MODULAR INVARIANTS OF FINITE GLUING GROUPS

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ABSTRACT. We use the gluing construction introduced by Jia Huang to explore the rings of invariants for a range of modular representations. We construct generating sets for the rings of invariants of the maximal parabolic subgroups of a finite symplectic group and their common Sylow p-subgroup. We also investigate the invariants of singular finite classical groups. We introduce parabolic gluing and use this construction to compute the invariant field of fractions for a range of representations. We use thin gluing to construct faithful representations of semidirect products and to determine the minimum dimension of a faithful representation of the semidirect product of a cyclic p-group acting on an elementary abelian p-group.

1. INTRODUCTION

In this paper we use the gluing construction introduced by Jia Huang [30] to explore the rings of invariants for a range of modular representations. The gluing construction was motivated, in part, by the work of Hewett [28], Kuhn and Mitchell [34], and Mui [37] on parabolic subgroups of a finite general linear group. In Section 5, we use gluing methods to construct generating sets for the rings of invariants of the maximal parabolic subgroups of a finite symplectic group and their common Sylow p-subgroup. Our work in that section relies on the results of Carlisle and Kropholler on the invariants of a finite symplectic group [2, §8]. We also use the gluing construction to investigate the invariants of singular finite classical groups (Section 6). In Section 7, we introduce parabolic gluing and use this construction to compute the invariant field of fractions for a range of representations. We use thin gluing to construct faithful representations of semidirect products (Theorem 2.3) and to determine the minimum dimension of a faithful representation of the semidirect product of a cyclic p-group acting on an elementary abelian p-group (Corollary 2.4).

Suppose V is a finite dimensional representation of a group G over a field $\mathbb{F}$. We view V as a left module over the group ring $\mathbb{F}G$. There is a natural right action of G on the dual $V^* = \hom_{\mathbb{F}}(V,\mathbb{F})$: for $\phi \in V^*$, $g \in G$, and $v \in V$, $(\phi \cdot g)(v) = \phi(g \cdot v)$. We use $\mathbb{F}[V]$ to denote the symmetric algebra on V^*. The action of G on V^* extends to an action by degree preserving $\mathbb{F}$-algebra automorphisms on $\mathbb{F}[V]$. The *ring of invariants* of the representation is the subalgebra $\mathbb{F}[V]^G := \{f \in \mathbb{F}[V] \mid f \cdot g = f, \forall g \in G\}$. The elements of $\mathbb{F}[V]$ represent polynomial functions on V and the elements of $\mathbb{F}[V]^G$ represent polynomial functions on the orbits V/G. If G is finite and $\mathbb{F}$ is algebraically closed, then $\mathbb{F}[V]^G$ is the ring of regular functions on the categorical quotient $V/\!\!/G$. For background material on the invariant theory of finite groups, see [2], [12], [19], [26] and [38]. For background material on modular representation theory we suggest [1]. We

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occasionally make reference to Steenrod operations; see [38, §8] for the definition in the context of invariant theory.

In Section 2 we introduce the gluing construction and define polynomial gluings, split gluings and thin gluings. In Section 3 we summarise the relevant properties of tensor products of algebras. In Section 4 we compute the image of the transfer for a polynomial gluing in terms of the image of the transfer of the factors of the gluing. Section 5 is devoted to the maximal parabolic subgroups of a finite symplectic group and Section 6 deals with finite singular classical groups. In Section 7 we introduce parabolic gluing and compute the invariant field of fractions for a range of representations. We conclude with Section 8, which introduces diagonal gluing and explores a number of examples.

2. The Gluing Construction

Let W_1 and W_2 denote representations over a field $\mathbb{F}$ of groups G_1 and G_2 respectively. The vector space of linear transformations $\hom_{\mathbb{F}}(W_2, W_1)$ is a left $\mathbb{F}G_1$/right $\mathbb{F}G_2$ bimodule: for $g_1 \in G_1$, $\varphi \in \hom_{\mathbb{F}}(W_2, W_1)$, $g_2 \in G_2$ and $v \in W_2$ we have $(g_1 \cdot \varphi \cdot g_2)(v) = g_1 \cdot (\varphi(g_2 \cdot v))$. Using the unique unital ring homomorphism from $\mathbb{Z}$ to $\mathbb{F}$, every $\mathbb{F}G_i$-module is also a $\mathbb{Z}G_i$-module. Let $\mathcal{M}$ denote a left $\mathbb{Z}G_1$/right $\mathbb{Z}G_2$ sub-bimodule of $\hom_{\mathbb{F}}(W_2, W_1)$. We use $G_1 \times_{\mathcal{M}} G_2$ to denote the semidirect product whose elements consist of triples $(g_1, \varphi, g_2) \in G_1 \times \mathcal{M} \times G_2$ with the product given by $(g_1, \varphi, g_2) \cdot (g_1', \varphi', g_2') = (g_1 g_1', g_1\varphi' + \varphi g_2', g_2 g_2')$. We refer to $G_1 \times_{\mathcal{M}} G_2$ as the *gluing of* G_1 *to* G_2 *through* $\mathcal{M}$. Note that, to perform this construction, we need $\mathcal{M}$ to be closed with respect to addition and with respect to the group actions. There is a natural action of $G_1 \times_{\mathcal{M}} G_2$ on $V := W_1 \oplus W_2$ given by $(g_1, \varphi, g_2)(w_1 \oplus w_2) = (g_1 w_1 + \varphi(w_2)) \oplus g_2 w_2$. If W_1 and W_2 are faithful, then V is a faithful representation of $G_1 \times_{\mathcal{M}} G_2$. If we choose bases for W_1 and W_2 and denote the resulting matrices by $[g_1]$, $[\varphi]$ and $[g_2]$, then the associated matrix group is given by

$$\left\{ \begin{pmatrix} [g_1] & [\varphi] \\ 0 & [g_2] \end{pmatrix} \in \mathrm{GL}_{m+n}(\mathbb{F}) \,\middle|\, g_1 \in G_1, g_2 \in G_2, \varphi \in \mathcal{M} \right\}$$

where $m = \dim(W_1)$ and $n = \dim(W_2)$. If W_1 and W_2 are both faithful, then the matrix group is isomorphic to $G_1 \times_{\mathcal{M}} G_2$.

Since $\mathcal{M}$ is a normal subgroup of $G_1 \times_{\mathcal{M}} G_2$ with quotient isomorphic to $G_1 \times G_2$, we can compute the ring of invariants $\mathbb{F}[V]^{G_1 \times_{\mathcal{M}} G_2}$ by first computing the invariants under the action of $\mathcal{M}$ and then computing the invariants under the action of $G_1 \times G_2$, in other words, $\mathbb{F}[V]^{G_1 \times_{\mathcal{M}} G_2} = (\mathbb{F}[V]^{\mathcal{M}})^{G_1 \times G_2}$. We will routinely identify $\mathbb{F}[W_2]$ with the subalgebra $\mathbb{F} \otimes \mathbb{F}[W_2] \subset \mathbb{F}[W_1] \otimes \mathbb{F}[W_2] = \mathbb{F}[W_1 \oplus W_2] = \mathbb{F}[V]$. Using this identification, we observe that $\mathbb{F}[W_2] \subset \mathbb{F}[V]^{\mathcal{M}}$. We will say that the gluing is *split* if there exists a graded subalgebra $A \subset \mathbb{F}[V]^{\mathcal{M}}$ such that

- $\mathbb{F}[V]^{\mathcal{M}} = A \otimes \mathbb{F}[W_2]$,
- A is an $\mathbb{F}(G_1 \times G_2)$ submodule of $\mathbb{F}[V]^{\mathcal{M}}$ and
- G_2 acts trivially on A.

When we write $\mathbb{F}[V]^{\mathcal{M}} = A \otimes \mathbb{F}[W_2]$ above, we mean that there is an algebra isomorphism from $A \otimes \mathbb{F}[W_2]$ to $\mathbb{F}[V]^{\mathcal{M}}$ which restricts to the inclusion on each of the factors. If the gluing is split, then $\mathbb{F}[V]^{G_1 \times_{\mathcal{M}} G_2} = A^{G_1} \otimes \mathbb{F}[W_2]^{G_2}$.

We are primarily interested in rings of invariants for finite groups. If $\mathcal{M}$ is non-zero and $\mathbb{F}$ has characteristic zero, then $\mathcal{M}$ is an infinite group. For the remainder of the paper, we assume the characteristic of $\mathbb{F}$ is a prime number p and that $\mathcal{M}$ is finite. This means that $\mathcal{M}$ is an elementary abelian p-group and a vector space over the prime field $\mathbb{F}_p$. Recent work on the rings of invariants of modular representation of elementary abelian p-groups includes [11], [39] and [40].

We will use $\{y_1,\ldots,y_m\}$ to denote a chosen basis for W_1^* and $\{x_1,\ldots,x_n\}$ to denote a chosen basis for W_2^*. Observe that $\mathbb{F}[x_1,\ldots,x_n]\subset\mathbb{F}[V]^{\mathcal{M}}$. We can use orbit products to construct additional $\mathcal{M}$-invariants: define $N_{\mathcal{M}}(y_j):=\prod\{y_j\cdot g\mid g\in\mathcal{M}\}$. The set

$$\mathcal{H}:=\{x_1,\ldots,x_n,N_{\mathcal{M}}(y_1),\ldots,N_{\mathcal{M}}(y_m)\}$$

is a homogeneous system of parameters for $\mathbb{F}[V]^{\mathcal{M}}$ (see, for example, [26, Example 6]). If $\mathcal{H}$ is a generating set for $\mathbb{F}[V]^{\mathcal{M}}$, we say that the chosen basis is a *Nakajima basis* for the $\mathbb{F}\mathcal{M}$-module V. Recall that $\mathcal{H}$ is a generating set for $\mathbb{F}[V]^{\mathcal{M}}$ if and only if the product of the degrees of the elements of $\mathcal{H}$ is equal to the order of $\mathcal{M}$ (see, for example, [19, 3.7.5]). The polynomials $N_{\mathcal{M}}(y_j)$ have a number of special properties. The $\mathcal{M}$-orbit of y_j is of the form y_j+U_j where U_j is an $\mathbb{F}_p$-subspace of $W_2^*=\mathrm{Span}_{\mathbb{F}}\{x_1,\ldots,x_n\}$. Thus

$$N_{\mathcal{M}}(y_j)=\prod_{u\in U_j}(y_j+u)=y_j^{p^{\ell_j}}+\sum_{k=0}^{\ell_j-1}d_{k-\ell_j,j}y_j^{p^k}$$

where $\ell_j=\dim_{\mathbb{F}_p}(U_j)$ and the $d_{i,j}$ are Dickson polynomials associated to U_j (see [49] or [20]). Therefore $N_{\mathcal{M}}(y_j)$ is additive as a function of y_j and is invariant under any subgroup of $\mathrm{GL}(W_2)$ which stabilises U_j.

Example 2.1. Suppose $\mathbb{F}=\mathbb{F}_q$, where $q=p^r$ for some r, and take $\mathcal{M}=\hom_{\mathbb{F}_q}(W_2,W_1)$. Then, for all j, we have $U_j=W_2^*$ and $\deg(N_{\mathcal{M}}(y_j))=q^n$. Therefore, since the order of $\mathcal{M}$ is q^{mn}, we have $\mathbb{F}_q[V]^{\mathcal{M}}=\mathbb{F}_q[x_1,\ldots,x_n,N_{\mathcal{M}}(y_1),\ldots,N_{\mathcal{M}}(y_m)]$. Hence any basis consistent with the decomposition $V^*=W_1^*\oplus W_2^*$ is Nakajima. Furthermore, any subgroup of $\mathrm{GL}(W_2)$ stabilises W_2^*. Thus G_2 acts trivially on $A=\mathbb{F}_q[N_{\mathcal{M}}(y_1),\ldots,N_{\mathcal{M}}(y_m)]$ and so the gluing is split giving $\mathbb{F}_q[V]^{G_1\times_{\mathcal{M}}G_2}=A^{G_1}\otimes\mathbb{F}_q[W_1]^{G_2}$. The algebra homomorphism from $\mathbb{F}_q[W_1]$ to A which takes y_j to $N_{\mathcal{M}}(y_j)$ is a G_1-equivariant isomorphism which takes elements of degree d to elements of degree $d\cdot q^n$. This means that $\mathbb{F}_q[V]^{G_1\times_{\mathcal{M}}G_2}$ is isomorphic to $\mathbb{F}_q[W_1]^{G_1}\otimes\mathbb{F}_q[W_2]^{G_2}$ by an isomorphism which is not degree preserving but does restrict to an isomorphism of the augmentation ideals.

Definition 2.2. *We say that a gluing is* **polynomial** *if there is a $(G_1\times G_2)$-equivariant isomorphism of $\mathbb{F}$-algebras from $\mathbb{F}[W_1\oplus W_2]$ to $\mathbb{F}[V]^{\mathcal{M}}$.*

In the case of a polynomial gluing, $\mathbb{F}[V]^{G_1\times_{\mathcal{M}}G_2}$ is isomorphic to $\mathbb{F}[W_1]^{G_1}\otimes\mathbb{F}[W_2]^{G_2}$. In most of the polynomial gluings we consider, the $\mathbb{F}$-algebra isomorphism $\psi:\mathbb{F}[W_1\oplus W_2]\to\mathbb{F}[V]^{\mathcal{M}}$ is an extension of the inclusion of $\mathbb{F}[W_2]$ into $\mathbb{F}[V]^{\mathcal{M}}$ and restricts to an isomorphism of $\mathbb{F}[W_1]$ to a subalgebra $A\subset\mathbb{F}[V]^{\mathcal{M}}$; in this case, the gluing is split and the induced map from $\mathbb{F}[W_1]$ to A is G_1-equivariant. Example 2.1 provides a canonical example of a family of split polynomial gluings.

Jia Huang's Gluing Lemma. Jia Huang's Gluing Lemma [30, §2] is an extension of Example 2.1. We reformulate his result in our notation. Suppose $q = p^r$, $\mathbb{F}_q \subseteq \mathbb{F}$ and M_1 is an $\mathbb{F}_q G_1$-submodule of W_1. Further suppose that M_2 is an $\mathbb{F}_q G_2$-submodule of W_2 with $\mathbb{F} \cdot M_2 = W_2$ and $\dim_{\mathbb{F}_q}(M_2) = \dim_{\mathbb{F}}(W_2) = n$. This means that every $\mathbb{F}_q$-basis for M_2 is an $\mathbb{F}$-basis for W_2. Therefore, every element of $\hom_{\mathbb{F}_q}(M_2, M_1)$ extends uniquely to an element of $\hom_{\mathbb{F}}(W_2, W_1)$, giving an isomorphism from $\hom_{\mathbb{F}_q}(M_2, M_1)$ to a left $\mathbb{F}_q G_1$/ right $\mathbb{F}_q G_2$ sub-bimodule of $\hom_{\mathbb{F}}(W_2, W_1)$; take $\mathcal{M}$ to be the image of this map. The gluing $G_1 \times_{\mathcal{M}} G_2$ is a split polynomial gluing. To see this, first observe that $U_j = M_2^*$ for each j, which means that $N_{\mathcal{M}}(y_j)$ has degree q^n and, as in Example 2.1, $\mathbb{F}[V]^{\mathcal{M}} = \mathbb{F}[x_1, \ldots, x_n, N_{\mathcal{M}}(y_1), \ldots, N_{\mathcal{M}}(y_m)]$. Then observe that, since any subgroup of $\mathrm{GL}(M_2)$ stabilises M_2^*, G_2 acts trivially on $A = \mathbb{F}_q[N_{\mathcal{M}}(y_1), \ldots, N_{\mathcal{M}}(y_m)]$.

