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GAP LOCALIZATION OF MULTIPLE TE-MODES BY ARBITRARILY WEAK DEFECTS

B.M.BROWN, V.HOANG, M. PLUM, M. RADOSZ AND I.WOOD

ABSTRACT. This paper considers the propagation of TE-modes in photonic crystal waveguides. The waveguide is created by introducing a linear defect into a periodic background medium. Both the periodic background problem and the perturbed problem are modeled by a divergence type equation. A feature of our analysis is that we allow discontinuities in the coefficients of the operator, which is required to model many photonic crystals. Using the Floquet-Bloch theory in negative order Sobolev spaces, we characterize the precise number of eigenvalues created by the line defect in terms of the band functions of the original periodic background medium for arbitrarily weak defects.

1. INTRODUCTION

Electromagnetic waves in periodically structured media, such as photonic crystals and metamaterials, are a subject of ongoing interest. Typically, the propagation of waves in such media exhibits *band-gaps*; see e.g. [11, 14]. These are intervals on the frequency or energy axis where propagation is forbidden. Mathematically, these correspond to gaps in the spectrum of the operator describing a problem with periodic background medium. The existence of these gaps for certain choices of material coefficients was proved in [6, 7, 10] and in [8] for the full Maxwell case.

In a previous paper [2], we studied the propagation of TE-polarized waves in two-dimensional photonic crystals that contain line defects and gave rigorous sufficient conditions which imply spectral localization in band gaps. Our results were restricted to the case where only one band function (see (14)) contributes to the edge of the band gap. In this paper, we deal with the general situation where multiple bands contribute to the edge of the gap. We also develop a new approach to characterize the precise number of eigenvalues created by the line defect in terms of the band functions of the original periodic structure.

Our results are applicable to non-smooth coefficients. This is motivated by physical applications, where, to produce the typical band-gap spectrum, the coefficient of the background medium is usually piecewise constant. See, for instance, [6, 7, 8]. In order to overcome the arising difficulties, we use Floquet-Bloch theory in negative function spaces [5]. Additionally, all our results do not depend on the precise geometry of the perturbation, e.g. the shape of the inclusions defined by the region within the periodicity cell where the perturbed material coefficients differ from the

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unperturbed ones. For a more detailed discussion of relevant background material, we refer to [2] and references therein.

The structure of our paper is as follows: In section 2 we give a brief description of the periodic problem and its perturbation by a line defect and formulate the operator-theoretic background. The following section 3 introduces the Floquet-Bloch theory in negative spaces with the technical proof provided in Appendix A. Section 4 contains some key preparatory lemmas and estimates. An upper estimate on the number of eigenvalues created in the band gap is given in section 5 while section 6 provides a lower bound and combines all results to our main statement (Theorem 6.4) on the precise number of eigenvalues. Compared to the results in our previous paper [2], the assumptions we make on the band functions of the fully periodic operator are more general. The results of [2] are strengthened in the sense that we allow for multiple bands coming together at the edge of a spectral band and the analysis is refined by providing upper and lower bounds on the number of eigenvalues created by the perturbation. In the case of exactly one band function touching the band edge, we get the existence of gap spectrum, as in [2], but now with the additional information that there is exactly one additional eigenvalue in the gap (of the reduced problem for fixed $\hat{\mathbf{x}}$ -quasimomentum).

We note that a variational method similar to the one here is used in [17] to prove generation of spectrum, though not the precise number of eigenvalues, in the band gaps of periodic Schrödinger operators under a slightly weaker sign condition on the perturbation than we require here.

2. THE OPERATOR THEORETIC FORMULATION

We consider the propagation of electromagnetic waves in a non-magnetic, inhomogeneous medium described by a varying dielectric function $\varepsilon(\mathbf{X})$ with $\mathbf{X} = (x, y, z)$. Assuming that the magnetic field \mathbf{H} has the form $\mathbf{H} = H(x, y)\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ denotes the unit vector in the z -direction, we look for time-harmonic solutions to Maxwell's equations. This leads to the equation

$$(1) \quad -\nabla \cdot \frac{1}{\varepsilon(\mathbf{x})} \nabla H = \lambda H$$

for the \mathbf{z} -component H of the magnetic field. Note that in the context of polarized waves, we assume that all fields and constitutive functions depend only on $\mathbf{x} = (x, y)$.

The periodic background medium is characterised by $\varepsilon_0(\mathbf{x})$, where for simplicity we assume that the unit square $[0, 1]^2$ is a cell of periodicity.

Let $\hat{\mathbf{x}} = (1, 0)$ and $\hat{\mathbf{y}} = (0, 1)$. We now introduce a line defect, which we assume to be aligned along the $\hat{\mathbf{x}}$ -axis and preserving the periodicity in the $\hat{\mathbf{x}}$ -direction. In addition, the defect is assumed to be localised in the $\hat{\mathbf{y}}$ -direction. The new (and perturbed) system is described by a dielectric function $\varepsilon_1(\mathbf{x})$, periodic in $\hat{\mathbf{x}}$ -direction (see Figure 1), i.e.

$$(2) \quad \varepsilon_1(\mathbf{x} + m\hat{\mathbf{x}}) = \varepsilon_1(\mathbf{x}) \quad (m \in \mathbb{Z}).$$

Before we go into the technical details of the definitions of the operators involved, we make brief comments on the physical picture. We will show that by introducing the line defect, the spectrum of the operator $-\nabla \cdot \varepsilon^{-1} \nabla$ is enlarged, i.e. the spectrum of $-\nabla \cdot \varepsilon_1^{-1} \nabla$ is a superset of that of $-\nabla \cdot \varepsilon^{-1} \nabla$. The additional spectrum is generally

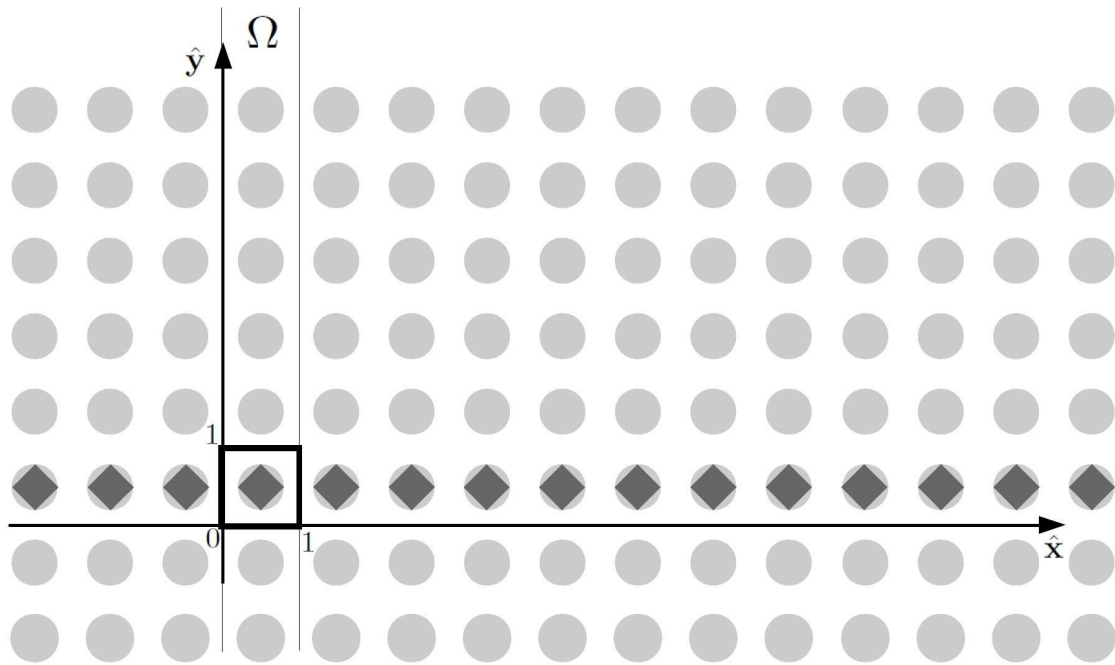


FIGURE 1. Illustration of the line defect and the strip $\Omega = (0, 1) \times \mathbb{R}$.

expected to be absolutely continuous. No rigorous proof of this for the divergence-type operator considered here is, however, available in the literature. The additional spectrum can be described by the Floquet-Bloch decomposition performed below: we introduce an unbounded period cell $\Omega = (0, 1) \times \mathbb{R}$ and corresponding cell operators depending on the quasimomentum k_x in $\hat{\mathbf{x}}$ -direction. The additional spectrum due to the perturbation consists of eigenvalues of the cell problem that depend on k_x . The spectrum of the operator on the whole space \mathbb{R}^2 is synthesized by taking the union of the spectra of the cell problems. The additional eigenvalues created by the perturbation will in general vary with k_x and create the additional continuous spectrum, which can be physically interpreted as the spectrum of guided modes propagating along the waveguide direction.

Assumption 1. *We make the following general assumptions on the material coefficients, valid throughout the paper:*

- (i) $\varepsilon_0, \varepsilon_1 \in L^\infty(\mathbb{R}^2)$.
- (ii) $\varepsilon_i \geq c_0 > 0$ for some constant c_0 and $i = 0, 1$.
- (iii) *The perturbation is nonnegative, i.e.*

$$(3) \quad \varepsilon_1(\mathbf{x}) - \varepsilon_0(\mathbf{x}) \geq 0.$$

- (iv) *There exists a nonempty ball $D \subset [0, 1]^2$ such that $\varepsilon_1 - \varepsilon_0 > 0$ on D .*

Since both the perturbed and unperturbed systems are periodic in the $\hat{\mathbf{x}}$ -direction, we can apply Bloch's theorem [16, 13] to reduce both problems to families of problems on the strip $\Omega := (0, 1) \times \mathbb{R}$, parametrized by the quasi-momentum $k_x \in [-\pi, \pi]$.

For fixed k_x we introduce the space of quasi-periodic H^1 -functions on Ω

$$H_{qp}^1(\Omega) := \{u \in H_{loc}^1(\mathbb{R}^2) : u|_{\Omega} \in H^1(\Omega) \text{ and } u(\mathbf{x}+(m,0)) = e^{ik_x m} u(\mathbf{x}), m \in \mathbb{Z}, \mathbf{x} \in \mathbb{R}^2\}.$$

For $u, v \in H_{qp}^1(\Omega)$ consider the sesquilinear form

$$(4) \quad B_0[u, v] = \int_{\Omega} \left(\frac{1}{\varepsilon_0(\mathbf{x})} \nabla u \overline{\nabla v} + u \bar{v} \right) d\mathbf{x}.$$

As ε_0 is bounded and bounded away from zero, we can introduce a new inner product on $H_{qp}^1(\Omega)$ given by

$$\langle u, v \rangle_{H_{qp}^1(\Omega)} := B_0[u, v]$$

which is equivalent to the standard inner product in $H^1(\Omega)$. When there is no danger of confusion, we denote the associated norm $\|\cdot\|_{H^1}$.

