c} \cdot v^+_{\omega_k^\vee}$.

(iii) Conversely, if $f^\vee_{i_1} \cdots f^\vee_{i_j} \cdot v^+_{\omega_k^\vee}$ is non-zero of weight $\mu^\vee$, then $s_{i_1} \cdots s_{i_j}$ is a reduced expression for the (unique) element $w_c \in W^P$ such that $w_c \cdot \omega_k^\vee = \mu^\vee$.

(iv) In particular, $\bar{w}_P, \dot{w}_P$ and their inverses act trivially on $v^+_{\omega_k^\vee}$ and we have that the lowest weight vector defined by $v^-_{\omega_k^\vee} = \bar{w}_0 \cdot v^+_{\omega_k^\vee}$ satisfies $v^-_{\omega_k^\vee} = \bar{w}^P \cdot v^+_{\omega_k^\vee} = f^\vee_{r_1} \cdots f^\vee_{r_\ell} \cdot v^+_{\omega_k^\vee}$, where $w^P = s_{r_1} \cdots s_{r_\ell}$ is the reduced expression fixed in equation (1.2.3). Moreover, if $f^\vee_{i_1} \cdots f^\vee_{i_j} \cdot v^+_{\omega_k^\vee} = v^-_{\omega_k^\vee}$, then $s_{i_1} \cdots s_{i_j} = w^P$ and this is a reduced expression.

Proof. **(i)** Recall the identity $s_j(\omega_k^\vee) = \omega_k^\vee$ for $j \neq k$, i.e. when $s_j \in W_P$. This implies that an arbitrary $w \in W$ acts on $\omega_k^\vee$ by its minimal coset representative w_c in W^P. Corollary 1.5.4 implies that $\bar{w} \cdot v^+_{\omega_k^\vee} = \bar{w}_c \cdot v^+_{\omega_k^\vee}$ and $\dot{w} \cdot v^+_{\omega_k^\vee} = \dot{w}_c \cdot v^+_{\omega_k^\vee}$.

(ii) We need to show that each factor of $\bar{w}_c = \bar{s}_{i_1} \cdots \bar{s}_{i_c}$ acts as $f_i^\vee$ on $v^+_{\omega_k^\vee}$. Considering Corollary 1.5.4, each $\bar{s}_i$ acts either by $f_i^\vee$, $-e_i^\vee$ or as the identity map. Clearly, none of these factors acts as the identity, as we could remove it from the product, which would contradict the minimality of the coset representative $\dot{w}_c$. Moreover, none of the factors acts as $-e_i^\vee$ either, because of the following argument:

Let $\bar{s}_i$ be the rightmost factor acting as $-e_i^\vee$. As $v^+_{\omega_k^\vee}$ is the highest weight vector, we have $e_i^\vee \cdot v^+_{\omega_k^\vee} = 0$ (as $e_i^\vee$ raises the height of the weight), so there must be a number of $\bar{s}_j$ in between $\bar{s}_i$ and $v^+_{\omega_k^\vee}$ acting as $f_j^\vee$. Let $f_j^\vee$ be the factor next to $e_i^\vee$. There are three cases: $j \neq i$ and $a_{ij} < 0$; $j \neq i$ and $a_{ij} = 0$; and $j = i$. When $j \neq i$ and $a_{ij} < 0$, Theorem 1.5.3 **(iv)** tells us that $e_i^\vee f_j^\vee \cdot v = 0$ in the representation, which is impossible. When $j \neq i$ and $a_{ij} = 0$, we know that $e_i^\vee$ and $f_j^\vee$ commute in the Lie algebra, so we can assume without loss of generality that only the case $j = i$ occurs. In the case $j = i$ we obtain $e_i^\vee f_j^\vee \cdot v = v$ according to Theorem 1.5.3 **(iii)**. However, this is in contradiction with the fact that w_c is a minimal coset representative. Thus, all of the factors $\bar{s}_i$ of $\bar{w}^P$ act as $f_i^\vee$.

For the equality $\dot{w}_c = (-1)^c f_{i_1}^\vee \cdots f_{i_c}^\vee \cdot v^+_{\omega_k^\vee}$, we use an analogous argument combined with the fact that $\dot{s}_i$ acts as either $e_i^\vee$, $-f_i^\vee$ or the identity due to Corollary 1.5.4.

(iii) Let $w = s_{i_1} \cdots s_{i_j}$ and let $w_c \in W^P$ denote the minimal length representative of wW_P. Given a reduced expression $w_c = s_{i'_1} \cdots s_{i'_c}$, parts **(i)** and **(ii)** imply that

$$\bar{w} \cdot v^+_{\omega_k^\vee} = \bar{w}_c \cdot v^+_{\omega_k^\vee} = f_{i'_1}^\vee \cdots f_{i'_c}^\vee \cdot v^+_{\omega_k^\vee}.$$

Thus, $\bar{w} \cdot v^+_{\omega_k^\vee}$ has weight $\omega_k^\vee - \alpha_{i'_1}^\vee - \cdots - \alpha_{i'_c}^\vee$ by Theorem 1.5.3 **(iii)**.

On the other hand, $f_{i_1}^\vee \cdots f_{i_j}^\vee \cdot v^+_{\omega_k^\vee} = \bar{s}_{i_1} \cdots \bar{s}_{i_j} \cdot v^+_{\omega_k^\vee} = \bar{w} \cdot v^+_{\omega_k^\vee}$ by Corollary 1.5.4, since none of the factors act as the zero map. This has two implications. Firstly, $w = s_{i_1} \cdots s_{i_j}$ is a reduced expression: else, one of the factors of $f_{i_1}^\vee \cdots f_{i_j}^\vee$ must act as the identity map, which contradicts Theorem 1.5.3 **(iii)**. Secondly, the weight of $\bar{w} \cdot v^+_{\omega_k^\vee}$ can also be written as $\omega_k^\vee - \alpha_{i_1}^\vee - \ldots - \alpha_{i_j}^\vee$.

We conclude that

$$\omega_k^\vee - \alpha_{i'_1}^\vee - \cdots - \alpha_{i'_c}^\vee = w(\omega_k^\vee) = \omega_k^\vee - \alpha_{i_1}^\vee - \ldots - \alpha_{i_j}^\vee.$$

Clearly, this can only hold when $c = \ell(w_c) = \ell(w) = j$.

Thus, we have $w = w_c \in W^P$ and we have already shown that $s_{i_1} \cdots s_{i_j}$ is a reduced expression for w. Moreover, $w(\omega_k^\vee) = \mu^\vee$ by definition and this determines w uniquely, proving **(iii)**.

(iv) As $w_P \in W_P$, **(i)** implies that $\bar{w}_P$, $\dot{w}_P$ and their inverses act trivially on $v^+_{\omega_k^\vee}$. As w^P is defined as the minimal coset representative of w_0, we conclude that $v^-_{\omega_k^\vee} = \bar{w}_0 \cdot v^+_{\omega_k^\vee} = \bar{w}^P \cdot v^+_{\omega_k^\vee}$ by **(i)**, and since $w^P = s_{r_1} \cdots s_{r_\ell}$ is the reduced expression we fixed in Section 1.2, **(ii)** implies that $v^-_{\omega_k^\vee} = \bar{s}_{r_1} \cdots \bar{s}_{r_\ell} \cdot v^+_{\omega_k^\vee} = f_{r_1}^\vee \cdots f_{r_\ell}^\vee \cdot v^+_{\omega_k^\vee}$. Part **(iii)** implies the last statement directly. □

2.2 Statement of the Laurent polynomial potential

In this section we state the main result of this chapter, Theorem 2.2.7, which is an explicit Laurent polynomial expression for $\mathcal{W}_{\mathcal{Z}_P^\vee}$ on an open, dense algebraic torus $\mathcal{Z}_P^\circ$ inside $\mathcal{Z}_P^\vee$, whenever $X = G/P$ is a *cominuscule* homogeneous space. Recall that $X = G/P$ is cominuscule when $P = P_k$ is maximal and $\omega_k^\vee$ is minuscule.

First, we define another subset of $U_-^\vee$:

Definition 2.2.1. Recall the reduced expression $w^P = s_{r_1} \cdots s_{r_\ell}$ fixed in equation (1.2.3). Let $U_-^\circ \subset U_-^\vee$ be the algebraic torus of elements u_- that can be written as

$$u_- = y_{r_\ell}^\vee(a_\ell) \cdots y_{r_1}^\vee(a_1) \tag{2.2.1}$$

with $a_i \in \mathbb{C}^*$.

Lemma 2.2.2. We have $U_-^\circ \subset U_-^P$ open and dense, where $U_-^P = U_-^\vee \cap B_+^\vee \bar{w}_P \bar{w}_0 B_+^\vee$ was defined in equation (1.4.4).

Proof. (See also Section 5.2 of [PR18].) Note that

$$B_+^\vee \dot{s}_i B_+^\vee = B_+^\vee x_i^\vee(1) y_i^\vee(-1) x_i^\vee(1) B_+^\vee = B_+^\vee y_i^\vee(-1) B_+^\vee,$$

so that $y_i^\vee(a) \in B_+^\vee \dot{s}_i B_+^\vee$, as the -1 can be scaled to any $a \in \mathbb{C}^*$ since $T^\vee \subset B_+^\vee$. Of course, $s_{r_\ell} \cdots s_{r_1} = (w^P)^{-1}$ is reduced, so the Bruhat lemma (see for example [Hum75], Lemma 29.3.A) implies that

$$U_-^\circ \subset B_+^\vee \dot{s}_{r_\ell} \cdots \dot{s}_{r_1} B_+^\vee = B_+^\vee (\bar{w}^P)^{-1} B_+^\vee$$

and it remains to show that $B_+^\vee (\bar{w}^P)^{-1} B_+^\vee = B_+^\vee \bar{w}_P \bar{w}_0 B_+^\vee$. This follows from the fact that $w_0 = w^P w_P$ and the fact that $\dot{s}_i$ and $\bar{s}_i$ only differ by a torus element.

It is clear that $U_-^\circ \subset U_-^P$ is an open subset and it is dense as both have dimension $\ell(w_P w_0)$ and U_-^P is irreducible [Lus94]. □

Definition 2.2.3. We define the open, dense algebraic torus $\mathcal{Z}_P^\circ \subset \mathcal{Z}_P^\vee$ as

$$\mathcal{Z}_P^\circ = B_-^\vee \bar{w}_0^{-1} \cap U_+^\vee T^P \bar{w}_P U_-^\circ \ \subset \ \mathcal{Z}_P^\vee.$$

Note that the fact that this variety is non-zero and an algebraic torus follows from Lemma 1.4.4 and the resulting isomorphism $\mathcal{Z}_P^\vee \to U_-^P \times T^P : z \mapsto (u_-, t)$.

The following is immediate from the definition and Lemma 1.4.4:

Corollary 2.2.4. Every $z \in \mathcal{Z}_P^\circ$ can be factorized in two ways: as $b_- \bar{w}_0^{-1}$ for $b_- \in B_-^\vee$; and as $z = u_+ t \bar{w}_P u_-$ with $u_- \in U_-^\circ$ of the form (2.2.1), $u_+ \in U_+^\vee$ and $t \in T^P$. Moreover, the latter decomposition is *unique* with u_+ determined by a choice of (u_-, t).

We will define the Laurent polynomial expression for the potential on this algebraic torus $\mathcal{Z}_P^\circ$. It turns out that this expression is indexed by the subexpressions of w' in w^P. Recall from equation (1.5.1) that $w' \in W$ is defined by $w^P = w'w''$, where $w^P, w'' \in W^P$ are the minimal coset representatives of $w_0 W_P$ and $w_P s_k W_P$ (with k such that $P = P_k$). Moreover, their lengths are denoted by $\ell(w^P) = \ell$, $\ell(w') = \ell'$ and $\ell(w'') = \ell''$ and satisfy $\ell = \ell' + \ell''$.

Definition 2.2.5. Let $\mathcal{I}$ be the set indexing reduced subexpressions for w' occurring inside the fixed reduced expression $w^P = s_{r_1} \cdots s_{r_\ell}$ of equation (1.2.3). In other words,

$$\mathcal{I} = \big\{(i_1, \dots, i_{\ell'}) \bigm| 1 \le i_1 < i_2 < \dots < i_{\ell'} \le \ell \text{ and } w' = s_{r_{i_1}} \cdots s_{r_{i_{\ell'}}}\big\}.$$

Remark 2.2.6. Note that the reduced expression $w' = s_{r_{i_1}} \cdots s_{r_{i_{\ell'}}}$ is *not* fixed, i.e. if $(i_j), (i'_j) \in \mathcal{I}$, then we do *not* necessarily have $r_{i_j} = r_{i'_j}$ for all j.

Theorem 2.2.7 (An explicit Laurent-polynomial Landau-Ginzburg model)**.** Let $X = G/P$ be a *cominuscule* complete homogeneous space with G a *simply-connected*, simple, (complex) linear algebraic group and $P = P_k$ a (maximal) parabolic subgroup. The restriction $\mathcal{W}_{\mathcal{Z}_P^\circ}$ of $\mathcal{W}_{\mathcal{Z}_P^\vee}$ to $\mathcal{Z}_P^\circ$ has the following Laurent polynomial expression:

$$\mathcal{W}_{\mathcal{Z}_P^\circ}(z) = \sum_{i=1}^{\ell} a_i + q \frac{\sum_{(i_j) \in \mathcal{I}} a_{i_1} \cdots a_{i_{\ell'}}}{\prod_{i=1}^{\ell} a_i}. \tag{2.2.2}$$

Here $z \in \mathcal{Z}_P^\circ$ is uniquely decomposed as $z = u_+ t \bar{w}_P u_-$ with $u_- = y_{r_\ell}^\vee(a_\ell) \cdots y_{r_1}^\vee(a_1) \in U_-^\circ$ as in Corollary 2.2.4, and $q \in \mathbb{C}^*$ is given by $q = \alpha_k^\vee(t)$ (with $t \in T^P$). Finally, the set $\mathcal{I}$ is given in Definition 2.2.5.

The proof of this statement follows in Sections 2.3 and 2.4. In Section 2.5 we rewrite the summation over $\mathcal{I}$ into a summation over subsets of a quiver associated to w^P by [CMP08], see Corollary 2.5.12. In Section 2.6 we apply Theorem 2.2.7 and its reformulation as Corollary 2.5.12 to all the cominuscule homogeneous spaces, leading to new Laurent polynomial potentials for the cominuscule homogeneous spaces of type D_n, E_6 and E_7, see Subsections 2.6.4, 2.6.5 and 2.6.6, respectively. We also work out an example for each of the families of cominuscule homogeneous spaces.

2.3 Proof of the Laurent polynomial expression

In Section 2.2 we stated the main result of this chapter in Theorem 2.2.7. This section is dedicated to proving this theorem. Before we get started on the proof, let us introduce the following notation:

Definition 2.3.1. Write $V(\omega_i^\vee)$ for the irreducible representation of $\mathfrak{g}^\vee$ with highest weight $\omega_i^\vee$, and denote by $v^+_{\omega_i^\vee}$ a choice of a highest weight vector. Denote by $v^-_{\omega_i^\vee} = \bar{w}_0 \cdot v^+_{\omega_i^\vee}$ the associated lowest weight vector. Note that the weight space of $v^+_{\omega_i^\vee}$ is one-dimensional, so that the projection of an arbitrary $v \in V(\omega_i^\vee)$ to this weight space (parallel to the other weight spaces) is a scalar multiple of $v^+_{\omega_i^\vee}$; we denote this scalar by $\langle v, v^+_{\omega_i^\vee}\rangle$.

Remark 2.3.2. Recall that every representation of a Lie algebra induces a representation of the associated simply-connected linear algebraic group. Here, the highest weight representation $V(\omega_i^\vee)$ of $\mathfrak{g}^\vee$ induces a representation of the universal cover $\widetilde{G}^\vee$ of $G^\vee$. Since this representation does not always descend to a representation of $G^\vee$, we need to work on $\widetilde{G}^\vee$ instead. Because we identified $U_+^\vee$ and $U_-^\vee$ with their universal covers, we consider the factors u_+ and u_- of $z = u_+ t\bar{w}_0 u_- \in \mathcal{Z}_P^\circ$ as elements of $\widetilde{G}^\vee$. The same holds for the one-parameter subgroups $x_j^\vee(a) \in U_+^\vee$ and $y_j^\vee(a) \in U_-^\vee$ for $j \in \{1, \dots, n\}$, which we also consider as elements of $\widetilde{G}^\vee$. However, the elements $\dot{s}_i$ and $\bar{s}_i$ of $G^\vee$ associated to $s_i \in W$ have multiple lifts to $\widetilde{G}^\vee$; we choose the lifts to be the elements obtained by taking the product of the one-parameter subgroups in $\widetilde{G}^\vee$, i.e.

$$\dot{s}_i = x_i^\vee(1) y_i^\vee(-1) x_i^\vee(1) \ \in \ \widetilde{G}^\vee \quad \text{and} \quad \bar{s}_i = x_i^\vee(-1) y_i^\vee(1) x_i^\vee(-1) \ \in \ \widetilde{G}^\vee.$$

Note that we abuse notation and denote these lifts in the same way as the original elements. Also note that with these choices we still have $\bar{s}_i = \dot{s}_i^{-1}$. The elements $\dot{w}, \bar{w} \in \widetilde{G}^\vee$ associated to $w \in W$ are similarly defined by $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_j}$ and $\bar{w} = \bar{s}_{i_1} \cdots \bar{s}_{i_j}$ respectively, where $w = s_{i_1} \cdots s_{i_j}$ is a reduced expression. This fixes the lift of $z = u_+ t\bar{w}_0 u_-$ up to a choice of lift of $t \in T^P \subset T^\vee$; the choices differ by a factor in

$\ker(\alpha_k^\vee) \subset \widetilde{T}^\vee$, so all the lifts have $\alpha_k^\vee(t) = q$ and we choose one arbitrarily (we will continue to abuse notation and also denote the lift of $t \in T^P$ by $t \in \widetilde{T}^\vee$).

Remark 2.3.3. Expressions of the form $\langle g \cdot v_{\omega_i^\vee}^+, v_{\omega_i^\vee}^+ \rangle$ are a priori only defined for $g \in \widetilde{G}^\vee$, so expressions of that form will always assume the group element g to be an element of the universal cover $\widetilde{G}^\vee$. Thus, the abuse of notation in Remark 2.3.2 should not give rise to ambiguity.

The proof of Theorem 2.2.7 requires a few intermediate results which we will prove in Section 2.4. Assuming for the moment that these hold, the proof of Theorem 2.2.7 is a straightforward computation:

Proof of Theorem 2.2.7. We want to find an expression for

$$\mathcal{W}_{\mathcal{Z}_P^\circ}(z) = \mathcal{E}^*(u_+^{-1}) + \mathcal{F}^*(u_-) = -\mathcal{E}^*(u_+) + \mathcal{F}^*(u_-)$$

in terms of the toric coordinates of $z \in \mathcal{Z}_P^\circ$. First, we calculate $\mathcal{F}^*(u_-)$:

Lemma 2.3.4. For $u_- \in U_-^\circ$ we have $\mathcal{F}^*(u_-) = \sum_{i=1}^{\ell} a_i$.

Thus, we only need to find the term involving the quantum parameter, which comes from $\mathcal{E}^*(u_+) = \sum_{i=1}^{n} (e_i^\vee)^*(u_+)$. We can rewrite each of the summands of this term as follows:

Lemma 2.3.5. For $z = u_+ t \bar{w}_P u_- \in \mathcal{Z}_P^\circ$ we have

$$(e_i^\vee)^*(u_+) = \alpha_i^\vee(t) \frac{\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \bar{s}_i \cdot v_{\omega_i^\vee}^+, v_{\omega_i^\vee}^+ \rangle}{\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \cdot v_{\omega_i^\vee}^+, v_{\omega_i^\vee}^+ \rangle}.$$

It turns out that *except* for $i = k$, these summands do not contribute to the sum:

Lemma 2.3.6. For $z = u_+ t \bar{w}_P u_- \in \mathcal{Z}_P^\circ$ and $i \neq k$ (where k is such that $P = P_k$) we have $(e_i^\vee)^*(u_+) = 0$.

Altogether, we have now found that

$$\mathcal{E}^*(u_+) = (e_k^\vee)^*(u_+) = q \frac{\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \bar{s}_k \cdot v_{\omega_k^\vee}^+, v_{\omega_k^\vee}^+ \rangle}{\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \cdot v_{\omega_k^\vee}^+, v_{\omega_k^\vee}^+ \rangle},$$

where $q = \alpha_k^\vee(t)$. Now we of course need to calculate the denominator and numerator of this quotient:

Lemma 2.3.7. For $u_- \in U_-^\circ$ we have $\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \cdot v_{\omega_k^\vee}^+, v_{\omega_k^\vee}^+ \rangle = (-1)^\ell \prod_{i=1}^\ell a_i$, where $\ell = \ell(w^P)$.

Lemma 2.3.8. For $u_- \in U_-^\circ$ we have

$$\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \bar{s}_k \cdot v_{\omega_k^\vee}^+, v_{\omega_k^\vee}^+ \rangle = (-1)^{\ell+1} \textstyle\sum_{(i_j) \in \mathcal{I}} a_{i_1} \cdots a_{i_{\ell'}},$$

where $\mathcal{I} = \{(i_1, \ldots, i_{\ell'}) \mid 1 \leq i_1 < i_2 < \ldots < i_{\ell'} \leq \ell \text{ and } w' = s_{r_{i_1}} \cdots s_{r_{i_{\ell'}}}\}$ and where we fixed the reduced expression $w^P = s_{r_1} \cdots s_{r_\ell}$ in Section 1.2.

Thus, we find

$$\mathcal{E}^*(u_+) = -q \frac{\sum_{(i_j) \in \mathcal{I}} a_{i_1} \cdots a_{i_{\ell'}}}{\prod_{i=1}^\ell a_i}.$$

We obtain the statement of Theorem 2.2.7 by inserting this together with the expression for $\mathcal{F}^*(u_-)$ into $\mathcal{W}_{\mathcal{Z}_P^\circ}(z) = -\mathcal{E}^*(u_+) + \mathcal{F}^*(u_-)$, . □

2.4 Proof of the intermediate results

What remains is proving the lemmas stated in the proof of Theorem 2.2.7.

Remark 2.4.1. The five intermediate lemmas in the proof of Theorem 2.2.7 are generalizations of Lemma 5.5 of [PR18]. Lemmas 2.3.4, 2.3.5 and 2.3.6 follow similar reasoning in the general case. The proof of Lemma 2.3.7 requires a modification using the general structure of minuscule representations, while Lemma 2.3.8 follows a different line of reasoning than its counterpart in [PR18].

Proof of Lemma 2.3.4. We want to show that $\mathcal{F}^*(u_-) = \sum_{i=1}^\ell a_i$ for $u_- \in U_-^\circ$.

