Three Essays in Financial Networks and Games

by

Dike Chukuwdi Henry

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Abstract

Chapter 1: Default and Punishment with Systemic Risk

This essay identifies substituting behaviors in an ex-ante financial system with cyclical default/feedback leading to potential breakdown. Here, firms in a debt network estimate potential default and as such, makes storage decision in order to avoid defaulting. They make these decision due to the potential damages associated with defaults. In doing so, firms optimize their savings strategy given their network/neighborhood effect. We observe properties and existence of a Nash equilibrium under instances where such ex-ante breakdown are caused in part by each owing firm. Equilibrium storage is dominated by the fraction of default not attributed to contagion. We also see a link between firms position in a default network and its storage. As a policy tool, equilibrium under harsh punishment are also socially efficient in achieving minimal default and systemic breakdown.

Chapter 2: Frictions in Financial Networks

This essay models transaction cost within an Eisenberg and Noe (2001) clearing system and identifies such clearing properties such as existence, uniqueness and methods of clearing. Further more, it adapts such transaction cost for decision making into the Demange (2016)'s threat index with default feedback and observes the behavior of the index/centrality rankings of each under changing transaction costs. We find under strict conditions, existential possibilities of switching in such rankings for firms involved.

Chapter 3: Strategic Interactions in Financial Networks

This essay models interactions of firms in a pre-trading(fixed network of lending/borrowing) period whereby firms set fixed lending rates given loan management cost. We show strategic substitution in the rate each firm sets and more fundamentally, propose that the rates charged to debtors by a creditor firm is likened to results from a private provision of public good in networks game. We then highlight specific core-periphery network properties in relation to interdependence and Nash rate charged by firms. For welfare policies, we find neutrality of intervention policies that create or reduce transaction cost and improvement based on policies that provide administrative subsidies thus creating an avenue for cost effective resource transfer policy. Lastly, we find significant relationship between a firms centrality measured by weaker negative externality and welfare improvement due to such subsidy.

Thesis Supervisor: Allouch Nizar.
Title: Professor of Economics

Thesis Supervisor: Alastair Bailey.
Title: Professor of Economics

Thesis Supervisor: Duncan Alfred.
Title: Lecturer in Economics

Declaration

I wish to declare that the Chapter 1 is jointly authored with my supervisor, Prof Nizar Allouch and Dr. Alfred Duncan. Additionally, an earlier version of this Chapter has been presented at several conferences including the SAET 2018 conference at The Academica Scinica, Taipei, Taiwan.

Additionally Chapter 2 is also authored alongside Prof Nizar Allouch.

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Contents

List of Figures

List of Tables

Chapter 1

Default and Punishments with Systemic Risk

1.1 Introduction

A reliably functioning financial system is one of the prominent goals monetary authorities of economies around the world struggle to maintain. Numerous incidents have left key players in the global financial sector at the brink of collapse or in worse cases, a systemic failure for a certain period.[1] There are however few exceptional cases in which early warning signals of such collapse spurred stakeholders to proactive interventions thereby, managing to elude a major catastrophe. Primary examples of market stabilisation through intervention by private firms include the case of Panic of 1907 (New York) . As explained by Tallman, Moen, et al. (1990), while signs of chaos were already looming on the streets of Manhattan, J.P Morgan, one of the greatest bankers at the time, decided to gather the big banks and manage to persuade them into signing for funds release to avert a potential crises. It was said that the amounts contributed was trivial in comparison to what the actual

[1] Popularly the 1930's and 2008 financial crises.

crises would have been. Also, another earlier incident with similar characteristics was the Barings bank collapse (1890, London) which originated in Argentina but its fear quickly escalated to other parts of the world.[2] In this case, the Bank of England was forced to lobby and succeeded in pulling in resources from numerous European powers to salvage the British financial market. Finally another important case was an incident with Long Term Capital Management (1998, New York) where, as Jorion (2000) recounts, the President William McDonough managed to convince 15 banks to bail out the Long-term Capital Management which had almost collapsed due to the devaluation of the Russian Rubles among many other reasons.

We build a model within the clearing system proposed by Eisenberg and Noe (2001) with a baseline model of systemic risk through default contagion. As a step forward, we uniquely infuse punishments as means of enforcing redemption of promises. These punishment vary directly to the default value of each firm which are treated in isolation, Where firms are able to estimate these defaults reliably, we grant the option of unlimited access to storage in order to avoid default. Our motivation links to the fact that studies have shown that defaults are damaging to firms and has even greater indirect consequences than its face value. Defaults and systemic breakdown has been proven to be significantly detrimental to firms involved. An example of cost linked to firms default can be found in the study carried out by Strebulaev and Zhao (2011) . Here, they used large data sourced from Default Risk Service (DRS) on firms that defaulted over 14 years in terms of observed prices of debt and equity. They found cost to be on average, as high as 21.7% of the market value of assets. Break down of these cost ranged from 14.7% for bond renegotiations to 30.5% for bankruptcies. We however for ease modify these assumptions immensely as would be explained in the next section. Studies like Glover (2016) carry out similar survey.

This work adapts financial networks contagion into a decision making model for

[2]see Mitchener and Weidenmier (2008).

the purpose of extracting public good behaviors from games of substitute in networks. This is so as to observe who(s) contributes to financial stability within a system and possibly to what extent would they be required to. Intervention by stakeholders which are not necessarily 3rd party regulators like the federal government can then be said to be of importance especially when signs of failures are imminent. In a lot of instances, it has been those private banks who have been convinced of possible loss (in the event the crises is allowed to proceed) that end up making attempts to help the situation. It has then been argued by some economists that the absence of similar intervention in the 2007 lead to the 2007-2008 financial crisis which at the end, still elicited interventions from the European powers to salvage failing economies like Greece, Portugal, Spain and the likes since the Eurozone Crises. In the field of behavioral financial economics, the agent based model have seen growth in not only depth, but breath as it has extended to the dimension of looking at participants in the financial system as interwoven networks whose interactions and activities has huge impact on the overall performance of system.

We show existence and uniqueness of Nash equilibrium and its link to the first wave default described in Eisenberg and Noe (2001). Equilibrium behavior of firms arise from an interaction based on strategic substitution such that firms benefit from storage decisions from their direct debtors. As such, equilibrium behavior are well linked to those found in public good games works such as Bergstrom, Blume, and Varian (1986), Bramoullé, Kranton, and D'Amours (2010), Allouch (2015), etc. Liquidity provision in a fragile system has been identified as possessing public good characteristics (Buiter, 2008). One which can be managed privately (by hoarding inherently liquid assets). Furthermore, Buiter (2008) suggested it would not be socially efficient for private banks and other financial institutions to hold liquid assets on their balance sheets in amounts sufficient to tie them over when markets become disorderly[3]. This suggests the ability for banks and other financial institutions to provide liquidity as a pubic good in the face of growing necessity. Also, we see a relationship between a firms position in a well bounded default network and the amount to which it stores.

For welfare properties, our Nash equilibrium is Pareto efficient and maximum social efficiency would be based on higher punishments. Lastly, we reveal instance of substitution behavior and equilibrium outcome which may link to the first wave default in instance where cost where convex to default amount of each firm. An intuition from this links to the fact that stochastic defaults of a firm are not substituted for and as such, when each firm focuses on covering for risk arising outside the network, then the system becomes increasingly less fragile. This provides a link between social optimality as firms individual choices ensures the stability in the system.

1.1.1 Related Literature

Financial networks and contagion has become an increasingly targeted topic for numerous researches since the start of the 21st century. Especially since 2008 events. This paper is built around the Eisenberg and Noe (2001) framework which set up a standard networks bounded by existing obligations which clear at a particular period. As its primary focus, clearing characteristics of the model was described while various methods of computing for the clearing payment was proposed, most notably, the fictitious default algorithm. This algorithm solves for the sequence by using a Gauss elimination approach from a starting point that all firms in the network presumably pay in full. Existence and uniqueness of equilibrium was established under mild conditions using fixed point theorems. One very important characteristic of this model was the concept of *Waves of Default* where other firms might fall short because some other connecting firm fell short. While the proposed sequence is fictitious

[3] In essence, characteristics of a looming crises.

in nature, intuitions can be very important for understanding contagion as well as strategic interactions in such network which we explore in this work. Additionally, the model gives us an initial insight into a prototype financial system financial system in which defaults exhibit cyclical behaviors. The clearing algorithm is such that so far as iterations stop in a total sequence less than the paying nodes, then systemic risk is avoided. In a case that it runs above, then evidence of feedback and as such, systemic risk is revealed.

As a modification to the Eisenberg and Noe (2001), Elsinger, Lehar, and Summer (2006) famously explores networks where certain agents are deemed of greater importance to others. Similar to the baseline model,[4] conditions for clearing and further characteristics where established. Also possible hints for its computations. Building further, Elliott, Golub, and Jackson (2014) explored strategic interaction and impact of shocks to networks with a primary focus on the dilemma between integration, diversification and contagion. Some important results reached is one that points to the inefficiency of networks due to high amount of one and lower amount of the other[5]. One form of these shocks (as it relates to the concept of first wave default in our model) can arise due to stochasticity of returns. Broadly, they are product of market irregularities which might affect the cashflow of firms. They as such, constitute the part of default which can arise to a firm outside his network interlinks. In the network, they materialise as contagion. In the light of contagion, Amini, Filipović, and Minca (2016) looked at interactions in an instance whereby firms facing default are forced to liquidate some amount to cover up and these liquidation is inversely proportional to the price of the assets so that the more assets, the lower its worth. It explored equilibrium amount of liquidation. A more general setting of this work was similarly covered by Feinstein (2017).

An important work on penalty in equilibrium analysis Dubey, Geanakoplos, and

[4]By baseline, we mean the Eisenberg and Noe (2001) model

[5]Integration or Diversification.

Shubik (2005) who famously captured punishment in general equilibrium analysis. Key assumptions of the model that we drift away from includes the notion that punishment may or may not occur even in the event of default, also agents where not aware of other agents commitments and as such, could not estimate a network. These limited information gave room for adverse selection and moral hazards in the network and also, punishment varied from contract to contract. Furthermore, we see in Demange (2016) a statement on destructiveness of systems and the ability for agents who cause such systemic risk to ignore the full effect of his actions on the system and as such aim to free-ride the provision of others to guarantee stability. We aim to, using our build up, to investigate into these kinds of behaviors. This links to the fact that *Defaults* due to ability for contagion/cascade in networks brings in characteristics of *Public Goods*. Public goods in networks with strategic interactions have been famously captured by Bramoullé et al. (2010) while private provision of such public goods was the focus of Allouch (2015). Best replies in both cases exhibit linearity (corresponding to those in Bergstrom et al. (1986), Bramoullé et al. (2010) , as well as Allouch (2015) to mention but a few) and also ability to free-ride contributions depends on the level of influence (using reverse centrality measures indicating vulnerability) on the network which otherwise would have led to what Allouch (2015) grouped as *specialist* and *free riders*. Uniquely in our model, best replies indicate that all players end up being specialist such that complete free-riders are avoided. Our equilibrium also reveals similar intuitions to components of the best replies of a default game given in Allouch and Jalloul (2016). However there are subtle differences as we discuss in subsequent sections.

1.2 The Model

We consider a two period economy. First the initial period which we denote as $t = 0$, and then the payment period denoted as $t = 1$. Let $\mathcal{N} = \{1, \ldots, n\}$ be a set which captures every firm that makes up the economy (nodes). At $t = 0$, firms are exogenously faced with existing promises of debt and repayment obligations to other firms and we denote the total obligation every firm $i \in \mathcal{N}$ has as L_i. For each firm $i \in \mathcal{N}$, let its set of debtors and creditors be $\mathcal{N}_i$ so that $\mathcal{N}_i = \{\mathcal{N}_i^{in} \cup \mathcal{N}_i^{out}\}$.[6] We denote the total portion of firm i's obligation that goes to firm j as L_{ij} and the relative liability of firm i to firm j as g_{ij} such that $g_{ij} = \dfrac{L_{ij}}{L_i}$. Also given $\sum_{j \in \mathcal{N}_i^{out}} L_{ij} = L_i$, then $\sum_{j \in \mathcal{N}^{out}} g_{ij} = 1$.

This is such that firm $j \in \mathcal{N}_i^{in}$ if and only if $g_{ji} > 0$ and firm $j \in \mathcal{N}_i^{out}$ if and only if $g_{ij} > 0$. Receiving as well as paying firms form a directed graph which we denote as $G(\mathcal{N}, g)$ so that $\mathcal{N}$ stands for the nodes(firms) while g represents obligations between 2 firms. The financial network graph $G(\mathcal{N}, g)$ is then captured using the relative liability/obligation matrix given as $\boldsymbol{G} = [g_{ij}] \in \mathbb{R}_+^{n \times n}$ whose elements contain relative obligations for each firm in the network. We allow the possibility for bilateral obligations such that liabilities cannot be netted off each other.

Furthermore, we hold that each firm $i \in \mathcal{N}$, has an initial endowment which we denote as y_i at $t = 0$ as well. The intuition as well as justification for this parameter could be the concept of a *Reserve Requirement* in banking systems. The quality of such endowment is that they are reserved solely for the purpose of meeting up with obligations. They cannot be used for any other purpose. Let π_i denote the parameter firm i pays given his obligation. This amount in turn is derived from the standard

[6] A firm i is a debtor if and only if $\mathcal{N}_i^{out} \neq \{\}$ as in a creditor only when $\mathcal{N}_i^{in} \neq \{\}$.

Eisenberg and Noe (2001) clearing mechanism given as:

$$\pi_i = \min \left\{ \sum_{j \in \mathcal{N}_i^{in}} g_{ji}\pi_j + y_i, L_i \right\}, \tag{1.1}$$

with π_j denoting the payment of a given firm j of which $g_{ji} > 0$ and hence, is directly connected to firm i. So in line with the standard framework, firms do not pay more than what they owe. The limited liability principle holds so that firms who are unable to meet fully their obligations split their assets among all creditor firms on pro-rata (In essence, in the case of a firm i, $g_{ij}\pi_j$ is given to each firm $j \in \mathcal{N}_i^{out}$). Let π_i^0 serve as the amount firm i pays given the parameters at $t = 0$, and the expected default estimable by the firm i to be:

$$\vartheta_i = L_i - \pi_i^0.$$

Where $\pi_i^0 = \min\left\{\sum_j g_{ji}\pi_j^0 + y_i, L_i\right\}$ denotes the estimated payment at $t = 0$.[7] This model assumes that $\forall\ i \in \mathcal{N}$, π_i^0 is known with full certainty.[8]. In our model, we assume that at $t = 0$, y_i captures the network endowment of firm i and hence, is not prone to any further shocks. Then given π_i^0 the following definition becomes relevant,

Definition 1.2.1. *Firm $i \in \mathcal{N}$ is a **defaulter** if $\vartheta_i > 0$.*

Then from the definition, let $\mathcal{D} = \{1, \ldots, d\}$ be a default set such that for any $i \in \mathcal{D}$, $\vartheta_i > 0$, then we assume that punishment is applied to such firms using the homogeneous functional form $\lambda_i(\vartheta_i)$ which captures the rate of punishment from default such that $\lambda_i(\vartheta_i)$ is twice differentiable in ϑ_i. As an integral part of our model, we also have that $\lambda_i = \Lambda$ for every $i \in \mathcal{D}$ such that punishment is applied in the

[7] The value π_i^0 is notably the same as the *initial stress* used in Paddrik, Rajan, and Young (2020).

[8] This is a subtle divergence from the structure of most contagion literature such as Elliott et al. (2014), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), Cabrales, Gottardi, and Vega-Redondo (2014) or Blume, Easley, Kleinberg, Kleinberg, and Tardos (2011) for instance who all studied networks where firms are susceptible to external shocks which could reduce their endowments (y_i).

same functional form and Λ intensity for each defaulting firm.

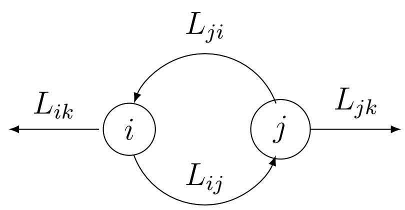


Figure 1-1: Network with 3 firms and 4 debt contracts (edges)

Given payments and possible defaults, it is not necessarily the case that based on π^0, all firms are defaulters. To further streamline our analysis, we focus our attention to the following situation;

Assumption 1.2.1. *For every firm $i \in \mathcal{N}$ the following holds true*

1. $\mathcal{N}_i^{out} \neq \{\}$,
2. $\pi_i^0 < L_i$ *so that Firm i is a defaulter.*

This implies that all firms have some financial obligation to which they estimate to fall short based on period the parameters at $t = 0$. We use the sets $\mathcal{N}$ and $\mathcal{D}$ interchangeably going forward. We grant in period $t = 0$ each firm $i \in \mathcal{D}$ the capacity to store an amount denoted as x_i to reduce default cost given their current position (π^0). Examples of such storage could include bank savings, vault cash set aside for payment, investment in collateral assets whose value are easily recoverable, etc.

These storage decisions of each firm in the network are common knowledge to every other firm in our economy. Another characteristics of x is its *irreversiblity* as firms are unable to retract their decision on storage once made. It is then possible to view π_i^0 as the output of fictitious default clearing mechanism at $t = 1$ assuming no further actions are taken by any firm in the network.

Payment system characterization

In this part we group the clearing payment for each firm i in $t = 1$ accounting for defaults and storage decisions in $t = 0$. Let δ_i be an indicator variable that takes value

$$\delta_i = \begin{cases} 1 \text{ if firm } i \text{ is a defaulter, and} \\ 0 \text{ otherwise.} \end{cases}$$

Also, let π_i^x be the payment of each firm $i \in \mathcal{D}$ in period $t = 1$. Then, (1.1) can be written as:

$$\pi_i^x = \delta_i \left(x_i + y_i + \sum_{j \in \mathcal{N}_i^{out}} g_{ji} \pi_j^x \right) + (1 - \delta_i) L_i \qquad \forall i. \tag{1.2}$$

The above equation then lets us group the inflow of a firm into those who can pay in full given $t = 0$ and those who cannot. The equality is guaranteed through limited liability of the clearing payment $\pi_{i \in \mathcal{N}}$. However, we know from assumption 1.2.1 that for all firm i for which $L_i > 0$, then $(1 - \delta_i) = 0$. So consequently, we have the (1.2) rewritten as below;

$$\pi_i^x = x_i + y_i + \sum_{j \in \mathcal{N}_i^{out}} g_{ji} \pi_j^x \qquad \forall i. \tag{1.3}$$

Let $\mathbf{I} = \{0, 1\}^{n \times n}$ as an identity matrix, $\mathbf{x} = (x_i)_{i \in \mathcal{N}} \in \mathbb{R}_+^n$ and $\mathbf{y} = (y_i)_{i \in \mathcal{N}}$ be column vectors referring to total obligation, storage and initial-endowments of each firm. Then we have the system in (1.3) given in vector form as follows;

$$\boldsymbol{\pi}^x = \mathbf{x} + \mathbf{y} + \boldsymbol{G}^T \boldsymbol{\pi}^x.$$

then collecting the like terms yields,

$$\left(\mathbf{I} - \boldsymbol{G}^T\right) \boldsymbol{\pi}^x = \mathbf{x} + \mathbf{y},$$

If the matrix $(\mathbf{I} - \boldsymbol{G}^T)$ is invertible , then we have the clearing payment at period

$t = 1$ then given as;

$$\boldsymbol{\pi}^x = (\mathbf{I} - \boldsymbol{G}^T)^{-1} \bullet (\mathbf{x} + \mathbf{y}) \tag{1.4}$$

For simplicity, let $\mathbf{w}$ be a square matrix such that $\mathbf{w} = [w_{ji}] \in \mathbb{R}_+^{d \times d} := (\mathbf{I} - \boldsymbol{G}^T)^{-1}$ and $\boldsymbol{w}_i = (w_{ji})_{j \in \mathcal{N}} \in \mathbb{R}_+^d$ be the $i - th$ row of the square matrix $\mathbf{w}$. We can then rewrite (1.4) as;

$$\boldsymbol{\pi}^x = \mathbf{w} \bullet (\mathbf{x} + \mathbf{y}),$$

so that for each firm $i \in \mathcal{D}$, we have

$$\pi_i^x = \boldsymbol{w}_i \bullet (\mathbf{x} + \mathbf{y}). \tag{1.5}$$

Corresponding to graph theory, w_{ji} is the sum of the number of backward walks of any length from i to j where each firm within the walk is a defaulter. We also hold the following assumption in relation to the graphical properties of the defaulting set stated as follows;

Definition 1.2.2. *A network graph $\mathcal{G}(\mathcal{N}, g)$ is* ***strongly connected*** *if for every pair of firms $\{0, n\} \subset \mathcal{D}$, there exist a* ***closed directed walk*** *(the sequence $0, g_{01}, 1, g_{12}, \ldots, g_{n-1,n}, n, g_{0,n}, 0$) from 0 to 0.*

As such we describe the structure of the default network as follows;

Assumption 1.2.2. *The default network $\boldsymbol{G}^T$ is* ***strongly connected.***

The nature of clearing system whereby it meets the assumption 1.2.2 limits the network to one exhibiting systemic risk such that defaults feedback to a less than 1 degree to source and thus a negative feedback mechanism is present. We focus on this network with strongly connected property for this paper as we aim to access a system where default feeds back to its primary source as is a unique property of systemic risk through contagion. The fig. 1-1 for example shows that $\mathcal{D} = \{i, j\}$ fits well into this description. To guarantee non-singluar matrix properties of $\boldsymbol{G}^T$, the following lemma becomes important;

Lemma 1.2.1. *Assume a default network $\boldsymbol{G}^T$ which is strongly connected, in so far as there exists $i \notin \mathcal{N}$ such that $\sum_{j \in \mathcal{N}_i^{out}} g_{ij} = 0$ and $\sum_{j \in \mathcal{N}_i^{in}} g_{ji} > 0$ (a sink node), then such default network is **regular** and as such $\mathbf{w} = (\mathbf{I} - \boldsymbol{G}^T)^{-1}$ is definite, and well defined with positive elements.*

Proof. See Appendix for proof. ■

Regularity here follows Eisenberg and Noe (2001) which in context, sink nodes guarantee that risk orbits are always a surplus set as it always has a non-zero equity. Its impact is also similar to **normality** under Allouch (2015). In recalling some of the key propositions of systemic risk in financial systems is the fact that firms face risk not only from external activities it participates in, but also as a result of other firms behaviors to which it is connected to and as such, we have chosen to focus on the latter. Cascading default primarily comes from a source firm(s). These firms in the network are said to have, among their total estimated default, a non-trivial portion which coming exclusively from its outside activities rather than systemic.[9] This is so that while the Fictitious default sequence captures different waves of default among firms in the network which is reflected in each iteration to arrive at the *Clearing Payment* (π^0), that of the First-wave/First iteration reflects those who default from non-systemic causes. In the light of that we can define the following term:

Definition 1.2.3 (Eisenberg and Noe (2001))**.** *A firm i is a **First-wave defaulter** if $L_i - \sum_{j \in \mathcal{N}_i^{in}} g_{ji} L_j - y_i > 0$.*

Thus implying that within the estimates at $t = 0$, firms who are first wave defaulters fail to meet up their obligations even when they receive full payment from

[9]In this case, activities leading up to $t = 0$.

debtors. On the other hand, these firms originate the default that then cascades through the system (thus instigating other iterations/waves). The first-wave default could come from multiple sources. They include;

1. **Internal Sources:** Administrative deficiency, Mismanagement of funds, settlement for litigation, other managerial diseconomies of scale, etc.

2. **External Sources:** Market shocks leading to stochasticity of returns, natural disasters, theft, etc.

The underlying point is that first-wave defaults are product of systematic risk or unsystematic risk a debtor firm faces. To complement assumption 1.2.1 for the sake of this model, the following assumption becomes relevant:

Assumption 1.2.3. *$\forall\, i \in \mathcal{N}$, firm i is First-wave defaulter.*

Magnitude of default amounts are allowed to vary as much among firms. What is of importance with the assumption is that all firms with debt obligations are not able to repay fully given they receive all they can from the network. This assumption now guarantees strict non-negativity of $x_{i\in\mathcal{D}}$. These assumptions are crucial as the first grantees default cyclicality but brings in mind questions as to normality. The assumption 1.2.3 on the other hand ensures continuity of substitution which is discussed in detail in the Appendix.

Additionally, an intuition behind $\boldsymbol{w}_{i\in\mathcal{D}}$ as contained in (1.5) as the parameter arises from the fact that when $\boldsymbol{G}^T$ is strongly connected, then a firm $i \in D$ has a total default which includes $\boldsymbol{w}_i$ times its first-wave default. Hence, default feedback which Eisenberg and Noe (2001) describes as the instance where *"a firm-A defaults causing other firms to default, which then makes firm-A default even more"*. Other literature captures such behaviors including Paddrik et al. (2020). This is within the framework seen as an evidence of systemic crises.

Payoff and Optimization Programme

Having described the payment and network structure, we then the punishment system $\lambda_i(\varphi_i)$ for each firm $i \in \mathcal{D}$ as the following payoff (cost) function:

$$C_i(\pi_i^x, x_i) = \frac{\Lambda}{2}(L_i - \pi_i^x)^2 + x_i. \tag{1.6}$$

The equation above then means that at $t = 1$, the total size of punishment a firm $i \in \mathcal{D}$ bears is an increasing function of its default amount within the period. This is split into to parts, the Benefit and Cost segments.

If we denote the Benefit firm i gets from storing as b_i, then is $b_i(x_i) = -0.5\Lambda(L_i - \pi_i^x)^2$. Therefore $x_i : x_i \to b_i$ is so that b_i is concave and continuously increasing in x_i. Our hypothesis here is that firms get more from storing while total default is still significant.[10] This relationship arises from the intuition that large amount of defaults become even more pronounced and as such attract greater consequences including fall in reputation of a firm due to bad publicity both from creditors and potential investors. There could be even spillover to other aspects of the society which would trace back to the defaulting firm. Other aspects such as bond renegotiation, bankruptcy renegotiation as well as other access for future credit could greatly be limited as a result of large amounts of defaults.

For the Cost part of C_i is simply x_i such that $x_i : x_i \to x_i$ is linear as we assume that storage at $t = 0$ is spending forgone and as such, becomes a past cost the firm bears. A defaulting firm i makes his storage decision following the programme given below:

$$\min_{x_i, \pi_i^x} C_i(x_i, \pi_i^x) = \frac{\Lambda}{2}(L_i - \pi_i^x)^2 + x_i, \tag{1.7}$$

[10] Note that we refer to the total default as opposed to simply its first wave. Hence, the system does not trace defaults of each firm from its origin (be it external or network based) but imposes penalty on the full shortfall.

subject to

$$\pi_i^x = \boldsymbol{w}_i \cdot (\mathbf{x} + \mathbf{y}), \tag{1.8}$$

We ignore the non-negativity of x_i given that assumption 1.2.3 binds our optimization which implies that $x_i \gneqq 0$ for each firm $i \in \mathcal{D}$. The first order condition is then:

$$C_i^{'}(x_i) = -w_{ii}\Lambda(L_i - \pi_i^x) + 1 = 0. \tag{1.9}$$

The left hand side of (1.9) is the marginal benefit of the action, and as such, captures the reduction in default punishment resulting from a marginal increase in the action x_i.

It is possible to view the parameter w_{ii} from different perspectives, however at this point, it is seen prominently as the value of additional payment a dollar action yields a firm i owing from the feedback loop that a default for the firm i also takes. For this reason, it then captures the marginal rate of transformation from the action x_i into the repayment π_i^x. This should not be confused with the total payments the system makes for a dollar action by firm i.[11] Finally, from the marginal rate of substitution from the action x_i to the repayments π_i^x is then $(\Lambda(L_i - \pi_i^x))^{-1}$.

1.3 General Characterization of Equilibrium

In this part, we present the optimal storage of each defaulting firm which is the solution to the optimisation progamme in the previous section. We have the statement below while keeping assumption 1.2.1 and assumption 1.2.3 in mind:

Proposition 1.3.1. *Let* $A_i = \frac{1}{w_{ii}\Lambda}\left(L_i - \frac{1}{w_{ii}\Lambda} - \sum_{j \in \mathcal{N}_i^{in}} w_{ji} y_j\right) - y_i$ *and* $B_{ij} = \dfrac{w_{ij}}{w_{ii}}$.

[11] Which would be the threat index as in Demange (2016).

For each firm $i \in \mathcal{D}$, the following best replies hold;

$$x_i = A_i - \Lambda^{-1} \cdot \sum_{j \in \mathcal{N}_i^{in}} B_{ij} x_j, \tag{1.10}$$

Proof. See Appendix for proof. ■

Remark 1.3.1. *Nash decisions $x_{i\in\mathcal{D}}$ under $\lambda_i = 1 \ \forall \ i \in \mathcal{D}$ arises from a game of strategic substitution captured in;*

$$\frac{dx_i}{dx_j} = -\frac{w_{ij}}{w_{ii}}.$$

Our best reply function in (1.10) is linear which has a form that is noticeably isomorphic to best replies found in general public goods in networks literature.[12]. In interpretation, the firm i reduces its effort given an increase in effort x_j of any other firm $j \in \mathcal{N}_i^{in}$ to which $g_{ji} > 0$. It is worth noting that the reason (1.10) does not hold as $x_i = \max\left\{A_i - \frac{1}{\Lambda} \cdot \sum_{j\in\mathcal{N}_i^{in}} B_{ij}x_j, 0\right\}$ is due to assumption 1.2.3 because the non-negativity constraint always holds[13]. The shape of the best reply from (1.23) is linear and the degree of change is $-\dfrac{w_{ij}}{w_{ii}}$. This is because from (1.10), firm i substitutes every 1$ amount of x_j relative to its liability proportion.

