



Kent Academic Repository

Clarkson, Peter and Jordaan, Kerstin (2021) *A generalized sextic Freud weight*. *Integral Transforms and Special Functions*, 32 (5-8). pp. 458-482. ISSN 1065-2469.

Downloaded from

<https://kar.kent.ac.uk/82162/> The University of Kent's Academic Repository KAR

The version of record is available from

<https://doi.org/10.1080/10652469.2020.1798949>

This document version

Author's Accepted Manuscript

DOI for this version

Licence for this version

UNSPECIFIED

Additional information

Versions of research works

Versions of Record

If this version is the version of record, it is the same as the published version available on the publisher's web site. Cite as the published version.

Author Accepted Manuscripts

If this document is identified as the Author Accepted Manuscript it is the version after peer review but before type setting, copy editing or publisher branding. Cite as Surname, Initial. (Year) 'Title of article'. To be published in *Title of Journal*, Volume and issue numbers [peer-reviewed accepted version]. Available at: DOI or URL (Accessed: date).

Enquiries

If you have questions about this document contact ResearchSupport@kent.ac.uk. Please include the URL of the record in KAR. If you believe that your, or a third party's rights have been compromised through this document please see our [Take Down policy](https://www.kent.ac.uk/guides/kar-the-kent-academic-repository#policies) (available from <https://www.kent.ac.uk/guides/kar-the-kent-academic-repository#policies>).

A Generalised Sextic Freud Weight

Peter A. Clarkson¹ and Kerstin Jordaan²

¹ School of Mathematics, Statistics and Actuarial Science,
University of Kent, Canterbury, CT2 7FS, UK

P.A.Clarkson@kent.ac.uk

² Department of Decision Sciences,
University of South Africa, Pretoria, 0003, South Africa

jordakh@unisa.ac.za

20 July 2020

Abstract

We discuss the recurrence coefficients of orthogonal polynomials with respect to a generalised sextic Freud weight

$$\omega(x; t, \lambda) = |x|^{2\lambda+1} \exp(-x^6 + tx^2), \quad x \in \mathbb{R},$$

with parameters $\lambda > -1$ and $t \in \mathbb{R}$. We show that the coefficients in these recurrence relations can be expressed in terms of Wronskians of generalised hypergeometric functions ${}_1F_2(a_1; b_1, b_2; z)$. We derive a nonlinear discrete as well as a system of differential equations satisfied by the recurrence coefficients and use these to investigate their asymptotic behaviour. We conclude by highlighting a fascinating connection between generalised quartic, sextic, octic and decic Freud weights when expressing their first moments in terms of generalised hypergeometric functions.

1 Introduction

In this paper we are concerned with semi-classical polynomials which are orthogonal with respect to a symmetric weight on an unbounded interval. Orthogonal polynomials find application in various branches of mathematics such as approximation theory, special functions, continued fractions and integral equations. Examples of classical orthogonal polynomials on infinite intervals include Hermite and Laguerre polynomials.

Although Szegő pioneered much of what is known on the theory of orthogonal polynomials on finite intervals, he did not carry his ideas over to infinite intervals, despite there being significant differences. It was only in the second half of the 20th century, starting with the work of Géza Freud on orthogonal polynomials on \mathbb{R} , that the study of Freud-type polynomials, and their generalisations, flourished. One of Freud's original aims was to extend the theory of best approximations using Jackson-Bernstein type estimates to the real line and, since a realistic expectation was that orthogonal expansions could serve as near-best approximations, a natural approach was to explore properties of orthogonal polynomials [35, 32].

Freud [14] studied polynomials $\{P_n(x)\}_{n=0}^{\infty}$ orthogonal on the real line with respect to a class of exponential-type weights, known as Freud weights, given by

$$w(x) = |x|^\rho \exp(-|x|^m), \quad m \in \mathbb{N},$$

with $\rho > -1$, for more details see [13, 34, 35, 22, 32]. Freud conjectured that the asymptotic behaviour of recurrence coefficients β_n in the recurrence relation

$$P_{n+1}(x) = xP_n(x) - \beta_n P_{n-1}(x), \tag{1.1}$$

with $P_{-1}(x) = 0$ and $P_0(x) = 1$, is given by

$$\lim_{n \rightarrow \infty} \frac{\beta_n}{n^{2/m}} = \left[\frac{\Gamma(\frac{1}{2}m)\Gamma(1 + \frac{1}{2}m)}{\Gamma(m+1)} \right]^{2/m}. \tag{1.2}$$

Freud [14] showed that if the limit exists for $m \in 2\mathbb{Z}$, then it is equal to the expression in (1.2) and proved existence of the limit (1.2) for $m = 2, 4, 6$ using a technique that gives rise to an infinite system of nonlinear equations, called

Freud equations. A general proof of Freud's conjecture was given by Lubinsky, Mhaskar and Saff [24]; see also [10, 13, 14, 26, 35]. Freud explored other properties, such as the asymptotic behaviour of the polynomials using the recurrence coefficients and the asymptotic behaviour of the greatest zero [15]. For recent contributions on the asymptotic behaviour of the recurrence coefficients associated with Freud-type exponential weights and zeros of the associated polynomials see, for example [1, 22, 24, 29, 30, 31, 34, 37].

Iserles and Webb [17] discuss orthogonal systems in $L^2(\mathbb{R})$ which give rise to a real skew-symmetric, tridiagonal, irreducible differentiation matrix. Such systems are important since they are stable by design and, if necessary, preserve Euclidean energy for a variety of time-dependent partial differential equations. Iserles and Webb [17] prove that there is a one-to-one correspondence between such an orthogonal system $\{\phi_n(x)\}_{n=0}^\infty$ and a sequence of polynomials $\{P_n(x)\}_{n=0}^\infty$ which are orthogonal with respect to a symmetric weight.

In this paper we consider polynomials orthogonal with respect to the *generalised sextic Freud weight*

$$\omega(x; t, \lambda) = |x|^{2\lambda+1} \exp(-x^6 + tx^2), \quad x, t \in \mathbb{R}, \quad (1.3)$$

with $\lambda > -1$. The weight (1.3) was briefly investigated by Freud [14] though lies in a more general framework that was given much earlier by Shohat [41]. The special case when $\lambda = -\frac{1}{2}$ and $t = 0$ was investigated by Sheen in his PhD thesis [39] and some asymptotics are given in [40]. Its and Kitaev [19], see also [12, §5], investigated the weight

$$\omega(x; \beta, q_1, q_2) = \exp\left\{-\beta\left(\frac{1}{2}x^2 + q_1x^4 + q_2x^6\right)\right\}, \quad (1.4)$$

with β, q_1 and q_2 parameters such that $q_1 < 0$ and $0 \leq 5q_2 < 4q_1^2$, in their study of the continuous limit for the Hermitian matrix model in connection with the nonperturbative theory of two-dimensional quantum gravity.

The paper is organised as follows: in §2, we review some properties of orthogonal polynomials with symmetric weight while we prove that the first moment of the generalised sextic Freud weight is a linear combination of generalised hypergeometric functions ${}_1F_2(a_1; b_1, b_2; z)$ and use this to derive an expression for the recurrence coefficients in terms of Wronskians of such generalised hypergeometric functions in §3. In §4 we derive a nonlinear difference equation satisfied by recurrence coefficients of generalised sextic Freud polynomials that is a special case of the second member of the discrete Painlevé I hierarchy. We also derive a system of differential equations that the recurrence coefficients satisfy and use the difference and differential equations to study the asymptotic behaviour of the coefficients when the degree n and parameter t tend to infinity. Finally, in §5 we derive the first moments of the generalised octic and decic Freud weights and show that the moments as well as the differential equations satisfied by the moments have a predictable structure as the order of the polynomial in the exponential factor of the Freud weight increases.

2 Orthogonal polynomials with symmetric weights

Let $P_n(x)$, $n \in \mathbb{N}$, be the monic orthogonal polynomial of degree n in x with respect to a positive weight $\omega(x)$ on the real line \mathbb{R} , such that

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)\omega(x)dx = h_n\delta_{m,n}, \quad h_n > 0,$$

where $\delta_{m,n}$ denotes the Kronecker delta. One of the most important properties of orthogonal polynomials is that they satisfy a three-term recurrence relationship of the form

$$P_{n+1}(x) = xP_n(x) - \alpha_n P_n(x) - \beta_n P_{n-1}(x),$$

where the coefficients α_n and β_n are given by the integrals

$$\alpha_n = \frac{1}{h_n} \int_{-\infty}^{\infty} xP_n^2(x)\omega(x)dx, \quad \beta_n = \frac{1}{h_{n-1}} \int_{-\infty}^{\infty} xP_{n-1}(x)P_n(x)\omega(x)dx,$$

with $P_{-1}(x) = 0$ and $P_0(x) = 1$. For symmetric weights, i.e. $\omega(x) = \omega(-x)$, then clearly $\alpha_n \equiv 0$. Hence for symmetric weights, the monic orthogonal polynomials $P_n(x)$, $n \in \mathbb{N}$, satisfy the three-term recurrence relation

$$P_{n+1}(x) = xP_n(x) - \beta_n P_{n-1}(x). \quad (2.1)$$

The relationship between the recurrence coefficient β_n and the normalisation constants h_n is given by

$$h_n = \beta_n h_{n-1}. \quad (2.2)$$

The coefficient β_n in the recurrence relation (2.1) can be expressed in terms of a determinant whose entries are given in terms of the moments associated with the weight $\omega(x)$. Specifically

$$\beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \quad (2.3)$$

where Δ_n is the Hankel determinant

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}, \quad n \geq 1, \quad (2.4)$$

with $\Delta_0 = 1$, $\Delta_{-1} = 0$, and μ_k , the k th moment, is given by the integral

$$\mu_k = \int_{-\infty}^{\infty} x^k \omega(x) dx.$$

For symmetric weights then clearly $\mu_{2k-1} \equiv 0$, for $k = 1, 2, \dots$.

For symmetric weights it is possible to write the Hankel determinant Δ_n in terms of the product of two Hankel determinants, as given in the following lemma. The decomposition depends on whether n is even or odd.

Lemma 2.1. *Suppose that \mathcal{A}_n and \mathcal{B}_n are the Hankel determinants given by*

$$\mathcal{A}_n = \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2n-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{2n-2} & \mu_{2n} & \cdots & \mu_{4n-4} \end{vmatrix}, \quad \mathcal{B}_n = \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2n} \\ \mu_4 & \mu_6 & \cdots & \mu_{2n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{2n} & \mu_{2n+2} & \cdots & \mu_{4n-2} \end{vmatrix}. \quad (2.5)$$

Then the determinant Δ_n (2.4) is given by

$$\Delta_{2n} = \mathcal{A}_n \mathcal{B}_n, \quad \Delta_{2n+1} = \mathcal{A}_{n+1} \mathcal{B}_n. \quad (2.6)$$

Proof. The result is easily obtained by matrix manipulation interchanging rows and columns. \square

Corollary 2.2. *For a symmetric weight, the recurrence coefficient β_n is given by*

$$\beta_{2n} = \frac{\mathcal{A}_{n+1} \mathcal{B}_{n-1}}{\mathcal{A}_n \mathcal{B}_n}, \quad \beta_{2n+1} = \frac{\mathcal{A}_n \mathcal{B}_{n+1}}{\mathcal{A}_{n+1} \mathcal{B}_n}, \quad (2.7)$$

where \mathcal{A}_n and \mathcal{B}_n are the Hankel determinants given by (2.5), with $\mathcal{A}_0 = \mathcal{B}_0 = 1$.