Recall from equation (1.1.5) that $(f_i^\vee)^*(y_j(a)) = a\delta_{ij}$. From this it follows that

$$(f_i^\vee)^*(u_-) = (f_i^\vee)^*\big(y_{r_\ell}^\vee(a_\ell) \cdots y_{r_1}^\vee(a_1)\big) = a_1 \delta_{i,r_1} + \ldots + a_\ell \delta_{i,r_\ell}.$$

Summing over all $i \in \{1, \ldots, n\}$, we find $\mathcal{F}^*(u_-) = \sum_{i=1}^n (f_i^\vee)^*(u_-) = \sum_{i=1}^\ell a_i$. □

Unfortunately, the other term, $\mathcal{E}^*(u_+) = \sum_{i=1}^n (e_i^\vee)^*(u_+)$, will not be as easy. We will first reformulate each of the terms $e_i^\vee(u_+)$:

Proof of Lemma 2.3.5. We want to show that

$$(e_i^\vee)^*(u_+) = \alpha_i^\vee(t) \frac{\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \bar{s}_i \cdot v_{\omega_i^\vee}^+, v_{\omega_i^\vee}^+ \rangle}{\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \cdot v_{\omega_i^\vee}^+, v_{\omega_i^\vee}^+ \rangle}.$$

for $z = u_+ t \bar{w}_P u_- \in \mathcal{Z}_P^\circ$.

First, note that the map $U_+^\vee \to \mathbb{C} : u \mapsto \langle u \bar{s}_i \cdot v_{\omega_i^\vee}^+, v_{\omega_i^\vee}^+ \rangle$ is equal to the unique homomorphism $(e_i^\vee)^*$ sending $x_i^\vee(a) = \exp(a\, e_i^\vee)$ to a and the other one-parameter-subgroups to zero, so

$$(e_i^\vee)^*(u_+) = \langle u_+ \bar{s}_i \cdot v_{\omega_i^\vee}^+, v_{\omega_i^\vee}^+ \rangle.$$

Next, we use the fact that u_+ is a factor in the decomposition of $z \in \mathcal{Z}_P^\circ$ as $z = u_+ t \bar{w}_P u_-$ to find a decomposition for u_+ itself. By definition, we have $z = b_- \bar{w}_0^{-1}$ for some $b_- \in B_-^\vee$ as $\mathcal{Z}_P^\circ \subset B_-^\vee \bar{w}_0^{-1}$. In Remark 2.3.2 we fixed lifts of $z = u_+ t \bar{w}_P u_-$ and the elements $\dot{s}_i$ and $\bar{s}_i$ to $\widetilde{G}^\vee$. Thus, there is a unique lift of $b_- \in B_-^\vee$, also denoted by $b_- \in \widetilde{B}_-^\vee$, such that $b_- \bar{w}_0^{-1} = z \in \widetilde{G}^\vee$. This gives

$$u_+ = b_- \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} t^{-1} \ \in \ \widetilde{G}^\vee. \tag{2.4.1}$$

Thus, we have to calculate $\langle b_- \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} t^{-1} \bar{s}_i \cdot v_{\omega_i^\vee}^+, v_{\omega_i^\vee}^+ \rangle$.

Now, $\bar{s}_i \cdot v_{\omega_i^\vee}^+$ has weight $\omega_i^\vee - \alpha_i^\vee$, so $t^{-1} \in \widetilde{T}^\vee$ acts on this vector by scalar multiplication with

$$[\omega_i^\vee - \alpha_i^\vee](t^{-1}) = \frac{\omega_i^\vee(t^{-1})}{\alpha_i^\vee(t^{-1})} = \frac{\alpha_i^\vee(t)}{\omega_i^\vee(t)} \ \in \ \mathbb{C}.$$

(Note that weights and roots are written additively.) We conclude

$$(e_i^\vee)^*(u_+) = \alpha_i^\vee(t) \frac{\langle b_- \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \bar{s}_i \cdot v_{\omega_i^\vee}^+, v_{\omega_i^\vee}^+ \rangle}{\omega_i^\vee(t)}.$$

Noting that $b_- \in \widetilde{B}_-^\vee$ sends a vector to a linear combination of vectors of *equal or* *lower* weight, we see that the only contribution of b_- to $\langle b_-\bar{w}_0^{-1}u_-^{-1}\bar{w}_P^{-1}\bar{s}_i \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee}\rangle$ will be the factor $\langle b_- \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee}\rangle$, so we find

$$(e_i^\vee)^*(u_+) = \alpha_i^\vee(t)\frac{\langle b_- \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee}\rangle}{\omega_i^\vee(t)}\,\langle \bar{w}_0^{-1}u_-^{-1}\bar{w}_P^{-1}\bar{s}_i \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee}\rangle. \tag{2.4.2}$$

Finally, we use the decomposition in (2.4.1) together with the fact that $u_+ \in U_+^\vee$ implies $\langle u_+ \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee}\rangle = 1$ to conclude that

$$\begin{aligned} 1 = \langle u_+ \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee}\rangle &= \langle b_-\bar{w}_0^{-1}u_-^{-1}\bar{w}_P^{-1}t^{-1} \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee}\rangle \\ &= \frac{\langle b_- \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee}\rangle}{\omega_i^\vee(t)}\,\langle \bar{w}_0^{-1}u_-^{-1}\bar{w}_P^{-1} \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee}\rangle, \end{aligned}$$

where we calculated the contributions of b_- and t^{-1} in an analogous way as above. Substituting this into (2.4.2), we obtain

$$(e_i^\vee)^*(u_+) = \alpha_i^\vee(t)\frac{\langle \bar{w}_0^{-1}u_-^{-1}\bar{w}_P^{-1}\bar{s}_i \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee}\rangle}{\langle \bar{w}_0^{-1}u_-^{-1}\bar{w}_P^{-1} \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee}\rangle},$$

as in the statement of the lemma. □

Lemma 2.3.6 claims that all of the summands of $\mathcal{E}^*(u_+) = \sum_{i=1}^n (e_i^\vee)^*(u_+)$ are zero, except for $i = k$ (where k is such that $P = P_k$).

Proof of Lemma 2.3.6. We need to show that $(e_i^\vee)^*(u_+) = 0$ for $z = u_+t\bar{w}_Pu_- \in \mathcal{Z}_P^\circ$ and $i \neq k$ (where k is such that $P = P_k$).

Considering the expression for $(e_i^\vee)^*(u_+)$ of Lemma 2.3.5, we need to show that

$$\langle \bar{w}_0^{-1}u_-^{-1}\bar{w}_P^{-1}\bar{s}_i \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee}\rangle = 0 \quad \text{for} \quad i \neq k.$$

Recall that we assumed $u_- \in U_-^\circ \subset U_-^P = U_-^\vee \cap B_+^\vee\bar{w}_P\bar{w}_0B_+^\vee$ (see Lemma 2.2.2). Thus, we have $u_-^{-1} \in B_+^\vee\bar{w}_0^{-1}\bar{w}_P^{-1}B_+^\vee$; in other words, there are $b_1, b_2 \in B_+^\vee$ such that $u_-^{-1} = b_1\bar{w}_0^{-1}\bar{w}_P^{-1}b_2$. Choosing lifts $b_1, b_2 \in \widetilde{B}_+^\vee$ such that $u_-^{-1} = b_1\bar{w}_0^{-1}\bar{w}_P^{-1}b_2$ as elements of $\widetilde{G}^\vee$ (again abusing notation), it follows that we have to show that

$$\langle \bar{w}_0^{-1} b_1 \bar{w}_0^{-1} \bar{w}_P^{-1} b_2 \bar{w}_P^{-1} \bar{s}_i \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee} \rangle = 0 \quad \text{for} \quad i \neq k.$$

Now, $\bar{w}_0^{-1} b_1 \bar{w}_0^{-1} \in \bar{w}_0^{-1} \widetilde{B}_+^\vee \bar{w}_0^{-1} = \widetilde{B}_-^\vee$, so using an analogous argument to the one in the proof of Lemma 2.3.5, we find that

$$\langle \bar{w}_0^{-1} b_1 \bar{w}_0^{-1} \bar{w}_P^{-1} b_2 \bar{w}_P^{-1} \bar{s}_i \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee} \rangle = \langle \bar{w}_0^{-1} b_1 \bar{w}_0^{-1} \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee} \rangle \langle \bar{w}_P^{-1} b_2 \bar{w}_P^{-1} \bar{s}_i \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee} \rangle.$$

As $\langle \bar{w}_0^{-1} b_1 \bar{w}_0^{-1} \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee} \rangle$ only contributes a scalar factor, we need to show that

$$\langle \bar{w}_P^{-1} b_2 \bar{w}_P^{-1} \bar{s}_i \cdot v^+_{\omega_i^\vee}, v^+_{\omega_i^\vee} \rangle = 0 \quad \text{for} \quad i \neq k. \tag{2.4.3}$$

In other words, it is enough to show that $\bar{w}_P^{-1} b_2 \bar{w}_P^{-1} \bar{s}_i \cdot v^+_{\omega_i^\vee}$ has no components of weight $\omega_i^\vee$. This is a straightforward argument with weights: $\bar{s}_i \cdot v^+_{\omega_i^\vee}$ has weight $\omega_i^\vee - \alpha_i^\vee$, so that $\bar{w}_P^{-1} \bar{s}_i \cdot v^+_{\omega_i^\vee}$ has weight $w_P\big(\omega_i^\vee - \alpha_i^\vee\big)$, noting that $w_P^{-1} = w_P$ as it is the longest element of the Weyl group $W_P = \langle s_i \mid i \neq k \rangle$. Now, as $b_2 \in \widetilde{B}_+^\vee$, all components of $b_2 \bar{w}_P^{-1} \bar{s}_i \cdot v^+_{\omega_i^\vee}$ will have weight $w_P\big(\omega_i^\vee - \alpha_i^\vee\big) + \alpha_+^\vee$ for some (possibly trivial) sum $\alpha_+^\vee$ of positive roots. Thus, we find that $\bar{w}_P^{-1} b_2 \bar{w}_P^{-1} \bar{s}_i \cdot v^+_{\omega_i^\vee}$ has components of weight

$$w_P\Big(w_P\big(\omega_i^\vee - \alpha_i^\vee\big) + \alpha_+^\vee\Big) = \omega_i^\vee - \alpha_i^\vee + w_P(\alpha_+^\vee),$$

again using $w_P^{-1} = w_P$. Therefore, one of these components has weight $\omega_i^\vee$ if and only if $w_P(\alpha_+^\vee) = \alpha_i^\vee$. However, since w_P is the longest element of the Weyl group $W_P = \langle s_i \mid i \neq k \rangle$, we know that it maps all the simple roots $\alpha_i^\vee$ with $i \neq k$ to negative roots, but then we must have that $\alpha_+^\vee = w_P(\alpha_i^\vee) \in \Pi_-^\vee$ is a *negative* root and definitely not a sum of positive roots, which gives a contradiction. Thus, all components have weight unequal to $\omega_i^\vee$, implying that (2.4.3) holds, which in turn implies the lemma. $\square$

Combining Lemmas 2.3.5 and 2.3.6, we conclude that

$$\mathcal{E}^*(u_+) = (e_k^\vee)^*(u_+) = q\frac{\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \bar{s}_k \cdot v^+_{\omega_k^\vee}, v^+_{\omega_k^\vee} \rangle}{\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \cdot v^+_{\omega_k^\vee}, v^+_{\omega_k^\vee} \rangle}. \tag{2.4.4}$$

Lemma 2.3.7 calculates the denominator of this quotient and Lemma 2.3.8 calculates its numerator:

Proof of Lemma 2.3.7. We need to show that for $u_- \in U_-^\circ$ we have

$$\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \cdot v^+_{\omega_k^\vee}, v^+_{\omega_k^\vee} \rangle = (-1)^\ell \prod_{i=1}^{\ell} a_i,$$

where $\ell = \ell(w^P)$.

Using Lemma 2.1.1 **(iv)** we find

$$\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \cdot v^+_{\omega_k^\vee}, v^+_{\omega_k^\vee} \rangle = \langle \bar{w}_0^{-1} u_-^{-1} \cdot v^+_{\omega_k^\vee}, v^+_{\omega_k^\vee} \rangle.$$

By definition of U_-° (see Definition 2.2.1), u_-^{-1} has a decomposition of the form

$$u_-^{-1} = y_{r_1}^\vee(-a_1) \cdots y_{r_\ell}^\vee(-a_\ell),$$

where the sequence of indices $(r_1, \ldots, r_\ell)$ is the same as the one used in the reduced expression $w^P = s_{r_1} \cdots s_{r_\ell}$ fixed in equation (1.2.3). Now, $y_i^\vee(a) = \exp(a\, f_i^\vee) = 1 + a\, f_i^\vee + \frac{1}{2} a^2 (f_i^\vee)^2 + \ldots$, but only the first two terms act non-trivially on the representation, since $(f_i^\vee)^2 \cdot v = 0$ for all $v \in V(\omega_k^\vee)$ according to Theorem 1.5.3 **(iv)**. We conclude that

$$u_-^{-1} \cdot v^+_{\omega_k^\vee} = (1 - a_1\, f_{r_1}^\vee) \cdots (1 - a_\ell\, f_{r_\ell}^\vee) \cdot v^+_{\omega_k^\vee}.$$

Note that this is a sum of vectors of different weights, the term of highest weight being $v^+_{\omega_k^\vee}$ (obtained by taking the term with all the identity factors), and the term of lowest weight being (see Lemma 2.1.1 **(iv)**)

$$(-a_1) \cdots (-a_\ell) f_{r_1}^\vee \cdots f_{r_\ell}^\vee \cdot v^+_{\omega_k^\vee} = (-1)^\ell \big(\textstyle\prod_{i=1}^{\ell} a_i\big) v^-_{\omega_k^\vee}.$$

Only the lowest weight term contributes a coefficient to $\langle \bar{w}_0^{-1} u_-^{-1} \cdot v^+_{\omega_k^\vee}, v^+_{\omega_k^\vee} \rangle$ since $\bar{w}_0^{-1} v^-_{\omega_k^\vee} = v^+_{\omega_k^\vee}$ and $\bar{w}_0^{-1}$ is a bijection. Thus, we obtain

$$\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \cdot v^+_{\omega_k^\vee}, v^+_{\omega_k^\vee} \rangle = (-1)^\ell \prod_{i=1}^\ell a_i$$

as we wanted to show. $\square$

Now we turn to the numerator of (2.4.4):

Proof of Lemma 2.3.8. We need to show that for $u_- \in U_-^\circ$ we have

$$\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \bar{s}_k \cdot v^+_{\omega_k^\vee}, v^+_{\omega_k^\vee} \rangle = (-1)^{\ell+1} \sum_{(i_j) \in \mathcal{I}} a_{i_1} \cdots a_{i_{\ell'}},$$

where $\mathcal{I} = \{(i_1, \ldots, i_{\ell'}) \mid 1 \leq i_1 < i_2 < \ldots < i_{\ell'} \leq \ell$ and $w' = s_{r_{i_1}} \cdots s_{r_{i_{\ell'}}}\}$ was defined in Definition 2.2.5 and where we fixed the reduced expression $w^P = s_{r_1} \cdots s_{r_\ell}$ in equation (1.2.3).

As we saw in the proof of Lemma 2.3.7, we only need to consider the lowest-weight term of the vector $u_-^{-1} \bar{w}_P^{-1} \bar{s}_k \cdot v^+_{\omega_k^\vee}$, as it is the only term mapped to $v^+_{\omega_k^\vee}$ by $\bar{w}_0^{-1}$. However, $\bar{w}_P^{-1} \bar{s}_k$ acts non-trivially on $v^+_{\omega_k^\vee}$, whereas in the proof of Lemma 2.3.7 $\bar{w}_P^{-1}$ acted trivially on $v^+_{\omega_k^\vee}$ by Lemma 2.1.1 **(iv)**.

Recall that we had fixed the reduced expression $w_P = s_{q_1} \cdots s_{q_{\ell_P}}$ in equation (1.2.3) and that $\bar{s}_i^{-1} = \dot{s}_i$, so we find that $\bar{w}_P^{-1} = \dot{s}_{q_{\ell_P}} \cdots \dot{s}_{q_1}$. Moreover, note that $\bar{s}_k \cdot v^+_{\omega_k^\vee} = f_k^\vee \cdot v^+_{\omega_k^\vee} = -\dot{s}_k \cdot v^+_{\omega_k^\vee}$ (see Corollary 1.5.4). All in all, we find that

$$\bar{w}_P^{-1} \bar{s}_k \cdot v^+_{\omega_k^\vee} = -\dot{s}_{q_{\ell_P}} \cdots \dot{s}_{q_1} \dot{s}_k \cdot v^+_{\omega_k^\vee}.$$

In Section 1.5 we have written $w'' \in W^P$ for the minimal coset representative of the coset $w_P s_k W_P$ (note that $w_P^{-1} = w_P$) and written $\ell'' = \ell(w'') \leq \ell(w^P) = \ell$ for its length. Since $s_{q_{\ell_P}} \cdots s_{q_1}$ is a reduced expression for $w_P \in W_P$ and $s_k \notin W_P$, we deduce that $s_{q_{\ell_P}} \cdots s_{q_1} s_k$ is a reduced expression for $w_P s_k$. Thus, by Lemma 2.1.1 **(i)** and **(ii)**, we deduce that $\bar{w}_P^{-1}\bar{s}_k \cdot v^+_{\omega_k^\vee} = -\dot{s}_{q_{\ell_P}} \cdots \dot{s}_{q_1}\dot{s}_k \cdot v^+_{\omega_k^\vee} = -\dot{w}_P\dot{s}_k \cdot v^+_{\omega_k^\vee}$ and that

$$\bar{w}_P^{-1}\bar{s}_k \cdot v^+_{\omega_k^\vee} = -\dot{w}_P\dot{s}_k \cdot v^+_{\omega_k^\vee} = -\dot{w}'' \cdot v^+_{\omega_k^\vee} = (-1)^{\ell''+1} f^\vee_{j_1} \cdots f^\vee_{j_{\ell''}} \cdot v^+_{\omega_k^\vee}, \tag{2.4.5}$$

where $w'' = s_{j_1} \cdots s_{j_{\ell''}}$ is a reduced expression.

Note that in the case $\ell'' = \ell$ (which only occurs for $\mathbb{CP}^n = \mathrm{Gr}(1, n+1) = A_n^{\mathrm{sc}}/P_1 \cong A_n^{\mathrm{sc}}/P_n$) the following arguments become trivial, see Remark 2.4.3 below.

Next, we need to multiply this vector by u_-^{-1}, which due to Theorem 1.5.3 **(iv)** reduces to multiplying by $(1 - a_1 f^\vee_{r_1}) \cdots (1 - a_\ell f^\vee_{r_\ell})$ in the representation. As we mentioned at the start of the proof, we only need to look at the coefficient in front of $v^-_{\omega_k^\vee}$ in this product. From Lemma 2.1.1 **(iv)** we know that $f^\vee_{i_1} \cdots f^\vee_{i_\ell} \cdot v^+_{\omega_k^\vee} = v^-_{\omega_k^\vee}$ if and only if $s_{i_1} \cdots s_{i_\ell} = w^P$ and this is a reduced expression. So we need exactly those terms of $(1 - a_1 f^\vee_{r_1}) \cdots (1 - a_\ell f^\vee_{r_\ell})$ that complete $f^\vee_{j_1} \cdots f^\vee_{j_{\ell''}} \cdot v^+_{\omega_k^\vee}$ to $f^\vee_{r_{i_1}} \cdots f^\vee_{r_{i_{\ell-\ell''}}} f^\vee_{j_1} \cdots f^\vee_{j_{\ell''}} \cdot v^+_{\omega_k^\vee}$ in such a way that the indices satisfy $s_{r_{i_1}} \cdots s_{r_{i_{\ell-\ell''}}} s_{j_1} \cdots s_{j_{\ell''}} = w^P$. However, $s_{j_1} \cdots s_{j_{\ell''}} = w''$ and $s_{r_1} \cdots s_{r_\ell} = w^P$, so we obtain a contributing term for every subexpression $s_{r_{i_1}} \cdots s_{r_{i_{\ell-\ell''}}}$ of the fixed reduced expression for w^P such that $s_{r_{i_1}} \cdots s_{r_{i_{\ell-\ell''}}} w'' = w^P$. Therefore, every subexpression of $w' = w^P (w'')^{-1}$ in the fixed reduced expression of w^P gives a contributing term.

Now, in Definition 2.2.5 we defined the set indexing these subexpressions as

$$\mathcal{I} = \{(i_1, \ldots, i_{\ell'}) \mid 1 \le i_1 < i_2 < \ldots < i_{\ell'} \le \ell \text{ and } w' = s_{r_{i_1}} \cdots s_{r_{i_{\ell'}}}\},$$

where $\ell' = \ell(w') = \ell(w^P) - \ell(w'') = \ell - \ell''$. In conclusion, for every $(i_1, \ldots, i_\ell) \in \mathcal{I}$, we obtain the following term

$$-(-a_{i_1}) \cdots (-a_{i_{\ell'}}) f^\vee_{r_{i_1}} \cdots f^\vee_{r_{i_{\ell'}}} (-1)^{\ell''} f^\vee_{j_1} \cdots f^\vee_{j_{\ell''}} \cdot v^+_{\omega_k^\vee} = (-1)^{\ell+1} a_{i_1} \cdots a_{i_{\ell'}}\, v^-_{\omega_k^\vee}$$

and we find

$$\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \bar{s}_k \cdot v^+_{\omega_k^\vee}, v^+_{\omega_k^\vee} \rangle = (-1)^{\ell+1} \sum_{(i_j) \in \mathcal{I}} a_{i_1} \cdots a_{i_{\ell'}},$$

as we wanted to show. □

This concludes the last of the intermediate results for the proof of Theorem 2.2.7.

Remark 2.4.2. Note that at no point in the proof of Lemma 2.3.8 do we fix a reduced expression for w', so if $(i_j), (i'_j) \in \mathcal{I}$, then we do *not* necessarily have $r_{i_j} = r_{i'_j}$ for all $j \in \{1, \dots, \ell'\}$.

Remark 2.4.3. Note that if $\ell'' = \ell$, we have $w'' = w^P$. In this case, equation (2.4.5) becomes

$$\bar{w}_P^{-1} \bar{s}_k \cdot v^+_{\omega_k^\vee} = -\dot{s}_{q_m} \cdots \dot{s}_{q_1} \dot{s}_k \cdot v^+_{\omega_k^\vee} = -\dot{w}^P \cdot v^+_{\omega_k^\vee} = (-1)^{\ell+1} f^\vee_{r_1} \cdots f^\vee_{r_\ell} \cdot v^+_{\omega_k^\vee} = (-1)^{\ell+1} v^-_{\omega_k^\vee},$$

Since $u_- \in U^\vee_-$, we find that u_-^{-1} acts trivially on this, so that

$$\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \bar{s}_k \cdot v^+_{\omega_k^\vee}, v^+_{\omega_k^\vee} \rangle = (-1)^{\ell+1}.$$

Of course, $w'' = w^P$ implies that $w' = w^P (w'')^{-1} = e$ is the identity element. Thus, subexpressions of w' inside w^P have zero length and there is only one such subexpression so we find $\mathcal{I} = \{\varnothing\}$. We conclude that $\sum_{(i_j)\in\mathcal{I}} \prod_{j=1}^{\ell'} a_{r_{i_j}} = 1$, taking the empty product to be 1. Thus,

$$\langle \bar{w}_0^{-1} u_-^{-1} \bar{w}_P^{-1} \bar{s}_k \cdot v^+_{\omega_k^\vee}, v^+_{\omega_k^\vee} \rangle = (-1)^{\ell+1} \sum_{(i_j)\in\mathcal{I}} \prod_{j=1}^{\ell'} a_{r_{i_j}}$$

also holds in case $\ell'' = \ell$.