Examples and Applications of Thin Gluing. We say that a gluing is *thin* if either $\dim_{\mathbb{F}}(W_1) = 1$ or $\dim_{\mathbb{F}}(W_2) = 1$. If $\dim_{\mathbb{F}}(W_1) = 1$, then the gluing is polynomial with $A = \mathbb{F}[N_{\mathcal{M}}(y_1)]$. If $\dim_{\mathbb{F}}(W_2) = 1$, then the representation of $\mathcal{M}$ is what is known as a hyperplane representation; the ring of invariants $\mathbb{F}[V]^{\mathcal{M}}$ is still a polynomial algebra but the generators are not necessarily orbit products of variables (see [9], [27] and [35]).

Theorem 2.3. *If a finite group G acts on an elementary abelian p-group E and $\mathbb{F}$ is a sufficiently large field of characteristic p, then there is a faithful representation of the semidirect product $G \ltimes E$ of dimension $|G| + 1$ over $\mathbb{F}$.*

Proof. The action of G on E makes E a module over the group ring $\mathbb{F}_p G$. Every $\mathbb{F}_p G$ module has an injective envelope. Since injective $\mathbb{F}_p G$-modules are projective, this means we can embed E into a free $\mathbb{F}_p G$-module, say $\mathcal{F} = \oplus_{i=1}^r \mathbb{F}_p G b_i$. Choose elements $c_1, \ldots, c_r \in \mathbb{F}$ so that $\{c_1, \ldots, c_r\}$ is linearly independent over $\mathbb{F}_p$. Then we can embed $\mathcal{F}$ into $\mathbb{F}G$ using the $\mathbb{F}_p G$-module map which takes b_i to c_i. Composing the map from E to $\mathcal{F}$ with the map from $\mathcal{F}$ to $\mathbb{F}G$ gives an $\mathbb{F}_p G$-module isomorphism from E to an $\mathbb{F}_p G$-submodule of $\mathbb{F}G$, say $\mathcal{E}$. If we take $G_1 = G$, $W_1 = \mathbb{F}G$, $G_2 = \mathbf{1} = \{1\}$ and $W_2 = \mathbb{F}$, the thin gluing $G \times_{\mathcal{E}} \mathbf{1}$ is isomorphic to $G \ltimes E$ and the associated representation is faithful with dimension $|G| + 1$. □

Corollary 2.4. *If the cyclic p-group of order p^r, C_{p^r}, acts on an elementary abelian p-group E and E contains a free $\mathbb{F}_p C_{p^r}$-submodule , then the minimum dimension of a faithful representation of $C_{p^r} \ltimes E$ over $\mathbb{F}$ is $p^r + 1$, for $\mathbb{F}$ sufficiently large.*

Proof. If E contains a free $\mathbb{F}_p C_{p^r}$-submodule then $C_{p^r} \ltimes E$ contains an element of order p^{r+1} (otherwise $C_{p^r} \ltimes E$ has exponent p^r). Since an element of p-power order in $\mathrm{GL}_{p^r}(\mathbb{F})$ has order at most p^r (see, for example, [12, Lemma 7.1.1]), the minimum dimension of a faithful representation is at least $p^r + 1$. If $\mathbb{F}$ is sufficiently large, then the existence of a faithful representation of dimension $p^r + 1$ is given by Theorem 2.3. □

Example 2.5. Specialise Example 2.1 by taking $W_2 = \mathbb{F}_q G_2$ and $W_1 = \mathbb{F}_q G_1$. Then $\mathcal{M} = \hom_{\mathbb{F}_q}(\mathbb{F}_q G_2, \mathbb{F}_q G_1)$ can be given the structure of a left module over $\mathbb{F}_q(G_1 \times G_2)$ in a natural way. With this choice of action, $\mathcal{M}$ is the principal module generated by the linear map which maps $1 \in \mathbb{F}_q G_2$ to $1 \in \mathbb{F}_q G_1$ and is zero on all other group elements. This means that $\mathcal{M}$ is isomorphic to $\mathbb{F}_q(G_1 \times G_2)$. Using this observation, it is easy to verify that $(G_1 \times G_2) \ltimes \mathbb{F}_q(G_1 \times G_2) \cong G_1 \times_{\mathcal{M}} G_2$. Therefore, the representation associated to $G_1 \times_{\mathcal{M}} G_2$ gives a faithful representation of $G \ltimes \mathbb{F}_q G$ of dimension $|G_1| + |G_2|$ whenever $G \cong G_1 \times G_2$. Taking $G_1 = G_2 = C_p$ gives an upper bound of $2p$ on on the minimum dimension of a faithful representation of $(C_p \times C_p) \ltimes \mathbb{F}_p(C_p \times C_p)$.

Remark 2.6. *Theorem 2.3 and Corollary 2.4 had their origins in a question from Alex Duncan related to his work with Christian Urech on representations of finite subgroups of Cremona groups.*

3. Properties of Tensor Products

In this section we summarise a number of properties of $\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2}$ which are inherited from $\mathbb{F}[W_1]^{G_1}$ and $\mathbb{F}[W_2]^{G_2}$. We expect that most of the results of this section will be familiar to the experts. Throughout we identify $\mathbb{F}[W_1]^{G_1}$ with $\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F} \subset \mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2}$ and $\mathbb{F}[W_2]^{G_2}$ with $\mathbb{F} \otimes \mathbb{F}[W_2]^{G_2} \subset \mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2}$. Let $\mathcal{A}$ denote the category of finitely generated commutative $\mathbb{F}$-algebras which are graded over the non-negative integers and have both a unit and an augmentation. Most of the results of this section are valid for objects in this category. One approach to this material would be to follow the model developed in [5] for Noetherian local rings. We have elected to take a more direct but ad hoc approach.

Lemma 3.1. *Suppose B and C are objects in $\mathcal{A}$. Then*

$$\dim_{\mathbb{F}}\left((B \otimes C)_+ / (B \otimes C)_+^2\right) = \dim_{\mathbb{F}}(B_+/B_+^2) + \dim_{\mathbb{F}}(C_+/C_+^2).$$

Proof. Observe that $(B \otimes C)_+ = (B_+ \otimes \mathbb{F}) \oplus (B_+ \otimes C_+) \oplus (\mathbb{F} \otimes C_+)$ and $(B \otimes C)_+^2 = (B_+^2 \otimes \mathbb{F}) \oplus (B_+ \otimes C_+) \oplus (\mathbb{F} \otimes C_+^2)$. □

Note that if B is an object in $\mathcal{A}$, then every set of minimal homogeneous generators for B projects to a homogeneous basis for the finite dimensional graded vector space B_+/B_+^2. Therefore, although B does not have a unique minimal homogeneous generating set, the degrees of a minimal homogeneous generating set are given by the dimensions of the homogeneous components of B_+/B_+^2. The following is a straightforward consequence of Lemma 3.1 and its proof.

Proposition 3.2. *If $\{f_1, \ldots, f_s\}$ is a minimal set of homogeneous generators for $\mathbb{F}[W_1]^{G_1}$ and $\{h_1, \ldots, h_k\}$ is a minimal set of homogeneous generators for $\mathbb{F}[W_2]^{G_2}$, then $\{f_1, \ldots, f_s, h_1, \ldots, h_k\}$ is a minimal set of homogeneous generators for $\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2}$.*

We will use Koszul complexes to relate properties of $B \otimes C$ to properties of B and C. We suggest [7, §1.6] as a good reference for the basic properties of Koszul complexes. Suppose B is an object in $\mathcal{A}$ and $f_1, \ldots, f_s$ is a minimal set of homogeneous generators for B. Let $K_B(\underline{f})$ denote the Koszul complex determined by the sequence $f_1, \ldots, f_s \in B$. Similarly, let $K_C(\underline{h})$ denote the Koszul complex determined by a minimal set of homogeneous generators $h_1, \ldots, h_k$ for C. We write $K_{B \otimes C}(\underline{f}, \underline{h})$ for the Koszul complex determined by $f_1 \otimes 1, \ldots, f_s \otimes 1, 1 \otimes h_1, \ldots, 1 \otimes h_k \in B \otimes C$. The following is essentially [7, Proposition 1.6.6].

Lemma 3.3. *There is an isomorphism of differential graded $(B\otimes C)$-algebras from $K_{B\otimes C}(\underline{f},\underline{h})$ to $K_B(\underline{f})\otimes K_C(\underline{h})$.*

Proof. The differential graded algebra $K_{B\otimes C}(\underline{f},\underline{h})$ is the exterior algebra on the free $(B\otimes C)$-module generated $\{e_i, b_j \mid i=1,\dots,s; j=1,\dots,k\}$ with the differential determined by $d(e_i)=f_i\otimes 1$ and $d(b_j)=1\otimes h_j$. Using the univeral property of the exterior algebra, the map taking e_i to $e_i\otimes 1$ and b_i to $1\otimes b_i$, extends to an algebra map from $K_{B\otimes C}(\underline{f},\underline{h})$ to $K_B(\underline{f})\otimes K_C(\underline{h})$. Using the Koszul sign convention to define the differential on $K_B(\underline{f})\otimes K_C(\underline{h})$, we see that the map is a map of differential graded algebras. Since the graded components of both algebras are free $(B\otimes C)$-modules of the same rank and the map takes a basis to a basis, we have the required isomorphism. □

It follows from Lemma 3.3 that the homology of $K_{B\otimes C}(\underline{f},\underline{h})$ is isomorphic to the homology of $K_B(\underline{f})\otimes K_C(\underline{h})$. Since the tensor product is over a field, the Künneth formula [48, Theorem 3.6.3] gives

$$H_n\left(K_B(\underline{f})\otimes K_C(\underline{h})\right)\cong\bigoplus_{i+j=n}H_i\left(K_B(\underline{f})\right)\otimes H_j\left(K_C(\underline{h})\right).$$

Since the Koszul homology is independent of the choice of generating set we write $H_*(B)$ for the homology of $K_B(\underline{f})$. Using this convention and the above observations, we have

$$H_n(B\otimes C)\cong\bigoplus_{i+j=n}H_i(B)\otimes H_j(C). \tag{1}$$

Recall that, for an object B in $\mathcal{A}$, the depth of B is $\mathrm{grade}(B_+,B)$, the length of a maximal B-regular sequence in B_+. Since the grade can be computed using the Koszul complex, see [7, Theorem 1.6.17], we have the following proposition.

Proposition 3.4. $\mathrm{depth}(\mathbb{F}[W_1]^{G_1}\otimes\mathbb{F}[W_2]^{G_2})=\mathrm{depth}(\mathbb{F}[W_1]^{G_1})+\mathrm{depth}(\mathbb{F}[W_2]^{G_2})$.

Since an object in $\mathcal{A}$ is Cohen-Macaulay if and only if the depth is equal to the Krull dimension, we get the following.

Proposition 3.5. $\mathbb{F}[W_1]^{G_1}\otimes\mathbb{F}[W_2]^{G_2}$ *is Cohen-Macaulay if and only if both* $\mathbb{F}[W_1]^{G_1}$ *and* $\mathbb{F}[W_2]^{G_2}$ *are Cohen-Macaulay.*

Suppose B is an object in $\mathcal{A}$ and $f_1,\dots,f_s$ is a minimal set of homogeneous generators for B. Denote $R:=\mathbb{F}[X_1,\dots,X_s]$ and consider the resulting presentation $\rho : R\longrightarrow B$ given by $\rho(X_i)=f_i$. We say that B is a *complete intersection* if $\ker(\rho)$ is generated by a regular sequence. We use $\dim(B)$ to denote the Krull dimension of B. The analogue of the following result is well-known in the setting of Noetherian local rings, see for example [7, §2.3] and [36, §21].

Proposition 3.6. *Suppose B is an integral domain. Using the above notation, B is a complete intersection if and only if* $\dim_{\mathbb{F}}(H_1(B))=s-\dim(B)$.

Proof. Note that, since B is an integral domain, $\ker(\rho)$ is a prime ideal. Let $\mu(\ker(\rho))$ denote the number of elements in a minimal generating set for $\ker(\rho)$. If B is a complete intersection then $\mu(\ker(\rho))=\mathrm{grade}(\ker(\rho),R)=\mathrm{height}(\ker(\rho))$. Conversely, since R is Cohen-Macaulay,

$\text{height}(\ker(\rho)) = \text{grade}(\ker(\rho), R)$ and if $\text{grade}(\ker(\rho), R) = \mu(\ker(\rho))$, then $\ker(\rho)$ is generated by a regular sequence, see [42, Theorem 16.21]. Observe that $K_B(\underline{f}) \cong K_R(\underline{X}) \otimes_R B$. Therefore $H_1(K_B(\underline{f})) \cong \text{Tor}_1^R(\mathbb{F}, B)$. The short exact sequence of R-modules $0 \to \ker(\rho) \to R \to B \to 0$ gives a long exact sequence ending in

$$\text{Tor}_1^R(\mathbb{F}, R) \to \text{Tor}_1^R(\mathbb{F}, B) \to \mathbb{F} \otimes_R \ker(\rho) \to \mathbb{F} \otimes_R R \to \mathbb{F} \otimes_R B \to 0.$$

Note that $\mathbb{F} \otimes_R R \cong \mathbb{F} \cong \mathbb{F} \otimes_R B$ and $\text{Tor}_1^R(\mathbb{F}, R) = 0$. Hence

$$H_1(B) \cong \text{Tor}_1^R(\mathbb{F}, B) \cong \mathbb{F} \otimes_R \ker(\rho) \cong \ker(\rho)/(R_+ \ker(\rho))$$

and $\dim_{\mathbb{F}}(H_1(B)) = \mu(\ker(\rho))$ (compare with [36, page 170]). Recall that $\text{height}(\ker(\rho)) = \dim(R) - \dim(B)$ (see, for example, [7, Theorem A.16]). Puting these ideas together, we see that B is a complete intersection if and only if $\dim_{\mathbb{F}}(H_1(B)) = \dim(R) - \dim(B) = s - \dim(B)$. □

Proposition 3.7. *$\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2}$ is a complete intersection if and only if both $\mathbb{F}[W_1]^{G_1}$ and $\mathbb{F}[W_2]^{G_2}$ are complete intersections.*

Proof. Since $\mathbb{F}[W_1]^{G_1}$, $\mathbb{F}[W_2]^{G_2}$ and $\mathbb{F}[W_1 \oplus W_2]^{G_1 \times G_2} \cong \mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2}$ are all integral domains, Proposition 3.6 applies. Using Equation 1, we have

$$\dim_{\mathbb{F}}\left(H_1(\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2})\right) = \dim_{\mathbb{F}}\left(H_1(\mathbb{F}[W_1]^{G_1})\right) + \dim_{\mathbb{F}}\left(H_1(\mathbb{F}[W_2]^{G_2})\right).$$

Suppose $\{f_1, \ldots, f_s\}$ is a minimal set of homogeneous generators for $\mathbb{F}[W_1]^{G_1}$ and $\{h_1, \ldots, h_k\}$ is a minimal set of homogeneous generators for $\mathbb{F}[W_2]^{G_2}$. Using Proposition 3.2, we see that $\{f_1, \ldots, f_s, h_1, \ldots, h_k\}$ is a minimal set of homogeneous generators for $\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2}$. Therefore $\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2}$ is a complete intersection if and only if

$$\dim_{\mathbb{F}}\left(H_1(\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2})\right) = (s + k) - \dim(\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2}).$$

Since G_1 and G_2 are finite groups and $\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2} \cong \mathbb{F}[W_1 \oplus W_2]^{G_1 \times G_2}$, we have

$$\dim(\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2}) = \dim_{\mathbb{F}}(W_1 \oplus W_2) = \dim(\mathbb{F}[W_1]^{G_1}) + \dim(\mathbb{F}[W_2]^{G_2})$$

and the result follows. □

Proposition 3.8. *$\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2}$ is a unique factorisation domain if and only if both $\mathbb{F}[W_1]^{G_1}$ and $\mathbb{F}[W_2]^{G_2}$ are unique factorisation domains.*

Proof. Again we identify $\mathbb{F}[W_1]^{G_1} \otimes \mathbb{F}[W_2]^{G_2}$ with $\mathbb{F}[W_1 \oplus W_2]^{G_1 \times G_2}$. Using a result of Nakajima (see [2, Corollary 3.9.3]), a ring of invariants of a finite group is a unique factorisation domain if and only if there are no non-trivial homomorphisms from the group to the units of the field taking the value one on every pseudoreflection. The pseudoreflections for the action of $G_1 \times G_2$ on $W_1 \oplus W_2$ are precisely the elements $(g_1, 1)$ and $(1, g_2)$ for g_1 a pseudoreflection for the action of G_1 on W_1 and g_2 a pseudoreflection for the action of G_2 on W_2. A homomorphism from $G_1 \times G_2$ to $\mathbb{F}^\times$ is determined uniquely by the restrictions to $G_1 \times \{1\}$ and $\{1\} \times G_2$. Therefore, any homomorphism $\phi : G_1 \times G_2 \to \mathbb{F}^\times$ which takes value one on every pseudoreflection will restrict to give homomorphisms $\phi_1 : G_1 \to \mathbb{F}^\times$ and $\phi_2 : G_2 \to \mathbb{F}^\times$ which take value one on every pseudoreflection. □

For a representation V of a finite group G a subset $\{f_1,\dots,f_s\}\subset\mathbb{F}[V]^G$ is called a *geometric separating set* if the elements of the set can be use to distinguish (separate) the G-orbits of $V\otimes_{\mathbb{F}}\overline{\mathbb{F}}$, where $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$ (see, for example, [22]).