Proof. Substituting (2.6) into (2.3) gives the result. \square

Remark 2.3. The expression of the Hankel determinant Δ_n for symmetric weights as a product of two determinants is given in [3, 25].

Lemma 2.4. *Suppose that $\omega_0(x)$ is a symmetric positive weight on the real line for which all the moments exist and $\omega(x; t) = \exp(tx^2) \omega_0(x)$, with $t \in \mathbb{R}$, is a weight such that all the moments also exist. Then the Hankel determinants \mathcal{A}_n and \mathcal{B}_n given by (2.5) can be written in terms of Wronskians, as follows*

$$\mathcal{A}_n = W \left(\mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}} \right), \quad \mathcal{B}_n = \mathcal{W} \left(\frac{d\mu_0}{dt}, \frac{d^2\mu_0}{dt^2}, \dots, \frac{d^n\mu_0}{dt^n} \right), \quad (2.8)$$

where

$$\mu_0(t; \lambda) = \int_{-\infty}^{\infty} \exp(tx^2) \omega_0(x) dx,$$

and $\mathcal{W}(\varphi_1, \varphi_2, \dots, \varphi_n)$ is the Wronskian given by

$$\mathcal{W}(\varphi_1, \varphi_2, \dots, \varphi_n) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \varphi_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}, \quad \varphi_j^{(k)} = \frac{d^k \varphi_j}{dt^k}.$$

Proof. If $\omega(x; t) = \exp(tx^2) \omega_0(x)$, with $t \in \mathbb{R}$ then

$$\mu_{2n} = \int_{-\infty}^{\infty} x^{2n} \exp(tx^2) \omega_0(x) dx = \frac{d^n}{dt^n} \int_{-\infty}^{\infty} \exp(tx^2) \omega_0(x) dx = \frac{d^n \mu_0}{dt^n}, \quad n = 1, 2, \dots,$$

and so it follows from (2.5) that \mathcal{A}_n and \mathcal{B}_n are given by (2.8). \square

Lemma 2.5. *If \mathcal{A}_n and \mathcal{B}_n are Wronskians given by (2.8), with $\mathcal{A}_0 = \mathcal{B}_0 = 1$, then*

$$\mathcal{A}_n \frac{d\mathcal{B}_n}{dt} - \mathcal{B}_n \frac{d\mathcal{A}_n}{dt} = \mathcal{A}_{n+1} \mathcal{B}_{n-1}, \quad \mathcal{B}_n \frac{d\mathcal{A}_{n+1}}{dt} - \mathcal{A}_{n+1} \frac{d\mathcal{B}_n}{dt} = \mathcal{A}_{n+1} \mathcal{B}_n. \quad (2.9)$$

Proof. See, for example, Vein and Dale [45, §6.5.1]. \square

Corollary 2.6. *For a symmetric weight, if \mathcal{A}_n and \mathcal{B}_n are Wronskians given by (2.8) then the recurrence coefficient β_n is given by*

$$\beta_{2n} = \frac{d}{dt} \ln \frac{\mathcal{B}_n}{\mathcal{A}_n}, \quad \beta_{2n+1} = \frac{d}{dt} \ln \frac{\mathcal{A}_{n+1}}{\mathcal{B}_n}.$$

Proof. This follows from (2.7) and (2.9). \square

Lemma 2.7. *Let $\omega_0(x)$ be a symmetric positive weight on the real line for which all the moments exist and let $\omega(x; t) = \exp(tx^2) \omega_0(x)$, with $t \in \mathbb{R}$, is a weight such that all the moments exist. Then the recurrence coefficient $\beta_n(t)$ satisfies the Volterra, or the Langmuir lattice, equation*

$$\frac{d\beta_n}{dt} = \beta_n(\beta_{n+1} - \beta_{n-1}). \quad (2.10)$$

We remark that the differential-difference equation (2.10) is also known as the discrete KdV equation, or the Kac-van Moerbeke lattice [20].

Proof. See, for example, Van Assche [44, Theorem 2.4] and Wang, Zhu and Chen [46]. \square

The weights of classical orthogonal polynomials satisfy a first-order ordinary differential equation, the *Pearson equation*

$$\frac{d}{dx} [\sigma(x)\omega(x)] = \tau(x)\omega(x), \quad (2.11)$$

where $\sigma(x)$ is a monic polynomial of degree at most 2 and $\tau(x)$ is a polynomial with degree 1. However for *semi-classical* orthogonal polynomials, the weight function $\omega(x)$ satisfies the Pearson equation (2.11) with either $\deg(\sigma) > 2$ or $\deg(\tau) \neq 1$ (cf. [16, 28]).

For example, the generalised sextic Freud weight (1.3) satisfies the Pearson equation (2.11) with

$$\sigma(x) = x, \quad \tau(x) = 2\lambda + 2 + 2tx^2 - 6x^6.$$

For further information about orthogonal polynomials see, for example [4, 18, 42].

3 Generalised sextic Freud weight

In this section we are concerned with the generalised sextic Freud weight

$$\omega(x; t, \lambda) = |x|^{2\lambda+1} \exp(-x^6 + tx^2), \quad \lambda > -1, \quad x, t \in \mathbb{R}. \quad (3.1)$$

Lemma 3.1. *For the generalised sextic Freud weight (3.1), the first moment is given by*

$$\begin{aligned} \mu_0(t; \lambda) &= \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(-x^6 + tx^2) dx = \int_0^{\infty} s^\lambda \exp(ts - s^3) ds \\ &= \frac{1}{3} \Gamma\left(\frac{1}{3}\lambda + \frac{1}{3}\right) {}_1F_2\left(\frac{1}{3}\lambda + \frac{1}{3}; \frac{1}{3}, \frac{2}{3}; \left(\frac{1}{3}t\right)^3\right) + \frac{1}{3} t \Gamma\left(\frac{1}{3}\lambda + \frac{2}{3}\right) {}_1F_2\left(\frac{1}{3}\lambda + \frac{2}{3}; \frac{2}{3}, \frac{4}{3}; \left(\frac{1}{3}t\right)^3\right) \\ &\quad + \frac{1}{6} t^2 \Gamma\left(\frac{1}{3}\lambda + 1\right) {}_1F_2\left(\frac{1}{3}\lambda + 1; \frac{4}{3}, \frac{5}{3}; \left(\frac{1}{3}t\right)^3\right), \end{aligned} \quad (3.2)$$

where ${}_1F_2(a_1; b_1, b_2; z)$ is the generalised hypergeometric function.

Proof. First we shall show that

$$\mu_0(t; \lambda) = \int_0^\infty s^\lambda \exp(ts - s^3) ds,$$

is a solution of the third order equation

$$\frac{d^3\varphi}{dt^3} - \frac{1}{3}t \frac{d\varphi}{dt} - \frac{1}{3}(\lambda + 1)\varphi = 0. \quad (3.3)$$

Following Muldoon [33], if we seek a solution of (3.3) in the form

$$\varphi(t) = \int_0^\infty e^{st} v(s) ds,$$

then

$$\begin{aligned} \frac{d^3\varphi}{dt^3} - \frac{1}{3}t \frac{d\varphi}{dt} - \frac{1}{3}(\lambda + 1)\varphi &= \int_0^\infty e^{st} \left\{ s^3 v(s) - \frac{1}{3}tsv(s) - \frac{1}{3}(\lambda + 1)v(s) \right\} ds \\ &= \int_0^\infty e^{st} \left\{ s^3 v(s) + \frac{1}{3}v(s) + \frac{1}{3}s \frac{dv}{ds} - \frac{1}{3}(\lambda + 1)v(s) \right\} ds = 0, \end{aligned}$$

using integration by parts and assuming that $\lim_{s \rightarrow \infty} sv(s)e^{st} = 0$. Therefore for $\varphi(t)$ to be a solution of (3.3) then $v(s)$ necessarily satisfies

$$s \frac{dv}{ds} + (3s^3 - \lambda)v = 0,$$

and so

$$v(s) = s^\lambda \exp(-s^3).$$

Hence $\mu_0(t; \lambda)$ satisfies (3.3). The general solution of equation (3.3) is given by

$$\varphi(t) = c_1 {}_1F_2\left(\frac{1}{3}\lambda + \frac{1}{3}; \frac{1}{3}, \frac{2}{3}; \left(\frac{1}{3}t\right)^3\right) + c_2 t {}_1F_2\left(\frac{1}{3}\lambda + \frac{2}{3}; \frac{2}{3}, \frac{4}{3}; \left(\frac{1}{3}t\right)^3\right) + c_3 t^2 {}_1F_2\left(\frac{1}{3}\lambda + 1; \frac{4}{3}, \frac{5}{3}; \left(\frac{1}{3}t\right)^3\right),$$

with c_1 , c_2 and c_3 arbitrary constants. This can be derived from the third order equation satisfied by ${}_1F_2(a_1; b_1, b_2; z)$ given in §16.8(ii) of the DLMF [36], i.e.

$$z^2 \frac{d^3w}{dz^3} + z(b_1 + b_2 + 1) \frac{d^2w}{dz^2} + (b_1 b_2 - z) \frac{dw}{dz} - a_1 w = 0, \quad (3.4)$$

which has general solution

$$\begin{aligned} w(z) &= c_1 {}_1F_2(a_1; b_1, b_2; z) + c_2 z^{1-b_1} {}_1F_2(1 + a_1 - b_1; 2 - b_1, 1 - b_1 + b_2; z) \\ &\quad + c_3 z^{1-b_2} {}_1F_2(1 + a_1 - b_2; 1 + b_1 - b_2, 2 - b_2; z), \end{aligned}$$

with c_1 , c_2 and c_3 constants. Note that making the transformation $w(z) = \varphi(t)$, with $z = \left(\frac{1}{3}t\right)^3$, in (3.4) gives

$$t^2 \frac{d^3\varphi}{dt^3} + 3t(b_1 + b_2 - 1) \frac{d^2\varphi}{dt^2} + [(3b_1 - 2)(3b_2 - 2) - \frac{1}{3}t^3] \frac{d\varphi}{dt} - a_1 t^2 \varphi = 0.$$

Consequently setting $a_1 = \frac{1}{3}(\lambda + 1)$, $b_1 = \frac{1}{3}$ and $b_2 = \frac{2}{3}$ we have

$$\begin{aligned} \mu_0(t; \lambda) &= \int_0^\infty s^\lambda \exp(ts - s^3) ds \\ &= c_1 {}_1F_2\left(\frac{1}{3}\lambda + \frac{1}{3}; \frac{1}{3}, \frac{2}{3}; \left(\frac{1}{3}t\right)^3\right) + c_2 t {}_1F_2\left(\frac{1}{3}\lambda + \frac{2}{3}; \frac{2}{3}, \frac{4}{3}; \left(\frac{1}{3}t\right)^3\right) + c_3 t^2 {}_1F_2\left(\frac{1}{3}\lambda + 1; \frac{4}{3}, \frac{5}{3}; \left(\frac{1}{3}t\right)^3\right), \end{aligned}$$

where c_1 , c_2 and c_3 are constants to be determined. Since ${}_1F_2(a_1; b_1, b_2; 0) = 1$ and

$$\mu_0(0; \lambda) = \int_0^\infty s^\lambda \exp(-s^3) ds = \frac{1}{3}\Gamma\left(\frac{1}{3}\lambda + \frac{1}{3}\right), \quad \frac{d\mu_0}{dt}(0; \lambda) = \frac{1}{3}\Gamma\left(\frac{1}{3}\lambda + \frac{2}{3}\right), \quad \frac{d^2\mu_0}{dt^2}(0; \lambda) = \frac{1}{3}\Gamma\left(\frac{1}{3}\lambda + 1\right)$$

then it follows that

$$c_1 = \frac{1}{3}\Gamma\left(\frac{1}{3}\lambda + \frac{1}{3}\right), \quad c_2 = \frac{1}{3}\Gamma\left(\frac{1}{3}\lambda + \frac{2}{3}\right), \quad c_3 = \frac{1}{6}\Gamma\left(\frac{1}{3}\lambda + 1\right),$$

which gives (3.2), as required. \square

Remarks 3.2.

