Some extensions of the component maintenance priority*

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Abstract. If a component in a binary system fails, preventive maintenance (PM) on other components may be conducted while the failed component is being repaired. This raises a question on which components should be selected for PM if maintenance resource is limited. The question can be answered by using the component maintenance priority (CMP) to prioritise components for PM. This paper extends the definition of the CMP to the cases of multi-state systems, continuum systems and non-coherent systems, respectively. It investigates the applications of the proposed measures for multi-state systems in optimisation of maintenance policies and proposes algorithms to minimise maintenance cost. A case study is used to instantiate the validity of the proposed measures.

Keywords: multistate system, system performance, importance measure, maintenance policy

1 Introduction

1.1 Background

Importance measures have been widely studied in the reliability literature to identify the weakest components in a system from various perspectives. They can provide valuable information for system design and maintenance for improving the performance of the system. The reader is referred to Kuo and Zhu [2, 3] for a comprehensive review of reliability importance measures and to Fu et al [4], Xu et al [5] and Dui et al [6, 7] for recent developments.

In terms of multistate systems, Levitin et al [8] considered a generalized concept of importance measures for multi-state systems and analyzed the importance change with some restrictions. Ramirez–Marquez and Coit [9] presented a composite importance measure: mean absolute deviation (MDV), which measures the expected absolute deviation in the reliability of a multi-state system. The MDV can be used to evaluate the effect of all component states on the system reliability. Borgonovo [10] proposed importance measures for basic events, and system structures and components. The

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author then used these measures to analyze the probability change of event trees. Dutuit and Rauzy [11] extended the importance measures to complex components, whose failures are modelled by a gate rather than a basic event. Borgonovo et al [12] introduced two new time-independent reliability importance measures and analysed the change of component importance with time. Lin and Yam [13] analysed the uncertainties associated with the transition rates in Markov models and proposed uncertainty importance measures based on the total sensitivity indices in multi-state systems. Xu et al [14] analysed the characteristic function-based moment-independent importance measure, which can be used to evaluate system uncertainty.

In terms of noncoherent systems and continuum systems, Andrews and Beeson [15] analysed the Birnbaum importance measure for noncoherent binary systems. Beeson and Andrews [16] extended four commonly used importance measures, based on the noncoherent extension of Birnbaum importance. Borgonovo [17] proposed the reliability importance of components in coherent and non-coherent systems. Vaurio [18] developed and compared the Birnbaum and criticality importance measures in non-coherent systems. Aliee et al [19] introduced a Boolean expression for the notion of criticality that allows the seamless extension of the Birnbaum importance to non-coherent systems. Besides, Kim and Baxter [20] defined the Birnbaum importance measure of the components in a continuum system with the states in the interval [0,1]. Liu et al [21] generalized the Griffith importance to the continuous-state systems by extending the system structure function. Cai et al [22] proposed the performance improvement to evaluate the change of the performance of continuum systems.

In terms of maintenance policies, Gao et al [23] introduced a conditional reliability importance, which meets the practical requirements such as maintenance and operating state monitoring. Wu and Coolen [24] proposed a new importance measure, which takes consideration of costs of repairing components and cost of repairing the system. Wu and Chan [25] discussed the contribution of an individual component to the performance utility of a multi-state system. Dui et al [26, 27] extended the integrated importance measure to evaluate how the transition of component states affects the system performance. Based on the integrated importance measure, Zhang et al [28] analyzed the component failure recognition and maintenance optimization for an offshore heave compensation system. Dui et al [29] introduced component joint importance measures for maintenances in a submarine blowout preventer system. Furthermore, considering the component maintenance cost and time, Dui et al [30] proposed a cost-based integrated importance measure to identify the component or group of components that may be selected for PM. Based on the stress-strength interference model, Lyu and Si [31] developed a dynamic importance measure to identify the dynamic weakness effectively for systems subjected to repeated and random load. Do and Bérenguer [32] proposed a novel time-dependent importance measure for multi-component systems and defined it as the ability to improve system reliability during a mission given the current conditions. Fu et al [33] proposed a new time-dependent importance measure and developed a system-lifetime maximization model to address the component reassignment problem for degrading components.

1.2 Research Questions and Novelty

The existing literature, however, lacks an importance measure for solving the following problems: Suppose the performance of a multistate system can be characterized by the system performance utility. When the system performance degrades to a state below a certain threshold, one needs to detect the failed components and then maintain them. This raises the following questions.

- If the degradation of a component is not self-announcing, how can the failed components be detected?
- After a component degrades from one state to another, how do other components affect the system performance?
- After the state degradation of a component causes the system to jump multiple states, how to prioritise the components to be maintained during the time of a failed component being repaired?
- While failed components are being repaired, which unfailed components should be selected for PM? How do we determine the number of components for PM and optimise the system performance?

Wu et al. [1] introduced an importance measure that prioritizes components for preventive maintenance while a failed component is being repaired and then used their proposed measure to find the optimal number of repairmen needed for maintaining the system. However, their work concentrates on binary systems. In the literature, there is little research on answering the following question: which components in a multi-state system (and continuum systems and non-coherent systems) have the top priority for preventive maintenance if some of the other components in the system are being repaired? To answer this question, this paper develops new importance measures to assess the component maintenance priority of a multistate system. It is then used to optimise the number of components for PM to maximize the expected system performance. The paper also generalizes the definition of the component maintenance priority to the cases of non-coherent binary and multi-state systems and continuum systems, respectively. As such, the novelty of this paper is on the introduction of new importance measures for those systems.

1.3 Overview

This remainder of this paper is structured as follows. Section 2 introduces the importance measures of component maintenance priority of multistate systems. Section 3 discusses the properties of the proposed importance measures and the optimization of some PM policies. Section 4 analyses some generalizations of the proposed measures in noncoherent systems and continuum systems. Section 5 uses a case study to show the validity of the proposed measures. Section 6 wraps up the findings of this paper.

2 Component maintenance priority in multistate systems

Notations

n	number of components in the system
$X_i(t)$	state of component i at time t, $X_i(t) = 0,1,2,\ldots,M_i$
a_m	performance level corresponding to state m of the system
$U(\boldsymbol{X}(t))$	expected performance of a system at time t
$\boldsymbol{X}(t)$	$X_1(t), X_2(t), \ldots X_n(t)$: state vector of the components
$\Phi(\boldsymbol{X}(t))$	system structure function with domain $\{0,1,\ldots,M_i\}^n$ and range $\{0,1,\ldots,M\}$
$(\cdot_i, \boldsymbol{X}(t))$	the state of the system $\left(X_1(t),\ldots,X_{i-1}(t),\cdot,X_{i+1}(t),\ldots X_n(t)\right)$
$\boldsymbol{\rho}_{im}(t)$	$\rho_{im}(t) = \Pr\{X_i(t) \geq m\}$
$\boldsymbol{I}_i^G(t)$	Griffith importance vector $\left(I_{i1}^G(t), I_{i2}^G(t), \ldots, I_{iM_i}^G(t)\right)$, where $I_{im}^G(t)$ is the Griffith importance of state m of component i

Assumptions

The following assumptions are made in this paper.

(1) The multi-state system is monotone and coherent.

(2) The state space of component i is $\{0,1,\ldots,M_i\}$ and that of the system is $\{0,1,\ldots,M\}$, where 0 represents the complete failure of the system or a component, and $M_i(M)$ is the perfect functioning state of component i (the system).

(3) All components (states) and the system (state) are statistically independent with each other.

(4) Each state of a component is characterized by a different level of performance. Precisely, the states of a component, i say, are numbered according to decreasing performance levels, from M_i to 0.

Let $a_0 \leq a_1 \leq \cdots \leq a_M$ be the performance levels corresponding to the state space $\{0,1,\ldots,M\}$ of a multistate system. Let $a_0 = 0$, without loss of generality, then the expected performance of the system can be defined by:

$$U(\boldsymbol{X}(t)) = \sum_{v=1}^{M} a_v \Pr\left(\Phi(\boldsymbol{X}(t)) = v\right) = \sum_{v=1}^{M} a_v \Pr\left(\Phi\left(X_1(t), X_2(t), \ldots X_n(t)\right) = v\right). \quad (1)$$

Recall that the Griffith importance of state m of component i is defined as [34]

$$I_{im}^G(t) = \sum_{v=1}^{M} (a_v - a_{v-1})\left[\Pr\left(\Phi(m_i, \boldsymbol{X}(t)) \geq v\right) - \Pr\left(\Phi((m-1)_i, \boldsymbol{X}(t)) \geq v\right)\right]. \quad (2)$$

For a binary system, Wu et al [1] defines the component maintenance priority (CMP) that prioritises the components to be maintained during the time of a failed component being repaired.

If component i has failed, the CMP of component j is defined by

$$I_{j|i}^{M}(t) = H_{j|i}\frac{\partial\phi(\lambda_i, \boldsymbol{p}_i(t))}{\partial p_j(t)}, \tag{3}$$

where $p_j(t)$is the reliability of component j, $H_{j|i} = \begin{cases} 1 & \text{if } \phi(1_1, \ldots, 1_{i-1}, 0_i, 1_{i+1}, \ldots, 1_n) = 0 \\ \phi(0_i, 0_j, \mathbf{1}_{ij}) & \text{if } \phi(1_1, \ldots, 1_{i-1}, 0_i, 1_{i+1}, \ldots, 1_n) = 1 \end{cases}$, $(0_i, 0_{j,} \ldots, \mathbf{1}_{i,j})$ represents that both components i and j stop working while all of the other components are working, $\lambda_i = \chi\{\phi(1_1, 1_2, \ldots, 1_{i-1}, 0_i, 1_{i+1}, \ldots, 1_n) = 0\}$, and $\phi(\boldsymbol{p}_i(t))$ is the system reliability as a function of $\boldsymbol{p}(t)$.

Eq. (3) represents the effect of component j on the system reliability, when component i has failed and repair needs performing on it. The CMP can be used to suggest which component may be selected for PM so that the reliability of the system can be maximally improved.

The CMP can be used to prioritise components in binary systems while a failed component is being repaired. For multistate systems, however, prioritising components or the states of components becomes more complicated. This is because the performance of a multistate system can be measured by either performance utility or merely degradation of states. In what follows, we consider the two cases for multistate systems.

Case I. Immediately after the system state degrades to a state below K, the system needs maintaining.

Case II. Only if the system has degraded k states (where $k > 1$), the system needs maintaining.

We assume the maintenance is imperfect, that is, the system cannot be restored to the perfect state.

2.1 Priority under Case I

Suppose that state K_i is the threshold state of component i. That is, once the state of a component degrades to a state below K_i, a certain symptom of performance immediately appears and can be detected, namely, the degradation from one state to another is self-announcing. Denote the detected state after maintenance by $(K_o)_i$, assuming the state of component i is below K_i.

