Quantum Surfing

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Abstract

Here we report a project in which time-dependent supersymmetry has been employed to derive a new potential and eigenfunctions that satisfy the Schrödinger equation. The supersymmetry method is outlined and we apply it to a wavefunction obeying the free-particle Schrödinger equation. This leads to an exactly soluble model in which a quantum particle is seen to "surf" on a time-dependent potential. The model can be solved and understood within both classical and quantum mechanics and the relationship between the two approaches is discussed. The mathematics of this formalism is accessible to a final year British undergraduate making supersymmetry derived Hamiltonians suitable as a final year theoretical physics research project.

I. INTRODUCTION

The Schrödinger equation forms the foundation of non-relativistic quantum mechanics. It has only very few physically meaningful exact solutions. Most of the familiar models in quantum theory are steady-state solutions which means that the space and time dependence of the problem can be separated. These, such as the harmonic oscillator and the one-electron atom form the basis of much of our understanding of the physics of nature. There are a few known solutions that are not steady state and which have some unusual properties such as being self-accelerating [1].

Non-stationary supersymmetric quantum theory has been derived [2–4] and extended [5] and provides a strategy for finding new solutions of the time-dependent Schrödinger equation if we know one solution. This work is a natural extension of the time independent supersymmetric methods discussed in very readable form by Cooper *et. al.* [6]. It is a very powerful technique but its implementation has been limited so far, Bagrov *et. al.* performed a number of examples in their papers [2–4] deriving the method, although these contain little physical interpretation of the results, and Zelaya and Rosas-Ortiz [7] have found interesting new potentials starting from the harmonic oscillator.

"Surfing" on the microscopic level has become an important idea in a number of areas of science. One example is rf-surfing where low energy ions are transported using inhomogeneous alternating electric fields. This method has been proposed [8], experimentally demonstrated [9] and is now in use [10] with direct applications in the development of linear accelerators [11]. A second example is electrons surfing on sound waves. Hermelin *et. al.* [12] and McNeil *et. al.* [13] demonstrated how electrons can surf on acoustic waves while remaining isolated from their surroundings with very high efficiency, which has implications for quantum information processing [14]. Such devices are under construction [15]. Another example is membrane induced hydrostatic migration of particles surfing on their own waves as discussed by Rallabandi *at. al.* [16]. They showed that self-surfing plays a role in small particles sedimenting under the influence of gravity close to an elastic membrane. The coupling between fluid flow and soft elastic surfaces is ubiquitous in biosciences and they also describe how surfing plays a role in the migration of bacteria while Horsthemke *et. al.* [17] describe bacteria surfing towards the body of a cell.

It is self-evident that surfing at the microscopic level plays a key role in many branches of science. Many of these effects occur at the level of atoms/particles which means that there is a need for a quantum mechanical theory describing surfing. In this paper we write down such a model theory. We make use of non-stationary supersymmetry to derive a potential and its associated wavefunction for the Schrödinger equation and explore its properties. We show that the chosen potential describes a particle that is effectively "surfing" close to the crest of the potential. This model is analysed within both quantum and classical mechanics and the connection between both approaches is discussed. Throughout this paper we will retain constants in equations, but diagrams and tabulated values are in atomic units with $m = 1/2$, $\hbar = 1$.

II. CALCULATION

Non-stationary supersymmetric quantum mechanics essentially involves employing a specific time-dependent transformation to reconstruct the Schrödinger equation. This procedure amounts to a process for finding potential and eigenfunctions that satisfy a new Schrödinger equation from the potential and eigenfunctions of a known Schrödinger equation. Furthermore such an approach defines a hierarchy of solutions. Once we have found the new potential and wavefunction we can use them as the input to a subsequent supersymmetry transformation. Here we outline the method, but refer the reader to the original literature for the calculational details [2–6]. Consider two different time-dependent one-dimensional Schrödinger equations

$$\begin{aligned} \left(i\hbar\frac{\partial}{\partial t} - \hat{H}_0\right)\psi(x,t) &= 0 \\ \left(i\hbar\frac{\partial}{\partial t} - \hat{H}_1\right)\phi(x,t) &= 0 \end{aligned} \tag{1}$$

with

$$\hat{H}_i = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_i(x,t) \tag{2}$$

