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Linear relations for Laurent polynomials and lattice equations

Andrew N W Hone[1,*] **and Joe Pallister**

School of Mathematics, Statistics & Actuarial Science, University of Kent, United Kingdom

E-mail: A.N.W.Hone@kent.ac.uk

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Abstract

A recurrence relation is said to have the Laurent property if all of its iterates are Laurent polynomials in the initial values with integer coefficients. Recurrences with this property appear in diverse areas of mathematics and physics, ranging from Lie theory and supersymmetric gauge theories to Teichmüller theory and dimer models. In many cases where such recurrences appear, there is a common structural thread running between these different areas, in the form of Fomin and Zelevinsky's theory of cluster algebras. Laurent phenomenon algebras, as defined by Lam and Pylyavskyy, are an extension of cluster algebras, and share with them the feature that all the generators of the algebra are Laurent polynomials in any initial set of generators (seed). Here we consider a family of nonlinear recurrences with the Laurent property, referred to as 'Little Pi', which was derived by Alman *et al* via a construction of periodic seeds in Laurent phenomenon algebras, and generalizes the Heideman–Hogan family of recurrences. Each member of the family is shown to be linearizable, in the sense that the iterates satisfy linear recurrence relations with constant coefficients. We derive the latter from linear relations with periodic coefficients, which were found recently by Kamiya *et al* from travelling wave reductions of a linearizable lattice equation on a six-point stencil. By making use of the periodic coefficients, we further show that the birational maps corresponding to the Little Pi

[1]Work partly carried out while on leave in the School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia.
[*]Author to whom any correspondence should be addressed.

family are maximally superintegrable. We also introduce another linearizable lattice equation on the same six-point stencil, and present the corresponding linearization for its travelling wave reductions. Finally, for both of the six-point lattice equations considered, we use the formalism of van der Kamp to construct a broad class of initial value problems with the Laurent property.

Keywords: Laurent property, Laurent phenomenon algebra, integrable lattice equation, linearization

Mathematics Subject Classification numbers: 11B37, 39A14, 13F60.

(Some figures may appear in colour only in the online journal)

1. Introduction

There continues to be a great deal of interest in nonlinear recurrences of the form

$$x_{n+m}x_n = P(x_{n+1}, \ldots, x_{m+n-1}), \tag{1}$$

for a polynomial P, with the surprising property that all of the iterates are Laurent polynomials in the initial data with integer coefficients, that is to say

$$x_n \in \mathbb{Z}[x_0^{\pm 1}, \ldots, x_{m-1}^{\pm 1}]$$

for all n. One of the first instances of this Laurent property was in the context of Somos sequences [26], which can be viewed as nonlinear analogues of Fibonacci or Lucas sequences in number theory [14]. The Laurent property is an essential feature of the generators in cluster algebras, a novel class of commutative algebras introduced by Fomin and Zelevinsky [19], which are defined by recursive relations of the same form as (1) but with the restriction that P should be a binomial expression of a specific kind. The same authors also considered a more general set of sufficient conditions which ensure that the above recurrence has the Laurent property, without requiring P to be a binomial [20]. More recently, this led to the introduction of the broader framework of Laurent phenomenon (LP) algebras [42].

Cluster algebras are the focus of much activity due to their connections with diverse areas of mathematics and physics, ranging from Lie theory and supersymmetric gauge theories to Teichmüller theory and dimer models [13, 17, 30]. The structure of a typical cluster algebra is very complicated, due to the complexity of the recursive process, called mutation, that produces the generators. However, there are certain subclasses of cluster algebras that are associated with discrete integrable systems of some kind, whose structure allows a more explicit description, and often these are the examples that are of most interest in applications to other areas. The simplest subclass consists of the finite type cluster algebras, which have a taxonomy that coincides with the Cartan–Killing classification of semisimple Lie algebras and finite root systems [21], and are associated with purely periodic dynamics, as was observed earlier by Zamolodchikov in the context of the thermodynamic Bethe ansatz for integrable quantum field theories [54].

Beyond finite type, the next interesting subclass of cluster algebras corresponds to discrete dynamical systems that admit linearization, in the sense that the variables satisfy linear recurrence relations with constant coefficients. The simplest example is the recurrence

$$x_n x_{n+2} = x_{n+1}^2 + 1, \tag{2}$$

arising from mutations of the Kronecker quiver (an orientation of the affine $A_1^{(1)}$ diagram), for which the iterates satisfy the linear relation

$$x_{n+2} - Cx_{n+1} + x_n = 0 \tag{3}$$

for all $n \in \mathbb{Z}$, where

$$C = \frac{x_n}{x_{n-1}} + \frac{x_{n-1}}{x_n} + \frac{1}{x_n x_{n-1}} \tag{4}$$

is a first integral (independent of n). The recurrence (2) is the simplest of the Q-systems appearing in the theory of quantum integrable models [10, 11, 40], which provide recursion relations for characters of representations of Yangian algebras, and can be obtained as reductions of discrete Hirota equations [48]. Moreover, the sequence of Laurent polynomials generated by (2), which begins with

$$x_1, x_2, \frac{x_2^2+1}{x_1}, \frac{x_1^2+x_2^4+2x_2^2+1}{x_1^2 x_2}, \frac{x_1^4+2x_1^2x_2^2+2x_1^2+x_2^6+3x_2^4+3x_2^2+1}{x_1^3 x_2^2}, \ldots, \tag{5}$$

has a combinatorial interpretation as the set of generating functions for the perfect matchings of certain graphs [47]. Also note that substituting $x_1 = x_2 = 1$ into (5) gives the sequence $1, 1, 2, 5, 13, \ldots$, which is a bisection of the Fibonacci numbers (missing out every other term).

The reader can easily verify by induction that (4) is a first integral for (2), and thence show directly that the linear relation (3) holds along each orbit, where $C = \text{const}$. However, here we sketch an alternative derivation which is a prototype for the methods used in the sequel. The key to the latter is that (4) can be rewritten using a 2×2 determinant, as

$$\begin{vmatrix} x_n & x_{n+1} \\ x_{n+1} & x_{n+2} \end{vmatrix} = 1.$$

As a consequence, the determinant of the 3×3 matrix

$$\mathbf{M}_n = \begin{pmatrix} x_n & x_{n+1} & x_{n+2} \\ x_{n+1} & x_{n+2} & x_{n+3} \\ x_{n+2} & x_{n+3} & x_{n+4} \end{pmatrix}$$

vanishes, i.e. $|\mathbf{M}_n| = 0$ for all n; this can be checked directly by substituting for the higher shifts of x_n in terms of the lower ones, using (2), but follows effortlessly from the Desnanot–Jacobi identity—see (24) below—by expanding the determinant of $\mathbf{M}_n$ in terms of 2×2 minors. It is then not hard to show that the kernel of $\mathbf{M}_n$ is spanned by a vector of the form $\mathbf{v} = (1, -C, 1)^{\mathrm{T}}$, where C is independent of n, and after solving a linear system for C the formula (4) results, while the first row of the matrix equation $\mathbf{M}_n\mathbf{v} = \mathbf{0}$ yields the linear relation (3).

Linearization in the above sense was found for dynamics of cluster variables obtained from affine Dynkin quivers of type A in [24], for types A and D via frieze patterns in [3], and in general for all affine types ADE in [41]. It has been further conjectured (and proved in certain cases) that linearizability holds for sequences of cluster variables obtained from mutation sequences obtained from box products $X\square Y$ of a finite type Dynkin quiver X and an affine Dynkin quiver Y [51]. A previously known example is provided by Q-systems, which correspond to taking $X = A_n$ for n arbitrary and $Y = A_1^{(1)}$, so (2) is included when $n = 1$. The case of a product of a pair of affine quivers X, Y is not linearizable, but is conjectured to be associated with systems that are integrable in the Liouville–Arnold sense [25].

In work by one of us with Fordy and Hone [23], concerning cluster algebras obtained from quivers that are mutation-periodic with period 1, in the sense of [24], we showed a further property of the affine type A recurrences, specified by a pair of coprime positive integers p, q as

$$x_n x_{n+p+q} = x_{n+p} x_{n+q} + 1, \tag{6}$$

namely that the iterates satisfy additional linear relations with periodic coefficients, of the form

$$\begin{aligned} x_{n+2q} - J_n x_{n+q} + x_n = 0, \quad J_{n+p} = J_n, \\ x_{n+2p} - K_n x_{n+p} + x_n = 0, \quad K_{n+q} = K_n. \end{aligned} \tag{7}$$

(The derivation of the latter relations is analogous to that for (3) sketched above, which is the case $p = q = 1$; see section 4 in [23].) Furthermore, another family of linearizable recurrences from period 1 quivers was found, of the form

$$x_n x_{n+2k} = x_{n+p} x_{n+q} + x_{n+k}, \quad p + q = 2k, \tag{8}$$

which (for $p = 1, q = 3$) includes Dana Scott's recurrence [26]

$$x_n x_{n+4} = x_{n+1} x_{n+3} + x_{n+2}, \tag{9}$$

and these also admit linear relations with periodic coefficients, given by

$$\begin{aligned} x_{n+3q} - J_{n+k} x_{n+2q} + J_n x_{n+q} - x_n = 0, \quad J_{n+p} = J_n, \\ x_{n+3p} - K_{n+k} x_{n+2p} + K_n x_{n+p} - x_n = 0, \quad K_{n+q} = K_n. \end{aligned} \tag{10}$$

(See section 5 in [23] for full details.) Analogous linear relations with periodic coefficients for affine quivers of types D and E, and associated Liouville integrable systems, appear in [50].

Apart from their relevance to the representation theory of affine quivers and associated cluster categories [41], there are other reasons why the recurrences (6) are of particular interest. In Teichmüller theory, they appear as the Ptolemy relations between lambda lengths of geodesic arcs on surfaces [17, 18] (in this case, an annulus with p points on one boundary and q points on the other), and, being associated with triangulated surfaces, they provide the main examples of cluster algebras of finite mutation type [15] (see section 2 for more details). Furthermore, for any p, q (not necessarily coprime), Fordy and Marsh obtained (6) from quivers that they named 'primitives' [24], since they are the building blocks for all quivers that are periodic with period 1 under cluster mutations, hence generate recurrences via cyclic sequences of mutations (once again, the reader is referred to section 2 for a more detailed explanation). In addition, all of the recurrences (6) can be rewritten in terms of a 2×2 determinant, so their integer solutions correspond to one of Coxeter's frieze patterns [7, 8, 46], while (as we shall see below) the solutions of (8) produce SL_3 friezes.

Here we are concerned with linearizable recurrences that exhibit the Laurent property, but go beyond the setting of cluster algebras. As well as providing new examples of discrete integrable systems, instances of the Laurent property that lie outside the framework of cluster algebras have recently been found in the context of Lie theory [28] and Teichmüller theory [53]. To begin with we will consider the family of recurrences

$$x_n x_{n+2k+l} = x_{n+2k} x_{n+l} + a x_{n+k} + a x_{n+k+l}, \tag{11}$$

with a fixed parameter a and positive integers k and l. These recurrences were named the 'Little Pi' family in [2], where they were shown to be generated by period 1 seeds in the setting of

LP algebras. They extend the Heideman–Hogan recurrences [33], corresponding to the case $l = 1$, for which detailed features of the linearization were proved in [36]. Thus our first aim here is to generalize the results of the latter work, and resolve some open conjectures from [52]. In particular, for (11) we obtain the constant coefficient linear relation

$$x_{n+6kl} - \mathcal{K}x_{n+4kl} + \mathcal{K}x_{n+2kl} - x_n = 0 \tag{12}$$

when $2k$ and l are coprime, and a counterpart relation

$$x_{n+6kl} - \mathcal{A}x_{n+5kl} + \mathcal{B}x_{n+4kl} - \mathcal{C}x_{n+3kl} + \mathcal{B}x_{n+2kl} - \mathcal{A}x_{n+kl} + x_n = 0. \tag{13}$$

if $\gcd(2k, l) = 2$; see theorems 4.4 and 4.7 below. (All other cases can be reduced to one of these.) In addition to the first integrals ($\mathcal{K}$ or $\mathcal{A}, \mathcal{B}, \mathcal{C}$) that appear as coefficients, we derive periodic quantities and associated linear relations with periodic coefficients.