Definition 1. If component i has degraded to a state worse than K_i, the CMP of component j is defined by

$$I_{j|i}^{M}(t) = H_{j|i} I_{j|i}(t), \tag{4}$$

where

$$H_{j|i} = \begin{cases} 1 & \text{if } \Phi((< K_i)_i, \boldsymbol{X}(t)) < K \\ \chi\left(\Phi\left((< K_i)_i, (< K_j)_{j}, \boldsymbol{X}(t)\right)\right) & \text{if } \Phi((< K_i)_i, \boldsymbol{X}(t)) \geq K \end{cases}$$

$$I_{j|i}(t) = \sum_{v=1}^{M}(a_v - a_{v-1})\left[\Pr\left(\Phi\left((K_o)_i, X_j(t), X(t)\right) \geq v\right) - \Pr\left(\Phi\left((K_o)_i, X_j(t) - 1, X(t)\right) \geq v\right)\right],$$

and $(< K_i)_i$ represents that the state of component i degrades to a state below its threshold state K_i.

$\Phi((< K_i)_i, \boldsymbol{X}(t)) < K$ represents that the state of the system is below K and component i is a critical component. $H_{j|i}$ ensures that critical components will not be selected for PM, given that component i is non-critical.

Denote

$$\boldsymbol{X}_i = ((k_1)_1, (k_2)_2, \dots, (k_{i-1})_{i-1}, *, (k_{i+1})_{i+1}, \dots, (k_n)_{\mathrm{n}}),$$

and

$$\boldsymbol{X}_{ij} = \left((k_1)_1, (k_2)_2, \dots, (k_{i-1})_{i-1}, *, (k_{i+1})_{i+1}, \dots (k_{j-1})_{j-1}, *, (k_{j+1})_{j+1}, \dots, (k_n)_{\mathrm{n}}\right).$$

$I_{j|i}(t)$ is the importance of component j, given that component i's state has downgraded. In Section 4.1, we will analyse the different expressions of Eq. (4), considering different maintenance policies.

Below we give an example to show how the CMP works.

Example 1. Suppose a multi-state system is composed of 4 multi-state components with the following system structure function

$$\Phi(X(t)) = \Phi(X_1(t), X_2(t), X_3(t), X_4(t)) = \min\{\max\{X_1(t), X_2(t), X_3(t)\}, X_4(t)\}.$$

Suppose both the state space of each component and that of the system are {0, 1, 2}. Assume both the performance values of each component and the system are 1, which means that when the states of component and system are smaller than 1, the component and system fail. Then we have the following two cases.

- Component 4 degrades to a state below 1.

  If component 4 has degraded to a state below 1, according to the structure function, we have $\Phi(X(t)) < 1$. Then $H_{j|4}$ =1 and $I_{j|4}^M(t) = I_{j|4}(t) = \sum_{v=1}^{M} a_v \left[\Pr\left(\Phi\left((K_o)_4, X_j(t), X(t)\right) = v\right) - \Pr\left(\Phi\left((K_o)_4, X_j(t) - 1, X(t)\right) = v\right)\right]$. As an example, we show how to computer $I_{1|4}^M(t)$ below.

  Assume $(K_o)_4 = 2$, $X_1(t) = 1$, then $I_{1|4}^M(t) = \sum_{v=1}^{2} a_v[\Pr(\Phi((2)_4, 1_1, X(t)) = v) - \Pr(\Phi((2)_4, 0_1, X(t)) = v)]$. Let $p_{im} = \Pr\{X_i(t) = m\}$, then $\Pr(\Phi((2)_4, 1_1, X(t)) = 1) = p_{21}(t)p_{31}(t) + p_{21}(t)p_{30}(t) + p_{20}(t)p_{31}(t) + p_{20}(t)p_{30}(t)$, and $\Pr(\Phi((2)_4, 0_1, X(t)) = 1) = p_{21}(t)p_{31}(t) + p_{21}(t)p_{30}(t) + p_{20}(t)p_{31}(t)$. Thus, we have $a_1[\Pr(\Phi((2)_4, 1_1, X(t)) = 1) - \Pr(\Phi((2)_4, 0_1, X(t)) = 1)] = a_1 p_{20}(t)p_{30}(t)$. Besides, $\Pr(\Phi((2)_4, 1_1, X(t)) = 2) = \Pr(\Phi((2)_4, 0_1, X(t)) = 2) = p_{22}(t) + p_{32}(t)$, so we have $a_2[\Pr(\Phi((2)_4, 1_1, X(t)) = 2) - \Pr(\Phi((2)_4, 0_1, X(t)) = 2)] = 0$. We can then obtain $I_{1|4}^M(t) = a_1 p_{20}(t)p_{30}(t)$.

- Component 1 degrades to a state below 1 and the states of the other components are higher than 1.

  If component 1 is the only component that has degraded to a state below 1, according to the structure function, we have $\Phi(X(t)) \geq 1$. Then $I_{4|1}^M(t) = 0$. Thus, one of components 2 and 3 can be selected for PM. We take component 2 for example to show how to computer $I_{2|1}^M(t)$ below.

Assume $(K_o)_1 = 0$ and $X_2(t) = 1$, then $I^M_{2|1}(t) = \sum_{v=1}^{2} a_v\big[\Pr\big(\Phi\big((0)_1, 1_2, X(t)\big) = v\big) - \Pr\big(\Phi\big((0)_1, 0_2, X(t)\big) = v\big)\big]$. We have: $\Pr\big(\Phi\big((0)_1, 1_2, X(t)\big) = 1\big) = p_{41}(t)p_{30}(t) + p_{41}(t)p_{31}(t) + p_{41}(t)p_{32}(t) + p_{42}(t)p_{30}(t) + p_{42}(t)p_{31}(t)$, and $\Pr\big(\Phi\big((0)_1, 0_2, X(t)\big) = 1\big) = p_{41}(t)p_{31}(t) + p_{41}(t)p_{32}(t) + p_{42}(t)p_{31}(t)$. Hence, $a_1\big[\Pr\big(\Phi\big((0)_1, 1_2, X(t)\big) = 1\big) - \Pr\big(\Phi\big((0)_1, 0_2, X(t)\big) = 1\big)\big] = a_1[p_{41}(t)p_{30}(t) + p_{42}(t)p_{30}(t)]$. Since $\Pr\big(\Phi\big((0)_1, 1_2, X(t)\big) = 2\big) = \Pr\big(\Phi\big((0)_1, 0_2, X(t)\big) = 2\big) = p_{42}(t)p_{32}(t)$, we have $a_2\big[\Pr\big(\Phi\big((0)_1, 1_2, X(t)\big) = 2\big) - \Pr\big(\Phi\big((0)_1, 0_2, X(t)\big) = 2\big)\big] = 0$, and $I^M_{2|1}(t) = a_1[p_{41}(t)p_{30}(t) + p_{42}(t)p_{30}(t)]$.

In the following, we investigate two scenarios and give the corresponding expressions of $I_{j|i}(t)$ to analyse the effect of component j on the system performance while component i is being maintained in multistate systems. Denote the threshold state of component j ($j \neq i$) by K_j.

Scenario 1. The state of the component which causes the system to downgrade to a state below K can be observed, but the states of other components cannot be detected.

Assume the state degradation of component i causes the system to downgrade to a state lower than K. Let the observed state of component i be $(K_o)_i$. Similarly to Eq. (1), we have

$$U\big((K_o)_i, X(t)\big) = \sum_{v=1}^{M} a_v \Pr\big(\Phi\big(X_1(t), \dots, X_{i-1}(t), K_o, X_{i+1}(t), \dots X_n(t)\big) = v\big).$$

Based on Eq. (2), the CMP of component j is

$$\begin{aligned} I_{j|i}(t) &= \frac{\partial U\big((K_o)_i, X(t)\big)}{\partial \rho_{jK_j}(t)} \\ &= \sum_{v=1}^{M} (a_v - a_{v-1}) \left[\Pr\left(\Phi\left((K_o)_i, K_j, X(t)\right) \geq v\right) - \Pr\left(\Phi\left((K_o)_i, (K-1)_j, X(t)\right) \geq v\right)\right]. \end{aligned} \tag{5}$$

Eq. (5) describes the effect of component j on the system performance when component i is maintained under Scenario 1.

Scenario 2. Assume component i causes the system to downgrade to a state below K. The state of component i can be detected, and other component states can be also detected.

Let the observed state of component i be $(K_o)_i$. Similarly to Eq. (2), we can use Eq. (6) to analyse the effect of component j on the system performance when component i is being maintained. The CMP of component j is

$$\begin{aligned} I_{j|i}(t) &= \frac{\partial U\big((K_o)_i, X(t)\big)}{\partial \rho_{j(K_o)_j}(t)} \\ &= \sum_{v=1}^{M} (a_v - a_{v-1}) \left[\Pr\left(\Phi\left((K_o)_i, (K_o)_j, X(t)\right) \geq v\right) - \Pr\left(\Phi\left((K_o)_i, (K_o-1)_j, , X(t)\right) \geq v\right)\right]. \end{aligned} \tag{6}$$

2.2 Priority under Case II

Based on the Case II, we have the following scenarios.

Scenario 3. Both system state and component state can be detected.

Assume that when the state degradation of component i causes the system to jump k states, the system fails. We define the following measure that prioritises the components to be maintained while a failed component is being repaired.

Definition 2. If component i has failed, the CMP of component j is defined by

$$I_{j|i}^{M}(t) = H_{j|i} I_{j|i}(t), \tag{7}$$

where

$$H_{j|i} = \begin{cases} 1 & \text{if } \phi\left(\mathrm{D}_{X_{i'}(t)\to X_i(t)}, \boldsymbol{X}(t)\right) \geq k \\ \chi\left(\phi\left(\mathrm{D}_{X_{i'}(t)\to X_i(t)}, \mathrm{D}_{X_{j'}(t)\to X_j(t)}, \boldsymbol{X}(t)\right) < k_1\right) & \text{if } \phi\left(\mathrm{D}_{X_{i'}(t)\to X_i(t)}, \boldsymbol{X}(t)\right) < k \end{cases}.$$

$\mathrm{D}_{X_{i'}(t)\to X_i(t)}$ represents that the state of component i degrades from state $X_{i'}(t)$ to $X_i(t)$, and $\mathrm{D}_{X_{j'}(t)\to X_j(t)}$ represents that the state of component j degrades from state $X_{j'}(t)$ to $X_j(t)$. Suppose component i's degrading from state $X_{i'}(t)$ to $X_i(t)$ causes the system to downgrade for more than k states, that is, the value of the function $\phi(.)$ to reduce for more than k states, then a maintenance is triggered; otherwise, no action will be taken. $I_{j|i}(t)$ is the importance of component j, given that component i's state has downgraded.

Scenario 4. The degradation of the system can be detected, but the state of a component cannot be detected.

Since the state degradation of a component cannot be detected, i.e., it is not self-announcing, we are not able to identify which component causes system to jump k states. Here we use the effect of all states of a component on the system performance.

Ramirez-Marquez and Coit [9] gave the following alternative composite importance measure, or *mean absolute deviation* (MAD), to measure the expected absolute deviation of the system reliability,

$$\mathrm{MAD}_i(t) = \sum_m p_{im}(t) \left| \Pr\left(\Phi\left(m_i, \boldsymbol{X}(t)\right) \geq d\right) - \Pr\left(\Phi(\boldsymbol{X}(t) \geq d)\right) \right|, \tag{8}$$

where d is a constant system demand, and $p_{im}(t)$ is the probability that component i is at state m at time t. $\mathrm{MAD}_i(t)$ is the expected absolute deviation of component i for system reliability.