Now we are going to postulate that an operator $\hat{A}$ exists such that

$$\hat{A}(i\hbar\frac{\partial}{\partial t} - \hat{H}_0)\psi(x,t) = (i\hbar\frac{\partial}{\partial t} - \hat{H}_1)\hat{A}\psi(x,t) \tag{3}$$

From equation (1) the left hand side of this is zero and so the right hand side must also be zero which implies

$$\phi(x,t) = \hat{A}\psi(x,t) \tag{4}$$

It turns out that such an operator does exist and it is written as a function of x and t as

$$\hat{A} = \hat{A}_0(x,t) + \hat{A}_1(t)\frac{\partial}{\partial x} \tag{5}$$

Here we find that $\hat{A}_1(t)$ has units of distance, while $A_0(x,t)$ is dimensionless and is given by

$$\hat{A}_0(x,t) = -\frac{1}{u(x,t)}\frac{\partial u(x,t)}{\partial x}\hat{A}_1(t) \tag{6}$$

and $u(x,t)$ is known as a transfer function and is a distinct solution of the same Schrödinger equation as $\psi(x,t)$. Then

$$V_1(x,t) = V_0(x,t) + i\hbar\frac{1}{\hat{A}_1(t)}\frac{\partial \hat{A}_1(t)}{\partial t} - \frac{\hbar^2}{m}\frac{\partial^2}{\partial x^2}(\log u) \tag{7}$$

$\hat{A}_1(t)$ is essentially arbitrary, but can be chosen to find the representation in which $V_1(x,t)$ is real if such a representation exists. Then

$$V_1(x,t) = V_0(x,t) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(\log|u(x,t)|^2) \tag{8}$$

We now have all we need to calculate new solutions of the Schrödinger equation from known solutions. The procedure is as follows. We choose two known solutions of the upper of equations (1) as $\psi(x,t)$ and $u(x,t)$, and the corresponding potential $V_0(x,t)$. First we calculate $V_1(x,t)$ from equation (7) and if it is not real we choose a value of $\hat{A}_1(t)$ to make it real. If this is not possible the calculation may be mathematically interesting, but there is very unlikely to be any physical applications of the results. Next we calculate $\hat{A}_0(x,t)$ from equation (6) and then $\hat{A}(x,t)$ from equation (5) Finally we find $\phi(x,t)$ from equation (4) and that is the wavefunction corresponding to the potential $V_1(x,t)$. This completes the calculation because that $V_1(x,t)$ and $\phi(x,t)$ are the potential and solutions of the lower of equations (1).

We have started from the following solution of the time-dependent free particle Schrödinger equation.

$$\Psi(x,t) = \sqrt{\frac{1}{n!}}\left(\frac{m}{\hbar\tau\pi}\right)^{1/4}\frac{2^{-n/2}}{(1+t^2/\tau^2)^{1/4}}e^{\left(-\frac{m\tau x^2}{(2\hbar(t^2+\tau^2)}\right)}e^{\left(-i(n+1/2)\arctan(\frac{t}{\tau})\right)}e^{\left(\frac{imx^2t}{2\hbar(t^2+\tau^2)}\right)}H_n\left(\left(\frac{m\tau}{\hbar(t^2+\tau^2)}\right)^{1/2}x\right). \tag{9}$$

The earliest reference to this wavefunction we have found is by Miller [19]. It has been discussed since by Bagrov *et. al* [3], Guerrero *et. al.* [20, 21] and Strange [22]. The symbols have their usual meanings. τ is a positive constant with dimensions of time. Throughout we have set $\tau = 1$. H_n is a Hermite polynomial and n is a non-negative integer quantum number.