Proposition 3.9. *If $\{f_1,\dots,f_s\}$ is a homogeneous geometric separating set for $\mathbb{F}[W_1]^{G_1}$ and $\{h_1,\dots,h_k\}$ is a homogeneous geometric separating set for $\mathbb{F}[W_2]^{G_2}$, then $\{f_1,\dots,f_s,h_1,\dots,h_k\}$ is a homogeneous geometric separating set for $\mathbb{F}[W_1]^{G_1}\otimes\mathbb{F}[W_2]^{G_2}$.*

Proof. We identify $\mathbb{F}[W_1]^{G_1}\otimes\mathbb{F}[W_2]^{G_2}$ with $\mathbb{F}[W_1\oplus W_2]^{G_1\times G_2}$. Let B_1 denote the subalgebra of $\mathbb{F}[W_1]$ generated by $\{f_1,\dots,f_s\}$ and let B_2 denote the subalgebra of $\mathbb{F}[W_2]$ generated by $\{h_1,\dots,h_k\}$. Then $B:=B_1\otimes B_2$ is the subalagebra of $\mathbb{F}[W_1\oplus W_2]$ generated by $\{f_1,\dots,f_s,h_1,\dots,h_k\}$. Using [22, Proposition 1.2], $\mathbb{F}[W_i]^{G_i}$ is the purely inseparable closure of B_i in $\mathbb{F}[W_i]$. Let $\overline{B}$ denote the purely inseparable closure of B in $\mathbb{F}[W_1\oplus W_2]$. Since $B\subseteq\mathbb{F}[W_1\oplus W_2]^{G_1\times G_2}$, we see that $\overline{B}\subseteq\mathbb{F}[W_1\oplus W_2]^{G_1\times G_2}$. Therefore, we need only show that every homogeneous element of $\mathbb{F}[W_1\oplus W_2]^{G_1\times G_2}$ lies in $\overline{B}$. Consider $f\in\mathbb{F}[W_1\oplus W_2]^{G_1\times G_2}$. Using Proposition 3.2, write $f=\sum_{j=1}^{\ell}\alpha_j\otimes\beta_j$ for some choice of $\alpha_j\in\mathbb{F}[W_1]$ and $\beta_j\in\mathbb{F}[W_2]$. Choose $N\in\mathbb{Z}^+$ so that $\alpha_j^{p^N}\in B_1$ and $\beta_j^{p^N}\in B_2$ for all j. Then, since taking the p^N-power is additive, we have $f^{p^N}=\sum_{j=1}^{\ell}\alpha_j^{p^N}\otimes\beta_j^{p^N}\in B_1\otimes B_2=B$, as required. □

Theorem 3.10. *Suppose $G_1\times_{\mathcal{M}}G_2$ is a polynomial gluing. Then*

(i) *$\mathbb{F}[V]^{G_1\times_{\mathcal{M}}G_2}$ is polynomial if and only if both $\mathbb{F}[W_1]^{G_1}$ and $\mathbb{F}[W_2]^{G_2}$ are polynomial;*
(ii) *$\mathbb{F}[V]^{G_1\times_{\mathcal{M}}G_2}$ is Cohen-Macaulay if and only if both $\mathbb{F}[W_1]^{G_1}$ and $\mathbb{F}[W_2]^{G_2}$ are Cohen-Macaulay;*
(iii) *$\mathbb{F}[V]^{G_1\times_{\mathcal{M}}G_2}$ is a complete intersection if and only if both $\mathbb{F}[W_1]^{G_1}$ and $\mathbb{F}[W_2]^{G_2}$ are complete intersections;*
(iv) *$\mathbb{F}[V]^{G_1\times_{\mathcal{M}}G_2}$ is unique factorisation domain if and only if both $\mathbb{F}[W_1]^{G_1}$ and $\mathbb{F}[W_2]^{G_2}$ are unique factorisation domains.*

Proof. Each of these properties is preserved by the gluing isomorphism. Therefore the results follow from Propositions 3.2, 3.5, 3.7 and 3.8. □

Theorem 3.11. *Suppose $G_1\times_{\mathcal{M}}G_2$ is a split polynomial gluing. Then*

$$\operatorname{depth}(\mathbb{F}[V]^{G_1\times_{\mathcal{M}}G_2})=\operatorname{depth}(\mathbb{F}[W_1]^{G_1})+\operatorname{depth}(\mathbb{F}[W_2]^{G_2}).$$

Furthermore, if $\psi:\mathbb{F}[W_1\oplus W_2]\to\mathbb{F}[V]^{\mathcal{M}}$ denotes an $\mathbb{F}$-algebra isomorphism which is an extension of the inclusion of $\mathbb{F}[W_2]$ into $\mathbb{F}[V]^{\mathcal{M}}$ and restricts to a G_1-equivariant isomorphism of $\mathbb{F}[W_1]$ to A which takes $\mathbb{F}[W_1]_+$ to A_+, then $\{\psi(f_1),\dots,\psi(f_s),h_1,\dots,h_k)\}$ is a geometric separating set for $\mathbb{F}[V]^{G_1\times_{\mathcal{M}}G_2}$ where $\{f_1,\dots,f_s\}$ is a homogeneous geometric separating set for $\mathbb{F}[W_1]^{G_1}$ and $\{h_1,\dots,h_k\}$ is a homogeneous geometric separating set for $\mathbb{F}[W_2]^{G_2}$.

Proof. While the map ψ is not degree preserving, it maps $\mathbb{F}[W_1\oplus W_2]_+^{G_1\times G_2}$ to $\mathbb{F}[V]_+^{G_1\times_{\mathcal{M}}G_2}$. Therefore, the results follow from Propositions 3.4 and 3.9. □

4. The Image of the Transfer

Suppose $G := G_1 \times_{\mathcal{M}} G_2$ is a polynomial gluing with gluing isomorphism $\psi : \mathbb{F}[V] \longrightarrow \mathbb{F}[V]^{\mathcal{M}}$ where $\mathbb{F}[V] = \mathbb{F}[W_1 \oplus W_2]$. We continue to identify $\mathbb{F}[W_i]$ with the appropriate subalgebra of $\mathbb{F}[V]$. Recall that the transfer map $\mathrm{Tr}^G : \mathbb{F}[V] \longrightarrow \mathbb{F}[V]^G$ is the morphism of $\mathbb{F}[V]^G$-modules defined by $\mathrm{Tr}^G(f) := \sum_{g\in G} f \cdot g$. The image of the transfer is an ideal in $\mathbb{F}[V]^G$. By a result of Bram Broer [6], since $\mathcal{M}$ is a p-group and $\mathbb{F}[V]^{\mathcal{M}}$ is polynomial, the image of $\mathrm{Tr}^{\mathcal{M}}$ is a principal ideal in $\mathbb{F}[V]^{\mathcal{M}}$. Let τ denote a generator for this ideal. There is a factorisation $\mathrm{Tr}^G = \mathrm{Tr}^{G_1\times G_2} \circ \mathrm{Tr}^{\mathcal{M}}$. Furthermore, since $\mathcal{M}$ is a normal subgroup of G, $G_1 \times G_2$ stabilises the image of $\mathrm{Tr}^{\mathcal{M}}$; to see this observe that $(\sum_{h\in\mathcal{M}} f\cdot h)g = \sum_{h\in\mathcal{M}}(f\cdot g)(g^{-1}hg)$. Therefore, for any $g \in G_1 \times G_2$, we see that $\tau \cdot g$ is a scalar multiple of τ. Thus we define a character of $G_1 \times G_2$ by $\tau \cdot g = \chi_\tau(g)\tau$. If $G_1 \times G_2$ is a p-group, then χ_τ is trivial and $\tau \in \mathbb{F}[V]^{G_1\times G_2}$.

Proposition 4.1. *Suppose $\{u_1, \ldots u_k\}$ is a generating set for the image of Tr^{G_1}, $\{v_1, \ldots, v_s\}$ is a generating set for the image of Tr^{G_2} and $\tau \in \mathbb{F}[V]^{G_1\times G_2}$ is a generator for the image of $\mathrm{Tr}^{\mathcal{M}}$. Then the image of Tr^G is the ideal generated by $\{\tau\psi(u_iv_j) \mid 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant s\}$.*

We give the proof of Proposition 4.1 after proving Lemma 4.3 below.

Example 4.2. In the context of Example 2.1, it follows from [41, Theorem 4.4] that $\tau = d_{n,n}(W_2^*)^m \in \mathbb{F}[V]^{G_1\times G_2}$. Hence, in this context, we can apply Proposition 4.1 to compute the image of Tr^G in terms of the image of Tr^{G_1} and the image of Tr^{G_2}.

For instance, let $\mathcal{U}(n,\mathbb{F}_p)$ be the group of $n\times n$ upper-triangular unipotent matrices over $\mathbb{F}_p$ and consider $G = \mathcal{U}(4,\mathbb{F}_p)$ and $G_1 = G_2 = \mathcal{U}(2,\mathbb{F}_p)$. Then $G = G_1 \times_{\mathcal{M}} G_2$ with $\mathcal{M} = \hom_{\mathbb{F}_p}(\mathbb{F}_p^2, \mathbb{F}_p^2)$. By [41, Theorem 4.4] we see that the image of Tr^G is the principal ideal generated by

$$\delta := d_{1,1}(x_1)\cdot d_{2,2}(x_1,x_2)\cdot d_{3,3}(x_1,x_2,y_1)$$

where $d_{i,j}$ denotes the Dickson invariants in the specified variables. In particular, $d_{1,1}(x_1) = x_1^{p-1}$, $d_{2,2}(x_1,x_2) = \det\begin{pmatrix} x_1 & x_1^p \\ x_2 & x_2^p\end{pmatrix}^{p-1}$ and

$$d_{3,3}(x_1,x_2,y_1) = \det\begin{pmatrix} x_1 & x_1^p & x_1^{p^2} \\ x_2 & x_2^p & x_2^{p^2} \\ y_1 & y_1^p & y_1^{p^2}\end{pmatrix}^{p-1}.$$

On the other hand, applying [41, Theorem 4.4] again we observe that the image of $\mathrm{Tr}^{\mathcal{M}}$ is the principal ideal generated by $\tau = d_{2,2}(x_1,x_2)^2$, which is $G_1\times G_2$-invariant. The image of Tr^{G_1} is generated by $u := d_{1,1}(x_1) = x_1^{p-1}$ and the image of Tr^{G_2} is generated by $v := d_{1,1}(y_1) = y_1^{p-1}$. Note that $\psi(u) = d_{1,1}(x_1)$ and

$$\psi(v) = \psi(y_1)^{p-1} = (y_1^{p^2} + d_{1,2}(x_1,x_2)y_1^p + d_{2,2}(x_1,x_2)y_1)^{p-1}.$$

Using Proposition 1.3(b) of [49], we see that $d_{3,3}(x_1,x_2,y_1) = -d_{2,2}(x_1,x_2)\psi(v)$. Therefore $\tau\cdot\psi(u)\cdot\psi(v) = d_{2,2}(x_1,x_2)^2\cdot d_{1,1}(x_1)\cdot\psi(v) = -d_{1,1}(x_1)\cdot d_{2,2}(x_1,x_2)\cdot d_{3,3}(x_1,x_2,y_1) = -\delta$.

Lemma 4.3. *Suppose $\{u_1,\dots u_k\}$ is a generating set for the image of Tr^{G_1} and $\{v_1,\dots,v_s\}$ is a generating set for the image of Tr^{G_2}. Then the image of $\mathrm{Tr}^{G_1\times G_2}$ is generated by $\{u_iv_j \mid 1\leqslant i\leqslant k, 1\leqslant j\leqslant s\}$.*

Proof. Let $I^{G_1\times G_2}$ denote the image of $\mathrm{Tr}^{G_1\times G_2}$, let I^{G_1} denote the image of Tr^{G_1} and let I^{G_2} denote the image of Tr^{G_2}. Observe that $\mathrm{Tr}^{G_1\times G_2}=\mathrm{Tr}^{\{1\}\times G_2}\circ\mathrm{Tr}^{G_1\times\{1\}}$. Choose $f_i\in\mathbb{F}[W_1]$ and $h_j\in\mathbb{F}[W_2]$ so that $\mathrm{Tr}^{G_1}(f_i)=u_i$ and $\mathrm{Tr}^{G_2}(h_j)=v_j$. Then

$$\mathrm{Tr}^{G_1\times G_2}(f_ih_j)=\mathrm{Tr}^{\{1\}\times G_2}\circ\mathrm{Tr}^{G_1\times\{1\}}(f_ih_j)=\mathrm{Tr}^{\{1\}\times G_2}(u_ih_j)=u_iv_j\in I^{G_1\times G_2}.$$

On the other hand, for any $f\in\mathbb{F}[V]$, write $f=\sum_{I,J}c_{I,J}x^Iy^J$ for exponent sequences I and J with $c_{I,J}\in\mathbb{F}$. Then

$$\mathrm{Tr}^{G_1\times G_2}(f)=\sum_{I,J}c_{I,J}\mathrm{Tr}^{G_1\times G_2}(x^Iy^J)=\sum_{I,J}c_{I,J}\mathrm{Tr}^{G_2}(x^I)\mathrm{Tr}^{G_1}(y^J)\in I^{G_1}I^{G_2}.$$

Hence $\mathrm{Tr}^{G_1\times G_2}(f)$ is in the ideal generated by $\{u_iv_j\mid 1\leqslant i\leqslant k, 1\leqslant j\leqslant s\}$. □

Proof of Proposition 4.1. Let I^G denote the image of Tr^G. Choose $f_i\in\mathbb{F}[W_1]$ and $h_j\in\mathbb{F}[W_2]$ so that $\mathrm{Tr}^{G_1}(f_i)=u_i$ and $\mathrm{Tr}^{G_2}(h_j)=v_k$. Choose $\alpha\in\mathbb{F}[V]$ so that $\mathrm{Tr}^{\mathcal{M}}(\alpha)=\tau$. Then

$$\begin{aligned}\mathrm{Tr}^G(\psi(f_ih_j)\alpha) &= \mathrm{Tr}^{G_1\times G_2}\circ\mathrm{Tr}^{\mathcal{M}}(\psi(f_ih_j)\alpha)=\mathrm{Tr}^{G_1\times G_2}(\psi(f_ih_j)\tau)\\ &= \tau\psi(\mathrm{Tr}^{G_1\times G_2}(f_ih_j))=\tau\psi(u_iv_j)\in I^G.\end{aligned}$$

Suppose $f=\mathrm{Tr}^G(f')$. Then $f=\mathrm{Tr}^{G_1\times G_2}(\mathrm{Tr}^{\mathcal{M}}(f'))=\mathrm{Tr}^{G_1\times G_2}(\widetilde{f}\tau)$ for some $\widetilde{f}\in\mathbb{F}[V]^{\mathcal{M}}$. Since $\tau\in\mathbb{F}[V]^{G_1\times G_2}$ and ψ is a $(G_1\times G_2)$-equivariant isomorphism, we get

$$f=\tau\mathrm{Tr}^{G_1\times G_2}(\psi(f''))=\tau\psi(\mathrm{Tr}^{G_1\times G_2}(f''))$$

for some $f''\in\mathbb{F}[V]$. Therefore f is in the ideal generated by $\{\tau\psi(u_iv_j)\mid 1\leqslant i\leqslant k, 1\leqslant j\leqslant s\}$. □

5. Maximal Parabolic Subgroups of Finite Symplectic Groups

For this section, we let V denote the defining representation of the symplectic group $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ with $q=p^r$. We choose an ordered basis $e_1,e_2,\dots,e_m,f_m,f_{m-1},\dots,f_1$ for V with dual basis $y_1,\dots,y_m,x_m,\dots,x_1$ so that the symplectic form is represented by the matrix

$$J=\begin{pmatrix}0 & Q\\ -Q & 0\end{pmatrix}\text{ with } Q=\begin{pmatrix}0&0&0&\cdots&0&0&1\\0&0&0&\cdots&0&1&0\\ &&&\vdots&&&\\0&1&0&\cdots&0&0&0\\1&0&0&\cdots&0&0&0\end{pmatrix}.$$

We identify $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ with the set of matrices $A\in\mathrm{GL}_{2m}(\mathbb{F}_q)$ satisfying $A^TJA=J$. Define $\xi_i:=y_m^{q^i}x_m-y_mx_m^{q^i}+\cdots+y_1^{q^i}x_1-y_1x_1^{q^i}$. Carlisle and Kropholler proved that $\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)}$ is the complete intersection generated by $\xi_1,\dots,\xi_{2m-1}$ and the Dickson invariants $d_{1,2m},\dots,d_{m,2m}$, see [2, Theorem 8.3.11]. Note that for $i>1$, ξ_i can be constructed by applying Steenrod operations to ξ_1. Therefore $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ is the subgroup of $\mathrm{GL}_{2m}(\mathbb{F}_q)$ which fixes ξ_1.