Based on the expected performance of a system, $U\left(X(t)\right)$, and the pre-specified performance utility threshold (i.e., w), we can obtain the expected absolute deviation of component i, as shown in Eq. (9).

$$\mathrm{UMAD}_i(t) = \sum_m p_{im}(t) \left| U\left(m_i, \boldsymbol{X}_i(t)\right) \geq w - U\left(\boldsymbol{X}(t)\right) \geq w \right|. \tag{9}$$

Let $\mathrm{UMAD}_i^*(t) = \max_i \{\mathrm{UMAD}_i(t)\}$, and the corresponding component of $\mathrm{UMAD}_i^*(t)$ is component i^*. As such, we introduce the following definition.

Definition 3. If component i^* has failed, the CMP of component j is defined by

$$I_{j|i^*}^{M}(t) = H_{j|i^*} \frac{UMAD_i^*(t)}{\sum_i UMAD_i(t)} I_{j|i^*}(t), \tag{10}$$

where $\frac{UMAD_i^*(t)}{\sum_i UMAD_i(t)}$ represents the ratio of component i^* in the expected absolute deviations of all components. $\text{UMAD}_i(t)$ is the expected absolute deviation of component i for system performance based on reference [9].

3 Linking maintenance policies

In this section, we will analyse how to determine the components for preventive maintenance in some maintenance policies.

3.1 Maintenance policies under Case I

Maintenance policy A. Once a component degrades to a state below its threshold state, the component must be maintained. Under this policy, the maintained component may be critical or non-critical. There are the following two situations

- If the maintained component is critical and fails, then the system fails. The preventive maintenance may be performed on other components.
- If the maintained component is non-critical and it fails, then the system still works. The preventive maintenance can be performed on the other non-critical components.

If the state of component i has degraded to a state below its threshold state K_i, then under maintenance policy A, the CMP of component j is defined by

$$I_{j|i}^M(t) = H_{j|i} I_{j|i}(t), \tag{11}$$

where

$$H_{j|i} = \begin{cases} 1, \text{if } \Phi\big((< K_i)_i, X(t)\big) < K \\ 1, \text{if } \Phi\big((< K_i)_i, X(t)\big) \geq K \text{ and } j \in \left\{ j | \Phi\left((< K_i)_i, \left(< K_j\right)_j, X(t) \right) \geq K \right\} \\ 0, \text{other} \end{cases}.$$

The symbol $(< K)_i$ represents that the state of component i degrades to a state below its threshold state K_i. The symbol $(< K)_j$ represents that the state of component j degrades to a state below its threshold state K_j. If the degradation of the state of component i degrades into below K_i causes the value of the system structure function $\Phi(\cdot)$ to reduce into below its threshold state K, i.e. $\Phi\big((< K)_i, X(t)\big) < K$, then component i is critical and the system stops working. Thus, the PM can be performed on all other components, $j \in \{1, \dots, i-1, i+1, \dots, n\}$. If $\Phi\big((< K)_i, X(t)\big) \geq K$, then component i is non- critical. Thus, the PM can be performed on the non-critical components, $j \in \left\{ j | \Phi\left((< K)_i, (< K)_j, X(t) \right) \geq K \right\}$.

For maintenance policy A, the Scenarios 1 and 2 are suitable by substituting equations (5) and (6) into Eq. (11), respectively. When component i is being repaired, component j with the maximal $I_{j|i}^M(t)$

should be first selected for PM so that the system performance can be improved. Then we should select the component order for PM following the ranking of the component $I_{j|i}^{M}(t)$.

Maintenance policy B. When the degradations of some components cause the system state to degrade to a state below its threshold state K, the system fails. Thus maintenance is needed. The corresponding components can be detected. Under this policy, the maintained components may consist of some critical components, or some non-critical components. Assume the set of degraded components that cause the system degrades into a state below its threshold K is $\{i_1, i_2, \dots, i_m\}$. Actually, the set of components $i_1, i_2, \dots, i_m$ is a cut set of the system. Under maintenance policy B, when components $i_1, i_2, \dots, i_m$ are being maintained, the system stops working, and the PM can be performed on all other components.

Under Scenario 1, when the observed states of components $i_1, i_2, \dots, i_m$ are $(K_o)_{i_1}, (K_o)_{i_2}, \dots, (K_o)_{i_m}$, respectively, if other component states corresponding to the performance cannot be observed, the CMP of component j is

$$I_{j|i_1,i_2,\dots,i_m}^{M}(t) = I_{j|i_1,i_2,\dots,i_m}(t) = \frac{\partial U\left((K_o)_{i_1},(K_o)_{i_2},\dots,(K_o)_{i_m},X(t)\right)}{\partial \rho_{jK_j}(t)} = \sum_{v=1}^{M}(a_v - a_{v-1})\left[\Pr\left(\Phi\left((K_o)_{i_1},(K_o)_{i_2},\dots,(K_o)_{i_m},K_j,X(t)\right) \geq v\right) - \Pr\left(\Phi\left((K_o)_{i_1},(K_o)_{i_2},\dots,(K_o)_{i_m},(K-1)_j,X(t)\right) \geq v\right)\right]. \tag{12}$$

Under Scenario 2, When the observed states of components $i_1, i_2, \dots, i_m$ are $(K_o)_{i_1}, (K_o)_{i_2}, \dots, (K_o)_{i_m}$, respectively, if other component states corresponding to the performance can be observed, the CMP of component j is

$$I_{j|i_1,i_2,\dots,i_m}^{M}(t) = I_{j|i_1,i_2,\dots,i_m}(t) = \frac{\partial U\left((K_o)_{i_1},(K_o)_{i_2},\dots,(K_o)_{i_m},X(t)\right)}{\partial \rho_{j(K_o)_j}(t)}$$
$$= \sum_{v=1}^{M}(a_v - a_{v-1})\left[\Pr\left(\Phi\left((K_o)_{i_1},(K_o)_{i_2},\dots,(K_o)_{i_m},(K_o)_j,X(t)\right) \geq v\right) - \Pr\left(\Phi\left((K_o)_{i_1},(K_o)_{i_2},\dots,(K_o)_{i_m},(K_o-1)_j,X(t)\right) \geq v\right)\right]. \tag{13}$$

Components $i_1, i_2, \dots, i_m$ in a cut set of the system degrade into states below their threshold states, so the system stops working. When components $i_1, i_2, \dots, i_m$ undergo maintenance, component j with the maximal $I_{j|i_1,i_2,\dots,i_m}^{M}(t)$ should be selected for PM so that the system performance can achieve the largest improvement.

3.2 Maintenance policies under Case II

Maintenance policy C. When the system has downgraded for k states, the system fails. Thus the system needs repairing. For different components, repairmen may have different ability. When component i is being maintained, it can be increased r_i states. Assume the state degradation of a component causes the system state to jump for k states. This maintained component may be critical or non-critical. So the PM of other components can be determined by $H_{j|i}$.

Based on Eq. (2), $I_{im}^{G}(t)=\sum_{v=1}^{M}(a_v-a_{v-1})\left[\Pr\left(\Phi\left(m_i,X(t)\right)\geq v\right)-\Pr\left(\Phi\left((m-1)_i,X(t)\right)\geq v\right)\right]$ represents the change of the system performance when component i is changed from state $m-1$ to state m. Then when component i is improved from state m to state $m+r_i$, the change of the system performance is

$$\begin{aligned}
I_{i(m\to m+r_i)}^{G}(t) &= I_{i(m+r_i)}^{G}(t)+I_{i(m+r_i-1)}^{G}(t)+\cdots+I_{i(m+1)}^{G}(t)\\
&=\sum_{q=m+1}^{m+r_i}\sum_{v=1}^{M}(a_v-a_{v-1})\left[\Pr\left(\Phi\left(q_i,X(t)\right)\geq v\right)-\Pr\left(\Phi\left((q-1)_i,X(t)\right)\geq v\right)\right]\\
&=\sum_{v=1}^{M}(a_v-a_{v-1})\sum_{q=m+1}^{m+r_i}\left[\Pr\left(\Phi\left(q_i,X(t)\right)\geq v\right)-\Pr\left(\Phi\left((q-1)_i,X(t)\right)\geq v\right)\right]\\
&=\sum_{v=1}^{M}(a_v-a_{v-1})\left[\Pr\left(\Phi\left(m_i+r_i,X(t)\right)\geq v\right)-\Pr\left(\Phi\left(m_i,X(t)\right)\geq v\right)\right]\\
&=\sum_{v=1}^{M}a_v\left[\Pr\left(\Phi\left(m_i+r_i,X(t)\right)=v\right)-\Pr\left(\Phi\left(m_i,X(t)\right)=v\right)\right].
\end{aligned}$$

When component i is being maintained, r_i states can be restored on it. So under Scenario 3, when component i has failed, we assume the observed state of component j is $(K_o)_j$. Then we have

$$I_{j|i}(t)=\sum_{v=1}^{M}a_v\left[\Pr\left(\Phi\left((K_o)_i,(K_o)_j+r_j,X(t)\right)=v\right)-\Pr\left(\Phi\left((K_o)_i,(K_o)_j,X(t)\right)=v\right)\right]. \quad (14)$$

If component i has failed, the CMP of component j is

$$I_{j|i}^{M}(t)=H_{j|i}\sum_{v=1}^{M}a_v\left[\Pr\left(\Phi\left((K_o)_i,(K_o)_j+r_j,X(t)\right)=v\right)-\Pr\left(\Phi\left((K_o)_i,(K_o)_j,X(t)\right)=v\right)\right]. \quad (15)$$

Under Scenario 4, component state cannot be observed. Then we have

$$I_{j|i^*}(t)=\sum_{v=1}^{M}a_v\left[\Pr\left(\Phi\left((K)_{i^*},K_j+r_j,X(t)\right)=v\right)-\Pr\left(\Phi\left((K)_{i^*},K_j,X(t)\right)=v\right)\right]. \quad (16)$$

If component i^* has failed, the CMP of component j is defined by

$$I_{j|i^*}^{M}(t)=H_{j|i^*}\frac{UMAD_i^*(t)}{\sum_i UMAD_i(t)}\sum_{v=1}^{M}a_v\left[\Pr\left(\Phi\left((K)_{i^*},K_j+r_j,X(t)\right)=v\right)-\Pr\left(\Phi\left((K)_{i^*},K_j,X(t)\right)=v\right)\right]. \quad (17)$$

3.3 Considering limited maintenance cost

Given the fixed maintenance budget C, we may determine the components for PM to maximize the expected system performance at time t.

a) When each component has the same maintenance cost, the components for PM can be determined following the ranking of component importance measures by $I_{j|i}^{M}(t)$ and $I_{j|i_1,i_2,\ldots,i_m}^{M}(t)$.

b) Under the situation that the cost of PM on different components differs, the component with a larger importance measure may also incur a larger PM cost. In this case, it is not always optimal to allocate the PM priority to the component with largest importance measures by $I_{j|i}^{M}(t)$ and $I_{j|i_1,i_2,\ldots,i_m}^{M}(t)$. Thus, we should use the integer programming models in subsections 4.3.1 and 4.3.2 to determine PM components.

