Production, Bankruptcy, and Financial Policies under Collateral Constraints*

Miguel León-Ledesma

University of Kent and CEPR

Jaime Orrillo

Universidade Católica de Brasília

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Abstract

We propose an extension of the incomplete markets general equilibrium model with production to situations in which firms can default (bankruptcy). In the model, firms are owned by a single individual whose roles as entrepreneur and consumer are anonymous. Assets are exogenously collateralised and are seized together with future output in case of default. Default is thus endogenous as output determines the deliveries from the assets issued. In turn, the receipts from each asset purchased are assumed to depend on a fraction of the firm's output. This fraction is anonymous and is treated as given by individuals who will anticipate it, in equilibrium, as a proportion of aggregate output, generating counter-cyclical default. We show that, in this context, the financial policies of the firm are, in general, not irrelevant. We establish the existence of equilibria and discuss the nature of the inefficiencies introduced by the presence of bankruptcy.

Keywords: General equilibrium; Incomplete markets; default; production economy.

JEL Codes: D52; D51; E44; C63.

*Miguel León-Ledesma: School of Economics and MaGHiC, University of Kent, and CEPR, Kennedy Building, Canterbury CT27FS, UK, m.a.leon-ledesma@kent.ac.uk. Jaime Orrillo: Universidade Católica de Brasília, Dept. de Mestrado de Doutorado Economia, QS 07, Lote 01 – EPCT, Office K–244, Brasilia–DF, ZIP: 71966-700, Brazil, orrillo@pos.ucb.br. We thank Alessandro Citanna, Alfred Duncan, Masaru Inaba, Felix Kubler, Wei Jiang, Mario Páscoa, and an anonymous referee for useful comments and discussions. Jaime Orrillo acknowledges CNPq for the financial support through projects 307126/2016-1 and 457098/2014-7. All remaining errors are the authors' sole responsibility.

1 Introduction

One of the key features of production activities is that they take time. In order to produce, firms have to carry out investments that are funded by either equity or debt. If there are financial frictions in the economy, this then leads to a close link between finance and the level of production of firms and, therefore, real economic activity. The impact of financial variables on the real economy has been the subject of an large literature in macroeconomics. This is especially so since the advent of the Global Financial Crisis. Although that particular crisis originated mainly in sub-prime lending for housing market investment, business default or bankruptcy can also lead to large disruptions in economic activity and to the amplification of shocks. This is even more relevant in the face of the unprecedented Covid-19 crisis that has put business activities in certain sectors under enormous stress. Figure 1 shows the evolution of business bankruptcy and delinquency rates in the US, which displays a clear counter-cyclical pattern. Both measures increase substantially during recession periods.[1] Thus, it is important to develop theoretical frameworks that put business default and bankruptcy at center stage in an economy with incomplete financial markets.

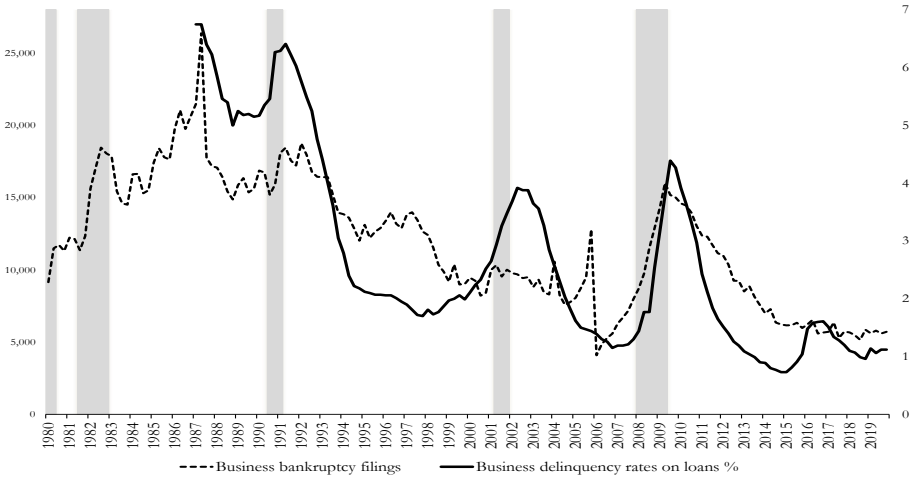

Figure 1: Business bankruptcy filings (left scale) and business delinquency rates (in percentage) on commercial banks (right scale) in the US, 1980-2019. Shaded areas are NBER recession dates.

Our paper has two main objectives. First, to introduce firm default in a general

[1]There was a surge and then a drop in bankruptcy filings between 2005 and 2006 prompted by the October 2005 implementation of the Bankruptcy Abuse Prevention and Consumer Protection Act.

equilibrium model of incomplete markets (GEI), collateral constraints, and production, and answer the question whether this setting is consistent with equilibrium. Note that the production aspect of our model has not been addressed in earlier studies with default and collateral. This is important, as he model can generate the counter-cyclical default observed in the data. Secondly, and our main objective here, to analyse the financial polices of the firm in the presence of collateral constraints and bankruptcy. If financial policies like the composition of liabilities between debt and equity matter, they can affect real variables such as investment leading to a transmission mechanism between finance and real economic activity.

We present two main results. First, we show that default and bankruptcy are consistent with the orderly functioning of markets.[2] That is to say, there is an equilibrium where firm default exists, although bankruptcy is a disequilibrium signal. Second, we show that, unless we impose very restrictive assumptions on the trading of portfolios and on the way borrowers repay their debts, the financial policies of the firm matter for real outcomes. Specifically, the Fisher Separation Theorem and the Modigliani-Miller Theorem are not satisfied in this context. Although our analysis of financial polices of the firm is carried out in the context of a one-good economy, we deal with the problems of both existence and efficiency in a more general multi-good multi-agent economy with two periods. This is because efficiency in a one-good economy turns out to be trivial. Even though equilibrium efficiency does not hold in an incomplete markets multi-good multi-agent economy, we do obtain a constrained efficiency result. Thus, for the sake of completeness, we retain the multi-good framework when dealing with equilibrium existence.

The model makes several key assumptions regarding ownership structure and repayment incentives:

- Firms are owned by a *single individual*[3] whose roles as both consumer and entrepreneur are anonymous. I.e. the assets they issue can be used to consume or produce, but the lender cannot distinguish between these two activities when buying assets. That is, they cannot see whether assets are financing their production or consumption activities.

[2]We clarify below the distinction between default and bankruptcy, which relates to the type of equilibrium analysed in different parts of the paper.

[3]Our analysis will be limited to the case of sole proprietorship and not to firms with open capital. The latter turns out to be more complicated in presence of incomplete markets even without bankruptcy, see DeMarzo (1988).

- Entrepreneurs have *unlimited liability* in the sense that the sole proprietor is liable for all business debts in case the business cannot pay its liabilities. Because of this, our model allows for the ex-post production to be seized together with the depreciated physical collateral committed ex-ante.

- Assets are *exogenously collateralised* as in Geanakoplos and Zame (2014). In the event of a default, our model allows the ex-post production to be seized together with the collateral. The dependence on second period output is a simple way of making deliveries on assets issued endogenous – leading to endogenous default – while maintaining exogenous collateral.[4] However, endogenous deliveries introduce a moral hazard problem that we discuss below.

In models without production, the existence of collateral provides insurance for lenders. However, with production and the dependence of deliveries on second period output, problems of asymmetric information reappear because deliveries depend on the level of production. More precisely, in case of a devaluation of the collateral value below payoff, we would have a moral hazard problem as entrepreneurs have an incentive to reduce production and let firms go bankrupt. In order to incentivize high production and prevent bankruptcy, we introduce a "default" insurance market similar to the one in Araujo et al. (2000). In Araujo et al (2000) this device is introduced to give an incentive for borrowers to offer high levels of collateral given that, in their model, collateral is endogenous.

On the other hand, in models with production, lenders could be over-pessimistic about repayments as the collateral could suffer a severe loss of value in the future. This would then lead to a trivial equilibrium with zero production. When there is trading in financial markets, we can resolve this problem by making asset receipts anonymous assuming a type of *tranching*. The production (and collateral) of a consumer/firm backs all the assets they issue. Because of anonymity, lenders have to anticipate a default rate on their whole portfolio that will depend on the average level of production of firms in the economy. This will depend on the state of nature and, hence, aggregate economic activity (i.e. default is counter-cyclical). In order to prevent a zero production equilibrium our default insurance market will once again play an important role as our model does not consider any intermediaries or insurance companies to protect lenders from eventual default.

[4]Financial constraints that are dependent on output are common in open economy models. See, for instance, Gente, León-Ledesma, and Nourry (2015).

