Extension Theory of Symmetric Second-Order Difference Operators

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Abstract

This thesis concerns itself with the extension theory of second-order difference equations in both a linear operator and linear relation framework.

In Chapter 1, we introduce the extension theory of linear operators by means of both the von Neumann and the Kreĭn-Vishik-Birman method. Chapter 2 is devoted to the construction of the non-negative, self-adjoint extensions of a particular class of second-order difference operators via the Kreĭn-Vishik-Birman theory, with particular emphasis on the Friedrichs extension. We determine an explicit characterisation of such extensions, before applying this result to a second-order difference equation whose solutions are the Stieltjes-Wigert polynomials.

Linear relations and their extensions are introduced in Chapter 3. In particular, a construction of the extremal maximal sectorial relations by Hassi et al. is considered. These results are utilised in Chapter 4 when we construct the extremal maximal sectorial extensions of the Discrete Laplacian with both the standard domain ℓ^2 and sequences in ℓ^2 whose first component equals 0.

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Introduction

Isolated, the subtlety of a Jacobi operator's importance may go unnoticed, however Jacobi operators — infinite tri-diagonal matrices acting in sequence spaces [53] — are inseparable from orthogonal polynomials, whose mystery and utility have been researched extensively — albeit not completely — for many decades. Indeed, the breadth of this research topic is staggering; one can pinpoint the influence of these operators within the classical moment problem [2, 50], continued fractions [39, 55] and random matrix theory [25], and only be scratching the surface. However, perhaps the most striking feature of the Jacobi operator is the following: it is the discretisation of a second-order differential operator. In fact, the symmetric Jacobi operators that will be focal to this thesis are precisely the analogue to Sturm-Liouville differential expressions on an interval [4, 6, 45].

Likewise, self-adjoint extensions of symmetric operators in Hilbert spaces have been studied systematically since von Neumann [46], laying down the operator theoretic foundations for quantum mechanics. Indeed, given a closed, symmetric operator in a Hilbert space, the method of von Neumann characterises the self-adjoint extensions via unitary maps between deficiency spaces. If, in addition, the operator is positive, then we may characterise all non-negative, self-adjoint extensions by means of the Kreĭn-Vishik-Birman (KVB) theory instead [5, 16, 41, 42, 54]. Having specified that the operator is now positive, one (natural!) benefit of the KVB theory immediately surfaces: there exist two distinguished extensions — the Friedrichs extension and the Kreĭn extension — such that all remaining extensions fall somewhere in between. In the sense of quadratic forms, the Friedrichs extension — first introduced in [30] — is the largest non-negative, self-adjoint extension, whilst the Kreĭn extension (also known as the Kreĭn-von Neumann extension [49]), the smallest [41, 46].

Kreĭn-Vishik-Birman theory has since been applied to numerous classes of operators: Brown and Evans determined the non-negative, self-adjoint extensions of Sturm-Liouville operators [20], whilst second-order elliptic partial differential operators were considered in [21], for example. Moreover, KVB theory was extended to dual pairs of operators and applied to elliptic PDEs in [31]. However, we — in the first half of this thesis — use the KVB theory to construct non-negative, self-adjoint extensions of a general positive, symmetric Jacobi operator. We should note that [19] utilises the von Neumann theory to determine boundary conditions associated with the Friedrichs and Kreĭn extensions of positive Jacobi operators.

However, if one resolved to investigate an operator, then it would be equally valid to consider its graph instead. Linear relations — arguably, first examined by Arens in [10] — are the gateway into this line of thought: pairs of elements in a product of Hilbert spaces. One drawback of both the von Neumann and KVB theory — and a quirk of operator theory in general — is that the adjoint of a non-densely defined operator is not defined. Linear relations overcome this issue and so it becomes natural to posit questions such as, 'How can one construct the self-adjoint extensions of a non-densely defined operator?'

Indeed, great strides into answering such questions have been made: methods for the construction of these extensions — in addition to general discussion on linear relations — can be attributed to papers such as [11, 24, 26, 32, 33]. This network of mathematicians present various results related to the sectorial extensions of a sectorial linear relation via several constructions, most notably through an association with sesquilinear forms and through a method of factorisation. Moreover, an analogous construction to the von Neumann theory exists in the context of linear relations [15].

Given the significance of both the Friedrichs and Kreĭn extension in the operator context, it is only reasonable to wonder if such concepts translate accordingly into the world of relations. Indeed, for a non-densely defined sectorial relation S, the Friedrichs extension S_F — in the form recognisable to this thesis — can be traced back to [47]. Conversely, the Kreĭn extension S_K of a non-negative, non-densely defined sectorial relation S enjoys the following form courtesy of [8, 24]: $S_K = ((S^{-1})_F)^{-1}$. These extensions were referred to as 'distinguished' above, but when we delve into extension theory of linear relations, this concept may be expressed more formally: they are examples of *extremal* sectorial extensions. As such, the second half of this thesis aims to construct sectorial extensions of the linear relations that are associated to a certain class of Jacobi operators.

With this division in mind, we choose to partition the thesis into two distinct halves. As such, we devote Chapter 1 to the presentation of the fundamental definitions and concepts necessary to appreciate the original research undertaken in Chapter 2. We introduce various results concerning linear operators acting in Hilbert spaces; the majority of these results are well known, but we endeavour to provide the reader with a full account of the relevant theory in the interest of self-containment. If it appears as though we have simply brandished terms such as the 'Friedrichs' or 'Kreĭn extension' of an operator, or the 'von Neumann' or 'KVB theory' indiscriminately, then Section 1.2 aims to elucidate these essential expressions so that they no longer appear esoteric. Although only the KVB theory is central to this thesis, we dedicate Appendix A to an example of Sturm-Liouville type where we apply both of these methods comprehensively. The purpose is twofold: we hope to familiarise the reader with the concepts fundamental to the thesis, in addition to providing a template that we may follow during Chapter 2. Finally, we close this first, introductory chapter by discussing relevant concepts and results from difference operator theory. The sections prior are somewhat general and abstract; here, we establish the types of operators that will be investigated in the remaining chapters of the thesis.

Chapter 2 opens with a short overview of the problem to be addressed. Essentially, we associate to a general positive Jacobi operator a sesquilinear form so that we may then invoke KVB theory. In particular, we conjecture a form domain for the Friedrichs extension before proving conclusively the necessary conditions that it must possess. As the Friedrichs extension forms the basis of KVB theory, we spend ample time convincing ourselves that the argument holds. Then, arguably, Section 2.5 is the focus of the first half of the thesis: it is here that we consolidate the chapter thus far, and arrive at the topic of research's main theorem. The format of this chapter is unsurprising — in essence, each section and subsection is dedicated to proving the next stage in the argument, before culminating with the result that characterises the operator domains of the non-negative, self-adjoint extensions of the positive operator introduced at the beginning of the chapter. We feel it prudent to emphasise that, whilst the results are intended to be the discrete analogues to those presented in [20], the form domain of the Friedrichs extension and, consequently, the result central to this chapter is, to our knowledge, new. Given the aforementioned, intimate relationship between Jacobi operators and orthogonal polynomials, we conclude this chapter — and half of the thesis — with an example. In particular, we apply our result to an expression whose solutions are the Stieltjes-Wigert polynomials.

We then move into the second half of the thesis with Chapter 3. The two halves are designed to be symmetric: in this chapter, we introduce the basic terminology and results pertinent to linear relations. Indeed, the decision to bisect the thesis becomes most obvious here as we simply *build* upon the concepts and theory presented in Chapter 1, once we realise that linear relations are simply the generalisation of graphs of operators. Then, we provide the definition of both the Friedrichs and Kreĭn extension of a linear relation, in addition to both the von Neumann theory (in this context, for completeness) and our construction method of choice: the theory presented in [33]. Whilst it might be alluring to draw further parallels between the chapters so far — both halves construct the Friedrichs and Kreĭn extension using a method that utilises sesquilinear forms — we do assert that the underlying foundation to both constructions is subtly different. Here, the Kreĭn extension has an explicit dependence on the Friedrichs extension as opposed to being 'just another' extension that one can construct, given a different parameter. With that said, however, Section 3.2.3 details a method in constructing *extremal* maximal sectorial extensions of a sectorial linear relation. Once we delve into this construction more deeply, it becomes apparent that *this* theory is more akin to the KVB theory — ultimately, we see that the Friedrichs extension and Kreĭn extension are the two ends of a containment string, as will become familiar.

Finally, we conclude this thesis by applying the results detailed in Chapter 3 to two explicit examples: the discrete Laplacian on two different domains. We begin by constructing both the Friedrichs and Kreĭn extension of a — perhaps obvious — sectorial relation. Although the results in Section 4.2 may not be surprising, the section offers an opportunity for the reader to become acquainted with the theory with a straightforward example and provides us a structure and methodology that we may follow in Sections 4.3 and 4.4. Indeed, these two sections form the bulk of this chapter as the nuances and minutiae of the theory surface throughout in this second, more complex example. Once we have exhausted the analysis of the first method, in Section 4.5, we continue by producing the extremal maximal sectorial relations of the two examples via the factorisation method detailed in Section 3.2.3. Whilst this chapter merely applies known results to two specific examples, we assert that this does have value as the results we utilise are exclusively abstract. In fact, we conclude this chapter – and the thesis — with an outlook, with this in mind. Indeed, we contemplate a class of bounded Jacobi operators and propose a way to generalise our calculations, in addition to drawing attention back to the more appropriate 'comparison' between this and the KVB theory. On the surface, the strongest through-line of the thesis may appear, simply, to be the Jacobi operator studied, but the countless parallels between the two halves will be emphasised, and the thesis, entire.

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Chapter 1

Preliminaries: Extensions of Linear Operators

1.1 Basic Definitions and Properties

The objective of this section will be to introduce various fundamental definitions and terminology that will be required in future sections of the thesis. In particular, we introduce the relevant spaces, two mappings in such spaces, and their relationship to one another. For a more in-depth account of the concepts provided here, we refer to Akhiezer and Glazman [3], Edmunds and Evans [27], Kato [37] and Kreyszig [43].

1.1.1 Hilbert Spaces

This section begins by detailing the spaces that will be of interest to us, in a general setting. In what follows, we may take the scalar field $\mathbb{K}$ to be either the set of real numbers or the set of complex numbers, denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively.

Definition 1.1.1. Let X be a vector space over the scalar field $\mathbb{K}$. The mapping

$$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$$

is called an *inner product* if, for all vectors x, y, $z \in X$ and all scalars $\lambda \in \mathbb{K}$, the following four conditions are satisfied:

(IP1) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(IP2) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$

(IP3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(IP4) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0.$

An *inner product space* is a vector space X equipped with an inner product that is defined on X: we denote this by $(X, \langle\cdot,\cdot\rangle)$. Moreover, the inner product on X naturally induces a *norm* on X:

$$\|x\| = \sqrt{\langle x, x\rangle} \geq 0, \qquad x \in X.$$

If a vector space X is equipped with a norm $\|\cdot\|$, then $(X, \|\cdot\|)$ is called a *normed space.*

We will be interested in a specific type of inner product space, so we now introduce the following definition.

Definition 1.1.2. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for all $\varepsilon > 0$, there exists an $N = N(\varepsilon)$ such that, for all n, $m > N$, we have $\|x_n - x_m\| < \varepsilon$, where $\|\cdot\|$ is the norm induced by the inner product.

If every Cauchy sequence $\{x_n\}$ in $(X, \langle\cdot,\cdot\rangle)$ converges to some $x \in X$, that is, there exists an $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$, then the inner product space is *complete.* Complete inner product spaces are called *Hilbert spaces.*

Remark. Inner product spaces may also be referred to as *pre-Hilbert spaces.* This terminology makes sense: a Hilbert space is merely a complete pre-Hilbert space.

Remark. If there is no danger of ambiguity, then, for brevity, when we refer to a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, we may omit the inner product in our notation. In other words, $H = (H, \langle\cdot,\cdot\rangle)$.

Throughout the thesis, we will often find it useful to decompose a Hilbert space H into a direct sum of two spaces. With this in mind, we introduce the orthogonal complement to a subspace M in H.

Definition 1.1.3. Let H be a Hilbert space and M a subspace of H. The *orthogonal complement* $M^\perp$ is the set of elements in H that are orthogonal to *every* element in M, i.e.,

$$\begin{aligned} M^\perp &= \{y \in H \mid y \perp M\} \\ &= \{y \in H \mid \langle y, x\rangle = 0 \text{ for all } x \in M\}. \end{aligned}$$

Remark. The orthogonal complement $M^\perp$ of M is a closed subspace.

Consequently, H admits the following decomposition: $H = \overline{M} \oplus M^\perp$. Thus, if M itself is closed, then $H = M \oplus M^\perp$. We often use the notation

$M^\perp = H \ominus M$ due to its generality: the subspaces in question are explicit and it is suggestive of the direct sum notation.

We conclude this section by presenting two inequalities that will be instrumental in the analysis performed in later sections: the first is known as the *Cauchy-Schwarz inequality*, whilst the second, the *triangle inequality.*

Lemma 1.1.4. *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and let $\| \cdot \|$ be the norm induced by the inner product. For all x, $y \in X$, we have*

$$|\langle x, y \rangle| \leq \|x\| \|y\| \qquad \textit{and} \qquad \|x + y\| \leq \|x\| + \|y\|.$$

1.1.2 Linear Operators in Hilbert Spaces

Operators acting in Hilbert spaces can have many properties that make them interesting to study, or useful in practice. This section will describe several desirable properties that the operators we will be concerned with may possess.

Definition 1.1.5. Let X and Y be vector spaces over the same field $\mathbb{K}$. We say that $T \colon \mathcal{D}(T) \to \mathcal{R}(T)$ is a *linear operator* from $\mathcal{D}(T) \subseteq X$ onto $\mathcal{R}(T) \subseteq Y$ (or, into Y) if, for all x, $y \in \mathcal{D}(T)$ and $\lambda \in \mathbb{K}$, we have

$$T(x + y) = Tx + Ty \qquad \text{and} \qquad T(\lambda x) = \lambda Tx.$$

The vector space $\mathcal{D}(T)$ is called the *domain* of T, whilst $\mathcal{R}(T)$ denotes the *range* of T: specifically, we have

$$\mathcal{R}(T) = \{ y \in Y \mid Tx = y \text{ for some } x \in \mathcal{D}(T) \}.$$

When the domain $\mathcal{D}(T)$ of an operator T is dense in the ambient Hilbert space, we have the following useful definition.

Definition 1.1.6. Let H_1 and H_2 be Hilbert spaces and $T \colon \mathcal{D}(T) \to H_2$ a linear operator, where $\mathcal{D}(T) \subseteq H_1$. The operator T is *densely defined* if $\mathcal{D}(T)$ is dense in H_1, that is, $\overline{\mathcal{D}(T)} = H_1$.

We will also make reference to the *kernel*, or *null space*, of a linear operator. In particular, the kernel of T is the set of all elements x in the domain of T that are mapped to the zero vector in Y, i.e.,

$$\ker T = \{ x \in \mathcal{D}(T) \mid Tx = 0 \}.$$

Throughout this thesis, we will be concerned with operators that act in Hilbert spaces. As such, the following definitions replace generic normed spaces $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ with Hilbert spaces $H_1 = (H_1, \langle \cdot, \cdot \rangle_{H_1})$ and

$H_2 = (H_2, \langle \cdot, \cdot \rangle_{H_2})$. The inner products $\langle \cdot, \cdot \rangle_{H_1}$ and $\langle \cdot, \cdot \rangle_{H_2}$ induce the norms $\| \cdot \|_{H_1}$ and $\| \cdot \|_{H_2}$ respectively. Then, we continue by presenting the next of our definitions.

Definition 1.1.7. Let $T \colon \mathcal{D}(T) \to H_2$ be a linear operator, where $\mathcal{D}(T) \subseteq H_1$. The operator T is *bounded* if there exists a constant $c \in \mathbb{R}$ such that

$$\|Tx\|_{H_2} \leq c\|x\|_{H_1},$$

for all $x \in \mathcal{D}(T)$. If no such c exists, then T is *unbounded* instead.

Remark. When T is a linear operator, T being bounded is equivalent to T being *continuous*.

When we are in possession of an operator T, it can be useful to speak of the inverse operator T^{-1}, and when such an operator even exists.

Definition 1.1.8. Let $T \colon \mathcal{D}(T) \to H_2$ be a linear operator, where $\mathcal{D}(T) \subseteq H_1$. If T is an injective operator, that is

$$x_1 \neq x_2 \implies Tx_1 \neq Tx_2, \qquad x_1,\, x_2 \in \mathcal{D}(T),$$

then $T^{-1} \colon \mathcal{R}(T) \to \mathcal{D}(T)$, where $Tx_0 \in \mathcal{R}(T) \subseteq H_2$ is mapped to $x_0 \in \mathcal{D}(T)$, is called the *inverse operator* of T.

In fact, the inverse operator T^{-1} of a linear operator T exists if and only if the kernel of T contains *only* the zero vector. More precisely, we have the following theorem.

Theorem 1.1.9 ([43, Thm. 2.6-10])**.** *Let H_1 and H_2 be Hilbert spaces over the same field $\mathbb{K}$. Let $T \colon \mathcal{D}(T) \to H_2$ be a linear operator with domain $\mathcal{D}(T) \subseteq H_1$ and range $\mathcal{R}(T) \subseteq H_2$. The inverse operator $T^{-1} \colon \mathcal{R}(T) \to \mathcal{D}(T)$ exists if and only if*

$$Tx = 0 \implies x = 0.$$

Furthermore, if T^{-1} exists, then it is a linear operator.

With the existence of the inverse operator fresh in one's mind, we make a short detour into spectral theory by presenting some basic definitions that will surface in later sections.

Let H be a non-empty complex Hilbert space and $T \colon \mathcal{D}(T) \to H$ a linear operator with domain $\mathcal{D}(T) \subseteq H$. For any $\lambda \in \mathbb{C}$, we can associate to T the operator

$$T_\lambda = T - \lambda I,$$

where I is the identity operator on $\mathcal{D}(T)$. When T_λ^{-1} exists, we call this operator the *resolvent operator* of T. This operator (often simply referred to as the *resolvent*) is denoted by $R_\lambda(T)$, providing the following notation:

$$R_\lambda(T) \coloneqq T_\lambda^{-1} = (T - \lambda I)^{-1}.$$

Since the resolvent has an explicit dependence on λ, we will be interested in finding regions of the complex plane where the resolvent possesses (or not!) certain properties. In fact, the subsequent definitions will precisely partition the complex plane into four disjoint sets which each depend on the value that λ takes.

Definition 1.1.10. Let H be a non-empty complex Hilbert space and let $T\colon \mathcal{D}(T) \to H$ be a densely defined linear operator with domain $\mathcal{D}(T) \subseteq H$. Let $\lambda \in \mathbb{C}$. If the following three conditions are satisfied, then λ is called a *regular value* of T:

(**R1**) $R_\lambda(T)$ exists,

(**R2**) $R_\lambda(T)$ is bounded,

(**R3**) $R_\lambda(T)$ is defined on a set which is dense in H.

The set

$$\rho(T) \coloneqq \{\lambda \in \mathbb{C} \mid \lambda \text{ is a regular value of } T\}$$

is called the *resolvent set* of T. Conversely, the set

$$\sigma(T) \coloneqq \mathbb{C} \setminus \rho(T)$$

is called the *spectrum* of T. A complex number $\lambda \in \sigma(T)$ is referred to as a *spectral value* of T.

It was alluded to briefly that the complex plane could be partitioned into four disjoint sets. We can further decompose the spectrum into three disjoint sets by investigating which of the conditions expressed in Definition 1.1.10 are (and are *not*) satisfied. Then, we arrive at the following definitions.

Definition 1.1.11. The *point spectrum*, or *discrete spectrum*, which we denote by $\sigma_p(T)$, is the set of $\lambda \in \mathbb{C}$ such that the resolvent $R_\lambda(T)$ does *not* exist: (**R1**) fails. An element λ of this set is called an *eigenvalue* of T.

The *continuous spectrum*, which we denote by $\sigma_c(T)$, is the set of $\lambda \in \mathbb{C}$ such that the resolvent $R_\lambda(T)$ exists and is defined on a set which is dense in H, but is *not* bounded: (**R1**) and (**R3**) are satisfied, but (**R2**) fails.

The *residual spectrum*, which we denote by $\sigma_r(T)$, is the set of $\lambda \in \mathbb{C}$ such that the resolvent $R_\lambda(T)$ exists but is *not* defined on a set that is dense in H: (**R1**) holds, but (**R3**) fails. Here, the resolvent operator can be either bounded or unbounded — the fulfilment of (**R2**) does not matter.

From these definitions, we then have the following decomposition of the complex plane:

$$\begin{aligned}\mathbb{C} &= \rho(T) \,\dot{\cup}\, \sigma(T) \\ &= \rho(T) \,\dot{\cup}\, \sigma_p(T) \,\dot{\cup}\, \sigma_c(T) \,\dot{\cup}\, \sigma_r(T),\end{aligned}$$

where $\dot{\cup}$ denotes the disjoint union between the sets.

Now that we have discussed the basic definitions involved in spectral theory, we continue by presenting more concepts relevant to linear operators that act in Hilbert spaces. For the remainder of this section, $H = (H, \langle \cdot, \cdot \rangle)$ will denote a complex Hilbert space, unless otherwise specified.

Definition 1.1.12. Let $T\colon \mathcal{D}(T) \to H$ be a densely defined linear operator, where $\mathcal{D}(T) \subseteq H$. The *adjoint operator* $T^*\colon \mathcal{D}(T^*) \to H$ is the operator with domain

$$\mathcal{D}(T^*) = \{y \in H \mid \exists z \in H \text{ such that } \langle Tx, y \rangle = \langle x, z \rangle \text{ for all } x \in \mathcal{D}(T)\}.$$

For each $y \in \mathcal{D}(T^*)$, the adjoint operator T^* is defined by the following equality: $T^* y = z$.

Remark. Whenever T is a densely defined operator, the adjoint T^* will be a linear operator.

The definition of the adjoint operator introduces an important equality between two inner products. From this, we obtain the following two definitions.

Definition 1.1.13. Let $T\colon \mathcal{D}(T) \to H$ be a densely defined linear operator, where $\mathcal{D}(T) \subseteq H$. The operator T is called *symmetric* if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all x, $y \in \mathcal{D}(T)$.

Definition 1.1.14. Let $T\colon \mathcal{D}(T) \to H$ be a densely defined linear operator, where $\mathcal{D}(T) \subseteq H$. The operator T is called *self-adjoint* if $T^* = T$.

Remark. From the two definitions above, it is immediate that a self-adjoint operator is also symmetric. However, a symmetric operator is not necessarily self-adjoint as $\mathcal{D}(T)$ may be a proper subset of $\mathcal{D}(T^*)$.

We continue by investigating the expression $\langle Tx, x \rangle$; in particular, we focus our attention on the value that this expression may take.

Definition 1.1.15. Let $T\colon \mathcal{D}(T) \to H$ be a linear operator, where $\mathcal{D}(T) \subseteq H$. The set

$$\Theta(T) = \{\langle Tx, x\rangle \in \mathbb{C} \mid x \in \mathcal{D}(T),\ \|x\| = 1\}$$

is called the *numerical range* of the operator T.

Since the numerical range $\Theta(T)$ of an operator T is a subset of the complex plane, we are able to further describe the operator depending on which region of the plane $\Theta(T)$ lies in. For example, it can be shown that when T is symmetric, the quantity $\langle Tx, x\rangle$ is entirely real. Indeed, let $\langle Tx, x\rangle = a + ib$ for some a, $b \in \mathbb{R}$. Then

$$\begin{aligned}
\langle Tx, x\rangle &= \overline{\langle x, Tx\rangle} && \text{by } \textbf{(IP3)}\\
&= \overline{\langle Tx, x\rangle} && \text{as } T \text{ is symmetric}\\
&= a - ib.
\end{aligned}$$

From this equality, it is clear that $b = 0$ or, in other words, $\langle Tx, x\rangle \in \mathbb{R}$. Knowing when $\Theta(T)$ is a subset of the real line is useful as we can then classify the operator further: the following definitions explore this statement.

Definition 1.1.16. Let $T\colon \mathcal{D}(T) \to H$ be a symmetric linear operator, where $\mathcal{D}(T) \subseteq H$. The operator T is said to be *bounded below* if there exists a constant $\gamma \in \mathbb{R}$ such that

$$\langle Tx, x\rangle \geq \gamma \langle x, x\rangle, \qquad \forall x \in \mathcal{D}(T).$$

The largest such γ is called the *lower bound*. Likewise, T is said to be *bounded above* if there exists a constant $\mu \in \mathbb{R}$ such that

$$\langle Tx, x\rangle \leq \mu \langle x, x\rangle, \qquad \forall x \in \mathcal{D}(T).$$

The smallest such μ is called the *upper bound*. If T is either bounded above or bounded below, then we say that T is *semi-bounded*.

In fact, we obtain useful terminology by specifically choosing γ or μ equal to zero in the above definitions. In particular, we have the following.

Definition 1.1.17. Let $T\colon \mathcal{D}(T) \to H$ be a symmetric linear operator, where $\mathcal{D}(T) \subseteq H$. Then

$$\begin{aligned}
&T \text{ is } \textit{positive } (\textit{non-negative}) \iff \langle Tx, x\rangle > 0\ (\geq 0), && \forall x \in \mathcal{D}(T) \setminus \{0\},\\
&T \text{ is } \textit{negative } (\textit{non-positive}) \iff \langle Tx, x\rangle < 0\ (\leq 0), && \forall x \in \mathcal{D}(T) \setminus \{0\}.
\end{aligned}$$

If T is positive (non-negative), then we write $T > 0$ (≥ 0). Similarly, when T is negative (non-positive), we write $T < 0$ (≤ 0).

In these cases, the numerical range lies entirely on the real axis, but we will also be interested in numerical ranges that lie within sectors, as expressed in the following definition.

Definition 1.1.18. Let $T\colon \mathcal{D}(T) \to H$ be a linear operator, where $\mathcal{D}(T) \subseteq H$. If the numerical range $\Theta(T)$ is contained within a sector $S_{\gamma,\theta}$ in the complex plane, where

$$S_{\gamma,\theta} = \{z \in \mathbb{C} \mid |\arg(z-\gamma)| \leq \theta\}, \qquad \gamma \in \mathbb{R},\ \theta \in \left[0, \frac{\pi}{2}\right),$$

then T is said to be *sectorial.* The constant γ is called the *vertex*, whilst θ is called the *semi-angle.*

Remark. If T is sectorial, then the sector in which the numerical range belongs to is not unique.

Finally, it can be useful to consider an operator T as the set of pairs (x, Tx). This concept will be explored further in Chapter 3, but for now we simply deliver the following definition.

Definition 1.1.19. Let H_1 and H_2 be Hilbert spaces over the same field $\mathbb{K}$ and $T\colon \mathcal{D}(T) \to H_2$ a linear operator, with $\mathcal{D}(T) \subseteq H_1$. The set

$$\mathcal{G}(T) = \{(x, y) \in H_1 \times H_2 \mid x \in \mathcal{D}(T),\ y = Tx\}$$

is known as the *graph* of T. If $\mathcal{G}(T)$ is a closed set in the space $H_1 \times H_2$ endowed with the norm $\|\cdot\|_{H_1 \times H_2}$, where

$$\|(x, y)\|_{H_1 \times H_2} = \|x\|_{H_1} + \|y\|_{H_2}, \qquad (x, y) \in H_1 \times H_2,$$

then T is called *closed.*

In fact, there are two other useful characterisations of a closed operator that we make use of, presented as follows. Let H_1 and H_2 be Hilbert spaces over the same field $\mathbb{K}$ and $T\colon \mathcal{D}(T) \to H_2$ a linear operator, with $\mathcal{D}(T) \subseteq H_1$. Firstly, let $\{x_n\}$ be a sequence in H_1. Then T is a closed operator if and only if

$$x_n \to x \text{ in } H_1 \text{ and } Tx_n \to y \text{ in } H_2 \implies x \in \mathcal{D}(T) \text{ and } Tx = y.$$

Alternatively, T is closed if and only if $(\mathcal{D}(T), \|\cdot\|_T)$ is a complete space, where $\|x\|_T = \|x\|_{H_1} + \|Tx\|_{H_2}$ for all $x \in \mathcal{D}(T)$.

Closed operators have several useful applications. One such application is demonstrated by that which we call the Rank-Nullity theorem, as adapted from [48, Prop. 1.6].

Theorem 1.1.20. *Let H_1 and H_2 be Hilbert spaces and $T\colon \mathcal{D}(T) \to H_2$ a closed, densely defined linear operator, where $\mathcal{D}(T) \subseteq H_1$. Then,*

$$H_2 = \overline{\mathcal{R}(T)} \oplus \ker T^*.$$

Remark. If $\ker T^* = \{0\}$, then $\overline{\mathcal{R}(T)} = H_2$. In other words, $\mathcal{R}(T)$ is dense in H_2 — we will frequently make use of this argument during later sections.

We have now presented all of the required definitions that will be used throughout the thesis with regards to general linear operators. The next section aims to achieve the same, but with so-called sesquilinear forms instead.

1.1.3 Operators and Forms

In Section 1.1.1 we introduced inner products and Hilbert spaces; here, we introduce sesquilinear forms and the properties that they may exhibit instead. The definitions presented will be in a structure comparable to that of Section 1.1.2 for maximal insight. We then conclude this section with an important representation theorem that showcases the relationship between linear operators and a certain class of sesquilinear forms: this relationship forms the basis of the results presented during the next chapter of the thesis.

Definition 1.1.21. Let X and Y be vector spaces over the scalar field $\mathbb{C}$. The mapping

$$\mathbf{a}\,[\cdot,\cdot] : X \times Y \to \mathbb{C}$$

is called a *sesquilinear form* if, for all vectors x, x_1, $x_2 \in X$ and y, y_1, $y_2 \in Y$ and all scalars λ, $\mu \in \mathbb{C}$, the following four conditions are satisfied:

(**SF1**) $$\mathbf{a}\,[x_1 + x_2, y] = \mathbf{a}\,[x_1, y] + \mathbf{a}\,[x_2, y]$$

(**SF2**) $$\mathbf{a}\,[x, y_1 + y_2] = \mathbf{a}\,[x, y_1] + \mathbf{a}\,[x, y_2]$$

(**SF3**) $$\mathbf{a}\,[\lambda x, y] = \lambda \mathbf{a}\,[x, y]$$

(**SF4**) $$\mathbf{a}\,[x, \mu y] = \overline{\mu}\mathbf{a}\,[x, y]\,.$$

Remark. It can be shown directly from the definition that an inner product is an example of a sesquilinear form.

Similarly to before, we will consider sesquilinear forms (often simply referred to as *forms*) in Hilbert spaces rather than in general vector spaces. In fact, the forms that will be of interest to us will map from $Q(\mathbf{a}) \times Q(\mathbf{a})$ to $\mathbb{C}$, where $Q(\mathbf{a}) \subseteq H$ denotes the *form domain* of $\mathbf{a}$, unless stated otherwise. Furthermore, we will make reference to the *quadratic form*, that is, the map $\mathbf{a}\colon Q(\mathbf{a}) \to \mathbb{C}$ defined by $\mathbf{a}\,[x] \coloneqq \mathbf{a}\,[x, x]$. With the domain of the form established, we arrive at the following definition.

Definition 1.1.22. Let $\mathbf{a}\colon Q(\mathbf{a})\times Q(\mathbf{a}) \to \mathbb{C}$ be a form, where $Q(\mathbf{a}) \subseteq H$. The form $\mathbf{a}$ is *densely defined* if its domain $Q(\mathbf{a})$ is dense in the ambient Hilbert space H.

Likewise, we may speak about a form being bounded, as discussed in the following definition.

Definition 1.1.23. Let $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$ be a form, where $Q(\mathbf{a}) \subseteq H$. The form $\mathbf{a}$ is said to be *bounded* if there exists a constant $c \in \mathbb{R}$ such that

$$|\mathbf{a}\,[x, y]| \leq c\|x\|_H \|y\|_H,$$

for all $x,\ y \in Q(\mathbf{a})$. If no such c exists, then $\mathbf{a}$ is called *unbounded* instead.

We have established that to a linear operator T, we can find an adjoint operator T^*: analogous definitions exist in the context of forms.

Definition 1.1.24. Let $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$ be a form, where $Q(\mathbf{a}) \subseteq H$. The *adjoint form* $\mathbf{a}^*$ is defined by

$$\mathbf{a}^*\,[x, y] := \overline{\mathbf{a}\,[y, x]},$$

for all $x,\ y \in Q(\mathbf{a}^*) = Q(\mathbf{a})$.

When we have equality between a form and its adjoint, we obtain the following definition.

Definition 1.1.25. Let $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$ be a form, where $Q(\mathbf{a}) \subseteq H$. The form $\mathbf{a}$ is called *symmetric* if $\mathbf{a}\,[x, y] = \mathbf{a}^*\,[x, y]$ for all $x,\ y \in Q(\mathbf{a})$.

We continue by introducing two further forms related to the form $\mathbf{a}$.

Definition 1.1.26. Let $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$ be a form, where $Q(\mathbf{a}) \subseteq H$. The *real* and *imaginary parts* of a form are defined, respectively, as follows:

$$\mathbf{a}_{\mathrm{Re}}\,[x, y] := \frac{1}{2}\left(\mathbf{a}\,[x, y] + \mathbf{a}^*\,[x, y]\right) \text{ and } \mathbf{a}_{\mathrm{Im}}\,[x, y] := \frac{1}{2i}\left(\mathbf{a}\,[x, y] - \mathbf{a}^*\,[x, y]\right),$$

for $x, y \in Q(\mathbf{a}_{\mathrm{Re}}) = Q(\mathbf{a}_{\mathrm{Im}}) = Q(\mathbf{a})$. Moreover, $\mathbf{a}\,[x, y] = \mathbf{a}_{\mathrm{Re}}\,[x, y] + i\mathbf{a}_{\mathrm{Im}}\,[x, y]$.

Symmetry of a form has two important features that we will often make use of: if a form $\mathbf{a}$ is symmetric, then $\mathbf{a}_{\mathrm{Re}}\,[x, y] = \mathbf{a}\,[x, y]$ for all $x,\ y \in Q(\mathbf{a})$ and the expression $\mathbf{a}\,[x, x]$ is real-valued for all $x \in Q(\mathbf{a})$. Furthermore, it is easy to show that for any sesquilinear form $\mathbf{a}$, both $\mathbf{a}_{\mathrm{Re}}$ and $\mathbf{a}_{\mathrm{Im}}$ are symmetric forms; this revelation unveils that for any sesquilinear form $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$, we have

$$\mathbf{a}\,[x, x] = \underbrace{\mathbf{a}_{\mathrm{Re}}\,[x, x]}_{\in\mathbb{R}} + i\,\underbrace{\mathbf{a}_{\mathrm{Im}}\,[x, x]}_{\in\mathbb{R}},$$

for all $x \in Q(\mathbf{a})$. Hence, for any sesquilinear form $\mathbf{a}$, we have

$$\operatorname{Re}(\mathbf{a}[x,x]) = \mathbf{a}_{\mathrm{Re}}[x,x] \qquad \text{and} \qquad \operatorname{Im}(\mathbf{a}[x,x]) = \mathbf{a}_{\mathrm{Im}}[x,x],$$

for all $x \in Q(\mathbf{a})$. In general, $\mathbf{a}_{\mathrm{Re}}[x,y]$ and $\mathbf{a}_{\mathrm{Im}}[x,y]$ are complex-valued, but by considering them as quadratic forms we justify our use of the terms real and imaginary.

We now continue by investigating the expression $\mathbf{a}[x,x]$ associated to a sesquilinear form; this is effectively the analogue of $\langle Tx, x\rangle$ from the operator setting. Forms also possess numerical ranges, and so we can further describe forms in relation to where this set lies in the complex plane.

Definition 1.1.27. Let $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$ be a form, where $Q(\mathbf{a}) \subseteq H$. The set

$$\Theta(\mathbf{a}) = \{\mathbf{a}[x,x] \mid x \in Q(\mathbf{a}),\ \|x\| = 1\}$$

is called the *numerical range* of the form $\mathbf{a}$.

Definition 1.1.28. Let $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$ be a form, where $Q(\mathbf{a}) \subseteq H$. The form $\mathbf{a}$ is said to be *bounded below* if there exists a constant $\gamma \in \mathbb{R}$ such that

$$\mathbf{a}[x,x] \geq \gamma\|x\|^2, \qquad \forall x \in Q(\mathbf{a}).$$

The largest such γ is called the *lower bound.* Likewise, $\mathbf{a}$ is said to be *bounded above* if there exists a constant $\mu \in \mathbb{R}$ such that

$$\mathbf{a}[x,x] \leq \mu\|x\|^2, \qquad \forall x \in Q(\mathbf{a}).$$

The smallest such μ is called the *upper bound.* If $\mathbf{a}$ is either bounded above or bounded below, then we say that $\mathbf{a}$ is *semi-bounded.*

By setting γ or μ equal to zero in the above definitions, we obtain the notion of positivity and negativity of a form.

Definition 1.1.29. Let $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$ be a form, where $Q(\mathbf{a}) \subseteq H$. Then

$$\begin{aligned} \mathbf{a} \text{ is } \textit{positive } (\textit{non-negative}) &\iff \mathbf{a}[x,x] > 0\ (\geq 0), \qquad \forall x \in Q(\mathbf{a}) \setminus \{0\},\\ \mathbf{a} \text{ is } \textit{negative } (\textit{non-positive}) &\iff \mathbf{a}[x,x] < 0\ (\leq 0), \qquad \forall x \in Q(\mathbf{a}) \setminus \{0\}. \end{aligned}$$

If $\mathbf{a}$ is positive (non-negative), then we write $\mathbf{a} > 0$ (≥ 0). Similarly, when $\mathbf{a}$ is negative (non-positive), we write $\mathbf{a} < 0$ (≤ 0).

Likewise, if the numerical range $\Theta(\mathbf{a})$ lies in a sector, then we have the notion of sectoriality with regards to forms.

Definition 1.1.30. Let $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$ be a form, where $Q(\mathbf{a}) \subseteq H$. If the numerical range $\Theta(\mathbf{a})$ is contained within a sector $S_{\gamma,\theta}$ in the complex plane, where

$$S_{\gamma,\theta} = \{z \in \mathbb{C} \mid |\arg(z-\gamma)| \leq \theta\}, \qquad \gamma \in \mathbb{R},\ \theta \in \left[0, \frac{\pi}{2}\right),$$

then $\mathbf{a}$ is said to be *sectorial*. The constant γ is called the *vertex*, whilst θ is called the *semi-angle*.

Remark. If $\mathbf{a}$ is sectorial, then the sector in which the numerical range belongs to lies is not unique.

Remark. Graphically, one can observe that if the numerical range of $\mathbf{a}$ lies within a sector with vertex $\gamma \in \mathbb{R}$ and semi-angle $\left[0, \frac{\pi}{2}\right)$, then

$$|\operatorname{Im} \mathbf{a}[x,x]| \leq \tan\alpha \left(\operatorname{Re} \mathbf{a}[x,x] - \gamma\|x\|^2\right) \tag{1.1}$$

must hold for all $x \in Q(\mathbf{a})$. Thus, we arrive at the following alternative characterisation for a sectorial form: a form $\mathbf{a}$ is sectorial if the inequality given by (1.1) holds for all $x \in Q(\mathbf{a})$.

We can also introduce the concept of closed forms, as expressed in the following definition.

Definition 1.1.31. Let $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$ be a sectorial form, where $Q(\mathbf{a}) \subseteq H$. The form $\mathbf{a}$ is said to be *closed* if $(Q(\mathbf{a}), \|\cdot\|_{\mathbf{a}})$ is a complete space, where

$$\|x\|_{\mathbf{a}} = \left(\mathbf{a}_{\mathrm{Re}}[x,x] - (\gamma - 1)\langle x, x\rangle\right)^{\frac{1}{2}}, \qquad x \in Q(\mathbf{a}),$$

and γ is chosen to ensure positivity of the norm. If $\mathbf{a}$ is sectorial, then γ can be chosen to be a vertex.

Remark. This definition of a closed form is comparable to the third characterisation of a closed operator given in Section 1.1.2.

We may also characterise closure via so-called form-convergence, as introduced in the following definition.

Definition 1.1.32. A sequence $\{x_n\}$ in $Q(\mathbf{a}) \subseteq H$ is said to be $\mathbf{a}$-*convergent* to $x \in H$ if

$$x_n \to x \text{ as } n \to \infty \text{ in } H \quad \text{and} \quad \mathbf{a}[x_n - x_m, x_n - x_m] \to 0 \text{ as } n, m \to \infty.$$

When $\{x_n\}$ is $\mathbf{a}$-convergent to x, we will write $x_n \to_{\mathbf{a}} x$.

Then, a form $\mathbf{a}$ is closed if

$$x_n \to_{\mathbf{a}} x \implies x \in Q(\mathbf{a}) \text{ and } \mathbf{a}\left[x_n - x, x_n - x\right] \to 0.$$

We make use of both of these characterisations of a closed form throughout the thesis.

It is worth noting that when a form is sectorial with vertex $\gamma > 0$, there exists an equivalence between $\|\cdot\|_{\mathbf{a}}$ and the norm given by $\|x\|_2 = (\mathbf{a}_{\mathrm{Re}}\left[x, x\right])^{\frac{1}{2}}$.

Definition 1.1.33. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in H are *equivalent* if there exists two positive constants c and C such that

$$c\|x\|_2 \leq \|x\|_1 \leq C\|x\|_2,$$

for all $x \in H$.

Let $\mathbf{a}$ be a sectorial form with vertex $\gamma > 0$. To show that the two norms $\|\cdot\|_{\mathbf{a}}$ and $\|\cdot\|_2$ are equivalent, we split the analysis into two distinct cases: $0 < \gamma \leq 1$ and $\gamma > 1$. In both cases, we make use of the following consequence of equation (1.1):

$$\mathbf{a}_{\mathrm{Re}}\left[x, x\right] \geq \gamma\|x\|_H^2, \qquad x \in Q(\mathbf{a}). \tag{1.2}$$

Firstly, let $0 < \gamma \leq 1$. Then $0 \leq -(\gamma - 1) < 1$, and so

$$\|x\|_2^2 = \mathbf{a}_{\mathrm{Re}}\left[x, x\right] \leq \mathbf{a}_{\mathrm{Re}}\left[x, x\right] - (\gamma - 1)\|x\|_H^2 = \|x\|_{\mathbf{a}}^2.$$

On the other hand,

$$\begin{aligned}
\|x\|_{\mathbf{a}}^2 &= \mathbf{a}_{\mathrm{Re}}\left[x, x\right] - (\gamma - 1)\|x\|_H^2 \\
&\leq \mathbf{a}_{\mathrm{Re}}\left[x, x\right] - \frac{\gamma - 1}{\gamma}\mathbf{a}_{\mathrm{Re}}\left[x, x\right] \\
&= \frac{1}{\gamma}\mathbf{a}_{\mathrm{Re}}\left[x, x\right] = \frac{1}{\gamma}\|x\|_2^2,
\end{aligned}$$

by means of equation (1.2). Thus, when $0 < \gamma \leq 1$, the two constants $c = 1$ and $C = \sqrt{\frac{1}{\gamma}}$ ensure an equivalence between the two norms.

If instead, $\gamma > 1$, then we immediately note that

$$\|x\|_{\mathbf{a}}^2 = \mathbf{a}_{\mathrm{Re}}\left[x, x\right] - (\gamma - 1)\|x\|_H^2 \leq \mathbf{a}_{\mathrm{Re}}\left[x, x\right] = \|x\|_2^2,$$

since $-\infty < -(\gamma - 1) < 0$. Conversely, an application of equation (1.2) yields

$$\begin{aligned}
\|x\|_2^2 = \mathbf{a}_{\mathrm{Re}}\left[x, x\right] &= \gamma\mathbf{a}_{\mathrm{Re}}\left[x, x\right] - (\gamma - 1)\mathbf{a}_{\mathrm{Re}}\left[x, x\right] \\
&\leq \gamma\mathbf{a}_{\mathrm{Re}}\left[x, x\right] - (\gamma - 1)\gamma\|x\|_H^2 \\
&= \gamma\left(\mathbf{a}_{\mathrm{Re}}\left[x, x\right] - (\gamma - 1)\|x\|_H^2\right) = \gamma\|x\|_{\mathbf{a}}^2.
\end{aligned}$$

As such,

$$\|x\|_{\mathbf{a}}^2 \le \|x\|_2^2 \le \gamma\|x\|_{\mathbf{a}}^2, \quad \text{or, in other words,} \quad \frac{1}{\gamma}\|x\|_{\mathbf{a}}^2 \le \frac{1}{\gamma}\|x\|_2^2 \le \|x\|_{\mathbf{a}}^2.$$

Together, these two inequalities show that

$$\frac{1}{\gamma}\|x\|_2^2 \le \|x\|_{\mathbf{a}}^2 \le \|x\|_2^2,$$

and so by setting $c = \sqrt{\frac{1}{\gamma}}$ and $C = 1$, we see that the two norms are equivalent when $\gamma > 1$.

Having now covered both cases, we may conclude that the two norms $\|\cdot\|_{\mathbf{a}}$ and $\|\cdot\|_2$ are, in fact, equivalent. When two norms are equivalent, convergence in one norm implies convergence in the other: the two norms are effectively interchangeable. As such, we opt to use the simpler norm when appropriate.

Now, with all of the relevant definitions and terminology in place, we continue by disclosing an important representation theorem that links a certain class of sesquilinear forms to self-adjoint operators.

Theorem 1.1.34 ([27, Thm. 2.4])**.** *Let* $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$ *be a closed, densely defined, symmetric form, where* $Q(\mathbf{a}) \subseteq H$*. Then, there exists a self-adjoint operator* $T\colon \mathcal{D}(T) \to \mathcal{R}(T) \subseteq H$ *whose domain* $\mathcal{D}(T)$ *is dense in* H *and can be characterised as follows:*

$$\mathcal{D}(T) = \{z \in Q(\mathbf{a}) \mid \exists f \in H \text{ such that } \mathbf{a}[z,x] = \langle f, x\rangle \ \ \forall x \in Q(\mathbf{a})\}.$$

Then, $f = Tz$*.*

This theorem works in reverse too: if we are in possession of a self-adjoint operator T with domain $\mathcal{D}(T)$, then there exists a closed, densely defined, symmetric form $\mathbf{a}$ which satisfies the following equality:

$$\langle Tz, x\rangle = \mathbf{a}[z,x], \qquad z \in \mathcal{D}(T),\ x \in Q(\mathbf{a}).$$

In essence, there is a one-to-one correspondence between self-adjoint operators T and closed, densely defined, symmetric forms $\mathbf{a}$ [27].

Remark. When we speak of the unique operator T that is related to the form $\mathbf{a}$ as in Theorem 1.1.34, we will write that T is *associated* to $\mathbf{a}$ (or vice versa).

One key concept that naturally arises from the relationship between a form and an operator is that of comparing two forms (and operators) to one another. Notably, we have the following definition.

Definition 1.1.35. Let $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$ and $\mathbf{b}\colon Q(\mathbf{b}) \times Q(\mathbf{b}) \to \mathbb{C}$ be two forms. If

$$Q(\mathbf{b}) \subseteq Q(\mathbf{a}) \qquad \text{and} \qquad \mathbf{a}\,[x,x] \leq \mathbf{b}\,[x,x] < \infty \text{ for all } x \in Q(\mathbf{b}),$$

then we write $\mathbf{a} \leq \mathbf{b}$. Moreover, let A and B be the operators associated to the forms $\mathbf{a}$ and $\mathbf{b}$, respectively, as in Theorem 1.1.34. We write $A \leq B$ if and only if $\mathbf{a} \leq \mathbf{b}$.

This relationship between sesquilinear forms and operators forms the basis of the original work that we present in Chapter 2. However, we first reserve the following section for definitions and fundamental theory with regards to extension theory.

1.2 Extension Theory of Linear Operators

At its core, this thesis aims to characterise extensions of operators that possess certain properties. The following subsections first introduce basic terminology before detailing two methods for characterising extensions: the former utilises so-called deficiency spaces, the latter, the relationship between operators and forms introduced in Theorem 1.1.34. Furthermore, Appendix A demonstrates both of the described methods with a comprehensive example.

1.2.1 Extensions of Linear Operators

Given an operator T in a Hilbert space, one might be interested in the following questions: what happens if we examine T on a smaller domain, does it 'make sense' to consider T on a larger domain? If so, then which of the properties that T enjoys are preserved when considering T on this new domain? The objective of this section will be to introduce various results from extension theory, as detailed in [43], so that we can make sense of these questions more formally.

Definition 1.2.1. Let H be a Hilbert space and $T\colon \mathcal{D}(T) \to H$ an operator, where $\mathcal{D}(T) \subseteq H$. Let R be a subset of $\mathcal{D}(T)$. The operator $\hat{T}\colon R \to H$, satisfying $\hat{T}x = Tx$ for all $x \in R$, is called a *restriction* of T.

Remark. If $\hat{T}$ is the restriction of T to the subset R, then we often use the following notation: $\hat{T} = T \restriction R$.

Definition 1.2.2. Let H be a Hilbert space and $T\colon \mathcal{D}(T) \to H$ an operator, where $\mathcal{D}(T) \subseteq H$. Let S satisfy $\mathcal{D}(T) \subset S \subseteq H$. Any operator $\tilde{T}\colon S \to H$, such that $\tilde{T}x = Tx$ for all $x \in \mathcal{D}(T)$ is called an *extension* of T. If $\tilde{T}$ is an extension of T, then we write $T \subset \tilde{T}$.

Remark. The relationship between restrictions and extensions is explicit: if $T \subset \tilde{T}$, then $\tilde{T} \restriction \mathcal{D}(T) = T$.

In general, it seems logical to find extensions of an operator T that is, in some sense, 'small'. Conversely, restrictions are often spoken about with regards to an operator that is, in some sense, 'big'. One such example of this idea is as follows: if an operator T is not closed, then we may attempt to find its closure $\overline{T}$. If $\overline{T}$ exists, then this closed operator is the smallest, closed extension of T. The existence of the closure of an operator is addressed in the following definition.

Definition 1.2.3. Let H_1 and H_2 be Hilbert spaces and $T\colon \mathcal{D}(T) \to H_2$ an operator, where $\mathcal{D}(T) \subseteq H_1$. Let $\mathcal{G}(T)$ denote the graph of T. If the closure of $\mathcal{G}(T)$ with respect to the graph norm $\|\cdot\|_{H_1 \times H_2}$ (that is, $\overline{\mathcal{G}(T)}$) is a graph, then T is said to be *closable.* If T is closable, then the unique operator $\overline{T}$ such that $\overline{\mathcal{G}(T)} = \mathcal{G}(\overline{T})$ is a closed operator and is referred to as the *closure* of T.

There exists an alternative formulation for the closability of an operator that we now present. Indeed, T is closable if and only if the following holds: if $\{x_n\}$ is a sequence in $\mathcal{D}(T)$ such that $x_n \to 0$ as $n \to \infty$ and $\{Tx_n\}$ in H_2 is convergent, then $Tx_n \to 0$ as $n \to \infty$.

In this thesis, we will be concerned with symmetric operators in Hilbert spaces. As such, the following theorem has practical consequences that we will often utilise.

Theorem 1.2.4 ([57, Thm. 5.4])**.** *If $T\colon \mathcal{D}(T) \to H$ is a symmetric operator in a Hilbert space H, where $\mathcal{D}(T) \subseteq H$, then T is closable. Moreover, $\overline{T}$ is also symmetric.*

Abstractly, we may assume that a symmetric operator is closed since its closure will always exist; in practice, however, it is paramount that we genuinely verify whether the operator is closed or not before applying any further theory. In line with the previous motivation, we continue by introducing two distinguished operators.

Definition 1.2.5. Let H be a Hilbert space and $T\colon \mathcal{D}(T) \to H$ a symmetric operator, where $\mathcal{D}(T) \subseteq H$. We refer to the closure $\overline{T}$ of T as the *minimal operator* and write $\overline{T} = T_{\min}$. On the other hand, the operator T^* is called the *maximal operator* and is denoted by $T_{\max}$.

Remark. For any linear operator T, we have $T^* = (\overline{T})^*$. Then, for T a symmetric operator, we have

$$(T_{\min})^* = T^* = T_{\max}.$$

Furthermore, $T \subset \overline{T} \subset T^*$ or, in other words, $T \subset T_{\min} \subset T_{\max}$.

This terminology is hopefully evocative: in practice, we aim to characterise either the restrictions of the maximal operator or the extensions of the minimal operator.

We conclude this section by noting that if a form $\mathbf{a}$ is *not* closed, then we have analogous definitions for both the closability of a form and its closure to those presented in the operator setting.

Definition 1.2.6. Let $\mathbf{a}\colon Q(\mathbf{a}) \times Q(\mathbf{a}) \to \mathbb{C}$ be a form, where $Q(\mathbf{a}) \subseteq H$. Let $\{x_n\}$ be a Cauchy sequence in $Q(\mathbf{a})$ that converges to 0 in H. The form $\mathbf{a}$ is said to be *closable* if and only if $\{x_n\}$ also converges in $Q(\mathbf{a})$ and

$$\lim_{n\to\infty} \mathbf{a}\,[x_n, x_n] = 0.$$

The form $\overline{\mathbf{a}}$ with form domain $Q(\overline{\mathbf{a}})$ is referred to as the *closure* of $\mathbf{a}$ and is defined as follows: $x \in Q(\overline{\mathbf{a}})$ if and only if there exists a sequence $\{x_n\}$ in $Q(\mathbf{a})$ such that

$$x_n \to x \text{ as } n \to \infty \text{ in } H \quad \text{and} \quad \mathbf{a}\,[x_n - x_m, x_n - x_m] \to 0 \text{ as } n,\, m \to \infty,$$

i.e., $x_n \to_{\mathbf{a}} x$. For x, $y \in Q(\overline{\mathbf{a}})$, we then set

$$\overline{\mathbf{a}}\,[x, y] := \lim_{n\to\infty} \mathbf{a}\,[x_n, y_n]\,,$$

where $\{x_n\}$, $\{y_n\}$ are sequences in $Q(\mathbf{a})$ that $\mathbf{a}$-converge to x and y, respectively.

Notably, we have the following relationship: a form $\mathbf{a}$ is closable if and only if the operator T associated to $\mathbf{a}$ has a closed extension. The form $\overline{\mathbf{a}}$ is then the form associated to this minimal, closed extension.

1.2.2 von Neumann Theory for Linear Operators

Typically, one objective in characterising the extensions of an operator T is to find those extensions that possess certain useful properties. In this section, we present the von Neumann theory: a way of characterising the self-adjoint extensions of a closed, symmetric operator T. In particular, we utilise isometric maps between the so-called deficiency spaces of T. We recommend [57] for a comprehensive account of the theory presented in this section.

Definition 1.2.7. Let H be a Hilbert space and $T\colon \mathcal{D}(T) \to H$ a closed, symmetric operator, where $\mathcal{D}(T) \subseteq H$. The closed subspaces $\mathcal{N}_+(T)$ and $\mathcal{N}_-(T)$ defined by

$$\mathcal{N}_+ \equiv \mathcal{N}_+(T) := \ker(T^* - iI) \quad \text{and} \quad \mathcal{N}_- \equiv \mathcal{N}_-(T) := \ker(T^* + iI),$$

where I is the identity operator on $\mathcal{D}(T^*)$, are called the *deficiency spaces* of T. The dimensions of these subspaces, denoted by $m_+(T)$ and $m_-(T)$ respectively, are called the *deficiency indices*.

Remark. We make special mention that the literature may express deficiency spaces with different or opposing signs to those used here. We justify our notation by recalling the operator T_λ associated to T from Section 1.1.1, i.e., $T_\lambda = T - \lambda I$ for $\lambda \in \mathbb{C}$: clearly, $\mathcal{N}_\pm(T) = \ker T^*_{\pm i}$.

We continue by presenting a useful decomposition of the domain of the adjoint operator T^*. This decomposition will form the basis of the theorem which characterises the extensions that we are interested in.

Theorem 1.2.8 ([57, Thm. 8.11])**.** *Let H be a Hilbert space and $T\colon \mathcal{D}(T) \to H$ a closed, symmetric operator, where $\mathcal{D}(T) \subseteq H$. Then*

$$\mathcal{D}(T^*) = \mathcal{D}(T) \dotplus \mathcal{N}_+ \dotplus \mathcal{N}_-,$$

where $\mathcal{N}_+$ and $\mathcal{N}_-$ denote the deficiency spaces of T. Moreover, if $f \in \mathcal{D}(T)$, $g_+ \in \mathcal{N}_+$ and $g_- \in \mathcal{N}_-$, then

$$T^*(f + g_+ + g_-) = Tf + ig_+ - ig_-.$$

Remark. The symbol $\dotplus$ denotes the *direct sum* between two sets. In particular, for two sets A and B, we have

$$A \dotplus B = \{a + b \mid a \in A,\, b \in B\}.$$

With this decomposition of the adjoint operator in mind, we proceed by presenting the main theorem of this section.

Theorem 1.2.9 ([57, Thm. 8.12])**.** *Let H be a Hilbert space and $T\colon \mathcal{D}(T) \to H$ a closed, symmetric operator, where $\mathcal{D}(T) \subseteq H$.*

1. *$\tilde{T}$ is a closed, symmetric extension of T if and only if the following holds:*

   There are closed subspaces $F_+ \subseteq \mathcal{N}_+$ and $F_- \subseteq \mathcal{N}_-$ and an isometric mapping $U\colon F_+ \to F_-$ such that:

$$\mathcal{D}(\tilde{T}) = \mathcal{D}(T) \dotplus \{g + Ug \mid g \in F_+\}$$

and

$$\begin{aligned}\tilde{T}(f+g+Ug) &= Tf + ig - iUg \\ &= T^*(f+g+Ug),\end{aligned}$$

for $f \in \mathcal{D}(T)$ and $g \in F_+$.

2. *$\tilde{T}$ is self-adjoint $\iff$ $F_+ = \mathcal{N}_+$ and $F_- = \mathcal{N}_-$. In this case, the deficiency indices $m_+(T)$ and $m_-(T)$ are equal, that is, $m_+(T) = m_-(T)$, and the mapping U is unitary.*

Remark. For a symmetric operator T, this theorem clearly demonstrates that $\tilde{T} = T^* \restriction \mathcal{D}(\tilde{T})$. Moreover, by reintroducing the maximal and minimal operator from Definition 1.2.5, we see that $T_{\min} \subset \tilde{T} \subset T_{\max}$.

Essentially, if there exists an isometric map — that is, a map U such that $\|Ux\| = \|x\|$ for all $x \in \mathcal{D}(U)$ — between closed subspaces of the two deficiency spaces, $\mathcal{N}_+(T)$ and $\mathcal{N}_-(T)$, then the domain $\mathcal{D}(\tilde{T})$ of the extension $\tilde{T}$ can be thought of as the domain of T plus a 'little bit more'. This is imprecise, but it serves as a basic interpretation for the theory; in fact, the notion of 'adding a bit' to an already formulated domain will be more prominent in the second theory we present during Section 1.2.4.

1.2.3 The Friedrichs and Kreĭn Extension

Historically, two specific self-adjoint extensions of a semi-bounded operator T were studied due to their 'maximal' and 'minimal' nature: namely, the Friedrichs extension and the Kreĭn extension, respectively. However, with the notion of size, there exists a powerful relationship between these extensions and all other self-adjoint extensions of T. Both the Friedrichs and Kreĭn extension will be introduced in this section, as presented in [19], along with a precise formulation of the aforementioned relationship. We also refer to Edmunds and Evans [27] for an in-depth account of the theory.

Definition 1.2.10. Let H be a Hilbert space and $T\colon \mathcal{D}(T) \to H$ a positive operator with lower bound $\gamma > 0$. The *Friedrichs extension* T_F is the extension of T with domain

$$\mathcal{D}(T_F) = \left\{ x \in \mathcal{D}(T_{\max}) \;\middle|\; \begin{array}{r} \exists x^{(k)} \in \mathcal{D}(T) \text{ such that } \|x - x^{(k)}\| \to 0 \text{ as} \\ k \to \infty \text{ and } \left\langle T(x^{(j)} - x^{(k)}), (x^{(j)} - x^{(k)}) \right\rangle \to 0 \\ \text{as } j,\, k \to \infty \end{array} \right\}.$$

Remark. The Friedrichs extension T_F is a positive self-adjoint operator with lower bound equal to γ.

With the Friedrichs extension now defined, we continue by presenting a useful decomposition of the maximal domain in terms of the Friedrichs domain and kernel elements of $T_{\max}$.

Lemma 1.2.11. *Let T be an operator with positive lower bound $\gamma > 0$. Then,*

$$\mathcal{D}(T_{\max}) = \mathcal{D}(T_F) \dot{+} \ker T_{\max}.$$

Proof. Let $u \in \mathcal{D}(T_{\max})$. As $T_{\min}$ is the closure of T, it possesses the same lower bound $\gamma > 0$ as T itself. Moreover, the lower bound of the Friedrichs extension T_F coincides with that of $T_{\min}$ too, so we may conclude that 0 lies in the resolvent set of T_F, since $\gamma > 0$. Then, we may set

$$u = T_F^{-1}(T_{\max}u) + (u - T_F^{-1}(T_{\max}u)), \qquad u \in \mathcal{D}(T_{\max}),$$

since the inverse of T_F exists. Now, we simply show that $T_F^{-1}(T_{\max}u) \in \mathcal{D}(T_F)$ and $u - T_F^{-1}(T_{\max}u) \in \ker T_{\max}$.

The former is easy to see: since the inverse of T_F exists, it is obvious that $T_F^{-1}(T_{\max}u) \in \mathcal{D}(T_F)$. To prove the latter, we begin by applying $T_{\max}$ to $u - T_F^{-1}(T_{\max}u)$. Indeed, for $u \in \mathcal{D}(T_{\max})$, we see that

$$\begin{aligned} T_{\max}\left[u - T_F^{-1}(T_{\max}u)\right] &= T_{\max}u - T_{\max}(T_F^{-1}(T_{\max}u)) \\ &= T_{\max}u - T_F(T_F^{-1}(T_{\max}u)), \end{aligned}$$

since $T_F x = T_{\max}x$ for $x \in \mathcal{D}(T_F)$. From this, it is clear to see that

$$T_{\max}\left[u - T_F^{-1}(T_{\max}u)\right] = T_{\max}u - T_{\max}u = 0.$$

Hence, $u - T_F^{-1}(T_{\max}u) \in \ker T_{\max}$, and so the decomposition must be valid. ∎

Since the Friedrichs extension is a self-adjoint (thus, symmetric) operator, we may associate to it a closed, densely defined symmetric form, say $\mathbf{t_F}$, by means of Theorem 1.1.34. Then, we state the following result.

Theorem 1.2.12 ([27])**.** *The Friedrichs extension T_F of some positive, symmetric operator T has a domain satisfying*

$$\mathcal{D}(T_F) = \mathcal{D}(T_{\max}) \cap Q(\mathbf{t_F}),$$

where $Q(\mathbf{t_F})$ denotes the form domain of $\mathbf{t_F}$.

We now continue by presenting a characterisation of the Kreĭn extension.

Definition 1.2.13. Let H be a Hilbert space and $T\colon \mathcal{D}(T) \to H$ a positive, symmetric operator. The *Kreĭn extension* T_K is the extension of T with domain

$$\mathcal{D}(T_K) = \mathcal{D}(T_{\min}) \dotplus \mathcal{N},$$

where $\mathcal{N} = \ker T_{\max}$.

Remark. The Kreĭn extension T_K is a non-negative, self-adjoint operator. We refer to [41] for proving that the Kreĭn extension is a self-adjoint operator; here, we show that T_K is non-negative. Indeed, we begin by letting $x \in \mathcal{D}(T_K)$. If $\mathcal{N} = \{0\}$, then it is immediate that T_K is not only non-negative, but positive, since T is positive. If, instead, we assume that $\mathcal{N}$ is non-trivial, then an element $x \in \mathcal{D}(T_K)$ can be decomposed into $x = x_0 + y$ for some $x_0 \in \mathcal{D}(T_{\min})$ and $y \in \mathcal{N} = \ker T_{\max}$. Then,

$$\begin{aligned}\langle T_K x, x\rangle &= \langle T_K(x_0 + y), x_0 + y\rangle \\ &= \langle T_K x_0, x_0 + y\rangle + \langle T_K y, x_0 + y\rangle .\end{aligned}$$

Observe that $\langle T_K y, x_0 + y\rangle = \langle 0, x_0 + y\rangle = 0$ since $T_K y = T_{\max} y = 0$ for $y \in \mathcal{N}$. Consequently, $\langle T_K x, x\rangle = \langle T_K x_0, x_0 + y\rangle$. Moreover, T_K is a self-adjoint operator so

$$\begin{aligned}\langle T_K x_0, x_0 + y\rangle &= \langle x_0, T_K(x_0 + y)\rangle \\ &= \langle x_0, T_K x_0\rangle + \langle x_0, T_K y\rangle \\ &= \langle x_0, T_K x_0\rangle .\end{aligned}$$

By recalling that $x_0 \in \mathcal{D}(T_{\min})$, we may replace T_K by $T_{\min}$. Thus, we assert that

$$\langle T_K x, x\rangle = \langle x_0, T_K x_0\rangle = \langle T_K x_0, x_0\rangle = \langle T_{\min} x_0, x_0\rangle .$$

Since $T_{\min}$ is a positive operator, we have

$$\langle T_K x, x\rangle = \langle T_{\min} x_0, x_0\rangle > 0$$

for all $x \in \mathcal{D}(T_K) \setminus \mathcal{N}$, showing that the numerical range at least lies on the positive axis. To see that 0 also lies in the numerical range, we simply take $x_0 = 0$ and a $y \neq 0$ such that $\|x\| = 1$. This choice of x shows that we can find an $x \neq 0$ with $\|x\| = 1$ such that $\langle Tx, x\rangle = 0$. As such, we may conclude that $0 \in \Theta(T_K)$. Then, it is clear that the numerical range $\Theta(T_K)$ lies in the interval $[0, \infty)$, proving that T_K is a non-negative operator.

With these two distinguished self-adjoint extensions now defined, we continue by presenting the following theorem by Kreĭn that details the relationship alluded to in the introduction to this section.

Theorem 1.2.14 ([5, Thm. 2.11])**.** *Let H be a Hilbert space and $T\colon \mathcal{D}(T) \to H$ a positive, symmetric operator. The set of all non-negative self-adjoint extensions of T is precisely the set of operators $\tilde{T}$ satisfying*

$$T_K \leq \tilde{T} \leq T_F,$$

where T_K and T_F are the Kreĭn extension and Friedrichs extension of T respectively.

Remark. In [41], the Friedrichs extension is referred to as the 'hard' extension, whilst the Kreĭn, the 'soft'.

In order to appreciate the operator inequality $T_K \leq \tilde{T} \leq T_F$, we first associate to these operators the forms $\mathbf{t_K}$, $\tilde{\mathbf{t}}$ and $\mathbf{t_F}$ respectively. Then,

$$\begin{aligned} T_K \leq \tilde{T} \leq T_F &\iff \mathbf{t_K} \leq \tilde{\mathbf{t}} \leq \mathbf{t_F} \\ &\iff \mathbf{t_K}[x,x] \leq \tilde{\mathbf{t}}[x,x] \text{ for all } x \in Q(\tilde{\mathbf{t}}) \text{ and} \\ &\qquad \tilde{\mathbf{t}}[x,x] \leq \mathbf{t_F}[x,x] \text{ for all } x \in Q(\mathbf{t_F}), \end{aligned}$$

where $Q(\mathbf{t_F}) \subseteq Q(\tilde{\mathbf{t}}) \subseteq Q(\mathbf{t_K})$. Thus, it is clear that the form associated to the Friedrichs extension has the smallest form domain. This idea is the crux of the theory that will be presented in the next section.

1.2.4 Kreĭn-Vishik-Birman Theory

In Section 1.2.2, we presented a theorem which characterised the self-adjoint extensions of a closed, symmetric operator. Here, we instead express an important correspondence between non-negative, self-adjoint extensions of a positive minimal operator $T_{\min}$ and certain non-negative, self-adjoint operators B. This section presents results from [41] and follows the presentation in [5].

We begin with the following theorem.