2.5 Reformulating the quantum term using quiver subsets

In the last two sections we proved that Theorem 2.2.7 gives a local Laurent polynomial expression for the potential constructed by Rietsch in [Rie08]. However, the drawback of the current expression is the effort required to find all the subexpressions of w' inside the fixed reduced expression of w^P. In this section we will use a quiver associated to

w^P to enumerate all these subexpressions. For this, we need to use the fact that both w^P and w' are *fully commutative*:

Definition 2.5.1. An element $w \in W$ is called *fully commutative* if every reduced expression of w can be obtained from a given reduced expression by commuting its factors.

Lemma 2.5.2. Both $w^P \in W^P$ and $w' \in W$ are fully commutative.

Proof. Full commutativity of w^P (and in fact of every element of W^P) follows from Theorem 6.1 of [Ste96]. Full commutativity of w' is now a consequence of the full commutativity of w^P due to Proposition 2.4 of [Ste96], which states that every element in W obtained from a fully commutative element by removing simple reflections at the right (or left) is fully commutative itself. □

In [CMP08] a quiver is associated to w^P using the full commutativity property. This quiver is a modification of the quiver introduced in [Per07] and closely related to the *heap* associated to (an expression of) a Weyl group element, see [Ste96], section 2.2, and references therein. It is defined as follows:

Definition 2.5.3 ([CMP08], Definition 2.1)**.** Given a fixed reduced expression for w^P, e.g. the reduced expression $w^P = s_{r_1} \cdots s_{r_\ell}$ fixed in equation (1.2.3).

- For $\beta \in \{\alpha_1, \ldots, \alpha_n\}$, let $m(\beta)$ be the number of occurrences of s_β in the reduced expression, i.e. $m(\beta) = \#\{j \mid s_{r_j} = s_\beta\}$.
- For (β, j) such that $1 \leq j \leq m(\beta)$, let $\mathscr{J}(\beta, j)$ be the index of the jth occurrence of s_β in the reduced expression (from left to right), i.e. let $\mathscr{J}(\beta, j)$ be the index such that $s_{r_{\mathscr{J}(\beta,j)}} = s_\beta$ and $\#\{\tilde{\jmath} \leq \mathscr{J}(\beta, j) \mid s_{r_{\tilde{\jmath}}} = s_\beta\} = j$. Also, set $\mathscr{J}(\beta, 0) = 0$ and $\mathscr{J}(\beta, m(i) + 1) = \infty$.

The quiver is now defined as follows:

- Draw for the jth occurrence of s_β in the reduced expression for w^P a vertex labeled (β, j), i.e. the vertices are (β, j) for $\beta \in \{\alpha_1, \ldots, \alpha_n\}$ and for $j \in \{1, \ldots, m(\beta)\}$.

- Draw an arrow from (β, j) to (β', j') if s_β and $s_{\beta'}$ do not commute and if the j'th occurrence of $s_{\beta'}$ is the is the first one to the right of the jth occurrence of s_β in the reduced expression for w^P and it occurs before the $(j+1)$th occurrence of s_β, i.e. draw an arrow from the vertex (β, j) to (β', j') if $(s_\beta s_{\beta'})^2 \neq 1$ and $\mathscr{J}(\beta', j'-1) < \mathscr{J}(\beta, j) < \mathscr{J}(\beta', j') < \mathscr{J}(\beta, j+1)$.

We will denote the resulting quiver by $Q_{\mathbb{X}^\vee}$.

Note that the resulting quiver does not depend on the reduced expression for w^P as it is fully commutative; it suffices to check that the quiver is the same after commuting two simple reflections.

Remark 2.5.4. In Definition 2.5.3, we associate the quiver $Q_{\mathbb{X}^\vee}$ to the *Langlands dual homogeneous space* $\mathbb{X}^\vee = P_k^\vee \backslash G^\vee$ which we introduced in Section 1.6. However, in [Spa21], we associated the quiver to X instead. We chose to do this because $\mathbb{X}^\vee$ played no other role in that paper (the results of which are presented in this chapter), although we noted that $P_k^\vee \backslash G^\vee$ would be a more appropriate choice as the Laurent polynomial $\mathcal{W}_{\mathcal{Z}_P^\circ}$ is defined on $\mathcal{Z}_P^\circ \subset G^\vee$. The Langlands dual homogeneous space $\mathbb{X}^\vee$ is closely related to the variety $X_{\mathrm{Lie}}^\vee$ of equation (1.4.1) and Theorem 1.4.2. In fact, in the articles [MR20, PR18, PRW16, PR13] it is shown that $X_{\mathrm{Lie}}^\vee$ is isomorphic to the *canonical mirror variety* $X_{\mathrm{can}}^\vee$, which is an open subvariety of $\mathbb{X}^\vee$, and an expression for the pull-back of $\mathcal{W}_{\mathrm{Lie}} : X_{\mathrm{Lie}}^\vee \times T^P \to \mathbb{C}$ (see Definition 1.4.5) to $\mathbb{X}^\vee$ is given. It is expected that such an isomorphic subvariety exists in general. Indeed, we will prove a similar isomorphism for the cominuscule homogeneous spaces of exceptional type (introduced in Section 1.7) in Chapter 3. We note that we are able to choose either X or $\mathbb{X}^\vee$, since they determine the same Weyl groups W and $W_P = \langle s_i \mid i \neq k \rangle \subset W$. This follows from the fact that the Weyl groups only depend on the Coxeter diagram underlying the Dynkin diagrams of G and $G^\vee$ and the vertex k, and these do not change under Langlands duality (see the discussion at the start of Section 1.2). Thus, X and $\mathbb{X}^\vee$ give rise to the same element w^P and thus the associated quivers are actually the same.

This quiver has as vertices the factors of the reduced expression of w^P, so every subexpression of w' inside the reduced expression will become a subset of vertices of this quiver.

Definition 2.5.5. We denote the set of subsets of vertices of the quiver $Q_{\mathbb{X}^\vee}$ that are associated to reduced subexpressions of w' inside the reduced expression of w^P fixed in Section 1.2 by $\mathcal{S}$. In other words,

$$\mathcal{S} = \Big\{ \big((\beta_1, j_1), \dots, (\beta_{\ell'}, j_{\ell'})\big) \;\Big|\; s_{\beta_1} \cdots s_{\beta_{\ell'}} = w' \Big\},$$

where $\ell' = \ell(w')$. Note that we implicitly order the vertices $(\beta_1, j_1), \dots, (\beta_{\ell'}, j_{\ell'})$ such that $\mathscr{J}(\beta_1, j_1) < \dots < \mathscr{J}(\beta_{\ell'}, j_{\ell'})$, but we will still refer to the elements of $\mathcal{S}$ as *subsets*.[2]

Remark 2.5.6. Recall from Definition 2.2.5 that we defined $\mathcal{I}$ as the set of sequences of subindices $(i_1, \dots, i_{\ell'})$ such that $s_{r_{i_1}} \cdots s_{r_{i_{\ell'}}} = w'$ is a reduced subexpression of w' in the fixed reduced expression $w^P = s_{r_1} \cdots s_{r_\ell}$. Note that $\mathscr{J}$ gives rise to a bijection between $\mathcal{S}$ and $\mathcal{I}$, which we will also denote by $\mathscr{J}$:

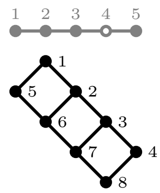
$$\mathscr{J} : \big((\beta_1, j_1), \dots, (\beta_{\ell'}, j_{\ell'})\big) \mapsto \big(\mathscr{J}(\beta_1, j_1), \dots, \mathscr{J}(\beta_{\ell'}, j_{\ell'})\big), \tag{2.5.1}$$

This bijection gives the translation between the subsets of vertices of the quiver $Q_{\mathbb{X}^\vee}$ (in $\mathcal{S}$) and their associated reduced subexpressions (in $\mathcal{I}$).

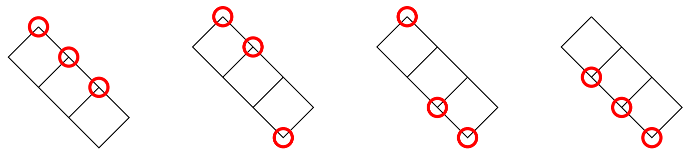
Example 2.5.7. To illustrate the quiver and the subsets associated to the subexpressions, consider the example of the Grassmannian $X = \mathrm{Gr}(4, 6) = \mathrm{SL}_6/P_4$ of type A_5. Fixing for w^P the reduced expression $w^P = (s_2 s_3 s_4 s_5)(s_1 s_2 s_3 s_4)$, we find that $Q_{\mathbb{X}^\vee}$ is of the form below. Here all edges are arrows are pointing downwards and the ith column of vertices contains the vertices (α_i, j) with j increasing from 1 to $m^P(\alpha_i)$ from top to bottom. Above the quiver, we have drawn the labeled Coxeter diagram in such a way that the ith vertex of the diagram is above the ith column of

[2]This is to distinguish them from the *sequences* of subindices that are the elements of $\mathcal{I}$ from Definition 2.2.5.

the quiver. The vertex labeled 4 is marked in the Coxeter diagram to signify that we are considering $X = \mathrm{SL}_6/P_4$, i.e. to signify that $k = 4$ in $X = G/P_k$. For each vertex (β, j), we also give the value of $\mathscr{J}(\beta, j)$.



The reduced expression $w' = s_2 s_3 s_4$ is unique and appears four times as a subexpression of w^P: $(\underline{\boldsymbol{s_2 s_3 s_4}} s_5)(s_1 s_2 s_3 s_4)$, $(\underline{\boldsymbol{s_2 s_3}} s_4 s_5)(s_1 s_2 s_3 \underline{\boldsymbol{s_4}})$, $(\underline{\boldsymbol{s_2}} s_3 s_4 s_5)(s_1 s_2 \underline{\boldsymbol{s_3 s_4}})$, and $(s_2 s_3 s_4 s_5)(s_1 \underline{\boldsymbol{s_2 s_3 s_4}})$. These correspond in $Q_{\mathbb{X}^\vee}$ to marking the vertices



respectively, where we suppressed the vertices. $\Diamond$

It turns out that we can use the quiver $Q_{\mathbb{X}^\vee}$ to find all the reduced subexpressions of w' in w^P using two operations that are straightforward when considered as operations on subsets of the quiver. First, we need the following observation:

Remark 2.5.8. From Definition 2.5.3, it is clear that we can only have an arrow $(i, j) \to (i', j')$ if $\mathscr{J}(i, j) < \mathscr{J}(i', j')$, so that any path $(i, j) \to (i_1, j_1) \to \ldots \to (i', j')$ between the two vertices correspond to simple reflections $s_{r_{\mathscr{J}(i,j)}}, s_{r_{\mathscr{J}(i_1,j_1)}}, \ldots, s_{r_{\mathscr{J}(i',j')}}$ that appear in that order in the reduced expression of w^P.

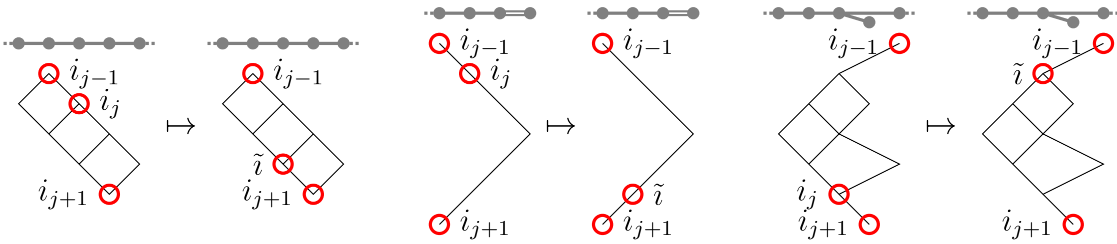
The two operations are given as follows:

Lemma 2.5.9. Suppose $(i_1, \ldots, i_{\ell'}) \in \mathcal{I}$, i.e. suppose it is a sequence of subindices such that $s_{r_{i_1}} \cdots s_{r_{i_{\ell'}}} = w'$ is a reduced subexpression inside the reduced expression $w^P = s_{r_1} \cdots s_{r_\ell}$. Denote by $S \in \mathcal{S}$ the corresponding subset obtained by (the inverse of) the bijection in (2.5.1).

(i) For every $\tilde{\imath}$ such that $r_{\tilde{\imath}} = r_{i_j}$ and $i_{j-1} < \tilde{\imath} < i_{j+1}$, the sequence of subindices obtained by replacing i_j with $\tilde{\imath}$, i.e. $(i_1, \dots, i_{j-1}, \tilde{\imath}, i_{j+1}, \dots, i_{\ell'})$, also gives a reduced subexpression of w' in w^P.

In terms of subsets in $\mathcal{S}$, this says that we can replace a vertex $(\beta, j) \in S$ with a vertex $(\beta, \tilde{\jmath})$ if every (β', j') with $\mathscr{J}(\beta, j) < \mathscr{J}(\beta', j') < \mathscr{J}(\beta, \tilde{\jmath})$ is *not* an element of S when $j < \tilde{\jmath}$ (or $\mathscr{J}(\beta, j) > \mathscr{J}(\beta', j') > \mathscr{J}(\beta, \tilde{\jmath})$ when $j > \tilde{\jmath}$).

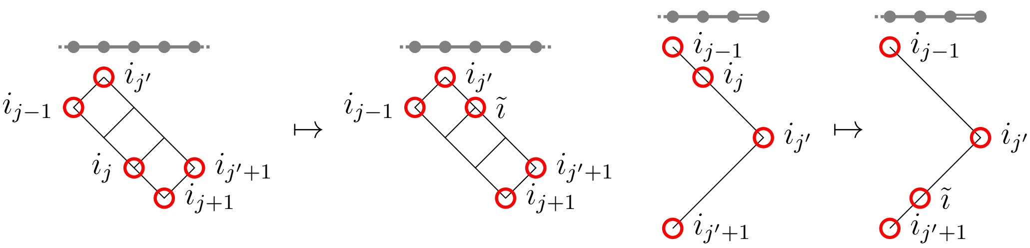
Examples of this operation are (using the conventions of Example 2.5.7):



(ii) For every $\tilde{\imath} < i_j$ with $r_{\tilde{\imath}} = r_{i_j}$ such that there exists a j' with $i_{j'} < \tilde{\imath} < i_{j'+1} \leq i_j$ and $(s_{r_{i_j}} s_{r_{i_{\tilde{\jmath}}}})^2 = 1$ for all $\tilde{\jmath} \in \{j'+1, \dots, j-1\}$, we have that the sequence of subindices $(i_1, \dots, i_{j'}, \tilde{\imath}, i_{j'+1}, \dots, i_{j-1}, i_{j+1}, \dots, i_{\ell'})$ is an element of $\mathcal{I}$ as well.

Similarly, for every $\tilde{\imath} > i_j$ with $r_{\tilde{\imath}} = r_{i_j}$ such that there exists a j' with $i_j \leq i_{j'} < \tilde{\imath} < i_{j'+1}$ and $(s_{r_{i_j}} s_{r_{i_{\tilde{\jmath}}}})^2 = 1$ for all $\tilde{\jmath} \in \{j+1, \dots, j'\}$, the sequence of subindices $(i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_{j'}, \tilde{\imath}, i_{j'+1}, \dots, i_{\ell'})$ is also an element of $\mathcal{I}$.

In terms of subsets in $\mathcal{S}$, this says that we can replace a vertex $(\beta, j) \in S$ with a vertex $(\beta, \tilde{\jmath})$ if for every path $(\beta, j) \to (\beta_1, j_1) \to \dots \to (\beta, \tilde{\jmath})$ when $j < \tilde{\jmath}$ (or $(\beta, \tilde{\jmath}) \to (\beta_1, j_1) \to \dots \to (\beta, j)$ when $j > \tilde{\jmath}$ respectively) there is no vertex $(\beta', j') \in S$ contained in the path such that $(s_\beta s_{\beta'})^2 \neq 1$. Examples of this operation are:



Proof. **(i)** By assumption, $(i_1, \dots, i_{j-1}, \tilde{\imath}, i_{j+1}, \dots, i_{\ell'})$ is an increasing sequence of subindices such that $s_{r_{i_1}} \cdots s_{r_{i_{j-1}}} s_{r_{\tilde{\imath}}} s_{r_{i_{j+1}}} \cdots s_{r_{i_{\ell'}}} = s_{r_{i_1}} \cdots s_{r_{i_{j-1}}} s_{r_{i_j}} s_{r_{i_{j+1}}} \cdots s_{r_{i_{\ell'}}} = w'$, and therefore a reduced expression of w' in w^P as well.

(ii) If $\tilde{\imath} < i_j$, we can commute the factor $s_{r_{i_j}}$ to the left in the given reduced subexpression to obtain the reduced expression $s_{r_{i_1}} \cdots s_{r_{i_{j'}}} s_{r_{i_j}} s_{r_{i_{j'+1}}} \cdots s_{r_{i_{j-1}}} s_{r_{i_{j+1}}} \cdots s_{r_{i_{\ell'}}}$ for w'. Moreover, this is a subexpression of w' in w^P since there is a $\tilde{\imath}$ with $i_{j'} < \tilde{\imath} < i_{j'+1}$ and $r_{\tilde{\imath}} = r_{i_j}$, so that $(i_1, \dots, i_{j'}, \tilde{\imath}, i_{j'+1}, \dots, i_{j-1}, i_j, \dots, i_{\ell'})$ is an increasing sequence of subindices.

The case $\tilde{\imath} > i_j$ is analogous, except that we commute the factor $s_{r_{i_j}}$ to the right. □

Remark 2.5.10. Note that the operation **(i)** is actually a special case of the operation **(ii)** where no commutation takes place.

It turns out that these operations suffice to obtain *all* the reduced subexpressions of w' in w^P. To show this, we introduce a total order on the set of reduced subexpressions by taking the lexicographical order $\prec$ on $\mathcal{I}$. In other words, we have $(i_1, \dots, i_{\ell'}) \prec (i'_1, \dots, i'_{\ell'})$ if and only if there exists a j such that $i_j < i'_j$ and $i_{j'} = i'_{j'}$ for $j' \in \{1, \dots, j-1\}$. Let $(i^*_1, \dots, i^*_{\ell'})$ be the minimal sequence associated to a reduced subexpression of w' in w^P.

Proposition 2.5.11. Every reduced subexpression of w' in w^P can be obtained using the operations of Lemma 2.5.9 on the minimal reduced subexpression $w' = s_{r_{i^*_1}} \cdots s_{r_{i^*_{\ell'}}}$.

Proof. We will show that every non-minimal reduced subexpression can be made smaller using one of the operations in Lemma 2.5.9. This gives a sequence of operations from any given reduced subexpression to the minimal one. Since it is evident that each operation is invertible, we obtain the statement.

Therefore, let $(i_1, \dots, i_{\ell'}) \in \mathcal{I}$ be non-minimal with respect to the lexicographical order. By definition, there exists a j such that $i_j > i^*_j$ and $i_{j'} = i^*_{j'}$ for all $j' \in \{1, \dots, j-1\}$. We distinguish two cases: $r_{i_j} = r_{i^*_j}$ and $r_{i_j} \neq r_{i^*_j}$.

In the case $r_{i_j} = r_{i^*_j}$, we can apply operation **(i)** to directly obtain the reduced subexpression $(i_1, \dots, i_{j-1}, i^*_j, i_{j+1}, \dots, i_{\ell'}) \prec (i_1, \dots, i_{j-1}, i_j, i_{j+1}, \dots, i_{\ell'})$.

Now, consider the case $r_{i_j} \neq r_{i^*_j}$. Suppose that the simple reflection $s_{r_{i^*_j}}$ occurs for the Nth time in the minimal reduced subexpression of w' in w^P, then there must exist some $j' > j$ such that $s_{r_{i_{j'}}}$ is the Nth occurrence of the same simple reflection in the subexpression $(i_1, \dots i_{\ell'})$. Indeed, w' is fully commutative so that each simple reflection appears the same number of times in each reduced expression. (Note that we have $j' > j$ since $s_{r_{i^*_{\tilde{\jmath}}}} = s_{r_{i_{\tilde{\jmath}}}}$ for $\tilde{\jmath} \in \{1, \dots, j-1\}$ and $s_{r_{i_j}} \neq s_{r_{i^*_j}}$.) We also know that the simple reflections $s_{r_{i_{\tilde{\jmath}}}}$ commute with $s_{r_{i_{j'}}}$ for $\tilde{\jmath} \in \{j, \dots, j'-1\}$ because of full commutativity of w', since these simple reflections have commuted with $s_{r_{i^*_j}} = s_{r_{i_{j'}}}$ going from $(i^*_1, \dots, i^*_{\ell'})$ to $(i_1, \dots, i_{\ell'})$. Thus, we can apply operation **(ii)** to obtain $(i_1, \dots, i_{j-1}, i^*_j, i_j, \dots, i_{j'-1}, i_{j'+1}, \dots, i_{\ell'}) \prec (i_1, \dots, i_{\ell'})$ as a reduced subexpression. □

Combining this with Theorem 2.2.7 and Remark 2.5.6, we conclude the following:

Corollary 2.5.12. Let $X = G/P$ be a *cominuscule* complete homogeneous space with G a *simply-connected*, simple, complex algebraic group and $P = P_k$ a (maximal) parabolic subgroup. The restriction $\mathcal{W}_{\mathcal{Z}^\circ_P}$ of $\mathcal{W}_{\mathcal{Z}^\vee_P}$ to $\mathcal{Z}^\circ_P$ has the following Laurent polynomial expression:

$$\mathcal{W}_{\mathcal{Z}^\circ_P}(z) = \sum_{i=1}^{\ell} a_i + q\frac{\sum_{S\in\mathcal{S}} \prod_{(\beta,j)\in S} a_{\mathscr{J}(\beta,j)}}{\prod_{i=1}^{\ell} a_i}. \tag{2.5.2}$$

Here $z \in \mathcal{Z}^\circ_P$ is uniquely decomposed as $z = u_+ t \bar{w}_P u_-$ with $u_- = y^\vee_{r_\ell}(a_\ell)\cdots y^\vee_{r_1}(a_1) \in U^\circ_-$ as in Corollary 2.2.4. Also, $q \in \mathbb{C}^*$ is given by $q = \alpha^\vee_k(t)$ (with $t \in T^P$), and the subindex relabeling $\mathscr{J}$ is defined in Definition 2.5.3. The set $\mathcal{S}$ is defined in Definition 2.5.5 and all its elements are obtained using the operations of Lemma 2.5.9.