3.3.1 Under Maintenance Policies A and C

When component i undergoes repair, we need to solve the following equation by fixing time t.

$$\max_{z_j} \sum_{j \neq i} I_{j|i}^{M}(t) \cdot z_j \,, \tag{18}$$

subject to

$$c_i + \sum_{j \neq i} c_j z_j \leq C \text{ and } z_j \in \{0,1\},$$

in which c_i is the repair cost for component i, c_j represents the maintenance cost for component j, and z_j is the decision variable representing whether component j should be maintained or not. Note that z_j can only take values from 0 and 1.

3.3.2 Under Maintenance Policy B

When components $i_1, i_2, \ldots, i_m$ are being repaired, we need to solve the following integer programming problem with given time t.

$$\max_{z_j} \sum_{j \neq i_1, i_2, \ldots, i_m} I_{j|i_1,i_2,\ldots,i_m}^{M}(t) \cdot z_j, \tag{19}$$

subject to $c_{i_1} + c_{i_2} + \cdots + c_{i_m} + \sum_{j \neq i_1,i_2,\ldots,i_m} c_j z_j \leq C$ and $z_j \in \{0,1\}$, where $c_{i_1}, c_{i_2}, \ldots, c_{i_m}$ are the repair costs of components $i_1, i_2, \ldots, i_m$, respectively.

For the above integer programming models, we assume that the optimal maintenance policies are $\{z_j^*, j \neq i\}$ and $\{z_j^*, j \neq i_1, i_2, \ldots, i_m\}$, then the set of optimal PM components is $\{j | z_j^* = 1\}$. Actually, $\sum_{j \neq i} z_j^*$ and $\sum_{j \neq i_1,i_2,\ldots,i_m} z_j^*$ are the number of maintained components.

Furthermore, if maintenance time is considered, then we assume that the time of PM on components is less than the repair time of failed components. Otherwise, the PM will delay the system operation, which will reduce system performance.

4 Generalizations of component maintenance priority

In this section, we will generalise the component maintenance priority to the non-coherent systems and continuum systems, respectively.

4.1 Priority for a noncoherent system

4.1.1 Noncoherent Binary Systems

A system is noncoherent if (1) its structure function is not monotone, or (2) some components are irrelevant, or both [14, 15]. For a noncoherent binary system, the failure of a component can cause the system to fail. The Birnbaum importance can be found in reference [16], for example.

For component i, let state $X_i(t)(\geq K_i)$ be the working state and state $X_i(t)(< K_i)$ be the failed state. For a noncoherent system, system state $\Phi\big(X(t)\big)$ is a function of component states. Let state $\Phi\big(X(t)\big)(\geq K)$ be the working state and state $\Phi\big(X(t)\big)(< K)$ be the failed state. Denote $p_i(t) = \Pr(X_i(t) \geq K_i)$, $q_i(t) = \Pr(X_i(t) < K_i)$, and $Q_{sys}(t) = \Pr(\Phi\big(X(t)\big) < K)$.

The Birnbaum importance of component i in a noncoherent system is [16]

$$I_i(t) = \frac{\partial \Pr\big(\Phi\big(p_i(t), X(t)\big) < K\big)}{\partial p_i(t)} + \frac{\partial \Pr\big(\Phi\big(q_i(t), X(t)\big) < K\big)}{\partial q_i(t)}.$$

If the failure of component i causes the system to fail, the CMP of component j is

$$I_{j|i}^M(t) = \frac{\partial \Pr\left(\Phi\left(p_i(t), p_j(t), X(t)\right) < K\right)}{\partial p_j(t)} + \frac{\partial \Pr\left(\Phi\left(p_i(t), q_j(t), X(t)\right) < K\right)}{\partial q_j(t)} + \frac{\partial \Pr\left(\Phi\left(q_i(t), p_j(t), X(t)\right) < K\right)}{\partial p_j(t)} + \frac{\partial \Pr\left(\Phi\left(q_i(t), q_j(t), X(t)\right) < K\right)}{\partial q_j(t)},$$

in which, $\frac{\partial \Pr\left(\Phi\left(p_i(t), p_j(t), X(t)\right) < K\right)}{\partial p_j(t)} + \frac{\partial \Pr\left(\Phi\left(p_i(t), q_j(t), X(t)\right) < K\right)}{\partial q_j(t)}$ represents the CMP of component j when the working state of component i causes the system to fail, and $\frac{\partial \Pr\left(\Phi\left(q_i(t), p_j(t), X(t)\right) < K\right)}{\partial p_j(t)} + \frac{\partial \Pr\left(\Phi\left(q_i(t), q_j(t), X(t)\right) < K\right)}{\partial q_j(t)}$ is the CMP of component j when the failed state of component i causes the system to fail.

Example 2. A noncoherent system consists of three components $\{1,2,3\}$. $Q_{sys}(t) = q_1(t)q_2(t) + q_1(t)q_3(t) + q_2(t)p_3(t) - q_1(t)q_2(t)q_3(t) - q_1(t)q_2(t)p_3(t)$.

We take component 3 for example. Since $\frac{\partial \Pr(\Phi(p_3(t), X(t)) < K)}{\partial p_3(t)} = q_2(t) - q_1(t)q_2(t)$, and $\frac{\partial \Pr(\Phi(q_3(t), X(t)) < K)}{\partial q_3(t)} = q_1(t) - q_1(t)q_2(t)$, we have $I_3(t) = p_1(t)q_2(t) + q_1(t)p_2(t)$.

We take the CMP of component 1 for example. The expression of $Q_{sys}(t)$ does not contain $p_3(t)p_1(t)$ and $q_3(t)p_1(t)$, So we have $\frac{\partial \Pr(\Phi(p_3(t),p_1(t),X(t))<K)}{\partial p_1(t)} = 0$, $\frac{\partial \Pr(\Phi(q_3(t),p_1(t),X(t))<K)}{\partial p_1(t)} = 0$. $\frac{\partial \Pr(\Phi(p_3(t),q_1(t),X(t))<K)}{\partial q_1(t)} = -q_2(t)p_3(t)$, and $\frac{\partial \Pr(\Phi(q_3(t),q_1(t),X(t))<K)}{\partial q_1(t)} = q_3(t) - q_2(t)q_3(t)$. Then we can obtain $I_{1|3}^{M}(t) = q_3(t) - q_2(t)$.

4.1.2 Noncoherent Multi-State Systems

In noncoherent multi-state systems, any degradation of component i may cause the system to degrade. Let Si_m be the set containing state m of component i, $S\bar{\imath}$ be the complementary set of S_i, and $[Si_m]$ be the set of system states covered by the set Si_m. Here γ may be one of the system states, or system performance. In a noncoherent system, some components may be irrelevant, so γ may not include the states of all components.

Then the expression for the system event can be written as $\gamma = [S\bar{\imath}] \cup [Si_0] \cup [Si_1] \cup \ldots \cup [Si_{M_i}]$. The sets $[Si_m]$ are mutually exclusive since each element in the sets represents a different state of component i. Therefore, the following expression holds for the probability of a system state, which can be obtained using the method used by Inagaki and Henley [35].

$$\Pr\{\gamma\} = \Pr\{[S\bar{\imath}] \cup [Si_1] \cup [Si_2] \cup \ldots \cup [Si_{M_i}]\} = \Pr\{[S\bar{\imath}]\} + \sum_{m=0}^{M_i} \Pr\{[Si_m]\} - \sum_{m=0}^{M_i} \Pr\{[S\bar{\imath}] \cap [Si_m]\}$$

$$= \Pr\{[S\bar{\imath}]\} + \sum_{m=0}^{M_i} \{\Pr\{[Si_m]\} - \Pr\{[S\bar{\imath}] \cap [Si_m]\}\}.$$

Because Si_m contains state m of component i, we can extract $p_{im}(t)$ from $\Pr\{[Si_m]\} - \Pr\{[S\bar{\imath}] \cap [Si_m]\}$, which can be denoted as $p_{im}(t)\left\{\{\Pr\{[Si_m]\} - \Pr\{[S\bar{\imath}] \cap [Si_m]\}\} | p_{im}(t) = 1\right\}$. Then we have

$$\Pr\{\gamma\} = \Pr\{[S\bar{\imath}]\} + \textstyle\sum_{m=0}^{M_i} p_{im}(t)\left\{\{\Pr\{[Si_m]\} - \Pr\{[S\bar{\imath}] \cap [Si_m]\}\} | p_{im}(t) = 1\right\}.$$

In a noncoherent multi-state system, the effect of state m of component i on the probability of the system state is

$$I_{im}(t) = \frac{\partial \Pr\{\gamma\}}{\partial p_{im}(t)} = \left\{\{\Pr\{[Si_m]\} - \Pr\{[S\bar{\imath}] \cap [Si_m]\}\} | p_{im}(t) = 1\right\},$$

and the effect of component i on the probability of the system state is

$$I_i(t) = \sum_{m=0}^{M_i} \frac{\partial \Pr\{\gamma\}}{\partial p_{im}(t)} = \sum_{m=0}^{M_i} \left\{\{\Pr\{[Si_m]\} - \Pr\{[S\bar{\imath}] \cap [Si_m]\}\} | p_{im}(t) = 1\right\}.$$

If the degradation of state m of component i causes the system state to degrade, which can be denoted as $\Pr\{p_{im}(t), \gamma\}$, then we have $\Pr\{p_{im}(t), \gamma\} = \Pr\{[Si_m]\} - \Pr\{[S\bar{\imath}] \cap [Si_m]\}$. If the degradation of state m of component i causes the system to degrade, the CMP of state m of component j is

$$I_{jl|im}(t) = \frac{\partial \Pr\{p_{im}(t), \gamma\}}{\partial p_{jl}(t)} = \frac{\partial \{\Pr\{[Si_m]\} - \Pr\{[S\bar{\imath}] \cap [Si_m]\}\}}{\partial p_{jl}(t)}$$

$$= \left\{\{\Pr\{[Si_m]\} - \Pr\{[S\bar{\imath}] \cap [Si_m]\}\} | p_{jl}(t) = 1\right\},$$

and the CMP of component j is

$$I_{j|im}(t) = \sum_{l=0}^{M_j} \frac{\partial \Pr\{p_{im}(t), \gamma\}}{\partial p_{jl}(t)} = \sum_{l=0}^{M_j} \left\{\{\Pr\{[Si_m]\} - \Pr\{[S\bar{\imath}] \cap [Si_m]\}\} | p_{jl}(t) = 1\right\}.$$

When the degradation of a component cannot be observed, we can use the average influence of component states on the system state degradation. When the degradation of component i causes the system to degrade, the CMP of component j is

$$I_{j|i}(t) = \sum_{m=0}^{M_i} \sum_{l=0}^{M_j} \frac{\partial \Pr\{p_{im}(t), \gamma\}}{\partial p_{jl}(t)} = \sum_{m=0}^{M_i} \sum_{l=0}^{M_j} \left\{\{\Pr\{[Si_m]\} - \Pr\{[S\bar{\imath}] \cap [Si_m]\}\} | p_{jl}(t) = 1\right\}.$$

Example 3. A noncoherent system consists of three components {1,2,3}, and each component has three states {0,1,2}. For a system event, the system has 4 sets: $\{p_{12}, p_{21}\}, \{p_{12}, p_{30}\}, \{p_{20}, p_{32}\}, \{p_{10}, p_{22}, p_{31}\}$.