The symplectic group $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ has a BN-pair of type C_m (see [45] or [13, §1.11]). To construct a maximal parabolic subgroup for $\mathrm{Sp}_{2m}(\mathbb{F}_q)$, we remove one of the m generating reflections from the associated Weyl group, $N/(B\cap N)$, and lift the resulting subgroup to get a subgroup of N, say $\widetilde{N}$. The resulting parabolic subgroup is generated by B and $\widetilde{N}$. We label the vertices of the Coxeter graph so that the edge of weight 4 joins vertex $m-1$ and m. We will refer to the maximal parabolic subgroup constructed by removing the reflection corresponding to vertex k as a maximal parabolic of type k and denote this subgroup by $\mathcal{G}_k$. The primary goal of this section is to compute $\mathbb{F}_q[V]^{\mathcal{G}_k}$. In the following, we use Q_k to represent the $k\times k$ submatrix of Q constructed by removing the first $m-k$ columns and the last $m-k$ rows. Therefore

$$J=\begin{pmatrix} 0 & 0 & 0 & Q_k\\ 0 & 0 & Q_{m-k} & 0\\ 0 & -Q_{m-k} & 0 & 0\\ -Q_k & 0 & 0 & 0\end{pmatrix}.$$

Let P_k denote the subgroup of $\mathrm{GL}_{2m}(\mathbb{F}_q)$ consisting of matrices of the form

$$\begin{pmatrix} I_k & C_1 & C_2 & A\\ 0 & I_{m-k} & 0 & B_1\\ 0 & 0 & I_{m-k} & B_2\\ 0 & 0 & 0 & I_k\end{pmatrix}$$

subject to the relations $C_1=-Q_kB_2^TQ_{m-k}$, $C_2=Q_kB_1^TQ_{m-k}$ and $A^TQ_k-Q_kA=B_2^TQ_{m-k}B_1-B_1^TQ_{m-k}B_2$. Observe that P_k is a p-group and a subgroup of $\mathrm{Sp}_{2m}(\mathbb{F}_q)$.

Proposition 5.1. *The maximal parabolic subgroup $\mathcal{G}_k$ is isomorphic to*

$$\big(\mathrm{GL}_k(\mathbb{F}_q)\times \mathrm{Sp}_{2m-2k}(\mathbb{F}_q)\big)\ltimes P_k$$

with the action of $\mathrm{GL}_k(\mathbb{F}_q)\times\mathrm{Sp}_{2m-2k}(\mathbb{F}_q)$ on P_k given by mapping $(A,B)\in\mathrm{GL}_k(\mathbb{F}_q)\times\mathrm{Sp}_{2m-2k}(\mathbb{F}_q)$ to

$$\begin{pmatrix} A & 0 & 0\\ 0 & B & 0\\ 0 & 0 & Q_k(A^{-1})^TQ_k\end{pmatrix}\in \mathrm{Sp}_{2m}(\mathbb{F}_q).$$

Proof. We use the BN-pair for $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ described in Chapter 8 of Taylor's book [45]. The Borel subgroup B is the group of upper triangular symplectic matrices and N is the group of symplectic monomial matrices. The Weyl group is generated by reflections $w_1,\dots,w_m$. For $i<m$, w_i can be lifted to the element $n_i\in N$ which takes e_i to $-e_{i+1}$, e_{i+1} to e_i, f_i to $-f_{i+1}$, f_{i+1} to f_i and fixes the other basis vectors. The generator w_m lifts to $n_m\in N$ which takes e_m to $-f_m$, f_m to e_m and fixes the other basis elements. The maximal parabolic subgroup $\mathcal{G}_k$ is generated by B and $\{n_1,\dots,n_m\}\setminus\{n_k\}$.

Direct calculation verifies that P_k is precisely the set of symplectic matrices of the given block-form and that the embedding of $\mathrm{GL}_k(\mathbb{F}_q)\times\mathrm{Sp}_{2m-2k}(\mathbb{F}_q)$ gives the set of symplectic matrices of that corresponding block-form. A further explicit calculation verifies that the embedding of $\mathrm{GL}_k(\mathbb{F}_q)\times\mathrm{Sp}_{2m-2k}(\mathbb{F}_q)$ normalises P_k. The removal of vertex k from the Coxeter graph gives a graph of type $A_{k-1}\times C_{m-k}$. The removal of the generator n_k separates the basis vectors into three sets: $\{e_1,\dots,e_k\}$, $\{e_{k+1},\dots,e_m,f_m,\dots,f_{k+1}\}$, $\{f_k,\dots,f_1\}$. We are left with the usual BN-pair for $\mathrm{GL}_k(\mathbb{F}_q)$ on the span of $\{e_1,\dots,e_k\}$, the usual BN-pair for $\mathrm{Sp}_{2m-2k}(\mathbb{F}_q)$ on the span of

$\{e_{k+1},\dots,e_m,f_m,\dots,f_{k+1}\}$, the action of $\mathrm{GL}_k(\mathbb{F}_q)$ on $\{f_k,\dots,f_1\}$ determined by the symplectic condition, extended by the normal subgroup P_k to recover the Borel subgroup. □

Let U_k denote the span of $\{e_1,\dots,e_k\}$ and let $\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}$ denote the pointwise stabiliser subgroup.

Corollary 5.2. *The maximal parabolic subgroup $\mathcal{G}_k$ is isomorphic to*

$$\mathrm{GL}_k(\mathbb{F}_q)\ltimes \mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}$$

with the action of $\mathrm{GL}_k(\mathbb{F}_q)$ as described in Proposition 5.1.

Proof. With our chosen basis, the matrices representing elements of $\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}$ are of the form

$$\begin{pmatrix} I_k & A & B\\ 0 & C & D\\ 0 & E & F\end{pmatrix}$$

where $\{A,E\}\subset\mathbb{F}_q^{k\times(2m-2k)}$, $\{B,F\}\subset\mathbb{F}_q^{k\times k}$, $C\in\mathbb{F}_q^{(2m-2k)\times(2m-2k)}$ and $D\in\mathbb{F}_q^{(2m-2k)\times k}$. The symplectic condition forces $E=0$, $F=I_k$ and $C\in\mathrm{Sp}_{2m-2k}(\mathbb{F}_q)$. If we restrict to $C=I_{2m-2k}$ and apply the symplectic condition, we recover P_k. Since the action of $\mathrm{Sp}_{2m-2k}(\mathbb{F}_q)$ normalises P_k, we see that $\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}$ is isomorphic to $\mathrm{Sp}_{2m-2k}(\mathbb{F}_q)\ltimes P_k$, and the result follows from Proposition 5.1. □

Notation 5.3. *For the rest of this section, we use $\widetilde{d}_{i,\ell}$ to denote the i^{th} Dickson invariant in the first ℓ variables taken from the ordered list $x_1,\dots,x_m,y_m,\dots,y_1$. For example, $\widetilde{d}_{i,m}$ is the i^{th} Dickson invariant in $x_1,\dots,x_m$. Let W_k denote the span of $\{x_1,\dots,x_m,y_m,\dots,y_1\}\setminus\{y_1,\dots,y_k\}$ and define*

$$N_k(t):=\prod_{v\in W_k}(t+v)=\sum_{j=0}^{2m-k} t^{q^{2m-k-j}}\widetilde{d}_{j,2m-k}\in\mathbb{F}_q[V][t].$$

Note that $N_k(x_i)=0$ and $N_k(y_i)=0$ if $i>k$.

Theorem 5.4. *The ring of invariants $\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}}$ is generated by $x_1,\dots,x_k$, $\xi_1,\dots,\xi_{2m-1}$, $N_k(y_1),\dots,N_k(y_k)$, and $\widetilde{d}_{i,2m-k}$ for $i=1,2,\dots,2m-k$. Furthermore, this ring is a complete intersection and, therefore, Cohen-Macaulay.*

Proof. Since $\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)}$ is Cohen-Macaulay and a complete intersection, it follows from [32, Theorem B] that $\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}}$ is Cohen-Macaulay and a complete intersection.

We will use Kemper's algorithm based on [32, Theorem 2.7] to compute a generating set for $\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}}$. Given a G-invariant polynomial, Kemper's algorithm produces a set of G_U-invariant polynomials. The algorithm uses only the input polynomials and the subspace U. Applying the algorithm to $\{d_{i,2m}\mid i=1,\dots,2m\}$ produces a generating set for $\mathbb{F}_q[V]^{\mathrm{GL}_{2m}(\mathbb{F}_q)_{U_k}}$. By comparing the order of $\mathrm{GL}_{2m}(\mathbb{F}_q)_{U_k}$ with product of the degrees, we see that $\mathbb{F}_q[V]^{\mathrm{GL}_{2m}(\mathbb{F}_q)_{U_k}}$ is the polynomial algebra generated by $\{N_k(y_i),\widetilde{d}_{j,2m-k}\mid i=1,\dots,k; j=1,2,\dots,2m-k\}$. Applying the algorithm to the ξ_j produces $\{x_1,\dots,x_k,\xi_1,\dots,\xi_{2m-1}\}$. We do not claim that this is a minimal generating set. □

Corollary 5.5. *For $k<m$, the set*

$$\mathcal{H}_k := \{x_1,\ldots,x_k,N_k(y_1),\ldots,N_k(y_k),\widetilde{d}_{1,2m-k},\ldots,\widetilde{d}_{2m-2k,2m-k}\}$$

is a homogeneous system of parameters.

Proof. It is sufficient to show that the ideal generated by $\mathcal{H}_k$ in $\mathbb{F}_q[V]$ is zero dimensional. Since $\mathbb{F}_q[V]$ is integral over $\mathbb{F}_q[V]^{\mathrm{GL}_{2m}(\mathbb{F}_q)_{U_k}}$ and, from the proof of Theorem 5.4, $\mathbb{F}_q[V]^{\mathrm{GL}_{2m}(\mathbb{F}_q)_{U_k}}$ is generated by $\mathcal{S} := \{N_k(y_i),\widetilde{d}_{j,2m-k} \mid i=1,\ldots,k; j=1,2,\ldots,2m-k\}$, we see that the ideal generated by $\mathcal{S}$ in $\mathbb{F}_q[V]$ is zero dimensional. Thus the ideal generated by $\mathcal{H}_k\cup\mathcal{S}$ is zero dimensional. However, it follows from the definition of $\widetilde{d}_{j,2m-k}$, see Notation 5.3, that for $j>2m-2k$, $\widetilde{d}_{j,2m-k}$ is in the ideal in $\mathbb{F}_q[V]$ generated by $\{x_1,\ldots,x_k\}$ (one way to see this is to compute the lead term of $\widetilde{d}_{j,2m-k}$ using a grevlex order with $y_1>y_2>\cdots>y_m>x_m>\cdots>x_1$). Therefore the ideal generated by $\mathcal{H}_k$ coincides with ideal generated by $\mathcal{H}_k\cup\mathcal{S}$ and so is zero dimensional. □

For $k=m$, we have

$$\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_m} = P_m = \left\{ \begin{pmatrix} I_m & A \\ 0 & I_m \end{pmatrix} \mid A = Q_m A^T Q_m \right\},$$

which is an elementary abelian p-group of order $q^{(m^2+m)/2}$.

Corollary 5.6. *The ring of invariants $\mathbb{F}_q[V]^{P_m}$ is the complete intersection generated by $x_1,\ldots,x_m$, $\xi_1,\ldots,\xi_{m-1}$, and $N_m(y_1),\ldots,N_m(y_m)$.*

Proof. By Theorem 5.4, $\mathbb{F}_q[V]^{P_m}$ is the complete intersection generated by $x_1,\ldots,x_m$, $\xi_1,\ldots,\xi_{2m-1}$, $N_m(y_1),\ldots,N_m(y_m)$, and $\widetilde{d}_{i,m}$ for $i=1,2,\ldots,m$. Since $\widetilde{d}_{i,m}\in\mathbb{F}_q[x_1,\ldots,x_m]$, these generators are redundant. An explicit calculation gives

$$\xi_m = \left(\sum_{j=1}^{m} x_j N_m(y_j)\right) - \left(\sum_{i=1}^{m-1} \xi_i \widetilde{d}_{m-i,m}\right).$$

Thus ξ_m is also a redundant generator. Similar relations can be constructed to eliminate ξ_j for $j>m$; one approach to doing this is to apply Steenrod operations to the relation for ξ_m. □

Remark 5.7. *The above result is essentially Theorem 5.4.8 of* [31].

The action of $\mathrm{GL}_k(\mathbb{F}_q)$ on V is the direct sum of the action as a vector and a covector on $\mathrm{Span}(e_1,\ldots,e_k)$ and $\mathrm{Span}(f_k,\ldots,f_1)$, in the sense of [3] or [14], and a trivial action on the remaining basis vectors. This means that we can use the results of [14] to compute generators for $\mathbb{F}_q[V]^{\mathrm{GL}_k(\mathbb{F}_q)}$. Our approach to the construction of generators for $\mathbb{F}_q[V]^{\mathcal{G}_k} = \left(\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}}\right)^{\mathrm{GL}_k(\mathbb{F}_q)}$ is to apply a "diagonal gluing" to the invariants of $\mathbb{F}_q[V]^{\mathrm{GL}_k(\mathbb{F}_q)}$. It is useful to start with the case $k=m$, which is closely related to the Sylow p-subgroup of $\mathrm{Sp}_{2m}(\mathbb{F}_q)$. We use USp_{2m} to denote the group of upper-triangular unipotent symplectic matrices, a Sylow p-subgroup for $\mathrm{Sp}_{2m}(\mathbb{F}_q)$. In the following theorem, we use $N(x_i)$ to denote the product of x_i+v as v runs over the span of $x_1,\ldots,x_{i-1}$.