To achieve our main goal of analysing the financial policies of the firm, we consider the finance economy derived from the corresponding multi-good economy. Specifically, we study under which conditions the Fisher Separation and the Modigliani-Miller Theorems hold in this sole proprietorship environment. Although we should not expect these theorems to hold in our setting because of the nonlinearity or nonadditivity of deliveries in relation to short positions,[5] it is still important to study the conditions leading to their failure. This is because the validity of these theorems is not only important from the perspective of corporate finance but also from a macroeconomic point of view (see Bisin, Gottardi and Ruta, 2009 and Gerbach, Haller and Müler, 2015). For instance, if bankruptcy prevents the Modigliani-Miller theorem from holding, then the debt to equity ratios of firms can affect investment, leading to amplification effects of macroeconomic shocks.

Two kinds of equilibria, as in Magill and Quinzii (1996), are defined to this end. The first one is the reduced-form equilibrium in which consumption and production activities are carried out by one individual whose roles as consumer and entrepreneur are indistinguishable. In this kind of setting a situation in which individuals do not honor their financial commitments is called "default" because the financial decisions as consumers and firms are not separated. The second kind of equilibrium is the extensive-form equilibrium where the financial accounts of firms are separated from those of consumers, who are the owners of the firms. In this case, we call "bankruptcy" a situation in which firms do not honor their debts. For the latter to be well defined, we slightly modify the classical definition of the reduced-form equilibrium which requires the decomposition of the debt portfolio. Once this is done, we define the extensive-form equilibrium requiring that both consumers and entrepreneurs (firms) decide separately their consumption and production activities as well as their financial decisions in order to maximise their objective functions: consumers maximise their utility functions subject to their budget constraints and firms maximise the present value of their dividends subject to technology. Under a hypothesis called separation of deliveries, we analyse the equivalence between the reduced-form equilibrium and the extensive-form equilibrium. From this equivalence we analyze the Modigliani-Miller Theorem and Fisher's Separation Theorem. The latter theorem is studied when the firm maximizes profits instead the present value of its dividends.

[5]Deliveries in models with collateral are defined to be the minimum between the claim and the value of the collateral. The latter is non additive with respect to short sales, since the collateral includes the production level of the firm, which is independent of the sale of assets.

Lastly, we analyze the efficiency properties of the equilibrium with endogenous bankruptcy and collateral. Because of the incomplete markets nature of the environment, we focus on analyzing constrained efficiency. We show that the equilibrium is constrained efficient. This is because, when we allow for lump-sum transfers among agents, the constrained efficient transfer is given by the equilibrium insurance premium from the insurance contract offered by lenders.

1.1 Related literature

Our paper is related to several streams of the literature on GEI models. The analysis of existence and non-triviality was first addressed in GEI models with default penalties and without production by Dubey et al. (1990, 2005) and Araujo et al. (1998), and in an exogenous collateral setting without production by Steinert and Torres-Martínez (2007). Our concept of non-triviality follows Araujo et al. (1998) where the repayment rate depends indirectly on aggregate output. Our existence argument follows Araujo et al. (2000) who proposed an endogenous collateral model with a continuum of agents but without production. In that model, borrowers choose the collateral level guaranteeing their debts and lenders offer default-insurance contracts. In our model, borrowers do not choose any collateral level (i.e. the collateral is exogenous), and a continuum of agents is not required because the budget sets are always convex, although deliveries are endogenous. The endogeneity of deliveries is due to the fact that they depend on the production level since, in case of default, borrowers will deliver the depreciated collateral plus the production level. We thus maintain the default-insurance contracts to obtain the market clearing condition in the truncated economy. This will be fundamental to obtain an equilibrium for the limit economy.[6]

The Fisher Separation Theorem in a model with incomplete markets was established by Magill and Quinzii (1996) in a sole proprietorship setting without default or bankruptcy. More recently, Carvalho, Divino and Orrillo (2007) extended the Magill and Quinzii (1996) result to a situation where agents are allowed not to honor their financial commitments. They obtain their result by assuming, as in Dubey, Geanakoplos and Shubik (2005), that agents are punished in proportion to their default. That is, the difference between the claim and the delivery which is assumed to be a decision variable. Consequently, the decision whether or not to default depends on the comparison between the desire for consuming and the utility penalty suffered if they default. Although this

[6]See León-Ledesma and Orrillo (2016) for details.

result involved default, it is only true provided that agents as entrepreneurs do not default. The Modigliani-Miller Theorem has been analyzed before in a bankruptcy setting with incomplete markets by Hellwig (1981). Hellwig (1981) was able to prove the theorem but only under very stringent conditions on the kind of securities traded by firms, essentially making the market structure equivalent to having assumed complete markets.[7]

Finally, our paper relates, albeit indirectly, to a literature in macroeconomics introducing the possibility of firm default in quantitative stochastic general equilibrium models. A leading example is Bernanke, Gertler and Gilchrist (1999) who develop a costly state verification general equilibrium model where firms differ in terms of productivity realizations.[8] Below a certain productivity threshold, firms default. As discussed in Gomes et al (2003), however, the model generates pro-cyclical defaults in the face of aggregate shocks. Khan, Senga and Thomas (2016) develop a model with firm heterogeneity where, depending on idiosyncratic and aggregate productivity shocks as well as aggregate credit shocks, firms can default on non-contingent debt contracts. In their model, credit shocks generate counter-cyclical default and a fall in aggregate TFP due to misallocation. Toda (2015) is perhaps closer to our approach. He develops a one-good economy with collateral constraints where financial intermediaries pool debt contracts and issue asset-backed securities. His focus is on the risk-sharing role of securities and shows that, in equilibrium, entrepreneurs will leverage to maximum, leading to the endogenous emergence of "subprime" loans. Our focus here is on the financial policies of firms without financial intermediaries and where default is endogenized by making deliveries depend on future output. Our model thus offers a simple mechanism of endogenous default with collateral. Finally, Magill and Quinzii (2015) present a 3-period OLG model of default with production under collateral constraints to analyze the effect of income shocks around the steady state. Contrary to traditional models of collateral constraints in macroeconomics such as Kiyotaki and Moore (1997), Magill and Quinzii (2015) allow borrowers to default as soon as the value of the collateral falls below the value of the loan just as we do. In their model, agents consume a perishable and a non-perishable good, but only the non-perishable good can be purchased as collateral. Our approach differs in many aspects. In our model there are only two periods and one

[7]See also Gottardi and Kubler (2015) for an analysis of complete markets in a collateral setting like ours but without production.

[8]Contributions to this literature are by now too vast to mention here, but see the excellent reviews of Quadrini (2011), Brunnermeier et al. (2012), Claessens and Kose (2017), and Duncan and Nolan (2018).

good, and deliveries in case of default depend on the level of production in the second period. Unlike Magill and Quinzii (2015), our utility and technology specifications are very general since we do not focus on the quantitative properties of the model. Finally, our main focus here is on the financial policies of the firm, rather than the response of the economy to unanticipated shocks. Although our model is not suited for quantitative analysis of business cycles, it allows for analytical proofs of the properties of equilibrium that are otherwise absent in most models of business cycles with financial frictions.

The paper is organised as follows. Section 2 presents the model. In Section 3, we establish the existence and non-triviality of the equilibrium. In Section 4, financial policies are analysed in a sole proprietorship setting. Section 5 analyses the efficiency of equilibrium. Finally, Section 6 concludes.

2 The Model

We now present the general model with many commodities which is used to prove existence and analyse efficiency. The model will be specialised later to one commodity to analyse financial policies. The model has two periods, $t = 0, 1$. There is only uncertainty in the second period. To avoid notational clutter, we will use the same symbol to denote both, sets and their cardinal. Uncertainty happens over S possible states of nature, $S = \{1, 2, \ldots, S\}$. There are L goods in both periods and, since we have S states of nature in the second period, the commodity space will be $R^{L(S+1)}$. The price system of commodities is the vector $p = (p_0, \tilde{p})$ belonging to $R_+^{L(S+1)}$ where $\tilde{p} = (p_1, \ldots, p_s) \in R_+^{(LS)}$. This notation is extended to other variables throughout the paper.

There are F firms and H consumers. We assume that the number of consumers is greater than or equal to the number of firms. That is, $H \geq F$.