The value $\frac{1}{w_{ii}\Lambda}$ is multiplied by the bracket components of A_i to serves as a counter to the effects of default feedback. This is as firm i intends to find the actual saving amount and default of firm i in its full scale also results from feedback firm i's initial default, then its actual requirement need be discounted by the feedback parameter w_{ii}.

[12]Notably Bramoullé et al. (2010) or Allouch (2015).

[13]It is always the case that $x_{i\in\mathcal{D}} = 0$.

1.3.1 Equilibrium for a defaulting Firm

With linear best replies as given in the previous equation, equilibrium decisions can be estimated for defaulting Firms. Similarly, for the firm i still holding our previous assumptions, we have the following statement,

Proposition 1.3.2. *Let Nash equilibrium storage vector be given as* $\mathbf{x}^* = (x_i^*)_{i\in\mathcal{D}} \in \mathbb{R}_+^d$. *Also, let* $\boldsymbol{B} = [B_{ij}] \in \mathbb{R}^{d\times d}$ *be a matrix while* $\boldsymbol{A} = (A_i)_{i\in\mathcal{D}} \in \mathbb{R}_+^d$ *be a column vector.*[14] *There exists a unique (interior) Nash equilibrium given as;*

$$\mathbf{x}^*(\mathcal{D}, \Lambda, \mathbf{y}) = \left(\mathbf{I} + \frac{1}{\Lambda}\cdot \boldsymbol{B}\right)^{-1} \cdot \boldsymbol{A}. \tag{1.11}$$

under the following necessary conditions:

1. *assumption 1.2.3 holds,*
2. $\frac{1}{\Lambda} \in \left]0, \left(\mu_{min}\left(\frac{\boldsymbol{B}+\boldsymbol{B}}{2}\right)\right)^{-1}\right]$ *where* $\mu_{min}(\boldsymbol{G})$ *is the minimum eigenvalue of a matrix* $\boldsymbol{G}$.

Proof. See Appendix for proof. ∎

More specific to the network we consider, we elaborate on proposition 1.3.2 by writing an initial lemma as stated below;

Lemma 1.3.1. *Assume a network* $\boldsymbol{G}^T$, *which is a directed ring network then it holds that for* $\mathcal{D} = \{0, 1, \dots, d\}$

$$w_{00} = w_{11} = \dots = w_{dd}. \tag{1.12}$$

Proof. See Appendix for proof. ∎

As a special case of lemma 1.3.1, take for example a directed ring network where $\mathcal{D} = \{i, j\}$, then $B_{ij} = \left(\frac{g_{ji}}{(1-g_{ij}g_{ji})} / \frac{1}{(1-g_{ij}g_{ji})}\right) = g_{ji}$. We however leave B_{ij} as it is

[14] Note that d is the cardinality of the set $\mathcal{D}$.

because of its ability to capture the dynamic nature of $\mathcal{D}$. As such we then have the following corollary from proposition 1.3.2 as;

Corollary 1.3.1 (Directed Ring Networks). *Assume a network $\boldsymbol{G}^T$, which is a directed ring network then there exists a unique Nash equilibrium given as;*

$$\mathbf{x}^*(\mathcal{D}, \Lambda, \mathbf{y}) = \left(\mathbf{I} + \frac{1}{\Lambda} \cdot \boldsymbol{G}^T\right)^{-1} \cdot \boldsymbol{A}, \tag{1.13}$$

in so far as $\frac{1}{\Lambda} \in \left]0, \left(\mu_{min}\left(\frac{\boldsymbol{G}+\boldsymbol{G}^T}{2}\right)\right)^{-1}\right]$.

Proof. See Appendix for proof. ■

This is so that in general, if firms with obligation are forced to avoid default at all cost and such information is available to all firms, each potential defaulter stores the amount that equals its first wave default.

A quick glance at the equation takes us back to definition 1.2.3. It is seen then that the firm i pays an amount that includes its *First wave default* and its reduction would be subject to the value of Λ as well as the magnitude of the work of lengths. Given the Nash equilibrium vector $\mathbf{x}^*(\mathcal{D}, \Lambda, \mathbf{y})$, we have the following equilibrium for each firm $i \in \mathcal{D}$ as written in the result below;

Proposition 1.3.3. *Nash equilibrium for a firm i case satisfies the following equation,*

$$x_i^* = L_i - \sum_{j \in \mathcal{N}_i^{in}} g_{ji} L_j - y_i - \frac{w_{ii}}{\Lambda} + \sum_{j \in \mathcal{N}_i^{in}} g_{ji} \frac{w_{jj}}{\Lambda}. \tag{1.14}$$

Proof. See Appendix for proof. ■

Observing (1.14), the Nash action of each player i can be decomposed in two segments namely:

1. The ***liability gap*** which for a firm $i \in \mathcal{D}$ we denote as l_i such that;

$$l_i = L_i - \sum_{j \in \mathcal{N}_i^{in}} g_{ji} L_j - y_i. \tag{1.15}$$

2. The ***centrality gap*** we denote as γ_i such that we have:

$$\gamma_i = w_{ii} - \sum_{j \in \mathcal{N}_i^{in}} g_{ji} w_{jj}, \tag{1.16}$$

Which in vector form for $\boldsymbol{Q} = (\gamma_i)_{i \in \mathcal{D}} \in \mathbb{R}_+^d$ is represented as;

$$\boldsymbol{Q} = (\mathbf{I} - \boldsymbol{G}^T) \cdot diag(\mathbf{w}) \tag{1.17}$$

The liability gap is synonymous to the value of *first-wave default* value of the Fictitious Default Algorithm (FDA) proposed by Eisenberg and Noe (2001). In its interpretation, it represents the amount of default by a firm which arises from circumstances outside the network (hence, not as a result of default contagion from debtor firms within the network). The value l_i of each firm $i \in \mathcal{D}$ is also a special case of the *stress* of a firm found in Paddrik et al. (2020)[15] with the contrast that why we assume unlimited access to x_i(though at cost of repayment), the greater l_i is, the greater the stress on the firm $i \in \mathcal{D}$ thus increasing its likelihood of default.

The centrality gap in a way reflects an additional benefit resulting from greater strategic dependence. To understand, take for example that $\boldsymbol{G}^T$ is a directed ring, for a firm i, its centrality gap is as follows;

$$\gamma_i = w_{ii} \left(1 - \sum_{j \in \mathcal{N}_i^{in}} g_{ji} \right).$$

This means that the stronger the firms *direct relative asset* for any firm $i \in \mathcal{D}$ (

[15]That is the instance to which a firm i receives all it is owed from its debtors in $\mathcal{N}_i^{in}$.

given as $\sum_{j\in\mathcal{N}_i^{in}} g_{ji}$), then the lower γ_i is.[16] The firm $i \in \mathcal{D}$ can now store an amount lower than its liability gap. Also the value w_{ii} is useful in a default network which is strongly connected as it hints to spread of influence of a firm. From lemma 1.3.1 the result on ring networks as as such because each firm has only one directed outgoing link. However, in order forms of strongly connected networks, it could vary depending on the spread of interlinks. This is because additional links creates additional loops. The more the loop the firm $i \in \mathcal{D}$ is a member of, the greater its w_{ii}. This is the case even when $\sum_{j\in\mathcal{N}_i^{out}} g_{ij}$ is the same weight $\forall$ firm $i \in \mathcal{D}$. Adapting l_i and γ_i into (1.14) gives us the following Nash equation;

$$x_i^* = l_i - \frac{1}{\Lambda}\gamma_i. \tag{1.18}$$

It is also worth noting from (1.18) that a sizeable Λ coupled up with γ_i for each firm $i \in \mathcal{D}$ guarantees that the value $\lambda^{-1}\gamma_i$ remains relatively small in size when compared to l_i. Additionally, the value $1 - \sum_{j\in\mathcal{N}_i^{in}} g_{ji}$ means that in so far as $\sum_{j\in\mathcal{N}_i^{in}} g_{ji} < 1$, then the greater the amount of direct relative asset owed to the firm i, then the greater it has to store close to but less than its first wave default. Should the reverse hold however, the firm $i \in \mathcal{D}$ would then be required to store above its first wave default. This in itself is due to the increased dependence leading to the firm i having to cover up for greater amount of lapse(defaults) by its defaulting debtor firms.

[16] assuming strictly that $\sum_{j\in\mathcal{N}_i^{in}} g_{ji} < 1$.

1.3.2 Relation to Public Good Problems

Recall our best reply in (1.10) as $x_i = A_i - \Lambda^{-1} \sum_{j \in \mathcal{N}_i^{in}} B_{ij} x_j$. In the case where assumption 1.2.3 was absent so that we had

$$x_i = \max \left\{ A_i - \frac{1}{\Lambda} \sum_{j \in \mathcal{N}_i^{in}} B_{ij} x_j, 0 \right\}, \tag{1.19}$$

then it will be likened to public good problems in network found in Bramoullé et al. (2010), (Bramoullé & Kranton, 2007) or Allouch (2015) etc which are specifically focused on private provision on public good and best replies revealing strategic substitution. Such substitution is a quality found in agents behavior to a privately provided public good. In this case, the item linked to the public good consumption for a firm i is its π_i^x. While the firm $i \in \mathcal{D}$ tries to ensure that $\pi_i^x \approx L_i$ by storing x_i, it then provides the resource x_i to its set $\mathcal{N}_i^{out}$ (hence, its direct creditors).

Also, it is common in public good problems for Nash equilibrium decision to include inactive firms[17]. In our model, assumption 1.2.3 rules out such possibility. However, since for sizeable values of Λ, we have the liability gap which dominates the equilibrium storage, then it is the case for each firm that while they are all active, they substitute/free-ride on the storage of their direct debtors as their total payment π_i^x is based on the sum of all decisions all firm $j \in \mathcal{N}_i^{in}$.

1.3.3 Comparative Study

We draw comparisons with Allouch and Jalloul (2016) as it is relatively close in concept. The set up was where firms face binary choices between defaulting ($\Psi_{i \in \mathcal{D}} = 0$) or not defaulting($\Psi_{i \in \mathcal{D}} = 1$) in $t = 1$. The key parameters included $T(\mathbf{a}_{\mathcal{N}_i^{in}})$ which defines the minimum amount required to avoid default, r represents the returns in

[17] Defined as any firm $i \in \mathcal{D}$ whose Nash decision $x_i = 0$.

$t = 1$ for investing/storing the fixed endowment x_i at $t = 0$ while $\bar{x}_i \equiv x_i$ comes from the utility of defaulting hence, satisfying the condition $U_i(x_i, 0) = U_i(0, \bar{x}_i)$. Given its best replies, its strategy is to not default when $(1+r)x_i - T(\mathbf{a}_{\mathcal{N}_i^{in}}) \geq \bar{x}_i$, otherwise, the firm defaults. Thus $\Psi_{i\in\mathcal{D}} = 1$ also implied x_i is stored with $\Psi_{i\in\mathcal{D}} = 0$ implying x_i is consumed at $t = 0$.

While payoffs and functional forms of the best replies differ significantly in both models,[18] we draw attention to the threshold value $T(\mathbf{a}_{\mathcal{N}_i^{in}})$.

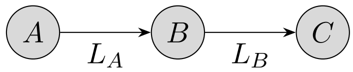


Figure 1-2: Line liability network with 3 firms.

Observe then from fig. 1-2 above that given at firm A and firm B are defaulters at $t - 0$, a unique strictly dominant Nash equilibrium always exits in a default game as in Allouch and Jalloul (2016). However, as such the threshold value for Firm A given as $T_A(\mathbf{a}_{\mathcal{N}_i^{in}}) = L_A - y_A$ which corresponds strictly to its first wave default/liability gap as in our equilibrium. In general, for $i \in \mathcal{D}$, we have that

$$T_i(\mathbf{a}_{j\in\mathcal{N}_i^{in}}) = L_i - y_i - \sum_{j\in\mathcal{N}_i^{in}} g_{j,i}(y_j + \sum_{k\in\mathcal{N}_j^{in}} g_{j,k}\pi_j^x) - \sum_{j\in\mathcal{N}_i^{in}} \Psi_j g_{j,i}\bar{x}_j. \tag{1.20}$$

Where $\Psi_j \in \{0, 1\}$ and as such is 0 if Firm j Nash decision is to default and 1 otherwise. This threshold value in (1.20) thus reveals the possibility whereby it corresponds to the first wave default even in a cyclical network. For example, if $j \in \mathcal{D} \cap \mathcal{N}_i^{in}$ and $\Psi_j = 1$, then $T_i(\mathbf{\Psi}_{\mathcal{N}_i^{in}} = \mathbf{1})$ is equal to the first wave default for Firm i. In a way, it can then be seen that results from our model point to solutions similar to $T_i(\mathbf{a}_{\mathcal{N}_i^{in}})$. However, an easily observable difference is that $T_i(\mathbf{a}_{\mathcal{N}_i^{in}})$ is a product of corner equilibrium choices of directly connected defaulting firms. Our

[18]Firms in Allouch and Jalloul (2016) are more likely to store if other firms connected to them store.

equilibrium x_i is a product of a continuous mapping best reply function of other connected firms storage which does not have to be a directly connected firm.

1.3.4 Efficiency and Equilibrium

Efficiency broadly can be split into individual as well as social efficiency. Given that $x_i^* = \underset{x_i \in \mathbb{R}_+}{\operatorname{argmax}} C_i(\pi_i^x, x_i)$, then the Nash equilibrium of each firm corresponds to the amount of storage x_i that yields minimum cost the firm $i \in \mathcal{D}$, which is its individually efficient storage. For efficiency of the system, we assume the planner who sets $\Lambda(\vartheta_i)$ binding on each contracts does so to act as a deterrent towards default. We also know that the centrality gap $\gamma_{i\in\mathcal{D}}$ becomes closer to zero the greater Λ is. Hence, since each firm i has unlimited access to x_i so that each are not cash constrained, then payments $\sum_{i\in\mathcal{N}} \pi_i^x$ is maximised when Λ is such that $\gamma_{i\in\mathcal{D}} \approx 0$.

1.4 Punishment and Non-interior equilibrium

So far, we have estimated the storage value $x_{i\in\mathcal{D}}$ through the mapping $x_i : x_i \to C_i(\pi_i^x, x_i)$ is convex (cup-shaped) and compact in x_i. However, should the shape be $x_i : x_i \to C_i(\pi_i^x, x_i)$ is concave, intuitions drawn might be valuable especially for robustness purpose. We do not attempt to go quite in-depth on this. Observe the interaction in its simplest form. Assume then that we have;

$$C_i(\pi_i^x, x_i) = \Lambda log(L_i - \pi_i^x) + x_i. \tag{1.21}$$

Then assuming assumption 1.2.1 and assumption 1.2.3 still binds, we present some little intuition summarised as follows;

Proposition 1.4.1. *Given a punishment function $C_i(\pi_i^x, x_i)$ such that $x_i : x_i \to C_i(\pi_i^x, x_i)$ is convex, if Λ is of significant magnitude, Nash equilibrium decisions $\forall$*

$i \in \mathcal{D}$ corresponds to the first wave default so that the following equation holds true $\forall\ i \in \mathcal{D}$,

$$x_i(\mathcal{D}, \mathbf{x}) = \begin{cases} \frac{1}{w_{ii}}\big(L_i - \sum_{j \in \mathcal{N}_i^{in}} w_{ij}(x_j + y_j)\big) - y_i & \text{if } C_i(x_i > 0) \leq C_i(x_i = 0) \\ 0 & \text{otherwise} \end{cases} \tag{1.22}$$

so far as $\forall\ i \in \mathcal{D},\ x_i \in]0, \mathbb{R}^+]$.

Proof. See Appendix for proof. ∎

In essence, what we show here simply put is that while local minimum exists at two critical values of $x_i(\sum_{j \in \mathcal{N}_i^{in}} x_j)\ \forall\ i \neq j$, then since the punishment Λ is homogeneous, then a value of Λ that forces $\min_{x_i} C_i(\pi_i^x, x_i) : x_i > 0$ would also force all other $x_j > 0 : j \neq i$. This then means that depending on Λ magnitude, it equilibrium would be binary: All default at $t = 1$ or all contribute an amount equivalent to its first wave default completely. A simple illustration of this problem can be found as follows.

Example 1.4.1. *Say we have the following;*

$$\begin{aligned} L_i &= 100 \\ w_{ii} &= 2 \\ \Lambda &= 1 \\ \sum_{j \in \mathcal{N}_i^{in}} w_{ij}(x_j + y_j) &= 30 \end{aligned}$$

then **at** $x_i = 0$ *Implies,*

$$C_i(\pi_i^x, x_i) = 1.69$$

and **and at** $x_i = \frac{1}{w_{ii}} \ldots$ *we have,*

$$C_i(\pi_i^x, x_i) = 25$$

$x_i^* = 0$ *minimizes* $C_i(\pi_i^x, x_i)$.

This the implies that the Nash profile arises from a new binary best reply, particularly $x_i = \{0, \vartheta_i(x_{\mathcal{N}_i^{in}})\}$ thus taking a similar game structure once again to Allouch and Jalloul (2016). Observe then that the *Centrality Gap* seizes to be a component of the equilibrium as there is no longer a trade off of choices such that each defaulting firm either does all or nothing.

1.5 Summary and Conclusions

So far we have introduced behavior of firms faced with ex-ante default and also corresponding penalties which is mapped by such defaults. Firms storing to avoid such default strategically substitute to arrive at the best amount of storage. Hence, their actions yields a public goods which can be substituted by their outgoing neighbours. Overall, when there is a sizeable amount of punishment in such ex-ante default system, we have seen that firms are best to store the amount corresponding to the amount of the default specifically attributed to each firm caused by factors external to the network. Lastly, we see that a firms position in a network can (especially when the punishment intensity is low) lead to greater or lesser burden of storage on the defaulting firm.

The model provides a basic framework into behaviors in well bounded default networks. Even at that, we distance ourselves from instances where potential kinks/discontinuity might arise. It would be useful to merge and investigate equilibrium under a best response which defaults are cyclical but not all firms default at first wave. Also shy away from restricted endowments of firms and implicitly assume in our equilibrium that such endowments are fully liquid. These can potentially be improved up thereby adopting methods from Amini et al. (2016) or Feinstein (2017) which can lead to a more sequential (algorithmic) Nash equilibrium. Also our model assumes homogeneous default rate applied to each firm. In reality, the size of Λ could depend on other factors such as the size of the firm as well as its network position. We believe that this idea provides a valuable extension to the model. Additionally, there could be other means to ensure that firms avoid default such as prudential regulation including reserve requirement which could be enforced by a regulator. Such instances are weakly accounted for in our model as one might say it would be the basis for our exogenous cash for each debtor firm. However, treating this policy as endogenous could shed additional light to firms behavior especially banks.

To conclude, our results arises from an environment where a planner uses an alternate approach monetary interventions or prudential regulations to achieve similar goals. Here firms are the decision maker as opposed to regulation. Broadly, in attempt to understand from various behaviors in response catalyst that could potentially lead to or avoid a default crises, interactions between firms/nodes involved under different settings provides valuable intuitions. Outcome of firms actions in themselves could serve as valuable hints and predictors to not only the anticipated cries but the individual firms roles in leading to or away from it.

1.6 Appendix

Existence properties and the impact of Assumption 1.2.3

Observing the best reply function at (1.10), we see that $x_j : x_j \to x_i$ by a coefficient defined by $B_i = \frac{w_{ij}}{w_{ii}}$. Also the parameters w_{ij}, w_{ii} are elements of the matrix $\mathbf{w}$. Assume then $\mathcal{N} = \{i, j, k\}$ where firm $k \in \{\mathcal{N}_i^{out} \cup \mathcal{N}_j^{out}\}$ such that firm k is a sink node. Then also assume $\mathcal{D} = \{i, j\}$, we have the following typically already shown in our proof of lemma 1.3.1;

$$\mathbf{w} = (\mathbf{I} - \boldsymbol{G}^T)^{-1} = \begin{pmatrix} \frac{1}{(1-g_{ji}g_{ij})} & \frac{g_{ji}}{(1-g_{ji}g_{ij})} \\ \frac{g_{ij}}{(1-g_{ji}g_{ij})} & \frac{1}{(1-g_{ji}g_{ij})} \end{pmatrix}$$

so that $w_{ii} = w_{jj} = (1 - g_{ij}g_{ji})^{-1}$. Then also assume only i is a first wave defaulter so that $L_j - g_{ij}L_i - y_j \geq 0$ but $L_j - g_{ij}\pi_i^0 - y_j < 0$. Then it means $\exists\ x_i : L_j - g_{ij}\pi_i^0(x_i) - y_j = 0$. In such a case, $\delta \cdot g_{ji} = 0$ and as such $w_{ii} = w_{jj} = (1 - (g_{ij} \cdot 0))^{-1} = 1$ or in matrix, we have;

$$\mathbf{w} = (\mathbf{I} - \boldsymbol{G}^T)^{-1} = \begin{pmatrix} 1 & 0 \\ g_{ij} & 1 \end{pmatrix}.$$

This causes a kink such that there exists a discontinuity within the best reply function for $i \in \mathcal{N}$ and so is it for any firm in the initial default loop. To evade this problem we then hold assumption 1.2.3 such that $\forall\ i \in \mathcal{D}$, $L_i - g_{ji}L_j - y_i < 0$. This then means that assuming $i, j \in \mathcal{N}$ are in default, $\mathbf{w}$ as well as $w_{ii} = w_{jj} = (1 - g_{ij}g_{ji})^{-1}$ are constant for all values, say $i, j \in \mathcal{D}$, $x_i \to A_i - B_{ji}x_j$ is linear and continuous and vice versa.

Proofs

Proof of Lemma 1.2.1

Assume no sink node exits, then the values in each column of g' sum to

$$\sum_i [\boldsymbol{G}^T]_i^j = 1 \quad \forall j$$

which is the same as the row sum of g. It the follows that when $\delta = J$ such that all assumption 1.2.1 holds,

$$(\mathbf{I} - \boldsymbol{G}^T) \qquad \text{is not invertible.}$$

When $(\mathbf{I} - \boldsymbol{G}^T)$ is invertible, then the spectral radius $\rho(\boldsymbol{G}^T) \in (-1, 1)$, and

$$(\mathbf{I} - \boldsymbol{G}^T)^{-1} = \mathbf{I} + \boldsymbol{G}^T + (\boldsymbol{G}^T)^2 + (\boldsymbol{G}^T)^3 + ...$$

However, with the existence of sink node, invertiblity is guaranteed as as $\boldsymbol{G}^T \neq \mathbf{1}$ because $\exists\ i \in \mathcal{D}$ such that $\sum_{j \in \mathcal{D}} g_{ij} < 1$. As such $\boldsymbol{G}^T \neq \mathbf{1}$ such that $\rho(\boldsymbol{G}^T) \in (-1, 1)$. This property also is the regularity lemma in Eisenberg and Noe (2001) where the clearing payment vector can simply be mapped to exogenous cash flow through the inversion in essence $\mathbf{y} : \mathbf{y} \to \boldsymbol{\pi}$ is given as simply;

$$\boldsymbol{\pi} = (\mathbf{I} - \boldsymbol{G}^T)^{-1} \bullet \mathbf{y},$$

when all paying firms are defaulting. □

Proof of Proposition 1.3.1

Given the FOC in (1.9) we have,

$$w_{ii}\Lambda(L_i - \pi_i^x) = 1$$

Then making π^x the subject of the formula gives

$$\pi_i^x = L_i - \frac{1}{w_{ii}\Lambda}$$

Then recall $\pi_i^x = w_{ii}(x_i + y_i) + w_{ji}(x_j + y_j)$ so that we then have

$$w_{ii}(x_i + y_i) + w_{ji}(x_j + y_j) = L_i - \frac{1}{w_{ii}\Lambda}$$

$\rightarrow$ then to make x_i the subject becomes

$$w_{ii}(x_i + y_i) = L_i - \frac{1}{w_{ii}\Lambda} - w_{ji}(x_j + y_j)$$

$\rightarrow$

$$x_i = \max\left\{\frac{1}{w_{ii}\Lambda}\left(L_i - \frac{1}{w_{ii}\Lambda} - \sum_{j\in\mathcal{N}_i^{in}} w_{ij}(x_j + y_j)\right) - y_i, 0\right\}. \tag{1.23}$$

and with assumption 1.2.3, we have,

$$x_i = \frac{1}{w_{ii}\Lambda}\left(L_i - \frac{1}{w_{ii}\Lambda} - \sum_{j\in\mathcal{N}_i^{in}} w_{ji}(x_j + y_j)\right) - y_i$$

□

Proof of Proposition 1.3.2

Given each firm $i \in \mathcal{D}$ has the best reply function as,

$$x_i = \frac{1}{w_{ii}\Lambda}\left(L_i - \frac{1}{w_{ii}\Lambda} - \sum_{j\in\mathcal{N}_i^{in}} w_{ji}(x_j + y_j)\right) - y_i,$$

let $\boldsymbol{B} = [B_{ij}]$ such that $B_{ij} = \frac{w_{ij}}{w_{ii}}$ then be the $d \times d$ zero diagonal matrix whose elements for $i, j \in \mathcal{D}$. $\boldsymbol{A} = (A_i)_{i\in\mathcal{D}}$ such that $A_i = \frac{1}{w_{ii}\Lambda}\left(L_i - \frac{1}{w_{ii}\Lambda} - \sum_{j\in\mathcal{N}_i^{in}} w_{ji}(y_j)\right) - y_i$. Then we have the vector form best reply given as;

$$\mathbf{x} = \boldsymbol{A} - \Lambda^{-1} \bullet \boldsymbol{B}\mathbf{x}. \tag{1.24}$$

Existence of $\mathbf{x}$ is the guaranteed under Brouwer's fixed point theorem as $[0, \boldsymbol{A}] \to [0, \boldsymbol{A}]$ intersects with the monotonic mapping $f : \mathbf{x} \to \boldsymbol{A} - \Lambda^{-1} \cdot \boldsymbol{B}\mathbf{x}$. For uniqueness however, it is guaranteed by positive definiteness of $(\mathbf{I} + \Lambda^{-1} \cdot \boldsymbol{B})$ which because $\boldsymbol{B}$ is directed, becomes that the scalar Λ as to be large enough such that the condition $\frac{1}{\Lambda} \in \left]0, \left(\mu_{min}\left(\frac{\boldsymbol{B}+\boldsymbol{B}^T}{2}\right)\right)^{-1}\right]$ is met. A more detailed explanation of this condition can be found in works such as Rosen (1965). □

Proof of Lemma 1.3.1

Take the same example in 1-1 for simplicity. Since both Firm i and Firm j are defaulters, then we have

$$\boldsymbol{G}^T = \begin{pmatrix} 0 & g_{ji} \\ g_{ij} & 0 \end{pmatrix}$$

then

$$\mathbf{w} = (\mathbf{I} - \boldsymbol{G}^T)^{-1} = \begin{pmatrix} \frac{1}{(1-g_{ji}g_{ij})} & \frac{g_{ji}}{(1-g_{ji}g_{ij})} \\ \frac{g_{ij}}{(1-g_{ji}g_{ij})} & \frac{1}{(1-g_{ji}g_{ij})} \end{pmatrix}$$

then notice from the diagonal that $w_{ii} = w_{jj} = (1 - g_{ji}g_{ij})^{-1}$. □

Proof of Corollary 1.3.1

Since we have for example given $\mathcal{D} = \{i, j\}$ that $w_{ii} = w_{jj} = \left(\left|\mathbf{I} + \frac{1}{\Lambda} \cdot \boldsymbol{G}^T\right|\right)^{-1}$, then for each firm i and firm $j \in \mathcal{N}_i^{in}$, $B_{ij} = \frac{w_{ij}}{w_{ii}} = g_{ji}$ this means that the Nash equilibrium vector holds as follows;

$$\mathbf{x}^* = (\mathbf{I} + \frac{1}{\Lambda} B)^{-1} \cdot A = \left(\mathbf{I} + \frac{1}{\Lambda} \cdot \boldsymbol{G}^T\right)^{-1} \cdot \boldsymbol{A}. \tag{1.25}$$

This means that in so far $\left(\mathbf{I} + \frac{1}{\Lambda} \cdot \boldsymbol{G}^T\right)$ is positive definite then $\mathbf{x}$ is uniquely defined. This is guaranteed so far as $\frac{1}{\Lambda} \in \left]0, \left(\mu_{min}\left(\frac{\boldsymbol{G}+\boldsymbol{G}^T}{2}\right)\right)^{-1}\right]$. □

Proof of Proposition 1.3.3

Without loss of generality, assume that $\mathcal{D} = \{i, j\}$ as in fig. 1-1. Then we have the Nash equilibrium equations as;

$$x_i = \frac{1}{w_{ii}}\left(L_i - \frac{1}{w_{ii}\Lambda} - w_{ji}(x_j + y_j)\right) - y_i$$

as well as,

$$x_j = \frac{1}{w_{jj}}\left(L_j - \frac{1}{w_{jj}\Lambda} - w_{ij}(x_i + y_i)\right) - y_j$$

Then substituting in x_i we have,

$$x_i = \frac{1}{w_{ii}}\left(L_i - \frac{1}{w_{ii}\Lambda} - w_{ji}\left[\frac{1}{w_{jj}}\left(L_j - \frac{1}{w_{jj}\Lambda} - w_{ij}(x_i + y_i)\right) - y_j + y_j\right]\right) - y_i$$