Ordinary difference equations can arise as reductions of two-dimensional lattice equations. For instance, the affine type A recurrences (6) are obtained from the four-point equation

$$\begin{vmatrix} u_{s,t} & u_{s+1,t} \\ u_{s,t+1} & u_{s+1,t+1} \end{vmatrix} = 1 \tag{14}$$

for (s, t) being coordinates on $\mathbb{Z}^2$ (or more generally, on a quadrilateral lattice), which is the relation for a frieze pattern [8]. To obtain (6), one should take the $(p, -q)$ travelling wave reduction

$$u_{s,t} = x_n, \quad n = ps + qt, \tag{15}$$

corresponding to a wave moving on the lattice with constant velocity $-q/p \in \mathbb{Q}$. Similarly, it was noted in [37] that the five-point lattice equation

$$\begin{vmatrix} u_{s,t-1} & u_{s+1,t} \\ u_{s-1,t} & u_{s,t+1} \end{vmatrix} = u_{s,t}, \tag{16}$$

which is the relation for a 2-frieze [45], reduces to (8) by substituting in (15) with the replacement $p \to p - k$, $q \to k$, to obtain the $(p - k, -k)$ travelling wave reduction. The authors of [37] also considered the Little Pi family (11) as a reduction of the six-point lattice equation

$$u_{s+1,t+2}u_{s,t} = u_{s+1,t}u_{s,t+2} + a(u_{s,t+1} + u_{s+1,t+1}). \tag{17}$$

(Note that, compared with [37], we have switched the order of the independent variables and introduced the parameter a.) By obtaining linear relations for the above lattice equation, they deduced linear recurrences with periodic coefficients for its $(l, -k)$ travelling wave reduction (11) (cf proposition 3.5 and corollary 3.7 below). In addition, they proved the Laurent property for the lattice equation (17), in the sense that for the initial value problem defined by

$$I = \{u_{s,0}, u_{s,1}, u_{0,t} : s, t \in \mathbb{N}\},$$

the iterates in the positive quadrant in $\mathbb{Z}^2$ are Laurent polynomials in the elements of this set.

In this paper we introduce a new six-point lattice equation, given by

$$(u_{s+1,t+2} + u_{s+1,t} + a)u_{s,t+1} = (u_{s,t+2} + u_{s,t} + a)u_{s+1,t+1} \tag{18}$$

and prove the Laurent property for both this and (17) with a much broader set of initial values than just I. We further show that (18) is linearizable, and this feature (as well as the Laurent property) extends to the family of $(l, -k)$ travelling wave reductions

$$(x_{n+2k+l} + x_{n+l} + a)x_{n+k} = (x_{n+2k} + x_n + a)x_{n+k+l}. \tag{19}$$

Our original motivation for introducing (18) was the fact that, when $k = 1$, the reduction (19) is the total difference of

$$x_{n+l+1}x_n = x_{n+l}x_{n+1} + a\sum_{i=1}^{l} x_{n+i} + b, \tag{20}$$

where the arbitrary parameter b is an integration constant. The latter family of recurrences was referred to as the 'extreme polynomial' in [2], where it was obtained from another set of period 1 seeds in LP algebras, and for $b = 0$ it was independently found in [52], where it was also shown to be linearizable and have the Laurent property (see [35] for further details). However, the recurrences (19) lie beyond the setting of LP algebras.

All of the lattice equations described above fit into the framework of partial differential, differential-difference and partial difference equations described by Demskoi and Tran [9], who considered the family of determinantal equations

$$|M| = \text{const}, \tag{21}$$

where $|M| = \det(M)$ is the determinant of an $N \times N$ matrix M of Casorati type, with entries specified by

$$M = (u_{s+i-1,t+j-1})_{1\leqslant i,j\leqslant N}$$

(up to shifts of indices) in the lattice case, or with appropriate modifications to Wronskian type entries in the case of partial differential/differential-difference equations. Equations of the form (21) are connected to 2D Toda lattices with appropriate boundary conditions, as well as Liouville's equation, and they are said to be Darboux integrable, meaning that they admit complete sets of first integrals that do not depend on one or the other of the independent variables s, t. If both s, t are taken as continuous variables, then the simplest example of (21) considered in [9] is the case $N = 2$, giving the partial differential equation

$$\begin{vmatrix} u & u_s \\ u_t & u_{st} \end{vmatrix} = \beta = \text{const}, \tag{22}$$

which, upon setting $\nu = -\log u$, is equivalent to Liouville's equation written in the form

$$\nu_{st} = -\beta \mathrm{e}^{2\nu}. \tag{23}$$

The SL_2 frieze relation (14) is already in the form (21) with $N = 2$, being the fully discrete analogue of (22), and has the consequence that the corresponding 3×3 determinant vanishes, i.e.

$$\begin{vmatrix} u_{s,t} & u_{s,t+1} & u_{s,t+2} \\ u_{s+1,t} & u_{s+1,t+1} & u_{s+1,t+2} \\ u_{s+2,t} & u_{s+2,t+1} & u_{s+2,t+2} \end{vmatrix} = 0.$$

The vanishing of the above determinant follows by applying the Dodgson condensation algorithm [12], based on the Desnanot–Jacobi identity for matrix minors [5] (this is also referred to as Sylvester's identity in [9]), that is

$$|M|\,|M_{1N}^{1N}| = |M_1^1|\,|M_N^N| - |M_N^1|\,|M_1^N| \tag{24}$$

in which a superscript i (subscript j) on a minor denotes that the ith row (jth column) is deleted. Using similar methods to [23], the right/left null vectors of the 3×3 matrix yield the linear relations

$$\begin{aligned} u_{s,t+2} - Ju_{s,t+1} + u_{s,t} = 0, \quad & \Delta_s J := J(s+1,t) - J(s,t) = 0, \\ u_{s+2,t} - Ku_{s+1,t} + u_{s,t} = 0, \quad & \Delta_t K := K(s,t+1) - K(s,t) = 0, \end{aligned} \tag{25}$$

where the coefficients $J = J(t)$, $K = K(s)$ are first integrals of (14) in the s, t directions respectively, and the two linear relations in (25) reduce to those in (7) after imposing the travelling wave reduction (15). In fact, there is another connection between (22) and its fully discrete version (14): by using canonical coordinates for the Poisson structure associated with (6) in the case $p = 1$, $q = 2m - 1$, Fordy showed that this map generates the Bäcklund transformation for m copies of Liouville's equation [22].

A similar application of Dodgson condensation using the 2-frieze relation (16) yields

$$\begin{vmatrix} u_{s,t-2} & u_{s+1,t-1} & u_{s+2,t} \\ u_{s-1,t-1} & u_{s,t} & u_{s+1,t+1} \\ u_{s-2,t} & u_{s-1,t+1} & u_{s,t+2} \end{vmatrix} = 1,$$

which is the relation for an SL_3 frieze on each of the sublattices obtained by restricting $s + t$ to have odd/even parity, and can be put in the standard form (21) by a linear change of coordinates. A further application of (24) shows that the corresponding 4×4 determinant vanishes for the 2-frieze relation, while in [37] it is shown that there is also a constant 3×3 determinant and a vanishing 4×4 determinant associated with (17), and in the sequel we prove an analogous result for the new lattice equation (18).

In the next section we give a very brief introduction to cluster algebras and LP algebras, explaining the main differences and how nonlinear recurrences of the form (1) can arise in that setting; we give full details for the particular case of the Little Pi family (11). Section 3 is devoted to an independent derivation of the linear recurrences with periodic coefficients found for the Little Pi family in [37], which we then use in section 4 to derive a constant coefficient relation of order $6kl$, of the form (12) or (13), for each pair of coprime positive integers k, l. Moreover, for each member of the Little Pi family, we prove that the corresponding birational map is maximally superintegrable: not only are the conditions of Liouville's theorem (or rather, its discrete version [43]) satisfied, but also the number of independent first integrals is one less than the dimension of the phase space. Section 5 is concerned with the new six-point lattice equation (18), including the proof of linearization both for the lattice equation and all its travelling wave reductions (19). Finally, in section 6 we show that the new lattice equation has the Laurent property for suitable band sets of initial values in $\mathbb{Z}^2$, of the kind described by van der Kamp in [39], and we use this to infer the Laurent property for its reductions (19). We apply the same approach to show the Laurent property for the lattice equation (17) with band sets of initial values, before making some concluding remarks.

2. Cluster algebras, LP algebras and recurrence relations

In this section we briefly review the construction of cluster algebras and LP algebras, and explain how particular recurrences of the form (1), having the Laurent property, appear in that context.

2.1. Construction of a cluster algebra

A cluster algebra is constructed from collections of objects called clusters: for a cluster algebra of rank m, a cluster is a set of m independent quantities called cluster variables. A seed in a cluster algebra consists of a cluster together with an $m \times m$ integer matrix, called an exchange matrix. There is a process called mutation which allows new seeds to be produced, using the exchange matrix. The most general definition of a cluster algebra, as in [19], includes an additional set of m quantities called coefficients, which also undergo mutation, but for the sake of simplicity, here we will focus on coefficient-free cluster algebras.

Definition 2.1. A seed $(\mathbf{x}, B)$ in a coefficient-free cluster algebra of rank m consists of a cluster $\mathbf{x}$, which is a collection of m algebraically independent elements (cluster variables) x_i, so $\mathbf{x} = \{x_1, \ldots, x_m\}$, together with an $m \times m$ exchange matrix $B = (b_{ij}) \in \mathrm{Mat}_m(\mathbb{Z})$ which is skew-symmetrizable, i.e. there is a diagonal matrix $D \in \mathrm{Mat}_m(\mathbb{Z}_{>0})$ such that DB is skew-symmetric.

Definition 2.2. For each seed $(\mathbf{x}, B)$ there is a mutation μ_k for each $k \in \{1, \ldots, m\}$, producing a new seed $\mu_k((\mathbf{x}, B)) = (\mathbf{x}', B')$. The process of mutation is defined in the following steps:

- Matrix mutation: the new exchange matrix $B' = (b'_{ij}) = \mu_k(B)$ is defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \mathrm{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise,} \end{cases} \tag{26}$$

  with $\mathrm{sgn}(a)$ being ± 1 for positive/negative $a \in \mathbb{R}$ and 0 for $a = 0$, and $[a]_+ = \max(a, 0)$.
- Cluster mutation: the new cluster $\mathbf{x}' = (x'_j) = \mu_k(\mathbf{x})$ is defined by the exchange relation

$$x'_k = \frac{\prod_{i=1}^N x_i^{[b_{ki}]_+} + \prod_{i=1}^N x_i^{[-b_{ki}]_+}}{x_k}, \tag{27}$$

  and $x'_j = x_j$ for $j \neq k$.

Remark 2.3. If the exchange matrix B is skew-symmetric, it can be associated with a quiver Q without one- or two-cycles, that is, a directed graph specified by the rule that b_{ij} is equal to the number of arrows $i \to j$ if it is non-negative, and minus the number of arrows $j \to i$ otherwise. There is an associated process of quiver mutation which modifies the arrows in Q to produce a new quiver $Q' = \mu_k(Q)$ associated with B', as specified by the rule (26).

Example 2.4. Take $m = 2$ and the skew-symmetric exchange matrix

$$B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},$$

which is associated with the Kronecker quiver. Given the initial cluster $\mathbf{x} = \{x_1, x_2\}$, applying the mutation μ_1 produces the new exchange matrix

$$B' = \mu_1(B) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix},$$

and the new cluster

$$\mathbf{x}' = \mu_1(\mathbf{x}) = \left\{ \frac{x_2^2 + 1}{x_1}, x_2 \right\}.$$

Example 2.5. Take $m = 4$ and the skew-symmetric exchange matrix

$$B = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 2 & -1 \\ 1 & -2 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}.$$

which is associated with one of the quivers considered in section 5 of [23]. Given the initial cluster $\mathbf{x} = \{x_1, x_2, x_3, x_4\}$, applying the mutation μ_1 produces the new exchange matrix

$$B' = \mu_1(B) = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix},$$

and the new cluster

$$\mathbf{x}' = \mu_1(\mathbf{x}) = \left\{ \frac{x_2x_4 + x_3}{x_1}, x_2, x_3, x_4 \right\}.$$

Definition 2.6. Two seeds are said to be mutation equivalent if one can be obtained from the other via a finite sequence of mutations. For a choice of initial seed $(\mathbf{x}, B)$, the cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}, B)$ is the subalgebra of $\mathbb{Q}(x_1, \ldots, x_m)$ generated by all cluster variables in seeds that are mutation equivalent to the initial seed. Evidently this does not depend on the choice of initial seed.

Any mutation is an involution, i.e. $\mu_k \circ \mu_k = \mathrm{id}$, but in general, arbitrary pairs of mutations do not commute with one another. Although the definition of mutation in a cluster algebra may appear very complicated at first sight, it has the remarkable feature that it generates Laurent polynomials in the cluster variables of any initial seed.

Theorem 2.7 ([19], *theorem 3.1*). *Each of the cluster variables in the cluster algebra is a Laurent polynomial in the cluster variables of an initial seed, i.e. for any seed* $(\{x_1, \ldots, x_m\}, B)$ *we have* $\mathcal{A} \subset \mathbb{Z}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$.

In general, cluster algebra mutations generate infinitely many distinct exchange matrices, and infinitely many cluster variables. As mentioned in the introduction, the simplest cluster algebras are those of finite type, which have only finitely many cluster variables, and were classified in [21]. These are a subset of the finite mutation type cluster algebras, as classified in [15, 16], for which the mutations produce only a finite set of exchange matrices.

2.2. Recurrence relations from period 1 seeds in cluster algebras

Given an initial cluster $\mathbf{x}$, any sequence of mutations produces a corresponding sequence of Laurent polynomials, but in general it is not possible to treat successive mutations as the iterations of a single birational map, because the exponents that appear in the exchange relation (27) are the entries of the matrix B, which changes with each subsequent mutation. However, in the case that B is skew-symmetric, Fordy and Marsh identified conditions on the matrix entries which ensure that a suitable sequence of cluster mutations is equivalent to iterating a single recurrence relation [24].