Each agent $h \in H$ is characterised by a utility function $U^h : R_+^{L(S+1)} \to R$ and an initial endowment vector of commodities $\omega^h \in R_+^{L(S+1)}$. Each firm f is characterised by their production set $Y^f \subset R^{L(S+1)}$. The elements of Y^f are called projects available to the firm and are denoted by $y = (y_o, \tilde{y}) \in Y^f \subset R^{L(S+1)}$. Lastly, a possible consumption plan for each consumer h is denoted by $(x_0, \tilde{x}) \in R_+^{L(S+1)}$

2.1 Ownership and financial structures

The ownership structure is characterized by a function $h : F \to H$. This assigns each firm f to its single owner $h(f)$ who is assumed to be an entrepreneur with unlimited

liability. We can also consider the function $f : H \to L$ which assigns a single firm $f(h)$ to each consumer h. Since $H \geq F$, there could be consumers who own no firms.

There are J real assets in the first period available for trading. Each asset j is sold at price π_j in the first period and promises a bundle A_s^j if state of nature s is revealed in the second period. These assets are collateralised as in Geanakoplos and Zame (2014). Thus, for each unit sold of asset j, it is necessary to purchase a bundle $C_j \in R_+^L$ backing its promise. The collateral is assumed to depreciate according to function $K_s : R_+^L \to R_+^L$ which depends on the state of nature. For the sake of simplicity, K_s will be assumed to be a non-singular linear transformation.

2.2 Deliveries and receipts

Since we are assuming no separation between consumption and production activities, then there is no separation either between financial decisions of consumers and entrepreneurs. In the event of default, individuals will give up, in addition to physical collateral, their production. This is because of the unlimited liability assumption. Hence, for every state s, the delivery or repayment of borrowers for each portfolio $\varphi \in R_+^J$ sold will be given by:

$$D_s(\varphi, y_s) = \sum_{j \in J} D_s^j(\varphi_j, y_s),$$

where $D_s^j(\varphi_j, y_s)$ is defined to be $\min\{p_s A_s^j \varphi_j, p_s(y_s + K_s C_j \varphi_j)\}$.

Note that the value of production y_s at price p_s is backing all J assets. That is, we are considering a type of *tranching.* Consequently, in case of default, lenders who purchase units of asset j will only expect to receive: a) a fraction $\overline{y}_s^j$ of the total production of the firm measured in nominal terms plus b) the value of the depreciated collateral which backed the issuance of asset j. That fraction, which is a pro rata of all its different sellers' production, is given for lenders but will be determined in equilibrium as a proportion of the aggregate output of the economy. Thus, the receipt for each unit of asset j purchased will be:

$$R_s^j = \min\{p_s A_s^j, \overline{y}_s^j + p_s K_s C_j\}.$$

In other words, lenders who suffer default will receive the value of depreciated collateral plus the average output level $\overline{y}_s^j$. For each portfolio purchased $\theta \in R_+^J$, its receipts at state s, denoted by $R_s\theta$, is defined as $\sum_{j \in J} R_s^j \theta_j$.

Remark 1: In the particular case in which the agents do not own any firms, so that

$y_s = 0$ and therefore $\overline{y}_s^j = 0$, deliveries and receipts will reduce to the same expression as in the exogenous collateral model of Geanakoplos and Zame (2014), where deliveries and receipts at sate s for each unit of security traded j is $\min\{p_s A_s^j, p_s K_s C_j\}$.

2.3 Default insurance

When a borrower defaults on asset j in state s, they deliver $D_s(\varphi_j, y_s) = p_s y_s + p_s K_s C_j \varphi_j$. Hence, the amount of default becomes $\left(p_s A_s^j \varphi_j - (p_s y_s + p_s K_s C_j \varphi_j)\right)$. Similarly, the investor (lender) who suffers default on asset j in state s will receive $R_s^j = \overline{y}_s^j + p_s K_s C_j$. Thus, the amount of this default, called default suffered, is $\left(p_s A_s^j - (\overline{y}_s^j + p_s K_s C_j)\right)\theta_j$ for each amount θ_j of asset purchased j.

As mentioned in the introduction, with production and the dependence of repayments on second period output, a moral hazard problem appears. Specifically, the problem is that borrowers will want their default to be as large as possible. This would occur if they, as entrepreneurs, do not produce. Knowing this, lenders will not have any incentive to invest as they would only receive the future value of depreciated collateral $p_s K_s C_j \theta_j$ in case of default. To prevent this zero production and lending equilibrium, we need a mechanism that creates incentives for entrepreneurs to have positive production. This is done in simple terms by introducing a "default" insurance market similar to the one in Araujo et al. (2000). In what follows, we describe such mechanism and, then, in the next section, look at how this mechanism impacts the decisions of agents.

2.3.1 The default insurance market

In the absence of intermediaries or insurance companies to protect from eventual default, it will be lenders who sell default insurance contracts to borrowers.[9]

Suppose that there are S state-contingent default-insurance contracts traded in a default insurance market. Each agent, who trades a portfolio $(\theta, \varphi) \in R_+^J \times R_+^J$, will add to their portfolio cost $(\pi(\theta - \varphi))$ the following overall insurance net premium:

$$\sum_{s\in S} \beta_s \sum_{j\in J} \left[\left(p_s A_s^j \varphi_j - (p_s y_s^{f(h)} + p_s K_s C_j \varphi_j) \right)^+ - \sum_{j\in J} \left(p_s A_s^j - (\overline{y}_s^j + p_s K_s C_j) \right)^+ \theta_j \right]$$

which depends both on the default incurred and the default suffered at each state s.

If we denote by $g_s(\theta, \varphi, y_s)$ the insurance net premium of state s given by expression:

$$\sum_{j\in J} \left(p_s A_s^j \varphi_j - (p_s y_s^{f(h)} + p_s K_s C_j \varphi_j) \right)^+ - \sum_{j\in J} \left(p_s A_s^j - (\overline{y}_s^j + p_s K_s C_j) \right)^+ \theta_j,$$

[9] A potential extension of this framework would be the introduction of intermediaries who specialise in insuring lenders.

then the overall insurance net premium is $\sum_{s\in S}\beta_s g_s(\theta,\varphi,y_s)$ with $\beta_s>0\forall s$. Which will be endogenously determined.

To be more precise, we can think of $\sum_{s\in S}\beta_s g_s(\theta,\varphi,y_s)$ as being the present value of the expected net default, where the expectation is taken with respect the probability vector $\rho=(\frac{\beta_1}{k},\ldots,\frac{\beta_S}{k})$. This vector is generated by the vector of (actuarially fair) state prices $\beta=(\beta_1,\ldots,\beta_S)$, where $k=\sum_{s\in S}\beta_s$ is interpreted as the discount factor.

2.4 Budget Feasibility

Given the price system $(\pi,p)\in R_+^{L(S+1)}\times R_+^J$ of assets and commodities, the average output level $\overline{y}\in R_+^{SJ}$ and the state prices $\beta\in R_+^S$; each agent $h\in H$ chooses $(x^h,\theta^h,\varphi^h,y^{f(h)})$, where x^h is consumption for agent h, subject to the following budget constraints:

$$p_o(x_o^h-y_o^{f(h)})+\pi\theta+\sum_{s\in S}\beta_s g_s(\theta^h,\varphi^h,y_s^{f(h)})\leq p_o\omega_o^h+\pi\varphi^h, \tag{1}$$

$$p_s x_s^h+D_s(\varphi^h,y_s^{f(h)})\leq p_s\omega_s^h+R_s\theta+p_s y_s^{f(h)}+p_s K_s(x_o^h-y_o^{f(h)}),s\in S \tag{2}$$

Budget constraint (1) implies that agent h uses their resources to purchase consumption goods and inputs for contingent future production plus collateralised assets and default insurance contracts. All this expenditure is funded by the value of the initial endowment plus the sales of collateralised assets. Budget constraint (2) states that, in each state of nature s in the second period, agent h cannot spend more on the purchase of commodities x_s and asset deliveries $D_s(\varphi,y_s)$ than the revenue they get from commodity sales (including initial endowments ω_s^h and depreciated collateral $p_sK_s(x_o-y_o)$) and the asset receipts $R_s\theta$.

In addition, the following collateral constraint is satisfied:

$$C\varphi:=\sum_{j\in J}C_j\varphi_j\leq x_o-y_o. \tag{3}$$

Hence, we define the budget set for each agent h to be the set

$$B^h(p,\pi,\overline{y},\beta)=\{(x,\theta,\varphi,y):(1),(2)\text{ and }(3)\text{ are satisfied}\}$$

2.4.1 Default insurance and asset prices

We now look at the effect that the insurance net premium has on asset prices. To do this, it is convenient to interpret the first-period budget constraint in terms of the

default-insurance using the discount factor k defined above and the probability vector ρ.