$\rightarrow$

$$x_i = \frac{L_i}{w_{ii}} - \frac{1}{w_{ii}^2\Lambda} - \frac{w_{ji}}{w_{ii}}\left[\frac{L_j}{w_{jj}} - \frac{1}{w_{jj}^2\Lambda} - \frac{w_{ij}}{w_{jj}}(x_i + y_i)\right] - y_i$$

so that,

$$x_i = \left(\frac{L_i}{w_{ii}} - \frac{1}{w_{ii}^2 \Lambda} - \frac{w_{ji}}{w_{ii}} \left[\frac{L_j}{w_{jj}} - \frac{1}{w_{jj}^2 \Lambda} - \frac{w_{ij}}{w_{jj}} y_i \right] - y_i \right) \left(\frac{w_{ii} w_{jj}}{(w_{ii} w_{jj} - w_{ij} w_{ji})} \right)$$

$\rightarrow$

$$x_i = \frac{w_{jj} L_i - w_{ji} L_j}{(w_{ii} w_{jj} - w_{ij} w_{ji})} - y_i + \frac{w_{ji} - w_{ii}}{\Lambda (w_{ii} w_{jj} - w_{ij} w_{ji})}$$

and since we have,

$$w_{ii} = w_{jj} = (1 - g_{ij} g_{ji})^{-1}$$

$$w_{ji} = g_{ji} w_{jj}, \; and,$$

$$w_{ji} = g_{ij} w_{ii}$$

Then we have:

$$x_i^* = L_i - g_{ji} L_j - y_i - \frac{w_{ii}}{\Lambda} + g_{ji} \frac{w_{ii}}{\Lambda},$$

which assume $\boldsymbol{G}^T$ is a directed ring network will then have the equation above written as:

$$x_i^* = L_i - g_{ji} L_j - y_i - \frac{w_{ii}}{\Lambda} (1 - g_{ji}).$$

So that in so far as $\Lambda > 0$ so that the last part of the equation does not become undefined, then x_i exists and is well defined. $\square$

Proof of Proposition 1.4.1

Say we have the following payoff for a firm $i \in \mathcal{D}$ given as follows;

$$C_i = \Lambda \log \left(L_i - w_{ii}(x_i + y_i) - \sum_{j \in \mathcal{N}_i^{in}} w_{ij}(x_j + y_j) + 1 \right) + x_i$$

and we have the following boundary of x_i as

$$\begin{aligned} x_i &= 0 \text{ and,} \\ x_i &= \frac{1}{w_{ii}} \left(L_i - \sum_{j \in \mathcal{N}_i^{in}} w_{ij}(x_j + y_j) \right) - y_i. \end{aligned}$$

Then When $x_i = 0$, we have

$$C_i = \Lambda \log \left(L_i - w_{ii}(y_i) - \sum_{j \in \mathcal{N}_i^{in}} w_{ij}(x_j + y_j) + 1 \right).$$

For $x_i = \frac{1}{w_{ii}} \left(L_i - \sum_{j \in \mathcal{N}_i^{in}} w_{ij}(x_j + y_j) \right) - y_i$, we have the following below;

$$\begin{aligned} C_i = \Lambda \log & \left(L_i - w_{ii} \left[\frac{1}{w_{ii}} \left(L_i - \sum_{j \in \mathcal{N}_i^{in}} w_{ij}(x_j + y_j) \right) - y_i + y_i \right] - \sum_{j \in \mathcal{N}_i^{in}} w_{ij}(x_j + y_j) + 1 \right) \\ & + \frac{1}{w_{ii}} \left(L_i - \sum_{j \in \mathcal{N}_i^{in}} w_{ij}(x_j + y_j) \right) - y_i \end{aligned}$$

$$C_i = \Lambda \log 1 + \frac{1}{w_{ii}} \left(L_i - \sum_{j \in \mathcal{N}_i^{in}} w_{ij}(x_j + y_j) \right) - y_i.$$

Therefore we have that

$$C = \frac{1}{w_{ii}} \left(L_i - \sum_{j \in \mathcal{N}_i^{in}} w_{ij}(x_j + y_j) \right) - y_i$$

If we assume that Λ and first wave default are substantially large such that $\mathbf{C}(\mathbf{x} > \mathbf{0}) \leq \mathbf{C}(\mathbf{x} = \mathbf{0})$ then we imply that;

$$\forall i, j \in \mathcal{N}, \; x_i = \frac{1}{w_{ii}} \left(L_i - \sum_{j \in \mathcal{N}_i^{in}} w_{ij}(x_j + y_j) \right) - y_i.$$

WLOG, assume that $\{i, j\} \in \mathcal{D}$ and $\boldsymbol{G}$ is a directed ring. The best replies for the Firm i becomes;

$$x_i = \frac{1}{w_{ii}}\left(L_i - w_{ij}\left[\frac{1}{w_{jj}}\left(L_j - w_{ji}(x_i + y_i)\right) - y_j + y_j\right]\right) - y_i$$

which gives;

$$x_i = \frac{1}{w_{ii}}\left(L_i - w_{ij}\left[\frac{L_j}{w_{jj}} - \frac{w_{ij}}{w_{jj}}(x_i + y_i)\right]\right) - y_i$$

⇒ So that collecting the like terms gives;

$$x_i^*\left(\frac{w_{ii}w_{jj} - w_{ji}w_{ij}}{w_{ii}w_{jj}}\right) = \frac{L_i}{w_{ii}} - \frac{w_{ij}L_j}{w_{jj}} - y_i\left(\frac{w_{ii}w_{jj} - w_{ji}w_{ij}}{w_{ii}w_{jj}}\right),$$

Which is the same as

$$x_i^* = \frac{w_{jj}L_i - w_{ij}L_j}{w_{ii}w_{jj} - w_{ij}w_{ji}} - y_i.$$

Then recall that the following holds,

$$w_{ii} = w_{jj} = (1 - g_{ij}g_{ji})$$
$$w_{ij} = g_{ji}w_{jj} \; and$$
$$w_{ji} = g_{ij}w_{ii}$$

Substituting into x_i^* above gives us;

$$x_i^* = \frac{(1 - g_{ij}g_{ji})^{-1}(L_i - g_{ji}L_j)}{(1 - g_{ij}g_{ji})^{-2}(1 - g_{ij}g_{ji})} - y_i$$

and finally The Nash for Firm i is ;

$$x_i^* = L_i - g_{ji}L_j - y_i \tag{1.26}$$

Implying *First wave default.*

Bibliography

Acemoglu, D., Ozdaglar, A., & Tahbaz-Salehi, A. (2015). Systemic risk and stability in financial networks. *The american economic review*, *105*(2), 564–608.

Allouch, N. (2015). On the private provision of public goods on networks. *Journal of Economic Theory*, *157*, 527–552.

Allouch, N., & Jalloul, M. (2016). Strategic defaults in financial networks. *Queen Marys University Working Paper series*.

Amini, H., Filipović, D., & Minca, A. (2016). Uniqueness of equilibrium in a payment system with liquidation costs. *Operations Research Letters*, *44*(1), 1–5.

Bergstrom, T., Blume, L., & Varian, H. (1986). On the private provision of public goods. *Journal of public economics*, *29*(1), 25–49.

Blume, L., Easley, D., Kleinberg, J., Kleinberg, R., & Tardos, É. (2011). Network formation in the presence of contagious risk. In *Proceedings of the 12th acm conference on electronic commerce* (pp. 1–10).

Bramoullé, Y., & Kranton, R. (2007). Public goods in networks. *Journal of Economic Theory*, *135*(1), 478–494.

Bramoullé, Y., Kranton, R., & D'Amours, M. (2010). Strategic interaction and networks.

Buiter, W. H. (2008). Central banks and financial crises.

Cabrales, A., Gottardi, P., & Vega-Redondo, F. (2014). Risk-sharing and contagion in networks.

Demange, G. (2016). Contagion in financial networks: a threat index. *Management Science*, *64*(2), 955–970.

Dubey, P., Geanakoplos, J., & Shubik, M. (2005). Default and punishment in general equilibrium. *econometrica*, *73*(1), 1–37.

Eisenberg, L., & Noe, T. H. (2001). Systemic risk in financial systems. *Management Science*, *47*(2), 236–249.

Elliott, M., Golub, B., & Jackson, M. O. (2014). Financial networks and contagion. *The American economic review*, *104*(10), 3115–3153.

Elsinger, H., Lehar, A., & Summer, M. (2006). Risk assessment for banking systems. *Management science*, *52*(9), 1301–1314.

Feinstein, Z. (2017). Financial contagion and asset liquidation strategies. *Operations Research Letters*, *45*(2), 109–114.

Glover, B. (2016). The expected cost of default. *Journal of Financial Economics*, *119*(2), 284–299.

Jorion, P. (2000). Risk management lessons from long-term capital management. *European financial management*, *6*(3), 277–300.

Mitchener, K. J., & Weidenmier, M. D. (2008). The baring crisis and the great latin american meltdown of the 1890s. *The Journal of Economic History*, *68*(2), 462–500.

Paddrik, M., Rajan, S., & Young, H. P. (2020). Contagion in derivatives markets. *Management Science*.

Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica: Journal of the Econometric Society*, 520–534.

Strebulaev, S. A. D. I. A., & Zhao, X. (2011). A market-based study of the cost of default.

Tallman, E. W., Moen, J. R., et al. (1990). Lessons from the panic of 1907. *Economic Review*, *75*, 2–13.

Chapter 2

Frictions in Financial Networks

2.1 Introduction

Networks in itself involves set of nodes connected by some link. While theoretical financial networks shows directed interlinks between nodes(which could be bilateral) by frequently a relative liability/asset ratio, the reality is that nodes may represent more unique and detailed property a mere asset/liability ratio may fail to reveal for analytical purpose. This means that simple network diagrams do not highlight the special properties of both the nodes involved or details on the links that bind two of such interconnected nodes. It is noticeable that, in networks bounded by debt or payment obligations, payments are assumed to be transferable between parties at no cost. As such the clearing payment system are designed with such implicit assumption. A core example is the fictitious default sequence proposed by Eisenberg and Noe (2001). However, the reality is that payments in many cases may not happen face to face or without the absence of barriers. Embedded within links between firms could include as impediments to payments. These strain of payments thus spur the need for an adequate intermediary to facilitate smoother transactions. Intermediaries might include banks and other monetary institutions. Take an example of a payment system existing across different economies with different currencies, exchange rates then comes into play. Even in cases where exchange rates are stable over period to payment, there are always tendencies for additional charges to exist per transaction when dealing with foreign currency operations. Impediments involving foreign currency transaction might hint a policy to reduce pressure on existing countries exchange. However in spite of such reason, what this means for the indebted firm is that payments made would have to account for that additional variable payment costs, hence, frictions to payments. Even in generic local transactions, there exists bank charges where banks take a certain quota for a transaction as more or less, commission for their services. In general, forms of transaction cost involved in movement of cash could include; Bank Transfer Charges, Broker/agency fees, Foreign exchange charges, Change in exchange rates in a network involving foreign exchange debts, recovery cost, other forms of commissions.

We henceforth use transaction cost and frictions interchangeably for the sake of this work. This work for this reason attempts to use the equilibrium properties with the transaction cost to observe the reaction threat indexes to frictions in the system. In a realistic sense, these forms of transaction cost could be caused by a host of factors. Factors in this case is grouped into two listed as follows;

1. **Market Induced Frictions:** Also known as Over-the-counter frictions. These forms refer to those introduced by the intermediaries who in turn raise the amount of transaction cost to vary with the demand intensity of these transactions. Other factors that potentially could be considered is the firms financial health, potential systemic risk as well as other factors including long-term reputation, market structure between intermediaries (competition vs collusion), etc.

2. **Regulatory Imposed Frictions:** These are set of transaction cost directly

imposed by a regulator which oversees the entire system. They are a product of various amount of sources and in many case may be a form of market regulation (a price ceiling on floor) between firms and their intermediaries.

For the lens of this essay, transaction cost is viewed as one which originates from regulatory imposed sources, hence not a direct product of market interactions between firms and intermediaries. The essay primarily captures a system where payments are made with certain cost proportional to the levels of payments and as such, though assets are liquid, an amount is paid to get them to their creditors. We begin with results on the clearing properties of the Eisenberg and Noe (2001) system as well as characteristics of the fictitious default system with transaction cost . Because Demange (2016) is primarily built upon Eisenberg and Noe (2001) equilibrium, we begin by identifying clearing characteristics in a system with transaction cost, then proceed into observing potential behaviors of threat indexes where these transaction costs hold. We show that a unique clearing payment still exist with transaction cost when similar criteria to those laid out in Eisenberg and Noe (2001) are met. We then proceed to show that in ranking threat indexes, the total relative link to direct creditors are the dominant part of a firms threat index compared to other firm. This result is then summarized into stylized topology such as ring, star as well as nested split graphs.

Related Literature

Financial frictions in most mainstream economic analysis has paid greater attention to its effects on general equilibrium. Various examples of works linked to general equilibrium would include Buss and Dumas (2013) who develops a general equilibrium model where agents perform everyday trading activities with transaction costs and as such, the roles in which transaction cost play in portfolio choices of investors, Hasman, Samartín Sáenz, and van Bommel (2009) proposes an overlapping generations model where agents assume intermediaries help to reduce transaction cost. Some important results they found is that transaction costs reduces cyclical effect in a system. This is notably similar with our results on transaction costs on threat index. Also, Duncan and Nolan (2018) summarizes groups of literature focused on financial frictions in DSGE models and examines loopholes. However, since the advent of increased study of systemic risk through financial networks which has in recent time gained increasing attention, there becomes the interest in understanding the importance of frictions in such partial equilibrium models. Eisenberg and Noe (2001) establishes a generic model of systemic risk to which some agent(s) initiates defaults and such defaults leads to other connected agent(s) defaulting. In extreme cases, these defaults even feeds back to the origin which serves as an easy prototype of a systemic risk/crises. In an extension of this work, models of frictions has been captured by authors such as Amini, Filipović, and Minca (2016) as well as Feinstein (2017) where potential defaulting agents incur liquidation cost on assets to meet up to such debts and the role in which systemic risk has to play.

Clearing payment equilibrium has been adapted to capture policy decision in works such as in Demange (2016) who uses these systemic properties to identify injection points of firms by using a centrality measure known as the *threat index*. As a ranking measure, e firms with higher threat index make the most valuable point to inject cash in order to increase payment in a defaulting system. This index is identical to the Bonacich Centrality measure for each firm in the default network.[1] This is because firms with higher threat indexes are the ones who can potentially reach the great number of other defaulting firms and with greater magnitude. Hence, a measure of power and influence as in standard network centrality literature.[2]

[1] See Bonacich (1987).
[2] More precisely Bonacich as well as Eigenvector Centrality.

2.2 Transaction Cost in Debt Clearing Systems.

Consider an economy with $\mathcal{N} = \{1, 2, ..., n\}$ set of firms. Firm i's endowment is z_i. The endowment of firm i denotes the cash flows arriving from outside the financial system. For each firm $i \in \mathcal{N}$, let the neighbourhood set be given as $\mathcal{N}_i$ such that $\mathcal{N}_i = \{\mathcal{N}_i^{out} \cup \mathcal{N}_i^{in}\}$ Also, let L_{ij} denote the liability that firm i owes firm j such that $L_{ij} \in]0, \mathbb{R}^+]$ if and only if firm $j \in \mathcal{N}_i^{out}$. Then, firm i's total liabilities is $L_i = \sum_{j \in \mathcal{N}_i^{out}} L_{ij}$. Meanwhile $\sum_{\mathcal{N}_i^{in}} L_{ji}$ is the total assets of firm i. Let $\boldsymbol{G} = [g_{ij}] \in \mathbb{R}_+^{n \times n}$ denote the matrix of relative liabilities, with entries $g_{ij} = \frac{L_{ij}}{L_i}$ representing the ratio of the liability firm i owes to firm j over the total amount of liabilities of firm i. The network formed from $\boldsymbol{G}$ is denoted as $G(\mathcal{N}, g)$. We also assume that edges (g's) form at least on cycle such that other agents not in the cycle are sink nodes. Also, we assume that z_i is readily available and firms, even in default do not liquidate. Standard with the model, all firms are also of equal status including sink nodes.

Let $\boldsymbol{\pi} = (\pi_i)_{i \in \mathcal{N}}$ denote the clearing payment vector, uniquely[3] defined as in Eisenberg and Noe (2001) such that for each agent i it holds that:

$$\pi_i = \min \left\{ z_i + \sum_{j \in \mathcal{N}_i^{in}} g_{ji} \pi_j ; L_i \right\} \tag{2.1}$$

As a key modification to this model, suppose for every edge, there exist a market friction or transaction cost $1 \geq \delta > 0$ representing a particular percentage of total payment made by an firm i to j which is deducted to cover this cost such that if firm j pays an amount we denote as π_j^{out} only $(1 - \delta) g_{ji} \pi_j^{out}$ reaches agent i. Let us denote this value as π_j^{in} such that we have,

$$\pi_j^{in} = (1 - \delta) \pi_j^{out}.$$

[3] Under mild assumptions.

Also, this then implies that a firm i needs to pay about $\frac{L_i}{1-\delta}$ to fully meet up to its obligation L_i if it can. Adapting this into the clearing mechanism, it becomes:

$$\pi_i^{out} = \min\left\{ z_i + (1-\delta) \sum_{j \in \mathcal{N}_i^{in}} g_{ji}\pi_j^{out}; \frac{L_i}{1-\delta} \right\}, \tag{2.2}$$

if we denote $\frac{L_i}{1-\delta} = \mathbb{L}_i$, then we have the (2.2) rewritten as;

$$\pi_i^{out} = \min\left\{ z_i + (1-\delta) \sum_{j \in \mathcal{N}_i^{in}} g_{ji}\pi_j^{out}; \mathbb{L}_i \right\}. \tag{2.3}$$

A simple interpretation of π_i^{in} would be that a firm i who does not default clears at $\pi_i^{in} = L_i$. We then define a set $\mathcal{D} = \{1, 2, \ldots, d\}$ such that a firm $i \in \mathcal{D}$ if at clearing, $\pi_i^{out} < \mathbb{L}_i$. Also we define a set $\mathcal{S} = \{1, 2, \ldots, s\}$ such that a firm $i \in \mathcal{S}$ $\iff$ firm $i \notin \mathcal{D}$. Hence, we bring in the concept of surplus set to include atleast one node who is safe and thus, in $\mathcal{S}$. In essence, the Eisenberg and Noe (2001) concept of the risk orbit applies so that for each firm i such that $\mathcal{N}_i^{out} \neq \{\}$, there exist a firm j which has the following properties;

1. $\mathcal{N}_j^{out} = \{\}$ so that the firm $j \in \mathcal{S}$ is a sink node.

2. There exist a directed path $i, g_{i0}, 0, g_{01}, 1, \ldots, n, g_{nj}, j$ from firm i to the firm j.

This is such that firm j has the potential to be paid less due to a default firm i.[4]

2.2.1 Clearing Payment with Homogeneous Frictions

The fixed point solution to (2.1) is ascertained through iteration which the fictitious default sequence solves for the clearing system as proposed by Eisenberg and Noe (2001). We adapt the proposed solution method to our model with $\delta \in]0, 1[$ so that

[4] This depends on the the payment of direct debtors to firm j. For example, a firm n for which $n \in \mathcal{N}_j^{in}$ and the firm n is also a within firm i's risk orbit.

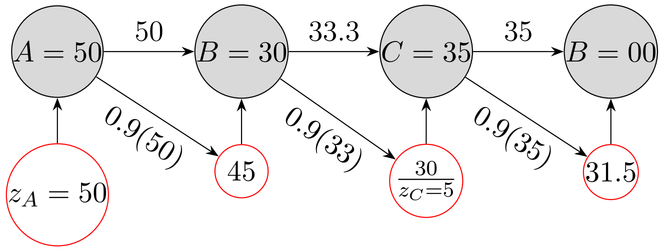


Figure 2-1: A hypothetical line network of a 3-paying Firm with 10% transaction cost: Arrows pointing downwards shows amount received by firms from debtors.

the clearing computation for the system that solves (2.3) for a given firm i. To show this solution, we divide the computation into 2 phases stated as follows:

Phase 1: This simply solves for the value of π_i^{out} for each firm $i \in \mathcal{N}$ so that it is simply given as;

$$\pi_i^{out} = \min \left\{ z_i + (1-\delta) \sum_{j \in \mathcal{N}_i^{in}} g_{ji} \pi_j^{out}; \mathbb{L}_i \right\}.$$

Phase 2: This phase is not particularly part of the default sequence iteration. This computes the value receivable by each firms creditors is shown in the equation below;

$$\pi_i^{in} = (1-\delta)\big(z_i + (1-\delta) \sum_{j \in \mathcal{N}_i^{in}} g_{ji} \pi_j^{out}\big), \qquad \forall \pi_i^{out} < \mathbb{L}_i,$$

so that opening the bracket we have:

$$\pi_i^{in} = (1-\delta) z_i + (1-\delta)^2 \sum_{j \in \mathcal{N}_i^{in}} g_{ji} \pi_j^{out}, \qquad \forall \pi_i^{out} < \mathbb{L}_i,$$

and we further denote $(1-\delta) z_i = \varepsilon_i$ the value firm i's creditors then receive would be:

$$\pi_i^{in} = \varepsilon_i + (1-\delta)^2 \sum_{j \in \mathcal{N}_i^{in}} g_{ji} \pi_j^{out}, \qquad \forall \pi_i^{out} < \mathbb{L}_i. \tag{2.4}$$

However for the firm i such that $\pi_i^{out} \geq \mathbb{L}_i$, it implies the firm $i \in \mathcal{S}$ and as such, $\pi_i^{in} = L_i$. This then applies to all firms who are safe. Additionally let us have the following definition;

Definition 2.2.1. $\forall\ i \in \mathcal{N}$, *let equity be defined as the residual asset after settling all obligations such that at clearing, equity is given as;*

$$eq_i \stackrel{\text{def}}{=} z_i + (1-\delta) \sum_{j \in \mathcal{N}_i^{in}} g_{ji} \pi_j^{out} - \mathbb{L}_i. \tag{2.5}$$

The (2.4) reveals that inflows from debtors are net of transaction cost while further transaction cost applies to payments to creditors as the movement of z_i to creditors is a one step movement from the firm i while the payment of total receipts to creditors yields a 2 step movement from each $j \in \mathcal{N}_i^{in}$ to another firm $k \in \mathcal{N}_i^{out}$.

2.3 Clearing Payment Properties

Fundamental clearing characteristics of a system with transaction cost relies of the following theorem which is hardly different from Eisenberg and Noe (2001) as follows;

Theorem 2.3.1. *Given a financial system including sink nodes whose edges contain transaction costs δ, There exist a clearing payment vector which satisfies the following conditions;*

1. *Limited Liability:* $\pi_i^{out} \leq z_i + (1-\delta) \sum_{j \in \mathcal{N}_i^{in}} g_{ji} \pi_j^{out}$ *as well as,*
2. *Absolute Priority:* $\pi_i^{out} = z_i + (1-\delta) \sum_{j \in \mathcal{N}_i^{in}} g_{ji} \pi_j^{out}$.

This clearing payment vector is unique in so far $\forall\ i \in \mathcal{N}$, the risk orbit is surplus set.

Proof. See Appendix for proof. ■

The result above simply shows that the clearing payment each firm $i \in \mathcal{N}$ makes to $\mathcal{N}_i^{out}$ is one which satisfies the condition of paying at most what it owes or what it has in pro-rata where the first option is not feasible. Regularity and existence of a uniquely defined clearing payment for all firm $i \in \mathcal{N}$ is a rule with very rare exceptions[5].

Shifting our focus to the fictitious default sequence, we observe that with existence of frictions, several firms are worse off in terms of value as a result of charges attributable to each higher level of payment they make. However, another intuition from the clearing system is that successive defaults by a firm implies that such firm avoids some value in transaction cost leakage. The reasoning is that those who are unable to meet up to their full obligation pay less and hence suffer less charges in total. This is a likely occurrence in networks with ring properties. From the definition definition 2.2.1, let equity of the system be captured in the column vector $\mathbf{eq} = (eq_i)_{i \in \mathcal{N}} \in \mathbb{R}^n_{++}$. Also, let us have other column vectors as $\boldsymbol{\pi}^{out} = (\pi_i^{out})_{i \in \mathcal{N}}$ and $\mathbf{z} = (z_i)_{i \in \mathcal{N}}$. Then we can rewrite (2.5) as

$$\mathbf{eq} = \mathbf{z} + (1 - \delta)\boldsymbol{G}^T \boldsymbol{\pi}^{out} - \boldsymbol{\pi}^{out}.$$

The total residual of the system is then $\sum_{i \notin \mathcal{S}} eq_i = \mathbf{1}^T \mathbf{eq}$. In the Eisenberg and Noe (2001) model, this amount does not depend on payment π. However, with friction, we introduce the following lemma;

Lemma 2.3.1. *Let iterations of the clearing system(Waves of defaults) be denoted as $K = \{1, \ldots, \kappa - 1, \kappa\}$ such that κ is the iteration that clears the system. Then with transaction cost ($\delta > 0$), then $\sum_{i=1}^n eq_i(1) < \ldots < \sum_{i=1}^n eq_i(\kappa)$ such that $\kappa : \kappa \to \mathbf{eq}$ is a strictly increasing concave function.*

Proof. See Appendix for proof. ■

[5]See Eisenberg and Noe (2001) for example of such exceptions.

The changing values of equity across waves of defaults gives us a hint that initial iterations of equity might not reveal signs of defaults feedback. Feedbacks occur strictly in networks where obligations are cyclical. The following results summarize this point;

Proposition 2.3.1 (Directed Ring Properties)**.** *Assume a network $G(\mathcal{N}, g)$ with a subgraph $G_1(\mathcal{N}_1, g)$ such that for every $\{0, n\} \subset \mathcal{N}_1$, there exists a closed walk given as $0, g_{01}, 1, \ldots, n-1, g_{n-1,n}, n, g_{n0}, 0$ between firm 0 and firm n. In so far as amount of leakage avoided as result of default at clearing is greater than the absolute value of the negative net equity of creditors at first wave, then defaults do not feedback.*

Proof. See Appendix for proof. ■

The network $G_1(\mathcal{N}_1, g)$ is as such *strongly connected* since a directed walk exists between every 2 firms within the network. This gives the opportunity for default feedback due to cyclical obligations and dependency. This happens when a firm, say firm i's initial default causes all firm in firm i's risk orbit (directed path) to default which makes firm i default even further. A common case of this kind of network is the directed ring network. The proposition 2.3.1 gives us the criteria for such feedback to be avoided. Hence, where it fails to hold, we have a firm defaulting more due to its initial default. We use the illustration below to show elaborate on this;

Example 2.3.1 (Illustration involving equity)**.** *Assuming we have a network as in fig. 2-2 below;*

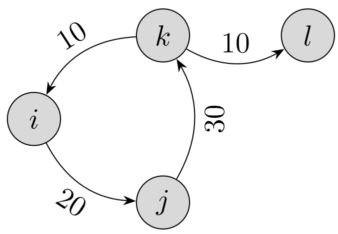


Figure 2-2: An Example of a 4-Firm Network

Other properties of the network are exogenous cash given as $Z = (5, 0, 10, 0)^T$ *while the liability is given* $L = (20, 30, 20, 0)^T$*, we have the following tables assuming* $\delta = 0$ *The clearing is solved in the table below;*

Table 2.1: Clearing without friction

$\delta = 0$				
Waves	Firms	Payment/Asset (π)	Equity(eq)	Net Equity
1st Wave	i	15	-5	5
	j	20	-10	
	k	40	20	
2nd Wave	i	15	0	5
	j	15	-5	
	k	30	10	
3rd Wave	i	15	0	5
	j	15	0	
	k	25	5	

assuming $\delta = 5\%$*, due to the friction, the liability becomes;*

$$\frac{L}{0.95} = (21.05, 31.58, 21.05, 0)^T$$

So that the payment system is then solved in the table below;

Table 2.2: Clearing with friction

$\delta = 0.05$				
Waves	Firms	Payment/Asset (π)	Equity(eq)	Net Equity
1st Wave	i	15	-6.05	1.32
	j	20	-11.58	
	k	40	18.95	
2nd Wave	i	15	0	2.2
	j	14.25	-5.75	
	k	29	7.95	
3rd Wave	i	15	0	2.48
	j	14.25	0	
	k	23.53	2.48	

From the table 2.2, \$2.48 − \$1.32 = \$1.16 is the equity gain from 1st wave through to the 3rd wave. This value is exactly the same as the amount of transaction cost Firms i and j did not pay as a result of default. In essence \$0.05(\$21.05 + \$31.58) − \$0.05(\$15+\$14.25) = \$2.63−\$1.4625 = \$1.16. The fact that the net equity (excluding firm l who is a sink node. Sink nodes are hence, always excluded) rises as more default increases satisfies our lemma 2.3.1.

Assuming no friction, if net equity was say −\$5 from the first wave, this means that it maintains the value such that the system falls short of the amount thus leading to default feedback. However, if from table 2.2 the net equity at 1st wave was −\$1,then it means that with gains of \$1.16 through the 3rd wave,[6] it leaves net equity at 3rd wave as \$0.16 which is positive for firm k and implies the system clears.