Definition 1. Let $H_{qp}^{-1}(\Omega)$ denote the dual space of $H_{qp}^1(\Omega)$. Let $\phi : H_{qp}^1(\Omega) \rightarrow H_{qp}^{-1}(\Omega)$ be defined by

$$(5) \quad (\phi u)[\varphi] = B_0[u, \varphi] \quad \text{for all } u, \varphi \in H_{qp}^1(\Omega),$$

where the $w[\varphi]$ -notation indicates the dual pairing, i.e. it is the action of the linear functional w on the function $\bar{\varphi}$.

ϕ is an isometric isomorphism, and hence the inner product on $H_{qp}^{-1}(\Omega)$ given by

$$\langle u, v \rangle_{H_{qp}^{-1}(\Omega)} := \langle \phi^{-1}u, \phi^{-1}v \rangle_{H_{qp}^1(\Omega)}$$

induces a norm which coincides with the usual operator sup-norm on $H_{qp}^{-1}(\Omega)$.

After this preparation, we now introduce the realisations of the operators in $H_{qp}^{-1}(\Omega)$ and define the operator $\mathfrak{L}_{\circ} : D(\mathfrak{L}_{\circ}) \rightarrow H_{qp}^{-1}(\Omega)$ by $D(\mathfrak{L}_{\circ}) := H_{qp}^1(\Omega) \subset H_{qp}^{-1}(\Omega)$ with

$$\mathfrak{L}_{\circ}u := \phi u - u.$$

Then $\mathfrak{L}_{\circ} + 1$ is bijective and both \mathfrak{L}_{\circ} and $\mathfrak{G}_{\circ} := (\mathfrak{L}_{\circ} + 1)^{-1}$ are self-adjoint, see [2, Proposition 4.1]. \mathfrak{L}_{\circ} corresponds to the fully periodic problem (1) with $\varepsilon = \varepsilon_0$.

The useful identity

$$(6) \quad \langle u, v \rangle_{H^{-1}} = \langle \phi^{-1}u, \phi^{-1}v \rangle_{H^1} = \langle u, \phi^{-1}v \rangle_{L^2} \quad \text{for } u \in L^2(\Omega), v \in H_{qp}^{-1}(\Omega)$$

follows from the definitions of ϕ and \mathfrak{L}_{\circ} .

Let (Λ_0, Λ_1) be a spectral gap for \mathfrak{L}_{\circ} and $\mu \in ((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$. Then $1/\mu \in \rho(\mathfrak{L}_{\circ} + 1)$, so

$$(7) \quad \frac{1}{\mu} \left[\frac{1}{\mu} - (\mathfrak{L}_{\circ} + 1) \right]^{-1} = ((I - \mu(\mathfrak{L}_{\circ} + 1))^{-1}) = (I - \mu\mathfrak{G}_{\circ}^{-1})^{-1}$$

is well-defined and maps $H_{qp}^{-1}(\Omega)$ bijectively onto $H_{qp}^1(\Omega)$. The operator $(I - \mu\mathfrak{G}_{\circ}^{-1})^{-1}$ is the solution operator to the problem

$$\langle u, \varphi \rangle_{L^2} - \mu \int_{\Omega} \left(\frac{1}{\varepsilon_0} \nabla u \overline{\nabla \varphi} + u \bar{\varphi} \right) d\mathbf{x} = f[\varphi], \quad \text{for all } \varphi \in H_{qp}^1(\Omega)$$

for a given $f \in H_{qp}^{-1}(\Omega)$.

We now examine the perturbed problem. Let the bilinear form B_1 and the operator $\mathfrak{L}_1 : H_{qp}^1(\Omega) \rightarrow H_{qp}^{-1}(\Omega)$ be defined by

$$(8) \quad B_1[u, \varphi] := \int_{\Omega} \left[\frac{1}{\varepsilon_1} \nabla u \overline{\nabla \varphi} + u \overline{\varphi} \right] dx \text{ and } ((\mathfrak{L}_1 + 1)u)[\varphi] = B_1(u, \varphi) \quad \text{for } u, \varphi \in H_{qp}^1(\Omega).$$

Moreover, we define $\mathfrak{G}_1 = (\mathfrak{L}_1 + 1)^{-1}$. Then $\mathfrak{G}_1 : H_{qp}^{-1}(\Omega) \rightarrow H_{qp}^1(\Omega)$ is a bounded non-negative operator (see [2, Lemma 4.3 & 4.4].)

Remark 1. *We note that just as in [5, Section 5], the spectra of the H^{-1} -realizations \mathfrak{L}_\circ and \mathfrak{L}_1 and the corresponding realizations of the operators in $L^2(\Omega)$ coincide.*

We will give conditions which ensure that localized modes, i.e. eigenvalues of the perturbed operator \mathfrak{L}_1 , appear in the band gap (Λ_0, Λ_1) of the unperturbed operator \mathfrak{L}_\circ under arbitrarily weak perturbations and use a Birman-Schwinger-type reformulation to find the eigenvalues λ of the operator \mathfrak{L}_1 in the band gap. For proofs of the results in this section and more details on the reformulation, see [2, Section 5].

Consider the operator

$$K := (\mathfrak{G}_\circ^{-1} \mathfrak{G}_1 - I) : H_{qp}^{-1}(\Omega) \rightarrow H_{qp}^{-1}(\Omega),$$

set $\mathcal{K} = \overline{\text{ran } K} \subseteq H_{qp}^{-1}(\Omega)$ and let $P : H_{qp}^{-1}(\Omega) \rightarrow \mathcal{K}$ be the orthogonal projection on \mathcal{K} . On \mathcal{K} , we introduce a new inner product given by

$$(9) \quad \langle f, g \rangle_{\mathcal{K}} := \langle Kf, g \rangle_{H^{-1}}.$$

The symmetry and definiteness of this inner product is shown in [2, Appendix A].

The following lemma gives useful estimates for K in terms of the size of the perturbation. In particular it shows that for small perturbations, the only dependence of the bound for $\|K\|$ on the perturbation ε_1 is through the term $\left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_{\infty}$.

Lemma 2.1. *The following estimates hold:*

(i)

$$\|K\| \leq \|\mathfrak{G}_1\|_{H^{-1} \rightarrow H^1} \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_{\infty}$$

(ii)

$$\|Ku\|_{H^{-1}}^2 \leq \|K\| \|u\|_{\mathcal{K}}^2 \quad (u \in \mathcal{K})$$

(iii) *Moreover, if $\eta := \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_{\infty} < 1/\|\mathfrak{G}_\circ\|_{H^{-1} \rightarrow H^1}$, then*

$$\|\mathfrak{G}_1\|_{H^{-1} \rightarrow H^1} \leq \frac{\|\mathfrak{G}_\circ\|_{H^{-1} \rightarrow H^1}}{1 - \eta \|\mathfrak{G}_\circ\|_{H^{-1} \rightarrow H^1}}.$$

Proof. See [2, Lemma 5.2]. □

Next for $\mu = (\lambda + 1)^{-1} \in ((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$ define

$$(10) \quad A_\mu := P(I - \mu \mathfrak{G}_\circ^{-1})^{-1} K : \mathcal{K} \rightarrow \mathcal{K}.$$

Lemma 2.2. *The equation $(\mathfrak{L}_1 - \lambda)u = 0$ with $\lambda \in (\Lambda_0, \Lambda_1)$ has a non-trivial solution u iff -1 is an eigenvalue of A_μ , where $\mu = (\lambda + 1)^{-1}$.*

Proof. See [2, Lemma 5.3]. □

To be able to use the variational characterisation of eigenvalues we need the following property of the operator A_μ .

Proposition 2.3. $A_\mu : \mathcal{K} \rightarrow \mathcal{K}$ is a compact, symmetric operator on \mathcal{K} .

Proof. See [2, Proposition 5.7]. \square

3. FLOQUET-BLOCH THEORY IN H^{-1}

For our results we will make use of Floquet-Bloch theory in $H_{qp}^{-1}(\Omega)$. We introduce the notation and state the results needed here. A fuller account with proofs of some properties of the Floquet-Bloch theory in H_{qp}^{-1} can be found in [5]. The Brillouin zone in our setting is the interval $[-\pi, \pi]$. This corresponds to our periodic cell in \hat{y} -direction which is the interval $[0, 1]$.

Definition 2. For all k in the Brioullin zone $[-\pi, \pi]$, we introduce an extension operator $E_k : L^2((0, 1)^2) \rightarrow L_{loc}^2(\Omega)$ with

$$(E_k u)(x, y + p) := e^{ikp} u(x, y)$$

for all $(x, y) \in (0, 1)^2$, $p \in \mathbb{Z}$.

The partial Floquet transform

$$U : L^2(\Omega) \rightarrow L^2((0, 1)^2 \times [-\pi, \pi])$$

is defined on functions with compact support by

$$(Uu)(x, y, k) := \frac{1}{(2\pi)^{1/2}} \sum_{n \in \mathbb{Z}} e^{ikn} u(x, y - n) \quad \text{for } (x, y) \in (0, 1)^2, k \in [-\pi, \pi]$$

and extended to $L^2(\Omega)$ by continuity.

U is an isometric isomorphism and

$$(11) \quad (U^{-1}v)(x, y) = \frac{1}{(2\pi)^{1/2}} \int_{-\pi}^{\pi} (E_k v(\cdot, \cdot, k))(x, y) dk$$

(see [13]).

Definition 3. Let $H_{qp}^1((0, 1)^2)$ denote the set of restrictions of functions $u \in H_{qp}^1(\Omega)$ to $(0, 1)^2$ endowed with the H^1 -inner product. For all $k \in [-\pi, \pi]$, let

$$\mathcal{H}_k := \{u \in H_{qp}^1((0, 1)^2) : E_k u \in H_{loc}^1(\Omega)\}.$$

Note that being an element of \mathcal{H}_k requires a weak form of semi-periodic boundary conditions on the boundary of $(0, 1)^2$. We denote by N_k the mapping

$$N_k : \mathcal{H}_0 \rightarrow \mathcal{H}_k, \quad (N_k u)(x, y) := e^{iky} u(x, y)$$

and extend it to a mapping $\mathcal{H}'_0 \rightarrow \mathcal{H}'_k$ between the dual spaces by

$$N_k u[\varphi] := u[N_k^{-1} \varphi] \text{ for all } u \in \mathcal{H}'_0, \varphi \in \mathcal{H}_k.$$

Let

$$\mathcal{H} = \{u \in L^2((0, 1)^2 \times [-\pi, \pi]) : \forall k \in [-\pi, \pi] \quad u(\cdot, \cdot, k) \in \mathcal{H}_k,$$

$$\left. \begin{array}{l} \text{the mapping } \left\{ \begin{array}{l} [-\pi, \pi] \rightarrow \mathbb{C} \\ k \mapsto \langle N_k^{-1} u(\cdot, \cdot, k), \varphi \rangle_{H^1((0, 1)^2)} \end{array} \right\} \text{ is measurable for all } \varphi \in \mathcal{H}_0, \\ \text{and } \|u\|_{\mathcal{H}} < \infty \end{array} \right\}$$

where, as usual, $\forall' k$ means for almost all k and the norm $\|\cdot\|_{\mathcal{H}}$ is induced by the inner product

$$\langle u, v \rangle_{\mathcal{H}} = \int_{-\pi}^{\pi} \langle u(\cdot, \cdot, k), v(\cdot, \cdot, k) \rangle_{H^1((0,1)^2)} dk.$$

\mathcal{H} can be viewed as the space of all functions $u(x, y, k) = (N_k v(k))(x, y)$ with $v \in L^2([-\pi, \pi], \mathcal{H}_0)$.