Definition 2.8. An exchange matrix B is said to be *cluster mutation-periodic* with period p if (for a suitable labelling of indices) $\mu_p \circ \mu_{p-1} \circ \ldots \circ \mu_1(B) = \rho^p(B)$, where ρ is the cyclic permutation $(1, 2, 3, \ldots, m) \mapsto (m, 1, 2, \ldots, m-1)$.

When B is skew-symmetric, it corresponds to a quiver Q, and in that context, the case of cluster mutation-periodicity with period $p = 1$ means that the action of mutation μ_1 on Q is the same as the action of ρ, which is such that the number of arrows $i \to j$ in Q is the same as the number of arrows $\rho^{-1}(i) \to \rho^{-1}(j)$ in $\rho(Q)$. This means that the cluster map $\varphi := \rho^{-1} \cdot \mu_1$ acts as the identity on Q (or equivalently, on B), but in general $\mathbf{x} \mapsto \varphi(\mathbf{x})$ has a non-trivial action on the cluster. Mutation-periodicity with period 1 implies that iterating this map is equivalent to iterating a single recurrence relation. The period 1 classification result of Fordy and Marsh can be paraphrased thus:

Theorem 2.9. *Let $(a_1, \ldots, a_{m-1})$ an $(m-1)$-tuple of integers that is palindromic, i.e. $a_j = a_{m-j}$ for all $j \in [1, m-1]$. Then the skew-symmetric exchange matrix $B = (b_{ij})$ with entries specified by*

$$b_{1,j+1} = a_j \quad \text{and} \quad b_{i+1,j+1} = b_{ij} + a_i[-a_j]_+ - a_j[-a_i]_+,$$

for all $i, j \in [1, m-1]$, is cluster mutation-periodic with period 1, and every period 1 skew-symmetric B arises in this way. Furthermore, the Laurent polynomials generated by the cyclic sequence of mutations $\cdots \circ \mu_3 \circ \mu_2 \circ \mu_1$ coincide with those produced by the cluster map φ, given by

$$\varphi : (x_1, \ldots, x_{m-1}, x_m) \mapsto \left(x_2, \ldots, x_m, \frac{\prod_{j=1}^{m-1} x_{j+1}^{[a_j]_+} + \prod_{j=1}^{m-1} x_{j+1}^{[-a_j]_+}}{x_1} \right),$$

corresponding to the iterates of the nonlinear recurrence relation

$$x_n x_{n+m} = \prod_{j:\, a_j>0} x_{n+j}^{a_j} + \prod_{j:\, a_j<0} x_{n+j}^{-a_j}. \tag{28}$$

The above result says that a skew-symmetric B matrix that is cluster-mutation periodic with period 1 is completely determined by the entries in its first row (or equivalently, its first column), and these form a palindrome after removing b_{11}. The entries a_j in the palindrome are precisely the exponents that appear in the exchange relation defining the cluster map φ, whose iterates are equivalent to those of the nonlinear recurrence relation (28). Thus (28) corresponds to a special sequence of mutations in a particular subclass of cluster algebras whose exchange matrices satisfy the requirements of theorem 2.9, up to mutation equivalence. Such a nonlinear recurrence is an example of a generalized T-system, in the terminology of Nakanishi, who introduced a more general notation of cluster-mutation periodicity in [49].

Remark 2.10. The case of rank 2 is very special: all of the cluster variables are generated by repeating the mutation sequence $\mu_2 \circ \mu_1$. Also, any 2×2 skew-symmetric exchange matrix is cluster mutation-periodic with period 1. The observant reader will already have noticed that the matrix B in example 2.4 produces the nonlinear recurrence (2). Hence, in this example, the whole cluster algebra is generated by iterating the recurrence (2). Similarly, example 2.5 corresponds to the Dana Scott recurrence (9). However, in that case the cluster algebra is much larger than what is obtained from the recurrence, because there are infinitely many sequences of mutations that do not correspond to iterations of the cluster map φ. For instance, applying the mutation μ_2 to the initial seed gives

$$\mu_2(B) = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ -1 & 2 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}, \quad \mu_2(\mathbf{x}) = \left\{ x_1, \frac{x_1x_4 + x_3^2}{x_2}, x_3, x_4 \right\},$$

where the Laurent polynomial $(x_1x_4 + x_3^2)/x_2$ is not one of the iterates of (9).

2.3. Construction of an LP algebra

There are various situations where birational transformations of the form (1) arise with more than two monomials on the right-hand side [28, 53], and Laurent phenomenon (LP) algebras provide a general framework for such situations which goes beyond the setting of cluster algebras [42]. Like cluster algebras, LP algebras are constructed from clusters: for an LP algebra of rank m, a cluster consists of m independent cluster variables. A seed in an LP algebra consists of a cluster together with m polynomials in the cluster variables, called exchange polynomials. Similarly, there is a process of mutation allowing the production of new seeds, using the exchange polynomials. Certain conditions must be imposed on the exchange polynomials which ensure that the Laurent property is preserved under arbitrary sequences of mutations, in the sense that all of the cluster variables so obtained are Laurent polynomials in the m cluster variables from the initial seed. There is also a concept of periodic seeds [2], analogous to the concept for cluster variables that was introduced in [24], which allows recurrence relations to be generated by particular sequences of mutations.

Definition 2.11. A seed $(\mathbf{x}, \mathbf{P})$ in an LP algebra of rank m consists of a cluster $\mathbf{x}$, which is a collection of m algebraically independent elements (cluster variables) x_i, so $\mathbf{x} = \{x_1, \ldots, x_m\}$, together with m exchange polynomials $\mathbf{P} = \{P_1, \ldots, P_m\}$. For each $i \in 1, \ldots, m$ it is required that

- P_i is irreducible in $\mathbb{Z}[x_1, \ldots, x_m]$;
- P_i does not contain the variable x_i.

Definition 2.12. For each seed $(\mathbf{x}, \mathbf{P})$ there is a mutation μ_k for each $k \in \{1, \ldots, m\}$, producing a new seed $\mu_k((\mathbf{x}, \mathbf{P})) = (\mathbf{x}', \mathbf{P}')$. The process of mutation is defined in the following steps:

(a) Define the exchange Laurent polynomials $\{\hat{P}_1, \ldots, \hat{P}_m\} \subset \mathbb{Z}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ to be the unique polynomials such that

1. $\hat{P}_j = P_j \prod_{1 \leqslant i \leqslant n, i \neq j} x_i^{a_i}$ for each j and $a_i \in \mathbb{Z}_{\leqslant 0}$ for each i;
2. for $i \neq j$,

$$\hat{P}_i\big|_{x_j \leftarrow \frac{P_j}{x}} \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_{j-1}^{\pm 1}, x^{\pm 1}, x_{j+1}^{\pm 1}, \ldots, x_m^{\pm 1}]$$

and this polynomial is not divisible by P_j in this ring.

(b) The new cluster is $\mathbf{x}' = \mu_k(\mathbf{x}) = \{x_1, \ldots, x_k', \ldots, x_m\}$ where $x_k' := \frac{\hat{P}_k}{x_k}$

(c) Now define polynomials

$$G_j := P_j\big|_{x_k \leftarrow \frac{\hat{P}_k|_{x_j \leftarrow 0}}{x_k'}}$$

(d) For each j, remove all common factors with $\hat{P}_k|_{x_j \leftarrow 0}$ from G_j in the unique factorization domain $\mathbb{Z}[x_1, \ldots, \hat{x}_k, \ldots, \hat{x}_j, \ldots, x_m]$, with the hats denoting omitted variables. Denote the polynomials obtained in this way by H_j.

(e) The new exchange polynomials are $P_j' = H_j M_j$, where M_j is the unique Laurent monomial in $\mathbb{Z}[x_1, \ldots, x_{k-1}, x_k', x_{k+1}, \ldots, x_m]$ such that P_j' is not divisible by any Laurent monomial in this ring.

(f) The new seed is $(\mathbf{x}', \mathbf{P}') = (\{x_1, \ldots, x_k', \ldots, x_m\}, \{P_1', \ldots, P_m'\})$

With the same definition of mutation equivalence as in the case of cluster algebras, the definition of an LP algebra is the following (again, it is independent of the choice of initial seed).

Definition 2.13. For a choice of initial seed $(\mathbf{x}, \mathbf{P})$, the LP algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{P})$ is the subalgebra of $\mathbb{Q}(x_1, \ldots, x_m)$ generated by all cluster variables in seeds that are mutation equivalent to the initial seed.

The somewhat convoluted construction of LP mutation ensures that the proof of the Laurent property of cluster algebras, via the Caterpillar lemma in [19], is still valid in the more general LP case.

Theorem 2.14 ([20], *theorem 5.1*). *Each of the cluster variables in the LP algebra is a Laurent polynomial in the cluster variables of an initial seed, i.e. for any seed* $(\{x_1, \ldots, x_m\}, \mathbf{P})$ *we have* $\mathcal{A} \subset \mathbb{Z}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$.

2.4. Recurrence relations from period 1 seeds in LP algebras

Following [2], we now show how the notion of periodic seeds for cluster algebras, introduced in [24], may be generalized to LP algebras, in the special case where the period is 1. Periodic seeds may be used to show that the iterates of certain recurrence relations correspond to mutations in an LP algebra, hence satisfying the conditions of theorem 2.14. This proves the Laurent property for these recurrences. In the definition above, we considered unordered seeds, but when we consider recurrence relations it is helpful to fix an ordering, which we identify by putting round brackets (,) instead of braces $\{,\}$ around the variables.

Definition 2.15 (period 1 seed). Let $(\mathbf{x}, \mathbf{P}) = ((x_1, \ldots, x_m), (P_1, \ldots, P_m))$ be a seed (where the cluster variables and exchange polynomials are ordered according to their subscript), let

$$(\mathbf{x}', \mathbf{P}') = \rho \circ \mu_1(\mathbf{x}, \mathbf{P}) = \big((x_2', \ldots, x_m', x_1'), (P_2', \ldots, P_m', P_1')\big)$$

be the seed obtained from it by applying the mutation μ_1 and then reordering the variables with a cyclic permutation ρ, and define $x_{m+1} = x_1'$. The seed $(\mathbf{x}, \mathbf{P})$ is called periodic with period 1 if

$$P_1' = \mathcal{S}P_m \quad \text{and} \quad P_i' = \mathcal{S}P_{i-1} \quad \text{for } 2 \leqslant i \leqslant m, \tag{29}$$

where the shift operator $\mathcal{S}$ increases the subscripts on each of the x_i appearing by one.

The importance of the above definition is due to the following result, which is corollary 2.5 in [2], but for completeness we sketch the proof here.

Proposition 2.16. *If* $(\mathbf{x}, \mathbf{P}) = ((x_1, \ldots, x_m), (P_1, \ldots, P_m))$ *is a period 1 seed with* $P_1 = P(x_2, \ldots, x_m)$*, then the iterates of recurrence*

$$x_n x_{n+m} = P(x_{n+1}, \ldots, x_{n+m-1}) \tag{30}$$

are Laurent polynomials $x_1, \ldots, x_m$ *with integer coefficients.*

Proof. Since $P_2' = \mathcal{S}P_1$ we have $P_2' = P(x_3, \ldots, x_{m+1})$. Applying the mutation μ_1 to the period 1 seed $(\mathbf{x}, \mathbf{P})$ gives

$$x_1' = \frac{P(x_2, \ldots, x_m)}{x_1},$$

and setting $x_{m+1} = x'_1$ agrees with the first iteration of the proposed recurrence (30). After applying the cyclic permutation ρ to reorder the variables and exchange polynomials, the new seed is

$$\rho \circ \mu_1(\mathbf{x}, \mathbf{P}) = \left((x_2, \ldots, x_m, x_{m+1}), (P'_2, \ldots, P'_m, P'_1)\right),$$

with new exchange polynomials given by (29). Now applying the mutation μ_2 gives a new cluster variable

$$x''_2 = \frac{P'_2(x_3, \ldots, x_{m+1})}{x_2} = \frac{P(x_3, \ldots, x_{m+1})}{x_2},$$

which is defined to be x_{m+2}, and produces the new seed

$$\rho \circ \mu_2(\mathbf{x}', \mathbf{P}') = \left((x_3, \ldots, x_m, x_{m+1}, x_{m+2}), (P''_3, \ldots, P''_m, P''_1, P''_2)\right),$$

where

$$P''_1 = \mathcal{S}P'_m, \quad P''_i = \mathcal{S}P'_{i-1} \quad \text{for } 2 \leqslant i \leqslant m.$$

Continuing to apply consecutive mutations μ_3, μ_4, and so on, one can see that this will give precisely the iterates of (30). Since these iterates are given by compositions of mutations they belong to the LP algebra $\mathcal{A}$ generated by the seed $(\mathbf{x}, \mathbf{P})$, hence are Laurent polynomials in the initial cluster variables by theorem 2.14. □

2.5. Little Pi from a period 1 seed

The Little Pi recurrences (11) are one of several examples of the form (30) found in [2] that can be shown to have the Laurent property by describing them in terms of successive mutations of a period 1 seed, as in proposition 2.16. In order to apply this result they construct the 'intermediate polynomials', that is to say, the other exchange polynomials that appear in the period 1 seed, such that the shifting conditions (29) hold. For Little Pi, this construction is split into four cases, which we list below.