(i) If $\lambda = -\frac{1}{2}$ then

$$\mu_0(t; -\frac{1}{2}) = \int_{-\infty}^{\infty} \exp(-x^6 + tx^2) dx = \pi^{3/2} 12^{-1/6} [\text{Ai}^2(\tau) + \text{Bi}^2(\tau)], \quad \tau = 12^{-1/3} t,$$

where $\text{Ai}(\tau)$ and $\text{Bi}(\tau)$ are the Airy functions. This result is equation 9.11.4 in the DLMF [36], which is due to Muldoon [33, p32], see also [38].

(ii) The generalised sextic Freud weight (3.1) is an example of a semi-classical weight for which the first moment $\mu_0(t; \lambda)$ satisfies a *third order equation*. In our earlier studies of semi-classical weights [5, 6, 7], the first moment has satisfied a second order equation. For example, for the quartic Freud weight

$$\omega(x; t) = |x|^{2\lambda+1} \exp(-x^4 + tx^2), \quad x, t \in \mathbb{R}, \quad (3.5)$$

the first moment is expressed in terms of parabolic cylinder functions $D_\nu(z)$, or equivalently in terms of the confluent hypergeometric function ${}_1F_1(a; b; z)$, see [5, 6, 7]. These are classical special functions that satisfy second order equations.

(iii) Equation (3.3) arises in association with threefold symmetric Hahn-classical multiple orthogonal polynomials [23] and in connection with Yablonskii–Vorob’ev polynomials associated with rational solutions of the second Painlevé equation [8].

The higher moment $\mu_k(t; \lambda)$ is given by

$$\mu_k(t; \lambda) = \int_{-\infty}^{\infty} x^k |x|^{2\lambda+1} \exp(-x^6 + tx^2) dx, \quad k = 0, 1, 2, \dots,$$

and so

$$\mu_{2k}(t; \lambda) = \frac{d^k}{dt^k} \mu_0(t; \lambda), \quad \mu_{2k+1}(t; \lambda) = 0, \quad (3.6)$$

with $\mu_0(t; \lambda)$ given by (3.2).

Lemma 3.3. *Suppose that $\Delta_n(t; \lambda)$ is the Hankel determinant given by*

$$\Delta_n(t; \lambda) = \det [\mu_{j+k}(t; \lambda)]_{j,k=0}^{n-1},$$

and $\mathcal{A}_n(t; \lambda)$ and $\mathcal{B}_n(t; \lambda)$ are the Hankel determinants given by

$$\mathcal{A}_n(t; \lambda) = \det [\mu_{2j+2k}(t; \lambda)]_{j,k=0}^{n-1}, \quad \mathcal{B}_n(t; \lambda) = \det [\mu_{2j+2k+2}(t; \lambda)]_{j,k=0}^{n-1}, \quad (3.7)$$

then

$$\Delta_{2n}(t; \lambda) = \mathcal{A}_n(t; \lambda) \mathcal{B}_n(t; \lambda), \quad \Delta_{2n+1}(t; \lambda) = \mathcal{A}_{n+1}(t; \lambda) \mathcal{B}_n(t; \lambda).$$

Lemma 3.4. *If $\mathcal{A}_n(t; \lambda)$ and $\mathcal{B}_n(t; \lambda)$ are given by (3.7) then $\mathcal{B}_n(t; \lambda) = \mathcal{A}_n(t; \lambda + 1)$.*

Proof. Since

$$\mu_{2k+2}(t; \lambda) = \int_0^\infty s^{\lambda+k+1} \exp(ts - s^3) ds = \mu_{2k}(t; \lambda + 1),$$

then the result immediately follows. □

Lemma 3.5. *For the generalised sextic Freud weight (3.1), the associated monic polynomials $P_n(x)$ satisfy the recurrence relation*

$$P_{n+1}(x) = xP_n(x) - \beta_n(t; \lambda)P_{n-1}(x), \quad n = 0, 1, 2, \dots, \quad (3.8)$$

with $P_{-1}(x) = 0$ and $P_0(x) = 1$, where

$$\begin{aligned} \beta_{2n}(t; \lambda) &= \frac{\mathcal{A}_{n+1}(t; \lambda) \mathcal{A}_{n-1}(t; \lambda + 1)}{\mathcal{A}_n(t; \lambda) \mathcal{A}_n(t; \lambda + 1)} = \frac{d}{dt} \ln \frac{\mathcal{A}_n(t; \lambda + 1)}{\mathcal{A}_n(t; \lambda)}, \\ \beta_{2n+1}(t; \lambda) &= \frac{\mathcal{A}_n(t; \lambda) \mathcal{A}_{n+1}(t; \lambda + 1)}{\mathcal{A}_{n+1}(t; \lambda) \mathcal{A}_n(t; \lambda + 1)} = \frac{d}{dt} \ln \frac{\mathcal{A}_{n+1}(t; \lambda)}{\mathcal{A}_n(t; \lambda + 1)}. \end{aligned}$$

where $\mathcal{A}_n(t; \lambda)$ is the Wronskian given by

$$\mathcal{A}_n(t; \lambda) = \mathcal{W} \left(\mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}} \right),$$

with

$$\begin{aligned} \mu_0(t; \lambda) &= \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(-x^6 + tx^2) dx \\ &= \frac{1}{3} \Gamma\left(\frac{1}{3}\lambda + \frac{1}{3}\right) {}_1F_2\left(\frac{1}{3}\lambda + \frac{1}{3}; \frac{1}{3}, \frac{2}{3}; \left(\frac{1}{3}t\right)^3\right) + \frac{1}{3} t \Gamma\left(\frac{1}{3}\lambda + \frac{2}{3}\right) {}_1F_2\left(\frac{1}{3}\lambda + \frac{2}{3}; \frac{2}{3}, \frac{4}{3}; \left(\frac{1}{3}t\right)^3\right) \\ &\quad + \frac{1}{6} t^2 \Gamma\left(\frac{1}{3}\lambda + 1\right) {}_1F_2\left(\frac{1}{3}\lambda + 1; \frac{4}{3}, \frac{5}{3}; \left(\frac{1}{3}t\right)^3\right). \end{aligned}$$

4 Equations satisfied by the recurrence coefficient

In this section we derive a discrete equation and a differential equation satisfied by recurrence coefficient $\beta_n(t; \lambda)$ and discuss this asymptotics of $\beta_n(t; \lambda)$ as $n \rightarrow \infty$ and $t \rightarrow \pm\infty$.

Lemma 4.1. *The recurrence coefficient $\beta_n(t; \lambda)$ satisfies the nonlinear discrete equation*

$$\begin{aligned} 6\beta_n(\beta_{n+2}\beta_{n+1} + \beta_{n+1}^2 + 2\beta_{n+1}\beta_n + \beta_{n+1}\beta_{n-1} + \beta_n^2 + 2\beta_n\beta_{n-1} + \beta_{n-1}^2 + \beta_{n-1}\beta_{n-2}) \\ - 2t\beta_n = n + \gamma_n, \end{aligned} \quad (4.1)$$

with $\gamma_n = (\lambda + \frac{1}{2})[1 - (-1)^n]$.

Proof. The fourth order nonlinear discrete equation (4.1) when $t = 0$ was derived by Freud [14]; see also Van Assche [43, §2.3]. It is straightforward to modify the proof for the case when $t \neq 0$; see also Wang, Zhu and Chen [46]. \square

Remark 4.2. The discrete equation (4.1) is a special case of $dP_I^{(2)}$, the second member of the discrete Painlevé I hierarchy which is given by

$$\begin{aligned} c_4\beta_n(\beta_{n+2}\beta_{n+1} + \beta_{n+1}^2 + 2\beta_{n+1}\beta_n + \beta_{n+1}\beta_{n-1} + \beta_n^2 + 2\beta_n\beta_{n-1} + \beta_{n-1}^2 + \beta_{n-1}\beta_{n-2}) \\ + c_3\beta_n(\beta_{n+1} + \beta_n + \beta_{n-1}) + c_2\beta_n = c_1 + c_0(-1)^n + n, \end{aligned} \quad (4.2)$$

with $c_j, j = 0, 1, \dots, 4$ constants. Cresswell and Joshi [9] show that:

- if $c_0 = 0$, then the continuum limit of (4.2) is equivalent to

$$\frac{d^4w}{dz^4} = 10w \frac{d^2w}{dz^2} + 5 \left(\frac{dw}{dz} \right)^2 - 10w^3 + z, \quad (4.3)$$

which is $P_I^{(2)}$, the second member of the first Painlevé hierarchy [21];

- if $c_0 \neq 0$, then the continuum limit of (4.2) is equivalent to

$$\frac{d^4w}{dz^4} = 10w^2 \frac{d^2w}{dz^2} + 10w \left(\frac{dw}{dz} \right)^2 - 6w^5 + zw + \alpha,$$

where α is a constant, which is $P_{II}^{(2)}$, the second member of the second Painlevé hierarchy [11].

We note that equation (4.2) with $c_1 = c_0 = 0$ is equation (8) in [19] and equation (4.3) is given in [2, 12, 19]. This is analogous to the situation for the general discrete Painlevé I equation

$$c_3\beta_n(\beta_{n+1} + \beta_n + \beta_{n-1}) + c_2\beta_n = c_1 + c_0(-1)^n + n, \quad (4.4)$$

with $c_j, j = 0, 1, 2, 3$ constants. If $c_0 = 0$ then the continuum limit of (4.4) is equivalent to the first Painlevé equation

$$\frac{d^2w}{dz^2} = 6w^2 + z,$$

whilst if $c_0 \neq 0$ then the continuum limit of (4.4) is equivalent to the second Painlevé equation

$$\frac{d^2w}{dz^2} = 2w^3 + zw + \alpha,$$

where α is a constant, see [9] for details.

Lemma 4.3. *The recurrence coefficient $\beta_n(t; \lambda)$ satisfies the system*

$$\frac{d^2\beta_n}{dt^2} - 3(\beta_n + \beta_{n+1})\frac{d\beta_n}{dt} + \beta_n^3 + 6\beta_n^2\beta_{n+1} + 3\beta_n\beta_{n+1}^2 - \frac{1}{3}t\beta_n = \frac{1}{6}(n + \gamma_n), \quad (4.5a)$$

$$\frac{d^2\beta_{n+1}}{dt^2} + 3(\beta_n + \beta_{n+1})\frac{d\beta_{n+1}}{dt} + \beta_{n+1}^3 + 6\beta_{n+1}^2\beta_n + 3\beta_{n+1}\beta_n^2 - \frac{1}{3}t\beta_{n+1} = \frac{1}{6}(n + 1 + \gamma_{n+1}), \quad (4.5b)$$

with $\gamma_n = (\lambda + \frac{1}{2})[1 - (-1)^n]$.