The probability of a system state is

$$\Pr\{\gamma\} = p_{11}(t)p_{21}(t) + p_{12}(t)p_{30}(t) + p_{20}(t)p_{32}(t) + p_{10}(t)p_{22}(t)p_{31}(t) - p_{11}(t)p_{21}(t)p_{30}(t).$$

We take component 3 for example. $I_{30}(t) = \frac{\partial \Pr\{\gamma\}}{\partial p_{30}} = p_{12}(t) - p_{11}(t)p_{21}(t)$, $I_{31}(t) = p_{10}(t)p_{22}(t)$, $I_{32}(t) = p_{20}(t)$. So we have $I_3(t) = I_{30}(t) + I_{31}(t) + I_{32}(t) = p_{12}(t) - p_{11}(t)p_{21}(t) + p_{10}(t)p_{22}(t) + p_{20}(t)$.

When considering the CMP, we take component 1 for example.

If a state of component 3 causes the system to degrade, then the CMP of a state of component 1 is $I_{12|30}(t) = \frac{\partial \Pr\{p_{30}(t), \gamma\}}{\partial p_{12}(t)} = p_{30}(t)$, $I_{11|30}(t) = p_{21}(t)p_{30}(t)$, $I_{10|31}(t) = p_{22}(t)p_{31}(t)$.

If the degradation of component 3 causes its system to degrade, then the CMP of component 1 is

$$I_{1|30}(t) = I_{12|30}(t) + I_{11|30}(t) = p_{30}(t) + p_{21}(t)p_{30}(t),\ I_{1|31}(t) = I_{10|31}(t) = p_{22}(t)p_{31}(t).$$

If component 3 causes the system to degrade, the CMP of component 1 is

$$I_{1|3}(t) = I_{1|30}(t) + I_{1|31}(t) = p_{30}(t) + p_{21}(t)p_{30}(t) + p_{22}(t)p_{31}(t).$$

4.2 Priority for a Continuum System

Baxter [36, 37] defined a continuum system, in which the states of an item (system or component) is any value in the interval $[0,1]$. The structure function of a continuum system is denoted by $\Phi: [0,1]^n \to [0,1]$, which is nondecreasing in each argument and satisfies $\Phi(0_1, 0_2, \ldots 0_n) = 0$ and $\Phi(1_1, 1_2, \ldots 1_n) = 1$.

For a continuum system, let $[0, \alpha)$ correspond to the failure states of the system, and $[\alpha, 1]$ correspond to the working states. Then Kim and Baxter [20] defined the Birnbaum importance measure of component i at level $\alpha \in (0,1]$ as

$$I_i^{CS}(t) = \Pr\big(\Phi\big(X(t)\big) \geq \alpha | X_i(t) \geq \delta_i^\alpha\big) - \Pr\big(\Phi\big(X(t)\big) \geq \alpha | X_i(t) < \delta_i^\alpha\big),$$

where δ_i^α denotes the corresponding key element for component i, and $0 < \delta_i^\alpha < 1$ for all α∈(0,1].

If component i causes the system to fail, the CMP of component j is

$$I_{j|i}^M(t) = H_{j|i} \frac{\partial \Pr\big(\Phi\big(X_i(t) < \delta_i^\alpha, X(t)\big) \geq \alpha\big)}{\partial \Pr\big(X_j(t) \geq \delta_j^\alpha\big)}$$

$$= H_{j|i}\{\Pr\big(\Phi\big(X_i(t) < \delta_i^\alpha, X(t)\big) \geq \alpha | X_j(t) \geq \delta_j^\alpha\big)$$

$$- \Pr\big(\Phi\big(X_i(t) < \delta_i^\alpha, X(t)\big) \geq \alpha | X_j(t) < \delta_j^\alpha\big)\},$$

where

$$H_{j|i} = \begin{cases} 1 & \text{if } \Phi\big(X_i(t) < \delta_i^\alpha, X(t)\big) < \alpha \\ \chi\left(\Phi\left(X_i(t) < \delta_i^\alpha, X_j(t) < \delta_j^\alpha, X(t)\right)\right) & \text{if } \Phi\big(X_i(t) < \delta_i^\alpha, X(t)\big) \geq \alpha \end{cases}.$$

If $\Phi\big(X_i(t) < \delta_i^\alpha, X(t)\big) \geq \alpha$ and $\Phi\left(X_i(t) < \delta_i^\alpha, X_j(t) < \delta_j^\alpha, X(t)\right) \geq \alpha$, then $\chi\left(\Phi\left(X_i(t) < \delta_i^\alpha, X_j(t) < \delta_j^\alpha, X(t)\right)\right) = 1$; otherwise, $\chi\left(\Phi\left(X_i(t) < \delta_i^\alpha, X_j(t) < \delta_j^\alpha, X(t)\right)\right) = 0$.

When any state of a continuum system is considered, let $f_i(\alpha)$ the probability density function (pdf) of $\Pr\big(\Phi\big(X(t)\big) \geq \alpha | X_i(t) \geq \delta_i^\alpha\big)$ and $g_i(\alpha)$ the pdf of $\Pr\big(\Phi\big(X(t)\big) \geq \alpha | X_i(t) < \delta_i^\alpha\big)$. Then $\int_0^1 (f_i(\alpha) - g_i(\alpha))d\alpha$ represent the effect of component i on the whole system.

Similarly, let $f_{ij}(\alpha)$ the pdf of $\Pr\big(\Phi\big(X_i(t) < \delta_i^\alpha, X(t)\big) \geq \alpha | X_j(t) \geq \delta_j^\alpha\big)$ and $g_{ij}(\alpha)$ the pdf of $\Pr\big(\Phi\big(X_i(t) < \delta_i^\alpha, X(t)\big) \geq \alpha | X_j(t) < \delta_j^\alpha\big)$. Then $H_{j|i} \int_0^1 (f_{ij}(\alpha) - g_{ij}(\alpha))d\alpha$ is the CMP of component j on the whole system.

5 Case studies

In this section, we apply the proposed method to an aircraft warning system and then illustrate its validity. The changes of the component maintenance priority under different scenarios with the increase of time are discussed, and then the priority is applied into three maintenance policies. Fig. 1 illustrates the components of the aircraft system [5, 38], which has 30 critical components, as listed in Table 1. Among these major components, we have four types of redundant components as follows. (a) Flight control computers (X38 and X39); (b) Hydraulic reservoirs (X45 and X47); (c) Motor driven pumps (X46 and X48); and (d) Generators (X60, X61, X65 and X66). The remaining components are critical components, and the failure of each critical component causes the entire system to fail.

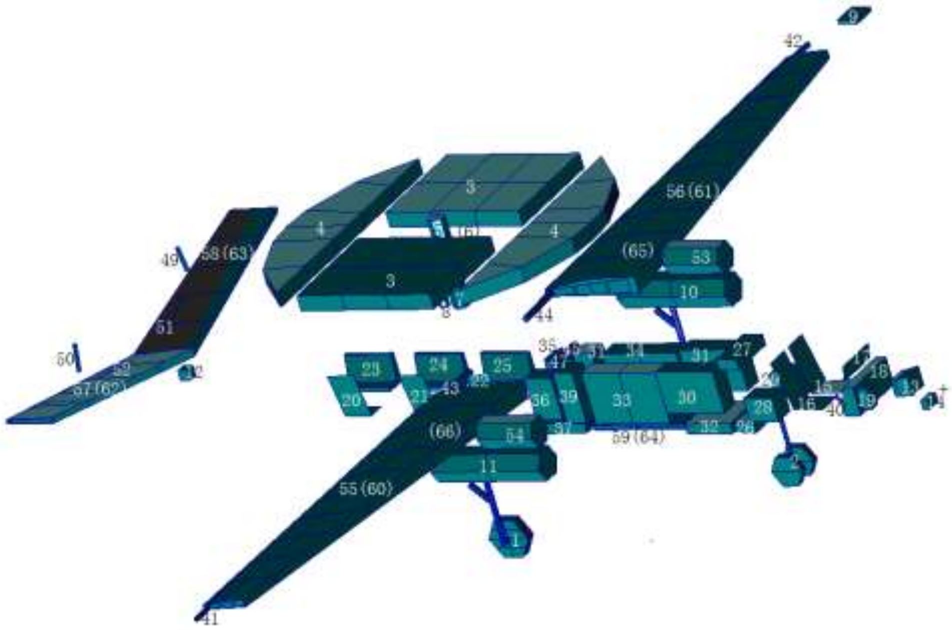

Fig. 1. Major components of an aircraft system

Table 1. The description of major components

No.	Code	Name	No.	Code	Name
1	X1	Main landing gear	16	X45	Hydraulic reservoir No. 1
2	X2	Nose landing gear	17	X46	Motor driven pump No. 1
3	X10	Left Engine	18	X47	Hydraulic reservoir No. 2
4	X11	Right Engine	19	X48	Motor driven pump No. 2
5	X17	Flight deck display	20	X51	Left stabilator actuator
6	X18	Operating panel	21	X52	Right stabilator actuator
7	X19	Forward power supply equipment	22	X55	Right w20ing fuel tank
8	X25	Instrument panel	23	X56	Left wing fuel tank
9	X26	Navigation equipment	24	X57	Right horizontal tail fuel tank
10	X31	Electrical apparatus	25	X58	Left horizontal tail fuel tank
11	X38	Flight control computer No. 1	26	X59	Forward fuselage fuel tank
12	X39	Flight control computer No. 2	27	X60	Generator No. 1
13	X40	Actuator near nose landing gear	28	X61	Generator No. 2
14	X43	Right flap actuator	29	X65	Generator No. 3
15	X44	Left flap actuator	30	X66	Generator No. 4

The aircraft system has 17 states, including complete failure state 0, intermediate states 1-15 and perfect state 16, as shown in Table 2. For example, the system state is 1 when components 47, 46, 38, and 66 are failed, while all other components are functioning. In Table 2, the performance of system state j is assumed to a_j (j=1,2,⋯,17), and a_j also increases with the increase of the system state.