Remark 2. \mathcal{H} can also be defined as the direct integral of the \mathcal{H}_k , which are then regarded as fibers over $k \in [-\pi, \pi]$ (see e.g. [18]).

By $\phi_k : \mathcal{H}_k \rightarrow \mathcal{H}'_k$ and $\phi_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{H}'$ we denote the canonical isometric isomorphisms (defined analogously to (5)).

Analogously to (6), we get

$$(12) \quad \langle u, v \rangle_{\mathcal{H}'_k} = \langle \phi_k^{-1} u, \phi_k^{-1} v \rangle_{\mathcal{H}_k} = \langle u, \phi_k^{-1} v \rangle_{L^2} \quad \text{for } u \in L^2((0, 1)^2), v \in \mathcal{H}'_k.$$

Let V be given by $V := U|_{H^1_{qp}(\Omega)}$. For $u, v \in H^1_{qp}(\Omega)$ we have $Vu, Vv \in \mathcal{H}$, and

$$\int_{-\pi}^{\pi} b_0[Vu(\cdot, \cdot, k), Vv(\cdot, \cdot, k)] dk = B_0[u, v],$$

where b_0 is defined as B_0 in (4) with the range of integration Ω replaced by $[0, 1]^2$ (see [5, Theorem 3.7]). The form b_0 induces the inner product on the space $H^1_{qp}((0, 1)^2)$ as well as on \mathcal{H}_k giving

$$(13) \quad \langle Vu, Vv \rangle_{\mathcal{H}} = \langle u, v \rangle_{H^1(\Omega)}.$$

Moreover, $V : H^1_{qp}(\Omega) \rightarrow \mathcal{H}$ is an isometric isomorphism (see [5, Theorem 3.8]), whence also its adjoint $V^* : \mathcal{H}' \rightarrow H^1_{qp}(\Omega)$ is. In particular, \mathcal{H} is a Hilbert space. The map

$$\hat{V} := (V^*)^{-1} : H^1_{qp}(\Omega) \rightarrow \mathcal{H}'$$

is an isometric isomorphism and $\hat{V}|_{L^2(\Omega)} = U$ (see [5, Lemma 3.9]). For $k \in [-\pi, \pi]$, let \mathcal{H}_k be the domain of the operator L_k defined in \mathcal{H}'_k by

$$L_k : \mathcal{H}_k \subseteq \mathcal{H}'_k \rightarrow \mathcal{H}'_k, \quad L_k u = \phi_k u - u.$$

This means that (cf. (5))

$$\langle (L_k + 1)u, \varphi \rangle = b_0[u, \varphi] \text{ for } u, \varphi \in \mathcal{H}_k.$$

Note that \mathcal{H}_k is dense in \mathcal{H}'_k since \mathcal{H}_k is dense in $L^2((0, 1)^2)$ and thus, by duality $L^2((0, 1)^2)$ is dense in \mathcal{H}'_k . Analogously to the case of \mathfrak{L}_0 , the operator L_k is self-adjoint. $(L_k + 1)^{-1}$ is compact since it is bounded from \mathcal{H}'_k to \mathcal{H}_k , which is compactly embedded in \mathcal{H}'_k .

It is possible to transform the spectral problem for the operators L_k which have k -dependent domains to a spectral problem for an operator family where the k -dependence is transferred to the differential expression (see, e.g. [4], for the transformation in a similar situation). This family is analytic of type (A) in the sense of Kato and using [12, Theorem VII.3.9 and Remark VII.3.10], we can obtain sequences of real-valued functions $\{\lambda_s(k)\}_{s \in \mathbb{N}}$ and eigenfunctions $\{\varphi_s(k)\}_{s \in \mathbb{N}}$, normalized in \mathcal{H}'_k . The functions $\lambda_s(k)$ and $\varphi_s(k)$ are all real-analytic functions in the variable k on $[-\pi, \pi]$ and are such that

$$(14) \quad (L_k + 1)\varphi_s(k) = \lambda_s(k)\varphi_s(k).$$

We note that the eigenvalues are not necessarily ordered by magnitude. We call the functions $\lambda_s(k)$ the band functions and $\varphi_s(k)$ the Bloch functions.

Throughout, we will need to make the following non-degeneracy assumption on the band functions:

Assumption 2. *The band functions λ_s are not constant as functions of $k \in [-\pi, \pi]$.*

For notational convenience, we also introduce

$$\psi_s(\cdot, k) := \frac{1}{\sqrt{\lambda_s(k) + 1}} \varphi_s(\cdot, k).$$

The set $\{\psi_s(\cdot, k)\}$ forms an orthonormal set in $L^2((0, 1)^2)$, which is also complete as the set of eigenfunctions of the self-adjoint realisation of the operators in $L^2((0, 1)^2)$. As a general rule, we will always extend the $\psi_s(k), \varphi_s(k)$ to the whole of Ω in a k -quasiperiodic manner, i.e.

$$\psi_s(\cdot + m\hat{\mathbf{y}}, k) = e^{ikm} \psi_s(\cdot, k).$$

In what follows, for $f \in H_{qp}^{-1}(\Omega)$ we denote by $(\hat{V}f)_k$ the element of \mathcal{H}'_k , defined by

$$(15) \quad [(\hat{V}f)_k][\varphi] := \langle (\phi_{\mathcal{H}}^{-1} \hat{V}f)(\cdot, k), \varphi \rangle_{\mathcal{H}_k} \text{ for } \varphi \in \mathcal{H}_k.$$

Lemma 3.1. *For almost all $k \in [-\pi, \pi]$ and $f \in H_{qp}^{-1}(\Omega)$*

$$(16) \quad \phi_k^{-1}(\hat{V}f)_k = (\phi_{\mathcal{H}}^{-1} \hat{V}f)(\cdot, k).$$

Proof. Let $w \in \mathcal{H}_k$. Then

$$\langle \phi_k^{-1}(\hat{V}f)_k, w \rangle_{\mathcal{H}_k} = (\hat{V}f)_k[w] = \langle (\phi_{\mathcal{H}}^{-1} \hat{V}f)(\cdot, k), w \rangle_{\mathcal{H}_k},$$

which proves the identity. \square

Having introduced the required notation, we are now able to state the results on expansions of functions in terms of the Bloch waves needed for this paper. The proofs can be found in Appendix A.

Proposition 3.2. (1) $\sigma(\mathfrak{L}_{\circ}) = \overline{\cup_k \sigma(L_k)}$.

(2) For $f \in H_{qp}^{-1}(\Omega)$ and $\lambda \notin \sigma(\mathfrak{L}_{\circ})$,

$$(17) \quad (V(\mathfrak{L}_{\circ} - \lambda)^{-1}f)(x, k) = (L_k - \lambda)^{-1}[(\hat{V}f)_k](x).$$

holds.

(3) For $g \in H_{qp}^{-1}(\Omega)$ and $\lambda \notin \sigma(\mathfrak{L}_{\circ})$ the equality

$$(\mathfrak{L}_{\circ} - \lambda)^{-1}g = \frac{1}{\sqrt{2\pi}} \sum_{s=1}^{\infty} \int_{-\pi}^{\pi} \frac{1}{\lambda_s(k) - \lambda} (\hat{V}g)_k[\psi_s(\cdot, k)] \psi_s(\cdot, k) dk$$

holds, where the series converges in $L^2(\Omega)$.

(4) For $\mu \in ((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$ and $u \in H_{qp}^{-1}(\Omega)$,

$$\begin{aligned} & \left\langle -\frac{1}{\mu} \left(\mathfrak{L}_{\circ} + 1 - \frac{1}{\mu} \right)^{-1} (Ku), Ku \right\rangle_{H^{-1}} \\ &= \int_{-\pi}^{\pi} \sum_{s=1}^{\infty} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk. \end{aligned}$$

(5) For $f \in H_{qp}^{-1}(\Omega)$,

$$(18) \quad \|f\|_{H^{-1}}^2 = \frac{1}{\sqrt{2\pi}} \sum_{s=1}^{\infty} \int_{-\pi}^{\pi} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \phi_k^{-1}(\hat{V}f)_k, \psi_s(k) \right\rangle_{H^1((0,1)^2)} \right|^2 dk.$$

We refer to Figure 2 for an overview of the spaces and mappings discussed here.

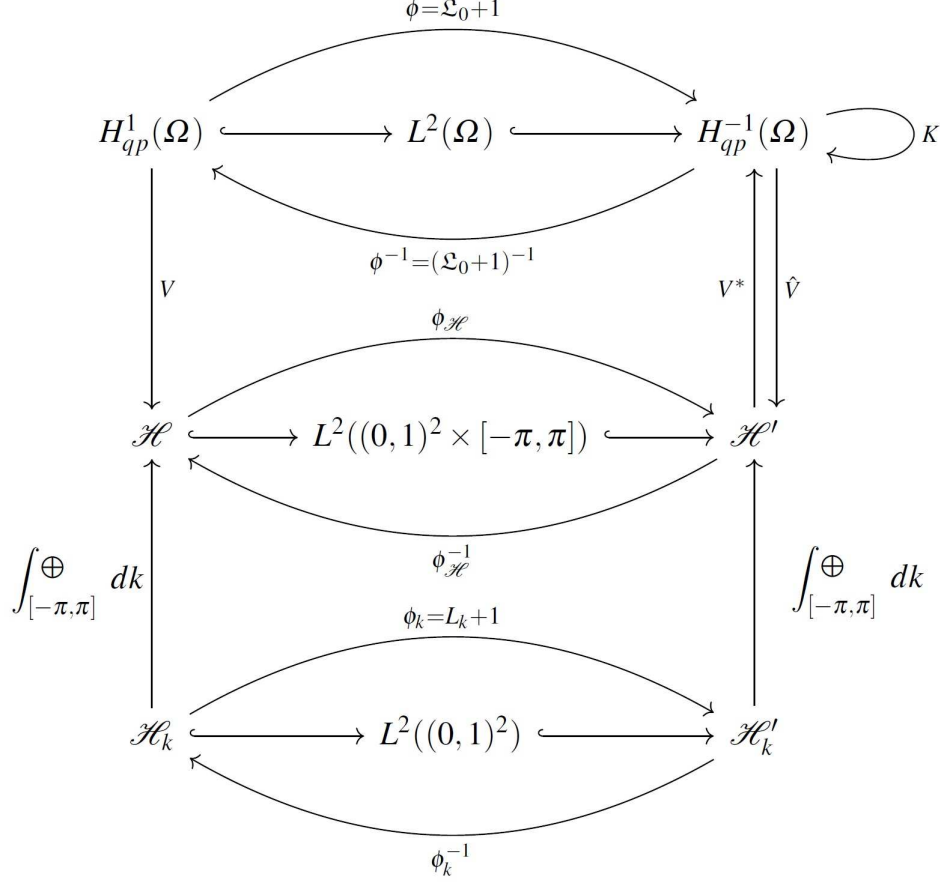


FIGURE 2. Spaces, isomorphisms and key mappings. The symbol $\int_{[-\pi, \pi]}^{\oplus} dk$ indicates the forming of a direct integral of the \mathcal{H}_k .