Theorem 1.2.15 ([5, Thm. 2.9])**.** *Let H be a Hilbert space and $T\colon \mathcal{D}(T) \to H$ a positive, symmetric operator, where $\mathcal{D}(T) \subseteq H$. There is a one-to-one correspondence between non-negative, self-adjoint extensions, T_B, of T and non-negative forms, $\mathbf{b}$, acting on subspaces $\mathcal{N}_B$ of $\mathcal{N} = \ker T^*$. If $\dim \mathcal{N} < \infty$, the word non-negative may be dropped in both places.*

By considering the kernel of the adjoint operator, $\mathcal{N}$, in conjunction with $Q(\mathbf{t_F})$ — that is, the domain of the form associated to the Friedrichs extension of the minimal operator $T_{\min}$ — it is possible to explicitly characterise all of the non-negative, self-adjoint extensions, $\tilde{T}$, of $T_{\min}$. Indeed, we summarise the pertinent results from [5] and present them in the following theorem that will be fundamental to many parts of the thesis.

Theorem 1.2.16. *Let $\tilde{T}$ be a non-negative, self-adjoint extension of $T_{\min}$. Then $\tilde{T} = T_B$ for some non-negative, self-adjoint operator B acting on $\mathcal{N}_B$, where T_B is the operator associated to the form $\mathbf{t_B}$ satisfying*

$$\mathbf{t_B}\left[u, v\right] = \mathbf{t_F}\left[u^F, v^F\right] + \mathbf{b}\left[u^N, v^N\right].$$

The form domain $Q(\mathbf{t_B})$ is given by

$$Q(\mathbf{t_B}) = Q(\mathbf{t_F}) \dot{+} Q(\mathbf{b}),$$

where $\mathbf{b}$ is the form associated to B and u, $v \in Q(\mathbf{t_B})$ are such that

$$\begin{cases} u = u^F + u^N, \\ v = v^F + v^N, \end{cases} \quad \text{for } u^F,\ v^F \in Q(\mathbf{t_F}) \text{ and } u^N,\ v^N \in Q(\mathbf{b}) \subseteq \mathcal{N}_B.$$

Remark. Theorem 1.2.16 strengthens the notion that the extensions of interest to us can be constructed by 'adding a bit' to a small, yet fundamental, starting domain.

This decomposition admits a useful consequence: when $\mathbf{b} = 0$ and $\mathcal{N}_B = \mathcal{N}$ we obtain the Kreĭn extension, whilst $\mathbf{b} = \infty$ yields the Friedrichs extension. Taking $\mathbf{b} = \infty$ is purely notational — we do so to reinforce the operator inequality presented in Theorem 1.2.14. When $\mathbf{b} = \infty$, we are to interpret this as taking $\mathcal{N}_B = \{0\}$ and setting $\infty\,[0, 0] = 0$. Since we are interested in the non-negative extensions of an operator T, the Kreĭn extension will then be the smallest extension in an operator sense.

In this thesis, we will utilise the Kreĭn-Vishik-Birman theory rather than the well-documented von Neumann theory of linear operators due to this explicit relationship with both the Friedrichs and Kreĭn extension. As such, we devote Appendix A to applying both the von Neumann theory and the Kreĭn-Vishik-Birman theory to a concrete example of Sturm-Liouville type. Crucially, the latter example forms a template that we may follow during Chapter 2.

1.3 Difference Operators

Throughout the thesis so far, we have provided general theory for an operator T that acts in a Hilbert space H. In this section, we will specify the type of operator that will be of interest to us, along with certain useful results that will be utilised in subsequent chapters.

1.3.1 Difference Operators and Jacobi Operators

We first introduce the Hilbert space fundamental to the thesis: the sequence space ℓ^2.

Definition 1.3.1. The sequence space ℓ^2 over the scalar field $\mathbb{C}$ is defined to be the set

$$\ell^2 = \left\{ x = \{x_n\}_{n=0}^{\infty} \;\middle|\; \sum_{n=0}^{\infty} |x_n|^2 < \infty \right\}$$

equipped with the inner product $\langle \cdot, \cdot \rangle_{\ell^2}$, where

$$\langle x, y \rangle_{\ell^2} = \sum_{n=0}^{\infty} x_n \overline{y}_n, \qquad x,\ y \in \ell^2.$$

Critically, $(\ell^2, \langle \cdot, \cdot \rangle_{\ell^2})$ forms a Hilbert space. As one would expect, the norm induced by this inner product is denoted by $\| \cdot \|_{\ell^2}$, where

$$\|x\|_{\ell^2} = \left(\sum_{n=0}^{\infty} |x_n|^2 \right)^{\frac{1}{2}}, \qquad x \in \ell^2.$$

We will omit the subscript ℓ^2 present on both the inner product and norm whenever it is clear what is meant.

With the Hilbert space fundamental to this thesis defined, we continue by introducing the operator of interest to us: the *forward difference* operator, which we denote by Δ. Indeed, we consider Δ on ℓ^2, where the n-th component of an element $u \in \ell^2$ can be described as follows:

$$(\Delta u)_n = u_{n+1} - u_n.$$

When there is no confusion possible, we will omit the brackets and simply write $\Delta u_n = u_{n+1} - u_n$.

It was alluded to in the introduction to the thesis, but we feel it worthwhile to reiterate here: the difference operator is the discrete analogue to the differential operator in the continuous case. Indeed, consider $u_n = f(x_n)$ for a sequence x_n in $\mathcal{D}(f)$. If $x_{n+1} = x_n + h$, then $\frac{\Delta u_n}{h} = \frac{f(x_n+h)-f(x_n)}{h}$ approximates the derivative of f at x_n for small h. Therefore, difference operators arise naturally when discretising differential equations.

It is worth remarking that the forward difference operator is a bounded linear operator. Indeed, let u, $v \in \ell^2$ and $\lambda \in \mathbb{C}$. Then

$$\Delta(\lambda u + v) = \begin{pmatrix} (\lambda u_1 + v_1) - (\lambda u_0 + v_0) \\ (\lambda u_2 + v_2) - (\lambda u_1 + v_1) \\ \vdots \\ (\lambda u_{n+1} + v_{n+1}) - (\lambda u_n + v_n) \\ \vdots \end{pmatrix}$$

$$= \lambda \begin{pmatrix} u_1 - u_0 \\ u_2 - u_1 \\ \vdots \\ u_{n+1} - u_n \\ \vdots \end{pmatrix} + \begin{pmatrix} v_1 - v_0 \\ v_2 - v_1 \\ \vdots \\ v_{n+1} - v_n \\ \vdots \end{pmatrix} = \lambda \Delta u + \Delta v.$$

Thus, we have confirmed that Δ acts linearly. To verify that the operator is also bounded, we must show that there exists a constant $c \in \mathbb{R}$ such that

$$\|\Delta u\| \leq c\|u\|, \qquad u \in \ell^2.$$

Consider the expression $\|\Delta u\|^2$. Then

$$\begin{aligned} \|\Delta u\|^2 &= \sum_{n=0}^{\infty} |\Delta u|^2 \\ &= \sum_{n=0}^{\infty} |u_{n+1} - u_n|^2 \\ &\leq \sum_{n=0}^{\infty} \left(|u_{n+1}| + |u_n|\right)^2, \end{aligned}$$

after an application of the triangle inequality. Let $\{\tilde{u}_n\}$ be the sequence such that $\tilde{u}_n = u_{n+1}$ for all $n \geq 0$. Then, after expanding the right-hand side of this equation, we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \left(|u_{n+1}| + |u_n|\right)^2 &= \sum_{n=0}^{\infty} |\tilde{u}_n|^2 + |u_n|^2 + 2|\tilde{u}_n||u_n| \\ &\leq 2\|u\|^2 + 2\sum_{n=0}^{\infty} |\tilde{u}_n||u_n|, \end{aligned}$$

since

$$\begin{aligned} \|\tilde{u}\|^2 = \sum_{n=0}^{\infty} |\tilde{u}_n|^2 = \sum_{n=0}^{\infty} |u_{n+1}|^2 &= \sum_{n=1}^{\infty} |u_n|^2 \\ &\leq \sum_{n=0}^{\infty} |u_n|^2 = \|u\|^2. \end{aligned}$$

On the other hand, invoking the Cauchy-Schwarz inequality yields

$$\begin{aligned} 2\sum_{n=0}^{\infty} |\tilde{u}_n||u_n| &\leq 2\|\tilde{u}\|\|u\| \\ &\leq 2\|u\|^2. \end{aligned}$$

By piecing together the shown inequalities, we see that

$$\|\Delta u\|^2 \leq 4\|u\|^2, \qquad u \in \ell^2,$$

or, in other words,

$$\|\Delta u\| \leq 2\|u\|, \qquad u \in \ell^2.$$

As we have now produced a constant $c \in \mathbb{R}$ such that $\|\Delta u\| \leq c\|u\|$ for all $u \in \ell^2$, we have shown that the operator Δ is a bounded operator in ℓ^2.

Now, let M be an operator with domain $\mathcal{D}(M) \subseteq \ell^2$. We say that M is a *second-order difference operator* if the n-th component of a sequence can be represented by the expression

$$(Mu)_n = a_n u_{n+1} + b_n u_n + c_n u_{n-1}, \qquad n \geq 0,$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $\mathbb{C}$ with a_n, $c_n \neq 0$ for all $n \geq 0$. The accompanying equation $Mu = 0$ is then referred to as a *second-order difference equation*, or a *second-order recurrence relation.*

In particular, this thesis will be concerned with second-order difference operators that satisfy the following:

$$(Mu)_n = -\Delta(p_{n-1}\Delta u_{n-1}) + q_n u_n, \qquad n \geq 0,$$

for two real sequences $\{p_n\}$ and $\{q_n\}$ with $p_n > 0$ for all $n \in \mathbb{N}_0$ and $p_{-1} \equiv 0$. By expanding $(Mu)_n$, we see that

$$(Mu)_n = -p_n u_{n+1} + (p_n + p_{n-1} + q_n)u_n - p_{n-1}u_{n-1}, \qquad n \geq 0, \tag{1.3}$$

demonstrating more clearly the second-order nature of the operator. In fact, this manipulation neatly illustrates the connection to so-called orthogonal polynomials — a topic we delve into more thoroughly during Section 2.6.

Second-order difference operators have an alternative representation that will often provide valuable insight during our analysis. Indeed, consider the symmetric, tri-diagonal matrix J given by

$$J = \begin{pmatrix} b_0 & a_0 & & & & & \\ a_0 & b_1 & a_1 & & & & \\ & a_1 & b_2 & a_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & a_{n-1} & b_n & a_n & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

for any given sequences $\{a_n\}$ and $\{b_n\}$. An operator T with domain $\mathcal{D}(T) \subseteq \ell^2$, such that $Tu = Ju$ for all $u \in \mathcal{D}(T)$, is referred to as a *Jacobi operator*.

By computing Ju, for some $u \in \ell^2$, we see that

$$
\begin{aligned}
Ju &= \begin{pmatrix} b_0 & a_0 & & & & & \\ a_0 & b_1 & a_1 & & & & \\ & a_1 & b_2 & a_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & a_{n-1} & b_n & a_n & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_n \\ \vdots \end{pmatrix} \\
&= \begin{pmatrix} b_0u_0 + a_0u_1 \\ a_0u_0 + b_1u_1 + a_1u_2 \\ a_1u_1 + b_2u_2 + a_2u_3 \\ \vdots \\ a_{n-1}u_{n-1} + b_nu_n + a_nu_{n+1} \\ \vdots \end{pmatrix}.
\end{aligned}
$$

It is then evident that

$$
(Ju)_n = \begin{cases} b_0u_0 + a_0u_1, & n = 0, \\ a_nu_{n+1} + b_nu_n + a_{n-1}u_{n-1}, & n \geq 1. \end{cases}
$$

By comparing this to equation (1.3), the relationship between the given sequences $\{p_n\}$, $\{q_n\}$ and $\{a_n\}$, $\{b_n\}$ becomes apparent. Namely, we have

$$
a_n = -p_n \qquad \text{and} \qquad b_n = p_n + p_{n+1} + q_n,
$$

for all $n \geq 0$. Here, and in what follows, we will assume that the sequences $\{p_n\}$ and $\{q_n\}$ are both real and $p_n > 0$ for all $n \in \mathbb{N}_0$, with $p_{-1} \equiv 0$.

Remark. If we take $n \in \mathbb{Z}$ instead of $\mathbb{N}_0$, then the associated Jacobi matrix will be infinite in both directions. The literature often makes reference to doubly-infinite Jacobi matrices: see, for example, [53].

Since the matrices we will investigate are infinite in only one direction, the first component of Ju plays a vital role in any analysis we undertake due to the recurrent nature of the operator. If we are investigating equations involving Ju, then we will often refer to this first equation as the *initial condition* prescribed by the first row or, alternatively, the *first row condition*.

We conclude this section by introducing three important lemmas that will be used periodically throughout the thesis. Firstly, we present *Jacobi's factorisation identity* [14].

Lemma 1.3.2. *For any sequences $\{x_n\}$ and $\{p_n\}$, the following equality holds:*

$$-\Delta(p_{n-1}\Delta x_{n-1}) + \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}x_n = -\frac{1}{g_n}\Delta\left[P_{n-1}\Delta\left(\frac{x_{n-1}}{g_{n-1}}\right)\right], \qquad (1.4)$$

where $P_n = p_n g_n g_{n+1}$ and g_n is any fixed sequence with $g_n \neq 0$ for all $n \in \mathbb{N}_0$.

Proof. Consider the right-hand side of equation (1.4). Upon applying simple manipulations, we see that

$$\begin{aligned}
&-\frac{1}{g_n}\left[p_n g_n g_{n+1}\Delta\left(\frac{x_n}{g_n}\right) - p_{n-1}g_{n-1}g_n\Delta\left(\frac{x_{n-1}}{g_{n-1}}\right)\right] \\
=&-\frac{1}{g_n}\left[p_n g_n g_{n+1}\left(\frac{x_{n+1}}{g_{n+1}} - \frac{x_n}{g_n}\right) - p_{n-1}g_{n-1}g_n\left(\frac{x_n}{g_n} - \frac{x_{n-1}}{g_{n-1}}\right)\right] \\
=&-\frac{1}{g_n}\left[p_n\left(x_{n+1}g_n - x_n g_{n+1}\right) - p_{n-1}\left(x_n g_{n-1} - x_{n-1}g_n\right)\right] \\
=&-p_n x_{n+1} + \frac{p_n x_n g_{n+1}}{g_n} + \frac{p_{n-1}x_n g_{n-1}}{g_n} - p_{n-1}x_{n-1}. \qquad (1.5)
\end{aligned}$$

In order to simplify this expression, consider the expression $-\Delta(p_{n-1}\Delta x_{n-1})$. Upon expanding this, we see that

$$-\Delta(p_{n-1}\Delta x_{n-1}) = -p_n x_{n+1} + (p_n + p_{n-1})\,x_n - p_{n-1}x_{n-1}. \qquad (1.6)$$

We may then insert this back into equation (1.5) and see that the right-hand side of the equation can be expressed as

$$\begin{aligned}
&-\Delta(p_{n-1}\Delta x_{n-1}) - (p_n + p_{n-1})\,x_n + \frac{p_n x_n g_{n+1}}{g_n} + \frac{p_{n-1}x_n g_{n-1}}{g_n} \\
=&-\Delta(p_{n-1}\Delta x_{n-1}) + \frac{x_n}{g_n}\left(p_n g_{n+1} - (p_n + p_{n-1})g_n + p_{n-1}g_{n-1}\right) \\
=&-\Delta(p_{n-1}\Delta x_{n-1}) + \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}x_n,
\end{aligned}$$

after a reverse application of equation (1.6), as required. ■

The second important lemma that we will frequently make use of is the *summation by parts* formula [14].

Lemma 1.3.3. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences and let k and m be fixed integers satisfying $0 \leq k \leq m$. Then*

$$\sum_{n=k}^{m}(x_n\Delta y_n) = x_{m+1}y_{m+1} - x_k y_k - \sum_{n=k}^{m}(y_{n+1}\Delta x_n). \qquad (1.7)$$

Proof. We will prove this lemma by means of induction. First, we must consider the base case. Upon letting $m = k$, we see that the left-hand side of equation (1.7) is given by $x_k\Delta y_k$, whilst the right, by

$$\begin{aligned}
x_{k+1}y_{k+1} - x_k y_k - y_{k+1}\Delta x_k &= x_{k+1}y_{k+1} - x_k y_k - y_{k+1}x_{k+1} + y_{k+1}x_k \\
&= -x_k y_k + y_{k+1}x_k \\
&= x_k\Delta y_k.
\end{aligned}$$

As both sides are equal, the base case holds.

We continue by assuming that the equation holds for $m = j \geq k$. We wish to prove that equation (1.7) holds for $m = j+1$, that is,

$$\sum_{n=k}^{j+1}(x_n \Delta y_n) = x_{j+2}y_{j+2} - x_k y_k - \sum_{n=k}^{j+1}(y_{n+1}\Delta x_n).$$

Immediately, we note that

$$\sum_{n=k}^{j+1}(x_n \Delta y_n) = x_{j+1}\Delta y_{j+1} + \sum_{n=k}^{j}(x_n \Delta y_n).$$

Then, we can use our inductive assumption on the right-hand side of this equation to conclude that

$$\begin{aligned}\sum_{n=k}^{j+1}(x_n \Delta y_n) &= x_{j+1}\Delta y_{j+1} + \left[x_{j+1}y_{j+1} - x_k y_k - \sum_{n=k}^{j}(y_{n+1}\Delta x_n)\right] \\ &= x_{j+1}y_{j+2} - x_k y_k - \sum_{n=k}^{j}(y_{n+1}\Delta x_n).\end{aligned}$$

By artificially introducing $x_{j+2}y_{j+2} - x_{j+2}y_{j+2} = 0$ into this equation, we see that

$$\begin{aligned}\sum_{n=k}^{j+1}(x_n \Delta y_n) &= x_{j+2}y_{j+2} - x_{j+2}y_{j+2} + x_{j+1}y_{j+2} - x_k y_k - \sum_{n=k}^{j}(y_{n+1}\Delta x_n) \\ &= x_{j+2}y_{j+2} - x_k y_k - y_{j+2}\Delta x_{j+1} - \sum_{n=k}^{j}(y_{n+1}\Delta x_n) \\ &= x_{j+2}y_{j+2} - x_k y_k - \sum_{n=k}^{j+1}(y_{n+1}\Delta x_n),\end{aligned}$$

as required. ■

Finally, consider the equation $Ju = f$ for some $f \in \ell^2$. The following lemma constructs a particular solution to this equation by means of the *variation of constants* formula, as seen in [28] and [51].

Lemma 1.3.4. *Let $\{\zeta, \eta\}$ be a fundamental system of solutions to the equation $Ju = 0$. Then, a particular solution $\tilde{u}$ to the equation $J\tilde{u} = f$ for some $f \in \ell^2$ is given by*

$$\tilde{u}_n = \zeta_n \sum_{r=0}^{n-1} \frac{\eta_r f_r}{W_r(\zeta,\eta)} - \eta_n \sum_{r=0}^{n-1} \frac{\zeta_r f_r}{W_r(\zeta,\eta)}, \qquad n \geq 0,$$

where

$$W_r(\zeta,\eta) = -a_r \begin{vmatrix} \zeta_r & \eta_r \\ \zeta_{r+1} & \eta_{r+1} \end{vmatrix}.$$

Remark. The function $W_r(\zeta, \eta)$ is known as the *Wronskian* of ζ and η.

With the fundamental Hilbert space and operators relevant to this thesis now described, the next section will be devoted to providing basic definitions and theory related to such operators.

1.3.2 Difference Operator Theory

In this section, we investigate equations of the form

$$(Mx)_n = \lambda x_n, \qquad n \in \mathbb{N}_0, \tag{1.8}$$

where $\lambda \in \mathbb{R}$ and M is a second-order difference expression such that

$$(Mx)_n = -\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n, \qquad n \geq 0,$$

for two real sequences $\{p_n\}$ and $\{q_n\}$ with $p_n > 0$ for all $n \in \mathbb{N}_0$ and $p_{-1} \equiv 0$. In particular, we begin by disclosing the following definitions [14].

Definition 1.3.5. For any real sequence $\{x_n\}$, we can construct a polygonal curve by plotting the sequence $\{x_n\}$ in the (n, x)-plane; if this curve crosses the n-axis, then we call that point a *node*.

Now, we introduce the notion of oscillatory and non-oscillatory solutions to equations of the form presented in (1.8).

Definition 1.3.6. Given a fixed $\lambda \in \mathbb{R}$, a real solution $x = \{x_n\}$ of equation (1.8) is said to be *oscillatory* if, for every $m \in \mathbb{N}_0$, there exists a node at some point $M \in \mathbb{R}$ where $M > m$. Conversely, the solution x is said to be *non-oscillatory* if there exists an $m \in \mathbb{N}_0$ such that there are no more nodes after this point m, or, in other words, either $x_n \geq 0$ or $x_n \leq 0$ for all $n > m$. If all solutions to equation (1.8) are non-oscillatory then equation (1.8) is called *non-oscillatory* [14].

Remark. To illustrate this concept, fix $\lambda > 0$ and consider the two differential expressions $M_1 f = -f''$ and $M_2 f = f''$, where f is a square integrable function on the interval $[0, \infty)$, i.e., $f \in L^2([0, \infty))$. We can then solve

$$-f'' = \lambda f \qquad \text{and} \qquad f'' = \lambda f$$

independently to see that the general solutions to these two equations are given by

$$f(x) = c_1 \sin(\sqrt{\lambda} x) + c_2 \cos(\sqrt{\lambda} x) \quad \text{and} \quad f(x) = d_1 e^{\sqrt{\lambda} x} + d_2 e^{-\sqrt{\lambda} x},$$

respectively. When the constants c_1, c_2, d_1 and d_2 are real, the trigonometric functions $\sin(\sqrt{\lambda}x)$ and $\cos(\sqrt{\lambda}x)$ would correspond to oscillatory solutions of M_1, whilst the exponential functions $e^{\sqrt{\lambda}x}$ and $e^{-\sqrt{\lambda}x}$, non-oscillatory solutions of M_2.

Remark. Let $p_n > 0$ for all $n \geq 0$. If equation (1.8) is non-oscillatory for some $\lambda \in \mathbb{R}$, then this is equivalent to the operator T associated to M being *bounded below*, as stated in [14, Thm. 2.1].

When equation (1.8) is non-oscillatory, say for some $\lambda \in \mathbb{R}$, we can further characterise two classes of noteworthy solutions: namely, the principal solution and non-principal solutions.

Definition 1.3.7. Let $u = \{u_n\}$ and $z = \{z_n\}$ be real solutions to a non-oscillatory equation of the form (1.8) for some $\lambda \in \mathbb{R}$. The *principal solution* u is the unique solution (up to constant multiples) with no nodes after some $m \in \mathbb{N}_0$ such that

$$\lim_{n\to\infty} \frac{u_n}{z_n} = 0$$

for all solutions z that are not a constant multiple of u. The solutions z are then called the *non-principal solutions.*

The principal solution can then be thought of as, in some sense, the smallest solution due to this limiting behaviour. We stress that there will always exist a unique principal solution (up to constant multiples) under this construction, as proved in [14, Thm. 2.4].

Remark. Recall the operator $M_2 f = f''$ for $f \in L^2([0,\infty])$ and fix $\lambda > 0$. Our previous remark demonstrated that the equation $f'' = \lambda f$ is non-oscillatory: any linear combination of the two non-oscillatory solutions $e^{\sqrt{\lambda}x}$ and $e^{-\sqrt{\lambda}x}$ is, again, non-oscillatory. Here, the principal solution u is given by $u(x) = e^{-\sqrt{\lambda}x}$, whilst the non-principal solutions v are of the form

$$v(x) = d_1 e^{\sqrt{\lambda}x} + d_2 e^{-\sqrt{\lambda}x},$$

for real constants $d_1 \neq 0$ and d_2.

Now, let T be an operator with domain $\mathcal{D}(T) \subseteq \ell^2$, where $Tx = Mx$ for all $x \in \mathcal{D}(T)$. We can explicitly characterise the operators $T_{\max}$ and $T_{\min}$ associated to this symmetric operator T. In particular, $T_{\max}$ is the operator with domain

$$\mathcal{D}(T_{\max}) = \left\{ x \in \ell^2 \;\middle|\; Mx \in \ell^2 \right\},$$

where $T_{\max} x = Mx$ for all $x \in D(T_{\max})$. Conversely, let T' be the restriction of $T_{\max}$ to the domain

$$\mathcal{D}(T') = \{x \in \mathcal{D}(T_{\max}) \mid x_n = 0 \text{ for all but a finite number of values of } n\}.$$

The minimal operator $T_{\min}$ is defined to be the closure of T' in ℓ^2.

Under this construction, it can be seen that $T_{\min}^* = T_{\max}$. Indeed, we begin by investigating the equality $\langle T'x, y\rangle = \langle x, (T')^*y\rangle$ for $x \in \mathcal{D}(T')$ and $y \in \mathcal{D}((T')^*)$. Since $(T')^* = \left(\overline{T'}\right)^*$, we have $(T')^* = T_{\min}^*$. As $x_n = 0$ for all $n > N$ for some $N \in \mathbb{N}$, formally, we have

$$\begin{aligned}\langle T'x, y\rangle &= \sum_{n=0}^{\infty} \left[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n\right] \overline{y}_n \\ &= -\sum_{n=0}^{N+1} \left[\Delta(p_{n-1}\Delta x_{n-1})\right] \overline{y}_n + \sum_{n=0}^{N} q_n x_n \overline{y}_n.\end{aligned}$$

We then isolate the first term in the equation above before applying the summation by parts formula. In particular, we see that

$$-\sum_{n=0}^{N+1} \left[\Delta(p_{n-1}\Delta x_{n-1})\right] \overline{y}_n = \sum_{n=0}^{N+1} p_n \Delta x_n \Delta \overline{y}_n,$$

after noticing that both boundary terms vanish since $p_{-1} \equiv 0$ and $x_n = 0$ for all $n > N$. Another application of the summation by parts formula then yields

$$\begin{aligned}\sum_{n=0}^{N+1} p_n \Delta x_n \Delta \overline{y}_n &= -p_0 x_0 \Delta \overline{y}_0 - \sum_{n=0}^{N+1} x_{n+1} \Delta(p_n \Delta \overline{y}_n) \\ &= -p_0 x_0 \Delta \overline{y}_0 - \sum_{n=1}^{N+1} x_n \Delta(p_{n-1} \Delta \overline{y}_{n-1}) \\ &= \sum_{n=0}^{N+1} x_n \left[-\Delta(p_{n-1}\Delta \overline{y}_{n-1})\right],\end{aligned}$$

since $-p_0 x_0 \Delta \overline{y}_0 = -x_0 \Delta(p_{-1}\Delta \overline{y}_{-1})$. We may then conclude that

$$\begin{aligned}\langle T'x, y\rangle &= \sum_{n=0}^{N+1} x_n \left[-\Delta(p_{n-1}\Delta \overline{y}_{n-1}) + q_n \overline{y}_n\right] \\ &= \sum_{n=0}^{\infty} x_n \left[-\Delta(p_{n-1}\Delta \overline{y}_{n-1}) + q_n \overline{y}_n\right].\end{aligned}$$

This equality is valid for any $y \in \ell^2$ with $My \in \ell^2$, and so $(T')^*y = My$ since $\langle T'x, y\rangle = \langle x, (T')^*y\rangle$. As such, $(T')^* = T_{\max}$, or, in other words, $T_{\min}^* = T_{\max}$, as required.

We now introduce another two important classes that an operator may belong to; these classes rely on the computation of the kernel of $T_{\max}$, so we begin by investigating the equation $T_{\max}x = 0$ for $x \in \mathcal{D}(T_{\max})$. Indeed, we expect two linearly independent solutions to the general equation

$$-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n = 0, \qquad n \in \mathbb{N}, \tag{1.9}$$

but we must determine whether or not those solutions lie in ℓ^2 and which of them further satisfy the initial condition prescribed by the first row condition, i.e., when $n = 0$.

In particular, one can use results from Chapter 5 of [12] — in particular, Theorem 5.4.1 — to conclude that either one or both of these solutions will belong to ℓ^2 (see also: [35]). From this, we can then define what it means for an operator to be of limit-point type in our setting.

Definition 1.3.8. An operator T is of *limit-point* type if only one solution to equation (1.9) lies in ℓ^2.

When an operator is of limit-point type, there exists precisely one non-negative, self-adjoint extension of $T_{\min}$: the Friedrichs extension. We now define the converse in our setting.

Definition 1.3.9. An operator T is of *limit-circle* type if both solutions to equation (1.9) lie in ℓ^2.

Remark. When the operator T is positive, we have equivalent definitions for what it means to be of limit-point type and of limit-circle type. Here, the kernel can either be zero or one-dimensional as the initial condition requires us to fix one of the constants in the general solution. Thus, T is of limit-point type if $\ker T^* = \{0\}$ whilst it is of limit-circle type if $\dim(\ker T^*) = 1$.

We have now disclosed all of the relevant definitions, theorems and results that will be necessary in parsing the next chapter of this thesis. Our objective will be to describe the non-negative, self-adjoint extensions of Jacobi operators that are associated to a positive difference expression M — that will be introduced at the beginning of the chapter — by means of the Kreĭn-Vishik-Birman theory.

Chapter 2

Non-negative, Self-adjoint Extensions of Jacobi Operators

2.1 An Introduction to the Problem

Let M be the second-order difference expression M given by

$$(Mx)_n = -\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n, \qquad x \in \ell^2,$$

where $\{p_n\}$ and $\{q_n\}$ are real sequences with $p_n > 0$ for all $n \in \mathbb{N}_0$ and $p_{-1} \equiv 0$. Recall that Δ represents the forward difference operator, that is,

$$\Delta x_n = x_{n+1} - x_n.$$

The objective of this chapter will be to characterise the non-negative, self-adjoint extensions of $T_{\min}$, the minimal operator associated to M. In what follows, we will assume that the operator enjoys a lower bound; in fact, we may assume, without loss of generality, that this lower bound is positive since we may simply shift q_n otherwise.

The first step to achieving this characterisation will be to find an expression that the sesquilinear form $\mathbf{t_F}$ associated to the Friedrichs extension T_F may take, before conjecturing its form domain, $Q(\mathbf{t_F})$. We can then proceed by using the known characterisations of the operator domain of the Friedrichs extension $\mathcal{D}(T_F)$ given in [14] to show that the operator associated to the form with domain $Q(\mathbf{t_F})$ is in fact T_F, confirming that our conjectured form domain is correct. This is the first time, to our knowledge, that the form associated to the Friedrichs extension — including its domain — has been explicitly constructed in the difference equation setting.

With the form domain of the Friedrichs extension established, we are then in a position to use results from the paper by Alonso-Simon [5], following the method presented in Brown-Evans [20], to describe all of the non-negative, self-adjoint extensions of $T_{\min}$ in terms of a non-principal solution to the equation $Mx = \lambda x$. In particular, we aim to characterise all such extensions of $T_{\min}$ by constructing analogous results to those formed in the continuous setting, i.e., those presented in [20]. We will then conclude this chapter by applying the theory and results attained to an example; namely, we investigate a second-order difference equation whose associated orthogonal polynomials are the Stieltjes-Wigert polynomials.

2.2 The Form Associated to the Friedrichs Extension

2.2.1 The Expression of the Form

As the first step in characterising all of the non-negative, self-adjoint extensions of $T_{\min}$ is to produce an expression that the associated form will take, this section will derive a suitable form by means of explicit calculations. We stress that Jacobi's factorisation identity — as presented in Lemma 1.3.2 — is crucial in determining the expression that the form may take as it is not clear that certain limits that arise exist if we do not first perform the transformation, as will become clear below.

Hence, Jacobi's factorisation identity may be expressed as

$$-\Delta(p_{n-1}\Delta x_{n-1})\overline{y}_n = -\frac{1}{g_n}\Delta\left[P_{n-1}\Delta\left(\frac{x_{n-1}}{g_{n-1}}\right)\right]\overline{y}_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}x_n\overline{y}_n,$$

where $P_n = p_n g_n g_{n+1}$, after multiplying both sides of equation (1.4) by some sequence $\overline{y}_n \in \ell^2$. If we proceed by summing both sides of this equation from 0 up to some fixed $k \in \mathbb{N}_0$, then we see that

$$\sum_{n=0}^{k} -\Delta(p_{n-1}\Delta x_{n-1})\overline{y}_n = \sum_{n=0}^{k}\left[-\Delta\left(P_{n-1}\Delta z_{n-1}\right)\overline{w}_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}x_n\overline{y}_n\right],$$

upon setting $z_n = \frac{x_n}{g_n}$ and $w_n = \frac{y_n}{g_n}$. By using the summation by parts formula described in Lemma 1.3.3, we may conclude that

$$\begin{aligned}\sum_{n=0}^{k} -\Delta\left(P_{n-1}\Delta z_{n-1}\right)\overline{w}_n &= \sum_{n=0}^{k} P_n\Delta z_n\Delta\overline{w}_n - P_k\Delta z_k\overline{w}_{k+1} + P_{-1}\Delta z_{-1}\overline{w}_0 \\ &= \sum_{n=0}^{k} P_n\Delta z_n\Delta\overline{w}_n - P_k\Delta z_k\overline{w}_{k+1},\end{aligned}$$

after recognising that $P_{-1} = p_{-1}g_{-1}g_0 = 0$. By substituting this into our equation so far, and then taking the limit of both sides as $k \to \infty$, we see that

$$\lim_{k\to\infty} \sum_{n=0}^{k} -\Delta(p_{n-1}\Delta x_{n-1})\overline{y}_n$$
$$= \lim_{k\to\infty} \sum_{n=0}^{k} \left[P_n \Delta z_n \Delta \overline{w}_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n} x_n \overline{y}_n \right] - \lim_{k\to\infty} P_k \Delta z_k \overline{w}_{k+1},$$

provided that both limits on the right-hand side of this equation exist individually. Hence, we arrive at the following equality, provided that the limits exist:

$$\langle T_{\max} x, y\rangle = \mathbf{t_F}\,[x,y] - \lim_{k\to\infty} p_k g_k g_{k+1} \Delta \left(\frac{x_k}{g_k}\right) \frac{\overline{y}_{k+1}}{g_{k+1}}, \tag{2.1}$$

where we have set

$$\mathbf{t_F}\,[x,y] := \sum_{n=0}^{\infty} p_n g_n g_{n+1} \Delta\left(\frac{x_n}{g_n}\right) \Delta\left(\frac{\overline{y}_n}{g_n}\right) + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n} \right] x_n \overline{y}_n. \tag{2.2}$$

Remark. We note that the term $p_k g_k g_{k+1} \Delta\left(\frac{x_k}{g_k}\right)$ inside of the limit in equation (2.1) can be expressed precisely as the Wronskian W_k, as introduced in Lemma 1.3.4, i.e.,

$$W_k(u,v) = p_k\left(u_k v_{k+1} - v_k u_{k+1}\right). \tag{2.3}$$

In particular, we see that

$$\begin{aligned} p_k g_k g_{k+1} \Delta\left(\frac{x_k}{g_k}\right) &= p_k g_k g_{k+1} \left(\frac{x_{k+1}}{g_{k+1}} - \frac{x_k}{g_k}\right) \\ &= p_k g_k g_{k+1} \left(\frac{g_k x_{k+1} - x_k g_{k+1}}{g_k g_{k+1}}\right) \\ &= p_k \left(g_k x_{k+1} - x_k g_{k+1}\right), \end{aligned}$$

resulting in

$$\lim_{k\to\infty} p_k g_k g_{k+1} \Delta\left(\frac{x_k}{g_k}\right) \frac{\overline{y}_{k+1}}{g_{k+1}} = \lim_{k\to\infty} W_k(g,x) \frac{\overline{y}_{k+1}}{g_{k+1}}.$$

We will see below that, for a suitable choice of g, this limit exists for all relevant sequences x and y.

With this expression of the form in mind, we must now conjecture a suitable form domain $Q(\mathbf{t_F})$ such that the form $\mathbf{t_F}$ with form domain $Q(\mathbf{t_F})$ possesses all of the necessary properties in order for us to conclude that there exists a self-adjoint operator associated to it.

2.2.2 Constructing the Form Domain

We begin by recalling Theorem 1.2.12: it states that the Friedrichs extension T_F of some positive, symmetric operator T has a domain satisfying

$$\mathcal{D}(T_F) = \mathcal{D}(T_{\max}) \cap Q(\mathbf{t_F}). \tag{2.4}$$

It is also known that $\mathcal{D}(T_F)$ has several explicit characterisations, as displayed in [14]. In particular, we wish to use the characterisation of $\mathcal{D}(T_F)$ which states that

$$\mathcal{D}(T_F) = \left\{ x \in \ell^2 \;\middle|\; \sum_{n=0}^{\infty} |(T_{\max}x)_n|^2 < \infty, \right.$$
$$\left. \sum_{n=m}^{\infty} p_n v_n v_{n+1} \left| \Delta \left(\frac{x_n}{v_n} \right) \right|^2 < \infty \text{ and } \lim_{n\to\infty} \frac{x_n}{v_n} = 0 \right\}, \tag{2.5}$$

where v_n is a non-principal solution to an equation of the form given in (1.8) that is non-oscillatory from some fixed $m \in \mathbb{N}_0$ onwards, for some $\lambda \in \mathbb{R}$. Without loss of generality, we may assume that $v_n > 0$ for all $n \geq m$.

Remark. We have specified that the equation will be non-oscillatory and the operator, positive. As such, the operator is bounded below by some strictly positive constant γ. Then, upon invoking [14, Thm. 2.1], we see that the equation is, again, non-oscillatory for all λ satisfying $\lambda \leq \gamma$. As such, here, and in what follows, we will set $\lambda = 0$.

With these two facts established, a reasonable conjecture for the form domain immediately surfaces. Effectively, the domain of the Friedrichs extension given in equation (2.5) consists of three conditions: the first of which simply demands that an $x \in \ell^2$ also lies in $\mathcal{D}(T_{\max})$. Then, the remaining two conditions *must* be present in $Q(\mathbf{t_F})$ for equation (2.4) to hold, and so we conjecture that $Q(\mathbf{t_F})$ is of the form

$$Q(\mathbf{t_F}) = \left\{ x \in \ell^2 \;\middle|\; \sum_{n=m}^{\infty} p_n g_n g_{n+1} \left| \Delta \left(\frac{x_n}{g_n} \right) \right|^2 < \infty \text{ and } \lim_{n\to\infty} \frac{x_n}{v_n} = 0 \right\}, \tag{2.6}$$

where $g_n = \begin{cases} g_n \in \mathbb{R}_{>0}, & n < m, \\ v_n > 0, & n \geq m. \end{cases}$

Now that we are in possession of a form $\mathbf{t_F}$ and its form domain $Q(\mathbf{t_F})$, we may begin to investigate what properties it may exhibit. In particular, we hope to utilise Theorem 1.1.34 as this would allow us to associate a self-adjoint operator to the form. The following section aims to verify that the form $\mathbf{t_F}$ given by equation (2.2), whose form domain $Q(\mathbf{t_F})$ is expressed in equation (2.6), is, indeed, in possession of the required properties in order for us to do so.

2.3 Properties of the Form $\mathbf{t_F}$

Our intention is to show that the form $\mathbf{t_F}$, with form domain $Q(\mathbf{t_F})$, that was constructed in Sections 2.2.1 and 2.2.2 is, in fact, associated to the Friedrichs extension T_F with domain $\mathcal{D}(T_F)$ as given in equation (2.5): we will do this by utilising Theorem 1.1.34. Recall that if $\mathbf{t}$ is a closed, densely defined, symmetric form, then we can uniquely associate it to a self-adjoint operator. This section explores these properties and verifies that it is valid to apply the theorem described above on the form that we have constructed.

2.3.1 Symmetry and Sectoriality of the Form

It is clear to see that the proposed form $\mathbf{t_F}$ given by equation (2.2), with domain $Q(\mathbf{t_F})$ expressed in equation (2.6), is, indeed, a sesquilinear form. Similarly, it is readily observed that $\mathbf{t_F}$ is a symmetric form since p_n, q_n and g_n are real for all $n \geq 0$: in particular,

$$\overline{P}_n = \overline{p_n g_n g_{n+1}} = p_n g_n g_{n+1} = P_n,$$

for all $n \in \mathbb{N}_0$.

Furthermore, as we have specified that the equation will be non-oscillatory and the operator positive, it will enjoy a positive lower bound. Since the form is real and has this positive lower bound, we can easily deduce that the form is sectorial since its numerical range will lie exclusively in an interval on the positive real axis.

2.3.2 Closure of the Form

In order to show that the form is closed, we are required to verify that $(Q(\mathbf{t_F}), \|\cdot\|_{\mathbf{t_F}})$ is a Hilbert space, where

$$\begin{aligned}\|x\|_{\mathbf{t_F}}^2 &= \mathbf{t_F}\,[x] \\ &= \sum_{k=0}^{\infty} p_k g_k g_{k+1} \left| \Delta \left(\frac{x_k}{g_k} \right) \right|^2 + \sum_{k=0}^{\infty} \left[q_k - \frac{\Delta(p_{k-1}\Delta g_{k-1})}{g_k} \right] |x_k|^2.\end{aligned}$$

Note that we have chosen the norm which corresponds to $\|\cdot\|_2$ in Section 1.1.3 due to the lower bound γ of the form being positive.

We begin by letting $x^{(n)}$ be a Cauchy sequence in $(Q(\mathbf{t_F}), \|\cdot\|_{\mathbf{t_F}})$. That is, given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all n, $m > N$, we have

$\|x^{(n)} - x^{(m)}\|^2_{\mathbf{t_F}} < \varepsilon^2$, or, in other words,

$$\sum_{k=0}^{\infty} p_k g_k g_{k+1} \left| \Delta \left(\frac{x_k^{(n)} - x_k^{(m)}}{g_k} \right) \right|^2 + \sum_{k=0}^{\infty} \left[q_k - \frac{\Delta(p_{k-1}\Delta g_{k-1})}{g_k} \right] |x_k^{(n)} - x_k^{(m)}|^2 < \varepsilon^2.$$

However, as $\mathbf{t_F}$ is lower semi-bounded with a positive γ, that is,

$$\mathbf{t_F}\,[x] \geq \gamma \|x\|^2_{\ell^2}, \qquad \gamma > 0,$$

we see that

$$\|x^{(n)} - x^{(m)}\|^2_{\ell^2} \leq \frac{1}{\gamma} \|x^{(n)} - x^{(m)}\|^2_{\mathbf{t_F}} \leq \frac{\varepsilon^2}{\gamma}.$$

From this, it is readily observed that $\gamma \|x^{(n)} - x^{(m)}\|^2_{\ell^2} < \varepsilon^2$, showing that $x^{(n)}$ is also a Cauchy sequence in ℓ^2. Hence $x^{(n)}$ converges to some limit x, say, as $n \to \infty$ in ℓ^2.

Now that we have found a candidate for the limit, we must show that the limit x lies in the space $Q(\mathbf{t_F})$ — x must satisfy the two conditions given in equation (2.6) for this to hold. Firstly, we must determine whether

$$\sum_{n=m}^{\infty} p_n g_n g_{n+1} \left| \Delta \left(\frac{x_n}{v_n} \right) \right|^2 < \infty$$

for such a sequence x. We begin by fixing $B \in \mathbb{N}$; then,

$$\sum_{k=0}^{B} p_k g_k g_{k+1} \left| \Delta \left(\frac{x_k^{(n)} - x_k^{(m)}}{g_k} \right) \right|^2 < \varepsilon^2.$$

Upon taking $m \to \infty$, we see that

$$\lim_{m\to\infty} \sum_{k=0}^{B} p_k g_k g_{k+1} \left| \Delta \left(\frac{x_k^{(n)} - x_k^{(m)}}{g_k} \right) \right|^2 = \sum_{k=0}^{B} p_k g_k g_{k+1} \left| \Delta \left(\frac{x_k^{(n)} - x_k}{g_k} \right) \right|^2 \leq \varepsilon^2,$$

since $x^{(m)} \to x$ in ℓ^2 and, in particular, $x_k^{(m)} \to x_k$ for all $k \in \mathbb{N}_0$. Hence,

$$\left(\sum_{k=0}^{B} p_k g_k g_{k+1} \left| \Delta \left(\frac{x_k^{(n)} - x_k}{g_k} \right) \right|^2 \right)_B$$

is an increasing sequence in B that is bounded above by ε^2, showing that the limit as $B \to \infty$ exists and, in particular,

$$\sum_{k=0}^{\infty} p_k g_k g_{k+1} \left| \Delta \left(\frac{x_k^{(n)} - x_k}{g_k} \right) \right|^2 \leq \varepsilon^2.$$

We may then conclude that the sequence $x^{(n)} - x$ satisfies the first condition for lying in the form domain $Q(\mathbf{t_F})$. If we define the semi-norm $|x|_{\mathbf{t_F}}$ such that

$$|x|^2_{\mathbf{t_F}} = \sum_{k=0}^{\infty} p_k g_k g_{k+1} \left| \Delta \left(\frac{x_k}{g_k} \right) \right|^2, \qquad x \in Q(\mathbf{t_F}),$$

then, by the triangle inequality, we have

$$|x|_{\mathbf{t_F}} = |x - x^{(n)} + x^{(n)}|_{\mathbf{t_F}} \leq |x - x^{(n)}|_{\mathbf{t_F}} + |x^{(n)}|_{\mathbf{t_F}}.$$

We have already shown that $|x - x^{(n)}|_{\mathbf{t_F}} \leq \varepsilon^2$, whilst $|x^{(n)}|_{\mathbf{t_F}} < \infty$ by virtue of $x^{(n)}$ belonging to $Q(\mathbf{t_F})$. As such, we can conclude that $|x|_{\mathbf{t_F}} < \infty$, or, in other words, the sequence x satisfies the first condition for lying in $Q(\mathbf{t_F})$, as required.

Next, we must show that $\lim_{k\to\infty} \frac{x_k}{v_k} = 0$; we will do so by showing that

$$S := \lim_{k\to\infty} S_k = 0,$$

where $S_k = \left|\frac{x_{k+1}}{v_{k+1}}\right|^2$. Let

$$S_k^{(n)} = \left| \frac{x^{(n)}_{k+1}}{v_{k+1}} \right|^2.$$

Since $x^{(n)} \in Q(\mathbf{t_F})$, we have $\lim_{k\to\infty} \frac{x_k^{(n)}}{v_k} = 0$. Hence, we may rewrite $S_k^{(n)}$ as the following telescopic sum:

$$\begin{aligned}
S_k^{(n)} &= \left| \sum_{m=k+1}^{\infty} \Delta \left(\frac{x_m^{(n)}}{v_m} \right) \right|^2 \\
&= \left| \sum_{m=k+1}^{\infty} \frac{(p_m v_m v_{m+1})^{\frac{1}{2}}}{(p_m v_m v_{m+1})^{\frac{1}{2}}} \Delta \left(\frac{x_m^{(n)}}{v_m} \right) \right|^2 \\
&= \left| \sum_{m=k+1}^{\infty} \frac{1}{(p_m v_m v_{m+1})^{\frac{1}{2}}} \cdot \left[(p_m v_m v_{m+1})^{\frac{1}{2}} \Delta \left(\frac{x_m^{(n)}}{v_m} \right) \right] \right|^2.
\end{aligned}$$

By the Cauchy-Schwarz inequality, we see that

$$\begin{aligned}
S_k^{(n)} &\leq \left(\sum_{m=k+1}^{\infty} \left| \frac{1}{(p_m v_m v_{m+1})^{\frac{1}{2}}} \right|^2 \right) \cdot \left(\sum_{m=k+1}^{\infty} \left| (p_m v_m v_{m+1})^{\frac{1}{2}} \Delta \left(\frac{x_m^{(n)}}{v_m} \right) \right|^2 \right) \\
&= \left(\sum_{m=k+1}^{\infty} \frac{1}{p_m v_m v_{m+1}} \right) \cdot \left(\sum_{m=k+1}^{\infty} p_m v_m v_{m+1} \left| \Delta \left(\frac{x_m^{(n)}}{v_m} \right) \right|^2 \right)
\end{aligned}$$

after recalling that p_n is a real sequence where $p_n > 0$ for all $n \in \mathbb{N}_0$ and v_n is a non-principal solution that has the *same* sign after some node. Using that $|x^{(n)} - x|_{\mathbf{t_F}} \to 0$, upon taking $n \to \infty$, we see that

$$S_k = \left| \frac{x_{k+1}}{v_{k+1}} \right|^2 \leq \left(\sum_{m=k+1}^{\infty} \frac{1}{p_m v_m v_{m+1}} \right) \cdot \left(\sum_{m=k+1}^{\infty} p_m v_m v_{m+1} \left| \Delta \left(\frac{x_m}{v_m} \right) \right|^2 \right).$$

Having already proven that $\sum_{m=k+1}^{\infty} p_m v_m v_{m+1} \left|\Delta\left(\frac{x_m}{v_m}\right)\right|^2 < \infty$, we now make use of [14, Thm. 2.4 (iii)] to conclude that

$$\sum_{m=k+1}^{\infty} \frac{1}{p_m v_m v_{m+1}} < \infty,$$

since v_n is the non-principal solution. Then, by taking $k \to \infty$, we see that

$$S \leq \lim_{k\to\infty} \left(\sum_{m=k+1}^{\infty} \frac{1}{p_m v_m v_{m+1}}\right) \cdot \left(\sum_{m=k+1}^{\infty} p_m v_m v_{m+1} \left|\Delta\left(\frac{x_m}{v_m}\right)\right|^2\right).$$

In particular, we have that

$$\lim_{k\to\infty} \sum_{m=k+1}^{\infty} \frac{1}{p_m v_m v_{m+1}} = 0 \text{ and } \lim_{k\to\infty} \sum_{m=k+1}^{\infty} p_m v_m v_{m+1} \left|\Delta\left(\frac{x_m}{v_m}\right)\right|^2 = 0,$$

and so $S = 0$, as required. As the sequence x satisfies both properties, we can conclude that $x \in Q(\mathbf{t_F})$. Note that Jacobi's factorisation identity has allowed us to introduce limits that we know exist.

With the candidate sequence firmly in $Q(\mathbf{t_F})$, it only remains to show that $x^{(n)}$ tends to x in the specified norm $\|\cdot\|_{\mathbf{t_F}}$. By recalling the semi-norm introduced earlier in this section, we see that

$$\|x^{(n)} - x\|_{\mathbf{t_F}}^2 = |x_k^{(n)} - x_k|_{\mathbf{t_F}}^2 + \sum_{k=0}^{\infty} \left[q_k - \frac{\Delta(p_{k-1}\Delta g_{k-1})}{g_k}\right] |x_k^{(n)} - x_k|^2.$$

Since g_k is a non-principal solution to the equation $(Mx)_k = 0$ for all k after some $m \in \mathbb{N}_0$, we have $-\Delta(p_{k-1}\Delta g_{k-1}) + q_k g_k = 0$ or, in other words,

$$\frac{-\Delta(p_{k-1}\Delta g_{k-1})}{g_k} + q_k = 0,$$

for all $k > m$. As such, there will exist a constant c such that

$$\sum_{k=0}^{\infty} \left[q_k - \frac{\Delta(p_{k-1}\Delta g_{k-1})}{g_k}\right] |x_k^{(n)} - x_k|^2 \leq c\|x^{(n)} - x\|_{\ell^2}^2$$

as the sum contains only a finite number of non-zero terms. With this in mind, we have

$$\|x^{(n)} - x\|_{\mathbf{t_F}}^2 \leq |x_k^{(n)} - x_k|_{\mathbf{t_F}}^2 + c\|x^{(n)} - x\|_{\ell^2}^2. \tag{2.7}$$

Upon taking $n \to \infty$ in equation (2.7), we finally see that $x^{(n)} \to x$ in the $\mathbf{t_F}$-norm as the two terms on the right-hand side individually tend to 0: the first we have already shown, the second because x was defined to be the limit of $x^{(n)}$ in ℓ^2. Hence, $(Q(\mathbf{t_F}), \|\cdot\|_{\mathbf{t_F}})$ is a Hilbert space and the form, closed.

2.3.3 Density of the Form Domain

We can verify that $\mathbf{t_F}$ is a densely defined form by finding a subset of $Q(\mathbf{t_F})$ that is dense in ℓ^2. Indeed, consider E_0, the set of sequences with finitely many non-zero terms, that is,

$$E_0 = \left\{x \in \ell^2 \;\middle|\; \exists N \in \mathbb{N}_0 \text{ such that } x_n = 0 \text{ for all } n > N\right\}.$$

Then, it is clear that E_0 is contained within $Q(\mathbf{t_F})$ and it is well known that the closure of E_0 with respect to the standard ℓ^2-norm is ℓ^2, i.e., $\overline{E_0} = \ell^2$. As such, $Q(\mathbf{t_F})$ contains a dense subset and so is, itself, dense in ℓ^2 with respect to the standard ℓ^2-norm.

Therefore, we have shown that the form $\mathbf{t_F}$ with form domain $Q(\mathbf{t_F})$ is a closed, densely defined, symmetric form. Consequently, we may invoke Theorem 1.1.34 and conclude that there exists a unique self-adjoint operator $\tilde{T}_F$, with domain $\mathcal{D}(\tilde{T}_F)$ and range $\mathcal{R}(\tilde{T}_F)$ in ℓ^2, associated to the form $\mathbf{t_F}$ that satisfies

$$\langle \tilde{T}_F x, y\rangle = \mathbf{t_F}\,[x, y]$$

for all $x \in \mathcal{D}(\tilde{T}_F)$ and $y \in Q(\mathbf{t_F})$. The next section aims to prove that the operator $\tilde{T}_F$ associated to the form $\mathbf{t_F}$ is, in fact, the Friedrichs extension T_F. This will be achieved by verifying that the form with the conjectured form domain gives rise to an operator whose domain is one of the known characterisations of the Friedrichs extension.

2.4 Verifying the Conjectured Form Domain

As we have established that there exists a self-adjoint operator associated to the form $\mathbf{t_F}$, this section intends to verify that this operator is, in fact, the Friedrichs extension of $T_{\min}$. We are in possession of several characterisations that the domain of the Friedrichs extension may take, so it is our hope that we can show that the domain that arises through the association coincides precisely with one such characterisation.

Firstly, we recall that the operator T associated with the form $\mathbf{a}$ is defined to have domain

$$\mathcal{D}(T) = \left\{x \in Q(\mathbf{a}) \;\middle|\; \exists f \in \ell^2 \text{ such that } \mathbf{a}\,[x, y] = \langle f, y\rangle \;\; \forall y \in Q(\mathbf{a})\right\}, \quad (2.8)$$

and, in such a case, $f = Tx$. We may also recall that, in our example, the Friedrichs extension exhibits a characterisation given by equation (2.5).

However, before we investigate whether or not we have constructed the Friedrichs extension, we must first ask whether $\tilde{T}_F$ truly is a restriction of the operator $T_{\max}$. This consists of verifying two properties: we must have $\mathcal{D}(\tilde{T}_F) \subseteq \mathcal{D}(T_{\max})$, and the equality $\tilde{T}_F x = T_{\max} x$ must hold for all $x \in \mathcal{D}(\tilde{T}_F)$, where $(T_{\max} x)_n = -\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n$ for all $n \geq 0$.

We begin by noting that, for all $x \in \mathcal{D}(\tilde{T}_F)$ and $y \in E_0 \subseteq Q(\mathbf{t_F})$, we have

$$\mathbf{t_F}\,[x, y] = \langle \tilde{T}_F x, y \rangle\,,$$

by means of equation (2.8). As $y \in E_0$, there exists an $N \in \mathbb{N}_0$ such that $y_n = 0$ for all $n > N$. Hence,

$$\begin{aligned}
\mathbf{t_F}\,[x, y] &= \sum_{n=0}^{N} p_n g_n g_{n+1} \Delta\left(\frac{x_n}{g_n}\right)\Delta\left(\frac{\overline{y}_n}{g_n}\right) + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right] x_n \overline{y}_n \\
&= \sum_{n=0}^{N} -\Delta\left(P_n \Delta z_n\right)\overline{w}_{n+1} + \left[q_n - \frac{\Delta\left(p_{n-1}\Delta g_{n-1}\right)}{g_n}\right] x_n \overline{y}_n \\
&\qquad\qquad\qquad + P_{N+1}\Delta z_{N+1}\overline{w}_{N+1} - P_0 \Delta z_0 \overline{w}_0,
\end{aligned}$$

after an application of the summation by parts formula, upon noting that $P_n = p_n g_n g_{n+1}$ and setting $z_n = \frac{x_n}{g_n}$ and $w_n = \frac{y_n}{g_n}$. We can simplify this equation by making use of Jacobi's factorisation identity in a manner comparable to that used in Section 2.2.1, in addition to noting that $\overline{w}_{N+1} = 0$ and $-\Delta(P_{-1}\Delta z_{-1})\overline{w}_0 = -P_0 \Delta z_0 \overline{w}_0$. Hence,

$$\begin{aligned}
\mathbf{t_F}\,[x, y] &= \sum_{n=-1}^{N-1} -\Delta\left(P_n \Delta z_n\right)\overline{w}_{n+1} + \sum_{n=0}^{N}\left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right] x_n \overline{y}_n \\
&= \sum_{n=0}^{N} -\Delta\left(P_{n-1}\Delta z_{n-1}\right)\overline{w}_n + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right] x_n \overline{y}_n \\
&= \sum_{n=0}^{N}\left[-\frac{1}{g_n}\Delta\left[P_{n-1}\Delta\left(\frac{x_{n-1}}{g_{n-1}}\right)\right] - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n} x_n\right]\overline{y}_n + q_n x_n \overline{y}_n \\
&= \sum_{n=0}^{N}\left[-\Delta\left(p_{n-1}\Delta x_{n-1}\right) + q_n x_n\right]\overline{y}_n.
\end{aligned}$$

Since $x \in \mathcal{D}(\tilde{T}_F)$, we have $\mathbf{t_F}\,[x, y] = \langle f, y \rangle$ for some $f \in \ell^2$ and, in particular,

$$\sum_{n=0}^{N}\left[-\Delta\left(p_{n-1}\Delta x_{n-1}\right) + q_n x_n\right]\overline{y}_n = \langle f, y \rangle\,.$$

Then, for an arbitrary $k \in \mathbb{N}$, we may choose $y = (\delta_{n,k})_n$, where $\delta_{n,k}$ is the Kronecker delta; this shows that we must take $f_k = -\Delta(p_{k-1}\Delta x_{k-1}) + q_k x_k$. As $f \in \ell^2$, we may then conclude that $\sum_{n=0}^{\infty} |-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n|^2 < \infty$, proving that $\mathcal{D}(\tilde{T}_F) \subseteq \mathcal{D}(T_{\max})$.

Now, for all $x \in \mathcal{D}(\tilde{T}_F)$ and $y \in E_0$,

$$\langle \tilde{T}_F x - T_{\max} x, y \rangle = 0.$$

Hence $\tilde{T}_F x - T_{\max} x \in E_0^{\perp}$, where $E_0^{\perp} = \{0\}$ as E_0 is dense in ℓ^2. Therefore $\tilde{T}_F x = T_{\max} x$, as required.

Now that we have ascertained that $\tilde{T}_F$ is a restriction of $T_{\max}$, the final step consists of proving that the two domains, $\mathcal{D}(\tilde{T}_F)$ and $\mathcal{D}(T_F)$, coincide. If we can successfully show that $\mathcal{D}(\tilde{T}_F) = \mathcal{D}(T_F)$, then we will have proven that the domain of the operator $\tilde{T}_F$ associated to the form $\mathbf{t_F}$ has a domain that can be described as that of the Friedrichs extension.

We begin the verification by first showing that $\mathcal{D}(\tilde{T}_F) \subseteq \mathcal{D}(T_F)$. In fact, showing this containment is trivial: both sets naturally contain the two conditions $\sum_{n=m}^{\infty} p_n g_n g_{n+1} \left| \Delta \left(\frac{x_n}{g_n} \right) \right|^2 < \infty$ and $\lim_{n\to\infty} \frac{x_n}{v_n} = 0$, and we have already shown that $\mathcal{D}(\tilde{T}_F) \subseteq \mathcal{D}(T_{\max})$. Hence, all that remains is to verify the converse, that is, $\mathcal{D}(T_F) \subseteq \mathcal{D}(\tilde{T}_F)$.

We begin by taking $x \in \mathcal{D}(T_F)$. Observe that the same two conditions are satisfied trivially, precisely as before. Thus, we are only required to show that for all $y \in Q(\mathbf{t_F})$ there exists an $f \in \ell^2$ such that $\mathbf{t_F}[x, y] = \langle f, y \rangle$. In fact, we have already shown that, under these circumstances, $f = \tilde{T}_F x = T_{\max} x$. Now, for $y \in Q(\mathbf{t_F})$ we have, by equation (2.1),

$$\langle T_{\max} x, y \rangle = \mathbf{t_F}[x, y] - \lim_{N\to\infty} p_N g_N g_{N+1} \Delta \left(\frac{x_N}{g_N} \right) \frac{\overline{y}_{N+1}}{g_{N+1}}.$$

In other words, we are looking to prove that $\lim_{N\to\infty} P_N \Delta z_N \overline{w}_{N+1} = 0$.

Then, upon considering the expression

$$\sum_{n=m+1}^{N} \left[-\Delta(p_{n-1} \Delta x_{n-1}) + q_n x_n \right] \overline{y}_n,$$

we see that this equals

$$\sum_{n=m+1}^{N} \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n} \right] x_n \overline{y}_n - P_N \Delta z_N \overline{w}_{N+1}$$
$$+ P_m \Delta z_m \overline{w}_{m+1} + \sum_{n=m+1}^{N} P_n \Delta z_n \Delta \overline{w}_n,$$

for $z_n = \frac{x_n}{g_n}$ and $w_n = \frac{y_n}{g_n}$, after applying both Jacobi's factorisation identity and the summation by parts formula once. Then, for sufficiently large m, we have that

$$-\Delta(p_{n-1}\Delta g_{n-1}) + q_n g_n = 0, \qquad n > m,$$

since $g_n = v_n$ for $n \geq m$, where v_n is a non-principal solution to the equation $Tv = 0$, as we have taken $\lambda = 0$.

Hence,

$$P_N \Delta z_N \overline{w}_{N+1} = \sum_{n=m+1}^{N} \Big[[\Delta(p_{n-1}\Delta x_{n-1}) - q_n x_n] \, \overline{y}_n + P_n \Delta z_n \Delta \overline{w}_n \Big] + P_m \Delta z_m \overline{w}_{m+1}, \quad (2.9)$$

after simply rearranging the equality above. By taking the limit as $N \to \infty$ of both sides of equation (2.9), we see that each expression on the right-hand side has a limit that exists and is finite, proving that $\lim_{N\to\infty} P_N \Delta z_N \overline{w}_{N+1} = L$ for some limit L. Specifically, we have that

$$\lim_{N\to\infty} \sum_{n=m+1}^{N} [\Delta(p_{n-1}\Delta x_{n-1}) - q_n x_n] \, \overline{y}_n = L_1,$$

as $x \in \mathcal{D}(T_{\max})$ and $y \in \ell^2$ and

$$\lim_{N\to\infty} \sum_{n=m+1}^{N} P_n \Delta z_n \Delta \overline{w}_n = \lim_{N\to\infty} \sum_{n=m+1}^{N} P_n \Delta \left(\frac{x_n}{g_n} \right) \Delta \left(\frac{\overline{y}_n}{g_n} \right) = L_2,$$

since $x \in \mathcal{D}(T_F)$ and $y \in Q(\mathbf{t_F})$. Finally, it is clear to see that

$$\lim_{N\to\infty} P_m \Delta z_m \overline{w}_{m+1} = L_3,$$

because $P_m \Delta z_m \overline{w}_{m+1}$ does not depend on N.

Now that we know that $\lim_{N\to\infty} P_N \Delta z_N \overline{w}_{N+1}$ exists and equals some L, we simply need to show that $L = 0$ in order to prove our initial claim; we will do this by using results stated in Section 4 of [14] to prove a statement similar to [14, Cor. 4.3].

Lemma 2.4.1. *Suppose that* $\sum_{n=0}^{\infty} P_n |\Delta z_n|^2 < \infty$ *and* $\sum_{n=0}^{\infty} P_n |\Delta w_n|^2 < \infty$ *for two sequences* $\{z_n\}$ *and* $\{w_n\}$*. Then*

$$\liminf_{n\to\infty} (P_n |\Delta z_n \overline{w}_{n+1}|) = 0.$$

Proof. Suppose, for a contradiction, that there exists an $N \in \mathbb{N}$ and $\varepsilon > 0$ such that $P_n |\Delta z_n \overline{w}_{n+1}| \geq \varepsilon$ for all $n \geq N$. Then, by [14, Lem. 4.1] we have that

$$\sum_{n=0}^{\infty} \frac{1}{P_n K(n)} = \sum_{n=0}^{\infty} \frac{1}{P_n K(n-1)} = \infty,$$

where $P_n = p_n g_n g_{n+1}$ and $K(n) = \sum_{j=n+1}^{\infty} \frac{1}{P_j}$. Hence,

$$\begin{aligned} \infty = \sum_{n=N}^{\infty} \frac{1}{P_n K(n-1)} &\leq \frac{1}{\varepsilon} \sum_{n=N}^{\infty} \frac{|\Delta z_n \overline{w}_{n+1}|}{K(n-1)} = \frac{1}{\varepsilon} \sum_{n=N}^{\infty} \frac{P_n^{\frac{1}{2}} |\Delta z_n| |w_{n+1}|}{P_n^{\frac{1}{2}} K(n-1)} \\ &\leq \frac{1}{\varepsilon} \left(\sum_{n=N}^{\infty} P_n |\Delta z_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=N}^{\infty} \frac{|w_{n+1}|^2}{P_n K^2(n-1)} \right)^{\frac{1}{2}}, \end{aligned}$$

by means of the Cauchy-Schwarz inequality. This leads to the desired contradiction as both expressions in the inequality are decidedly finite; specifically, we have that

$$\left(\sum_{n=N}^{\infty} P_n|\Delta z_n|^2\right)^{\frac{1}{2}} < \infty$$

as part of the statement of the lemma and

$$\left(\sum_{n=N}^{\infty} \frac{|w_{n+1}|^2}{P_n K(n-1)^2}\right)^{\frac{1}{2}} < \infty$$

due to [14, Lem. 4.2]. Therefore, it is the case that

$$\liminf_{n\to\infty}(P_n|\Delta z_n \overline{w}_{n+1}|) = 0,$$

as required. ∎

With Lemma 2.4.1 in hand, we are now able to complete the argument which proves that $\mathcal{D}(T_F) \subseteq \mathcal{D}(\tilde{T}_F)$. First, recall that

$$\lim_{N\to\infty} P_N \Delta z_N \overline{w}_{N+1} = L,$$

for some limit L. Then, as the limit exists and has a subsequence converging to 0, we may assert that $L = 0$. Hence, $\mathcal{D}(T_F) \subseteq \mathcal{D}(\tilde{T}_F)$, and, in fact, the two domains $\mathcal{D}(T_F)$ and $\mathcal{D}(\tilde{T}_F)$ coincide.

This completes the proof and verifies that the form $\mathbf{t_F}$ with form domain $Q(\mathbf{t_F})$ represents the form associated to the Friedrichs extension T_F of the minimal operator $T_{\min}$. As such, the next section aims to construct the non-negative, self-adjoint extensions of $T_{\min}$ since the form fundamental to the Kreĭn-Vishik-Birman theory has been determined explicitly.