2.6 Laurent polynomial potentials for all cominuscule homogeneous spaces

Theorem 2.2.7 allows us to calculate Laurent polynomial potentials for the cominuscule homogeneous spaces listed in Table 1.5.1, and Corollary 2.5.12 gives us a tractable way to find all the terms. In this section, we will give reduced expressions for w^P and w', the quivers $Q_{\mathbb{X}^\vee}$ for all the cominuscule homogeneous varieties and we will work out the sets $\mathcal{S}$ and the resulting Laurent polynomial expressions for representative examples.

The Laurent polynomials we obtain for quadrics (type B_n and D_n) and Lagrangian Grassmannians (type C_n) are identical to those found in [PRW16] (Propositions 2.2 and 3.11) and [PR13] (Proposition A.1). This is to be expected, as Theorem 2.2.7 is obtained by a generalization of the methods used there.

However, to the best of our knowledge, the Laurent polynomials for orthogonal Grassmannians (type D_n) for general n, the Cayley plane (type E_6) and the Freudenthal variety (type E_7) have not yet been given. Moreover, all the potentials have a uniform structure resembling Givental's Laurent polynomial potential for projective complete intersections [Giv96], namely they are the sum of the toric coordinates plus a quantum term consisting of a homogeneous polynomial divided by the product of all the toric coordinates.

2.6.1 The Grassmannian

Let $X = \mathrm{Gr}(k,n) = \mathrm{SL}_n/P_k$ be the Grassmannian of k-planes in $\mathbb{C}^n$, considered as a homogeneous space for the special linear group, the simply-connected complex Lie group of type A_{n-1}. Note that the parabolic subgroup is given by

$$P_k = \begin{bmatrix} \mathrm{GL}_k & \mathrm{Mat}_{k\times(n-k)} \\ 0 & \mathrm{GL}_{n-k} \end{bmatrix} \cap \mathrm{SL}_n.$$

We make two assumptions on k: Firstly, we assume that $k \notin \{1, n-1\}$: for $k=1$ and $k=n-1$ we find $w'=1$. Secondly, we assume without loss of generality that $k \geq n-k$: for the remaining cases apply the Dynkin diagram bijection $i \mapsto n-i$. The longest Weyl group element has minimal coset representative

$$w^P = (s_{n-k}s_{n+1-k}\cdots s_{n-1})(s_{n-1-k}s_{n-k}\cdots s_{n-2})\cdots(s_1 s_2\cdots s_k),$$

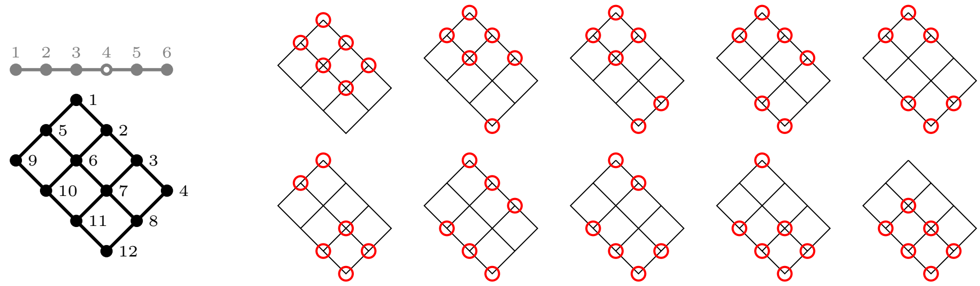
having $n-k$ products in parentheses each with k factors. On the other hand, we find for w' the reduced expression

$$w' = (s_{n-k} \cdots s_{n-2})(s_{n-k-1} \cdots s_{n-3}) \cdots (s_2 \cdots s_k),$$

having $n-k-1$ products in parentheses each with $k-1$ factors.

The quiver $Q_{\mathbb{X}^\vee}$ can be written as a $(n-k-1)\times(k-1)$-rectangle and the reduced subexpression for w' that is minimal in the lexicographical order is the $(n-k-2)\times(k-2)$-rectangle obtained by removing the bottom row and rightmost column.

Example 2.6.1. Consider $\mathrm{Gr}(4,7)$, which is homogeneous for SL_7 of type A_6. We find that $w^P = (s_3s_4s_5s_6)(s_2s_3s_4s_5)(s_1s_2s_3s_4)$ and $w' = (s_3s_4s_5)(s_2s_3s_4)$. The quiver is of the following form, using the conventions of Example 2.5.7:



The ten subsets associated to reduced subexpressions of w' in w^P are drawn to the right. So, taking $z = u_+ t\bar{w}_P u_-$ with $(u_-)^{-1} = y_3^\vee(-a_1)y_4^\vee(-a_2)\cdots y_4^\vee(-a_{10})$ and $q = \alpha_4^\vee(t)$, we find

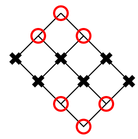
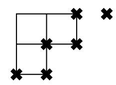
$$\mathcal{W}_{\mathcal{Z}_P^\circ}(z) = \sum_{i=1}^{12} a_i + q\frac{P(a_i)}{a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}},$$

where

$$\begin{aligned} P(a_i) = {} & a_1a_2a_3a_5a_6a_7 + a_1a_2a_3a_5a_6a_{12} + a_1a_2a_5a_6a_8a_{12} + a_1a_2a_3a_5a_{11}a_{12} \\ & + a_1a_2a_5a_8a_{11}a_{12} + a_1a_5a_7a_8a_{11}a_{12} + a_1a_2a_3a_{10}a_{11}a_{12} + a_1a_2a_8a_{10}a_{11}a_{12} \\ & + a_1a_7a_8a_{10}a_{11}a_{12} + a_6a_7a_8a_{10}a_{11}a_{12}, \end{aligned}$$

with the summands written in order of the subsets.

Note that there is a clear bijection between the subsets of the quiver $Q_{\mathbb{X}^\vee}$ and Young diagrams that fit inside a 2×3-rectangle: the unmarked vertices correspond to the contour of the Young diagram. For example:

$a_1a_2a_5a_8a_{11}a_{12} \quad \leftrightarrow$  $\leftrightarrow$ 

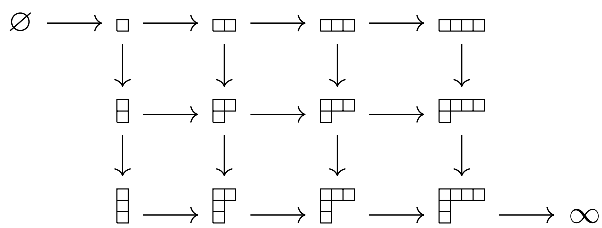
This bijection works in general, so we obtain an alternative description of the quantum term for a general Grassmannian $\mathrm{Gr}(k, n)$ that sums over the Young diagrams that fit inside a $(n - k - 1) \times (k - 1)$-rectangle. $\Diamond$

Remark 2.6.2. Other Laurent polynomial Landau-Ginzburg models have already been given for Grassmannians. Particularly relevant is the potential $L : (\mathbb{C}^*)^{k(n-k)} \times \mathbb{C}^* \to \mathbb{C}$ given in [EHX97], equation (B.25) (see also [BCFKvS98], Conjecture 4.2.2, as well as [MR20], Section 6.3), which is described as follows.

For $i \in \{1, \ldots, n - k\}$ and $j \in \{1, \ldots, k\}$, write $[i, j]$ for the Young diagram corresponding to the partition $(j, 1, 1, \ldots, 1)$ of length i; for example, $[2, 3]$ corresponds to the diagram . Denote the set of these diagrams by Λ_s. Now, denote the coordinates of $(\mathbb{C}^*)^{k(n-k)}$ by $z_{[i,j]}$ for $[i, j] \in \Lambda_s$, denote the coordinate of the remaining factor $\mathbb{C}^*$ by $z_\infty = q$ and finally set $z_\varnothing = 1$. Consider the quiver with as vertices $\Lambda_s^* = \Lambda_s \cup \{\varnothing, \infty\}$ and as arrows

$$\{[i, j] \to [i, j + 1],\ [i, j] \to [i + 1, j]\} \ \cup\ \{\varnothing \to [1, 1]\} \ \cup\ \{[n - k, k] \to \infty\}.$$

For example, for $G(4, 7)$ we obtain:



Denoting $[1, 0] = \varnothing$ and $[n-k, k+1] = \infty$ (so that $z_{[1,0]} = z_{\varnothing} = 1$ and $z_{[n-k,k+1]} = z_{\infty} = q$), consider for $[i, j] \in \Lambda_s^*$ the following quotients:

$$T_{[i,j]} = \frac{z_{[i+1,j]} + z_{[i,j+1]}}{z_{[i,j]}}, \tag{2.6.1}$$

where $z_{[i,j]} = 0$ for $[i, j] \notin \Lambda_s^*$. In other words, for each vertex $[i, j] \in \Lambda_s^*$, $T_{[i,j]}$ consists of the sum of the coordinates at the ends of the outgoing arrows divided by the coordinate of the vertex. For example,

$$T_{\ydiagram{1}} = \frac{z_{\ydiagram{2}} + z_{\ydiagram{1,1}}}{z_{\ydiagram{1}}}, \quad T_{[n-k,1]} = \frac{z_{[n-k,2]}}{z_{[n-k,1]}}, \quad T_{[1,k]} = \frac{z_{[2,k]}}{z_{[1,k]}}, \quad T_{\varnothing} = T_{[1,0]} = z_{\ydiagram{1}},$$
$$T_{[n-k,k]} = \frac{q}{z_{[n-k,k]}} \quad \text{and} \quad T_{\infty} = T_{[n-k,k+1]} = 0.$$

The Laurent polynomial potential of [EHX97] is the following:

$$L(z_{[i,j]}) = \sum_{[i,j] \in \Lambda_s^*} T_{[i,j]}, \tag{2.6.2}$$

where the quantum term is $T_{[n-k,k]} = \frac{q}{z_{[n-k,k]}}$. In the example of $\mathrm{Gr}(4, 7)$, this becomes

$$L(z_{[i,j]}) = z_{\ydiagram{1}} + \frac{z_{\ydiagram{2}} + z_{\ydiagram{1,1}}}{z_{\ydiagram{1}}} + \ldots + \frac{z_{\ydiagram{4,1,1}}}{z_{\ydiagram{3,1,1}}} + \frac{z_{\ydiagram{4,1,1}}}{z_{\ydiagram{4,1}}} + \frac{q}{z_{\ydiagram{4,1,1}}},$$

which has a total of thirteen terms (twelve plus a quantum term).

The potential L of (2.6.2) is shown in [MR20], Theorem 4.6, to be a local Laurent polynomial expression for the Landau-Ginzburg model used there. That model is also shown in Proposition 6.7 of [MR20] to be isomorphic to Rietsch's Lie-theoretic Landau-Ginzburg model given in [Rie08], see also Theorem 1.4.2 here. By construction, the Laurent polynomial potential $\mathcal{W}_{\mathcal{Z}_P^\circ}$ of Theorem 2.2.7 and Corollary 2.5.12 here is a local expression for Rietsch's Lie-theoretic Landau-Ginzburg model. Thus, both $\mathcal{W}_{\mathcal{Z}_P^\circ}$ and L are local Laurent polynomial expressions for the same model. However, it is clear that $\mathcal{W}_{\mathcal{Z}_P^\circ}$ and L are not isomorphic: the quantum term of L is a Laurent monomial, namely $T_{[n-k,k]} = \frac{q}{z_{[n-k,k]}}$, whereas the quantum term of $\mathcal{W}_{\mathcal{Z}_P^\circ}$ is not.

On the other hand, it is straightforward to find a birational map Φ such that $\Phi^* L = \mathcal{W}_{\mathcal{Z}_P^\circ}$. First, consider the following bijection: for $[i,j] \in \Lambda_s \setminus \{[n-k,k]\}$, let $\phi([i,j]) = i \cdot k - j + 1$, and let $\phi(\varnothing) = (n-k-1)k + 1$. Clearly, ϕ is a bijection $\Lambda_s^* \setminus \{[n-k,k], \infty\} \to \{1, \dots, k(n-k)\}$. Define $\Phi : (\mathbb{C}^*)^{k(n-k)} \to (\mathbb{C}^*)^{k(n-k)}$ to be the birational map such that $\Phi^*(a_i) = T_{\phi^{-1}(i)}$. By definition, we find (recalling $T_\infty = 0$)

$$\begin{aligned}
\Phi^* \mathcal{W}_{\mathcal{Z}_P^\circ} &= \Phi^* \left(\sum_{i=1}^{k(n-k)} a_i + q \frac{P(a_j)}{a_1 \cdots a_{k(n-k)}} \right) \\
&= \sum_{[i,j] \in \Lambda_s^* \setminus \{[n-k,k]\}} T_{[i,j]} + q \frac{P(T_{[i',j']})}{T_\varnothing T_{[1,1]} T_{[1,2]} \cdots T_{[n-k,k-1]}},
\end{aligned}$$

where P is the homogeneous polynomial in the numerator of the quantum term of Corollary 2.5.12. Thus, what remains to be shown is that the quantum term simplifies to $T_{[n-k,k]} = \frac{q}{z_{[n-k,k]}}$, which is a straightforward computation for any given k and n.

For example, in the case $\mathrm{Gr}(4,7)$ we can simplify the following products:

$$T_\varnothing T_{\ydiagram{1}} = z_{\ydiagram{1}} \frac{z_{\ydiagram{2}} + z_{\ydiagram{1,1}}}{z_{\ydiagram{1}}} = z_{\ydiagram{2}} + z_{\ydiagram{1,1}}, \quad T_{\ydiagram{4}} T_{\ydiagram{4,1}} = \frac{z_{\ydiagram{4,1}}}{z_{\ydiagram{4}}} \frac{z_{\ydiagram{4,1,1}}}{z_{\ydiagram{4,1}}} = \frac{z_{\ydiagram{4,1,1}}}{z_{\ydiagram{4}}}, \quad T_{\ydiagram{1,1,1}} T_{\ydiagram{2,1,1}} T_{\ydiagram{3,1,1}} = \frac{z_{\ydiagram{4,1,1}}}{z_{\ydiagram{1,1,1}}}.$$

Thus, the denominator becomes

$$\begin{aligned}
&(T_\varnothing T_{\ydiagram{1}})(T_{\ydiagram{2}} T_{\ydiagram{3}} T_{\ydiagram{1,1}} T_{\ydiagram{2,1}} T_{\ydiagram{3,1}})(T_{\ydiagram{4}} T_{\ydiagram{4,1}})(T_{\ydiagram{1,1,1}} T_{\ydiagram{2,1,1}} T_{\ydiagram{3,1,1}}) \\
&\quad = (z_{\ydiagram{2}} + z_{\ydiagram{1,1}}) \left(\frac{z_{\ydiagram{3}} + z_{\ydiagram{2,1}}}{z_{\ydiagram{2}}} \frac{z_{\ydiagram{4}} + z_{\ydiagram{3,1}}}{z_{\ydiagram{3}}} \frac{z_{\ydiagram{2,1}} + z_{\ydiagram{1,1,1}}}{z_{\ydiagram{1,1}}} \frac{z_{\ydiagram{3,1}} + z_{\ydiagram{2,1,1}}}{z_{\ydiagram{2,1}}} \frac{z_{\ydiagram{4,1}} + z_{\ydiagram{3,1,1}}}{z_{\ydiagram{3,1}}} \right) \left(\frac{z_{\ydiagram{4,1,1}}}{z_{\ydiagram{4}}} \right) \left(\frac{z_{\ydiagram{4,1,1}}}{z_{\ydiagram{1,1,1}}} \right).
\end{aligned}$$

It is more work to simplify $P(T_{[i',j']})$, but in the end we find

$$P(T_{[i',j']}) = (z_{\ydiagram{2}} + z_{\ydiagram{1,1}}) \left(\frac{z_{\ydiagram{3}} + z_{\ydiagram{2,1}}}{z_{\ydiagram{2}}} \frac{z_{\ydiagram{4}} + z_{\ydiagram{3,1}}}{z_{\ydiagram{3}}} \frac{z_{\ydiagram{2,1}} + z_{\ydiagram{1,1,1}}}{z_{\ydiagram{1,1}}} \frac{z_{\ydiagram{3,1}} + z_{\ydiagram{2,1,1}}}{z_{\ydiagram{2,1}}} \frac{z_{\ydiagram{4,1}} + z_{\ydiagram{3,1,1}}}{z_{\ydiagram{3,1}}} \right) \left(\frac{z_{\ydiagram{4,1,1}}}{z_{\ydiagram{4}}} \right) \left(\frac{1}{z_{\ydiagram{1,1,1}}} \right).$$

So the quotient is indeed $T_{[n-k,k]} = \frac{q}{z_{[n-k,k]}}$.

2.6.2 The quadric

Let $X = Q_d$ be the d-dimensional quadric. Note that both odd- and even-dimensional quadrics are homogeneous for Spin_{d+2}, and that the parabolic subgroup is associated to the first vertex of the Dynkin diagram. Note, however, that Spin_{d+2} is of a different type depending on whether $d+2 = 2n+1$ is odd (type B_n) or $d+2 = 2n$ is even (type D_n), but the resulting homogeneous spaces are nonetheless similar enough to be considered at the same time. We find for w^P the reduced expressions:

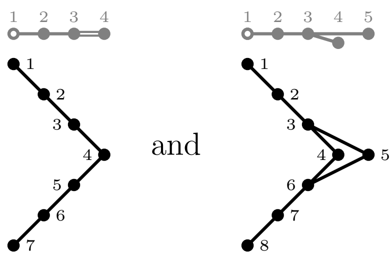
$$w^P = \begin{cases} s_1 s_2 \cdots s_{n-1}(s_n)s_{n-1}s_{n-2}\cdots s_1, & \text{for } d = 2n-1, \\ s_1 s_2 \cdots s_{n-2}(s_{n-1}s_n)s_{n-2}s_{n-3}\cdots s_1, & \text{for } d = 2n-2. \end{cases}$$

For both odd and even quadrics we find $w' = s_1$ and in both cases the simple reflection s_1 only appears as the first and the last factor, so it is easy enough to find the Laurent polynomial potential without using Corollary 2.5.12. We find the same potential in both cases, namely:

$$\mathcal{W}_{\mathcal{Z}_P^\circ}(z) = \sum_{i=1}^{d} a_i + q\frac{a_1 + a_d}{\prod_{i=1}^{d} a_i},$$

where d is the dimension of the quadric, and we decomposed $z = u_+ t\bar{w}_P u_-$ with $u_- = y_1^\vee(-a_1)y_2^\vee(-a_2)\cdots y_1^\vee(-a_d)$ and $q = \alpha_1^\vee(t)$. Note that this Laurent polynomial expression is indeed identical to the ones obtained in [PRW16], Propositions 2.2 and 3.11. For completeness' sake, let us consider two examples of quadrics and draw the associated quivers:

Example 2.6.3. Consider the odd quadric Q_7 of type B_4 and the even quadric Q_8 of type D_5. These have for w^P the reduced expressions $s_1s_2s_3(s_4)s_3s_2s_1$ and $s_1s_2s_3(s_4s_5)s_3s_2s_1$ respectively. Thus, we find the quivers



Also in the quivers it is clear that there are only two subsets associated to $w' = s_1$ each containing one vertex: the top and the bottom one. $\Diamond$

Remark 2.6.4. Note that we have chosen to draw the Coxeter diagram of type D_5 as  instead of as  or . This is to avoid confusing the vertex $(4, 1)$ with either $(5, 1)$ or $(3, 1)$ and $(3, 2)$, respectively. We will continue to do this with the Coxeter diagrams of type D_n and E_n. This causes the quivers for the orthogonal Grassmannians (Subsection 2.6.4), the Cayley plane (Subsection 2.6.5) and the Freudenthal variety (Subsection 2.6.6) to lose their symmetry, but this is purely aesthetic.

2.6.3 The Lagrangian Grassmannian

Let $X = \mathrm{LG}(n, 2n) = \mathrm{Sp}_{2n}/P_n$ be the Grassmannian of Lagrangian subspaces in $\mathbb{C}^{2n}$ equipped with the standard symplectic form, considered as a homogeneous space for the symplectic group of type C_n. Note that the parabolic subgroup is given by

$$P_n = \begin{bmatrix} \mathrm{GL}_n & \mathrm{Mat}_{n\times n} \\ 0 & \mathrm{GL}_n \end{bmatrix} \cap \mathrm{Sp}_{2n}.$$

The coset representative for the longest Weyl group element has reduced expression

$$w^P = (s_n)(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n)\cdots(s_1 s_2 \cdots s_n),$$

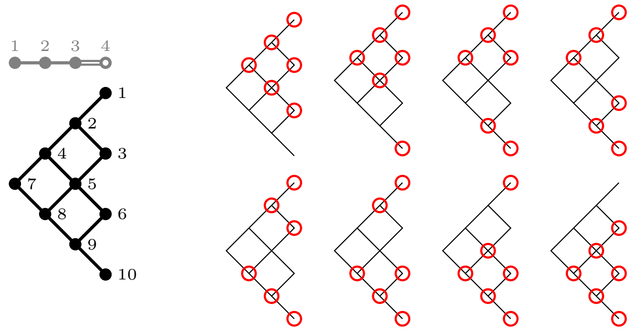
Similarly, we find for w'

$$w' = (s_n)(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n)\cdots(s_2 s_3 \cdots s_n).$$

The quiver $Q_{\mathbb{X}^\vee}$ can be written as the triangle that is the left half of an $n \times n$-square, and the reduced subexpression for w' that is minimal with respect to the lexicographical order is the triangle that is the left half of the $(n-1) \times (n-1)$-square obtained after removing the bottom row and the rightmost column. Note that the

Laurent polynomial expression of Theorem 2.2.7 coincides with the description given in Proposition A.1 of [PR13].