4. PREPARATORY RESULTS

Our strategy consists in following $\kappa_m(\mu)$, the m -th lowest negative eigenvalue (if it exists) of the operator A_μ , introduced in (10), as μ varies. The following standard variational characterisations (see, for example, [9]) hold:

$$(19) \quad \kappa_m(\mu) = \max_{\text{codim } L=m-1} \inf_{\phi \in L} \frac{\langle \phi, A_\mu \phi \rangle_{\mathcal{K}}}{\langle \phi, \phi \rangle_{\mathcal{K}}} = \min_{\text{dim } L=m} \max_{\phi \in L} \frac{\langle \phi, A_\mu \phi \rangle_{\mathcal{K}}}{\langle \phi, \phi \rangle_{\mathcal{K}}}$$

where the first max and the min are taken over subspaces L of \mathcal{K} .

Lemma 4.1. *For μ in the spectral gap $((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$, the mapping $\mu \mapsto \kappa_m(\mu)$ is continuous and increasing.*

The proof is virtually identical to that of Lemma 6.1 in [2] and will be omitted here. We remind the reader that Λ_1 is the lowest point of a spectral band and lies at the top edge of a gap. The solutions of the equation $\lambda_s(k) = \Lambda_1$ will play an important role in our analysis. We first introduce the following sets:

$$\begin{aligned}\Sigma &= \{(s, k) \in \mathbb{N} \times [-\pi, \pi] : \lambda_s(k) = \Lambda_1\}, \\ S_k &= \{s \in \mathbb{N} : (s, k) \in \Sigma\}, \\ S &= \{s \in \mathbb{N} : \text{there is a } k \text{ with } (s, k) \in \Sigma\} = \bigcup_k S_k.\end{aligned}$$

We will next see that the set Σ is finite. In the following we denote the elements of Σ by (s_j, k_j) , $j = 1, \dots, n$ and set $\psi_j = \psi_{s_j}(k_j)$ and $\varphi_j = \varphi_{s_j}(k_j)$.

Lemma 4.2. *The set Σ is a non-empty finite set. Moreover, $\lambda_s(k) \rightarrow \infty$ as $s \rightarrow \infty$, uniformly in $k \in [-\pi, \pi]$.*

Proof. We first note the coincidence of the spectra of the L^2 and H^{-1} realisations (see [5, Section 5]), so it is enough to consider the L^2 -realisation of the operator. The result then follows from [4, Proposition 3.2 and its proof]. \square

Corollary 4.3. *When the band functions are re-ordered by magnitude there is an $s_0 \in \mathbb{N}$ such that for all $s \geq s_0$ and for all $k \in [-\pi, \pi]$ we have $\lambda_s(k) \geq \Lambda_1$, while for all $s < s_0$ and for all $k \in [-\pi, \pi]$, $\lambda_s(k) \leq \Lambda_0$ holds.*

Proof. The assertion follows from continuity of the band functions, existence of the spectral gap and Lemma 4.2. \square

Lemma 4.4. *The set Σ is isolated in the sense that there is $\eta > 0$ such that for all $s \notin S$, $|\lambda_s(k) - \Lambda_1| \geq \eta$ for all $k \in [-\pi, \pi]$.*

Proof. The proof is the same as that of [3, Lemma 3.7], noting that analyticity and non-constancy of the band function in the one-dimensional variable k are sufficient to avoid Assumption 3.3 in [3] in the proof. \square

Noting that $\varepsilon_0 - \varepsilon_1$ is compactly supported in Ω , for $\psi \in H_{loc}^1(\Omega)$, let $(\mathfrak{L}_0 - \mathfrak{L}_1)\psi$ be the element of $H_{qp}^{-1}(\Omega)$ defined by

$$(20) \quad [(\mathfrak{L}_0 - \mathfrak{L}_1)\psi][\varphi] := \int_{\Omega} \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right) \nabla \psi \nabla \bar{\varphi} \text{ for all } \varphi \in H_{qp}^1(\Omega).$$

Moreover, letting $H_c^1(\Omega)$ denote the functions in $H_{qp}^1(\Omega)$ with compact support, we define for $\psi \in H_{loc}^1(\Omega)$, the element $\mathfrak{L}_0\psi$ of $(H_c^1(\Omega))'$ by

$$(21) \quad [\mathfrak{L}_0\psi][\varphi] := \int_{\Omega} \frac{1}{\varepsilon_0} \nabla \psi \nabla \bar{\varphi} \text{ for all } \varphi \in H_c^1(\Omega).$$

Define

$$(22) \quad \mathcal{L} = \{u \in \mathcal{K} : \forall j = 1, \dots, n. \quad [(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_j][\mathfrak{G}_1 u] = 0\},$$

where the action is interpreted as in (20).

Remark 3. Observe that the action of Ku on any $\varphi \in H_{qp}^1$ can be written as

$$Ku[\varphi] = \int_{\Omega} \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right) \nabla \mathfrak{G}_1 u \overline{\nabla \varphi}.$$

Since $\left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right)$ has compact support, the action of Ku can be extended to any H_{loc}^1 -function φ . Hence we shall define

$$Ku[\varphi] := \int_{\Omega} \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right) \nabla \mathfrak{G}_1 u \overline{\nabla \varphi} \quad (\varphi \in H_{loc}^1).$$

Then recalling (20) we get

$$(23) \quad Ku[\psi_j] = \int_{\Omega} \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right) \nabla \mathfrak{G}_1 u \overline{\nabla \psi_j} = [(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_j][\mathfrak{G}_1 u].$$

Lemma 4.5. The codimension of \mathcal{L} satisfies $\text{codim } \mathcal{L} = n$.

Proof. For $u \in \mathcal{K}$,

$$[(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_j][\mathfrak{G}_1 u] = \int_{\Omega} \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right) \nabla \psi_j \overline{\nabla \mathfrak{G}_1 u}.$$

Let $\theta \in C^\infty(\Omega)$ be compactly supported in the $\hat{\mathbf{y}}$ -direction with $\theta = 1$ on $[0, 1]^2$. Then

$$(24) \quad \begin{aligned} \overline{[(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_j][\mathfrak{G}_1 u]} &= \int_{\Omega} \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right) \overline{\nabla(\theta\psi_j)} \nabla \mathfrak{G}_1 u \\ &= Ku[\theta\psi_j] = \langle Ku, \phi(\theta\psi_j) \rangle_{H^{-1}} \\ &= \langle Ku, P\phi(\theta\psi_j) \rangle_{H^{-1}} = \langle u, P\phi(\theta\psi_j) \rangle_{\mathcal{K}}. \end{aligned}$$

Hence, $\mathcal{L} = \text{span}\{P\phi(\theta\psi_j) : j = 1, \dots, n\}^\perp$ and we need to show that

$$\dim(\text{span}\{P\phi(\theta\psi_j) : j = 1, \dots, n\}) = n.$$

Assume $\sum \alpha_j P\phi(\theta\psi_j) = 0$ in \mathcal{K} . As K is symmetric and non-negative in $H_{qp}^{-1}(\Omega)$, this is equivalent to $\Psi = \sum \alpha_j \phi(\theta\psi_j) \in \ker K$. Now, $K\Psi = 0$ is equivalent to $\mathfrak{G}_0\Psi = \mathfrak{G}_1\Psi$. Let $v := \mathfrak{G}_0\Psi$. Then $(\mathfrak{L}_0 + 1)v = (\mathfrak{L}_1 + 1)v = \Psi$, so

$$0 = ((\mathfrak{L}_0 - \mathfrak{L}_1)v)[v] = \int \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right) |\nabla v|^2$$

and thus $\nabla v|_D = 0$ where D is the ball from Assumption 1 (iv). Hence $\mathfrak{L}_0 v|_D = \mathfrak{L}_1 v|_D = 0$ (in the sense that for any $\varphi \in C_c^\infty(D)$ we have $[\mathfrak{L}_0 v][\varphi] = [\mathfrak{L}_1 v][\varphi] = 0$). Moreover, $v = \phi^{-1}\Psi = \sum \alpha_j \theta\psi_j$. Therefore, for any $\varphi \in C_c^\infty(D)$,

$$\begin{aligned} [\mathfrak{L}_0 v][\varphi] &= \sum \alpha_j \int_{\Omega} \frac{1}{\varepsilon_0} \nabla(\theta\psi_j) \nabla \overline{\varphi} \\ &= \sum \alpha_j \int_D \frac{1}{\varepsilon_0} \nabla(\theta\psi_j) \nabla \overline{\varphi} \\ &= \sum \alpha_j \int_D \frac{1}{\varepsilon_0} \nabla \psi_j \nabla \overline{\varphi} \\ &= \sum \alpha_j \int_{\Omega} \frac{1}{\varepsilon_0} \nabla \psi_j \nabla \overline{\varphi} \\ &= \sum \alpha_j \Lambda_1 \psi_j[\varphi] = \Lambda_1 v[\varphi]. \end{aligned}$$

So $\mathfrak{L}_\circ v|_D = \Lambda_1 v|_D$, and hence $v|_D = 0$. By unique continuation, see [1], $v = 0$ and as the ψ_j are linearly independent over D (see [3, Proposition 3.8]), we get $\alpha_j = 0$ for all j . \square

5. UPPER BOUND ON THE NUMBER OF EIGENVALUES

The main result in this section will require the following additional non-degeneracy assumption on the band functions $\lambda_s(k)$.

Assumption 3. *There are $\alpha > 0$ and $\delta > 0$ such that for all $(\hat{s}, \hat{k}) \in \Sigma$ and $k \in [-\pi, \pi]$ satisfying $|k - \hat{k}| \leq \delta$,*

$$\lambda_{\hat{s}}(k) \geq \Lambda_1 + \alpha|k - \hat{k}|^2$$

holds.

Remark 4. *The assumption is true if the zero of $\lambda_{\hat{s}}(\hat{k}) - \Lambda_1$ is only of order 2. Non-degeneracy assumptions of a similiar form are common in the mathematical and physical literature (see e.g. [15] and references therein) and are believed to be “generically” true. In other words, it is believed that degeneracy of the band function can be removed by a small perturbation of the coefficients of the differential operator.*

The next lemma provides a uniform bound on contributions to the Rayleigh quotient away from points in Σ .

Lemma 5.1. *Let $\mu \in ((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$. If*

$$(25) \quad |1 - \mu(\lambda_s(k) + 1)|^{-1} \leq c_1$$

uniformly for (s, k) in a set $\tilde{S} \times J \subseteq \mathbb{N} \times [-\pi, \pi]$, then

$$\left| \sum_{s \in \tilde{S}} \int_J \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \right| \leq C \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_\infty \|u\|_{\mathcal{K}}^2,$$

where $C > 0$ is a constant depending only on c_1 , but not μ .