We write $\sum_{s\in S}\beta_s g_s(\theta,\varphi,y_s)$ as the difference between two terms. The first is the expected present value of future default incurred by agents evaluated at probabilities ρ and denoted E_ρ. These are the agents that issue the assets and hence we denote them by *short*:

$$\pi_{short}(\varphi,y)=kE_\rho[\sum_{j\in J}\left(pA^j\varphi_j-(py^{f(h)}+pKC_j\varphi_j)\right)^+]].$$

The second term is the expected present value of future losses due to default suffered (by those who buy the assets and hence *long*):

$$\pi_{long}\theta=kE_\rho[\sum_{j\in J}\left(pA^j-(\overline{y}^j+pKC_j)\right)^+\theta_j].$$

Hence, (1) we can be re-written as:

$$p_o(x_o-y_o)+(\pi-\pi_{long})\theta\leq p_o\omega_o^h+[\pi\varphi-\pi_{long}(\varphi,y)]. \tag{1'}$$

Expression (1') means that the investors who purchase assets receive a discount on the price of every unit of asset purchased. The issuers of the security, on the other hand, face a reduction in the value of their total borrowing. This reduction $\pi_{long}(\varphi,y)$ depends on the level of production they decide and also on the portfolio φ sold. Hence, the higher the level of production, the higher the amount of borrowing, thus encouraging borrowers to increase their production.

Finally, we can summarise our production economy with exogenous collateral and default by the following array:

$$\mathcal{E}=[(U^h,\omega^h,Y^{f(h)})_{h\in H},(A,C,K)]$$

which consists of household-producers, endowments, and collateralised assets which depreciate according to function K. Note that if $Y^{f(h)}=\{0\}$ then an agent h owning firm f is just a consumer.

2.5 Collateral Equilibrium with Production

Let $R^m=(R_+^{L(S+1)}\times R_+^J\times R_+^J\times R_+^{L(S+1)})^H$ and $R^n:=R_+^{L(S+1)}\times R_+^J\times R_+^{SJ}$.

Definition 1. *The allocation* $(x^h,\theta^h,\varphi^h,y^{f(h)})_{h\in H}\in R^m$ *together with a price system and an average output level* $(p,\pi,\overline{y},\beta)\in R_+^n$ *is a production-collateral equilibrium for economy* $\mathcal{E}$ *if the following conditions are satisfied:*

1. *Choices are optimal. That is, for each* $h \in H$,

$$(x^h, \theta^h, \varphi^h, y^{f(h)}) \text{ maximise } U^h(x^h) \text{ subject to } B^h(p, \pi, \overline{y}).$$

2. *Market clearing conditions hold:*

$$\sum_H \left(x_0^h - \omega_0^h - y_0^{f(h)}\right) = 0,$$

$$\sum_H \left(x_s^h - K_s(\omega_o^h) - y_s^{f(h)} - \omega_s^h\right) = 0, s \in S,$$

$$\sum_{h \in J} \theta^h = \sum_{h \in J} \varphi^h.$$

3. *For each* $s \in S$ *and for each* j,

$$\overline{y}_s^j \sum_{h \in H} \theta_j^h = \sum_{h \in H} p_s y_s^{f(h)}.$$

4. $\forall j, s$, *we have:*

$$\sum_{h \in \mathcal{S}_s^j} \left(p_s A_s^j - (\overline{y}_s^j + p_s K_s C_j)\right)^+ \theta_j^h = \sum_{h \in \mathcal{D}_s^j} \left(p_s A_s \varphi_j^h - (p_s y_s^{f(h)} + p_s K_s C_j \varphi_j^h)\right)^+,$$

where $\mathcal{D}_s^j = \{h \in H : p_s A_s \varphi_j^h > (p_s y_s^{f(h)} + p_s K_s C_j \varphi_j^h)\}$ *is the set of agents that incur default in state of nature* s *on asset* j, *and* $\mathcal{S}_s^j = \{h \in H : p_s A_s > (\overline{y}_s^j + p_s K_s C_j)\}$ *is the set of agents that suffer default in state of nature* s *on asset* j.

Remark 2: Items 1 and 2 are the usual conditions in the classical definition of equilibrium. Item 3 states that, in equilibrium, the average output level $\overline{y}_s^j$, treated as given by individuals, is perfectly anticipated as a proportion of the aggregate output of the economy. Finally, Item 4 states that the default incurred by asset issuers must be equal to the default suffered by lenders.

3 Equilibrium Existence

We now tackle the problem of proving equilibrium existence and non-triviality. In order to do so, we need to add four extra assumptions. The first three are standard in models of economies with production (see, for instance, Magill and Quinzii, 1996 and Drèze, Minelli and Tirelli, 2008):

H1 The initial endowments of each individual $h \in H$ are strictly positive: $\omega^h \in R_{++}^{L(S+1)}$;

H2 $\forall h \in H$, $U^h : R^L_+ \times R^{LS}_+ \to R$ is concave, continuous and strictly increasing;

H3 $\forall v \in R^{S+1}_+$, where v is any vector of non-negative constants, the set $(v+\sum_{h\in H} Y^{f(h)})\cap R^{L(S+1)}_+$ is compact;

H4 $C_j \in R^L_+ \setminus \{0\}$.

H4 simply states that collateral must be non zero as is normally assumed in classic collateral models without production (see Geanakoplos and Zame, 2014). Also note that, since returns on asset purchases are positive and the net price of a purchase of assets $q := \pi - kE_\rho[pA^j - (\overline{y}^j + pKC_j)]^+$ is independent of agents, then from (1') and H2 (particularly the strict monotonicity of U^h) it follows that both $\pi - \pi_{long} >> 0$ and $p_s >> 0, \forall s \geq 0$ make up a necessary condition for non-arbitrage. Otherwise, agents could buy infinitely many units, increasing unboundedly their second period consumption and, therefore, the consumer's problem would not have a solution.

3.1 Non-Triviality of Equilibrium

It is well known that collateral requirements partially protect lenders because receipts are always non-zero due to H4 above. However, this does not preclude the possibility that lenders have over-pessimistic beliefs about receipts because the collateral could experience a large depreciation in the second period.

In our model, we can find a trivial equilibrium in the following way: suppose that the prices of assets are $\pi = 0$, the average output level $\overline{y} = 0$, and the collateral requirements are small enough so that the receipts $R^j_s = \min\{p_s A^j_s, \overline{y}^j_s + p_s K_s C_j\} \to 0$, implying $\theta^h_j = 0, \forall h$. Thus, $\varphi^h_s = 0, \forall h$ which implies that $D^h_s(\varphi^h_j, y^{f(h)}_s) = 0, \forall s, j$ since $y^{f(h)}_s = 0, \forall h$ since we have fixed $\overline{y}^j_s$ to be zero for every j, s. This way our model reduces to the model with durable goods and without trading in the financial markets. Existence in this type of setting has been proven by Geanakoplos and Zame (2014).

If there is trading in the asset markets in equilibrium (i.e., $\sum_{h\in H} \theta^h_j = \sum_{h\in H} \varphi^h_j \neq 0$), then lenders will rationally anticipate the average level of production as $\overline{y}^j_s = \frac{\sum_{h\in H} p_s y^{f(h)}_s}{\sum_{h\in H} \varphi^h_j}$. However, when assets are not traded, the expected average level of production $\overline{y}^j_s$ cannot be determined from the earlier formula. This issue was firstly addressed in GEI models with default penalties and without production by Dubey et al. (1990) and Araujo et al. (1998) and in an exogenous collateral setting without production by Steinert and Torres-Martínez (2004).

We next adapt the concept of non-triviality of equilibria to our context following Araujo et al. (1998). We first define a no-trade equilibrium as the one in which $\theta^h = 0 = \varphi^h, \forall h$. Secondly, we define the payment rate t_s^j of asset j in state s as being $\min\{1, \frac{p_s K_s C_j + \overline{y}_s^j}{p_s A_s^j}\}$. Hence, R_s^j can be written as $R_s^j = t_s^j p_s A_s^j, \forall j, s$.

Notice that t_s^j could be equal to 1 independently of the endogenous expected output level $\overline{y}_s^j$. This occurs when the future value of the depreciated collateral is large enough so that $p_s K_s C_j \geq p_s A_s^j$. Otherwise, the endogenous expected output level $\overline{y}_s^j$ must be large enough to make $p_s K_s C_j + \overline{y}_s^j > p_s A_s^j$. Consequently, when collateral is sufficiently valued, it eliminates any investors' pessimism with respect to the payment rate. Otherwise, the expectation about the payment rate of the economy depends on its aggregate level of output (GDP) so that a high aggregate output level would also eliminate any investors' pessimism.