For convenience, we slightly change the notation compared with the above discussion, where we followed [20] in labelling a cluster of size m with indices from 1 to m. To be consistent with [2], below we label the initial cluster variables x_i and exchange polynomials P_i with indices $0 \leqslant i \leqslant m - 1$. Note that the inclusion of the coefficient a in (11) means that the Laurent property takes the form

$$x_n \in \mathbb{Z}[a, x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_{2k+l-1}^{\pm 1}],$$

but in fact in the next section we will take $a \to 1$. (More details of the Laurent phenomenon over a ring of coefficients are provided in [20].) Only the polynomials P_j for $j \in J := \{0, k, 2k, l, k + l\}$ are given here. To find the intermediate polynomial P_i for any i, take the largest $j \in J$ with $j \leqslant i$ and shift P_j up by $i - j$, so that $P_i = \mathcal{S}^{i-j}P_j$. Note that in all cases we have $P_0 = P$.

- For $l > 2k$:

$$P_k = ax_0x_{2k} + ax_{2k}x_l + x_0x_{3k}x_{k+l} + a^2_{k+l},$$

$$P_{2k} = ax_0x_{3k} + ax_{l-k}x_{k+l} + x_0x_{l-2k}x_{k+l} + a^2x_0,$$

$$P_l = ax_kx_{l-k} + ax_{l-k}x_{k+l} + x_0x_{l-2k}x_{k+l} + a^2x_0,$$

$$P_{k+l} = ax_0 + ax_l + x_kx_{l-k}.$$

- For $l = 2k$:

$$P_k = ax_0x_{2k} + ax_{2k}^2 + x_0x_{3k}^2 + a^2x_{3k},$$

$$P_{2k} = ax_k^2 + ax_kx_{3k} + x_0^2x_{3k} + a^2x_0,$$

$$P_{3k} = ax_0 + ax_{2k} + x_k^2.$$

- For $2k > l > k$:

$$P_k = ax_0x_{2k} + ax_{2k}x_l + x_0x_{3k}x_{k+l} + a^2x_{k+l},$$

$$P_{k+l} = x_0x_{l-k}x_{k+l} + x_0x_{2l-k}x_{k+l} + x_{l-k}x_kx_{2l} + x_{l-k}x_{k+l}x_{2l} + ax_0x_{2l},$$

$$P_{2k} = ax_kx_{3k-l} + ax_kx_{3k} + x_0x_kx_{3k} + a^2x_{2k-l},$$

$$P_{k+l} = ax_0 + ax_l + x_kx_{l-k}.$$

- For $k > l$:

$$P_l = x_{2l}x_k + x_{2l}x_{k+l} + x_0x_{2k} + x_0x_{k+2l},$$

$$P_k = x_0x_{k+l}x_{2k-l} + x_0x_{k+l}x_{2k} + x_0x_{k-l}x_{2k} + x_lx_{k-l}x_{2k} + ax_{k-l}x_{k+l},$$

$$P_{k+l} = x_lx_k + x_{k+2l}x_l + x_0x_{k+2l} + x_kx_{2l},$$

$$P_{2k} = ax_{k-l} + ax_k + x_0x_{2k-l}.$$

3. Linear relations with periodic coefficients for Little Pi

Henceforth we shall work with the Little Pi family of recurrences in the form

$$x_nx_{n+2k+l} = x_{n+2k}x_{n+l} + x_{n+k} + x_{n+k+l}, \tag{31}$$

which is obtained from (11) after rescaling $x_n \to ax_n$. These recurrences generalize the family found by Heideman and Hogan [33], which is the case $l = 1$. In order to find linear relations, we begin by showing that the 3×3 matrix

$$\Psi_n := \begin{bmatrix} x_n & x_{n+2k} & x_{n+4k} \\ x_{n+l} & x_{n+2k+l} & x_{n+4k+l} \\ x_{n+2l} & x_{n+2k+2l} & x_{n+4k+2l} \end{bmatrix} \tag{32}$$

has a non-zero periodic determinant. For convenience we set

$$z_n := x_n + x_{n+l},$$

and note the following two identities, which are a consequence of (31):

$$z_nx_{n+2k+l} = x_{n+l}z_{n+2k} + z_{n+k}, \tag{33}$$

$$z_nx_{n+2k} = x_nz_{n+2k} - z_{n+k}. \tag{34}$$

Lemma 3.1. *The* 3×3 *determinant*

$$\delta_n := |\Psi_n| = \begin{vmatrix} x_n & x_{n+2k} & x_{n+4k} \\ x_{n+l} & x_{n+2k+l} & x_{n+4k+l} \\ x_{n+2l} & x_{n+2k+2l} & x_{n+4k+2l} \end{vmatrix}$$

has period k.

Proof. First observe that (31) can be rewritten as

$$\begin{vmatrix} x_n & x_{n+2k} \\ x_{n+l} & x_{n+2k+l} \end{vmatrix} = z_{n+k},$$

so using Dodgson condensation, as given by (24) with $N = 3$, we may write

$$x_{n+2k+l}\delta_n = \begin{vmatrix} z_{n+k} & z_{n+3k} \\ z_{n+k+l} & z_{n+3k+l} \end{vmatrix} =: \delta'_{n+k}. \tag{35}$$

Upon scaling the first column by x_{n+3k+l}, we see that the 2×2 determinant δ'_{n+k} in (35) satisfies

$$x_{n+3k+l}\delta'_{n+k} = \begin{vmatrix} z_{n+k}x_{n+3k+l} & z_{n+3k} \\ z_{n+k+l}x_{n+3k+l} & z_{n+3k+l} \end{vmatrix}.$$

Then we can use (33) and (34) on the left column to obtain

$$\begin{aligned} x_{n+3k+l}\delta'_{n+k} &= \begin{vmatrix} x_{n+k+l}z_{n+3k} + z_{n+2k} & z_{n+3k} \\ x_{n+k+l}z_{n+3k+l} - z_{n+2k+l} & z_{n+3k+l} \end{vmatrix} \\ &= \begin{vmatrix} z_{n+2k} & z_{n+3k} \\ -z_{n+2k+l} & z_{n+3k+l} \end{vmatrix}, \end{aligned} \tag{36}$$

and by the same token, but instead manipulating the right column in (35), we have

$$x_{n+k+l}\delta'_{n+k} = \begin{vmatrix} z_{n+k} & -z_{n+2k} \\ z_{n+k+l} & z_{n+2k+l} \end{vmatrix}. \tag{37}$$

Shifting up $n \to n + k$ and comparing with (36) we arrive at

$$\frac{\delta'_{n+k}}{x_{n+2k+l}} = \frac{\delta'_{n+2k}}{x_{n+3k+l}},$$

so these ratios are periodic with period k, which is the required result. □

Lemma 3.2. *For each n the determinant* $\delta_n = |\Psi_n|$ *is non-zero, considered as an element of* $\mathbb{Q}(x_0, x_1, \ldots, x_{2k+l-1})$*, the ambient field of fractions in the initial data for* (31).

Proof. Without assuming the Laurent property, *a priori* the iterates of (31) are rational functions of the initial data with rational numbers as coefficients, and the same is true for the determinant δ_n. Let us consider the case of substituting real positive initial values $x_n > 0$ for $n = 0, \ldots, 2k + l - 1$. It follows by induction that $x_n > 0$ for all $n \in \mathbb{Z}$, hence also $z_n > 0$ for all n. If δ_n vanishes for some n then δ'_{n+k} vanishes, by (35), but then

$$z_{n+k}z_{n+2k+l} + z_{n+2k}z_{n+k+l} = 0$$

by (37), which is a contradiction. Hence δ_n is a non-zero rational function. □

We now consider the corresponding 4×4 matrix

$$\hat{\Psi}_n := \begin{bmatrix} x_n & x_{n+2k} & x_{n+4k} & x_{n+6k} \\ x_{n+l} & x_{n+2k+l} & x_{n+4k+l} & x_{n+6k+l} \\ x_{n+2l} & x_{n+2k+2l} & x_{n+4k+2l} & x_{n+6k+2l} \\ x_{n+3l} & x_{n+2k+3l} & x_{n+4k+3l} & x_{n+6k+3l} \end{bmatrix},$$

and use Dodgson condensation once more, with $N = 4$ in (24), to calculate

$$|\hat{\Psi}_n| = \frac{\delta_{n+k}\delta_{n+2k+l} - \delta_{n+k+l}\delta_{n+2k}}{z_{n+3k+l}},$$

and then by periodicity of δ_n we have the

Corollary 3.3. *The 4×4 determinant $|\hat{\Psi}_n|$ is identically zero.*

Given that $|\hat{\Psi}_n| = 0$, we obtain linear relations with periodic coefficients by considering the right and left kernels (i.e. the kernel of $\hat{\Psi}_n$ and that of its transpose).

Remark 3.4. The kernel of $\hat{\Psi}_n$ is one-dimensional, since if it were of dimension greater than one then Ψ_n we would have a non-trivial kernel, contradicting lemma 3.2.

Proposition 3.5. *The iterates of (31) satisfy the linear relations*

$$x_{n+6k} + K_n^{(3)} x_{n+4k} + K_n^{(2)} x_{n+2k} - x_n = 0, \tag{38}$$

$$x_{n+3l} + \gamma_n x_{n+2l} + \beta_n x_{n+l} + \alpha_n x_n = 0, \tag{39}$$

with periodic coefficients: $K_n^{(2)}$ and $K_n^{(3)}$ have period l, α_n has period k, and β_n and γ_n have period $2k$.

Proof. Let $(K_n^{(1)}, K_n^{(2)}, K_n^{(3)}, 1)^{\mathrm{T}}$ be in the kernel of $\hat{\Psi}_n$. (We are justified in scaling the last entry to 1 due to lemma 3.2.) From the first three rows of

$$\hat{\Psi}_n (K_n^{(1)}, K_n^{(2)}, K_n^{(3)}, 1)^{\mathrm{T}} = 0 \tag{40}$$

we get the matrix equation

$$\begin{bmatrix} x_n & x_{n+2k} & x_{n+4k} \\ x_{n+l} & x_{n+2k+l} & x_{n+4k+l} \\ x_{n+2l} & x_{n+2k+2l} & x_{n+4k+2l} \end{bmatrix} \begin{bmatrix} K_n^{(1)} \\ K_n^{(2)} \\ K_n^{(3)} \end{bmatrix} = - \begin{bmatrix} x_{n+6k} \\ x_{n+6k+l} \\ x_{n+6k+2l} \end{bmatrix}, \tag{41}$$

and by Cramer's rule

$$K_n^{(1)} = \frac{-\delta_{n+2k}}{\delta_n} = -1$$

The last three rows of (40) give

$$\begin{bmatrix} x_{n+l} & x_{n+2k+l} & x_{n+4k+l} \\ x_{n+2l} & x_{n+2k+2l} & x_{n+4k+2l} \\ x_{n+3l} & x_{n+2k+3l} & x_{n+4k+3l} \end{bmatrix} \begin{bmatrix} K_n^{(1)} \\ K_n^{(2)} \\ K_n^{(3)} \end{bmatrix} = - \begin{bmatrix} x_{n+6k+l} \\ x_{n+6k+2l} \\ x_{n+6k+3l} \end{bmatrix}. \tag{42}$$

The equations (41) and (42) imply that $K_n^{(2)}$ and $K_n^{(3)}$ both have period l. Now set

$$\hat{\Psi}_n^{\mathrm{T}} (\alpha_n, \beta_n, \gamma_n, 1)^{\mathrm{T}} = 0, \tag{43}$$

and analogous arguments to the preceding ones give

$$\alpha_n = -\frac{\delta_{n+l}}{\delta_n} \tag{44}$$

and the result that α_n is k-periodic, and β_n and γ_n are $2k$-periodic. □

We can derive further relations between the coefficients in (38) and (39) by using (41) and (42), as well as the corresponding equations for the left kernel of $\hat{\Psi}_n$.

Lemma 3.6. *The periodic coefficients in* (38) *are related to one another by* $K^{(2)}_{n+k} = -K^{(3)}_n$.