Proof. Note that the order of integration over J and summation over s can be exchanged by the monotone convergence theorem. We have

$$\begin{aligned} & \int_J \sum_{s \in \tilde{S}} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \\ & \geq -C_\mu \int_J \sum_{s \in \tilde{S}} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \\ & \geq -C_\mu \int_{-\pi}^{\pi} \sum_s \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \\ & = -C_\mu \|Ku\|_{H^{-1}}^2 \geq -C_\mu \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_\infty \|u\|_{\mathcal{K}}^2, \end{aligned}$$

where the equality follows from Proposition 3.2 (5) and the final inequality from Lemma 2.1. \square

Before stating the first main result we introduce an auxilliary function f , which will play a crucial role in the estimates of the Rayleigh quotient, and prove some identities and estimates involving f .

For \tilde{k} such that $k_j + \tilde{k} \in [-\pi, \pi]$ ($j = 1, \dots, n$) and $u \in H_{qp}^{-1}(\Omega)$ let

$$(26) \quad f(\tilde{k}, u) := \sum_{j=1}^n \sum_{s \in S_{k_j}} \left| \left\langle (\hat{V}Ku)_{k_j + \tilde{k}}, \varphi_s(\cdot, k_j + \tilde{k}) \right\rangle_{H^{-1}([0,1]^2)} \right|^2.$$

Lemma 5.2. *The function f from (26) can be represented as follows:*

$$\begin{aligned} f(\tilde{27}) &= \sum_{j=1}^n \sum_{s \in S_{k_j}} \frac{1}{(\lambda_s(k_j + \tilde{k}) + 1)^2} |(\hat{V}Ku)_{k_j + \tilde{k}}[\varphi_s(\cdot, k_j + \tilde{k})]|^2 \\ (28) &= \sum_{j=1}^n \sum_{s \in S_{k_j}} \frac{1}{(\lambda_s(k_j + \tilde{k}) + 1)^2} \left| \left\langle \phi_{k_j + \tilde{k}}^{-1}(\hat{V}Ku)_{k_j + \tilde{k}}, \varphi_s(\cdot, k_j + \tilde{k}) \right\rangle_{H^1([0,1]^2)} \right|^2 \\ f(\tilde{29}) &= \frac{1}{\sqrt{2\pi}} \sum_{j=1}^n \sum_{s \in S_{k_j}} \frac{1}{(\lambda_s(k_j + \tilde{k}) + 1)^2} \left| Ku[\varphi_s(\cdot, k_j + \tilde{k})] \right|^2, \end{aligned}$$

where the action is considered as an $H^{-1}((0, 1)^2) - H^1((0, 1)^2)$ -pairing. Moreover,

$$(30) \quad f(\tilde{k}, u) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^n \sum_{s \in S_{k_j}} \frac{1}{(\lambda_s(k_j + \tilde{k}) + 1)^2} \left| [(\mathfrak{L}_0 - \mathfrak{L}_1)\varphi_s(\cdot, k_j + \tilde{k})][\mathfrak{G}_1 u] \right|^2.$$

Proof. We have

$$\begin{aligned} (\hat{V}Ku)_{k_j + \tilde{k}}[\varphi_s(k_j + \tilde{k})] &= \left\langle (\hat{V}Ku)_{k_j + \tilde{k}}, \phi_{k_j + \tilde{k}} \varphi_s(k_j + \tilde{k}) \right\rangle_{H^{-1}([0,1]^2)} \\ &= (\lambda_s(k_j + \tilde{k}) + 1) \left\langle (\hat{V}Ku)_{k_j + \tilde{k}}, \varphi_s(k_j + \tilde{k}) \right\rangle_{H^{-1}([0,1]^2)}, \end{aligned}$$

which proves (27).

From (15) and Lemma 3.1 it follows that

$$(\hat{V}Ku)_{k_j + \tilde{k}}[\varphi_s(\cdot, k_j + \tilde{k})] = \left\langle \phi_{k_j + \tilde{k}}^{-1}(\hat{V}Ku)_{k_j + \tilde{k}}, \varphi_s(\cdot, k_j + \tilde{k}) \right\rangle_{H^1([0,1]^2)},$$

so (28) holds.

We next prove (29). In order to make use of the explicit form of the Floquet transform on compactly supported functions, we let Θ_N be a cut-off function with $\Theta_N(y) = \Theta_1(y/N)$ and $\Theta_1 \in \mathbb{C}_c^\infty(\mathbb{R})$ with $\Theta_1(y) = 1$ for $|y| \leq 1$ and $\Theta_1(y) = 0$ for $|y| \geq 2$. Applying the Floquet transform V in $H_{qp}^1(\Omega)$ to the function $\Theta_N(\mathfrak{L}_0 +$

$1)^{-1}Ku$ we get

$$\begin{aligned}
& \left\langle V(\Theta_N(\mathfrak{L}_\circ + 1)^{-1}Ku)(\cdot, k_j + \tilde{k}), \varphi \right\rangle_{H^1((0,1)^2)} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{p \in \mathbb{Z}} e^{i(k_j + \tilde{k})p} \left\langle \Theta_N(\mathfrak{L}_\circ + 1)^{-1}Ku \left(\cdot - \begin{pmatrix} 0 \\ p \end{pmatrix} \right), \varphi \right\rangle_{H^1((0,1)^2)} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{p \in \mathbb{Z}} e^{i(k_j + \tilde{k})p} \left\langle \Theta_N(\mathfrak{L}_\circ + 1)^{-1}Ku, \varphi \left(\cdot + \begin{pmatrix} 0 \\ p \end{pmatrix} \right) \right\rangle_{H^1((0,1) \times (-p, -p+1))} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{p \in \mathbb{Z}} \langle \Theta_N(\mathfrak{L}_\circ + 1)^{-1}Ku, \varphi \rangle_{H^1((0,1) \times (-p, -p+1))} \\
&= \frac{1}{\sqrt{2\pi}} \langle \Theta_N(\mathfrak{L}_\circ + 1)^{-1}Ku, \varphi \rangle_{H^1(\Omega)}.
\end{aligned}$$

We now argue that in the limit $N \rightarrow \infty$, we can move Θ_N to the other side of the inner product. Observe that

$$\langle \Theta_N(\mathfrak{L}_\circ + 1)^{-1}Ku, \varphi_s \rangle_{H^1(\Omega)} = \langle \Theta_N(\mathfrak{L}_\circ + 1)^{-1}Ku, \varphi_s \rangle_{L^2} + \langle \varepsilon_0^{-1} \nabla(\Theta_N(\mathfrak{L}_\circ + 1)^{-1}Ku), \nabla \varphi_s \rangle_{L^2}.$$

Clearly, the first term allows moving Θ_N to the right and it remains to show that

$$\lim_{N \rightarrow \infty} \langle \varepsilon_0^{-1} \nabla(\Theta_N(\mathfrak{L}_\circ + 1)^{-1}Ku), \nabla \varphi_s \rangle_{L^2} = \lim_{N \rightarrow \infty} \langle \varepsilon_0^{-1} \nabla(\mathfrak{L}_\circ + 1)^{-1}Ku, \nabla(\Theta_N \varphi_s) \rangle_{L^2}.$$

Therefore, it suffices to show that

$$\lim_{N \rightarrow \infty} \langle \varepsilon_0^{-1} \nabla(\Theta_N)(\mathfrak{L}_\circ + 1)^{-1}Ku, \nabla \varphi_s \rangle_{L^2} = \lim_{N \rightarrow \infty} \langle \varepsilon_0^{-1} \nabla(\mathfrak{L}_\circ + 1)^{-1}Ku, \nabla(\Theta_N) \varphi_s \rangle_{L^2}$$

and we will see that both limits vanish. Now,

$$\begin{aligned}
\langle \varepsilon_0^{-1}(\mathfrak{L}_\circ + 1)^{-1}Ku, (\nabla \Theta_N) \nabla \varphi_s \rangle_{L^2(\Omega)} &\leq \| \varepsilon_0^{-1}(\mathfrak{L}_\circ + 1)^{-1}Ku \|_{L^2(\Omega)} \| (\nabla \Theta_N) \nabla \varphi_s \|_{L^2(\Omega)} \\
&\leq \| \varepsilon_0^{-1}(\mathfrak{L}_\circ + 1)^{-1}Ku \|_{L^2(\Omega)} \frac{C}{\sqrt{N}} \| \nabla \varphi_s \|_{L^2((0,1)^2)},
\end{aligned}$$

as

$$\| (\nabla \Theta_N) \nabla \varphi_s \|_{L^2(\Omega)}^2 = \int_{\text{supp}(\nabla \Theta_N)} |\nabla \Theta_N|^2 |\nabla \varphi_s|^2 \leq \frac{C}{N^2} N \| \nabla \varphi_s \|_{L^2((0,1)^2)}^2.$$

The other term can be estimated in a similar manner.

Using (15), Lemma 3.1 and (17),

$$\begin{aligned}
(\hat{V}Ku)_{k_j + \tilde{k}}[\varphi] &= \langle (\phi_{\mathcal{H}}^{-1} \hat{V}f)(\cdot, k), \varphi \rangle_{H^1} \\
&= \langle (\phi_{k_j + \tilde{k}}^{-1} (\hat{V}Ku)_{k_j + \tilde{k}}, \varphi \rangle_{H^1} \\
&= \left\langle (V(\mathfrak{L}_\circ + 1)^{-1}Ku)(\cdot, k_j + \tilde{k}), \varphi \right\rangle_{H^1},
\end{aligned}$$

which implies that

$$\begin{aligned}
(\hat{V}Ku)_{k_j + \tilde{k}}[\varphi] &= \frac{1}{\sqrt{2\pi}} \lim_{N \rightarrow \infty} \langle (\mathfrak{L}_\circ + 1)^{-1}Ku, \Theta_N \varphi \rangle_{H^1(\Omega)} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{N \rightarrow \infty} Ku[\Theta_N \varphi_s] = \frac{1}{\sqrt{2\pi}} Ku[\varphi_s],
\end{aligned}$$

where the last equality follows from compactness of the support of Ku . Equation (29) now follows from (28). To obtain (30), we use Remark 3. \square

Lemma 5.3. *Let \mathcal{L} be the space defined in (22). For $u \in \mathcal{L}$ the function $f(\tilde{k}, u)$ satisfies the estimate*

$$|f(\tilde{k}, u)| \leq C|\tilde{k}|^2 \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_{\infty} \|u\|_{\mathcal{K}}^2.$$

Proof. We use (29). First note the following:

$$\frac{Ku[\varphi_s(\cdot, k_j + \tilde{k})]}{\lambda_s(k_j + \tilde{k}) + 1} = \frac{\left\langle Ku, (L_{k_j + \tilde{k}} + 1)\varphi_s(\cdot, k_j + \tilde{k}) \right\rangle_{H^{-1}((0,1)^2)}}{(\lambda_s(k_j + \tilde{k}) + 1)} = \left\langle Ku, \varphi_s(\cdot, k_j + \tilde{k}) \right\rangle_{H^{-1}((0,1)^2)}.$$

In particular, using (23), for $u \in \mathcal{L}$ we obtain

$$\frac{1}{(\lambda_s(k_j + \tilde{k}) + 1)} Ku[\varphi_s(\cdot, k_j + \tilde{k})] = \left\langle Ku, \varphi_s(\cdot, k_j + \tilde{k}) - \varphi_s(\cdot, k_j) \right\rangle_{H^{-1}((0,1)^2)}.$$