Theorem 5.8. *The ring of invariants $\mathbb{F}_q[V]^{USp_{2m}}$ is the complete intersection generated by $\xi_1,\ldots,\xi_{2m-2}$, $N(x_1),\ldots,N(x_m)$ and $N_m(y_m),N_{m-1}(y_{m-1}),\ldots,N_1(y_1)$.*

Proof. Let H denote the upper triangular unipotent subgroup of $\mathrm{GL}_m(\mathbb{F}_q)$. Observe that USp_{2m} is isomorphic to $H\ltimes P_m$ with the action given by restricting the action of $\mathrm{GL}_m(\mathbb{F}_q)$ given in Proposition 5.1 to the subgroup H. Furthermore, this restriction gives a vector/covector action of H on V. Bonnafé and Kemper [3] showed that $\mathbb{F}_q[V]^H$ is a complete intersection generated by $N_H(y_i)$, $N(x_i)$ for $i=1,\ldots,m$ and additional invariants which they denote by u_j for $j=2-m,\ldots,m-2$. Let S denote the algebra generated by $x_1,\ldots,x_m$, $N_m(y_1),\ldots,N_m(y_m)$. Referring to Corollary 5.6, we see that $\{x_1,\ldots,x_m,N_m(y_1),\ldots,N_m(y_m)\}$ is a homogeneous system of parameters for $\mathbb{F}_q[V]^{P_m}$ and that $\mathbb{F}_q[V]^{P_m}$ is a finitely generated free S-module. Furthermore, the S-module generators can be taken to be monomials in the ξ_i. Therefore, since the ξ_i are H-invariant, adjoining the ξ_i to a generating set for S^H gives a generating set for $\mathbb{F}_q[V]^{USp_{2m}}=(\mathbb{F}_q[V]^{P_m})^H$. The algebra homomorphism from $\mathbb{F}_q[V]$ to S which fixes x_i and maps y_i to $N_m(y_i)$ is an H-equivariant isomorphism of algebras. Thus we can construct generators for S^H by substituting $N_m(y_i)$ for y_i in the generators for $\mathbb{F}_q[V]^H$. Note that $N_H(N_m(y_i))=N_i(y_i)$. Substituting $N_m(y_i)$ for y_i in u_j gives

$$\begin{aligned}
\widetilde{u}_0 &= \sum_{i=1}^{m} x_iN_m(y_i)\\
\widetilde{u}_j &= \sum_{i=1}^{m} x_i^{q^j}N_m(y_i)\\
\widetilde{u}_{-j} &= \sum_{i=1}^{m} x_iN_m(y_i)^{q^j}
\end{aligned}$$

for $j=1,\ldots,m$. These invariants can be written in terms of the ξ_i and the Dickson invariants in $x_1,\ldots,x_m$:

$$\begin{aligned}
\widetilde{u}_0 &= \sum_{i=1}^{m} \xi_i\widetilde{d}_{m-i,m}\\
\widetilde{u}_j &= \sum_{i=0}^{m-j-1} \xi_{m-i-j}^{q^j}\widetilde{d}_{i,m}-\sum_{i=m-j+1}^{m}\xi_{i+j-m}^{q^{m-i}}\widetilde{d}_{i,m}\\
\widetilde{u}_{-j} &= \sum_{i=0}^{m} \xi_{m+j-i}\widetilde{d}_{i,m}^{q^j}
\end{aligned}$$

for $j=1,\ldots,m$; see Lemma 5.10 below for details. Since the Dickson invariants $\widetilde{d_{i,m}}$ can be written as polynomials in $N(x_1),\ldots,N(x_m)$, the $\widetilde{u}_{2-m},\ldots,\widetilde{u}_{m-2}$ are redundant, as long as we include $\xi_1,\ldots,\xi_{2m-2}$ in our generating set (the invariants $\widetilde{u}_{-m}$, $\widetilde{u}_{1-m}$, $\widetilde{u}_{m-1}$ and $\widetilde{u}_m$ require ξ_{2m-1} to rewrite). Define $\mathcal{H}:=\{N(x_1),\ldots,N(x_m),N_m(y_m),\ldots,N_1(y_1)\}$ and let A denote the algebra generated $\mathcal{H}$. Note that A is the ring of invariants for the upper triangular unipotent subgroup of $\mathrm{GL}_{2m}(\mathbb{F}_q)$, each $d_{i,2m}\in A$ and $\mathcal{H}$ is a homogeneous system of parameters. Furthermore, $\mathbb{F}_q[V]^{USp_{2m}}$ is a finite A-module and the module generators can be taken to be monomials in the ξ_i. The Bonnafé-Kemper relations [3, page 105] translate into relations which allow us to rewrite powers of the ξ_i in terms of elements of A and powers of ξ_j of lower total degree. In particular, the translation of relation R_{m-j} rewrites $\xi_{2j+1}^{q^{m-j-1}}$ for $j = 0,\dots,m-2$, the translation of R^-_{m-j} rewrites $\xi_{2j}^{q^{m-j}}$ for $j = 1,\dots,m-2$ and the R^+_1 rewrites ξ_{2m-2}^{q}; compare with [3, Theorem 2.4]. Using these relations, we see that $\{\xi_1^{a_1}\cdots\xi_{2m-2}^{a_{2m-2}} \mid a_i < q^{\lceil m-i/2\rceil}\}$ is a set of A-module generators for $\mathbb{F}_q[V]^{USp_{2m}}$. Observe that the product of the degrees of the elements in $\mathcal{H}$ is $q^{m(2m-1)}$, the order of USp_{2m} is q^{m^2} and the number of module generators is $q^{m(m-1)}$. Therefore $\mathbb{F}_q[V]^{USp_{2m}}$ is Cohen-Macaulay, see [19, Theorem 3.7.1] or [12, Corollary 3.1.4]. Since $\mathbb{F}_q[V]^{USp_{2m}}$ is Cohen-Macaulay, the elements of $\mathcal{H}$ form a regular sequence. Let $\mathfrak{h}$ denote the ideal in $\mathbb{F}_q[V]^{USp_{2m}}$ generated by the elements of $\mathcal{H}$. The ring $\mathbb{F}_q[V]^{USp_{2m}}$ is a complete intersection if and only if $\mathbb{F}_q[V]^{USp_{2m}}/\mathfrak{h}$ is a complete intersection (this follows from [7, Theorem 2.3.4]). However, using the rewriting relations $\mathbb{F}_q[V]^{USp_{2m}}/\mathfrak{h}$ is isomorphic to $\mathbb{F}_q[\xi_1,\dots,\xi_{2m-2}]/\langle\xi_1^{q^{m-1}},\xi_2^{q^{m-1}},\xi_3^{q^{m-2}},\dots,\xi_{2m-2}^{q^2}\rangle$, which is a complete intersection. □

The above is consistent with the calculation of $\mathbb{F}_q[V]^{USp_4}$ in [25]. Since USp_{2m} is a Sylow p-subgroup for all of the standard parabolic subgroups of $\mathrm{Sp}_{2m}(\mathbb{F}_q)$, the following is a consequence of [10].

Corollary 5.9. *If G is any parabolic subgroup of $\mathrm{Sp}_{2m}(\mathbb{F}_q)$, then $\mathbb{F}_q[V]^G$ is Cohen-Macaulay.*

In the following, it is useful to take $\xi_0 = 0$ and $\xi_{-i}^{q^j} = -\xi_i^{q^{j-i}}$ for $j \geq i > 0$.

Lemma 5.10. *For $k \leq m$ and $i, j \geq 0$, we have*

$$\sum_{s=1}^{k} x_s^{q^i} N_k(y_s)^{q^j} = \sum_{\ell=0}^{2m-k} \xi_{2m-k-i-\ell+j}^{q^i}\widetilde{d}_{\ell,2m-k}^{q^j}.$$

Proof. We expand the right hand side giving

$$\begin{aligned}
\sum_{\ell=0}^{2m-k} \xi_{2m-k-i-\ell+j}^{q^i}\widetilde{d}_{\ell,2m-k}^{q^j} &= \sum_{\ell=0}^{2m-k}\left(\sum_{s=1}^{m}\left(x_s y_s^{q^{2m-k-i-\ell+j}} - y_s x_s^{q^{2m-k-i-\ell+j}}\right)\right)^{q^i}\widetilde{d}_{\ell,2m-k}^{q^j} \\
&= \sum_{\ell=0}^{2m-k}\left(\sum_{s=1}^{m} x_s^{q^i} y_s^{q^{2m-k-\ell+j}} - y_s^{q^i} x_s^{q^{2m-k-\ell+j}}\right)\widetilde{d}_{\ell,2m-k}^{q^j} \\
&= \sum_{s=1}^{m}(x_s^{q^i}N_k(y_s)^{q^j} - y_s^{q^i}N_k(x_s)^{q^j}) \\
&= \sum_{s=1}^{k} x_s^{q^i}N_k(y_s)^{q^j},
\end{aligned}$$

as required. □

Theorem 5.11. *The ring of invariants $\mathbb{F}_q[V]^{\mathcal{G}_k}$ is generated by $\xi_1,\dots,\xi_{2m-1}$, $d_{1,2m},\dots,d_{k,2m}$, $\widetilde{d}_{1,k},\dots,\widetilde{d}_{k,k}$ and $\widetilde{d}_{1,2m-k},\dots,\widetilde{d}_{m-k,2m-k}$.*

Proof. Using Corollary 5.2, we have $\mathbb{F}_q[V]^{\mathcal{G}_k} = (\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}})^{\mathrm{GL}_k(\mathbb{F}_q)}$. A generating set for $\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}}$ was given in Theorem 5.4. Let S denote the subalgebra of $\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}}$ generated by $\{x_1,\ldots,x_k,N_k(y_1),\ldots,N_k(y_k)\}$ and let W denote the span of $\{e_1,\ldots,e_k,f_k,\ldots,f_1\}$ so that $\mathbb{F}_q[W]=\mathbb{F}_q[y_1,\ldots,y_k,x_k,\ldots,x_1]$. The action of $\mathrm{GL}_k(\mathbb{F}_q)$ on W is a vector/covector action and the algebra homomorphism from $\mathbb{F}_q[W]$ to S which takes y_i to $N_k(y_i)$ and fixes x_i is $\mathrm{GL}_k(\mathbb{F}_q)$-equivariant. Therefore, we can construct generators for $S^{\mathrm{GL}_k(\mathbb{F}_q)}$ by substituting $N_k(y_i)$ for y_i in the generators of $\mathbb{F}_q[W]^{\mathrm{GL}_k(\mathbb{F}_q)}$ given in [14]. Let A denote the subalgebra of $\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}}$ generated by $\{\widetilde{d}_{1,2m-k},\ldots,\widetilde{d}_{2m-2k,2m-k}\}$. It follows from Corollary 5.5 that $\{x_1,\ldots,x_k,N_k(y_1),\ldots,N_k(y_k),\widetilde{d}_{1,2m-k},\ldots,\widetilde{d}_{2m-2k,2m-k}\}$ is a homogeneous system of parameters for $\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}}$ and, therefore, $\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}}$ is a free finitely generated $(S\otimes A)$-module. Furthermore, the $(S\otimes A)$-module generators can be taken to be monomials in the ξ_i and $\widetilde{d}_{j,2m-k}$ for $j>2m-2k$. Since the $\widetilde{d}_{j,2m-k}$ are $\mathrm{GL}_k(\mathbb{F}_q)$-invariant, we have $(S\otimes A)^{\mathrm{GL}_k(\mathbb{F}_q)} = S^{\mathrm{GL}_k(\mathbb{F}_q)}\otimes A$. Since the ξ_i are also $\mathrm{GL}_k(\mathbb{F}_q)$-invariant, to construct a generating set for $\mathbb{F}_q[V]^{\mathcal{G}_k} = (\mathbb{F}_q[V]^{\mathrm{Sp}_{2m}(\mathbb{F}_q)_{U_k}})^{\mathrm{GL}_k(\mathbb{F}_q)}$, we need only adjoin $\xi_1,\ldots,\xi_{2m-1}$ and $\widetilde{d}_{i,2m-k}$ for $i=1,2,\ldots,2m-k$ to a generating set for $S^{\mathrm{GL}_k(\mathbb{F}_q)}$. It was shown in [14] that $\mathbb{F}_q[W]^{\mathrm{GL}_k(\mathbb{F}_q)}$ is generated by Dickson invariants in the x_i, Dickson invariants in the y_i and the Bonnafé-Kemper invariants u_j for $j=1-k,\ldots,k-1$. Let $\bar{d}_{\ell,k}$ denote the polynomial constructed by substituting $N_k(y_i)$ for y_i in the Dickson invariants in the y_i. Then $S^{\mathrm{GL}_k(\mathbb{F}_q)}$ is generated by $\{\bar{d}_{i,k},\widetilde{d}_{i,k},\widetilde{u}_j \mid i=1,\ldots,k;\ j=1-k,\ldots,k-1\}$ with $\widetilde{u}_0=\sum_{i=1}^k x_iN_k(y_i)$, $\widetilde{u}_\ell=\sum_{i=1}^k x_i^{q^\ell}N_k(y_i)$ and $\widetilde{u}_{-\ell}=\sum_{i=1}^m x_iN_k(y_i)^{q^\ell}$ for $\ell>0$.

For the remainder of this proof we use H to denote the parabolic subgroup of $\mathrm{GL}_{2m}(\mathbb{F}_q)$ associated to the partition $(k,2m-2k,k)$ and observe that $\mathcal{G}_k$ is a subgroup of H. We claim that $\mathbb{F}_q[V]^H$ is generated by

$$\{\bar{d}_{i,k},\widetilde{d}_{i,k},\widetilde{d}_{j,2m-k} \mid i=1,\ldots,k;\ j=1,\ldots,2m-2k\}.$$

First note that we have $2m$ elements in this set and the product of the degrees equals the order of H. Thus we only need to show the set is a homogenerous system of parameters. We could do this directly but prefer to take an alternate approach. The set $\{\widetilde{d}_{i,k},\widetilde{d}_{j,2m-k} \mid i=1,\ldots,k;\ j=1,\ldots,2m-2k\}$ is the generating set for the parabolic subgroup of $\mathrm{GL}_{2m-k}(\mathbb{F}_q)$ associated to the partition $(2m-2k,k)$ constructed by Kuhn and Mitchell [34, Theorem 2.2]. We can then form H as the polynomial gluing of this group with $\mathrm{GL}_k(\mathbb{F}_q)$. This shows that the given set is a generating set for $\mathbb{F}_q[V]^H$ but this also means that we can replace the $\bar{d}_{i,k}$ with $d_{i,2m}$ in our generating set for $\mathbb{F}_q[V]^{\mathcal{G}_k}$ by using the Kuhn-Mitchell generators for $\mathbb{F}_q[V]^H$.

To complete the proof we need to show that $\widetilde{u}_j$ and $\widetilde{d}_{i,2m-k}$ for $i>m-k$ are redundant. To show that the $\widetilde{u}_j$ are redundant we use Lemma 5.10: taking $i=0$ and $j=0$ shows that $\widetilde{u}_0$ is redundant, taking $i=0$ and $j>0$ shows that $\widetilde{u}_{-j}$ is redundant, and taking $i>0$ and $j=0$ shows that $\widetilde{u}_i$ is redundant.

Define $R:=\mathbb{F}_q[y_{k+1},\ldots,y_m,x_m,\ldots,x_1]$, let $\overline{W}_k$ denote the span of $\{x_1,\ldots,x_k\}$ and define $\overline{N}_k(t):=\prod_{u\in\overline{W}_k}(t-u)$. Let $\overline{d}_{i,2m-2k}$ denote the element of R constructed by substituting $\overline{N}_k(y_j)$ for y_j and $\overline{N}_k(x_j)$ for x_j into the i^{th} Dickson invariant in the variables $\{y_{k+1},\ldots,y_m,x_m,\ldots,x_{k+1}\}$.