Example 2.6.5. Consider $\mathrm{LG}(4,8)$ which is homogeneous for Sp_8 of type C_4. We find that $w^P = (s_4)(s_3s_4)(s_2s_3s_4)(s_1s_2s_3s_4)$, $w' = (s_4)(s_3s_4)(s_2s_3s_4)$ and the following quiver:



To the right we have drawn the eight subsets corresponding to reduced subexpressions of w' inside the reduced expression for w^P. So, considering the decomposition $z = u_+t\bar{w}_Pu_-$ with $(u_-)^{-1} = y_4^\vee(-a_1)y_3^\vee(-a_2)\cdots y_4^\vee(-a_{10})$ and $q = \alpha_4^\vee(t)$, we find

$$\mathcal{W}_{\mathcal{Z}_P^\circ}(z) = \sum_{i=1}^{10} a_i + q\frac{P(a_i)}{a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}},$$

where

$$\begin{aligned} P(a_i) = {} & a_1a_2a_3a_4a_5a_6 + a_1a_2a_3a_4a_5a_{10} + a_1a_2a_3a_4a_9a_{10} + a_1a_2a_4a_6a_9a_{10} \\ & + a_1a_2a_3a_8a_9a_{10} + a_1a_2a_6a_8a_9a_{10} + a_1a_5a_6a_8a_9a_{10} + a_3a_5a_6a_8a_9a_{10} \end{aligned}$$

are the eight reduced expression for w' obtained from the diagrams above. $\diamondsuit$

2.6.4 The orthogonal Grassmannian

Let $X = \mathrm{OG}(n,2n) = \mathrm{Spin}_{2n}/P_n$ be the Grassmannian of maximal isotropic subspaces in $\mathbb{C}^{2n}$ equipped with the standard quadratic form, considered as a homogeneous space for the spin group of type D_n. We fix for the minimal coset representative w^P of the longest element the reduced expression

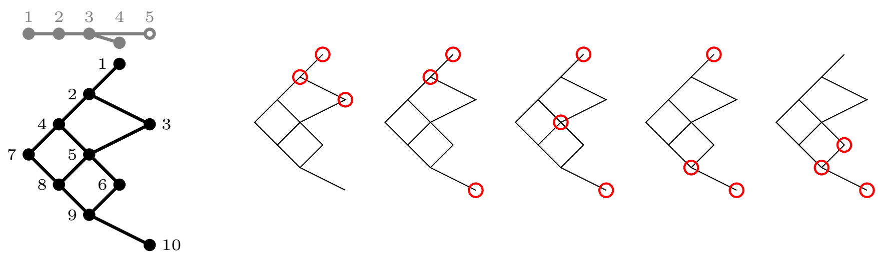
$$w^P = \begin{cases} s_{n-1}(s_{n-2})s_n(s_{n-3}s_{n-2})s_{n-1}\cdots(s_1s_2\cdots s_{n-2})s_n, & \text{for } n \text{ odd,} \\ s_n(s_{n-2})s_{n-1}(s_{n-3}s_{n-2})s_n\cdots(s_1s_2\cdots s_{n-2})s_n, & \text{for } n \text{ even.} \end{cases}$$

Now, for w' we find the reduced expression

$$w' = \begin{cases} s_{n-1}(s_{n-2})s_n(s_{n-3}s_{n-2})s_{n-1}\cdots(s_3s_4\cdots s_{n-2})s_n, & \text{for } n \text{ odd,} \\ s_n(s_{n-2})s_{n-1}(s_{n-3}s_{n-2})s_n\cdots(s_3s_4\cdots s_{n-2})s_n, & \text{for } n \text{ even.} \end{cases}$$

Note that these expressions are very similar to those of Lagrangian Grassmannians except that the expressions for the orthogonal Grassmannians alternate between s_n and s_{n-1}. The quiver $Q_{\mathbb{X}^\vee}$ is therefore the left half of an $(n-1)\times(n-1)$-square with the longest column split over two columns. The subset that corresponds to the reduced subexpression for w' that is minimal with respect to the lexicographical order is in this case the triangle that is the left half of the $(n-3)\times(n-3)$-square obtained after removing the two bottom rows and the two rightmost columns.

Example 2.6.6. Consider $\mathrm{OG}(5,10)$ which is homogeneous for Spin_{10} of type D_5. We find that $w^P = s_4(s_3)s_5(s_2s_3)s_4(s_1s_2s_3)s_5$ and $w' = s_4(s_3)s_5$. We find the quiver



To the right we have drawn the five subsets corresponding to reduced subexpressions of w' inside the reduced expression for w^P. So, we find

$$\mathcal{W}_{\mathcal{Z}_P^\circ}(z) = \sum_{i=1}^{10} a_i + q\frac{a_1a_2a_3 + a_1a_2a_{10} + a_1a_5a_{10} + a_1a_9a_{10} + a_6a_9a_{10}}{a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}},$$

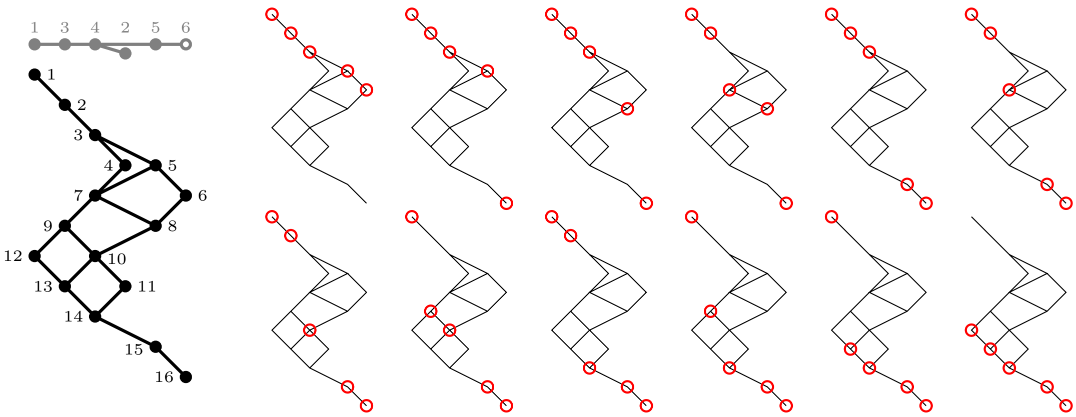
where $z = u_+t\bar{w}_Pu_-$ with $(u_-)^{-1} = y_4^\vee(-a_1)y_3^\vee(-a_2)\cdots y_5^\vee(-a_{10})$ and $q = \alpha_5^\vee(t)$. $\Diamond$

2.6.5 The Cayley plane

Let $X = \mathbb{OP}^2 = E_6^{\mathrm{sc}}/P_6$ be the Cayley plane, considered as a homogeneous space for the simply-connected Lie group E_6^{sc} of type E_6. We fix for w^P the reduced expression

$$w^P = s_1 s_3 s_4 s_2 s_5 s_6 s_4 s_5 s_3 s_4 s_2 s_1 s_3 s_4 s_5 s_6$$

and we find for w' the (unique) reduced expression $w' = s_1 s_3 s_4 s_5 s_6$. Thus, the quiver is as follows:



To the right we have drawn the twelve subsets corresponding to subexpressions of w'. We find for $z = u_+ t \bar{w}_P u_-$ with $(u_-)^{-1} = y_1^\vee(-a_1) y_3^\vee(-a_2) \cdots y_6^\vee(-a_{16})$

$$\mathcal{W}_{\mathcal{Z}_P^\circ}(z) = \sum_{i=1}^{16} a_i + q \frac{P(a_i)}{\prod_{i=1}^{16} a_i},$$

with $q = \alpha_6^\vee(t)$ and where (in order of the drawn subsets)

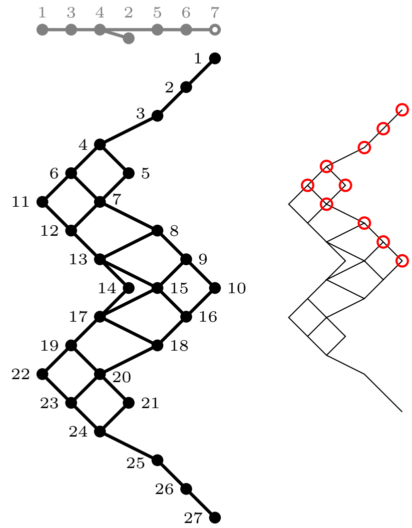
$$\begin{aligned} P(a_i) = {} & a_1a_2a_3a_5a_6 + a_1a_2a_3a_5a_{16} + a_1a_2a_3a_8a_{16} + a_1a_2a_7a_8a_{16} \\ & + a_1a_2a_3a_{15}a_{16} + a_1a_2a_7a_{15}a_{16} + a_1a_2a_{10}a_{15}a_{16} + a_1a_9a_{10}a_{15}a_{16} \\ & + a_1a_2a_{14}a_{15}a_{16} + a_1a_9a_{14}a_{15}a_{16} + a_1a_{13}a_{14}a_{15}a_{16} + a_{12}a_{13}a_{14}a_{15}a_{16}. \end{aligned}$$

2.6.6 The Freudenthal variety

Let $X = E_7^{\mathrm{sc}}/P_7$ be the Freudenthal variety, considered as a homogeneous space for the simply-connected Lie group E_7^{sc} of type E_7. We fix for w^P the reduced expression

$$w^P = s_7s_6s_5s_4s_2s_3s_4s_5s_6s_7s_1s_3s_4s_2s_5s_6s_4s_5s_3s_4s_2s_1s_3s_4s_5s_6s_7,$$

and w' has reduced expression $w' = s_7s_6s_5s_4(s_2s_3)s_4s_5s_6s_7$. The quiver $Q_{\mathbb{X}^\vee}$ is of the form



To the right we have drawn the subset corresponding to the reduced subexpression of w' that is minimal with respect to the lexicographical order. However, in this case, there are 78 reduced subexpressions, so instead of giving all the corresponding subsets, we will simply list in lexicographical order the elements of $\mathcal{I}$ consisting of sequences of subindices $(i_1, \ldots, i_{10})$ of $w^P = s_{r_1} \cdots s_{r_\ell}$ such that $w' = s_{i_1} \cdots s_{i_{10}}$, see Table 2.6.1.

We find for $z = u_+t\bar{w}_Pu_-$ with $(u_-)^{-1} = y_7^\vee(-a_1)y_6^\vee(-a_2)\cdots y_7^\vee(-a_{27})$ that

$$\mathcal{W}_{\mathcal{Z}_P^\circ}(z) = \sum_{i=1}^{27} a_i + q\frac{P(a_i)}{\prod_{i=1}^{27} a_i},$$

with $q = \alpha_7^\vee(t)$ and $P(a_i) = \sum_{(i_j)\in\mathcal{I}} \prod_{j=1}^{10} a_{i_j}$ is a homogeneous polynomial of degree 10 with 78 terms.

$$
\begin{aligned}
\mathcal{I} = \{&(1,2,3,4,5,6,7,8,9,10), (1,2,3,4,5,6,7,8,9,27), (1,2,3,4,5,6,7,8,16,27),\\
&(1,2,3,4,5,6,7,8,26,27), (1,2,3,4,5,6,7,15,16,27), (1,2,3,4,5,6,7,15,26,27),\\
&(1,2,3,4,5,6,7,18,26,27), (1,2,3,4,5,6,7,25,26,27), (1,2,3,4,5,6,13,15,16,27),\\
&(1,2,3,4,5,6,13,15,26,27), (1,2,3,4,5,6,13,18,26,27), (1,2,3,4,5,6,13,25,26,27),\\
&(1,2,3,4,5,6,17,18,26,27), (1,2,3,4,5,6,17,25,26,27), (1,2,3,4,5,6,20,25,26,27),\\
&(1,2,3,4,5,6,24,25,26,27), (1,2,3,4,5,12,13,15,16,27), (1,2,3,4,5,12,13,15,26,27),\\
&(1,2,3,4,5,12,13,18,26,27), (1,2,3,4,5,12,13,25,26,27), (1,2,3,4,5,12,17,18,26,27),\\
&(1,2,3,4,5,12,17,25,26,27), (1,2,3,4,5,12,20,25,26,27), (1,2,3,4,5,12,24,25,26,27),\\
&(1,2,3,4,5,19,20,25,26,27), (1,2,3,4,5,19,24,25,26,27), (1,2,3,4,5,23,24,25,26,27),\\
&(1,2,3,4,6,14,17,18,26,27), (1,2,3,4,6,14,17,25,26,27), (1,2,3,4,6,14,20,25,26,27),\\
&(1,2,3,4,6,14,24,25,26,27), (1,2,3,4,6,21,24,25,26,27), (1,2,3,4,12,14,17,18,26,27),\\
&(1,2,3,4,12,14,17,25,26,27), (1,2,3,4,12,14,20,25,26,27), (1,2,3,4,12,14,24,25,26,27),\\
&(1,2,3,4,12,21,24,25,26,27), (1,2,3,4,14,19,20,25,26,27), (1,2,3,4,14,19,24,25,26,27),\\
&(1,2,3,4,14,23,24,25,26,27), (1,2,3,4,19,21,24,25,26,27), (1,2,3,4,21,23,24,25,26,27),\\
&(1,2,3,7,12,14,17,18,26,27), (1,2,3,7,12,14,17,25,26,27), (1,2,3,7,12,14,20,25,26,27),\\
&(1,2,3,7,12,14,24,25,26,27), (1,2,3,7,12,21,24,25,26,27), (1,2,3,7,14,19,20,25,26,27),\\
&(1,2,3,7,14,19,24,25,26,27), (1,2,3,7,14,23,24,25,26,27), (1,2,3,7,19,21,24,25,26,27),\\
&(1,2,3,7,21,23,24,25,26,27), (1,2,3,13,14,19,20,25,26,27), (1,2,3,13,14,19,24,25,26,27),\\
&(1,2,3,13,14,23,24,25,26,27), (1,2,3,13,19,21,24,25,26,27), (1,2,3,13,21,23,24,25,26,27),\\
&(1,2,3,17,19,21,24,25,26,27), (1,2,3,17,21,23,24,25,26,27), (1,2,3,20,21,23,24,25,26,27),\\
&(1,2,8,13,14,19,20,25,26,27), (1,2,8,13,14,19,24,25,26,27), (1,2,8,13,14,23,24,25,26,27),\\
&(1,2,8,13,19,21,24,25,26,27), (1,2,8,13,21,23,24,25,26,27), (1,2,8,17,19,21,24,25,26,27),\\
&(1,2,8,17,21,23,24,25,26,27), (1,2,8,20,21,23,24,25,26,27), (1,2,15,17,19,21,24,25,26,27),\\
&(1,2,15,17,21,23,24,25,26,27), (1,2,15,20,21,23,24,25,26,27), (1,2,18,20,21,23,24,25,26,27),\\
&(1,9,15,17,19,21,24,25,26,27), (1,9,15,17,21,23,24,25,26,27), (1,9,15,20,21,23,24,25,26,27),\\
&(1,9,18,20,21,23,24,25,26,27), (1,16,18,20,21,23,24,25,26,27), (10,16,18,20,21,23,24,25,26,27)\}
\end{aligned}
$$

Table 2.6.1: The full list of all the 78 elements of $\mathcal{I}$ for the Freudenthal variety E_7^{sc}.

Chapter 3

Mirror symmetry for exceptional cominuscule homogeneous spaces

In this chapter, we will consider the exceptional cominuscule homogeneous spaces, the Cayley plane and the Freudenthal variety. For these varieties, we will present Plücker coordinate expressions for the mirror model for homogeneous spaces given by [Rie08], which we introduced in Section 1.4. We will call these models the *canonical mirror models*, following [PRW16].

We follow the general approach as used in [PR13, PRW16]. In particular, we will use a presentation of the coordinate ring of U_-^P derived from the results in [GLS11] (which we introduced in Section 1.6) and express it in terms of Plücker coordinates using representation theory. However, the varieties considered in these references are homogeneous under Sp_{2n} and Spin_{2n}, and the representation theory of these groups is well-established (although by no means trivial). For the groups of type E_6 and E_7, the representations have a more complicated structure. This is remedied by the fact that we are only considering two groups, and not a whole family, so it is manageable to make direct computations. Moreover, in contrast with [PR13, PRW16], we will perform most calculations restricted to the open, dense torus U_-° (or more precisely: its transpose) to simplify the computations. This is reminiscent of our approach in Chapter 2.

The implications of the canonical models have been discussed in the introduction to Chapter 1.

We will start by formulating the canonical mirror models in Section 3.1. In Section 3.2 we continue with an overview of the involved isomorphisms and give a type-independent criterion relating the generators of the coordinate ring of U_-^P given by [GLS11] and Plücker coordinates. Subsequently, we express the coordinate rings of (the transpose of) U_-^P for the Cayley plane and the Freudenthal variety in terms of Plücker coordinates in Sections 3.3 and 3.4, respectively. In these sections, we also give a list of Plücker relations on the mirror varieties of the Cayley plane and the Freudenthal variety that express the remaining Plücker coordinates in terms of the generating ones, see Remarks 3.3.4 and 3.4.4. Using the Plücker coordinate presentations of these coordinate rings, we prove the isomorphism between the Lie-theoretic and the canonical Landau-Ginzburg models in Section 3.5.

3.1 The canonical models for cominuscule E_n^{sc}/P_n

In this section we will define the *canonical mirror models* $(X_{\mathrm{can}}^\vee, \mathcal{W}_{\mathrm{can}})$ for the Cayley plane and the Freudenthal variety, given in terms of Plücker coordinates. The terminology "canonical" follows that of [PRW16]. This chapter will revolve around the following two results:

Theorem 3.1.1. For cominuscule E_n^{sc}/P_n, the variety $X_{\mathrm{can}}^\vee$ is isomorphic to the open Richardson variety $X_{\mathrm{Lie}}^\vee = \mathcal{R}_{w_P,w_0}^\vee$.

Theorem 3.1.2. For cominuscule E_n^{sc}/P_n, the pull-back of $\mathcal{W}_{\mathrm{can}}$ under the isomorphism of Theorem 3.1.1 equals $\mathcal{W}_{\mathrm{Lie}}$. In other words, the Landau-Ginzburg model $(X_{\mathrm{can}}^\vee, \mathcal{W}_{\mathrm{can}})$ is isomorphic to Rietsch's Lie theoretic Landau-Ginzburg model $(X_{\mathrm{Lie}}^\vee, \mathcal{W}_{\mathrm{Lie}})$.

As we noted in Remark 2.5.4, it is expected that canonical mirrors exist more generally, and they have indeed already been constructed for Grassmannians, Lagrangian Grassmannians, and quadrics in [MR20, PR13, PR18, PRW16]. The potentials for the exceptional cominuscule homogeneous spaces are similar to these cases. However, there is a notable difference: in the case of Grassmannians, the potential consisted of a sum of quotients of Plücker coordinates (see equation (6.4) of [MR20]); in both the cases of quadrics and Lagrangian Grassmannians, some of the quotients had instead homogeneous quadratic polynomials as their numerators and denominators (see equations (11) and (28) of [PRW16], and equation (7) of [PR13]); however, in the case of the exceptional cominuscule homogeneous spaces, we find cubic and quartic homogeneous polynomials as well.

We will denote the homogeneous polynomials appearing in the numerators and denominators by q_i; this should not cause any confusion with quantum parameters as cominuscule homogeneous spaces always have $\dim_{\mathbb{Z}} H_2(X, \mathbb{Z}) = 1$ and thus also only a single quantum parameter, which we denote by q. (See the discussion at the end of Section 1.3.) Also note that we will no longer use (q_i) to denote the sequence of indices of the reduced expression of w_P; the q_i will from now on be reserved for the homogeneous polynomials in Plücker coordinates defined below.

The canonical model for the Cayley plane. The homogeneous polynomials in Plücker coordinates appearing in the denominators are the following:

$$\begin{aligned}
&q_{12} = p_1 p''_{11} - p_0 p''_{12}, \qquad q_{16} = p'_7 p'_9 - p_8 p'_8, \qquad q_{20} = p''_5 p_{15} - p''_4 p_{16},\\
&q_{24} = p_0(p'_{11} p_{13} - p'_{12} p''_{12}) - p_1(p''_{10} p_{13} - p''_{11} p'_{12}) + p_2(p''_{10} p''_{12} - p'_{11} p''_{11});
\end{aligned} \tag{3.1.1}$$

and the following will be numerators:

$$\begin{aligned}
&q_{13} = p_2 p''_{11} - p_0 p_{13}, \qquad q_{17} = p'_7 p'_{10} - p_8 p''_9, \qquad q_{21} = p''_6 p_{15} - p'_5 p_{16},\\
&q_{25} = p_0 p'_{11} p_{14} - p_1 p''_{10} p_{14} + p_3(p''_{10} p''_{12} - p'_{11} p''_{11}).
\end{aligned} \tag{3.1.2}$$

Using these coordinates, define the following rational map $\mathbb{X}^\vee = \widetilde{P}^\vee \backslash \widetilde{G}^\vee \to \mathbb{C}$, called the *canonical potential*:

$$\mathcal{W}_{\mathrm{can}} = \frac{p_1}{p_0} + \frac{p'_9}{p_8} + \frac{q_{13}}{q_{12}} + \frac{q_{17}}{q_{16}} + \frac{q_{21}}{q_{20}} + \frac{q_{25}}{q_{24}} + q\frac{p''_5}{p_{16}}. \tag{3.1.3}$$

The denominators of the rational map of equation (3.1.3) define principal divisors on $\mathbb{X}^\vee$ which we will denote as follows:

$$\begin{aligned}
&D^{(1)}_0 = \{p_0 = 0\}, \quad D^{(1)}_8 = \{p_8 = 0\}, \quad D^{(1)}_{16} = \{p_{16} = 0\},\\
&D^{(2)}_{12} = \{q_{12} = 0\}, \quad D^{(2)}_{16} = \{q_{16} = 0\}, \quad D^{(2)}_{20} = \{q_{20} = 0\},\\
&D^{(3)}_{24} = \{q_{24} = 0\}.
\end{aligned}$$

Take $D = \sum D^{(j)}_i$ and let $X^\vee_{\mathrm{can}} = \mathbb{X}^\vee \setminus D$ be the complement of this anticanonical divisor. The restriction of $\mathcal{W}_{\mathrm{can}}$ to $X^\vee_{\mathrm{can}}$ is regular and will also be denoted by $\mathcal{W}_{\mathrm{can}}$. The pair $(X^\vee_{\mathrm{can}}, \mathcal{W}_{\mathrm{can}})$ forms the canonical LG model for the Cayley plane.

The canonical model for the Freudenthal variety. The canonical potential for the Freudenthal variety will have the following denominators:

$$\begin{aligned}
q_{18} &= p_1p_{17} - p_0p_{18}, \qquad q_{36} = p_{10}p_{26} - p_9p_{27},\\
q_{27} &= p_0p_{27} - p_1p_{26} + p_2p_{25} - p_3p_{24} + p_4p_{23} - p''_5p''_{22} + p''_6p''_{21},\\
q_{45} &= (p_9p_{11} - p_{10}p'_{10})p_{25} + (-p'_8p_{11} + p'_9p_{10})p_{26} + (p'_8p'_{10} - p_9p'_9)p_{27},\\
q'_{36} &= (p_0p'_{14} - p_1p''_{13} + p_2p''_{12})p'_{22} + (-p_0p_{13} + p_1p'_{12} - p_2p''_{11})p_{23}\\
&\quad + (p_0p_{12} - p_1p'_{11} + p_2p''_{10})p_{24} - p_2p''_9p_{25} + p_1p''_9p_{26} - p_0p''_9p_{27},\\
q_{54} &= (p''_5p''_7 - p'_6p''_6)(p''_{20}p''_{22} - p'_{21}p''_{21}) - (p_4p''_7 - p'_5p''_6)(p''_{20}p_{23} - p''_{21}p'_{22})\\
&\quad + (p_4p'_6 - p'_5p''_5)(p'_{21}p_{23} - p'_{22}p''_{22}) + q'_{27}q''_{27},
\end{aligned} \tag{3.1.4}$$

where

$$\begin{aligned}
q'_{27} &= -p_0p_{27} + p_1p_{26} - p_2p_{25} + p_3p_{24},\\
q''_{27} &= -p''_6p''_{21} - p'_7p'_{20} + p''_7p''_{20} + p'_8p'_{19} - p_9p_{18} + p_{10}p_{17};
\end{aligned}$$

and the following numerators:

$$
\begin{aligned}
q_{19} &= p_2p_{17} - p_0p'_{19}, \qquad q_{28} = p'_5p_{23} - p'_6p''_{22} + p''_7p''_{21}, \qquad q_{37} = p_{11}p_{26} - p'_{10}p_{27},\\
q_{46} &= (p_9p_{12} - p_{10}p'_{11})p_{25} + (-p'_8p_{12} + p_{10}p''_{10})p_{26} + (p'_8p'_{11} - p_9p''_{10})p_{27},\\
q'_{37} &= (p_0p'_{15} - p_1p_{14} + p_3p''_{12})p'_{22} + (-p_0p''_{14} + p_1p'_{13} - p_3p''_{11})p_{23}\\
&\quad + p_3p''_{10}p_{24} + (p_0p_{12} - p_1p'_{11} - p_3p''_9)p_{25} + p_1p''_{10}p_{26} - p_0p''_{10}p_{27},\\
q_{55} &= (p''_5p''_8 - p''_6p'_7)(p''_{20}p''_{22} - p'_{21}p''_{21}) - p_4p''_8(p''_{20}p_{23} - p''_{21}p'_{22}) + p_4p'_7(p'_{21}p_{23} - p'_{22}p''_{22})\\
&\quad + p'_5p_{24}(-p''_5p'_{21} + p''_6p''_{20}) + p_4p_{24}(-p_4p_{23} + p'_5p'_{22} + p''_5p''_{22} - p''_6p''_{21})\\
&\quad + (p_4p_{24} - p'_7p'_{21} + p''_8p''_{20})(-p_0p_{27} + p_1p_{26} - p_2p_{25} + p_3p_{24}). \qquad (3.1.5)
\end{aligned}
$$

The *canonical potential* is the following map $\mathbb{X}^\vee \to \mathbb{C}$:

$$\mathcal{W}_{\text{can}} = \frac{p_1}{p_0} + \frac{q_{19}}{q_{18}} + \frac{q_{28}}{q_{27}} + \frac{q_{37}}{q_{36}} + \frac{q'_{37}}{q'_{36}} + \frac{q_{46}}{q_{45}} + \frac{q_{55}}{q_{54}} + q\frac{p_{10}}{p_{27}} \qquad (3.1.6)$$

The sum of the loci where the denominators of this potential are zero forms an anticanonical divisor D; we will write $X^\vee_{\text{can}} = \mathbb{X}^\vee \setminus D$ for its complement. We will also write $\mathcal{W}_{\text{can}}$ for the regular restriction of the potential to $X^\vee_{\text{can}}$. The pair $(X^\vee_{\text{can}}, \mathcal{W}_{\text{can}})$ is the canonical LG model for the Freudenthal variety.