For further properties, we proceed introduce the following terminology;

Definition 2.3.1. *A given firm i is called a **Fragile firm** $\iff$ for the firm i,* $L_i - \left(z_i + \sum_{j \in \mathcal{N}_j^{in}} g_{ji}\pi_j\right) = eq_i(\delta|\delta = 0) \approx 0.$

As such, firms are classified as fragile when they have a near or even zero value after all their obligations are met. This definition is contextual as such threshold value might vary depending on the nature of the system. Then as a means of understanding the effect on such firm, we initiate the following proposition:

Proposition 2.3.2. *Let the total default of a firm i be $\vartheta_i(\delta) = \mathbb{L}_i - \pi_i^{out}$ Given transaction cost, a firm i who is a fragile firm would now have its total default amount as follows;*

$$\vartheta_i \approx \begin{cases} \delta\left(z_i + \sum_{j \in \mathcal{N}_i^{in}} g_{ji}L_j\right), & \text{if } \pi_j^{in} = L_j \ \forall \ j \in \mathcal{N}_i^{in}, \\ \delta\left(z_i + \sum_{j \in \mathcal{N}_i^{in}} g_{ji}\pi_j^{in}\right) + \sum_{j \in \mathcal{N}_i^{in}} g_{ji}\vartheta_j, & \text{if } \pi_j^{in} < L_j. \end{cases} \tag{2.6}$$

Proof. No further proof required. ■

[6]Usually the wave equivalent to the total number of firms in the loop.

Given the concept of a fragile firm, assume then that equity without friction is zero hence $eq_i(\delta = 0) = 0$. Then the statement above explains that a firm i defaults by the amount equal to δ times its total assets[7] in unique instance where firm i still receives its payment from debtors ($\mathcal{N}_i^{in}$) in full. Otherwise, it defaults by the amount equivalent to δ times its total asset as well as the total sum of default borne by the firm i from its $\mathcal{N}_i^{in}$.

2.4 Intervention Targeting with Market Frictions

We highlight the effect of transaction cost in decision making models. To do this let $\boldsymbol{x} = (x_i)_{i\in\mathcal{D}} \in \mathbb{R}_+^n$ such that x_i represents any exogenous cash amount added to the endowment z_i of the firm i which could be possibly be an outside intervention. We adopt the problem in Demange (2016). Assume a planner who has a value function denoted as $V(\boldsymbol{x})$ who estimates systemic defaults wishes to intervene in other to reduce or eliminate total default. Here, $V(\boldsymbol{x})$ is piece-wise linear and concave. Also $V(\boldsymbol{x})$ is sub-modular such that for all firm $i \in \mathcal{D}$, $x_i^{'} \geq x_i$ and $x_{-i}^{'} \geq x_{-i}$ implies that $V(x_i^{'}, x_{-i}^{'}) - V(x_i, x_{-i}^{'}) \leq V(x_i^{'}, x_{-i}) - V(x_i, x_{-i})$.

Assuming no fragile defaulting firm i, $V'(x_i) = \mu_i$ where μ_i is the threat index for the firm $i \in \mathcal{D}$ given as;

$$\mu_i = 1 + \sum_{j\in\{\mathcal{D}\cap\mathcal{N}_i^{out}\}} g_{ij}\mu_j. \tag{2.7}$$

When we then include frictions to the model, we then have(2.7) rewritten as;

$$\mu_i = 1 + (1-\delta) \sum_{j\in\{\mathcal{D}\cap\mathcal{N}_i^{out}\}} g_{ij}\mu_j. \tag{2.8}$$

Let us then have $\boldsymbol{\mu} = (\mu_i)_{i\in\mathcal{D}} \in \mathbb{R}_+^d$ as the vector of threat indexes of each firm. The

[7] Same as *delta* times firm i's total liability

equation above in vector form is written as follows;

$$\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta) = (\mathbf{I} - (1-\delta) \cdot \boldsymbol{G}_{\mathcal{D}\times\mathcal{D}})^{-1} \cdot \mathbf{1}_{\mathcal{D}} \tag{2.9}$$

Observe then that the equation above corresponds to the Bonacich Centrality of the network derived from the relative liability matrix of defaulting firms $\boldsymbol{G}_{\mathcal{D}\times\mathcal{D}}$. Hence, the threat points to the defaulting firm whose \$1 increase in asset(liquid cash) will lead to the greatest total payment in the system. This implies a multiplier effect such that in so far there exist a firm j such that $j \in \mathcal{D} \cap \mathcal{N}_i^{out}$, then the overall system payment from a \$1 increase to z_i would be greater than \$1.

Also, there are parallels that can be drawn with the threat index and *principal component* of a matrix $\boldsymbol{G}$ as proposed in Galeotti, Golub, and Goyal (2017). Though we have the contrast of such properties being obtainable in symmetric (adjacency) matrix, the intuitions are similar in that in games of strategic compliments (for example, the (2.1)), the greater the principal component of a firm, the greater its importance in intervention targeting. Similarly, the threat index here gives the same interpretation. Moreover, the amplifying factor in Galeotti et al. (2017) is synonymous to the discounted value $(1-\delta)$ in our model.

2.5 Endogenous Threat Index Rankings

We observe the impact of transaction costs to threat index rankings of firm in different default network topology. Before we proceed, let us introduce the following definition;

Definition 2.5.1. *A* ***direct relative liability*** *for any firm $i \in \mathcal{D}$ is the sum of relative liability to all defaulting firms. If we denote the direct relative liability for* *any firm $i \in \mathcal{D}$ as g_i, then we have;*

$$g_i = \sum_{j \in \{\mathcal{D} \cap \mathcal{N}_i^{out}\}} g_{ij}.$$

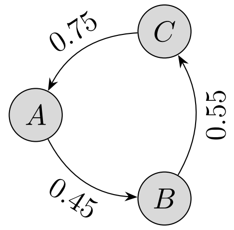


Figure 2-3: An Example of a 3-paying Firm Network which arrow shows relative liability

Keeping this in mind, we highlight some overall characteristics of the threat index under transaction cost in the following lemma;

Lemma 2.5.1. *In so far as there exists one or more sink-nodes such that $\mathcal{D} \subsetneq \mathcal{N}$, even with networks with cyclical properties $(\mathbf{I} - (1-\delta) \cdot \boldsymbol{G}_{\mathcal{D}\times\mathcal{D}})$ is always invertible.*

To which we then have the theorem as follows;

Theorem 2.5.1. *Given $\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta) = (\mathbf{I} - (1-\delta) \cdot \boldsymbol{G}_{\mathcal{D}\times\mathcal{D}})^{-1} \cdot \mathbf{1}_{\mathcal{D}}$, then as $\delta \to 1$ such that $\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta) \to \mathbf{1}_{\mathcal{D}}$, the direct relative liability g_i of each firm $i \in \mathcal{D}$ dominates the value μ_i.*

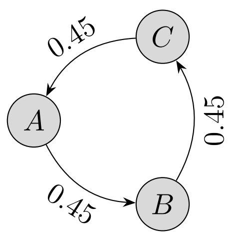
This result as such has significant implications especially as it pertains the ranking of each firms threat index. We explore some of implications of our theorem above by examining its implication in stylized network. Let $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta))$ be the rank of threat index $\forall$ elements of $\mathcal{D}$ while $\boldsymbol{g}(\mathcal{D}, \boldsymbol{G})) = (g_i)_{i\in\mathcal{D}}$ be the direct relative liability vector so that $Rank(\boldsymbol{g}(\mathcal{D}, \boldsymbol{G})))$ is the rank of each firms direct relative liability in ascending order. We highlight an important corollary built on the theorem 2.5.1 above as follows;

Corollary 2.5.1. *Assuming the threat index vector* $\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta)$ *is proportional to the the direct relative liability* $\boldsymbol{g}(\mathcal{D}, \boldsymbol{G})$*, then assume 2 values of* δ*, say* δ_1 *and* δ_2 *such that* $\delta_2 \gneq \delta_1$*. Then we have that* $\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta_1) = \boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta_2)$*.*

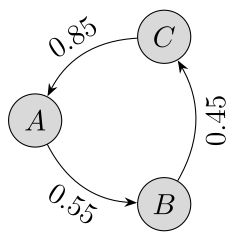
Proof. Particularly intuitive as since for each firm $i \in \mathcal{D}$, g_i dominates $\mu_i(\boldsymbol{G}, \delta)$ the greater the value of δ. ∎

This corollary then forms the most important intuition of our model as it means that where the $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta))$ is not proportional to $Rank(\boldsymbol{g}(\mathcal{D}, \boldsymbol{G}))$, then there exists the probability that given $\delta_2 \gneq \delta_1$, then $\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta_1) = \boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta_2)$ implying that threat index rankings are then altered by the different size of transaction cost. We discuss examples and conditions of this in stylised networks as follows;

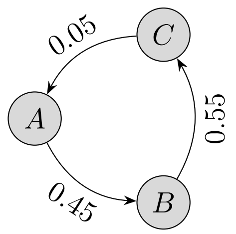
2.5.1 Sample 1: Directed Ring Networks



(a) Perfect Cycle



(b) Stronger Cycle



(c) Weaker Cycle

Figure 2-4: Three forms of Directed Ring Networks.

Take an example from fig. 2-4a, we have $\boldsymbol{g}(\mathcal{D}, \boldsymbol{G}) = (0.45, 0.45, 0.45)^T$

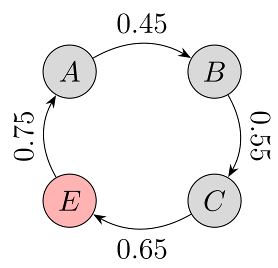
which is proportional to the threat index $\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}) = (1.82, 1.82, 1.82)^T$. Hence, $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta))$ remains constant for any $\delta \in]0, 1[$.

In the fig. 2-4b , $\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}) = (2.28, 2.32, 2.93)^T$ while $\boldsymbol{g}(\mathcal{D}, \boldsymbol{G}) = (0.55, 0.45, 0.85)^T$. Since $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta = 0)) \Rightarrow C > B > A$ while $Rank(\boldsymbol{g}(\mathcal{D}, \boldsymbol{G})) \Rightarrow C > A > B$, then $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta))$ is robust to δ. To confirm, take for example $\delta = 0.5$, then we have $\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta = 0.5) = (1.37, 1.36, 1.58)^T$ such that $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta = 0)) \Rightarrow C > A > B$ which then corresponds to $Rank(\boldsymbol{g}(\mathcal{D}, \boldsymbol{G}))$.

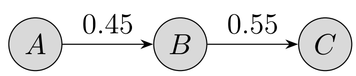
Lastly, for the fig. 2-4c , $\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}) = (1.72, 1.60, 1.09)^T$ while $\boldsymbol{g}(\mathcal{D}, \boldsymbol{G})) = (0.45, 0.55, 0.05)^T$. Since $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta = 0)) \Rightarrow A > B > C$ while $Rank(\boldsymbol{g}(\mathcal{D}, \boldsymbol{G})) \Rightarrow B > A > C$, then $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta))$ is robust to δ. To confirm, take for example $\delta = 0.7$, then we have $\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta = 0.7) = (1.16, 1.17, 1.02)^T$ such that $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta = 0)) \Rightarrow B > A > C$ which also corresponds to $Rank(\boldsymbol{g}(\mathcal{D}, \boldsymbol{G}))$.

2.5.2 Sample 2: Non-Cyclical Defaults

Here, we refer back to our construct of the default network $G(\mathcal{D}, g)$ with its corresponding matrix $\boldsymbol{G}_{d\times d}$ pointing to the fact that for a firm $i \in \mathcal{D}$ and firm $j \in \{\mathcal{D}' \cap \mathcal{N}_i^{out}\}$ where $\mathcal{D}'$ is the set of non-defaulting firms, $g_{ij}(\boldsymbol{G}_{d\times d}) = 0$.



(a) Ring network which firm $E \notin \mathcal{D}$.



(b) Default Network in line form

Figure 2-5: Shows a ring network with non-cyclical default being represented in Line form.

Because links to and from a non-defaulting firm are excluded from $\boldsymbol{G}_{d\times d}$, then if original network $G(\mathcal{N}, g)$ is a ring network of debtor firms such that one or more firm do not default, then the network of the default set $G(\mathcal{D}, g)$ is a line network. An example of this is reflected in fig. 2-5 where we see the representation between $G(\mathcal{N}, g)$ which is a ring as in fig. 2-5a and then the transformation to a line for $G(\mathcal{D}, g)$ as in fig. 2-5a.

For the threat index for defaulting firms in fig. 2-5 , we have that $\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}) = (1.70, 1.55, 1)^T$ while $\boldsymbol{g}(\mathcal{D}, \boldsymbol{G})) = (0.45, 0.55, 0)^T$ and it means $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta = 0)) \Rightarrow A > B > C$ while $Rank(\boldsymbol{g}(\mathcal{D}, \boldsymbol{G})) \Rightarrow B > A > C$. Again, if we have $\delta = 0.9$ for example, we then have $\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta = 0.6) = (1.219, 1.220, 1)^T$ such that $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta = 0)) \Rightarrow B > A > C$.

The combination of the ring and line default networks can improve our understanding of a special form of network known as Nested Split Graphs. Split graphs are increasingly studied as they relate to understanding qualities relating to subgroup behavior within network. As Belhaj, Bervoets, and Deroïan (2016) rightly points out, the core-periphery network is a special case of a Nested split graph. Here, we have 2 partition set namely the Core and the Periphery.

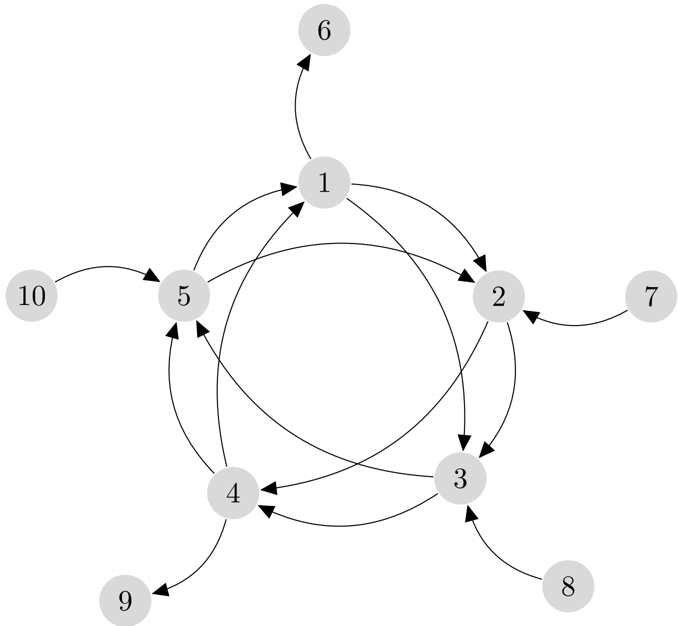


Figure 2-6: Core-periphery default network.

For directed network properties, we identify core set of firms as those whom are directly connected to other core firms while connected to one or more periphery. Periphery firms on the other hand are only connected to their corresponding core and as such are not connected to other periphery firms.

If we take an illustration from fig. 2-7 above, we observe that $\mu_6 = \mu_9 = 1$ and remains constant irrespective of δ. Subsequently, for the rest of the subgroup of defaulting firms $\{6,9\}' \subsetneq \mathcal{D}$, the criterion on the $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta = 0))$ and its proportionality to $Rank(\boldsymbol{g}(\mathcal{D}, \boldsymbol{G}))$ determines the responsiveness of $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta))$ to the size of δ.

2.5.3 Endogenous Defaults and the Threat Index

Let $\mathcal{D}' = \mathcal{S} \subsetneq \mathcal{N}$ represent the non-defaulting set of firms. Then we have the following statement;

Proposition 2.5.1. *Assume network graph $G(\mathcal{N}, g, \delta)$ which strongly connected*[8]*. If at $\delta = 0$, we have that $\mathcal{S} \neq \{\}$, $\exists\ 0 < \delta \leq 1$ to which $\mathcal{S}(\delta) = \{\}$ so that default becomes cyclical.*

This is because even otherwise safe systems become fragile in the face of high transaction costs on payments as more resources are then required to meet up with firms obligations.

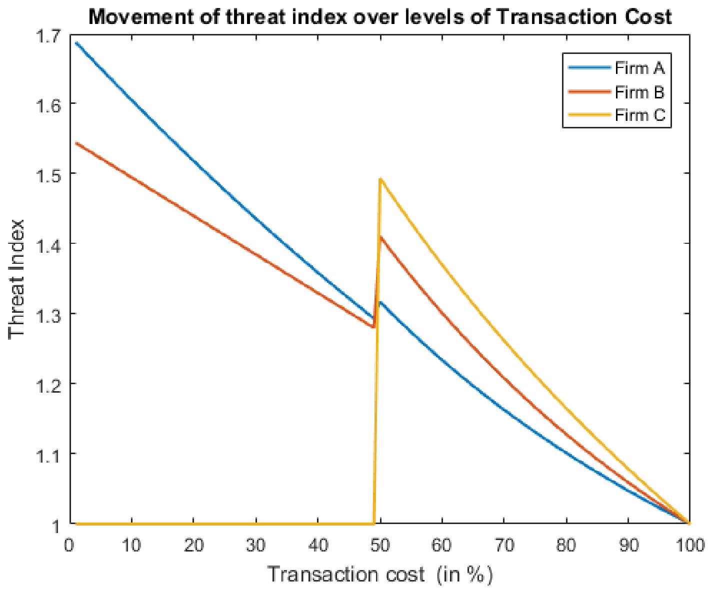


Figure 2-7: Where a full loop of firms default due to transaction cost: Observe the sharp rise in the threat index of Firm A and Firm B when Firm C joins in default due to an increase in δ.

This possibility is earlier introduced in section 3 where we examined the ability for a fragile system to face systemic crises when frictions are introduced and as such, exhibit feedback characteristics. Its implication then is that $\mathcal{D}$ is responsive in a discreet manner to the value of δ. This in turn affects the value of the threat index of each firm $i \in \mathcal{D}$ $\mu_i(\mathcal{D}, \boldsymbol{G}, \delta)$ as well as its rank $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta))$. An example can be seen from fig. 2-5a whose threat index relationship to δ is given in fig. 2-7. Observe then that at at $\delta = 0.5$, firm E now joins the default set and if

[8]Strongly connected graph here refers to a case where $\forall$ firms $0, n \in \mathcal{N}$, there exist a directed path(cycle) $0, g_{01}, 1, g_{12}, \ldots, g_{n-1,n}, n, g_{0,n}, 0)$.

your compare firm A, firm B and firm C's index as shown in the figure, firm C, $Rank(\boldsymbol{\mu}(\mathcal{D}, \boldsymbol{G}, \delta = 0.5)) \Rightarrow C > B > A$. Also observable is the snap switch between μ_A and μ_B in terms of ranking. This is the implication of a change in the composition of the default set.

2.6 Conclusions

The implication of our results align with the policy purpose of the threat index as in Demange (2016). As such since a regulator who has limited intervention potential is best to target the firm/bank with the greatest threat index, then the main implication of our result is that at varying level of homogeneous transaction cost, different firms might meet such criteria whom otherwise would not. Though contagion tend to grow weaker with frictions, increased risk of initial defaults as consequence of frictions could alter significantly which firm becomes the most desirable for monetary intervention. Overall, since cash policies are optimally targeted at banks/firms with greater default impact, frictions could put an otherwise bank of lower importance to a higher ranking position.

As is noticeable by now, our concept of frictions used so far assumes homogeneity which is easily not the case in real life. Additional intuitions should be obtainable from setting a system with heterogeneous transaction cost. Also, the model could be extended to an inter-temporal setting where firms might anticipate frictions and adopt prudential behaviours can be considered. Here, assuming such frictions remain exogenous to the firm, it is concerned with such friction to the extent it might lead to it defaulting to its creditors. This would have a lot to do with default punishment as well as a dynamic model to capture forecast trends that might predict the introduction of frictions to the payment system. lastly, we do fail to capture transaction cost as a product of market equilibrium decision between firms and financial intermediaries. We believe that further work on this aspect has the prospect of improving our knowledge on firms interactions and decisions given such market induced transaction cost.

2.7 Appendix

Proof of Theorem 2.3.1

With (2.2), we have $\pi_i^{out} = \min\left\{ z_i + (1-\delta)\sum_j g_{ji}\pi_j^{out}; \frac{L_i}{1-\delta} \right\}$ which still follows the Knaster-Tarski fixed point theorem such that if we use $\wedge$ to denote "the least between": the monotone mapping $f : \boldsymbol{\pi}^{out} \rightarrow (\mathbf{z} + (1-\delta)\boldsymbol{G}^T\boldsymbol{\pi}^{out}) \wedge \frac{\mathbf{L}}{1-\delta}$ of a complete lattice $[0, \frac{\mathbf{L}}{1-\delta}] \rightarrow [0, \frac{\mathbf{L}}{1-\delta}]$ consists of greater an least fixed points ($\underline{\boldsymbol{\pi}}^{out}$, and $\overline{\boldsymbol{\pi}}^{out}$).

And since the set $[0, \frac{\mathbf{L}}{1-\delta}]$ has the property that it is both convex, compact and f is a continuous mapping, the existence of fixed point still follows the classical Brouwer fixed point theorem. Uniqueness of a clearing payment vector implies the greater and least $\boldsymbol{\pi}^{out}$ are equal to each other as such $\boldsymbol{\pi}^{out+} = \boldsymbol{\pi}^{out-}$. This is the case all firms affected by debtor firms are a surplus set (hence, atleast on of such firm has a positive initial cash value, i.e, given a cyclical network n, $\mathbf{1}_n^{'}\mathbf{z}_n >> 0.$). This is always the case as long as a sink node exists because intuitively, it is a creditor firm with no obligation and so fare as $\sum_{i\in\mathcal{N}} z_i > 0$, then the net value of the sink node is greater than zero.

Since $(1-\delta)$ is infused into a uniquely clearing EN model, then we recall that the EN clearing is likened to a game of strategic complimentarity as in Ballester, Calvó-Armengol, and Zenou (2006) such that $(\mathbf{I}-\boldsymbol{G}^T)$ is positive definite such as to guarantee uniquely defined payment vector without friction. Let $\lambda_{max}(\boldsymbol{G}^T)$ be the maximum eigenvalue of $\boldsymbol{G}^T$. Then $(\mathbf{I}-\boldsymbol{G}^T)$ being positive definite implies $\lambda_{max}(\frac{\boldsymbol{G}^T+\boldsymbol{G}}{2}) < 1$. $(\mathbf{I}-(1-\delta)\boldsymbol{G}^T)$ then depends on the condition that $(1-\delta) \in \left]0, \frac{1}{\lambda_{max}(\frac{\boldsymbol{G}^T+\boldsymbol{G}}{2})}\right[$. Since we then have $(1-\delta) \in]0,1[$ and $\lambda_{max}(\frac{G^T+G}{2}) < 1$ such that $\frac{1}{\lambda_{max}(\frac{G^T+G}{2})} > 1$, then clearing payment is always unique as transaction cost always satisfies such criteria. □

Proof of Lemma 2.3.1

Recall the first iteration of the payments in the fictitious default sequence assumes that $\forall$ firm $i \in n$, $\pi_i^{out} = \mathbb{L}_i$ and as such the total transaction cost absorbed based on that assumption is $\delta \sum_{i=1}^{n} \mathbb{L}_i$.

But then the reason for a higher iteration is because of the existence of firms who default in the first wave. We group firms into default and no default categories by using the operator given as:

$$\psi_i(\kappa) = \begin{cases} 1 \text{ if firm } i\text{, up to the } \kappa^{th} - iteration\text{, is a defaulter , and} \\ 0 \text{ otherwise.} \end{cases}$$

Then the total value of transaction cost for leaving the system based on the second iteration(second default wave) would be given as:

$$\delta \sum_{i=1}^{n} (1-\psi_i(2))\mathbb{L}_i(2) + \delta \sum_{i=1}^{n} \psi_i(2)\pi_i^{out}(2) < \delta \sum_{i=1}^{n} \mathbb{L}_i. \tag{2.10}$$

So that up until clearing, the adjusted level of transaction cost over all iterations is captured in the expression written as:

$$\begin{aligned} &\delta \sum_{i=1}^{n} (1-\psi_i(\kappa))\mathbb{L}_i(\kappa) + \delta \sum_{i=1}^{n} \psi_i(\kappa)\pi_i^{out}(\kappa) \\ &< \delta \sum_{i=1}^{n} (1-\psi_i(\kappa-1))\mathbb{L}_i(\kappa-1) + \delta \sum_{i=1}^{n} \psi_i(\kappa-1)\pi_i^{out}(\kappa-1) < \dots \\ &< \delta \sum_{i=1}^{n} (1-\psi_i(2))\mathbb{L}_i(2) + \delta \sum_{i=1}^{n} \psi_i(2)\pi_i^{out}(2) < \delta \sum_{i=1}^{n} \mathbb{L}_i. \end{aligned}$$

Proof of Proposition 2.3.1

From lemma 2.3.1 is that waves of default imply lesser overall payment ($\sum_{i=1}^{n} \pi_i^{out}$) payment and as such, lower transaction cost.

As such Equity is strictly increasing with respect to π^{out} and as such order preserving/strictly monotone. The implication on a closed walk graph $G_1(\mathcal{N}_1, g)$ is that since $\sum_{i\in n} eq_i(k=n) - \sum_{i\in n} eq_i(k=0) = \delta(\sum_{i\in\mathcal{D}} \mathbb{L}_i - \sum_{i\in\mathcal{D}} \pi_i^{out})$, it then means that $\sum_{i\in n} eq_i(k=n) = \delta(\sum_{i\in\mathcal{D}} \mathbb{L}_i - \sum_{i\in\mathcal{D}} \pi_i^{out}) + \sum_{i\in n} eq_i(k=0)$. Recall also that because π^{out} satisfies the Eisenberg and Noe (2001) limited liability condition, then it is the case that $\delta(\sum_{i\in\mathcal{D}} \mathbb{L}_i - \sum_{i\in\mathcal{D}} \pi_i^{out})$ is strictly positive. This means then that $\sum_{i\in n} eq_i(k=n) < 0 \iff |-ve \sum_{i\in n} eq_i(k=0)| > \delta(\sum_{i\in\mathcal{D}} \mathbb{L}_i - \sum_{i\in\mathcal{D}} \pi_i^{out})$. $\square$

Proof of Lemma 2.5.1

Recall that networks contains sink nodes. This is such that if we have a default set D given the entire network set $\mathcal{N}$, then it is the case that $\mathcal{D} \neq \mathcal{N}$ even when possible loops default. Also, because there is atleast a sink node, then atleast a firm in the *loop*(D) defaults to a non-defaulting firm such that $\exists i \in \mathcal{D}$ such that $\sum_{j\in\mathcal{D}} g_{ij} < 1$. This then implies that $\boldsymbol{G}_{\mathcal{D}\times\mathcal{D}} \cdot \mathbf{1}_{\mathcal{D}} << \mathbf{1}_{\mathcal{D}}$ and $\boldsymbol{G}^{(p)}_{\mathcal{D}\times\mathcal{D}} \cdot \mathbf{1}_{\mathcal{D}} << \mathbf{1}_{\mathcal{D}}$ where $p = 1, 2, \ldots$.

Additionally, the spectral radius $\rho(\boldsymbol{G}_{\mathcal{D}\times\mathcal{D}})$, $\rho(\boldsymbol{G}^2_{\mathcal{D}\times\mathcal{D}})$ ... are all less than 1(See Demange Appendix on proof). This is also the same as in $\rho(\boldsymbol{G}^T_{\mathcal{D}\times\mathcal{D}})$ because atleast the row(s) belonging to the value receivable are always less than 1. And since $\delta \in [0,1]$, then $(1-\delta)\boldsymbol{G}^T_{\mathcal{D}\times\mathcal{D}}$ where $\delta > 0$ can only reduce the value of the g's and as such, is just as invertible. $\square$

Proof of Theorem 2.5.1

So we have the parameter $\boldsymbol{\mu}(\mathcal{D},\delta) = ((\mathbf{I} - (1-\delta)\cdot \boldsymbol{G}_{\mathcal{D}\times\mathcal{D}})^{-1})\cdot \mathbf{1}$. Let us start by assuming that $\delta = 0$, then we have the following transformation;

$$\boldsymbol{\mu}(\mathcal{D},\delta) = \left((\mathbf{I} - \boldsymbol{G}_{\mathcal{D}\times\mathcal{D}})^{-1}\right)\cdot \mathbf{1} = \left(\mathbf{I} + \boldsymbol{G}_{\mathcal{D}\times\mathcal{D}} + \boldsymbol{G}^2_{\mathcal{D}\times\mathcal{D}} + \boldsymbol{G}^3_{\mathcal{D}\times\mathcal{D}} + \ldots\right)\cdot \mathbf{1}. \quad (2.11)$$

Recall that following theorem 2.5.1, $(\mathbf{I} - \boldsymbol{G}_{\mathcal{D}\times\mathcal{D}})$ is invertible for the following reasons already given in the proof of theorem 2.5.1. Now since it is already proven that the sum of rows in $\boldsymbol{G}_{\mathcal{D}\times\mathcal{D}}$ is strictly < 1 , then we know through power series that higher exponents of $\boldsymbol{G}_{\mathcal{D}\times\mathcal{D}}$ imply lower values of each elements and as such, lower row sums. This is so that if $\delta > 0$ then we have,

$$\boldsymbol{\mu}(\mathcal{D},\delta) = \left(\mathbf{I} + (1-\delta)\cdot \boldsymbol{G}_{\mathcal{D}\times\mathcal{D}} + (1-\delta)^2\cdot \boldsymbol{G}^2_{\mathcal{D}\times\mathcal{D}} + (1-\delta)^3\cdot \boldsymbol{G}^3_{\mathcal{D}\times\mathcal{D}} + \ldots\right)\cdot \mathbf{1} \quad (2.12)$$

And as δ is also strictly less than 1, then it means that the higher order parts die out even faster such that as $\delta \to 1$, $(1-\delta)\cdot \boldsymbol{G}_{\mathcal{D}\times\mathcal{D}}$ dominates all other parts and as such the rank of the threat index is purely based on $\boldsymbol{\mu}(\mathcal{D},\delta) = (\mathbf{I} + (1-\delta)\cdot \boldsymbol{G}_{\mathcal{D}\times\mathcal{D}})\cdot \mathbf{1}$. □

Proof of Proposition 2.5.1

Take a firm $i \in \mathcal{S}$ when $\delta = 0$ with the clearing function given earlier as in (2.3) as $\pi_i^{out} = \min\left\{z_i + (1-\delta)\sum_j g_{ji}\pi_j^{out}; \mathbb{L}_i\right\}$. Then as $\delta \to 1$, $\mathbb{L}_i \to +\infty$ and $(1-\delta)\sum_j g_{ji}\pi_j^{out} \to 0$ while z_i remains constant. And as z_i is clearly defined and less than ∞, then $\infty - z_i > 0$ so that firm i is no longer able to meet up its liability, hence, $\mathcal{S} \cup \mathcal{D} = \mathcal{D}$ such that all debtor are now in the default set. □

Bibliography

Amini, H., Filipović, D., & Minca, A. (2016). Uniqueness of equilibrium in a payment system with liquidation costs. *Operations Research Letters*, *44*(1), 1–5.