Following Araujo et al. (1998), thus, we can state the following definition:

Definition 2. *An equilibrium is said to be non-trivial if there is trading in the asset markets or, when assets are not traded, the payment rates $t_s^j = 1, \forall s, j$.*

We are now ready to state our existence result:

Theorem 1. *Under assumptions H1-H4, for every collateral-production economy $\mathcal{E}$ a non-trivial production-collateral equilibrium exists.*

Technically, the proof of Theorem 1 follows the lines of Araujo et al (2000) and Araujo et al (2005), who use the generalised game methodology due to Debreu (1952). This methodology consists of first defining a sequence of truncated economies where the truncating concerns the set of macro variables (commodity prices, asset prices, state prices, and the average output level) and individual choice variables (consumption, asset purchases, asset sales and production). Next, we associate a generalised game to each truncated economy. We prove that such a game has an equilibrium and, for n large enough, such an equilibrium is also an equilibrium of the truncated economy. We then show that the sequence of equilibria is uniformly bounded so that it will have a subsequence of equilibria converging to a certain limit. Finally, we prove that the limit obtained corresponds to an equilibrium of our original economy. All details can be founded in the Online Appendix.

4 Financial Policies

We now tackle two core questions for our paper. Firstly, in the context of our model, we want to determine whether the financial policies of the firm have an impact on real variables in the economy. That is, whether or not they are relevant. Secondly, we also study whether, as it is standard procedure in many models in macroeconomics, it is possible to separate the roles of agents as, on the one had, consumers and, on the other hand, producers who only decide their production and funding policies. The former is known as the Modigliani-Miller Theorem and the latter as the Fisher Separation Theorem.

To do so, we specialise our analysis by considering the finance economy associated to the multi-goods economy of the previous section. In this representation of the model, we will distinguish two kinds of equilibria. First, a reduced-form equilibrium where consumption and production activities are not separated. In other words, the roles of an agent as consumer and entrepreneur are undistinguishable. Second, an extensive-form equilibrium where, in addition to separating consumption and production roles, the financial accounts of consumers and firms are also separated.

4.1 The Finance Economy

Following Magill and Quinzii (1996), the finance economy is obtained by fixing the equilibrium commodity prices $p \in R_+^{L(S+1)}$ of the corresponding multi-goods economy. Consequently, any consumption or production plan involving the L goods is replaced by an income stream. Thus, the commodity space is now R^{1+S}; the utility function of each agent h is $U^h : R_+^{1+S} \to R$; and the technology describing all the feasible investment projects is $Y^{f(h)} \subset R^{1+S}$. All income streams will be denoted by lowercase letters, which are also used to denote the values of collateral, $c = (c_o, c_1, ..., c_S)$, and the value of promises that are now represented by $r = (r_1, r_2, ..., r_S)$ where each $r_s \in R_+^J$ stands for the payoff of the portfolio containing J assets. Finally, k_s stands for the value depreciation of goods used for consumption, production, or collateral.

Each element y belonging to $Y^{f(h)}$ describes the income streams $y = (y_o, y_1, \ldots, y_S)$ generated by investment projects available to firm $f(h)$. We assume that the set of investment projects $Y^{f(h)}$ is represented by a transformation $T : R^{1+S} \to R$. Since, by definition, $Y^{f(h)} = \{y \in R^{(S+1)} : T(y) \leq 0\}$ then $y \in Y^{f(h)}$ if and only if $T(y) \leq 0$. The transformation T is assumed to be non-decreasing, quasi-convex, and differentiable. The utility function is also assumed to be differentiable and to have a degree of concavity

so that the Kuhn-Tucker conditions to solve an optimisation problem associated to the consumer or the firm are necessary and sufficient. Lastly, to prevent boundary solutions, we assume that for any $x \in R^{1+S}_{++}$ the indifference curve $\{x' \in R^{1+S}_{+} : U^h(x') \geq U^h(x)\}$ is included in R^{1+S}_{++}.

We can write agent h's problem as follows:

$$\max_{(x,y,\theta,\varphi)} U^h(x_o, x_s)$$

s.t.

$$x_o - y_o - \omega^h_o + \pi(\theta - \varphi) + \sum_{s\in S} \beta_s \sum_{j\in J} g^j_s(\varphi_j, \theta_j, y_s) \leq 0, \tag{2'}$$

$$x_s - y_s - \omega^h_s \leq \sum_{j\in J} R^j_s \theta_j - \sum_{j\in J} D^j_s(\varphi_j, y_s) + k_s(x_o - y_o), s \in S, \tag{3'}$$

$$c_o\varphi \leq x_o - y_o \tag{4'}$$

$$T(y_o, y_s) \leq 0 \tag{5'}$$

where R^j_s, $D^j_s(\varphi_j, y_s)$ and $g^j_s(\theta, \varphi, y_s)$ are defined as

$$R^j_s = \min\{r^j_s, \overline{y}^j_s + c_{sj}\},$$

$$D^j_s(\varphi_j, y_s) = \min\{r^j_s\varphi_j, y_s + c_{sj}\varphi_j\},$$

$$g_s(\theta, \varphi, y_s) = \sum_{j\in J} g^j_s(\theta_j, \varphi_j, y_s),$$

with

$$g^j_s(\varphi_j, \theta_j, y_s) = \left(r^j_s\varphi_j - (y^{f(h)}_s + c_{sj}\varphi_j)\right)^+ - \left(r^j_s - (\overline{y}^j_s + c_{sj})\right)^+ \theta_j.$$

Thus, we can re-write g^j_s as follows:

$$g^j_s(\varphi_j, \theta_j, y_s) = r^j_s\varphi_j - D^j_s(\varphi_j, y_s) - (r^j_s - R^j_s)\theta_j$$

In the rest of the paper, we will use the symbol $\partial_x f$ to denote the partial derivative of function f with respect to variable x if the function is differentiable. If f is convex or concave, then the symbol will denote the super-gradient or sub-gradient when f is not differentiable. More precisely, $\partial_x D^j_s$ is the sub-gradient with respect to x, since the function $\min\{.,.\}$ is convex. Finally, if x is a vector of R^n, then the symbol x_{-j} denotes vector x without the $j-$coordinate.

4.2 Reduced-form equilibrium

The following notion of equilibrium is practically the same as in Definition 1, but now it is given in nominal terms and the commodity market clearing conditions are implied by the asset market clearing conditions. More precisely, we have:

Definition 3. *An allocation* $\left((x^h, \theta^h, \varphi^h); (y^{f(h)})\right)_{h\in H} \in R_+^{L(S+1)} \times R_+^J \times R_+^J \times R_+^{L(S+1)}$ *together with* $(\pi, \beta, \overline{y})$ *is said to be a reduced-form equilibrium with production and default if the following conditions are satisfied:*

1. *Asset Markets clear:*
   $\sum_H \theta^h = \sum_{h\in H} \varphi^h$.

2. *For each* $h \in H$, $\left((x^h, \theta^h, \varphi^h); y^{f(h)}\right)$ *maximizes* $U^h(x_o, x_{-o})$ *subject to budget constraints (2'), (3') and (4').*

3. *For every* j, s, *one has:*
$$\overline{y}_s^j \sum_{h\in H} \theta_j^h = \sum_{h\in H} y_s^{f(h)}$$

4. $\forall j, s$, *we have:*
$$\sum_{h\in \mathcal{S}_s^j} \left(r_s^j - (\overline{y}_s^j + c_{sj})\right)^+ \theta_j^h = \sum_{h\in \mathcal{D}_s^j} \left(r_s^j \varphi_j^h - (y_s^{f(h)} + c_{sj}\varphi_j^h)\right)^+,$$

   where $\mathcal{D}_s^j = \{h \in H : r_s^j \varphi_j^h > (y_s^{f(h)} + c_{sj}\varphi_j^h)\}$ *is the set of agents that incur default in state of nature* s *on asset* j, *and* $\mathcal{S}_s^j = \{h \in H : r_s^j > (\overline{y}_s^j + c_{sj})\}$ *is the set of agents that suffer default in state of nature* s *on asset* j.

It is useful to note that, since in a reduced-form equilibrium default and bankruptcy cannot be distinguished, the financial roles of agents cannot be distinguished either. For this reason, we will keep using the term "default" for the reduced-form equilibrium since it is the individuals, as borrowers, who do not honour their financial commitments. Later, we will be able to separate the non-payment decisions of consumers and firms. When we do this, we will use the term "default" for consumers and "bankruptcy" for firms (entrepreneurs). However, individuals as investors (lenders) will only suffer default regardless of who is not honoring their financial commitments.