3.2 Overview of the isomorphisms

The proof of Theorem 3.1.1 requires an auxiliary variety U^P_-, which we defined in Section 1.4. Namely, we will construct the isomorphism as the following composition of maps:

$$X^\vee_{\text{Lie}} \xrightarrow{\varphi} U^P_- \xrightarrow{\pi} \mathbb{X}^\vee \supset X^\vee_{\text{can}}. \qquad (3.2.1)$$

Recall from equation (1.4.4) that

$$U^P_- = U^\vee_- \cap B^\vee_+ \bar{w}_P \bar{w}_0 B^\vee_+.$$

The map $\pi : U_-^P \to \mathbb{X}^\vee$ is the restriction of the quotient map $\widetilde{G}^\vee \to \widetilde{P}^\vee \backslash \widetilde{G}^\vee = \mathbb{X}^\vee$ to U_-^P. (Note that we identify $U_-^\vee \subset G^\vee$ and its universal cover in $\widetilde{G}^\vee$.)

The map $\varphi : X_{\mathrm{Lie}}^\vee \xrightarrow{\sim} U_-^P$ is an isomorphism that cannot be expressed so explicitly. It is given by mapping $u_+ \bar{w}_P B_-^\vee \in X_{\mathrm{Lie}}^\vee$ (with $u_+ \in U_+^\vee$) to the unique $u_- \in U_-^P$ such that $u_+ \bar{w}_P u_- \in \mathcal{Z}_P^\vee$. Here, the variety $\mathcal{Z}_P^\vee$ was defined in equation (1.4.2) as

$$\mathcal{Z}_P^\vee = B_-^\vee \bar{w}_0^{-1} \cap U_+^\vee T^P \bar{w}_P U_-^\vee \ \subset \ G^\vee,$$

and we noted in Lemma 1.4.4 that each $z \in \mathcal{Z}_P^\vee$ has a unique decomposition as $z = u_+ t \bar{w}_P u_-$ with $u_+ \in U_+^\vee$, $t \in T^P$ and $u_- \in U_-^P$. The map can alternatively be obtained by composing the isomorphism $X_{\mathrm{Lie}}^\vee \times T^P \to \mathcal{Z}_P^\vee$ given in equation (1.4.3) and discussed in Remark 1.4.1 with the isomorphism $\mathcal{Z}_P^\vee \to U_-^P \times T^P$ sending z to the unique (u_-, t) such that $z = u_+ t \bar{w}_P u_-$ for some $u_+ \in U_+^\vee$, and then restricting the resulting map $X_{\mathrm{Lie}}^\vee \times T^P \to U_-^P \times T^P$ to the unit element in T^P.

To show that π is an isomorphism between U_-^P and $X_{\mathrm{can}}^\vee \subset \mathbb{X}^\vee$ we use the description of the coordinate ring of $U_+^P = (U_-^P)^T$ given by [GLS11], which we discussed in Section 1.6. The first step in proving the isomorphism is to express the coordinate ring in terms of the Plücker coordinates.

It turns out that there is an easy criterion for a generalized Plücker coordinate p_i defined in equation (1.6.2) to coincide with a coordinate p_m^{GLS} used by [GLS11] under transposition. Recall from Section 1.6 that p_m^{GLS} was defined as the dual map associated to a basis element $e_{w^P}^\vee(m)$ for $\mathfrak{u}_{-\beta_{w^P}^\vee(m)}^\vee$, where $\beta_{w^P}^\vee(m)$ is the mth positive root mapped to a negative root by w^P, see equation (1.6.4).

Remark 3.2.1. This criterion holds for general cominuscule homogeneous spaces, and we will prove it in this context. Thus, until the end of this section, $X = G/P_k$ refers to a general cominuscule homogeneous space.

We first need the following characterization of the set $\Pi_{w^P}^\vee = \{\beta_{w^P}^\vee(m) \mid m = 1, \ldots, \ell\}$, and of its complement $\Pi_P^\vee$ which coincides with the positive roots mapped to negative roots by w_P:

Lemma 3.2.2. Every positive root $\alpha^\vee = \sum_{i=1}^n d_i\alpha_i^\vee \in \Pi^\vee = \Pi^\vee_{w^P} \sqcup \Pi^\vee_P$ has $d_k \in \{0,1\}$. Moreover, $\alpha^\vee \in \Pi^\vee_P$ if and only if $d_k = 0$; and $\alpha^\vee \in \Pi^\vee_{w^P}$ if and only if $d_k = 1$.

Proof. Since $\omega_k^\vee$ is minuscule, we know that the longest root $\alpha_0^\vee \in \Phi^\vee$ has coefficient 1 in front of $\alpha_k^\vee$ in the decomposition of $\alpha_0^\vee$ in terms of simple roots. (See Section 1.5.) Thus, we know that $d_k \in \{0,1\}$.

Now, it is well known that there is a sequence of (not necessarily distinct) simple roots $(\alpha^\vee_{i_j})_{j=1}^h$, where $h = \mathrm{ht}(\alpha^\vee) = \sum_{i=1}^n d_i$ is the height of $\alpha^\vee$, such that we have $\alpha^\vee = \sum_{j=1}^h \alpha^\vee_{i_j}$ and such that for every $h' \leq h$ the partial sum $\sum_{j=1}^{h'} \alpha^\vee_{i_j} \in \Phi^\vee$ is a root as well. (See Corollary 10.2 of [Hum78].)

Suppose $d_k = 0$, then we clearly have that $i_j \neq k$ for all j. Since $\{\alpha_i^\vee \mid i \neq k\} \subset \Pi^\vee_P$ and $\Pi^\vee_P$ is bracket closed, every partial sum lies in $\Pi^\vee_P$ and thus $\alpha^\vee$ as well. Conversely, since $\Pi^\vee_P$ can be considered as the positive roots of the root system with as Dynkin diagram the diagram of $G^\vee$ with the kth vertex removed, we have that $\alpha^\vee \in \Pi^\vee_P$ implies that $d_k = 0$.

As $\Pi^\vee_{w^P}$ is the complement of $\Pi^\vee_P$ in $\Pi^\vee$, the statement follows. □

Recall that we defined a $\mathbb{N}_0\Delta^\vee$-grading on the completed universal enveloping algebra $\widehat{\mathcal{U}}_+^\vee$ of $\mathfrak{u}_+^\vee$ in Section 1.6. Here, $\mathbb{N}_0\Delta^\vee \cong \mathbb{N}_0^n$ is the positive root lattice of all non-negative integral linear combinations of the simple roots $\alpha_i^\vee$. We defined $\widehat{\mathcal{U}}_+^\vee(\alpha^\vee) \subset \widehat{\mathcal{U}}_+^\vee$ as the subspace spanned by pure tensors $e^\vee_{i_1} \cdots e^\vee_{i_j}$ such that exactly d_i of the factors have index i, where $\alpha^\vee = \sum_{i=1}^n d_i\alpha_i^\vee \in \mathbb{N}_0\Delta^\vee$. We noted there that $\mathfrak{u}^\vee_{+\alpha^\vee} \subset \widehat{\mathcal{U}}_+^\vee(\alpha^\vee)$ for $\alpha^\vee \in \Pi^\vee$, so we directly conclude that:

Corollary 3.2.3. For $j \in \{1,\ldots,\ell_0\}$, $e^\vee_{w^P}(j) \in \widehat{\mathcal{U}}_+^\vee(\beta^\vee_{w^P}(j))$. Let $d_i \in \mathbb{N}_0$ such that $\beta^\vee_{w^P}(j) = \sum_{i=1}^n d_i\alpha_i^\vee$. If $j \in \{1,\ldots,\ell\}$, then $d_k = 1$. Else, i.e. $j \in \{\ell+1,\ldots,\ell_0\}$, we have $d_k = 0$.

We also need the following simple observation regarding the action of the elements of the completed universal enveloping algebra on $V(-w_0 \cdot \omega_k^\vee)$:

Lemma 3.2.4. If $u \in \widehat{\mathcal{U}}_+^\vee(\alpha^\vee)$ for $\alpha^\vee \in \mathbb{N}_0\Delta$ and $v \in V(-w_0 \cdot \omega_k^\vee)$ a vector of weight μ, then $u \cdot v$ is either a vector of weight $\mu + \alpha^\vee$ or the zero vector.

Proof. By Theorem 1.5.3, we know that the statement holds for the Chevalley generators $e_i^\vee \in \widehat{\mathcal{U}}_+^\vee(\alpha_i^\vee)$. By definition $u \in \widehat{\mathcal{U}}_+^\vee(\alpha^\vee)$ for $\alpha^\vee = \sum_{i=1}^n d_i \alpha_i^\vee$ can be decomposed into a linear combination of pure tensors in which each $e_i^\vee$ appears d_i times. The statement now follows by induction. $\square$

Now, recall that the Plücker coordinates $p_i : g \mapsto v_0^*(g \cdot v_i)$ are defined using the minuscule representation $V(-w_0 \cdot \omega_k^\vee)$ and its dual, where $-w_0 \cdot \omega_k^\vee = -w^P \cdot \omega_k^\vee$ and this either equals $\omega_k^\vee$ or $\omega_{\sigma_0(k)}^\vee$ for $\sigma_0 : \{1, \dots, n\} \to \{1, \dots, n\}$ the Dynkin diagram symmetry; see the discussion of Plücker coordinates of Section 1.6. The criterion is given as follows:

Proposition 3.2.5. If v_i is of weight $-\omega_k^\vee + \alpha^\vee(v_i)$ and $\alpha^\vee(v_i) = \beta_{w^P}^\vee(m) \in \Pi_{w^P}^\vee$, then $u \mapsto p_i(u^T)$ coincides up to a constant with p_m^{GLS} on U_+^P.

Proof. Let $u \in U_+^P \hookrightarrow \widehat{\mathcal{U}}_+^\vee$ and consider the decomposition of u with respect to the PBW basis. If the coefficient in front of $e_{w^P}^\vee(m)$ is c, then we have by definition that $p_m^{\mathrm{GLS}}(u) = c$.

On the other hand, we have that $p_i(u^T)$ is the coefficient in front of v_0 of the projection of $u^T \cdot v_i$ to $\mathbb{C}v_0 \subset V(-w_0 \cdot \omega_k^\vee)$. Clearly, it suffices to show that $e_{w^P}^\vee(m)^T$ is the only transposed PBW basis element mapping v_i to a non-zero multiple of v_0. Equivalently, we need to show that $e_{w^P}^\vee(m)$ is the only PBW basis element mapping v_0 to a non-zero multiple of v_i. One can see this equivalence as follows: For Chevalley generators, Theorem 1.5.3 implies that $e_j^\vee$ maps v to v' if and only if $(e_i^\vee)^T = f_j^\vee$ maps v' to v; by induction, the same holds for all pure tensors; finally, each PBW basis element is a linear combination of these pure tensors, so that the equivalence follows.

By Lemma 3.2.4, only the elements of $\widehat{\mathcal{U}}_+^\vee\big(\beta_{w^P}^\vee(m)\big)$ map v_0 to $\mathbb{C}v_i$. Thus, we can restrict ourselves to PBW basis elements $e_{w^P}^\vee(\mathbf{m})$ lying in $\widehat{\mathcal{U}}_+^\vee\big(\beta_{w^P}^\vee(m)\big)$. Note that $e_{w^P}^\vee(m) \in \mathfrak{u}_{+\beta_{w^P}^\vee(m)}^\vee \subset \widehat{\mathcal{U}}_+^\vee\big(\beta_{w^P}^\vee(m)\big)$. Moreover, Lemma 3.2.2 implies that $d_k = 1$ in the decomposition $\beta_{w^P}^\vee(m) = \sum_{i=1}^n d_i \alpha_i^\vee$.

First, let us consider $e_{w^P}^\vee(\mathbf{m}) \in \widehat{\mathcal{U}}_+^\vee\big(\beta_{w^P}^\vee(m)\big)$ not equal to $e_{w^P}^\vee(m)$. Since for $j \neq m$ we have $e_{w^P}^\vee(j) \notin \widehat{\mathcal{U}}_+^\vee\big(\beta_{w^P}^\vee(m)\big)$, this assumption means that we are considering those

$e^\vee_{w^P}(\mathbf{m})$ with $\mathbf{m} = (m_1, m_2, \dots, m_{\ell_0})$ satisfying $\sum_{j=1}^{\ell_0} m_j \geq 2$. Corollary 3.2.3 implies that of the coefficients $m_1, \dots, m_\ell$ (where $\ell = \ell(w^P)$) exactly one is non-zero, and that coefficient equals 1, as these are the only coefficients contributing to $d_k = 1$. Thus, at least one of $m_{\ell+1}, \dots, m_{\ell_0}$ is non-zero; let $j \in \{\ell+1, \dots, \ell_0\}$ be the largest index such that $m_j \neq 0$. Recalling the definition of the PBW basis elements from equation (1.6.5), we find that $e^\vee_{w^P}(\mathbf{m}) \cdot v_0 = \big(e^\vee_{w^P}(1)\big)^{m_1} \cdots \big(e^\vee_{w^P}(j)\big)^{m_j} \cdot v_0$. However, $e^\vee_{w^P}(j) \cdot v_0 = 0$: we have by Corollary 3.2.3 that $e^\vee_{w^P}(j) \in \widehat{\mathcal{U}}^\vee_+\big(\beta^\vee_{w^P}(m)\big)$ where $\beta^\vee_{w^P} = \sum_{j'=1}^n d'_{j'} \alpha^\vee_i$ with $d'_k = 0$, which implies that $e^\vee_{w^P}(j)$ can be written as a linear combination of pure tensors involving the Chevalley generators $e^\vee_{i'}$ for $i' \neq k$, but by Theorem 1.5.3 we know that $e^\vee_{i'} \cdot v_0 = 0$ for $i' \neq k$.

Thus, what remains to be shown is that $e^\vee_{w^P}(m) \cdot v_0$ is a non-zero multiple of v_i. Suppose it acts trivially, then we find that $\mathbb{C}v_i \cap \widehat{\mathcal{U}}^\vee_+ v_0 = \{0\}$, which contradicts the irreducibility of $V(-w_0 \cdot \omega^\vee_k)$. $\square$

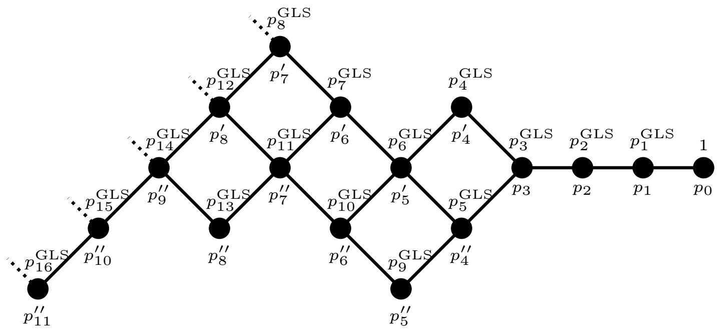
Using this criterion, we will be able to identify all of the generators of the coordinate ring of U^P_- given by [GLS11] with generalized Plücker coordinates.

Moreover, we can identify the generalized minors defined in (1.6.7) with the denominators of the potentials for the cominuscule exceptional homogeneous spaces as given in (3.1.1) and (3.1.4). We will identify these functions by checking that their expressions in terms of the coordinates $\{a_i\}_{i=1}^{\ell}$ on the algebraic torus U°_- of Definition 2.2.1 agree.

3.3 The coordinate ring of U^P_- for the Cayley plane

Recall the conventions for the Cayley plane E_6^{sc}/P_6 from Section 1.7. In particular, we have $\ell = \ell(w^P) = 16$. Applying this to the coordinate ring of U^P_+ described in Proposition 1.6.3 (see Proposition 8.5 of [GLS11]), we find that it is generated by the dual PBW elements p^{GLS}_m, $m \in \{1, \dots, 16\}$, and localized at the generalized minors $\Delta_{\omega^\vee_j, (w^P)^{-1}(\omega^\vee_j)}$, $j \in \{1, \dots, 6\}$. Using the criterion in Proposition 3.2.5, we find that the generators p^{GLS}_m can be expressed in terms of Plücker coordinates as follows:

Lemma 3.3.1. On $U_+^P \subset \widetilde{G}^\vee$, we have the following identifications:



(3.3.1)

Here, a vertex marked with $p_i^{(j)}$ below and p_m^{GLS} above denotes that $p_i^{(j)}(u_+^T)$ equals $p_m^{\mathrm{GLS}}(u_+)$ up to a constant for $u_+ \in U_-^P$.

For example, we have that $p_7''(u_+^T)$ equals $p_{11}^{\mathrm{GLS}}(u_+)$ up to a constant for $u_+ \in U_+^P$. In particular, the rightmost vertex tells us that $p_0(u_+^T)$ is constant, and it is easy to see that this constant is 1. Note that the diagram of equation (3.3.1) is a subdiagram of the one in equation (1.7.1). Upon removing the rightmost vertex, this subdiagram is in fact isomorphic to the Hasse diagram of $\mathrm{OG}(5, 10) = \mathrm{PSO}_{10}/P_5 = D_5^{\mathrm{sc}}/P_5$. This variety is also known as a *spinor variety*. See [IM05], Section 4, for its role in the geometry of the Cayley plane. Note that the appearance of this variety should not be too much of a surprise, as we obtain the Dynkin diagram of type D_5 upon removing the sixth vertex from the diagram of type E_6; in other words W_P is of type D_5.

Proof. We can compute the weights of the minuscule fundamental weight representation $V(-w_0 \cdot \omega_6^\vee)$ directly using diagram (1.7.1) and Theorem 1.5.3. On the other hand, the positive roots $\beta_{w^P}^\vee(m)$, $m \in \{1, \ldots, 16\}$ are straightforward to compute using the expression in equation (1.6.4). A comparison of the results gives the following equalities:

$$\begin{array}{llll}
\alpha^\vee(v_1) = \beta_{w^P}^\vee(1) & \alpha^\vee(v_2) = \beta_{w^P}^\vee(2) & \alpha^\vee(v_3) = \beta_{w^P}^\vee(3) & \alpha^\vee(v_4') = \beta_{w^P}^\vee(4) \\
\alpha^\vee(v_4'') = \beta_{w^P}^\vee(5) & \alpha^\vee(v_5') = \beta_{w^P}^\vee(6) & \alpha^\vee(v_5'') = \beta_{w^P}^\vee(9) & \alpha^\vee(v_6') = \beta_{w^P}^\vee(7) \\
\alpha^\vee(v_6'') = \beta_{w^P}^\vee(10) & \alpha^\vee(v_7') = \beta_{w^P}^\vee(8) & \alpha^\vee(v_7'') = \beta_{w^P}^\vee(11) & \alpha^\vee(v_8') = \beta_{w^P}^\vee(12) \\
\alpha^\vee(v_8'') = \beta_{w^P}^\vee(13) & \alpha^\vee(v_9'') = \beta_{w^P}^\vee(14) & \alpha^\vee(v_{10}'') = \beta_{w^P}^\vee(15) & \alpha^\vee(v_{11}'') = \beta_{w^P}^\vee(16)
\end{array}$$

Using Proposition 3.2.5, we find that $p_i^{(j)}(u_+^T)$ equals $p_m^{\mathrm{GLS}}(u_+)$ up to a constant as indicated in (3.3.1) for $u_+ \in U_+^P$. The only exception is the identification $p_0(u_-) = 1$, but this identity is obvious from the fact that $u_+^T \in U_-^\vee$ is a lower unipotent element and thus acts trivially on the lowest weight vector v_0. $\square$

To simplify expressions in the following, we will refer to the Plücker coordinates appearing in the above as

$$p_{\alpha^\vee} = p_i^{(j)} \text{ when } \alpha^\vee = \alpha^\vee(v_i^{(j)}) \in \Pi_{w^P}^\vee. \tag{3.3.2}$$

Note that the set $\{p_{\alpha^\vee} \mid \alpha^\vee \in \Pi_{w^P}^\vee\}$ consists of the Plücker coordinates appearing in Lemma 3.3.1 (it does not include p_0). Moreover, we will write $p_{\alpha^\vee}^T : U_+^P \to \mathbb{C}$ for the map $u_+ \mapsto p_{\alpha^\vee}(u_+^T)$.

Lemma 3.3.2. The generalized minors defined in equation (1.6.7) can be expressed on U_+^P in terms of Plücker coordinates as

$$\begin{aligned}
&\Delta_{\omega_1^\vee,(w^P)^{-1}(\omega_1^\vee)}(u_+) = p_{16}(u_+^T), \quad \Delta_{\omega_2^\vee,(w^P)^{-1}(\omega_2^\vee)}(u_+) = q_{12}(u_+^T),\\
&\Delta_{\omega_3^\vee,(w^P)^{-1}(\omega_3^\vee)}(u_+) = q_{20}(u_+^T), \quad \Delta_{\omega_4^\vee,(w^P)^{-1}(\omega_4^\vee)}(u_+) = q_{24}(u_+^T),\\
&\Delta_{\omega_5^\vee,(w^P)^{-1}(\omega_5^\vee)}(u_+) = q_{16}(u_+^T), \quad \Delta_{\omega_6^\vee,(w^P)^{-1}(\omega_6^\vee)}(u_+) = p_8(u_+^T),
\end{aligned} \tag{3.3.3}$$

where the quadratic and cubic expressions q_i in Plücker coordinates were defined in (3.1.1).