As the φ_s depend analytically on k ,

$$\left\| \varphi_s(\cdot, k_j + \tilde{k}) - \varphi_s(\cdot, k_j) \right\|_{H^{-1}((0,1)^2)} \leq C|\tilde{k}|,$$

and we get for $u \in \mathcal{L}$ that

$$|f(\tilde{k}, u)| \leq C|\tilde{k}|^2 \|Ku\|_{H^{-1}((0,1)^2)}^2 \leq C|\tilde{k}|^2 \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_{\infty} \|u\|_{\mathcal{K}}^2,$$

completing the proof. \square

Lemma 5.4. *There exists $C > 0$ such that $f(0, u) \geq C\|u\|_{\mathcal{K}}^2$ for all $u \in \mathcal{L}^{\perp} = \text{span}\{P\phi(\theta\psi_j) : j = 1 \dots n\}$, where $\theta \in C^{\infty}(\Omega)$ is any function compactly supported in the \hat{y} -direction with $\theta = 1$ on $[0, 1]^2$.*

Proof. Using (30) and (24), we have

$$\begin{aligned} f(0, u) &= \frac{1}{\sqrt{2\pi}} \sum_{j=1}^n \sum_{s \in S_{k_j}} \frac{1}{(\Lambda_1 + 1)^2} |[(\mathfrak{L}_0 - \mathfrak{L}_1)\varphi_s(\cdot, k_j)][\mathfrak{G}_1 u]|^2 \\ &= \frac{1}{\sqrt{2\pi}} \sum_{j=1}^n \frac{1}{\Lambda_1 + 1} |\langle u, P\phi(\theta\psi_j) \rangle_{\mathcal{K}}|^2. \end{aligned}$$

Now let $u = \sum_{\mu=1}^n \alpha_{\mu} P\phi(\theta\Psi_{\mu})$ and set $\theta_{\mu} = P\phi(\theta\Psi_{\mu})$. Then

$$f(0, u) = \frac{1}{\sqrt{2\pi}(\Lambda_1 + 1)} \sum_j \sum_{\mu, \nu} \alpha_{\mu} \bar{\alpha}_{\nu} \langle \theta_{\mu}, \theta_j \rangle_{\mathcal{K}} \langle \theta_j, \theta_{\nu} \rangle_{\mathcal{K}} = \frac{\alpha^* G \alpha}{\sqrt{2\pi}(\Lambda_1 + 1)} \geq \frac{\lambda_{\min}(G) \|\alpha\|^2}{\sqrt{2\pi}(\Lambda_1 + 1)},$$

where G is an $n \times n$ -matrix with entries

$$G_{\mu, \nu} = \sum_j \langle \theta_{\mu}, \theta_j \rangle_{\mathcal{K}} \langle \theta_j, \theta_{\nu} \rangle_{\mathcal{K}}.$$

Then $G = \tilde{G}^2$ where $\tilde{G}_{\gamma, \beta} = \langle \theta_{\gamma}, \theta_{\beta} \rangle_{\mathcal{K}}$. By the proof of Lemma 4.5, the set $\{\theta_j : j = 1, \dots, n\}$ is linearly independent, so \tilde{G} is a positive definite Hermitian matrix and also its square G is.

Now,

$$\|u\|_{\mathcal{K}}^2 = \langle Ku, u \rangle_{H^{-1}} = \sum_{i, j} \alpha_i \bar{\alpha}_j \langle K\theta_i, \theta_j \rangle_{H^{-1}} = \sum_{i, j} \alpha_i \bar{\alpha}_j \langle \theta_i, \theta_j \rangle_{\mathcal{K}} = \alpha^* \tilde{G} \alpha \leq \lambda_{\max}(\tilde{G}) \|\alpha\|^2.$$

Thus,

$$f(0, u) \geq \frac{1}{\sqrt{2\pi}(\Lambda_1 + 1)} \frac{\lambda_{\min}(G)}{\lambda_{\max}(\tilde{G})} \|u\|_{\mathcal{K}}^2.$$

□

We now state the main result of this section.

Theorem 5.5. *Let Assumptions 1, 2 and 3 hold. Then there exists $c > 0$ such that if $\left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_{\infty} < c$, then the operator \mathfrak{L}_1 has at most $n = |\Sigma|$ eigenvalues in the spectral gap (Λ_0, Λ_1) of the operator \mathfrak{L}_0 .*

Proof. We start by noting an equality for the Rayleigh quotient. Let $u \in \mathcal{K}$. Then by using proposition 3.2 (4),

$$\langle A_{\mu} \mathfrak{L}_1 u \rangle_{\mathcal{K}} = \int_{-\pi}^{\pi} \sum_{s=1}^{\infty} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk$$

for $\mu \in ((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$.

By continuity of the band function λ_s we have, for each $s \in \mathbb{N}$, either $\lambda_s(k) \leq \Lambda_0$ for all $k \in [-\pi, \pi]$ or $\lambda_s(k) \geq \Lambda_1$ for all $k \in [-\pi, \pi]$. In the first case, $1/[(1 - \mu(\lambda_s(k) + 1))(\lambda_s(k) + 1)] \geq 0$ while in the second case, we have the reverse inequality. Therefore, with s_0 as in Corollary 4.3,

$$\begin{aligned} \langle A_{\mu} u, u \rangle_{\mathcal{K}} & \stackrel{(32)}{=} \int_{-\pi}^{\pi} \sum_{s \geq s_0} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \\ & = \int_{-\pi}^{\pi} \sum_{s \in S} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \\ & \quad + \int_{-\pi}^{\pi} \sum_{s \geq s_0, s \notin S} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk. \end{aligned}$$

We first consider the second sum. By Lemma 4.4 and Lemma 5.1, it can be bounded below by

$$(33) \quad -C \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_{\infty} \|u\|_{\mathcal{K}}^2.$$

Now, we turn our attention to the first sum. We remind the reader that the set Σ consists of the elements (s_j, k_j) with $j = 1, \dots, n$. We split the domain of integration into balls of radius δ around the points k_j and the complement of the union of these balls in $[-\pi, \pi]$, where δ is chosen as in Assumption 3. Then

$$\begin{aligned} & \int_{-\pi}^{\pi} \sum_{s \in S} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \\ & = \sum_{s \in S} \left[\sum_{\substack{j=1 \\ s_j=s}}^n \int_{B_{\delta}(k_j)} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \right. \\ & \quad \left. + \int_{R_s} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \right] \end{aligned}$$

where $R_s := [-\pi, \pi] \setminus \cup_{\substack{j=1 \\ s_j=s}}^n B_\delta(k_j)$. On R_s we again use that $(1 - \mu(\lambda_s(k) + 1))^{-1}$ is uniformly bounded (with respect to s and k), since the continuous function $\lambda_s(\cdot) - \Lambda_1$ is positive and therefore positively bounded away from 0 on the compact set R_s . Using Lemma 5.1 again, the sum of the last integrals can be bounded below by (33).

It remains to estimate the sum of the integrals over $B_\delta(k_j)$. Exchanging the order of the sums which can only add negative terms (if $s \in S_{k_j}$ for several j) and then shifting the integration variable yields

$$\begin{aligned}
& \sum_{s \in S} \sum_{\substack{j=1 \\ s_j=s}}^n \int_{B_\delta(k_j)} \frac{1}{(1 - \mu(\lambda_s(k) + 1))(\lambda_s(k) + 1)} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \\
& \geq \sum_{j=1}^n \sum_{s \in S_{k_j}} \int_{B_\delta(k_j)} \frac{1}{(1 - \mu(\lambda_s(k) + 1))(\lambda_s(k) + 1)} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \\
& = \sum_{j=1}^n \sum_{s \in S_{k_j}} \int_{B_\delta(0)} \frac{1}{(1 - \mu(\lambda_s(k_j + \tilde{k}) + 1))(\lambda_s(k_j + \tilde{k}) + 1)} \left| \left\langle \psi_s(k_j + \tilde{k}), \phi_{k_j + \tilde{k}}^{-1}(\hat{V}Ku(k_j + \tilde{k})) \right\rangle_{H^1(0,1)^2} \right|^2 d\tilde{k} \\
& = \sum_{j=1}^n \sum_{s \in S_{k_j}} \int_{B_\delta(0)} \frac{1}{(1 - \mu(\lambda_s(k_j + \tilde{k}) + 1))(\lambda_s(k_j + \tilde{k}) + 1)^2} \left| \left\langle \varphi_s(k_j + \tilde{k}), \phi_{k_j + \tilde{k}}^{-1}(\hat{V}Ku(k_j + \tilde{k})) \right\rangle_{H^1(0,1)^2} \right|^2 d\tilde{k} \\
& \geq \int_{B_\delta(0)} \frac{1}{1 - \mu(\Lambda_1 + \alpha\tilde{k}^2 + 1)} f(\tilde{k}, u) d\tilde{k},
\end{aligned}$$

where in the last step we have used Assumption 3 and Equation (28). Now, for $u \in \mathcal{L}$, by Lemma 5.3,

$$\begin{aligned}
\int_{B_\delta(0)} \frac{1}{1 - \mu(\Lambda_1 + \alpha\tilde{k}^2 + 1)} f(\tilde{k}, u) d\tilde{k} & \geq C \int_{B_\delta(0)} \frac{|\tilde{k}|^2}{1 - \mu(\Lambda_1 + \alpha\tilde{k}^2 + 1)} d\tilde{k} \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_\infty \|u\|_{\mathcal{K}}^2 \\
& \geq -\tilde{C} \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_\infty \|u\|_{\mathcal{K}}^2.
\end{aligned}$$

Combining all our results, we get that for $u \in \mathcal{L}$

$$\langle A_\mu u, u \rangle_{\mathcal{K}} \geq -C \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_\infty \|u\|_{\mathcal{K}}^2$$

for some $C > 0$, independent of $\mu \in ((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$. Therefore, if $C \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_\infty < 1$ the Rayleigh quotient is larger than -1 on the space \mathcal{L} with $\text{codim } \mathcal{L} = n$. By the variational characterisation of the eigenvalues in (19) we have $\kappa_{n+1}(\mu) > -1$ for all $\mu \in ((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$. Therefore, using Lemmas 2.2 and 4.1, we see that no more than n eigenvalues of the operator \mathfrak{L}_1 are created in the gap. \square

6. LOWER BOUND ON THE NUMBER OF EIGENVALUES

Lemma 6.1. *Let $\mu \in ((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$. For all $u \in \mathcal{K}$,*

$$\langle A_\mu u, u \rangle_{\mathcal{K}} \geq \frac{\sqrt{2\pi}}{1 - \mu(\Lambda_1 + 1)} \|Ku\|_{H^{-1}}^2.$$

holds.