Proof. From the first two rows of (40) we have

$$\begin{bmatrix} x_{n+2k} & x_{n+4k} \\ x_{n+2k+l} & x_{n+4k+l} \end{bmatrix} \begin{bmatrix} K^{(2)}_n \\ K^{(3)}_n \end{bmatrix} = \begin{bmatrix} x_n & x_{n+6k} \\ x_{n+l} & x_{n+6k+l} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Then solving for $K^{(2)}_n$ and $K^{(3)}_n$ yields

$$K^{(2)}_n = \frac{x_n x_{n+4k+l} - x_{n+l} x_{n+4k} + z_{n+5k}}{z_{n+3k}},$$

$$K^{(3)}_n = \frac{x_{n+2k+l} x_{n+6k} - x_{n+2k} x_{n+6k+l} - z_{n+k}}{z_{n+3k}},$$

from which we get

$$z_{n+5k} K^{(2)}_{n+2k} + z_{n+3k} K^{(3)}_n = z_{n+7k} - z_{n+k}.$$

The sequence of z_j satisfy the same linear equation as the x_j, obtained by replacing each $x_j \to z_j$ in the matrix equation (41), due to the l-periodicity of $K^{(2)}$ and $K^{(3)}$, so we have

$$z_{n+5k} K^{(2)}_{n+2k} + z_{n+3k} K^{(3)}_n = -K^{(2)}_{n+k} z_{n+3k} - -K^{(3)}_{n+k} z_{n+5k}.$$

Assuming that $K^{(2)}_{n+k} + K^{(3)}_n \neq 0$ implies

$$\frac{K^{(2)}_{n+2k} + K^{(3)}_{n+k}}{K^{(2)}_{n+k} + K^{(3)}_n} = -\frac{z_{n+3k}}{z_{n+5k}},$$

and the left-hand side above is periodic with period l so the right-hand side should be too, i.e.

$$\frac{z_{n+3k}}{z_{n+5k}} = \frac{z_{n+3k+l}}{z_{n+5k+l}} \iff \begin{vmatrix} z_{n+3k} & z_{n+3k+l} \\ z_{n+5k} & z_{n+5k+l} \end{vmatrix} = 0,$$

and this determinant is δ'_{n+3k} from (35), but by the proof of lemma 3.2 this cannot be identically zero, which gives a contradiction. Hence $K^{(2)}_{n+k} + K^{(3)}_n = 0$ as required. □

Corollary 3.7. *The linear relation* (38) *with l-periodic coefficients has the form*

$$x_{n+6k} - K_{n+k} x_{n+4k} + K_n x_{n+2k} - x_n = 0$$

with $K_n := K^{(2)}_n$.

Remark 3.8. The latter results were previously obtained via a different method, using the travelling wave reduction of (11), in [37] (see corollary 3.2 and proposition 3.3 therein).

We close this section by proving some conjectures made for $l = 1$ in [52], and extending them to arbitrary l.

Proposition 3.9. *The periodic coefficients of the linear relation (39) satisfy the following set of identities:*

$$\alpha_n = \beta_n + \gamma_{n+k} - 1, \tag{45}$$

$$\alpha_{n+l}(\gamma_n + \gamma_{n+k}) = \beta_{n+l} + \beta_{n+k+l}, \tag{46}$$

$$\prod_{i=0}^{k-1} \alpha_{n+i} = (-1)^k. \tag{47}$$

Proof. From the left kernel analogue of (42) we have

$$\begin{bmatrix} x_{n+l} & x_{n+2l} \\ x_{n+2k+l} & x_{n+2k+2l} \end{bmatrix} \begin{bmatrix} \beta_n \\ \gamma_n \end{bmatrix} = - \begin{bmatrix} x_n & x_{n+3l} \\ x_{n+2k} & x_{n+2k+3l} \end{bmatrix} \begin{bmatrix} \alpha_n \\ 1 \end{bmatrix},$$

so we can express β_n and γ_n as

$$\beta_n = \frac{\alpha_n(x_{n+2k}x_{n+2l} - x_n x_{n+2k+2l}) + z_{n+k+2l}}{z_{n+k+l}},$$

$$\gamma_n = \frac{(x_{n+2k+l}x_{n+3l} - x_{n+l}x_{n+2k+3l}) + \alpha_n z_{n+k}}{z_{n+k+l}}.$$

Upon shifting $\beta_n \to \beta_{n+k}$ we can equate the bracketed terms above as

$$\alpha_{n+l}\gamma_n z_{n+k+l} - \alpha_n\alpha_{n+l}z_{n+k} = \beta_{n+l}z_{n+k+2l} - z_{n+k+3l}. \tag{48}$$

Now if we write the z_j in terms of the x_i and replace the x_{n+k+4l} that appears as

$$x_{n+k+4l} = -\alpha_{n+l}x_{n+k+l} - \beta_{n+k+l}x_{n+k+2l} - \gamma_{n+k+l}x_{n+k+3l},$$

then (48) becomes

$$\begin{aligned} &- \alpha_n\alpha_{n+l}x_{n+k} + (\alpha_{n+l}\gamma_n - \alpha_n\alpha_{n+l} - \alpha_{n+l})x_{n+k+l} \\ &+ (\alpha_{n+l}\gamma_n - \beta_{n+l} - \beta_{n+k+l})x_{n+k+2l} + (1 - \beta_{n+l} - \gamma_{n+k+l})x_{n+k+3l} = 0. \end{aligned} \tag{49}$$

Since the kernel of $\hat{\Psi}_n$ is one-dimensional we can scale and equate coefficients in (49) and an appropriate shift of (43) to get three equations, namely

$$\begin{aligned} \alpha_n &= \beta_n + \gamma_{n+k} - 1, \;\; \gamma_{n+k}\alpha_{n+l} \\ &= \beta_{n+l} + \beta_{n+k+l} - \alpha_{n+l}\gamma_n, \;\; \alpha_{n+l} = \beta_{n+l} + \gamma_{n+k+l} - 1. \end{aligned}$$

where the third of these is simply a shift of the first, and these rearrange to give (45) and (46). The identity (47) follows from (44) and the fact that δ_n has period k. □

4. Linear relations with constant coefficients

In this section we derive the linearization of the Little Pi family (31), in the form of linear relations with constant coefficients, which were not previously considered in [37]. The key is to use monodromy arguments, similar to those employed in [23] in the case of the cluster algebra recurrences (6) and (8).

We start by defining the sequences of matrices

$$L_n := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -K_n \\ 0 & 1 & K_{n+k} \end{bmatrix}, \quad \tilde{L}_n := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_n & -\beta_n & -\gamma_n \end{bmatrix}, \tag{50}$$

which vary with overall periods l and $2k$, respectively, and allow the linear relations (38) and (39) to be rewritten in matrix form as

$$\Psi_n L_n = \Psi_{n+2k}, \quad \tilde{L}_n \Psi_n = \Psi_{n+l},$$

where as before Ψ_n is given by (32). The point of this is that if we define the pair of monodromy matrices

$$M_n := L_n L_{n+2k} L_{n+4k} \cdots L_{n+2k(l-1)}, \quad \tilde{M}_n := \tilde{L}_{n+(2k-1)l} \cdots \tilde{L}_{n+2l} \tilde{L}_{n+l} \tilde{L}_n \tag{51}$$

then right multiplication by M_n will shift Ψ_n by $2k$ upwards l times, that is

$$\Psi_n M_n = \Psi_{n+2kl}, \tag{52}$$

and left multiplication by $\tilde{M}_n$ will shift Ψ_n by l upwards $2k$ times, so that

$$\tilde{M}_n \Psi_n = \Psi_{n+2kl}. \tag{53}$$

Remark 4.1. If $d := \gcd(k, l) > 1$ then the recurrence (31) splits into d copies of itself, so without loss of generality we can take $d = 1$. Then with $d = 1$, if l is odd then $\gcd(2k, l) = 1$ and $\mathrm{lcm}(2k, l) = 2kl$, while if l is even then $\gcd(2k, l) = 2$ and $\mathrm{lcm}(2k, l) = kl$, and we need to deal with these two different cases separately.

4.1. *The case* $\gcd(2k, l) = 1$

Here $\mathrm{lcm}(2k, l) = 2kl$, so l is odd, and with the monodromy matrices M_n and $\tilde{M}_n$ defined as in (51) above, we see that due to the l-periodicity of L_n and the cyclic property of the trace, the quantity $\mathcal{K} := \mathrm{tr}(M_n)$ has period l, and similarly $\mathrm{tr}(\tilde{M}_n)$ has period $2k$. Now from (52) and (53) we have

$$\mathcal{K} = \mathrm{tr}(M_n) = \mathrm{tr}(\Psi_n^{-1}\Psi_{n+2kl}) = \mathrm{tr}(\Psi_{n+2kl}\Psi_n^{-1}) = \mathrm{tr}(\tilde{M}_n),$$

so $\mathcal{K}$ has period $\gcd(2k, l) = 1$, hence is a first integral for (31), independent of n. The same argument applies to the quantity $\tilde{K} := \mathrm{tr}\,(M_n^{-1}) = \mathrm{tr}\,(\tilde{M}_n^{-1})$, which we will now show is equal to $\mathcal{K}$.

Proposition 4.2. *The trace of the monodromy matrix* M_n *satisfies* $\mathcal{K} = \mathrm{tr}(M_n) = \mathrm{tr}\,(M_n^{-1})$.

Proof. The result holds in the case $l = 1$, since we have $\mathcal{K} = \mathrm{tr}(M_n) = \mathrm{tr}(L_n) = K_{n+k}$, $\tilde{K} = \mathrm{tr}(L_n^{-1}) = K_n$, and K_n has period 1, hence $\tilde{K} = \mathcal{K}$. (Another proof for $l = 1$ is given in [36].) Rewriting $\mathcal{K}$ as $\mathrm{tr}(\tilde{M}_n)$, and similarly for $|| = \mathrm{tr}(\tilde{M}_n^{-1})$, and (setting $n = 0$

without loss of generality) this implies an algebraic relation between the $5k$ quantities $\alpha_0, \ldots, \alpha_{k-1}, \beta_0, \ldots, \beta_{2k-1}, \gamma_0, \ldots, \gamma_{2k-1}$, namely that

$$\mathrm{tr}\,(\tilde{L}_{2k-1}\tilde{L}_{2k-2}\cdots\tilde{L}_0) = \mathrm{tr}\,(\tilde{L}_0^{-1}\tilde{L}_1^{-1}\cdots\tilde{L}_{2k-1}^{-1}) \tag{54}$$

must hold as a consequence of the relations in proposition 3.9 for $l = 1$. Due to periodicity there are $2k$ independent relations of the form (45), as well as k independent relations of the form (46), together with (47), but in fact there are only $3k$ independent relations in total, so these equations define an affine variety of dimension $2k$. Now for odd $l > 1$ note that $\mathcal{K} = \tilde{\mathcal{K}}$ holds if and only if

$$\mathrm{tr}\,(\tilde{L}_{\sigma(2k-1)}\tilde{L}_{\sigma(2k-2)}\cdots\tilde{L}_{\sigma(0)}) = \mathrm{tr}\,(\tilde{L}_{\sigma(0)}^{-1}\tilde{L}_{\sigma(1)}^{-1}\cdots\tilde{L}_{\sigma(2k-1)}^{-1}), \tag{55}$$

where σ is the permutation of the indices $0, 1, \ldots, 2k-1$ defined by

$$\sigma(i) = il \mod 2k,$$

which satisfies the properties

$$\sigma(i+k) - \sigma(i) \equiv k\,(\bmod\ 2k), \quad \sigma(i+1) - \sigma(i) \equiv l\,(\bmod\ 2k).$$

With all indices read mod $2k$ (or mod k in the case of α_j), it follows from these properties that σ acts by permuting the coordinates $\alpha_j, \beta_j, \gamma_j$ in the identities for $l = 1$ in proposition 3.9, so that the identities for each odd l are just

$$\alpha_{\sigma(n)} = \beta_{\sigma(n)} + \gamma_{\sigma(n+k)} - 1,$$
$$\alpha_{\sigma(n+1)}(\gamma_{\sigma(n)} + \gamma_{\sigma(n+k)}) = \beta_{\sigma(n+1)} + \gamma_{\sigma(n+k+1)},$$

and similarly for (47). In other words, the identities for odd $l > 1$ are just permutations of those for $l = 1$, so the algebraic relation (55) holds as an immediate consequence of the relation (54) when $l = 1$. □

Remark 4.3. There is an implicit assumption in the above proof, namely that when $l = 1$ the map from the initial values $x_0, x_1, \ldots, x_{2k}$ for (31) to the variety defined by the relations in proposition 3.9 is surjective, which ensures that the identity $\mathrm{tr}(M_n) = \mathrm{tr}\,(M_n^{-1})$ must be an algebraic consequence of these relations. In particular, it is enough to check that for $l = 1$ there is collection of $2k$ independent $2k$-periodic functions of the initial data (e.g. either of the sets $\beta_0, \ldots, \beta_{2k-1}$ or $\gamma_0, \ldots, \gamma_{2k-1}$ should be functionally independent). While this is a straightforward but laborious task for any given k, we do not know of a simple verification that is valid for all k. However, for any l, a direct algebraic proof of proposition 4.2 is provided by the argument used to prove theorem 4.5 in [35], which we will revisit in the proof of theorem 5.4 below.