Proof. Following Magnus [27, Example 5], from the Langmuir lattice (2.10) we have

$$\begin{aligned} \frac{d\beta_{n-1}}{dt} &= \beta_{n-1}(\beta_n - \beta_{n-2}) \\ &= \beta_{n-1}^2 + 3\beta_{n-1}\beta_n + \beta_{n-1}\beta_{n+1} + \beta_n^2 + 2\beta_n\beta_{n+1} + \beta_{n+1}^2 + \beta_{n+1}\beta_{n+2} - \frac{n + \gamma_n}{6\beta_n} - \frac{1}{3}t, \end{aligned} \quad (4.6a)$$

$$\frac{d\beta_n}{dt} = \beta_n(\beta_{n+1} - \beta_{n-1}), \quad (4.6b)$$

$$\frac{d\beta_{n+1}}{dt} = \beta_{n+1}(\beta_{n+2} - \beta_n), \quad (4.6c)$$

$$\begin{aligned} \frac{d\beta_{n+2}}{dt} &= \beta_{n+2}(\beta_{n+3} - \beta_{n+1}) \\ &= -\beta_{n-1}\beta_n - \beta_n^2 - 2\beta_n\beta_{n+1} - \beta_n\beta_{n+2} - \beta_{n+1}^2 - 3\beta_{n+1}\beta_{n+2} - \beta_{n+2}^2 + \frac{n + 1 + \gamma_{n+1}}{6\beta_{n+1}} + \frac{1}{3}t. \end{aligned} \quad (4.6d)$$

where we have used the discrete equation (4.1) to eliminate β_{n+3} and β_{n-2} . Solving (4.6b) and (4.6c) for β_{n+2} and β_{n-1} gives

$$\beta_{n+2} = \beta_n + \frac{1}{\beta_{n+1}} \frac{d\beta_{n+1}}{dt}, \quad \beta_{n-1} = \beta_{n+1} - \frac{1}{\beta_n} \frac{d\beta_n}{dt},$$

and substitution into (4.6a) and (4.6d) yields the system (4.5) as required. \square

Lemma 4.4. *The recurrence coefficient $\beta_n(t; \lambda)$ has the asymptotics, as $t \rightarrow \infty$*

$$\beta_{2n}(t; \lambda) = \frac{n}{2t} + \frac{3\sqrt{3}n(2n - 2\lambda - 1)}{8t^{5/2}} + \mathcal{O}(t^{-4}), \quad (4.7a)$$

$$\beta_{2n+1}(t; \lambda) = \frac{1}{3}\sqrt{3}t - \frac{4n - 2\lambda + 1}{4t} - \frac{\sqrt{3}(36n^2 - 72\lambda n + 12\lambda^2 - 24\lambda + 5)}{32t^{5/2}} + \mathcal{O}(t^{-4}). \quad (4.7b)$$

and as $t \rightarrow -\infty$

$$\beta_{2n}(t; \lambda) = -\frac{n}{t} - \frac{3n[10n^2 + 6(2\lambda + 1)n + 3\lambda^2 + 3\lambda + 2]}{t^4} + \mathcal{O}(t^{-7}), \quad (4.8a)$$

$$\beta_{2n+1}(t; \lambda) = -\frac{n + \lambda + 1}{t} - \frac{3(n + \lambda + 1)[10n^2 + (8\lambda + 14)n + (\lambda + 3)(\lambda + 2)]}{t^4} + \mathcal{O}(t^{-7}). \quad (4.8b)$$

Proof. First we consider $\beta_1(t; \lambda)$ which is given by

$$\beta_1(t; \lambda) = \frac{\mu_2(t; \lambda)}{\mu_0(t; \lambda)} = \frac{\int_0^\infty s^{\lambda+1} \exp(ts - s^3) ds}{\int_0^\infty s^\lambda \exp(ts - s^3) ds},$$

and satisfies the equation

$$\frac{d^2\beta_1}{dt^2} + 3\beta_1 \frac{d\beta_1}{dt} + \beta_1^3 - \frac{1}{3}t\beta_1 = \frac{1}{3}(\lambda + 1). \quad (4.9)$$

Since $\mu_0(t; \lambda)$ given by (3.2) involves the sum of three generalised hypergeometric functions then its asymptotics are not as straightforward as for a classical special function, as was the case for the generalised quartic Freud weight we discussed in [6, 7] which involved parabolic cylinder functions. Using Laplace's method it follows that as $t \rightarrow \infty$

$$\begin{aligned} \mu_0(t; \lambda) &= \int_0^\infty s^\lambda \exp(ts - s^3) ds = t^{(\lambda+1)/2} \int_0^\infty \xi^\lambda \exp\{t^{3/2}\xi(1 - \xi^2)\} d\xi \\ &= 3^{-1/4 - \lambda/2} t^{\lambda/2 - 1/4} \sqrt{\pi} \exp\left(\frac{2}{9}\sqrt{3}t^{3/2}\right) \left[1 + \mathcal{O}(t^{-3/2})\right] \\ \mu_2(t; \lambda) &= \mu_0(t; \lambda + 1) = 3^{-3/4 - \lambda/2} t^{\lambda/2 + 1/4} \sqrt{\pi} \exp\left(\frac{2}{9}\sqrt{3}t^{3/2}\right) \left[1 + \mathcal{O}(t^{-3/2})\right] \end{aligned}$$

and so

$$\beta_1(t; \lambda) = \frac{\mu_2(t; \lambda)}{\mu_0(t; \lambda)} = \frac{1}{3}\sqrt{3t} \left[1 + \mathcal{O}(t^{-3/2}) \right], \quad \text{as } t \rightarrow \infty.$$

Hence we suppose that as $t \rightarrow \infty$

$$\beta_1(t; \lambda) = \frac{1}{3}\sqrt{3t} + \frac{a_1}{t} + \frac{a_2}{t^{5/2}} + \mathcal{O}(t^{-4}).$$

Substituting this into (4.9) and equating coefficients of powers of t gives

$$a_1 = \frac{1}{4}(2\lambda - 1), \quad a_2 = -\sqrt{3}(12\lambda^2 - 24\lambda + 5)/32,$$

and so

$$\beta_1(t; \lambda) = \frac{1}{3}\sqrt{3t} + \frac{2\lambda - 1}{4t} - \frac{\sqrt{3}(12\lambda^2 - 24\lambda + 5)}{32t^{5/2}} + \mathcal{O}(t^{-4}).$$

Also using Watson's Lemma it follows that as $t \rightarrow -\infty$

$$\mu_0(t; \lambda) = \Gamma(\lambda + 1)(-t)^{-\lambda-1} [1 + \mathcal{O}(t^{-3})], \quad \mu_2(t; \lambda) = \Gamma(\lambda + 2)(-t)^{-\lambda-2} [1 + \mathcal{O}(t^{-3})],$$

and so

$$\beta_1(t; \lambda) = -\frac{\lambda + 1}{t} [1 + \mathcal{O}(t^{-3})], \quad \text{as } t \rightarrow -\infty.$$

Hence we suppose that as $t \rightarrow -\infty$

$$\beta_1(t; \lambda) = -\frac{\lambda + 1}{t} + \frac{b_1}{t^4} + \frac{b_2}{t^7} + \mathcal{O}(t^{-10}).$$

Substituting this into (4.9) and equating coefficients of powers of t gives

$$b_1 = -3(\lambda + 1)(\lambda + 2)(\lambda + 3), \quad b_2 = -9(\lambda + 1)(\lambda + 2)(\lambda + 3)(3\lambda^2 + 21\lambda + 38).$$

Then using the Langmuir lattice (2.10) it can be shown that as $t \rightarrow \infty$

$$\begin{aligned} \beta_2(t; \lambda) &= \frac{1}{2t} - \frac{3\sqrt{3}(2\lambda - 1)}{8t^{5/2}} + \mathcal{O}(t^{-4}), & \beta_3(t; \lambda) &= \frac{1}{3}\sqrt{3t} + \frac{2\lambda - 5}{4t} - \frac{\sqrt{3}(12\lambda^2 - 96\lambda + 41)}{32t^{5/2}} + \mathcal{O}(t^{-4}), \\ \beta_4(t; \lambda) &= \frac{1}{t} - \frac{3\sqrt{3}(2\lambda - 3)}{4t^{5/2}} + \mathcal{O}(t^{-4}), & \beta_5(t; \lambda) &= \frac{1}{3}\sqrt{3t} + \frac{2\lambda - 9}{4t} - \frac{\sqrt{3}(12\lambda^2 - 168\lambda + 149)}{32t^{5/2}} + \mathcal{O}(t^{-4}), \end{aligned}$$

and as $t \rightarrow -\infty$

$$\begin{aligned} \beta_2(t; \lambda) &= -\frac{1}{t} - \frac{9(\lambda + 2)(\lambda + 3)}{t^4} + \mathcal{O}(t^{-7}), & \beta_3(t; \lambda) &= -\frac{\lambda + 2}{t} - \frac{3(\lambda + 2)(\lambda + 3)(\lambda + 10)}{t^4} + \mathcal{O}(t^{-7}), \\ \beta_4(t; \lambda) &= -\frac{2}{t} - \frac{18(\lambda + 3)(\lambda + 6)}{t^4} + \mathcal{O}(t^{-7}), & \beta_5(t; \lambda) &= -\frac{\lambda + 3}{t} - \frac{3(\lambda + 3)(\lambda^2 + 21\lambda + 74)}{t^4} + \mathcal{O}(t^{-7}). \end{aligned}$$

From these we can see a pattern emerging for the asymptotics of $\beta_n(t; \lambda)$ as $t \rightarrow \pm\infty$, which are different depending on whether n is even or odd.