Ballester, C., Calvó-Armengol, A., & Zenou, Y. (2006). Who's who in networks. wanted: The key player. *Econometrica*, *74*(5), 1403–1417.

Belhaj, M., Bervoets, S., & Deroïan, F. (2016). Efficient networks in games with local complementarities. *Theoretical Economics*, *11*(1), 357–380.

Bonacich, P. (1987). Power and centrality: A family of measures. *American journal of sociology*, *92*(5), 1170–1182.

Buss, A., & Dumas, B. (2013). Financial-market equilibrium with friction.

Demange, G. (2016). Contagion in financial networks: a threat index. *Management Science*, *64*(2), 955–970.

Duncan, A., & Nolan, C. (2018). Financial frictions in macroeconomic models. In *Oxford research encyclopedia of economics and finance.*

Eisenberg, L., & Noe, T. H. (2001). Systemic risk in financial systems. *Management Science*, *47*(2), 236–249.

Feinstein, Z. (2017). Financial contagion and asset liquidation strategies. *Operations Research Letters*, *45*(2), 109–114.

Galeotti, A., Golub, B., & Goyal, S. (2017). Targeting interventions in networks. *arXiv preprint arXiv:1710.06026*.

Hasman, A., Samartín Sáenz, M., & van Bommel, J. (2009). Financial intermediaries

and transaction costs.

Chapter 3

Strategic Interactions in Financial Networks

3.1 Introduction

Financial behaviours can be studied in several ways including interactions which are a function of the link between parties(firms) involved. Links commonly modeled are those which captures contractual obligation of one party which represents an asset to the other party. Because a contract can only be created by the mutual consent of the two potential firms involved, links project such attributes among a host of properties. Also with these links comes exposure. Sometimes these exposure could be advantageous to firms for example in a firms ability to external shocks and promotion of stability (Elliott, Golub, & Jackson, 2014). However, where these links are binding and difficult to sever, they could be a source of negative spillover to one or more firms.

We explore endogenous interest(lending) rate determination in fixed lending/borrowing network where firms account for administrative overheads associated with levels of total lending and a percentage of borrowing. Inter-firm lending rate, as core of the contract clause, is the additional amount over a period of time to which a lending firm charges a borrowing firm for such transaction. In this case, firms give significant attention to cost arising from its debt management procedure. It is not a novel observation that the vast amount of businesses engage in credit financing. Many of such businesses have both creditors (suppliers, banks etc) as well as debtors (customers,retailers, etc). In order to effectively keep track and ensure a smooth settling process, administrative resources are then devoted towards different aspects of such overall loans. One major assumption we make is to relate the total amount of loans[1] to total administrative cost. The intuition here remains that large amount of loans implies large volume of transaction to which raises risk of significant amount of loss to both the personnel[2] involved and the business at large. Such delicate nature then spurs the need for remuneration to match up to such risk, hence greater expenses. Other types of market frictions could also add to this cost. This is so that assuming a single clearing period, all liabilities are settled such that the lending firm regains its cash and some premium.

As a main result, we show that the optimal lending rate a firm charges to its debtors is on derived from a game of strategic substitution. such substitution behaviours are in line with major public goods in network literature such as Bramoullé, Kranton, and D'amours (2014), Allouch (2015), etc. We then show that Nash equilibrium exists and is uniquely defined in pure strategies given our network game. We then discuss special equilibrium properties in networks such as the Core-periphery network which are specifically well bounded. We find that core firms are closely dependent on links from other core firms while giving less priority to periphery firms. Periphery firms on the other hand substitute from rates charged by their core interlinks.

Adopting a utilitarian welfare approach, we show welfare neutrality of firms,

[1] We discount those from creditors as we show later in the model.

[2] Accountant, Debt Administrator, Legal teams, etc.

whose equilibrium lending rate is positive, to intermediation policies. Given such neutrality, we hold that Pareto improvement of welfare can be achieved at zero cost to Planner by leaking out little amount from the paying system (such that active firms are fixed) and proportionally splitting it so as to subside for loan management expenses. Lastly, we then show quality of firms least externality based centrality is vital for targeting firms with subsidy policies.

Related Literature

Rates agreed upon by such parties become an asset in that they contribute potentially to profit of firms from an inter-firm lending in a given trading period. Sometime these rates are determined based on set regulatory benchmarks, for example in case of banks premiums are added to the inter-banks offered rate (e.g Libor as shown by Eisl, Jankowitsch, and Subrahmanyam (2017), Coulter, Shapiro, and Zimmerman (2017), Duffie and Dworczak (2014), Abrantes-Metz, Kraten, Metz, and Seow (2012) and Eaglesham (2013)). There might be other benchmark at typical firms level which for example might be the risk-free interest rate used in CAPM analysis. We however pay less attention to such parameter. Instead we focus on situations where by firms primarily make decisions as to their lending rate. An existing work in this is found in Aldasoro, Gatti, and Faia (2017) which models an endogenous interest premium in a system in which firms given potential cascading defaults/systemic risk. Hence, the additional premium is determined while accounting for estimated default probability.

Earlier discussion has made reference to strategic interactions and more sepcifically, a substitution relationship. Strategic interdependence here can be traced to network properties of our debtors and/or creditor firms as a form of financial network. Most financial network literature such as Morris (2000), Morris and Shin (2001), Allen and Babus (2008), Babus (2016), Bhattacharya, Gale, Barnett, and Singleton (1985) have focused on systemic risk as well as other issues to do with risk contagion and financial network stability e.g Caballero and Simsek (2013), Cabrales, Gottardi, and Vega-Redondo (2014), Nier, Yang, Yorulmazer, and Alentorn (2007), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), Greenwood, Landier, and Thesmar (2015)(on banks), Galeotti, Ghiglinoy, and Goyal (2016), König, Tessone, and Zenou (2009), Bilkic, Gries, et al. (2014), Gollier, Koehl, and Rochet (1997), etc. Others on network influence and power as in Demange (2016) as well as Aldasoro and Angeloni (2015) while a host of literature pay attention to liquidation as well as network financing such as Allouch and Jalloul (2016), Amini, Filipović, and Minca (2016), Feinstein (2017), Rogers and Veraart (2013) and Elsinger, Lehar, and Summer (2006) to mention but a few. However, it is noted that since systemic risk and stability is the key focus of most of the works mentioned above, strategic interaction plays less importance.

Additionally, little attempt have been given to link interactions in debt networks to public good games which yields best replies revealing strategic substitution. However, those on contagion in the previous par graph rely on strategic complements. Public good games with strategic substitution are found in in Allouch (2015) and Allouch and King (2018a)(which shows equilibrium in a fully bounded action profile). Also importantly is Bramoullé and Kranton (2007) and Bramoullé et al. (2014) where by interaction mostly based on strategic substitution is identified. Such games of public good provision and more specifically private public good provision can materialize in different ways in financial networks. That being said, the underlying ideas has not been aimed at identifying such behaviors in financial network games. A contrast of this work from seminal works including Bramoullé and Kranton (2007) and Allouch (2015) is the close attention paid to interactions to undirected network (with Bramoullé et al. (2014) providing initial intuitions as to weighted and direct network). While there are observable differences, intuitions are very useful in observing such behaviors in financial networks which are uni-directional and weighted in nature.

Our results on neutrality is also in distinction to neutrality of income redistribution. For income redistribution, neutrality holds such that wealth transfer between active agents in a public good game leads to no change in aggregate public good provision and individual consumption. These forms of neutrality is discussed well in Bergstrom, Blume, and Varian (1986), Wells (2004) as well as Allouch (2015) though we point out that our intervention are not particularly redistributive in nature. lastly, our targeting criterion has a lot of similarities to works like key player concepts in works like Ballester, Calvó-Armengol, and Zenou (2006), Galeotti, Golub, and Goyal (2020), Belhaj, Bervoets, and Deroïan (2016) as well as Belhaj and Deroïan (2019).

3.2 The Model

Assume a three period economy consisting of $\hat{\mathcal{N}} = \{1, \ldots, n\}$ set of firms. We denote the set of periods denoted as $\mathcal{T}$ such that $\mathcal{T} = \{t-1, t, t+1\}$. For every firm $i \in \hat{\mathcal{N}}$, its neighborhood is denoted as $\mathcal{N}_i$ and $\mathcal{N}_i = \{\mathcal{N}_i^{out} \cup \mathcal{N}_i^{in}\} \subset \hat{\mathcal{N}}$ where $\mathcal{N}_i^{out}$ represents firm $i \in \mathcal{N}$'s debtors and $\mathcal{N}_i^{in}$ represents firm $i \in \mathcal{N}$'s creditors. Debtors are those whom a firm lends to and creditors are who the firm loans from. The amount to be borrowed by each firm $i \in \hat{\mathcal{N}}$ is given as $b_i : b_i > 0 \quad \forall\, i \in \hat{\mathcal{N}}$. This interaction forms a borrowing network $G(\hat{\mathcal{N}}, \hat{g})$ with g representing links between firms.

Each firm strictly lends to each other based on the network $G(\hat{\mathcal{N}}, \hat{g})$ in t and as such $G(\hat{\mathcal{N}}, \hat{g})$ indicate borrowing/lending contract in this model. The network $G(\hat{\mathcal{N}}, \hat{g})$ is set at $t-1 \in \mathcal{T}$ so that it is exogenous to period t and $t+1$. At $t+1$ links are dissolved (cleared). Given that if firm $j \in \mathcal{N}_i^{in}$, then $\hat{g}_{ji} > 0$ while $\hat{g}_{ji} = 0$ otherwise. Hence, $\sum_{j \in \mathcal{N}_i^{in}} \hat{g}_{ji} = 1$ and $\sum_{j \in \mathcal{N}_i^{in}} \hat{g}_{ji} b_i = b_i$. A firm while lending charges an extra amount it sets at $t \in \mathcal{T}$ denoted as r_i. This rate r_i remains fixed for the rest of $\mathcal{T}$ once set. Also, we assume that firms then incur additional cost to manage debtors and creditors accounts and repayment procedures which we capture under loan management cost.

In the economy, a typical firm i's lending is then given as;

$$b_{-i} = \sum_{j \in \mathcal{N}_i^{out}} \hat{g}_{ij} b_j, \qquad \forall\, i \in \hat{\mathcal{N}}. \tag{3.1}$$

A sample balance sheet at $t+1$ for any firm in $\hat{\mathcal{N}}$ is represented in the table below;

Table 3.1: Firms balance Sheet at $t+1$

Assets	Amount	Liability	Amount
Debtors	$xxxx$	Creditors	$xxxx$
		Profit	$xxxx$

So then assume a firm $i \in \hat{\mathcal{N}}$ who is scheduled at t to lend to as well as borrow from other firms within the same system (Thus creating incoming and/or outgoing links). Then profit for the firm $i \in \hat{\mathcal{N}}$ intuitively drawn from its balance sheet marks its payoff which we show subsequently.

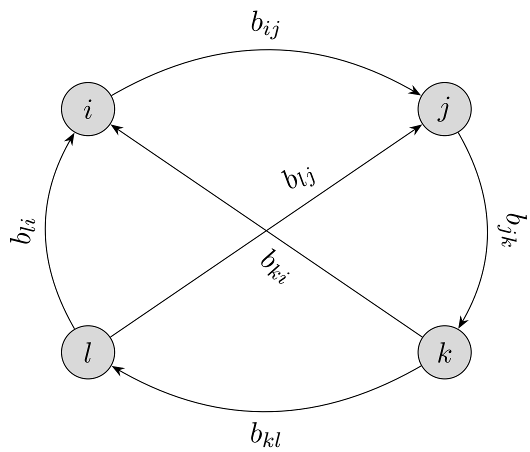
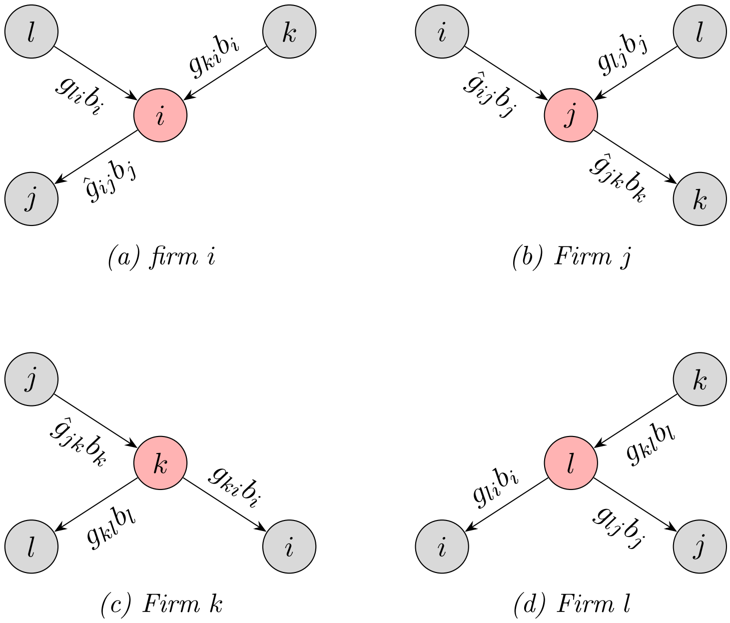


Figure 3-1: A Bounded Borrowing Network:
Arrows (edges) point to direction of borrowers and originate from lenders.

We observe that the firm $i \in \hat{\mathcal{N}}$ is only concerned about his lenders and borrowers



(a) firm i *(b) Firm j*

(c) Firm k *(d) Firm l*

Figure 3-2: Decomposed Network to capture pivotal links. We see that for decision making purpose, it is direct incoming and outgoing link that are useful to firms decision. This is as Loan management cost is spent mainly through managing a firms asset/liabilities.

as opposed to the entire network. Since we ignore default risk it then implies that regardless of the nature of borrowing network, each firm i can identify its position given a star which carries its lenders and borrowers.

Take an example of a system $\hat{\mathcal{N}} = \{i, j, k, l\}$ as in fig. 3-1. Observe also, the break down in $fig.\ 3-2$. It shows that from any directed network of borrowing and lending, for example the network in $fig.\ 3-1$, relevant sub-networks can be derived. Such sub-networks as in $fig.\ 3-2$ captures each individual firm $i \in \hat{\mathcal{N}}$ lending and borrowing.

To translate this into a firm $i \in \hat{\mathcal{N}}$ payoff, we additionally assume each firm $i \in \hat{\mathcal{N}}$ has included in its cost, loan management/administrative expenses. This loan management cost is split into 2 main parts. The first a homogeneous constant κ which measures the level of efficiency in managing overall debtors and creditors accounts and recovery process. In itself, a higher κ implies lesser efficiency in loan management while a lesser κ implies greater efficiency. Such efficiency could arise from specialization, technical know-how, technological progress and other factors that imply positive economies of scale for the firm. The second part is the endogenous loan size parameter which we denote as μ_i for the given firm $i \in \hat{\mathcal{N}}$. More formally, we define $\mu_{i\in\hat{\mathcal{N}}}$ as follows;

$$\mu_i(r_i, r_{-i\in\mathcal{N}_i^{in}}) = b_{-i} \bullet r_i + a \sum_{j\in\mathcal{N}_i^{in}} r_j \left(\hat{g}_{ji} b_i\right), \tag{3.2}$$

where the parameter $a \in \mathbb{R}_+$ captures the degree to which interest from debtors increases administrative cost. Let $\mathbf{r} = (r_i)_{i\in\mathcal{N}} \in \mathbb{R}_+^n$ be the lending rate vector for firms, we assume that the firm $i \in \hat{\mathcal{N}}$ has the following variable loan management cost;

$$\kappa \bullet f(\mu_i(r_i)) \tag{3.3}$$

such that we then have the following important assumption;

Assumption 3.2.1. $\forall$ *firm* $i \in \hat{\mathcal{N}}$, *we hold that;*

$$\frac{\partial f(\mu_i(r_i))}{\partial r_i} > 0, \quad \frac{\partial^2 f(\mu_i(r_i))}{\partial r_i^2} > 0.$$

We assume the mapping $r_i : r_i \to f(\mu_i)$ is convex due to the fact that we assume that cost exponentially rises as total variable loan obligation to and from the firm rises. The (3.3) captures the variable loan management cost which is weighed using the parameter κ that would usually assume a very small value. Loan management cost as defined here includes both the firm i's debtors management as well as its creditors the value $'a'$ discounts the total creditor portion.

3.2.1 Payoffs and Strategic Substitution

We then hold that the firm $i \in \hat{\mathcal{N}}$, $f(\mu_i) = (\mu_i)^2$ which fits well into assumption 3.2.1. Then the firm $i \in \hat{\mathcal{N}}$, has the following payoff function,

$$P_i(r_i|r_j,\ldots) = b_{-i} \bullet r_i - \sum_{j \in \mathcal{N}_i^{in}} r_j \left(\hat{g}_{ji} b_i\right) - \kappa(\mu_i)^2. \tag{3.4}$$

To elaborate, the payoff captures variable components of a firm $i \in \hat{\mathcal{N}}$ since for example, lends out a total of b_{-i} and at $t+1$, gets back $b_{-i} \bullet (1+r_i)$ from its debtors. So what it gets at $t+1$ is $\overline{b_{-i}} + b_{-i} r_i$. Additionally, it pays $\sum_{j \in \mathcal{N}_i^{in}} (1+r_j)\left(\hat{g}_{ji} b_i\right)$ to its creditors such that it is broken into $\overline{\sum_{j \in \mathcal{N}_i^{in}} \left(\hat{g}_{ji} b_i\right)} + \sum_{j \in \mathcal{N}_i^{in}} r_j \left(\hat{g}_{ji} b_i\right)$. We thus define the firm $i \in \hat{\mathcal{N}}$ payoff as one that captures only the parts which are multiples of the action profile $\mathbf{r}$.

To optimize P_i, the Lagrange equation is given as;

$$\max_{r_i \geq 0} Q_i(r_i) = b_{-i} \bullet r_i - \sum_{j \in \mathcal{N}_i^{in}} \hat{g}_{ji} b_i \left(r_j\right) - \kappa(\mu_i)^2 - \xi r_i \tag{3.5}$$

With the complementary slackness condition $\xi r_i = 0$ from the non-negativity constraint of $r_{i \in \hat{\mathcal{N}}}$. Then given $\xi = 0$ satisfies the condition, the first order condition equates the marginal benefit to marginal cost as shown below;

$$\frac{\partial Q_i}{\partial r_i} = 0 \Rightarrow 2b_{-i}(\mu_i) = \frac{b_{-i}}{\kappa}.$$

We then make r_i the subject of the formula using μ_i as in (3.2) so that we have the firm i's optimal lending rate as,

$$r_i = \frac{1}{2\kappa b_{-i}} - a \sum_{j \in \mathcal{N}_i^{in}} \frac{\hat{g}_{ji} b_i}{b_{-i}} r_j. \tag{3.6}$$

Let us have $\pi_i = \frac{1}{2\kappa b_{-i}}$ and $g_{ji} = \frac{\hat{g}_{ji} b_i}{b_{-i}} = \frac{b_{ji}}{b_{-i}}$ $\forall\ j \in \mathcal{N}_i^{in}$, we have;

$$r_i = \pi_i - a \sum_{j \in \mathcal{N}_i^{in}} g_{ji} r_j. \tag{3.7}$$

The linear reaction curve (best reply) for the firm i when $r_{i \in \hat{\mathcal{N}}} \in [0, \mathbb{R}^+]$ is given as;

$$r_i = \max \left\{ \pi_i - a \sum_{j \in \mathcal{N}_i^{in}} g_{ji} r_j, 0 \right\}. \tag{3.8}$$

The lending rate π_i reflects the autarkic amount charged to each firm j such that $j \in \mathcal{N}_i^{in}$. firm i is desires a greater r_i if it expects to lend in greater deal compared to its borrowing and thus submits its rate accordingly. However, the magnitude of its rate charged depends on its best reply. π_i additionally reveals the Engels curve for the firm i. Also, strategic substitution properties is captured in $\frac{\delta r_i}{\delta r_j} = -a g_{ji}$ for $j \in \mathcal{N}_i^{in}$.

Let $\mathbf{G} = [g_{ji}]$ be a zero-diagonal matrix and the game arising from (3.8) be denoted as $\Gamma(\mathbf{G}, \mathbf{r}, a)$. We make distinction between participating firms and those who do not participate in $\Gamma(\mathbf{G}, \mathbf{r}, a)$. This is because financial networks could posses cyclical interconnection as we see in line works within Eisenberg and Noe (2001) framework. Assume a subset $\mathcal{S} \subset \hat{\mathcal{N}}$. We have the formal definition;

Definition 3.2.1. *A firm $i \in \hat{\mathcal{N}}$ is a sink-node* $\iff$ $\mathcal{N}_i^{out} = \{\}$.

A Sink-node is a debtor to one or more firms but not a creditor to any other firm. Let the set $\mathcal{N} = \{1, \ldots, n\}$ be so that $\mathcal{S} \cup \mathcal{N} \subseteq \hat{\mathcal{N}}$ and $\forall$ firm $i \in \mathcal{N}$, $\mathcal{N}_i^{out} \neq \{\}$ and also $\mathcal{N}_i^{in} \neq \{\}$. This distinction is important for example if we have a firm i such that $\mathcal{N}_i^{out} = \{\}$, then $g_{ji} = \infty$ as $b_{-i} = 0$. It means are unable to define firm i's best reply as it makes no decision. Furthermore, we could have also the firm i such that $\mathcal{N}_i^{in} = \{\}$. Let $\mathbf{G}_i$ represent the $i - th$ row of the matrix $\mathbf{G}$, we would have $\mathbf{G}_i = (0)_{i \in \mathcal{N}}$ leading to a pure strategy Nash equilibrium $r_i = \pi_i$. This is described as *strategic dominance* as its lending rate is made in isolation. To avoid these instances, we introduce another important but common concept to directed networks as follows;

Definition 3.2.2. *A directed graph $G(\mathcal{N}, g)$ is* strongly connected *(SC) if and only if for every $\{0, n\} \in \mathcal{N}$, there exist a* ***closed directed walk*** *(the sequence $0, g_{01}, 1, g_{12}, \ldots, g_{n-1,n}, n, g_{0,n}, 0)$ from 0 to 0.*

Then going further, we will rely on the assumption written below;

Assumption 3.2.2. *The graph $G(\mathcal{N}, g)$ is strongly connected so that the set $\forall$ firm $i \in \mathcal{N}$, firm i is a strongly connected firm(SCF).*

This as such ensures that we avoid dominant equilibrium outcomes or undefined best replies given sink nodes (for any firm $i \in \mathcal{S}$, $b_{-i} = 0$ such that $r_i = \infty$).

3.3 Pure Strategy Solutions

We define in this section the shape and characteristics of equilibrium under such game $\Gamma(\mathbf{G}, \mathbf{r}, a)$.

3.3.1 Uniqueness and Stability

We present the existence of the equilibrium and conditions for uniqueness. To support our next few results, we define a key attribute which is positive definiteness a directed network as follows;

Definition 3.3.1. *Let $\mathbf{M}$ be a matrix and $\nu_1(M), \ldots, \nu_n(M)$ be the eigenvalues of the matrix $(\mathbf{M})$. Then M is positive definite if and only if it holds that;*

$$\nu_1\left(\frac{\mathbf{M} + \mathbf{M}^T}{2}\right), \ldots, \nu_n\left(\frac{\mathbf{M} + \mathbf{M}^T}{2}\right) > 0. \tag{3.9}$$

This definition is useful given the vast amount of public good in network literature emphasises symmetric matrix. Let the minimum eigenvalue of a matrix $\mathbf{M}$ be denoted as $\nu_{min}(\mathbf{M})$, we have the following lemma;

Lemma 3.3.1. *The matrix* $(\mathbf{I} + a\mathbf{G})$ *is positive definite in so far* $a \in \left]0, \frac{1}{\left|\nu_{min}(\frac{\mathbf{G}+\mathbf{G}^T}{2})\right|}\right[$.

Proof. See Appendix for proof. ∎

We summarize the properties of the equilibrium under $\Gamma(\mathbf{G}, \mathbf{r}, a)$ with the following proposition;

Proposition 3.3.1. *Given the parameter 'a' meets the boundary conditions as in lemma 3.3.1, there always exists a unique Nash equilibrium in pure strategies for the game* $\Gamma(\mathbf{G}, \mathbf{r}, a)$ *and the unique Nash equilibrium is always asymptotically stable.*

Proof. From Rosen (1965) concept of *diagonal strict concavity*, we understand that a sufficient condition for the payoff $P(\mathbf{r})$ to be diagonally strictly concave, then $\mathbf{H}(\mathbf{r}, \mathbf{1}) + \mathbf{H}(\mathbf{r}, \mathbf{1})^T$ must be negative definite where $\mathbf{H}(\mathbf{r}, \mathbf{1})$ is the Jacobian with respect to $\mathbf{r}$ of $P^{'}(\mathbf{r})$. Since it hold that the Jacobian $\mathbf{H}(\mathbf{r}, \mathbf{1}) = -(\mathbf{I} + a\mathbf{G})$, then the condition is achieve should $(\mathbf{I} + a\mathbf{G})$ be positive definite which lemma 3.3.1 satisfies. It is then shown that Nash equilibrium is unique if and only if lemma 3.3.1 is satisfied.

∎

This is so that each firm capture the amount charged by their creditors on borrowings in order to determine the rate charged to debtors without consideration for their own power.[3]

[3]The magnitude to which a firm $i \in \mathcal{N}$ rate charged r_i affects all other firms outcome.

3.3.2 Analysis of Equilibrium

We denote $\boldsymbol{\pi} = (\pi_i)_{i \in \mathcal{N}} \in \mathbb{R}^n_+$ as the autarkic-rate column vector while $\mathbf{r}^* = (r^*_i)_{i \in \mathcal{N}} \in \mathbb{R}^n_+$ is the Nash equilibrium vector. Following the best reply in (3.8), draw distinction between active and inactive firms in the definition below.

Definition 3.3.2. *A firm $i \in \mathcal{N}$ is thus defined as* active *if and only if $r^*_i(\mathcal{N}, a) \in]0, \mathbb{R}_+]$ and non-active if $r^*_i(\mathcal{N}, a) = 0$.*

Let the set of active firms be denoted with the set $\mathcal{A} \subseteq \mathcal{N}$ and hence non-active firms be $\mathcal{N} - \mathcal{A} \subsetneq \mathcal{N}$. Then using intuitions from Bergstrom et al. (1986), Bramoullé et al. (2014) and more closely, Allouch (2015), we have the following;

Proposition 3.3.2. *A set of rates vector $\mathbf{r}^*(\mathcal{A}, a)$ with active firms $\mathcal{A} \neq \{\}$ is a Nash equilibrium if and only if the following conditions hold true;*

1.

$$(\mathbf{I} + a\mathbf{G})_{\mathcal{A} \times \mathcal{A}} \bullet \mathbf{r}^*_{\mathcal{A}} = \boldsymbol{\pi}_{\mathcal{A}}$$

2.

$$a\mathbf{G}_{\mathcal{N}-\mathcal{A} \times \mathcal{A}} \bullet \mathbf{r}^*_{\mathcal{A}} \geq \boldsymbol{\pi}_{\mathcal{N}-\mathcal{A}}$$

Proof. See Appendix for proof. ■

The proposition above translates to the fact that firms become non-active when targets are achieved by simply charging a zero rate and thus, substitute for rates charged of active firms in such a way that the outcome is the same or is greater than the outcome from the non-active firms' autarkic rate charged to debtors. It also holds then that Nash equilibrium for the game has to include at least one active firms such that A cannot be a null set. We show a simple algorithm in the appendix which efficiently computes this equilibrium[4]. Furthermore, we draw the following statement from the proposition 3.3.2 as follows;

Corollary 3.3.1. *Assume that* $\forall\ i, j \in \mathcal{A}$, $b_{-i} = b_{-j}$ *so that* $\boldsymbol{\pi} = \pi \bullet \mathbf{1}_{\mathcal{A}}$. *This means that* $\mathbf{r}^*(\mathcal{A}, -a) = \pi \bullet \boldsymbol{b}(\mathcal{A}, -a)$ *so that* $\forall$ *firm* $i \in \mathcal{A}$*;*

$$r_i^*(\mathbf{G}, \mathcal{A}, a) = \pi \bullet \beta_i(\mathcal{A}, -a),$$

where $\beta_i(\mathcal{A}, -a)$ *refers the Bonacich independence index*[5] *or simply independence index of an active firm* i *implying* $\boldsymbol{b}(\mathcal{A}, -a) = (\beta_i(\mathcal{A}, -a))_{i \in \mathcal{A}} \in \mathbb{R}_+^n$.