We have chosen $u(x,t) = \Psi(x,t)$ with one value of the quantum number n, $\psi(x,t)$ as $\Psi(x,t)$ with a different value of the quantum number, l say, and $V_0(x,t) = 0$. The supersymmetry transformation then yields a singular potential and wavefunction which both depend on the numerical values of both the quantum numbers. We have then used this potential and wavefunction as the new $V_0(x,t)$, $u(x,t)$ (choosing $n = 1, l = 2$) and $\psi(x,t)$ (choosing $n = 1, l = q$) for a second transformation. It might be thought that choosing either $n = 0$ or $l = 0$ would yield simpler results, but it produces a flat and uninteresting potential with a stationary spreading eigenfunction. For both transformations $\hat{A}_1(t) \propto \sqrt{t^2+\tau^2}$, the actual constant of proportionality can be determined numerically to normalise the new eigenfunction. The resulting potential only depends on the quantum numbers associated with $u(x,t)$ and is given by

$$V(x,t) = \frac{2\tau\hbar(4m^2x^4\tau^2 + 8mx^2\tau(t^2+\tau^2)\hbar - (t^2+\tau^2)^2\hbar^2)}{(t^2+\tau^2)(2mx^2\tau + (t^2+\tau^2)\hbar)^2}. \tag{10}$$

This potential retains its shape, but stretches and decreases in magnitude with increasing time as shown in Figure 1. Note that time only appears quadratically so the potential does not distinguish between positive and negative times.

A. Quantum mechanical solution

The supersymmetry transformation yields many wavefunctions corresponding to the potential above. This requires the third quantum number q which must differ from the other quantum numbers used in the supersymmetry transformation. As expected the ground state ($q = 0$) has a particle located close to the bottom of the central potential well. Here we examine the first excited state ($q = 3$) which has wavefunction

$$\phi(x,t) = \sqrt{\frac{m\tau}{3\hbar(t^2+\tau^2)}}\left(\frac{m}{\tau\hbar}\right)^{5/4} e^{\left(\frac{i}{2}\left(\frac{mx^2}{(t-i\tau)\hbar} - 7\arctan\left(\frac{t}{\tau}\right)\right)\right)} \frac{4x\tau^2(2mx^2\tau + 3(t^2+\tau^2)\hbar)}{(\pi + \pi t^2/\tau^2)^{1/4}\,(2mx^2\tau + (t^2+\tau^2)\hbar)} \tag{11}$$

The associated probability density is also shown in Figure 1. At $t = 0$ it has a maximum just below the potential

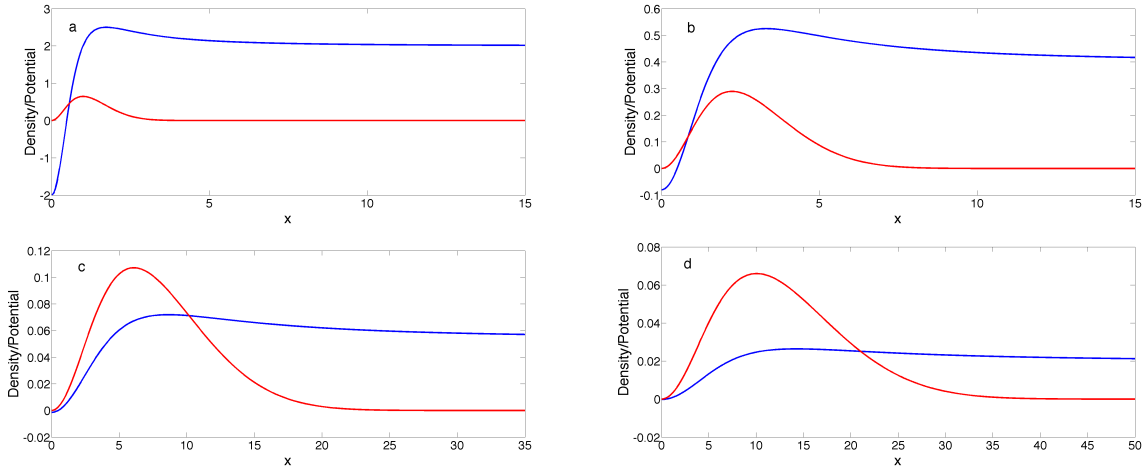

FIG. 1: The potential of equation (10) (blue line), and the probability density of equation (11) (red line) at times: a $t = 0$; b $t = 2$; c $t = 6$; d $t = 10$. The potential and probability density are symmetric in the coordinate x so we only show results for $x \geq 0$.