Now suppose that $u_n(t; \lambda) = \beta_{2n}(t; \lambda)$ and $v_n(t; \lambda) = \beta_{2n+1}(t; \lambda)$, which from (4.5) satisfy

$$\frac{d^2 u_n}{dt^2} - 3(u_n + v_n) \frac{du_n}{dt} + u_n^3 + 6u_n^2 v_n + 3u_n v_n^2 - \frac{1}{3} t u_n = \frac{1}{3} n, \quad (4.10a)$$

$$\frac{d^2 v_n}{dt^2} + 3(u_n + v_n) \frac{dv_n}{dt} + v_n^3 + 6v_n^2 u_n + 3v_n u_n^2 - \frac{1}{3} t v_n = \frac{1}{3} (n + 1 + \lambda). \quad (4.10b)$$

If we suppose that as $t \rightarrow \infty$

$$u_n = \frac{\tilde{a}_1}{t} + \frac{\tilde{a}_2}{t^{5/2}} + \mathcal{O}(t^{-4}), \quad v_n = \frac{1}{3}\sqrt{3t} + \frac{\tilde{b}_1}{t} + \frac{\tilde{b}_2}{t^{5/2}} + \mathcal{O}(t^{-4}),$$

with $\tilde{a}_1, \tilde{a}_2, \tilde{b}_1$ and \tilde{b}_2 constants, then substituting into (4.10) and equating coefficients of powers of t gives (4.7). Also if we suppose that as $t \rightarrow -\infty$

$$u_n = \frac{\tilde{c}_1}{t} + \frac{\tilde{c}_2}{t^4} + \mathcal{O}(t^{-7}), \quad v_n = \frac{\tilde{d}_1}{t} + \frac{\tilde{d}_2}{t^4} + \mathcal{O}(t^{-7}),$$

with $\tilde{c}_1, \tilde{c}_2, \tilde{d}_1$ and \tilde{d}_2 constants, then substituting into (4.10) and equating coefficients of powers of t gives (4.8). \square

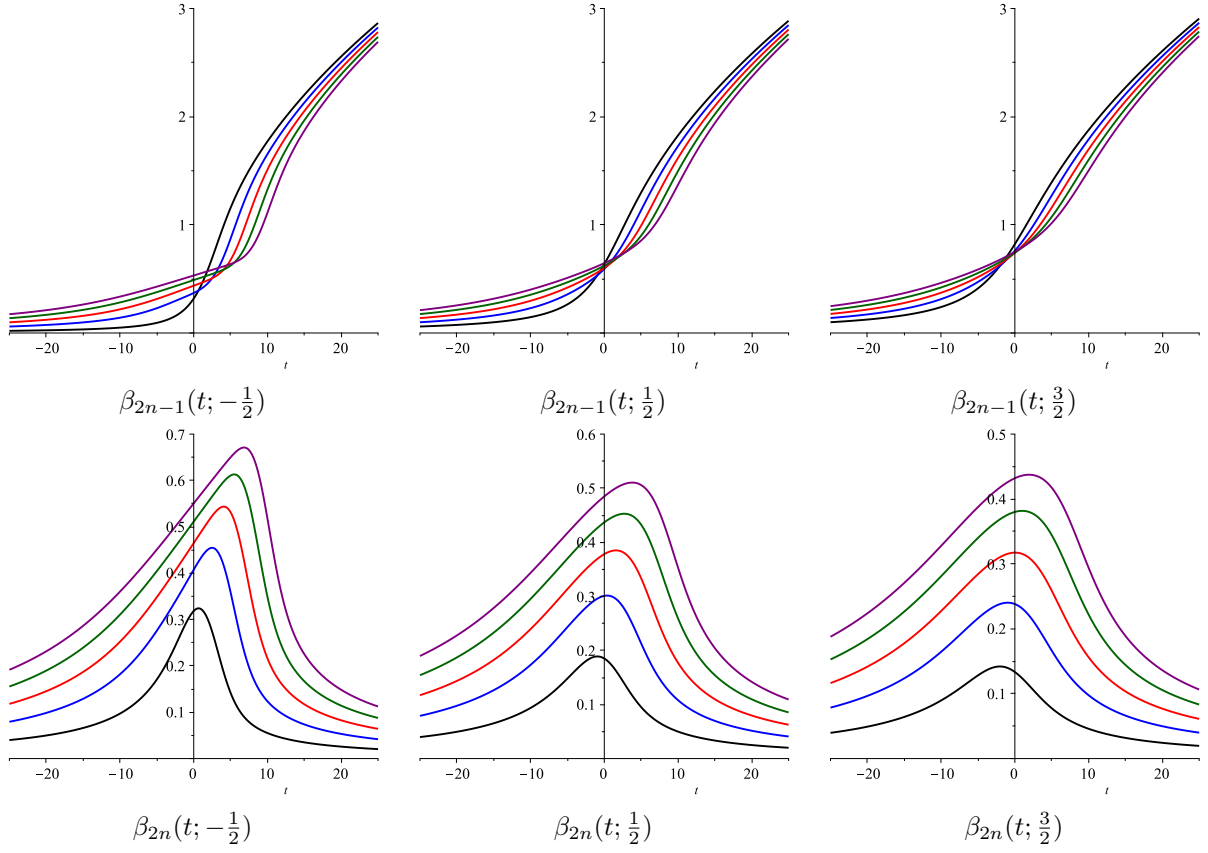


Figure 4.1: Plots of the recurrence coefficients $\beta_{2n-1}(t; \lambda)$ and $\beta_{2n}(t; \lambda)$, $n = 1, 2, \dots, 5$, with $\lambda = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$, for $n = 1$ (black), $n = 2$ (red), $n = 3$ (blue), $n = 4$ (green) and $n = 5$ (purple).

Plots of $\beta_n(t; \lambda)$, for $n = 1, 2, \dots, 10$, with $\lambda = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ are given in Figure 4.1. We see that there is completely different behaviour for $\beta_n(t; \lambda)$ as $t \rightarrow \infty$, depending on whether n is even or odd, which is reflected in Lemma 4.4. From these plots we make the following conjecture.

Conjecture 4.5.

1. The recurrence coefficient $\beta_{2n+1}(t; \lambda)$ is a monotonically increasing function of t .
2. $\beta_{2n+2}(t; \lambda) > \beta_{2n}(t; \lambda)$, for all t .
3. The recurrence coefficient $\beta_{2n}(t; \lambda)$ has one maximum at $t = t_{2n}^*$, with $t_{2n+2}^* > t_{2n}^*$.

Remarks 4.6.

(i) From the Langmuir lattice (2.10) we have

$$\frac{1}{\beta_{2n+1}} \frac{d\beta_{2n+1}}{dt} = \beta_{2n+2} - \beta_{2n},$$

and so $\beta_{2n+2}(t; \lambda) > \beta_{2n}(t; \lambda)$ if and only if $\beta_{2n+1}(t; \lambda)$ is a monotonically increasing function of t since $\beta_{2n+1}(t; \lambda) > 0$.

(ii) Also from the Langmuir lattice we have

$$\frac{1}{\beta_{2n}} \frac{d\beta_{2n}}{dt} = \beta_{2n+1} - \beta_{2n-1}.$$

and so $\beta_{2n}(t; \lambda)$ has a maximum when $\beta_{2n+1}(t; \lambda) = \beta_{2n-1}(t; \lambda)$. Since $\beta_{2n}(t; \lambda) \rightarrow 0$ as $t \rightarrow \pm\infty$ and $\beta_{2n}(t; \lambda) > 0$ then it is a maximum rather than a minimum.

Freud [14] proved the following result, see also [43, §2.3].

Lemma 4.7. *For the weight*

$$w(x) = |x|^\rho \exp(-|x|^6), \quad m \in \mathbb{N},$$

the recurrence coefficient $\beta_n(\rho)$ has the following asymptotic behaviour as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \frac{\beta_n(\rho)}{n^{1/3}} = \frac{1}{\sqrt[3]{60}}.$$

Corollary 4.8. *For the sextic Freud weight (3.1), the recurrence coefficient $\beta_n(t; \lambda)$ has the following asymptotic behaviour as $n \rightarrow \infty$*

$$\lim_{n \rightarrow \infty} \frac{\beta_n(t; \lambda)}{n^{1/3}} = \frac{1}{\sqrt[3]{60}}.$$

Proof. Whilst Lemma 4.7 applies to the sextic Freud weight (3.1) in the case $t = 0$, it is straightforward to extend the proof, for example in [43, §2.3], to the case when $t \neq 0$. \square

Theorem 4.9. *The recurrence coefficient $\beta_n(t; \lambda)$ in the three-term recurrence relation for the sextic Freud weight (3.1) has the formal asymptotic expansions, for fixed t and fixed λ , if n is even*

$$\beta_n(t; \lambda) = \frac{n^{1/3}}{\kappa} + \frac{t\kappa}{90n^{1/3}} - \frac{(2\lambda + 1)\kappa^2}{90n^{2/3}} - \frac{4t(2\lambda + 1)\kappa}{135n^{4/3}} - \frac{[t^3 - 135(7\lambda^2 + 7\lambda + 2)]\kappa^2}{36450n^{5/3}} + \mathcal{O}(n^{-2}), \quad (4.11a)$$

and if n is odd

$$\beta_n(t; \lambda) = \frac{n^{1/3}}{\kappa} + \frac{t\kappa}{90n^{1/3}} + \frac{(2\lambda + 1)\kappa^2}{60n^{2/3}} + \frac{7t(2\lambda + 1)\kappa}{270n^{4/3}} - \frac{[2t^3 + 135(6\lambda^2 + 6\lambda + 1)]\kappa^2}{72900n^{5/3}} + \mathcal{O}(n^{-2}), \quad (4.11b)$$

with $\kappa = \sqrt[3]{60}$, as $n \rightarrow \infty$.

Proof. The recurrence coefficient β_n satisfies the nonlinear discrete equation (4.1), which for $\lambda \neq -\frac{1}{2}$ has a $(-1)^n$ term which suggests an even-odd dependence in $\beta_n(t; \lambda)$. This dependence needs to be taken into account to obtain an asymptotic approximation. Therefore we suppose that

$$\beta_n = \begin{cases} u_n, & \text{if } n \text{ even,} \\ v_n, & \text{if } n \text{ odd,} \end{cases} \quad (4.12)$$

where from Corollary 4.8

$$\lim_{n \rightarrow \infty} \frac{u_n}{\sqrt[3]{n}} = \frac{1}{\sqrt[3]{60}}, \quad \lim_{n \rightarrow \infty} \frac{v_n}{\sqrt[3]{n}} = \frac{1}{\sqrt[3]{60}},$$

then (u_n, v_n) satisfy

$$6u_n(u_{n+2}v_{n+1} + v_{n+1}^2 + 2v_{n+1}u_n + v_{n+1}v_{n-1} + u_n^2 + 2u_nv_{n-1} + v_{n-1}^2 + v_{n-1}u_{n-2}) - 2tu_n = n, \quad (4.13a)$$

$$6v_n(v_{n+2}u_{n+1} + u_{n+1}^2 + 2u_{n+1}v_n + u_{n+1}u_{n-1} + v_n^2 + 2v_nu_{n-1} + u_{n-1}^2 + u_{n-1}v_{n-2}) - 2tv_n = n + 2\lambda + 1. \quad (4.13b)$$

We remark that the transformation (4.12) was used by Cresswell and Joshi [9] when they derived the continuum limit of (4.2). Now suppose that

$$u_n = \frac{n^{1/3}}{60^{1/3}} + \sum_{j=0}^5 \frac{a_j}{n^{j/3}} + \mathcal{O}(n^{-2}), \quad v_n = \frac{n^{1/3}}{60^{1/3}} + \sum_{j=0}^5 \frac{b_j}{n^{j/3}} + \mathcal{O}(n^{-2}), \quad (4.14a)$$

where $a_j, b_j, j = 0, 1, \dots, 5$, are constants to be determined. Then

$$u_{n\pm 1} = \frac{n^{1/3}}{\kappa} + a_0 + \frac{a_1}{n^{1/3}} + \frac{a_2\kappa \pm \frac{1}{3}}{\kappa n^{2/3}} + \frac{a_3}{n} + \frac{a_2 \mp \frac{1}{3}a_1}{n^{4/3}} + \frac{(a_5 \mp \frac{2}{3}a_2)\kappa - \frac{1}{9}}{\kappa n^{5/3}} + \mathcal{O}(n^{-2}), \quad (4.14b)$$

$$v_{n\pm 1} = \frac{n^{1/3}}{\kappa} + b_0 + \frac{b_1}{n^{1/3}} + \frac{b_2\kappa \pm \frac{1}{3}}{\kappa n^{2/3}} + \frac{b_3}{n} + \frac{b_2 \mp \frac{1}{3}b_1}{n^{4/3}} + \frac{(b_5 \mp \frac{2}{3}b_2)\kappa - \frac{1}{9}}{\kappa n^{5/3}} + \mathcal{O}(n^{-2}), \quad (4.14c)$$