4.3 Modified reduced-form equilibrium

Since our objective is to compare the different types of equilibria defined earlier, we will assume that φ can be decomposed in two terms: $\varphi = \xi + \phi$, where ξ and ϕ are the amounts of borrowing of agents as consumers and firms respectively. Given this, and changing x_o for $x_o + c_o\xi$ and y_o for $y_o - c_o\phi$, we can write agent h's problem as:

$$\max_{(x,y,\theta,\xi,\phi)} U^h(x_o + c_o\xi, x_{-o})$$

s.t.

$$x_o + c_o\xi - (y_o - c_o\phi) - \omega_o^h + \pi(\theta - \xi) - \pi\phi + \sum_{s\in S}\beta_s \sum_{j\in J} g_s^j(\xi + \phi, \theta, y_s) \leq 0, \qquad (1'')$$

$$x_s - y_s - \omega_s^h \leq \sum_{j\in J} R_s^j\theta_j - \sum_{j\in J} D_s^j(\xi_j + \phi_j, y_s) + k_s(x_o - y_o) + c_s(\xi + \phi), s \in S, \qquad (2'')$$

$$T(y_o - c_o\phi, y_s) \leq 0 \qquad (3'')$$

where x_o and y_o are now consumption and (legitimate) inputs which are assumed to depreciate completely (i.e., $k_s(x_o - y_o) = 0$). $c_o\xi$ is the collateral used for consumption (e.g. a car) and $c_o\phi$ is the collateral used in the production process (e.g. machinery). Finally, the value of collateral in the second period is defined as $c_s \in R^J_{++}$.

Definition 4. *An allocation* $\left((x^h, \theta^h, \xi^h); (y^{f(h)}, \phi^{f(h)})\right)_{h\in H} \in R_+^{L(S+1)} \times R_+^J \times R_+^J \times R_+^{L(S+1)}$ *together with* $(\pi, \beta, \overline{y})$ *is said to be a reduced-form equilibrium with production and default*[10] *if the following conditions are satisfied:*

1. *Asset Markets clear:*

   $\sum_H \theta^h = \sum_{h\in H} \xi^h + \phi^{f(h)}$.

2. *For each* $h \in H$, $\left((x^h, \theta^h, \xi^h); (y^{f(h)}, \phi^{f(h)})\right)$ *maximizes* $U^h(x_o + c_o\xi, x_{-o})$ *subject to budget constraints (1"), (2") and (3").*

3. *For every* j, s, *one has:*

$$\overline{y}_s^j \sum_{h\in H} \theta_j^h = \sum_{h\in H} y_s^{f(h)}$$

[10]Note that we mean "default" because it is agent h who does not honor financial commitments.

4. Item 4 of Definition 3 holds with $\varphi^h = \xi^h + \phi^{f(h)}$. That is:

$$\sum_{h\in\mathcal{S}_s^j}\left(r_s^j - (\overline{y}_s^j + c_{sj})\right)^+\theta_j^h = \sum_{h\in\mathcal{D}_s^j}\left(r_s^j(\xi^h+\phi^{f(h)}) - (y_s^{f(h)} + c_{sj}(\xi^h+\phi^{f(h)}))\right)^+.$$

The right hand side of Item 4 above can be written as:

$$\sum_{h\in\mathcal{D}_s^j}[r_s^j(\xi^h+\phi^{f(h)}) - D_s^j(\xi^h+\phi^{f(h)}, y_s)].$$

We write the Lagrangian associated to the individual's problem defined above as:

$$L = U^h(x_o + c_o\xi, x_{-o}) - \alpha_o^h[x_o + c_o\xi - (y_o - c_o\phi) - \omega_o^h + \pi(\theta-\xi)$$

$$-\pi\phi + \sum_{s\in S}\beta_s\sum_{j\in J}g_s^j(\xi_j+\phi_j,\theta_j,y_s)] - \sum_{s\in S}\alpha_s^h[x_s - y_s - \omega_s^h - c_s(\xi+\phi)$$

$$+\sum_{j\in J}D_s^j(\xi_j+\phi_j,y_s) - \sum_{j\in J}R_s^j\theta_j] - \nu T(y_o - c_o\phi, y_s).$$

where α_o is the Lagrange multiplier of the first period budget constraint (1"), α_s are the Lagrange multipliers of the state-contingent second period budget constraint (2"), and ν is the Lagrange multiplier of the technological constraint (3").

At an *interior solution*, the first order conditions for this problem are:

1. x_o :

$$\partial_{x_o}U^h(x_o^h + c_o\xi^h, x_{-o}^h) = \alpha_o^h \tag{4}$$

2. x_s :

$$\partial_{x_s}U^h(x_o^h + c_o\xi^h, x_{-o}^h) = \alpha_s^h \tag{5}$$

3. ξ_j :

$$\partial_o U^h(x_o^h + c_o\xi^h, x_{-o}^h)c_o - \alpha_o^h(c_{oj} - \pi_j) + \sum_{s\in S}\partial_{\xi_j}g_s^j(\xi_j+\phi_j,\theta_j,y_s)+$$

$$+\sum_{s\in S}\alpha_s^h[c_{sj} - \partial_{\xi_j}D_s^j(\xi_j^h + \phi_j^{f(h)}, y_s)] = 0, \tag{6}$$

4. θ_j :

$$\alpha_o^h\pi_j + \sum_{s\in S}\beta_s\partial_{\theta_j}g_s^j(\xi_j+\phi_j,\theta_j,y_s) = \sum_{s\in S}\alpha_s^h R_s^j, s\in S. \tag{7}$$

5. y_o :

$$\alpha_o = \nu\partial_{y_o}T(y_o^f(h) - c_o\phi^{f(h)}, y_{-o}^{f(h)}) \tag{8}$$

6. y_s :

$$\alpha_s^h \left(1 - \sum_{j \in J} \partial_{y_s} D_s^j(\xi_j^h + \phi_j^{f(h)}, y_s^{f(h)})\right) - \alpha_o^h \beta_s \sum_{j \in J} \partial_{y_s} g_s^j(\xi_j + \phi_j, \theta_j, y_s)$$

$$= \nu \partial_s T(y_o^f(h) - c_o \phi^{f(h)}, y_{-o}^{f(h)}), s \in S. \tag{9}$$

7. ϕ_j :

$$\alpha_o[\pi_j - c_{oj} - \sum_{s \in S} \beta_s \partial_{\phi_j} g_s^j(\xi_j + \phi_j, \theta_j, y_s)] +$$

$$\sum_{s \in S} \alpha_s^h [c_{sj} - \partial_{\phi_j} D_s^j(\xi_j^h + \phi_j^{f(h)}, y_s)] + \nu \partial_o T(y_o^{f(h)} - c_o \phi^{f(h)}, y_{-o}^{f(h)}) c_{oj} = 0. \tag{10}$$

4.4 Separation of Deliveries

Our objective is to define an equilibrium where firms' financial polices imply both the separation of repayments (deliveries) and the separation of the net insurance premium. Hence, from this point, we will focus on analyzing the financial decisions of consumers and firms as borrowers and, therefore, their deliveries. Note that both "default" and "bankruptcy" are the decision of borrowers. However, investors will suffer default whenever either consumers default or firms go bankrupt.

To get there, we need define adequately the firm's problem in a context where their financial decisions will now be separated from its owners'. The following hypothesis allows us to do just that. Namely, to compare the extensive-form equilibrium (to be defined later) with the reduced-form equilibrium and, thus, to understand whether or not the Fisher Separation Theorem holds.

H5 Separation of Deliveries. For any $\phi_j > 0$

$$D_s^j(\xi_j + \phi_j, y_s) = d_s^j \xi_j + D_s^j(\phi_j, y_s) \tag{I},$$

where $d_s^j = \min\{r_s^j, c_{sj}\}$.

There are sufficient conditions (in terms of the economy's fundamentals) for H5 to be satisfied. For instance, if we assume that the maximum quantity of production possibilities of the entire economy (guaranteed by H3 with $L = 1$) is less than the minimum of all the value promised by assets issued. This sufficient condition implies, in particular, that if there is bankruptcy for asset j and state s, i.e. $y_s + c_{sj}\phi_j \leq r_s^j \phi_j$, then there is default for asset j and state s. Thus (I) follows. On the other hand, if there is

no default ($r_s^j \leq c_{sj}$), there will be no bankruptcy, and therefore (I) is satisfied once again.

Remark 3: Notice that there could be default (i.e., $c_{sj} < r_s^j$) without having bankruptcy. This is true since the firm could produce a y_s large enough such that $y_s + c_{sj}\phi_j > r_s^j\phi_j$ so that (I) no longer holds.