peak. This is characteristic of many of the excited states of this potential. They are localised and normalisable, but have a peak in a region of relatively high potential. In quantum mechanics we expect the wavefunction to be small close to potential peaks. To decide whether or not this behaviour is unusual we note that at small x the potential is harmonic so we may compare these states with those of the harmonic oscillator. The ground states in both cases are similar. However the first excited states are not. In the harmonic oscillator there is about 80% probability of finding the particle inside the classically allowed region, while for the present case it is only 32%. Inside the well this wavefunction is unlike any harmonic state. Thus, the behaviour of a particle in this potential is generally very different from a harmonic oscillator.

As we see in Fig. 1 the probability density maximum remains in the same position relative to the potential peak at all times, it moves along with the potential. This is unexpected given that $\partial V/\partial x$ is always positive at the peak in the probability density indicating that there is a force towards the origin, albeit a rapidly decreasing one.

A probability density map is shown in Figure 2. The density is symmetric about $t = 0$. Clearly the particle moves towards $x = 0$ at negative times and back towards $x = \infty$ for $t > 0$. Both the potential and density are only well localised for a short time around $t = 0$. It is clear that the probability density spreads rapidly, as is typical of quantum mechanical wavepackets. However the rate of broadening is determined by the potential.

We now show expectation values for kinematic properties of the particle for $t \geq 0$. For $t \leq 0$ the magnitudes and formulae remain the same, but, numerically, $< p >$ and $d < x > /dt$ change sign. The standard expectation values of position and momentum are zero from the symmetry of the system. For more meaningful numbers we note that this is a single particle calculation and that particle moves either to the left or right for $t > 0$. If we assume it moves to the right we can renormalise the probability density in the $x \geq 0$ region and calculate observables for that region only. In Table I we show these values. These confirm our picture of a particle moving away from the origin. With a rapidly decreasing potential the particle moves with a velocity that tends to a constant asymptotically. Evidently $< p >= m\, d < x > /dt$ as required. The numbers in this table are fit very accurately with the simple formulae that are shown in the last line of Table I. In Table II we show energetic quantities for this particle. Energy is not conserved as the potential is time-dependent and the expectation values of the kinetic and potential energies both decrease with time. We note that as time increases the expectation value of kinetic energy decreases while the magnitude of the

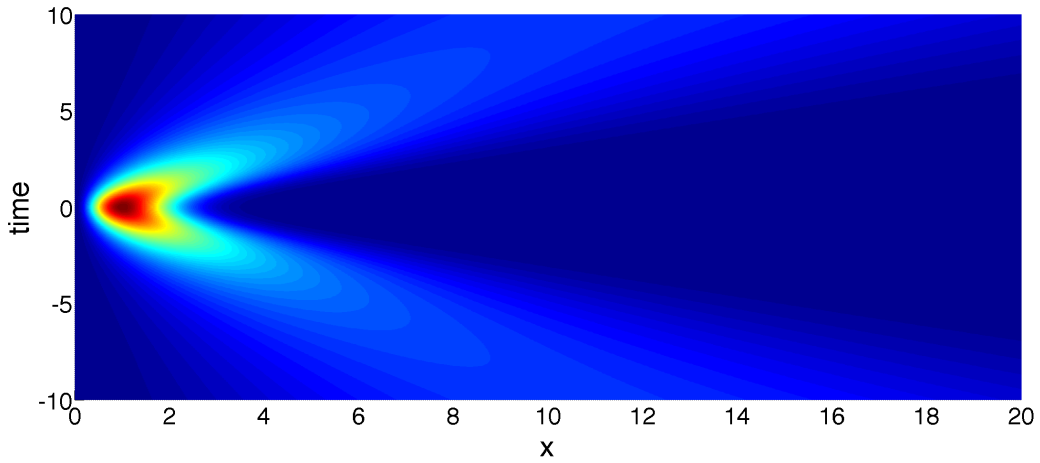

FIG. 2: The probability density for positive and negative times for the wavefunction of equation (11), for positive x only. Clearly the particle moves towards the origin for $t \leq 0$ and away from the origin at $t \geq 0$