Proof. As in the proof of Theorem 5.5, we have that (32) holds for $u \in \mathcal{K}$. This leads to the estimate

$$\begin{aligned} \langle A_\mu u, u \rangle_{\mathcal{K}} &\geq \frac{1}{1 - \mu(\Lambda_1 + 1)} \int_{-\pi}^{\pi} \sum_{s \geq s_0} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \\ &\geq \frac{1}{1 - \mu(\Lambda_1 + 1)} \int_{-\pi}^{\pi} \sum_s \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \\ &\geq \frac{\sqrt{2\pi}}{1 - \mu(\Lambda_1 + 1)} \|Ku\|_{H^{-1}}^2 \end{aligned}$$

where the last inequality follows from Proposition 3.2 (5). \square

Corollary 6.2. *Let $\mu \in ((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$ and suppose that $\left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_{\infty}$ sufficiently small. Then*

$$\inf_{u \in \mathcal{K} \setminus \{0\}} \frac{\langle A_\mu u, u \rangle}{\|u\|_{\mathcal{K}}^2} > -1.$$

Proof. This follows from Lemma 6.1 together with Lemma 2.1. \square

Remark 5. *This shows that for a fixed μ in the spectral gap, the size of the perturbation has to reach a threshold before it is possible for μ to lie in the spectrum.*

Theorem 6.3. *Let Assumptions 1 and 2 hold. For any ε_1 such that $\left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_{\infty}$ is sufficiently small, at least $n = |\Sigma|$ eigenvalues are created in the spectral gap.*

Proof. By Corollary 6.2, if $\left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_{\infty}$ is sufficiently small, we can find $\mu' \in ((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$ such that

$$(34) \quad \kappa_1(\mu') = \inf_{u \neq 0} \frac{\langle A_{\mu'} u, u \rangle_{\mathcal{K}}}{\|u\|_{\mathcal{K}}^2} > -1.$$

We next give an upper bound on the Rayleigh quotient using equality (31) and decomposing the sum over $s \in \mathbb{N}$ into three parts: one over $s < s_0$, one over $s \geq s_0$ with $s \notin S$, and one over $s \in S$. (Note that $s \geq s_0$ for all $s \in S$). By Lemma 5.1 the first sum is bounded from above by $C \|u\|_{\mathcal{K}}^2$ as long as μ stays away from $(\Lambda_0 + 1)^{-1}$. The second sum is bounded from above by 0. Therefore,

$$\langle A_\mu u, u \rangle_{\mathcal{K}} \leq C \|u\|_{\mathcal{K}}^2 + \int_{-\pi}^{\pi} \sum_{s \in S} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk$$

Now we split up the integration over $[-\pi, \pi]$ into a part over the intervals $B_\delta(k_j)$ and a remainder, as before in the proof of Theorem 5.5. We get

$$\begin{aligned} &\langle A_\mu u, u \rangle_{\mathcal{K}} \\ &\leq C \|u\|_{\mathcal{K}}^2 + \sum_{s \in S} \left[\sum_{\substack{j=1 \\ s_j^1=s}}^n \int_{B_\delta(k_j)} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \right. \\ &\quad \left. + \int_{R_s} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk \right] \end{aligned}$$

and using Lemma 5.1 to estimate the integral over R_s , we continue the estimate as follows:

$$\begin{aligned}
&\leq C \|u\|_{\mathcal{K}}^2 + \sum_{s \in S} \sum_{\substack{j=1 \\ s_j=s}}^n \int_{B_\delta(k_j)} \frac{1}{(1 - \mu(\lambda_s(k) + 1))(\lambda_s(k) + 1)} \left| \langle \psi_s(k), \phi_k^{-1}(\hat{V}Ku)_k \rangle_{H^1} \right|^2 dk \\
&\leq C \|u\|_{\mathcal{K}}^2 + \frac{1}{n} \sum_{j=1}^n \sum_{s \in S_{k_j}} \int_{B_\delta(0)} \frac{\left| \langle \psi_s(k_j + \tilde{k}), \phi_k^{-1}(\hat{V}Ku(k_j + \tilde{k})) \rangle_{H^1} \right|^2}{(1 - \mu(\lambda_s(k_j + \tilde{k}) + 1))(\lambda_s(k_j + \tilde{k}) + 1)} d\tilde{k} \\
&\leq C \|u\|_{\mathcal{K}}^2 + \frac{1}{n} \int_{B_\delta(0)} \frac{1}{1 - \mu(\Lambda_1 + \beta\tilde{k}^2 + 1)} f(\tilde{k}, u) d\tilde{k}.
\end{aligned}$$

In the last but one inequality we use the fact that any $s \in S$ can be at most in n sets S_{k_j} ; in the last line, due to analyticity, we have for $|\tilde{k}| < \delta$ that $\lambda_s(k_j + \tilde{k}) \leq \Lambda_1 + \beta\tilde{k}^2$ for some $\beta > 0$. For any function

$$(35) \quad u = \sum_{i=1}^n \xi_i P\phi(\theta\Psi_i) \in \mathcal{L}^\perp$$

with coefficients $(\xi_i)_{i=1}^n \in \mathbb{C}^n$, we have from Lemma 5.4 and continuity of f that $f(\tilde{k}, u)$ is bounded below on $B_\delta(0)$. Thus the Rayleigh quotient satisfies the following estimate:

$$\frac{\langle A_\mu u, u \rangle_{\mathcal{K}}}{\|u\|_{\mathcal{K}}^2} \leq C + \frac{C}{n} \int_{B_\delta(0)} \frac{1}{1 - \mu(\Lambda_1 + \beta\tilde{k}^2 + 1)} d\tilde{k} \quad \text{for some } C > 0.$$

To show that the Rayleigh quotient tends to $-\infty$ as $\mu \rightarrow (\Lambda_1 + 1)^{-1}$, it is therefore sufficient for

$$\int_0^\delta \frac{1}{\mu(\Lambda_1 + \beta\tilde{k}^2 + 1) - 1} d\tilde{k}$$

to diverge in the limit as $\mu \searrow (\Lambda_1 + 1)^{-1}$. We have

$$\begin{aligned}
\int_0^\delta \frac{1}{\mu(\Lambda_1 + \beta\tilde{k}^2 + 1) - 1} d\tilde{k} &= (\mu\beta(\mu(\Lambda_1 + 1) - 1))^{-\frac{1}{2}} \arctan \left(\delta \sqrt{\frac{\mu\beta}{\mu(\Lambda_1 + 1) - 1}} \right) \\
&\rightarrow +\infty \text{ as } \mu \searrow (\Lambda_1 + 1)^{-1}.
\end{aligned}$$

Therefore,

$$\max_{u \in \mathcal{L}^\perp \setminus \{0\}} \frac{\langle A_\mu u, u \rangle_{\mathcal{K}}}{\|u\|_{\mathcal{K}}^2} \rightarrow -\infty \text{ as } \mu \searrow (\Lambda_1 + 1)^{-1}.$$

As $\text{codim } \mathcal{L} = n$, the variational characterisation of the eigenvalues (19) implies $\kappa_n(\mu) \rightarrow -\infty$ as $\mu \searrow (\Lambda_1 + 1)^{-1}$, and combined with Lemma 4.1 and (34) this means that at least n eigenvalues are created in the gap. \square

Theorem 5.5 and Theorem 6.3 together yield the following result.

Theorem 6.4. *Let Assumptions 1, 2 and 3 hold, i.e.*

- (i) $\varepsilon_0, \varepsilon_1 \in L^\infty(\mathbb{R}^2)$.
- (ii) $\varepsilon_i \geq c_0 > 0$ for some constant c_0 and $i = 0, 1$.
- (iii) *The perturbation is nonnegative, i.e.*

$$\varepsilon_1(\mathbf{x}) - \varepsilon_0(\mathbf{x}) \geq 0.$$

- (iv) *There exists a ball D such that $\varepsilon_1 - \varepsilon_0 > 0$ on D .*

- (v) The band functions λ_s are not constant as functions of $k \in [-\pi, \pi]$.
- (vi) There are $\alpha > 0$ and $\delta > 0$ such that for all $(\hat{s}, \hat{k}) \in \Sigma$ and $k \in [-\pi, \pi]$ satisfying $|k - \hat{k}| \leq \delta$,

$$\lambda_{\hat{s}}(k) \geq \Lambda_1 + \alpha|k - \hat{k}|^2.$$

Moreover, let $\left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_{\infty} > 0$ be sufficiently small. Then the number of eigenvalues of the operator \mathfrak{L}_1 in the gap (Λ_0, Λ_1) equals n , the finite number of solution pairs (s, k) of the equation $\Lambda_1 = \lambda_s(k)$.

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APPENDIX A. PROOF OF PROPOSITION 3.2

From [5, Theorem 4.3 & Theorem 4.7], we have $\sigma(\mathfrak{L}_o) = \overline{\cup_k \sigma(L_k)}$, as required for Proposition 3.2 (1).

For Proposition 3.2 (2), let $v \in \mathcal{H}$ be defined by $v(\cdot, k) = (L_k - \lambda)^{-1}(\hat{V}f)_k$. Then in the proof of [5, Theorem 4.3] it is shown that $(\mathfrak{L}_o - \lambda)u = f$, where $u = V^{-1}v$. Thus both sides of (17) equal $v(\cdot, k)$ and the statement is true.

To prove Proposition 3.2 (3) let $f \in L^2(\Omega)$ and use the decomposition (see [13], [5])

$$f(\cdot) = \frac{1}{\sqrt{2\pi}} \sum_{s=1}^{\infty} \int_{-\pi}^{\pi} \langle Uf, \psi_s(\cdot, k) \rangle_{L^2} \psi_s(\cdot, k) dk,$$

where the series converges in $L^2(\Omega)$. Thus for $g \in H_{qp}^{-1}(\Omega)$,

$$(\mathfrak{L}_o - \lambda)^{-1}g(\cdot) = \frac{1}{\sqrt{2\pi}} \sum_{s=1}^{\infty} \int_{-\pi}^{\pi} \langle U(\mathfrak{L}_o - \lambda)^{-1}g, \psi_s(\cdot, k) \rangle_{L^2} \psi_s(\cdot, k) dk$$

and using Proposition 3.2 (2) and that $U|_{H_{qp}^1(\Omega)} = V$ we get

$$(36) \quad (\mathfrak{L}_o - \lambda)^{-1}g(\cdot) = \frac{1}{\sqrt{2\pi}} \sum_{s=1}^{\infty} \int_{-\pi}^{\pi} \langle (L_k - \lambda)^{-1}\hat{V}g(k), \psi_s(\cdot, k) \rangle_{L^2} \psi_s(\cdot, k) dk.$$

Now, with $\phi_k = (L_k + 1) : \mathcal{H}_k \rightarrow \mathcal{H}'_k$, using (12) and that ϕ_k and $(L_k - \lambda)^{-1}$ commute we have

$$\begin{aligned} \langle (L_k - \lambda)^{-1}(\hat{V}g)_k, \psi_s(\cdot, k) \rangle_{L^2} &= \langle (L_k - \lambda)^{-1}(\hat{V}g)_k, \phi_k \psi_s(\cdot, k) \rangle_{\mathcal{H}'_k} \\ &= \langle (\hat{V}g)_k, (L_k - \lambda)^{-1} \phi_k \psi_s(\cdot, k) \rangle_{\mathcal{H}'_k} \\ &= \frac{1}{\lambda_s(k) - \lambda} \langle (\hat{V}g)_k, \phi_k \psi_s(\cdot, k) \rangle_{\mathcal{H}'_k} \\ &= \frac{1}{\lambda_s(k) - \lambda} (\hat{V}g)_k[\psi_s(\cdot, k)]. \end{aligned}$$