2.5 Non-negative, Self-adjoint Extensions of $T_{\min}$

Now that we are in possession of the Friedrichs extension, we may begin our attempt in characterising the non-negative, self-adjoint extensions of $T_{\min}$ by means of Theorem 1.2.16. If $T_{\min}$ is of limit-point type, then there exists precisely one non-negative, self-adjoint extension — the Friedrichs extension — so we will instead assume that $T_{\min}$ is of limit-circle type. Thus, we are now able to characterise all of the non-negative, self-adjoint extensions of $T_{\min}$. Indeed, we note that since $\dim \mathcal{N} = 1$, distinguishing between subspaces $\mathcal{N}_B$ of $\mathcal{N}$ is unnecessary: $\mathcal{N}_B$ is either $\{0\}$ or $\mathcal{N}$ itself. Moreover, when $\mathcal{N}_B = \{0\}$, we obtain the Friedrichs extension after letting B be any self-adjoint operator (although formally we choose $B = \infty$, as in Section 1.2.4), so we only need to consider the case when $\mathcal{N}_B = \mathcal{N}$. Hence, we choose to modify Theorem 1.2.16 and restate it as follows: apart from the Friedrichs extension, all non-negative, self-adjoint extensions T_B of $T_{\min}$ are associated to a form $\mathbf{t_B}$ which satisfies

$$\mathbf{t_B}[u,v] = \mathbf{t_F}\left[u^F, v^F\right] + \mathbf{b}\left[u^N, v^N\right], \tag{2.10}$$

whose domain is given by

$$Q(\mathbf{t_B}) = Q(\mathbf{t_F}) \dotplus \mathcal{N}. \tag{2.11}$$

Here, $\mathbf{b}$ is the form associated to the operator B, which acts in $\mathcal{N} = \ker T_{\max}$. Furthermore, elements u, $v \in Q(\mathbf{t_B})$ may be decomposed into

$$\begin{cases} u = u^F + u^N, \\ v = v^F + v^N, \end{cases} \quad \text{where } u^F,\ v^F \in Q(\mathbf{t_F}) \text{ and } u^N,\ v^N \in \mathcal{N}. \tag{2.12}$$

With this adapted theorem in mind, we are now ready to construct the non-negative, self-adjoint extensions of $T_{\min}$ by explicitly characterising the operator domains.

Let $\{\zeta,\ \eta\}$ form a fundamental system of solutions to the recurrence relation

$$(T_{\max}x)_n = -\Delta\left(p_{n-1}\Delta x_{n-1}\right) + q_n x_n = 0, \tag{2.13}$$

where ζ is the principal solution and η is a non-principal solution of equation (2.13).

Remark. Note that we have again chosen $\lambda = 0$ in equation (2.13) since the lower bound γ is positive.

Throughout this section, we will use the characterisation of $\mathbf{t_F}$ which is given by

$$\mathbf{t_F}[x,y] = \sum_{n=0}^{\infty} p_n g_n g_{n+1} \Delta\left(\frac{x_n}{g_n}\right)\Delta\left(\frac{\overline{y}_n}{g_n}\right) + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right] x_n \overline{y}_n,$$

with domain

$$Q(\mathbf{t_F}) = \left\{ x \in \ell^2 \;\middle|\; \sum_{n=m}^{\infty} p_n g_n g_{n+1} \left|\Delta\left(\frac{x_n}{g_n}\right)\right|^2 < \infty \text{ and } \lim_{n\to\infty} \frac{x_n}{\eta_n} = 0 \right\}.$$

Here, η_n is a non-principal solution to equation (2.13) which is non-oscillatory after some fixed $m \in \mathbb{N}_0$ and $g_n = \begin{cases} g_n \in \mathbb{R}_{>0}, & n < m, \\ \eta_n, & n \geq m. \end{cases}$

Remark. We have simply chosen the sequence v in equations (2.5) and (2.6) to be a non-principal solution η. Furthermore, without loss of generality, we may assume that $\eta_n > 0$ for all $n \geq m$.

Since $\dim \mathcal{N} = 1$, we may let the sequence $\{\psi_n\}_0^\infty$ form a basis of $\mathcal{N}$; moreover, we may choose to normalise ψ in such a way that $\psi_0 = 1$. Then, as $B \colon \mathcal{N} \to \mathcal{N}$ is a non-negative operator, it is clear that if $u^N = c\psi \in \mathcal{N}$, for some $c \in \mathbb{C}$, then $Bu^N = \beta c\psi$ for some $\beta \geq 0$.

Our first step in the process of characterising the extensions is to determine how we may decompose elements of $Q(\mathbf{t_B})$ in line with equations (2.10), (2.11) and (2.12). It is immediately clear that any sequence u can be expressed as $u = u - c\psi + c\psi$. Then, we may ask whether there exists a unique constant c such that $u - c\psi \in Q(\mathbf{t_F})$, as $c\psi$ will clearly lie in $\mathcal{N}$.

We begin by assuming that there exist two constants c_1 and c_2 such that $u - c_1\psi,\ u - c_2\psi \in Q(\mathbf{t_F})$. As $Q(\mathbf{t_F})$ is a vector space, we then have that $(u - c_1\psi) - (u - c_2\psi) \in Q(\mathbf{t_F})$. Hence, $(c_2 - c_1)\psi \in Q(\mathbf{t_F})$. If $\psi \notin Q(\mathbf{t_F})$, then it must be true that $c_1 = c_2$. So, for a contradiction, assume that $\psi \in Q(\mathbf{t_F})$ and consider the expression $\mathbf{t_F}[\psi, \psi]$. As ψ belongs to the kernel of the maximal operator, we must have that ψ also belongs to the maximal domain $\mathcal{D}(T_{\max})$; then, we may invoke Theorem 1.2.12, that is

$$\mathcal{D}(T_F) = Q(\mathbf{t_F}) \cap \mathcal{D}(T_{\max}),$$

to conclude that ψ must belong to $\mathcal{D}(T_F)$. Now, as $T_F\psi = T_{\max}\psi = 0$, we can conclude that $\ker T_F \neq \{0\}$; hence, $0 \in \sigma_p(T_F)$. Since we have assumed that $\psi \in Q(\mathbf{t_F})$, we arrive at the following two facts since $\mathbf{t_F}$ is a positive form:

$$\langle T_F\psi, \psi\rangle = \mathbf{t_F}[\psi, \psi] \qquad \text{and} \qquad \mathbf{t_F}[\psi, \psi] \geq c\|\psi\|^2,$$

for some constant $c > 0$. However, as $T_F\psi = 0$, we immediately observe that $0 \geq c\|\psi\|^2$, arriving at a contradiction. Hence $\psi \notin Q(\mathbf{t_F})$, showing that we must, indeed, have that $c_1 = c_2$. Therefore, we have shown that if $u \in Q(\mathbf{t_B})$, then there exists a unique c_1 such that $u - c_1\psi \in Q(\mathbf{t_F})$ and $c_1\psi \in \mathcal{N}$.

With this in mind, we may use the notation given in equation (2.12) to conclude that if $u = u^F + u^N$, then $u^F = u - c_u\psi$ and $u^N = c_u\psi$, where we have relabelled c_1 as c_u to illustrate its dependence on u. Similarly, we have that if $v = v^F + v^N \in Q(\mathbf{t_B})$, where $v^N \in Q(\mathbf{t_F})$ and $v^N \in \mathcal{N}$, then $v^F = v - c_v\psi$ and $v^N = c_v\psi$.

Remark. Since $u - c_u\psi \in Q(\mathbf{t_F})$, we must have that $\lim_{n\to\infty} \frac{u_n - c_u\psi_n}{v_n} = 0$. In practice, we can use this equality to find the constant c_u.

With the decomposition of u, $v \in Q(\mathbf{t_B})$ established, we may now begin to investigate expressions of the form given in equation (2.10). In fact, our goal is to equate two expressions that $\mathbf{t_B}$ may exhibit in order to produce an identity that will form the basis for our characterisation of the domain of the operator T_B. In particular, we have that $\mathbf{t_B}[u,v]$ can be written as

$$\begin{aligned}
&\mathbf{t_F}\left[u^F, v^F\right] + \mathbf{b}\left[u^N, v^N\right] \\
=&\sum_{n=0}^{\infty}\left[p_n g_n g_{n+1}\Delta\left(\frac{u_n - c_u\psi_n}{g_n}\right)\Delta\left(\overline{\frac{v_n - c_v\psi_n}{g_n}}\right)\right. \\
&\quad\left.+\left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right](u_n - c_u\psi_n)(\overline{v_n - c_v\psi_n})\right] + \mathbf{b}\left[u^N, v^N\right] \\
=&\sum_{n=0}^{\infty}\left[p_n g_n g_{n+1}\Delta\left(\frac{u_n}{g_n}\right)\Delta\left(\frac{\overline{v}_n}{g_n}\right) + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right]u_n\overline{v}_n\right] \\
&-\overline{c}_v\sum_{n=0}^{\infty}\left[p_n g_n g_{n+1}\Delta\left(\frac{u_n}{g_n}\right)\Delta\left(\frac{\overline{\psi}_n}{g_n}\right) + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right]u_n\overline{\psi}_n\right] \\
&\qquad - c_u\sum_{n=0}^{\infty}\left[p_n g_n g_{n+1}\Delta\left(\frac{\psi_n}{g_n}\right)\Delta\left(\overline{\frac{v_n - c_v\psi_n}{g_n}}\right)\right. \\
&\qquad\quad\left.+\left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right]\psi_n(\overline{v_n - c_v\psi_n})\right] + \mathbf{b}\left[u^N, v^N\right].
\end{aligned}$$

For convenience, we may then label each summation by I_n, where $n = 1$, 2 or 3, resulting in

$$\begin{aligned}
\mathbf{t_B}[u,v] &= I_1 + I_2 + I_3 + \mathbf{b}\left[u^N, v^N\right] \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We hope to make use of the limiting behaviour of elements in $Q(\mathbf{t_F})$ to simplify the result given above. In particular, after splitting the result into the four suggested expressions above, we aim to show that

$$\begin{aligned}
I_3 \coloneqq -c_u\sum_{n=0}^{\infty}&\left[p_n g_n g_{n+1}\Delta\left(\frac{\psi_n}{g_n}\right)\Delta\left(\overline{\frac{v_n - c_v\psi_n}{g_n}}\right)\right. \\
&\qquad\left.+\left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right]\psi_n(\overline{v_n - c_v\psi_n})\right] = 0.
\end{aligned}$$

Indeed, if we begin by applying the summation by parts formula to

$$\sum_{n=0}^{k} p_n g_n g_{n+1}\Delta\left(\frac{\psi_n}{g_n}\right)\Delta\left(\overline{\frac{v_n - c_v\psi_n}{g_n}}\right),$$

then we see that this equals

$$\begin{aligned}
&\left(\overline{\frac{v_{k+1} - c_v\psi_{k+1}}{g_{k+1}}}\right)p_{k+1}g_{k+1}g_{k+2}\Delta\left(\frac{\psi_{k+1}}{g_{k+1}}\right) \\
&\qquad\qquad - \sum_{n=0}^{k+1}\left(\overline{\frac{v_n - c_v\psi_n}{g_n}}\right)\Delta\left[p_{n-1}g_{n-1}g_n\Delta\left(\frac{\psi_{n-1}}{g_{n-1}}\right)\right],
\end{aligned}$$

after shifting the index once. We can insert this back into the formula we have for I_3 and take the limit as $k \to \infty$ to obtain

$$\begin{aligned} I_3 = -c_u \Bigg\{ \sum_{n=0}^{\infty} \Bigg[-\left(\overline{\frac{v_n - c_v\psi_n}{g_n}}\right) \Delta \left[p_{n-1}g_{n-1}g_n \Delta \left(\frac{\psi_{n-1}}{g_{n-1}}\right)\right] \\ + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right] \psi_n (\overline{v_n - c_v\psi_n}) \Bigg] \\ + \lim_{k\to\infty} \left(\overline{\frac{v_k - c_v\psi_k}{g_k}}\right) p_k g_k g_{k+1} \Delta \left(\frac{\psi_k}{g_k}\right) \Bigg\}. \end{aligned}$$

In order to simplify this sum, we first recall Jacobi's factorisation identity:

$$-\Delta(p_{n-1}\Delta\psi_{n-1}) = -\frac{1}{g_n}\Delta\left[p_{n-1}g_{n-1}g_n\Delta\left(\frac{\psi_{n-1}}{g_{n-1}}\right)\right] - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\psi_n.$$

Hence,

$$\begin{aligned} -\Delta(p_{n-1}\Delta\psi_{n-1}) + q_n\psi_n = -\frac{1}{g_n}\Delta\left[p_{n-1}g_{n-1}g_n\Delta\left(\frac{\psi_{n-1}}{g_{n-1}}\right)\right] \\ + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right]\psi_n. \end{aligned}$$

Now, since ψ belongs to the kernel of $T_{\max}$, we have that $-\Delta(p_{n-1}\Delta\psi_{n-1}) + q_n\psi_n = 0$. Then

$$0 = -\frac{1}{g_n}\Delta\left[p_{n-1}g_{n-1}g_n\Delta\left(\frac{\psi_{n-1}}{g_{n-1}}\right)\right] + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right]\psi_n,$$

or, in other words,

$$-\frac{1}{g_n}\Delta\left[p_{n-1}g_{n-1}g_n\Delta\left(\frac{\psi_{n-1}}{g_{n-1}}\right)\right] = -\left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right]\psi_n. \tag{2.14}$$

If we multiply both sides of equation (2.14) by $(\overline{v_n - c_v\psi_n})$, then we obtain an identity that aids us in simplifying I_3. Indeed, it is readily observed that the entire summation collapses and we are left with

$$I_3 = -c_u \lim_{k\to\infty}\left(\overline{\frac{v_k - c_v\psi_k}{g_k}}\right) p_k g_k g_{k+1}\Delta\left(\frac{\psi_k}{g_k}\right).$$

In order to evaluate this limit, first recall that $v_k = v_k^F + v_k^N$, where $v^F \in Q(\mathbf{t_F})$ and $v^N = c_v\psi \in \mathcal{N}$. Then, as $g_k \neq 0$ for all $k \in \mathbb{N}_0$, we may divide both sides of $v_k = v_k^F + v_k^N$ by g_k and take the limit as $k \to \infty$. Hence,

$$\begin{aligned} \lim_{k\to\infty}\frac{v_k}{g_k} &= \lim_{k\to\infty}\frac{v_k^F + c_v\psi_k}{g_k} \\ &= \lim_{k\to\infty}\frac{v_k^F}{g_k} + c_v\lim_{k\to\infty}\frac{\psi_k}{g_k}. \end{aligned}$$

As $v^F \in Q(\mathbf{t_F})$, we have that $\lim_{k\to\infty} \frac{v_k^F}{g_k} = 0$, so it is immediate that

$$\lim_{k\to\infty} \frac{v_k}{g_k} = c_v \lim_{k\to\infty} \frac{\psi_k}{g_k}.$$

Hence

$$\lim_{k\to\infty} \left(\overline{\frac{v_k - c_\varphi \psi_k}{g_k}} \right) = 0,$$

and so $I_3 = 0$ provided that $\lim_{k\to\infty} p_k g_k g_{k+1} \Delta \left(\frac{\psi_k}{g_k}\right)$ is finite.

In order to verify that this is indeed the case, we first recall that

$$\begin{aligned} p_k g_k g_{k+1} \Delta \left(\frac{\psi_k}{g_k} \right) &= p_k \left(g_k \psi_{k+1} - \psi_k g_{k+1} \right) \\ &= W_k(g, \psi), \end{aligned}$$

where $W_k(g, \psi)$ denotes the Wronskian as expressed in equation (2.3). It is sufficient to show that $W_k(g, \psi) = W_{k+1}(g, \psi)$ for all $k \geq m$, since this will immediately confirm that $\lim_{k\to\infty} p_k g_k g_{k+1} \Delta \left(\frac{\psi_k}{g_k}\right)$ exists and is finite.

Firstly, as $g_k = \eta_k$ for all $k \geq m$, we have that

$$-\Delta(p_{k-1}\Delta g_{k-1}) + q_k g_k = 0, \qquad k > m,$$

since η_k is the non-principal solution to the recurrence relation with $\lambda = 0$. Upon rearranging this equation, observe that

$$\begin{aligned} g_{k+1} &= \frac{p_k g_k + p_{k-1} g_k - p_{k-1} g_{k-1} + q_k g_k}{p_k} \\ &= g_k + \frac{p_{k-1}}{p_k} \Delta g_{k-1} + \frac{q_k}{p_k} g_k, \end{aligned}$$

for all $k > m$. On the other hand, $-\Delta(p_{k-1}\Delta\psi_{k-1}) + q_k\psi_k = 0$ for all $k \in \mathbb{N}_0$, as $\psi \in \ker T_{\max}$, and so we see that

$$\psi_{k+1} = \psi_k + \frac{p_{k-1}}{p_k}\Delta\psi_{k-1} + \frac{q_k}{p_k}\psi_k.$$

We may use these equations to find expressions for g_{k+2} and ψ_{k+2} and insert them into

$$W_{k+1}(g, \psi) = p_{k+1} \left(g_{k+1}\psi_{k+2} - \psi_{k+1} g_{k+2} \right).$$

After noting that the terms involving q_{k+1} cancel out, we see that

$$\begin{aligned} W_{k+1}(g, \psi) &= p_{k+1} \left[g_{k+1} \left(\psi_{k+1} + \frac{p_k}{p_{k+1}} \Delta\psi_k \right) - \psi_{k+1} \left(g_{k+1} + \frac{p_k}{p_{k+1}} \Delta g_k \right) \right] \\ &= p_k \left(g_{k+1}\Delta\psi_k - \psi_{k+1}\Delta g_k \right) \\ &= p_k (g_k\psi_{k+1} - \psi_k g_{k+1}) \\ &= W_k(g, \psi). \end{aligned}$$

Then, as $W_{k+1}(g,\psi) = W_k(g,\psi)$ for all $k \geq m$, it must be true that

$$\lim_{k\to\infty} p_k g_k g_{k+1} \Delta\left(\frac{\psi_k}{g_k}\right)$$

is finite. Therefore $I_3 = 0$, as claimed.

We now use a second approach to calculate $\mathbf{t_B}[u,v]$ for when $u \in \mathcal{D}(T_B)$. Indeed, we know that if $u \in \mathcal{D}(T_B)$, then

$$\begin{aligned}\mathbf{t_B}[u,v] &= \langle T_B u, v\rangle \\ &= \sum_{n=0}^{\infty} \left[-\Delta(p_{n-1}\Delta u_{n-1})\right]\overline{v}_n + q_n u_n \overline{v}_n,\end{aligned}$$

since T_B is a restriction of $T_{\max}$. Then, by using Jacobi's factorisation identity, we see that

$$\begin{aligned}\mathbf{t_B}[u,v] = \sum_{n=0}^{\infty} -\frac{1}{g_n}\Delta\left[p_{n-1}g_{n-1}g_n\Delta\left(\frac{u_{n-1}}{g_{n-1}}\right)\right]\overline{v}_n \\ + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right] u_n\overline{v}_n.\end{aligned}$$

After an application of the summation by parts formula, we obtain

$$\begin{aligned}\mathbf{t_B}[u,v] = \sum_{n=0}^{\infty}\left[p_n g_n g_{n+1}\Delta\left(\frac{u_n}{g_n}\right)\Delta\left(\frac{\overline{v}_n}{g_n}\right) + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right]u_n\overline{v}_n\right] \\ - \lim_{k\to\infty} p_k g_k g_{k+1}\Delta\left(\frac{u_k}{g_k}\right)\frac{\overline{v}_{k+1}}{g_{k+1}}.\end{aligned}$$

We are now in possession of two different expressions for $\mathbf{t_B}[u,v]$. Hence, we may equate the two expressions to see that, for $u \in \mathcal{D}(T_B)$, we have

$$I_1 + I_2 + I_4 = I_1 - \lim_{k\to\infty} p_k g_k g_{k+1}\Delta\left(\frac{u_k}{g_k}\right)\frac{\overline{v}_{k+1}}{g_{k+1}},$$

or, in other words,

$$\begin{aligned}-\overline{c}_v\sum_{n=0}^{\infty}\left[p_n g_n g_{n+1}\Delta\left(\frac{u_n}{g_n}\right)\Delta\left(\frac{\overline{\psi}_n}{g_n}\right) + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right]u_n\overline{\psi}_n\right] \\ + \mathbf{b}\left[u^N, v^N\right] = -\lim_{k\to\infty} p_k g_k g_{k+1}\Delta\left(\frac{u_k}{g_k}\right)\frac{\overline{v}_{k+1}}{g_{k+1}}. \quad (2.15)\end{aligned}$$

With equation (2.15) in mind, we hope to make further simplifications using an argument similar to that used previously. In particular, we may use summation by parts on I_2 to conclude that

$$\begin{aligned}I_2 = -\overline{c}_v\Bigg\{\sum_{n=0}^{\infty}\Bigg[-\Delta\left[p_{n-1}g_{n-1}g_n\Delta\left(\frac{\overline{\psi}_{n-1}}{g_{n-1}}\right)\right]\frac{u_n}{g_n} \\ + \left[q_n - \frac{\Delta(p_{n-1}\Delta g_{n-1})}{g_n}\right]u_n\overline{\psi}_n\Bigg] \\ + \lim_{k\to\infty} p_{k+1}g_{k+1}g_{k+2}\Delta\left(\frac{\overline{\psi}_{k+1}}{g_{k+1}}\right)\frac{u_{k+1}}{g_{k+1}}\Bigg\}.\end{aligned}$$

Since ψ belongs to the kernel of the maximal operator, we are able to use equation (2.14) again to conclude that

$$I_2 = -\overline{c}_v \lim_{k\to\infty} p_{k+1}g_{k+1}g_{k+2}\Delta\left(\frac{\overline{\psi}_{k+1}}{g_{k+1}}\right)\frac{u_{k+1}}{g_{k+1}}.$$

We can then shift the index $k+1$ down to k in I_2 to conclude that

$$\begin{aligned}
-\overline{c}_v \lim_{k\to\infty} p_k g_k g_{k+1}\Delta\left(\frac{\overline{\psi}_k}{g_k}\right)\frac{u_k}{g_k} + \mathbf{b}\left[u^N, v^N\right]& \\
= -\lim_{k\to\infty} p_k g_k g_{k+1}\Delta\left(\frac{u_k}{g_k}\right)\frac{\overline{v}_{k+1}}{g_{k+1}}.& \quad (2.16)
\end{aligned}$$

Our final step to producing a characterisation of the non-negative, self-adjoint extensions is to try to succinctly write the equality above. Recall that

$$v_n = v_n^F + v_n^N = (v_n - c_v\psi_n) + c_v\psi_n$$

for $v^F \in Q(\mathbf{t_F})$ and $v^N = c_v\psi \in \mathcal{N}$. Then, as before,

$$\begin{aligned}
\lim_{k\to\infty}\frac{v_{k+1}}{g_{k+1}} &= \lim_{k\to\infty}\left[\frac{v_{k+1} - c_v\psi_{k+1}}{g_{k+1}} + \frac{c_v\psi_{k+1}}{g_{k+1}}\right] \\
&= \lim_{k\to\infty}\frac{c_v\psi_{k+1}}{g_{k+1}}.
\end{aligned}$$

By inserting this back into equation (2.16), we see that

$$\begin{aligned}
-\overline{c}_v \lim_{k\to\infty} p_k g_k g_{k+1}\Delta\left(\frac{\overline{\psi}_k}{g_k}\right)\frac{u_k}{g_k} + \mathbf{b}\left[u^N, v^F\right]& \\
= -\overline{c}_v \lim_{k\to\infty} p_k g_k g_{k+1}\Delta\left(\frac{u_k}{g_k}\right)\frac{\overline{\psi}_{k+1}}{g_{k+1}}.&
\end{aligned}$$

Finally, we note that

$$\mathbf{b}\left[u^N, v^N\right] = \beta c_u \overline{c}_v \|\psi\|^2,$$

for some $\beta \geq 0$, which leads us to the identity

$$\beta c_u\|\psi\|^2 = \lim_{k\to\infty} p_k g_k g_{k+1}\left[\frac{u_k}{g_k}\Delta\left(\frac{\overline{\psi}_k}{g_k}\right) - \frac{\overline{\psi}_{k+1}}{g_{k+1}}\Delta\left(\frac{u_k}{g_k}\right)\right].$$

With the analysis complete, we now have a characterisation of the non-negative, self-adjoint extensions T_B of $T_{\min}$. In particular, we present our result in the following theorem.

Theorem 2.5.1. *Let $T_{\min}$ be the closed, symmetric operator with positive lower bound associated to the difference expression*

$$(Mx)_n = -\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n,$$

where $\{p_n\}$ and $\{q_n\}$ are two real sequences with $p_n > 0$ for all $n \in \mathbb{N}_0$ and $p_{-1} \equiv 0$. Excluding the Friedrichs extension, the non-negative, self-adjoint extensions T_B of $T_{\min}$ have a domain that is defined by

$$\mathcal{D}(T_B) := \left\{ u \in \mathcal{D}(T_{\max}) \;\middle|\; \begin{aligned} \lim_{k\to\infty} p_k g_k g_{k+1} \Bigg[\frac{u_k}{g_k} \Delta\left(\frac{\overline{\psi}_k}{g_k}\right) \\ -\frac{\overline{\psi}_{k+1}}{g_{k+1}} \Delta\left(\frac{u_k}{g_k}\right)\Bigg] = \beta c_u \|\psi\|^2 \end{aligned} \right\},$$

where ψ is a basis of $\mathcal{N}$ with $\psi_0 = 1$ and $B\psi = \beta\psi$ for $\beta \geq 0$. Here,

$$g_n = \begin{cases} g_n \in \mathbb{R}_{>0}, & n < m, \\ v_n, & n \geq m, \end{cases}$$

where v_n is a non-principal solution to the difference expression M which is positive for $n \geq m$. The operator T_B acts as follows on an element $u \in \mathcal{D}(T_B)$:

$$T_B u = Mu.$$

Remark. As the kernel of $T_{\max}$ is 1-dimensional, it is clear from Lemma 1.2.11 that an element $u \in \mathcal{D}(T_{\max})$ decomposed into

$$u = u - c_u\psi + c_u\psi, \qquad \psi \in \ker T_{\max},$$

satisfies $u - c_u\psi \in \mathcal{D}(T_F) \subseteq Q(\mathbf{t_F})$. Hence, every $u \in \mathcal{D}(T_{\max})$ can be decomposed into $u = u^F + u^N$, and so elements in $\mathcal{D}(T_B)$ do not require any further restrictions placed upon them.

It is worth restating that our result excludes the Friedrichs extension. This is not too surprising: various descriptions of the Friedrichs domain already exist and are well established — this was even the basis of our work! However, we note that it is, in fact, possible to obtain the Friedrichs domain explicitly by means of a corollary. Formally, taking $\beta = \infty$ in the limit condition only makes sense if we, additionally, enforce that $c_u = 0$. Indeed, this is consistent with our construction: c_u *should* equal zero, as there is no contribution from $\ker T_{\max}$ for an element decomposed as in equation (2.11). Then, we arrive at the following result.

Corollary 2.5.2. *The domain of the Friedrichs extension is given by*

$$\mathcal{D}(T_F) := \left\{ u \in \mathcal{D}(T_{\max}) \;\middle|\; \lim_{k\to\infty} \frac{u_k}{g_k} = 0 \right\}.$$

Remark. This description of the Friedrichs extension is precisely one of the constructions expressed in [14], and therefore coincides with the Friedrichs domain as given in equation (2.5).

2.6 An Example: the Stieltjes-Wigert Polynomials

With the main result of this chapter now established, we continue by presenting a comprehensive example of the theory. We begin by declaring that the minimal operator $T_{\min}$ will be associated to the second-order difference expression of the form

$$(Mx)_n = -\Delta(p_{n-1}\Delta x_{n-1}) + \tilde{q}_n x_n,$$

where $\{p_n\}$ and $\{\tilde{q}_n\}$ are two real sequences with $p_n > 0$ for all $n \in \mathbb{N}_0$ and $p_{-1} \equiv 0$. We will present the sequences $\{p_n\}$ and $\{\tilde{q}_n\}$ shortly, but for now it is enough to say that the two linearly independent solutions to the equation $Mx = \lambda x$, for $\lambda \in \mathbb{R}$, will be given by specific variations of the *Stieltjes-Wigert polynomials.* For more background on these polynomials, we refer to [22], [40] and [52], with special mention to the papers by Christiansen [23] and Wang and Wong [56].

The Stieltjes-Wigert polynomials of the first kind are the solutions $S_n(x;q)$ to the recurrence relation

$$\begin{aligned} -q^{2n+1}xS_n(x;q) &= (1-q^{n+1})S_{n+1}(x;q) \\ &\quad - [1+q-q^{n+1}]S_n(x;q) + qS_{n-1}(x;q), \qquad (2.17) \end{aligned}$$

where

$$S_n(x;q) = \frac{1}{(q;q)_n} {}_1\phi_1 \left(\begin{matrix} q^{-n} \\ 0 \end{matrix} ; q, -q^{n+1}x \right),$$

for $0 < q < 1$. We note that ${}_r\phi_s$ is the *basic hypergeometric series* defined by

$$\begin{aligned} {}_r\phi_s &\left(\begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s \end{matrix} ; q, z \right) \\ &\qquad := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_k}{(b_1, b_2, \cdots, b_s; q)_k} (-1)^{(1+s-r)k} q^{(1+s-r)\binom{k}{2}} \frac{z^k}{(q;q)_k}, \end{aligned}$$

where

$$(a_1, a_2, \cdots, a_r; q)_k = (a_1;q)_k (a_2;q)_k \cdots (a_r;q)_k,$$

and $(a;q)_k$ is the *q-Pochhammer symbol* defined as

$$(a;q)_k := \prod_{m=0}^{k-1} (1 - aq^m).$$

Remark. The expression $(a;q)_0$ is to be interpreted as 0 for all a and $0 < q < 1$.

Alternatively, $S_n(x;q)$ can be expressed as

$$S_n(x;q) = \frac{1}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2} x^k, \tag{2.18}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}, & k \leq n, \\ 0, & k > n, \end{cases}$$

denotes the *q-binomial coefficient* [23].

We will be concerned with the Stieltjes-Wigert polynomials normalised by $P_n(x) = \sqrt{q^n(q;q)_n} S_n(x;q)$, that is,

$$P_n(x) = \sqrt{\frac{q^n}{(q;q)_n}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2} x^k. \tag{2.19}$$

It can be shown [23] that the Stieltjes-Wigert polynomials given by equation (2.19) are orthonormal with respect to the weight function

$$w(x) = \frac{q^{1/8}}{\sqrt{2\pi \log q^{-1}}} \frac{1}{\sqrt{x}} \exp\left(\frac{(\log x)^2}{2\log q}\right)$$

on the interval $(0,\infty)$. In this case, the moments s_n are given by

$$\begin{aligned} \int_0^\infty x^n w(x)\; dx &= s_n \\ &= q^{-\binom{n+1}{2}}. \end{aligned}$$

With these moments in mind, we are able to construct the Stieltjes-Wigert polynomials of the second kind, $Q_n(x)$, by following the standard method presented in [23]. First, let $P_n(x) = \sum_{k=0}^{\infty} c_k x^k$ for coefficients

$$c_k = \sqrt{\frac{q^n}{(q;q)_n}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2}.$$

If $Q_n(x)$ is such that $Q_0(x) = 0$ and

$$Q_n(x) = \sum_{m=0}^{n-1} \left(\sum_{k=m+1}^{n} c_k s_{k-m-1} \right) x^m, \qquad n \geq 1,$$

then we obtain

$$\begin{aligned} Q_n(x) &= \sqrt{\frac{q^n}{(q;q)_n}} \sum_{m=0}^{n-1} \left(\sum_{k=m+1}^{n} s_{k-m-1} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2} \right) x^m \\ &= \sqrt{\frac{q^n}{(q;q)_n}} \sum_{m=0}^{n-1} \left(\sum_{k=m+1}^{n} q^{-\binom{k-m}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2} \right) x^m. \end{aligned}$$

We may then simplify the exponent of q by noting that

$$\begin{aligned}
-\binom{k-m}{2}+k^2 &= -\frac{(k-m)(k-m-1)}{2}+k^2 \\
&= -\frac{m(m+1)}{2}+\frac{k(k+1)}{2}+km \\
&= -\binom{m+1}{2}+\binom{k+1}{2}+km \\
&= -\binom{m+1}{2}+\binom{k}{2}+(m+1)k.
\end{aligned}$$

Hence, the Stieltjes-Wigert polynomials of the second kind that we will be considering will be those of the form

$$Q_n(x)=\sqrt{\frac{q^n}{(q;q)_n}}\sum_{m=0}^{n-1} q^{-\binom{m+1}{2}}\left(\sum_{k=m+1}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+(m+1)k}\right)x^m, \quad (2.20)$$

when $n \geq 1$, and $Q_0(x)=0$. Equally, we may also find it useful to define $Q_n(x)=\sqrt{q^n(q;q)_n}\tilde{S}_n(x;q)$, where

$$\tilde{S}_n(x;q)=\frac{1}{(q;q)_n}\sum_{m=0}^{n-1} q^{-\binom{m+1}{2}}\left(\sum_{k=m+1}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+(m+1)k}\right)x^m.$$

With the Stieltjes-Wigert polynomials of the first and second kind now defined, we can continue by finding the symmetric three-term recurrence relation that $P_n(x)$ and $Q_n(x)$ satisfy. In particular, we are aiming to find two real sequences $\{a_n\}$ and $\{b_n\}$ such that

$$xP_n = a_{n-1}P_{n-1}+b_nP_n+a_nP_{n+1}, \quad (2.21)$$

for all $n \in \mathbb{N}_0$. We begin by rewriting the recurrence relation presented in equation (2.17) as

$$S_{n+1}=-\frac{q^{2n+1}x}{1-q^{n+1}}S_n+\frac{1+q-q^{n+1}}{1-q^{n+1}}S_n-\frac{q}{1-q^{n+1}}S_{n-1}. \quad (2.22)$$

Upon setting $\gamma_n=\sqrt{q^n(q;q)_n}$, we see that $P_n=\gamma_n S_n(x;q)$. By substituting this into equation (2.21), we see that

$$S_{n+1}=\frac{\gamma_n}{\gamma_{n+1}}\frac{1}{a_n}xS_n-\frac{\gamma_n}{\gamma_{n+1}}\frac{b_n}{a_n}S_n-\frac{\gamma_{n-1}}{\gamma_{n+1}}\frac{a_{n-1}}{a_n}S_{n-1}. \quad (2.23)$$

As $S_n(x;q)$ is a polynomial of degree n, we may deduce a_n by comparing the leading coefficients of both sides of this equation. In particular, we see that

$$\frac{(-1)^{n+1}q^{(n+1)^2}}{(q;q)_{n+1}}=\frac{(-1)^n q^{n^f 2}}{(q;q)_n}\frac{\sqrt{q^n(q;q)_n}}{\sqrt{q^{n+1}(q;q)_{n+1}}}\frac{1}{a_n},$$

after extracting the relevant coefficients from equation (2.18). Hence

$$\begin{aligned}
a_n &= \frac{(-1)^n}{(-1)^{n+1}} \frac{q^{n^2}}{q^{(n+1)^2}} \frac{(q;q)_{n+1}}{(q;q)_n} \frac{\sqrt{(q;q)_n}}{\sqrt{(q;q)_{n+1}}} \frac{\sqrt{q^n}}{\sqrt{q^{n+1}}} \\
&= -q^{-2n-\frac{3}{2}} \sqrt{1-q^{n+1}}.
\end{aligned}$$

With the sequence $\{a_n\}$ now in hand, we can easily determine b_n by comparing equation (2.22) to equation (2.23). As such, we see that

$$\frac{1+q-q^{n+1}}{1-q^{n+1}} = -\frac{\gamma_n}{\gamma_{n+1}} \frac{b_n}{a_n},$$

or

$$b_n = -\frac{1+q-q^{n+1}}{1-q^{n+1}} \frac{\gamma_{n+1}}{\gamma_n} a_n.$$

Hence

$$\begin{aligned}
b_n &= \frac{1+q-q^{n+1}}{1-q^{n+1}} \frac{\sqrt{(q;q)_{n+1}}}{\sqrt{(q;q)_n}} \frac{\sqrt{q^{n+1}}}{\sqrt{q^n}} q^{-2n-\frac{3}{2}} \sqrt{1-q^{n+1}} \\
&= [1+q-q^{n+1}] q^{-2n-1} \\
&= q^{-2n-1} + q^{-2n} - q^{-n}.
\end{aligned}$$

With the sequences $\{a_n\}$ and $\{b_n\}$ now determined, we assert that the polynomials $P_n(x)$ and $Q_n(x)$ given by equations (2.19) and (2.20) respectively are the two linearly independent solutions to the recurrence relation given by

$$\begin{aligned}
xU_n = -q^{-2n+\frac{1}{2}} \sqrt{1-q^n} U_{n-1} \\
+ [q^{-2n-1} + q^{-2n} - q^{-n}] U_n - q^{-2n-\frac{3}{2}} \sqrt{1-q^{n+1}} U_{n+1}.
\end{aligned}$$

We can now transform the right-hand side of equation (2.21) into something of the form $-\Delta(p_{n-1}\Delta x_{n-1}) + \tilde{q}_n x_n$, for real sequences $\{p_n\}$ and $\{\tilde{q}_n\}$. In particular, we note that

$$-\Delta(p_{n-1}\Delta x_{n-1}) + \tilde{q}_n x_n = -p_{n-1} x_{n-1} + [p_n + p_{n-1} + \tilde{q}_n] x_n - p_n x_{n+1}.$$

Then, by direct comparison, we see that

$$\begin{cases} a_n = -p_n, \\ b_n = p_n + p_{n-1} + \tilde{q}_n, \end{cases} \implies \begin{cases} p_n = -a_n, \\ \tilde{q}_n = b_n + a_n + a_{n-1}, \end{cases}$$

or, in other words,

$$\begin{cases} p_n = q^{-2n-\frac{3}{2}} \sqrt{1-q^{n+1}}, \\ \tilde{q}_n = q^{-2n-1} + q^{-2n} - q^{-n} - q^{-2n-\frac{3}{2}} \sqrt{1-q^{n+1}} - q^{-2n+\frac{1}{2}} \sqrt{1-q^n}. \end{cases}$$

Here, we draw special attention to the fact that $p_n > 0$ for all $n \in \mathbb{N}_0$ — this is crucial in our analysis, as it was consistently required of our sequence $\{p_n\}$ in the sections prior. By solving the equation

$$-\Delta(p_{n-1}\Delta x_{n-1}) + \tilde{q}_n x_n = 0$$

with the above sequences $\{p_n\}$ and $\{\tilde{q}_n\}$, we can immediately deduce that the solutions are given by $P_n(0)$ and $Q_n(0)$. By studying the behaviour of these two solutions, we will be able to confirm that this expression is of the limit-circle type, i.e., both solutions lie in ℓ^2.

First, we note that

$$P_n(0) = \sqrt{\frac{q^n}{(q;q)_n}} \quad \text{and} \quad Q_n(0) = \sqrt{\frac{q^n}{(q;q)_n}} \left(\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+k} \right),$$

for $n \geq 0$, where $Q_0(0) = 0$. We can then utilise [23, Remark 3.1] to simplify the polynomial $Q_n(0)$. In particular, we see that as

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^k = (x;q)_n,$$

we have

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+k} = (q;q)_n - 1.$$

Therefore, for $n \geq 1$, we can write $Q_n(0)$ as follows:

$$\begin{aligned} Q_n(0) &= \sqrt{\frac{q^n}{(q;q)_n}} \left[(q;q)_n - 1\right] \\ &= P_n(0) \left[(q;q)_n - 1\right]. \end{aligned}$$

We will be able to conclude that $\{P_n(0)\}_n$ lies in ℓ^2 by the ratio test. In particular, we note that

$$\begin{aligned} \left| \frac{(P_{n+1}(0))^2}{(P_n(0))^2} \right| &= \left| \frac{q^{n+1}}{(q;q)_{n+1}} \frac{(q;q)_n}{q^n} \right| \\ &= \frac{q}{1-q^{n+1}} \\ &\to q \quad \text{as} \quad n \to \infty, \end{aligned}$$

since $0 < q < 1$. As this expression tends to q, and q is strictly less than 1, we may conclude that $\sum_{n=0}^{\infty} \frac{q_n}{(q;q)_n} < \infty$ by the ratio test. Hence, $\{P_n(0)\} \in \ell^2$. In order to determine whether $Q_n(0)$ lies in ℓ^2 or not, we first note that

$$(q;q)_n = \prod_{k=1}^{n} (1-q^k).$$

Since $0 < q < 1$, we have that $0 < 1 - q^k < 1$ for all $k \in \mathbb{N}$. Then, it is clear that $(q;q)_n$ is a decreasing function. Thus, $(q;q)_n - 1$ is also a decreasing function with

$$(q;q)_n - 1 \in (-1, 0)$$

for all $n \geq 1$. Upon noting that $P_n(0) > 0$ for all $n \in \mathbb{N}_0$, we immediately see that $Q_n(0) < 0$ for all $n \geq 1$. Furthermore, $P_n(0) > |Q_n(0)|$ for all $n \in \mathbb{N}_0$, hence $\{Q_n(0)\} \in \ell^2$ by the comparison test. As both solutions $\{P_n(0)\}$ and $\{Q_n(0)\}$ lie in ℓ^2, we can conclude that the difference expression is of limit-circle type.

Non-principal solutions play an instrumental role in the characterisation of the extensions, so we must now show that the minimal operator $T_{\min}$ associated to the expression whose solutions are the Stieltjes-Wigert polynomials is bounded below. In fact, we must show more in order to further our analysis: we need this operator to have a *positive* lower bound, else we are unable to apply the theory to this example. Indeed, we could then invoke [14, Thm. 2.1] to conclude that the equation $Mx = 0$ is non-oscillatory — certainly then would there exist principal and non-principal solutions.

Let $u \in \mathcal{D}(T_{\min})$ and consider the expression $\langle T_{\min} u, u\rangle$. Then,

$$\begin{aligned}\langle T_{\min} u, u\rangle &= \sum_{n=0}^{\infty} a_{n-1} u_{n-1} \overline{u}_n + b_n |u_n|^2 + a_n u_{n+1} \overline{u}_n \\ &= \sum_{n=0}^{\infty} \Big[-q^{-2n+\frac{1}{2}} \sqrt{1-q^n}\, u_{n-1} \overline{u}_n + (q^{-2n-1} + q^{-2n} - q^{-n}) |u_n|^2 \\ &\qquad\qquad -q^{-2n-\frac{3}{2}} \sqrt{1-q^{n+1}}\, u_{n+1} \overline{u}_n \Big] =: \sum_{n=0}^{\infty} s_n.\end{aligned} \tag{2.24}$$

Now,

$$\begin{aligned}\left| q^{-2n+\frac{1}{2}} \sqrt{1-q^n}\, u_{n-1} \overline{u}_n \right| &= \left| q^{-n+1} u_{n-1} q^{-n-\frac{1}{2}} \sqrt{1-q^n}\, \overline{u}_n \right| \\ &\leq \frac{1}{2} \left[q^{-2n+2} |u_{n-1}|^2 + q^{-2n-1}(1-q^n)|u_n|^2 \right],\end{aligned}$$

by means of the binomial formula. Likewise,

$$\begin{aligned}|q^{-2n-\frac{3}{2}} \sqrt{1-q^{n+1}}\, u_{n+1} \overline{u}_n| &= \left| q^{-n-1} u_{n+1} q^{-n-\frac{1}{2}} \sqrt{1-q^{n+1}}\, \overline{u}_n \right| \\ &\leq \frac{1}{2} \left[q^{-2n-2} |u_{n+1}|^2 + q^{-2n-1}(1-q^{n+1})|u_n|^2 \right].\end{aligned}$$

These two inequalities are essential as we can then estimate s_n as follows:

$$
\begin{aligned}
s_n \geq & -\frac{1}{2}\left[q^{-2n+2}|u_{n-1}|^2 + q^{-2n-1}(1-q^n)|u_n|^2\right] \\
& \qquad\qquad + \left[q^{-2n-1} + q^{-2n} - q^{-n}\right]|u_n|^2 \\
& \qquad\qquad\qquad - \frac{1}{2}\left[q^{-2n-2}|u_{n+1}|^2 + q^{-2n-1}(1-q^{n+1})|u_n|^2\right] \\
= & -\frac{q^{-2n+2}}{2}|u_{n-1}|^2 + \left[-\frac{q^{-2n-1}(1-q^n)}{2} + q^{-2n-1} + q^{-2n}\right. \\
& \qquad \left. - q^{-n} - \frac{q^{-2n-1}(1-q^{n+1})}{2}\right]|u_n|^2 - \frac{q^{-2n-2}}{2}|u_{n+1}|^2 \\
= & -\frac{q^{-2n+2}}{2}|u_{n-1}|^2 + \left[q^{-2n} + \frac{q^{-n-1}}{2}(1-q)\right]|u_n|^2 - \frac{q^{-2n-2}}{2}|u_{n+1}|^2.
\end{aligned}
$$

Since $u \in \mathcal{D}(T_{\min})$, we now specify that $u_n = 0$ for all $n > N$ for some $N \in \mathbb{N}$. From equation (2.24), it is clear that $s_n = 0$ for all $n > N$. Hence, for this particular u, we have

$$
\begin{aligned}
\langle T_{\min} u, u\rangle = & \sum_{n=0}^{N} s_n \\
\geq & \sum_{n=0}^{N}\left[-\frac{q^{-2n+2}}{2}|u_{n-1}|^2 + \left[q^{-2n} + \frac{q^{-n-1}}{2}(1-q)\right]|u_n|^2\right. \\
& \qquad\qquad\qquad\qquad \left. - \frac{q^{-2n-2}}{2}|u_{n+1}|^2\right].
\end{aligned}
$$

Since this sum has a finite number of terms, we can reorder and collect the terms involving $|u_n|^2$ together. Indeed, we may then deduce that

$$
\begin{aligned}
\langle T_{\min} u, u\rangle \geq & \sum_{n=0}^{N} \frac{q^{-n-1}}{2}(1-q)|u_n|^2 \\
\geq & \sum_{n=0}^{N} \frac{1-q}{2q}|u_n|^2 \\
\geq & \frac{1-q}{2q}\|u\|^2,
\end{aligned}
$$

since $0 < q < 1$; note that $0 < \frac{1-q}{2q} < \infty$ for such q. Hence, $T_{\min}$ possesses a strictly positive lower bound $\gamma = \frac{1-q}{2q}$, and so we have now confirmed that it is valid to apply our result to this example.

Recall that the sequence u is the principal solution if $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$ for any solution v that is not a multiple of u. Then, consider the following limit:

$$
\lim_{n\to\infty} \frac{Q_n(0)}{P_n(0)} = \lim_{n\to\infty} (q;q)_n - 1.
$$

If this limit is *not* 0, then $Q_n(0)$ is not the principal solution. However, we have already shown that $(q;q)_n - 1 \in (-1, 0)$ and is a decreasing function, so

$$\lim_{n\to\infty} (q;q)_n - 1 \in [-1, 0).$$

As this limit is decidedly not 0, it must be true that $Q_n(0)$ is not the principal solution. Likewise, if we instead consider $\lim_{n\to\infty} \frac{P_n(0)}{Q_n(0)}$ then we see that

$$\lim_{n\to\infty} \frac{P_n(0)}{Q_n(0)} = \lim_{n\to\infty} \frac{1}{(q;q)_n - 1} \in (-\infty,\ -1]$$

showing that $P_n(0)$ is also not the principal solution. Now that we have candidates for a non-principal solution, we are almost able to construct the non-negative, self-adjoint extensions of the operator T_{min} whose difference expression has solutions that are the Stieltjes-Wigert polynomials.

The final step we must take before we are able to do so is to determine the kernel of the maximal operator T_{max}, that is, find the sequence $\{\psi_n\}$ — with $\psi_0 = 1$ — that forms a basis of $\ker T_{\text{max}}$. Note that the general solution to the equation $-\Delta(p_{n-1}\Delta u_{n-1}) + \tilde{q}_n u_n = 0$ is given by $u_n = c_1 P_n(0) + c_2 Q_n(0)$. Then, as the kernel element must adhere to the initial condition given by $b_0 u_0 + a_0 u_1 = 0$, we see that

$$\begin{aligned} b_0 u_0 + a_0 u_1 &= q^{-1}\left[c_1 + 0\right] - q^{-\frac{3}{2}}\sqrt{1-q}\left[c_1\sqrt{\frac{q}{1-q}} - c_2\sqrt{\frac{q}{1-q}}q\right] \\ &= c_2, \end{aligned}$$

upon recalling that $Q_0(0) = 0$. Therefore, $c_2 = 0$ and $c_1 \in \mathbb{C}$ is arbitrary. However, since we require the kernel element ψ to have its first component $\psi_0 = 1$, we may set $c_1 = 1$ as $P_0(0) = 1$.

Recall that the condition within the domain of the extension is given by

$$\lim_{k\to\infty} p_k g_k g_{k+1} \left[\frac{u_k}{g_k}\Delta\left(\frac{\overline{\psi}_k}{g_k}\right) - \frac{\overline{\psi}_{k+1}}{g_{k+1}}\Delta\left(\frac{u_k}{g_k}\right)\right] = \beta c_u \|\psi\|^2,$$

for $u \in \mathcal{D}(T_{\text{max}})$ and $\beta \geq 0$. Then, with all of the pieces necessary to construct the extensions now in hand, we specify the following: both the sequence $\{g_n\}$ and the kernel element ψ may be chosen to be $P(0)$, i.e.,

$$g_n = P_n(0) \qquad \text{and} \qquad \psi_n = P_n(0),$$

for all $n \in \mathbb{N}_0$. Hence, the condition can be written as

$$\lim_{k\to\infty} p_k P_k(0) P_{k+1}(0) \left[-\Delta\left(\frac{u_k}{P_k(0)}\right)\right] = \beta c_u \|P(0)\|^2, \tag{2.25}$$

for $\beta \geq 0$, since $\Delta(c) = 0$ for any constant c. We can simplify this condition by first recognising that

$$-\Delta\left(\frac{u_k}{P_k(0)}\right) = \frac{u_k P_{k+1}(0) - P_k(0) u_{k+1}}{P_k(0) P_{k+1}(0)}.$$

We also note that $\|P(0)\|^2$ may be written in terms of a basic hypergeometric series. In particular, we have

$$\begin{aligned}\|P(0)\|^2 &= \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} \\ &= {}_1\phi_0\left(\begin{matrix}0\\ -\end{matrix};q,q\right).\end{aligned}$$

Remark. We may then use the identity

$${}_1\phi_0\left(\begin{matrix}a\\ -\end{matrix};q,z\right) = \frac{(az;q)_\infty}{(z;q)_\infty},$$

expressed in [18], to conclude that

$$\begin{aligned}{}_1\phi_0\left(\begin{matrix}0\\ -\end{matrix};q,q\right) &= \prod_{n=0}^{\infty} \frac{1 - 0 \cdot q^{n+1}}{1 - q^{n+1}} \\ &= \frac{1}{\prod_{n=1}^{\infty}(1-q^n)}.\end{aligned}$$

The denominator is of precisely the form expressed in the *Pentagonal Number Theorem* [9], originally proved by Euler, which states that

$$\begin{aligned}\prod_{n=1}^{\infty}(1-x^n) &= \sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{k(3k-1)}{2}} \\ &= 1 + \sum_{k=1}^{\infty}(-1)^k\left(x^{\frac{k(3k+1)}{2}} + x^{\frac{k(3k-1)}{2}}\right).\end{aligned}$$

Hence, for $x = q$, the reciprocal of this formula produces an alternative expression for $\|P(0)\|^2$.

Furthermore, upon recalling Lemma 1.2.11, i.e.,

$$\mathcal{D}(T_{\max}) = \mathcal{D}(T_F) \dotplus \ker T_{\max},$$

we can simplify our current result. Indeed, a sequence $u \in \mathcal{D}(T_F)$ can be written as $u = (u - \tilde{c}_u\psi) + \tilde{c}_u\psi$ where $u - \tilde{c}_u\psi \in \mathcal{D}(T_F)$ and $\tilde{c}_u\psi \in \ker T_{\max}$. As $\mathcal{D}(T_F) \subseteq Q(\mathbf{t_F})$, we must have

$$\lim_{n\to\infty} \frac{u_n - \tilde{c}_u P_n(0)}{P_n(0)} = 0,$$

or, in other words, $\lim_{n\to\infty} \frac{u_n}{P_n(0)} = \tilde{c}_u$. Finally, we may then write equation (2.25) as

$$\lim_{k\to\infty} q^{-2k-\frac{3}{2}} \sqrt{1-q^{k+1}} \left[u_k P_{k+1}(0) - P_k(0) u_{k+1}\right] = \beta_1 \phi_0 \left(\begin{matrix} 0 \\ - \end{matrix} ; q, q \right) \lim_{k\to\infty} \frac{u_k}{P_k(0)}$$

or, more compactly,

$$\lim_{k\to\infty} W_k(u, P(0)) = \beta_1 \phi_0 \left(\begin{matrix} 0 \\ - \end{matrix} ; q, q \right) \lim_{k\to\infty} \frac{u_k}{P_k(0)},$$

after observing that the left-hand side is precisely the Wronskian as presented in equation (2.3). Hence, the domains of the non-negative, self-adjoint extensions T_B of $T_{\min}$ — excluding the Friedrichs extension — are of the form

$$\mathcal{D}(T_B) = \left\{ u \in \mathcal{D}(T_{\max}) \;\middle|\; \lim_{k\to\infty} W_k(P(0), u) = \beta_1 \phi_0 \left(\begin{matrix} 0 \\ - \end{matrix} ; q, q \right) \lim_{k\to\infty} \frac{u_k}{P_k(0)} \right\},$$

for some $\beta \geq 0$.

Although we have neglected the Friedrichs extension in this example so far, we are able to determine an explicit characterisation of $\mathcal{D}(T_F)$ by means of Corollary 2.5.2. Indeed, we complete this chapter — and half of the thesis — with the following: the domain of the Friedrichs extension is given by

$$\begin{aligned} \mathcal{D}(T_F) &= \left\{ u \in \mathcal{D}(T_{\max}) \;\middle|\; \lim_{k\to\infty} \frac{u_k}{P_k(0)} = 0 \right\} \\ &= \left\{ u \in \mathcal{D}(T_{\max}) \;\middle|\; \lim_{k\to\infty} \sqrt{\frac{(q;q)_k}{q^k}} u_k = 0 \right\}, \end{aligned}$$

for $0 < q < 1$.

Chapter 3

Preliminaries: Extensions of Linear Relations

3.1 Linear Relations in Hilbert Spaces

Chapter 1 was devoted to the introductory material necessary for understanding Chapter 2; likewise, this chapter presents the fundamental definitions and theory required in Chapter 4. Due to the parallels between the topics covered in the two halves of this thesis, we choose to follow the delivery of Chapter 1, making reference to any similarities and differences that naturally arise. The definitions and theory presented can be found in [24], [33] and [34], amongst others.

Firstly, let H_1 and H_2 be Hilbert spaces. Now, if we equip the space $H_1 \times H_2$ with the inner product

$$\left\langle (x,y),(x',y')\right\rangle_{H_1\times H_2} = \left\langle x,x'\right\rangle_{H_1} + \left\langle y,y'\right\rangle_{H_2}, \qquad (x,y),\ (x',y') \in H_1 \times H_2,$$

then it can be shown that $(H_1 \times H_2, \langle\cdot,\cdot\rangle_{H_1\times H_2})$ forms a Hilbert space. This may seem familiar so far; indeed, to a linear operator $T\colon \mathcal{D}(T) \to H_2$, where $\mathcal{D}(T) \subseteq H_1$, we can associate its graph, as described in Definition 1.1.19. Then, we can think of an operator as the set of pairs (x, Tx) in $H_1 \times H_2$, for $x \in \mathcal{D}(T)$. Here, the second component is entirely dictated by the first: T is a linear operator, so x is mapped to the unique element $Tx \in \mathcal{R}(T) \subseteq H_2$.

If T is a *multi-valued operator* instead, that is, x may be mapped to more than one element in H_2, then does it still make sense to consider the graph of T? We introduce linear relations (often, the 'linear' is dropped) as a means of answering this question.

Definition 3.1.1. Let H_1 and H_2 be Hilbert spaces over the complex field $\mathbb{C}$. Any subspace S of $H_1 \times H_2$ is called a *linear relation* from H_1 to H_2.

Remark. Let $\mathcal{G}(T)$ be the graph associated to a linear operator $T\colon \mathcal{D}(T) \to H_2$, where $\mathcal{D}(T) \subseteq H_1$. It is then clear that a graph is an example of a linear relation. In fact, we can say more: linear relations are a generalisation of the graph.

Now that we have defined the term 'linear relation', we continue by presenting the way in which we are to interpret the interactions between them. Let $S \subseteq H_1 \times H_2$ and $T \subseteq H_1 \times H_2$ be two linear relations. The sum of S and T, denoted by $S + T$, is the relation given by

$$S + T = \{(x, y + z) \in H_1 \times H_2 \mid (x, y) \in S,\ (x, z) \in T\},$$

whereas their *componentwise* sum, denoted by $S \mathbin{\widehat{+}} T$, is the relation given by

$$S \mathbin{\widehat{+}} T = \big\{(x + x', y + y') \in H_1 \times H_2 \bigm| (x, y) \in S,\ (x', y') \in T\big\}.$$

Moreover, the product of $S \subseteq H_1 \times H_2$ and $U \subseteq H_2 \times H_3$, denoted by US, is the relation given by

$$US = \{(x, z) \in H_1 \times H_3 \mid (x, y) \in S,\ (y, z) \in U\},$$

and, for $\lambda \in \mathbb{C}$, the relation

$$\lambda S = \{(x, \lambda y) \in H_1 \times H_2 \mid (x, y) \in S\},$$

demonstrates how we are to interpret scalar multiplication.

When we speak of an operator, there are many additional notions that come along with it: operators have domains, ranges, perhaps an inverse, etc. The following definition covers what will be the companions to these concepts.

Definition 3.1.2. Let $S \subseteq H_1 \times H_2$ be a linear relation. The *domain* of S is given by the set

$$\mathcal{D}(S) = \{x \in H_1 \mid (x, y) \in S \text{ for some } y \in H_2\},$$

whilst the *range* of S is given by the set

$$\mathcal{R}(S) = \{y \in H_2 \mid (x, y) \in S \text{ for some } x \in H_1\}.$$

The *inverse relation* S^{-1} is given by the set

$$S^{-1} = \{(y, x) \in H_2 \times H_1 \mid (x, y) \in S\}.$$

Remark. Note that the inverse relation *always* exists, unlike the inverse operator.

Immediately, we have the following relationship between the domain and range of a relation S and its inverse S^{-1}:

$$\mathcal{D}(S^{-1}) = \mathcal{R}(S) \qquad \text{and} \qquad \mathcal{R}(S^{-1}) = \mathcal{D}(S).$$

Furthermore, we may speak about when a relation is densely defined, precisely as in the operator setting.

Definition 3.1.3. Let $S \subseteq H_1 \times H_2$ be a linear relation. If $\mathcal{D}(S)$ is dense in H_1, then S is a *densely defined* linear relation.

Remark. Let T be a linear operator and set $S = \mathcal{G}(T)$. It can be seen that the definitions above are consistent to those presented in the operator setting. As we progress through this chapter, it is worth keeping this example in mind in order to appreciate the power that linear relations have.

Given an operator T, it is often useful to analyse its kernel, i.e., the set of elements in $\mathcal{D}(T)$ that are mapped to zero; this concept translates into the relations setting as one might expect. In particular, we express this set formally in the following definition, along with another crucial set: the multi-valued part.

Definition 3.1.4. Let $S \subseteq H_1 \times H_2$ be a linear relation. The *kernel* of S is given by the set

$$\ker S = \{x \in H_1 \mid (x, 0) \in S\},$$

whilst the *multi-valued part* of S is given by the set

$$\operatorname{mul} S = \{y \in H_2 \mid (0, y) \in S\}.$$

These two sets exhibit the following useful relationship, similar to the domain and range of S and its inverse:

$$\ker S^{-1} = \operatorname{mul} S \qquad \text{and} \qquad \operatorname{mul} S^{-1} = \ker S.$$

Remark. If T is a linear operator and $S = \mathcal{G}(T)$, then $\operatorname{mul} S = \{0\}$. Conversely, if $\operatorname{mul} S = \{0\}$, then S is the graph of some linear operator T. This makes sense: we do not expect a non-trivial multi-valued part to an operator that is not multi-valued!

It is useful to know when an operator is closed (or, at least, closable) as we are then able to apply certain theory or techniques to any analysis undertaken. We also have the notion of closed linear relations, as expressed in the following definition.

Definition 3.1.5. Let $S \subseteq H_1 \times H_2$ be a linear relation. The *closure* $\overline{S}$ of S is given by the set closure of S in $H_1 \times H_2$. If $S = \overline{S}$, then we say that S is a *closed* linear relation.

Computing the multi-valued part of a closed relation has a useful consequence: we may decompose a relation into the orthogonal sum of a graph and a purely multi-valued relation. Indeed, let S be a closed linear relation in $H_1 \times H_2$. Then, S has the componentwise orthogonal sum decomposition

$$S = S_s \oplus S_{\text{mul}},$$

where

$$S_s = S \ominus S_{\text{mul}} \quad \text{and} \quad S_{\text{mul}} = \{0\} \times \operatorname{mul} S.$$

Alternatively, we may express S_s as the set

$$S_s = \{(x, Py) \mid (x, y) \in S\},$$

where P is the orthogonal projection onto $(\operatorname{mul} S)^{\perp}$ [33]. In particular, S_s is called the *operator part*, or *single-valued part*, of S and is the graph of some operator, whilst S_{mul} is an entirely multi-valued relation, known as the *multi-valued part* of S.

Remark. Note that S_{mul} and $\operatorname{mul} S$ are different objects that are both referred to as the multi-valued part — we hope that it is clear which will be meant due to context, but we endeavour to be explicit if any confusion is possible.

There remains only one crucial relation left to present in this section: the adjoint relation. We remarked after Definition 1.1.12 that if T was a densely defined operator, then T^* would also be a linear operator. However, if T is *not* densely defined, then T^* is *not* a linear operator: instead, it will be multi-valued. One notable benefit of investigating linear relations is that the adjoint relation S^* of some linear relation S will *always*, again, be a linear relation — we may apply the same theory to both S and S^* indiscriminately. In particular, we may sensibly define the adjoint of a non-densely defined operator through linear relations.

Definition 3.1.6. Let $S \subseteq H_1 \times H_2$ be a linear relation. The adjoint relation S^* is the linear relation defined by

$$S^* = \left\{(x', y') \in H_2 \times H_1 \;\middle|\; \langle y', x \rangle_{H_1} = \langle x', y \rangle_{H_2} \text{ for all } (x, y) \in S\right\}.$$

Remark. The adjoint relation S^* is, in fact, a closed relation. Furthermore, from [15, Prop. 1.1], we have that for any relation S, its closure $\overline{S}$ is given by $\overline{S} = S^{**}$.

The following result demonstrates a useful relationship that will arise during the next chapter. In particular, we are able to determine when the adjoint relation is, in fact, the graph of an operator.

Lemma 3.1.7. *Let $S \subseteq H_1 \times H_2$ a linear relation. Then*

$$\operatorname{mul} S^* = (\mathcal{D}(S))^\perp.$$

In particular, if S is densely defined, then the adjoint relation S^* is the graph of an operator.

Proof. Let $S \subseteq H_1 \times H_2$ be a linear relation. We begin the proof by computing $(\mathcal{D}(S))^\perp$. Then,

$$\begin{aligned}(\mathcal{D}(S))^\perp &= \{x \in H_1 \mid (x, y) \in S\}^\perp \\ &= \left\{z \in H_1 \;\middle|\; \langle z, x\rangle_{H_1} = 0 \text{ for all } x \in \mathcal{D}(S)\right\}.\end{aligned}$$

On the other hand, if z lies in $\operatorname{mul} S^*$, then the corresponding element in S^* is of the form $(0, z)$. Hence,

$$\begin{aligned}\operatorname{mul} S^* &= \left\{z \in H_1 \;\middle|\; \langle z, x\rangle_{H_1} = \langle 0, y\rangle_{H_2} \text{ for all } (x, y) \in S\right\} \\ &= \left\{z \in H_1 \;\middle|\; \langle z, x\rangle_{H_1} = 0 \text{ for all } x \in \mathcal{D}(S)\right\}.\end{aligned}$$

As the two sets are equal, we may conclude that $\operatorname{mul} S^* = (\mathcal{D}(S))^\perp$.

Now, let S be a densely defined linear relation. To show that S^* is the graph of an operator, we simply need to show that $\operatorname{mul} S^* = \{0\}$. Since $\mathcal{D}(S) \subseteq H_1$, we may decompose H_1 into the orthogonal sum

$$H_1 = \overline{\mathcal{D}(S)} \oplus (\mathcal{D}(S))^\perp,$$

as discussed after Definition 1.1.3. Then, as $\mathcal{D}(S)$ is dense in H_1 — that is, $\overline{\mathcal{D}(S)} = H_1$ — we may conclude that $(\mathcal{D}(S))^\perp = \{0\}$. Hence $\operatorname{mul} S^* = \{0\}$, showing that S^* is, indeed, the graph of an operator. ■

With the adjoint relation now defined, we discuss what it means for a linear relation to be symmetric or self-adjoint. In what follows, we set $H_1 = H_2 = H$.

Definition 3.1.8. Let $S \subseteq H \times H$ be a linear relation. If $S \subseteq S^*$, then S is called *symmetric*. If $S = S^*$, then S is called *self-adjoint*.

Remark. In other words, $S \subseteq H \times H$ is a symmetric linear relation if $\langle y', x\rangle_H = \langle x', y\rangle_H$ for all (x, y), $(x', y') \in S$.

In Chapter 1, we made the distinction between positive and non-negative operators. Here, we are only interested in non-negative relations, as expressed in the following definition.

Definition 3.1.9. Let $S \subseteq H \times H$ be a linear relation. If $\langle y, x\rangle \geq 0$ for all $(x, y) \in S$, then S is said to be *non-negative.*

Throughout the thesis so far, symmetry is rarely mentioned without sectoriality. Then, in the final definition of this section, we introduce the concept of sectoriality in the relations setting.

Definition 3.1.10. Let $S \subseteq H \times H$ be a linear relation. If

$$|\operatorname{Im}\langle y, x\rangle| \leq \tan\alpha \left(\operatorname{Re}\langle y, x\rangle - \gamma\|x\|^2\right), \qquad \gamma \in \mathbb{R},\ \alpha \in \left[0, \frac{\pi}{2}\right),$$

for all $(x, y) \in S$, then S is said to be *sectorial.* Furthermore, γ and α are referred to as the vertex and semi-angle, respectively.

Remark. Note that this definition is in line with the alternative description of a sectorial form as given in the remark following Definition 1.1.30.

Remark. If S is a non-negative relation, then S is also sectorial with vertex $\gamma = 0$ and any semi-angle α: for simplicity, we choose $\alpha = 0$.

We conclude this section by reiterating that the definitions presented here are valid for whenever S is the graph of an operator T, further strengthening the claim made at the start of this chapter: relations are a generalisation of the graph. The next section will be devoted to presenting the extension theory necessary for the remainder of the thesis.

3.2 Extension Theory of Linear Relations

In Section 1.2, we presented two methods of constructing extensions of operators: the von Neumann theory and the Kreĭn-Vishik-Birman theory. Here, we aim to deliver the analogous theory in the context of linear relations. Firstly, the von Neumann theory will be described so that we may, again, document the similarities between operators and linear relations. Then, Sections 3.2.2 and 3.2.3 document, in depth, the theory required to appreciate Chapter 4.

3.2.1 von Neumann Theory for Linear Relations

When discussing the von Neumann theory in Section 1.2.2, we noted that the deficiency spaces of an operator T were fundamental in the construction of the closed, symmetric extensions of T; crucially, we decomposed the domain of the adjoint operator into a particular sum and made use of isometric maps between subspaces of the deficiency spaces. Here, the idea is the same. Then, this section details the von Neumann theory with respect to linear relations, as found in [15].

First, we need to discuss how we are to interpret the notion of an extension in the linear relations setting.

Definition 3.2.1. Let $S \subseteq H_1 \times H_2$ be a linear relation. Any subspace $\tilde{S}$ that satisfies $S \subseteq \tilde{S} \subseteq H_1 \times H_2$ is called an *extension* of S.

As per the von Neumann theory, our objective is as follows: we wish to construct the closed, symmetric extensions $\tilde{S}$ of a closed, symmetric relation S. Thus, let S be a symmetric relation. Immediately, we note that $\overline{S}$ is, again, a closed relation and is the smallest closed extension of S. Likewise, we have that $(\overline{S})^* = S^*$. As such, abstractly, we may simply assume that S is a closed relation to begin with.

Definition 3.2.2. Let $S \subseteq H \times H$ be a linear relation. The closed subspaces $\mathcal{N}_+$ and $\mathcal{N}_-$ defined by

$$\begin{aligned}\mathcal{N}_+ \equiv \mathcal{N}_+(S) &:= \{(x, y) \in S^* \mid (-iy, ix) = (x, y)\} \\ &= \{(x, ix) \mid x \in \mathcal{D}(S^*)\} \cap S^*\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}_- \equiv \mathcal{N}_-(S) &:= \{(x, y) \in S^* \mid (-iy, ix) = (-x, -y)\} \\ &= \{(x, -ix) \mid x \in \mathcal{D}(S^*)\} \cap S^*\end{aligned}$$

are called the *deficiency spaces* of S. The dimensions of these subspaces, denoted by $m_+(S)$ and $m_-(S)$ respectively, are called the *deficiency indices*.