Table 2. Aircraft system states and the corresponding performance levels

j	State description	a_j	j	State description	a_j
1	X47, X48, X39, X66	0.252	5	X45, X46, X38	0.360
1	X47, X48, X39, X65	0.252	5	X47, X48, X39	0.360
1	X47, X48, X38, X66	0.252	5	X47, X48, X38	0.360
1	X47, X48, X38, X65	0.252	6	X46, X39	0.400
1	X47, X46, X39, X66	0.252	6	X46, X38	0.400
1	X47, X46, X39, X65	0.252	6	X48, X39	0.400
1	X47, X46, X38, X66	0.252	6	X48, X38	0.400
1	X47, X46, X38, X65	0.252	7	X45, X39	0.450
1	X45, X48, X39, X66	0.252	7	X45, X38	0.450
1	X45, X48, X39, X65	0.252	7	X47, X39	0.450
1	X45, X48, X38, X66	0.252	7	X47, X38	0.450
1	X45, X48, X38, X65	0.252	8	X39	0.500
1	X45, X46, X39, X66	0.252	8	X38	0.500
1	X45, X46, X39, X65	0.252	9	X47, X48, X66	0.504
1	X45, X46, X38, X66	0.252	9	X47, X48, X65	0.504
1	X45, X46, X38, X65	0.252	9	X47, X46, X66	0.504
2	X48, X39, X66	0.280	9	X47, X46, X65	0.504
2	X48, X39, X65	0.280	9	X45, X48, X66	0.504
2	X48, X38, X66	0.280	9	X45, X48, X65	0.504
2	X48, X38, X65	0.280	9	X45, X46, X66	0.504
2	X46, X39, X66	0.280	9	X45, X46, X65	0.504
2	X46, X39, X65	0.280	10	X48, X66	0.560
2	X46, X38, X66	0.280	10	X48, X65	0.560
2	X46, X38, X65	0.280	10	X46, X66	0.560
3	X47, X39, X66	0.315	10	X46, X65	0.560
3	X47, X39, X65	0.315	11	X45, X66	0.630
3	X47, X38, X66	0.315	11	X45, X65	0.630
3	X47, X38, X65	0.315	11	X47, X66	0.630
3	X45, X39, X66	0.315	11	X47, X65	0.630
3	X45, X39, X65	0.315	12	X66	0.700
3	X45, X38, X66	0.315	12	X65	0.700
3	X45, X38, X65	0.315	13	X47, X48	0.720
4	X39, X66	0.350	13	X47, X46	0.720
4	X39, X65	0.350	13	X45, X48	0.720
4	X38, X66	0.350	13	X45, X46	0.720
4	X38, X65	0.350	14	X48	0.800
5	X47, X46, X39	0.360	14	X46	0.800
5	X47, X46, X38	0.360	15	X47	0.900
5	X45, X48, X39	0.360	15	X45	0.900
5	X45, X48, X38	0.360	16	Perfect state	1.000
5	X45, X46, X39	0.360	0	Complete failure state	0.000

We assume that the failure time of all components follows the Weibull distribution $W(t;\eta,\beta)$. The scale parameter η and the shape parameter β of each component's failure time are listed in Table 3, respectively.

Table 3. The scale and shape parameters of each component's failure time

No.	Code	η	β	No.	Code	η	β
1	X1	250	2	16	X45	420	2
2	X2	230	2	17	X46	600	3
3	X10	3600	2	18	X47	420	2
4	X11	3600	2	19	X48	600	3
5	X17	800	3	20	X51	560	2
6	X18	350	2	21	X52	560	2
7	X19	560	2	22	X55	2200	2
8	X25	800	3	23	X56	2200	2
9	X26	350	2	24	X57	2800	2
10	X31	180	3	25	X58	2800	2
11	X38	250	2	26	X59	1200	3
12	X39	250	2	27	X65	560	2
13	X40	190	2	28	X66	560	2
14	X43	600	2	29	X65	560	2
15	X44	600	2	30	X66	560	2

5.1 The priority changes with different cases

In this section, the priority changes with the increase of time is discussed under four different scenarios when the critical component 10 is failed. To illustrate the priority change tendency clearly, we select three types of components, which are components 11 & 12, component 13, and components 17 & 19, respectively. The evaluation equation of priority depends on different cases and scenarios. The priority in Scenario 1 under Case I can be evaluated by Eqs. (4) and (5). The priority in Scenario 2 under Case I can be evaluated by Eqs. (4) and (6). The priority in Scenario 3 under Case II can be evaluated by Eq. (7). The priority in Scenario 4 under Case II can be evaluated by Eq. (10).

(1) Case 1: The priority changes under Scenario 1

The priority changes under Scenario 1 is shown in Fig. 2. No matter what K is, we can find the priority decreases with the time increase. Once K is determined, the priority of components 11&12 is less than that of component 13. However, the priority of components 17&19 is higher than that of other components.

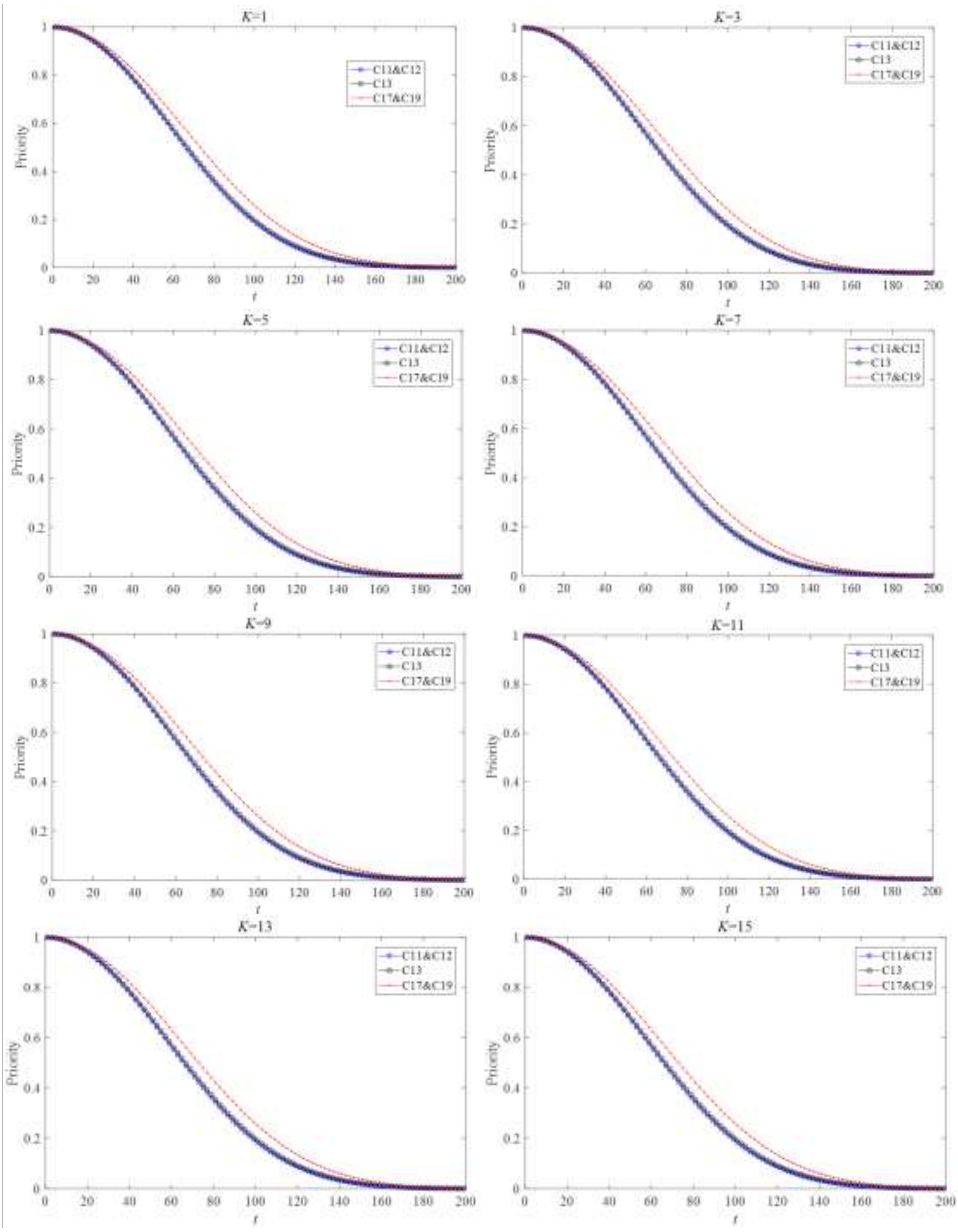

Fig. 2. Priority of other components with different state *K* under Scenario 1

(2) Case 2: The priority changes under Scenario 2

The priority changes under Scenario 2 is shown in Fig. 3. The priority tendency under Scenario 2 is similar to that of Scenario 1. Once *K* is determined, the priority of components 17&19 is a bit less than that of component 13. However, the priority of components 11&12 is lower than that of other components.

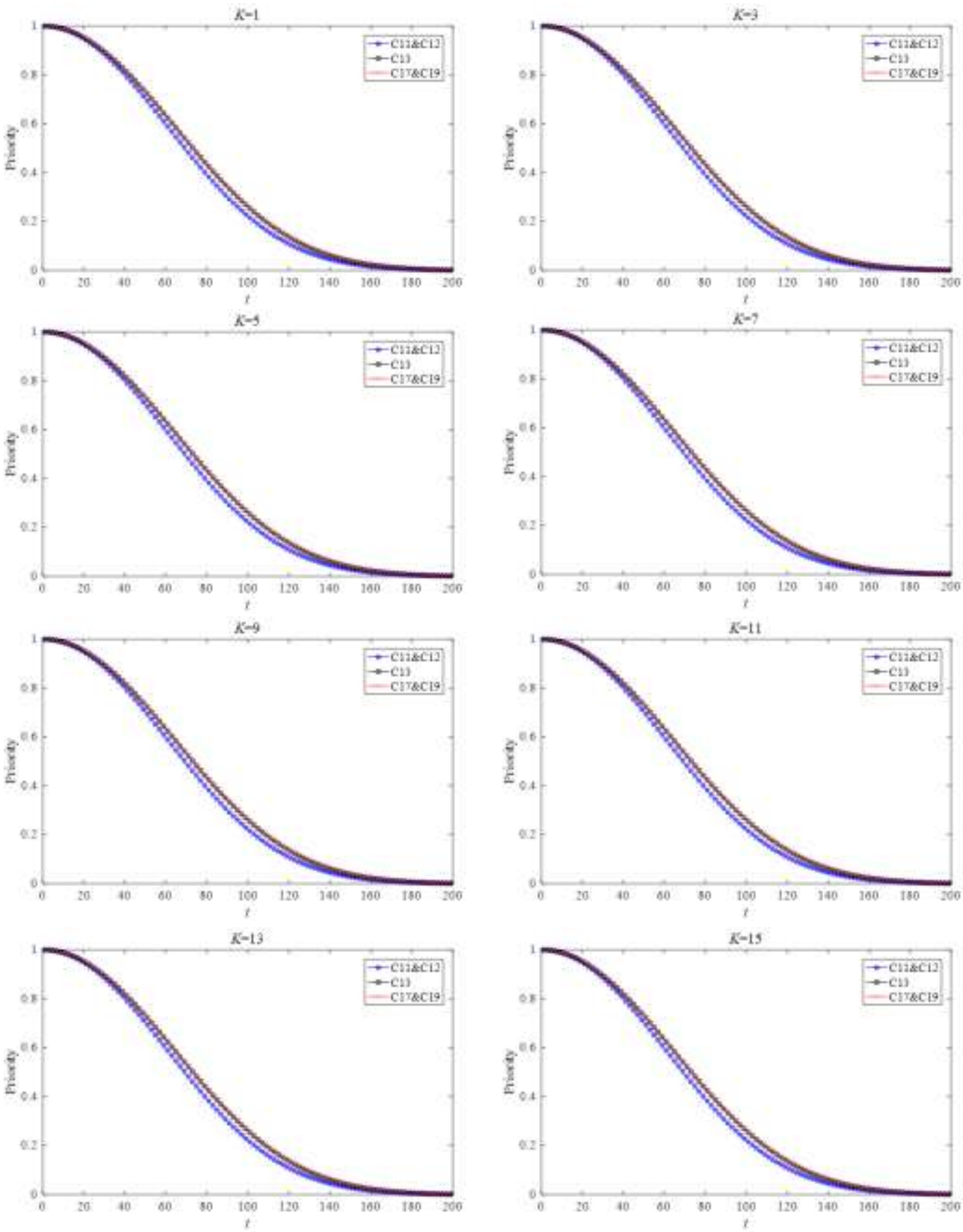

Fig. 3. Priority of other components with different state *K* under Scenario 2

(3) Case 3: The priority changes under Scenario 3

The priority changes under Scenario 3 are shown in Fig. 4. The priority tendency under Scenario 3 is similar to that of Scenarios 1&2. The rank of component priority under Scenario 3 is the same as that of Scenario 2, but the differences of component priority become smaller.