$$u_{n\pm 2} = \frac{n^{1/3}}{\kappa} + a_0 + \frac{a_1}{n^{1/3}} + \frac{a_2\kappa \pm \frac{2}{3}}{\kappa n^{2/3}} + \frac{a_3}{n} + \frac{a_2 \mp \frac{2}{3}a_1}{n^{4/3}} + \frac{(a_5 \mp \frac{4}{3}a_2)\kappa - \frac{4}{9}}{\kappa n^{5/3}} + \mathcal{O}(n^{-2}), \quad (4.14d)$$

$$v_{n\pm 2} = \frac{n^{1/3}}{\kappa} + b_0 + \frac{b_1}{n^{1/3}} + \frac{b_2\kappa \pm \frac{2}{3}}{\kappa n^{2/3}} + \frac{b_3}{n} + \frac{b_2 \mp \frac{2}{3}b_1}{n^{4/3}} + \frac{(b_5 \mp \frac{4}{3}b_2)\kappa - \frac{4}{9}}{\kappa n^{5/3}} + \mathcal{O}(n^{-2}), \quad (4.14e)$$

with $\kappa = \sqrt[3]{60}$. Substituting (4.14) into (4.13) and equating powers of n gives

$$\begin{aligned} a_0 = b_0 = 0, \quad a_1 = b_1 = \frac{t\kappa}{90}, \quad a_2 = -\frac{(2\lambda+1)\kappa^2}{90}, \quad b_2 = \frac{(2\lambda+1)\kappa^2}{60}, \\ a_3 = b_3 = 0, \quad a_4 = -\frac{4t(2\lambda+1)\kappa}{135}, \quad b_4 = \frac{7t(2\lambda+1)\kappa}{270}, \\ a_5 = -\frac{[t^3 - 135(7\lambda^2 + 7\lambda + 2)]\kappa^2}{36450}, \quad b_5 = -\frac{[2t^3 + 135(6\lambda^2 + 6\lambda + 1)]\kappa^2}{72900}, \end{aligned}$$

with $\kappa = \sqrt[3]{60}$, and so if n is even then

$$\beta_n = \frac{n^{1/3}}{\kappa} + \frac{t\kappa}{90n^{1/3}} - \frac{(2\lambda+1)\kappa^2}{90n^{2/3}} - \frac{4t(2\lambda+1)\kappa}{135n^{4/3}} - \frac{[t^3 - 135(7\lambda^2 + 7\lambda + 2)]\kappa^2}{36450n^{5/3}} + \mathcal{O}(n^{-2}), \quad (4.15a)$$

whilst if n is odd then

$$\beta_n = \frac{n^{1/3}}{\kappa} + \frac{t\kappa}{90n^{1/3}} + \frac{(2\lambda+1)\kappa^2}{60n^{2/3}} + \frac{7t(2\lambda+1)\kappa}{270n^{4/3}} - \frac{[2t^3 + 135(6\lambda^2 + 6\lambda + 1)]\kappa^2}{72900n^{5/3}} + \mathcal{O}(n^{-2}), \quad (4.15b)$$

as required. \square

Plots of $\beta_n(t; \frac{1}{2})$, for $n = 1, 2, \dots, 100$, with $t = 0, 1, 2, 3, 5, 10$ are given in Figure 4.2 and plots of $\beta_n(2; \lambda)$, for $n = 1, 2, \dots, 100$, with $\lambda = 0, \frac{1}{2}, 1, 2, 3, 5$ are given in Figure 4.3. In these plots, the blue dots \bullet are $\beta_n(t; \lambda)$ for n even and the red dots \bullet are $\beta_n(t; \lambda)$ for n odd. The solid lines are the asymptotics (4.15) and the dashed line is

$$\frac{n^{1/3}}{\kappa} + \frac{t\kappa}{90n^{1/3}}, \quad (4.16)$$

with $\kappa = \sqrt[3]{60}$.

Remark 4.10. In [46], Wang, Zhu and Chen state that

$$\beta_n(t; \lambda) \sim \frac{2^{4/3}t}{\Theta_n(t; \lambda)} + \frac{\Theta_n(t; \lambda)}{45 \times 2^{7/3}}, \quad \text{as } n \rightarrow \infty,$$

where

$$\Theta_n^3(t; \lambda) = 48600 \left[2n + 2\lambda + 1 + \sqrt{(2n + 2\lambda + 1)^2 - \frac{32}{405}t^3} \right].$$

From this it can be shown that as $n \rightarrow \infty$

$$\beta_n(t; \lambda) = \frac{n^{1/3}}{\kappa} + \frac{t\kappa}{90n^{1/3}} + \frac{(2\lambda+1)\kappa^2}{360n^{2/3}} - \frac{t(2\lambda+1)\kappa}{540n^{4/3}} + \mathcal{O}(n^{-5/3}),$$

with $\kappa = \sqrt[3]{60}$, though this is not given in [46], which is the average of the asymptotic expressions for $\beta_n(t; \lambda)$ for n even and odd given by (4.11).

5 Higher Freud weights

In this section we discuss generalised higher Freud weights of the form

$$\omega(x; t) = |x|^{2\lambda+1} \exp(-x^{2m} + tx^2), \quad \lambda > -1, \quad x, t \in \mathbb{R}, \quad (5.1)$$

in the cases when $m = 4$ and $m = 5$ and show how some of the results for the generalised sextic Freud weight (3.1) can be extended to these higher weights.

In Lemma 3.1 we showed that, when $m = 3$ in (5.1), the first moment of the generalised sextic Freud weight is given by

$$\begin{aligned} \mu_0(t; \lambda) &= \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(-x^6 + tx^2) dx = \int_0^{\infty} s^\lambda \exp(ts - s^3) ds \\ &= \frac{1}{3} \Gamma\left(\frac{1}{3}\lambda + \frac{1}{3}\right) {}_1F_2\left(\frac{1}{3}\lambda + \frac{1}{3}; \frac{1}{3}, \frac{2}{3}; \left(\frac{1}{3}t\right)^3\right) + \frac{1}{3} t \Gamma\left(\frac{1}{3}\lambda + \frac{2}{3}\right) {}_1F_2\left(\frac{1}{3}\lambda + \frac{2}{3}; \frac{2}{3}, \frac{4}{3}; \left(\frac{1}{3}t\right)^3\right) \\ &\quad + \frac{1}{6} t^2 \Gamma\left(\frac{1}{3}\lambda + 1\right) {}_1F_2\left(\frac{1}{3}\lambda + 1; \frac{4}{3}, \frac{5}{3}; \left(\frac{1}{3}t\right)^3\right), \end{aligned}$$

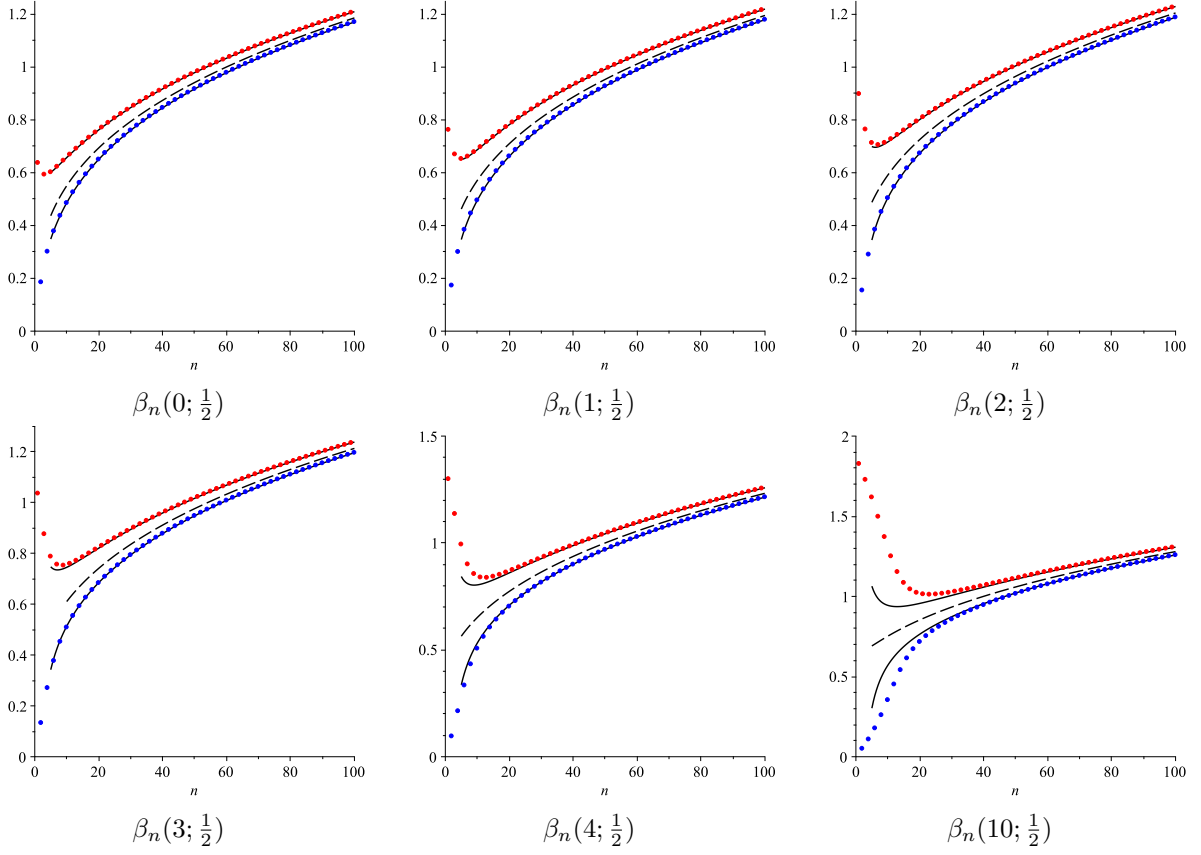


Figure 4.2: Plots of $\beta_n(t; \frac{1}{2})$, for $n = 1, 2, \dots, 100$, with $t = 0, 1, 2, 3, 5, 10$. The blue dots \bullet are $\beta_n(t; \frac{1}{2})$ for n even and the red dots \bullet are $\beta_n(t; \frac{1}{2})$ for n odd. The solid lines are the asymptotics (4.15) and the dashed line is (4.16).

where ${}_1F_2(a_1; b_1, b_2; z)$ is the generalised hypergeometric function and satisfies the third-order equation

$$\frac{d^3\varphi}{dt^3} - \frac{1}{3}t \frac{d\varphi}{dt} - \frac{1}{3}(\lambda + 1)\varphi = 0.$$

Recall that when $m = 2$ in (5.1), the first moment of the generalised quartic Freud weight

$$\omega(x; t) = |x|^{2\lambda+1} \exp(-x^4 + tx^2), \quad \lambda > -1, \quad x, t \in \mathbb{R},$$

is given by (cf. [7])

$$\begin{aligned} \mu_0(t; \lambda) &= \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(-x^4 + tx^2) dx = \frac{\Gamma(\lambda)}{2^{(\lambda+1)/2}} \exp(\frac{1}{8}t^2) D_{-\lambda-1}(-\frac{1}{2}\sqrt{2}t) \\ &= \frac{1}{2}\Gamma(\frac{1}{2}\lambda + \frac{1}{2}) {}_1F_1(\frac{1}{2}\lambda + \frac{1}{2}; \frac{1}{2}; \frac{1}{4}t^2) + \frac{1}{2}t\Gamma(\frac{1}{2}\lambda + 1) {}_1F_1(\frac{1}{2}\lambda + 1; \frac{3}{2}; \frac{1}{4}t^2), \end{aligned} \quad (5.2)$$

where ${}_1F_1(a; b; z)$ is the confluent hypergeometric function, which is equivalent to the Kummer function $M(a, b, z)$. The relationship between the parabolic cylinder function $D_\nu(\zeta)$ and the Kummer function $M(a, b, z)$ is given in [36, §13.6]. Further $\mu_0(t; \lambda)$ given by (5.2) satisfies the second-order equation

$$\frac{d^2\varphi}{dt^2} - \frac{1}{2}t \frac{d\varphi}{dt} - \frac{1}{2}(\lambda + 1)\varphi = 0.$$