Theorem 4.4. *If* $\gcd(2k, l) = 1$ *then the iterates of* (31) *satisfy the constant coefficient linear relation*

$$x_{n+6kl} - \mathcal{K}x_{n+4kl} + \mathcal{K}x_{n+2kl} - x_n = 0$$

where $\mathcal{K} = \mathrm{tr}(M_n)$ *is a first integral.*

Proof. Note that $|L_n| = 1$, hence $|M_n| = 1$, so by the Cayley–Hamilton theorem applied to M_n we have

$$M_n^3 - \mathrm{tr}\,(M_n)M_n^2 + cM_n - I = 0 \tag{56}$$

for some c. Premultiplying by M_n^{-3} in (56) gives

$$M_n^{-3} - cM_n^{-2} + \mathrm{tr}\,(M_n)M_n^{-1} - I = 0,$$

so from the Cayley–Hamilton theorem for M_n^{-1} we see that $c = \mathrm{tr}\,(M_n^{-1}) = \mathrm{tr}\,(M_n) = \mathcal{K}$. Multiplying (56) by Ψ_n from the left yields

$$\Psi_{n+6kl} - \mathcal{K}\Psi_{n+4kl} + \mathcal{K}\Psi_{n+2kl} - \Psi_n = 0$$

and the top leftmost entry of this matrix equation gives the required linear relation for x_n. $\square$

4.2. The case $\gcd(2k, l) = 2$

In this case $\mathrm{lcm}(2k, l) = kl$, with l even and k odd. With the same definition (51) for M_n, each matrix in the product appears twice in the same order relative to its neighbours, so M_n is a perfect square, and we can define the square root $M_n^* = M_n^{1/2}$ by the same product with half as many factors, and similarly for $\tilde{M}_n^* = \tilde{M}_n^{1/2}$. The total shift for Ψ_n is now kl instead of $2kl$, so we have

$$M_n^* := L_n L_{n+2k} \cdots L_{n+k(l-4)} L_{n+k(l-2)},$$
$$\tilde{M}_n^* := \tilde{L}_{n+(k-1)l} \tilde{L}_{n+(k-2)l} \cdots \tilde{L}_{n+l} \tilde{L}_n,$$

with

$$\Psi_{n+kl} = \Psi_n M_n^* = \tilde{M}_n^* \Psi_n \tag{57}$$

Again $\mathrm{tr}\,(M_n^*)$ has period $2k$ and $\mathrm{tr}\,(\tilde{M}_n^*)$ has period l, but now this implies that $\mathcal{K}_n := \mathrm{tr}\,(M_n^*) = \mathrm{tr}\,(\tilde{M}_n^*)$ has period $\gcd(2k, l) = 2$, and similarly for $\tilde{\mathcal{K}}_n := \mathrm{tr}\,((M_n^*)^{-1})$. The analogue of proposition 4.2 requires more work in this case. We begin with

Proposition 4.5. *The trace of* M_n^* *satisfies* $\mathcal{K}_n = \mathrm{tr}(M_n^*) = \mathrm{tr}\,((M_{n+1}^*)^{-1})$.

Proof. When $k = 1$ in terms of $\tilde{M}_n^*$ we have

$$\mathcal{K}_n = \mathrm{tr}\,(\tilde{L}_n) = -\gamma_n, \quad \tilde{\mathcal{K}}_n = \mathrm{tr}\,(\tilde{L}_n^{-1}) = -\frac{\beta_n}{\alpha_n}.$$

Now due to (47) we have $\alpha_n = -1$ and using (45) we get $\beta_n = -\gamma_{n+1}$, hence $\mathcal{K}_n = \tilde{\mathcal{K}}_{n+1}$, and the result holds in this case. This implies an algebraic identity between the entries of the monodromy matrix M_n^* and its shift, namely that

$$\mathrm{tr}\,(L_0 L_2 \cdots L_{l-2}) = \mathrm{tr}\,(L_{l-1}^{-1} L_{l-3}^{-1} \cdots L_1^{-1})$$

(where we set $n = 0$ without loss of generality), which is just a tautology in terms of the l quantities $K_0, K_1, \ldots, K_{l-1}$ that appear in the entries. Similarly to the argument in the proof of proposition 4.2, we have that for $k > 1$ the required identity of traces for M_n^* and $(M_{n+1}^*)^{-1}$ just corresponds to a permutation σ of the indices of the quantities K_j, given by $\sigma(j) = 2jk \bmod l$, so the relation $\mathcal{K}_n = \tilde{\mathcal{K}}_{n+1}$ holds for all k. $\square$

Proposition 4.6. *The iterates of* (31) *satisfy a linear relation with period 2 coefficients, given by*

$$x_{n+3kl} - \mathcal{K}_n x_{n+2kl} + \mathcal{K}_{n+1} x_{n+kl} - x_n = 0 \tag{58}$$

Proof. This follows by the Cayley–Hamilton theorem, as in the proof of theorem 4.4, but with different traces appearing. □

Theorem 4.7. *When* $\gcd(2k, l) = 2$, *the iterates of* (31) *satisfy the constant coefficient relation*

$$x_{n+6kl} - \mathcal{A}x_{n+5kl} + \mathcal{B}x_{n+4kl} - \mathcal{C}x_{n+3kl} + \mathcal{B}x_{n+2kl} - \mathcal{A}x_{n+kl} + x_n = 0$$

where

$$\mathcal{A} = \mathcal{K}_n + \mathcal{K}_{n+1}, \quad \mathcal{B} = \mathcal{K}_n\mathcal{K}_{n+1} + \mathcal{K}_n + \mathcal{K}_{n+1}, \quad \mathcal{C} = \mathcal{K}_n^2 + \mathcal{K}_{n+1}^2 + 2.$$

Proof. Let $\mathcal{S}$ be the shift operator such that $\mathcal{S}(f_n) = f_{n+1}$ for any function of n. Then applying the operator

$$\mathcal{S}^{3kl} - \mathcal{K}_{n+1}\mathcal{S}^{2kl} + \mathcal{K}_n\mathcal{S}^{kl} - 1$$

to equation (58) gives the required result. □

4.3. Superintegrability of Little Pi

The birational map

$$\varphi : (x_0, \ldots, x_{2k+l-2}, x_{2k+l-1}) \mapsto (x_1, \ldots, x_{2k+l-1}, x_{2k+l})$$

defined by the Little Pi recurrence (31) is measure-preserving, in the sense that

$$\varphi^*\Omega = (-1)^l\Omega,$$

where Ω is the volume form

$$\Omega = \frac{\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{2k+l-1}}{x_0x_1 \cdots x_{2k+l-1}}.$$

We can use this to show that the map φ is maximally superintegrable, in the sense that it admits an (anti-) invariant Poisson structure, and the number of independent first integrals is one less than the dimension of the phase space.

In the case $\gcd(2k, l) = 1$, it appears that the l-periodic quantities $K_0, \ldots, K_{l-1}$ are independent of one another, hence any cyclically symmetric functions of these quantities are first integrals: so this provides l independent first integrals for (31). Similarly, subject to the relations in proposition 3.9 one can take $2k$ independent $2k$-periodic quantities, and cyclically symmetric functions of these provide $2k$ independent first integrals. However, in total this should give exactly $2k + l - 1$ independent first integrals $I_1, I_2, \ldots, I_{2k+l-1}$, since the identity $\mathrm{tr}\,(M_n) = \mathrm{tr}\,(\tilde{M}_n)$ gives a relation between these two sets of cyclically symmetric functions. Then by a result from [6], taking all but one of these first integrals together with the covolume form

$$V = x_0 \cdots x_{2k+l-1}\,\frac{\partial}{\partial x_0} \wedge \cdots \wedge \frac{\partial}{\partial x_{2k+l-1}}$$

(i.e. the $(2k + l)$-multivector field that contracts with Ω to give 1) yields a Poisson bracket defined by

$$\{\, f, g \,\} = V(\mathrm{d}f, \mathrm{d}g, \mathrm{d}I_1, \mathrm{d}I_2, \ldots, \mathrm{d}I_{2k+l-2}),$$

and this bracket is invariant/anti-invariant under the action of φ, according to the parity of l, that is

$$\varphi^*\{\, f, g \,\} = (-1)^l\,\{\, \varphi^* f, \varphi^* g \,\}$$

for any pair of functions f, g on the $(2k+l)$-dimensional phase space. By construction, the first integrals $I_1, I_2, \ldots, I_{2k+l-2}$ are Casimirs for this bracket, but the additional first integral I_{2k+l-1} is not, so it defines a non-trivial Hamiltonian vector field.

As an example, we take the simplest case $k = l = 1$, when φ is defined by

$$x_{n+3}x_n = x_{n+2}x_{n+1} + x_{n+2} + x_{n+1}. \tag{59}$$

The quantity $K_0 = \mathcal{K}$ is a first integral, which can be written as

$$\mathcal{K} = \frac{x_6 - x_0}{x_4 - x_2},$$

by theorem 4.4, and then rewritten as a function of the initial values x_0, x_1, x_2 by using (59). In fact, by theorem 1.2 in [36], the explicit expression is

$$\mathcal{K} = P^{(0)} + P^{(1)} + P^{(2)},$$

where

$$P^{(0)} = 1 + \frac{x_0}{x_2} + \frac{x_2}{x_0}, \quad P^{(1)} = \left(1 + \frac{x_2}{x_0}\right)\frac{x_0 + x_1}{x_1x_2} + \left(1 + \frac{x_0}{x_2}\right)\frac{x_1 + x_2}{x_0x_1},$$

and

$$P^{(2)} = \frac{1}{x_1x_2} + \frac{1}{x_0x_1} + \frac{1}{x_0x_2}.$$

Also, by proposition 3.9 we have

$$\alpha_0 = -1, \quad \gamma_0 = -\beta_1, \quad \gamma_1 = -\beta_0,$$

and then from (39) we can find two independent 2-periodic quantities by solving for β_0, β_1 in terms of the x_j from the pair of linear equations

$$\begin{aligned} &x_3 - \beta_1\, x_2 + \beta_0\, x_1 - x_0 = 0, \\ &x_4 - \beta_0\, x_3 + \beta_1\, x_2 - x_1 = 0. \end{aligned}$$

Then the two symmetric functions

$$I_1 = \beta_0\beta_1, \quad I_2 = \beta_0 + \beta_1$$

provide two independent first integrals, but they are related to $\mathcal{K}$ by

$$\mathcal{K} = \mathrm{tr}\,(M_0) = \mathrm{tr}\,(\tilde{M}_0) = \mathrm{tr}\,(\tilde{L}_1\tilde{L}_0) = \beta_0\beta_1 - \beta_0 - \beta_1 = I_1 - I_2.$$

Finally, contracting the covolume form

$$V = x_0x_1x_2\,\frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$$

with the one-form $\mathrm{d}I_1$ gives the Poisson bracket

$$\{\, f, g \,\} = V(\mathrm{d}f, \mathrm{d}g, \mathrm{d}I_1),$$

which is anti-invariant under the action of the map φ defined by (59), so it is invariant under the doubled map φ^2, and this is a superintegrable map in three dimensions. The flow of the Hamiltonian vector field

$$\frac{\mathrm{d}}{\mathrm{d}t} = \{\, \cdot, I_2 \,\}$$

commutes with the map, and its level sets are curves defined by $I_1 = \text{const}$, $I_2 = \text{const}$.

5. Linearization and reductions of a six-point lattice equation

In this section we consider the new six-point lattice equation (18), which can be rewritten as an equality of two 2×2 determinants, in the form

$$\begin{vmatrix} u_{s,t+1} & u_{s,t+2} \\ u_{s+1,t+1} & u_{s+1,t+2}+a \end{vmatrix} = \begin{vmatrix} u_{s,t}+a & u_{s,t+1} \\ u_{s+1,t} & u_{s+1,t+1} \end{vmatrix},$$

or in the form of a conservation law, as

$$\Delta_s\, a u_{s,t+1} = \Delta_t \begin{vmatrix} u_{s,t} & u_{s,t+1} \\ u_{s+1,t} & u_{s+1,t+1} \end{vmatrix}. \tag{60}$$

By imposing the constraint

$$u_{s,t} = u_{s+k,t-l}$$

for integers k, l, one obtains the $(l, -k)$ travelling wave reduction

$$u_{s,t} = x_n \quad n = ls + kt, \tag{61}$$

which produces the family of recurrences (19). Upon making use of the conservation law (60), we can write the reduction as

$$(\mathcal{S}^k - 1) \begin{vmatrix} x_n & x_{n+k} \\ x_{n+l} & x_{n+k+l} \end{vmatrix} = (\mathcal{S}^l - 1)\, a x_{n+k},$$

and as both sides are a total difference this can be integrated to give

$$\sum_{j=0}^{k-1} \begin{vmatrix} x_{n+j} & x_{n+j+k} \\ x_{n+j+l} & x_{n+j+k+l} \end{vmatrix} - \sum_{j=0}^{l-1} a x_{n+j+k} = b, \tag{62}$$

where b is an integration constant. In other words, b is a first integral for (19). The particular case $k = 1$, given by (20), is the 'extreme polynomial' family found from period 1 seeds in LP algebras in [2], whose linearization was studied in [35] for $b = 0$.

For $k > 1$, the recurrences (62) are not of the type (1) that can arise from periodic seeds in LP algebras, and none of the recurrences (19) are of this type. Nevertheless, these recurrences turn out to have the Laurent property for any k, as a consequence of the fact that the lattice equation (18) has the Laurent property.

5.1. Linearization of the new lattice equation

Similarly to the results on linearization of the lattice equation (17) obtained in [37], the iterates of the new six-point equation (18), or equivalently (60), satisfy two types of linear relation, with coefficients that are independent of one or the other of the lattice variables s, t.

Proposition 5.1. *The solutions $u_{s,t}$ of the lattice equation (18) satisfy the linear relations*

$$u_{s,t+3} - (J(t+1)+1)u_{s,t+2} + (J(t)+1)u_{s,t+1} - u_{s,t} = 0, \tag{63}$$

$$u_{s+3,t} + A(s)u_{s+2,t} + B(s)u_{s+1,t} + C(s)u_{s,t} = 0, \tag{64}$$

where $J(t)$ is independent of s, and $A(s), B(s)$ and $C(s)$ are independent of t.