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time	$< x >$	$d < x > /dt$	$d^2 < x > /dt^2$	$< p >$
0	1.31	0.00	1.31	0.00
2	2.93	1.17	0.117	0.586
4	5.40	1.27	0.019	0.635
6	7.96	1.29	0.0058	0.645
8	10.56	1.30	0.0025	0.650
10	13.16	1.30	0.0013	0.651
	$1.31(t^2+\tau^2)^{1/2}$	$\frac{1.31t}{(t^2+\tau^2)^{1/2}}$	$\frac{1.31\tau^2}{(t^2+\tau^2)^{3/2}}$	$\frac{0.655t}{(t^2+\tau^2)^{1/2}}$

TABLE I: The expectation values of position, velocity, acceleration and momentum as a function of time, with $\tau = 1$.

momentum increases. Classically this isn't possible, but quantum mechanically the momentum is dependent on the gradient of the wavefunction while the kinetic energy is dependent upon its curvature, so the two are not directly related.

This model can be interpreted as a potential that is stretching and decreasing in magnitude which is carrying a particle along with it.

time	$< \frac{\hat{p}^2}{2m} >$	$< \hat{V}(x) >$	$< \hat{E} = i\hbar\frac{\partial}{\partial t} >$
0	1.14	1.83	2.97
2	0.65	0.37	1.02
4	0.57	0.11	0.68
6	0.55	0.05	0.60
8	0.54	0.03	0.57
10	0.54	0.02	0.56

TABLE II: The expectation values for energies as a function of time.

B. Classical mechanical solution

Now we turn to a description of the same system in terms of classical mechanics. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{2\tau\hbar(4m^2x^4\tau^2 + 8mx^2\tau(t^2+\tau^2)\hbar - (t^2+\tau^2)^2\hbar^2)}{(t^2+\tau^2)(2mx^2\tau + (t^2+\tau^2)\hbar)^2} \tag{12}$$

This potential is simplified if we assume

$$x = \frac{y}{\tau}\sqrt{t^2+\tau^2}. \tag{13}$$

with y =constant. This limits the available solutions to the equations of motion, but is consistent with the quantum mechanical formula for $< x >$ shown in Table I. With this assumption Hamilton's equations yield the following equation of motion:

$$\frac{d^2x}{dt^2} = \frac{4\tau^3\hbar(4m^2y^4 + 8my^2\tau\hbar - \tau^2\hbar^2)}{ym\tau(t^2+\tau^2)^{3/2}(2my^2+\tau\hbar)^2}.$$

This can be integrated trivially to give

$$y^2 = \frac{4\tau\hbar(4m^2y^4 + 8my^2\tau\hbar - \tau^2\hbar^2)}{m(2my^2+\tau\hbar)^2}$$

Integration constants are zero for consistency with Equation (13). Values of y that satisfy this expression are

$$y_1 = \pm 0.496753 \qquad y_2 = \pm 3.042346$$

Then substitution into Equation (13) yields the motion as a function of time in terms of the original coordinate x. It is easiest to distinguish between positive and negative times to understand these solutions.

At negative times the y_1 solution represents a particle ahead of the wave crest falling towards the origin, but the increasing depth of the well exerts a greater opposing force, decelerating the particle and preventing it reaching the bottom of the well. This results from a near cancellation of the force away from the origin, due to the rapidly increasing potential, by $\partial V/\partial x$ which is directed towards the origin. y_1 represents the point at which this cancellation results in an acceleration equal to that of the potential peak. The particle comes to rest instantaneously at $t = 0$. At positive times the particle is behind the wave peak and the forces retain the same time dependence while the velocity changes sign, and the particle accelerates away from the origin, but the collapsing potential means the force becomes small quickly and the particle attains a constant velocity asymptotically.