Remark. To see why this construction seems reasonable, let T be a closed symmetric operator and recall that $\mathcal{N}_\pm(T) = \ker(T^* \mp iI)$ from Definition 1.2.7. Immediately, we observe that

$$\begin{aligned}\mathcal{N}_\pm(T) &= \{x \in \mathcal{D}(T^*) \mid (T^* \mp iI)x = 0\} \\ &= \{x \in \mathcal{D}(T^*) \mid T^*x = \pm ix\}.\end{aligned}$$

We may then convert these calculations into the graph setting since there is a one-to-one correspondence between operators and graphs. Essentially, as $\mathcal{N}_\pm(\mathcal{G}(T)) \subseteq \mathcal{G}(T^*)$ for $\mathcal{G}(T)$ the graph of T, we see that $\mathcal{N}_\pm(\mathcal{G}(T))$ is such that

$$\begin{aligned}\mathcal{N}_\pm(\mathcal{G}(T)) &= \{(x, T^*x) \mid x \in \mathcal{D}(T^*) \text{ and } T^*x = \pm ix\} \\ &= \{(x, \pm ix) \mid x \in \mathcal{D}(T^*)\} \cap \mathcal{G}(T^*).\end{aligned}$$

Thus, when T is an operator, the two definitions coincide.

In Theorem 1.2.8, we demonstrated how one could decompose the domain of the adjoint operator T^* of a closed, symmetric operator T into

$$\mathcal{D}(T^*) = \mathcal{D}(T) \dotplus \mathcal{N}_+(T) \dotplus \mathcal{N}_-(T).$$

We now disclose the analogous decomposition by means of the following theorem.

Theorem 3.2.3 ([15, Thm. 1.2])**.** *Let $S \subseteq H \times H$ be a closed symmetric relation whose deficiency spaces are given by $\mathcal{N}_+(S)$ and $\mathcal{N}_-(S)$. Then*

$$S^* = S \oplus \mathcal{N}_+(S) \oplus \mathcal{N}_-(S).$$

Since S is closed, symmetric relation, [15, Prop. 1.1] gives that

$$S \subset \tilde{S} \subset \tilde{S}^* \subset S^*,$$

for any symmetric extension $\tilde{S}$ of S. Then, we are able to characterise the closed symmetric extensions $\tilde{S}$ by noting that they are all to be restrictions of S^*. As such, we have the following theorem.

Theorem 3.2.4 ([15, Thm. 1.6])**.** *Let $S \subseteq H \times H$ be a closed, symmetric linear relation. If $\tilde{S}$ is a closed, symmetric extension of S, then $\tilde{S} = S \oplus \mathcal{N}$, where $\mathcal{N}$ is a subspace of $\mathcal{N}_+(S) \oplus \mathcal{N}_-(S)$ that satisfies*

$$\mathcal{N} = \{(x, ix) + J(x, ix) \mid (x, ix) \in \mathcal{D}(J) \subseteq \mathcal{N}_+(S)\},$$

where $J\colon \mathcal{D}(J) \to \mathcal{R}(J) \subseteq \mathcal{N}_-(S)$ is some linear isometry with closed domain $\mathcal{D}(J) \subseteq \mathcal{N}_+(S)$. The reverse also holds: for every space $\mathcal{N}$ of this form, there exists a unique closed, symmetric extension $\tilde{S}$ of S satisfying $\tilde{S} = S \oplus \mathcal{N}$.

Remark. This theorem is the analogue of Theorem 1.2.9.

For completeness, we conclude this section by presenting the following definition along with one final corollary to the theorem above.

Definition 3.2.5. Let $S \subseteq H \times H$ be a symmetric linear relation. If the only symmetric extension of S is S itself, then S is said to be *maximal symmetric.*

In the operator case, equality between the deficiency indices allowed us to classify different types of extensions. This concept translates accordingly in the context of relations, as demonstrated in the following corollary.

Corollary 3.2.6. *Let $S \subseteq H \times H$ be a linear relation. If precisely one of the deficiency indices $m_+(S)$ or $m_-(S)$ is equal to zero, then S is a maximal symmetric relation. If both are equal to zero, that is, $m_+(S) = m_-(S) = 0$, then S is self-adjoint.*

The von Neumann theory is instantly recognisable no matter the setting due to its reliance on the decomposition that uses the adjoint and the deficiency spaces. We also hope that this section has given some insight into why linear relations have value: the von Neumann theory is not only transferable, but also more general when considering relations. The next section, however, aims to construct the closed, *sectorial* extensions of a sectorial relation S by means of sesquilinear forms.

3.2.2 The Friedrichs Extension of a Sectorial Relation

In the previous section, we presented the von Neumann theory for linear relations; Sections 1.2.2 and 3.2.1 can be thought of as companion sections due to the more than intimate connection between the theory presented. For the purpose of symmetry, this section can be thought of as the analogue to Section 1.2.3 as, here, we construct both the Friedrichs and Kreĭn extension of a sectorial linear relation. Although the theory utilises an association between sesquilinear forms and sectorial relations, we assert that the Friedrichs extension will *not* form the basis for any extension other than the Kreĭn extension — the Kreĭn-Vishik-Birman theory for linear operators is only tangentially related, this time. We remark that the theory presented in this section may be found in [33].

We have established that a relation can be symmetric or sectorial, and Definition 3.2.5 introduced the notion of *maximal* symmetric relations. The following definition completes the set by discussing maximal *sectorial* relations.

Definition 3.2.7. Let $S \subseteq H \times H$ be a sectorial linear relation. If the only sectorial extension of S is S itself, then S is said to be *maximal sectorial.*

The theory that we present here, and in what follows, concerns itself with sectorial linear relations whose vertex is at the origin, i.e., $\gamma = 0$. Then, we introduce an important theorem that will form the bulk of the work undertaken in the final chapter of the thesis.

Theorem 3.2.8 ([33, Thm. 4.3])**.** *Let* $\mathbf{a}$ *be a closed sectorial form in a Hilbert space* H *with vertex* $\gamma = 0$ *and semi-angle* α*, where* $\alpha \in \left[0, \frac{\pi}{2}\right)$*. Then, there exists a unique maximal sectorial relation* S *in* H *with* $\gamma = 0$ *and semi-angle* α *such that*

$$\mathcal{D}(S) \subseteq Q(\mathbf{a}), \tag{3.1}$$

and, for every $(x, y) \in S$ *and* $k \in Q(\mathbf{a})$*,*

$$\mathbf{a}\left[x, k\right] = \langle y, k \rangle \, . \tag{3.2}$$

Remark. The converse of this theorem is also true: for every maximal sectorial relation S with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$, there exists a unique, closed sectorial form $\mathbf{a}$ that satisfies both equation (3.1) and (3.2).

Remark. Note that if $\mathbf{a}$ is a densely defined form, then S is the graph of a maximal sectorial operator.

The proof of this theorem is insightful: it constructs a candidate relation S before proving that S does indeed possess the desired properties. Since we follow the steps outlined in this construction during Chapter 4, we choose to present the method of constructing the maximal sectorial extension S here.

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\mathbf{a}$ a closed sectorial form with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$. Denote by $Q(\mathbf{a}) \subseteq H$ its form domain. Then, by Definition 1.1.31, $(Q(\mathbf{a}), \|\cdot\|_{\mathbf{a}})$ is a Hilbert space, where

$$\|x\|_{\mathbf{a}} = \left(\mathbf{a}_{\mathrm{Re}}\,[x, x] + \langle x, x \rangle\right)^{\frac{1}{2}}, \qquad x \in Q(\mathbf{a}).$$

This norm is induced by an inner product, say $\langle \cdot, \cdot \rangle_{\mathbf{a}}$.

Let $\hat{\mathbf{a}}$ be the form defined by

$$\hat{\mathbf{a}}\,[x, y] = \mathbf{a}\,[x, y] + \langle x, y \rangle\,,$$

for x, $y \in Q(\hat{\mathbf{a}}) = Q(\mathbf{a})$. This form will also be sectorial with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$; this is immediate, since the numerical range $\Theta(\hat{\mathbf{a}})$ is simply a translation of $\Theta(\mathbf{a})$ in the complex plane by precisely 1 to the right.

Remark. If we suppress the entries of the form $\hat{\mathbf{a}}$, then we will write $\hat{\mathbf{a}} = \mathbf{a} + 1$: the 1 signifies the ℓ^2-inner product.

Our first objective is to verify that $\hat{\mathbf{a}}$ is a bounded form in $(Q(\mathbf{a}), \|\cdot\|_{\mathbf{a}})$; if it is, then we may invoke the Riesz Representation Theorem (see, for example, [43, Thm. 3.8-4]) and associate to it a bounded linear operator $B\colon Q(\mathbf{a}) \to Q(\mathbf{a})$ such that

$$\hat{\mathbf{a}}\,[x, y] = \langle Bx, y \rangle_{\mathbf{a}}\,, \qquad x,\ y \in Q(\mathbf{a}). \tag{3.3}$$

Then, we wish to find a real constant c such that

$$|\hat{\mathbf{a}}\,[x, y]\,| \leq c\|x\|_{\mathbf{a}}\|y\|_{\mathbf{a}},$$

for all x, $y \in Q(\mathbf{a})$. Since

$$\|x\| \leq \|x\|_{\mathbf{a}}, \qquad x \in Q(\mathbf{a}), \tag{3.4}$$

we immediately observe that

$$\begin{aligned} |\hat{\mathbf{a}}\,[x, y]\,| = |\mathbf{a}\,[x, y] + \langle x, y \rangle\,| &\leq |\mathbf{a}\,[x, y]\,| + |\,\langle x, y \rangle\,| \\ &\leq |\mathbf{a}\,[x, y]\,| + \|x\|\|y\|, \\ &\leq |\mathbf{a}\,[x, y]\,| + \|x\|_{\mathbf{a}}\|y\|_{\mathbf{a}}, \end{aligned}$$

after an application of the Cauchy-Schwarz inequality. In order to estimate $|\mathbf{a}[x,y]|$, we first disclose the following lemma [33].

Lemma 3.2.9. *Let H be a Hilbert space and let $\mathbf{a}$ be a sectorial form with vertex $\gamma = 0$ and semi-angle α, where $\alpha \in \left[0, \frac{\pi}{2}\right)$. Then, the following estimate holds for all x, $y \in Q(\mathbf{a})$:*

$$|\mathbf{a}[x,y]| \leq (1+\tan\alpha)\mathbf{a}_{\mathrm{Re}}[x,x]^{\frac{1}{2}}\,\mathbf{a}_{\mathrm{Re}}[y,y]^{\frac{1}{2}}.$$

With this lemma in mind, it is clear that

$$\begin{aligned}
|\hat{\mathbf{a}}[x,y]| &\leq (1+\tan\alpha)\mathbf{a}_{\mathrm{Re}}[x,x]^{\frac{1}{2}}\,\mathbf{a}_{\mathrm{Re}}[y,y]^{\frac{1}{2}} + \|x\|_{\mathbf{a}}\|y\|_{\mathbf{a}} \\
&\leq (1+\tan\alpha)\left(\mathbf{a}_{\mathrm{Re}}[x,x] + \langle x,x\rangle\right)^{\frac{1}{2}}\left(\mathbf{a}_{\mathrm{Re}}[y,y] + \langle y,y\rangle\right)^{\frac{1}{2}} + \|x\|_{\mathbf{a}}\|y\|_{\mathbf{a}} \\
&= (2+\tan\alpha)\|x\|_{\mathbf{a}}\|y\|_{\mathbf{a}}.
\end{aligned}$$

Thus, the form $\hat{\mathbf{a}}$ is bounded in $(Q(\mathbf{a}), \|\cdot\|_{\mathbf{a}})$ and so there exists a unique, bounded linear operator B such that equation (3.3) holds. Consequently, we see that following equality holds for all $x \in Q(\mathbf{a})$:

$$\mathrm{Re}\,\langle Bx,x\rangle_{\mathbf{a}} = \mathrm{Re}\,(\hat{\mathbf{a}}[x,x]) = \mathrm{Re}\,(\mathbf{a}[x,x]) + \mathrm{Re}\,\langle x,x\rangle = \mathbf{a}_{\mathrm{Re}}[x,x] + \langle x,x\rangle.$$

Then, it is clear that

$$\|x\|_{\mathbf{a}}^2 = \mathbf{a}_{\mathrm{Re}}[x,x] + \langle x,x\rangle = \mathrm{Re}\,\langle Bx,x\rangle_{\mathbf{a}} \leq |\langle Bx,x\rangle_{\mathbf{a}}| = |\hat{\mathbf{a}}[x,x]|,$$

for all $x \in Q(\mathbf{a})$. We may then invoke [33, Lem. 4.1] to deduce that the operator B is invertible. Moreover, B^{-1} is a bounded operator in $Q(\mathbf{a})$ and satisfies $\|B^{-1}\|_{\mathbf{a}} \leq 1$.

Now, fix $\omega \in H$ and consider the linear mapping $k \mapsto \langle k,\omega\rangle$; this map is a linear functional from $Q(\mathbf{a}) \to \mathbb{C}$ that is defined for all $k \in Q(\mathbf{a})$. From the Cauchy-Schwarz inequality and equation (3.4), it follows that

$$|\langle k,\omega\rangle| \leq \|k\|\|\omega\| \leq \|\omega\|\|k\|_{\mathbf{a}}, \qquad k \in Q(\mathbf{a}).$$

Since we have found a real constant c such that $|\langle k,\omega\rangle| \leq c\|k\|_{\mathbf{a}}$ for all $k \in Q(\mathbf{a})$, we may conclude that the mapping $k \mapsto \langle k,\omega\rangle$ is a bounded linear functional. Therefore, by the Riesz Representation Theorem for linear functionals (see, for example, [43, Thm. 3.8-1]), there exists a unique element $\hat{\omega} \in Q(\mathbf{a})$ such that for all $k \in Q(\mathbf{a})$:

$$\langle k,\omega\rangle = \langle k,\hat{\omega}\rangle_{\mathbf{a}} \quad \text{and} \quad \|\hat{\omega}\|_{\mathbf{a}} \leq \|\omega\|. \tag{3.5}$$

Note that the inequality $\|\hat{\omega}\|_{\mathbf{a}} \leq \|\omega\|$ holds since

$$\begin{aligned}\|\hat{\omega}\|_{\mathbf{a}} = \sup_{\substack{k\in Q(\mathbf{a}),\\ \|k\|_{\mathbf{a}}=1}} |\langle k,\hat{\omega}\rangle_{\mathbf{a}}| &= \sup_{\substack{k\in Q(\mathbf{a}),\\ \|k\|_{\mathbf{a}}=1}} |\langle k,\omega\rangle| \\ &\leq \sup_{\substack{k\in Q(\mathbf{a}),\\ \|k\|=1}} |\langle k,\omega\rangle| \leq \sup_{\substack{k\in \ell^2,\\ \|k\|=1}} |\langle k,\omega\rangle| = \|\omega\|,\end{aligned}$$

where we have used the equality given in (3.5) in conjunction with inequality (3.4).

Since the operator B is invertible, we have

$$\langle \omega, k\rangle = \langle \hat{\omega}, k\rangle_{\mathbf{a}} = \left\langle BB^{-1}\hat{\omega}, k\right\rangle_{\mathbf{a}},$$

for all $k \in Q(\mathbf{a})$. Then, from equation (3.3), we have

$$\left\langle BB^{-1}\hat{\omega}, k\right\rangle_{\mathbf{a}} = \hat{\mathbf{a}}\left[B^{-1}\hat{\omega}, k\right] = \mathbf{a}\left[B^{-1}\hat{\omega}, k\right] + \left\langle B^{-1}\hat{\omega}, k\right\rangle.$$

By piecing together these two equalities, we see that

$$\mathbf{a}\left[B^{-1}\hat{\omega}, k\right] = \left\langle \omega - B^{-1}\hat{\omega}, k\right\rangle, \qquad k \in Q(\mathbf{a}). \tag{3.6}$$

Upon recalling equation (3.2), the form $\mathbf{a}$ aims to satisfy $\mathbf{a}[x,k] = \langle y, k\rangle$ for all $(x,y) \in S$ and $k \in Q(\mathbf{a})$. A direct comparison to equation (3.6) gives us an indication of how to continue.

Indeed, define the linear mapping A from H to $Q(\mathbf{a})$ by $A\omega = B^{-1}\hat{\omega}$. If we are to interpret A as a mapping from H to H instead — that is, we embed $B^{-1}\hat{\omega}$ in H — we obtain the following estimate:

$$\begin{aligned}\|A\omega\| &\leq \|A\omega\|_{\mathbf{a}} && \text{by equation (3.4)}\\ &= \|B^{-1}\hat{\omega}\|_{\mathbf{a}} && \text{since } A\omega = B^{-1}\hat{\omega}\\ &\leq \|\hat{\omega}\|_{\mathbf{a}} && \text{since } \|B^{-1}\|_{\mathbf{a}} \leq 1\\ &\leq \|\omega\| && \text{by equation (3.5).}\end{aligned}$$

Hence, the operator A is, in fact, a bounded linear operator on H.

This operator A is fundamental in the construction of the maximal sectorial relation S associated to the closed sectorial form $\mathbf{a}$. In particular, when considering the operator A from H to H, the maximal sectorial relation S — also from H to H — with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$, is given by

$$S = \{(A\omega, \omega - A\omega) \mid \omega \in H\}. \tag{3.7}$$

Furthermore, the relation $S + I$ has a few notable properties that we wish to draw attention to. Firstly, as I is defined on all of H, we are to interpret the domain of $S + I$ by

$$\mathcal{D}(S+I) = \mathcal{D}(S) = \{A\omega \mid \omega \in H\}.$$

Then, upon recalling how one is to interpret the addition of two linear relations, we see that

$$\begin{aligned} S + I &= \{(A\omega, \omega - A\omega + A\omega) \mid \omega \in H\} \\ &= \{(A\omega, \omega) \mid \omega \in H\}. \end{aligned}$$

It is then immediate that $\mathcal{R}(S+I) = H$ and $\ker(S+I) = \{0\}$. Finally, we remark that when S is a linear relation, the *resolvent relation* is given by the relation $(S - \lambda I)^{-1}$ where $\lambda \in \mathbb{C}$ [13]. Then, upon letting $\lambda = -1$, we see that

$$(S+I)^{-1} = \{(\omega, A\omega) \mid \omega \in H\},$$

demonstrating that $(S+I)^{-1}$ coincides precisely with the graph of the operator A.

In essence, when we are in possession of the bounded linear operator A, as constructed above, we can easily express the maximal sectorial relation S by means of equation (3.7). In order to verify that this relation does, indeed, enjoy all of the relevant properties, we divert the reader's attention to the proof of Theorem 3.2.8 as stated in [33] i.e., [33, Thm. 4.3].

With the main theorem of this section described, we continue by introducing the Friedrichs extension S_F and the Kreĭn extension S_K of a sectorial relation S. Firstly, however, we require the following result.

Lemma 3.2.10 ([33, Lem. 4.2]). *Let $S \subseteq H \times H$ be a sectorial relation with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$. The form $\mathbf{a_S}$ with*

$$\mathbf{a_S}[x, z] = \langle y, z \rangle, \qquad (x, y),\ (z, w) \in S,$$

and form domain $Q(\mathbf{a_S}) = \mathcal{D}(S)$ is well-defined, sectorial and closable.

Since the form $\mathbf{a_S}$ is closable, denote by $\mathbf{a_{S_F}}$ its closure. As this new form is both closed and sectorial, we may associate to it a unique maximal sectorial relation S_F, say, by means of Theorem 3.2.8. We then invoke [33, Lem. 7.1] to conclude that this construction of S_F does, indeed, give rise to an extension of S. The maximal sectorial relation S_F constructed in this manner *is* the Friedrichs extension of S.

Perhaps the most striking similarity with the Kreĭn-Vishik-Birman theory is that the Friedrichs extension is *necessary* for the construction of the Kreĭn extension, albeit not in quite the same way. Here, we define the Kreĭn extension S_K of a sectorial linear relation S to be the maximal sectorial relation satisfying the following:

$$S_K = \left(\left(S^{-1}\right)_F\right)^{-1}.$$

This definition may seem complicated at first, but it may be unravelled methodically. In particular, we take the inverse relation S^{-1} of S and associate to it the form $\mathbf{a_{S^{-1}}}$, before finding its closure $\mathbf{a_{S_F^{-1}}}$. To this form, we may associate a unique maximal sectorial relation: this will be $(S^{-1})_F$. Afterwards, we merely need to take the inverse of this relation, that is, $\left(\left(S^{-1}\right)_F\right)^{-1}$ — this will give rise to the Kreĭn extension S_K of S.

Remark. This construction of the Kreĭn extension coincides with that of [8], which investigates the non-negative, self-adjoint extensions of positive, symmetric operators. Note that the Kreĭn extension is referred to as the *von Neumann* extension there.

3.2.3 Extremal Maximal Sectorial Relations

Whilst the previous section introduced a way of associating a sesquilinear form $\mathbf{a}$ to a sectorial relation S, it did not yield a practical method of constructing *all* of the sectorial extensions of S: we merely obtained the Friedrichs extension and Kreĭn extension. This section aims to rectify this. We claim that there exists an approximate comparison to the Kreĭn-Vishik-Birman theory for linear operators in the context of linear relations. The comparison may not be perfect, but the theory presented in this section — extracted from [33] — constructs the Friedrichs extension, the Kreĭn extension and — crucially — all extensions in between. Moreover, we may then associate to these extensions a sesquilinear form, strengthening the initial claim.

Definition 3.2.11. Let $S \subseteq H \times H$ be a sectorial relation with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$ and let $\mathbf{a_{S_K}}$ be the form with domain $Q(\mathbf{a_{S_K}})$ associated to the Kreĭn extension of S. Let $\tilde{S}$ be a maximal sectorial extension of S. Then $\tilde{S}$ is an *extremal* maximal sectorial extension of S if the closed form $\mathbf{a_{\tilde{S}}}$ associated to $\tilde{S}$ satisfies the following two conditions:

$$Q(\mathbf{a_{\tilde{S}}}) \subseteq Q(\mathbf{a_{S_K}}) \quad \text{and} \quad \mathbf{a_{\tilde{S}}}\left[x, y\right] = \mathbf{a_{S_K}}\left[x, y\right] \text{ for all } x,\, y \in Q(\mathbf{a_{\tilde{S}}}).$$

Remark. In particular, we note that S_F and S_K are extremal maximal sectorial extensions of a sectorial relation S.

We now make a short detour to explore this concept in the operator setting as presented in Chapters 1 and 2. Let T be a positive, sectorial — thus, symmetric — operator and T_K its Kreĭn extension. Then, by Theorem 1.2.16, the form $\mathbf{t_K}$ associated to T_K is such that

$$\begin{aligned}\mathbf{t_K}\left[u, v\right] &= \mathbf{t_F}\left[u^F, v^F\right] + \mathbf{b_1}\left[u^N, v^N\right] \\ &= \mathbf{t_F}\left[u^F, v^F\right],\end{aligned}$$

for elements u, v in $Q(\mathbf{t_K})$, where $u = u^F + u^N$ and $v = v^F + v^N$ for u^F, $v^F \in Q(\mathbf{t_F})$ and u^N, $v^N \in Q(\mathbf{b_1})$. Recall that in the construction of the Kreĭn extension, we must take the form $\mathbf{b_1} = 0$ and $Q(\mathbf{b_1}) = \mathcal{N} = \ker T_{\max}$. Now let $\mathbf{t_B}$ be the form associated to a non-negative, self-adjoint extension T_B of T, i.e.,

$$\mathbf{t_B}[u, v] = \mathbf{t_F}\left[u^F, v^F\right] + \mathbf{b_2}\left[u^N, v^N\right],$$

where u and v are to be decomposed as above. It is clear that any form $\mathbf{t_B}$ will satisfy $Q(\mathbf{t_B}) \subseteq Q(\mathbf{t_K})$ since $Q(\mathbf{t_K})$ has the largest possible domain: we have taken the entirety of $\ker T_{\max}$ for the form domain of $\mathbf{b_1} = 0$.

On the other hand, if we wish for the equality

$$\begin{aligned}\mathbf{t_B}[u, v] = \mathbf{t_F}\left[u^F, v^F\right] + \mathbf{b_2}\left[u^N, v^N\right] &= \mathbf{t_K}[u, v] \\ &= \mathbf{t_F}\left[u^F, v^F\right]\end{aligned}$$

to hold for all u, $v \in Q(\mathbf{t_B})$, then we must set $\mathbf{b_2}\left[u^N, v^N\right] = 0$ for all u^N, $v^N \in Q(\mathbf{b_2})$. Thus, if $\mathbf{t_B}$ is an extremal maximal sectorial extension, then we must fix $\mathbf{b_2} = 0$ and merely require that $Q(\mathbf{b_2})$ is a subspace of $\mathcal{N}$.

In fact, by analysing the dimension of $\mathcal{N}$, we are able to say more. If $\dim \mathcal{N} = 1$, as in Chapter 2, then there are only two subspaces $\mathcal{N}_B$ of $\mathcal{N}$: either we take $\mathcal{N}_B = \{0\}$ or $\mathcal{N}_B = \mathcal{N}$. The former produces the Friedrichs extension, whilst the latter, the Kreĭn. This shows that, when $\dim \mathcal{N} = 1$, these two extensions are the *only* extremal maximal sectorial extensions of T. Conversely, if $\dim \mathcal{N} \geq 2$, then there *will* exist other extremal extensions, as we may simply choose any subspace of $\mathcal{N}$ that is neither non-trivial nor $\mathcal{N}$ itself. We note that this remains true for the examples of relations we will consider in Chapter 4.

The remainder of this section will be devoted to detailing the construction of the extremal maximal sectorial extensions of a sectorial relation S. Let H be a Hilbert space and $S \subseteq H \times H$ a sectorial relation with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$. On $\mathcal{R}(S) \times \mathcal{R}(S)$, define the map

$$\left\langle x', y' \right\rangle_{\mathcal{R}(S)} = \frac{1}{2}\left(\left\langle x', y \right\rangle + \left\langle x, y' \right\rangle\right), \qquad (x, x'), \, (y, y') \in S. \tag{3.8}$$

The following lemma proves that such a map is well-defined, in addition to being a semi-inner product.

Lemma 3.2.12. *The map $\langle \cdot, \cdot \rangle_{\mathcal{R}(S)}$ as defined by equation (3.8) is a well-defined, semi-inner product.*

Proof. Let (x, x'), (x_0, x'), (y, y') and (y_0, y') lie in S. First, we show that $\langle \cdot, \cdot \rangle_{\mathcal{R}(S)}$ is well-defined, that is, the elements in S with the same second component but *different* first components do *not* produce a different result. In other words, we require both

$$\langle x', y' \rangle_{\mathcal{R}(S)} = \frac{1}{2} \left(\langle x', y \rangle + \langle x, y' \rangle \right) = \frac{1}{2} \left(\langle x', y \rangle + \langle x_0, y' \rangle \right)$$

and

$$\langle x', y' \rangle_{\mathcal{R}(S)} = \frac{1}{2} \left(\langle x', y \rangle + \langle x, y' \rangle \right) = \frac{1}{2} \left(\langle x', y_0 \rangle + \langle x, y' \rangle \right)$$

to hold, i.e., $\langle x, y' \rangle = \langle x_0, y' \rangle$ and $\langle x', y \rangle = \langle x', y_0 \rangle$, and, in particular,

$$\langle x - x_0, y' \rangle = 0 \quad \text{and} \quad \langle x', y - y_0 \rangle = 0. \tag{3.9}$$

Note that we may associate the form $\mathbf{a_S}$ to S by means of Lemma 3.2.10. This form, after applying Lemma 3.2.9, admits a useful consequence:

$$\begin{aligned} |\langle x', y \rangle| = |\mathbf{a_S}[x, y]| &\leq (1 + \tan \alpha) \left((\mathbf{a_S})_{\mathrm{Re}}[x, x] \right)^{\frac{1}{2}} \left((\mathbf{a_S})_{\mathrm{Re}}[y, y] \right)^{\frac{1}{2}} \\ &= (1 + \tan \alpha) \left(\mathrm{Re} \langle x', x \rangle \right)^{\frac{1}{2}} \left(\mathrm{Re} \langle y', y \rangle \right)^{\frac{1}{2}}, \end{aligned} \tag{3.10}$$

for all (x, x'), $(y, y') \in S$, since

$$(\mathbf{a_S})_{\mathrm{Re}}[x, x] = \mathrm{Re}\left(\mathbf{a_S}[x, x]\right) = \mathrm{Re} \langle x', x \rangle, \qquad (x, x') \in S.$$

Furthermore, since S is a subspace, it is clear that $(x - x_0, 0)$ and $(y - y_0, 0)$ must also lie in S. Then, for (y, y'), $(x - x_0, 0) \in S$, we have

$$|\langle y', x - x_0 \rangle| \leq (1 + \tan \alpha) \left(\mathrm{Re} \langle y', y \rangle \right)^{\frac{1}{2}} \left(\mathrm{Re} \langle 0, x - x_0 \rangle \right)^{\frac{1}{2}},$$

whilst for (x, x'), $(y - y_0, 0) \in S$, we have

$$|\langle x', y - y_0 \rangle| \leq (1 + \tan \alpha) \left(\mathrm{Re} \langle x', x \rangle \right)^{\frac{1}{2}} \left(\mathrm{Re} \langle 0, y - y_0 \rangle \right)^{\frac{1}{2}}.$$

Since $\langle 0, x - x_0 \rangle = \langle 0, y - y_0 \rangle = 0$, the left-hand side of these inequalities must also equal zero, proving that the conditions given in (3.9) are satisfied. Thus, the map is well-defined.

Upon recalling Definition 1.1.1, a semi-inner product differs to an inner product by only one property: one of the conditions present in (**IP4**) states that $\langle x, x \rangle = 0 \iff x = 0$, but for a semi-inner product, this does *not* have to hold. Then, it is easy to see that $\langle \cdot, \cdot \rangle_{\mathcal{R}(S)}$ satisfies the first three properties and $\langle x', x' \rangle_{\mathcal{R}(S)} \geq 0$ for all $x' \in \mathcal{R}(S)$ by virtue of $\langle \cdot, \cdot \rangle$ being an inner product. ■

We now show that there may exist an $(x, x') \in S$ — thus, an $x' \in \mathcal{R}(S)$ — such that $\langle x', x' \rangle_{\mathcal{R}(S)} = 0$. Indeed, let $(x, x') \in S$, then

$$\langle x', x' \rangle_{\mathcal{R}(S)} = \frac{1}{2} \left(\langle x', x \rangle + \langle x, x' \rangle \right) = \mathrm{Re} \langle x', x \rangle.$$

Hence, $\langle x', x'\rangle_{\mathcal{R}(S)} = 0$ if and only if $\operatorname{Re}\langle x', x\rangle = 0$. In fact, we observe that $\langle x', x'\rangle_{\mathcal{R}(S)} = 0$ if and only $\langle x', x\rangle = 0$, since for a sectorial relation S, we have $|\operatorname{Im}\langle x', x\rangle| \leq (\tan\alpha)\operatorname{Re}\langle x', x\rangle$ for all $(x, x') \in S$. Thus, the existence of an element $(x, x') \in S$ satisfying $\langle x', x\rangle = 0$ demonstrates that the map $\langle \cdot, \cdot\rangle_{\mathcal{R}(S)}$ is not an inner product, but rather a semi-inner product on $\mathcal{R}(S) \times \mathcal{R}(S)$.

This argument is pivotal in the construction of the extremal maximal sectorial extensions of S; it demonstrates precisely which elements in S prevent $\langle \cdot, \cdot\rangle_{\mathcal{R}(S)}$ from being an inner product. Then, let $\mathfrak{R}_0$ be the set of those elements, i.e.,

$$\mathfrak{R}_0 = \left\{x' \in \mathcal{R}(S) \;\middle|\; \text{there exists } (x, x') \in S \text{ such that } \left\langle x', x\right\rangle = 0\right\}. \tag{3.11}$$

There exists an alternative characterisation of $\mathfrak{R}_0$ that we frequently make use of, as expressed in the following lemma.

Lemma 3.2.13 ([33, Lem. 8.1])**.** *Let S be a sectorial linear relation with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$. The set $\mathfrak{R}_0$ as described by equation (3.11) admits the following representation:*

$$\mathfrak{R}_0 = \mathcal{R}(S) \cap \operatorname{mul} S^*.$$

Since $\mathfrak{R}_0$ is contained within $\mathcal{R}(S)$, we naturally obtain the quotient space $\mathcal{R}(S)/\mathfrak{R}_0$. By factoring out these terms, we assert that the space $\mathcal{R}(S)/\mathfrak{R}_0$ endowed with the inner product

$$\left\langle [x'], [y']\right\rangle_{\mathcal{R}(S)/\mathfrak{R}_0} = \frac{1}{2}\left(\left\langle x', y\right\rangle + \left\langle x, y'\right\rangle\right), \qquad (x, x'),\, (y, y') \in S. \tag{3.12}$$

forms a pre-Hilbert space, where $[x']$ and $[y']$ denote the equivalence classes containing x' and y' respectively. Let $(H_S, \langle \cdot, \cdot\rangle_{H_S})$ denote the completion of the pre-Hilbert space $\mathcal{R}(S)/\mathfrak{R}_0$.

Now, define the form $\mathbf{b}'$ on $\mathcal{R}(S)/\mathfrak{R}_0$ by

$$\mathbf{b}'\left[[x'], [y']\right] = \frac{i}{2}\left(\left\langle x, y'\right\rangle - \left\langle x', y\right\rangle\right), \qquad (x, x'),\, (y, y') \in S, \tag{3.13}$$

where $[x'], [y'] \in Q(\mathbf{b}') = \mathcal{R}(S)/\mathfrak{R}_0$. Notably, this is a well-defined symmetric form: it is well-defined through an argument similar to that presented in the proof of Lemma 3.2.12. Furthermore, $\mathbf{b}'$ is a bounded form on H_S. Indeed, for (x, x'), $(y, y') \in S$, we see that

$$\begin{aligned}\left|\mathbf{b}'\left[[x'], [y']\right]\right| &= \left|\frac{i}{2}\left(\left\langle x', y\right\rangle - \left\langle x, y'\right\rangle\right)\right| \\ &\leq \frac{1}{2}\left(\left|\left\langle x', y\right\rangle\right| + \left|\left\langle y', x\right\rangle\right|\right).\end{aligned}$$

An application of inequality (3.10) then yields

$$|\mathbf{b}'\left[[x'],[y']\right]| \leq (1+\tan\alpha)\left(\operatorname{Re}\left\langle x',x\right\rangle\right)^{\frac{1}{2}}\left(\operatorname{Re}\left\langle y',y\right\rangle\right)^{\frac{1}{2}}. \tag{3.14}$$

Furthermore, the inner product given in equation (3.12) gives rise to the following equality:

$$\left\langle [x'],[x']\right\rangle_{\mathcal{R}(S)/\mathfrak{N}_0} = \operatorname{Re}\left\langle x',x\right\rangle, \qquad (x,x')\in S.$$

Then, we may insert this equation back into the inequality (3.14) to conclude that

$$\begin{aligned}|\mathbf{b}'\left[[x'],[y']\right]| &\leq (1+\tan\alpha)\|[x']\|_{\mathcal{R}(S)/\mathfrak{N}_0}\|[y']\|_{\mathcal{R}(S)/\mathfrak{N}_0}\\ &= (1+\tan\alpha)\|[x']\|_{H_S}\|[y']\|_{H_S},\end{aligned}$$

since $\|[x']\|_{\mathcal{R}(S)/\mathfrak{N}_0} = \|[x']\|_{H_S}$ for $[x']\in\mathcal{R}(S)/\mathfrak{N}_0$. As such, $\mathbf{b}'$ is a well-defined symmetric form that is bounded on $\mathcal{R}(S)/\mathfrak{N}_0$. Its closure $\overline{\mathbf{b}'}$ (henceforth called $\mathbf{b}$) is then a well-defined, closed symmetric form that is bounded on H_S, and so we may associate to it a bounded self-adjoint operator upon invoking the Riesz Representation Theorem. In particular, there exists a self-adjoint operator B_S, that is bounded on H_S, such that

$$\mathbf{b}\left[[x'],[y']\right] = \left\langle B_S[x'],[y']\right\rangle_{H_S}, \qquad (x,x'),\ (y,y')\in S. \tag{3.15}$$

This operator B_S is fundamental in the construction of the extremal maximal sectorial extensions of S, as we will see in the subsequent definitions and theorems.

First, we define the linear relation $U\subseteq H\times H_S$ by

$$U = \left\{(x,[x'])\in H\times H_S \;\middle|\; (x,x')\in S\right\}. \tag{3.16}$$

The relation U is merely a modification of S: the second component is now the corresponding element in $\mathcal{R}(S)/\mathfrak{N}_0\subseteq H_S$. Next, define the linear relation $V\subseteq H_S\times H$ by

$$V = \left\{((I+iB_S)[x'],x')\in H_S\times H \;\middle|\; (x,x')\in S\right\}, \tag{3.17}$$

where I is the identity operator on H_S and B_S is defined as in equation (3.15).

Remark. If B_S is the zero operator on H_S, then $VU=S$.

The relations U and V defined in this way enjoy several useful properties. Firstly, U is the graph of an operator. This is evident upon letting $(0,x')\in S$: it is clear that x' will then lie in $\mathfrak{N}_0$ and so $[x']=0$. Moreover,

$$\begin{aligned}\operatorname{mul}V &= \left\{x'\in\mathcal{R}(S)\;\middle|\;(I+iB_S)[x']=0 \text{ and } (x,x')\in S\right\}\\ &= \left\{x'\in\mathcal{R}(S)\;\middle|\;[x']=0 \text{ and } (x,x')\in S\right\},\end{aligned}$$

since $\ker(I + iB_S) = \{0\}$. Indeed, let $x \in \ker(I + iB_S)$. Then, since both I and B_S are self-adjoint operators, for any $y \in H_S$, we have

$$0 = \langle (I + iB_S)x, y \rangle_{H_S} = \langle x, (I - iB_S)y \rangle_{H_S} = \langle x, y \rangle_{H_S} + i \langle x, B_s y \rangle_{H_S}.$$

Upon specifying $y = x$, we see that

$$0 = \underbrace{\langle x, x \rangle_{H_S}}_{\geq 0} + i \underbrace{\langle x, B_s x \rangle_{H_S}}_{\in \mathbb{R}},$$

since $\langle \cdot, \cdot \rangle_{H_S}$ is an inner product. For this equality to hold, we must have that both terms equal zero and, in particular, $\langle x, x \rangle = 0$. Hence, x must equal 0, and so $\ker(I + iB_S) = \{0\}$.

Upon further inspection, the set $\operatorname{mul} V$ is precisely the set $\mathfrak{R}_0$ since, for $x' \in \mathcal{R}(S)$, we have $[x'] = 0 \iff x' \in \mathfrak{R}_0$. Furthermore, the relation V^* is the graph of an operator, however we first need the following lemma before we can prove this statement.

Lemma 3.2.14. *Let M be a dense subspace of a Hilbert space H and $T \colon H \to H$ a bounded linear operator. Then,*

$$\overline{\mathcal{R}(T \restriction M)} = \overline{\mathcal{R}(T)}.$$

Proof. Verifying that $\overline{\mathcal{R}(T \restriction M)} \subseteq \overline{\mathcal{R}(T)}$ is immediate: as $\mathcal{R}(T \restriction M) \subseteq \mathcal{R}(T)$, we have $\overline{\mathcal{R}(T \restriction M)} \subseteq \overline{\mathcal{R}(T)}$.

Conversely, to prove that $\overline{\mathcal{R}(T)} \subseteq \overline{\mathcal{R}(T \restriction M)}$, we begin by letting x be an element in $\overline{\mathcal{R}(T)}$. Then, there exists a sequence in $\mathcal{R}(T)$ that converges to x or, in other words, there exists a sequence $\{y_n\}$ in $\mathcal{D}(T) = H$ such that $Ty_n \to x$ as $n \to \infty$.

Fix an $n \in \mathbb{N}$. Since M is dense in H, there exists a sequence $\{z_{n,k}\}_k$ in M such that $\|z_{n,k} - y_n\| \to 0$ as $k \to \infty$. In particular, we may choose a subsequence of $z_{n,k}$ such that $\|z_{n,k} - y_n\| \leq \frac{1}{k}$ for all n and k. We need to show that there exists a sequence in $\mathcal{R}(T \restriction M)$ that converges to x, i.e., there exists a sequence $\{p_n\}$ in M such that $Tp_n \to x$ as $n \to \infty$.

Let $p_n = z_{n,n}$. Then,

$$\begin{aligned}
\|Tp_n - x\| = \|Tz_{n,n} - x\| &= \|Tz_{n,n} - Ty_n + Ty_n - x\| \\
&\leq \|Tz_{n,n} - Ty_n\| + \|Ty_n - x\| \\
&\leq \|T\| \|z_{n,n} - y_n\| + \|Ty_n - x\| \\
&\leq \frac{\|T\|}{n} + \|Ty_n - x\|,
\end{aligned}$$

since T is a bounded operator. Then, upon taking $n \to \infty$, we see that

$$\|Tp_n - x\| \leq \frac{\|T\|}{n} + \|Ty_n - x\| \to 0,$$

verifying that there exists a sequence in $\mathcal{R}(T \upharpoonright M)$ that converges to $x \in \mathcal{R}(T)$, as required. ■

With this lemma in hand, we may now prove the following statement.

Lemma 3.2.15. *The adjoint relation V^* of the linear relation V, as defined in equation (3.17), is the graph of an operator.*

Proof. To prove that V^* is the graph of an operator, we must first prove that V is densely defined, that is, $\overline{\mathcal{D}(V)} = H_S$. Since

$$\mathcal{D}(V) = \left\{ (I + iB_S)[x'] \;\middle|\; (x, x') \in S \right\},$$

this is equivalent to showing that

$$\overline{\mathcal{D}(V)} = \overline{\mathcal{R}((I + iB_S) \upharpoonright \mathcal{R}(S)/\mathfrak{R}_0)} = H_S.$$

Via the construction of the Hilbert space H_S, we immediately recall that the space $\mathcal{R}(S)/\mathfrak{R}_0$ is a dense subset of H_S. Then, with Lemma 3.2.14 in mind, we aim to show that $\mathcal{R}(I + iB_S)$ is a dense subset of H_S instead.

The Rank-Nullity theorem as given in Theorem 1.1.20 is fundamental in the proof: since $I + iB_S$ is a densely defined, bounded (thus closed) operator on H_S, we may conclude that

$$H_S = \overline{\mathcal{R}(I + iB_S)} \oplus \ker{(I + iB_S)^*}.$$

Therefore, if $\ker{(I + iB_S)^*} = \{0\}$, then it must be true that $\overline{\mathcal{R}(I + iB_S)} = H_S$. However, we have already shown that the kernel of $I + iB_S$ is trivial, and so we may follow an analogous argument to confirm that $\ker{(I + iB_S)^*} = \{0\}$ too. Thus, we may conclude that $\overline{\mathcal{R}(I + iB_S)} = H_S$ and so $\mathcal{D}(V)$ is dense in H_S after invoking Lemma 3.2.14.

Since $\mathcal{D}(V)$ is dense in H_S, it is then immediate that V^* is the graph of an operator upon recalling Lemma 3.1.7. ■

Furthermore, the linear relations U and V are such that $V \subseteq U^*$ and $U \subseteq V^*$, as detailed in the following lemma.

Lemma 3.2.16. *The linear relations U and V, as defined in equations (3.16) and (3.17) respectively, satisfy:*

$$V \subseteq U^* \qquad \textit{and} \qquad U \subseteq V^*.$$

Proof. First, we examine the expression $\langle [x'], (I+iB_S)[y'] \rangle_{H_S}$ for elements (x, x') and $(y, y') \in S$. Then,

$$\begin{aligned}
\left\langle [x'], (I+iB_S)[y'] \right\rangle_{H_S} &= \left\langle [x'], [y'] \right\rangle_{H_S} + \left\langle [x'], iB_S[y'] \right\rangle_{H_S} \\
&= \frac{1}{2}\left(\left\langle x', y \right\rangle + \left\langle x, y' \right\rangle \right) - i\overline{\langle B_S[y'], [x'] \rangle}_{H_S} \\
&= \frac{1}{2}\left(\left\langle x', y \right\rangle + \left\langle x, y' \right\rangle \right) - i\overline{\mathbf{b}\,[[y'], [x']]} \\
&= \frac{1}{2}\left(\left\langle x', y \right\rangle + \left\langle x, y' \right\rangle - \overline{\langle y, x' \rangle} + \overline{\langle y', x \rangle} \right) \\
&= \left\langle x, y' \right\rangle, \qquad (3.18)
\end{aligned}$$

courtesy of equations (3.12), (3.15) and (3.13) respectively.

Now, let $(z, z') \in S$ so that $((I+iB_S[z']), z') \in V$. Moreover, observe that U^* is of the form

$$U^* = \left\{ ([f'], g) \in H_S \times H \;\middle|\; \langle g, x \rangle = \left\langle [f'], [x'] \right\rangle_{H_S} \text{ for all } (x, [x']) \in U \right\}.$$

Then, the element $((I+iB_S)[z'], z')$ lies in U^* if and only if the equality $\langle z', x \rangle = \langle (I+iB_S)[z'], [x] \rangle_{H_S}$ holds for all $(x, [x']) \in U$. By unravelling the right-hand side of this expression, we see that this is, in fact, true. Indeed,

$$\begin{aligned}
\left\langle (I+iB_S)[z'], x' \right\rangle_{H_S} &= \overline{\langle [x'], (I+iB_S)[z'] \rangle}_{H_S} \\
&= \overline{\langle x, z' \rangle} \\
&= \left\langle z', x \right\rangle,
\end{aligned}$$

by equation (3.18), showing that $V \subseteq U^*$.

Likewise, let $(z, z') \in S$ such that $(z, [z']) \in U$ and observe that

$$V^* = \left\{ (f, [g']) \in H \times H_S \;\middle|\; \begin{array}{r} \left\langle [g'], (I+iB_S)[x'] \right\rangle_{H_S} = \left\langle f, x' \right\rangle \\ \text{for all } ((I+iB_S)[x'], x') \in V. \end{array} \right\}$$

Then, $(z, [z']) \in V^*$ if and only if the equality $\langle [z'], (I+iB_S)[x'] \rangle_{H_S} = \langle z, x' \rangle$ holds for all $((I+iB_S)[x'], x') \in V$. This time, the result is immediate from equation (3.18), and so we may conclude that showing that $U \subseteq V^*$. ∎

Since the adjoint S^* of any relation S is closed and for any two relations S and T, we have

$$S \subseteq T \implies T^* \subseteq S^*,$$

we see that Lemma 3.2.16 admits a useful consequence. The relation V^* is the closed graph of an operator, thus any restriction of V is closable: in particular, U is closable. Then, as $V \subseteq U^*$, we have $\overline{U} = U^{**} \subseteq V^*$. This inclusion demonstrates that U^{**} is also the closed graph of an operator. Furthermore, since the range of U satisfies $\mathcal{R}(U) = \mathcal{R}(S)/\mathfrak{R}_0$, it is clear that $\mathcal{R}(U)$ is dense in H_S by its very construction; we may then conclude that $\mathcal{R}(U^{**})$ is dense in H_S, since $\mathcal{R}(U) \subseteq \mathcal{R}(U^{**})$. Moreover, it is then immediate that $\ker U^* = \{0\}$ by the Rank-Nullity theorem as given in Theorem 1.1.20.

The self-adjoint operator B_S and the relations U and V^* are fundamental to describing the extremal maximal sectorial extensions of S — the properties that we described are imperative to the construction. Essentially, if we are in possession of some closed linear operator $T\colon H \to H_S$ whose graph $\mathcal{G}(T)$ satisfies

$$U \subseteq \mathcal{G}(T) \subseteq V^*,$$

then we can construct the extremal maximal sectorial extension of S associated to T. Formally, we have the following theorem.

Theorem 3.2.17 ([33, Thm. 8.4])**.** *Let H be a Hilbert space and $S \subseteq H \times H$ a sectorial relation with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$. Furthermore, let B_S, U and V be defined by equations (3.15), (3.16) and (3.17) respectively. There is a one-to-one correspondence between extremal maximal sectorial extensions $\tilde{S}$ of S and closed linear operators T whose graph $\mathcal{G}(T)$ satisfies*

$$U \subseteq \mathcal{G}(T) \subseteq V^*.$$

If $\tilde{S}$ is associated to such an operator T, then

$$\tilde{S} = \mathcal{G}(T)^* \mathcal{G}(I + iB_S)\mathcal{G}(T) \subseteq H \times H,$$

where $\mathcal{G}(T)^*$ is the adjoint relation of $\mathcal{G}(T)$. Furthermore, T induces the closed form $\tilde{\mathfrak{s}}$ which satisfies

$$\tilde{\mathfrak{s}}\,[x, y] = \langle (I + iB_S)Tx, Ty \rangle_{H_S}$$

for elements x, $y \in Q(\tilde{\mathfrak{s}}) = \mathcal{D}(T)$.

Since it was previously shown that the Friedrichs and Kreĭn extensions of S were extremal, one can ask how these extensions are to be constructed using Theorem 3.2.17. Observe that since $U \subseteq U^{**} \subseteq V^*$, U^{**} is the smallest possible closed relation that would make an appropriate choice for $\mathcal{G}(T)$. Likewise, the largest valid closed relation is V^* itself. Taking the graph of the operator T to be either of these relations proves fruitful, as the following theorem details.

Theorem 3.2.18 ([33, Thm. 8.3])**.** *Let H be a Hilbert space and $S \subseteq H \times H$ a sectorial relation with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$. Furthermore,* *let B_S, U and V be defined by equations (3.15), (3.16) and (3.17) respectively. The Friedrichs extension S_F of S is given by*

$$S_F = U^* \mathcal{G}(I + iB_S)U^{**}, \tag{3.19}$$

and the form $\mathbf{s_F}$ associated to S_F is given by

$$\mathbf{s_F}[x,y] = \left\langle (I+iB_S)\tilde{U}^{**}x, \tilde{U}^{**}y \right\rangle_{H_S}, \quad x, y \in Q(\mathbf{s_F}) = \mathcal{D}(U^{**}), \tag{3.20}$$

where $\tilde{U}^{**}$ is the operator associated to U^{**}. The Kreĭn extension S_K of S is given by

$$S_K = V^{**}\mathcal{G}(I + iB_S)V^*, \tag{3.21}$$

and the form $\mathbf{s_K}$ associated to S_K is given by

$$\mathbf{s_K}[x,y] = \left\langle (I+iB_S)\tilde{V}^{*}x, \tilde{V}^{*}y \right\rangle_{H_S}, \quad x, y \in Q(\mathbf{s_K}) = \mathcal{D}(V^{*}), \tag{3.22}$$

where $\tilde{V}^*$ is the operator associated to V^*.

We conclude this chapter by remarking that the Friedrichs and Kreĭn extensions of a sectorial relation S are likely themselves *relations* rather than graphs of an operator. This is clear: there is no guarantee that U will be densely defined so U^* would not be the graph of an operator. Conversely, it is clear that V^{**} may not be the graph on an operator — $\operatorname{mul} V = \mathfrak{R}_0$, after all. However, equations (3.20) and (3.22) together are most illuminating. Since $U^{**} \subseteq V^*$, it must be true that $\mathcal{D}(U^{**}) \subseteq \mathcal{D}(V^*)$. Then, when we consider the *forms* associated to these two extremal extensions, we quickly uncover that the form associated to the Friedrichs extension has the smallest feasible domain, whilst the Kreĭn, the largest! This revelation is consistent with the construction of the Friedrichs and Kreĭn extension as in Chapter 1, and so our decision to study linear relations is, once again, vindicated.

Chapter 4

Maximal Sectorial Extensions of the Discrete Laplacian

4.1 The Discrete Laplacian

The main results presented in [33], as expressed during Section 3.2.3, show how one can construct all of the extremal maximal sectorial extensions of a sectorial relation S. This result is abstract and general. In this section, we apply such results to two particular examples as a means of providing insight into the theory. The main motivation is as follows: if our sectorial relations are of a specific form, then can we say more about their extremal maximal sectorial extensions?

In all that follows, we will work in the ambient Hilbert space $H = \ell^2$, and make reference to the subspace of ℓ^2 whose elements have first component equal to zero, i.e.,

$$\ell_0^2 = \left\{ x \in \ell^2 \;\middle|\; x_0 = 0 \right\}.$$

Consider the operator $J \colon \ell^2 \to \ell^2$ such that

$$(Jx)_n = -\Delta(\Delta x_{n-1}) = -x_{n+1} + 2x_n - x_{n-1},$$

for all $n \geq 0$. This linear operator — known as the *Discrete Laplacian*, at least, up to a shift — has many favourable properties: it is closed, densely defined, and bounded. Furthermore, J is self-adjoint and its spectrum $\sigma(J)$ is precisely the closed interval $[0, 4]$ as shown in the following lemma.

Remark. In Chapter 2, our attention was on operators of the form

$$(Tx)_n = -\Delta\left(p_{n-1}\Delta x_{n-1}\right) + q_n x_n, \qquad x \in \ell^2,$$

for two real sequences $\{p_n\}$ and $\{q_n\}$ with $p_n > 0$ for all $n \in \mathbb{N}_0$ and $p_{-1} \equiv 0$: we can express J in this form too. To be consistent with this convention, we should take $p_n = 1$ for all $n \in \mathbb{N}_0$, whilst $\{q_n\}$ is the sequence with $q_n = 0$ for $n > 1$ and $q_0 = 1$. This is clear upon expanding $(Tx)_n$: indeed,

$$-\Delta\left(p_{n-1}\Delta x_{n-1}\right) + q_n x_n = -p_n x_{n+1} + (p_n + p_{n-1} + q_n)x_n - p_{n-1}x_{n-1}.$$

However, we may equivalently set $p_n = 1$ for all $n \geq -1$ and $q_n = 0$ for all $n \in \mathbb{N}_0$. Both of these forms have value, and so we draw special attention to it here.

Lemma 4.1.1. *The operator $J\colon \ell^2 \to \ell^2$, where $(Jx)_n = -\Delta(\Delta x_{n-1})$ for all $n \geq 0$, is self-adjoint. Moreover, the spectrum of J is given by $\sigma(J) = [0, 4]$.*

Remark. Although this result is well known, we choose to present the proof in full detail because it introduces fundamental techniques and concepts that will be used throughout the chapter to come.

Proof. First, we show that J is self-adjoint. Then, consider the equality

$$\langle Jx, y\rangle = \langle x, J^*y\rangle\,,$$

for $x \in \mathcal{D}(J) = \ell^2$ and $y \in \mathcal{D}(J^*)$. We aim to use the summation by parts formula, as expressed in Lemma 1.3.3, on the left-hand side twice to obtain an expression for J^* before determining the elements for which it is valid. Then, for some fixed $N \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{n=0}^{N}(Jx)_n\overline{y}_n &= -\sum_{n=0}^{N}\Delta(\Delta x_{n-1})\overline{y}_n\\ &= -\left[\Delta x_N\overline{y}_{N+1} - \Delta x_{-1}\overline{y}_0 - \sum_{n=0}^{N}\Delta x_n\Delta\overline{y}_n\right]\\ &= -\Delta x_N\overline{y}_{N+1} + \Delta x_{-1}\overline{y}_0 + \sum_{n=0}^{N}\Delta x_n\Delta\overline{y}_n.\end{aligned}$$

Additionally, we have

$$\begin{aligned}\sum_{n=0}^{N}\Delta x_n\Delta\overline{y}_n &= x_{N+1}\Delta\overline{y}_{N+1} - x_0\Delta\overline{y_0} - \sum_{n=0}^{N}x_{n+1}\Delta(\Delta\overline{y}_n)\\ &= x_{N+1}\Delta\overline{y}_{N+1} - x_0\Delta\overline{y_0} - \sum_{n=1}^{N+1}x_n\Delta(\Delta\overline{y}_{n-1}).\end{aligned}$$

Note that

$$-x_0\Delta(\Delta\overline{y}_{-1}) = \Delta x_{-1}\overline{y}_0 - x_0\Delta\overline{y}_0,$$

so we may conclude that

$$\sum_{n=0}^{N}(Jx)_n\overline{y}_n = -\Delta x_N\overline{y}_{N+1} + x_{N+1}\Delta\overline{y}_{N+1} - \sum_{n=0}^{N+1}x_n\Delta(\Delta\overline{y}_{n-1}). \tag{4.1}$$

Observe that for any two sequences $\{x_n\}$ and $\{y_n\}$ that lie in ℓ^2, we have

$$\begin{aligned} |\Delta x_N \overline{y}_{N+1}| &\leq |x_{N+1}\overline{y}_{N+1}| + |x_N \overline{y}_{N+1}| \\ &\leq |x_{N+1}|^2 + 2|\overline{y}_{N+1}|^2 + |x_N|^2, \end{aligned}$$

by means of the binomial formula. Then, the terms on right-hand side tend to 0 as $N \to \infty$ by virtue of the sequences lying in ℓ^2. Hence

$$\lim_{N\to\infty} \Delta x_N \overline{y}_{N+1} = 0,$$

Likewise, $\lim_{N\to\infty} x_{N+1}\Delta\overline{y}_{N+1} = 0$. Then, upon taking $N \to \infty$ in equation (4.1), we see that

$$\langle Jx, y\rangle = -\sum_{n=0}^{\infty} x_n \Delta(\Delta\overline{y}_{n-1}) = \sum_{n=0}^{\infty} x_n (Jy)_n.$$

As this equality holds for any $y \in \ell^2$, we may conclude that $J = J^*$.

In order to determine the spectrum of J, we first recall the spectral equation:

$$Jx = \lambda x, \qquad \lambda \in \mathbb{C}.$$

Then, we are able to determine the spectrum of J by using subordinacy theory — we will investigate the growth of the fundamental solutions to this equation, in conjunction with [38, Thm. 3]. We can fully characterise the spectrum of J after checking which, if any, of the solutions are *subordinate*: essentially, we wish to identify the solutions that decay. In particular, we have the following three fundamental statements:

- if there exists a decaying solution to the equation $Jx = \lambda x$ *and* it satisfies the initial condition, then λ belongs to the spectrum. Moreover, λ is an eigenvalue.
- if there exists a decaying solution to the equation $Jx = \lambda x$ but it does *not* satisfy the initial condition, then λ does not belong to the spectrum.
- if there does not exist a decaying solution to the equation $Jx = \lambda x$ then λ belongs to the spectrum, however λ is *not* an eigenvalue.

Then, with this result in mind, we merely have to analyse the growth of the solutions to the spectral equation in order to prove that $\sigma(J) = [0, 4]$. First, we note that

$$\begin{cases} -x_1 + 2x_0 = \lambda x_0, & n = 0, \\ -x_{n+1} + 2x_n - x_{n-1} = \lambda x_n, & n \geq 1, \end{cases}$$

implies that

$$\begin{cases} x_1 = (2-\lambda)x_0, & n = 0, \\ x_{n+1} = (2-\lambda)x_n - x_{n-1}, & n \geq 1. \end{cases}$$

Then, we may introduce *transfer matrices* to conclude that, for $n \geq 0$,

$$\begin{aligned} \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 2-\lambda \end{pmatrix}^n \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}. \end{aligned}$$

Since the characteristic polynomial of the transfer matrix $M = \begin{pmatrix} 0 & 1 \\ -1 & 2-\lambda \end{pmatrix}$ is given by

$$\det(M - \lambda I) = (-z)(2-\lambda-z) + 1 = z^2 - z(2-\lambda) + 1,$$

the eigenvalues $z_\pm$ are precisely the solutions to the characteristic equation

$$z^2 - z(2-\lambda) + 1 = 0.$$

Then,

$$z_\pm = \frac{2-\lambda \pm \sqrt{4 - 4\lambda + \lambda^2 - 4}}{2} = \frac{2-\lambda \pm \sqrt{\lambda(\lambda-4)}}{2},$$

and, notably, satisfy $z_+ z_- = 1$. Clearly, if $\lambda \in \mathbb{C} \setminus \{0, 4\}$, then the two eigenvalues are distinct.

Since $z_+ z_- = 1$, either

$$|z_+| = |z_-| = 1 \qquad \text{or} \qquad |z_+| > 1 \text{ and } |z_-| < 1. \tag{4.2}$$

Note that we may simply relabel the solutions if this is not the case. We begin by assuming that $|z_+| = |z_-| = 1$. Then, as $|z_+| = 1$, we may set $z_+ = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Hence,

$$z_- = \frac{1}{z_+} = \frac{1}{e^{i\theta}} = e^{-i\theta} = \overline{z}_+.$$

Furthermore, $z_- = \overline{z}_+$ implies that

$$z_+ + z_- = z_+ + \overline{z}_+ = 2 \operatorname{Re} z_+$$

and

$$z_+ + z_- = \frac{2-\lambda + \sqrt{\lambda(\lambda-4)}}{2} + \frac{2-\lambda - \sqrt{\lambda(\lambda-4)}}{2} = 2 - \lambda.$$

Together, these two equalities show that

$$|z_+| = |z_-| = 1 \implies \lambda = 2 - 2 \operatorname{Re} z_+. \tag{4.3}$$

In particular, λ is necessarily real-valued — if $\lambda \in \mathbb{C}$, then we must have that $|z_+| > 1$ and $|z_-| < 1$ instead. However, this is clearly not true for every $\lambda \in \mathbb{R}$: we must determine which values of λ ensure that $|z_+| = |z_-| = 1$. In fact, this is true precisely when $\lambda \in [0,4]$. Indeed, it is clear to see that when $\lambda = 0$ and $\lambda = 4$, we have $z_\pm = 1$ and $z_\pm = -1$ respectively. Furthermore, if $0 < \lambda < 4$, then $\sqrt{\lambda(\lambda-4)} < 0$, and so

$$\left|\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right| = \left|\frac{2-\lambda+i\sqrt{\lambda(4-\lambda)}}{2}\right| = \sqrt{\frac{(2-\lambda)^2}{4}+\frac{\lambda(4-\lambda)}{4}}$$
$$= \sqrt{\frac{4-4\lambda+\lambda^2+4\lambda-\lambda^2}{4}} = 1.$$

On the other hand, if $\lambda \in \mathbb{R} \setminus [0,4]$, then $\sqrt{\lambda(\lambda-4)} > 0$; this shows that z_+ is entirely real, and so equation (4.3) will not hold. Hence $|z_\pm| \neq 1$. As such, we have accounted for the entire complex plane and so we may conclude that the four sets $I_1 = \mathbb{C} \setminus [0,4]$, $I_2 = (0,4)$, $I_3 = \{0\}$ and $I_4 = \{4\}$ exhibit radically different behaviour, and so they must be considered individually.

If $\lambda \in I_1 = \mathbb{C} \setminus [0,4]$, then we know from (4.2) that there exists a decaying solution. However, the initial condition given by $x_1 = (2-\lambda)x_0$ will *not* be satisfied, as a self-adjoint operator cannot have complex eigenvalues. Hence, if $\lambda \in I_1$, then λ does not belong to the spectrum of J. On the other hand, if $\lambda \in I_2 = (0,4)$, then neither solution will decay. As such, it is immediate that these values of λ belong to spectrum of J. Finally, when $\lambda = 0$ or $\lambda = 4$, we have $\sqrt{\lambda(\lambda-4)} = 0$ and so the eigenvalues of the transfer matrix M are repeated. However, we readily observe that the general solution to the spectral equation is given by $x_n = c_1 + c_2 n$ and $x_n = (-1)^n(c_1 + c_2 n)$, for constants c_1 and c_2, respectively. Since neither of the fundamental solutions decay, we see that $\lambda = 0$ and $\lambda = 4$ also belong to the spectrum. Thus, having now accounted for all values of $\lambda \in \mathbb{C}$, we may conclude that $\sigma(J) = [0,4]$, as required. ■

Remark. The spectrum of J is clearly real. However, [38, Thm. 3] also shows that $\sigma(J)$ is entirely continuous, i.e., $\sigma(J) = \sigma_c(J)$ and J has no eigenvalues.

Let $\tilde{J}$ be the restriction of J to the domain $\mathcal{D}(\tilde{J}) = \ell_0^2$, that is, $J \restriction \ell_0^2 = \tilde{J}$. Then, the two relations of interest to us are defined as follows:

$$S_1 = \left\{(x, Jx) \in \ell^2 \times \ell^2 \;\middle|\; x \in \ell^2\right\}$$

and

$$S_2 = \left\{(x, \tilde{J}x) \in \ell^2 \times \ell^2 \;\middle|\; x \in \ell_0^2\right\},$$

where $\tilde{J} = J \restriction \ell_0^2$ and $(Jx)_n = -\Delta(\Delta x_{n-1})$ for $n \geq 0$. Observe that S_1 and S_2 are the graphs of the operator J and $\tilde{J}$ respectively. Furthermore, we stress that we are to interpret S_2 as a relation from ℓ^2 to ℓ^2 and, in particular, $S_2 \subseteq S_1$ since $\ell_0^2 \subseteq \ell^2$.

These relations were chosen specifically: S_1 is perhaps an obvious choice for some second-order difference operator J, whereas the adjoint relation of S_2 will be multi-valued, since ℓ_0^2 is not dense in ℓ^2 — we will be able to see how *linear relations* play a part in the theory. The chapter to come will be divided as follows. First, we will construct the Friedrichs extension, $S_{1,F}$, and Kreĭn extension, $S_{1,K}$, of S_1 by utilising Lemma 3.2.10 and Theorem 3.2.8, before working through analogous computations for $S_{2,F}$ and $S_{2,K}$ — the Friedrichs and Kreĭn extensions of S_2. Once we are in possession of these extensions, we aim to utilise Theorem 3.2.17 to both corroborate our findings and express *all* extremal maximal sectorial extensions of S_1 and S_2. Since we merely apply the theory to these specific examples, we conclude this chapter by reflecting upon the theory and computations presented; in particular, this outlook addresses more general class of second-order difference operators J and potential future works.

4.2 The Friedrichs and Kreĭn Extension of S_1

During this section, we aim to construct both the Friedrichs extension and the Kreĭn extension of S_1, where

$$S_1 = \left\{ (x, Jx) \in \ell^2 \times \ell^2 \;\middle|\; x \in \ell^2 \right\}, \tag{4.4}$$

for the second-order difference operator J, where $(Jx)_n = -\Delta(\Delta x_{n-1})$ for $n \geq 0$. Whilst the results of S_1 will be of little surprise to experts in the field, the constructions demonstrate how to proceed in the simple case and will prove enlightening for the more interesting, complicated example S_2.

First we must verify that this relation is actually a reasonable choice: can we apply the relevant theory to S_1? To begin with, we show that S_1 is a sectorial relation with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{2}{\pi}\right)$ — we may then associate to it a well-defined, closable sectorial form via Lemma 3.2.10.

Lemma 4.2.1. *The relation S_1, as defined by equation (4.4), is sectorial with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$.*

Proof. In order to show that S_1 is sectorial with the required vertex and semi-angle, we simply need to show that

$$|\operatorname{Im} \langle Jx, x \rangle| \leq (\tan \alpha) \operatorname{Re} \langle Jx, x \rangle$$

for all $(x, Jx) \in S_1$. To do this, we apply the summation by parts formula given in Lemma 1.3.3 to the expression $\langle Jx, x\rangle$. Indeed, observe that

$$\begin{aligned}\sum_{n=0}^{N}(Jx)_n\overline{x}_n &= -\sum_{n=0}^{N}\Delta(\Delta x_{n-1})\overline{x}_n \\ &= -\left[\Delta x_N\overline{x}_{N+1} - \Delta x_{-1}\overline{x}_0 - \sum_{n=0}^{N}\Delta x_n\Delta\overline{x}_n\right] \\ &= -\Delta x_N\overline{x}_{N+1} + \sum_{n=0}^{N}\left|\Delta x_n\right|^2 + \left|x_0\right|^2,\end{aligned}$$

after recalling that x_{-1} is defined to be 0. By taking $N \to \infty$, we see that

$$\langle Jx, x\rangle = \sum_{n=0}^{\infty}\left|\Delta x_n\right|^2 + \left|x_0\right|^2,$$

since $\lim_{N\to\infty}\Delta x_N\overline{x}_{N+1} = 0$.