Proof. Dividing both sides of (18) by $u_{s,t+1}u_{s+1,t+1}$, we immediately find a quantity which is invariant under shifts in the s direction, since the equation becomes the total difference

$$\Delta_s J = 0,$$

where $J = J(t)$ is defined by

$$J(t) := \frac{u_{s,t+2} + u_{s,t} + a}{u_{s,t+1}}. \tag{65}$$

The definition of J rearranges to give an inhomogeneous linear relation for $u_{s,t}$, that is

$$u_{s,t+2} - J(t)u_{s,t+1} + u_{s,t} + a = 0, \tag{66}$$

and by applying the difference operator Δ_t to this we are led to the homogeneous relation (63). Since the coefficients of the latter are fixed under shifting s, we can write down four shifts of the relation in the form of a matrix linear system, that is

$$\begin{bmatrix} u_{s,t} & u_{s,t+1} & u_{s,t+2} & u_{s,t+3} \\ u_{s+1,t} & u_{s+1,t+1} & u_{s+1,t+2} & u_{s+1,t+3} \\ u_{s+2,t} & u_{s+2,t+1} & u_{s+2,t+2} & u_{s+2,t+3} \\ u_{s+3,t} & u_{s+3,t+1} & u_{s+3,t+2} & u_{s+3,t+3} \end{bmatrix} \begin{bmatrix} -1 \\ J(t)+1 \\ -J(t+1)-1 \\ 1 \end{bmatrix} = 0. \tag{67}$$

Thus we see that the 4×4 matrix above has determinant zero, so we may take a vector $(C, B, A, 1)$ in the left kernel, and then it is apparent that the entries of this vector are invariant under shifting t. This kernel gives the second linear relation (64). □

In the course of the proof, we observed the following result, which is also proved for (17) in [37].

Corollary 5.2. *For any solution $u_{s,t}$ of (18), the corresponding 4×4 Casorati matrix has vanishing determinant, that is*

$$\begin{vmatrix} u_{s,t} & u_{s,t+1} & u_{s,t+2} & u_{s,t+3} \\ u_{s+1,t} & u_{s+1,t+1} & u_{s+1,t+2} & u_{s+1,t+3} \\ u_{s+2,t} & u_{s+2,t+1} & u_{s+2,t+2} & u_{s+2,t+3} \\ u_{s+3,t} & u_{s+3,t+1} & u_{s+3,t+2} & u_{s+3,t+3} \end{vmatrix} = 0.$$

5.2. Linear relations for travelling wave reductions

Upon applying the reduction (61), the linear relations (63) and (64) for the lattice equation (18) reduce to linear recurrences with periodic coefficients for (19).

Proposition 5.3. *The iterates of the equation (19) for the $(l, -k)$ travelling reduction of (18) satisfy linear recurrence relations with periodic coefficients, given by*

$$x_{n+3k} - (J_{n+k} + 1)x_{n+2k} + (J_n + 1)x_{n+k} - x_n = 0, \tag{68}$$

$$x_{n+3l} + A_n x_{n+2l} + B_n x_{n+l} + C_n x_n = 0, \tag{69}$$

where the coefficient J_n is periodic with period l, and A_n, B_n, C_n are periodic with period k.

Proof. The travelling wave reduction (61) applied to (18) gives the nonlinear recurrence (19), and under this reduction the quantity $J(t)$ defined by (65) becomes

$$J_n := \frac{x_{n+2k} + x_n + a}{x_{n+k}} \tag{70}$$

which is periodic with period l, while the coefficients $A(s), B(s)$ and $C(s)$ in (64) become k-periodic quantities, denoted A_n, B_n, C_n. The linear relations (68) and (69) can also be constructed directly from the observation that J_n defined by (70) has period l. (See [35] for details in the case $k = 1$.) □

Making appropriate adjustments compared with the case of Little Pi, we redefine

$$\Psi_n := \begin{bmatrix} x_n & x_{n+k} & x_{n+2k} \\ x_{n+l} & x_{n+k+l} & x_{n+2k+l} \\ x_{n+2l} & x_{n+k+2l} & x_{n+2k+2l} \end{bmatrix},$$

and set

$$L_n := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -J_n - 1 \\ 0 & 1 & J_{n+k} + 1 \end{bmatrix}, \quad \tilde{L}_n := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -A_n & -B_n & -C_n \end{bmatrix},$$

so that we have the linear matrix equations

$$\Psi_{n+k} = \Psi_n L_n, \quad \Psi_{n+l} = \tilde{L}_n \Psi_n. \tag{71}$$

As before, we can make the assumption $\gcd(k, l) = 1$, since otherwise (19) splits into several copies of lower dimension.

Theorem 5.4. *When* $\gcd(k, l) = 1$, *the iterates of* (19) *satisfy the constant coefficient linear relation*

$$x_{n+3kl} - \mathcal{K} x_{n+2kl} + \mathcal{K} x_{n+kl} - x_n = 0 \tag{72}$$

where $\mathcal{K}$ *is the trace of the monodromy matrix,*

$$\mathcal{K} = \mathrm{tr}\,(M_n), \quad M_n := L_n L_{n+k} \cdots L_{n+k(l-1)},$$

which is a first integral, as well as the linear relation

$$x_{n+(2k+1)l} - x_{n+2kl} - (\mathcal{K} - 1)(x_{n+(k+1)l} - x_{n+kl}) + x_{n+l} - x_n = 0. \tag{73}$$

Proof. From the matrix linear relations (71) we have

$$\Psi_{n+kl} = \Psi_n M_n = \tilde{M}_n \Psi_n,$$

where the second monodromy matrix is

$$\tilde{M}_n := \tilde{L}_{n+(k-1)l} \tilde{L}_{n+(k-2)l} \cdots \tilde{L}_{n+l} \tilde{L}_n.$$

Then since the entries of L_n and $\tilde{L}_n$ have periods l and k respectively, it follows that $\mathcal{K} = \mathrm{tr}\,(M_n) = \mathrm{tr}\,(\tilde{M}_n)$ has period $\gcd(k, l) = 1$, by assumption. By an analogous permutation argument to the one used in the proof of proposition 4.2 and in the proof of proposition

4.5, we have that $\operatorname{tr}(M_n) = \operatorname{tr}(M_n^{-1})$, and then the relation (72) follows by applying the Cayley–Hamilton theorem to M_n, just as in the proof of theorem 4.4. However, we can get a stronger result by considering (70), and defining the matrices

$$\Phi_n = \begin{bmatrix} x_n & x_{n+k} \\ x_{n+l} & x_{n+k+l} \end{bmatrix}, \quad L_n^* = \begin{bmatrix} 0 & -1 \\ 1 & J_n \end{bmatrix}, \quad C^* = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

which are related by the inhomogeneous equation

$$\Phi_{n+k} = \Phi_n L_n^* - aC^*.$$

Then paraphrasing the steps of the proof of theorem 4.5 in [35], we introduce the 2×2 monodromy matrix

$$M_n^* = L_n^* L_{n+k}^* \cdots L_{n+k(l-1)}^*,$$

and find a matrix equation of the form

$$\Phi_{n+2kl} - \kappa\, \Phi_{n+kl} + \Phi_n = \tilde{C}_n^*, \tag{74}$$

with

$$\kappa = \operatorname{tr}(M_n^*),$$

where (like those of L_n^*) the entries of the matrix $\tilde{C}_n^*$ are periodic with period l. The top leftmost entry of (74) gives the equation

$$x_{n+2kl} - \kappa\, x_{n+kl} + x_n = \tilde{J}_n, \tag{75}$$

for some l-periodic quantity $\tilde{J}_n$, and if we apply the operator $\mathcal{S}^{kl} - 1$ then we obtain (72) together with the relation

$$\operatorname{tr}(M_n) = \mathcal{K} = \operatorname{tr}(M_n^*) + 1; \tag{76}$$

this also gives an independent proof that $\operatorname{tr}(M_n) = \operatorname{tr}(M_n^{-1})$. However, we can instead apply the operator $\mathcal{S}^l - 1$ to (75), giving the homogeneous linear relation (73), which is of lower order than (72) when $k > 1$. □

Remark 5.5. For $k = 1$, the quantity $\kappa = \operatorname{tr}(M_n^*)$ is given by the explicit formula

$$\kappa = \prod_{i=0}^{l-1} \left(1 - \frac{\partial^2}{\partial J_i \partial J_{i+1}}\right) \prod_{n=0}^{l-1} J_n,$$

and the formula for $k > 1$ is obtained by a permutation of indices. The term of each distinct homogeneous degree in the above expression corresponds to a first integral of the dressing chain for one-dimensional Schrödinger operators (see [35] and references). With minor modifications, the preceding argument shows that the trace of the 3×3 monodromy matrix M_n in (51), defined by a product over matrices L_n as in (50), is related via (76) to the trace of a 2×2 monodromy matrix M_n^* expressed in terms of l-periodic entries $J_n = K_n - 1$, and this gives an independent proof of proposition 4.2, deriving $\operatorname{tr}(M_n) = \operatorname{tr}(M_n^{-1})$ as an equality between cyclically symmetric functions of $K_0, K_1, \ldots, K_{l-1}$.

Remark 5.6. The birational map φ in dimension $2k+l$ defined by (19) is measure-preserving, with the volume form

$$\Omega = \frac{\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{2k+l-1}}{x_k x_{k+1} \cdots x_{k+l-1}}$$

such that

$$\varphi^* \Omega = (-1)^l \Omega.$$

There should be l independent l-periodic quantities J_n, which appear as coefficients in (68), but it is unclear how many of the k-periodic quantities appearing in (69) should be independent, so the question of superintegrability of (19) remains open.

6. Laurent property for linearizable lattice equations

In this section we discuss the Laurent property for both of the six-point lattice equations (17) and (18). Following van der Kamp [39], we construct a family of bands of initial values, as well as some special sets, that give well-defined solutions on the whole $\mathbb{Z}^2$ lattice. Given suitable conditions on the initial values, linear relations with coefficients fixed in one lattice direction can be used to prove the Laurent property. We show that the band sets of initial values satisfy the necessary criteria.

Definition 6.1. Let I denote a set of initial values for a lattice equation, and let $\mathcal{L}$ denote the ring of Laurent polynomials generated by I, that is

$$\mathcal{L} := \mathbb{Z}[I, I^{-1}]$$

where $I^{-1} = \{1/u : u \in I\}$. For this I, given an additional set of coefficients A appearing in the lattice equation, we say that a two-dimensional lattice equation satisfies the Laurent property, or is Laurent, if

$$u_{s,t} \in \mathcal{L}[A]$$

for all $s, t \in \mathbb{Z}^2$.

In the equation (18) there is a single coefficient a, so we have $A = \{a\}$, and we write $\mathcal{L}[a]$ for the ring of Laurent polynomials associated with an initial value set I.

6.1. Construction of band sets of initial values

In [39] an algorithm is given which finds (in almost all cases) a unique solution to a lattice equation on an arbitrary stencil, given a band of initial values I. We apply this to the six-point domino-shaped stencil that (18) is defined on.

First we define the lines L_1 and L_2, each with positive rational gradient, such that $L_2 = L_1 + (1, -2)$. For a given pair of lines L_1, L_2 related in this way, the associated band set of initial values $I = I(L_1, L_2)$ consists of all the lattice points lying between these two lines, including the points on L_1 but not those on L_2. An example with gradient $1/3$ is shown in figure 1, where the points in the band set I are coloured yellow. We will consider each of the points on L_1 to be the top left corner of a six-point domino, hence L_2 will pass through the points diagonally opposite. By the results of [39], taking initial values between these lines and on L_1 (but not on L_2) allows us to find a unique solution of (18) for each choice of gradient. The first step is to calculate the values on L_2, drawn in blue, using the yellow initial values. We then shift the

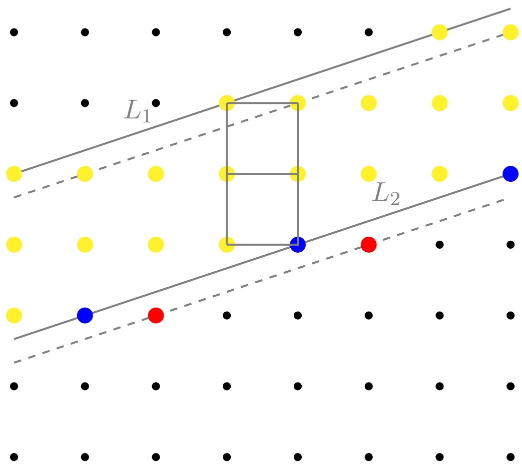

Figure 1. Initial values on the band with gradient 1/3.

lines perpendicularly to their direction until they pass through another point of the domino, corresponding to the dashed lines in the figure. The new L_2 will pass through the next points to be calculated, drawn in red. This process is continued until we fill the whole lattice below L_1. We can also shift the lines in the opposite direction to fill the whole lattice above L_1.

6.2. The Laurent property for lattice equation (18)

To prove the Laurent property we will use the linear relation (66), but first we must prove that the coefficients $J(t)$ belong to the ring of Laurent polynomials.

Lemma 6.2. *If we have an $\tilde{s}$ such that $u_{\tilde{s},t+1} \in I$ and*

$$\{u_{\tilde{s},t}, u_{\tilde{s},t+2}\} \subset \mathcal{L}[a]$$

then $J(t) \in \mathcal{L}[a]$.