5.1 The generalised octic Freud weight

Lemma 5.1. *For the generalised octic Freud weight*

$$\omega(x; t) = |x|^{2\lambda+1} \exp(-x^8 + tx^2), \quad \lambda > -1, \quad x, t \in \mathbb{R}, \quad (5.3)$$

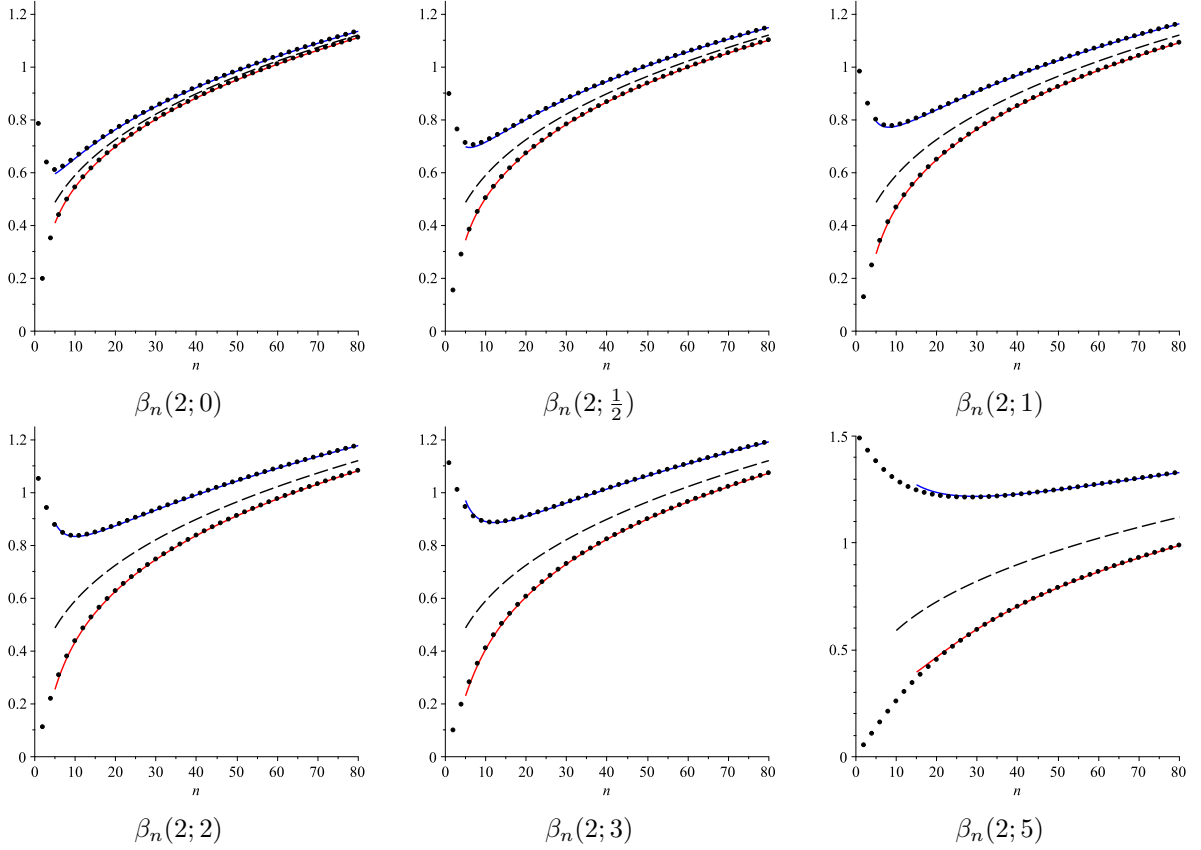


Figure 4.3: Plots of $\beta_n(2; \lambda)$, for $n = 1, 2, \dots, 100$, with $\lambda = 0, \frac{1}{2}, 1, 2, 3, 5$. The blue dots \bullet are $\beta_n(2; \lambda)$ for n even and the red dots \bullet are $\beta_n(2; \lambda)$ for n odd. The solid lines are the asymptotics (4.15) and the dashed line is (4.16).

then the first moment is given by

$$\begin{aligned} \mu_0(t; \lambda) &= \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(-x^8 + tx^2) dx = \int_0^{\infty} s^\lambda \exp(-s^4 + ts) ds \\ &= \frac{1}{4} \Gamma\left(\frac{1}{4}\lambda + \frac{1}{4}\right) {}_1F_3\left(\frac{1}{4}\lambda + \frac{1}{4}; \frac{1}{4}, \frac{1}{2}, \frac{3}{4}; \left(\frac{1}{4}t\right)^4\right) + \frac{1}{4} t \Gamma\left(\frac{1}{4}\lambda + \frac{1}{2}\right) {}_1F_3\left(\frac{1}{4}\lambda + \frac{1}{2}; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; \left(\frac{1}{4}t\right)^4\right) \\ &\quad + \frac{1}{8} t^2 \Gamma\left(\frac{1}{4}\lambda + \frac{3}{4}\right) {}_1F_3\left(\frac{1}{4}\lambda + \frac{3}{4}; \frac{3}{4}, \frac{5}{4}, \frac{3}{2}; \left(\frac{1}{4}t\right)^4\right) + \frac{1}{24} t^3 \Gamma\left(\frac{1}{4}\lambda + 1\right) {}_1F_3\left(\frac{1}{4}\lambda + 1; \frac{5}{4}, \frac{3}{2}, \frac{7}{4}; \left(\frac{1}{4}t\right)^4\right), \end{aligned}$$

where ${}_1F_3(a_1; b_1, b_2, b_3; z)$ is the generalised hypergeometric function. Further $\mu_0(t; \lambda)$ satisfies the fourth-order equation

$$\frac{d^4\varphi}{dt^4} - \frac{1}{4}t \frac{d\varphi}{dt} - \frac{1}{4}(\lambda + 1)\varphi = 0.$$

Proof. The proof is analogous that for the generalised sextic Freud weight (3.1) in Lemma 3.1. \square

Lemma 5.2. The recurrence coefficient $\beta_n(t; \lambda)$ for the generalised octic Freud weight (5.3) has the asymptotics as $t \rightarrow \infty$

$$\begin{aligned} \beta_{2n}(t; \lambda) &= \frac{n}{3t} + \frac{2^{5/3} n(3n - 1 - 2\lambda)}{9t^{7/3}} + \mathcal{O}(t^{-11/3}), \\ \beta_{2n+1}(t; \lambda) &= \left(\frac{1}{4}t\right)^{1/3} - \frac{3n - \lambda + 1}{3t} - \frac{2^{2/3} [36n^2 - 12n(4\lambda - 1) + 6\lambda^2 - 18\lambda + 7]}{27t^{7/3}} + \mathcal{O}(t^{-11/3}), \end{aligned}$$

and as $t \rightarrow -\infty$

$$\beta_{2n}(t; \lambda) = -\frac{n}{t} + \mathcal{O}(t^{-5}), \quad \beta_{2n+1}(t; \lambda) = -\frac{n + \lambda + 1}{t} + \mathcal{O}(t^{-5}).$$

Proof. The proof is similar to that for the generalised sextic Freud weight (3.1) in Lemma 4.4, though with one important difference. For the generalised sextic weight we were able to use the system (4.10) to derive the leading asymptotics for $\beta_n(t; \lambda)$ as $t \rightarrow \pm\infty$. However for the generalised octic Freud weight (5.3) we don't have the analog of (4.10). Instead we can use induction applied to the Langmuir lattice (2.10). Suppose that $u_n(t; \lambda) = \beta_{2n}(t; \lambda)$ and $v_n(t; \lambda) = \beta_{2n+1}(t; \lambda)$, then from the Langmuir lattice (2.10) we obtain

$$u_{n+1} = u_n + \frac{d}{dt} \ln v_n, \quad v_{n+1} = v_n + \frac{d}{dt} \ln u_{n+1}, \quad (5.4)$$

with

$$u_0 = 0, \quad v_0(t; \lambda) = \beta_1(t; \lambda) = \frac{1}{3}\sqrt{3t} + \frac{\lambda - 1}{3t} - \frac{2^{2/3}(6\lambda^2 - 18\lambda + 7)}{27t^{7/3}} + \mathcal{O}(t^{-11/3}).$$

Here $\beta_1(t; \lambda) = \frac{d}{dt} \ln \mu_0(t; \lambda)$ satisfies the third-order equation

$$\frac{d^3 \beta_1}{dt^3} + 4\beta_1 \frac{d^2 \beta_1}{dt^2} + 3 \left(\frac{d\beta_1}{dt} \right)^2 + 6\beta_1^2 \frac{d\beta_1}{dt} + \beta_1^4 - \frac{1}{4}t\beta_1 = \frac{1}{4}(\lambda + 1).$$

□

5.2 The generalised decic Freud weight

Lemma 5.3. *For the generalised decic Freud weight*

$$\omega(x; t) = |x|^{2\lambda+1} \exp(-x^{10} + tx^2), \quad \lambda > -1, \quad x, t \in \mathbb{R},$$

then the first moment is given by

$$\begin{aligned} \mu_0(t; \lambda) &= \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(-x^{10} + tx^2) dx = \int_0^{\infty} s^\lambda \exp(-s^5 + ts) ds \\ &= \frac{1}{5} \Gamma\left(\frac{1}{5}\lambda + \frac{1}{5}\right) {}_1F_4\left(\frac{1}{5}\lambda + \frac{1}{5}; \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \left(\frac{1}{5}t\right)^5\right) + \frac{1}{5} t \Gamma\left(\frac{1}{5}\lambda + \frac{2}{5}\right) {}_1F_4\left(\frac{1}{5}\lambda + \frac{2}{5}; \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{6}{5}; \left(\frac{1}{5}t\right)^5\right) \\ &\quad + \frac{1}{10} t^2 \Gamma\left(\frac{1}{5}\lambda + \frac{3}{5}\right) {}_1F_4\left(\frac{1}{5}\lambda + \frac{3}{5}; \frac{3}{5}, \frac{4}{5}, \frac{6}{5}, \frac{7}{5}; \left(\frac{1}{5}t\right)^5\right) + \frac{1}{30} t^3 \Gamma\left(\frac{1}{5}\lambda + \frac{4}{5}\right) {}_1F_4\left(\frac{1}{5}\lambda + \frac{4}{5}; \frac{4}{5}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}; \left(\frac{1}{5}t\right)^5\right) \\ &\quad + \frac{1}{120} t^4 \Gamma\left(\frac{1}{5}\lambda + 1\right) {}_1F_4\left(\frac{1}{5}\lambda + 1; \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}; \left(\frac{1}{5}t\right)^5\right) \end{aligned}$$

where ${}_1F_4(a_1; b_1, b_2, b_3, b_4; z)$ is the generalised hypergeometric function. Further $\mu_0(t; \lambda)$ satisfies the fifth-order equation

$$\frac{d^5 \varphi}{dt^5} - \frac{1}{5}t \frac{d\varphi}{dt} - \frac{1}{5}(\lambda + 1) \varphi = 0.$$