Proof. Because we have the following;

$$\boldsymbol{b}(\mathcal{A}, -a) \stackrel{\text{def}}{=} (\mathbf{I} + a\mathbf{G})^{-1}_{\mathcal{A}\times\mathcal{A}} \bullet \mathbf{1}_{\mathcal{A}}. \tag{3.10}$$

■

This implies that Nash equilibrium rate is of each firm is directly proportional to their independence index. The independence index is so named because $\mathbf{G} = [g_{ji}]$ accounts for the strength of incoming links. Also since in the series, $(\mathbf{I} - a\mathbf{G})_{\mathcal{A}\times\mathcal{A}} \bullet \mathbf{1}_{\mathcal{A}}$ dominates $((\mathbf{I} - a\mathbf{G})_{\mathcal{A}\times\mathcal{A}} \bullet \boldsymbol{\pi}_{\mathcal{A}}$ dominates the Nash equilibrium $\mathbf{r}(\mathcal{A}, -a))$, then the greater the strength of g_{ji} for each firm i, the lower its $\beta_i(\mathcal{A}, -a)]$. This then hints as to which firm i charging less amount in lending rate. We explore some special network properties in relation to this in the next section.

[4]A simple computational algorithm takes $2^{|\mathcal{N}|} - 1$ iterations representing possible combination of active firms. It is noteworthy that even if we relax assumption 3.2.2 so that $\hat{\mathcal{N}} = \{\mathcal{N}, \mathcal{S}\}$, Equilibrium is simply obtainable by computing for $\mathcal{N}$. Hence, for each firm $i \in \mathcal{N}$ such that a firm $j \in \mathcal{S} \cap \mathcal{N}_i^{out}$, then $b_{-i} = b_{ij} + \ldots$

[5]So as not to confuse it with Bonacich Centrality which is $\beta_i(\mathbf{G}^T, a)$ for a firm i.

3.3.3 Equilibrium and Inactive Firms

Our proposition 3.3.2 shows that Nash equilibrium could be such that $\mathcal{N} - \mathcal{A} \neq \{\}$. A firm $i \in \mathcal{N} - \mathcal{A}$ thus has an $r_i = 0$ as its equilibrium rate charged to its debtors. We draw a swift distinction between inactive firms in our model and the concept of *free-riders* found in major public goods in networks papers such as Bramoullé and Kranton (2007), Bramoullé et al. (2014) as well as Allouch (2015). To understand this is to understand the best replies given in (3.8) as an outcome of the payoff. Observe that loan management is a main objective of the firm and as such, strategic substitution arises in a bid to reduce such management cost. So while a firm who borrows cannot influence (directly) the rate to which it is charged, it can charge a corresponding rate to its debtors to balance and optimize loan management expenses. For this reason, charging a zero rate to debtor thus arises from the fact the present loan management cost is quite substantial that a positive rate would be even more harmful to the firm.

The idea here is that an inactive firm $i \in \mathcal{N} - \mathcal{A}$ is not necessarily *free-riding* the provision of other firms but on the other hand, is simply avoiding any further cost as a result of its own decision since its creditors has increased such cost to the maximum.

3.4 Core-Periphery Networks

We explore in a unique way, further properties of our equilibrium. More specifically we aim to discuss and show unique network properties of active firms and what the implication might be in terms of the equilibrium lending rate each firm charges.

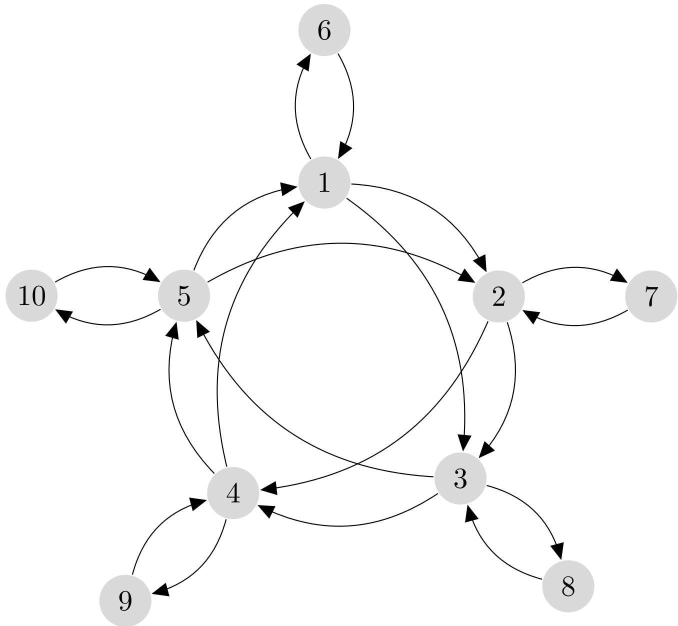


Figure 3-3: Core-periphery network bilateral links between periphery and core sets.

To do this, we examine a stylized case of a network which is shown in fig. 3-5 which contains a core-periphery network where each firm in the core-periphery lends and borrows to others. Core-periphery network has been widely stylized within the inter-bank network literature especially within the line of financial contagion and systemic risk. Recent examples of such studies include Chiu, Eisenschmidt, and Monnet (2020), Lux, Fricke, et al. (2012), Van Lelyveld et al. (2014) as well as Sui, Tanna, and Zhou (2020). We describe a core-periphery network as one which has 2 groups of firms, the core firms whose set we denote as Cr and the periphery set which we denote as Pr. For a directed graph $G(\mathcal{N}, g)$ such that $\Omega = \{Cr, Pr\} = \mathcal{N}$. Also, assume $\mathcal{N}(\Omega, \mathbf{r}^*) = \mathcal{A}$ such that all firms in the core-periphery network are actively charging lending rates at equilibrium. The core-periphery network has the following graph form;

$$\mathbf{G}(Cr, Pr) = \left[\begin{array}{c|c} Cr \times Cr & Cr \times Pr \\ \hline Pr \times Cr & Pr \times Pr \end{array}\right] = \left[\begin{array}{c|c} \mathbf{G}(\mathbf{CC}) & \mathbf{G}(\mathbf{CP}) \\ \hline \mathbf{G}(\mathbf{PC}) & \mathbf{G}(\mathbf{PP}) \end{array}\right] \tag{3.11}$$

For a network to be deemed core-periphery, it means it can be grouped into the block partition as shown above.

Assumption 3.4.1 (Block Matrix Properties)**.** *Given* $\mathbf{G}(Cr, Pr)$ *which is strictly unidirectional let* $\Omega = \{Cr, Pr\}$*. We have the following conditions;*

1. $|Pr| = |Cr|$,

2. $\mathbf{G}(\mathbf{PP}) = \mathbf{0}$,

3. $\mathbf{G}(\mathbf{PC}) = \varrho \bullet \mathbf{I}$,

4. $\mathbf{G}(\mathbf{CP}) = \theta \bullet \mathbf{I}$.

This indicates that we allow for bilateral relationships.[6]. We are able to compute the Nash Equilibrium using the block-partition matrix as in (3.11) to compute each firms Nash Equilibrium:

Proposition 3.4.1. *Let* $G(\Omega, g)$ *be a directional graph whose topology is core-periphery in nature. Also assume* $\Omega = \{Cr, Pr\}$, $\boldsymbol{\pi} = (\boldsymbol{\pi}_{Cr}, \boldsymbol{\pi}_{Pr})^T$ *and assumption 3.4.1 holds. In so far as* $a \in \left]0, \frac{1}{\left|\nu_{min}(\frac{\mathbf{G}+\mathbf{G}^T}{2})\right|}\right[$, *we have the following Nash Equilibrium;*

$$\mathbf{r}^*_{Cr}(\mathbf{G}, \Omega, a) = \left(\mathbf{I} + \frac{a}{1 - a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^{-1} \frac{(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr})}{1 - a^2\theta\varrho}, \tag{3.12}$$

[6]One could possibly argue that bilateral liabilities would not hold given that it can simply be netted off. However, because each firm makes separate lending rate decision, bilateral links need be exactly as they are as contractual properties might differ as we show for example in fig. 3-5 where periphery and core have between them, such bilateral relationships.

$$\mathbf{r}^*_{Pr}(\mathbf{G}, \Omega, a) = \boldsymbol{\pi}_{Pr} - a\varrho \bullet \mathbf{r}^*_{Cr}(\mathbf{G}, \Omega, a). \tag{3.13}$$

Proof. See Appendix for proof. ■

The (3.13) above then provides us with an initial intuition as we can see that for any firm $i \in Cr$ and a firm $j \in Pr | j \in \{\mathcal{N}_i \cap Cr\}$, then it holds that;

$$ag_{ij} \bullet r^*_i(\mathbf{G}, \Omega, a) + r^*_j(\mathbf{G}, \Omega, a) = \pi_j, \tag{3.14}$$

thus implying a direct strategic substitution relationship between Nash lending rate decision of each core and its corresponding periphery. The greater the Nash rate the core charges, the less its corresponding periphery lending rate is and vice versa.

Furthermore, (3.12) shows the core set Nash Equilibrium is then modified into a measure which includes the value $\frac{a}{1-a^2\theta\varrho}$. This parameter takes the form of a *new attenuation parameter* such that it replaces the initial attenuation parameter a. The new attenuation parameter is greater as $\frac{a}{1-a^2\theta\varrho} > a$ and since the Bonacich expression in (3.12) takes the power series $\mathbf{I} - \frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC}) + \left(-\frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^2 + \left(-\frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^3 + \ldots$, it implies that greater substitution from distant neighbours. However, this time, the weight of relationship between the core and periphery set of firms now determines how much of such weight is accounted for in the Nash equilibrium . To show, since we have that $1 - a^2\theta\varrho \downarrow$ if either θ or ϱ rises, then it means that the link $\frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})$ is strengthened to such rise in θ and/or ϱ and weakened when the reverse holds.

Lastly, the density of the core set is accounted for in the matrix $\mathbf{G}(\mathbf{CC})$ such that the diversification of core firms within the core network is crucial in determining the rate to which the core firms would charge.

3.4.1 Equitable Partition

We are able to understand further special properties of the core periphery relation under possible stylized partition network property. We hold the following assumption for this part of the paper;

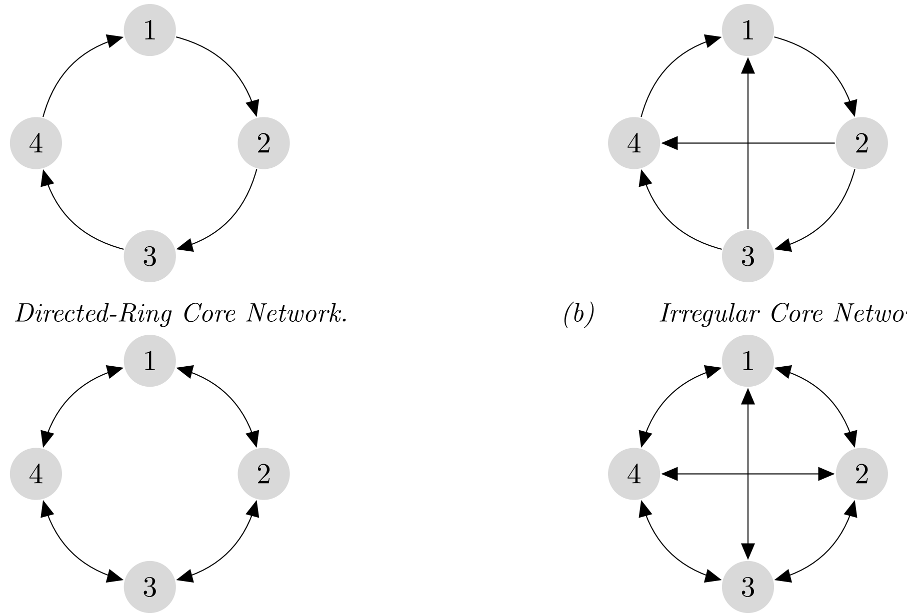
Assumption 3.4.2. *Given* $\Omega = \{Cr, Pr\}$*, we hold that* $\mathbf{G}(\mathbf{CC}) \bullet \mathbf{1} = \rho \bullet \mathbf{1}$*, while* $\mathbf{G}(\mathbf{PC}) = \varrho \bullet \mathbf{I}$ *and* $\mathbf{G}(\mathbf{CP}) = \theta \bullet \mathbf{I}$ *for* $\rho, \varrho, \theta \in]0, \mathbb{R}_{++}]$.

This yields a core-periphery network which has both out-equitable and in-equitable properties as defined using Kada (2020) as well as Deng, Sato, and Wu (2007) as follows;

Definition 3.4.1 (Equitable Partition)**.** *Consider* $\Omega = \{Cr, Pr\}$ *where* $\mathbf{G}(\mathbf{CC}) \bullet \mathbf{1} = \rho \bullet \mathbf{1}$ *and* $\mathbf{G}(\mathbf{PP}) \bullet \mathbf{1} = 0 \bullet \mathbf{1}$ *arising from* $\mathbf{G}(\mathbf{PP}) = \mathbf{0}$ *(from assumption 3.4.1) so that* Cr *and* Pr *are* Partitions*. If we have that* $\mathbf{G}(\mathbf{PC}) = \varrho \bullet \mathbf{I}$ *and* $\mathbf{G}(\mathbf{CP}) = 0 \bullet \mathbf{I}$ *then* Ω *is 'out-equitable' while if we have that* $\mathbf{G}(\mathbf{CP}) = \theta \bullet \mathbf{I}$ *and* $\mathbf{G}(\mathbf{PC}) = 0 \bullet \mathbf{I}$*, then* Ω *is 'in-equitable'. Where both* $\mathbf{G}(\mathbf{PC}) = \varrho \bullet \mathbf{I}$ *and* $\mathbf{G}(\mathbf{CP}) = \theta \bullet \mathbf{I}$ *holds simultaneously then* Ω *is simply an 'Equitable' partition.*

Since core network are directed ring-network, observe the examples as in fig. 3-5a as well as a complete bi-directional core-network shown in fig. 3-6b. Also from the fig. 3-6 , observe that all but fig. 3-5b are partitioned cores[7] as each core-firm has same amount of incoming and outgoing link in each case.

[7]For example, $\rho = 1$ for fig. 3-5a, $\rho = 2$ in fig. 3-6a while $\rho = 3$ in fig. 3-6b. Because $\mathbf{G}(\mathbf{CC}) \bullet \mathbf{1} = (2, 1, 1, 2)^T$ for fig. 3-5b, we are unable to define ρ in such case.



(a) *Directed-Ring Core Network.* (b) *Irregular Core Network.*

(c) *Bilateral-Ring Core Network.* (d) *Complete Core Network.*

Figure 3-4: Sample Core Networks. The figure a, c and d involve cores with fit the partition criteria.

We begin with an added assumption which satisfies both the ring core-network, core network with regular ring properties based on proposition 3.4.1 to show some realization in the following statement below;

Proposition 3.4.2. *Let* $\mathbf{G}$ *bi-directional graph of SCF which are core-periphery in nature as in* $\Omega = \{Cr, Pr\}$ *and assumption 3.4.2 holds. In so far as 'a' is within threshold, we have the following Nash Equilibrium;*

$$\begin{aligned} \mathbf{r}^*_{Cr}(\mathbf{G}, \Omega, a) &= (1 + a\rho - a^2\theta\varrho)^{-1}(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr}), \\ \mathbf{r}^*_{Pr}(\mathbf{G}, \Omega, a) &= \boldsymbol{\pi}_{Pr} - a\varrho \cdot \mathbf{r}^*_{Cr}(\mathbf{G}, \Omega, a), \end{aligned} \tag{3.15}$$

Proof. See Appendix for proof. ■

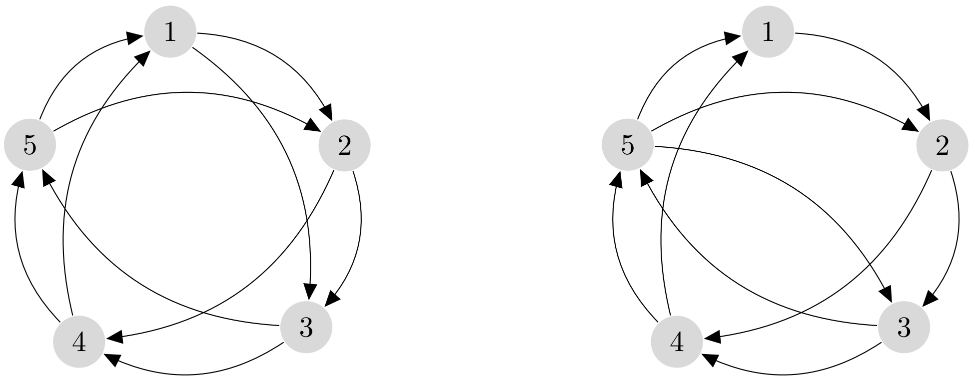
This is such that if $\boldsymbol{\pi} = \pi \cdot \mathbf{1}$, each within a partition charges identical lending rates. Observe further intuitions from this proposition,

Remark 3.4.1. *If $\boldsymbol{\pi} = \pi \boldsymbol{\cdot} \mathbf{1}$ and $\theta = \varrho$, the expression $r^*_{Pr}(\mathbf{G}, \Omega, a) > r^*_{Cr}(\mathbf{G}, \Omega, a)$ always holds true in so far $\rho > 0$.*

The remark above points to the fact that if a borrowing network meets the criteria for core-periphery relationship where links between each periphery and its core are identical and each firm lends same total amount, then one can presume core firms would charge a lower lending rate as compared to the peripheries.

To discuss more on the condition that $\mathbf{G}(\mathbf{CC}) \boldsymbol{\cdot} \mathbf{1} = \rho \boldsymbol{\cdot} \mathbf{1}$ in assumption 3.4.2, we illustrate this in the examples below;

Example 3.4.1. *Assuming the following networks with homogeneous links such that each edge is weighted $\alpha \in]0, \mathbb{R}^+[$ below;*



(a) Regular Core Network *(b) Irregular Core Network*

Figure 3-5: Core Network with homogeneous links.

We have the sub-matrix of core interconnections as;

$$\mathbf{G}_1(\mathbf{CC}) = \alpha \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \textit{and} \qquad \mathbf{G}_2(\mathbf{CC}) = \alpha \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{G}_1(\mathbf{CC}) \cdot \mathbf{1}_{|Cr|} = \mathbf{G}_2(\mathbf{CC}) \cdot \mathbf{1}_{|Cr|} = \alpha(2, 2, 2, 2, 2)^T$$

This implies that,

$$\mathbf{G}_1(\mathbf{CC})/\Omega = \mathbf{G}_2(\mathbf{CC})/\Omega = \rho = \alpha \cdot 2$$

Example 3.4.2. *Let us take another set of networks, this time with heterogeneous links as follows;*

(a) Directed Core Network. *(b) Directed Core Network.*

Figure 3-6: Core Networks with heterogeneous links.

We have the sub-matrix of core interconnections as;

$$\mathbf{G}_1(\mathbf{CC}) = \begin{bmatrix} 0 & 0 & 0.4 & 0.4 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}, \textit{and} \qquad \mathbf{G}_2(\mathbf{CC}) = \begin{bmatrix} 0 & 0 & 0.4 & 0.8 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$$

$$\mathbf{G}_1(\mathbf{CC}) \cdot \mathbf{1}_{|Cr|} = (0.8, 0.8, 0.8, 0.8)^T,$$
$$\mathbf{G}_2(\mathbf{CC}) \cdot \mathbf{1}_{|Cr|} = (0.8, 0.8, 1.2, 0.4)^T.$$

This implies that,

$$\mathbf{G}_1(\mathbf{CC})/\Omega = \rho = 0.8$$

$$\mathbf{G}_2(\mathbf{CC})/\Omega = ??$$

As such the core network in fig. 3-6b does not satisfy the assumption 3.4.2.

3.5 Intervention and Welfare Policies

In this section, we define outcomes based on Nash lending rates and then observe welfare properties of the model. More precisely, we highlight various possible policy initiative to which a maximising Planner could adopt and its estimate the overall impact. To study welfare, we adopt the standard utilitarian approach. As such, we introduce the following definition;

Definition 3.5.1. *The welfare from the game* $\Gamma(\mathbf{G}, \mathbf{r}, a)$ *is defined specially for firms who charge a positive amount as;*

$$W(\mathbf{r}, \mathcal{A}, a) \stackrel{\text{def}}{=} \sum_{i \in \mathcal{A}} P_i, \tag{3.16}$$

This implies we use welfare is the aggregate payoff of all firms who charge a positive amount. To define such payoff, we write the following lemma;

Lemma 3.5.1. *Assume* $\mathcal{N}$ *and the game* $\Gamma(\mathbf{G}, \mathbf{r}, a)$*,* $\forall$ *firm* $i \in \mathcal{A}$*, payoff given Nash equilibrium is as follows;*

$$\mathbf{P}_{\mathcal{A}} = diag(\mathbf{B}) \cdot \left((\mathbf{I} + a\mathbf{G})^{-1} \cdot \boldsymbol{\pi}_{\mathcal{A}}\right) - \boldsymbol{K}, \tag{3.17}$$

where $\boldsymbol{K} = \left(\frac{2+a}{4\kappa a}\right)_{i \in \mathcal{A}}$ *and* $\mathbf{B} = \left(\frac{(a+1)}{a} \cdot b_{-i}\right)_{i \in \mathcal{A}}$ *are both column vectors.*

Proof. See Appendix for proof. ∎

It is important to note the implication of (3.17). We see here that firms utility for charging is mainly dependent on their individual Nash equilibrium rate. This means that if we were to observe (3.17) and our best reply in (3.5) we then have an idea of kinds of policy implications for the model which we explore in the coming sections.

3.5.1 Transaction Cost and Welfare Neutrality

In this part, we explore the possibility of intervention policies and their welfare impact. Usually in payment systems, movement of a cash could face barriers such as foreign exchange conversion cost (if 2 firms are located at different economic regions), transaction cost like bank charges, etc. If we assume a system where firms incur transaction cost on total payment which we denote as λ, let us have a case where a regulator decides to grant λb_{-i} to each $i \in \mathcal{N}$, such policies are done so far as they keep the active set $\mathcal{A}$ the same which means the network graph $\mathbf{G}_{\mathcal{A}}$ should remain unchanged, unchanged. Given λb_{-i} for all firm $i \in \mathcal{N}$. The initial mark of the policy λ is such that payoff is written as;

$$P_i^{\lambda}(r_i) = \lambda \left(b_{-i} \bullet r_i - \sum_{j \in \mathcal{N}_i^{in}} (\hat{g}_{ji} b_i)\, r_j \right) - \kappa \left(\lambda \mu_i\right)^2 \tag{3.18}$$

The diagram fig. 3-7 shows a Planner P whose objective is to maximise $\sum_{i \in \mathcal{A}} P_i$. More specifically, the fig. 3-7a represents and instance where a Planner increases payment made by each firm to another (for example, through elimination of a prevailing transaction cost like bank charges) while the fig. 3-7b shows a case where even without friction, the Planner grants each lender an extra amount to loan its potential debtors given strictly that debtors are not opposed to such additional loans. The arrows show the policy action. In terms of equilibrium, we introduce the following lemma;

Lemma 3.5.2. *The active set $\mathcal{A}$ remains fixed $\forall\ \lambda \lesseqgtr 1$ even though $\mathbf{r}^{\lambda} = \lambda^{-1} \bullet \mathbf{r}$.*

Proof. For the Nash equilibrium given such policies, we have for all active firms that;

$$r_i^{\lambda} = \frac{\pi_i}{\lambda} - a \sum_j \frac{\lambda b_{ji}}{\lambda b_{-i}} r_j^{\lambda} = \frac{\pi_i}{\lambda} - a \sum_j \frac{b_{ji}}{b_{-i}} r_j^{\lambda}. \tag{3.19}$$

This is so that rewriting in vector form, our Nash for active firms is given as;

$$\mathbf{r}_{\mathcal{A}}^{\lambda} = (\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \frac{\boldsymbol{\pi}_{\mathcal{A}}}{\lambda} = \lambda^{-1} \bullet \mathbf{r}_{\mathcal{A}}.$$

For this set combination $\mathcal{A}$ and $\mathcal{N} - \mathcal{A}$ to not be the Nash equilibrium set would mean that the following has to hold;

$$\frac{a}{\lambda} \mathbf{G}_{\mathcal{N}-\mathcal{A}\times\mathcal{A}} \bullet \mathbf{r}_{\mathcal{A}} < \frac{\boldsymbol{\pi}_{\mathcal{N}-\mathcal{A}}}{\lambda}.$$

However, multiplying the equation above by λ gives the condition as;

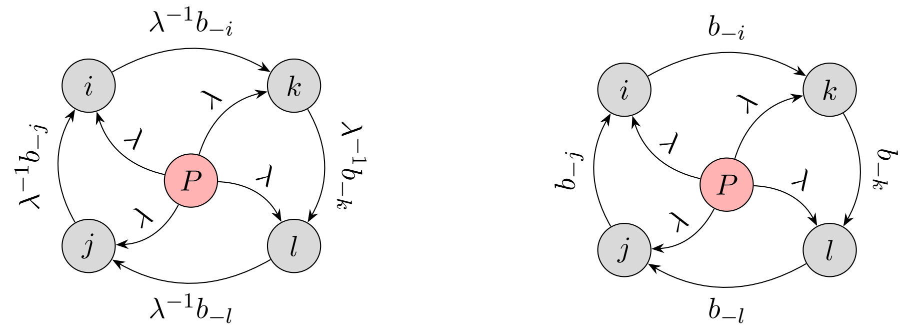
$$a\mathbf{G}_{\mathcal{N}-\mathcal{A}\times\mathcal{A}} \bullet \mathbf{r}_{\mathcal{A}} < \boldsymbol{\pi}_{\mathcal{N}-\mathcal{A}},$$

Which is a contradiction to the original equilibrium of $a\mathbf{G}_{\mathcal{N}-\mathcal{A}\times\mathcal{A}}\mathbf{r}_{\mathcal{A}} \geq \boldsymbol{\pi}_{\mathcal{N}-\mathcal{A}}$. ∎

Some explanation of this lemma is that since transaction cost λ is homogeneous and since inactive firms are such that $a\mathbf{G}_{\mathcal{N}-\mathcal{A}\times\mathcal{A}} \bullet \mathbf{r}_{\mathcal{A}}^{\lambda} \geq \boldsymbol{\pi}_{\mathcal{N}-\mathcal{A}}^{\lambda}$, then it means that while it is that $r_{i\in\mathcal{A}}^{\lambda} = \lambda^{-1} r_i$, it is also the case that $\pi_{i\in\mathcal{N}-\mathcal{A}}^{\lambda} = \lambda\pi_i$. So if because π rises and falls at equal magnitude for each firm, set of active firms remains constant. As such the magnitude of transaction cost or intermediate intervention is not relevant in terms of what the composition of active set would be at Nash equilibrium. In that light, we summarise the effect of such homogeneous intervention policy as follows;

Proposition 3.5.1. *Given the homogeneous policy λ, $\Delta W(\mathbf{r}^{\lambda}, \mathcal{A}, a) = 0$, hence welfare is neutral.*

Proof. See Appendix for proof. ■



(a) Intervention in a system with existing frictions.

(b) Intervention in a system without existing frictions.

Figure 3-7: Directions of a Planner P's intervention to 4 firms

We then move to observe the impact of mutually exclusive policy $\lambda_i b_{-i}$ $\forall\ i \in \mathcal{A}$ such that $\lambda_i \lesseqqgtr \lambda_j$ for all $i, j \in \mathcal{A}$. Observe here that policies are restricted to active set, we assume strictly that such policy intervention is such that leaves active set unchanged. For simplicity, one can initially assume the policy λ_i is applied to a single firm while holding others fixed as shown in fig. 3-8 where this time a regulator increases only one firms borrowing . In practice, it could be through eliminating transaction cost for a single firm while leaving other constant as shown in the figure below;

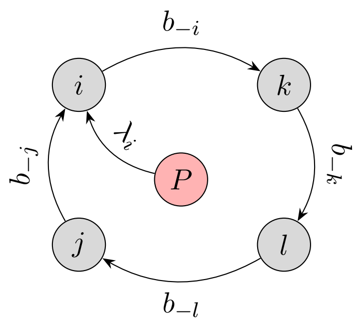


Figure 3-8: Ring network with 4 firms where a regulator decides to increase total firm i's lending to $\lambda_i b_{-i}$.

More broadly, the concept of the policy is that links of firms could be increased at heterogeneous proportion. The impact of such policy on welfare goes as follows;

Theorem 3.5.1. *Given a policy* $(\lambda_i, \lambda_j, \ldots)$ *so that* $\lambda_i \lesseqgtr \lambda_j$ *for all* $i, j \in \mathcal{A}$ *we have the following outcome;*

$$\Delta W^{\lambda}(\mathbf{r}^{\lambda}, \mathcal{A}, a) = 0 \tag{3.20}$$

Hence such policy is welfare neutral.

Proof. See Appendix for proof. ■

This means that it is not possible for a regulator to improve the welfare of active players by simply increasing/reducing one or more active firm network intensity even if it is by varying amounts. Welfare Neutral policies are also found in major public literature such as Bergstrom et al. (1986) and Warr (1983) who both showed neutrality to aggregate provision of public good and individual consumption of private good in so far as wealth redistribution does not change the set of active players involved. In an extension to this, Allouch (2015) adds that small transfers that leave active set the same are also neutral only when such transfers are made between the active set themselves. To contrast with our results yield neutrality without transfer policies. Because each firms utility is based on their individual Nash equilibrium, payoffs are neutral which leaves overall welfare unchanged. Additionally, intervention are not be restricted to active firms and due to the homogeneous nature of intervention, the magnitude of λ is pertinent in influencing the outcome in so far rates charged by creditor firms are limited to non-negative rates.