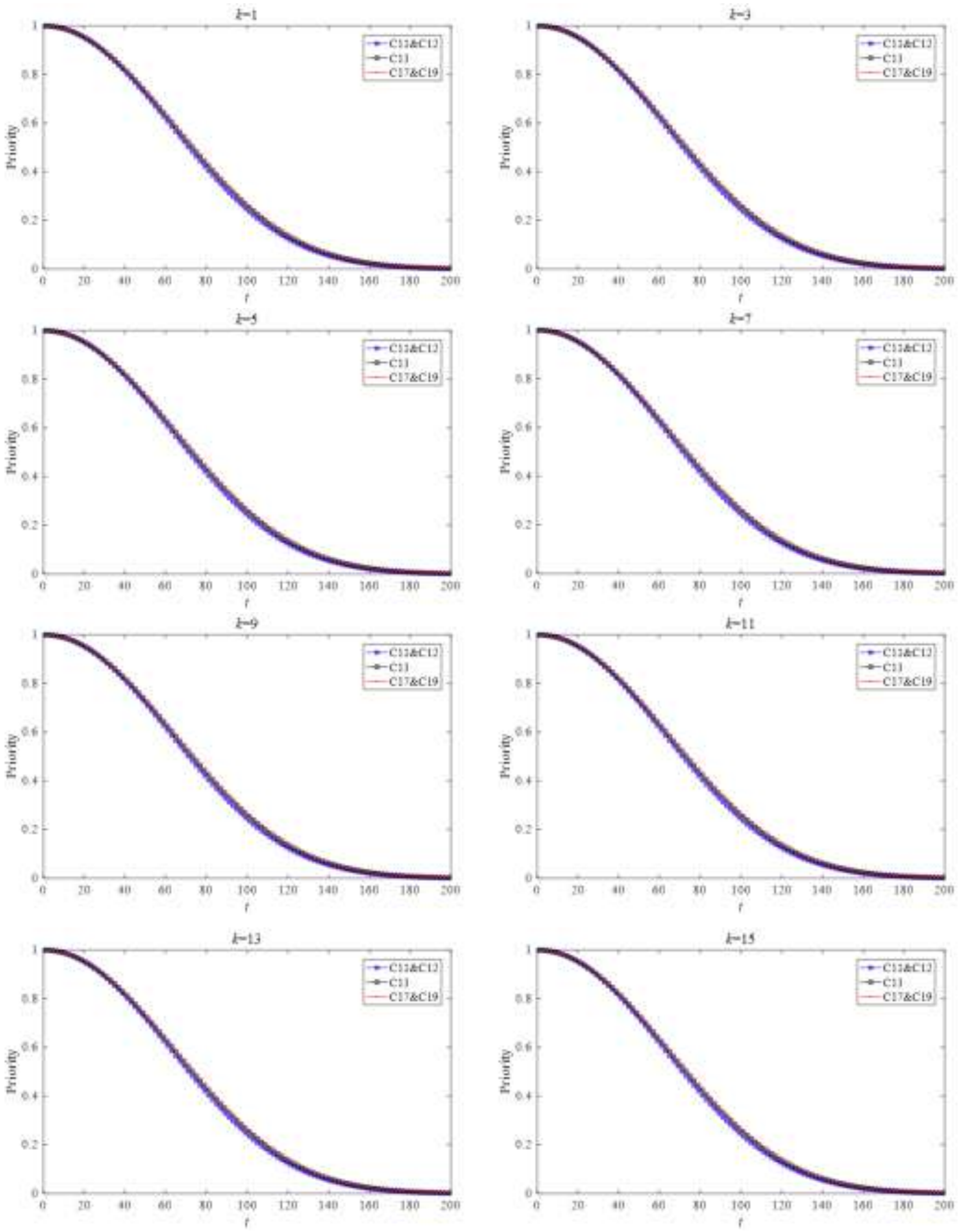

Fig. 4. Priority of other components with different state *K* under Scenario 3

(4) Case 4: The priority changes under Scenario 4

The priority changes under Scenario 4 is shown in Fig. 5. The priority tendency under Scenario 4 is different from that of the other three scenarios because the critical components are changing when time goes by. Whatever the value of *K* is, the change tendency of four types of components has its features. The top point of component priority may appear at different time points when the value of *K* is different, such as the top point appears at t=115 when K=13 while it appears at t=155 when K=1. However, the priority of component 13 remains zero at first, and then jumps to a high value with a decent tendency. Moreover, the priority of components 11&12 and 17&19 increase first, respectively, and then decreases after they reach the top point.

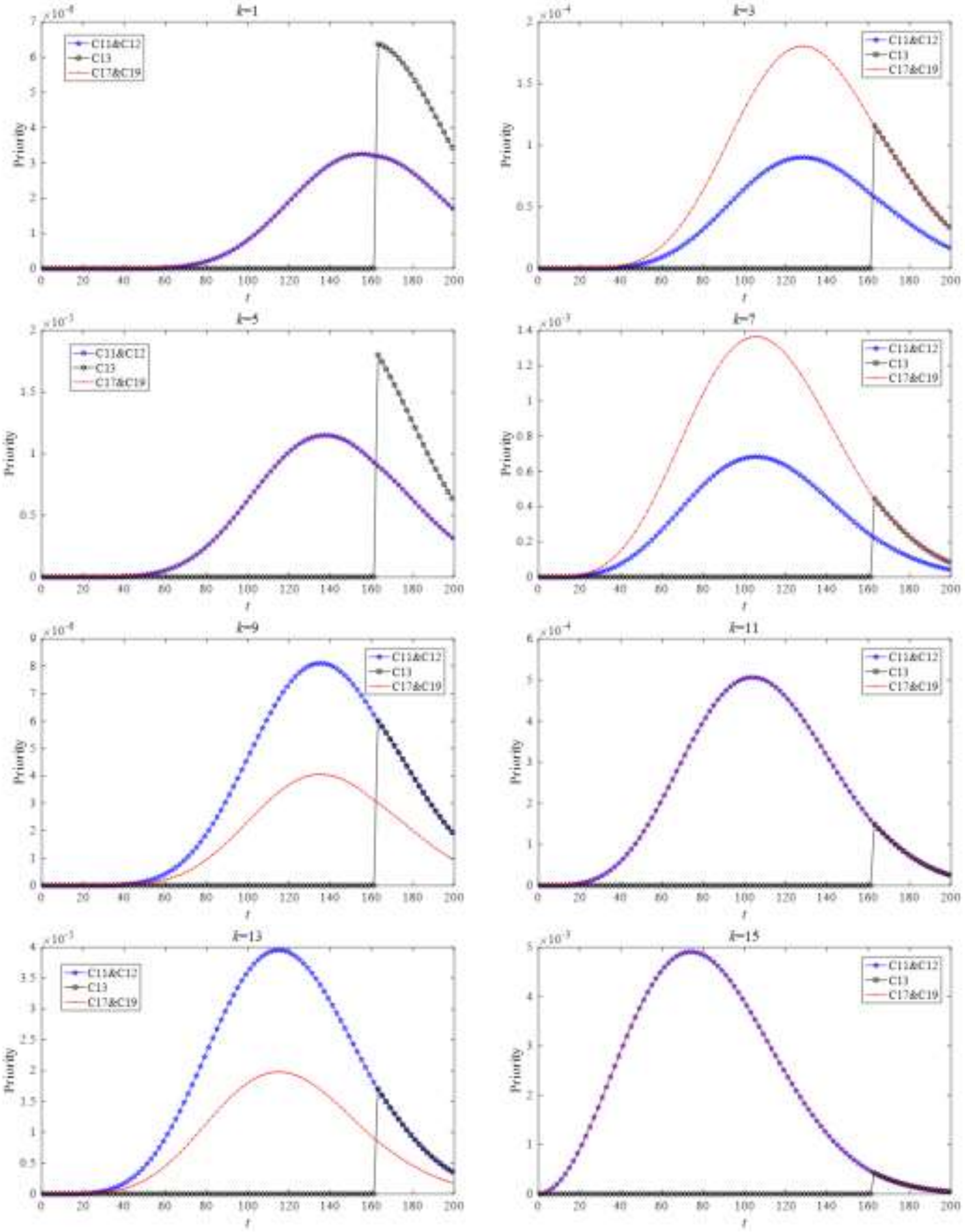

Fig. 5. Priority of other components with different state *K* under Scenario 4

Through the analyses of the numerical results for different scenarios, we can find that the ranks of the component priority remain unchanged with the increase of time under Scenarios 1, 2, and 3 once the critical component is known. However, the ranks of the component priority under Scenario 4 may have different changes as k and t change. The reasons for changing the rank are that the critical component changes with the increase of time because the system state is known while the component state is unknown.

5.2 Discussions about priority -based Maintenance Policies

Section 4 discusses three maintenance policies, i.e., Maintenance policies A, B, and C. The priority of conducting maintenance on components can be determined by the component priority, while the component priority need to be determined by the integer programming method through the Matlab, with the consideration of the limited maintenance budget. The maintenance scheme depends on the importance level of the failed components and the state of components needs repairing. If the states of components that need to be repaired are known, maintenance policies A and B should use the priority under Scenario 1, and maintenance policy C should use the priority under Scenario 3. If the states of these components are unknown, the maintenance policies A and B should use the priority under Scenario 2, and maintenance policy C should use the priority under Scenario 4. Therefore, there are two numerical experiments to illustrate the maintenance policies, depending on whether the maintenance cost is considered. Experiment 1 illustrates the PM scheme without consideration of maintenance cost, and Experiment 2 illustrates the PM scheme with consideration of maintenance cost. In each numerical experiment, maintenance policies A and B include four cases, as shown in Table 4; while maintenance policy C includes Case I, Case II, and Case IV, respectively. Assume the replacement cost of each component is [5, 8, 100, 100, 20, 4, 10, 15, 6, 8, 20, 20, 7, 4, 4, 10, 15, 10, 15, 12, 12, 20, 20, 12, 12, 8, 25, 25, 25, 25], and t=100.