An immediate consequence of H5 is that the net insurance premium is given by

$$g_s^j(\xi_j + \phi_j, \theta_j, y_s) = (r_s^j - d_s^j)\xi_j - (r_s^j - R_s^j)\theta_j + \left(r_s^j\phi_j - D_s^j(\phi_j, y_s)\right). \qquad (II)$$

The following corollary is thus straightforward:

Corollary 1. Under H5, *the following relations hold:*

1. $\partial_x D_s^j(\xi_j + \phi_j, y_s) = \partial_x D_s^j(\phi_j, y_s), x \in \{\phi_j, y_s\}$,
2. $\partial_{\xi_j} D_s^j(\xi_j + \phi_j, y_s) = d_s^j$,
3. $\partial_{\phi_j} g_s^j(\xi_j + \phi_j, \theta_j, y_s) = r_s^j - \partial_{\phi_j} D_s^j(\phi_j, y_s)$,
4. $\partial_{y_s} g_s^j(\xi_j + \phi_j, \theta_j, y_s) = -\partial_{y_s} D_s^j(\phi_j, y_s)$,
5. $\partial_{\xi_j} g_s^j(\xi_j + \phi_j, \theta_j, y_s) = r_s^j - d_s^j$,
6. $\partial_{\theta_j} g_s^j(\xi_j + \phi_j, \theta_j, y_s) = -(r_s^j - R_s^j)$.

4.5 Extensive-form equilibrium

When there is no default or bankruptcy, we can define an equilibrium where consumption and production activities are separated. This kind of equilibrium is called "profit-maximising equilibrium" where firms maximize their profits (see Magill and Quinzii, 1996). However, in our setting, due to the presence of collateral and the possibility of bankruptcy, firms, in general, do not maximize profits unless some conditions are satisfied which we will discuss later on.

4.5.1 Separation of Financial Policies and the Firm's problem

Suppose that H5 holds. Then, given the present-value vector $\beta^h = (\frac{\alpha_o^h}{\alpha_o^h}, \ldots, \frac{\alpha_S^h}{\alpha_o^h}) \in R_{++}^{1+S}$, we can formulate the following problem for the firm to solve:

$$\max_{y,\phi} \pi\phi + (y_o - c_o\phi) - \sum_{s\in S}\beta_s \sum_{j\in J}\left(r_s^j\phi_j - D_s^j(\phi_j, y_s)\right) + \sum_{s\in S}\beta_s^h\left(y_s + c_s\phi - \sum_{j\in J} D_s^j(\phi_j, y_s)\right) \qquad (III)$$

s.t .

$$T(y_o - c_o\phi, y_s) \leq 0.$$

That is, the firm $f(h)$ maximizes the present value of all its cash flows discounted by the present-value vector β^h of its owner h.

Note that the term $\sum_{s\in S}\beta_s\sum_{j\in J}\left(r_s^j\phi_j - D_s^j(\phi_j, y_s)\right)$ is the part of the net insurance premium, given by (II), paid by the firm. If we denote by μ the Lagrange multiplier associated to the technological constraint above, the first order conditions for the firm's problem are:

1. y_o :

$$1 = \mu\partial_o T(y_o^f(h) - c_o\phi^{f(h)}, y_{-o}^{f(h)}) \tag{11}$$

2. y_s :

$$\beta_s^h + (\beta_s - \beta_s^h)\sum_{j\in J}\partial_{y_s}D_s^j(\phi_j, y_s) = \mu\partial_s T(y_o^{f(h)} - c_o\phi^{f(h)}, y_{-o}^{f(h)}) \tag{12}$$

3. ϕ_j :

$$\begin{aligned}\pi_j - c_{oj} - \sum_{s\in S}\left(\beta_s r_s^j - \beta_s^h c_{sj}\right) + \sum_{s\in S}(\beta_s - \beta_s^h)\partial_{\phi_j}D_s^j(\phi_j, y_s)\\ +\mu\partial_o T(y_o^f(h) - c_o\phi^{f(h)}, y_{-o}^{f(h)})c_{oj} = 0\end{aligned} \tag{13}$$

Remark 4: Using Corollary 1 we can verify that (8), (9) and (10) together are equivalent to (11), (12) and (13) together. This motivates the following notion of equilibrium where consumption and production activities are separated.

Definition 5. *An allocation* $[(x^h, \theta^h, \xi^h); (y^{f(h)}, \phi^{f(h)})]_{h\in H} \in R_+^{L(S+1)} \times R_+^J \times R_+^J \times R_+^{L(S+1)} \times R_+^J$ *together with* $(\pi, \beta, \overline{y})$ *is said to be an extensive-form equilibrium with default and bankruptcy if the following conditions are satisfied:*

1. *Asset Markets clear:*

$$\sum_H \theta^h = \sum_{h\in H}\left(\xi^h + \phi^{f(h)}\right).$$

2. *For each* $h \in H, (x^h, \theta^h, \xi^h)$ *maximises* U^h *subject to budget constraints (1"), (2") with* $(y^{f(h)}, \phi^{f(h)})$ *taken as given;*

3. *The vector* $(y^{f(h)}, \phi^{f(h)})$ *maximises the firm's problem given in (III)*

4. *Items 3 and 4 of Definition 4 are satisfied under separation of deliveries given by (I) and separation of the payment of the default-insurance premium given by (II).*

Remark 5: Let $\triangle^{f(h)} \in R^{1+S}$ be the dividend policy of firm $f(h)$. If $\triangle^{f(h)}$ is defined as the pair $\triangle_o^{f(h)} = y_o^{f(h)} + (\pi - c_o)\phi^{f(h)} - \sum_{s\in S}\beta_s \sum_{j\in J}[r_s^j \phi_j^{f(h)} - (y_s^{f(h)} + c_{sj}\phi_j^{f(h)})]^+$ and $\triangle_s^{f(h)} = y_s + c_s\phi^{f(h)} - \sum_{j\in J}\min\{r_s^j\phi^{f(h)}, y_s + c_{sj}\phi_j^{f(h)}\}, s \in S$, then we say that portfolio-dividend policy $(\triangle^{f(h)}, \phi^{f(h)})$ balances the investment project $y^{f(h)}$. If we interpret $\triangle^{f(h)}$ as the stream of payments made by the firm $f(h)$ to its owner h, then we can think of Item 2 of Definition 4 as h maximising U^h subject to (1") and (2") where the new income is now $\tilde{w}_s^h = \omega_s^h + \triangle_s^{f(h)}, s = 0, 1, \ldots, S$. Similarly, Item 3 we be viewed as $f(h)$ maximising the present value of its dividends $\sum_{s=0}^S \beta_s^h \triangle_s^{f(h)}$ subject to all portfolio-dividend policies balancing all the feasible invest projects. That is, in this kind of equilibrium, the financial accounts of both firms and their owners ares separated. See Magill and Quinzii (1996) for an ampler discussion on this kind of equilibrium in a setting without default or bankruptcy.

Consequently, if we assume separation in deliveries (H5), one obtains the following theorem:

Theorem 2. *Under separation of deliveries (H5), an extensive-form equilibrium with default and bankruptcy is equivalent to a reduced-form equilibrium with default.*

Proof. The proof of this theorem follows from Remark 2 and the equivalence between the first order conditions associated to h's problem defined in Item 2 of Definition 4 and (4), (5), (6) and (7) together. □

Theorem 2 is the simplest version of the well-known Modigliani-Miller Theorem which states that the precise nature of a firm's financial policy does not matter provided that it finances the firm's productive activities. The result of Theorem 2 (comparison of equilibria), however, depends strongly on the separation of deliveries which fails, for instance, when there is default without bankruptcy. This situation could happen when the collateral value has fallen below the payoff but the level of production is sufficiently high so that $c_s + y_s > r_s$. In this case, there would be no bankruptcy and the separation of deliveries would not be satisfied, making Theorem 2 invalid. Therefore, the financial policies of firms would, in general, be relevant.

4.6 Fisher and Modigliani-Miller

We now discuss the two fundamental results in corporate finance: Fisher's Separation Theorem and Modigliani-Miller's Theorem. In order to get the Fisher Theorem, we need to prove that the reduced-form equilibrium is equivalent to the extensive-form

equilibrium where the objective of the firm is the maximization of profits. To obtain this, it is necessary that, for every asset j, $r_s\phi_j = D_s^j(\phi_j, y_s)$. That is to say, that there is no bankruptcy. Thus, combining (11) and (13) the firm's problem of Definition 4 would reduce to:

$$\max_{y,\phi}(y_o - c_o\phi) + \sum_{s\in S}\beta_s^h y_s$$

s.t .

$$T(y_o - c_o\phi, y_s) \leq 0.$$

which is the legitimate profit maximization problem.

Otherwise, if there is strict bankruptcy (i.e., $\exists j : r_s^j\phi_j > y_s + c_{sj}\phi_j$), then Fisher's separation theorem cannot hold anymore unless the firm goes bankrupt in only one asset j and $\beta_s = \beta_s^h$.