Proof. We will begin by considering the generalized minors. The action of $(\dot{w}^P)^{-1}$ on the highest weight vector $v_{\omega_j^\vee}^+$ of $V(\omega_j^\vee)$ is straightforward to calculate using the action of $(w^P)^{-1}$ on the weight $\omega_j^\vee$ and the following equality:

$$\bar{s}_i \cdot v_{\mu^\vee} = \tfrac{1}{m!}(f_i^\vee)^m \cdot v_{\mu^\vee} \quad \text{when } s_i(\mu^\vee) = \mu^\vee - m\alpha_i^\vee, \tag{3.3.4}$$

where $v_{\mu^\vee} \in V(\omega_j^\vee)$ is assumed to be of weight $\mu^\vee$. As the expression can get quite large, we will use the following abbreviation:

$$(f_{i_1}^\vee)^{m_1}(f_{i_2}^\vee)^{m_2}\cdots = f^\vee_{i_1^{m_1}i_2^{m_2}\dots} \tag{3.3.5}$$

These calculations yield (cf. equation (1.7.2)):

$$\begin{aligned}
(\dot{w}^P)^{-1}\cdot v^+_{\omega_1^\vee} &= f^\vee_{6542345613452431}\cdot v^+_{\omega_1^\vee}, & (\dot{w}^P)^{-1}\cdot v^+_{\omega_2^\vee} &= \tfrac{1}{2} f^\vee_{6^25423451342}\cdot v^+_{\omega_2^\vee},\\
(\dot{w}^P)^{-1}\cdot v^+_{\omega_3^\vee} &= \tfrac{1}{2^5} f^\vee_{6^25^24^223^245612^2345243}\cdot v^+_{\omega_3^\vee}, & (\dot{w}^P)^{-1}\cdot v^+_{\omega_4^\vee} &= \tfrac{1}{2^8}\tfrac{1}{3^2} f^\vee_{6^35^34^22^23^24^25612^23^24524}\cdot v^+_{\omega_4^\vee},\\
(\dot{w}^P)^{-1}\cdot v^+_{\omega_5^\vee} &= \tfrac{1}{2^4} f^\vee_{6^25^24^22^234561345}\cdot v^+_{\omega_5^\vee}, & (\dot{w}^P)^{-1}\cdot v^+_{\omega_6^\vee} &= f^\vee_{65423456}\cdot v^+_{\omega_6^\vee}.
\end{aligned}$$

We now need to find the coefficient in front of $v^+_{\omega_j^\vee}$ of the product $u_+(\dot{w}^P)^{-1}\cdot v^+_{\omega_j^\vee}$ to calculate the minors.

Instead of calculating these coefficients for general $u_+ \in U_+^P$, we will restrict to the open, dense algebraic torus $U_+^\circ \subset U_+^P$ that is the image of U_-° under the transposition isomorphism $\cdot^T : U_-^P \to U_+^P$. Recall from Definition 2.2.1 that the elements of U_-° have a specific decomposition; after transposing these elements, we find that every $u_+ \in U_+^\circ$ has a decomposition of the form

$$u_+ = x_{r_1}^\vee(a_1)x_{r_2}^\vee(a_2)\cdots x_{r_{16}}^\vee(a_{16}), \quad \text{where } a_i \in \mathbb{C}^*. \tag{3.3.6}$$

Note that the sequence $(r_1, r_2, \dots, r_{16}) = (1,3,4,2,5,4,3,1,6,5,4,3,2,4,5,6)$ is the same as that of the fixed reduced expression for w^P (see (1.7.2)), and thus the inverse sequence compared to that of $(w^P)^{-1}$.

Since $x_i^\vee(a)$ acts on $v \in V(\omega_j^\vee)$ as

$$x_i^\vee(a)\cdot v = \big(1 + a\, e_i^\vee + \tfrac{1}{2}a^2(e_i^\vee)^2 + \tfrac{1}{3!}a^3(e_i^\vee)^3 + \dots\big)\cdot v, \tag{3.3.7}$$

we need to find all terms of u_+ that cancel each the factors $f_i^\vee$ of $(\dot{w}^P)^{-1}\cdot v^+_{\omega_j^\vee}$. However, Theorem 1.10 of [FZ99] tells us that the toric coordinates are related to the generalized minors by an invertible monomial transformation; in other words, there is exactly one term in the action of u_+ that cancels out the $f_i^\vee$. In the following, we will argue that this term is the one whose $e_i^\vee$ have the opposite sequence of indices (counting doubles and triples).

First, consider $u_+(\dot{w}^P)^{-1} \cdot v^+_{\omega_1^\vee} = u_+ f^\vee_{6542345613452431} \cdot v^+_{\omega_1^\vee}$. As $(\dot{w}^P)^{-1} \cdot v^+_{\omega_1^\vee}$ acts with 16 factors $f_i^\vee$, lowering the weight by subtracting 16 simple roots, we need to act with 16 factors $e_i^\vee$ to get back to a vector of the correct weight. However, since the representation in question is minuscule, we know that $(e_i^\vee)^2$ acts trivially on the representations for all $i \in \{1, \ldots, 6\}$. (See Theorem 1.5.3.) Thus, the decomposition of u_+ in equation (3.3.6) only has one term with 16 factors acting non-trivially:

$$\left(\prod_{j=1}^{16} a_j\right) e^\vee_{1342543165432456}.$$

For $u_+(\dot{w}^P)^{-1} \cdot v^+_{\omega_6^\vee} = u_+ f^\vee_{65423456} \cdot v^+_{\omega_6^\vee}$, the computation is facilitated by the Hasse diagram in (1.7.1): we find that $(\dot{w}^P)^{-1} \cdot v^+_{\omega_6^\vee} = v_8$. (Note that in interpreting diagram (1.7.1) as the weight spaces of the minuscule representation $V(\omega_6^\vee)$, we have put the highest weight vector $v^+_{\omega_6^\vee}$ on the right and denoted it v_0, which is the opposite of the way we interpreted it for $V(\omega_1^\vee)$.) The diagram shows us that both $e^\vee_{65423456}$ and $e^\vee_{65432456}$ map v_8 back to $v_0 = v^+_{\omega_6^\vee}$. However, the decomposition of u_+ tells us only the second occurs, so the only term in u_+ acting non-trivially is

$$\left(\prod_{j=9}^{16} a_j\right) e^\vee_{65432456}.$$

The arguments for $u_+(\dot{w}^P)^{-1} \cdot v^+_{\omega_2^\vee} = u_+ \frac{1}{2} f^\vee_{6^2 5423451342} \cdot v^+_{\omega_2^\vee}$ are more involved. First, note that $s_i(\omega_2^\vee) = \omega_2^\vee - \delta_{i2}\alpha_2^\vee$ implies that of $\omega_2^\vee - \alpha_i^\vee$ only $i = 2$ is a weight occurring in the representation $V(\omega_2^\vee)$. Moreover, it tells us that $\omega_2^\vee - 2\alpha_2^\vee$ is *not* a weight of the representation. Thus, the last factor in any term of u_+ acting non-trivially must be $e_2^\vee$. There are only two indices j such that $r_j = 2$ in the decomposition in (3.3.6), namely $j = 4$ and $j = 13$. However, only the $j = 4$ factor $x_2^\vee(a_i)$ can contribute a factor $e_2^\vee$ non-trivially: we will need the second factor to cancel out the second factor

$f_2^\vee$ in $(\dot{w}^P)^{-1} \cdot v^+_{\omega_2^\vee}$. Thus, we reduced $u_+(\dot{w}^P)^{-1} \cdot v^+_{\omega_2^\vee}$ to

$$a_4\, e_2^\vee\, x^\vee_{r_5}(a_5) x^\vee_{r_6}(a_6) \cdots x^\vee_{r_{16}}(a_{16})\, (\dot{w}^P)^{-1} \cdot v^+_{\omega_2^\vee}.$$

For the second step, we have $s_i(\omega_2^\vee - \alpha_2^\vee) = \omega_2^\vee - \alpha_2^\vee - \delta_{i4}\alpha_4^\vee$ for $i \neq 2$ (s_2 will raise the weight back to $\omega_2^\vee$), so that the next factor has to be $e_4^\vee$. Now, $r_j = 4$ for $j = 3, 6, 11, 14$, but we need $j > 5$ by the above. Moreover, if we take $j = 11$ or $j = 14$ we will not have sufficient factors to reach $v^+_{\omega_2^\vee}$ from $(\dot{w}^P)^{-1} \cdot v^+_{\omega_2^\vee}$ (we need ten more factors). Thus, we reduced $u_+(\dot{w}^P)^{-1} \cdot v^+_{\omega_2^\vee}$ to

$$a_4 a_6\, e_{24}^\vee\, x^\vee_{r_7}(a_7) x^\vee_{r_8}(a_8) \cdots x^\vee_{r_{16}}(a_{16})\, (\dot{w}^P)^{-1} \cdot v^+_{\omega_2^\vee}.$$

In the third step, we have $s_i(\omega_2^\vee - \alpha_2^\vee - \alpha_4^\vee) = \omega_2^\vee - \alpha_2^\vee - \delta_{i3}\alpha_3^\vee - \alpha_4^\vee - \delta_{i5}\alpha_5^\vee$ for $i \neq 4$, so there are two "paths": using an $e_3^\vee$ or an $e_5^\vee$. For the latter path, we are looking for a j with $r_j = 5$ and $j > 6$. This means that $j = 10$ or $j = 15$, but this would give a problem: as $(\dot{w}^P)^{-1} \cdot v^+_{\omega_2^\vee} = \frac{1}{2} f^\vee_{6^2 5423451342} \cdot v^+_{\omega_2^\vee}$, we need at least one factor $e_1^\vee$, but $r_j \neq 1$ for $j > 10$. Thus, we need to take the path with $e_3^\vee$. We find that $j = 7$ gives the only possible factor $e_3^\vee$ using an argument similar to the second step. Continuing to apply similar arguments, we conclude that only the term

$$\tfrac{1}{2} a_4 a_6 a_7 a_8 a_{10} a_{11} a_{12} a_{13} a_{14} a_{15} (a_{16})^2\, e^\vee_{24315432456^2}$$

of u_+ gives a non-trivial contribution. One can easily see that

$$\tfrac{1}{2}\, e^\vee_{24315432456^2} \cdot \tfrac{1}{2} f^\vee_{6^2 5423451342} \cdot v^+_{\omega_2^\vee} = v^+_{\omega_2^\vee},$$

so that

$$\Delta_{\omega_2^\vee, (w^P)^{-1}(\omega_2^\vee)}(u_+) = a_4 a_6 a_7 a_8 a_{10} a_{11} a_{12} a_{13} a_{14} a_{15} a_{16}^2.$$

Analogous arguments to those used in the previous three calculations imply that $u_+(\dot{w}^P)^{-1} \cdot v^+_{\omega_i^\vee}$ for $i = 3, 4, 5$ only have the claimed contribution as well.

Altogether, we find for $u_+ \in U_+^\circ$

$$\begin{aligned}
\Delta_{\omega_1^\vee,(w^P)^{-1}(\omega_1^\vee)}(u_+) &= a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16},\\
\Delta_{\omega_2^\vee,(w^P)^{-1}(\omega_2^\vee)}(u_+) &= a_4a_6a_7a_8a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}^2,\\
\Delta_{\omega_3^\vee,(w^P)^{-1}(\omega_3^\vee)}(u_+) &= a_2a_3a_4a_5a_6a_7a_8^2a_9a_{10}a_{11}a_{12}^2a_{13}a_{14}^2a_{15}^2a_{16}^2,\\
\Delta_{\omega_4^\vee,(w^P)^{-1}(\omega_4^\vee)}(u_+) &= a_3a_4a_5a_6a_7^2a_8^2a_9a_{10}a_{11}^2a_{12}^2a_{13}^2a_{14}^2a_{15}^3a_{16}^3,\\
\Delta_{\omega_5^\vee,(w^P)^{-1}(\omega_5^\vee)}(u_+) &= a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}^2a_{14}^2a_{15}^2a_{16}^2,\\
\Delta_{\omega_6^\vee,(w^P)^{-1}(\omega_6^\vee)}(u_+) &= a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}.
\end{aligned}$$

To prove the equalities in (3.3.3), the next step is to evaluate the Plücker coordinate expressions (defined in equation (3.1.1)) on the transpose of $u_+ \in U_+^\circ$. We will find that they have the same monomial expressions in the toric coordinates $\{a_i\}$ of U_+°. Recall from Section 1.6 that the generalized Plücker coordinates are defined on the fundamental weight representation $V(\omega_1^\vee)$, which is minuscule. As we saw in the above, this simplifies calculations considerably.

Recall from equation (1.6.2) that $p_i(g) = v_0^*(g \cdot v_i)$, where the v_i have been defined using diagram (1.7.1). Thus, we need to find the terms of $u_+^T \cdot v_i$ that are multiples of v_0. As $V(\omega_1^\vee)$ is minuscule, we know that each $y_i^\vee(a_j)$ in the decomposition of $u_+^T \in U_-^\circ$ (see (2.2.1)) acts as $1 + a_j f_i^\vee$ on the representation. Now, every path from v_i to v_0 in diagram (1.7.1) gives a sequence $(b_1, \dots, b_i)$ of indices such that $\bar{s}_{b_1} \cdots \bar{s}_{b_i} \cdot v_i = v_0$. By Lemma 2.1.1, we find that each $\bar{s}_{b_j}$ acts as $f_{b_j}^\vee$, so that $f_{b_1}^\vee \cdots f_{b_i}^\vee \cdot v_i = v_0$. Thus, for each sequence of indices $(b_1, \dots, b_i)$ obtained from a path from v_i to v_0, the term $f_{b_1}^\vee \cdots f_{b_i}^\vee$ in u_+^T will contribute its coefficient to $p_i^{(j)}(u_+^T)$.

To find these coefficients, apply the following algorithm: determine all the paths $(b_1, \dots, b_i)$ from v_i to v_0 in diagram (1.7.1); find all the subexpressions of the form $s_{b_1} \cdots s_{b_i}$ in $(w^P)^{-1}$; for each of the subexpressions, take the respective coefficients in the decomposition of $u_+^T \in U_-^\circ$. Note that we can use the quiver $Q_{\mathbb{X}^\vee}$ of Definition 2.5.3 and Proposition 2.5.11 (applied to $(w^P)^{-1}$; in other words flipping all the arrows) to find all these subexpressions.

Let us illustrate with $p'_7(u_+^T)$ for $u_+^T \in U_-^\circ$. We find two paths in diagram (1.7.1): $(6,5,4,2,3,4,5)$ and $(6,5,4,3,2,4,5)$. The first gives the following subexpressions:

$$\underline{\boldsymbol{s_6s_5s_4s_2s_3s_4s_5}}s_6s_1s_3s_4s_5s_2s_4s_3s_1, \quad \underline{\boldsymbol{s_6s_5s_4s_2s_3s_4}}s_5s_6s_1s_3s_4\underline{\boldsymbol{s_5}}s_2s_4s_3s_1,$$

$$\underline{\boldsymbol{s_6s_5s_4s_2s_3}}s_4s_5s_6s_1s_3\underline{\boldsymbol{s_4s_5}}s_2s_4s_3s_1, \quad \underline{\boldsymbol{s_6s_5s_4s_2}}s_3s_4s_5s_6s_1\underline{\boldsymbol{s_3s_4s_5}}s_2s_4s_3s_1,$$

The second does not give any subexpressions in $(w^P)^{-1}$. We now compare it with the decomposition of u_+^T as

$$\begin{aligned} u_+^T = {} & y_6^\vee(a_{16})y_5^\vee(a_{15})y_4^\vee(a_{14})y_2^\vee(a_{13})y_3^\vee(a_{12})y_4^\vee(a_{11})y_5^\vee(a_{10})y_6^\vee(a_9) \\ & \cdot y_1^\vee(a_8)y_3^\vee(a_7)y_4^\vee(a_6)y_5^\vee(a_5)y_2^\vee(a_4)y_4^\vee(a_3)y_3^\vee(a_2)y_1^\vee(a_1). \end{aligned}$$

to find the contributing terms (in order of the subexpressions):

$$\begin{aligned} p'_7(u_+^T) &= a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16} + a_5a_{11}a_{12}a_{13}a_{14}a_{15}a_{16} \\ &\quad + a_5a_6a_{12}a_{13}a_{14}a_{15}a_{16} + a_5a_6a_7a_{13}a_{14}a_{15}a_{16} \\ &= \big((a_5+a_{10})a_{11}a_{12} + a_5a_6(a_7+a_{12})\big)a_{13}a_{14}a_{15}a_{16}. \end{aligned}$$

Note the similarities between these calculations and the calculations for the local Laurent polynomial expression for Rietsch's potential in the examples in Section 2.6.

Once we obtained all toric expressions for the Plücker coordinates, calculating the expressions of equation (3.1.1) is straightforward. We find the following for $u_+ \in U_+^\circ$

$$\begin{aligned} p_8(u_+^T) &= a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}, \\ p_{16}(u_+^T) &= a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}, \\ q_{12}(u_+^T) &= a_4a_6a_7a_8a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}^2, \\ q_{16}(u_+^T) &= a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}^2a_{14}^2a_{15}^2a_{16}^2, \\ q_{20}(u_+^T) &= a_2a_3a_4a_5a_6a_7a_8^2a_9a_{10}a_{11}a_{12}^2a_{13}a_{14}^2a_{15}^2a_{16}^2, \\ q_{24}(u_+^T) &= a_3a_4a_5a_6a_7^2a_8^2a_9a_{10}a_{11}^2a_{12}^2a_{13}^2a_{14}^2a_{15}^3a_{16}^3. \end{aligned}$$

Comparing these with the expressions for the minors, we conclude that the equalities in (3.3.3) hold, at least for $u_+ \in U_+^\circ$. However, $U_+^\circ \subset U_+^P$ is open and dense (this follows from Lemma 2.2.2 using that the transposition is an anti-isomorphism between U_-° and U_+°, and between U_-^P and U_+^P) and U_+^P is irreducible, so these equalities extend to all of U_-^P. □

Again, we will write $p_i^T : u_+ \mapsto p_i(u_+^T)$ for $i = 8, 16$ and $q_i^T : u_+ \mapsto q_i(u_+^T)$ for $i = 12, 16, 20, 24$. Combining Lemmas 3.3.1 and 3.3.2 with Proposition 1.6.3 giving the expression for the coordinate ring of U_+^P, we conclude the following:

Corollary 3.3.3. The coordinate ring of the affine variety U_+^P is given by

$$\mathbb{C}[U_+^P] = \mathbb{C}[p_{\alpha^\vee}^T \mid \alpha^\vee \in \Pi_{w^P}^\vee][(p_8^T)^{-1}, (p_{16}^T)^{-1}, (q_{12}^T)^{-1}, (q_{16}^T)^{-1}, (q_{20}^T)^{-1}, (q_{24}^T)^{-1}].$$

This implies that the coordinate ring of the affine variety $U_-^P = (U_+^P)^T$ is given by

$$\mathbb{C}[U_-^P] = \mathbb{C}[p_{\alpha^\vee} \mid \alpha^\vee \in \Pi_{w^P}^\vee][p_8^{-1}, p_{16}^{-1}, q_{12}^{-1}, q_{16}^{-1}, q_{20}^{-1}, q_{24}^{-1}].$$

This in turn implies that $\pi(U_-^P) \subseteq X_{\mathrm{can}}^\vee$.

Recall that the generating Plücker coordinates are the ones appearing in Lemma 3.3.1. and that the (generalized) Plücker coordinates are defined in equation (1.6.2). The quadratic and cubic expressions q_i are given in equation (3.1.1).

Remark 3.3.4. Note that p_8, p_{16}, q_{12}, q_{16}, q_{20} and q_{24} are not a priori generated by the generators of the coordinate ring. Recall from Corollary 3.3.3 that the coordinate ring is generated by the Plücker coordinates $p_{\alpha^\vee}$ for $\alpha^\vee \in \Pi_{w^P}^\vee$, i.e.

$$p_1, p_2, p_3, p_4', p_4'', p_5', p_5'', p_6', p_6'', p_7', p_7'', p_8', p_8'', p_9'', p_{10}'', p_{11}''.$$

(Note that $p_0 \equiv 1$ on $U_-^\vee$.) However, it turns out that there are "Plücker relations" expressing the remaining Plücker coordinates in terms of these. We have found these using the toric expressions for the Plücker coordinates restricted to U_-°; as $U_-^\circ \subset U_-^P$

is dense and U_-^P is irreducible, this suffices. The relations are given as follows:

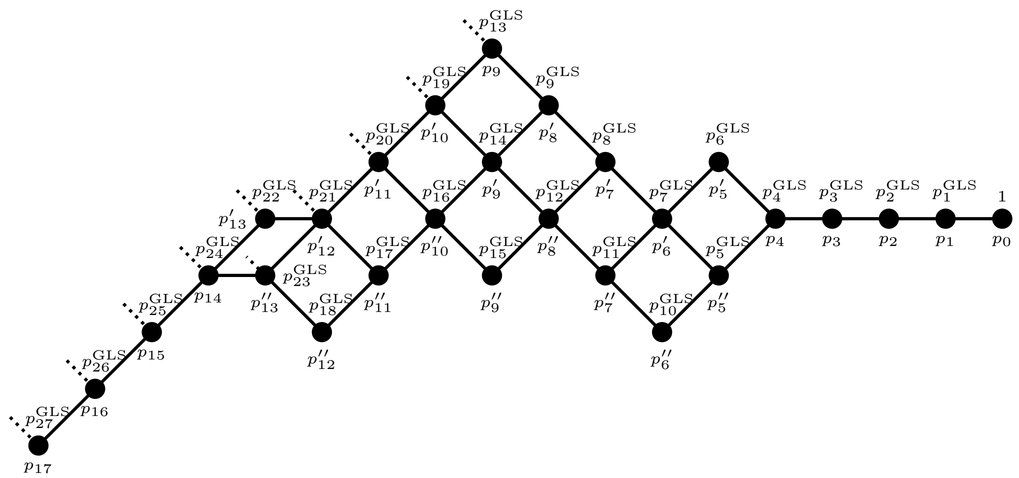
$$
\begin{aligned}
p_8 &= p_1p'_7 - p_2p'_6 + p_3p'_5 - p'_4p''_4, & p'_9 &= p_1p'_8 - p_2p''_7 + p_3p''_6 - p'_4p''_5,\\
p'_{10} &= p_1p''_9 - p_2p''_8 + p''_4p''_6 - p'_5p''_5, & p'_{11} &= p_1p''_{10} - p_3p''_8 + p''_4p''_7 - p''_5p'_6,\\
p'_{12} &= p_2p''_{10} - p_3p''_8 + p''_4p'_8 - p''_5p''_7, & p''_{12} &= p_1p''_{11} - p''_4p'_8 + p''_5p'_7 - p'_4p''_8,\\
p_{13} &= p_2p''_{11} - p'_4p''_9 + p'_5p'_8 - p''_6p'_7, & p_{14} &= p_3p''_{11} - p'_4p''_{10} + p'_6p'_8 - p'_7p''_7,\\
p_{15} &= p''_4p''_{11} - p'_5p''_{10} + p'_6p''_9 - p'_7p''_8, & p_{16} &= p''_5p''_{11} - p''_6p''_{10} + p''_7p''_9 - p'_8p''_8.
\end{aligned}
$$

Thus, the localized expressions are in fact expressible in terms of the generators.