Table 4. Four cases for the numerical experiment

Case #	The importance level of failed components	The state of components that need to be repaired
I	Critical	Unknown
II	Critical	Known
III	Non-critical	Unknown
IV	Non-critical	Known

(1) Experiment 1 without the consideration of maintenance cost

If we do not consider the maintenance cost, the PM is determined based on the ranks of the component priority, which includes eleven cases. The results of Experiment 1 are shown in Table 5.

Table 5. Results of Experiment 1 without the consideration of maintenance cost

Policy	Case	PM Scheme
A	I	[26, 3, 4, 24, 25, 5, 8, 22, 23, 17, 19, 14, 15, 7, 20, 21, 27, 28, 29, 30, 6, 9, 18, 16, 1, 2, 13, 11, 12]
A	II	[1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 20, 21, 22, 23, 24, 25, 26, 15, 17, 19, 27, 28, 29, 30, 18, 16, 11, 12]
A	III	[17, 19, 18, 16]
A	IV	[17, 19, 18, 16]
B	I	[26, 3, 4, 24, 25, 5, 8, 23, 14, 7, 20, 21, 27, 28, 29, 30, 6, 18, 16, 1, 2, 13, 11]
B	II	[1, 2, 3, 4, 5, 6, 7, 8, 13, 14, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 18, 16, 11]

B	III	[16, 18]
B	IV	[16, 18]
C	I	[17, 19, 18, 16]
C	II	[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 15, 16, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 11, 12, 17, 19]
C	IV	[1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 20, 21, 22, 23, 24, 25, 26, 15, 17, 19, 27, 28, 29, 30, 18, 16, 11, 12]

From Table 5, we can find that the PM scheme is the ranks of the corresponding components. The PM scheme needs to determine which components should be replaced, but we need to determine the selection priority of components according to the ranks of component priority.

(2) Experiment 2 with the consideration of maintenance cost

If we consider the maintenance cost and the PM is determined based on the ranks of component priority, there are eleven cases. The limited maintenance cost of each case is known and listed in Table 6. The PM scheme can be determined by the 0-1 integer programming tool in Matlab, and the results of Experiment 2 are shown in Table 6. From Table 6, we can find that the PM scheme is determined without the maintenance order because we need to replace the determined components with the limited cost.

Table 6. Results of Experiment 2 with the consideration of maintenance cost

Policy	Case	C	PM scheme
A	I	308	[1, 2, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30]
A	II	208	[1, 2, 6, 7, 8, 9, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]
A	III	60	[17, 18, 19]
A	IV	50	[17, 19]
B	I	388	[1, 2, 5, 6, 7, 8, 11, 13, 14, 16, 18, 20, 21, 23, 24, 25, 26, 27, 28]
B	II	288	[1, 2, 5, 6, 7, 8, 13, 14, 16, 18, 20, 21, 23, 24, 25, 26, 30]
B	III	115	[19]
B	IV	120	[16, 18]
C	I	307	[1, 2, 5, 6, 7, 8, 9, 10, 14, 15, 16, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]
C	II	208	[1, 2, 6, 7, 8, 9, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]
C	IV	320	[1, 2, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30]

6 Conclusions

This paper develops importance measures of component maintenance priority for multistate systems, and investigates the component maintenance policy with maintenance cost and resource constraint being considered. Besides, the proposed importance measures are generalized to non-coherent binary and multi-state systems and continuum systems, respectively.

If the state of a degraded component cannot be observed, the proposed measures can identify the failed components. After the state of a component degrades, the proposed measures can evaluate how other components affect the system performance. When failed components are being maintained, the proposed measures can determine the priority of components that should be selected for preventive maintenance. Considering the limited maintenance cost, the proposed measures can optimise the number of components for preventive maintenance to maximize the expected system performance.

In real systems, the transition rates of component states are the key indices for reliability evaluations. The integrated importance measure considers the effect of transition rates on the system reliability. Thus, in future work, we will develop the integrated importance measure of component maintenance priority for multistate systems, and investigates the component maintenance policy.

Acknowledgments

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References

[1] S. Wu, Y. Chen, Q. Wu, Z Wang, Linking component importance to optimisation of preventive maintenance policy, Reliability Engineering and System Safety 2016; 146: 26-32.

[2] W. Kuo, X. Zhu, Relations and Generalizations of Importance Measures in Reliability, IEEE Transactions on Reliability 2012; 61(3): 659-674.

[3] W. Kuo, X. Zhu, Importance Measures in Reliability, Risk, and Optimization: Principles and Applications. UK: Wiley, 2012.

[4] Y. Fu, T. Yuan, X. Zhu, Importance-measure based methods for component reassignment problem of degrading components, Reliability Engineering & System Safety 2019, doi: https://doi.org/10.1016/j.ress.2019.106501

[5] Z. Xu , J. E. Ramirez-Marquez , Y. Liu , T. Xiahou , A New Resilience-Based Component Importance Measure for Multi-State Networks, Reliability Engineering and System Safety 2019, doi: https://doi.org/10.1016/j.ress.2019.106591

[6] H. Dui, S. Li, L. Xing, H. Liu. System performance-based joint importance analysis guided maintenance for repairable systems, Research Reliability Engineering & System Safety 2019, 186, 162-175.

[7] H. Dui, S. Si, RCM Yam, Importance measures for optimal structure in linear consecutive- k -out-of-n systems, Reliability Engineering and System Safety 2018; 169: 339–350.

[8] G. Levitin, L. Podofillini, E. Zio, Generalized importance measures for multi-state elements based on performance level restrictions, Reliability Engineering & System Safety 2003; 82: 287–298.

[9] J. E. Ramirez-Marquez, D. W. Coit, Composite importance measures for multi-State systems with multi-state components, IEEE Transactions on Reliability 2005; 54: 517-529.

[10] E. Borgonovo, Differential, criticality and Birnbaum importance measures: An application to basic event, groups and SSCs in event trees and binary decision diagrams, Reliability Engineering and System Safety 2007; 92: 1458-1467.

[11] Y. Dutuit, A. Rauzy. On the extension of importance measures to complex components, Reliability Engineering & System Safety 2015; 142: 161-168.

[12] E. Borgonovo, H. Aliee, M. Glaß, J. Teich, A new time-independent reliability importance measure, European Journal of Operational Research 2016; 254: 427-442.

[13] Y. Lin, RCM Yam, Uncertainty importance measures of dependent transition rates for transient and steady state probabilities, Reliability Engineering & System Safety 2017; 165: 402-409.

[14] X. Xu, Z. Lu, X. Luo, A kernel estimate method for characteristic function-based uncertainty importance measure, Applied Mathematical Modelling 2017; 42: 58-70.

[15] J. D. Andrews, S. Beeson, Birnbaum's Measure of Component Importance for Noncoherent Systems, IEEE Transactions on Reliability 2003; 52: 213-219.

[16] S. Beeson, J. D. Andrews, Importance Measures for Non-Coherent-System Analysis, IEEE Transactions on Reliability 2003; 52: 301-310.

[17] E. Borgonovo, The reliability importance of components and prime implicants in coherent and non-coherent systems including total-order interactions, European Journal of Operational Research 2010; 204: 485-495.

[18] J. K. Vaurio, Importances of components and events in non-coherent systems and risk models, Reliability Engineering & System Safety 2016; 147: 117-122.

[19] H. Aliee, E. Borgonovo, M. Glaß, J. Teich, On the Boolean extension of the Birnbaum importance to non-coherent systems, Reliability Engineering & System Safety 2017; 160: 191-200.

[20] C. Kim, L. A. Baxter, Reliability importance for continuum structure functions, Journal of applied probability 1987; 24(3): 779-785.

[21] Y. Liu, S. Si, L. Cui, Z. Wang, S. Sun, A Generalized Griffith Importance Measure for Components With Multiple State Transitions, IEEE Transactions on Reliability 2016; 65(2): 662-673.

[22] Z. Cai, S. Si, Y. Liu, J. Zhao, Maintenance Optimization of Continuous State Systems Based on Performance Improvement, IEEE Transactions on Reliability 2017; 66(11): 1-15.

[23] X. L. Gao, L. R. Cui, J. L. Li, Analysis for joint importance of components in a coherent system, European Journal of Operational Research 2007; 182: 282-299.

[24] S. Wu, Frank P. A. Coolen, A cost-based importance measure for system components: An extension of the Birnbaum importance, European Journal of Operational Research 2013; 225: 189-195.
[25] S. Wu, L. Chan, Performance utility-analysis of multi-state systems, IEEE Transactions on Reliability 2003; 52: 14-21.
[26] H. Dui, S. Si, M. Zuo, S. Sun, Semi-Markov process-based integrated importance measure for multi-state systems, IEEE Transactions on Reliability 2015; 64: 754-765.
[27] H. Dui, L. Chen, S. Wu, Generalized integrated importance measure for system performance evaluation: application to a propeller plane system. Eksploatacja i Niezawodnosc-Maintenance and Reliability 2017; 19: 279-286.
[28] C. Zhang, Y. Qian, H. Dui, S. Wang, J. Shi, Component failure recognition and maintenance optimization for offshore heave compensation systems based on importance measures, Journal of Loss Prevention in the Process Industries 2020; 63, 103996.
[29] H. Dui, C. Zhang, X. Zheng, Component joint importance measures for maintenances in submarine blowout preventer system, Journal of Loss Prevention in the Process Industries 2020; 63, 104003.
[30] H. Dui, S. Si, RCM Yam, A cost-based integrated importance measure of system components for preventive maintenance, Reliability Engineering and System Safety 2017; 168: 98-104.
[31] D. Lyu, S. Si, Dynamic importance measure for the K-out-of-n: G system under repeated random load, Reliability Engineering & System Safety 2020; 195, 106720.
[32] P. Do, C. Bérenguer, Conditional reliability-based importance measures, Reliability Engineering & System Safety 2020; 193, 106633.
[33] Y. Fu, T. Yuan, X. Zhu, Importance-measure based methods for component reassignment problem of degrading components, Reliability Engineering & System Safety, 2019; 190, 106501.
[34] W. S. Griffith, Multi-state reliability models, Journal of Applied Probability 1980; 17: 735–744.
[35] T. Inagaki, E. J. Henley, Probabilistic evaluation of prime implicants and top-events for noncoherent systems, IEEE Transactions on Reliability 1980; 29(5): 361-367.
[36] L. A. Baxter, Continuum structures I, Journal of Applied Probability 1984; 21: 802-815.
[37] L. A. Baxter, Continuum structures II, Mathematical Proceedings of the Cambridge Philosophical Society 1986; 99: 331-338.
[38] Y. Pei, W. Wang, and L. Li, Ranking the Vulnerable Components of Aircraft by Considering Performance Degradations, Journal of Aircraft 2016; 53: 1400-1410.