3.5.2 Resource Allocation

To access a possible impact of theorem 3.5.1, we observe a policy change of $\Delta\ \kappa_i$ for $i \in \mathcal{A}$ (i.e, firms who charge at Nash equilibrium). Let us have the following definition;

Definition 3.5.2. *For any firm $i \in \mathcal{N}$, we have it that*

- *Subsidy* $\Rightarrow \frac{\Delta\kappa_i}{\kappa_i} = \gamma_i^-$
- *Tax burden* $\Rightarrow \frac{\Delta\kappa_i}{\kappa_i} = \gamma_i^+$

Assume then that $\gamma_i = \gamma\ \forall$ firm $i \in \mathcal{A}$ so that the policy is applied in homogeneous proportion to all active firms. Payoff of each firm i is written as;

$$\forall\, i \in \mathcal{N} \quad P_i^{\gamma}(r_i) = b_{-i} \bullet r_i - \sum_{j \in \mathcal{N}_i^{in}} (\hat{g}_{ji} b_i)\, r_j - (1+\gamma)\kappa(\mu_i)^2 \tag{3.21}$$

We summarise the effect in the following results;

Lemma 3.5.3. *Given γ, welfare differential is as follows;*

$$\Delta W^{\gamma}(\mathbf{r}^{\gamma}, \mathcal{A}, a) = \mathbf{1}^T \mathbf{P}_{\mathcal{A}} \bullet \frac{-\gamma}{(1+\gamma)}. \tag{3.22}$$

Remark 3.5.1. *This implies that if $\gamma \in [-1, 0[$, then $\Delta W^{\gamma}(\mathbf{r}^{\gamma}, \mathcal{A}, a) > 0$ while if $\gamma \in [0, 1[$ then $\Delta W^{\gamma}(\mathbf{r}^{\gamma}, \mathcal{A}, a) < 0$ and its interpretation is simply that subsidies improves welfare while taxes reduce welfare.*

Note that Active firms $\mathcal{A}$ also remains fixed $\forall\ \gamma \in]0, 1[$. Results in this case are clearly unsurprising as lighter burden means firms are less sensitive to the volume of indebtedness given its fixed debt. Examples of such policies could be through providing outsourcing facility to a portion of debts or maybe policies to reduce call rates or providing free training of labour force involved in such area. When however, this policy applies in a heterogeneous manner to firms, it then becomes isomorphic to resource transfers which we explore in details subsequently.

In lemma 3.5.2 as well as theorem 3.5.1, it is noted that given a policy $\lambda_i \lesseqgtr 1$ such that lending becomes $\lambda_i b_{-1}$ for any $i \in \mathcal{A}$, $\Delta W(\mathbf{G}, a) = 0$ in so far as the active firms $\mathcal{A}$ remains fixed. Given our results above, we have the following results;

Proposition 3.5.2. *Given the game* $\Gamma(\mathbf{G}, \mathcal{A}, a)$ *there exists* $\Delta W(\mathbf{r}, \mathcal{A}, \gamma, \lambda) \in]0, \mathbb{R}_{++}[$ *(not necessarily Pareto) at zero cost to a Planner in so far as there exists* $\sum_{i \in \mathcal{A}} \lambda_i b_{-i}$ *such that* $\mathcal{A}$ *remains fixed.*

Proof. Strictly holding $\mathcal{A}$ fixed, let $\sum_{i \in \mathcal{A}}(1-\lambda_i)b_{-1}$ be the amount the regulator charges for intermediate payments from each firm i (building from proposition 3.5.1) , then this is the case so far $\sum_{i \in \mathcal{A}}(1-\lambda_i)b_{-1} = \gamma \sum_{i \in \mathcal{A}} (\kappa_i(\mu_i)^2)$ which then guarantees Pareto improvement among active firms. For non-Pareto improvement, subsidised administrative cost $\gamma_i \kappa_i(\bullet)^2$ need not apply to all firms in $\mathcal{A}$. In this case, the criteria shown in theorem 3.6.1 becomes useful. ∎

This comes from the fact that so far as active set remains fixed, the regulator can instead of eliminating transaction cost, create one at no cost to overall welfare. This also grants resources to subsidise one or more firms in a way that improves welfare. Pareto improvement is possible if $X = \sum_{i \in \mathcal{A}}(1-\lambda_i)b_{-1}$ is split such that $\gamma \sum_{i \in \mathcal{A}} \kappa_i(\mu_i)^2 \leq X$. Observe now that γ is constant so that its effect on welfare corresponds to lemma 3.5.3. This is a unique form of transfer compared to those found in mainstream public good in networks literature such as Allouch (2015), Allouch and King (2018b), etc. This is because in this case, transfers could be simply from one firm to another through different variables the firm faces.

3.6 Intervention Targeting

We project in this section the relationship between Bonacich externality measures and firms quality, especially in terms of marginal welfare given a resource constrained Planner. We here generalise the Planner to one who wishes to grant loan management cost subsidy in order to maximise overall welfare $(\Delta W^{\gamma}(\mathbf{r}^{\gamma}, \mathcal{A}, a))_{max}$ of active firms $\mathcal{A}$. Then if the set $\Phi(\mathcal{A})$ represents the possible combinations of firms, the Planner has $|\Phi(\mathcal{A})| = 2^{|\mathcal{A}|} - 1$ amount of alternative actions as to the distribution of subsidy intervention in order to achieve $(\Delta W^{\gamma}(\mathbf{r}^{\gamma}, \mathcal{A}, a))_{max}$. This is such that the earlier discussed "γ $\forall$ firm $i \in \mathcal{A}$" is a strategy element in $\Phi(\mathcal{A})$ arising from the $C(|\mathcal{A}|, |\mathcal{A}|)$ combination, where $C(a,b) = \frac{a!}{(a-b)!b!}$. On the other extreme, let $\phi \subset \Phi$ be the subset arising the combination $C(|\mathcal{A}|, 1)$, This then means that $|\phi| = |\mathcal{A}|$ such that the Planner calculates the total welfare from subsidising for a single firm $i \in \mathcal{A}$. We then wish to show the qualities of the firm $i \in \mathcal{A}$ which yields the greatest payoff from the strategy subset ϕ. Literature in recent times have, within network spillover problems come up with various targeting criterion; The *Key-Player* concept introduced in Ballester et al. (2006), The highest threat index (which is the Bonacich centrality) introduced in Demange (2016) as well as the *top Principal Components* as another eigenvalue related measure used in Galeotti et al. (2020).

We begin with a naive scenario. Assume a Planner with unlimited finance but one who wishes to subsidise administrative cost by a γ amount a single selected firm so as to maximise overall network welfare. Formally, we define the Planners problem within the strategy $\phi \subset \Phi$ is stated as;

$$\max_{\gamma} \{P_i^{\gamma} - P_i | i = 1, \ldots, n\} \qquad s.t \quad \gamma^{-} = \gamma_i | i \in \{\mathcal{A}\}. \tag{3.23}$$

The choice firm $i \in \mathcal{A}$ then has a payoff is written as;

$$P_i^{\gamma}(r_i) = \left(b_{-i} \bullet r_i - \sum_{j \in \mathcal{N}_i^{in}} (\hat{g}_{ji} b_i) \, r_j \right) - (1+\gamma)\kappa(\mu_i)^2 \tag{3.24}$$

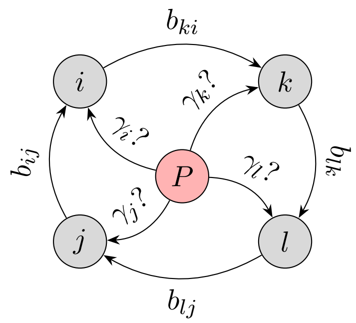


Figure 3-9: Ring network with 4 firms to which the Planner makes a decision which to subsidise.

Hence the question is which firm should the Planner subsidise for? Observe the following equation of the measure of a firm $i \in \mathcal{N}$;[8]

$$\beta_i(\mathbf{G}^T, -a) \stackrel{\text{def}}{=} \sum_{k=0}^{+\infty}(-a)^k \sum_{j=1}^{n} \left(\left(\mathbf{G}^T\right)^k\right)_{ij} \tag{3.25}$$

This is such that $\boldsymbol{b}(\mathbf{G}^T, -a) = (\mathbf{I}+a\mathbf{G}^T)^{-1}\boldsymbol{\cdot}\mathbf{1} = \left(\beta_i(\mathbf{G}^T, -a)\right)_{i\in\mathcal{N}} \in \mathbb{R}^n_+$. The measure above is related to the Bonacich centrality used to capture prestige and network influence as proposed by Bonacich (1987). However, it measures the weakness of firms link to its debtors. This means that the greater $\beta_i(\mathbf{G}^T, -a)$ is for a firm i, the smaller the weight of the direct link to $\mathcal{N}_i^{out}$. Going further, $\beta_i(\mathbf{G}^T, -a)$ is referred to as the *externality index* for firm i. We as such present the following results.

Theorem 3.6.1. *Assume that $b_{-i} = b_{-j}$ $\forall$ $i, j \in \mathcal{A}$. The welfare differential $\Delta W(\mathbf{r}^\gamma, \mathcal{A})$ is at maximum if and only if subsidy γ_i such that for firm i ;*

$$\beta_i(\mathbf{G}^T_{\mathcal{A}}, -a) \geq \beta_{j\neq i}(\mathbf{G}^T_{\mathcal{A}}, -a),$$

Hence firm i has the largest externality index.

Proof. See Appendix for proof. ■

[8]We still hold in this part that $\mathcal{N} = \mathcal{A}$.

This result shows the relationship between externalities on outgoing links based on weighted interconnections and ability to improve overall welfare overall from intervention related to subsidy. To summarise this point, recall that we can also write firm i's centrality measure as below,

$$\beta_i(\mathbf{G}_{\mathcal{A}}^T, -a) = 1 - a \sum_{j \in \mathcal{N}_i^{out}} g_{ij} \beta_j(\mathbf{G}_{\mathcal{A}}^T, -a). \tag{3.26}$$

This means that for every unit increase in π_i, it negatively impacts each $r_{j \in \{\mathcal{N}_i^{out} \cap \mathcal{A}\}}$. Thus a negative externality. Then given that lending rates charged by active firm serve as a form on negative externality, the subsidy should be given to the firm who produces the least externality in the network. This is as subsidy here increases strategic substitution since it increases the potential r_i for any firm whose $\kappa(\mu_i)^2$ is reduced. This serves as an identifier for *pressure points* of our model in contrast to other network targeting works.

A more practical and justifiable scenario would be where the Planner has limited resource. In this instance, the Planner wishes to maximise total welfare and as such, measures the impact of channeling subsidy to a single firm versus splitting proportionally across all active firms. In order to select the firm to consider allocating resource to, let us rewrite the problem of the Planner from (3.23) as follows;

$$\max_{\gamma_i | i \in \mathcal{A}} \left\{ P_i^{\gamma_i} - P_i | i = 1, \ldots, n \right\}, \tag{3.27}$$

$$s.t \quad \gamma_i = \gamma_i | i \in \{\mathcal{A}\} \quad and,$$
$$\gamma_i \bullet \kappa_i(\mu_i)^2 \leq X$$

It follows then that $\gamma_i \leq -\frac{X}{\kappa_i(\mu_i)^2}$ where X represents the cash endowment of the regulator. In this case, we then derive another corollary from theorem 3.6.1 as,

Corollary 3.6.1. *Assuming a regulator who is cash constrained and* $b_{-i} = b_{-j}\ \forall\ i, j \in \mathcal{A}$*, the welfare differential* $\Delta W^{\gamma_i}(\mathbf{r}^{\gamma_i}, \mathcal{A}, a)|i \in \mathcal{A}$ *is at maximum if and only if subsidy* γ *is applied to firm* i *which meets the following criteria,*

$$\beta_i(\mathbf{G}_{\mathcal{A}}^T, -a) \bullet \frac{-\gamma_i}{1+\gamma_i} \geq \beta_{j \neq i}(\mathbf{G}_{\mathcal{A}}^T, -a) \bullet \frac{-\gamma_j}{1+\gamma_j}.$$

Proof. Let $\eta = \frac{2+a}{4a}$ and $\omega = \frac{a+1}{a}$. Since γ_i is not necessarily homogeneous across firms, then $\forall$ firm i such that $P_i = \ldots + (1+\gamma_i) \bullet \kappa(\mu_i)^2$, $\Delta W(\mathbf{r}^{\gamma}, \mathcal{A}, a) = \frac{-\gamma_i \omega}{2\kappa(1+\gamma_i)} \bullet \beta_i(\mathbf{G}_{\mathcal{A}}^T, -a) + \frac{\eta\gamma_i}{1+\gamma_i} = \frac{\gamma_i}{1+\gamma_i}\left(\eta - \frac{\omega}{2\kappa} \bullet \beta_i(\mathbf{G}_{\mathcal{A}}^T, -a)\right)$ and we also hold that $\frac{\gamma_i}{1+\gamma_i} \to +\infty$ as $\gamma_i \to -1$ while keeping active set $\mathcal{A}$ strictly fixed. ■

The intuition then from our results is that welfare due to individual subsidy especially when the regulator has limited funds are best allocated to firms with a combination of greater proportional reduction in loan management expenses as well as lower negative spillover effects. An example of the Planner making this decision can be observed below;

Example 3.6.1 (Individual vs Group Targeting)**.** *Assuming the following debt network below;*

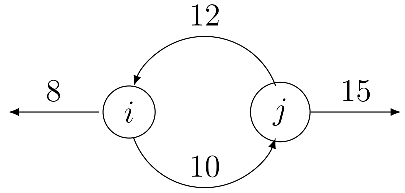


Figure 3-10: Network with 3 firms and 4 debt contracts (edges)

Other parameters are as follows, $a = 0.8$, $\kappa = 0.04$. *This means we have* $\boldsymbol{\pi} =$

$(0.699, 0.46)^T$ *and*

$$\mathbf{G} = \begin{bmatrix} 0 & 0.67 \\ 0.37 & 0 \end{bmatrix}.$$

So that $\mathbf{r}^* = (0.54, 0.3)^T$, $\mathbf{b}(\mathbf{G}^T, -a) = (0.8368, 0.5515)^T$ *and* $\mathbf{P} = (19.198, 15.553)^T$. *Which leaves the initial welfare* $\mathbf{1}^T\mathbf{P} = 34.751$.

Assume then that a Planner has \$2 to distribute. First we have the loan management cost as;

$$\kappa(\mu_i(\mathbf{r}^*))^2 = 6.35 \text{ and,}$$
$$\kappa(\mu_j(\mathbf{r}^*))^2 = 6.17.$$

We have $\Phi = \{\phi_1, \phi_2, \phi_3\}$ *where* $\phi_1 = \{i, j\}$, $\phi_2 = \{i\}$ *and* $\phi_3 = \{j\}$.

For the strategy ϕ_1, $\gamma_i = \gamma_j = \gamma$. *This gives the value as* $\gamma = -0.1587$. *Strategy* ϕ_2 *gives* $\gamma_i = -0.3149$ *while Strategy* ϕ_3 *gives* $\gamma_j = -0.324$.

Strategy 1(ϕ_1): Where $\gamma = -0.1587$.

We have the welfare improvement then as;

$$\begin{aligned} \Delta W^\gamma(\mathbf{r}^\gamma, \mathcal{A}, a) &= \mathbf{1}^T\mathbf{P}_\mathcal{A} \bullet \frac{-\gamma}{(1+\gamma)}, \\ &= 34.751 \bullet \frac{0.1587}{0.8413}, \\ &= 6.56. \end{aligned}$$

Strategy 2(ϕ_2): Where $\gamma_i = -0.3149$.

The welfare improvement is;

$$\begin{aligned}\Delta W^{\gamma}(\gamma_i, \mathcal{A}, a) &= \frac{-\gamma_i \omega}{2\kappa(1+\gamma_i)} \bullet \beta_i(\mathbf{G}^T_{\mathcal{A}}, -a) + \frac{\eta\gamma_i}{1+\gamma_i}, \\ &= \frac{0.7085}{0.0548} \bullet (0.8368) - \frac{0.2755}{0.6851}, \\ &= 10.41.\end{aligned}$$

Strategy 3(ϕ_3): Where $\gamma_j = -0.324$.

The welfare improvement is;

$$\begin{aligned}\Delta W^{\gamma}(\gamma_j, \mathcal{A}, a) &= \frac{-\gamma_j \omega}{2\kappa(1+\gamma_j)} \bullet \beta_j(\mathbf{G}^T_{\mathcal{A}}, -a) + \frac{\eta\gamma_j}{1+\gamma_j}, \\ &= \frac{0.729}{0.0508} \bullet (0.5515) - \frac{0.2835}{0.6760}, \\ &= 7.4928.\end{aligned}$$

Here, we see that the optimal intervention would be to spend the $2 on subsidising firm i's loan management cost which in itself, gives a total welfare improvement that supersedes splitting proportionately among both active firms. Also noticeable, is the fact that firm i has a greater externality index $\beta_i(\mathbf{G}^T_{\mathcal{A}}, -a)$ in comparison to firm j which corresponds to our results. On a final note, it is worth pointing out that the sub-strategy combination $C(|\mathcal{A}|, b)$, where $1 < b < |\mathcal{A}|$, strategies are known as *group strategy.* This is even more distinct when the number of active firms exceeds 2 ($|\mathcal{A}| > 2$). Our analysis still implies the Planner weighs these strategy and indeed, the optimal could be found within such strategy. However, we have focused primarily on individual firms quality which makes it a suitable target. Group based intervention remain unexplored but relevant.

3.7 Concluding Remarks

We have shown strategic substituting behaviour of firms arising from firms making an inter-temporal lending rate decision so as to make maximum profit in the face of Loan management cost. Such Loan management cost depends on the level of firms efficiency in managing overall debtors as well as creditors. The outcome of this is a substitute game with mostly a unique equilibrium. Our best replies are very likened to notable works such as Blume, Easley, Kleinberg, Kleinberg, and Tardos (2011), Allouch (2015) as well as Bramoullé et al. (2014) without boundaries and Allouch and King (2018a) with boundaries but with slightly different weight and directional properties. We identify neutrality and welfare improving policies given various types of intervention. One main intuition from our model is that resources can be redirected from within and to the same firm such that the Planner improves welfare while suffering little to no additional cost. Lastly, we established that interventions targeted at firms who have a relatively higher degree of network centrality based on weak link to debtors yields the most efficient welfare based outcomes. This is because then, raising such firms lending rate yields lower negative spillover to debtor firms.

This work primarily pays more attention to cost coming from loan management and as such gives intuition towards strategic substitute under the assumption that the firm incurs additional cost on the basis of additional volume of loans. A possible critique of this idea would be that to a significant degree, the number of debtors and creditors are also key drivers of loan management cost as well and our model seems to ignore this prospect. One reason for ignoring this is that it would mean that firms then make decision as to how many incoming and outgoing link to establish, which goes against our fixed network environment as we assume that such decision are made exogenous to the model, hence the network environment we have . As with regards to decisions on lending rates, given that there are host of other factors that might influence a lending rate charged, then it is easily predicted that other forms of interaction including games of complementarity could arise if the focus is on other factors. Also, because we assume a one-shot decision making, we ignore instances where firms could work to increase administrative efficiency. This in itself could lead to new problems including moral hazard (for example, a personnel might not reveal his/her true efficiency as it might alter remuneration). We believe this would make for a vital extension to the model. Another line of extension is linked to welfare whereby the Planner weights firms by order of importance such that payoffs are given weights. This could also shed a more realistic lights to impacts of policies to firms.

3.8 Appendix

Proof of Lemma 3.3.1

Intuitions on this concept is briefly discussed in (Bramoullé et al., 2014). Additionally, it should be noted that because $\mathbf{G}$ is a directed graph, then $(\mathbf{I} + a\mathbf{G})$ being positive definite implies

$$1 + a\,\nu_{min}\left(\frac{\mathbf{G} + \mathbf{G}^T}{2}\right) > 0, \tag{3.28}$$

hence the condition. □

Proof of Proposition 3.3.2

Given (3.8), then for the active set $\mathcal{A}$, we would have for firm $i \in \mathcal{A}$ the following;

$$r_{i\in\mathcal{A}} = \pi_i - a \sum_{j\in\mathcal{N}_i^{in}, j\in\mathcal{A}} \frac{b_{ji}}{b_{-i}} r_j. \tag{3.29}$$

Intuitively, any firm $l \in \mathcal{N} - \mathcal{A}$ would be such that the following holds;

$$r_{l\in\mathcal{N}-\mathcal{A}} = \pi_l - a \sum_{j\in\mathcal{N}_l^{in}, j\in\mathcal{A}} \frac{b_{jl}}{b_{-l}} r_j \leq 0,$$

Which then translates to;

$$a \sum_{j\in\mathcal{N}_l^{in}, j\in\mathcal{A}} \frac{b_{jl}}{b_{-l}} r_j \geq \pi_l. \tag{3.30}$$

Writing (3.29) and (3.30) in vector form for the full set $\mathcal{N}$ completes the proof.

□

Proof of Proposition 3.4.1

Holding $\mathcal{N} = \mathcal{A}$, since our Nash equilibrium is $\mathbf{r}(\mathbf{G}, a) = (\mathbf{I} + a\mathbf{G})^{-1} \cdot \boldsymbol{\pi}$, we then solve for $\mathbf{r}(\mathbf{G}, a)$ below as follows;

$$\mathbf{r}(\mathbf{G}, \Omega, a) = \begin{bmatrix} (\mathbf{I} + a\mathbf{G}(\mathbf{CC})) & a\mathbf{G}(\mathbf{CP}) \\ a\mathbf{G}(\mathbf{PC}) & \mathbf{I} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \boldsymbol{\pi}_{Cr} \\ \boldsymbol{\pi}_{Pr} \end{bmatrix} \tag{3.31}$$

Using block matrix inversion concept to solve for $\begin{bmatrix} (\mathbf{I} + a\mathbf{G}(\mathbf{CC})) & a\mathbf{G}(\mathbf{CP}) \\ a\mathbf{G}(\mathbf{PC}) & \mathbf{I} \end{bmatrix}$, assume without loss of generality that $(\mathbf{I} + a\mathbf{G}(\mathbf{CC})) = A$, $a\mathbf{G}(\mathbf{CP}) = F$ while $a\mathbf{G}(\mathbf{PC}) = E$ so that

$$\begin{bmatrix} (\mathbf{I} + a\mathbf{G}(\mathbf{CC})) & a\mathbf{G}(\mathbf{CP}) \\ a\mathbf{G}(\mathbf{PC}) & \mathbf{I} \end{bmatrix} = \begin{bmatrix} A & F \\ E & \mathbf{I} \end{bmatrix}.$$

From the Helmert-Wolf blocking inversion method,[9] we have the following;

[9]See Wolf (1978).

$$\begin{bmatrix} A & F \\ E & \mathbf{I} \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}F(\mathbf{I} - EA^{-1}F)^{-1}EA^{-1} & -A^{-1}F(\mathbf{I} - EA^{-1}F)^{-1} \\ -(\mathbf{I} - EA^{-1}F)^{-1}EA^{-1} & (\mathbf{I} - EA^{-1}F)^{-1} \end{bmatrix}$$

This is so that we have $\Rightarrow$

$$\begin{aligned} \begin{bmatrix} A & F \\ E & \mathbf{I} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \boldsymbol{\pi}_{Cr} \\ \boldsymbol{\pi}_{Pr} \end{bmatrix} &= \begin{bmatrix} (A^{-1} + A^{-1}F(\mathbf{I} - EA^{-1}F)^{-1}EA^{-1})\,\boldsymbol{\pi}_{Cr} & -\left(A^{-1}F(\mathbf{I} - EA^{-1}F)^{-1}\right)\boldsymbol{\pi}_{Pr} \\ -\left((\mathbf{I} - EA^{-1}F)^{-1}EA^{-1}\right)\boldsymbol{\pi}_{Cr} & \left((\mathbf{I} - EA^{-1}F)^{-1}\right)\boldsymbol{\pi}_{Pr} \end{bmatrix} \\ &= \begin{bmatrix} A^{-1}\boldsymbol{\pi}_{Cr} - A^{-1}F(\mathbf{I} - EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Pr} - EA^{-1}\boldsymbol{\pi}_{Cr}) \\ (\mathbf{I} - EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Pr} - EA^{-1}\boldsymbol{\pi}_{Cr}) \end{bmatrix} \end{aligned}$$

We then focus on the first line for which we have the following expression $\Rightarrow$

$$\begin{aligned} A^{-1}\boldsymbol{\pi}_{Cr} - A^{-1}F(\mathbf{I} - EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Pr} - EA^{-1}\boldsymbol{\pi}_{Cr}) &= A^{-1}(\mathbf{I} - EA^{-1}F)^{-1}(\mathbf{I} - EA^{-1}F)\boldsymbol{\pi}_{Cr} \\ &\quad - A^{-1}F(\mathbf{I} - EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Pr} - EA^{-1}\boldsymbol{\pi}_{Cr}) \\ &= A^{-1}(\mathbf{I} - EA^{-1}F)^{-1}\boldsymbol{\pi}_{Cr} \\ &\quad - A^{-1}(\mathbf{I} - EA^{-1}F)^{-1}EA^{-1}F\boldsymbol{\pi}_{Cr} \\ &\quad - A^{-1}F(\mathbf{I} - EA^{-1}F)^{-1}\boldsymbol{\pi}_{Pr} \\ &\quad + A^{-1}F(\mathbf{I} - EA^{-1}F)^{-1}EA^{-1}\boldsymbol{\pi}_{Cr} \\ &= A^{-1}(\mathbf{I} - EA^{-1}F)^{-1}\boldsymbol{\pi}_{Cr} \\ &\quad - A^{-1}F(\mathbf{I} - EA^{-1}F)^{-1}\boldsymbol{\pi}_{Pr} \\ &= A^{-1}(\mathbf{I} - EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Cr} - F\boldsymbol{\pi}_{Pr}) \end{aligned}$$

$$\mathbf{r}^*_{Cr}(\mathbf{G}, \Omega, a) = A^{-1}(\mathbf{I} - EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Cr} - F\boldsymbol{\pi}_{Pr})$$
$$\mathbf{r}^*_{Pr}(\mathbf{G}, \Omega, a) = (\mathbf{I} - EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Pr} - EA^{-1}\boldsymbol{\pi}_{Cr}).$$

Since $F = a\mathbf{G}(\mathbf{CP})$, $E = a\mathbf{G}(\mathbf{PC})$, and $(\mathbf{I} + a\mathbf{G}(\mathbf{CC})) = A$ we then have the following;

$$\mathbf{r}^*_{Cr}(\mathbf{G}, \Omega, a) = (\mathbf{I} + a\mathbf{G}(\mathbf{CC}))^{-1}\left(\mathbf{I} - a\mathbf{G}(\mathbf{PC})(\mathbf{I} + a\mathbf{G}(\mathbf{CC}))^{-1}a\mathbf{G}(\mathbf{CP})\right)^{-1}(\boldsymbol{\pi}_{Cr} - a\mathbf{G}(\mathbf{CP})\boldsymbol{\pi}_{Pr}),$$
$$\mathbf{r}^*_{Pr}(\mathbf{G}, \Omega, a) = \left(\mathbf{I} - a\mathbf{G}(\mathbf{PC})(\mathbf{I} + a\mathbf{G}(\mathbf{CC}))^{-1}a\mathbf{G}(\mathbf{CP})\right)^{-1}(\boldsymbol{\pi}_{Pr} - a\mathbf{G}(\mathbf{PC})(\mathbf{I} + a\mathbf{G}(\mathbf{CC}))^{-1}\boldsymbol{\pi}_{Cr}).$$

Since for a matrix $\mathbf{Z}$, $(\mathbf{I} - \theta \bullet \mathbf{I}) \bullet \mathbf{Z} = (1 - \theta) \bullet \mathbf{Z}$, we then have the following;

$$\begin{aligned}
\mathbf{r}^*_{Cr}(\mathbf{G}, \Omega, a) &= (\mathbf{I} + a\mathbf{G}(\mathbf{CC}))^{-1}\left(\mathbf{I} - a^2\theta\varrho(\mathbf{I} + a\mathbf{G}(\mathbf{CC}))^{-1}\right)^{-1}(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr}),\\
&= \left((\mathbf{I} + a\mathbf{G}(\mathbf{CC}))\left(\mathbf{I} - a^2\theta\varrho(\mathbf{I} + a\mathbf{G}(\mathbf{CC}))^{-1}\right)\right)^{-1}(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr}),\\
&= \left(\mathbf{I} + a\mathbf{G}(\mathbf{CC}) - a^2\theta\varrho \bullet \mathbf{I}\right)^{-1}(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr}),\\
&= \left((1 - a^2\theta\varrho) \bullet \mathbf{I} + a\mathbf{G}(\mathbf{CC})\right)^{-1}(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr}),\\
&= \left(\mathbf{I} + \frac{a}{1 - a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^{-1}\frac{(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr})}{1 - a^2\theta\varrho}.
\end{aligned}$$

$$
\begin{aligned}
\mathbf{r}^*_{Pr}(\mathbf{G},\Omega,a) &= \left(\mathbf{I}-a^2\theta\varrho(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\right)^{-1}(\boldsymbol{\pi}_{Pr}-a\varrho(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\boldsymbol{\pi}_{Cr}),\\
&= \left((\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\left(\mathbf{I}+a\mathbf{G}(\mathbf{CC})-a^2\theta\varrho\bullet\mathbf{I}\right)\right)^{-1}\boldsymbol{\pi}_{Pr}\\
&- a\varrho\left(\left(\mathbf{I}-a^2\theta\varrho(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\right)(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))\right)^{-1}\boldsymbol{\pi}_{Cr},\\
&= (\mathbf{I}+a\mathbf{G}(\mathbf{CC}))\left(\mathbf{I}+a\mathbf{G}(\mathbf{CC})-a^2\theta\varrho\bullet\mathbf{I}\right)^{-1}\boldsymbol{\pi}_{Pr}\\
&- a\varrho\left(\mathbf{I}+a\mathbf{G}(\mathbf{CC})-a^2\theta\varrho\bullet\mathbf{I}\right)^{-1}\boldsymbol{\pi}_{Cr},\\
&= \left(\mathbf{I}+a\mathbf{G}(\mathbf{CC})-a^2\theta\varrho\bullet\mathbf{I}\right)^{-1}((\mathbf{I}+a\mathbf{G}(\mathbf{CC}))\boldsymbol{\pi}_{Pr}-a\varrho\bullet\boldsymbol{\pi}_{Cr})\\
&= \left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)^{-1}((\mathbf{I}+a\mathbf{G}(\mathbf{CC}))\boldsymbol{\pi}_{Pr}-a\varrho\bullet\boldsymbol{\pi}_{Cr})\\
&= \left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)^{-1}(a\mathbf{G}(\mathbf{CC})\boldsymbol{\pi}_{Pr}+\boldsymbol{\pi}_{Pr}-a\varrho\bullet\boldsymbol{\pi}_{Cr})\\
&= \left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)^{-1}\\
&\left(\left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)\boldsymbol{\pi}_{Pr}-(1-a^2\theta\varrho)\bullet\boldsymbol{\pi}_{Pr}+\boldsymbol{\pi}_{Pr}-a\varrho\bullet\boldsymbol{\pi}_{Cr}\right)\\
&= \left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)^{-1}(a^2\theta\varrho\bullet\boldsymbol{\pi}_{Pr}-a\varrho\bullet\boldsymbol{\pi}_{Cr})\\
&+ ((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})^{-1}\left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)\boldsymbol{\pi}_{Pr}\\
&= \boldsymbol{\pi}_{Pr}+\left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)^{-1}(a\varrho(a\theta\bullet\boldsymbol{\pi}_{Pr}-\boldsymbol{\pi}_{Cr}))\\
&= \boldsymbol{\pi}_{Pr}-a\varrho\bullet\left(\mathbf{I}+\frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^{-1}\frac{\boldsymbol{\pi}_{Cr}-a\theta\bullet\boldsymbol{\pi}_{Pr}}{1-a^2\theta\varrho}.
\end{aligned}
$$

□

*

Proof of Proposition 3.4.2 From proposition 3.4.1, recall we have the vector of Bonacich centrality grouped in Core and Periphery vector as follows;

$$
\mathbf{r}_{Cr}(\mathbf{G},\Omega,a) = \left(\mathbf{I}+\frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^{-1}\frac{(\boldsymbol{\pi}_{Cr}-a\theta\boldsymbol{\pi}_{Pr})}{1-a^2\theta\varrho}, \tag{3.32}
$$

as well as,

$$\mathbf{r}_{Pr}(\mathbf{G},\Omega,a) = \boldsymbol{\pi}_{Pr} - \left(\mathbf{I} + \frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^{-1} \frac{(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr})}{1-a^2\theta\varrho} \tag{3.33}$$

Given the assumption that $\mathbf{G}(\mathbf{CC}) \bullet \mathbf{1} = \rho \bullet \mathbf{1}$, then such regularity means that we have the following Bonacich centrality based Nash equilibrium vectors;

$$\begin{aligned}
\mathbf{r}_{Cr}(\mathbf{G},\Omega,a) &= \left(1 + \frac{a\rho}{1-a^2\theta\varrho}\right)^{-1} \frac{(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr})}{1-a^2\theta\varrho}, \\
&= \left(\frac{1-a^2\theta\varrho}{1-a^2\theta\varrho + a\rho}\right) \frac{(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr})}{1-a^2\theta\varrho}, \\
&= (1 + a\rho - a^2\theta\varrho)^{-1}(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr}).
\end{aligned}$$

Then for the Periphery set we have;

$$\mathbf{r}_{Pr}(\mathbf{G},\Omega,a) = \boldsymbol{\pi}_{Pr} - a\varrho(1 + a\rho - a^2\theta\varrho)^{-1}(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr})$$

□

Proof of Lemma 3.5.1

Recall that $\hat{g}_{ji} = \frac{b_{ji}}{b_i}$.