Define

$$\overline{\xi}_i := \sum_{j=k+1}^{m} \overline{N}_k(x_j)\overline{N}_k(y_j)^{q^i} - \overline{N}_k(y_j)\overline{N}_k(x_j)^{q^i}.$$

The action of $\mathcal{G}_k$ on R is a polynomial gluing of $\mathrm{Sp}_{2m-2k}(\mathbb{F}_q)$ and $\mathrm{GL}_k(\mathbb{F}_q)$. Using the Carlisle-Kropholler generators for the symplectic invariants, the gluing construction and Theorem 3.10, we see that the ring $R^{\mathcal{G}_k}$ is the complete intersection generated by $\widetilde{d}_{i,k}$ for $i=1,\ldots,k$, $\overline{d}_{i,2m-2k}$ for $i=1,\ldots,m-k$ and $\overline{\xi}_i$ for $i=1,\ldots,2m-2k-1$. Using the expansion of $\overline{N}_k(t)$ in terms of the Dickson invariants and the fact that $\overline{N}_k(x_i)=0$ for $i\le k$ gives

$$\begin{aligned}
\overline{\xi}_i &= \sum_{j=1}^{m}\left(\left(\sum_{\ell=0}^{k} x_j^{q^{k-\ell}}\widetilde{d}_{\ell,k}\right)\left(\sum_{s=0}^{k} y_j^{q^{k-s}}\widetilde{d}_{s,k}\right)^{q^i} - \left(\sum_{\ell=0}^{k} y_j^{q^{k-\ell}}\widetilde{d}_{\ell,k}\right)\left(\sum_{s=0}^{k} x_j^{q^{k-s}}\widetilde{d}_{s,k}\right)^{q^i}\right)\\
&= \sum_{j=1}^{m}\sum_{\ell=0}^{k}\sum_{s=0}^{k}\left(x_j^{q^{k-\ell}}y_j^{q^{k-s+i}} - x_j^{q^{k-s+i}}y_j^{q^{k-\ell}}\right)\widetilde{d}_{\ell,k}\widetilde{d}_{s,k}^{q^i}
\end{aligned}$$

which gives

$$\overline{\xi}_i = \sum_{\ell>s-i}\xi_{\ell-s+i}^{q^{k-\ell}}\widetilde{d}_{\ell,k}\widetilde{d}_{s,k}^{q^i} + \sum_{\ell<s-i}\xi_{s-\ell-i}^{q^{k-s+i}}\widetilde{d}_{\ell,k}\widetilde{d}_{s,k}^{q^i}. \tag{2}$$

Therefore the $\overline{\xi}_i$ lie in the algebra generated by the ξ_i and the $\widetilde{d}_{i,k}$. Since $\widetilde{d}_{i,2m-k}\in R^{\mathcal{G}_k}$, these invariants can be re-written as polynomials in $\widetilde{d}_{i,k}$ for $i=1,\ldots,k$, $\overline{d}_{i,2m-2k}$ for $i=1,\ldots,m-k$ and ξ_i for $i=1,\ldots,2m-2k-1$. Using the Kuhn-Mitchel generators for the parabolic subgroup of $\mathrm{GL}_{2m-k}(\mathbb{F}_q)$ associated to the partition $(2m-2k,k)$ and comparing degrees, $\overline{d}_{i,2m-2k}$ for $i=1,\ldots,m-k$ can be be rewritten using $\widetilde{d}_{i,2m-k}$ for $i=1,\ldots,m-k$ and $\widetilde{d}_{i,k}$ for $i=1,\ldots,k$. □

Corollary 5.12. *The ring $\mathbb{F}_q[V]^{\mathcal{G}_1}$ is a complete intersection.*

Proof. By Theorem 5.11, $\mathbb{F}_q[V]^{\mathcal{G}_1}$ is generated by $\xi_1,\ldots,\xi_{2m-1}$, $\widetilde{d}_{1,2m-1},\ldots,\widetilde{d}_{m-1,2m-1}$, $d_{1,2m}$ and $\widetilde{d}_{1,1}=x_1^{q-1}$. Define

$$\mathcal{H} := \{x_1^{q-1},\xi_1,\ldots,\xi_{m-1},\widetilde{d}_{1,2m-1},\ldots,\widetilde{d}_{m-1,2m-1},d_{1,2m}\}$$

and let $\mathfrak{h}$ denote the ideal in $\mathbb{F}_q[V]^{\mathcal{G}_1}$ generated by $\mathcal{H}$. We will show that $\xi_{2m-1}^{q-1}\in\mathfrak{h}$ and $\xi_{2m-j}^{q^{j-1}}\in\mathfrak{h}$ for $j=2,\ldots,m$. From this it follows that $\mathcal{H}$ is a homogeneous system of parameters and that the resulting module generators are monomials in $\xi_m,\ldots,\xi_{2m-1}$. Since the order of the group is $q^{m^2}(q-1)\prod_{i=1}^{m-1}(q^{2i}-1)$, the product of the degrees of the elements of $\mathcal{H}$ is

$$q^{(3m-1)m/2}(q-1)^2\prod_{i=1}^{m-1}(q^{2i}-1),$$

and the ring is Cohen-Macaulay, we need $(q-1)q^{m(m-1)/2}$ module generators. Therefore, there are no further relations and $\mathbb{F}_q[V]^{\mathcal{G}_1}$ is a complete intersection.

Lemma 5.10 with $k=1$, $i=0$ and $j=0$ gives

$$x_1N_1(y_1) = \sum_{\ell=0}^{2m-1}\xi_{2m-1-\ell}\widetilde{d}_{\ell,2m-1} = \xi_{2m-1} + \sum_{\ell=1}^{m-1}\xi_{2m-1-\ell}\widetilde{d}_{\ell,2m-1} + \sum_{s=1}^{m-1}\xi_s\widetilde{d}_{2m-s-1,2m-1}.$$

Therefore $x_1N_1(y_1) - \xi_{2m-1} \in \mathfrak{h}$. Furthermore $d_{1,2m} = \widetilde{d}^q_{1,2m-1} - N_1(y_1)^{q-1}$. Hence $\xi^{q-1}_{2m-1} \equiv_{\mathfrak{h}} (x_1N_1(y_1))^{q-1} \in \mathfrak{h}$.

Equation 2 above with $k=1$ and $i \in \{m,\dots,2m-2\}$ gives $\overline{\xi}_i = \xi_i^q - \xi_{i+1}x_1^{q-1} - \xi_{i-1}x_1^{(q-1)q^i} + \xi_i x_1^{(q-1)(q^i+1)}$. Therefore $\xi_i^q \equiv_{\mathfrak{h}} \overline{\xi}_i$. Using the notation of the proof of Theorem 5.11 with $k=1$, $R^{\mathcal{G}_1}$ is a complete intersection with the relations coming through the gluing isomorphism from the Carlisle-Kropholler relations for $\mathrm{Sp}_{2m-2k}(\mathbb{F}_q)$. Define

$$\overline{\mathcal{H}} := \{x_1^{q-1}, \overline{\xi_1}, \dots, \overline{\xi}_{m-2}, \overline{d}_{1,2m-2}, \dots, \overline{d}_{m-2,2m-2}\}.$$

an let $\overline{\mathfrak{h}}$ denote the ideal in $R^{\mathcal{G}_1}$ generated by $\overline{H}$. Then $\overline{\mathfrak{h}} \subseteq \mathfrak{h} \cap R^{\mathcal{G}_1}$. For $i = m,\dots,2m-2$, we can use the Carlisle-Kropholler relations to show that $\overline{\xi}_i^{q^{2m-2-i}} \in \overline{\mathfrak{h}} \subset \mathfrak{h}$. Therefore $\xi_{2m-j}^{q^{j-1}} \equiv_{\mathfrak{h}} \overline{\xi}_{2m-j}^{q^{j-2}} \in \mathfrak{h}$ for $j = 2,\dots,m$, as required. □

Example 5.13. *Consider the type 2 parabolic for $m=2$. By Theorem 5.11, the invariants are generated by $\xi_1, \xi_2, \xi_3, \widetilde{d}_{1,2}, \widetilde{d}_{2,2}, d_{1,4}, d_{2,4}$ which have degrees $q+1, q^2+1, q^3+1, q^2-q, q^2-1, q^4-q^3, q^4-q^2$. There are two natural choices for a homogeneous system of parameters:*

$$\mathcal{H}_1 := \{\xi_1, \xi_2, d_{1,4}, d_{2,4}\},$$
$$\mathcal{H}_2 := \{\widetilde{d}_{1,2}, \widetilde{d}_{2,2}, d_{1,4}, d_{2,4}\}.$$

For the first system of parameters, we need $q(q^2+1)(q+1)$ module generators and for the second we need $q^2(q-1)^2(q+1)$ generators. Using the second system of paprameters, the module generators can be taken to be monomials in the ξ_i. Using the Carlisle-Kropholler relation for $\mathbb{F}_q[V]^{\mathrm{Sp}_4(\mathbb{F}_q)}$, we have

$$\xi_3^q + d_{1,4}\xi_2^q + d_{2,4}\xi_1^q = \xi_1(\xi_1^{q^2+1} - \xi_2^{q+1} + \xi_1^q\xi_3)^{q-1} = \xi_1 d_{4,4}.$$

This is essentially the translation of the relation T_1^* from [14] into this context. Let $\mathfrak{h}$ denote the ideal in $\mathbb{F}_q[V]^{G_2}$ generated by $\mathcal{H}_2$. Since $d_{4,4} \in \mathbb{F}_q[\widetilde{d}_{1,2}, \widetilde{d}_{2,2}, d_{1,4}, d_{2,4}]$, the relation above shows $\xi_3^q \in \mathfrak{h}$. The translation of the relation T_1 from [14] shows that $\xi_1^{q^2} \in \mathfrak{h}$. The translation of the relation T_{00} shows that $(\xi_2^{q+1} - \xi_1^q\xi_3)^{q-1} \in \mathfrak{h}$. Conjecture 16 from [14] suggests two additional relations, $T_{1,0}$ and $T_{0,1}$. These relations would translate to relations in degrees q^4-q and $q^4-q^3+q^2-1$. We can use the the relation T_{00} to show that $\xi_2^{q^2-1}$ is congruent modulo $\mathfrak{h}$ to a linear combination of smaller monomials in the ξ_i (using the grevlex order with $\xi_3 > \xi_2 > \xi_1$). We conjecture that the relation in degree q^4-q can be used to show that $\xi_3\xi_2^{q^2-q-1}$ is also congruent to a linear combination of smaller monomials and that the relation in degree $q^4-q^3+q^2-1$ shows that $\xi_1^q\xi_2^{q^2-q-1}$ is congruent to a linear combination of smaller monomials. If true, by counting independent monomials, these conjectures would show that $\mathbb{F}_q[V]^{\mathcal{G}_2}$ for $m=2$ is not a complete intersection. We have confirmed these conjectures for $q=3$ and $q=5$ using Magma [4].

Remark 5.14. *To an arbitrary parabolic subgroup of $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ we can associate a partition of $2m$: $\lambda = (r_1,\dots,r_\ell, 2m-2k, r_\ell,\dots,r_1)$ with $r_1+\cdots+r_\ell = k$. Let G_λ denote the parabolic subgroup of $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ associated to λ and let GL_λ denote the parabolic subgroup of $\mathrm{GL}_{2m}(\mathbb{F}_q)$*

associated to the partition λ. The ring $\mathbb{F}_q[V]^{\mathrm{GL}_\lambda}$ is a polynomial algebra generated by a collection of Dickson invariants. Since $\mathbb{F}_q[V]^{G_\lambda}$ is Cohen-Macaulay, it is a free $\mathbb{F}_q[V]^{\mathrm{GL}_\lambda}$-module. We conjecture that the module generators can be taken to be monomials in the ξ_i and that $\mathbb{F}_q[V]^{G_\lambda}$ is generated by $\xi_1,\ldots,\xi_{2m-1}$ and a subset of a generating set for $\mathbb{F}_q[V]^{\mathrm{GL}_\lambda}$.

6. Singular Finite Classical Groups

In this section $\mathbb{F}$ will denote a finite field and β an alternating, symmetric or Hermitian form on V. We suggest [45] or [46] for background material on finite classical groups. Define $G_\beta := \{g \in \mathrm{GL}(V) \mid \beta(gv,gw) = \beta(v,w)\,\forall v,w,\in V\}$. Take W_1 to be the radical of β, in other words, $W_1 := \{u \in V \mid \beta(u,v) = 0\,\forall v \in V\}$. Take $G_1 := \mathrm{GL}(W_1)$, $W_2 := V/W_1$ and G_2 to be the image of G_β in $\mathrm{GL}(W_2)$. It is a straight-forward observation that G_β is the polynomial gluing $G_1 \times_{\mathcal{M}} G_2$ with $\mathcal{M} = \hom_{\mathbb{F}}(W_2,W_1)$. Furthermore, β induces a non-degenerate form on W_2. This construction reproduces the content of [31, §4.7]. Following the convention of Section 2, we denote $m = \dim(W_1)$ and $n = \dim(W_2)$.

If β is an alternating form, then n is even, G_2 is isomorphic to $\mathrm{Sp}_n(\mathbb{F})$ and G_β is isomorphic to $\mathrm{GL}_m(\mathbb{F}) \times_{\mathcal{M}} \mathrm{Sp}_n(\mathbb{F})$. By the work of Carlisle and Kropholler [2, §8.3], we know that $\mathbb{F}[W_2]^{\mathrm{Sp}_n(\mathbb{F})}$ is a complete intersection and a unique factorization domain. Therefore, using Theorem 3.10, we see that $\mathbb{F}[V]^{G_\beta}$ is a complete intersection and a unique factorization domain. Note that in this case, we can identify G_β with the subgroup of $\mathrm{GL}_{n+m}(\mathbb{F}_q)$ which fixes $\xi := x_1x_2^q - x_1^q x_2 + \cdots + x_{n-1}x_n^q - x_{n-1}^q x_n$ with $\mathbb{F} = \mathbb{F}_q$.

If β is Hermitian, then G_2 is isomorphic to the unitary group $\mathrm{U}_n(\mathbb{F})$ and G_β is isomorphic to $\mathrm{GL}_m(\mathbb{F}) \times_{\mathcal{M}} \mathrm{U}_n(\mathbb{F})$. By the work of Chu and Jow [17], we know that $\mathbb{F}[W_2]^{\mathrm{U}_n(\mathbb{F})}$ is a complete intersection and a unique factorization domain. Therefore, using Theorem 3.10, we see that $\mathbb{F}[V]^{G_\beta}$ is a complete intersection and a unique factorization domain. Note that in this case, we can identify G_β with the subgroup of $\mathrm{GL}_{n+m}(\mathbb{F}_{q^2})$ which fixes $\xi := x_1^{q+1} + \cdots + x_n^{q+1}$ with $\mathbb{F} = \mathbb{F}_{q^2}$.

If β is symmetric and the characteristic of $\mathbb{F}$ is not 2, then β is the polarization of the quadratic form $Q(v) := \beta(v,v)/2$ and G_2 is isomorphic to the orthogonal group $\mathrm{O}_n(\mathbb{F},Q)$. While there are some partial results and conjectures concerning $\mathbb{F}[W_2]^{\mathrm{O}_n(\mathbb{F},Q)}$, see [15] and [16], there is no published refereed account of the general result. In characteristic 2, there is a breakdown in the correspondence between quadratic forms and symmetric forms. We do have the work of Kropholler et al. [33] which computes the ring of invariants for the orthogonal group associated to a non-singular quadratic form on vector space over the field $\mathbb{F}_2$; in this context, the polarization of the quadratic form is a possibly degenerate alternating form.