Clearly, $\langle Jx, x\rangle$ is an entirely real, positive quantity, verifying that

$$0 = |\operatorname{Im}\langle Jx, x\rangle| \le (\tan\alpha)\operatorname{Re}\langle Jx, x\rangle$$

for all $(x, Jx) \in S_1$. Thus S_1 is a sectorial relation with vertex $\gamma = 0$ and any semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$. ■

Remark. We will take $\alpha = 0$ since this provides the most information: under this construction, the numerical range lies exclusively on the positive real axis.

In fact, the computations within this proof lead nicely into the following section. We have shown that we can associate a well-defined, closable sectorial form to S_1 by means of Lemma 3.2.10; the closure of this form is instrumental in the construction of the form $\mathbf{s_{1,F}}$ associated to the Friedrichs extension $S_{1,F}$. The next section aims to construct $S_{1,F}$ by following this argument.

4.2.1 The Friedrichs Extension of S_1

Since the linear relation S_1 defined by equation (4.4) is a sectorial relation, we continue by defining the form $\mathbf{s_1}$ associated to it. In particular, the form $\mathbf{s_1}$ has form domain $Q(\mathbf{s_1}) = \mathcal{D}(S_1) = \ell^2$ and is defined by

$$\begin{aligned}\mathbf{s_1}\left[x, y\right] &= \langle Jx, y\rangle \\ &= \sum_{n=0}^{\infty}\Delta x_n\Delta\overline{y}_n + x_0\overline{y}_0, \qquad (x, Jx),\ (y, Jy) \in S_1,\end{aligned}$$

after an application of the summation by parts formula, as we demonstrated in the previous section. The closure of this well-defined sectorial form is then $\mathbf{s_{1,F}}$: the form associated to the Friedrichs extension. The following lemma shows that $\mathbf{s_1}$ is already closed, i.e., $\mathbf{s_1} = \mathbf{s_{1,F}}$.

Lemma 4.2.2. *The form* $\mathbf{s_1}$ *with domain* $Q(\mathbf{s_1}) = \ell^2$ *is a closed form.*

Proof. Let $x \in \ell^2$ and let $\{x_n\}$ be a sequence in $Q(\mathbf{s_1})$ such that $x_n \to_{\mathbf{s_1}} x$. The proof then consists of two steps: we must show that $x \in Q(\mathbf{s_1})$ and that $\mathbf{s_1}[x_n - x, x_n - x] \to 0$ as $n \to \infty$.

The first step is trivial: since the form domain of $\mathbf{s_1}$ *is* ℓ^2, there is nothing to show. Now, in order to verify that the second condition holds, we will consider the expression $|\mathbf{s_1}[x,y]|$ and show that it is bounded. Hence,

$$\begin{aligned}
|\mathbf{s_1}[x,y]| &= \left|\sum_{n=0}^{\infty}\left[\Delta x_n \Delta \overline{y}_n\right] + x_0\overline{y}_0\right| \\
&= \left|\sum_{n=0}^{\infty}\left[x_{n+1}\overline{y}_{n+1} - x_{n+1}\overline{y}_n - x_n\overline{y}_{n+1} + x_n\overline{y}_n\right] + x_0\overline{y}_0\right| \\
&\le \sum_{n=0}^{\infty}\left[\left|x_{n+1}\overline{y}_{n+1}\right| + |x_{n+1}\overline{y}_n| + \left|x_n\overline{y}_{n+1}\right| + |x_n\overline{y}_n|\right] + |x_0\overline{y}_0| \\
&= \sum_{n=0}^{\infty}\left[|x_n\overline{y}_n| + |x_{n+1}\overline{y}_n| + \left|x_n\overline{y}_{n+1}\right| + |x_n\overline{y}_n|\right].
\end{aligned}$$

We may then set $\tilde{x}_n = x_{n+1}$ and $\tilde{y}_n = y_{n+1}$ and use the Cauchy-Schwarz inequality to conclude that

$$\begin{aligned}
|\mathbf{s_1}[x,y]| &\le 2\|x\|\|y\| + \|\tilde{x}\|\|y\| + \|x\|\|\tilde{y}\| \\
&\le 4\|x\|\|y\|.
\end{aligned}$$

Using this inequality, we can see that

$$|\mathbf{s_1}[x_n - x, x_n - x]| \le 4\|x_n - x\|^2.$$

However, we know that $x_n \to x$ in ℓ^2 since $x_n \to_{\mathbf{s_1}} x$, and so $\|x_n - x\|^2 \to 0$, proving that

$$|\mathbf{s_1}[x_n - x, x_n - x]| \to 0$$

as $n \to \infty$. Hence, $\mathbf{s_1}$ is a closed form. ■

With this lemma in hand, we may then conclude that $\mathbf{s_1} = \mathbf{s_{1,F}}$. As such, the maximal sectorial relation associated to $\mathbf{s_{1,F}}$ is the Friedrichs extension of S_1. As was discussed during Section 3.2.2, we may construct the unique maximal relation associated to a closed sectorial form by following the steps outlined in the proof of Theorem 3.2.8; the remainder of this section is devoted to following this construction in a manner consistent with Section 3.2.2.

Immediately we note that $\left(Q(\mathbf{s_{1,F}}), \|\cdot\|_{\mathbf{s_{1,F}}}\right)$ is a Hilbert space, where the norm $\|\cdot\|_{\mathbf{s_{1,F}}}$ is induced by the inner product given by

$$\langle x, y\rangle_{\mathbf{s_{1,F}}} = \mathbf{s_{1,F}}[x,y] + \langle x, y\rangle, \qquad x,\, y \in Q(\mathbf{s_{1,F}}) = \ell^2,$$

since $(\mathbf{s_{1,F}})_{\mathrm{Re}} = \mathbf{s_{1,F}}$. Let $\hat{\mathbf{s}}_{\mathbf{1},\mathbf{F}}$ be the form such that

$$\hat{\mathbf{s}}_{\mathbf{1},\mathbf{F}} = \mathbf{s_{1,F}} + 1 \qquad \text{and} \qquad Q(\hat{\mathbf{s}}_{\mathbf{1},\mathbf{F}}) = Q(\mathbf{s_{1,F}}) = \ell^2.$$

This form is bounded in $\big(Q(\mathbf{s_{1,F}}), \|\cdot\|_{\mathbf{s_{1,F}}}\big)$, so there exists a bounded linear operator $B_1\colon Q(\mathbf{s_{1,F}}) \to Q(\mathbf{s_{1,F}})$ such that

$$\hat{\mathbf{s}}_{\mathbf{1},\mathbf{F}}\,[x, y] = \langle B_1 x, y\rangle_{\mathbf{s_{1,F}}}\,, \qquad x,\, y \in Q(\mathbf{s_{1,F}}). \tag{4.5}$$

If we rewrite both sides of equation (4.5), then it can be seen that

$$\hat{\mathbf{s}}_{\mathbf{1},\mathbf{F}}\,[x, y] = \mathbf{s_{1,F}}\,[x, y] + \langle x, y\rangle$$

and

$$\langle B_1 x, y\rangle_{\mathbf{s_{1,F}}} = \mathbf{s_{1,F}}\,[B_1 x, y] + \langle B_1 x, y\rangle\,.$$

Therefore, we may take B_1 to be the identity operator on $Q(\mathbf{s_{1,F}}) = \ell^2$. Then, with the operator B_1 firmly established, all that remains is to construct the operator A_1 as described in Section 3.2.2. Once we have done so, the unique maximal sectorial relation associated to $\mathbf{s_{1,F}}$ will be given by the set whose form is given in equation (3.7).

Let k, $\hat{\omega} \in Q(\mathbf{s_{1,F}})$ and $\omega \in \ell^2$ and recall equation (3.5), that is,

$$\langle k, \omega\rangle = \langle k, \hat{\omega}\rangle_{\mathbf{s_{1,F}}}\,.$$

Our objective will be to express $\hat{\omega}$ in terms of ω, knowing in advance that

$$A_1\omega = B_1^{-1}\hat{\omega} = \hat{\omega}.$$

In particular, we have that

$$\begin{aligned}\sum_{n=0}^{\infty} k_n \overline{\omega}_n &= \mathbf{s_{1,F}}\,[k, \hat{\omega}] + \langle k, \hat{\omega}\rangle \\ &= \sum_{n=0}^{\infty} \big[\Delta k_n \Delta\overline{\hat{\omega}}_n + k_n \overline{\hat{\omega}}_n\big] + k_0\overline{\hat{\omega}}_0.\end{aligned} \tag{4.6}$$

Thus, we need to isolate k_n by making use of the summation by parts formula. Then, the finite sum $\sum_{n=0}^{N} \Delta k_n \Delta\overline{\hat{\omega}}_n$ can be rewritten as

$$\begin{aligned}\sum_{n=0}^{N} \Delta k_n \Delta\overline{\hat{\omega}}_n &= \Delta\overline{\hat{\omega}}_{N+1} k_{N+1} - \Delta\overline{\hat{\omega}}_0 k_0 - \sum_{n=0}^{N}\big[k_{n+1}\Delta(\Delta\overline{\hat{\omega}}_n)\big] \\ &= \Delta\overline{\hat{\omega}}_{N+1} k_{N+1} - \Delta\overline{\hat{\omega}}_0 k_0 - \sum_{n=1}^{N+1}\big[k_n\Delta(\Delta\overline{\hat{\omega}}_{n-1})\big] \\ &= \Delta\overline{\hat{\omega}}_{N+1} k_{N+1} - \sum_{n=0}^{N+1}\big[k_n\Delta(\Delta\overline{\hat{\omega}}_{n-1})\big] - k_0\overline{\hat{\omega}}_0,\end{aligned}$$

by shifting the indices in the summation and recalling that both $k_{-1} = 0$ and $\overline{\hat{\omega}}_{-1} = 0$. If we let $N \to \infty$, then we see that

$$\sum_{n=0}^{\infty} \Delta k_n \Delta \overline{\hat{\omega}}_n = -\sum_{n=0}^{\infty} \left[k_n \Delta(\Delta \overline{\hat{\omega}}_{n-1}) \right] - k_0 \overline{\hat{\omega}}_0,$$

since

$$\lim_{N \to \infty} \Delta \overline{\hat{\omega}}_{N+1} k_{N+1} = 0.$$

By inserting this result back into equation (4.6), we see that

$$\sum_{n=0}^{\infty} k_n \overline{\omega}_n = \sum_{n=0}^{\infty} k_n \left[-\Delta(\Delta \overline{\hat{\omega}}_{n-1}) + \overline{\hat{\omega}}_n \right],$$

or, in other words, $\omega_n = \begin{cases} -\hat{\omega}_1 + 3\hat{\omega}_0, & n = 0, \\ -\hat{\omega}_{n+1} + 3\hat{\omega}_n - \hat{\omega}_{n-1}, & n \geq 1. \end{cases}$

We are now in possession of a second-order recurrence relation which expresses ω_n in terms of $\hat{\omega}_n$, for all $n \in \mathbb{N}_0$. We aim to solve the associated homogeneous recurrence relation before constructing a particular solution to the system. By using the variation of constants technique we will, in fact, derive an expression for $\hat{\omega}_n$ in terms of ω_n, just as we require.

Then, we begin by noting that the associated homogeneous equations is given by:

$$\begin{cases} \hat{\omega}_1 = 3\hat{\omega}_0, & n = 0, \\ \hat{\omega}_{n+1} = 3\hat{\omega}_n - \hat{\omega}_{n-1}, & n \geq 1. \end{cases}$$

We can proceed by rewriting this system of equations by, once again, introducing transfer matrices. Then, for $n \geq 0$, we have

$$\begin{aligned} \begin{pmatrix} \hat{\omega}_n \\ \hat{\omega}_{n+1} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \hat{\omega}_{n-1} \\ \hat{\omega}_n \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}^n \begin{pmatrix} \hat{\omega}_0 \\ \hat{\omega}_1 \end{pmatrix}. \end{aligned}$$

Since this recurrence relation has constant coefficients, the general solution is given by

$$\hat{\omega}_n = c_1 \lambda_+^n + c_2 \lambda_-^n, \qquad n \geq 0,$$

where λ_+ and λ_- are the eigenvalues of the transfer matrix $M = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$, and c_1 and c_2 are constants to be determined. By solving the characteristic equation

$$\det(M - \lambda I) = 0$$

for λ, we see that the eigenvalues of M are given by $\lambda_\pm = \frac{3 \pm \sqrt{5}}{2}$.

Now that we are in possession of the homogeneous solution, we are able to construct a particular solution by using the variation of constants technique as described in Lemma 1.3.4.

Remark. Note that we can express our recurrence relation in terms of a Jacobi operator. In particular, we see that the homogeneous problem is equivalent to

$$\begin{pmatrix} 3 & -1 & & & \\ -1 & 3 & -1 & & \\ & -1 & 3 & -1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \hat{\omega}_0 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

This equation has a fundamental system of solutions given by $\{\varphi_+, \varphi_-\}$ where $(\varphi_\pm)_n = \lambda_\pm^n$. Then, a particular solution $\tilde{\omega}$ of the equation $J\tilde{\omega} = \omega$ can be constructed using the variation of constants formula. In particular,

$$\tilde{\omega}_n = \lambda_+^n \sum_{r=0}^{n-1} \frac{\lambda_-^r \omega_r}{W_r(\varphi_+, \varphi_-)} - \lambda_-^n \sum_{r=0}^{n-1} \frac{\lambda_+^r \omega_r}{W_r(\varphi_+, \varphi_-)},$$

where $W_r(\varphi_+, \varphi_-)$ denotes the Wronskian between φ_+ and φ_-, i.e.,

$$W_r(\varphi_+, \varphi_-) = \begin{vmatrix} \lambda_+^r & \lambda_-^r \\ \lambda_+^{r+1} & \lambda_-^{r+1} \end{vmatrix}.$$

Note that, in our case, $W_r(\varphi_+, \varphi_-) = -\sqrt{5}$ for all $r \geq 0$. Furthermore, when $n = 0$ the summation from $r = 0$ to $n - 1$ collapses, and is to be interpreted as 0.

Now that we are in possession of the general solution to the homogeneous system of equations *and* the particular solution, we may assert that $\hat{\omega}$ is of the form

$$\hat{\omega}_n = c_1 \lambda_+^n + c_2 \lambda_-^n - \frac{1}{\sqrt{5}} \left[\lambda_+^n \sum_{r=0}^{n-1} \lambda_-^r \omega_r - \lambda_-^n \sum_{r=0}^{n-1} \lambda_+^r \omega_r \right], \qquad n \geq 0,$$

where c_1 and c_2 are constants to be determined.

Before we find c_1 and c_2, it is imperative to state that we require $\hat{\omega}$ to lie in $Q(\mathbf{s_{1,F}}) = \ell^2$; since $|\lambda_+| > 1$, we must be mindful of the growth of particular terms. As such, we find it sensible to collect terms as follows:

$$\hat{\omega}_n = \lambda_+^n \left[c_1 - \frac{1}{\sqrt{5}} \sum_{r=0}^{n-1} \lambda_-^r \omega_r \right] + \lambda_-^n \left[c_2 + \frac{1}{\sqrt{5}} \sum_{r=0}^{n-1} \lambda_+^r \omega_r \right].$$

Then, as λ_+ is dominant in the first term, we must choose c_1 appropriately to ensure that $\hat{\omega} \in Q(\mathbf{s_{1,F}})$. If we choose $c_1 = \frac{1}{\sqrt{5}} \sum_{r=0}^{\infty} \lambda_-^r \omega_r$, then

$$\begin{aligned}
\lambda_+^n \left[\frac{1}{\sqrt{5}} \sum_{r=0}^{\infty} \lambda_-^r \omega_r - \frac{1}{\sqrt{5}} \sum_{r=0}^{n-1} \lambda_-^r \omega_r \right] &= \lambda_+^n \left[\frac{1}{\sqrt{5}} \sum_{r=n}^{\infty} \lambda_-^r \omega_r \right] \\
&= \frac{1}{\sqrt{5}} \sum_{r=n}^{\infty} \lambda_-^{r-n} \omega_r < \infty,
\end{aligned}$$

showing that the problematic term now lies in ℓ^2. Therefore

$$\hat{\omega}_n = \left[\frac{1}{\sqrt{5}} \sum_{r=0}^{\infty} \lambda_-^r \omega_r \right] \lambda_+^n + c_2 \lambda_-^n - \frac{1}{\sqrt{5}} \left[\lambda_+^n \sum_{r=0}^{n-1} \lambda_-^r \omega_r - \lambda_-^n \sum_{r=0}^{n-1} \lambda_+^r \omega_r \right], \tag{4.7}$$

for $n \geq 0$. Finally, we can find c_2 by using the initial condition given by $-\hat{\omega}_1 + 3\hat{\omega}_0 = \omega_0$. After substituting $n = 0$ and $n = 1$ into the above expression for $\hat{\omega}_n$, we may eventually conclude that

$$c_2 = -\lambda_-^2 \left[\frac{1}{\sqrt{5}} \sum_{r=0}^{\infty} \lambda_-^r \omega_r \right].$$

Hence,

$$\hat{\omega}_n = \left[\frac{1}{\sqrt{5}} \sum_{r=0}^{\infty} \lambda_-^r \omega_r \right] (\lambda_+^n - \lambda_-^{n+2}) - \frac{1}{\sqrt{5}} \left[\lambda_+^n \sum_{r=0}^{n-1} \lambda_-^r \omega_r - \lambda_-^n \sum_{r=0}^{n-1} \lambda_+^r \omega_r \right],$$

for all $n \geq 0$, or, alternatively,

$$\hat{\omega}_n = \frac{\lambda_+^n}{\sqrt{5}} \sum_{r=n}^{\infty} \lambda_-^r \omega_r - \frac{\lambda_-^n}{\sqrt{5}} \left[\lambda_-^2 \sum_{r=0}^{\infty} \lambda_-^r \omega_r - \sum_{r=0}^{n-1} \lambda_+^r \omega_r \right]. \tag{4.8}$$

Now that we have found $\hat{\omega}$ in terms of ω we can assert that the maximal sectorial relation $S_{1,F}$ associated to $\mathbf{s_{1,F}}$ is given by

$$S_{1,F} = \left\{ (A_1\omega, \omega - A_1\omega) \;\middle|\; \omega \in \ell^2 \right\},$$

where $(A_1\omega)_n = \hat{\omega}_n$ as given by equation (4.8).

Whilst this is a valid representation of the Friedrichs extension $S_{1,F}$ of S_1, we conclude this section by finding an alternative — arguably, more useful — representation that it enjoys. First, we note that $(J + I)\hat{\omega} = \omega$. Then, since J is a non-negative operator, $J + I$ is strictly positive and so is, additionally, invertible. As such, $\hat{\omega} = (J + I)^{-1}\omega$. Then,

$$\begin{aligned}
S_{1,F} &= \left\{ \left((J + I)^{-1}\omega, \omega - (J + I)^{-1}\omega \right) \;\middle|\; \omega \in \ell^2 \right\} \\
&= \left\{ \left((J + I)^{-1}\omega, \left(I - (J + I)^{-1} \right) \omega \right) \;\middle|\; \omega \in \ell^2 \right\}.
\end{aligned}$$

Let $(J+I)^{-1}\omega = x$ for some element $x \in \mathcal{R}((J+I)^{-1}) = \mathcal{D}(J+I) = \ell^2$. Then

$$(J + I)x = \omega, \qquad x \in \ell^2.$$

We can then insert this sequence into $S_{1,F}$ to see that

$$\begin{aligned}S_{1,F} &= \left\{\left(x, \left(I-(J+I)^{-1}\right)(J+I)x\right) \;\middle|\; x \in \ell^2\right\} \\ &= \left\{(x, (J+I)x - Ix) \;\middle|\; x \in \ell^2\right\} \\ &= \left\{(x, Jx) \;\middle|\; x \in \ell^2\right\}.\end{aligned}$$

Hence, the Friedrichs extension of S_1 is S_1 itself, that is, $S_{1,F} = S_1$. This is believable: S_1 is the graph of a densely defined operator that is of limit-point type, and therefore *as an operator* coincides with its Friedrichs extension. Then, the graph of this extension is precisely the linear relation $S_{1,F}$.

Remark. If we were to investigate this theory in the continuous setting instead, then the shift operator Δ would correspond to the differential operator in $L^2([0,\infty))$. Notably, the form $\mathbf{a}$ associated to the Laplacian ∇^2 with Dirichlet boundary conditions in one-dimension, that is, functions that vanish at $x = 0$, would be given by

$$\mathbf{a}[f,g] = \left\langle \nabla^2 f, g\right\rangle = \int_0^\infty f''\overline{g}\; dx = \int_0^\infty f'\overline{g}'\; dx,$$

after an application of integration by parts. Naively perhaps, it may then seem natural to begin with the form

$$\tilde{\mathbf{s}}_{\mathbf{1}}[x,y] = \sum_{n=0}^{\infty} \Delta x_n \Delta \overline{y}_n, \qquad x,\, y \in Q(\tilde{\mathbf{s}}_{\mathbf{1}}) = \ell^2,$$

instead — effectively 'replacing' any instances of the first derivative with the shift operator. With this form in mind, we can now construct the maximal sectorial relation associated to $\tilde{\mathbf{s}}_{\mathbf{1}}$ by following the argument above closely. In particular, we arrive at the following system of equations that must be solved instead:

$$\omega_n = \begin{cases} -\hat{\omega}_1 + 2\hat{\omega}_0, & n = 0, \\ -\hat{\omega}_{n+1} + 3\hat{\omega}_n - \hat{\omega}_{n-1}, & n \geq 1. \end{cases}$$

Many computations can be repeated without fear, but we draw particular attention to the new initial condition $-\hat{\omega}_1 + 2\hat{\omega}_0 = \omega_0$. In fact, this is the only detail that we must be mindful of: all that changes is the constant c_2 in equation (4.7). Then, we assert that the maximal sectorial relation associated to form $\tilde{\mathbf{s}}_{\mathbf{1}}$ is given by equation (3.7), where $\tilde{A}_1\omega = \hat{\omega}$ and

$$\hat{\omega}_n = \frac{\lambda_+^n}{\sqrt{5}} \sum_{r=n}^{\infty} \lambda_-^r \omega_r + \frac{\lambda_-^n}{\sqrt{5}} \left[\lambda_- \sum_{r=0}^{\infty} \lambda_-^r \omega_r + \sum_{r=0}^{n-1} \lambda_+^r \omega_r\right], \qquad n \geq 0.$$

To more closely mimic the case of the differential operator with a vanishing boundary condition at 0, in Sections 4.3 and 4.4 we will consider a Jacobi operator whose domain consists of those sequences with first component equal to zero. Unlike in the differential operator situation however, this leads to a non-densely defined operator. Then, we must make use of linear relations in order to analyse its extensions.

The remark above serves two purposes: it simultaneously highlights the difference between the chosen sectorial form and, perhaps, a more natural form as well as demonstrating the intimate connection between the continuous case. However, as we have only constructed the Friedrichs extension of S_1 so far, we devote the next section to finding the Kreĭn extension of the relation S_1.

4.2.2 The Kreĭn Extension of S_1

This section aims to construct the Kreĭn extension of the relation S_1 as given at the beginning of Section 4.2 by means of the definition detailed in Section 3.2.2, i.e.,

$$S_K = \left((S^{-1})_F\right)^{-1}.$$

Upon untangling this definition, we see that the first step in constructing the Kreĭn extension of a given relation S is to find the inverse relation S^{-1}. Then, for the relation S_1 given by

$$S_1 = \left\{(x, Jx) \in \ell^2 \times \ell^2 \;\middle|\; x \in \ell^2\right\},$$

where $(Jx)_n = -\Delta(\Delta x_{n-1})$ for all $n \geq 0$, we see that

$$S_1^{-1} = \left\{(Jx, x) \in \ell^2 \times \ell^2 \;\middle|\; x \in \ell^2\right\}.$$

Furthermore, since S_1 is sectorial, it is clear that S_1^{-1} is too; thus, we may associate to this relation a well-defined, closable sectorial form. As such, we introduce the form

$$\mathbf{s_1^{-1}}\left[Jx, Jy\right] = \langle x, Jy\rangle, \qquad (Jx, x),\, (Jy, y) \in S_1^{-1}, \tag{4.9}$$

where $Q(\mathbf{s_1^{-1}}) = \mathcal{D}(S_1^{-1}) = \mathcal{R}(J)$, by means of Lemma 3.2.10. Since this form is closable, let $\mathbf{s_{1,F}^{-1}}$ be the form with domain $Q(\mathbf{s_{1,F}^{-1}})$ such that $\overline{\mathbf{s_1^{-1}}} = \mathbf{s_{1,F}^{-1}}$. In particular, we have that

$$\mathbf{s_{1,F}^{-1}}\left[Jx, Jy\right] = \langle x, Jy\rangle \tag{4.10}$$

for Jx, $Jy \in \mathcal{R}(J)$. Note that this form is not explicitly specified for elements in $Q(\mathbf{s_{1,F}^{-1}}) \setminus \mathcal{R}(J)$.

It is difficult to investigate $\mathbf{s_{1,F}^{-1}}$ in its current form since the left-hand side of equation (4.10) involves Jx and Jy. As such, we aim to show that the operator J^{-1} exists on an appropriate domain since the expression $\mathbf{s}_{\mathbf{1},\mathbf{F}}^{-\mathbf{1}}[u,v]$ is considerably more manageable.

From Theorem 1.1.20, the Rank-Nullity theorem states that

$$\begin{aligned}\ell^2 &= \overline{\mathcal{R}(J)} \oplus \ker J^* \\ &= \overline{\mathcal{R}(J)} \oplus \ker J,\end{aligned}$$

since J is a densely defined, self-adjoint operator on ℓ^2. First, we determine the kernel of J. From the proof of Lemma 4.1.1, we know that the general solution to the equation $-\Delta(\Delta x_{n-1}) = 0$ is given by $x_n = c_1 + c_2 n$ for all $n \geq 0$, where c_1, $c_2 \in \mathbb{C}$. Since 1 remains constant and n grows as $n \to \infty$, we are forced to choose $c_1 = c_2 = 0$ in order to ensure that x_n lies in ℓ^2. As such, it is clear that $\ker J = \{0\}$. Then, as the kernel of J — thus J^* — is trivial, we see that $\overline{\mathcal{R}(J)} = \ell^2$. In other words, $\mathcal{R}(J)$ is dense in ℓ^2.

This argument serves multiple purposes once we begin to consider J^{-1} as the resolvent operator of J at $\lambda = 0$. In particular, we have

$$R_0(J) = J_0^{-1} = J^{-1},$$

and so we are able to determine what properties J^{-1} possesses upon referencing Definition 1.1.11. Immediately we assert that J^{-1} exists as an operator from $\mathcal{R}(J)$ to $\mathcal{D}(J)$ since $\ker J = \{0\}$, where $\mathcal{R}(J)$ is dense in ℓ^2: both (**R1**) and (**R3**) hold. Then, as $0 \in \sigma(J)$, it must be true that (**R2**) fails, else we arrive at a contradiction. As such, J^{-1} is an unbounded operator. Furthermore, since J is a closed operator, J^{-1} is also a closed operator. Then, as J^{-1} is closed but unbounded, we may invoke the Closed Graph Theorem (see, for example, [43, Thm. 4.13-2]) to conclude that $\mathcal{R}(J)$ is *not* a closed set in ℓ^2 and so $\mathcal{R}(J) \neq \ell^2$.

Now that we have established that J^{-1} exists on $\mathcal{R}(J)$, if $Jx = u$ for some $x \in \mathcal{D}(J) = \ell^2$ and $u \in \mathcal{R}(J)$, then we may write $x = J^{-1}u$. In particular, equation (4.10) can be rewritten as

$$\mathbf{s}_{\mathbf{1},\mathbf{F}}^{-\mathbf{1}}[u,v] = \left\langle J^{-1}u, v \right\rangle, \qquad u,\, v \in \mathcal{R}(J).$$

In fact, this equality actually holds for all $u \in \mathcal{R}(J)$ and $v \in Q(\mathbf{s}_{\mathbf{1},\mathbf{F}}^{-\mathbf{1}})$, as we now show. Let $u \in \mathcal{R}(J)$ and $v \in Q(\mathbf{s}_{\mathbf{1},\mathbf{F}}^{-\mathbf{1}})$. By Definition 1.2.6, there exist sequences $w_n \in \ell^2$ such that $Jw_n \to v$ in ℓ^2 and

$$\mathbf{s}_{\mathbf{1},\mathbf{F}}^{-\mathbf{1}}[u,v] = \lim_{n\to\infty} \mathbf{s}_{\mathbf{1}}^{-\mathbf{1}}[u, Jw_n] = \lim_{n\to\infty} \left\langle J^{-1}u, Jw_n \right\rangle = \left\langle J^{-1}u, v \right\rangle.$$

Hence

$$\mathbf{s}_{\mathbf{1},\mathbf{F}}^{-\mathbf{1}}[u,v] = \left\langle J^{-1}u, v \right\rangle, \qquad u \in \mathcal{R}(J),\ v \in Q(\mathbf{s}_{\mathbf{1},\mathbf{F}}^{-\mathbf{1}}). \tag{4.11}$$

As this representation of the form is more suitable for our analysis, we now continue by following the construction presented in Section 3.2.2 to find the Friedrichs extension $(S_1^{-1})_F$ of S_1^{-1}. Note that we choose to omit the brackets in future: $(S_1^{-1})_F = S_{1,F}^{-1}$.

We begin by noting that $(Q(\mathbf{s_{1,F}^{-1}}), \|\cdot\|_{\mathbf{s_{1,F}^{-1}}})$ is a Hilbert space, where $\|\cdot\|_{\mathbf{s_{1,F}^{-1}}}$ is the norm induced by the inner product given by

$$\langle x, y\rangle_{\mathbf{s_{1,F}^{-1}}} = \mathbf{s_{1,F}^{-1}}\,[x,y] + \langle x, y\rangle\,, \qquad x,\, y \in Q(\mathbf{s_{1,F}^{-1}}).$$

Here, we remark that $(\mathbf{s_{1,F}^{-1}})_{\mathrm{Re}} = \mathbf{s_{1,F}^{-1}}$ since the closure of a real form is again real — $\mathbf{s_1^{-1}}$ is clearly real by means of equation (4.9). Next, we introduce the new form $\hat{\mathbf{s}}_{\mathbf{1,F}}^{-1} = \mathbf{s_{1,F}^{-1}} + 1$ with domain $Q(\hat{\mathbf{s}}_{\mathbf{1,F}}^{-1}) = Q(\mathbf{s_{1,F}^{-1}})$. Then, in accordance with Theorem 3.2.8, there exists a bounded operator B_1 on $Q(\mathbf{s_{1,F}^{-1}})$ such that

$$\hat{\mathbf{s}}_{\mathbf{1,F}}^{-1}\,[x,y] = \langle B_1 x, y\rangle_{\mathbf{s_{1,F}^{-1}}}\,,$$

for all $x,\, y \in Q(\mathbf{s_{1,F}^{-1}})$. Upon rewriting this equality, we see that

$$\begin{aligned}\mathbf{s_{1,F}^{-1}}\,[x,y] + \langle x, y\rangle &= \langle B_1 x, y\rangle_{\mathbf{s_{1,F}^{-1}}} \\ &= \mathbf{s_{1,F}^{-1}}\,[B_1 x, y] + \langle B_1 x, y\rangle\,,\end{aligned}$$

and so we may set B_1 as the identity operator on $Q(\mathbf{s_{1,F}^{-1}})$.

Now, let $k,\, \hat{\omega} \in Q(\mathbf{s_{1,F}^{-1}})$ and $\omega \in \ell^2$ and consider the following equality:

$$\langle k, \omega\rangle = \langle k, \hat{\omega}\rangle_{\mathbf{s_{1,F}^{-1}}}\,.$$

We hope to determine the relationship between ω and $\hat{\omega}$ by expanding both sides of this equality. In fact, if we specify that $k \in \mathcal{R}(J) \subseteq Q(\mathbf{s_{1,F}^{-1}})$, then we may explicitly unravel the right-hand side of this equation. In particular, for $k \in \mathcal{R}(J)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} k_n \overline{\omega}_n &= \sum_{n=0}^{\infty} (J^{-1}k)_n \overline{\hat{\omega}}_n + \sum_{n=0}^{\infty} k_n \overline{\hat{\omega}}_n \\ &= \sum_{n=0}^{\infty} \big((J^{-1}+I)k\big)_n \overline{\hat{\omega}}_n. \qquad (4.12)\end{aligned}$$

Now, introduce a new sequence h such that $h = (J^{-1}+I)k$; clearly, h lies in $\mathcal{R}((J^{-1}+I) \restriction \mathcal{R}(J))$. Then, as $k \in \mathcal{R}(J)$, there exists a $u \in \mathcal{D}(J) = \ell^2$ such that $Ju = k$. Hence

$$h = (J^{-1}+I)k = (J^{-1}+I)Ju = (J+I)u,$$

and so h also lies in $\mathcal{R}(J+I)$, that is,

$$\mathcal{R}((J^{-1}+I) \restriction \mathcal{R}(J)) \subseteq \mathcal{R}(J+I). \qquad (4.13)$$

As $(J+I)$ is invertible, we may conclude that $(J+I)^{-1}h = u$. Then, by applying J to both sides of this equation, we see that

$$J(J+I)^{-1}h = Ju \implies (J+I-I)(J+I)^{-1}h = Ju.$$

Hence

$$(I-(J+I)^{-1})h = k.$$

If we insert this equality into the left-hand side of equation (4.12), we see that

$$\sum_{n=0}^{\infty}((I-(J+I)^{-1})h)_n\overline{\omega}_n = \sum_{n=0}^{\infty} h_n\overline{\hat{\omega}}_n \tag{4.14}$$

for $h \in \mathcal{R}((J^{-1}+I) \restriction \mathcal{R}(J))$. Then, by utilising inner product notation, we see that

$$\begin{aligned}\sum_{n=0}^{\infty}((I-(J+I)^{-1})h)_n\overline{\omega}_n &= \left\langle (I-(J+I)^{-1})h, \omega \right\rangle \\ &= \left\langle h, (I-(J+I)^{-1})^*\omega \right\rangle \\ &= \left\langle h, (I-(J+I)^{-1})\omega \right\rangle,\end{aligned}$$

after recalling that J is a self-adjoint operator. By combining this with the right-hand side of equation (4.14), we may conclude that

$$\left\langle h, \left(I-(J+I)^{-1}\right)\omega \right\rangle = \langle h, \hat{\omega} \rangle,$$

implying that $\left(I-(J+I)^{-1}\right)\omega = \hat{\omega}$ *if* the set X_1, where

$$X_1 = \mathcal{R}((J^{-1}+I) \restriction \mathcal{R}(J)), \tag{4.15}$$

is dense in ℓ^2. The following lemma will show more than this: we show that X_1 is, in fact, ℓ^2.

Lemma 4.2.3. *The set X_1, as defined by equation (4.15), is equal to ℓ^2.*

Proof. Firstly, we show that $X_1 = \mathcal{R}(J+I)$. In fact, equation (4.13) shows that $X_1 \subseteq \mathcal{R}(J+I)$, so we begin by verifying the converse.

Let $x \in \mathcal{R}(J+I)$. Then, there exists a $y \in \mathcal{D}(J+I) = \mathcal{D}(J)$ such that $x = (J+I)y$ and $z = Jy$ for some $z \in \ell^2$. As such, we have

$$x = (J+I)y = (J+I)J^{-1}z = (J^{-1}+I)z.$$

In fact, as $z \in \mathcal{R}(J)$, we see that $x \in \mathcal{R}((J^{-1}+I) \restriction \mathcal{R}(J))$, as required.

Now, as the spectrum of $J+I$ is simply a shift of $\sigma(J)$, we assert that $\sigma(J+I) = [1,5]$. In particular, $-1 \notin \sigma(J+I)$, and so we conclude that the resolvent $R_{-1}(J) = (J + I)^{-1}$ is a bounded operator that is defined on a set that is dense in ℓ^2. Since $(J + I)^{-1}$ is bounded, it is necessarily a closed operator, and so $\mathcal{D}((J + I)^{-1}) = \mathcal{R}(J + I)$ is closed via the Closed Graph Theorem. As such, $\mathcal{R}(J + I) = \ell^2$ since this is the only set that is both dense *and* closed, as required. ■

Therefore, we are now able to determine the operator $A_1 \colon \ell^2 \to Q(\mathbf{s}_{\mathbf{1,F}}^{\mathbf{-1}})$ as defined through equality $A_1\omega = \hat{\omega}$. In particular, we have that

$$A_1 = I - (J + I)^{-1}.$$

With this relationship in mind, we note that the Friedrichs extension $S_{1,F}^{-1}$ of S_1^{-1} is given by the set

$$\begin{aligned} S_{1,F}^{-1} &= \left\{(A_1\omega, \omega - A_1\omega) \middle| \; \omega \in \ell^2\right\} \\ &= \left\{\left((I - (J + I)^{-1})\omega, \omega - (I - (J + I)^{-1})\omega\right) \;\middle|\; \omega \in \ell^2\right\} \\ &= \left\{\left((I - (J + I)^{-1})\omega, (J + I)^{-1}\omega\right) \;\middle|\; \omega \in \ell^2\right\}. \end{aligned}$$

To find $S_{1,K}$ we simply take the inverse of this relation. Hence,

$$(S_{1,F}^{-1})^{-1} = S_{1,K} = \left\{\left((J + I)^{-1}\omega, (I - (J + I)^{-1})\omega\right) \;\middle|\; \omega \in \ell^2\right\}.$$

However, as $(J+I)^{-1}$ is a linear operator that maps into $\mathcal{D}(J+I) = \ell^2$, we may set $(J + I)^{-1}\omega = x$ for some sequence $x \in \ell^2$ and, in particular, $\omega = (J + I)x$. By making this substitution, we are able to simplify $S_{1,K}$ considerably. Then,

$$\begin{aligned} S_{1,K} &= \left\{\left(x, (I - (J + I)^{-1})(J + I)x\right) \;\middle|\; x \in \ell^2\right\} \\ &= \left\{(x, Jx) \;\middle|\; x \in \ell^2\right\} = S_1. \end{aligned}$$

Notably, the Friedrichs extension $S_{1,F}$ and Kreĭn extension $S_{1,K}$ of S_1 coincide, that is:

$$S_{1,F} = S_{1,K} = S_1.$$

We conclude by remarking that the arguments presented in Section 4.2 serve as a template for the following section, where we aim to construct both the Friedrichs extension and the Kreĭn extension of a different, yet similar, linear relation.

4.3 The Friedrichs Extension of S_2

This section exists as a counterpart to Section 4.2.1; we aim to construct the Friedrichs extension of S_2, where

$$S_2 = \left\{(x, \tilde{J}x) \in \ell^2 \times \ell^2 \;\middle|\; x \in \ell_0^2\right\}, \tag{4.16}$$

for the second-order difference operator $\tilde{J}$, where $(\tilde{J}x)_n = -\Delta(\Delta x_{n-1})$ and

$$\ell_0^2 = \left\{ x \in \ell^2 \mid x_0 = 0 \right\}.$$

Since this section will follow the structure of Section 4.2.1 closely, we must first verify that this relation is, in fact, sectorial with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$. Since the arguments remain the same, we present condensed versions of the proofs when appropriate; however, we endeavour to call attention to any notable differences between the two sections for maximal insight.

Lemma 4.3.1. *The relation S_2, as defined by equation (4.16), is sectorial with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$.*

Proof. In order to show that S_2 is sectorial with the required vertex and semi-angle, we simply need to show that

$$\left| \operatorname{Im} \left\langle \tilde{J}x, x \right\rangle \right| \leq (\tan \alpha) \operatorname{Re} \left\langle \tilde{J}x, x \right\rangle$$

for all $(x, \tilde{J}x) \in S_2$. However, we have already shown that this inequality holds for all elements in S_1 during Lemma 4.2.1. Since S_2 is a subset of S_1, it is then immediate that S_2 is a sectorial relation with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$. ∎

Remark. Once again, we will take $\alpha = 0$ as this provides the most information: the numerical range lies exclusively on the positive real axis.

Since S_2 is a sectorial relation, we may associate to it a well-defined, closable sectorial form $\mathbf{s_2}$ by means of Lemma 3.2.10. If we continue by taking the closure of this form, then we will be in possession of $\mathbf{s_{2,F}}$ — the form associated to the Friedrichs extension of S_2. This time, however, the form $\mathbf{s_2}$ has form domain $Q(\mathbf{s_2}) = \mathcal{D}(S_2) = \ell_0^2$ and is defined by

$$\begin{aligned} \mathbf{s_2}[x, y] &= \left\langle \tilde{J}x, y \right\rangle \\ &= \sum_{n=0}^{\infty} \Delta x_n \Delta \overline{y}_n, \qquad (x, \tilde{J}x), (y, \tilde{J}y) \in S_2, \end{aligned}$$

after an application of the summation by parts formula. Although the form domain is different to that of $\mathbf{s_1}$, the following lemma proves that $\mathbf{s_2}$ is *also* already closed, that is, $\mathbf{s_2} = \mathbf{s_{2,F}}$.

Lemma 4.3.2. *The form $\mathbf{s_2}$ with domain $Q(\mathbf{s_2}) = \ell_0^2$ is a closed form.*

Proof. Let $x \in \ell^2$ and let $\{x_n\}$ be a sequence in $Q(\mathbf{s_2})$ such that $x_n \to_{\mathbf{s_2}} x$. As before, we must show that $x \in Q(\mathbf{s_2})$ and that $\mathbf{s_2}[x_n - x, x_n - x] \to 0$ as $n \to \infty$.

Firstly, if $x_n \to x \in \ell^2$, then it is clear that the first component of x equals 0. As such, it is immediate that $x \in Q(\mathbf{s_2})$. Then, it is easy to verify that $\mathbf{s_2}\left[x_n - x, x_n - x\right] \to 0$ as $n \to \infty$ by following the proof of Lemma 4.2.2. ∎

This lemma allows us to conclude that $\mathbf{s_2} = \mathbf{s_{2,F}}$ and so the maximal sectorial relation associated to $\mathbf{s_{2,F}}$ is the Friedrichs extension of S_2. This time, $\left(Q(\mathbf{s_{2,F}}), \|\cdot\|_{\mathbf{s_{2,F}}}\right)$ is the Hilbert space of concern to us, where $\|\cdot\|_{\mathbf{s_{2,F}}}$ is the norm induced by the inner product given by

$$\langle x, y\rangle_{\mathbf{s_{2,F}}} = \mathbf{s_{2,F}}\left[x, y\right] + \langle x, y\rangle, \qquad x,\, y \in Q(\mathbf{s_{2,F}}) = \ell^2_0.$$

Let $\hat{\mathbf{s}}_{\mathbf{2,F}}$ be the form such that

$$\hat{\mathbf{s}}_{\mathbf{2,F}} = \mathbf{s_{2,F}} + 1 \qquad \text{and} \qquad Q(\hat{\mathbf{s}}_{\mathbf{2,F}}) = Q(\mathbf{s_{2,F}}) = \ell^2_0.$$

This form is bounded in $\left(Q(\mathbf{s_{2,F}}), \|\cdot\|_{\mathbf{s_{2,F}}}\right)$, so there exists a bounded linear operator $B_2\colon Q(\mathbf{s_{2,F}}) \to Q(\mathbf{s_{2,F}})$ such that

$$\hat{\mathbf{s}}_{\mathbf{2,F}}\left[x, y\right] = \langle B_2 x, y\rangle_{\mathbf{s_{2,F}}}, \qquad x,\, y \in Q(\mathbf{s_{2,F}}). \tag{4.17}$$

Observe that the left-hand side of equation (4.17) can be expressed as

$$\hat{\mathbf{s}}_{\mathbf{2,F}}\left[x, y\right] = \mathbf{s_{2,F}}\left[x, y\right] + \langle x, y\rangle,$$

whilst the right,

$$\langle B_2 x, y\rangle_{\mathbf{s_{2,F}}} = \mathbf{s_{2,F}}\left[B_2 x, y\right] + \langle B_2 x, y\rangle.$$

By comparing these two expressions, it is then clear that we may set B_2 to be the identity operator on $Q(\mathbf{s_{2,F}}) = \ell^2_0$. Thus, with the operator B_2 in hand, all that remains is to construct the operator A_2 — the unique maximal sectorial relation associated to $\mathbf{s_{2,F}}$ may then be obtained from equation (3.7).

Let $k,\, \hat{\omega} \in Q(\mathbf{s_{2,F}})$ and $\omega \in \ell^2$. As before, we wish to use the equality

$$\langle k, \omega\rangle = \langle k, \hat{\omega}\rangle_{\mathbf{s_{2,F}}}$$

to express $\hat{\omega}$ in terms of ω. Then, A_2 will be the operator such that

$$A_2\omega = B_2^{-1}\hat{\omega} = \hat{\omega}.$$

Although we will follow the same calculations as before, we must be mindful and accommodate the additional condition that $k_0 = \hat{\omega}_0 = 0$. As such,

$$\begin{aligned}\sum_{n=1}^{\infty} k_n\overline{\omega}_n &= \mathbf{s_{2,F}}\left[k, \hat{\omega}\right] + \langle k, \hat{\omega}\rangle \\ &= \sum_{n=0}^{\infty}\left[\Delta k_n \Delta\overline{\hat{\omega}}_n + k_n\overline{\hat{\omega}}_n\right] \\ &= \sum_{n=0}^{\infty} k_n\left[-\Delta(\Delta\overline{\hat{\omega}}_{n-1}) + \overline{\hat{\omega}}_n\right],\end{aligned}$$

after an application of the summation by parts formula. This alteration complicates the analysis slightly because we are not able to accrue any information about ω_0. However, we may still conclude that

$$\omega_n = \begin{cases} \omega_0, & n = 0, \\ -\hat{\omega}_{n+1} + 3\hat{\omega}_n, & n = 1, \\ -\hat{\omega}_{n+1} + 3\hat{\omega}_n - \hat{\omega}_{n-1}, & n \geq 2. \end{cases} \tag{4.18}$$

In order to circumvent this issue, we must make use of the fact that A_2 is a linear operator. In particular, we know that

$$\begin{aligned} \langle k, \omega \rangle &= \langle k, \hat{\omega} \rangle_{\mathbf{s_{2,F}}} \\ &= \mathbf{s_{2,F}}\left[k, \hat{\omega}\right] + \langle k, \hat{\omega} \rangle \end{aligned}$$

for all k, $\hat{\omega} \in Q(\mathbf{s_{2,F}})$ and $\omega \in \ell^2$. Then, by setting $k = \hat{\omega}$ we see that

$$\begin{aligned} \langle \hat{\omega}, \omega \rangle &= \mathbf{s_{2,F}}\left[\hat{\omega}, \hat{\omega}\right] + \langle \hat{\omega}, \hat{\omega} \rangle \\ &= \sum_{n=0}^{\infty} |\Delta \hat{\omega}_n|^2 + \|\hat{\omega}\|^2. \end{aligned}$$

If we set $\omega = (1, 0, 0, \dots)$, then it is readily observed that

$$0 = \sum_{n=0}^{\infty} |\Delta \hat{\omega}_n|^2 + \|\hat{\omega}\|^2,$$

which can only be consistent if $\hat{\omega} = 0$. Thus,

$$A_2\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

and, by linearity,

$$A_2\left(\begin{pmatrix} c \\ 0 \\ 0 \\ \vdots \end{pmatrix}\right) = cA_2\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \tag{4.19}$$

for $c \in \mathbb{C}$. Since we may decompose any sequence $\omega \in \ell^2$ into

$$\omega = \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \omega_0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \vdots \end{pmatrix},$$

by linearity, it must be true that

$$A_2\left(\begin{pmatrix}\omega_0\\ \omega_1\\ \omega_2\\ \vdots\end{pmatrix}\right) = A_2\left(\begin{pmatrix}\omega_0\\ 0\\ 0\\ \vdots\end{pmatrix}\right) + A_2\left(\begin{pmatrix}0\\ \omega_1\\ \omega_2\\ \vdots\end{pmatrix}\right) = A_2\left(\begin{pmatrix}0\\ \omega_1\\ \omega_2\\ \vdots\end{pmatrix}\right).$$

Thus, all that remains is to show how A_2 acts on an element of ℓ^2 whose first component is equal to zero. Note that we can express the remaining equations in (4.18) concisely with a Jacobi operator as follows:

$$\begin{pmatrix}0 & 0 & & & \\ 0 & 3 & -1 & & \\ & -1 & 3 & -1 & \\ & & \ddots & \ddots & \ddots\end{pmatrix}\begin{pmatrix}0\\ \hat{\omega}_1\\ \hat{\omega}_2\\ \vdots\end{pmatrix} = \begin{pmatrix}0\\ \omega_1\\ \omega_2\\ \vdots\end{pmatrix}. \tag{4.20}$$

If we introduce the notation $\omega'_n = \hat{\omega}_{n+1}$, then this system becomes slightly more familiar:

$$\begin{cases} 3\omega'_0 - \omega'_1 = \omega_1, & n = 0,\\ -\omega'_{n+1} + 3\omega'_n - \omega'_{n-1} = \omega_{n+1}, & n \geq 1.\end{cases} \tag{4.21}$$

This system of equations is subtly different to the previous case: there is a shift in indices on the right-hand side of the equation. However, we do not need to alter our method in solving this system greatly. In fact, we can simply recycle our previous calculations and read off the solution to the homogeneous problem. The particular solution will differ slightly, but it is not too difficult to confirm that

$$\omega'_n = c_1\lambda_+^n + c_2\lambda_-^n - \frac{1}{\sqrt{5}}\left[\lambda_+^n\sum_{r=0}^{n-1}\lambda_-^r\omega_{r+1} - \lambda_-^n\sum_{r=0}^{n-1}\lambda_+^r\omega_{r+1}\right], \qquad n \geq 0.$$

Since $|\lambda_+| > 1$, we have to choose the constant c_1 in a manner that ensures that $\omega' \in \ell^2$. As such, we may follow a parallel argument to before to deduce that $c_1 = \frac{1}{\sqrt{5}}\sum_{r=0}^{\infty}\lambda_-^r\omega_{r+1}$. We can then proceed to find c_2 by using the initial condition given by $3\omega'_0 - \omega'_1 = \omega_1$. After some simple calculations we see that $c_2 = -\frac{1}{\sqrt{5}}\sum_{r=1}^{\infty}\lambda_-^r\omega_{r+1}$. Therefore, we have that

$$\omega'_n = \left[\frac{1}{\sqrt{5}}\sum_{r=0}^{\infty}\lambda_-^r\omega_{r+1}\right](\lambda_+^n - \lambda_-^{n+2}) - \frac{1}{\sqrt{5}}\left[\lambda_+^n\sum_{r=0}^{n-1}\lambda_-^r\omega_{r+1} - \lambda_-^n\sum_{r=0}^{n-1}\lambda_+^r\omega_{r+1}\right]$$

for $n \geq 0$. Upon recalling that $\omega'_n = \hat{\omega}_{n+1}$, we may finally claim that

$$\hat{\omega}_n = \begin{cases} 0 & n = 0,\\ \frac{\lambda_+^{n-1}}{\sqrt{5}}\sum\limits_{r=n-1}^{\infty}\lambda_-^r\omega_{r+1} - \frac{\lambda_-^{n-1}}{\sqrt{5}}\left[\lambda_-^2\sum\limits_{r=0}^{\infty}\lambda_-^r\omega_{r+1} - \sum\limits_{r=0}^{n-2}\lambda_+^r\omega_{r+1}\right], & n \geq 1,\end{cases} \tag{4.22}$$

after collecting together powers of λ_+ and λ_-.

With $\hat{\omega}$ now expressed in terms of ω, we can finally assert that the maximal sectorial relation $S_{2,F}$ associated to $\mathbf{s_{2,F}}$ is given by

$$S_{2,F} = \left\{(A_2\omega, \omega - A_2\omega) \;\middle|\; \omega \in \ell^2\right\},$$

where $A_2\omega = \hat{\omega}$, as prescribed by equation (4.22).

Although we have constructed the Friedrichs extension $S_{2,F}$ of S_2, the form it currently takes is not particularly insightful: we are currently unable to make any direct comparison between $S_{1,F}$ and $S_{2,F}$. Thus, we conclude this section by decomposing $S_{2,F}$ into its operator part $(S_{2_F})_s$ and multi-valued part $(S_{2,F})_{\text{mul}}$ — it is then much easier for us to make connections between the two relations, allowing us to note any differences that may surface.

We begin by determining the multi-valued part $(S_{2,F})_{\text{mul}}$ of $S_{2,F}$. Recall that

$$(S_{2,F})_{\text{mul}} = \{0\} \times \operatorname{mul} S_{2,F},$$

where

$$\begin{aligned} \operatorname{mul} S_{2,F} &= \left\{\omega - A_2\omega \;\middle|\; A_2\omega = 0,\ \omega \in \ell^2\right\} \\ &= \left\{\omega \in \ell^2 \;\middle|\; A_2\omega = 0\right\}. \end{aligned}$$

In other words, $\operatorname{mul} S_{2,F} = \ker A_2$.

Let $\omega \in \ker A_2$. Since A_2 is a linear operator and $A_2\omega = \hat{\omega}$, we must have that $\hat{\omega} = 0$. To find A_2, recall that we solved the system of equations given by (4.21), i.e.,

$$\begin{cases} 3\omega_0' - \omega_1' = \omega_1, & n = 0, \\ -\omega_{n+1}' + 3\omega_n' - \omega_{n-1}' = \omega_{n+1}, & n \geq 1, \end{cases}$$

where $\omega_n' = \hat{\omega}_{n+1}$ for all $n \geq 0$. As $\hat{\omega} = 0$, we must have $\omega' = 0$. Hence, from the system of equations, $\omega_n = 0$ for all $n \geq 1$. As such, ω_0 is free, and so we may conclude that

$$\operatorname{mul} S_{2,F} = \ker A_2 = \operatorname{span}\{e_0\},$$

where $e_0 = (1, 0, 0, \dots)$. Alternatively, one may immediately observe that $e_0 \in \ker A_2$ by means of equation (4.19). Hence,

$$(S_{2,F})_{\text{mul}} = \{0\} \times \operatorname{span}\{e_0\},$$

for $e_0 = (1, 0, 0, \dots)$.

Now, we continue by constructing the operator part $(S_{2,F})_s$ of $S_{2,F}$. In particular, we have

$$(S_{2,F})_s = \left\{(A_2\omega, P(\omega - A_2\omega)) \;\middle|\; \omega \in \ell^2\right\},$$

where P projects onto $(\operatorname{mul} S_{2,F})^\perp = (\operatorname{span}\{e_0\})^\perp = \ell_0^2$. Since A_2 maps into ℓ_0^2, we can rewrite the operator part as

$$(S_{2,F})_s = \left\{(A_2\omega, \omega - \omega_0 e_0 - A_2\omega)) \;\middle|\; \omega \in \ell^2\right\},$$

after noting that the projection of an ℓ^2-element onto ℓ_0^2 simply sets its first component to zero. Now, as $\ell^2 = \ell_0^2 \oplus \operatorname{span}\{e_0\}$, we see that

$$(S_{2,F})_s = \left\{(A_2\omega, \omega - A_2\omega)) \;\middle|\; \omega \in \ell_0^2\right\},$$

since an ω of the form $\omega = ce_0$, returns a contribution of $(0,0)$: we may as well remove such elements from the analysis. As $(S_{2,F})_s$ will be the graph of an operator T, say, we assert that

$$\begin{aligned}(S_{2,F})_s &= \left\{(A_2\omega, \omega - A_2\omega)) \;\middle|\; \omega \in \ell_0^2\right\} \\ &= \{(x, Tx) \mid x \in \mathcal{D}(T)\}.\end{aligned}$$

We continue by determining this operator T.

Denote by L the *left shift* operator, that is,

$$L((u_0, u_1, u_2, \dots)) = (u_1, u_2, u_3, \dots), \qquad \mathcal{D}(L) = \ell^2,$$

and R the *right shift* operator, where

$$R((u_0, u_1, u_2, \dots)) = (0, u_1, u_2, u_3, \dots), \qquad \mathcal{D}(R) = \ell^2.$$

If $\omega \in \ell^2$, then $LR\omega = \omega$, that is LR is the identity on ℓ^2. Conversely, RL is *not* the identity on ℓ^2 as $RL\omega \neq \omega$ — we are unable to recover the first component ω_0. However, if $\omega \in \ell_0^2$ instead, then $RL\omega = LR\omega = \omega$. Then, by reintroducing the operator J from Section 4.2, that is, $(Jx)_n = -\Delta(\Delta x_{n-1})$ for $n \geq 0$, where $x \in \ell^2$, we note that equation (4.20) can be written as

$$(J + I)L\hat{\omega} = L\omega \qquad \text{or} \qquad R(J + I)L\hat{\omega} = \omega.$$

Since we are concerned with ω and $\hat{\omega}$ that lie in ℓ_0^2, both $(J + I)$ and L are invertible. As such,

$$\hat{\omega} = R(J + I)^{-1}L\omega,$$

and, in particular, $A_2 = R(J+I)^{-1}L$ on ℓ_0^2. Since $\mathcal{R}(L \restriction \ell_0^2) = \ell^2$ and $J + I$ is bijective, we see that $\mathcal{R}(A_2) = \ell_0^2$. Then, for $x \in \mathcal{R}(A_2)$, we have

$$\begin{aligned}(S_{2,F})_s &= \left\{(x, R(J+I)Lx - x) \;\middle|\; x \in \ell_0^2\right\} \\ &= \left\{(x, (R(J+I)L - RL)x) \;\middle|\; x \in \ell_0^2\right\} \\ &= \left\{(x, RJLx) \;\middle|\; x \in \ell_0^2\right\},\end{aligned}$$

showing that the operator T is, in fact, the operator RJL with domain $\mathcal{D}(RJL) = \ell_0^2$. With the operator and multi-valued parts of $S_{2,F}$ now defined, we may finally conclude that

$$S_{2,F} = \left\{(x, RJLx) \;\middle|\; x \in \ell_0^2\right\} \oplus (\{0\} \times \operatorname{span}\{e_0\}).$$

The operator part of $S_{2,F}$ is in some way comparable to that of $S_{1,F}$: they are both heavily dependent on the operator J, shifts notwithstanding. Additionally, in our closing remark of Section 4.2.1, we mentioned that the operator to come — that is, $\tilde{J}$ — was *not* densely defined and that relations would be critical in any analysis undertaken: hopefully this is apparent with the advent of a non-trivial multi-valued part!

Now that we are in possession of the Friedrichs extension of S_2, we finally turn our attention to $S_{2,K}$: the Kreĭn extension of S_2. We aim to follow the same format as Section 4.2.2, but note that the difference in initial domains makes the analysis considerably more involved.

4.4 The Kreĭn Extension of S_2

As before, we construct the Kreĭn extension $S_{2,K}$ of the sectorial relation S_2 by methodically unravelling the relation $((S_2^{-1})_F)^{-1}$. Therefore, the first step in doing so is to find the inverse relation S_2^{-1}. Recall that S_2 is given by

$$S_2 = \left\{(x, \tilde{J}x) \in \ell^2 \times \ell^2 \;\middle|\; x \in \ell_0^2\right\},$$

where $(\tilde{J}x)_n = -\Delta(\Delta x_{n-1})$ for $n \geq 0$; hence,

$$S_2^{-1} = \left\{(\tilde{J}x, x) \in \ell^2 \times \ell^2 \;\middle|\; x \in \ell_0^2\right\}.$$

Since S_2 is sectorial, S_2^{-1} is also sectorial, and so we may associate to it the form

$$\mathbf{s_2^{-1}}\left[\tilde{J}x, \tilde{J}y\right] = \left\langle x, \tilde{J}y\right\rangle, \qquad (\tilde{J}x, x),\, (\tilde{J}y, y) \in S_2^{-1},$$

where $Q(\mathbf{s_2^{-1}}) = \mathcal{D}(S_2^{-1}) = \mathcal{R}(\tilde{J})$. As before, we have that $\mathbf{s_2^{-1}}$ is a well-defined, sectorial and closable form. Let $\mathbf{s_{2,F}^{-1}}$ be the form with domain $Q(\mathbf{s_{2,F}^{-1}})$ defined as the closure of $\mathbf{s_2^{-1}}$. Therefore,

$$\mathbf{s_{2,F}^{-1}}\left[\tilde{J}x, \tilde{J}y\right] = \left\langle x, \tilde{J}y\right\rangle, \qquad (\tilde{J}x, x),\, (\tilde{J}y, y) \in S_2^{-1}.$$

Note that we have not defined the form explicitly for elements that lie in $Q(\mathbf{s_{2,F}^{-1}}) \setminus \mathcal{R}(\tilde{J})$.

Although we have an expression for the form on certain elements, we are able to rewrite it in a more useful manner by showing that the inverse of $\tilde{J}$

exists on $\mathcal{R}(\tilde{J})$. The following lemma proves this, in addition to showing that $\mathcal{R}(\tilde{J})$ is, in fact, a dense set in ℓ^2.

Lemma 4.4.1. *The operator $\tilde{J}\colon \ell^2_0 \to \ell^2$, defined by $(\tilde{J}x)_n = -\Delta(\Delta x_{n-1})$ for $n \geq 0$, has an inverse that is densely defined.*

Proof. If $\ker \tilde{J} = \{0\}$, then $\tilde{J}^{-1}$ exists as an operator from $\mathcal{R}(\tilde{J})$ to ℓ^2_0. In fact, we have previously shown that $\ker J = \{0\}$ — we refer to Section 4.2.2 for more details — and so the kernel of $\tilde{J}$ must also be trivial since $\tilde{J}$ is merely a restriction of J.

To show that $\tilde{J}^{-1}$ is densely defined, we need to show that $\mathcal{R}(\tilde{J})$ is a dense set in ℓ^2. From the Rank-Nullity theorem, as stated in Theorem 1.1.20, we are able to decompose ℓ^2 into

$$\ell^2 = \overline{\mathcal{R}(\tilde{J})} \oplus \ker \tilde{J}^*, \tag{4.23}$$

since $\tilde{J}$ is certainly densely defined on ℓ^2_0. Then, $\mathcal{R}(\tilde{J})$ is dense in ℓ^2 if and only if $\ker \tilde{J}^* = \{0\}$.

To find $\tilde{J}^*\colon \ell^2 \to \ell^2_0$, consider the following equality:

$$\left\langle \tilde{J}x, y \right\rangle = \left\langle x, \tilde{J}^* y \right\rangle, \qquad x \in \mathcal{D}(\tilde{J}),\ y \in \mathcal{D}(\tilde{J}^*). \tag{4.24}$$

Then, as in the proof of Lemma 4.1.1, the left-hand side of this equation can be expressed as

$$\begin{aligned}\sum_{n=0}^{\infty} -\Delta(\Delta x_{n-1})\overline{y}_n &= \sum_{n=0}^{\infty} x_n \left[-\Delta(\Delta \overline{y}_{n-1})\right] \\ &= \sum_{n=1}^{\infty} x_n \left[-\Delta(\Delta \overline{y}_{n-1})\right],\end{aligned}$$

after an application of the summation by parts formula, upon recalling that $x_0 = 0$. Conversely, we may express the right-hand side of equation (4.24) as

$$\left\langle x, \tilde{J}^* y \right\rangle = \sum_{n=0}^{\infty} x_n \overline{(\tilde{J}^* y)_n} = \sum_{n=1}^{\infty} x_n \overline{(\tilde{J}^* y)_n}.$$

By equating these two expressions, we see that $(\tilde{J}^* y)_n = -\Delta(\Delta y_{n-1})$ for $n \geq 1$, for any $y \in \ell^2$. Whilst we obtain no information about $(\tilde{J}y)_0$ from this argument, we know that $\tilde{J}^*$ maps into ℓ^2_0; then, we are able to conclude that

$$(\tilde{J}^* y)_n = \begin{cases} 0, & n = 0, \\ -\Delta(\Delta y_{n-1}), & n \geq 1, \end{cases}$$

for $y \in \ell^2$. Then, we deduce that $\ker \tilde{J}^* = \{0\}$ because there does not exist a non-zero solution to the equation $-\Delta(\Delta y_{n-1}) = 0$ that lies in ℓ^2. As such, the decomposition presented in (4.23) informs us that $\ell^2 = \overline{\mathcal{R}(\tilde{J})}$. Hence $\mathcal{R}(\tilde{J})$ is dense in ℓ^2 and $\tilde{J}^{-1}\colon \mathcal{R}(J) \to \ell^2_0$ is a densely defined operator. ■

Now that we have established that $\tilde{J}^{-1}$ exists, we see that if $\tilde{J}x = u$ for $x \in \ell_0^2$ and $u \in \mathcal{R}(\tilde{J})$, then we have that $x = \tilde{J}^{-1}u$ instead. As such, we have that

$$\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}[u,v] = \left\langle \tilde{J}^{-1}u, v \right\rangle, \qquad u \in \mathcal{R}(\tilde{J}),\ v \in Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}),$$

by means of an argument parallel to that of equation (4.11). With this form in mind we are now able to construct the Friedrichs extension $(S_2^{-1})_F$ of S_2^{-1}. Note that we will, once again, omit brackets: $(S_2^{-1})_F = S_{2,F}^{-1}$.

This time, $(Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}), \|\cdot\|_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}})$ is the Hilbert space of concern to us, where $\|\cdot\|_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}}$ is the norm induced by the inner product given by

$$\langle x, y\rangle_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}} = \mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}[x,y] + \langle x, y\rangle, \qquad x,\ y \in Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}).$$

Then, we begin to construct the Friedrichs extension $S_{2,F}^{-1}$ by introducing the form $\hat{\mathbf{s}}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}} = \mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}} + 1$ with domain $Q(\hat{\mathbf{s}}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}) = Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}})$. We then assert that there exists a bounded linear operator operator $B_2\colon Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}) \to Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}})$ such that

$$\hat{\mathbf{s}}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}[x,y] = \langle B_2x, y\rangle_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}},$$

for all $x,\ y \in Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}})$. In other words,

$$\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}[x,y] + \langle x, y\rangle = \mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}[B_2x,y] + \langle B_2x, y\rangle,$$

suggesting that B_2 may be taken to be the identity operator on $Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}})$.

Then, for $k,\ \hat{\omega} \in Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}})$ and $\omega \in \ell^2$, we consider the following equality

$$\langle k, \omega\rangle = \langle k, \hat{\omega}\rangle_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}}, \tag{4.25}$$

where we wish to identify the relationship between ω and $\hat{\omega}$ again. In particular, choose $k \in \mathcal{R}(\tilde{J})$. Then, there exists an $m \in \ell_0^2$ such that $\tilde{J}m = k$. Therefore, we may rewrite the right-hand side of this equality to see that

$$\begin{aligned}\langle k, \hat{\omega}\rangle_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}} &= \sum_{n=0}^{\infty}(\tilde{J}^{-1}k)_n\overline{\hat{\omega}}_n + \sum_{n=0}^{\infty}k_n\overline{\hat{\omega}}_n \\ &= \sum_{n=0}^{\infty}(\tilde{J}^{-1}\tilde{J}m)_n\overline{\hat{\omega}}_n + (\tilde{J}m)_n\overline{\hat{\omega}}_n \\ &= \sum_{n=0}^{\infty}((\tilde{J} + \tilde{I})m)_n\overline{\hat{\omega}}_n,\end{aligned}$$

where $\tilde{I}\colon \ell_0^2 \to \ell^2$ simply embeds an element of ℓ_0^2 into ℓ^2. On the other hand,

$$\langle k, \omega\rangle = \sum_{n=0}^{\infty}k_n\overline{\omega}_n = \sum_{n=0}^{\infty}(\tilde{J}m)_n\overline{\omega}_n.$$

Then,

$$\sum_{n=0}^{\infty}(\tilde{J}m)_n\overline{\omega}_n = \sum_{n=0}^{\infty}((\tilde{J}+\tilde{I})m)_n\overline{\hat{\omega}}_n,$$

for $m \in \ell_0^2$, $\hat{\omega} \in Q(\mathbf{s_{2,F}^{-1}})$ and $\omega \in \ell^2$.