At $t \to -\infty$ the y_2 solution moves with constant velocity towards the origin behind the potential peak. As t increases towards zero the potential grows and the contributions to the force from $\partial V/\partial x$, and the much larger contribution due to the potential increasing with time, both point away from the origin causing the particle to decelerate. The peak in the potential also decelerates and this value of y represents the point at which the the force due to the changing potential and $\partial V/\partial x$ add to give a deceleration equal to that of the potential peak. The particle comes instantaneously to rest at $t = 0$ and then the forces accelerate the particle away from the origin, again at the same rate as the potential peak moves away from the origin. For $t > 0$ the potential collapses rapidly leading to a rapid decrease in the forces and again the particle attains a constant velocity asymptotically.

Both solutions occur at both positive and negative times because the potential depends on t quadratically. There are equivalent solutions on the negative side of the x-axis. At negative times the y_1 solution is motion analogous to that of a classical surfer while at positive times the y_2 solution is similar to a classical surfer. At positive times the y_1 solution, and at negative times the y_2 solution, are not surfing in a classical sense. However these solutions exist and represent a particle being pulled along by the wave, but behind the crest rather than in front of it. One could describe this as a generalised form of surfing. Finally in this section we point out that we have found some solutions of the equations of motion, but they are subject to the limitation of equation (13). There is likely to be other more complex solutions when this restriction is lifted.

III. DISCUSSION

The non-stationary supersymmetry method has led to the potential shown in Figure 1. This potential broadens at a rate determined by the broadening of the initial wavefunction, (Eq 9) which we used to initiate the supersymmetry

calculation. It also leads to a solution of the new Schrödinger equation which broadens, but now the broadening is controlled by the potential. The peaks in the particle probability density and the expectation value of position are are close to the crest of the potential. They remain in the same position relative to the potential peak at all times.

Classical analysis yields two trajectories caused by balancing of forces due to the rate of change of potential with position and the rate of change of potential with time. At negative times one classical trajectory, and at positive times the other, may be regarded as classical surfing. The reverse cases can be regarded as a generalised form of surfing.

The quantum theory produces a single probability density peak with the same time dependence as the potential and sits just below its peak. This peak can also be regarded as surfing in a generalised sense and we surmise that, at least as a first approximation, it represents a weighted average of the classical behaviours. Hence we can claim to have a model describing quantum surfing.

We have examined the simplest solution of the Schrödinger equation with the potential of equation (10) which displays surfing. The ground state has the particle around the centre of the potential well at all times. The state studied here is the first excited state. Increasing the third quantum number to $q > 3$ gives higher energy states of this potential which have more nodes and peaks. Inside the potential well many of these states are what we would expect based on our knowledge of the harmonic oscillator, but some are not. In either case they are frequently matched onto localised functions outside the potential well. This means there is often a substantial probability of finding the particle "surfing" outside the well. In quantum mechanics wavefunction broadening is ubiquitous and causes difficulties in the interpretation of quantum phenomena in terms of classical physics. Here we have been able to make use of the broadening to develop an analytic quantum mechanical model of surfing, which is physics that is known classically, but for which no first principles quantum model has previously been recognised. Unfortunately, at present it seems not possible to predict which initials states will yield interesting potentials and wavefunctions after a supersymmetry transformation and so finding them is just a trial and error process.

We recommend time-dependent supersymmetry as an excellent starting place for British undergraduate final year projects. This procedure of deriving a new Hamiltonian and eigenfunctions for the Schrödinger equation requires detailed study of very mathematical research papers[3, 5] which deepen and reinforce understanding of some of the formal aspects of quantum theory. This project involved the use of Matlab and Maple or Mathematica and so students gain experience in applying mathematical software and finally calculation and exploration of observables requires students to perform some original mathematics and to think creatively about its interpretation. This paper reports an example of such a project.

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