Assume $\mathcal{N} = \mathcal{A}$. This means we can rewrite (3.4) as follows

$$P_i = b_{-i}r_i - \sum_{j\in\mathcal{N}_i^{in}} b_{ji}r_j - \kappa \bullet (\mu_i)^2 \tag{3.34}$$

Also, from (3.8),

$$r_i = \pi_i - a\sum_{j\in\mathcal{N}_i^{in}} \frac{b_{ji}}{b_{-i}} r_j$$

yielding;

$$b_{-i}\pi_i = b_{-i}r_i + a \sum_{j\in\mathcal{N}_i^{in}} b_{ji}r_j \tag{3.35}$$

Also from (3.35),

$$\sum_{j\in\mathcal{N}_i^{in}} b_{ji}r_j = \frac{b_{-i}\pi_i - b_{-i}r_i}{a} \tag{3.36}$$

then substituting (3.35) and (3.36) in (3.34) yields;

$$P_i = b_{-i}r_i - \frac{b_{-i}\pi_i + b_{-i}r_i}{a} - \kappa \bullet (\mu_i)^2$$

which is also;

$$P_i = b_{-i}r_i\frac{(a+1)}{a} - \frac{b_{-i}\pi_i}{a} - \kappa \bullet (b_{-i}\pi_i)^2$$

Given that we have $\pi_i = \frac{1}{2\kappa b_{-i}}$, we then have our payoff as ;

$$P_{i\in\mathcal{A}} = \frac{b_{-i}(a+1)}{a}r_i - \frac{2+a}{4\kappa a}. \tag{3.37}$$

Let $\omega = \frac{(a+1)}{a}$ and $\eta = \frac{2+a}{4\kappa a}$, given (4), we have the expression with respect to firm $i \in \mathcal{A}$ Bonacich centrality as;

$$P_{i\in\mathcal{A}} = \omega b_{-i}\left((\mathbf{I} + a\mathbf{G})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}\right)_i - \eta \tag{3.38}$$

In vector for, this becomes;

$$\mathbf{P}_{\mathcal{A}} = diag(\mathbf{B}) \bullet \left((\mathbf{I} + a\mathbf{G})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}\right) - \boldsymbol{K}$$

such that $\boldsymbol{K} = [\eta]^{\mathcal{A}\times 1}$ and $\mathbf{B} = [\omega \bullet b_{-i}]^{\mathcal{A}\times 1}$. $\square$

Proof of Proposition 3.5.1

So we have that given $\lambda = (1+\varepsilon)$, we have $P_i^{\varepsilon}(r_i) = \lambda\left(b_{-i} \bullet r_i - \sum_{j\in\mathcal{N}_i^{in}} (\hat{g}_{ji} b_i) r_j\right) - \kappa\left(\lambda\mu_i\right)^2 + \xi r_i$. If we were to take the differential with respect to r_i; we end up with the best reply as follows;

$$r_i(\lambda) = \frac{\pi_i}{\lambda} - a\sum_j \frac{\lambda b_{ji}}{\lambda b_{-i}} r_j = \frac{\pi_i}{\lambda} - a\sum_j \frac{b_{ji}}{b_{-i}} r_j.$$

This is so that rewriting in vector form, our Nash for active firms is given as;

$$\mathbf{r}(\lambda) = (\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \frac{\boldsymbol{\pi}_{\mathcal{A}}}{\lambda}.$$

We can simply deduce from (3.18) that the vector payoff for active firms is as follows;

$$\mathbf{P}_{\mathcal{A}}^{\lambda} = \lambda diag(\mathbf{B}) \bullet \left((\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \frac{\boldsymbol{\pi}_{\mathcal{A}}}{\lambda}\right) - \boldsymbol{K} = \mathbf{P}_{\mathcal{A}}$$

This is because granting εb_{-i} to each firm $i \in \mathcal{N}$ yields equation (1) and (6). Hence payoff is homogeneous of degree zero, i.e $\mathbf{P}_{\mathcal{A}}^{\lambda}(\lambda b_{-i}) = \mathbf{P}_{\mathcal{A}}(b_{-i})$. As such, welfare differential

$$W(\mathbf{r}^*, \mathcal{A}) - W^{\lambda}(\mathbf{r}^{\lambda}, \mathcal{A}) = \mathbf{1}^T(\mathbf{P}_{\mathcal{A}} - \mathbf{P}_{\mathcal{A}}^{\lambda}) = 0.$$

□

Proof of Theorem 3.5.1

Assume the Planner decides to change a firm i's total lending by a parameter λ and let us have it that the policy intervention λ_i such that payoffs is written as;

$$P_i^{\lambda}(r_i) = \lambda_i b_{-i} \bullet r_i - \sum_{j\in\mathcal{N}_i^{in}} (\hat{g}_{ji} b_i) r_j - k\left(\lambda_i b_{-i} \bullet r_i + a\sum_{j\in\mathcal{N}_i^{in}} (\hat{g}_{ji} b_i) r_j\right)^2 \tag{3.39}$$

The addition of λb_i to firm i is strictly conditional on the following;

1. $\mathcal{A}(\lambda) = \mathcal{A}$, and

2. $a \in \left]0, \frac{1}{\left|\frac{\mathbf{G}(\lambda)+\mathbf{G}(\lambda)^T}{2}\right|}\right[$.

The Nash equilibrium for firm i given λb_{-i} is ;

$$r_i^\lambda = \frac{\pi_i}{\lambda_i} - a \sum_{j \in \mathcal{N}_i^{in}, j \in \mathcal{A}} \frac{g_{ji}}{\lambda_i} r_j^\lambda$$

While the equilibrium for all firm $j | j \in \mathcal{N}_i^{out} \cap \mathcal{A}$ is

$$r_j^\lambda = \pi_j - a \sum_{k \in \left(\mathcal{N}_k^{in} - \{i\}\right) \cap \mathcal{A}} g_{kj} r_k^\lambda - a \lambda_i g_{ij} r_i^\lambda$$

The vector payoff for active firms is then;

$$\mathbf{P}_\mathcal{A}^\lambda = diag(\mathbf{B}^\lambda) \boldsymbol{\cdot} \left((\mathbf{I} + a\mathbf{G}_\mathcal{A}^\lambda)^{-1} \boldsymbol{\cdot} \boldsymbol{\pi}_\mathcal{A}^\lambda\right) - \boldsymbol{K}$$

where $\mathbf{B}^\lambda = (\omega \lambda_i b_i, \omega b_j, \omega b_k, \ldots)^T$, $\boldsymbol{\pi}^\lambda = \left(\lambda_i^{-1} \pi_i, \pi_j, \pi_k \ldots\right)^T$ and lastly,

$$\mathbf{G}_\mathcal{A}^\lambda = \begin{bmatrix} 0 & \frac{g_{ji}}{\lambda_i} & \ldots & \frac{g_{ni}}{\lambda_i} \\ \lambda_i g_{ij} & \ldots & \ldots & g_{ji} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_i g_{in} & g_{nj} & \ldots & 0 \end{bmatrix}$$

We then show that $diag(\mathbf{B}^\lambda) \boldsymbol{\cdot} \left((\mathbf{I} + a\mathbf{G}_\mathcal{A}^\lambda)^{-1} \boldsymbol{\cdot} \boldsymbol{\pi}_\mathcal{A}^\lambda\right) = diag(\mathbf{B}) \boldsymbol{\cdot} ((\mathbf{I} + a\mathbf{G}_\mathcal{A})^{-1} \boldsymbol{\cdot} \boldsymbol{\pi}_\mathcal{A})$. First we have that

$$diag(\mathbf{B}^{\lambda}) \bullet \left((\mathbf{I}+a\mathbf{G}_{\mathcal{A}}^{\lambda})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}^{\lambda}\right) = \begin{bmatrix} \omega\lambda_i b_i & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \omega b_n \end{bmatrix} \times \begin{bmatrix} m_{ii} & \frac{m_{ji}}{\lambda_i} & \dots & \frac{m_{ni}}{\lambda_i} \\ m_{ij} * \lambda_i & \dots & \dots & m_{ni} \\ \vdots & \vdots & \vdots & \vdots \\ m_{in} * \lambda_i & m_{nj} & \dots & m_{nn} \end{bmatrix} \times \begin{bmatrix} \frac{\pi_i}{\lambda_i} \\ \pi_j \\ \vdots \\ \pi_n \end{bmatrix}$$

This is then the same as;

$$diag(\mathbf{B}^{\lambda}) \bullet \left((\mathbf{I}+a\mathbf{G}_{\mathcal{A}}^{\lambda})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}^{\lambda}\right) = \begin{bmatrix} \omega\lambda_i b_{-i} & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \omega b_{-n} \end{bmatrix} \times \begin{bmatrix} \frac{1}{\lambda_i}\left(m_{ii}\pi_i + m_{ji}\pi_j + \ldots + m_{ni}\pi_n\right) \\ m_{ij}\pi_i + \ldots + \ldots + m_{ni}\pi_n \\ \vdots \\ m_{in}\pi_i + m_{nj}\pi_j + \ldots + m_{nn}\pi_n \end{bmatrix}$$

$$= \begin{bmatrix} \omega b_i\left(m_{ii}\pi_i + m_{ji}\pi_j + \ldots + m_{ni}\pi_n\right) \\ \omega b_j\left(m_{ij}\pi_i + \ldots + \ldots + m_{ni}\pi_n\right) \\ \vdots \\ \omega b_n\left(m_{in}\pi_i + m_{nj}\pi_j + \ldots + m_{nn}\pi_n\right) \end{bmatrix}$$

$$= diag(\mathbf{B}) \bullet \left((\mathbf{I}+a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}\right)$$

Say then we have $\lambda_i \neq \lambda_j \neq \ldots \neq \lambda_n$, we have our Nash equilibrium as;

$$(\mathbf{I}+a\mathbf{G}_{\mathcal{A}}^{\lambda})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}^{\lambda} = \begin{bmatrix} m_{ii} & \frac{m_{ji}*\lambda_j}{\lambda_i} & \dots & \frac{m_{ni}*\lambda_n}{\lambda_i} \\ \frac{m_{ij}*\lambda_i}{\lambda_j} & \dots & \dots & \frac{m_{ni}*\lambda_n}{\lambda_j} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{m_{in}*\lambda_i}{\lambda_n} & \frac{m_{nj}*\lambda_j}{\lambda_n} & \dots & m_{nn} \end{bmatrix} \times \begin{bmatrix} \frac{\pi_i}{\lambda_i} \\ \frac{\pi_j}{\lambda_j} \\ \vdots \\ \frac{\pi_n}{\lambda_n} \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_i}\left(m_{ii}\pi_i + m_{ji}\pi_j + \ldots + m_{ni}\pi_n\right) \\ \frac{1}{\lambda_j}\left(m_{ij}\pi_i + \ldots + \ldots + m_{ni}\pi_n\right) \\ \vdots \\ \frac{1}{\lambda_j}\left(m_{in}\pi_i + m_{nj}\pi_j + \ldots + m_{nn}\pi_n\right) \end{bmatrix}$$

Which when multiplied by $diag(\mathbf{B}^{\xi})$ still yields the same expression that

$$diag(\mathbf{B}^{\lambda}) \bullet \left((\mathbf{I}+a\mathbf{G}_{\mathcal{A}}^{\lambda})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}^{\lambda}\right) = diag(\mathbf{B}) \bullet \left((\mathbf{I}+a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}\right).$$

□

Proof of Lemma 3.5.3

then best replies are ;

$$r_i^\gamma = \frac{\pi_i}{(1+\gamma)} - a \sum_{j \in \mathcal{N}_i^{in}, j \in \mathcal{A}} g_{ji} r_j^\gamma$$

While the vector payoff for active firms is then;

$$\mathbf{P}_{\mathcal{A}}^\gamma = diag(\mathbf{B}) \bullet \left((\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \frac{\boldsymbol{\pi}_{\mathcal{A}}}{(1+\gamma)} \right) - \frac{\boldsymbol{K}}{(1+\gamma)} = \frac{1}{(1+\gamma)} \bullet \mathbf{P}_{\mathcal{A}}$$

as such, welfare differential

$$W^\gamma(\mathbf{r}^\gamma, \mathcal{A}) - W(\mathbf{r}^*, \mathcal{A}) = \mathbf{1}^T \mathbf{P}_{\mathcal{A}} \bullet \frac{\gamma}{(1+\gamma)}$$

□

*

Proof of Theorem 3.6.1 The best replies for the firm i which is subsidised for is ;

$$r_i^\gamma = \frac{\pi_i}{(1+\gamma)} - a \sum_{j \in \mathcal{N}_i^{in}, j \in \mathcal{A}} g_{ji} r_j^\gamma$$

While the vector payoff for active firms is then;

$$\mathbf{P}_{\mathcal{A}}^\gamma = diag(\mathbf{B}) \bullet \left((\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}^\gamma \right) - \boldsymbol{K}^\gamma$$

Where $\boldsymbol{\pi}_{\mathcal{A}}^\gamma = \left(\frac{\pi_i}{1+\gamma}, \pi_j, \ldots \right)^T$, while $\boldsymbol{K}^\gamma = \left(\frac{\eta}{1+\gamma}, \eta, \ldots \right)^T$. As such, payoff vector differential;

$$\mathbf{P}^{\gamma}(\mathbf{r}^{\gamma},\mathcal{A}) - \mathbf{P}(\mathbf{r}^{*},\mathcal{A}) = diag(\mathbf{B}) \bullet \left((\mathbf{I}+a\mathbf{G}_{\mathcal{A}})^{-1} \bullet (\boldsymbol{\pi}^{\gamma}_{\mathcal{A}} - \boldsymbol{\pi}_{\mathcal{A}})\right) - (\boldsymbol{K}^{\gamma} + \boldsymbol{K}) \quad (3.40)$$

Where $\boldsymbol{\pi}^{\gamma}_{\mathcal{A}} - \boldsymbol{\pi}_{\mathcal{A}} = \left(\frac{\pi_i\gamma}{1+\gamma}, 0, \ldots, 0\right)^T$, while $\boldsymbol{K}^{\gamma} - \boldsymbol{K} = \left(\frac{\eta\gamma}{1+\gamma}, 0, \ldots, 0\right)^T$ We can then expand (3.40) as such;

$$\mathbf{P}^{\gamma}(\mathbf{r}^{\gamma},\mathcal{A}) - \mathbf{P}(\mathbf{r},\mathcal{A}) = \begin{bmatrix} \omega b_{-i} & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \omega b_{-n} \end{bmatrix} \bullet (\mathbf{I}+a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \begin{bmatrix} -\frac{\pi_i\gamma}{1+\gamma} \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{\eta\gamma}{1+\gamma} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \omega b_{-i} & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \omega b_{-n} \end{bmatrix} \bullet \begin{bmatrix} -m_{ii}\frac{\pi_i\gamma}{1+\gamma} \\ -m_{ij}\frac{\pi_i\gamma}{1+\gamma} \\ \vdots \\ -m_{ik}\frac{\pi_i\gamma}{1+\gamma} \end{bmatrix} - \begin{bmatrix} -\frac{\eta\gamma}{1+\gamma} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -m_{ii}\omega\frac{b_{-i}\pi_i\gamma}{1+\gamma} \\ -m_{ij}\omega\frac{b_{-j}\pi_i\gamma}{1+\gamma} \\ \vdots \\ -m_{ik}\omega\frac{b_{-n}\pi_i\gamma}{1+\gamma} \end{bmatrix} - \begin{bmatrix} -\frac{\eta\gamma}{1+\gamma} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This then means that since $\pi_i = \frac{1}{2\kappa b_{-i}}$, we have;

$$\Delta W(\mathbf{r}^{\gamma},\mathcal{A}|i) = \frac{-\gamma\omega}{2\kappa(1+\gamma)}\left(m_{ii} + m_{ij}\frac{b_{-j}}{b_{-i}} + \ldots + m_{in}\frac{b_{-n}}{b_{-i}}\right) + \frac{\eta\gamma}{1+\gamma} \quad (3.41)$$

This means that if $b_{-i} = b_{-j}\ \forall\ i,j \in \mathcal{A}$, then we have that the equation above becomes;

$$\begin{aligned}
\Delta W(\mathbf{r}^{\gamma}, \mathcal{A}|i) &= \frac{-\gamma\omega}{2\kappa(1+\gamma)}\,(m_{ii} + m_{ij} + \ldots + m_{in}) + \frac{\eta\gamma}{1+\gamma} \\
&= \frac{-\gamma\omega}{2\kappa(1+\gamma)} \bullet \beta_i(\mathbf{G}^T_{\mathcal{A}}, -a) + \frac{\eta\gamma}{1+\gamma} \qquad (3.42) \\
&> 0 \qquad \text{in so far } \gamma < 0.
\end{aligned}$$

Observe also that $\frac{-\gamma\omega}{2\kappa(1+\gamma)}$ as well as $\frac{\eta\gamma}{1+\gamma}$ is common to every active firm. This means that the firm i such that $\beta_i(\mathbf{G}^T_{\mathcal{A}}, -a)$ is greatest achieves the highest value of $\Delta W(\mathbf{r}^{\gamma}, \mathcal{A}|i)$.

□

Pseudo-Code for Computation

Algorithm 1 Nash Equilibrium Lending Rate Algorithm

procedure DEFINE PARAMETERS

  $\mathcal{A}(k) \subset \mathcal{N}$, $\mathcal{N} - \mathcal{A}(k) \subset \mathcal{N}$, $\mathcal{N} - \mathcal{A}(k) \cap \mathcal{A}(k) = \emptyset$, $\mathcal{N} - \mathcal{A}(k) \cup \mathcal{A}(k) = \mathcal{N}$.

  $max_k = 2^{|\mathcal{N}|} - 1$ (*loop*)

loop:

  if $k = 1 : 1 : max_k$ **then**

    $\mathbf{r}^*_{\mathcal{A}(k)} = (\mathbf{I} + a\mathbf{G}_{\mathcal{A}(k),\mathcal{A}(k)})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}(k)}$.

    $\mathbf{r}^*_{\mathcal{N}-\mathcal{A}(k)} = \mathbf{0}$.

End If:

  $\mathbf{r}^*_{\mathcal{A}(k)} \geq \mathbf{0}$ and,

  $a\mathbf{G}_{\mathcal{N}-\mathcal{A}(k),\mathcal{A}(k)} \bullet \mathbf{r}^*_{\mathcal{A}(k)} \geq \boldsymbol{\pi}_{\mathcal{N}-\mathcal{A}(k)}$.

Else:

  goto *loop*.

Bibliography

Abrantes-Metz, R. M., Kraten, M., Metz, A. D., & Seow, G. S. (2012). Libor manipulation? *Journal of Banking & Finance*, *36*(1), 136–150.

Acemoglu, D., Ozdaglar, A., & Tahbaz-Salehi, A. (2015). Systemic risk and stability in financial networks. *The american economic review*, *105*(2), 564–608.

Aldasoro, I., & Angeloni, I. (2015). Input–output-based measures of systemic importance. *Quantitative Finance*, *15*(4), 589–606.

Aldasoro, I., Gatti, D. D., & Faia, E. (2017). Bank networks: Contagion, systemic risk and prudential policy. *Journal of Economic Behavior & Organization*, *142*, 164–188.

Allen, F., & Babus, A. (2008). Networks in finance.

Allouch, N. (2015). On the private provision of public goods on networks. *Journal of Economic Theory*, *157*, 527–552.

Allouch, N., & Jalloul, M. (2016). Strategic defaults in financial networks. *Queen Marys University Working Paper series*.

Allouch, N., & King, M. (2018a). *Constrained public goods in networks* (Tech. Rep.). School of Economics Discussion Papers.

Allouch, N., & King, M. (2018b). A network approach to welfare.

Amini, H., Filipović, D., & Minca, A. (2016). Uniqueness of equilibrium in a payment system with liquidation costs. *Operations Research Letters*, *44*(1), 1–5.

Babus, A. (2016). The formation of financial networks. *The RAND Journal of*

Economics, *47*(2), 239–272.

Ballester, C., Calvó-Armengol, A., & Zenou, Y. (2006). Who's who in networks. wanted: The key player. *Econometrica*, *74*(5), 1403–1417.

Belhaj, M., Bervoets, S., & Deroïan, F. (2016). Efficient networks in games with local complementarities. *Theoretical Economics*, *11*(1), 357–380.

Belhaj, M., & Deroïan, F. (2019). Group targeting under networked synergies. *Games and Economic Behavior*, *118*, 29–46.

Bergstrom, T., Blume, L., & Varian, H. (1986). On the private provision of public goods. *Journal of public economics*, *29*(1), 25–49.

Bhattacharya, S., Gale, D., Barnett, W., & Singleton, K. (1985). Preference shocks, liquidity, and central bank policy. *Liquidity and crises*, *35*.

Bilkic, N., Gries, T., et al. (2014). *Destructive agents, finance firms, and systemic risk* (Tech. Rep.). Paderborn University, CIE Center for International Economics.

Blume, L., Easley, D., Kleinberg, J., Kleinberg, R., & Tardos, É. (2011). Network formation in the presence of contagious risk. In *Proceedings of the 12th acm conference on electronic commerce* (pp. 1–10).

Bonacich, P. (1987). Power and centrality: A family of measures. *American journal of sociology*, *92*(5), 1170–1182.

Bramoullé, Y., & Kranton, R. (2007). Public goods in networks. *Journal of Economic Theory*, *135*(1), 478–494.

Bramoullé, Y., Kranton, R., & D'amours, M. (2014). Strategic interaction and networks. *American Economic Review*, *104*(3), 898–930.

Caballero, R. J., & Simsek, A. (2013). Fire sales in a model of complexity. *The Journal of Finance*, *68*(6), 2549–2587.

Cabrales, A., Gottardi, P., & Vega-Redondo, F. (2014). Risk-sharing and contagion in networks.

Chiu, J., Eisenschmidt, J., & Monnet, C. (2020). Relationships in the interbank market. *Review of Economic Dynamics*, *35*, 170–191.

Coulter, B., Shapiro, J., & Zimmerman, P. (2017). A mechanism for libor. *Review of Finance*, *22*(2), 491–520.

Demange, G. (2016). Contagion in financial networks: a threat index. *Management Science*, *64*(2), 955–970.

Deng, A., Sato, I., & Wu, Y. (2007). Homomorphisms, representations and characteristic polynomials of digraphs. *Linear algebra and its applications*, *423*(2-3), 386–407.

Duffie, D., & Dworczak, P. (2014). *Robust benchmark design* (Tech. Rep.). National Bureau of Economic Research.

Eaglesham, J. (2013). Bank made huge bet, and profit, on libor. *Wall Street Journal*, *9*.

Eisenberg, L., & Noe, T. H. (2001). Systemic risk in financial systems. *Management Science*, *47*(2), 236–249.

Eisl, A., Jankowitsch, R., & Subrahmanyam, M. G. (2017). The manipulation potential of libor and euribor. *European Financial Management*, *23*(4), 604–647.

Elliott, M., Golub, B., & Jackson, M. O. (2014). Financial networks and contagion. *The American economic review*, *104*(10), 3115–3153.

Elsinger, H., Lehar, A., & Summer, M. (2006). Risk assessment for banking systems. *Management science*, *52*(9), 1301–1314.

Feinstein, Z. (2017). Financial contagion and asset liquidation strategies. *Operations Research Letters*, *45*(2), 109–114.

Galeotti, A., Ghiglinoy, C., & Goyal, S. (2016). Financial linkages, portfolio choice and systemic risk.

Galeotti, A., Golub, B., & Goyal, S. (2020). Targeting interventions in networks. *Econometrica*, *88*(6), 2445–2471.

Gollier, C., Koehl, P.-F., & Rochet, J.-C. (1997). Risk-taking behavior with limited liability and risk aversion. *Journal of Risk and Insurance*, 347–370.

Greenwood, R., Landier, A., & Thesmar, D. (2015). Vulnerable banks. *Journal of Financial Economics*, *115*(3), 471–485.

Kada, O. (2020). Characteristic polynomials and zeta functions of equitably partitioned graphs. *Linear Algebra and its Applications*, *588*, 471–488.

König, M., Tessone, C. J., & Zenou, Y. (2009). A dynamic model of network formation with strategic interactions.

Lux, T., Fricke, D., et al. (2012). Core-periphery structure in the overnight money market: evidence from the e-mid trading platform. *Computational Economics*.

Morris, S. (2000). Contagion. *The Review of Economic Studies*, *67*(1), 57–78.

Morris, S., & Shin, H. S. (2001). Global games: Theory and applications.

Nier, E., Yang, J., Yorulmazer, T., & Alentorn, A. (2007). Network models and financial stability. *Journal of Economic Dynamics and Control*, *31*(6), 2033–2060.

Rogers, L. C., & Veraart, L. A. (2013). Failure and rescue in an interbank network. *Management Science*, *59*(4), 882–898.

Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica: Journal of the Econometric Society*, 520–534.

Sui, P., Tanna, S., & Zhou, D. (2020). Financial contagion in a core-periphery interbank network. *The European Journal of Finance*, *26*(7-8), 691–710.

Van Lelyveld, I., et al. (2014). Finding the core: Network structure in interbank markets. *Journal of Banking & Finance*, *49*, 27–40.

Warr, P. G. (1983). The private provision of a public good is independent of the distribution of income. *Economics letters*, *13*(2-3), 207–211.

Wells, S. J. (2004). Financial interlinkages in the united kingdom's interbank market and the risk of contagion.

Wolf, H. (1978). The helmert block method, its origin and development. In *Proceedings of the second international symposium on problems related to the redefinition of north american geodetic networks* (pp. 319–326).