We continue by using the summation by parts formula on both the left and right-hand sides of this equation twice, similar to its use in Lemma 4.1.1. For the left-hand side, we see that

$$\sum_{n=0}^{\infty}(\tilde{J}m)_n\overline{\omega}_n = -\sum_{n=1}^{\infty} m_n\Delta(\Delta\overline{\omega}_{n-1}),$$

since $m_0 = 0$, whilst the right-hand side gives that

$$\begin{aligned}\sum_{n=0}^{\infty}((\tilde{J}+\tilde{I})m)_n\overline{\hat{\omega}}_n &= -\sum_{n=1}^{\infty} m_n\Delta(\Delta\overline{\hat{\omega}}_{n-1}) + \sum_{n=1}^{\infty} m_n\overline{\hat{\omega}}_n \\ &= \sum_{n=1}^{\infty} m_n\left[-\Delta(\Delta\overline{\hat{\omega}}_{n-1}) + \overline{\hat{\omega}}_n\right].\end{aligned}$$

Hence, for arbitrary $m \in \ell_0^2$, we have

$$\langle m, J\omega\rangle = \langle m, (J+I)\hat{\omega}\rangle\,,$$

demonstrating that

$$J\omega = (J+I)\hat{\omega} - \alpha e_0, \tag{4.26}$$

for $\alpha \in \mathbb{C}$, where e_0 is the sequence $(1, 0, 0, \dots)$. Since we are looking for the linear operator A_2 such that $A_2\omega = \hat{\omega}$, the α in equation (4.26) will be dependent on ω but, crucially, unique. In other words, if we fix $\omega \in \ell^2$, then there exists precisely one $\hat{\omega} \in Q(\mathbf{s_{2,F}^{-1}})$ such that the inequality in equation (3.5) holds.

In order to determine the value that α takes for each ω, we continue by utilising a different decomposition of ℓ^2 to that given in (4.23); we find that α does not take the same value for all $\omega \in \ell^2$, so we must decompose ℓ^2 into distinct parts to accommodate this. In particular, we note that $\tilde{J}+\tilde{I}$ is a densely defined operator on ℓ_0^2, so we see that

$$\ell^2 = \overline{\mathcal{R}(\tilde{J}+\tilde{I})} \oplus \ker(\tilde{J}+\tilde{I})^*,$$

upon invoking the Rank-Nullity theorem. In fact, upon investigating this decomposition further we arrive at the following lemma.

Lemma 4.4.2. *The space ℓ^2 can be decomposed into*

$$\ell^2 = \mathcal{R}(\tilde{J}+\tilde{I}) \oplus \operatorname{span}\{\varphi_-\},$$

where $\varphi_- = (1, \lambda_-, \lambda_-^2, \dots) \in \ell^2$, for $\lambda_- = \frac{3-\sqrt{5}}{2}$.

Proof. To prove this lemma, we begin by showing that $\mathcal{R}(\tilde{J}+\tilde{I})$ is a closed set. Let $\{v^{(n)}\}$ be a sequence in $\mathcal{R}(\tilde{J}+\tilde{I})$ such that $v^{(n)} \to v$ in ℓ^2. Then, we show that $v \in \mathcal{R}(\tilde{J}+\tilde{I})$.

Since $v^{(n)} \in \mathcal{R}(\tilde{J}+\tilde{I})$, there exists a $u^{(n)} \in \ell_0^2$ such that $v^{(n)} = (\tilde{J}+\tilde{I})u^{(n)}$. In fact, if E denotes the embedding of a sequence of ℓ_0^2 into ℓ^2, then it is clear to see that $v^{(n)} = (J+I)Eu^{(n)}$. Since $J+I$ is invertible, $(J+I)^{-1}v^{(n)} = Eu^{(n)}$. Hence,

$$Eu^{(n)} = (J+I)^{-1}v^{(n)} \to (J+I)^{-1}v =: \tilde{u}$$

as $n \to \infty$, since $(J+I)^{-1}$ is bounded (see: Lemma 4.2.3). Since $u^{(n)} \in \ell_0^2$, its first component, $u_0^{(n)}$, is equal to zero, and so $\tilde{u}_0$ must equal zero too. Since $\tilde{u}_0 = 0$, there exists a $u \in \ell_0^2$ such that $Eu = \tilde{u}$ and, in particular, $u^{(n)} \to u$ as $n \to \infty$. Then, we may conclude that

$$v = (J+I)\tilde{u} = (J+I)Eu = (\tilde{J}+\tilde{I})u,$$

verifying that $v \in \mathcal{R}(\tilde{J}+\tilde{I})$. Hence $\overline{\mathcal{R}(\tilde{J}+\tilde{I})} = \mathcal{R}(\tilde{J}+\tilde{I})$.

We continue the proof by calculating the kernel of $(\tilde{J}+\tilde{I})^*$. First, we note that $\tilde{J}+\tilde{I}\colon \ell_0^2 \to \ell^2$, and so $(\tilde{J}+\tilde{I})^*\colon \ell^2 \to \ell_0^2$. Consider the equation

$$\left\langle (\tilde{J}+\tilde{I})x, y \right\rangle = \left\langle x, (\tilde{J}+\tilde{I})^* y \right\rangle,$$

for $x \in \mathcal{D}(\tilde{J}+\tilde{I}) = \ell_0^2$ and $y \in \mathcal{D}((\tilde{J}+\tilde{I})^*)$. Then, upon applying the summation by parts formula twice to the left-hand side of this equation, we see that

$$\sum_{n=0}^{\infty} ((\tilde{J}+\tilde{I})x)_n \overline{y}_n = \sum_{n=1}^{\infty} x_n \left[-\Delta(\Delta \overline{y}_{n-1}) + \overline{y}_n \right].$$

This equality clearly holds for all $y \in \ell^2$. Then we may compare this to $\left\langle x, (\tilde{J}+\tilde{I})^* y \right\rangle$ to see that $((\tilde{J}+\tilde{I})^* y)_n = -\Delta(\Delta y_{n-1}) + y_n$ for all $n \geq 1$. In order to determine the first component, we merely recall that $(\tilde{J}+\tilde{I})^*$ maps into ℓ_0^2, resulting in

$$((\tilde{J}+\tilde{I})^* y)_n = \begin{cases} 0, & n = 0, \\ -\Delta(\Delta y_{n-1}) + y_n, & n \geq 1, \end{cases}$$

for $y \in \ell^2$. All that now remains is to determine $\ker(\tilde{J}+\tilde{I})^*$. In fact, we have performed these calculations before: the general solution to

$$-\Delta(\Delta y_{n-1}) + y_n = 0$$

is given by $y_n = c_1 \lambda_+^n + c_2 \lambda_-^n$, for constants c_1 and c_2, where $\lambda_\pm = \frac{3 \pm \sqrt{5}}{2}$. However, we must set $c_1 = 0$ to ensure that $y \in \ell^2$. In fact, we may immediately conclude that $y_n = c_2 \lambda_-^n$, for $n \geq 0$, as there is no additional first row condition that must be adhered to. Hence,

$$\ker(\tilde{J}+\tilde{I})^* = \operatorname{span}\{\varphi_-\},$$

where $\varphi_- = (1, \lambda_-, \lambda_-^2, \dots)$.

This shows that we may decompose ℓ^2 into $\ell^2 = \mathcal{R}(\tilde{J}+\tilde{I}) \oplus \operatorname{span}\{\varphi_-\}$, as required. ■

With this decomposition in mind, we may take ω in $\mathcal{R}(\tilde{J}+\tilde{I})$ and $\operatorname{span}\{\varphi_-\}$ separately and determine how A_2 acts on each set individually.

4.4.1 Determining A_2 on $\mathcal{R}(\tilde{J}+\tilde{I})$

Note that for any sequence $x = (x_0, x_1, x_2, \dots)$, we may write $x = \tilde{x} + x_0 e_0$, where $\tilde{x} = (0, x_1, x_2, \dots)$ and $e_0 = (1, 0, 0, \dots)$. Hence, for $\omega \in \mathcal{R}(\tilde{J}+\tilde{I})$ and $\hat{\omega} \in Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}})$, we have

$$\omega = \tilde{\omega} + \omega_0 e_0 \qquad \text{and} \qquad \hat{\omega} = \tilde{\hat{\omega}} + \hat{\omega}_0 e_0,$$

when decomposed as above. Then, equation (4.26) implies that

$$\begin{aligned}
\hat{\omega} &= J\omega - J\hat{\omega} + \alpha e_0 \\
&= J(\omega - \hat{\omega}) + \alpha e_0 \\
&= \tilde{J}(\tilde{\omega} - \tilde{\hat{\omega}}) + (\omega_0 - \hat{\omega}_0)Je_0 + \alpha e_0,
\end{aligned}$$

since $\tilde{\omega}$, $\tilde{\hat{\omega}} \in \ell_0^2$. If we can show that $\omega_0 = \hat{\omega}_0$ and $\alpha = 0$, then $\hat{\omega} = \tilde{J}(\tilde{\omega} - \tilde{\hat{\omega}})$. This equality is critical upon recalling that $\mathcal{R}(\tilde{J}) \subseteq Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}})$: we will have found the unique element $\hat{\omega} \in Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}})$ such that $A_2\omega = \hat{\omega}$, provided that the following inequality holds:

$$\|\hat{\omega}\|_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}} \leq \|\omega\|. \tag{4.27}$$

Let $\alpha = 0$. Then, $J\omega = (J+I)\hat{\omega}$ and, in particular, $(J+I)^{-1}J\omega = \hat{\omega}$, since $J+I$ is invertible. Since we have specified that $\omega \in \mathcal{R}(\tilde{J}+\tilde{I})$ and we know that $\tilde{J}+\tilde{I} = (J+I) \restriction \ell_0^2$, there exists a $v \in \ell_0^2$ such that $\omega = (J+I)v$. Therefore,

$$\begin{aligned}
\hat{\omega} &= (J+I)^{-1}J\omega \\
&= (J+I)^{-1}(J+I-I)\omega \\
&= \omega - (J+I)^{-1}\omega \\
&= \omega - v.
\end{aligned} \tag{4.28}$$

As $v \in \ell_0^2$, this equality shows that we must, in fact, have $\hat{\omega}_0 = \omega_0$ whenever $\alpha = 0$. With this candidate in mind, all that remains is to verify that inequality (4.27) holds.

Since $\hat{\omega} \in \mathcal{R}(\tilde{J})$,

$$\begin{aligned}\|\hat{\omega}\|^2_{\mathbf{s}^{-1}_{\mathbf{2},\mathbf{F}}} &= \mathbf{s}^{-1}_{\mathbf{2},\mathbf{F}}[\hat{\omega}, \hat{\omega}] + \langle \hat{\omega}, \hat{\omega} \rangle \\ &= \left\langle \tilde{J}^{-1}\hat{\omega}, \hat{\omega} \right\rangle + \langle \hat{\omega}, \hat{\omega} \rangle .\end{aligned}$$

From equation (4.28), we see that $\hat{\omega} = \omega - v$ for $v \in \ell^2_0$. It is then immediate that $\omega - \hat{\omega} \in \ell^2_0$, and so equation (4.26) implies that $\hat{\omega} = \tilde{J}(\omega - \hat{\omega})$. In fact, we may invert $\tilde{J}$ and conclude that $\tilde{J}^{-1}\hat{\omega} = \omega - \hat{\omega}$. Hence,

$$\begin{aligned}\|\hat{\omega}\|^2_{\mathbf{s}^{-1}_{\mathbf{2},\mathbf{F}}} &= \langle \omega - \hat{\omega}, \hat{\omega} \rangle + \langle \hat{\omega}, \hat{\omega} \rangle \\ &= \langle \omega, \hat{\omega} \rangle \\ &= \langle \omega, \omega - v \rangle \\ &= \|\omega\|^2 - \langle \omega, v \rangle .\end{aligned}$$

Since $\omega = (J + I)v$, and $(J + I)$ is a positive operator, we may then conclude that

$$\begin{aligned}\|\hat{\omega}\|^2_{\mathbf{s}^{-1}_{\mathbf{2},\mathbf{F}}} &= \|\omega\|^2 - \langle (J + I)v, v \rangle \\ &\leq \|\omega\|^2,\end{aligned}$$

verifying that inequality (4.27) holds for all $\omega \in \mathcal{R}(\tilde{J} + \tilde{I})$ and $\hat{\omega} \in \mathcal{R}(\tilde{J})$. Consequently, we are now able to assert that

$$\begin{aligned}A_2\omega &= (J + I)^{-1}J\omega \\ &= (I - (J + I)^{-1})\omega, \end{aligned} \tag{4.29}$$

for $\omega \in \mathcal{R}(\tilde{J} + \tilde{I})$.

4.4.2 Determining A_2 on span $\{\varphi_-\}$

Now that we have established how the operator A_2 is to act on an element $\omega \in \mathcal{R}(\tilde{J} + \tilde{I})$, we continue by considering $\omega \in \operatorname{span}\{\varphi_-\}$ instead. As before, we begin by investigating the equality given by equation (4.25), except all instances of ω will be replaced by $c\varphi_-$; in fact, we may look at the case when the coefficient c is equal to 1 due to the linearity of A_2. Then, we immediately obtain the following relationship from equation (4.26):

$$J\varphi_- = (J + I)\hat{\omega} - \alpha e_0. \tag{4.30}$$

We may rewrite equation (4.30) by noting that $J\varphi_- = (J + I)\varphi_- - \varphi_-$ and making use of the following lemma.

Lemma 4.4.3. *The ℓ^2-solution to the equation $(J+I)x = e_0$, where $e_0 = (1,0,0,\dots)$, is given by $x_n = \lambda_-^{n+1}$, where $\lambda_- = \frac{3-\sqrt{5}}{2}$, i.e., $x = \lambda_-\varphi_-$.*

Proof. Solving the equation $(J+I)x = e_0$ is equivalent to solving the following system of equations:

$$\begin{cases} -x_1 + 3x_0 = 1, & n = 0, \\ -x_{n+1} + 3x_n - x_{n-1} = 0, & n \geq 1. \end{cases}$$

Then, the general solution to the equation $-x_{n+1}+3x_n-x_{n-1} = 0$ is given by $x_n = c_1\lambda_+^n + c_2\lambda_-^n$ for constants c_1, $c_2 \in \mathbb{C}$, where $\lambda_\pm = \frac{3\pm\sqrt{5}}{2}$. To ensure that the solution to the system of equations lies in ℓ^2, we must take $c_1 = 0$. Then, by substituting our solution x_n into the initial condition $-x_1 + 3x_0 = 1$, we obtain

$$-c_2\lambda_- + 3c_2 = 1.$$

Upon rearranging this for c_2, we see that $c_2 = \frac{1}{(3-\lambda_-)}$ and, in particular, $c_2 = \lambda_+^{-1} = \lambda_-$. It is then clear that $x_n = \lambda_-^{n+1}$ is the ℓ^2-solution to the equation $(J+I)x = e_0$. ■

Since $x = \lambda_-\varphi_-$, we note that $(J+I)x = e_0 \iff \lambda_-(J+I)\varphi_- = e_0$. Hence, we may rewrite equation (4.30) as

$$J\varphi_- = (J+I)\hat{\omega} - \alpha\lambda_-(J+I)\varphi_-.$$

Then, in conjunction with the fact that $J\varphi_- = (J+I)\varphi_- - \varphi_-$, we may conclude that

$$(J+I)\varphi_- - \varphi_- = (J+I)\hat{\omega} - \alpha\lambda_-(J+I)\varphi_-$$

or, in other words,

$$(J+I)(\hat{\omega} - (\alpha\lambda_- + 1)\varphi_-) = -\varphi_-.$$

Since we are trying to uncover the relationship between φ_- and $\hat{\omega}$, we proceed by asking whether or not there exists an ℓ^2-solution x to the equation

$$(J+I)x = -\varphi_-.$$

If there *does*, then $x = \hat{\omega}-(\alpha\lambda_-+1)\varphi_-$ and, in particular, $\hat{\omega} = x+(\alpha\lambda_-+1)\varphi_-$. The following lemma shows that there is, indeed, such a sequence x that solves this equation.

Lemma 4.4.4. *The ℓ^2-solution to the equation $(J+I)x = -\varphi_-$, where $\varphi_- = (1,\lambda_-,\lambda_-^2,\dots)$ and $\lambda_- = \frac{3-\sqrt{5}}{2}$, is given by $x_n = -\frac{\lambda_-^n(n+1)}{\sqrt{5}}$.*

Proof. We begin the proof in a manner equivalent to that of Lemma 4.4.3, i.e., by noting that $(J+I)x = -\varphi_-$ is equivalent to solving the following system of equations:

$$\begin{cases} -x_1 + 3x_0 = -1, & n = 0, \\ -x_{n+1} + 3x_n - x_{n-1} = -\lambda_-^n, & n \geq 1. \end{cases} \tag{4.31}$$

Furthermore, we know that the general solution to $-x_{n+1} + 3x_n - x_{n-1} = 0$ is given by the sequence $x_n = c_1\lambda_+^n + c_2\lambda_-^n$ for constants $c_1,\ c_2 \in \mathbb{C}$, where $\lambda_\pm = \frac{3\pm\sqrt{5}}{2}$. This time, however, we must construct a particular solution to equation (4.31) before we can determine the constants c_1 and c_2. Then, from the variation of constants formula, we see that the particular solution $\tilde{x}$ is given by

$$\tilde{x}_n = \frac{1}{W(\varphi_+, \varphi_-)} \left[\lambda_+^n \sum_{r=0}^{n-1} \lambda_-^r(-\lambda_-^r) - \lambda_-^n \sum_{r=0}^{n-1} \lambda_+^r(-\lambda_-^r) \right], \qquad n \geq 0,$$

where $\varphi_\pm = (1, \lambda_\pm, \lambda_\pm^2, \dots)$ and $W(\varphi_+, \varphi_-)$ is the Wronskian of the two sequences. In particular, we have

$$\begin{aligned} W(\varphi_+, \varphi_-) &= \begin{vmatrix} \lambda_+^n & \lambda_-^n \\ \lambda_+^{n+1} & \lambda_-^{n+1} \end{vmatrix} \\ &= (\lambda_+\lambda_-)^n\lambda_- - (\lambda_+\lambda_-)^n\lambda_+ \\ &= 1\cdot\lambda_- - 1\cdot\lambda_+ \\ &= -\sqrt{5}. \end{aligned}$$

Hence,

$$\begin{aligned} \tilde{x}_n &= \frac{1}{\sqrt{5}} \left[\lambda_+^n \sum_{r=0}^{n-1} \lambda_-^{2r} - \lambda_-^n \sum_{r=0}^{n-1} 1 \right] \\ &= \frac{1}{\sqrt{5}} \left[\lambda_+^n \left(\frac{\lambda_-^{2n} - 1}{\lambda_-^2 - 1} \right) - \lambda_-^n n \right], \end{aligned}$$

by noting that the first summation is a geometric series, since $|\lambda_-^2| < 1$. With this particular solution in mind, we may then assert that the solution x_n is given by

$$x_n = c_1\lambda_+^n + c_2\lambda_-^n + \frac{1}{\sqrt{5}} \left[\lambda_+^n \left(\frac{\lambda_-^{2n} - 1}{\lambda_-^2 - 1} \right) - \lambda_-^n n \right], \qquad n \geq 0,$$

for constants c_1 and c_2. However, this x_n does not not necessarily lie in ℓ^2 *or* satisfy the initial condition: we must choose c_1 and c_2 with these conditions in mind.

As we require x to lie in ℓ^2, we need to choose c_1 appropriately such that the coefficient of the growing term, λ_+^n, vanishes. In fact, we find it more suitable to rewrite x_n as

$$\begin{aligned} x_n &= \lambda_+^n\left(c_1 + \frac{\lambda_-^{2n}-1}{\sqrt{5}(\lambda_-^2-1)}\right) + \lambda_-^n\left(c_2 - \frac{n}{\sqrt{5}}\right) \\ &= \lambda_+^n\left(c_1 - \frac{1}{\sqrt{5}(\lambda_-^2-1)}\right) + \lambda_-^n\left(c_2 - \frac{n}{\sqrt{5}} + \frac{1}{\sqrt{5}(\lambda_-^2-1)}\right) \\ &= \lambda_+^n\left(c_1 + \frac{\lambda_+}{5}\right) + \lambda_-^n\left(c_2 - \frac{n}{\sqrt{5}} - \frac{\lambda_+}{5}\right), \end{aligned}$$

because we may simply read off the value of c_1 that ensures that the growth of λ_+^n will not pose an issue. In particular, we may set $c_1 = -\frac{\lambda_+}{5}$.

In order to find c_2, we make use of the initial row condition $-x_1+3x_0 = -1$. Then, we substitute our x_n into this equation to arrive at

$$-\lambda_-\left(c_2 - \frac{1}{\sqrt{5}} - \frac{\lambda_+}{5}\right) + 3\left(c_2 - \frac{\lambda_+}{5}\right) = -1.$$

Hence, upon rearranging this equation for c_2, we obtain

$$c_2 = \frac{-1 - \frac{\lambda_-}{\sqrt{5}} - \frac{1}{5} + \frac{3\lambda_+}{5}}{(3-\lambda_-)}.$$

Fortunately, c_2 can be simplified drastically. Indeed, manipulating the right-hand side of this equation eventually reveals that

$$c_2 = \frac{1}{5\lambda_+} = \frac{\lambda_-}{5}.$$

Now that we have obtained the constant c_2 explicitly, we can assert that the ℓ^2-solution of the equation $(J+I)x = -\varphi_-$ is given by

$$x_n = \lambda_-^n\left[\frac{\lambda_-}{5} - \frac{n}{\sqrt{5}} - \frac{\lambda_+}{5}\right] = -\frac{\lambda_-^n(n+1)}{\sqrt{5}}, \qquad n \geq 0,$$

where $\lambda_- = \frac{3-\sqrt{5}}{2}$. ■

With the ℓ^2-solution of the equation $(J+I)x = -\varphi_-$ now in hand, we are one step closer to determining the relationship between φ_- and $\hat{\omega}$. In particular, if $x = \hat{\omega} - (\alpha\lambda_- + 1)\varphi_-$, then, for $n \geq 0$, we have

$$\begin{aligned} \hat{\omega}_n &= x_n + (\alpha\lambda_- + 1)\lambda_-^n \\ &= \left[\left(\alpha\lambda_- + 1 - \frac{1}{\sqrt{5}}\right) - \frac{n}{\sqrt{5}}\right]\lambda_-^n. \end{aligned} \tag{4.32}$$

For $\omega \in \mathcal{R}(\tilde{J}+\tilde{I})$, we were easily able to identify the unique α such that $\hat{\omega} \in \mathcal{R}(\tilde{J})$, and so it was clear that $\hat{\omega} \in Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}})$. Unfortunately, that is not possible for $\omega \in \operatorname{span}\{\varphi_-\}$: there does not exist a sequence $y \in \ell_0^2$ such that $\tilde{J}y = \hat{\omega}$ — this will be proven shortly. Instead, since $\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}$ is a closed form, we must construct a sequence $f^{(N)} \in \mathcal{R}(\tilde{J})$ such that, for a unique α, the following two conditions hold:

$$f^{(N)} \to \hat{\omega} \text{ as } N \to \infty \text{ in } \ell^2 \tag{4.33}$$

and

$$\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}\left[f^{(N)} - f^{(M)}, f^{(N)} - f^{(M)}\right] \to 0 \text{ as } N,\, M \to \infty. \tag{4.34}$$

Together, these two conditions ensure that $\hat{\omega}$ will lie in $Q(\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}})$, as expressed in Definition 1.2.6. Furthermore, for this unique α, we will require an analogous inequality to (4.27) to hold, i.e., we must have that

$$\|\hat{\omega}\|_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}} \leq \|\varphi_-\|_{\ell^2}. \tag{4.35}$$

Then, we begin by proving the following lemma.

Lemma 4.4.5. *For any $\alpha \in \mathbb{C}$, there does not exist a sequence $y \in \ell_0^2$ such that $\tilde{J}y = \hat{\omega}$, where $\hat{\omega}$ is the sequence defined by equation (4.32).*

Proof. Rather than solve $\tilde{J}y = \hat{\omega}$ for $y \in \ell_0^2$, note that we may instead solve the equation $Jy = \hat{\omega}$ for $y \in \ell^2$, and *then* impose the condition $y_0 = 0$. Hence, we are looking for a sequence $y \in \ell^2$ such that

$$y_0 = 0 \qquad \text{and} \qquad \begin{cases} -y_1 + 2y_0 = \hat{\omega}_0, & n = 0, \\ -y_{n+1} + 2y_n - y_{n-1} = \hat{\omega}_n, & n \geq 1. \end{cases}$$

Then, the general solution to $-y_{n+1} + 2y_n - y_{n-1} = \hat{\omega}_n$ is given by

$$y_n = c_1 + c_2 n + 1 \cdot \sum_{r=0}^{n-1} r\hat{\omega}_r - n \sum_{r=0}^{n-1} 1 \cdot \hat{\omega}_r, \qquad n \geq 0,$$

after recalling the variation of constants formula presented in Lemma 1.3.4. Since we have an explicit expression for $\hat{\omega}$, i.e.,

$$\hat{\omega}_n = \left[\left(\alpha\lambda_- + 1 - \frac{1}{\sqrt{5}}\right) - \frac{n}{\sqrt{5}}\right]\lambda_-^n, \qquad n \geq 0,$$

we can simplify y_n by expanding the summations, after recalling that $|\lambda_-| < 1$. In particular, we require the following two results: let $|x| < 1$, then

$$\sum_{r=0}^{n-1} rx^r = \frac{nx^n}{x-1} - \frac{x(x^n - 1)}{(x-1)^2}$$

and

$$\sum_{r=0}^{n-1} r^2 x^r = \frac{n^2 x^n}{x-1} - \frac{2nx^{n+1}}{(x-1)^2} + \frac{(x^n - 1)(x+1)x}{(x-1)^3}.$$

These identities are a consequence of the formula for a geometric series, i.e.,

$$\sum_{r=0}^{n-1} x^r = \frac{x^n - 1}{x - 1}, \qquad |x| < 1,$$

upon differentiating both sides with respect to x, before multiplying by x and then simplifying.

With the general solution in place, we must now choose the constants c_1 and c_2 such that the various conditions are satisfied. Firstly, we choose c_1 and c_2 such that $y \in \ell^2$. Then, with the above results in mind, we simplify the sequence y_n as follows:

$$\begin{aligned}
y_n &= c_1 + c_2 n + \sum_{r=0}^{n-1} r\hat{\omega}_r - n \sum_{r=0}^{n-1} \hat{\omega}_r \\
&= c_1 + c_2 n + \left(\alpha\lambda_- + 1 - \frac{1}{\sqrt{5}}\right) \sum_{r=0}^{n-1} r\lambda_-^r - \frac{1}{\sqrt{5}} \sum_{r=0}^{n-1} r^2 \lambda_-^r \\
&\qquad - n \left[\left(\alpha\lambda_- + 1 - \frac{1}{\sqrt{5}}\right) \sum_{r=0}^{n-1} \lambda_-^r - \frac{1}{\sqrt{5}} \sum_{r=0}^{n-1} r\lambda_-^r \right] \\
&= c_1 + c_2 n + A \sum_{r=0}^{n-1} r\lambda_-^r - B \sum_{r=0}^{n-1} r^2 \lambda_-^r - n \left[A \sum_{r=0}^{n-1} \lambda_-^r - B \sum_{r=0}^{n-1} r\lambda_-^r \right],
\end{aligned}$$

where $A = \alpha\lambda_- + 1 - \frac{1}{\sqrt{5}}$ and $B = \frac{1}{\sqrt{5}}$. Hence,

$$\begin{aligned}
y_n &= c_1 + c_2 n + A \left(\frac{n\lambda_-^n}{\lambda_- - 1} - \frac{\lambda_-(\lambda_-^n - 1)}{(\lambda_- - 1)^2} \right) \\
&\quad - B \left(\frac{n^2 \lambda_-^n}{\lambda_- - 1} - \frac{2n\lambda_-^{n+1}}{(\lambda_- - 1)^2} + \frac{(\lambda_-^n - 1)(\lambda_- + 1)\lambda_-}{(\lambda_- - 1)^3} \right) \\
&\quad - n \left[A \left(\frac{\lambda_-^n - 1}{\lambda_- - 1} \right) - B \left(\frac{n\lambda_-^n}{\lambda_- - 1} - \frac{\lambda_-(\lambda_-^n - 1)}{(\lambda_- - 1)^2} \right) \right],
\end{aligned}$$

after inserting the finite sum results in. Collecting powers of n then yields

$$\begin{aligned}
y_n = &\left[c_1 - \frac{A\lambda_-^{n+1}}{(\lambda_- - 1)^2} + \frac{A\lambda_-}{(\lambda_- - 1)^2} - \frac{B\lambda^{n+1}(\lambda_- + 1)}{(\lambda_- - 1)^3} + \frac{B\lambda_-(\lambda_- + 1)}{(\lambda_- - 1)^3} \right] \\
&\qquad + n \left[c_2 + \frac{A}{\lambda_- - 1} + \frac{B\lambda_-^{n+1}}{(\lambda_- - 1)^2} + \frac{B\lambda_-}{(\lambda_- - 1)^2} \right],
\end{aligned}$$

after observing that the terms involving n^2 cancel out. We remark that both $\{\lambda_-^n\}$ and $\{n\lambda_-^n\}$ are sequences that lie in ℓ^2. Then, to ensure that $y \in \ell^2$, we must take

$$c_1 = -\frac{A\lambda_-}{(\lambda_- - 1)^2} - \frac{B\lambda_-(\lambda_- + 1)}{(\lambda_- - 1)^3} \quad \text{and} \quad c_2 = -\frac{A}{\lambda_- - 1} - \frac{B\lambda_-}{(\lambda_- - 1)^2},$$

i.e.,

$$c_1 = \frac{\alpha(\sqrt{5} - 3)}{2} + \frac{1}{\sqrt{5}} \quad \text{and} \quad c_2 = \frac{\alpha(\sqrt{5} - 1)}{2} + \frac{1}{\sqrt{5}},$$

upon recalling that $A = \alpha\lambda_- + 1 - \frac{1}{\sqrt{5}}$ and $B = \frac{1}{\sqrt{5}}$. Hence, for $n \geq 0$,

$$\begin{aligned}
y_n &= \left[-\frac{A\lambda_-^{n+1}}{(\lambda_- - 1)^2} - \frac{B\lambda_-^{n+1}(\lambda_- + 1)}{(\lambda_- - 1)^3} \right] + n \left[\frac{B\lambda_-^{n+1}}{(\lambda_- - 1)^2} \right] \\
&= -\left[\frac{\left(\alpha\lambda_- + 1 - \frac{1}{\sqrt{5}}\right)\lambda_-^{n+1}}{(\lambda_- - 1)^2} + \frac{\lambda_-^{n+1}(\lambda_- + 1)}{\sqrt{5}(\lambda_- - 1)^3} \right] + n \left[\frac{\lambda_-^{n+1}}{\sqrt{5}(\lambda_- - 1)^2} \right] \\
&= -\left[\left(\alpha\lambda_- + 1 - \frac{1}{\sqrt{5}}\right)\lambda_-^n + \frac{\lambda_-^n(\lambda_- + 1)}{\sqrt{5}(\lambda_- - 1)} \right] + \frac{n\lambda_-^n}{\sqrt{5}},
\end{aligned}$$

is the ℓ^2-solution to the equation $Jy = \hat{\omega}$, disregarding the initial condition.

Since $y \in \ell^2$ for *any* value of α, we need to show that there does not exist an α such that both $y_0 = 0$ and $-y_1 + 2y_0 = \hat{\omega}_0$ hold simultaneously. We begin by noting that

$$y_0 = c_1 = \frac{\alpha(\sqrt{5} - 3)}{2} + \frac{1}{\sqrt{5}} = -\alpha\lambda_- + \frac{1}{\sqrt{5}}.$$

Therefore, if we take

$$\alpha = \frac{1}{\sqrt{5}\lambda_-} = \frac{1}{2} + \frac{3\sqrt{5}}{10}$$

then we ensure that $y_0 = 0$. Now that we have a candidate value for α, we must verify that $-y_1 + 2y_0 = \hat{\omega}_0$ does *not* hold. By substituting this value of α into the first row condition, we see that

$$-y_1 + 2y_0 = -(c_1 + c_2 - \hat{\omega}_0) + 2 \cdot 0 = -\frac{1}{2} - \frac{3\sqrt{5}}{10} + \hat{\omega}_0.$$

Clearly, $-y_1 + 2y_0 \neq \hat{\omega}_0$, and so there is no single value of α such that both conditions are simultaneously satisfied. As such, we may conclude that there does not exist an ℓ_0^2-solution to the equation $\tilde{J}y = \hat{\omega}$, as required. ■

4.4.3 Constructing an Approximation for $\hat{\omega}$ in $\mathcal{R}(\tilde{J})$

Now that we have established that there does not exist an ℓ_0^2-solution to the equation $\tilde{J}y = \hat{\omega}$, we continue by constructing a sequence $f^{(N)}$ in $\mathcal{R}(\tilde{J})$ that approximates $\hat{\omega}$. We begin by considering the equation $\tilde{J}z = f$ for $f \in E_0$, where

$$E_0 = \left\{ x \in \ell^2 \;\middle|\; \exists N \in \mathbb{N}_0 \text{ such that } x_n = 0 \text{ for all } n > N \right\}.$$

We aim to arrive at general conditions that ensure that both z lies in ℓ_0^2 and an initial condition is satisfied: crucially, we guarantee that f will lie in $\mathcal{R}(\tilde{J})$. Then, the general solution to this equation is given by

$$z_n = 1 \cdot \left[c_1 + \sum_{r=0}^{n-1} r f_r \right] + n \cdot \left[c_2 - \sum_{r=0}^{n-1} f_r \right].$$

To ensure that $z \in \ell^2$, we must take $c_1 = -\sum_{r=0}^{\infty} r f_r$ and $c_2 = \sum_{r=0}^{\infty} f_r$; note that these expressions are finite since $f_r = 0$ for all r after some N. Then,

$$\begin{aligned} z_n &= 1 \cdot \left[-\sum_{r=0}^{\infty} r f_r + \sum_{r=0}^{n-1} r f_r \right] + n \cdot \left[\sum_{r=0}^{\infty} f_r - \sum_{r=0}^{n-1} f_r \right] \\ &= -1 \cdot \left[\sum_{r=n}^{\infty} r f_r \right] + n \cdot \left[\sum_{r=n}^{\infty} f_r \right] \\ &= \sum_{r=n}^{\infty} (n-r) f_r. \end{aligned} \tag{4.36}$$

In addition to the above, we must deduce two further conditions: we must have that $z_0 = 0$, i.e, $z \in \ell_0^2$, and we require the first row condition given by $-z_1 = f_0$ to be satisfied. These conditions are simple to formulate by virtue of equation (4.36): we merely require

$$-\sum_{r=0}^{\infty} r f_r = 0 \qquad \text{and} \qquad -\sum_{r=1}^{\infty} (1-r) f_r = f_0. \tag{4.37}$$

If we rewrite the latter condition as $\sum_{r=0}^{\infty}(1-r)f_r = 0$ then we may subtract these conditions from each other to produce two, less complicated expressions. In particular, the following two conditions are equivalent to those derived in (4.37):

$$\sum_{r=0}^{\infty} f_r = 0 \qquad \text{and} \qquad \sum_{r=0}^{\infty} r f_r = 0. \tag{4.38}$$

Thus, if an $f \in E_0$ satisfies both equations given in (4.38), then there exists a $z \in \ell_0^2$, where z is defined by equation (4.36), such that $\tilde{J} z = f$. Then, $f \in \mathcal{R}(\tilde{J})$.

With these calculations in mind, our candidate sequence $f^{(N)} \in E_0$ will be of the form

$$f_n^{(N)} = \begin{cases} \hat{\omega}_n, & 0 \le n \le N, \\ F_N, & N+1 \le n \le 2N, \\ G_N, & 2N+1 \le n \le 3N, \\ 0, & 3N+1 \le n, \end{cases}$$

where F_N and G_N are functions of N to be found. In fact, we may solve both of the conditions presented in (4.38) simultaneously to obtain these functions. To begin with, we note that

$$\begin{aligned} \sum_{r=0}^{\infty} f_r = 0 &\implies \sum_{r=N+1}^{2N} F_N + \sum_{r=2N+1}^{3N} G_N = -\sum_{r=0}^{N} \hat{\omega}_r \\ &\implies N F_N + N G_N = -\sum_{r=0}^{N} \hat{\omega}_r. \end{aligned} \tag{4.39}$$

Similarly, the second condition shows that

$$\begin{aligned}\sum_{r=0}^{\infty} r f_r = 0 &\implies \sum_{r=N+1}^{2N} rF_N + \sum_{r=2N+1}^{3N} rG_N = -\sum_{r=0}^{N} r\hat{\omega}_r \\ &\implies \frac{N(3N+1)}{2}F_N + \frac{N(5N+1)}{2}G_N = -\sum_{r=0}^{N} r\hat{\omega}_r. \end{aligned} \tag{4.40}$$

From equation (4.39), we have $NF_N = -NG_N - \sum_{r=0}^{N} \hat{\omega}_r$. We may then insert this into equation (4.40) and isolate G_N. After performing this calculation, we see that

$$G_N = \sum_{r=0}^{N} \left[\frac{3N+1-2r}{2N^2}\right] \hat{\omega}_r. \tag{4.41}$$

With G_N now defined, we can use equation (4.39) to conclude that

$$F_N = -\sum_{r=0}^{N} \left[\frac{5N+1-2r}{2N^2}\right] \hat{\omega}_r. \tag{4.42}$$

As such, our candidate for the sequence $f^{(N)}$ that approximates $\hat{\omega}$ is given by

$$f_n^{(N)} = \begin{cases} \hat{\omega}_n, & 0 \le n \le N, \\ -\sum_{r=0}^{N} \left[\frac{5N+1-2r}{2N^2}\right] \hat{\omega}_r, & N+1 \le n \le 2N, \\ \sum_{r=0}^{N} \left[\frac{3N+1-2r}{2N^2}\right] \hat{\omega}_r, & 2N+1 \le n \le 3N, \\ 0, & 3N+1 \le n. \end{cases} \tag{4.43}$$

With the sequence $f^{(N)}$ defined, we may also use equation (4.36) to determine the sequence $z^{(N)}$ that solves $\tilde{J}z^{(N)} = f^{(N)}$. In particular, we see that

$$z_n^{(N)} = \begin{cases} \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=N+1}^{2N} (n-r)F_N \\ \qquad\qquad\displaystyle + \sum_{r=n}^{N} (n-r)\hat{\omega}_r, & 0 \le n \le N, \\ \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=n}^{2N} (n-r)F_N, & N+1 \le n \le 2N, \\ \displaystyle\sum_{r=n}^{3N} (n-r)G_N, & 2N+1 \le n \le 3N, \\ 0, & 3N+1 \le n. \end{cases} \tag{4.44}$$

4.4.4 The Sequence $f^{(N)}$ Approximates $\hat{\omega}$ in ℓ^2

Now that we have explicit expressions for F_N and G_N, we must ask whether $f^{(N)}$ really does approximate $\hat{\omega}$ in ℓ^2, i.e., does $f^{(N)} \to \hat{\omega}$ as $N \to \infty$? However, as we are looking for a unique $\alpha \in \mathbb{C}$, we can first find an appropriate value for α now in order to streamline future calculations.

If F_N and G_N are defined as in equations (4.42) and (4.41), then the finite nature of the sum allows us to calculate the expressions themselves explicitly. In particular, define the following in MAPLE:

```
>  W:=(a,n)->(a*L+1-1/sqrt(5)-n/sqrt(5))*L^n
>  F:=(a,N)->-sum((5*N+1-2*r)/(2*N^2)*W(a,r),r=0..N)
>  G:=(a,N)->sum((3*N+1-2*r)/(2*N^2)*W(a,r),r=0..N),
```

where W, F, G, a and L signify $\hat{\omega}_n$, F_N, G_N, α and λ_- respectively. By using the 'collect' command on F_N and G_N, that is,

```
>  collect(F(a,N),N)
>  collect(G(a,N),N),
```

we can group both expressions in terms of decreasing powers of n. Indeed,

$$
\begin{aligned}
F_N = &-\frac{2(9\sqrt{5}-15)\left(\frac{3-\sqrt{5}}{2}\right)^{N+1}}{5(\sqrt{5}-1)^3} \\
&+\frac{1}{N}\left[-\frac{2(45\alpha\sqrt{5}+26\sqrt{5}-105\alpha-60)\left(\frac{3-\sqrt{5}}{2}\right)^{N+1}}{5(\sqrt{5}-1)^3}\right. \\
&\qquad\left.+\frac{2(75\alpha\sqrt{5}+20\sqrt{5}-175\alpha-50)}{5(\sqrt{5}-1)^3}\right] \\
&+\frac{1}{N^2}\left[-\frac{2(35\alpha\sqrt{5}+9\sqrt{5}-75\alpha-25)\left(\frac{3-\sqrt{5}}{2}\right)^{N+1}}{5(\sqrt{5}-1)^3}\right. \\
&\qquad\left.+\frac{2(65\alpha\sqrt{5}+12\sqrt{5}-145\alpha-30)}{5(\sqrt{5}-1)^3}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
G_N = &\frac{2(3\sqrt{5}-5)\left(\frac{3-\sqrt{5}}{2}\right)^{N+1}}{5(\sqrt{5}-1)^3} \\
&+\frac{1}{N}\left[\frac{2(15\alpha\sqrt{5}+12\sqrt{5}-35\alpha-30)\left(\frac{3-\sqrt{5}}{2}\right)^{N+1}}{5(\sqrt{5}-1)^3}\right. \\
&\qquad\left.-\frac{2(45\alpha\sqrt{5}+12\sqrt{5}-105\alpha-30)}{5(\sqrt{5}-1)^3}\right] \\
&+\frac{1}{N^2}\left[\frac{2(35\alpha\sqrt{5}+9\sqrt{5}-75\alpha-25)\left(\frac{3-\sqrt{5}}{2}\right)^{N+1}}{5(\sqrt{5}-1)^3}\right. \\
&\qquad\left.-\frac{2(65\alpha\sqrt{5}+12\sqrt{5}-145\alpha-30)}{5(\sqrt{5}-1)^3}\right].
\end{aligned}
$$

Clearly, both F_N and G_N tend to 0 for any $\alpha \in \mathbb{C}$ as $N \to \infty$, however both NF_N and NG_N do *not*. Then, it is our intention to choose an α such that both NF_N and NG_N tend to zero as $N \to \infty$. In fact, upon solving both

$$\frac{2(75\alpha\sqrt{5} + 20\sqrt{5} - 175\alpha - 50)}{5(\sqrt{5}-1)^3} = 0$$

and

$$-\frac{2(45\alpha\sqrt{5} + 12\sqrt{5} - 105\alpha - 30)}{5(\sqrt{5}-1)^3} = 0$$

simultaneously for α, we obtain one, single candidate: $\alpha = -\frac{1}{2} - \frac{\sqrt{5}}{10}$. As such, all calculations performed during this section will now use this value of α.

Remark. It may seem miraculous that the same α ensures that NF_N and NG_N both tend to 0, but this is merely a consequence of their form. In particular, observe that

$$F_N = -\left[G_N - \frac{1}{N}\sum_{r=0}^{N} \hat{\omega}_r\right].$$

Then, we can explicitly calculate $\sum_{r=0}^{N} \hat{\omega}_r$ by using the geometric series identities presented in Lemma 4.4.5. Hence,

$$\sum_{r=0}^{N} \hat{\omega}_r = \frac{(5-\sqrt{5})(N+1)\left(\frac{3-\sqrt{5}}{2}\right)^N}{10}.$$

Then, it is clear that $\sum_{r=0}^{N} \hat{\omega}_r$ and $\frac{1}{N}\sum_{r=0}^{N} \hat{\omega}_r$ both tend to 0 as $N \to \infty$. As such, the behaviour of NF_N as $N \to \infty$ coincides precisely with that of NG_N.

We now continue by proving that $f^{(N)}$ tends to $\hat{\omega}$ in ℓ^2, that is, we must show that $\|f^{(N)} - \hat{\omega}\|^2 \to 0$ as $N \to \infty$. Then,

$$\begin{aligned}
\|f^{(N)} - \hat{\omega}\|^2 &= \sum_{n=0}^{\infty} |f_n^{(N)} - \hat{\omega}_n|^2 \\
&= \sum_{n=N+1}^{2N} |F_N - \hat{\omega}_n|^2 + \sum_{n=2N+1}^{3N} |G_N - \hat{\omega}_n|^2 \\
&\leq 2\left[\sum_{n=N+1}^{2N} |F_N|^2 + |\hat{\omega}_n|^2\right] + 2\left[\sum_{n=2N+1}^{3N} |G_N|^2 + |\hat{\omega}_n|^2\right],
\end{aligned}$$

since $|x-y|^2 \leq 2(|x|^2 + |y|^2)$ for any two complex numbers x and y. Moreover,

$$\begin{aligned}
\|f^{(N)} - \hat{\omega}\|^2 &\leq 2N\left(|F_N|^2 + |G_N|^2\right) + 2\left[\sum_{n=N+1}^{3N} |\hat{\omega}_n|^2\right] \\
&\leq 2N\left(|F_N|^2 + |G_N|^2\right) + 2\left[\sum_{n=N+1}^{\infty} |\hat{\omega}_n|^2\right].
\end{aligned}$$

As both NF_N and NG_N tend to 0 as $N \to \infty$ and $\hat{\omega} \in \ell^2$, the entire expression must also tend to 0. Thus, we may conclude that $f^{(N)} \to \hat{\omega}$ as $N \to \infty$ in ℓ^2, and so the condition given in (4.33) is satisfied.

4.4.5 The Sequence $f^{(N)}$ is a Cauchy Sequence in $\|\cdot\|_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-1}}$

With the condition given in (4.33) now satisfied, we must continue by verifying that

$$\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}\left[f^{(N)}-f^{(M)},f^{(N)}-f^{(M)}\right]\to 0 \text{ as } N,M\to\infty,$$

where $f^{(N)}$ and $f^{(M)}$ are sequences defined by equation (4.43). We will assume that $N>M$ here and in what follows.

Due to our construction of the sequence $f^{(N)}$, we know that there exists a $z^{(N)}\in\mathcal{R}(\tilde{J})$ such that $\tilde{J}z^{(N)}=f^{(N)}$. Then, we may rewrite the form as

$$\begin{aligned}\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}\left[f^{(N)}-f^{(M)},f^{(N)}-f^{(M)}\right]&=\left\langle\tilde{J}^{-1}\left(f^{(N)}-f^{(M)}\right),f^{(N)}-f^{(M)}\right\rangle\\&=\left\langle z^{(N)}-z^{(M)},f^{(N)}-f^{(M)}\right\rangle.\end{aligned}$$

The two sequences $z^{(N)}-z^{(M)}$ and $f^{(N)}-f^{(M)}$ that appear in the form above depend on the relationship between N and M; specifically, there are seven distinct cases that must be considered individually before we can confirm that the condition given in (4.34) is, indeed, satisfied.

Case I: $M<\frac{N}{3}$

It is prudent to first find expressions that both $f^{(N)}-f^{(M)}$ and $z^{(N)}-z^{(M)}$ take. In particular, we can use equation (4.43) to determine $\left(f^{(N)}-f^{(M)}\right)_n$, whilst $\left(z^{(N)}-z^{(M)}\right)_n$ must be obtained using equation (4.44). Hence, if $M<\frac{N}{3}$, then we have

$$\left(f^{(N)}-f^{(M)}\right)_n=\begin{cases}0, & 0\le n\le M,\\ \hat{\omega}_n-F_M, & M+1\le n\le 2M,\\ \hat{\omega}_n-G_M, & 2M+1\le n\le 3M,\\ \hat{\omega}_n, & 3M+1\le n\le N,\\ F_N, & N+1\le n\le 2N,\\ G_N, & 2N+1\le n\le 3N,\\ 0, & 3N+1\le n,\end{cases}$$

whilst $\left(z^{(N)}-z^{(M)}\right)_n$ is given by equation (B.2) as found in Appendix B. We can then insert these expressions into the form to obtain

$$\begin{aligned}\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}}\left[f^{(N)}-f^{(M)},f^{(N)}-f^{(M)}\right]&=\left\langle z^{(N)}-z^{(M)},f^{(N)}-f^{(M)}\right\rangle\\&=A+B+C+D+E,\end{aligned}$$

where

$$\begin{aligned}
A = &\sum_{n=M+1}^{2M}\left[\sum_{r=2N+1}^{3N}(n-r)G_N + \sum_{r=N+1}^{2N}(n-r)F_N + \sum_{r=3M+1}^{N}(n-r)\hat{\omega}_r\right. \\
&\left. + \sum_{r=2M+1}^{3M}(n-r)(\hat{\omega}_r - G_M) + \sum_{r=n}^{2M}(n-r)(\hat{\omega}_r - F_M)\right](\hat{\omega}_n - F_M), \\
B = &\sum_{n=2M+1}^{3M}\left[\sum_{r=2N+1}^{3N}(n-r)G_N + \sum_{r=N+1}^{2N}(n-r)F_N + \sum_{r=3M+1}^{N}(n-r)\hat{\omega}_r\right. \\
&\left. + \sum_{r=n}^{3M}(n-r)(\hat{\omega}_r - G_M)\right](\hat{\omega}_n - G_M), \\
C = &\sum_{n=3M+1}^{N}\left[\sum_{r=2N+1}^{3N}(n-r)G_N + \sum_{r=N+1}^{2N}(n-r)F_N + \sum_{r=n}^{N}(n-r)\hat{\omega}_r\right]\hat{\omega}_n, \\
D = &\sum_{n=N+1}^{2N}\left[\sum_{r=2N+1}^{3N}(n-r)G_N + \sum_{r=n}^{2N}(n-r)F_N\right]F_N \qquad \text{and} \\
E = &\sum_{n=2N+1}^{3N}\left[\sum_{r=n}^{3N}(n-r)G_N\right]G_N.
\end{aligned}$$

Our aim is to show that as N and M tend to infinity, all five of these terms individually tend to 0. If this is indeed the case, then the sequence $f^{(N)}$ is Cauchy with respect to the form $\mathbf{s}^{-1}_{\mathbf{2},\mathbf{F}}$, whenever $M < \frac{N}{3}$.

The strategy that we will employ when proving that the form is Cauchy for $M < \frac{N}{3}$ is as follows. Firstly, we need to ensure that the denominator of the expressions A to E do not approach zero for large N and M; this is necessary as the form would almost certainly *not* tend to 0 under this circumstance. We then compute the summation using MAPLE and analyse the output. In particular, as $N > M$, expressions that multiply $\left(\frac{3-\sqrt{5}}{2}\right)^N$ decay faster than both expressions multiplied by $\left(\frac{3-\sqrt{5}}{2}\right)^M$ and rational functions in N and M — note that we only experience positive powers of $\frac{3-\sqrt{5}}{2}$. As such, we may treat anything involving $\left(\frac{3-\sqrt{5}}{2}\right)^N$ as 0 and inspect the remaining expression. Then, we can continue by further treating $\left(\frac{3-\sqrt{5}}{2}\right)^M$ as 0 due to its speed of decay — terms involving this factor will certainly tend to 0 providing that the term does not diverge as $N \to \infty$. All that then remains is to confirm that the surviving terms — solely rational functions in N and M — tend to 0 as N and M tend to infinity.

For optimal clarity, we choose to illustrate the method for a single term: the first term in the expression A above, i.e.,

$$T_1 \coloneqq \sum_{n=M+1}^{2M}\left[\sum_{r=2N+1}^{3N}(n-r)G_N\right](\hat{\omega}_n - F_M).$$

It does not take MAPLE very long to compute this summation: indeed, we are able to extract the denominator of T_1, upon factoring, with relative ease. In particular, $\operatorname{denom}(T_1) = 80NM$. Clearly, $\operatorname{denom}(T_1) = 0$ if and only if either N or M are equal to 0: this situation is of no concern to us since N and M are to grow larger.

The calculations show that every term in the expression T_1 is of the form

$$-\frac{1}{80NM}\left[aN^bM^c\lambda_-^{dN+eM+f}\right] = \tilde{a}N^{b-1}M^{c-1}\lambda_-^{dN+eM},$$

for some constants a, $\tilde{a} \in \mathbb{R}$ and b, c, d, e, $f \in \{0, 1, \ldots, 4\}$: note that we may extract any value of f into the coefficient $\tilde{a}$.

Remark. We stress that not every combination of exponents exists.

Then, all terms with $d \geq 1$ vanish as N tends to infinity. In fact, setting these terms to be zero leaves terms only of the form

$$\tilde{a}M^{c-1}\lambda_-^{eM} \qquad \text{or} \qquad \frac{\tilde{a}M^c\lambda_-^{eM}}{N},$$

with $\tilde{a} \in \mathbb{R}$ and c, $e \in \{0, 1, 2\}$.

Now, we may discard all terms with $e = 1$ and $e = 2$ since these terms will tend to 0. Then, the only the terms remaining are of the form $\frac{\tilde{a}}{M}$ or $\frac{\tilde{a}}{N}$, for $\tilde{a} \in \mathbb{R}$. Clearly, as M and N tend to infinity, these terms tend to 0, verifying that the entire expression T_1 tends to 0. Of course, this is merely one of many terms: we must ensure that each term tends to 0. Fortunately, this method is transferable to the other expressions and produces favourable results — we are then able to conclude that the first case holds as anticipated.

Case II: $M = \frac{N}{3}$

Although the strategy remains the same during Case II, it is easier to prove that the sequence $f^{(N)}$ is Cauchy with respect to the form when $M = \frac{N}{3}$ as the expression is merely in terms of M. Effectively, we must show that the form tends to 0 as M tends to infinity. In this case, $f^{(N)} - f^{(M)}$ is remarkably similar to that of Case I; in particular, one of the intervals collapses and we are left with

$$\left(f^{(N)} - f^{(M)}\right)_n = \begin{cases} 0, & 0 \leq n \leq M, \\ \hat{\omega}_n - F_M, & M+1 \leq n \leq 2M, \\ \hat{\omega}_n - G_M, & 2M+1 \leq n \leq N, \\ F_N, & N+1 \leq n \leq 2N, \\ G_N, & 2N+1 \leq n \leq 3N, \\ 0, & 3N+1 \leq n. \end{cases}$$

Note that the sequence $z^{(N)} - z^{(M)}$ is given by equation (B.3) in Appendix B.

Then, we consider the analogous term to that given in Case I in order to demonstrate the method more concretely; as $M = \frac{N}{3}$, we will analyse

$$T_2 := \sum_{n=M+1}^{2M} \left[\sum_{r=6M+1}^{9M} (n-r) G_{3M} \right] (\hat{\omega}_n - F_M),$$

for the purpose of comparison. In fact, since T_2 is an expression only involving M, it is enough to simply take M to infinity immediately. Indeed, if we use the MAPLE code expressed earlier in this section for $\hat{\omega}_n$, F_N and G_N, in conjunction with the 'limit' command, then we see that T_2 tends to 0 as $M \to \infty$. In particular, if L denotes λ_- and a denotes α, then

```
>  W:=(a,n)->(a*L+1-1/sqrt(5)-n/sqrt(5))*L^n
>  F:=(a,N)->-sum((5*N+1-2*r)/(2*N^2)*W(a,r),r=0..N)
>  G:=(a,N)->sum((3*N+1-2*r)/(2*N^2)*W(a,r),r=0..N),
>  sum(sum((n-r)*G(-1/2-sqrt(5)/10,3*M), r=6*M+1..9*M)
*(W(-1/2-sqrt(5)/10,n)-F(-1/2-sqrt(5)/10,M)),n=M+1..2*M)
>  limit(%,M=infinity)
```

will yield 0 upon setting $\alpha = -\frac{1}{2} - \frac{\sqrt{5}}{10}$, as required.

Case III: $\frac{N}{3} < M < \frac{N}{2}$

When $\frac{N}{3} < M < \frac{N}{2}$, the sequence $f^{(N)} - f^{(M)}$ is such that

$$\left(f^{(N)} - f^{(M)}\right)_n = \begin{cases} 0, & 0 \le n \le M, \\ \hat{\omega}_n - F_M, & M+1 \le n \le 2M, \\ \hat{\omega}_n - G_M, & 2M+1 \le n \le N, \\ F_N - G_M, & N+1 \le n \le 3M, \\ F_N, & 3M+1 \le n \le 2N, \\ G_N, & 2N+1 \le n \le 3N, \\ 0, & 3N+1 \le n, \end{cases}$$

whilst $z^{(N)} - z^{(M)}$ is given by equation (B.4) in Appendix B.

Since M does not equal some constant multiplied by N, we find ourselves more in line with Case I. However, if we are to follow the same strategy as in Case I, then we must be mindful of a minor, yet non-trivial, difference. Unlike in Case I, M grows at the same rate as N: at no point can we fix a value of M and take N to infinity. In practice, this simply means that we must estimate M using the case-inequality after we have let powers of $\frac{3-\sqrt{5}}{2}$ tend to 0 — we will only be left with rational functions in N and M, so this is reasonable. In particular, we know that $M < \frac{N}{2}$ and $\frac{1}{M} < \frac{3}{N}$ so we must ensure that the numerator of this rational function has a power of N that is less than the denominator so that when N tends to infinity, the function tends to 0. As such, we consider the analogous expression in this case in order to demonstrate the difference.

Consider the expression

$$T_3 \coloneqq \sum_{n=M+1}^{2M} \left[\sum_{r=2N+1}^{3N} (n-r)G_N \right] (\hat{\omega}_n - F_M).$$

Again, we observe that $\operatorname{denom}(T_3) = 80NM$, and so there are no problematic values of N and M that we must be wary of. Then, if we expand T_3 and set all terms involving λ_-^N and λ_-^M to be 0, we are left with

$$-\frac{(3+\sqrt{5})(5N-3M)}{20NM}.$$

Since we have an explicit bound on M in terms of N, if we are to appropriately estimate this expression by N, then we observe that the denominator is always of a greater power of N than the numerator. As such, as $N \to \infty$, this expression will tend to 0, as we require. Again, we must check that all terms tend to 0 before we can conclude that the sequence is Cauchy with respect to the form, but it is not too difficult with this process in mind.

Case IV: $M = \frac{N}{2}$

The remaining four cases are all similar to a previous case: Cases IV and VI are similar to Case II, whilst V and VII are similar to III. This is because the even cases are all special versions of the case before it — identical to the interplay between Case I and II — whilst the odd cases all have an M which is bounded by a multiple of N. As such, we employ the appropriate strategy when we are to compute the expression $\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}} \left[f^{(N)} - f^{(M)}, f^{(N)} - f^{(M)} \right]$.

However, we only choose to present the sequence $f^{(N)} - f^{(M)}$ here since there is nothing more to discuss: the method remains the same, and there are no surprises that we must draw attention to. As such, when $M = \frac{N}{2}$,

$$\left(f^{(N)} - f^{(M)}\right)_n = \begin{cases} 0, & 0 \le n \le M, \\ \hat{\omega}_n - F_M, & M+1 \le n \le N, \\ F_N - G_M, & N+1 \le n \le 3M, \\ F_N, & 3M+1 \le n \le 2N, \\ G_N, & 2N+1 \le n \le 3N, \\ 0, & 3N+1 \le n, \end{cases}$$

whilst $z^{(N)} - z^{(M)}$ is given by equation (B.5) in Appendix B.

Case V: $\frac{N}{2} < M < \frac{2N}{3}$

When $\frac{N}{2} < M < \frac{2N}{3}$, the sequence $f^{(N)} - f^{(M)}$ is given by

$$\left(f^{(N)} - f^{(M)}\right)_n = \begin{cases} 0, & 0 \leq n \leq M, \\ \hat{\omega}_n - F_M, & M+1 \leq n \leq N, \\ F_N - F_M, & N+1 \leq n \leq 2M, \\ F_N - G_M, & 2M+1 \leq n \leq 3M, \\ F_N, & 3M+1 \leq n \leq 2N, \\ G_N, & 2N+1 \leq n \leq 3N, \\ 0, & 3N+1 \leq n. \end{cases}$$

Case VI: $M = \frac{2N}{3}$

When $M = \frac{2N}{3}$, the sequence $f^{(N)} - f^{(M)}$ is given by

$$\left(f^{(N)} - f^{(M)}\right)_n = \begin{cases} 0, & 0 \leq n \leq M, \\ \hat{\omega}_n - F_M, & M+1 \leq n \leq N, \\ F_N - F_M, & N+1 \leq n \leq 2M, \\ F_N - G_M, & 2M+1 \leq n \leq 2N, \\ G_N, & 2N+1 \leq n \leq 3N, \\ 0, & 3N+1 \leq n. \end{cases}$$

Case VII: $\frac{2N}{3} < M < N$

When $\frac{2N}{3} < M < N$, the sequence $f^{(N)} - f^{(M)}$ is given by

$$\left(f^{(N)} - f^{(M)}\right)_n = \begin{cases} 0, & 0 \leq n \leq M, \\ \hat{\omega}_n - F_M, & M+1 \leq n \leq N, \\ F_N - F_M, & N+1 \leq n \leq 2M, \\ F_N - G_M, & 2M+1 \leq n \leq 2N, \\ G_N - G_M, & 2N+1 \leq n \leq 3M, \\ G_N, & 3M+1 \leq n \leq 3N, \\ 0, & 3N+1 \leq n. \end{cases}$$

Note that the sequences $z^{(N)} - z^{(M)}$ in Case V, VI and VII can be found in Appendix B: they are given by equations (B.6), (B.7) and (B.8), respectively.

With all seven cases now explored thoroughly, we may finally conclude that the sequence $f^{(N)}$ given by equation (4.43) not only approximates $\hat{\omega}$ but also is a Cauchy sequence with respect to the form $\mathbf{s}_{\mathbf{2},\mathbf{F}}^{\mathbf{-1}}$. This is crucial, as it demonstrates that the two conditions given by (4.33) and (4.34) are satisfied.

4.4.6 The Inequality $\|\hat{\omega}\|_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{\mathbf{-1}}} \leq \|\varphi_{-}\|_{\ell^2}$ Holds

We have now proven that $f^{(N)} \to \hat{\omega}$ in ℓ^2 and that $f^{(N)}$ is a Cauchy sequence with respect to the norm given by the form $\mathbf{s}_{\mathbf{2},\mathbf{F}}^{\mathbf{-1}}$. All that remains is to verify that the inequality given by (4.35) holds, that is,

$$\|\hat{\omega}\|_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{\mathbf{-1}}} \leq \|\varphi_{-}\|_{\ell^2},$$

for $(\varphi_{-})_n = \lambda_{-}^{n} = \left(\frac{3-\sqrt{5}}{2}\right)^n$.

To begin with, we first note that

$$\|\hat{\omega}\|_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{\mathbf{-1}}} = \lim_{N\to\infty} \|f^{(N)}\|_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{\mathbf{-1}}}.$$

Then,

$$\begin{aligned}
\|f^{(N)}\|^2_{\mathbf{s}_{\mathbf{2},\mathbf{F}}^{\mathbf{-1}}} &= \mathbf{s}_{\mathbf{2},\mathbf{F}}^{\mathbf{-1}}\left[f^{(N)}, f^{(N)}\right] + \left\langle f^{(N)}, f^{(N)}\right\rangle \\
&= \left\langle \tilde{J}^{-1} f^{(N)}, f^{(N)}\right\rangle + \left\langle f^{(N)}, f^{(N)}\right\rangle \\
&= \left\langle z^{(N)}, f^{(N)}\right\rangle + \left\langle f^{(N)}, f^{(N)}\right\rangle.
\end{aligned}$$

We continue by analysing both $\langle z^{(N)}, f^{(N)}\rangle$ and $\langle f^{(N)}, f^{(N)}\rangle$ separately. Immediately, we observe that $\langle z^{(N)}, f^{(N)}\rangle$ can be expressed as

$$\begin{aligned}
&\sum_{n=0}^{N}\left[\sum_{r=2N+1}^{3N}(n-r)G_N + \sum_{r=N+1}^{2N}(n-r)F_N + \sum_{r=n}^{N}(n-r)\hat{\omega}_r\right]\overline{\hat{\omega}}_n \\
&\qquad + \sum_{n=N+1}^{2N}\left[\sum_{r=2N+1}^{3N}(n-r)G_N + \sum_{r=n}^{2N}(n-r)F_N\right]\overline{F}_N \\
&\qquad\qquad + \sum_{n=2N+1}^{3N}\left[\sum_{r=n}^{3N}(n-r)G_N\right]\overline{G}_N.
\end{aligned}$$

By evaluating this expression in MAPLE and taking N to infinity, we see that

$$\lim_{N\to\infty}\left\langle z^{(N)}, f^{(N)}\right\rangle \approx 0.140498.$$

On the other hand,

$$\begin{aligned}
\left\langle f^{(N)}, f^{(N)}\right\rangle &= \sum_{n=0}^{N}|\hat{\omega}_n|^2 + \sum_{n=N+1}^{2N}|F_N|^2 + \sum_{n=2N+1}^{3N}|G_N|^2 \\
&= \sum_{n=0}^{N}|\hat{\omega}_n|^2 + N\left(F_N^2 + G_N^2\right).
\end{aligned}$$

Again, taking the limit of this expression as N goes to infinity yields

$$\lim_{N\to\infty} \left\langle f^{(N)}, f^{(N)} \right\rangle \approx 0.093665.$$

In contrast, we can compute $\|\varphi_-\|^2 = \sum_{n=0}^{\infty} |\lambda_-^n|^2$ directly after noting that this is precisely a geometric series. Then,

$$\begin{aligned} \|\varphi_-\|^2 &= \sum_{n=0}^{\infty} \lambda_-^{2n} \\ &= \frac{1}{1-\lambda_-^2}, \end{aligned} \tag{4.45}$$

and so $\|\varphi_-\|^2 = \frac{1}{2} + \frac{3\sqrt{5}}{10} \approx 1.1708203$. It is then clear that the inequality presented in (4.35) holds, as $0.140498 + 0.093665 \leq 1.1708203$.

4.4.7 The Relation $S_{2,K}$

We have now shown that all three conditions hold for the particular, unique, value of $\alpha = -\frac{1}{2} - \frac{\sqrt{5}}{10}$; all that remains is to construct $S_{2,K}$ using

$$S_{2,K} = (S_{2,F}^{-1})^{-1} = \left\{ (\omega - A_2\omega, A_2\omega) \in \ell^2 \times \ell^2 \;\mid\; \omega \in \ell^2 \right\},$$

where $A_2\omega = \hat{\omega}$ for $\hat{\omega} \in Q(\mathbf{s}_{\mathbf{2,F}}^{\mathbf{-1}})$. Note that an arbitrary $\omega \in \ell^2$ may be expressed as

$$\omega = \omega - c\varphi_- + c\varphi_-,$$

for any constant c. Then, we aim to find the unique constant $c = c_\omega$ such that

$$\omega = \underbrace{(\omega - c_\omega\varphi_-)}_{\in\mathcal{R}(\tilde{J}+\tilde{I})} + \underbrace{(c_\omega\varphi_-)}_{\in \operatorname{span}\{\varphi_-\}},$$

upon recalling Lemma 4.4.2 — ℓ^2 may be decomposed into the orthogonal sum $\mathcal{R}(\tilde{J}+\tilde{I}) \oplus \operatorname{span}\{\varphi_-\}$. Indeed, if $\omega - c_\omega\varphi_-$ is orthogonal to $c_\omega\varphi_-$, then

$$\langle \omega - c_\omega\varphi_-, c_\omega\varphi_- \rangle = 0,$$

and, in particular,

$$\overline{c}_\omega \langle \omega, \varphi_- \rangle - c_\omega \overline{c}_\omega \langle \varphi_-, \varphi_- \rangle = 0.$$

Hence, if we set

$$c_\omega = \frac{\langle \omega, \varphi_- \rangle}{\|\varphi_-\|^2},$$

then ω takes the following form:

$$\omega = \underbrace{\left(\omega - \frac{\langle \omega, \varphi_- \rangle}{\|\varphi_-\|^2}\varphi_-\right)}_{\in\mathcal{R}(\tilde{J}+\tilde{I})} + \underbrace{\left(\frac{\langle \omega, \varphi_- \rangle}{\|\varphi_-\|^2}\varphi_-\right)}_{\in \operatorname{span}\{\varphi_-\}}. \tag{4.46}$$

With this decomposition in mind, we must now determine how A_2 acts on a general element $\omega \in \ell^2$. Then, we may summarise all of our previous calculations as follows.