Proof. As for Lemma 5.1 above, the proof is analogous to that for the generalised sextic Freud weight (3.1) in Lemma 3.1. □

Lemma 5.4. *The recurrence coefficient $\beta_n(t; \lambda)$ for the generalised decic Freud weight (5.3) has the asymptotics as $t \rightarrow \infty$*

$$\begin{aligned} \beta_{2n}(t; \lambda) &= \frac{n}{4t} + \frac{5^{5/4} n(4n - 1 - 2\lambda)}{32 t^{9/4}} + \mathcal{O}(t^{-7/2}), \\ \beta_{2n+1}(t; \lambda) &= \left(\frac{1}{5}t\right)^{1/4} - \frac{8n - 2\lambda + 3}{8t} - \frac{5^{5/4} [40n^2 - 20(2\lambda - 1) + 4\lambda^2 - 16\lambda + 9]}{128 t^{9/4}} + \mathcal{O}(t^{-7/2}), \end{aligned}$$

and as $t \rightarrow -\infty$

$$\beta_{2n}(t; \lambda) = -\frac{n}{t} + \mathcal{O}(t^{-6}), \quad \beta_{2n+1}(t; \lambda) = -\frac{n + \lambda + 1}{t} + \mathcal{O}(t^{-6}).$$

Proof. The proof is very similar to the proof of Lemma 5.2 above. □

6 Discussion

In this paper we studied generalised sextic Freud weights, the associated orthogonal polynomials and the recurrence coefficients. We also investigated the interesting structural connections between the moments of the weight when the order of the polynomial in the exponential factor of the weight is increased. Further analysis of this interesting class of generalised higher order Freud polynomials and their properties, such as asymptotic expressions for the polynomials and their greatest zeros, is currently in progress. It is important to note that our technique of expressing the Hankel determinants Δ_{2n} and Δ_{2n+1} , for symmetric weights such as the generalised Freud weights, in terms of smaller Hankel determinants \mathcal{A}_n and \mathcal{B}_n , as was done in §2, had several benefits. The method resulted in expressions for β_{2n} and β_{2n+1} in terms of \mathcal{A}_n that allowed the derivation of nonlinear discrete and nonlinear differential equations for β_n which do not appear to exist when using Δ_n . Furthermore, from a computational point of view, the numerical evaluation of β_n seems to be much faster when using the determinants \mathcal{A}_n and \mathcal{B}_n rather than Δ_n .

Acknowledgements

We gratefully acknowledge the support of a Royal Society Newton Advanced Fellowship NAF\R2\180669. PAC thanks Evelyne Hubert, Arieh Iserles, Ana Loureiro and Walter Van Assche for their helpful comments and illuminating discussions. PAC would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme “Complex analysis: techniques, applications and computations” when the work on this paper was undertaken. This work was supported by EPSRC grant number EP/R014604/1. We also thank the referees for helpful suggestions and additional references.

References

- [1] S.M. Alsulami, P. Nevai, J. Szabados and W. Van Assche, A family of nonlinear difference equations: existence, uniqueness, and asymptotic behaviour of positive solutions, *J. Approx. Theory*, **193** (2015) 39–55.
- [2] É. Brézin, E. Marinari and G. Parisi, A nonperturbative ambiguity free solution of a string model, *Phys. Lett. B*, **242** (1990) 35–38.
- [3] X.-K. Chang, X.-B. Hu and S.-H. Li, Moment modification, multippeakons, and nonisospectral generalizations, *J. Differential Equations*, **265** (2018) 3858–3887.
- [4] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
- [5] P.A. Clarkson and K. Jordaan, The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation, *Constr. Approx.*, **39** (2014) 223–254.
- [6] P.A. Clarkson and K. Jordaan, Properties of generalized Freud weight, *J. Approx. Theory*, **25** (2018) 148–175.
- [7] P.A. Clarkson, K. Jordaan and A. Kelil, A generalized Freud weight, *Stud. Appl. Math.*, **136** (2016) 288–320.
- [8] P.A. Clarkson and E.L. Mansfield, The second Painlevé equation, its hierarchy and associated special polynomials, *Nonlinearity*, **16** (2003) R1–R26.
- [9] C. Cresswell and N. Joshi, The discrete first, second and thirty-fourth Painlevé hierarchies, *J. Phys. A*, **32** (1999) 655–669.
- [10] S.B. Damelin, Asymptotics of recurrence coefficients for orthonormal polynomials on the line – Magnus’s method revisited, *Math. Comp.*, **73** (2004) 191–209.
- [11] H. Flaschka and A.C. Newell, Monodromy- and spectrum preserving deformations. I, *Commun. Math. Phys.*, **76** (1980) 65–116.
- [12] A.S. Fokas, A.R. Its and A.V. Kitaev, Discrete Painlevé equations and their appearance in quantum-gravity, *Commun. Math. Phys.*, **142** (1991) 313–344.
- [13] G. Freud, *Orthogonal Polynomials*, Akadémiai Kiadó/Pergamon Press, Budapest/Oxford, 1971.
- [14] G. Freud, On the coefficients in the recursion formulae of orthogonal polynomials, *Proc. R. Irish Acad., Sect. A*, **76** (1976) 1–6.
- [15] G. Freud, On the greatest zero of an orthogonal polynomial, *J. Approx. Theory*, **46** (1986) 16–24.
- [16] E. Hendriksen and H. van Rossum, Semi-classical orthogonal polynomials, in: *Orthogonal Polynomials and Applications*, C. Brezinski, A. Draux, A.P. Magnus, P. Maroni and A. Ronveaux (Editors), *Lect. Notes Math.*, vol. **1171**, Springer-Verlag, Berlin, pp. 354–361, 1985.
- [17] A. Iserles and M. Webb, Orthogonal systems with a skew-symmetric differentiation matrix, *Found. Comput. Math.*, **19** (2019) 1191–1221.
- [18] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and its Applications, vol. **98**, Cambridge University Press, Cambridge, 2005.
- [19] A.R. Its and A.V. Kitaev, Mathematical aspects of the nonperturbative 2D quantum gravity, *Modern Phys. Lett. A*, **5** (2079–2083) 1990.
- [20] M. Kac and P. van Moerbeke, On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices, *Advances in Math.*, **16** (1975) 160–169.

- [21] N.A. Kudryashov, The first and second Painlevé equations of higher order and some relations between them, *Phys. Lett. A*, **224** (1997) 353–360.
- [22] E. Levin and D.S. Lubinsky, *Orthogonal Polynomials with Exponential Weights*, CMS Books in Mathematics, Springer-Verlag, New York, 2001.
- [23] A.F. Loureiro and W. Van Assche, Threefold symmetric Hahn-classical multiple orthogonal polynomials, *Anal. Appl. (Singapore)*, **18** (2020) 271–332.
- [24] D. Lubinsky, H. Mhaskar and E. Saff, A proof of Freud’s conjecture for exponential weights, *Constr. Approx.*, **4** (1988) 65–83.
- [25] S. Lyu, Y. Chen and E. Fan, Asymptotic gap probability distributions of the Gaussian unitary ensembles and Jacobi unitary ensembles, *Nuclear Phys. B*, **926** (2018) 639–670.
- [26] A.P. Magnus, A proof of Freud’s conjecture about the orthogonal polynomials related to $|x|^p \exp(-x^{2m})$, for integer m , in: *Orthogonal Polynomials and Applications*, C. Brezinski, A. Draux, A.P. Magnus, P. Maroni and A. Ronveaux (Editors), *Lect. Notes Math.*, vol. **1171**, Springer-Verlag, Berlin, pp. 362–372, 1985.
- [27] A.P. Magnus, Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials, *J. Comput. Appl. Math.*, **57** (1995) 215–237.
- [28] P. Maroni, Prolégomènes à l’étude des polynômes orthogonaux semi-classiques, *Ann. Mat. Pura Appl. (4)*, **149** (1987) 165–184.
- [29] A. Máté, P. Nevai and V. Totik, Asymptotics for the zeros of orthogonal polynomials, *SIAM J. Math. Anal.*, **17** (1986) 745–751.
- [30] A. Máté, P. Nevai and V. Totik, Asymptotics for the zeros of orthogonal polynomials associated with infinite intervals, *J. London Math. Soc.*, **33** (1986) 303–310.
- [31] A. Máté, P. Nevai and T. Zaslavsky, Asymptotic expansions of ratios of coefficients of orthogonal polynomials with exponential weights, *Trans. Amer. Math. Soc.*, **287** (1985) 495–505.
- [32] H.N. Mhaskar, *Introduction to the Theory of Weighted Polynomial Approximation*, World Scientific, Singapore, 1996.
- [33] M.E. Muldoon, Higher monotonicity properties of certain Sturm-Liouville functions. V, *Proc. Roy. Soc. Edinburgh Sect. A*, **77** (1977/78) 23–37.
- [34] P. Nevai, Two of my favorite ways of obtain obtaining asymptotics for orthogonal polynomials, in: *Linear Functional Analysis and Approximation*, P.L. Butzer, R.L. Stens and B. Sz-Nagy (Editors), *Internat. Ser. Numer. Math.* vol. **65**, Birkhauser Verlag, Basel, pp. 417–436, 1984.
- [35] P. Nevai, Géza Freud, orthogonal polynomials and Christoffel functions. A case study, *J. Approx. Theory*, **48** (1986) 3–167.
- [36] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark (Editors), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
- [37] E.A. Rakhmanov, On asymptotic properties of polynomials orthogonal on the real axis, *Math. Sb. (N.S.)*, **119** (1982) 163–203.
- [38] W.H. Reid, Integral representations for products of Airy functions, *Z. Angew. Math. Phys.*, **46** (1995) 159–170.
- [39] R.C. Sheen, Orthogonal polynomials associated with $\exp(-x^6/6)$, PhD Thesis, Ohio State University (1984), 114pp.
- [40] R.C. Sheen, Plancherel-Rotach-type asymptotics for orthogonal polynomials associated with $\exp(-x^6/6)$, *J. Approx. Theory*, **50** (1987) 232–293.
- [41] J. Shohat, A differential equation for orthogonal polynomials, *Duke Math. J.*, **5** (1939) 401–417.
- [42] G. Szegő, *Orthogonal Polynomials*, AMS Colloquium Publications, vol. **23**, Amer. Math. Soc., Providence RI, 1975.
- [43] W. Van Assche, Discrete Painleve equations for recurrence coefficients of orthogonal polynomials, in: *Difference Equations, Special Functions and Orthogonal Polynomials*, S. Elaydi, J. Cushing, R. Lasser, V. Papageorgiou, A. Ruffing and W. Van Assche (Editors), World Scientific, Hackensack, NJ, pp. 687–725, 2007.
- [44] W. Van Assche, *Orthogonal Polynomials and Painlevé Equations*, Australian Mathematical Society Lecture Series, Cambridge University Press, Cambridge, 2018.
- [45] P.R. Vein and P. Dale, *Determinants and Their Applications in Mathematical Physics*, Springer-Verlag, New York, 1999.
- [46] D. Wang, M. Zhu and Y. Chen, On semi-classical orthogonal polynomials associated with a Freud-type weight, *Math Meth Appl Sci.*, **43** (2020) 5295–5313.