Example 6.1. *Take q odd, $n = 3$ and consider the quadratic form $\Delta = x_2^2 - x_1x_3$. Then G_2 is $\mathrm{O}_3(\mathbb{F}_q)$. The order of $\mathrm{O}_3(\mathbb{F}_q)$ is $2q(q^2-1)$ and $\mathbb{F}_q[W_2]^{\mathrm{O}_3(\mathbb{F}_q)}$ is a polynomial algebra with generators in degrees* 2, $q+1$ *and* $q(q-1)$*, see* [18] *or* [43]. *The generator in degree* 2 *is* Δ *and* *the generator in degree $q+1$ can be constructed by applying a Steenrod operation to Δ. Define*

$$E := \left\{ \begin{pmatrix} 1 & 2c & c^2 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid c \in \mathbb{F}_q \right\}.$$

Then E is a Sylow p-subgroup for $\mathrm{O}_3(\mathbb{F}_q)$. The ring of invariants $\mathbb{F}_q[W_2]^E$ is the hypersurface generated by Δ and the orbit products of the variables, see [29] *and, for the case $q = p$,* [21]. *Note that we can identify W_2 with the second symmetric power representation of $SL_2(\mathbb{F}_q)$. The action of $SL_2(\mathbb{F}_q)$ on W_2 is not faithful; the image of $SL_2(\mathbb{F}_q)$ in W_2 is isomorphic to $SL_2(\mathbb{F}_q)/\langle -1\rangle$ and has index 4 in $\mathrm{O}_3(\mathbb{F}_q)$ with coset representatives*

$$\left\{ \begin{pmatrix} (-1)^{e_1} & 0 & 0 \\ 0 & (-1)^{e_2} & 0 \\ 0 & 0 & (-1)^{e_1} \end{pmatrix} \mid e_1, e_2 \in \{0,1\} \right\}.$$

The ring of invariants of $\mathbb{F}_q[W_2]^{SL_2(\mathbb{F}_q)}$ is a hypersurface with generators in degrees 2, $q+1$, $q(q-1)/2$ and $q(q+1)/2$, see [29] *and, for the case $q = p$,* [21]. *It follows from Theorem 3.10 that the ring of invariants for $\mathrm{GL}_m(\mathbb{F}_q) \times_{\mathcal{M}} \mathrm{O}_3(\mathbb{F}_q)$ is a polynomial algebra and that the ring of invariants for $\mathrm{GL}_m(\mathbb{F}_q) \times_{\mathcal{M}} E$ and $\mathrm{GL}_m(\mathbb{F}_q) \times_{\mathcal{M}} (SL_2(\mathbb{F}_q)/\langle -1\rangle)$ are both hypersurfaces.*

Example 6.2. *Take q odd, $n = 4$ and consider the quadratic form $u = x_2x_3 - x_1x_4$. Then G_2 is $\mathrm{O}_4^+(\mathbb{F}_q)$. The order of $\mathrm{O}_4^+(\mathbb{F}_q)$ is $2q^2(q^2-1)^2$ and $\mathbb{F}_q[W_2]^{\mathrm{O}_4^+(\mathbb{F}_q)}$ is a hypersurface with generators in degrees 2, $q+1$, q^2+1, q^3-q^2 and q^3-q, see* [15]. *The generator in degree 2 is u and the generators in degrees $q+1$ and q^2+1 can be constructed by applying Steenrod operations to u. Define*

$$E := \left\{ \begin{pmatrix} 1 & c_1 & c_2 & c_1c_2 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid (c_1, c_2) \in \mathbb{F}_q^2 \right\}.$$

Then E is a Sylow p-subgroup for $\mathrm{O}_4^+(\mathbb{F}_q)$. The ring of invariants $\mathbb{F}_q[W_2]^E$ is a complete intersection generated by u, an element in degree $q+1$ constructed by applying a Steenrod operation to u, and the orbit products of the variables: $N_E(x_4)$ of degree q^3, $N_E(x_2)$ and $N_E(x_2)$ of degree q, and x_1 (see [25]*). Let E_1 denote the subgroup of E of order q corresponding to taking $c_2 = 0$. It is easy to show that $\mathbb{F}_q[W_2]^{E_1}$ is the hypersurface generated by u and the E_1-orbit products of the variables; this calculation can also be interpreted as the $n = 2$ case of* [3, Theorem 2.4]. *Let E_2 denote the subgroup of E of order q corresponding to taking $c_1 = 0$. Since E_2 is conjugate to E_1, $\mathbb{F}_q[V]^{E_2}$ is also a hypersurface. Note that there is vector/covector action of $\mathrm{GL}_2(\mathbb{F}_q)$ on W_2 with E_1 as the image of the Sylow p-subgroup. It follows from* [14] *that $\mathbb{F}_q[W_2]^{\mathrm{GL}_2(\mathbb{F}_q)}$ is generated by u, $u_1 = x_2^qx_3 - x_1^qx_4$, $u_{-1} = x_2x_3^q - x_1x_4^q$, and Dickson invariants: $d_{1,2}$ and $d_{2,2}$ in the variables x_1 and x_2, and $d_{1,2}^*$ and $d_{2,2}^*$ in the variables x_3 and x_4. Furthermore, while $\mathbb{F}_q[W_2]^{\mathrm{GL}_2(\mathbb{F}_q)}$ is Gorenstein, it is not a complete intersection. Let B denote the upper triangular subgroup of $\mathrm{O}_4^+(\mathbb{F}_q)$. It follows from* [3, Theorem 2.4] *that $\mathbb{F}_q[W_2]^B$ is the complete intersection generated by u, u_1, u_{-1}, x_1^{q-1}, and $N_E(x_i)^{q-1}$ for $i = 2,3,4$. There are two equivalence classes of maximal parabolic subgroups for $\mathrm{O}_4^+(\mathbb{F}_q)$. One is represented by $\mathrm{GL}_2(\mathbb{F}_q) \ltimes E_2$ using the vector/covector* *action of* $\mathrm{GL}_2(\mathbb{F}_q)$ *and the other is represented by* $C_2 \ltimes B$*, where* C_2 *acts by exchanging* x_2 *and* x_3*. The ring* $\mathbb{F}_q[W_1]^{C_2\ltimes B}$ *is the complete intersection generated by* u*,* u_1*,* u_{-1}*,* x_1^{q-1}*,* $N_E(x_4)^{q-1}$*,* $N_E(x_2)^{q-1}+N_E(x_2)^{q-1}$ *and* $N_E(x_2)^{q-1}N_E(x_2)^{q-1}$*. The ring* $\mathbb{F}_q[W_2]^{\mathrm{GL}_2(\mathbb{F}_q)\ltimes E_2}$ *is Cohen-Macaulay and is generated by* $d_{1,2}$*,* $d_{2,2}$ *and the five generators for* $\mathbb{F}_q[W_1]^{\mathrm{O}_4^+(\mathbb{F}_q)}$*. If* H *is any of the above subgroups of* $\mathrm{O}_4^+(\mathbb{F}_q)$*, the results of Section 3 can be used to deduce properties of the ring of invariants of* $\mathrm{GL}_m(\mathbb{F}_q)\times_{\mathcal{M}} H$*.*

Recall that a group G acts on V as a *rigid reflection group* if every isotropy subgroup acts on V as a reflection group (see, for example, [23]).

Theorem 6.3. *If* $\mathbb{F}[W_2]^{G_2}$ *has a homogeneous geometric separating set of size* n *then* $\mathbb{F}[V]^{G_\beta}$ *has a homogeneous geometric separating set of size* $m+n$ *and* G_β *acts on* V *as a rigid reflection group.*

Proof. Since $G_1\times_{\mathcal{M}} G_2$ is a split polynomial gluing and $\mathbb{F}[W_1]^{G_1}=\mathbb{F}[W_1]^{\mathrm{GL}(W_1)}$ is a polynomial algebra, it follows from Theorem 3.11 that $\mathbb{F}[V]^{G_\beta}$ has a homogeneous geometric separating set of size $m+n$. It then follows from [23] that G_β acts on V as a rigid reflection group. □

Example 6.4. *In this example, we consider an alternating form with* $n=4$*. We will show that* $\mathbb{F}_q[W_2]^{\mathrm{Sp}_4(\mathbb{F}_q)}$ *has a homogeneous geometric separating set of size* 4 *and therefore, using the above theorem,* $\mathrm{GL}_m(\mathbb{F}_q)\times_{\mathcal{M}}\mathrm{Sp}_4(\mathbb{F}_q)$ *acts on* V *as a rigid reflection group. Using the work of Carlisle and Kropholler* [2, Section 8.3] *we know that* $\mathbb{F}_q[W_2]^{\mathrm{Sp}_4(\mathbb{F}_q)}$ *is the hypersurface generated by* $d_{1,4},d_{2,4},\xi_1,\xi_2,\xi_3$ *subject to the relation*

$$\xi_3^q+d_{1,4}\xi_2^q+d_{2,4}\xi_1^q=\xi_1(\xi_1^{q^2+1}-\xi_2^{q+1}+\xi_1^q\xi_3)^{q-1}.$$

Write $\xi_1(\xi_1^{q^2+1}-\xi_2^{q+1}+\xi_1^q\xi_3)^{q-1}=\alpha_0+\xi_1^q(\alpha_1\xi_3+\cdots+\alpha_{q-1}\xi_3^{q-1})$ *for* $\alpha_i\in\mathbb{F}_q[\xi_1,\xi_2]$ *and define* $f:=d_{2,4}-(\alpha_1\xi_3+\cdots+\alpha_{q-1}\xi_3^{q-1})$ *so that* $\xi_3^q=\alpha_0-d_{1,4}\xi_2^q-f\xi_1^q$*. We will show that* $\{f,d_{1,4},\xi_1,\xi_2\}$ *is a homogeneous geometric separating set for* $\mathbb{F}_q[W_1]^{\mathrm{Sp}_4(\mathbb{F}_q)}$*. Let* $\overline{\mathbb{F}}_q$ *denote the algebraic closure of* $\mathbb{F}_q$ *and define* $\overline{W}_2:=W_2\otimes_{\mathbb{F}_q}\overline{\mathbb{F}}_q$*. Since* ξ_3^q *and* ξ_3 *separate the same points in* $\overline{W}_2$*, the value of* ξ_3 *at any point* $w\in\overline{W}_2$ *is determined by the values of* $\{f,d_{1,4},\xi_1,\xi_2\}$*. It then follows from the definition of* f *that the values of* $\{f,d_{1,4},\xi_1,\xi_2\}$ *on* w *determine the value of* $d_{2,4}$ *on* w*. The conclusion then follows from the fact that the generating set is a geometric separating set.*

7. Parabolic Gluing

Consider a vector space W over $\mathbb{F}$ with a flag

$$\mathcal{F}=(F_0(W)=\{0\},F_1(W),F_2(W),\ldots,F_\ell(W)=W)$$

so that $F_i(W)$ is a proper subspace of $F_{i+1}(W)$. Let $P_{\mathcal{F}}$ denote the parabolic subgroup of $\mathrm{GL}(W)$ consisting of invertible linear transformations which stabilise the flag. Define $\mathcal{M}_{\mathcal{F}}$ to be the subset of $\hom_{\mathbb{F}}(W,W)$ consisting of linear transformations consistent with the flag $\mathcal{F}$, i.e.,

$$\mathcal{M}_{\mathcal{F}}=\{\phi\in\hom_{\mathbb{F}}(W,W)\mid \phi(F_i(W))\subseteq F_i(W)\,\text{for}\ i=1,\ldots,\ell\}.$$

Choose G_1 and G_2 to be subgroups of $P_{\mathcal{F}}$ and take $W_1 = W_2 = W$. Then $\mathcal{M}_{\mathcal{F}}$ is a left $\mathbb{F}G_1$/ right $\mathbb{F}G_2$ sub-bimodule of $\hom_{\mathbb{F}}(W,W)$ and we may form the *parabolic gluing* of G_1 to G_2 through $\mathcal{M}_{\mathcal{F}}$, which we denote by $G_1 \times_{\mathcal{F}} G_2$.

For the rest of this section, we assume $\mathbb{F} = \mathbb{F}_q$ so that $G_1 \times_{\mathcal{F}} G_2$ is a finite group. If we choose a basis for W which is consistent with the flag $\mathcal{F}$ then the matrices representing $\mathcal{M}_{\mathcal{F}}$ are block upper-triangular with diagonal blocks of size $n_i \times n_i$ where $n_i = \dim(F_i(W)/F_{i-1}(W))$. Denote $r_k = n_1 + \cdots + n_k$. We continue to denote the basis for W_1^* by $\{y_1,\ldots,y_n\}$ and the basis for W_2^* by $\{x_1,\ldots,x_n\}$ and assume the bases are consistent with the flag. Taking the dual of the surjection from W to $W/F_i(W)$ gives an inclusion of $(W/F_i(W))^*$ into W^*. Define $\widetilde{U}_i$ to be the image of $(W/F_{i-1}(W))^*$ in W_2^*. Thus $\widetilde{U}_1 = \mathrm{Span}_{\mathbb{F}_q}\{x_1,\ldots,x_n\}$, $\widetilde{U}_2 = \mathrm{Span}_{\mathbb{F}_q}\{x_{n_1+1},\ldots,x_n\}$, $\widetilde{U}_3 = \mathrm{Span}_{\mathbb{F}_q}\{x_{n_1+n_2+1},\ldots,x_n\}$, and so on. Define $N_i(t) := \prod_{u\in\widetilde{U}_i}(t+u)$. Then

$$\mathcal{H} = \{x_1,\ldots,x_n\} \cup \left(\bigcup_{i=1}^{\ell}\{N_i(y_j) \mid r_{i-1} < j \leq r_i\}\right)$$

is a generating set for $\mathbb{F}_q[V]^{\mathcal{M}_{\mathcal{F}}}$ and any basis for V^* consistent with the flag is a Nakajima basis for V as an $\mathbb{F}_q\mathcal{M}_{\mathcal{F}}$-module. Since G_2 stabilises the image of $(W/F_{i-1}(W))^*$ in W_2^*, we see that G_2 acts trivially on

$$A = \bigotimes_{i=1}^{\ell}\mathbb{F}_q[N_i(y_j) \mid r_{i-1} < j \leq r_i]$$

and $\mathbb{F}_q[V]^{\mathcal{M}} = A \otimes \mathbb{F}_q[W_2]$. However, in general, the algebra isomorphism from $\mathbb{F}_q[W_1]$ to A which takes y_j to $N_{\mathcal{M}_{\mathcal{F}}}(y_j)$ is not G_1-equivariant. For example, if we take $n=2$ and choose the partition $(1,1)$ then $A = \mathbb{F}_q[N_1(y_1),N_2(y_2)]$ where $N_1(y_1)$ has degree q^2 and $N_2(y_2)$ has degree q. If G_1 contains an element g such that $y_1 \cdot g = y_1 + y_2$, then

$$N_1(y_1)\cdot g = N_1(y_1) + N_2(y_2)^q + \left(d_{1,2}(x_1,x_2) + x_2^{q(q-1)}\right)N_2(y_2).$$

Remark 7.1. *The algebra isomorphism*

$$\mathbb{F}_q[W_1/F_i(W_1)] = \mathbb{F}_q[y_{r_i+1},\ldots,y_n] \to \mathbb{F}_q[N_{i+1}(y_{r_i+1})\ldots,N_{i+1}(y_n)]$$

which takes y_k to $N_{i+1}(y_k)$ is G_1-equivariant – compare with Example 2.1.

We will say that a sequence of homogeneous polynomials $f_1,f_2,\ldots,f_n \in \mathbb{F}_q[W_1]$ is *consistent* with the flag $\mathcal{F}$ if $f_j \in \mathbb{F}_q[y_{r_i+1},\ldots,y_n]$ whenever $r_i < j \leq r_{i+1}$. In this case we write $\widetilde{f}_j$ for the polynomial formed by substituting $N_{i+1}(y_k)$ for y_k in f_j.

Theorem 7.2. *Suppose $\mathbb{F}_q[W_1]^{G_1} = \mathbb{F}_q[f_1,\ldots,f_n]$ where the sequence of homogeneous polynomials $f_1,\ldots,f_n$ is consistent with $\mathcal{F}$ and $\mathbb{F}_q[W_2]^{G_2} = \mathbb{F}_q[h_1,\ldots,h_n]$ for homogeneous polynomials $h_1,\ldots,h_n$. Then $\mathbb{F}_q[V]^{G_1\times_{\mathcal{F}}G_2} = \mathbb{F}_q[\widetilde{f}_1,\ldots,\widetilde{f}_n,h_1,\ldots,h_n]$.*

Proof. First observe that, since G_1 stabilises the flag $\mathcal{F}$, we have $\{\widetilde{f}_1,\ldots,\widetilde{f}_n\} \subset \mathbb{F}[V]^{G_1\times_{\mathcal{F}}G_2}$. Furthermore, if $r_i < j \leq r_{i+1}$, then $\deg(\widetilde{f}_j) = q^{n-i}\deg(f_j)$.