Let $\omega \in \ell^2$. Then, $\omega = \mu + \eta$ for $\mu \in \mathcal{R}(\tilde{J} + \tilde{I})$ and $\eta \in \operatorname{span}\{\varphi_-\}$: specifically,

$$\mu = \omega - \frac{\langle \omega, \varphi_- \rangle}{\|\varphi_-\|^2}\varphi_- \qquad \text{and} \qquad \eta = \frac{\langle \omega, \varphi_- \rangle}{\|\varphi_-\|^2}\varphi_-,$$

where $(\varphi_-)_n = \lambda_-^n = \left(\frac{3-\sqrt{5}}{2}\right)^n$, by means of equation (4.46). Then, for $\mu \in \mathcal{R}(\tilde{J} + \tilde{I})$, equation (4.29) shows that

$$A_2\mu = \left[I - (J+I)^{-1}\right]\mu,$$

whilst for $\eta \in \operatorname{span}\{\varphi_-\}$, i.e., $\eta = c\varphi_-$, we have

$$(A_2\eta)_n = c\left((\alpha\lambda_- + 1)\lambda_-^n - \frac{(n+1)\lambda_-^n}{\sqrt{5}}\right),$$

for $\alpha = -\frac{1}{2} - \frac{\sqrt{5}}{10}$, due to equation (4.32). Furthermore, recall that, by Lemma 4.4.4, the ℓ^2-solution to the equation $(J+I)x = -\varphi_-$ is given by $x_n = -\frac{\lambda_-^n(n+1)}{\sqrt{5}}$. Hence,

$$A_2\eta = c\left((\alpha\lambda_- + 1)\varphi_- + x\right).$$

As A_2 is a linear operator, we assert that

$$A_2\omega = A_2\mu + A_2\eta.$$

Hence, for a general $\omega \in \ell^2$, as decomposed in equation (4.46), we have

$$\begin{aligned}
A_2\omega &= \left[I - (J+I)^{-1}\right](\omega - c_\omega\varphi_-) + c_\omega\left[(\alpha\lambda_- + 1)\varphi_- + x\right] \\
&= \left[I - (J+I)^{-1}\right]\omega - c_\omega\varphi_- - c_\omega(J+I)^{-1}(-\varphi_-) \\
&\qquad\qquad\qquad\qquad\qquad + c_\omega\left[(\alpha\lambda_- + 1)\varphi_- + x\right] \\
&= \left[I - (J+I)^{-1}\right]\omega - c_\omega\varphi_- - c_\omega x + c_\omega(\alpha\lambda_- + 1)\varphi_- + c_\omega x \\
&= \left[I - (J+I)^{-1}\right]\omega + c_\omega\alpha\lambda_-\varphi_-.
\end{aligned}$$

With the linear mapping A_2 defined, we may now assert that $S_{2,K}$ is the following relation:

$$\begin{aligned}
S_{2,K} &= \left\{(\omega - A_2\omega, A_2\omega) \;\middle|\; \omega \in \ell^2\right\} \\
&= \left\{ \begin{pmatrix} (J+I)^{-1}\omega - c_\omega\alpha\lambda_-\varphi_-, \\ \left[I - (J+I)^{-1}\right]\omega + c_\omega\alpha\lambda_-\varphi_- \end{pmatrix} \;\middle|\; \omega \in \ell^2 \right\},
\end{aligned} \tag{4.47}$$

where $(\varphi_-)_n = \lambda_-^n = \left(\frac{3-\sqrt{5}}{2}\right)^n$, $\alpha = -\frac{1}{2} - \frac{\sqrt{5}}{10}$ and $c_\omega = \frac{\langle \omega, \varphi_- \rangle}{\|\varphi_-\|^2}$.

Whilst equation (4.47) displays a valid representation of $S_{2,K}$, we are able to manipulate the relation upon defining $u \coloneqq (J+I)^{-1}\omega - c_\omega \alpha \lambda_- \varphi_-$. By means of Lemma 4.4.3, we note that $\lambda_-(J+I)\varphi_- = e_0$. Then, we readily observe that

$$\begin{aligned} u &= (J+I)^{-1}\omega - c_\omega \alpha (J+I)^{-1} e_0 \\ &= (J+I)^{-1}(\omega - c_\omega \alpha e_0). \end{aligned}$$

Hence $\omega = (J+I)u + c_\omega \alpha e_0$.

Since $c_\omega = \frac{\langle \omega, \varphi_- \rangle}{\|\varphi_-\|^2}$ involves ω, we now aim to rewrite c_ω strictly in terms of u. Then, consider the expression $\langle \omega, \varphi_- \rangle$. Since $J+I$ is self-adjoint, we have that

$$\begin{aligned} \langle \omega, \varphi_- \rangle &= \langle (J+I)u, \varphi_- \rangle + c_\omega \alpha \langle e_0, \varphi_- \rangle \\ &= \langle u, (J+I)\varphi_- \rangle + c_\omega \alpha \cdot 1 \\ &= \frac{1}{\lambda_-} \langle u, e_0 \rangle + c_\omega \alpha \\ &= \frac{u_0}{\lambda_-} + c_\omega \alpha, \end{aligned}$$

and so we may conclude that

$$c_\omega = \frac{u_0}{\lambda_- \|\varphi_-\|^2} + \frac{c_\omega \alpha}{\|\varphi_-\|^2} \implies c_\omega = \frac{u_0}{\lambda_-(\|\varphi_-\|^2 - \alpha)}.$$

All of these quantities are known explicitly: $\lambda_- = \frac{3-\sqrt{5}}{2}$, $\alpha = -\frac{1}{2} - \frac{\sqrt{5}}{10}$ and $\|\varphi_-\|^2 = \frac{1}{1-\lambda_-^2}$ by means of equation (4.45). Then, it is easily verified that $c_\omega = -\frac{u_0}{\alpha}$. Hence,

$$\begin{aligned} \omega &= (J+I)u - u_0 e_0 \\ &= (J+I)u - \langle u, e_0 \rangle e_0. \end{aligned}$$

With this calculation in mind, we may then declare that, for the appropriate ω and u,

$$(\omega - A_2\omega, A_2\omega) = (u, \omega - u),$$

where $\omega - u$ is such that

$$\begin{aligned} \omega - u &= (J+I)u - \langle u, e_0 \rangle e_0 - u \\ &= Ju - \langle u, e_0 \rangle e_0. \end{aligned}$$

Hence

$$S_{2,K} = \left\{ (u, Ju - \langle u, e_0 \rangle e_0) \;\middle|\; u = (J+I)^{-1}(\omega - c_\omega \alpha e_0) \text{ for } \omega \in \ell^2 \right\}.$$

Given the relationship between ω and u, we now ask which space the sequence u belongs to. In fact, since $J + I\colon \ell^2 \to \ell^2$ is a bijective operator (see: Lemma 4.2.3), if the operator $T\colon \ell^2 \to \ell^2$ such that $\omega \mapsto \omega - c_\omega \alpha e_0$ is bijective, then we may deduce that u can be an arbitrary element in ℓ^2 — this is clear, upon taking the composition of the two operators. The following lemma proves that this is, indeed, the case.

Lemma 4.4.6. *The operator $T\colon \ell^2 \to \ell^2$ such that*

$$\omega \mapsto \omega - \frac{\langle \omega, \varphi_- \rangle}{\|\varphi_-\|^2} \alpha e_0$$

is a bijective, linear operator.

Proof. Let x, $y \in \ell^2$ and $\lambda \in \mathbb{C}$. Proving that T is linear is easy: we must simply verify that $T(\lambda x + y) = \lambda T x + T y$. Indeed,

$$\begin{aligned} T(\lambda x + y) &= (\lambda x + y) - \frac{\langle \lambda x + y, \varphi_- \rangle}{\|\varphi_-\|^2} \alpha e_0 \\ &= \lambda x - \lambda \frac{\langle x, \varphi_- \rangle}{\|\varphi_-\|^2} \alpha e_0 + y - \frac{\langle y, \varphi_- \rangle}{\|\varphi_-\|^2} \alpha e_0 \\ &= \lambda T x + T y, \end{aligned}$$

confirming that T is a linear operator.

To verify that T is bijective, we first note that we may write T as the operator $I - S$, for $S\omega = \frac{\langle \omega, \varphi_- \rangle}{\|\varphi_-\|^2} \alpha e_0$. Then, to show that T is bijective, we make use of the Neumann series — see, for example, [43, Thm. 7.3-1]. In particular, if $\|S\| < 1$, then $(I - S)^{-1} = T^{-1}$ exists on the whole of ℓ^2, and so T is bijective. Indeed,

$$\|S\omega\| = \left\| \frac{\langle \omega, \varphi_- \rangle}{\|\varphi_-\|^2} \alpha e_0 \right\| \leq \frac{\|\omega\| \|\varphi_-\|}{\|\varphi_-\|^2} |\alpha| \|e_0\| = \frac{\|\omega\| |\alpha|}{\|\varphi_-\|},$$

after an application of the Cauchy-Schwarz inequality. This inequality demonstrates that

$$\|S\| \leq \frac{|\alpha|}{\|\varphi_-\|}, \qquad \text{or} \qquad \|S\|^2 \leq \frac{|\alpha|^2}{\|\varphi_-\|^2}.$$

The second inequality above is more useful to us as equation (4.45) gives an explicit expression for $\|\varphi_-\|^2$. Additionally, it is easily verified that $\alpha^2 = \frac{\lambda_+}{5}$ for $\lambda_+ = \frac{3+\sqrt{5}}{2}$. Together, these two expressions show that

$$\|S\|^2 \leq \frac{\lambda_+(1 - \lambda_-^2)}{5} = \frac{\lambda_+ - \lambda_-}{5} = \frac{\sqrt{5}}{5} < 1,$$

since $\lambda_+ \lambda_- = 1$. As this inequality demonstrates that $\|S\| < 1$, we may conclude that T is a bijective, linear operator, as required. ∎

Having now discerned that T is bijective, we see that $(J+I)T\colon \ell^2 \to \ell^2$ is also bijective, and so we may conclude that the Kreĭn extension $S_{2,K}$ of S_2 takes the following form:

$$S_{2,K} = \left\{ (u, Ju - \langle u, e_0 \rangle e_0) \;\middle|\; u \in \ell^2 \right\}.$$

It is worth noting that this relation does not have a multi-valued part. Indeed, this is obvious upon setting $u = 0$: $J0 - \langle 0, e_0 \rangle e_0 = 0$, and so there can not exist an element of the form $(0, x) \in S_{2,K}$ for $x \neq 0$. Consequently, we can easily express $Ju - \langle u, e_0 \rangle e_0$ as a Jacobi operator $\hat{J}$. In particular,

$$S_{2,K} = \left\{ (u, \hat{J}u) \;\middle|\; u \in \ell^2 \right\},$$

where

$$Ju - \langle u, e_0 \rangle e_0 = \hat{J}u = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \end{pmatrix}, \tag{4.48}$$

for $u = (u_0, u_1, u_2, \dots) \in \ell^2$.

Now that we have constructed both the Friedrichs and Kreĭn extension of S_2, we conclude this section by remarking that $S_{2,F}$ does *not* equal $S_{2,K}$. This differs to S_1 quite drastically, as $S_1 = S_{1,F} = S_{1,K}$ in that case. Thus, we continue by asking whether or not there exist *other* extremal maximal sectorial extensions of S_1 and S_2. The next section addresses this question by means of Theorem 3.2.17.

4.5 Extremal Maximal Sectorial Extensions of S_1 and S_2

During Sections 4.2, 4.3 and 4.4, we constructed the Friedrichs and Kreĭn extensions of two linear relations S_1 and S_2. In particular, we constructed maximal sectorial relations after making use of the association between sectorial relations and closed forms. However, in Section 3.2.3, we also described how one might construct *all* extremal maximal sectorial extensions by means of the linear relations U and V expressed in equations (3.16) and (3.17) respectively. This section serves two purposes. On one hand, we aim to characterise all of the extremal maximal sectorial extensions of S_1 and S_2 by means of Theorem 3.2.17. Moreover, we aim to explicitly construct the Friedrichs and Kreĭn extensions of both S_1 and S_2, verifying that the two methods coincide.

4.5.1 Extremal Maximal Sectorial Extensions of S_1

We begin by recalling that the linear relation S_1 is given by

$$S_1 = \left\{ (x, Jx) \in \ell^2 \times \ell^2 \;\middle|\; x \in \ell^2 \right\},$$

where $(Jx)_n = -\Delta(\Delta x_{n-1})$ for all $n \geq 0$. The set $\mathfrak{R}_0$, as described in equation (3.11), is fundamental to the construction of the extremal maximal extensions of S_1; in particular,

$$\begin{aligned} \mathfrak{R}_0 &= \left\{ x' \in \mathcal{R}(S_1) \;\middle|\; \text{there exists } (x, x') \in S_1 \text{ such that } \left\langle x', x \right\rangle = 0 \right\} \\ &= \left\{ Jx \;\middle|\; \text{there exists } x \in \ell^2 \text{ such that } \langle Jx, x \rangle = 0 \right\}. \end{aligned}$$

It is clear that $\langle Jx, x \rangle = 0$ if and only if $x = 0$, and so we readily observe that $\mathfrak{R}_0 = \{0\}$.

Remark. Alternatively, since $\operatorname{mul} S_1^* = \operatorname{mul} S_1 = \{0\}$, it is immediate that $\mathfrak{R}_0 = \{0\}$ upon invoking Lemma 3.2.13.

Remark. Since $\mathfrak{R}_0 = \{0\}$, the element $[Jx]$ is effectively the same as Jx. However, we will keep the bracket notation so that there is no ambiguity between the spaces and inner products that we will work with.

We continue by defining $(H_{S_1}, \langle \cdot, \cdot \rangle_{H_{S_1}})$ to be the completion of the inner product space $\left(\mathcal{R}(S_1)/\mathfrak{R}_0, \langle \cdot, \cdot \rangle_{\mathcal{R}(S_1)/\mathfrak{R}_0} \right)$, where

$$\left\langle [x'], [y'] \right\rangle_{\mathcal{R}(S_1)/\mathfrak{R}_0} = \frac{1}{2} \left(\left\langle x', y \right\rangle + \left\langle x, y' \right\rangle \right), \qquad (x, x'),\, (y, y') \in S_1.$$

Here, $[x']$ and $[y']$ denote the equivalence classes containing x' and y' respectively. Then, for (x, Jx), $(y, Jy) \in S_1$, it is clear that

$$\begin{aligned} \langle [Jx], [Jy] \rangle_{H_{S_1}} &= \frac{1}{2} \left(\langle Jx, y \rangle + \langle x, Jy \rangle \right) \\ &= \frac{1}{2} \left(\langle Jx, y \rangle + \langle Jx, y \rangle \right) \\ &= \langle Jx, y \rangle, \end{aligned}$$

since J is self-adjoint. Alternatively, we note that

$$\langle [Jx], [Jy] \rangle_{H_{S_1}} = \langle x, Jy \rangle, \qquad (x, Jx),\, (y, Jy) \in S_1,$$

using the symmetry of J.

Let $\mathbf{b_1'}$ be the symmetric form with $Q(\mathbf{b_1'}) = \mathcal{R}(S_1)/\mathfrak{R}_0$ defined by

$$\mathbf{b_1'} \left[[Jx], [Jy] \right] = \frac{i}{2} \left(\langle x, Jy \rangle - \langle Jx, y \rangle \right), \qquad (x, Jx),\, (y, Jy) \in S_1.$$

In this case, the form $\mathbf{b_1'}$ simplifies significantly. Indeed, for all (x, Jx), $(y, Jy) \in S_1$, we have

$$\begin{aligned}\mathbf{b_1'}\left[[Jx],[Jy]\right] &= \frac{i}{2}\left(\langle x, Jy\rangle - \langle Jx, y\rangle\right)\\ &= \frac{i}{2}\left(\langle x, Jy\rangle - \langle x, Jy\rangle\right)\\ &= 0,\end{aligned}$$

since J is self-adjoint. By taking the closure of $\mathbf{b_1'}$, we obtain the closed form $\mathbf{b_1}$, where $\mathbf{b_1}$ is an everywhere defined, bounded symmetric form on H_{S_1} that is identically equal to 0. Then, there exists a bounded self-adjoint operator B_{S_1} on H_{S_1} such that

$$\mathbf{b_1}\left[[Jx],[Jy]\right] = \langle B_{S_1}[Jx],[Jy]\rangle_{H_{S_1}}, \quad (x, Jx), (y, Jy) \in S_1.$$

Clearly $B_{S_1} = 0$, i.e., B_{S_1} is the zero operator.

Now, define U_1 to be the linear relation from ℓ^2 to H_{S_1} with

$$U_1 = \left\{(x,[Jx]) \in \ell^2 \times H_{S_1} \;\middle|\; (x, Jx) \in S_1\right\}, \tag{4.49}$$

and V_1 from H_{S_1} to ℓ^2 with

$$V_1 = \left\{([Jx], Jx) \in H_{S_1} \times \ell^2 \;\middle|\; (x, Jx) \in S_1\right\}. \tag{4.50}$$

With these two linear relations in mind, we are able to determine all extremal maximal sectorial extensions of S_1 by means of Theorem 3.2.17. In particular, if we are in possession of a closed linear operator T whose graph satisfies $U_1 \subseteq \mathcal{G}(T) \subseteq V_1^*$, then

$$\begin{aligned}\tilde{S} &= \mathcal{G}(T)^*\mathcal{G}(I + iB_{S_1})\mathcal{G}(T)\\ &= \mathcal{G}(T)^*\mathcal{G}(T)\end{aligned}$$

is the extremal maximal sectorial extension of S_1 associated to T. Furthermore, by Theorem 3.2.18, the Friedrichs extension $S_{1,F}$ and Kreĭn extension $S_{1,K}$ of S_1 are given by

$$\begin{aligned}S_{1,F} &= U_1^*\mathcal{G}(I + iB_{S_1})U_1^{**}\\ &= U_1^*U_1^{**}\end{aligned}$$

and

$$\begin{aligned}S_{1,K} &= V_1^{**}\mathcal{G}(I + iB_{S_1})V_1^*\\ &= V_1^{**}V_1^*,\end{aligned}$$

respectively. Then, we must determine the following relations: U_1^*, U_1^{**}, V_1^* and V_1^{**}. In fact, the following lemma shows that these relations, in this case, are intimately connected.

Lemma 4.5.1. *Let U_1 and V_1 be defined as in equations (4.49) and (4.50). Then $U_1 = V_1^* = U_1^{**}$ and $U_1^* = V_1^{**}$.*

Proof. The latter equality is immediate provided that $U_1 = V_1^*$, so we focus on proving the former. Then, we begin by noting that V_1^* is of the form

$$V_1^* = \left\{ (z, [y]) \in \ell^2 \times H_{S_1} \;\middle|\; \langle [y], [Jx] \rangle_{H_{S_1}} = \langle z, Jx \rangle \text{ for all } ([Jx], Jx) \in V_1 \right\}.$$

As $[y] \in H_{S_1}$, there exists a sequence $[p_n]$ in $\mathcal{R}(S_1)/\{0\}$ such that $[p_n] \to [y]$ as $n \to \infty$ in H_{S_1}. In particular, as $\mathcal{R}(S_1) = \mathcal{R}(J)$, there exists a sequence q_n in ℓ^2 such that $[Jq_n] \to [y]$. Then,

$$\langle z, Jx \rangle = \langle [y], [Jx] \rangle_{H_{S_1}} = \lim_{n\to\infty} \langle [Jq_n], [Jx] \rangle_{H_{S_1}} = \lim_{n\to\infty} \langle Jq_n, x \rangle .$$

Since J is self-adjoint, we note that $\langle z, Jx \rangle = \langle Jz, x \rangle$. As $x \in \ell^2$ is arbitrary, Jq_n must tend to Jz weakly in ℓ^2 for equality to hold: we choose to denote this by $Jq_n \rightharpoonup Jz$. Moreover, for $[Jx] \in \mathcal{D}(V_1)$,

$$\begin{aligned} \langle [y] - [Jz], [Jx] \rangle_{H_{S_1}} &= \langle [y - Jz], [Jx] \rangle_{H_{S_1}} \\ &= \lim_{n\to\infty} \langle [Jq_n - Jz], [Jx] \rangle_{H_{S_1}} \\ &= \lim_{n\to\infty} \langle [J(q_n - z)], [Jx] \rangle_{H_{S_1}} \\ &= \lim_{n\to\infty} \langle J(q_n - z), x \rangle \\ &= \lim_{n\to\infty} \langle Jq_n - Jz, x \rangle . \end{aligned}$$

As $Jq_n \rightharpoonup Jz$ in ℓ^2, $\lim_{n\to\infty} \langle Jq_n - Jz, x \rangle = 0$ for all $x \in \ell^2$. Thus, $[y] = [Jz]$. With this result in mind, we can then assert that

$$V_1^* = \left\{ (z, [Jz]) \in \ell^2 \times H_{S_1} \;\middle|\; \begin{array}{r} z \in \ell^2 \text{ such that } \langle [Jz], [Jx] \rangle_{H_{S_1}} = \langle z, Jx \rangle \\ \text{for all } ([Jx], Jx) \in V_1 \end{array} \right\}.$$

However, the extra condition required on $z \in \ell^2$ is unimportant for our purposes: clearly, $V_1^* \subseteq U_1$. Moreover, from Lemma 3.2.16, we have that $V_1 \subseteq U_1^*$. By taking the adjoint of both sides, we readily observe that $U_1^{**} \subseteq V_1^*$. Hence,

$$U_1^{**} \subseteq V_1^* \subseteq U_1 \subseteq U_1^{**},$$

since U_1^{**} is the closure of U_1. From this string of containments, we conclude equality. Hence

$$U_1 = V_1^* = U_1^{**} \qquad \text{and} \qquad U_1^* = V_1^{**},$$

as required. ∎

This lemma has several notable consequences. Firstly, since $U_1 = V_1^*$, there are no other extremal maximal extensions of S_1 — there are no additional choices for graphs that lie in between the two. Furthermore,

$$S_{1,K} = V_1^{**}V_1^* = U_1^*V_1^* = U_1^*U_1^{**} = S_{1,F},$$

verifying that the Friedrichs and Kreĭn extension coincide in this example. We then devote the remainder of this section to the construction of the Friedrichs extension $S_{1,F}$ by means of the composition $U_1^*U_1^{**} = U_1^*U_1$ in order to ensure that the two methods do, indeed, coincide.

By definition, U_1^* is of the form

$$U_1^* = \left\{ ([y], z) \in H_{S_1} \times \ell^2 \;\middle|\; \langle z, x\rangle = \langle [y], [Jx]\rangle_{H_{S_1}} \text{ for all } (x, [Jx]) \in U_1 \right\}.$$

Then, as $[y] \in H_{S_1}$, there exists a sequence q_n in ℓ^2 such that $[Jq_n] \to [y]$ as $n \to \infty$ in H_{S_1}. Hence,

$$\langle z, x\rangle = \langle [y], [Jx]\rangle_{H_{S_1}} = \lim_{n\to\infty} \langle [Jq_n], [Jx]\rangle_{H_{S_1}} = \lim_{n\to\infty} \langle Jq_n, x\rangle .$$

As such, we require $Jq_n \rightharpoonup z$ in ℓ^2 for equality to hold. Then,

$$U_1^* = \left\{ ([y], z) \in H_{S_1} \times \ell^2 \;\middle|\; \begin{array}{l} \text{there exists a sequence } q_n \text{ in } \ell^2 \text{ such} \\ \quad \text{that } Jq_n \rightharpoonup z \text{ in } \ell^2 \text{ and } [Jq_n] \to [y] \text{ in } H_{S_1} \end{array} \right\}.$$

Now, in order to find $S_{1,F}$, we simply need to determine the elements in U_1^* whose first component is $[Jx]$. If such an element could be expressed as $([Jx], z)$, say, then $S_{1,F}$ would consist of the elements (x, z), for $x \in \ell^2$: this is simply the composition of two linear relations. In other words,

$$U_1^*U_1 = \left\{ (x, z) \in \ell^2 \times \ell^2 \;\middle|\; \begin{array}{l} \text{there exists a sequence } q_n \text{ in } \ell^2 \text{ such} \\ \quad \text{that } Jq_n \rightharpoonup z \text{ in } \ell^2 \text{ and } [Jq_n] \to [Jx] \text{ in } H_{S_1} \end{array} \right\}.$$

Let $y \in \ell^2$. If we consider the expression $\langle z - Jx, y\rangle$ for $x \in \ell^2$ and $z \in \mathcal{R}(U_1^*)$, then we see that

$$\begin{aligned}
\langle z - Jx, y\rangle &= \lim_{n\to\infty} \langle Jq_n - Jx, y\rangle \\
&= \lim_{n\to\infty} \langle J(q_n - x), y\rangle \\
&= \lim_{n\to\infty} \langle [J(q_n - x)], [Jy]\rangle_{H_{S_1}} \\
&= \lim_{n\to\infty} \langle [Jq_n] - [Jx], [Jy]\rangle_{H_{S_1}} .
\end{aligned}$$

As $[Jq_n] \to [Jx]$, we see that $\lim_{n\to\infty} \langle [Jq_n] - [Jx], [Jy]\rangle = 0$ for all $y \in \ell^2$. Hence, $z = Jx$, and so,

$$U_1^*U_1 = \left\{ (x, Jx) \in \ell^2 \times \ell^2 \;\middle|\; \begin{array}{l} \text{there exists a sequence } q_n \text{ in } \ell^2 \text{ such that} \\ \quad Jq_n \rightharpoonup Jx \text{ in } \ell^2 \text{ and } [Jq_n] \to [Jx] \text{ in } H_{S_1} \end{array} \right\}.$$

By choosing q_n to be x for all n, the limits are trivially satisfied. Thus, it is clear that there exists a sequence for all $x \in \ell^2$. Hence,

$$S_{1,F} = U_1^* U_1 = \left\{ (x, Jx) \in \ell^2 \times \ell^2 \;\middle|\; x \in \ell^2 \right\},$$

and

$$S_{1,F} = S_{1,K} = S_1.$$

Observe that this result coincides with that of Section 4.2. However, Theorem 3.2.18 unveils more: we can explicitly construct the form $\mathbf{s_{1,F}}$ associated to $S_{1,F}$ by means of equation (3.20). In particular, since $U^{**} = U$, we have

$$\begin{aligned} \mathbf{s_{1,F}}\,[x,y] &= \langle [Jx], [Jy] \rangle_{H_{S_1}} \\ &= \langle Jx, y \rangle, \end{aligned}$$

for x, $y \in \ell^2$. This form coincides precisely with the form constructed during Lemma 4.2.2, verifying that the two methods produce identical results for this example.

4.5.2 Extremal Maximal Sectorial Extensions of S_2

With the analysis of S_1 now complete, we continue by constructing the extremal maximal sectorial extensions of S_2. Recall that the linear relation S_2 is given by

$$S_2 = \left\{ (x, \tilde{J}x) \in \ell^2 \times \ell^2 \;\middle|\; x \in \ell_0^2 \right\},$$

where $(\tilde{J}x)_n = -\Delta(\Delta x_{n-1})$ for all $n \geq 0$. As in Section 4.5.1, the first step in constructing such extensions is to determine the set $\mathfrak{R}_0$, where

$$\begin{aligned} \mathfrak{R}_0 &= \left\{ x' \in \mathcal{R}(S_2) \;\middle|\; \text{there exists } (x, x') \in S_2 \text{ such that } \langle x', x \rangle = 0 \right\} \\ &= \left\{ \tilde{J}x \;\middle|\; \text{there exists } x \in \ell_0^2 \text{ such that } \left\langle \tilde{J}x, x \right\rangle = 0 \right\}. \end{aligned}$$

Indeed, $\mathfrak{R}_0 = \{0\}$, as $\left\langle \tilde{J}x, x \right\rangle = 0$ if and only if $x = 0$.

Then, proceed by defining $(H_{S_2}, \langle \cdot, \cdot \rangle_{H_{S_2}})$ to be the completion of the inner product space $\left(\mathcal{R}(S_2)/\mathfrak{R}_0, \langle \cdot, \cdot \rangle_{\mathcal{R}(S_2)/\mathfrak{R}_0} \right)$, where

$$\langle [x'], [y'] \rangle_{\mathcal{R}(S_2)/\mathfrak{R}_0} = \frac{1}{2} \left(\langle x', y \rangle + \langle x, y' \rangle \right), \qquad (x, x'),\ (y, y') \in S_2.$$

In other words,

$$\begin{aligned} \left\langle [\tilde{J}x], [\tilde{J}y] \right\rangle_{H_{S_2}} &= \frac{1}{2} \left(\left\langle \tilde{J}x, y \right\rangle + \left\langle x, \tilde{J}y \right\rangle \right) \\ &= \left\langle \tilde{J}x, y \right\rangle \end{aligned}$$

for $(x, \tilde{J}x)$, $(y, \tilde{J}y) \in S_2$ since $\tilde{J}$ is a restriction of the self-adjoint operator J. Alternatively, we may note that

$$\left\langle [\tilde{J}x], [\tilde{J}y] \right\rangle_{H_{S_2}} = \left\langle x, \tilde{J}y \right\rangle, \qquad (x, \tilde{J}x),\, (y, \tilde{J}y) \in S_2.$$

Let $\mathbf{b_2'}$ be the symmetric form with $Q(\mathbf{b_2'}) = \mathcal{R}(S_2)/\mathfrak{R}_0$ defined by

$$\mathbf{b_2'}\left[[\tilde{J}x], [\tilde{J}y]\right] = \frac{i}{2}\left(\left\langle x, \tilde{J}y \right\rangle - \left\langle \tilde{J}x, y \right\rangle\right), \qquad (x, \tilde{J}x),\, (y, \tilde{J}y) \in S_2.$$

We may simplify the form $\mathbf{b_2'}$ significantly once again. Indeed, for all $(x, \tilde{J}x)$, $(y, \tilde{J}y) \in S_2$, we have

$$\begin{aligned} \mathbf{b_2'}\left[[\tilde{J}x], [\tilde{J}y]\right] &= \frac{i}{2}\left(\left\langle x, \tilde{J}y \right\rangle - \left\langle \tilde{J}x, y \right\rangle\right) \\ &= \frac{i}{2}\left(\langle x, Jy \rangle - \langle Jx, y \rangle\right) \\ &= 0, \end{aligned}$$

since J is self-adjoint. We continue by taking the closure of $\mathbf{b_2'}$ — this produces $\mathbf{b_2}$, the everywhere defined, bounded symmetric form on H_{S_2}. Then, the bounded self-adjoint operator B_{S_2} on H_{S_2} such that

$$\mathbf{b_2}\left[[\tilde{J}x], [\tilde{J}y]\right] = \left\langle B_{S_2}[\tilde{J}x], [\tilde{J}y] \right\rangle_{H_{S_2}}, \quad (x, \tilde{J}x),\, (y, \tilde{J}y) \in S_2,$$

is clearly the zero operator.

Having now determined the operator B_{S_2}, we continue by defining U_2 to be the linear relation from ℓ^2 to H_{S_2} such that

$$U_2 = \left\{ (x, [\tilde{J}x]) \in \ell^2 \times H_{S_2} \;\middle|\; (x, \tilde{J}x) \in S_2 \right\},$$

and V_2 from H_{S_2} to ℓ^2 with

$$V_2 = \left\{ ([\tilde{J}x], \tilde{J}x) \in H_{S_2} \times \ell^2 \;\middle|\; (x, \tilde{J}x) \in S_2 \right\}.$$

We are now in possession of all of the components necessary to construct the extremal maximal sectorial extensions of S_2, courtesy of Theorem 3.2.17. In essence, if we are in possession of a closed linear operator T whose graph satisfies $U_2 \subseteq \mathcal{G}(T) \subseteq V_2^*$, then

$$\begin{aligned} \tilde{S} &= \mathcal{G}(T)^* \mathcal{G}(I + iB_{S_2}) \mathcal{G}(T) \\ &= \mathcal{G}(T)^* \mathcal{G}(T), \end{aligned}$$

is the extremal maximal sectorial relation associated to T. In addition, we see that the Friedrichs extension $S_{2,F}$ and Kreĭn extension $S_{2,K}$ of S_1 are given by equations (3.19) and (3.21) respectively, i.e.,

$$S_{2,F} = U_2^* U_2^{**} \qquad \text{and} \qquad S_{2,K} = V_2^{**} V_2^*.$$

First, we construct the Friedrichs extension $S_{2,F}$. We begin by noting that U_2 is a closed linear relation, i.e., $U_2^{**} = U_2$. Indeed, U_2 is the graph of an operator that maps an x in ℓ_0^2 to $[\tilde{J}x]$ in H_{S_2}; we can show that this operator is bounded by noting that

$$\begin{aligned}\|[\tilde{J}x]\|_{H_{S_2}}^2 &= \left\langle [\tilde{J}x], [\tilde{J}x] \right\rangle_{H_{S_2}} \\ &= \left\langle \tilde{J}x, x \right\rangle,\end{aligned}$$

for all $x \in \ell_0^2$. Since Δ is a bounded operator — in particular, $\|\Delta u\| \leq 2\|u\|$ for $u \in \ell^2$ — we see that

$$\left\langle \tilde{J}x, x \right\rangle \leq 4\|x\|^2,$$

after applying the summation by parts formula. Then, as $\|[\tilde{J}x]\|_{H_{S_2}} \leq 2\|x\|$, we can conclude that the operator that maps x into $[\tilde{J}x]$ is bounded, thus the associated graph is closed.

Now that we have established that $U_2^{**} = U_2$, we may conclude that the Friedrichs extension takes the form $S_{2,F} = U_2^* U_2$ instead. Thus, all that remains is to find the elements of the form $([\tilde{J}x], z) \in U_2^*$ for $x \in \ell_0^2$, since the elements in $S_{2,F}$ will then be of the form (x, z).

Let $([\tilde{J}x], z) \in U_2^*$. By definition,

$$U_2^* = \left\{ ([y], z) \in H_{S_2} \times \ell^2 \;\middle|\; \langle z, u \rangle = \left\langle [y], [\tilde{J}u] \right\rangle_{H_{S_2}} \text{ for all } (u, [\tilde{J}u]) \in U_2 \right\},$$

so we can analyse the inner product in order to discern the possible values of z. In particular, $z \in \ell^2$ must satisfy

$$\begin{aligned}\langle z, u \rangle &= \left\langle [\tilde{J}x], [\tilde{J}u] \right\rangle_{H_{S_2}} \\ &= \left\langle \tilde{J}x, u \right\rangle\end{aligned}$$

for all $u \in \ell_0^2$. Since ℓ_0^2 is *not* dense in ℓ^2, we may only conclude that

$$z = \tilde{J}x + ce_0, \qquad c \in \mathbb{C},$$

where $e_0 = (1, 0, 0, \dots)$. On the other hand, clearly $([\tilde{J}x], \tilde{J}x + ce_0) \in U_2^*$ for any $c \in \mathbb{C}$ — it is not hard to show that the inner product condition is satisfied. Thus, we assert that $S_{2,F}$ is of the form

$$S_{2,F} = \left\{ (x, \tilde{J}x + ce_0) \;\middle|\; x \in \ell_0^2,\ c \in \mathbb{C} \right\}.$$

We may then decompose this multi-valued relation into its operator and multi-valued parts in order to conclude that the method of constructing $S_{2,F}$ detailed in Section 4.3 coincides with the construction above.

Since $\tilde{J}$ is a linear operator, it is clear that the multi-valued part of $S_{2,F}$ is given by

$$(S_{2,F})_{\text{mul}} = \{0\} \times \text{span}\,\{e_0\}.$$

Conversely, the operator part can be expressed as

$$(S_{2,F})_s = \left\{(x, P(\tilde{J}x + ce_0) \;\middle|\; x \in \ell_0^2\right\},$$

where P is the orthogonal projection onto $(\text{mul}\, S_{2,F})^{\perp} = \ell_0^2$. In fact, upon reintroducing the left and right shift operators L and R, respectively, it can be shown that

$$P(\tilde{J}x + ce_0) = RJLx, \qquad x \in \ell_0^2.$$

Hence

$$(S_{2,F})_s = \left\{(x, RJLx) \;\middle|\; x \in \ell_0^2\right\}.$$

As such, the operator and multi-valued parts of $S_{2,F}$ coincide precisely with those constructed in Section 4.3.

Furthermore, it is worth verifying that the form associated to $S_{2,F}$, as defined through equation (3.20), corresponds to that which we are to expect. Indeed, $\mathbf{s_{2,F}}$ is of the form

$$\mathbf{s_{2,F}}\,[x, y] = \left\langle \tilde{U}^{**}x, \tilde{U}^{**}y \right\rangle_{H_{S_2}}, \qquad x,\, y \in \mathcal{D}(U^{**}),$$

where $\tilde{U}^{**}$ is the operator associated to (the graph) U^{**}. However, since $U^{**} = U$, we note that, for $x,\, y \in \mathcal{D}(U) = \ell_0^2$, we have

$$\begin{aligned}
\mathbf{s_{2,F}}\,[x, y] &= \left\langle \tilde{U}x, \tilde{U}y \right\rangle_{H_{S_2}} \\
&= \left\langle [\tilde{J}x], [\tilde{J}y] \right\rangle_{H_{S_2}} \\
&= \left\langle \tilde{J}x, y \right\rangle \\
&= \sum_{n=0}^{\infty} \Delta x_n \Delta \overline{y}_n,
\end{aligned}$$

after an application of the summation by parts formula. In fact, this is precisely the form as expressed during the beginning of Section 4.3. As such, we have verified that the two methods in constructing $S_{2,F}$ coincide precisely.

With the Friedrichs extension of S_2 now in hand, we conclude this section by constructing the Kreĭn extension $S_{2,K}$ by means of Theorem 3.2.18. In particular,

$$S_{2,K} = V_2^{**}V_2^{*} \quad \text{for} \quad V_2 = \left\{([\tilde{J}x], \tilde{J}x) \in H_{S_2} \times \ell^2 \;\middle|\; (x, \tilde{J}x) \in S_2\right\}.$$

Our first step in constructing $S_{2,K}$ is to find expressions for V_2^* and V_2^{**}; indeed, by definition,

$$V_2^* = \left\{ (y,[z]) \in \ell^2 \times H_{S_2} \;\middle|\; \left\langle [z], [\tilde{J}x] \right\rangle_{H_{S_2}} = \langle y, Jx \rangle \text{ for all } ([\tilde{J}x], \tilde{J}x) \in V_2 \right\},$$

whilst

$$V_2^{**} = \left\{ ([a], b) \in H_{S_2} \times \ell^2 \;\middle|\; \langle b, y \rangle = \langle [a], [z] \rangle_{H_{S_2}} \text{ for all } (y,[z]) \in V_2^* \right\}.$$

Abstractly, we know that

$$U_2 \subseteq U_2^{**} \subseteq V_2^* \qquad \text{and} \qquad V_2^* \text{ is the graph of an operator.}$$

However, we have previously shown that $S_{2,F}$ does *not* equal $S_{2,K}$ and so it cannot be true that $U_2^{**} = V_2^*$. In fact, we very specifically have that $U_2 = U_2^{**} \subset V_2^*$. Then, since $\mathcal{D}(U_2) = \ell_0^2$, the only way to extend $\mathcal{D}(V_2^*)$ linearly is to additionally consider the span of the sequence $e_0 = (1, 0, 0, \dots)$. Therefore, there exists $[z_0] \in H_{S_2}$ such that

$$\begin{aligned} V_2^* &= \left\{ (x, [\tilde{J}x]) \in \ell^2 \times H_{S_2} \;\middle|\; x \in \ell_0^2 \right\} \\ &\quad \dot{+} \operatorname{span}\left\{ (e_0, [z_0]) \;\middle|\; \left\langle [z_0], [\tilde{J}y] \right\rangle_{H_{S_2}} = \left\langle e_0, \tilde{J}y \right\rangle \text{ for all } ([\tilde{J}y], \tilde{J}y) \in V_2 \right\} \\ &= \left\{ (x, [\tilde{J}x]) \in \ell^2 \times H_{S_2} \;\middle|\; x \in \ell_0^2 \right\} \\ &\quad \dot{+} \operatorname{span}\left\{ (e_0, [z_0]) \;\middle|\; \left\langle [z_0], [\tilde{J}y] \right\rangle_{H_{S_2}} = -\overline{y}_1 \text{ for all } ([\tilde{J}y], \tilde{J}y) \in V_2 \right\}. \end{aligned} \tag{4.51}$$

Let $x + ce_0 \in \mathcal{D}(V_2^*)$, where $x \in \ell_0^2$ and $c \in \mathbb{C}$. Since V_2^* is the graph of an operator, $[\tilde{J}x] + c[z_0]$ will be the associated second component in V_2^*. In order to find $S_{2,K}$, we must look for elements of the form $([\tilde{J}x] + c[z_0], b) \in V_2^{**}$: $S_{2,K}$ will then consist of all pairs $(x + ce_0, b)$. From the inner product in the definition of V_2^{**}, we see that

$$\langle b, y \rangle = \left\langle [\tilde{J}x] + c[z_0], [z] \right\rangle_{H_{S_2}}$$

for all $(y, [z]) \in V_2^*$. However, we have established that elements of V_2^* are of the form $(u + de_0, [\tilde{J}u] + d[z_0])$ and so we note that

$$\langle b, u + de_0 \rangle = \left\langle [\tilde{J}x] + c[z_0], [\tilde{J}u] + d[z_0] \right\rangle_{H_{S_2}}$$

for all $(u + de_0, [\tilde{J}u] + d[z_0]) \in V_2^*$. Upon unravelling this equation, we see that

$$\begin{aligned} \langle b, u \rangle + \overline{d} b_0 &= \left\langle [\tilde{J}x], [\tilde{J}u] \right\rangle_{H_{S_2}} + \overline{d} \left\langle [\tilde{J}x], [z_0] \right\rangle_{H_{S_2}} \\ &\quad + c \left\langle [z_0], [\tilde{J}u] \right\rangle_{H_{S_2}} + c\overline{d} \left\langle [z_0], [z_0] \right\rangle_{H_{S_2}} \\ &= \left\langle \tilde{J}x, u \right\rangle - \overline{d} x_1 - c\overline{u}_1 + c\overline{d} \|[z_0]\|_{H_{S_2}}^2, \end{aligned}$$

since $\left\langle [z_0], [\tilde{J}y] \right\rangle_{H_{S_2}} = -\overline{y}_1$ for $y \in \ell_0^2$, as demonstrated by equation (4.51).

Since

$$\langle b, u \rangle + \overline{d} b_0 = \left\langle \tilde{J}x, u \right\rangle - \overline{d} x_1 - c\overline{u}_1 + c\overline{d} \| [z_0] \|_{H_{S_2}}^2$$

for all $(y, [z]) = (u + de_0, [\tilde{J}u] + d[z_0]) \in V_2^*$, we can find b by specifying the element y. Initially, let $u = 0$ and $d = 1$. Then,

$$\begin{aligned} b_0 &= -x_1 + c \| [z_0] \|_{H_{S_2}}^2 \\ &= (\tilde{J}x)_0 + c \| [z_0] \|_{H_{S_2}}^2. \end{aligned} \tag{4.52}$$

Conversely, let $d = 0$ and $u = e_k$ for $k \geq 1$, where $(e_k)_n = \delta_{k,n}$ — note that we are unable to set $u = e_0$ as $u \in \ell_0^2$. Then

$$\langle b, e_k \rangle = \left\langle \tilde{J}x, e_k \right\rangle - c(e_k)_1.$$

In particular,

$$b_1 = (\tilde{J}x)_1 - c \text{ when } k = 1 \qquad \text{and} \qquad b_k = (\tilde{J}x)_k \text{ when } k \geq 2.$$

This, in conjunction with equation (4.52), completely describes the component b in the element $([\tilde{J}x] + c[z_0], b) \in V_2^{**}$. However, we established that $S_{2,K}$ was the graph of an operator, and so there exists a T such that $T(x + ce_0) = b$ for $x \in \ell_0^2$ and $c \in \mathbb{C}$. In fact, by piecing together the components of b, we see that

$$b = T(x + ce_0) = \begin{pmatrix} \| [z_0] \|_{H_{S_2}}^2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} c \\ x_1 \\ x_2 \\ \vdots \end{pmatrix}.$$

If we can show that $\| [z_0] \|_{H_{S_2}}^2 = 1$, then the operator T corresponds precisely with the Jacobi operator $\hat{J}$ as presented in equation (4.48), verifying that the relation we have constructed is, in fact, $S_{2,K}$. Fortunately this is, indeed, the case, as demonstrated in the following lemma.

Lemma 4.5.2. *The element* $[z_0] \in H_{S_2}$ *that satisfies*

$$\left\langle [z_0], [\tilde{J}y] \right\rangle_{H_{S_2}} = -\overline{y}_1, \qquad y \in \ell_0^2, \tag{4.53}$$

has a norm of 1*, i.e.,* $\| [z_0] \|_{H_{S_2}} = 1$.

Proof. We split the proof of this lemma into two parts: we individually show that

$$\| [z_0] \|_{H_{S_2}} \leq 1 \qquad \text{and} \qquad \| [z_0] \|_{H_{S_2}} \geq 1 \tag{4.54}$$

both hold, since we may then conclude that $\|[z_0]\|_{H_{S_2}} = 1$. However, we first preface the proof with a few useful results.

To begin with, we note that

$$\|[z_0]\|_{H_{S_2}} = \sup_{\substack{[h]\in H_{S_2},\\ [h]\neq 0}} \frac{\left|\langle [z_0],[h]\rangle_{H_{S_2}}\right|}{\|[h]\|_{H_{S_2}}} = \sup_{\substack{[h]\in H_{S_2},\\ \|[h]\|_{H_{S_2}}=1}} \left|\langle [z_0],[h]\rangle_{H_{S_2}}\right|$$

via the Riesz Representation Theorem. In fact, since $\mathcal{R}(\tilde{J})/\{0\}$ is dense in H_{S_2} by its very construction, we see that

$$\|[z_0]\|_{H_{S_2}} = \sup_{\substack{[\tilde{J}y]\in H_{S_2},\\ [\tilde{J}y]\neq 0}} \frac{\left|\left\langle [z_0],[\tilde{J}y]\right\rangle_{H_{S_2}}\right|}{\|[\tilde{J}y]\|_{H_{S_2}}} = \sup_{\substack{[\tilde{J}y]\in H_{S_2},\\ \|[\tilde{J}y]\|_{H_{S_2}}=1}} \left|\left\langle [z_0],[\tilde{J}y]\right\rangle_{H_{S_2}}\right|,$$

and so

$$\|[z_0]\|_{H_{S_2}} = \sup_{\substack{[\tilde{J}y]\in H_{S_2},\\ [\tilde{J}y]\neq 0}} \frac{|y_1|}{\|[\tilde{J}y]\|_{H_{S_2}}} = \sup_{\substack{[\tilde{J}y]\in H_{S_2},\\ \|[\tilde{J}y]\|_{H_{S_2}}=1}} |y_1|\,, \tag{4.55}$$

due to equation (4.53).

Furthermore, we remark that the operator $J\colon \ell^2 \to \ell^2$, where $(Jx)_n = -\Delta(\Delta x_{n-1})$, can be expressed as

$$J = (I-L)(I-R),$$

where L and R are the left and right shift operators, respectively. Hence, for $y \in \ell_0^2$, we have

$$\begin{aligned}\|[\tilde{J}y]\|^2_{H_{S_2}} &= \left\langle [\tilde{J}y],[\tilde{J}y]\right\rangle_{H_{S_2}} = \left\langle \tilde{J}y, y\right\rangle = \langle Jy, y\rangle \\ &= \langle (I-L)(I-R)y, y\rangle = \langle (I-R)y, (I-R)y\rangle = \|(I-R)y\|^2,\end{aligned}$$

since $(I-L)^* = I-R$. With these results in mind, we now continue by proving that the inequalities in (4.54) are both valid.

Let $y \in \ell_0^2$, then $((I-R)y)_1 = y_1 - y_0 = y_1$. Hence,

$$|y_1| = |((I-R)y)_1| = \sqrt{|((I-R)y)_1|^2} \leq \|(I-R)y\| = \|[\tilde{J}y]\|_{H_{S_2}}.$$

Then, from equation (4.55), we see that

$$\|[z_0]\|_{H_{S_2}} = \sup_{\substack{[\tilde{J}y]\in H_{S_2},\\ \|[\tilde{J}y]\|_{H_{S_2}}=1}} |y_1| \leq \sup_{\substack{[\tilde{J}y]\in H_{S_2},\\ \|[\tilde{J}y]\|_{H_{S_2}}=1}} \|[\tilde{J}y]\|_{H_{S_2}} = 1,$$

as required.

Conversely, let $q \in \mathbb{R}$ with $|q| < 1$ and define $y = y(q) \in \ell_0^2$ to be the sequence with

$$y_n = y(q)_n = \begin{cases} 0, & n = 0, \\ q^n, & n \geq 1. \end{cases}$$

Then,

$$((I - R)y)_n = \begin{cases} 0, & n = 0, \\ q, & n = 1, \\ q^n - q^{n-1} = q^{n-1}(q-1), & n \geq 2. \end{cases}$$

Since $\|[\tilde{J}y]\|_{H_{S_2}}^2 = \|(I-R)y\|^2$, observe that

$$\begin{aligned} \|(I-R)y\|^2 &= \sum_{n=0}^{\infty} |((I-R)y)_n|^2 \\ &= q^2 + \sum_{n=2}^{\infty} q^{2(n-1)}(q-1)^2 \\ &= q^2 + (q-1)^2 \sum_{n=1}^{\infty} q^{2n}. \end{aligned}$$

As $|q| < 1$, we are able to compute this sum explicitly as it is a geometric series. In particular,

$$\begin{aligned} \|(I-R)y\|^2 &= q^2 + (q-1)^2 \frac{q^2}{1-q^2} \\ &= q^2 \left[\frac{1 - q^2 + (q-1)^2}{1-q^2} \right] \\ &= \frac{2q^2}{1+q}. \end{aligned}$$

If we let q tend to 1 from below, then

$$y_1 = q \to 1 \qquad \text{and} \qquad \|[\tilde{J}y]\|_{H_{S_2}}^2 = \frac{2q^2}{1+q} \to 1.$$

As such,

$$\begin{aligned} \|[z_0]\|_{H_{S_2}} &= \sup_{\substack{[\tilde{J}y] \in H_{S_2}, \\ [\tilde{J}y] \neq 0}} \frac{|y_1|}{\|[\tilde{J}y]\|_{H_{S_2}}} = \sup_{\substack{[\tilde{J}y] \in H_{S_2}, \\ [\tilde{J}y] \neq 0}} \frac{|y_1|}{\|(I-R)y\|} \\ &\geq \sup_{\substack{y(q) \in \ell_0^2 \\ 0<|q|<1}} \frac{|y_1|}{\|(I-R)y\|} = \sup_{0<|q|<1} \frac{|q|}{\sqrt{\frac{2q^2}{1+q}}} = \sup_{|q|<1} \sqrt{\frac{1+q}{2}} = 1, \end{aligned}$$

i.e., $\|[z_0]\|_{H_{S_2}} \geq 1$.

Since we have shown that both $\|[z_0]\|_{H_{S_2}} \leq 1$ and $\|[z_0]\|_{H_{S_2}} \geq 1$ hold for all $y \in \ell_0^2$, we may conclude that $\|[z_0]\|_{H_{S_2}} = 1$, as required. ■

Now that we have shown that $\|[z_0]\|_{H_{S_2}} = 1$, we may finally conclude that the operator T corresponds precisely with the Jacobi operator $\hat{J}$ since the top left component of $\hat{J}$ equals 1. In other words,

$$S_{2,K} = \left\{ (x, \hat{J}x) \;\middle|\; x \in \ell^2 \right\},$$

as expected.

We conclude this section by finding the form $\mathbf{s_{2,K}}$ associated to the Kreĭn extension by means of equation (3.22). Indeed, let u, $v \in \mathcal{D}(V_2^*) = \ell^2$. Then,

$$u = x + ce_0 \quad \text{and} \quad v = y + de_0,$$

for x, $y \in \ell_0^2$ and c, $d \in \mathbb{C}$. Then, as $\tilde{V_2}^*$ is the operator associated to V_2^*, we have

$$\begin{aligned}
\mathbf{s_{2,K}}\,[u, v] = \left\langle \tilde{V_2}^* u, \tilde{V_2^*} v \right\rangle_{H_{S_2}} &= \left\langle [\tilde{J}x] + c[z_0], [\tilde{J}y] + d[z_0] \right\rangle_{H_{S_2}} \\
&= \left\langle [\tilde{J}x], [\tilde{J}y] \right\rangle_{H_{S_2}} + c \left\langle [z_0], [\tilde{J}y] \right\rangle_{H_{S_2}} \\
&\qquad + \overline{d} \left\langle [\tilde{J}x], [z_0] \right\rangle_{H_{S_2}} + c\overline{d} \left\langle [z_0], [z_0] \right\rangle_{H_{S_2}} \\
&= \left\langle \tilde{J}x, y \right\rangle - c\overline{y}_1 - \overline{d}x_1 + c\overline{d}.
\end{aligned}$$

Our objective is to rewrite the right-hand side of this equality in terms of u and v since $u = (c, x_1, x_2, \dots)$ and $v = (d, y_1, y_2, \dots)$. Since $x_0 = y_0 = 0$, we note that

$$\left\langle \tilde{J}x, y \right\rangle = \sum_{n=0}^{\infty} -\Delta(\Delta x_{n-1})\overline{y}_n = \sum_{n=0}^{\infty} \Delta x_n \Delta \overline{y}_n,$$

after an application of the summation by parts formula. Now, since

$$\Delta u_n = \begin{cases} x_1 - c, & n = 0, \\ u_{n+1} - u_n = x_{n+1} - x_n = \Delta x_n, & n \geq 1, \end{cases}$$

we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \Delta x_n \Delta \overline{y}_n &= \sum_{n=1}^{\infty} \Delta x_n \Delta \overline{y}_n + x_1 \overline{y}_1 \\
&= \sum_{n=0}^{\infty} \Delta u_n \Delta \overline{v}_n - \Delta u_0 \Delta \overline{v}_0 + x_1 \overline{y}_1 \\
&= \sum_{n=0}^{\infty} \Delta u_n \Delta \overline{v}_n - (x_1 - c)(\overline{y}_1 - \overline{d}) + x_1 \overline{y}_1 \\
&= \sum_{n=0}^{\infty} \Delta u_n \Delta \overline{v}_n + c\overline{y}_1 + \overline{d}x_1 - c\overline{d}.
\end{aligned}$$

We may then substitute this equality into the expression for $\mathbf{s_{2,K}}[u,v]$. Thus, the form associated to the Kreĭn extension is given by,

$$\begin{aligned}\mathbf{s_{2,K}}[u,v] &= \left[\sum_{n=0}^{\infty} \Delta u_n \Delta \overline{v}_n + c\overline{y}_1 + \overline{d}x_1 - c\overline{d}\right] - c\overline{y}_1 - \overline{d}x_1 + c\overline{d} \\ &= \sum_{n=0}^{\infty} \Delta u_n \Delta \overline{v}_n,\end{aligned}$$

for u, $v \in \ell^2$. A direct comparison to the form associated to the Friedrichs extension yields the following: the expression for the forms are identical. Indeed, the forms only differ in their domain: $Q(\mathbf{s_{2,F}}) = \ell^2_0$, whilst $Q(\mathbf{s_{2,K}}) = \ell^2$. This is consistent with the work presented in Chapters 1 and 2 — the form domain of the $\mathbf{s_{2,F}}$ is minimal, whilst the form domain of $\mathbf{s_{2,K}}$ is maximal. Moreover, we note that the form $\mathbf{s_{2,K}}$ corresponds precisely with the form that we claimed 'seemed more natural', presented towards the end of Section 2.2. Finally, we make special note that

$$Q(\mathbf{s_{2,K}}) = Q(\mathbf{s_{2,F}}) \dotplus \operatorname{span}\{e_0\}, \tag{4.56}$$

where $\operatorname{span}\{e_0\} = \operatorname{mul} S_2^*$. Indeed, we will revisit this decomposition in the next, and final, section of the thesis where we discuss this example more generally and provide an outlook for future work.

4.6 Concluding Remarks

Now that we have constructed all of the extremal maximal sectorial extensions of both S_1 and S_2, we conclude Chapter 4 — and, indeed, the thesis — with a few final observations in an effort to provide direction for possible future works.

Firstly, recall that if $T_{\min}$ is a positive, symmetric operator, then the form domain of the Kreĭn extension is given by

$$Q(\mathbf{t_K}) = Q(\mathbf{t_F}) \dotplus \ker T_{\max},$$

as described in Theorem 1.2.16. Then, we draw attention back to the decomposition of the form domain of $\mathbf{s_{2,K}}$ presented in equation (4.56) — note that the kernel of S_2^* is trivial. If such a result exists, then this decomposition may give an indication as to how one might generalise the Kreĭn-Vishik-Birman theory to linear relations. Indeed, for a sectorial relation S with vertex $\gamma = 0$ and semi-angle $\alpha \in \left[0, \frac{\pi}{2}\right)$, it might seem reasonable to conjecture that

$$Q(\mathbf{s_K}) = Q(\mathbf{s_F}) \dotplus \ker S^* \dotplus \operatorname{mul} S^*.$$

Such a decomposition would certainly be true for the graph of an appropriate operator T, since $\operatorname{mul}\mathcal{G}(T) = \{0\}$, after all.

Finally, we deliver a closing remark in an attempt to generalise the two examples S_1 and S_2 discussed earlier in this chapter. Although Chapter 4 explicitly considers one specific Jacobi operator, we may extend the theory to a general second-order difference operator J, provided that we make the appropriate assumptions. Indeed, let J be the operator such that

$$(Jx)_n = -\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n, \quad x \in \mathcal{D}(J) = \ell^2,$$

where $\{p_n\}$ and $\{q_n\}$ are two real sequences with $p_n > 0$ for all $n \in \mathbb{N}_0$ and $p_{-1} \equiv 0$. Furthermore, we impose that $\{p_n\}$ and $\{q_n\}$ are *bounded* sequences. The consequence of this requirement is clear: if we are to apply the summation by parts formula to expressions of the form

$$\sum_{n=0}^{\infty} \left[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n\right] \overline{y}_n,$$

then the limiting behaviour of the boundary term will be of no concern to us. Moreover, this assumption ensures that J is self-adjoint and so we can prove that the spectrum of J will be contained within a closed interval $[a, b]$. Note that it may not be the whole interval — typically, periodic Jacobi matrices have gaps in the spectrum [36]. Since the operator will have a lower bound $\gamma = a$, we may conclude that there exists a principal solution and non-principal solutions by means of [14, Thm. 2.1]: the former will play the role of φ_-. In fact, we may shift q_n in such a way that guarantees $a = 0$: this ensures that $0 \in \sigma(J)$.

We must make one final assumption in order for this method to work seamlessly: we require $\ker J = \{0\}$. This assumption is crucial once we recall the Rank-Nullity theorem: $\ell^2 = \overline{\mathcal{R}(J)} \oplus \ker J^*$. Indeed, since J is self-adjoint, we immediately note that $\ker J = \{0\}$ guarantees that $\mathcal{R}(J)$ is dense in ℓ^2. This, in conjunction with the fact that 0 will lie in the spectrum of J after the shift, gives that $\mathcal{R}(J)$ is dense in ℓ^2 but is *not* ℓ^2 itself — see Section 4.2.2 for more details.

Then, under this construction, the graph of J will have precisely one self-adjoint extension — the Friedrichs extension. Conversely, the graph of J restricted to ℓ_0^2, considered as a linear relation in $\ell^2 \times \ell^2$, possesses two extremal maximal relations: the Friedrichs extension and the Kreĭn extension. Then, we finish by remarking that these extensions can be constructed analogously to the extensions of S_1 and S_2: the procedures detailed in Sections 4.2, 4.3 and 4.4 detail a comprehensive account of the steps one should follow.

Appendices

A An Example of Sturm-Liouville Type

In Sections 1.2.2 and 1.2.4, we discussed two methods for constructing the self-adjoint extensions of a closed, symmetric operator: the von Neumann theory and the Kreĭn-Vishik-Birman theory, respectively. In this appendix, we apply the theory to a concrete example in order to demonstrate the two methods explicitly. Results presented in this appendix may be found in, for example, [3], [7] or [29].

Let $L^2([0,1])$ denote the Hilbert space consisting of square integrable functions over the closed interval $[0,1]$ whose inner product is given by

$$\langle f, g\rangle_{L^2([0,1])} = \int_0^1 f\overline{g}\, dx, \qquad f,\, g \in L^2([0,1]),$$

and consider the expression

$$M_\nu f = -f'' + \left(\nu^2 - \frac{1}{4}\right) x^{-2} f, \qquad \nu \in (0,1)\,, \tag{A.1}$$

over $[0,1]$. Note that when $\nu = \frac{1}{2}$, the differential expression $M_{1/2}$ reduces to $M_{1/2} f = -f''$.

Remark. The expression M_ν is of Sturm-Liouville type, i.e., $M_\nu f$ is of the form $-(pf')' + qf$ for functions p and q. This can be seen upon taking $p(x) = 1$ and $q(x) = \left(\nu^2 - \frac{1}{4}\right) x^{-2}$.

We may associate to M_ν the operator T_ν whose domain is characterised by those functions f in $L^2([0,1])$ whose first and second derivative are, again, in $L^2([0,1])$ and both f and f' vanish at the endpoints of the interval $[0,1]$. For those $f \in \mathcal{D}(T_\nu)$, define $T_\nu f = M_\nu f$. Note that we may succinctly express the domain of T_ν by

$$\mathcal{D}(T_\nu) = \left\{ f \in L^2([0,1]) \;\middle|\; \begin{array}{l} f',\, f'' \in L^2([0,1]) \text{ and} \\ \qquad f(0) = f'(0) = f(1) = f'(1) = 0 \end{array} \right\}.$$

Remark. We stress that this is a suitable domain to study as T_ν is, in fact, the minimal operator associated to M_ν. For more details, see [7, Prop. 3.1 (ii)].

Whilst this notation is sufficient, we find it useful to introduce so-called *Sobolev spaces* as a means of being consistent with the literature referenced throughout the forthcoming sections. In particular, the domains of interest to us can be expressed explicitly in terms of these spaces. Then, for $n \in \mathbb{N}_0$, we define $H^n([0,1])$ and $H_0^n([0,1])$ as follows [1]:

$$H^n([0,1]) \coloneqq \left\{ f \in L^2([0,1]) \;\middle|\; f^{(k)} \in L^2([0,1]) \text{ for } k \le n \right\},$$

where $f^{(k)} = \frac{d^k}{dx^k} f$, and

$$H_0^n([0,1]) \coloneqq \left\{ f \in H^n([0,1]) \;\middle|\; f^{(k)}(0) = f^{(k)}(1) = 0 \text{ for } k \le n-1 \right\}.$$

Furthermore, $H^n([0,1])$ is a Hilbert space when it is equipped with the inner product

$$\langle f, g \rangle_{H^n([0,1])} = \sum_{k=0}^{n} \left\langle f^{(k)}, g^{(k)} \right\rangle_{L^2([0,1])}$$

for elements f, $g \in H^n([0,1])$; when $n = 0$, we see that $H^0([0,1]) = L^2([0,1])$. We now omit writing the interval with regards to the Sobolev spaces defined above; in particular, we stress that $H^n = H^n([0,1])$ and $H_0^n = H_0^n([0,1])$, unless otherwise specified.

Remark. With Sobolev spaces now defined, we may conclude that

$$\mathcal{D}(T_\nu) = H_0^2([0,1]) = H_0^2.$$

Now that we have an explicit characterisation of the minimal operator, it is only natural to question what form the maximal operator $(T_\nu)_{\max}$ is of due to its importance within the theory. However, we note that since T_ν is a closed, symmetric operator, we have that $(T_\nu)_{\max} = T_\nu^*$ and so it is sufficient to construct the adjoint operator T_ν^* for use in both the von Neumann theory and the Kreĭn-Vishik-Birman theory. Then, let $f \in \mathcal{D}(T_\nu) = H_0^2$ and $g \in \mathcal{D}(T_\nu^*)$ and consider the equality $\langle T_\nu f, g \rangle = \langle f, T_\nu^* g \rangle$. Formally,

$$\begin{aligned}
\langle T_\nu f, g \rangle &= \int_0^1 -f'' \overline{g} \, dx + \int_0^1 \left(\nu^2 - \frac{1}{4} \right) x^{-2} f \overline{g} \, dx \\
&= -\left[f' \overline{g} \right]_0^1 + \int_0^1 f' \overline{g}' \, dx + \int_0^1 \left(\nu^2 - \frac{1}{4} \right) x^{-2} f \overline{g} \, dx \\
&= -\left[f' \overline{g} \right]_0^1 + \left[f \overline{g}' \right]_0^1 - \int_0^1 f \overline{g}'' \, dx + \int_0^1 \left(\nu^2 - \frac{1}{4} \right) x^{-2} f \overline{g} \, dx,
\end{aligned}$$

after two applications of the integration by parts formula.

Remark. There is a subtlety here that we must draw attention to: functions $f \in \mathcal{D}(T_\nu)$ and $g \in \mathcal{D}(T_\nu^*)$ are well-behaved in such a way that both boundary terms, i.e., $[f'\overline{g}]_0^1$ and $[f\overline{g}']_0^1$, actually vanish. This follows from [7, Lem. 3.3] and [7, Prop. 3.1], but we will explore this result more thoroughly when the argument resurfaces later on in this appendix.

Remark. We also note that whenever we are to evaluate a boundary term at $x = 0$, we should really consider the *limit* of the expression as x tends to 0 from the right. This is due to the possibility of certain expressions blowing up at $x = 0$ — this idea will be explored shortly.

We may then conclude that

$$\langle T_\nu f, g\rangle = \int_0^1 f\left[-\overline{g}'' + \left(\nu^2 - \frac{1}{4}\right)x^{-2}\overline{g}\right]\, dx, \qquad f \in \mathcal{D}(T_\nu),\ g \in \mathcal{D}(T_\nu^*).$$

We do not demand much of an element $g \in \mathcal{D}(T_\nu^*)$ for this equality to hold: we merely require that g and $M_\nu g$ lie in $L^2([0,1])$. Hence, T_ν^* is the operator whose domain is given by

$$\mathcal{D}(T_\nu^*) = \left\{g \in L^2([0,1]) \;\middle|\; M_\nu g \in L^2([0,1])\right\},$$

where $T_\nu^* g = M_\nu g$ as described in equation (A.1), with $(T_\nu)_{\max} = T_\nu^*$.

Finally, we note that the operator T_ν exhibits different characteristics depending on the value of $\nu \in (0,1)$. For any $\nu \in (0,1)$, T_ν is regular at the right endpoint $x = 1$. On the other hand, T_ν is of limit-circle type for all $\nu \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ at the left endpoint $x = 0$, and is regular at this endpoint when $\nu = \frac{1}{2}$ — this is because $q(x)$ blows up at $x = 0$ for $\nu \neq \frac{1}{2}$ [29]. Whilst both of these classifications effectively mean that all solutions of the equation $M_\nu f = 0$ belong to $L^2([0,1])$ near the appropriate endpoint, we must be careful about how we describe any boundary conditions that are imposed. This appendix will frequently make reference to the extension of T_ν with Dirichlet boundary conditions, say $(T_\nu)_D$, so we continue this section by defining this domain with the above classifications in mind.

In essence, $(T_\nu)_D$ consists of all functions that vanish at both endpoints. Then, when an endpoint is regular, it is safe to simply evaluate a function at the endpoint as one might expect. For example, $T_{1/2}$ is regular at both $x = 0$ and $x = 1$ and so $(T_{1/2})_D$ has a domain that may be written explicitly as

$$\mathcal{D}((T_{1/2})_D) = \left\{f \in L^2([0,1]) \;\middle|\; M_{1/2} f \in L^2([0,1]),\, f(0) = 0 \text{ and } f(1) = 0\right\}.$$

However, when $\nu \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, we must adapt the condition $f(0) = 0$ since T_ν is of limit-circle type at $x = 0$. We can do so by first introducing the symplectic form $[\cdot,\cdot](\cdot)\colon \mathcal{D}((T_\nu)_{\max}) \times \mathcal{D}((T_\nu)_{\max}) \times [0,1]$ that acts as follows:

$$[f, g](x) \coloneqq f(x)\overline{g}'(x) - f'(x)\overline{g}(x), \qquad f,\, g \in \mathcal{D}((T_\nu)_{\max}).$$

Then, the condition $f(0) = 0$ must be replaced with $[f, x^{\nu+\frac{1}{2}}](0) = 0$, where

$$\begin{aligned} [f, x^{\nu+\frac{1}{2}}](0) = 0 \quad &\iff \quad \lim_{x\to 0^+} [f, x^{\nu+\frac{1}{2}}](x) = 0 \\ &\iff \quad \lim_{x\to 0^+} \left\{ \left(\nu - \frac{1}{2}\right) x^{\nu-\frac{1}{2}} f(x) - f'(x) x^{\nu+\frac{1}{2}} \right\} = 0, \end{aligned}$$

where the function $x^{\nu+\frac{1}{2}}$ was chosen because it is the principal solution to the equation $M_\nu f = 0$ [29]. This is a believable replacement: when $\nu = \frac{1}{2}$, one may observe that $[f, x](0) = 0$ reduces down to the expected condition. Indeed,

$$\begin{aligned} [f, x](0) = 0 \quad &\iff \quad \lim_{x\to 0^+} f(x) - x f'(x) = 0 \\ &\iff \quad \lim_{x\to 0^+} f(x) = 0 \\ &\iff \quad f(0) = 0, \end{aligned}$$

since T_ν is regular. Therefore, whenever $\nu \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, the domain of the extension with Dirichlet boundary conditions can be expressed as

$$\mathcal{D}((T_\nu)_D) = \left\{ f \in L^2([0,1]) \;\middle|\; \begin{array}{l} M_\nu f \in L^2([0,1]), \\ \quad [f, x^{\nu+\frac{1}{2}}](0) = 0 \text{ and } f(1) = 0 \end{array} \right\}.$$

Remark. If the operator was of limit-circle type at the right endpoint, then we would consider the limit as x approaches 1 from the left instead.