3.4 The coordinate ring of U_-^P for the Freudenthal variety

The approach for the Freudenthal variety E_7^{sc}/P_7 is analogous to the Cayley plane. Recall the from Section 1.7 that $\ell = \ell(w^P) = 27$. Thus, Proposition 1.6.3 (see Proposition 8.5 of [GLS11]) tells us that the coordinate ring of U_+^P is generated by the dual PBW elements p_m^{GLS}, $m \in \{1, \ldots, 27\}$, and localized at the generalized minors $\Delta_{\omega_j^\vee, (w^P)^{-1}(\omega_j^\vee)}$, $j \in \{1, \ldots, 7\}$. The generators are expressed as follows:

Lemma 3.4.1. On $U_+^P = (U_-^P)^T \subset \widetilde{G}^\vee$, we have the following identifications:



(3.4.1)

Here, a vertex marked with $p_i^{(j)}$ below (or left) and p_m^{GLS} above (or right) denotes that $p_i^{(j)}(u_+^T)$ equals $p_m^{\mathrm{GLS}}(u_+)$ up to a constant for $u_+ \in U_+^P$.

Note that the diagram of equation (3.4.1) is a subdiagram of the one in equation (1.7.3). Upon removing the rightmost vertex, this subdiagram is isomorphic to the Hasse diagram of the Cayley plane $\mathbb{OP}^2$ (see equation (1.7.1)).

Proof. This is proven by computing the weight of each of the basis vectors $v_i^{(j)}$ and the positive roots $\beta^\vee_{w^P}(m)$ that are mapped to negative roots by w^P $(m \in \{1, \dots, 27\})$, and using Proposition 3.2.5. Compare Lemma 3.3.1. $\square$

Lemma 3.4.2. The generalized minors defined in equation (1.6.7) can be expressed on U_+^P in terms of Plücker coordinates as

$$\begin{aligned} &\Delta_{\omega_1^\vee,(w^P)^{-1}(\omega_1^\vee)}(u_+) = q_{18}(u_+^T), \quad \Delta_{\omega_2^\vee,(w^P)^{-1}(\omega_2^\vee)}(u_+) = q_{27}(u_+^T),\\ &\Delta_{\omega_3^\vee,(w^P)^{-1}(\omega_3^\vee)}(u_+) = q'_{36}(u_+^T), \quad \Delta_{\omega_4^\vee,(w^P)^{-1}(\omega_4^\vee)}(u_+) = q_{54}(u_+^T),\\ &\Delta_{\omega_5^\vee,(w^P)^{-1}(\omega_5^\vee)}(u_+) = q_{45}(u_+^T), \quad \Delta_{\omega_6^\vee,(w^P)^{-1}(\omega_6^\vee)}(u_+) = q_{36}(u_+^T),\\ &\Delta_{\omega_7^\vee,(w^P)^{-1}(\omega_7^\vee)}(u_+) = p_{27}(u_+^T), \end{aligned} \tag{3.4.2}$$

where the quadratic, cubic and quartic expressions q_i in Plücker coordinates were defined in (3.1.4).

Proof. The arguments are analogous to the proof of Lemma 3.3.2. Restricting to $u_+ \in U_+^\circ = \{x^\vee_{r_1}(a_1)x^\vee_{r_2}(a_2)\cdots y^\vee_{r_{27}}(a_{27}) \mid a_i \in \mathbb{C}^*\}$, where the sequence of indices equals $(r_1, r_2, \dots, r_{27}) = (7,6,5,4,3,2,4,5,6,1,3,4,2,5,7,4,3,1,6,5,4,2,3,4,5,6,7)$, we find that

$$\begin{aligned} \Delta_{\omega_1^\vee,(w^P)^{-1}(\omega_1^\vee)}(u_+) &= \frac{a_{27}}{a_{15}}\left(\textstyle\prod_{i=10}^{27} a_i\right),\\ \Delta_{\omega_2^\vee,(w^P)^{-1}(\omega_2^\vee)}(u_+) &= \frac{a_{18}a_{23}a_{24}a_{25}a_{26}a_{27}}{a_{10}}\left(\textstyle\prod_{i=6}^{27} a_i\right),\\ \Delta_{\omega_3^\vee,(w^P)^{-1}(\omega_3^\vee)}(u_+) &= \frac{a_{26}a_{27}}{a_6a_{14}a_{15}a_{19}}\left(\textstyle\prod_{i=13}^{27} a_i\right)\left(\textstyle\prod_{j=5}^{27} a_j\right),\\ \Delta_{\omega_4^\vee,(w^P)^{-1}(\omega_4^\vee)}(u_+) &= \frac{a_{25}a_{26}a_{27}}{a_{10}a_{11}a_{19}a_{20}}\left(\textstyle\prod_{i_1=17}^{27} a_{i_1}\right)\left(\textstyle\prod_{i_2=8}^{27} a_{i_2}\right)\left(\textstyle\prod_{i_3=4}^{27} a_{i_3}\right),\\ \Delta_{\omega_5^\vee,(w^P)^{-1}(\omega_5^\vee)}(u_+) &= a_9a_{22}a_{24}a_{25}a_{26}a_{27}\left(\textstyle\prod_{i_1=14}^{27} a_{i_1}\right)\left(\textstyle\prod_{i_2=3}^{27} a_{i_2}\right),\\ \Delta_{\omega_6^\vee,(w^P)^{-1}(\omega_6^\vee)}(u_+) &= a_{15}\left(\textstyle\prod_{i_1=19}^{27} a_{i_1}\right)\left(\textstyle\prod_{i_2=2}^{27} a_{i_2}\right),\\ \Delta_{\omega_7^\vee,(w^P)^{-1}(\omega_7^\vee)}(u_+) &= \textstyle\prod_{i=1}^{27} a_i. \end{aligned}$$

We find the same expressions in toric coordinates for the homogeneous polynomials in Plücker coordinates, so that we obtain the equalities in (3.4.2) on U_+°, and therefore on U_+^P as well. $\square$

Using the same notation as in (3.3.2), Lemmas 3.4.1 and 3.4.2 imply:

Corollary 3.4.3. The coordinate ring of the affine variety U_+^P is given by:

$$\mathbb{C}[U_+^P] = \mathbb{C}[p^T_{\alpha^\vee} \mid \alpha^\vee \in \Pi^\vee_{w^P}][(p^T_{27})^{-1}, (q^T_{18})^{-1}, (q^T_{27})^{-1}, (q^T_{36})^{-1}, (q'^T_{36})^{-1}, (q^T_{45})^{-1}, (q^T_{54})^{-1}].$$

This implies that the coordinate ring of the affine variety $U_-^P = (U_+^P)^T)$ is given by

$$\mathbb{C}[U_-^P] = \mathbb{C}[p_{\alpha^\vee} \mid \alpha^\vee \in \Pi^\vee_{w^P}][p_{27}^{-1}, q_{18}^{-1}, q_{27}^{-1}, q_{36}^{-1}, (q'_{36})^{-1}, q_{45}^{-1}, q_{54}^{-1}].$$

Consequently, $\pi(U_-^P) \subseteq X^\vee_{\text{can}}$.

Recall that the (generalized) Plücker coordinates $p_i^{(j)}$ are defined in equation (1.6.2) and the quadratic, cubic and quartic expressions q_i are given in equation (3.1.4).

Remark 3.4.4. Just as in the Remark 3.3.4, the localized expressions p_{27}, q_{18}, q_{36}, (q'_{36}), q_{45} and q_{54} are not a priori generated by the generators of the coordinate ring. Recall that the generators were $p_{\alpha^\vee}$ for $\alpha^\vee \in \Pi^\vee_{w^P}$, i.e.

$$p_1, p_2, p_3, p_4, p'_5, p''_5, p'_6, p''_6, p'_7, p''_7, p'_8, p''_8, p_9, p'_9, p''_9, p'_{10}, p''_{10},$$
$$p'_{11}, p''_{11}, p'_{12}, p''_{12}, p'_{13}, p''_{13}, p_{14}, p_{15}, p_{16}, p_{17}.$$

We can express the remaining Plücker coordinates in terms of these using the following "Plücker relations":

$$p_{10} = p_1p_9 - p_2p'_8 + p_3p'_7 - p_4p'_6 + p'_5p''_5,$$
$$p_{11} = p_1p'_{10} - p_2p'_9 + p_3p''_8 - p_4p''_7 + p'_5p''_6,$$
$$p_{12} = p_1p'_{11} - p_2p''_{10} + p_3p''_9 - p''_5p''_7 + p'_6p''_6,$$
$$p_{13} = p_1p'_{12} - p_2p''_{11} + p_4p''_9 - p''_5p''_8 + p''_6p'_7,$$

$$p'_{14} = p_1 p''_{13} - p_2 p''_{12} + p'_5 p''_9 - p'_6 p_{8,2} + p'_7 p''_7,$$

$$p''_{14} = p_1 p'_{13} - p_3 p''_{11} + p_4 p''_{10} - p''_5 p'_9 + p''_6 p'_8,$$

$$p'_{15} = p_1 p_{14} - p_3 p''_{12} + p'_5 p''_{10} - p'_6 p'_9 + p''_7 p'_8,$$

$$p''_{15} = p_2 p'_{13} - p_3 p'_{12} + p_4 p'_{11} - p''_5 p'_{10} + p''_6 p_9,$$

$$p'_{16} = p_1 p_{15} - p_4 p''_{12} + p'_5 p''_{11} - p'_7 p'_9 + p'_8 p''_8,$$

$$p''_{16} = p_2 p_{14} - p_3 p''_{13} + p'_5 p'_{11} - p'_6 p'_{10} + p''_7 p_9$$

$$p'_{17} = p_1 p_{16} - p''_5 p''_{12} + p'_6 p''_{11} - p'_7 p''_{10} + p'_8 p''_9,$$

$$p''_{17} = p_2 p_{15} - p_4 p''_{13} + p'_5 p'_{12} - p'_7 p'_{10} + p''_8 p_9,$$

$$p_{18} = p_1 p_{17} - p''_6 p''_{12} + p''_7 p''_{11} - p''_8 p''_{10} + p'_9 p''_9,$$

$$p'_{18} = p_2 p_{16} - p''_5 p''_{13} + p'_6 p'_{12} - p'_7 p'_{11} + p_9 p''_9,$$

$$p''_{18} = p_3 p_{15} - p_4 p_{14} + p'_5 p'_{13} - p'_8 p'_{10} + p_9 p'_9,$$

$$p'_{19} = p_2 p_{17} - p''_6 p''_{13} + p''_7 p'_{12} - p''_8 p'_{11} + p''_9 p'_{10},$$

$$p''_{19} = p_3 p_{16} - p''_5 p_{14} + p'_6 p'_{13} - p'_8 p'_{11} + p_9 p''_{10},$$

$$p'_{20} = p_3 p_{17} - p''_6 p_{14} + p''_7 p'_{13} - p'_9 p'_{11} + p'_{10} p''_{10},$$

$$p''_{20} = p_4 p_{16} - p''_5 p_{15} + p'_7 p'_{13} - p'_8 p'_{12} + p_9 p''_{11},$$

$$p'_{21} = p_4 p_{17} - p''_6 p_{15} + p''_8 p'_{13} - p'_9 p'_{12} + p'_{10} p''_{11},$$

$$p''_{21} = p'_5 p_{16} - p'_6 p_{15} + p'_7 p_{14} - p'_8 p''_{13} + p_9 p''_{12},$$

$$p'_{22} = p''_5 p_{17} - p''_6 p_{16} + p''_9 p'_{13} - p''_{10} p'_{12} + p'_{11} p''_{11},$$

$$p''_{22} = p'_5 p_{17} - p''_7 p_{15} + p''_8 p_{14} - p'_9 p''_{13} + p'_{10} p''_{12},$$

$$p_{23} = p'_6 p_{17} - p''_7 p_{16} + p''_9 p_{14} - p''_{10} p''_{13} + p'_{11} p''_{12},$$

$$p_{24} = p'_7 p_{17} - p''_8 p_{16} + p''_9 p_{15} - p''_{11} p''_{13} + p'_{12} p''_{12},$$

$$p_{25} = p'_8 p_{17} - p'_9 p_{16} + p''_{10} p_{15} - p''_{11} p_{14} + p''_{12} p'_{13},$$

$$p_{26} = p_9 p_{17} - p'_{10} p_{16} + p'_{11} p_{15} - p'_{12} p_{14} + p'_{13} p''_{13}.$$

We can express p_{27} in terms of the generating Plücker coordinates as well, but it is more conveniently expressed in terms of the other Plücker coordinates as follows:

$$p_{27} = p_1p_{26} - p_2p_{25} + p_3p_{24} - p_4p_{23} + p'_5p'_{22} + p''_5p''_{22} - p'_6p'_{21} + p'_7p'_{20} - p'_8p'_{19} + p_9p_{18} - p_{10}p_{17}.$$

Substituting the Plücker relations then gives an expression in terms of the generating Plücker coordinates.

3.5 Proof of the isomorphism of LG-models on cominuscule E_n^{sc}/P_n

Recall that we want to show that $\pi : U_-^P \to \mathbb{X}^\vee$ is an isomorphism to $X_{\mathrm{can}}^\vee \subset \mathbb{X}^\vee$

Remark 3.5.1. The following proof follows the same rationale as the proof of Theorem 3.9 in Section 8 of [PR13].

Proof of Theorem 3.1.1. Corollaries 3.3.3 and 3.4.3 showed us that $\pi(U_-^P) \subseteq X_{\mathrm{can}}^\vee$ for cominuscule E_n^{sc}/P_n.

Now, to show that $\pi : U_-^P \to X_{\mathrm{can}}^\vee$ is an isomorphism, it suffices to show that $\pi^* : \mathbb{C}[X_{\mathrm{can}}^\vee] \to \mathbb{C}[U_-^P]$ is an isomorphism since both U_-^P and $X_{\mathrm{can}}^\vee$ are affine varieties (for U_-^P, this follows from Proposition 1.6.3; for $X_{\mathrm{can}}^\vee$, this follows from the fact that it is the complement of an ample divisor). The fact that π^* is injective follows from the fact that $\pi : U_-^P \to X_{\mathrm{can}}^\vee$ is dominant, which follows from the fact that U_-^P and $X_{\mathrm{can}}^\vee$ have the same dimension and that π is an open map (as it is the quotient map $u_- \mapsto \widetilde{P}^\vee u_-$).

Surjectivity of $\pi^* : \mathbb{C}[X_{\mathrm{can}}^\vee] \to \mathbb{C}[U_-^P]$ is an easy check that each of the generators of $\mathbb{C}[U_-^P]$ as given in Corollaries 3.3.3 and 3.4.3 is a well-defined function on $X_{\mathrm{can}}^\vee$, and that each localized expression is non-zero on $X_{\mathrm{can}}^\vee$. This is clear from the fact that they are all generalized Plücker coordinates or expressions in these, and from the definition of $X_{\mathrm{can}}^\vee$. □

Remark 3.5.2. Note that this proof also shows that every coset in $X_{\mathrm{can}}^\vee \subset \widetilde{P}^\vee \backslash \widetilde{G}^\vee$ can uniquely be represented by an element of $U_-^P = U_-^\vee \cap B_+^\vee \bar{w}_P \bar{w}_0 B_+^\vee$.

Now we know that $\pi : U_-^P \to X_{\text{can}}^\vee$ is an isomorphism, showing that $\pi \circ \varphi$ is an isomorphism of Landau-Ginzburg models reduces to checking that $(\pi \circ \varphi)^*(\mathcal{W}_{\text{can}}) = \mathcal{W}_{\text{Lie}}$.

Remark 3.5.3. We are abusing notation here: a more proper way to write this would be $\big((\pi \circ \varphi) \times \mathrm{Id}\big)^*(\mathcal{W}_{\text{can}}) = \mathcal{W}_{\text{Lie}}$, as $\mathcal{W}_{\text{can}} : X_{\text{can}}^\vee \times T^P \to \mathbb{C}$ and $\mathcal{W}_{\text{Lie}} : X_{\text{Lie}}^\vee \times T^P \to \mathbb{C}$. We will continue to use this notation as it should not give rise to ambiguities.

The proof of Theorem 3.1.2 is obtained by restricting to the torus $U_-^\circ \subset U_-^P$ of Definition 2.2.1 and showing that $(\varphi^{-1})^*(\mathcal{W}_{\text{Lie}}) = \pi^*(\mathcal{W}_{\text{can}})$ on $U_-^\circ \times T^P$. This is most conveniently done separately for the Cayley plane and the Freudenthal variety:

Proof of Theorem 3.1.2 for the Cayley plane. Applying the results of Theorem 2.2.7 to the Cayley plane, we see that $(\varphi^{-1})^*(\mathcal{W}_{\text{Lie}})$ can be written as follows on $U_-^\circ \times T^P$:

$$[(\varphi^{-1})^*(\mathcal{W}_{\text{Lie}})](u_-, t) = a_1 + a_2 + \ldots + a_{16} + q\frac{\sum_{S\in\mathcal{S}}\prod_{(\beta,j)\in S} a_{\mathscr{J}(\beta,j)}}{a_1 \cdots a_{16}},$$

where $u_- = y_{r_{16}}^\vee(a_{16}) y_{r_{15}}^\vee(a_{15}) \cdots y_{r_1}^\vee(a_1)$ and $q = \alpha_6^\vee(t)$. (Recall that $(r_1, \ldots, r_{16})$ is the sequence of indices in the reduced expression of w^P in equation (1.7.2).) The relabeling $\mathscr{J}$ is given in Definition 2.5.3 and the set $\mathcal{S}$ is given in Definition 2.5.5.

On the other hand we need to calculate the terms of $\mathcal{W}_{\text{can}}$ as given in equation (3.1.3) applied to an element of U_-°. The algorithm to compute these is the same as the one explained at the end of the proof of Lemma 3.3.2. The computations are straightforward, consisting of elementary operations on polynomials (addition, multiplication, and so on). As this is somewhat cumbersome, we used a computer algebra program and simply state the results here:

$$\frac{p_1}{p_0} = a_9 + a_{16}, \qquad \frac{p_9'}{p_8} = a_1 + a_8, \qquad \frac{q_{13}}{q_{12}} = a_5 + a_{10} + a_{15},$$
$$\frac{q_{17}}{q_{16}} = a_2 + a_7 + a_{12}, \qquad \frac{q_{21}}{q_{20}} = a_4 + a_{13}, \qquad \frac{q_{25}}{q_{24}} = a_3 + a_6 + a_{11} + a_{14},$$
$$q\frac{p_5''}{p_{16}} = q\frac{\sum_{S\in\mathcal{S}}\prod_{(\beta,j)\in S} a_{\mathscr{J}(\beta,j)}}{a_1 \cdots a_{16}},$$

where we suppressed the arguments of the Plücker coordinates (they are all applied to u_-). Clearly, we have $(\varphi^{-1})^*(\mathcal{W}_{\mathrm{Lie}}) = \pi^*(\mathcal{W}_{\mathrm{can}})$ on $U_-^\circ \times T^P$, but since $U_-^\circ \subset U_-^P$ is open and dense and U_-^P is irreducible, we conclude that the equality holds on all of $U_-^P \times T^P$. The theorem now follows in the case of the Cayley plane. $\square$

Proof of Theorem 3.1.2 for the Freudenthal variety. In this case, the results of Theorem 2.2.7 yield the following expression for $(\varphi^{-1})^*(\mathcal{W}_{\mathrm{Lie}})$ on $U_-^\circ \times T^P$:

$$[(\varphi^{-1})^*(\mathcal{W}_{\mathrm{Lie}})](u_-, t) = a_1 + a_2 + \ldots + a_{27} + q\frac{\sum_{S\in\mathcal{S}} \prod_{(\beta,j)\in S} a_{\mathscr{J}(\beta,j)}}{a_1 \cdots a_{27}},$$

where $u_- = y_{r_{27}}^\vee(a_{27}) y_{r_{26}}^\vee(a_{26}) \cdots y_{r_1}^\vee(a_1)$ and $q = \alpha_7^\vee(t)$. (Recall that $(r_1, \ldots, r_{27})$ is the sequence of indices in the reduced expression of w^P in equation (1.7.4).)

The terms of $\mathcal{W}_{\mathrm{can}}$ applied to an element of U_-° can be computed using a computer algebra program to be the following:

$$\frac{p_1}{p_0} = a_1 + a_{15} + a_{27}, \qquad \frac{q_{19}}{q_{18}} = a_2 + a_9 + a_{19} + a_{26}, \qquad \frac{q_{28}}{q_{27}} = a_6 + a_{13} + a_{22}$$
$$\frac{q_{37}}{q_{36}} = a_{10} + a_{18}, \qquad \frac{q'_{37}}{q'_{36}} = a_3 + a_8 + a_{14} + a_{20} + a_{25}, \qquad \frac{q_{46}}{q_{45}} = a_5 + a_{11} + a_{17} + a_{23}$$
$$\frac{q_{55}}{q_{54}} = a_4 + a_7 + a_{12} + a_{16} + a_{21} + a_{24}, \qquad q\frac{p_{10}}{p_{27}} = q\frac{\sum_{S\in\mathcal{S}} \prod_{(\beta,j)\in S} a_{\mathscr{J}(\beta,j)}}{a_1 \cdots a_{27}}$$

Here all the Plücker coordinates are applied to $u_- \in U_-^\circ$, so this is suppressed. This shows that $(\varphi^{-1})^*(\mathcal{W}_{\mathrm{Lie}}) = \pi^*(\mathcal{W}_{\mathrm{can}})$ on $U_-^\circ \times T^P$, implying that the equality holds on $U_-^P \times T^P$, as $U_-^\circ \subset U_-^P$ is open and dense and U_-^P is irreducible. $\square$

Thus, we have shown that the canonical models we defined in Section 3.1 for the exceptional cominuscule homogeneous spaces are isomorphic to the Lie-theoretic Landau-Ginzburg models given in [Rie08] (discussed in Section 1.4 of this thesis). From this we conclude that the canonical models are indeed Landau-Ginzburg models, satisfying

$$qH^*(X)_{\mathrm{loc}} \cong \mathbb{C}[X_{\mathrm{can}} \times T^P]/\langle \partial \mathcal{W}_{\mathrm{can}} \rangle,$$

where $\langle \partial \mathcal{W}_{\text{can}} \rangle$ is generated by the derivatives of $\mathcal{W}_{\text{can}}$ along X_{can}. Due to the geometric Satake correspondence, this isomorphism is given by associating to $\sigma_i \in H^*(X)$ the Plücker coordinate $p_i \in \mathbb{C}[X_{\text{can}}]$.

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