Proof. Since $J(t)$ defined by (65) is independent of s we may shift it in the s direction until the index value $\tilde{s}$ appears, and then we have

$$J(t) = \frac{u_{\tilde{s},t+2} + u_{\tilde{s},t} + a}{u_{\tilde{s},t+1}} \in \mathcal{L}[a]$$

as required. □

Theorem 6.3. *For a given initial value set I, if lemma 6.2 holds for each t, and if for each s there is some $\tilde{t}$ such that*

$$\{u_{s,\tilde{t}}, u_{s,\tilde{t}+1}\} \subset \mathcal{L}[a],$$

then (18) has the Laurent property for this I.

Proof. For each s we use induction on t and the relation

$$u_{s,t+2} = J(t)u_{s,t+1} - u_{s,t} - a$$

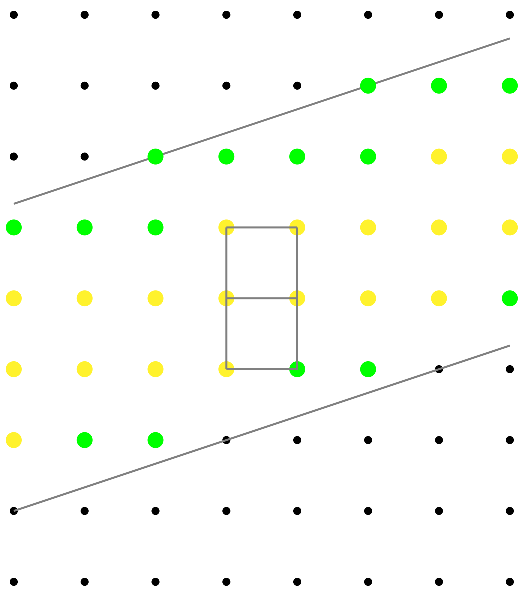

Figure 2. The points marked green are in the Laurent ring.

noting that, by the previous lemma, $J(t)$ is in the Laurent ring. The base case for the induction is given by

$$u_{s,\tilde{t}+2} = J(\tilde{t})u_{s,\tilde{t}+1} - u_{s,\tilde{t}} - a.$$

This proves Laurentness for $t > \tilde{t} + 1$, and the proof for $t < \tilde{t}$ is similar. □

Theorem 6.4. *The Laurent property for* (18) *holds for the band sets of initial values I, as described in subsection* 6.1.

Proof. To calculate $u_{s,t}$ we only have to divide by $u_{s+1,t+1}$ and vice versa, so we know all the values we calculate are Laurent polynomials until we have to divide at one of the blue points in figure 1, and the corresponding points above L_1. These we mark in green in figure 2. We draw L_2' parallel to and below L_2 through the first non-green point and L_1' parallel to and above L_1 through the last green point. Equivalently

$$L_2' = L_2 + (-1, -1), \quad L_1' = L_1 + (-1, 1),$$

hence $L_2' = L_1' + (1, -4)$. Since the minimal distance between L_1' and L_2' is $\sqrt{17} > 4$ any line that intersects I will intersect at least four elements of $\mathcal{L}$. For lemma 6.2 we take horizontal lines with height t and see that they intersect at least four green or yellow points, at least one of which will be yellow. Hence $J(t) \in \mathcal{L}$ for all t. For theorem 6.3 we take vertical lines for each s and see that these intersect at least two green or yellow points. Hence the conditions of the preceding theorem hold and we have the Laurent property for these initial values. □

In the special case where the gradient is 0 it is prescribed in [39] that we should take an extra line of initial values perpendicular to L_1 and L_2, as shown in figure 3, and this case also has the Laurent property. However, we note that Laurentness does not hold for all well-posed initial value problems, for example the yellow set shown in figure 4. In this case one can see

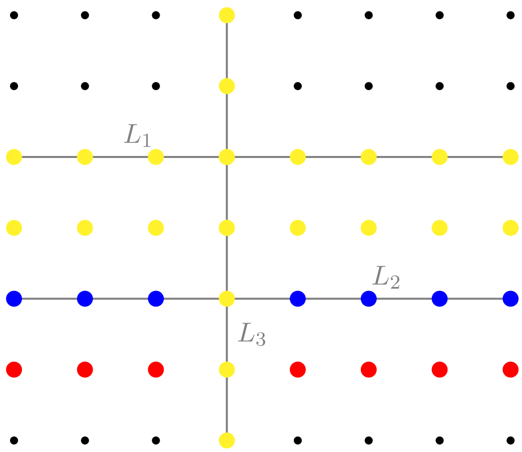

Figure 3. Initial values on the band with gradient 0 with Laurentness.

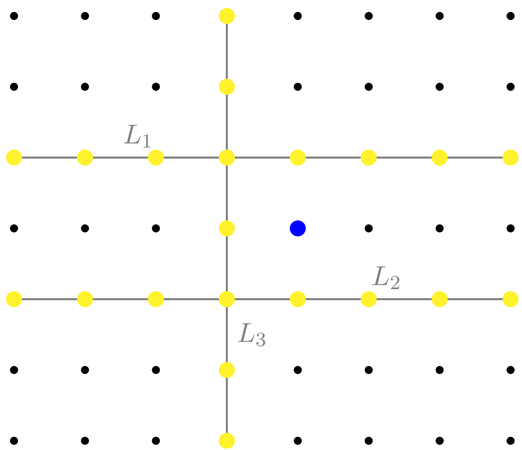

Figure 4. An example of initial values without the Laurent property.

from the form of (18) that to calculate the value of $u_{s,t}$ at the blue node we must divide by a polynomial (not a monomial) in the surrounding initial values.

Note that the Laurent property for the reductions (19) is easily seen from (70). In fact, since J_n has period l, the only initial variables that can appear in the denominator are $x_k, x_{k+1}, \dots, x_{k+l-1}$. In particular, setting each of these to be 1 will give a polynomial sequence in the remaining initial values. So we have

Corollary 6.5. *The equation* (19) *has the Laurent property in the form*

$$x_n \in \mathbb{Z}[a, x_0, \dots, x_{k-1}, x_k^{\pm 1}, \dots, x_{k+l-1}^{\pm 1}, x_{k+l}, \dots, x_{2k+l-1}] \quad \forall n \in \mathbb{Z}.$$

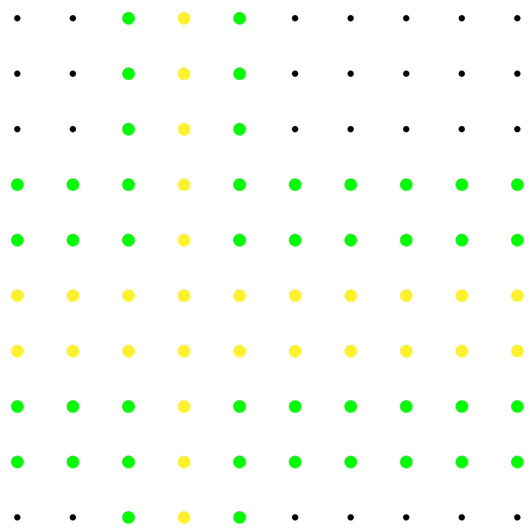

Figure 5. The yellow dots are initial values and the green are Laurent in the initial values.

6.3. The Laurent property for the lattice Little Pi

The Laurent property for the lattice equation (17) was proved in [37] for points (s, t) in the positive quadrant with the initial value set

$$I = \{u_{s,0}, u_{s,1}, u_{0,t} : s, t \in \mathbb{N}\}$$

(note that we switched s and t compared with the original reference). We have drawn the above set I in yellow in figure 5, extended to include indices s, t in the whole of $\mathbb{Z}$. Again we can define the associated Laurent ring $\mathcal{L}$ (without the coefficient a, since we set $a \to 1$ here), and provide a different proof of Laurentness, similar to the proof for (18).

From theorem 2.1 and proposition 2.6 in [37], respectively, we have that

$$u_{s,t+6} - \beta(t+1)u_{s,t+4} + \beta(t)u_{s,t+2} - u_{s,t} = 0 \tag{77}$$

with $\beta = \beta(t)$ (independent of s) being given by

$$\beta(t) = \frac{1 + u_{0,t}u_{0,t+3} + u_{0,t+1}u_{0,t+4} + u_{0,t+2}u_{0,t+5}}{u_{0,t+2}u_{0,t+3}} \tag{78}$$

Note that in this expression s has been set to zero, but due to the fact that β is s-independent the same formula is valid with each term $u_{0,j}$ replaced by $u_{s,j}$ for $j = t, t+1, \ldots, t+5$. Similarly to the proof of theorem 6.4, we colour the values that only require division by elements of I in green in figure 5. Due to the shape of (17) we end up with more green vertices than we had for (18).

Proposition 6.6. *The lattice equation* (17) *has the Laurent property for the initial values*

$$I = \{u_{s,0}, u_{s,1}, u_{0,t} : s, t \in \mathbb{Z}\}.$$

Proof. Again we fix s and use induction on t. The induction starts with the vertical line of six values in $\mathcal{L}$, shown in yellow and green in figure 5. We can see from (78) that, for this I,

$\beta(t) \in \mathcal{L}$ for all t, and we increase t by solving (77) for $u_{s,t+6}$ to obtain Laurent polynomials for all $t > 0$, while to extend to $t < 0$ we solve for $u_{s,t}$ instead. □

For the band sets of initial values we have to work harder.

Lemma 6.7. *For a set of initial values I, suppose that there is an $\tilde{s}$ such that*

$$\{u_{\tilde{s},t}, u_{\tilde{s},t+1}, u_{\tilde{s},t+4}, u_{\tilde{s},t+5}\} \subset \mathcal{L}$$

and

$$\{u_{\tilde{s},t+2}, u_{\tilde{s},t+3}\} \subset I;$$

then $\beta(t) \in \mathcal{L}$.

Proof. In the expression for $\beta(t)$ in (78), we shift s until $u_{\tilde{s},t+2}$ and $u_{\tilde{s},t+3}$ appear in the denominator, and the terms in the numerator belong to $\mathcal{L}$ by assumption, so the result follows. □

Theorem 6.8. *For a given initial set I, if the conditions of lemma 6.7 hold for all t, and if for all s there is a $\tilde{t}$ such that*

$$\{u_{s,\tilde{t}}, u_{s,\tilde{t}+1}, u_{s,\tilde{t}+2}, u_{s,\tilde{t}+3}, u_{s,\tilde{t}+4}, u_{s,\tilde{t}+5}\} \subset \mathcal{L},$$

then equation (17) has the Laurent property.

Proof. The proof is the same as for proposition 6.6. □

Theorem 6.9. *The equation (17) has the Laurent property if I is a band of initial values.*

Proof. Similarly to the proof of theorem 6.4 we have

$$L'_1 = L_1 + (-1, -2), \quad L'_2 = L_2 + (1, -2),$$

so $L'_2 = L'_1 + (3, -6)$ and the minimal distance between them is $\sqrt{45} > 6$. Hence for any vertical or horizontal line intersecting the lattice we have at least six consecutive values in $\mathcal{L}$, and at least two of these will be neighbours and in I. □

7. Concluding remarks

We have constructed linear relations with constant coefficients for the Little Pi family (11), and given an alternative derivation of the linear relations with periodic coefficients found in [37], using them to show that these systems are maximally superintegrable. For the new family of recurrences (19) we have also derived linear relations with constant and periodic coefficients, but the question of superintegrability remains open.

Each of the two families of nonlinear recurrences we have considered provides travelling wave solutions of a partial difference equation defined on the same six-point stencil in $\mathbb{Z}^2$, namely the equation (17) introduced by Kamiya *et al*, and the new equation (18), respectively. As was shown in [37], the lattice equation (17) is integrable in the sense that it admits linearization, and here we have shown the same for (18).

While, in the continuous setting, the theory of solitons and integrable partial differential equations is very well established, discrete integrable systems are much less well understood. For example, for partial difference equations defined on quadrilateral lattices, the classification of equations on four-point stencils was pioneered by Adler *et al* [1], based on the condition of 3D consistency around a cube (see also [4] for systems of quad-equations); for another approach to classification, based on generalized symmetries, see [27]. The recent monograph [34] surveys the current state of the art in discrete integrable systems. As far as we are aware, there are no classification results for integrable 2D lattice equations on stencils with five or more points, so it would be interesting to find other integrable equations on the same six-point stencil as (17) and (18).

The recurrences and lattice equations considered here all possess the Laurent property, but go beyond the setting of cluster algebras. The role of the Laurent property in discrete integrable systems is rather intriguing: on the one hand, most recurrences with the Laurent property are not integrable; and on the other hand, most discrete integrable systems do not possess the Laurent property, when written in their standard coordinates. However, the Laurent property is an essential feature of discrete Hirota equations for tau functions [38, 44], and, in a suitable algebro-geometric setting, there is an expectation that all discrete integrable systems should admit 'Laurentification' [31, 32], that is, a lift to a new set of coordinates (tau functions or their analogues) in which the Laurent property does hold. For a recent review of cluster algebras in the context of discrete integrable systems, see [29].

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ORCID iDs

Andrew N W Hone https://orcid.org/0000-0001-9780-7369

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