Remark. Although this appendix will not be concerned with the case when $\nu = 0$, we note that T_0 is of limit-circle type at $x = 0$ and is regular at $x = 1$ and so one can expect to apply a similar analysis to all $\nu \in \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ without too much difficulty. However, since the form of the kernel elements change, this case requires more details than we feel necessary to make entirely rigorous and hope that the example presented demonstrates the theory in an approachable yet informative manner.

With the operator T_ν and the extension with Dirichlet boundary conditions $(T_\nu)_D$ now defined, we continue by finding the self-adjoint extensions by means of the von Neumann theory and the Kreĭn-Vishik-Birman theory.

A.1 via the von Neumann Theory

Recall the von Neumann theory from Section 1.2.2. Here, our objective will be to explicitly construct the unitary matrix associated to the extension of T_ν with Dirichlet boundary conditions, i.e., $(T_\nu)_D$. We will concern ourselves with $\nu \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, but we endeavour to draw attention to the case when

$\nu = \frac{1}{2}$ whenever appropriate. Consequently, our calculations will make use of the following description of the domain of $(T_\nu)_D$:

$$\mathcal{D}((T_\nu)_D) = \left\{ f \in L^2([0,1]) \;\middle|\; \begin{array}{l} M_\nu f \in L^2([0,1]), \\ \quad [f, x^{\nu+\frac{1}{2}}](0) = 0 \text{ and } f(1) = 0 \end{array} \right\}.$$

Furthermore, recall that the adjoint operator T_ν^* has domain

$$\mathcal{D}(T_\nu^*) = \left\{ g \in L^2([0,1]) \mid M_\nu g \in L^2([0,1]) \right\},$$

and acts as follows on elements: $T_\nu^* g = M_\nu g$ for $g \in \mathcal{D}(T_\nu^*)$.

The von Neumann theory revolves around computing the deficiency spaces $\mathcal{N}_+$ and $\mathcal{N}_-$ of the operator T_ν, so we begin by letting $f \in \mathcal{D}(T_\nu^*)$. Then, we can compute $\mathcal{N}_+ = \ker(T_\nu^* - iI)$ by solving the differential equation

$$-f'' + \left(\nu^2 - \frac{1}{4}\right) x^{-2} f - if = 0, \qquad \nu \in (0,1).$$

For $\nu \neq \frac{1}{2}$, the general solution to this equation is given by

$$f(x) = c_1\sqrt{x}\,\text{BesselJ}(\nu, \sqrt{i}x) + c_2\sqrt{x}\,\text{BesselY}(\nu, \sqrt{i}x), \qquad c_1, c_2 \in \mathbb{C},$$

where BesselJ and BesselY denote the Bessel function of the first kind and second kind respectively. It can be shown that the deficiency spaces of T_ν both have dimension 2, i.e.,

$$m_+(T_\nu) = m_-(T_\nu) = 2,$$

and so both linearly independent solutions will lie in $L^2([0,1])$ — see, for example, [7, Prop. 3.1 (i)]. As such, we may conclude that

$$\mathcal{N}_+ = \text{span}\left\{ \sqrt{x}\,\text{BesselJ}(\nu, \sqrt{i}x),\ \sqrt{x}\,\text{BesselY}(\nu, \sqrt{i}x) \right\}.$$

Similarly, for $f \in \mathcal{D}(T_\nu^*)$, we can compute $\mathcal{N}_- = \ker(T_\nu^* + iI)$ by solving the differential equation

$$-f'' + \left(\nu^2 - \frac{1}{4}\right) x^{-2} f + if = 0, \qquad \nu \in (0,1).$$

This time, the general solution for $\nu \neq \frac{1}{2}$ is given by

$$f(x) = d_1\sqrt{x}\,\text{BesselJ}(\nu, \sqrt{-i}x) + d_2\sqrt{x}\,\text{BesselY}(\nu, \sqrt{-i}x), \qquad d_1, d_2 \in \mathbb{C},$$

and so the deficiency space $\mathcal{N}_-$ may be expressed as

$$\mathcal{N}_- = \text{span}\left\{ \sqrt{x}\,\text{BesselJ}(\nu, \sqrt{-i}x),\ \sqrt{x}\,\text{BesselY}(\nu, \sqrt{-i}x) \right\}.$$

As the deficiency indices are equal, we are able to construct self-adjoint extensions of T_ν by investigating isometric maps from one deficiency space to the other.

Remark. When $\nu = \frac{1}{2}$, the deficiency spaces can be given by

$$\mathcal{N}_+ = \operatorname{span}\left\{\sin(\sqrt{i}x),\, \cos(\sqrt{i}x)\right\}$$

and

$$\mathcal{N}_- = \operatorname{span}\left\{\sin(\sqrt{-i}x),\, \cos(-\sqrt{i}x)\right\}$$

instead. If we begin with the differential equation $-f'' \pm if = 0$, then it may seem more natural for the basis elements to be in terms of exponential functions. However, the form expressed above is more consistent here since the Bessel functions evaluated at $\nu = \frac{1}{2}$ reduce down to sine and cosine functions.

With the two deficiency spaces now explicitly realised, we can begin to investigate isometric maps between the two spaces. In fact, if $T\colon H_1 \to H_2$ is a linear operator where H_1 and H_2 are Hilbert spaces with $\dim H_1 = n < \infty$ and $\dim H_2 = m < \infty$, then T can be represented by an $m \times n$ matrix U [17]. With this result in mind, we assert that if we are in possession of orthonormal bases for both $\mathcal{N}_+$ and $\mathcal{N}_-$, then an isometric map $U\colon \mathcal{N}_+ \to \mathcal{N}_-$ can be represented by a 2×2 *unitary* matrix.

Remark. A unitary matrix U is an $n \times n$ matrix such that

$$UU^\star = U^\star U = I,$$

where I denotes the $n \times n$ identity matrix and $U^\star$, the conjugate transpose of U, i.e., $U^\star = (\overline{U})^\top$. Critically, unitary matrices are isometric maps. To see this, let U be a unitary matrix and $z \in \mathbb{C}^n$. Then,

$$\|Uz\|^2 = \langle Uz, Uz\rangle = \langle z, U^\star Uz\rangle = \langle z, z\rangle = \|z\|^2,$$

demonstrating that U is, indeed, distance preserving.

More specifically, we have the following argument: for a self-adjoint extension $\tilde{T_\nu}$ of T_ν, we may decompose an element $g \in \mathcal{D}(\tilde{T_\nu})$ into

$$g = g_0 + g_+ + Ug_+, \qquad g_0 \in \mathcal{D}(T_\nu),\ g_+ \in \mathcal{N}_+,$$

for some isometric mapping $U\colon \mathcal{N}_+ \to \mathcal{N}_-$, after invoking Theorem 1.2.9. Upon letting $\{e_1, e_2\}$ be an orthonormal basis of $\mathcal{N}_+$, we can then express an element $g_+ \in \mathcal{N}_+$ as $g_+ = \alpha e_1 + \beta e_2$ for constants $\alpha, \beta \in \mathbb{C}$. Likewise, if $\{f_1, f_2\}$ is an orthonormal basis of $\mathcal{N}_-$, then an element $g_- \in \mathcal{N}_-$ can be expressed as $g_- = \gamma f_1 + \delta f_2$ for constants $\gamma, \delta \in \mathbb{C}$. The 2×2 unitary matrix U will then be the matrix that maps the constants α and β to the new constants γ and δ:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}. \tag{A.2}$$

Once we are in possession of the entries of the matrix U, we may then assert that the domain of a self-adjoint extension $\tilde{T_\nu}$ of T_ν is given by

$$\mathcal{D}(\tilde{T_\nu}) = \mathcal{D}(T_\nu) \dot{+} \left\{ \begin{array}{r} \alpha e_1 + \beta e_2 + (A\alpha + B\beta) f_1 \\ + (C\alpha + D\beta) f_2 \end{array} \middle| \ \alpha, \beta \in \mathbb{C} \right\}, \qquad \text{(A.3)}$$

where $\{e_1, e_2\}$ forms an orthonormal basis of $\mathcal{N}_+$ and $\{f_1, f_2\}$ forms an orthonormal basis of $\mathcal{N}_-$.

As this section is interested in the construction of the unitary matrix associated to the extension with Dirichlet boundary conditions, we recall that

$$\mathcal{D}((T_\nu)_D) = \left\{ f \in L^2([0,1]) \ \middle| \ \begin{array}{l} M_\nu f \in L^2([0,1]), \\ \quad [f, x^{\nu+\frac{1}{2}}](0) = 0 \text{ and } f(1) = 0 \end{array} \right\}. \qquad \text{(A.4)}$$

Essentially, we hope to find unique constants A, B, C and D by solving the boundary condition at both endpoints simultaneously: this will yield the matrix U. Firstly, however, we must construct orthonormal bases for the deficiency spaces $\mathcal{N}_+$ and $\mathcal{N}_-$.

When we wish to construct an orthonormal basis of a set, it is customary to use the Gram-Schmidt orthogonalisation process [44]. This process is constructive: if $\{v_1, v_2, \dots, v_n\}$ forms a basis of an inner product space $(V, \langle \cdot, \cdot \rangle)$, then the set $\{w_1, w_2, \dots, w_n\}$, where

$$w_1 = v_1 \qquad \text{and} \qquad w_n = v_n - \sum_{k=1}^{n-1} \frac{\langle v_n, w_k \rangle}{\langle w_k, w_k \rangle} w_k, \qquad n \geq 2,$$

forms an orthogonal basis of V. Furthermore, the set $\{e_1, e_2, \dots, e_n\}$, where

$$e_n = \frac{w_n}{\|w_n\|}, \qquad n \geq 1,$$

forms an orthonormal basis of V.

Fix $\nu \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ — the case when $\nu = \frac{1}{2}$ may be considered similarly. To begin with, we have that $\{v_1, v_2\}$ forms a basis of $\mathcal{N}_+$, where

$$v_1(x) = \sqrt{x}\,\mathrm{BesselJ}(\nu, \sqrt{i}x) \quad \text{and} \quad v_2(x) = \sqrt{x}\,\mathrm{BesselY}(\nu, \sqrt{i}x).$$

After applying the Gram-Schmidt process, it is clear that $\{v_1, v_2 - c_1 v_1\}$ will form an orthogonal basis of $\mathcal{N}_+$, where

$$c_1 = \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} = \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle}$$

Upon normalising v_1 and $v_2 - c_1 v_1$, we obtain an orthonormal basis $\{e_1, e_2\}$ for $\mathcal{N}_+$, where

$$e_1(x) = \frac{v_1(x)}{\|v_1\|} \quad \text{and} \quad e_2(x) = \frac{v_2(x) - c_1 v_1(x)}{\|v_2 - c_1 v_1\|}.$$

Likewise, if $\{\tilde{v}_1,\, \tilde{v}_2\}$ forms a basis of $\mathcal{N}_-$, where

$$\tilde{v}_1(x) = \sqrt{x}\,\mathrm{BesselJ}(\nu, \sqrt{-i}x) \quad \text{and} \quad \tilde{v}_2(x) = \sqrt{x}\,\mathrm{BesselY}(\nu, \sqrt{-i}x),$$

then $\{\tilde{v}_1, \tilde{v}_2 - d_1\tilde{v}_1\}$ forms an orthogonal basis of $\mathcal{N}_-$, where

$$d_1 = \frac{\langle \tilde{v}_2, \tilde{w}_1\rangle}{\langle \tilde{w}_1, \tilde{w}_1\rangle} = \frac{\langle \tilde{v}_2, \tilde{v}_1\rangle}{\langle \tilde{v}_1, \tilde{v}_1\rangle}.$$

Then $\{f_1,\, f_2\}$ forms an orthonormal basis of $\mathcal{N}_-$, where

$$f_1(x) = \frac{\tilde{v}_1(x)}{\|\tilde{v}_1\|} \quad \text{and} \quad f_2(x) = \frac{\tilde{v}_2(x) - d_1\tilde{v}_1(x)}{\|\tilde{v}_2 - d_1\tilde{v}_1\|}.$$

With two orthonormal bases now in hand, all that remains is to construct the 2×2 unitary matrix $(U_\nu)_D$ that corresponds to the extension with Dirichlet boundary conditions as given in equation (A.4).

As we have established that an element $g \in \mathcal{D}((T_\nu)_D)$ can be decomposed into

$$g(x) = g_0(x) + \alpha e_1(x) + \beta e_2(x) + \gamma f_1(x) + \delta f_2(x),$$

for $g_0 \in \mathcal{D}(T_\nu)$ and constants $\alpha,\, \beta,\, \gamma,\, \delta \in \mathbb{C}$ by means of equation (A.3), we continue by investigating $g(x)$ with reference to the boundary conditions at both $x = 0$ and $x = 1$. Immediately, we note that $g_0(0) = g_0(1) = 0$ since $g_0 \in \mathcal{D}(T_\nu) = H_0^2$, whilst regularity at the right endpoint ensures that we may safely evaluate each term at $x = 1$. At $x = 0$, however, we must make use of the limiting condition and so, together, this reduces to solving the two equations

$$\lim_{x\to 0^+} \left\{ \left(\nu + \frac{1}{2}\right) x^{\nu - \frac{1}{2}} \left[\alpha e_1(x) + \beta e_2(x) + \gamma f_1(x) + \delta f_2(x)\right] \right.$$
$$\left. - \left[\alpha e_1'(x) + \beta e_2'(x) + \gamma f_1'(x) + \delta f_2'(x)\right] x^{\nu + \frac{1}{2}} \right\} = 0$$

and

$$\alpha e_1(1) + \beta e_2(1) + \gamma f_1(1) + \delta f_2(1) = 0$$

simultaneously for γ and δ.

Remark. When $\nu = \frac{1}{2}$, we simply replace the first equation with

$$\alpha e_1(0) + \beta e_2(0) + \gamma f_1(0) + \delta f_2(0) = 0,$$

for appropriate functions e_1, e_2, f_1 and f_2, since T_ν is regular at $x = 0$.

As γ and δ will depend on α and β, we are in possession of a system of two equations with two variables to find. Upon solving this system, we find that there exists a unique solution where γ and δ are linear combinations of

α and β, say $\gamma = a_1\alpha + a_2\beta$ and $\delta = b_1\alpha + b_2\beta$. We may then compare this to equation (A.2) and immediately recognise that the constants line up neatly; we can construct the matrix $(U_\nu)_D$ by reading off the coefficients of α and β, that is,

$$(U_\nu)_D = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix},$$

where the method for finding these constants is detailed explicitly in the subsection containing MAPLE calculations below. This matrix $(U_\nu)_D$ is then the 2×2 unitary matrix that corresponds to the extension $\tilde{T}_\nu$ of T_ν with Dirichlet boundary conditions, i.e., $(T_\nu)_D$.

MAPLE Calculations of the Unitary Matrices

This section will detail the method for constructing the unitary matrix associated to the extension with Dirichlet boundary conditions by means of explicit calculations in MAPLE. Crucially, after defining $e1(\nu, x)$ and $e2(\nu, x)$ to be the orthonormal basis elements of $\mathcal{N}_+$, and $f1(\nu, x)$ and $f2(\nu, x)$ to be those of $\mathcal{N}_-$, we set

```
>  V:=(nu,x,alpha,beta)->alpha*e1(nu,x)+beta*e2(nu,x)
```

and

```
>  W:=(nu,x,Gamma,delta)->Gamma*f1(nu,x)+delta*f2(nu,x).
```

Under this construction, an element $g \in \mathcal{D}((T_\nu)_D)$ can be decomposed into $g(x) = g_0(x) + V(\nu, x, \alpha, \beta) + W(\nu, x, \Gamma, \delta)$. We may then create functions that mimic the boundary conditions, knowing in advance that $g_0 \in H_0^2$. In particular, we define

```
>  B1:=(nu,alpha,beta,Gamma,delta)-> limit(
(nu+1/2)*x^(nu-1/2)*(V(nu,x,alpha,beta)+W(nu,x,Gamma,delta))
-(x^(nu+1/2)*(diff(V(nu,x,alpha,beta),x)
+diff(W(nu,x,Gamma,delta),x))),x=0,right)
```

and

```
>  B2:=(nu,alpha,beta,Gamma,delta)->
simplify(subs(x=1,evalf(V(nu,x,alpha,beta)
+W(nu,x,Gamma,delta))))
```

as these functions may be used to simulate $[g, x^{\nu+\frac{1}{2}}](0) = 0$ and $g(1) = 0$ respectively: the Dirichlet boundary conditions. We now solve the two functions $B1$ and $B2$ equal to 0 simultaneously for Γ and δ in terms of α and β using the 'solve' command as follows:

```
>  SolSet:=simplify(solve({B1(nu,alpha,beta,Gamma,delta)=0,
B2(nu,alpha,beta,Gamma,delta)=0},{Gamma,delta})).
```

At this point, we would have to specify the value of $\nu \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ in order for Maple to return a sensible result. Indeed, for a chosen ν, we can then use the 'collect' command, that is,

```
>   Coll1:=collect(SolSet[1],{alpha,beta})
>   Coll2:=collect(SolSet[2],{alpha,beta}),
```

and immediately read off the entries of $(U_\nu)_D$ as this command endeavours to express the constants γ and δ as $\gamma = a_1 e_1 + a_2 e_2$ and $\delta = b_1 f_1 + b_2 f_2$. In particular, we see that $(U_\nu)_D = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C and D are given by the outputs of

```
>   A:= coeff(rhs(Coll1),alpha)
>   B:= coeff(rhs(Coll1),beta)
>   C:= coeff(rhs(Coll2),alpha)
>   D:= coeff(rhs(Coll2),beta)
```

respectively. Rather fortuitously, the matrix $(U_\nu)_d$ is *diagonal* for all ν in the interval $\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$! To illustrate this more tangibly, we present the following few examples:

$$(U_{1/5})_D = \begin{pmatrix} -0.995 + 0.101i & 0 \\ 0 & -0.951 + 0.309i \end{pmatrix},$$

$$(U_{1/3})_D = \begin{pmatrix} -0.989 - 0.149i & 0 \\ 0 & -0.866 + 0.5i \end{pmatrix},$$

$$(U_{3/4})_D = \begin{pmatrix} -0.627 - 0.779i & 0 \\ 0 & -0.383 + 0.924i \end{pmatrix}.$$

Note that the use of the 'evalf' command and rounding makes the entries of the matrices approximate.

Remark. When $\nu = \frac{1}{2}$, we can apply a similar analysis. After making the appropriate changes to the MAPLE code — i.e., suitably replacing basis elements and the initial condition $B1$ — we see that the matrix $(U_{1/2})_D$ is given by

$$(U_{1/2})_D = \begin{pmatrix} -0.327 - 0.945i & 0 \\ 0 & -1 \end{pmatrix}.$$

We conclude the case when $\nu = \frac{1}{2}$ by remarking that the matrix $(U_{1/2})_D$ being diagonal is a consequence of the initial bases chosen for $\mathcal{N}_+$ and $\mathcal{N}_-$ — had we chosen exponential functions instead, then the matrix would *not* be diagonal.

We have now successfully produced the unitary matrix $(U_\nu)_D$ corresponding to the Dirichlet extension of the operator T_ν by means of the von Neumann theory, thus concluding this section.

A.2 via the Kreĭn-Vishik-Birman Theory

In Section A.1, we constructed the extension of the second-order differential operator T_ν that possessed Dirichlet boundary conditions. In this section, we aim to construct the domains of both the Friedrichs extension and Kreĭn extension of the same operator by using the definitions and results presented in Section 1.2.3. Furthermore, we construct these extensions explicitly using the Kreĭn-Vishik-Birman theory.

Recall the expression M_ν from the beginning of Appendix A, that is,

$$M_\nu f = -f' + \left(\nu^2 - \frac{1}{2}\right) x^{-2} f, \qquad \nu \in (0,1).$$

Then, the operator T_ν is such that $T_\nu f = M_\nu f$ for elements $f \in \mathcal{D}(T_\nu) = H_0^2$. Furthermore, as $T_\nu^* = (T_\nu)_{\max}$, we stress that

$$\mathcal{D}((T_\nu)_{\max}) = \left\{ f \in L^2([0,1]) \;\middle|\; M_\nu f \in L^2([0,1]) \right\}. \tag{A.5}$$

Remark. We once again remark that $T_\nu = (T_\nu)_{\min}$ here: the language presented in Section 1.2.3 remains consistent.

We begin by constructing the form $(\mathbf{t}_\nu)_{\mathbf{F}}$ associated to the Friedrichs extension due to its fundamental placement within the theory. By investigating the expression $\langle (T_\nu)_{\max} f, g\rangle$, we aim to arrive at an expression that the form $(\mathbf{t}_\nu)_{\mathbf{F}}$ may take and an appropriate form domain $Q((\mathbf{t}_\nu)_{\mathbf{F}})$ for it to act upon. Then, for $f \in \mathcal{D}((T_\nu)_{\max})$ and $g \in Q((\mathbf{t}_\nu)_{\mathbf{F}})$, we have

$$\begin{aligned}\langle (T_\nu)_{\max} f, g\rangle &= \int_0^1 -f''\overline{g}\; dx + \int_0^1 \left(\nu^2 - \frac{1}{4}\right) x^{-2} f\overline{g}\; dx \\ &= -[f'\overline{g}]_0^1 + \int_0^1 f'\overline{g}' + \left(\nu^2 - \frac{1}{4}\right) x^{-2} f\overline{g}\; dx,\end{aligned}$$

after an application of the integration by parts formula. Due to the association between operators and forms detailed in Section 1.1.3, we require $[f'\overline{g}]_0^1 = 0$ so that we may set

$$(\mathbf{t}_\nu)_{\mathbf{F}}[f,g] = \int_0^1 f'\overline{g}' + \left(\nu^2 - \frac{1}{4}\right) x^{-2} f\overline{g}\; dx,$$

as this would at least ensure that $(\mathbf{t}_\nu)_{\mathbf{F}}$ — with an appropriate form domain — would be a symmetric, sesquilinear form. We now attempt to find this form domain, i.e., $Q((\mathbf{t}_\nu)_{\mathbf{F}})$.

Let

$$\tilde{H}_0^2([0,1]) = \left\{ f \in H^2([0,1]) \;\middle|\; f(0) = f'(0) = 0 \right\}.$$

Unlike $H_0^2([0,1])$, this space involves boundary conditions at only the left endpoint. Although equation (A.5) gives a description of the domain of the maximal operator, we find it useful to invoke [7, Prop. 3.1 (iii)] to express $\mathcal{D}((T_\nu)_{\max})$ as

$$\mathcal{D}((T_\nu)_{\max}) = \tilde{H}_0^2([0,1]) \dot{+} \operatorname{span}\{u, v\},$$

where $u = x^{\nu+\frac{1}{2}}$ and $v = x^{-\nu+\frac{1}{2}}$, i.e., $\ker(T_\nu)_{\max} = \operatorname{span}\{u, v\}$.

Remark. Although we do not consider the case when $\nu = 0$ in our analysis, it is worth remarking that, when $\nu = 0$, we should take $u = x^{\frac{1}{2}}$ and $v = x^{\frac{1}{2}} \ln(x)$.

From this decomposition, it is clear that we may express an $f \in \mathcal{D}((T_\nu)_{\max})$ as $f = \tilde{f}_0 + \alpha u + \beta v$ for $\tilde{f}_0 \in \tilde{H}_0^2([0,1])$. Then, the boundary term $[f'\overline{g}]_0^1$ may be evaluated safely since the functions u and v are elements of $\ker(T_\nu)_{\max}$. Then, to ensure that the boundary term vanishes for every $f \in \mathcal{D}((T_\nu)_{\max})$, we must impose $g(1) = 0$. Upon specifying $f = v$, observe that the boundary term then becomes $[v'\overline{g}]_0^1$, and so we must further impose that $g(0) = 0$. Then, as

$$\begin{aligned}(\mathbf{t}_\nu)_{\mathbf{F}}[g,g] &= \int_0^1 |g'|^2 + \left(\nu^2 - \frac{1}{4}\right) x^{-2}|g|^2 \, dx \\ &= \int_0^1 |g'|^2 \, dx + \left(\nu^2 - \frac{1}{4}\right) \int_0^1 \left|\frac{g}{x}\right|^2 \, dx,\end{aligned}$$

it is clear that we require that both g' and $\frac{g}{x}$ are elements of $L^2([0,1])$ for these calculations to make sense. In other words,

$$\begin{aligned}Q((\mathbf{t}_\nu)_{\mathbf{F}}) &= \left\{g \in L^2([0,1]) \;\middle|\; g', \frac{g}{x} \in L^2([0,1]) \text{ and } g(0) = g(1) = 0\right\} \\ &= \left\{g \in H_0^1 \;\middle|\; \frac{g}{x} \in L^2([0,1])\right\},\end{aligned}$$

is not only a viable domain for the form $(\mathbf{t}_\nu)_{\mathbf{F}}$, but also the biggest domain such that these calculations make sense. In fact, the condition $\frac{g}{x} \in L^2([0,1])$ is superfluous: one can show that

$$Q((\mathbf{t}_\nu)_{\mathbf{F}}) = H_0^1$$

by means of Hardy's inequality as in the proof of [7, Prop. 3.2 (i)]. Finally, we note that this domain ensures that $(\mathbf{t}_\nu)_{\mathbf{F}}$ is a closed form — critical in associating an operator to it.

With Theorem 1.2.12 in mind, we can immediately construct the Friedrichs extension. In particular, since $\mathcal{D}(T_F) = \mathcal{D}(T_{\max}) \cap Q(\mathbf{t}_{\mathbf{F}})$, we readily observe that

$$\begin{aligned}\mathcal{D}((T_\nu)_F) &= \left\{f \in H_0^1 \;\middle|\; M_\nu f \in L^2([0,1])\right\} \\ &= \left\{f \in L^2([0,1]) \;\middle|\; f', M_\nu f \in L^2([0,1]) \text{ and } f(0) = f(1) = 0\right\}.\end{aligned} \tag{A.6}$$

Conversely, we can easily construct the Kreĭn extension $(T_\nu)_K$ by means of Definition 1.2.13. Since

$$\mathcal{D}(T_K) = \mathcal{D}(T_{\min}) \dotplus \mathcal{N},$$

all that remains is to determine $\mathcal{N} = \ker (T_\nu)_{\max}$. This is simple: we merely solve $(T_\nu)_{\max} f = 0$ and so it is immediate that

$$\mathcal{N} = \operatorname{span}\left\{x^{\nu+\frac{1}{2}},\, x^{-\nu+\frac{1}{2}}\right\}.$$

Hence, we may conclude that

$$\mathcal{D}((T_\nu)_K) = H_0^2 \dotplus \operatorname{span}\left\{x^{\nu+\frac{1}{2}},\, x^{-\nu+\frac{1}{2}}\right\}. \tag{A.7}$$

Although we are now in possession of descriptions of both the Friedrichs extension and the Kreĭn extension, we feel it prudent to illustrate the construction of these extensions through the use of Theorem 1.2.16.

First, we construct the Friedrichs extension; this corresponds to the form $(\mathbf{t}_\nu)_\infty = (\mathbf{t}_\nu)_\mathbf{F} + \infty$ with form domain $Q((\mathbf{t}_\nu)_\infty) = Q((\mathbf{t}_\nu)_\mathbf{F})$, since we must take $\mathcal{N}_B = \{0\}$ when $\mathbf{b} = \infty$. The self-adjoint operator B associated to $\mathbf{b} = \infty$ is the zero operator (which we choose to denote by ∞ so that $(T_\nu)_\infty$ makes sense), so we must define $\infty\,[0,0] = 0$. Then, the domain of $(T_\nu)_F$ is given by

$$\mathcal{D}((T_\nu)_F) = \left\{ z \in H_0^1 \;\middle|\; \begin{array}{l} \exists f \in L^2([0,1]) \text{ such that} \\ \qquad\qquad (\mathbf{t}_\nu)_\mathbf{F}\,[z,y] = \langle f, y\rangle \;\; \forall y \in H_0^1 \end{array} \right\}.$$

For these z, we define $f = (T_\nu)_F z$. Our aim is to determine $(T_\nu)_F$ explicitly and so we begin by investigating the expression $(\mathbf{t}_\nu)_\mathbf{F}\,[z,y]$ for $z \in \mathcal{D}((T_\nu)_F)$ and $y \in Q((\mathbf{t}_\nu)_\mathbf{F}) = H_0^1$. Observe that

$$\begin{aligned}
(\mathbf{t}_\nu)_\mathbf{F}\,[z,y] &= \int_0^1 z'\overline{y}' + \left(\nu^2 - \frac{1}{4}\right) x^{-2} z\overline{y}\; dx \\
&= [z'\overline{y}]_0^1 + \int_0^1 \left[-z'' + \left(\nu^2 - \frac{1}{4}\right) x^{-2} z\right] \overline{y}\; dx \\
&= \int_0^1 \left[-z'' + \left(\nu^2 - \frac{1}{4}\right) x^{-2} z\right] \overline{y}\; dx,
\end{aligned}$$

since $\overline{y}(0) = \overline{y}(1) = 0$. Then, for those z and y, we have

$$\langle f, y\rangle = (\mathbf{t}_\nu)_\mathbf{F}\,[z,y] = \int_0^1 (M_\nu z)\overline{y}\; dx,$$

where $M_\nu z$ on the right-hand side of this equality is to be interpreted in the distributional sense. It then remains to show that $M_\nu z$ lies in $L^2([0,1])$.

Using the Riesz Representation Theorem, for an element $g \in L^2([0,1])$, the norm of g satisfies the following equality:

$$\|g\|_{L^2([0,1])} = \sup_{\substack{y \in L^2([0,1]),\\ y \neq 0}} \frac{|\langle g, y\rangle|}{\|y\|_{L^2([0,1])}}. \tag{A.8}$$

This result is critical in the argument to showing that $M_\nu z \in L^2([0,1])$; in particular, we aim we prove that $\|M_\nu z\|_{L^2([0,1])} < \infty$. Thus, we begin by noting that

$$\|M_\nu z\|_{L^2([0,1])} = \sup_{\substack{y \in L^2([0,1]),\\ y \neq 0}} \frac{|\int_0^1 (M_\nu z)\overline{y}\ dx|}{\|y\|_{L^2([0,1])}},$$

by means of equation (A.8). Then, as H_0^1 is a dense set in $L^2([0,1])$ with respect to the $L^2([0,1])$ norm, we have

$$\sup_{\substack{y \in L^2([0,1]),\\ y \neq 0}} \frac{|\int_0^1 (M_\nu z)\overline{y}\ dx|}{\|y\|_{L^2([0,1])}} = \sup_{\substack{y \in H_0^1,\\ y \neq 0}} \frac{|\int_0^1 (M_\nu z)\overline{y}\ dx|}{\|y\|_{L^2([0,1])}}.$$

For $y \in H_0^1$, we have $\int_0^1 (M_\nu z)\overline{y}\ dx = \langle f, y\rangle$ for some $f \in L^2([0,1])$, and so

$$\begin{aligned}\sup_{\substack{y \in H_0^1,\\ y \neq 0}} \frac{|\int_0^1 (M_\nu z)\overline{y}\ dx|}{\|y\|_{L^2([0,1])}} &= \sup_{\substack{y \in H_0^1,\\ y \neq 0}} \frac{|\langle f, y\rangle|}{\|y\|_{L^2([0,1])}} \\ &= \sup_{\substack{y \in L^2([0,1]),\\ y \neq 0}} \frac{|\langle f, y\rangle|}{\|y\|_{L^2([0,1])}} = \|f\|_{L^2([0,1])}.\end{aligned}$$

Since $f \in L^2([0,1])$, we may finally conclude that

$$\|M_\nu z\|_{L^2([0,1])} = \|f\|_{L^2([0,1])} < \infty,$$

thus $M_\nu z \in L^2([0,1])$ and $(T_\nu)_F z = M_\nu z$.

Since we have shown that $M_\nu z$ must lie in $L^2([0,1])$, we may now assert that the Friedrichs extension has domain

$$\mathcal{D}((T_\nu)_F) = \left\{ z \in H_0^1 \;\middle|\; M_\nu z \in L^2([0,1]) \right\}$$

where $(T_\nu)_F z = M_\nu z$ for all $z \in \mathcal{D}((T_\nu)_F)$. It is worth remarking that this is precisely the domain as given in equation (A.6), i.e., the two constructions coincide! In fact, we can show more: we can prove that the extension with Dirichlet boundary conditions as expressed in equation (A.4) coincides with the Friedrichs extension. However, we first require an alternative description of $\mathcal{D}((T_\nu)_F)$. In particular, [7, Prop. 3.2 (ii)] demonstrates that

$$\mathcal{D}((T_\nu)_F) = H_0^2 \dotplus \operatorname{span}\left\{ x^{\nu+\frac{1}{2}}(x-1),\, x^2(x-1) \right\}.$$

Remark. If $\nu = 0$, then the domain of $(T_0)_F$ may be expressed as

$$\mathcal{D}((T_0)_F) = H_0^2 \dotplus \operatorname{span}\left\{x^{\frac{1}{2}}(x-1),\, x^{\frac{1}{2}}\ln(x)(x-1)\right\}$$

instead.

With this new description of the domain of the Friedrichs extension of T_ν in hand, we present the following lemma.

Lemma A.1. *The Friedrichs extension of T_ν and the extension of T_ν with Dirichlet boundary conditions coincide.*

Proof. For the purposes of this lemma, let

$$\mathcal{D}((T_\nu)_F) = H_0^2 \dotplus \operatorname{span}\left\{x^{\nu+\frac{1}{2}}(x-1),\, x^2(x-1)\right\},$$

and recall that

$$\mathcal{D}((T_\nu)_D) = \left\{ f \in L^2([0,1]) \;\middle|\; \begin{array}{l} M_\nu f \in L^2([0,1]), \\ \quad [f, x^{\nu+\frac{1}{2}}](0) = 0 \text{ and } f(1) = 0 \end{array} \right\}.$$

The first step in proving that the two domains coincide will be to show that $\mathcal{D}((T_\nu)_F) \subseteq \mathcal{D}((T_\nu)_D)$. Then, as both $\mathcal{D}((T_\nu)_F)$ and $\mathcal{D}((T_\nu)_D)$ are the domains of a self-adjoint extension of T_ν, they must coincide. To see this, we argue the following: were they *not* the same, then $(T_\nu)_D$ would be an extension of $(T_\nu)_F$. This cannot be true: a non-trivial extension of a self-adjoint extension is no longer self-adjoint.

Let $z \in \mathcal{D}((T_\nu)_F)$, that is, $z = z_0 + \alpha\tilde{u} + \beta\tilde{v}$ for $z_0 \in H_0^2$, $\tilde{u} = x^{\nu+\frac{1}{2}}(x-1)$ and $\tilde{v} = x^2(x-1)$. Then, we must show that $M_\nu z \in L^2([0,1])$ and both boundary conditions in $\mathcal{D}((T_\nu)_D)$) hold. Immediately, we note that

$$M_\nu z = M_\nu z_0 + \alpha\left[-(2\nu+1)x^{\nu-\frac{1}{2}}\right] + \beta\left[2 - 6x + \left(\nu^2 - \frac{1}{4}\right)(x-1)\right],$$

so if we can show that each term individually lies in $L^2([0,1])$, then $M_\nu z$ must too. First, observe that if $f \in H_0^2$, then f' admits the integral representation $f'(x) = \int_0^x f''(t)\,dt$. Therefore, by the Cauchy-Schwarz inequality, we have

$$|f'(x)|^2 \leq \left(\int_0^x |f''(t)|\,dt\right)^2 \leq x\int_0^x |f''(t)|^2\,dt = x \cdot o(1) = o(x),$$

demonstrating that $f'(x) = o(x^{\frac{1}{2}})$. Then, since $z_0 \in H_0^2$, it is true that both $z_0'' \in L^2([0,1])$ and $z_0 = o(x^{\frac{3}{2}})$; the latter condition ensures that $\frac{z_0}{x^2} \in L^2([0,1])$ and so, together, we may conclude that $M_\nu z_0 \in L^2([0,1])$. The remaining two terms are immediate: we simply need to ensure that each exponent of x, when squared, is greater than -1. This is clear, since $2\nu - 1 > -1$ for all $\nu \in (0,1)$. Furthermore, it is easy to show that $z(1) = 0$; indeed,

$$z(1) = z_0(1) + \alpha \cdot 1 \cdot 0 + \beta \cdot 1 \cdot 0 = 0,$$

since $z_0 \in H_0^2$. Then, all that remains is to verify that $[z, x^{\nu+\frac{1}{2}}](0) = 0$. We choose to split the analysis into three parts: we will take $z = z_0$, $z = \tilde{u}$ and $z = \tilde{v}$ separately.

Consider the expression $[z_0, x^{\nu+\frac{1}{2}}](0)$, i.e.,

$$[z_0, x^{\nu+\frac{1}{2}}](0) = \lim_{x \to 0^+} \left\{ \left(\nu + \frac{1}{2}\right) x^{\nu-\frac{1}{2}} z_0(x) - z_0'(x) x^{\nu+\frac{1}{2}} \right\}.$$

This limit may be evaluated once we recall that

$$z_0 \in H_0^2 \qquad \Longrightarrow \qquad z_0 = o(x^{\frac{3}{2}}) \quad \text{and} \quad z_0' = o(x^{\frac{1}{2}}).$$

Indeed, with these properties in mind, it is clear that

$$x^{\nu-\frac{1}{2}} z_0(x) = o(x^{\nu+1}) \qquad \text{and} \qquad z_0'(x) x^{\nu+\frac{1}{2}} = o(x^{\nu+1}),$$

and so the limits tend to 0 as x tends to 0, proving that $[z_0, x^{\nu+\frac{1}{2}}](0) = 0$.

Next, consider the expression $[\tilde{u}, x^{\nu+\frac{1}{2}}](0)$ and note that

$$\begin{aligned} [\tilde{u}, x^{\nu+\frac{1}{2}}](0) &= [x^{\nu+\frac{1}{2}}(x-1), x^{\nu+\frac{1}{2}}](0) \\ &= [x^{\nu+\frac{3}{2}}, x^{\nu+\frac{1}{2}}](0) - [x^{\nu+\frac{1}{2}}, x^{\nu+\frac{1}{2}}](0). \end{aligned}$$

It is readily observed that $[x^{\nu+\frac{1}{2}}, x^{\nu+\frac{1}{2}}](0) = 0$ since the symplectic form is, effectively, a Wronskian-type expression. Hence,

$$[\tilde{u}, x^{\nu+\frac{1}{2}}](0) = [x^{\nu+\frac{3}{2}}, x^{\nu+\frac{1}{2}}](0).$$

Then, since $x^{\nu+\frac{3}{2}} = o(x^{\frac{3}{2}})$, we may conclude that $[\tilde{u}, x^{\nu+\frac{1}{2}}](0) = 0$ after mirroring the argument for $z = z_0$. Finally, since $x^2(x-1) = O(x^2)$ — and in particular, $x^2(x-1) = o(x^{\frac{3}{2}})$ — it is immediate that

$$[\tilde{v}, x^{\nu+\frac{1}{2}}](0) = [x^2(x-1), x^{\nu+\frac{1}{2}}](0) = 0.$$

Therefore, we have proved that an element $z \in \mathcal{D}((T_\nu)_F)$ satisfies all conditions required to lie in $\mathcal{D}((T_\nu)_D)$, and so we may conclude that the two domains must actually coincide. ■

Next, we wish to construct the Kreĭn extension $(T_\nu)_K$ by means of the Kreĭn-Vishik-Birman theory. In particular, we take $\mathbf{b} = 0$ and

$$\mathcal{N}_B = \mathcal{N} = \operatorname{span}\{x^{\nu+\frac{1}{2}},\, x^{-\nu+\frac{1}{2}}\},$$

where the self-adjoint operator B associated to $\mathbf{b}$ is the 2×2 zero matrix. Then, the domain of $(T_\nu)_K$ is given by

$$\mathcal{D}((T_\nu)_K) = \left\{ z \in H_0^1 \dotplus \mathcal{N} \;\middle|\; \begin{array}{l} \exists f \in L^2([0,1]) \text{ such that} \\ \quad \mathbf{t_K}[z,y] = \langle f, y \rangle \;\; \forall y \in H_0^1 \dotplus \mathcal{N} \end{array} \right\},$$

and, for those f, we define $f = (T_\nu)_K z$.

Let $z \in \mathcal{D}((T_\nu)_K)$ and $y \in Q((\mathbf{t}_\nu)_\mathbf{K}) = H_0^1 \dotplus \mathcal{N}$ and consider the expression $(\mathbf{t}_\nu)_\mathbf{K}[z,y]$. We first note that the elements z and y can be decomposed into

$$z(x) = z_0(x) + \alpha u(x) + \beta v(x) \quad \text{and} \quad y(x) = y_0(x) + \gamma u(x) + \delta v(x),$$

where z_0, $y_0 \in H_0^1$ and α, β, γ, $\delta \in \mathbb{C}$. Hence,

$$\begin{aligned} (\mathbf{t}_\nu)_\mathbf{K}[z,y] = (\mathbf{t}_\nu)_\mathbf{F}[z_0,y_0] &= \int_0^1 z_0'\overline{y}_0' + \left(\nu^2 - \frac{1}{4}\right) x^{-2} z_0 \overline{y}_0 \, dx \\ &= [z_0'\overline{y}_0]_0^1 + \int_0^1 \left[-z_0'' + \left(\nu^2 - \frac{1}{4}\right) x^{-2} z_0 \right] \overline{y}_0 \, dx \\ &= \int_0^1 (M_\nu z_0)\overline{y}_0 \, dx \\ &= \langle M_\nu z_0, y_0 \rangle , \end{aligned}$$

after noting that $y_0 \in H_0^1$. From this equality, we assert that $M_\nu z_0$ must lie in $L^2([0,1])$ since we may repeat the argument from the previous example. Then, it is clear that $M_\nu z$ must also lie in $L^2([0,1])$ since

$$M_\nu z = M_\nu(z_0 + \alpha u + \beta v) = M_\nu z_0.$$

As we are looking for the element $f \in L^2([0,1])$ such that $(\mathbf{t}_\nu)_\mathbf{K}[z,y] = \langle f, y \rangle$, we continue by choosing y to be in H_0^1; in particular, we set $y = y_0$, that is, $\gamma = \delta = 0$. Note that, from the representation theorem, we have

$$\langle (T_\nu)_K z, y \rangle = (\mathbf{t}_\nu)_\mathbf{K}[z,y], \qquad z \in \mathcal{D}((T_\nu)_K),\, y \in H_0^1,$$

whilst the calculations above show that

$$(\mathbf{t}_\nu)_\mathbf{K}[z,y] = \langle M_\nu z, y \rangle = \langle f, y \rangle , \qquad z \in \mathcal{D}((T_\nu)_K),\, y \in H_0^1.$$

Then, since H_0^1 is dense in $L^2([0,1])$ with respect to the L^2-norm, we may combine these two equalities to conclude that

$$\langle (T_\nu)_K z, y \rangle = \langle M_\nu z, y \rangle = \langle f, y \rangle , \qquad z \in \mathcal{D}((T_\nu)_K),\, y \in L^2([0,1]),$$

In other words, $(T_\nu)_K z = M_\nu z$, and so the domain of the Kreĭn extension can be expressed as

$$\mathcal{D}((T_\nu)_K) = \left\{ z \in H_0^1 \dotplus \mathcal{N} \;\middle|\; (\mathbf{t}_\nu)_\mathbf{K}[z,y] = \langle M_\nu z, y \rangle \;\; \forall y \in H_0^1 \dotplus \mathcal{N} \right\}. \tag{A.9}$$

Since this produces a different characterisation of $(T_\nu)_K$ to that of equation (A.7), the following lemma confirms that the two constructions do, indeed, coincide.

Lemma A.2. *The two descriptions of $\mathcal{D}((T_\nu)_K)$ as given in equations (A.7) and (A.9) coincide.*

Proof. In order to prove that the two descriptions of $\mathcal{D}((T_\nu)_K)$ are equivalent, we apply a similar proof to that of Lemma A.1. In particular, we only verify one containment before noting that $(T_\nu)_K$ is actually a self-adjoint extension of T_ν. Both domains must then coincide, else we are in possession of a non-trivial extension — such an extension is no longer self-adjoint, leading to the desired contradiction.

For brevity, denote by $\mathcal{D}(K_1)$ the domain of the Kreĭn extension as given in equation (A.7) and by $\mathcal{D}(K_2)$, that of equation (A.9), i.e.,

$$\mathcal{D}(K_1) = H_0^2 \dotplus \mathcal{N}$$

and

$$\mathcal{D}(K_2) = \left\{ z \in H_0^1 \dotplus \mathcal{N} \;\middle|\; (\mathbf{t}_\nu)_{\mathbf{K}}[z,y] = \langle M_\nu z, y\rangle \;\; \forall y \in H_0^1 \dotplus \mathcal{N} \right\},$$

where $\mathcal{N} = \operatorname{span}\{u,\, v\}$ for $u = x^{\nu+\frac{1}{2}}$ and $v = x^{-\nu+\frac{1}{2}}$. In order to prove that $\mathcal{D}(K_1) \subseteq \mathcal{D}(K_2)$, we begin by letting $z \in H_0^2 \dotplus \mathcal{N}$. Then, we need to show that

$$z \in H_0^1 \dotplus \mathcal{N} \quad \text{and} \quad (\mathbf{t}_\nu)_{\mathbf{K}}[z,y] = \langle M_\nu z, y\rangle \;\text{ for all } y \in H_0^1 \dotplus \mathcal{N}.$$

Immediately, we note that $H_0^2 \subset H_0^1$ and so the first condition is trivially satisfied. Then, let $y \in H_0^1 \dotplus \mathcal{N}$ and consider the expression $(\mathbf{t}_\nu)_{\mathbf{K}}[z,y]$ for $z \in H_0^2 \dotplus \mathcal{N}$. Since we may decompose z and y into $z = z_0 + \alpha u + \beta v$ and $y = y_0 + \gamma u + \delta v$ for $z \in H_0^2$ and $y \in H_0^1$, we may recycle previous calculations without fear. In particular,

$$\begin{aligned}
(\mathbf{t}_\nu)_{\mathbf{K}}[z,y] = (\mathbf{t}_\nu)_{\mathbf{F}}[z_0,y_0] &= \int_0^1 z_0'\overline{y}_0' + \left(\nu^2 - \frac{1}{4}\right) x^{-2} z_0 \overline{y}_0 \; dx \\
&= [z_0'\overline{y}_0]_0^1 + \int_0^1 \left[-z_0'' + \left(\nu^2 - \frac{1}{4}\right) x^{-2} z_0 \right] \overline{y}_0 \; dx \\
&= \int_0^1 (M_\nu z)\overline{y}_0 \; dx.
\end{aligned}$$

Thus, $(\mathbf{t}_\nu)_{\mathbf{K}}[z,y] = \langle M_\nu z, y\rangle$ provided that $M_\nu z \in L^2([0,1])$. However, this is clear after applying the argument concerning the distributional derivative, and so we may conclude that $\mathcal{D}(K_1) \subseteq \mathcal{D}(K_2)$. Then, as $\mathcal{D}(K_1)$ and $\mathcal{D}(K_2)$ are the domains of a self-adjoint extension of T_ν, we must have that $\mathcal{D}(K_1) = \mathcal{D}(K_2)$, as required. ■

We are now in possession of various descriptions of both the Friedrichs extension $(T_\nu)_F$ and the Kreĭn extension $(T_\nu)_K$ of T_ν, and so we choose to close this appendix by asking one final question: if the Dirichlet boundary conditions correspond to $(T_\nu)_F$, then can we find which boundary conditions correspond to $(T_\nu)_K$ instead?

To begin our construction of the Kreĭn extension in terms of boundary conditions, let $z \in \mathcal{D}((T_\nu)_K)$ where

$$\mathcal{D}(K_1) = H_0^2 \dotplus \operatorname{span}\{u,\, v\}$$

where $u = x^{\nu+\frac{1}{2}}$ and $v = x^{-\nu+\frac{1}{2}}$. Then, for $z_0 \in H_0^2$ and $\alpha,\, \beta \in \mathbb{C}$, we have

$$\begin{aligned} z(x) &= z_0(x) + \alpha u(x) + \beta v(x) \\ &= z_0(x) + \alpha x^{\nu+\frac{1}{2}} + \beta x^{-\nu+\frac{1}{2}}. \end{aligned}$$

Rearranging this equality gives insight into how we may determine α and β in terms of limits. Indeed,

$$\begin{aligned} \beta v(x) = z(x) - z_0(x) - \alpha u(x) \qquad &\Longrightarrow \qquad \beta = \lim_{x\to 0^+} \frac{z(x) - z_0(x) - \alpha u(x)}{v(x)} \\ &\Longrightarrow \qquad \beta = \lim_{x\to 0^+} z(x) x^{\nu-\frac{1}{2}}, \end{aligned}$$

since $z_0 = o(x^{\frac{3}{2}})$. Likewise,

$$\begin{aligned} \alpha &= \lim_{x\to 0^+} \frac{z(x) - \beta v(x)}{u(x)} \\ &= \lim_{x\to 0^+} \frac{z(x) - \left[\lim_{y\to 0^+} z(y) y^{\nu-\frac{1}{2}}\right] x^{-\nu+\frac{1}{2}}}{x^{\nu+\frac{1}{2}}}, \end{aligned}$$

after substituting in our expression for β.

Next, it is clear that

$$\begin{aligned} z'(x) &= z_0'(x) + \alpha u'(x) + \beta v'(x) \\ &= z_0'(x) + \left(\nu + \frac{1}{2}\right) \alpha x^{\nu-\frac{1}{2}} + \left(-\nu + \frac{1}{2}\right) \beta x^{-\nu-\frac{1}{2}}. \end{aligned}$$

upon differentiating $z(x)$. If we continue by substituting $x = 1$ into both z and z', then we see that

$$z(1) = \alpha + \beta \qquad \text{and} \qquad z'(1) = \left(\nu + \frac{1}{2}\right)\alpha + \left(-\nu + \frac{1}{2}\right)\beta.$$

Hence, the boundary conditions associated to the Kreĭn extension can be expressed as

$$z(1) = \lim_{x\to 0^+} \frac{z(x) - \left[\lim_{y\to 0^+} z(y) y^{\nu-\frac{1}{2}}\right] x^{-\nu+\frac{1}{2}}}{x^{\nu+\frac{1}{2}}} + \lim_{x\to 0^+} z(x) x^{\nu-\frac{1}{2}}$$

and

$$z'(1) = \left(\nu + \frac{1}{2}\right) \lim_{x\to 0^+} \frac{z(x) - \left[\lim_{y\to 0^+} z(y) y^{\nu-\frac{1}{2}}\right] x^{-\nu+\frac{1}{2}}}{x^{\nu+\frac{1}{2}}} + \left(-\nu + \frac{1}{2}\right) \lim_{x\to 0^+} z(x) x^{\nu-\frac{1}{2}}.$$

We recognise that this result is not particularly illuminating, and so we conclude this example by setting $\nu = \frac{1}{2}$ in the boundary conditions found above. In particular — once we recall that $T_{1/2}$ is regular at both endpoints — the two conditions above reduce down to

$$\begin{aligned} z(1) &= \lim_{x\to 0^+} \frac{z(x) - \left[\lim_{y\to 0^+} z(y)\right]}{x} + \lim_{x\to 0^+} z(x) \\ &= \lim_{x\to 0^+} \frac{z(x) - z(0)}{x} + z(0) \\ &= z'(0) + z(0) \end{aligned}$$

and

$$\begin{aligned} z'(1) &= 1 \cdot \lim_{x\to 0^+} \frac{z(x) - \left[\lim_{y\to 0^+} z(y)\right]}{x} + 0 \cdot \lim_{x\to 0^+} z(x) \\ &= z'(0), \end{aligned}$$

respectively, and so the domain of $(T_{1/2})_K$ may be expressed as

$$\mathcal{D}((T_{1/2})_K) = \left\{ z \in H_0^2 \dotplus \operatorname{span}\{u,\, v\} \;\middle|\; \begin{array}{r} M_\nu z \in L^2([0,1]),\ z'(1) = z'(0) \\ \text{and } z(1) = z'(0) + z(0) \end{array} \right\},$$

where $u = x$ and $v = 1$.

We have now completed the intended examples of this section. In particular, for the operator T_ν, we have shown that the Friedrichs extension and the Dirichlet extension coincide precisely and we have determined explicit boundary conditions for the Kreĭn extension. The Kreĭn-Vishik-Birman theory is fundamental to the thesis and so we hope that this example serves as a practical introduction to the theory.

B Sequences $z^{(N)} - z^{(M)}$ for the Kreĭn Extension $S_{2,K}$

In order to construct the Kreĭn extension $S_{2,K}$ of the relation

$$S_2 = \left\{ (x, \tilde{J}x) \in \ell^2 \times \ell^2 \;\middle|\; x \in \ell^2_0 \right\},$$

where $(\tilde{J}x)_n = -\Delta(\Delta x_{n-1})$, we need to determine how the linear operator A_2 acts on an arbitrary element $\omega \in \ell^2$. In Section 4.4, it was shown that if $\omega \in \operatorname{span}\{\varphi_-\}$, where $(\varphi_-)_n = \lambda_-^n$ and $\lambda_- = \frac{3-\sqrt{5}}{2}$, then there does not exist an α such that $\hat{\omega} \in \mathcal{R}(\tilde{J})$. As such, we must approximate $\hat{\omega}$ by some sequence $f^{(N)}$ in $\mathcal{R}(\tilde{J})$ such that, amongst others, the following condition holds:

$$\mathbf{s}_{\mathbf{2},\mathbf{F}}^{-\mathbf{1}} \left[f^{(N)} - f^{(M)}, f^{(N)} - f^{(M)} \right] \to 0 \text{ as } N,\, M \to \infty. \tag{B.1}$$

We have shown that the sequence $f^{(N)}$ takes the form

$$f_n^{(N)} = \begin{cases} \hat{\omega}_n, & 0 \le n \le N, \\ F_N, & N+1 \le n \le 2N, \\ G_N, & 2N+1 \le n \le 3N, \\ 0, & 3N+1 \le n, \end{cases}$$

whilst $z^{(N)}$ is of the form

$$z_n^{(N)} = \begin{cases} \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=N+1}^{2N} (n-r)F_N \\ \qquad\qquad\qquad + \displaystyle\sum_{r=n}^{N} (n-r)\hat{\omega}_r, & 0 \le n \le N, \\ \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=n}^{2N} (n-r)F_N, & N+1 \le n \le 2N, \\ \displaystyle\sum_{r=n}^{3N} (n-r)G_N, & 2N+1 \le n \le 3N, \\ 0, & 3N+1 \le n, \end{cases}$$

where

$$F_N = -\sum_{r=0}^{N} \left[\frac{5N+1-2r}{2N^2} \right] \hat{\omega}_r \quad \text{and} \quad G_N = \sum_{r=0}^{N} \left[\frac{3N+1-2r}{2N^2} \right] \hat{\omega}_r.$$

In order to show that the condition given in (B.1) holds, we require the expressions $f^{(N)} - f^{(M)}$ and $z^{(N)} - z^{(M)}$ explicitly. However, due to the interplay between N and M in the above intervals, we note that there are *seven* distinct cases that must be considered. Since the expressions are large and unwieldy, we choose to display the expressions $z^{(N)} - z^{(M)}$ in this appendix.

Case I: $M < \frac{N}{3}$

The sequence $z^{(N)} - z^{(M)}$ takes the following form when $M < \frac{N}{3}$:

$$\left(z^{(N)} - z^{(M)}\right)_n = \begin{cases} \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=N+1}^{2N} (n-r)F_N + \sum_{r=3M+1}^{N} (n-r)\hat{\omega}_r + \sum_{r=2M+1}^{3M} (n-r)(\hat{\omega}_r - G_M) & \\ \displaystyle\qquad + \sum_{r=M+1}^{2M} (n-r)(\hat{\omega}_r - F_M), & 0 \leq n \leq M, \\ \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=N+1}^{2N} (n-r)F_N + \sum_{r=3M+1}^{N} (n-r)\hat{\omega}_r + \sum_{r=2M+1}^{3M} (n-r)(\hat{\omega}_r - G_M) & \\ \displaystyle\qquad + \sum_{r=n}^{2M} (n-r)(\hat{\omega}_r - F_M), & M+1 \leq n \leq 2M, \\ \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=N+1}^{2N} (n-r)F_N + \sum_{r=3M+1}^{N} (n-r)\hat{\omega}_r + \sum_{r=n}^{3M} (n-r)(\hat{\omega}_r - G_M), & 2M+1 \leq n \leq 3M, \\ \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=N+1}^{2N} (n-r)F_N + \sum_{r=n}^{N} (n-r)\hat{\omega}_r, & 3M+1 \leq n \leq N, \\ \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=n}^{2N} (n-r)F_N, & N+1 \leq n \leq 2N, \\ \displaystyle\sum_{r=n}^{3N} (n-r)G_N, & 2N+1 \leq n \leq 3N, \\ 0, & 3N+1 \leq n. \end{cases} \tag{B.2}$$

Case II: $M = \frac{N}{3}$

The sequence $z^{(N)} - z^{(M)}$ takes the following form when $M = \frac{N}{3}$:

$$\left(z^{(N)} - z^{(M)}\right)_n = \begin{cases} \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=N+1}^{2N} (n-r)F_N + \sum_{r=2M+1}^{N} (n-r)(\hat{\omega}_r - G_M) \\ \qquad\qquad\qquad\qquad \displaystyle + \sum_{r=M+1}^{2M} (n-r)(\hat{\omega}_r - F_M), & 0 \leq n \leq M, \\ \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=N+1}^{2N} (n-r)F_N + \sum_{r=2M+1}^{N} (n-r)(\hat{\omega}_r - G_M) \\ \qquad\qquad\qquad\qquad \displaystyle + \sum_{r=n}^{2M} (n-r)(\hat{\omega}_r - F_M), & M+1 \leq n \leq 2M, \\ \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=N+1}^{2N} (n-r)F_N + \sum_{r=n}^{N} (n-r)(\hat{\omega}_r - G_M), & 2M+1 \leq n \leq N, \\ \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=n}^{2N} (n-r)F_N, & N+1 \leq n \leq 2N, \\ \displaystyle\sum_{r=n}^{3N} (n-r)G_N, & 2N+1 \leq n \leq 3N, \\ 0, & 3N+1 \leq n. \end{cases} \tag{B.3}$$

Whilst this case realistically only involves one variable — either M or N depending on which substitution is undertaken — we choose to display $z^{(N)} - z^{(M)}$ in the form above in order to demonstrate the similarities between it and Case I: essentially, the interval $[3M+1, N]$ collapses, since $3M + 1 = N + 1$.

Case III: $\frac{N}{3} < M < \frac{N}{2}$

The sequence $z^{(N)} - z^{(M)}$ takes the following form when $\frac{N}{3} < M < \frac{N}{2}$:

$$
\left(z^{(N)} - z^{(M)}\right)_n = \begin{cases}
\begin{aligned} &\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=3M+1}^{2N} (n-r)F_N + \sum_{r=N+1}^{3M} (n-r)(F_N - G_M) + \sum_{r=2M+1}^{N} (n-r)(\hat{\omega}_r - G_M) \\ &\qquad + \sum_{r=M+1}^{2M} (n-r)(\hat{\omega}_r - F_M), \end{aligned} & 0 \le n \le M, \\
\begin{aligned} &\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=3M+1}^{2N} (n-r)F_N + \sum_{r=N+1}^{3M} (n-r)(F_N - G_M) + \sum_{r=2M+1}^{N} (n-r)(\hat{\omega}_r - G_M) \\ &\qquad + \sum_{r=n}^{2M} (n-r)(\hat{\omega}_r - F_M), \end{aligned} & M+1 \le n \le 2M, \\
\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=3M+1}^{2N} (n-r)F_N + \sum_{r=N+1}^{3M} (n-r)(F_N - G_M) + \sum_{r=n}^{N} (n-r)(\hat{\omega}_r - G_M), & 2M+1 \le n \le N, \\
\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=3M+1}^{2N} (n-r)F_N + \sum_{r=n}^{3M} (n-r)(F_N - G_M), & N+1 \le n \le 3M, \\
\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=n}^{2N} (n-r)F_N, & 3M+1 \le n \le 2N, \\
\sum_{r=n}^{3N} (n-r)G_N, & 2N+1 \le n \le 3N, \\
0, & 3N+1 \le n.
\end{cases} \tag{B.4}
$$

Case IV: $M = \frac{N}{2}$

The sequence $z^{(N)} - z^{(M)}$ takes the following form when $M = \frac{N}{2}$:

$$\left(z^{(N)} - z^{(M)}\right)_n = \begin{cases} \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=3M+1}^{2N} (n-r)F_N + \sum_{r=N+1}^{3M} (n-r)(F_N - G_M) \\ \qquad\qquad \displaystyle + \sum_{r=M+1}^{N} (n-r)(\hat{\omega}_r - F_M), & 0 \le n \le M, \\ \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=3M+1}^{2N} (n-r)F_N + \sum_{r=N+1}^{3M} (n-r)(F_N - G_M) \\ \qquad\qquad \displaystyle + \sum_{r=n}^{N} (n-r)(\hat{\omega}_r - F_M), & M+1 \le n \le N, \\ \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=3M+1}^{2N} (n-r)F_N + \sum_{r=n}^{3M} (n-r)(F_N - G_M), & N+1 \le n \le 3M, \\ \displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=n}^{2N} (n-r)F_N, & 3M+1 \le n \le 2N, \\ \displaystyle\sum_{r=n}^{3N} (n-r)G_N, & 2N+1 \le n \le 3N, \\ 0, & 3N+1 \le n. \end{cases} \tag{B.5}$$

Case V: $\frac{N}{2} < M < \frac{2N}{3}$

The sequence $z^{(N)} - z^{(M)}$ takes the following form when $\frac{N}{2} < M < \frac{2N}{3}$:

$$
\left(z^{(N)} - z^{(M)}\right)_n = \begin{cases}
\begin{aligned}&\sum_{r=2N+1}^{3N}(n-r)G_N + \sum_{r=3M+1}^{2N}(n-r)F_N + \sum_{r=2M+1}^{3M}(n-r)(F_N - G_M) + \sum_{r=N+1}^{2M}(n-r)(F_N - F_M)\\ &\qquad + \sum_{r=M+1}^{N}(n-r)(\hat{\omega}_r - F_M),\end{aligned} & 0 \le n \le M,\\
\begin{aligned}&\sum_{r=2N+1}^{3N}(n-r)G_N + \sum_{r=3M+1}^{2N}(n-r)F_N + \sum_{r=2M+1}^{3M}(n-r)(F_N - G_M) + \sum_{r=N+1}^{2M}(n-r)(F_N - F_M)\\ &\qquad + \sum_{r=n}^{N}(n-r)(\hat{\omega}_r - F_M),\end{aligned} & M+1 \le n \le N,\\
\sum_{r=2N+1}^{3N}(n-r)G_N + \sum_{r=3M+1}^{2N}(n-r)F_N + \sum_{r=2M+1}^{3M}(n-r)(F_N - G_M) + \sum_{r=n}^{2M}(n-r)(F_N - F_M), & N+1 \le n \le 2M,\\
\sum_{r=2N+1}^{3N}(n-r)G_N + \sum_{r=3M+1}^{2N}(n-r)F_N + \sum_{r=n}^{3M}(n-r)(F_N - G_M), & 2M+1 \le n \le 3M,\\
\sum_{r=2N+1}^{3N}(n-r)G_N + \sum_{r=n}^{2N}(n-r)F_N, & 3M+1 \le n \le 2N,\\
\sum_{r=n}^{3N}(n-r)G_N, & 2N+1 \le n \le 3N,\\
0, & 3N+1 \le n.
\end{cases} \tag{B.6}
$$

Case VI: $M = \frac{2N}{3}$

The sequence $z^{(N)} - z^{(M)}$ takes the following form when $M = \frac{2N}{3}$:

$$
\left(z^{(N)} - z^{(M)}\right)_n = \begin{cases}
\displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=2M+1}^{2N} (n-r)(F_N - G_M) + \sum_{r=N+1}^{2M} (n-r)(F_N - F_M) \\
\qquad\qquad\qquad + \displaystyle\sum_{r=M+1}^{N} (n-r)(\hat{\omega}_r - F_M), & 0 \le n \le M, \\
\displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=2M+1}^{2N} (n-r)(F_N - G_M) + \sum_{r=N+1}^{2M} (n-r)(F_N - F_M) \\
\qquad\qquad\qquad + \displaystyle\sum_{r=n}^{N} (n-r)(\hat{\omega}_r - F_M), & M+1 \le n \le N, \\
\displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=2M+1}^{2N} (n-r)(F_N - G_M) + \sum_{r=n}^{2M} (n-r)(F_N - F_M), & N+1 \le n \le 2M, \\
\displaystyle\sum_{r=2N+1}^{3N} (n-r)G_N + \sum_{r=n}^{2N} (n-r)(F_N - G_M), & 2M+1 \le n \le 2N, \\
\displaystyle\sum_{r=n}^{3N} (n-r)G_N, & 2N+1 \le n \le 3N, \\
0, & 3N+1 \le n.
\end{cases} \tag{B.7}
$$

Case VII: $\frac{2N}{3} < M < N$

The sequence $z^{(N)} - z^{(M)}$ takes the following form when $\frac{2N}{3} < M < N$:

$$
\left(z^{(N)} - z^{(M)}\right)_n = \begin{cases}
\displaystyle\sum_{r=3M+1}^{3N} (n-r)G_N + \sum_{r=2N+1}^{3M} (n-r)(G_N - G_M) + \sum_{r=2M+1}^{2N} (n-r)(F_N - G_M) + \sum_{r=N+1}^{2M} (n-r)(F_N - F_M) \\
\qquad\qquad\qquad + \displaystyle\sum_{r=M+1}^{N} (n-r)(\hat{\omega}_r - F_M), & 0 \leq n \leq M, \\
\displaystyle\sum_{r=3M+1}^{3N} (n-r)G_N + \sum_{r=2N+1}^{3M} (n-r)(G_N - G_M) + \sum_{r=2M+1}^{2N} (n-r)(F_N - G_M) + \sum_{r=N+1}^{2M} (n-r)(F_N - F_M) \\
\qquad\qquad\qquad + \displaystyle\sum_{r=n}^{N} (n-r)(\hat{\omega}_r - F_M), & M+1 \leq n \leq N, \\
\displaystyle\sum_{r=3M+1}^{3N} (n-r)G_N + \sum_{r=2N+1}^{3M} (n-r)(G_N - G_M) + \sum_{r=2M+1}^{2N} (n-r)(F_N - G_M) + \sum_{r=n}^{2M} (n-r)(F_N - F_M), & N+1 \leq n \leq 2M, \\
\displaystyle\sum_{r=3M+1}^{3N} (n-r)G_N + \sum_{r=2N+1}^{3M} (n-r)(G_N - G_M) + \sum_{r=n}^{2N} (n-r)(F_N - G_M), & 2M+1 \leq n \leq 2N, \\
\displaystyle\sum_{r=3M+1}^{3N} (n-r)G_N + \sum_{r=n}^{3M} (n-r)(G_N - G_M), & 2N+1 \leq n \leq 3M, \\
\displaystyle\sum_{r=n}^{3N} (n-r)G_N, & 3M+1 \leq n \leq 3N, \\
0, & 3N+1 \leq n.
\end{cases}
\tag{B.8}
$$

With the final expression for $z^{(N)} - z^{(M)}$ now defined, we thus conclude this appendix.

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