For an ideal $I \subset \mathbb{F}_q[V]$, let $\mathcal{V}(I)$ denote the variety in $\overline{V} = V \otimes \overline{\mathbb{F}_q}$ cut out by I. Since $\mathbb{F}_q[W_1]^{G_1} = \mathbb{F}_q[f_1,\ldots,f_n]$, we know that $\prod_{j=1}^n \deg(f_j) = |G_1|$ and $\mathcal{V}(\langle f_1,\ldots,f_n\rangle) = \mathcal{V}(\langle y_1,\ldots,y_n\rangle)$ (see Corollary 3.2.6 and Lemma 2.6.3 of [12]). Similarly, since $\mathbb{F}_q[W_2]^{G_2} = \mathbb{F}_q[h_1,\ldots,h_n]$, we have $\prod_{j=1}^n \deg(h_j) = |G_2|$ and $\mathcal{V}(\langle h_1,\ldots,h_n\rangle) = \mathcal{V}(\langle x_1,\ldots,x_n\rangle)$. Observe that if $w \in \mathcal{V}(\langle x_1,\ldots,x_n\rangle)$ then $\widetilde{f}_j(w) = (f_j(w))^{q^{n-i}}$ whenever $r_i < j < r_{i+1}$. Hence

$$\mathcal{V}(\langle \widetilde{f}_1,\ldots,\widetilde{f}_n,h_1,\ldots,h_n\rangle) = \mathcal{V}(\langle y_1,\ldots,y_n,x_1,\ldots,x_n\rangle)$$

and $\{\widetilde{f}_1,\ldots,\widetilde{f}_n,h_1,\ldots,h_n\}$ is a homogeneous system of parameters. It follows from the construction of $\widetilde{f}_j$ that

$$\prod_{j=1}^n \deg(\widetilde{f}_j)\deg(h_j) = \left(\prod_{j=1}^n \deg(f_j)\deg(h_j)\right)\prod_{i=1}^{\ell}(q^{n-i-1})^{n_i} = |G_1|\cdot|G_2|\cdot|\mathcal{M}_{\mathcal{F}}| = |G_1 \times_{\mathcal{F}} G_2|.$$

Therefore $\mathbb{F}_q[V]^{G_1\times_{\mathcal{F}}G_2} = \mathbb{F}_q[\widetilde{f}_1,\ldots,\widetilde{f}_n,h_1,\ldots,h_n]$. □

Example 7.3. *Suppose $G_1 = P_{\mathcal{F}}$. By choosing a basis for W consistent with the flag we can think of the elements of $P_{\mathcal{F}}$ as block upper-triangular matrices. Let $\widetilde{U}_{\mathcal{F}}$ denote the subgroup consisting of the elements whose block-diagonal entries are 1_{n_i}. Then $\widetilde{U}_{\mathcal{F}}$ is a p-group and a normal subgroup of $P_{\mathcal{F}}$. A generating set for $\mathbb{F}_q[W_1]^{P_{\mathcal{F}}}$ is given by*

$$\bigcup_{i=1}^{\ell}\left\{d_{s,n_i}\left(N_{\widetilde{U}_{\mathcal{F}}}(y_j)\mid r_{i-1} < j \le r_i\right)\mid s=1,\ldots,n_i\right\}$$

where the Dickson invariants d_{s,n_i} are evaluated on the indicated n_i-tuple of orbit products (the resulting polynomial is independent of the order). To see this observe that the polynomials are invariant, homogeneous, algebraically independent and the product of the degrees is the order of $P_{\mathcal{F}}$. This construction is essentially the result of iterated polynomial gluing – compare with [28], [34], *and* [37]. *Since $d_{s,n_i}(N_{\widetilde{U}_{\mathcal{F}}}(y_j)) \in \mathbb{F}_q[y_{r_{i-1}+1},\ldots,y_n]$ whenever $r_{i-1} < j \le r_i$, the hypotheses of Theorem 7.2 are satisfied whenever $\mathbb{F}_q[W_2]^{G_2}$ is a polynomial algebra.*

Theorem 7.4. *Suppose $\mathbb{F}_q(W_1)^{G_1} = \mathbb{F}_q(f_1,\ldots,f_n)$ where $f_1,\ldots,f_n$ is a sequence of homogeneous polynomials consistent with $\mathcal{F}$. Then $\mathbb{F}_q(V)^{G_1\times_{\mathcal{F}}G_2} = \mathbb{F}_q(\widetilde{f}_1,\ldots,\widetilde{f}_n)\otimes\mathbb{F}_q(W_2)^{G_2}$.*

Proof. As in the proof of Theorem 7.2, first observe that $\{\widetilde{f}_1,\ldots,\widetilde{f}_n\} \subset \mathbb{F}_q[V]^{G_1\times_{\mathcal{F}}G_2}$. Also observe that the image of $\widetilde{f}_j - f_j^{q^{n-i}}$ lies in the ideal $\langle x_1,\ldots,x_n\rangle$ whenever $r_i < j \le r_{i+1}$. Therefore

$$\mathbb{F}_q(\widetilde{f}_1,\ldots,\widetilde{f}_n)\otimes\mathbb{F}_q(W_2)^{G_2} \subset \mathbb{F}_q(V)^{G_1\times_{\mathcal{F}}G_2}.$$

Since G_1 and G_2 are subgroups of $P_{\mathcal{F}}$, we have $\mathbb{F}_q(W_1)^{P_{\mathcal{F}}} \subset \mathbb{F}_q(W_1)^{G_1} = \mathbb{F}_q(f_1,\ldots,f_n)$ and $\mathbb{F}_q(W_2)^{P_{\mathcal{F}}} \subset \mathbb{F}_q(W_2)^{G_2}$. It then follows from Example 7.3 that

$$\mathbb{F}_q(V)^{P_{\mathcal{F}}\times_{\mathcal{F}}P_{\mathcal{F}}} \subset \mathbb{F}_q(\widetilde{f}_1,\ldots,\widetilde{f}_n)\otimes\mathbb{F}_q(W_2)^{G_2} \subset \mathbb{F}_q(V)^{G_1\times_{\mathcal{F}}G_2} \subset \mathbb{F}_q(V).$$

Therefore $\mathbb{F}_q(\widetilde{f}_1,\ldots,\widetilde{f}_n)\otimes\mathbb{F}_q(W_2)^{G_2} \subset \mathbb{F}_q(V)$is a Galois extension with Galois group H satisfying $P_{\mathcal{F}}\times_{\mathcal{F}}P_{\mathcal{F}} \ge H \ge G_1\times_{\mathcal{F}}G_2$ and $\mathbb{F}_q(V)^H = \mathbb{F}_q(\widetilde{f}_1,\ldots,\widetilde{f}_n)\otimes\mathbb{F}_q(W_2)^{G_2}$. To complete the proof, we need to show that $H \le G_1\times_{\mathcal{F}}G_2$. Suppose $(h_1,\varphi,h_2)\in H$ with $h_i \in P_{\mathcal{F}}$ and $\varphi \in \mathcal{M}_{\mathcal{F}}$. Since $\widetilde{f}_j \in \mathbb{F}_q(V)^H$, we have $\widetilde{f}_j\cdot(h_1,\varphi,h_2) = \widetilde{f}_j$. It then follows from Remark 7.1 that $f_j\cdot h_1 = f_j$. Since this true for every j, by Galois Theory, we have $h_1 \in G_1$. For any $f \in \mathbb{F}_q(W_2)^{G_2} \subset \mathbb{F}_q(V)^H$, we have $f = f \cdot (h_1, \varphi, h_2) = f \cdot h_2$ and, therefore, by another application of Galois Theory, $h_2 \in G_2$. Since $h_1 \in G_1$ and $h_2 \in G_2$, we have $(h_1, \varphi, h_2) \in G_1 \times_{\mathcal{F}} G_2$, as required. □

Example 7.5. *Choose a basis consistent with the filtration $\mathcal{F}$ and let G_1 be a subgroup of $\mathrm{GL}(W_1)$ represented by a subgroup of the upper triangular unipotent matrices using this basis. Then G_1 is a p-group and a subgroup of $P_{\mathcal{F}}$. Using the Campbell-Chuai construction from* [8] *gives a generating set $f_1, f_2, \ldots, f_n$ for the field of fractions $\mathbb{F}_q(W_1)^{G_1}$ which is consistent with $\mathcal{F}$. If G_2 is any subgroup of $P_{\mathcal{F}}$ then the hypotheses of Theorem 7.4 are satisfied and $\mathbb{F}_q(V)^{G_1 \times_{\mathcal{F}} G_2} = \mathbb{F}_q(\widetilde{f}_1, \ldots, \widetilde{f}_n) \otimes \mathbb{F}_q(W_2)^{G_2}$.*

8. Diagonal Gluing

Suppose we have a gluing with $G = G_1 = G_2$. Embed G in $G \times G$ using the diagonal map: $g \mapsto (g, g)$. Restricting the gluing to the image of the diagonal map gives a subgroup of $G \times_{\mathcal{M}} G$ which we denote by $G_{\mathcal{M}}$ and refer to as the *diagonal gluing* of G through $\mathcal{M}$. Note that $\hom_{\mathbb{F}}(W_2, W_1)$ is a left $\mathbb{F}G$-module with the action given by $\varphi \mapsto g \cdot \varphi \cdot g^{-1}$ and $\mathcal{M}$ is an $\mathbb{F}G$ submodule. Furthermore, $G_{\mathcal{M}}$ is isomorphic to the semi-direct product $G \ltimes \mathcal{M}$. There is an action of $G_{\mathcal{M}}$ on $V = W_1 \oplus W_2$ given by $(g, \varphi) \cdot (w_1 \oplus w_2) = (g \cdot w_1 + \varphi(w_2)) \oplus (g \cdot w_2)$. Since $\mathcal{M}$ is a normal subgroup of $G_{\mathcal{M}}$, we have $\mathbb{F}[V]^{G_{\mathcal{M}}} = (\mathbb{F}[V]^{\mathcal{M}})^G$. If there is a G-equivariant algebra isomorphism $\psi : \mathbb{F}[V] \to \mathbb{F}[V]^{\mathcal{M}}$ then the gluing is *polynomial* and ψ induces an isomorphism from $\mathbb{F}[V]^G$ to $\mathbb{F}[V]^{G_{\mathcal{M}}}$.

Example 8.1. *Take $\mathbb{F} = \mathbb{F}_q$, $G = \mathrm{GL}_n(\mathbb{F}_q)$, $W_1 = W_2 = \mathbb{F}_q^n$, and $\mathcal{M} = \hom_{\mathbb{F}_q}(W_1, W_2)$. Then we have a split polynomial gluing, see Example 2.1. In general, computing $\mathbb{F}_q[\mathbb{F}_q^n \oplus \mathbb{F}_q^n]^{GL_n(\mathbb{F}_q)}$ is a difficult problem. However, the field of fractions $\mathbb{F}_q(\mathbb{F}_q^n \oplus \mathbb{F}_q^n)^{\mathrm{GL}_n(\mathbb{F}_q)}$ is rational over $\mathbb{F}_q$ and a generating set is given in Section 3 of* [44]. *Applying the gluing isomorphism gives a generating set for $\mathbb{F}_q(\mathbb{F}_q^n \oplus \mathbb{F}_q^n)^{\mathrm{GL}_n(\mathbb{F}_q)_{\mathcal{M}}}$ proving that this field is also rational over $\mathbb{F}_q$.*

Example 8.2. *Take $\mathbb{F} = \mathbb{F}_q$, $G = \mathrm{GL}_n(\mathbb{F}_q)$, $W_1 = \mathbb{F}_q^n$, $W_2 = (\mathbb{F}_q^n)^*$, and $\mathcal{M} = \hom_{\mathbb{F}_q}(W_1, W_2)$. Then again it follows from Example 2.1 that we have a split polynomial gluing. A generating set for $\mathbb{F}_q[\mathbb{F}_q^n \oplus (\mathbb{F}_q^n)^*]^{\mathrm{GL}_n(\mathbb{F}_q)}$ was computed in* [14]*; the ring is Cohen-Macaulay but not a complete intersection. Applying the gluing isomorphism gives a generating set for $\mathbb{F}_q[\mathbb{F}_q^n \oplus (\mathbb{F}_q^n)^*]^{\mathrm{GL}_n(\mathbb{F}_q)_{\mathcal{M}}}$ and shows that this ring is also Cohen-Macaulay but not a complete intersection.*

Example 8.3. *Take $\mathbb{F} = \mathbb{F}_q$, $G = C_p$ (the cyclic group of order p with $q = p^r$), $W_1 = V_m$, $W_2 = V_n$ (indecomposable $\mathbb{F}_q C_p$-modules) and $\mathcal{M} = \hom_{\mathbb{F}_q}(W_2, W_1)$. It follows from Example 2.1 that the gluing is split polynomial. Generating sets for $\mathbb{F}_q[V_m \oplus V_n]^{C_p}$ are known for $m, n \leq 4$, see* [47]. *In each case, applying the gluing isomorphism gives a generating set for $\mathbb{F}_q[V_m \oplus V_n]^{(C_p)_{\mathcal{M}}}$. By the celebrated formula of Ellinsgrud and Skjelbred (see* [24] *or* [19, §3.9.2]*) the depth of $\mathbb{F}_q[V_m \oplus V_n]^{C_p}$ is 4 as long as $m + n > 3$. Since the the gluing preserves the augmentation ideal, this also holds for $\mathbb{F}_q[V_m \oplus V_n]^{(C_p)_{\mathcal{M}}}$.*

Example 8.4. *Take $\mathbb{F} = \mathbb{F}_q$, $G = C_p$ and $W_1 = W_2 = V_n$. Let $\mathbf{1}$ denote the one dimensional submodule of $\hom_{\mathbb{F}_q}(V_n, V_n)$ given by scalar multiples of the identity function. The ring $\mathbb{F}_q[V_n \oplus V_n]^{\mathbf{1}}$*

is polynomial for $n = 1$, a hypersurface for $n = 2$ and not Cohen-Macaulay for $n > 2$. Therefore the gluing in not polynomial for $n > 1$. However, the field of fractions $\mathbb{F}_q(V_n \oplus V_n)^{\mathbf{1}}$ is rational over $\mathbb{F}_q$. Define $u_j := y_1 x_j - y_j x_1$ for $j = 2, 3, \ldots, n$ and $N := y_1^q - y_1 x_1^{q-1}$. Using the Campbell-Chuai construction from [8], we have $\mathbb{F}_q[V_n \oplus V_n]^{\mathbf{1}}[x_1^{-1}] = \mathbb{F}_q[x_1, \ldots, x_n, N, u_2, \ldots, u_n][x_1^{-1}]$. Let g denote a generator for C_p and choose bases for W_1 and W_2 so that $x_j g = x_j + x_{j-1}$, $y_j g = y_j + y_{j-1}$ for $j > 1$, $x_1 g = x_1$ and $y_1 g = y_1$. Then $\mathrm{Span}_{\mathbb{F}_q}\{u_2, \ldots, u_n\}$ is isomorphic as an $\mathbb{F}_q C_p$-module to V_{n-1}^* and $N \in \mathbb{F}_q[V_n \oplus V_n]^{\mathbf{1}}$. Therefore $\mathbb{F}_q[V_n \oplus V_n]^{(C_p)_{\mathbf{1}}}[x_1^{-1}]$ is isomorphic to $\mathbb{F}_q[V_{n-1} \oplus V_1 \oplus V_n]^{C_p}[x_1^{-1}]$ where the isomorphism is induced by the C_p-equivariant map taking $\mathrm{Span}_{\mathbb{F}_q}\{u_2, \ldots, u_n\} \oplus \mathbb{F}_q N \oplus V_n^*$ to $V_{n-1}^* \oplus V_1^* \oplus V_n^*$.

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