Finally, to obtain the Modigliani-Miller Theorem, which relates to the indeterminacy of the financial policies of firm, we need that ϕ does not affect either the present value of dividends of the firm or the budget set of the entrepreneur. Clearly, this will not be true unless there is no bankruptcy or default as discussed above.

5 A remark on efficiency

We finish by studying the efficiency properties of equilibrium in the model.[11] We compare the equilibrium allocation to other feasible allocations whose portfolios do not necessarily result from trading competitively in asset markets. That is, in alternative allocations, agents pay participation fees that may differ from the market portfolio cost. Equivalently, we allow for (balanced) lump-sum transfers across agents which are added to the usual market portfolio cost.[12] These transfers, a priori, do not necessarily have to coincide with the net default insurance premium considered in our model.

Our key result states that an equilibrium allocation is efficient among all feasible allocations that provide income across states through the same spot prices. That is, given equilibrium prices. Yet, equilibrium allocations might be dominated when spot prices are free to vary and only the structure of asset returns is taken as given. In other words, our result shows that the equilibrium is *constrained* efficient, since a social planner would not be able to eliminate market incompleteness. The concept of constrained *inefficiency*

[11] For an analysis of constrained efficiency in two-period GEI economies with production but no default or collateral constraints see Carvajal and Polemarchakis (2011).

[12] The transfers are *balanced* on aggregate in the sense that they sum up to zero.

implies that market incompleteness can induce allocations that can be improved upon in the Pareto sense if agents acted differently in the same set of (incomplete) markets.[13]

Proposition 1. *Let* $[(\tilde{x}^h, \tilde{\theta}^h, \tilde{\varphi}^h, \tilde{y}^{f(h)})_{h\in H}, (\tilde{p}, \tilde{\pi}, \tilde{\overline{y}}, \tilde{\beta})]$ *be a collateral-production equilibrium. Allocation* $(\tilde{x}^h, \tilde{\theta}^h, \tilde{\varphi}^h, \tilde{y}^{f(h)})_{h\in H}$ *is efficient among all allocations* $(x^h, \theta^h, \varphi^h, y^{f(h)})_{h\in H}$ *for which there are transfers* W^h *across agents and the following items hold:*

1. $\sum_H \left(x_0^h - \omega_0^h - y_0^{f(h)}\right) = 0, \sum_H \left(x_s^h - K_s(\omega_o^h) - y_s^{f(h)} - \omega_s^h\right) = 0, s \in S, \sum_{h\in J} \theta^h = \sum_{h\in J} \varphi^h$.

2. $\tilde{p}_s(x_s^h - y_s^{f(h)} - K_s(x_o^h - y_o^{f(h)}) - \omega_s^h) \leq \tilde{R}_s\theta - \tilde{D}_s(\varphi^h, y^{f(h)}), s \in S$; *where* $\tilde{D}_s(\varphi^h, y_s^{f(h)}) = \min\{\tilde{p}_s A_s^j \varphi_j^h, \tilde{p}_s(y_s^{f(h)} + K_s C_j \varphi_j^h)\}$, *and* $\tilde{R}_s^j = \min\{\tilde{p}_s A_s^j, \tilde{\overline{y}}_s{}^j + \tilde{p}_s K_s C_j\}, j \in J$;

3. $\tilde{p}_o(x_o^h - y_o^{f(h)} - \omega_o^h) + \tilde{\pi}(\theta^h - \varphi^h) + W^h \leq 0$,

4. $C\varphi := \sum_{j\in J} C_j \varphi_j^h \leq x_o^h - y_o^{f(h)}$,

5. $\sum_{h\in H} W^h = 0$, *and*

6. $\tilde{\overline{y}}_s{}^j \sum_{h\in H} \theta_j^h = \sum_{h\in H} \tilde{p}_s y_s^{f(h)}, \forall s, \forall j$.

Proof. Suppose the above does not hold. Say that $(x^h, \theta^h, \varphi^h, y^{f(h)})_{h\in H}$, together with some transfer fraction, W^h satisfies: i) all items in the proposition; ii) $U^h(x^h) \geq U^h(\tilde{x}^h), \forall h \in H$; and iii) $\exists h' : U^{h'}(x^{h'}) > U^{h'}(\tilde{x}^{h'})$. Thus, for h', the first-period budget constraint must be violated. That is,

$$\tilde{p}_o(x_o^{h'} - y_o^{f(h')} - \omega_o^{h'}) + \tilde{\pi}(\theta^{h'} - \varphi^{h'}) + \sum_{s\in S} \tilde{\beta}_s \tilde{g}_s(\theta^{h'}, \varphi^{h'}, y_s^{f(h')}) > 0, \tag{14}$$

From continuity and monotonicity of preferences (H2), we can assume[14] , without loss of generality, that $U^h(x^h) > U^h(\tilde{x}^h), \forall h$. Therefore, (I) holds for all h.

Adding over h in (14), and using Item 1 above, one has that:

$$\sum_{s\in S} \tilde{\beta}_s \sum_{h\in H} \tilde{g}_s(\theta^h, \varphi^h, y_s^{f(h)}) > 0,$$

which implies that $\sum_{h\in H} g_s(\theta^h, \varphi^h, y_s^{f(h)}) > 0$ for some s.

[13] See Dávila et al (2012) in the context of the neoclassical growth model with uninsurable risk.

[14] Notice that under continuity and monotonicity of preferences the concepts of "Pareto efficiency" and "Pareto weak efficiency" are the same. This result can easily extend to the constrained efficiency addressed here.

Substituting $\tilde{D}_s(\varphi^h, y_s^{f(h)})$ and $\tilde{R}_s^j$ from Item 2 above into $\tilde{g}_s(\theta^h, \varphi^h, y_s^{f(h)})$, yields:

$$\sum_{h\in H} \tilde{R}_s\theta^h > \sum_{h\in H} \tilde{D}_s(\varphi^h, y_s^{f(h)}). \tag{15}$$

Item 2 and (15) imply that

$$\tilde{p}_s \sum_{h\in H} (x_s^h - y_s^{f(h)} - K_s(x_o^h - y_o^{f(h)}) - \omega_s^h) = \sum_{h\in H} \tilde{R}_s\theta^h - \sum_{h\in H} \tilde{D}_s(\varphi^h, y_s^{f(h)}) > 0.$$

But this yields $\sum_{h\in H}(x_s^h - y_s^{f(h)} - K_s(x_o^h - y_o^{f(h)}) - \omega_s^h) > 0$, for some s, contradicting Item 1. Hence, the result follows.

□

The proposition then shows that the weakly constrained efficient transfer among agents is given by the equilibrium net insurance premium. Thus, following Coase's (1960) externalities approach, the inefficiency due to default can be corrected when a market for default externalities in each state of nature is created with prices β_s. Although the inefficiency due to default can been corrected, the one caused by market incompleteness cannot be corrected by a social planner as in the classic GEI literature. This makes the equilibrium allocation become dominated when spot prices are free to differ from the equilibrium levels. The inefficiency due to the incompleteness of asset markets could potentially be mitigated by making the asset structure in our model endogenous, but we do not explore this subject here.

6 Conclusions

We present an incomplete markets general equilibrium model of default and bankruptcy in an setting involving exogenous collateral and, importantly, endogenous production. The collateral applies to all the commodities available for consumption and production in the economy. Because the owners of firms have unlimited liability, in case of default, lenders can seize the value of collateral and future production. This makes deliveries and default/bankruptcy decisions endogenous. Because of this endogeneity, we make receipts anonymous in the sense that lenders have a portfolio of assets issued by producers but cannot anticipate the identity of the firm(s) defaulting. Thus, lenders anticipate a default rate that depends on the average level of production (default is counter-cyclical). That is, we assume a type of tranching because lenders, in case of default, only expect to receive a fraction of the output of the firm plus the value of collateral. Finally, in order to avoid a zero production equilibrium, we introduce an default insurance mechanism.

Within this setting we study the financial policies of the firm. We conclude that the two key theorems in corporate finance, namely Fisher's Separation Theorem and the Modigliani-Miller Theorem, are not satisfied unless we impose very restrictive assumptions on the trading of portfolios and on the way borrowers repay their debts. Thus, in general, the separation between the financial decisions of firms and their owners is not possible, and the financial policies of firms will be relevant for real outcomes in the economy.

We also show that, under usual assumptions on utilities and technologies our economy has a non-trivial equilibrium. In analysing the non-triviality of equilibrium, we show that a high aggregate level of economic activity could eliminate any investors' pessimism. Finally, we show that, because of the existence of an insurance premium, the resulting equilibrium is constrained efficient.

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