Inserting this in (36) gives Proposition 3.2 (3). We next show Proposition 3.2 (4). Noting that for $\lambda = \frac{1}{\mu} - 1$ we have

$$-\frac{1}{\mu} \frac{1}{\lambda_s(k) - \lambda} = \frac{1}{1 - \mu(\lambda_s(k) + 1)}$$

and using (15), by Proposition 3.2 (3), we have

$$\begin{aligned} & \left\langle -\frac{1}{\mu} (\mathfrak{L}_\circ + 1 - \frac{1}{\mu})^{-1} (Ku), Ku \right\rangle_{H^{-1}} \\ &= \frac{1}{2\sqrt{\pi}} \left\langle \sum_{s=1}^{\infty} \int_{-\pi}^{\pi} \frac{1}{1 - \mu(\lambda_s(k) + 1)} (\hat{V}Ku)_k [\psi_s(\cdot, k)] \psi_s(\cdot, k) dk, Ku(\cdot) \right\rangle_{H^{-1}} \\ &= \frac{1}{\sqrt{2\pi}} \lim_{l \rightarrow \infty} \left\langle \sum_{s=1}^l \int_{-\pi}^{\pi} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \left\langle (\phi_{\mathcal{H}}^{-1} \hat{V}Ku)(\cdot, k), \psi_s(\cdot, k) \right\rangle_{H^1} \psi_s(\cdot, k) dk, \phi^{-1} Ku(\cdot) \right\rangle_{L^2(\Omega)}. \end{aligned}$$

Next let

$$\chi_l(\cdot, k) = \sum_{s=1}^l \frac{1}{1 - \mu(\lambda_s(k) + 1)} \left\langle (\phi_{\mathcal{H}}^{-1} \hat{V}Ku)(\cdot, k), \psi_s(\cdot, k) \right\rangle_{H^1((0,1)^2)} \psi_s(\cdot, k)$$

Then using the formula (11) for the inverse Floquet transform and the isometry property of U we get

$$\begin{aligned} \left\langle -\frac{1}{\mu} (\mathfrak{L}_\circ + 1 - \frac{1}{\mu})^{-1} (Ku), Ku \right\rangle_{H^{-1}} &= \lim_{l \rightarrow \infty} \left\langle \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} E_k \chi_l(\cdot, k) dk, \phi^{-1} Ku \right\rangle_{L^2(\Omega)} \\ &= \lim_{l \rightarrow \infty} \langle U^{-1} \chi_l, \phi^{-1} Ku \rangle_{L^2(\Omega)} \\ &= \lim_{l \rightarrow \infty} \langle \chi_l, U \phi^{-1} Ku \rangle_{L^2((0,1)^2 \times [-\pi, \pi])}. \end{aligned}$$

Therefore, by Proposition 3.2 (2) using that $U|_{H_{qp}^1(\Omega)} = V$, and by (12) we get

$$\begin{aligned} & \left\langle -\frac{1}{\mu} (\mathfrak{L}_\circ + 1 - \frac{1}{\mu})^{-1} (Ku), Ku \right\rangle_{H^{-1}} \\ &= \lim_{l \rightarrow \infty} \int_{-\pi}^{\pi} \left\langle \chi_l(k), \phi_k^{-1} (\hat{V}Ku)_k \right\rangle_{L^2(0,1)^2} dk \\ &= \lim_{l \rightarrow \infty} \int_{-\pi}^{\pi} \left\langle \phi_k^{-1} \chi_l(k), \phi_k^{-1} (\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} dk \\ &= \lim_{l \rightarrow \infty} \int_{-\pi}^{\pi} \overline{(\hat{V}Ku)_k} [\phi_k^{-1} \chi_l(k)] \\ &= \lim_{l \rightarrow \infty} \int_{-\pi}^{\pi} \left\langle \phi_k^{-1} \chi_l(k), (\phi_{\mathcal{H}}^{-1} \hat{V}Ku)(k) \right\rangle_{H^1(0,1)^2} \\ &= \lim_{l \rightarrow \infty} \int_{-\pi}^{\pi} \sum_{s=1}^l \frac{\left\langle (\phi_{\mathcal{H}}^{-1} \hat{V}Ku)(k), \psi_s(k) \right\rangle_{H^1(0,1)^2} \cdot \left\langle \psi_s(k), (\phi_{\mathcal{H}}^{-1} \hat{V}Ku)(k) \right\rangle_{H^1(0,1)^2}}{(1 - \mu(\lambda_s(k) + 1))(\lambda_s(k) + 1)} dk \\ (37) \quad &= \lim_{l \rightarrow \infty} \int_{-\pi}^{\pi} \sum_{s=1}^l \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1} (\hat{V}Ku)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk. \end{aligned}$$

We now wish to interchange the order of taking the limit and integrating. To do this note that

$$\chi_l(\cdot, k) = \sum_{s=1}^l \frac{\lambda_s(k) + 1}{1 - \mu(\lambda_s(k) + 1)} \left\langle (\phi_{\mathcal{H}}^{-1} \hat{V} K u)(\cdot, k), \frac{\psi_s(\cdot, k)}{\sqrt{\lambda_s(k) + 1}} \right\rangle_{H^1((0,1)^2)} \frac{\psi_s(\cdot, k)}{\sqrt{\lambda_s(k) + 1}}$$

and set

$$\chi(\cdot, k) = \sum_{s=1}^{\infty} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \left\langle (\phi_{\mathcal{H}}^{-1} \hat{V} K u)(\cdot, k), \psi_s(\cdot, k) \right\rangle_{H^1((0,1)^2)} \psi_s(\cdot, k).$$

Since the set $\left\{ \frac{\psi_s(k)}{\sqrt{\lambda_s(k) + 1}} \right\}$ is an orthonormal basis in \mathcal{H}_k and $(\phi_{\mathcal{H}}^{-1} \hat{V} K u)(k) \in \mathcal{H}_k$, the series converges in \mathcal{H}_k . In particular, we have that for every $k \in [-\pi, \pi]$

$$\chi_l(\cdot, k) \rightarrow \chi(\cdot, k) \text{ in } H^1((0, 1)^2) \text{ as } l \rightarrow \infty.$$

Moreover, by Bessel's inequality

$$\int_{(0,1)^2} |\chi(x, k) - \chi_l(x, k)|^2 dx \leq \int_{(0,1)^2} |\chi(x, k)|^2 dx$$

and as a function of k the right hand side lies in $L^1(-\pi, \pi)$. By Fubini's theorem, we have

(38)

$$\int_{-\pi}^{\pi} \left(\int_{(0,1)^2} |\chi(x, k) - \chi_l(x, k)|^2 dx \right) dk = \int_{(0,1)^2} \left(\int_{-\pi}^{\pi} |\chi(x, k) - \chi_l(x, k)|^2 dk \right) dx$$

and by dominated convergence the LHS of (38) tends to 0 and so the RHS of (38) also does. This implies that

$$(39) \quad \int_{-\pi}^{\pi} \chi_l(\cdot, k) dk \rightarrow \int_{-\pi}^{\pi} \chi(\cdot, k) dk \text{ in } L^2((0, 1)^2).$$

Therefore, using the Cauchy-Schwarz inequality we have that

$$\begin{aligned} & \left| \int_{-\pi}^{\pi} \left\langle (\phi_{\mathcal{H}}^{-1} \hat{V} K u)(k), \phi_k^{-1}(\chi - \chi_l)(k) \right\rangle_{H^1(0,1)^2} dk \right| \\ & \leq \int_{-\pi}^{\pi} \left| \left\langle (\phi_{\mathcal{H}}^{-1} \hat{V} K u)(k), (\chi - \chi_l)(k) \right\rangle_{H^1(0,1)^2} \right| dk \rightarrow 0 \end{aligned}$$

as $l \rightarrow \infty$ and so we can exchange the order of summation over s and integration over k in (37). This gives

$$\begin{aligned} & \left\langle -\frac{1}{\mu} (\mathfrak{L}_{\circ} + 1 - \frac{1}{\mu})^{-1} (K u), K u \right\rangle_{H^{-1}} \\ & = \int_{-\pi}^{\pi} \sum_{s=1}^{\infty} \frac{1}{1 - \mu(\lambda_s(k) + 1)} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V} K u)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk. \end{aligned}$$

proving Proposition 3.2 (4).

Finally, we consider Proposition 3.2 (5). For $f \in H_{qp}^{-1}(\Omega)$, we have

$$\begin{aligned} \|f\|_{H^{-1}}^2 & = \langle \phi^{-1} f, \phi^{-1} f \rangle_{H^1} = \langle (\mathfrak{L}_{\circ} + 1)^{-1} f, \phi^{-1} f \rangle_{H^1} \\ & = \frac{1}{\sqrt{2\pi}} \left\langle \sum_{s=1}^{\infty} \int_{-\pi}^{\pi} \frac{1}{\lambda_s(k) + 1} (\hat{V} f)_k [\psi_s(\cdot, k)] \psi_s(\cdot, k) dk, \phi^{-1} f(\cdot) \right\rangle_{H^1} \end{aligned}$$

where we have used Proposition 3.2 (3) for $\lambda = -1$.

Next, let

$$\tilde{\chi}(\cdot, k) = \sum_{s=1}^{\infty} \frac{1}{\lambda_s(k) + 1} \left\langle (\phi_{\mathcal{H}}^{-1} \hat{V} f)(\cdot, k), \psi_s(\cdot, k) \right\rangle_{H^1((0,1)^2)} \psi_s(\cdot, k).$$

By a similar argument to the proof of Proposition 3.2 (4), we can swap the order of summation and integration and then using the formula (11) for the inverse Floquet transform and the isometry property of V we get

$$\|f\|_{H^{-1}}^2 = \left\langle \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} E_k \tilde{\chi}(\cdot, k) dk, \phi^{-1} f \right\rangle_{H^1} = \frac{1}{\sqrt{2\pi}} \langle V^{-1} \tilde{\chi}, \phi^{-1} f \rangle_{H^1} = \frac{1}{\sqrt{2\pi}} \langle \tilde{\chi}, V \phi^{-1} f \rangle_{\mathcal{H}}.$$

Therefore, using Proposition 3.2 (2)

$$\begin{aligned} \|f\|_{H^{-1}}^2 &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \left\langle \tilde{\chi}(k), \phi_k^{-1}(\hat{V} f)_k \right\rangle_{\mathcal{H}_k} dk \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \left\langle \sum_{s=1}^{\infty} \frac{1}{\lambda_s(k) + 1} \left\langle (\phi_{\mathcal{H}}^{-1} \hat{V} f)(\cdot, k), \psi_s(\cdot, k) \right\rangle_{H^1((0,1)^2)} \psi_s(\cdot, k), \phi_k^{-1}(\hat{V} f)_k \right\rangle_{\mathcal{H}_k} dk \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sum_{s=1}^{\infty} \frac{1}{\lambda_s(k) + 1} \left| \left\langle \psi_s(k), \phi_k^{-1}(\hat{V} f)_k \right\rangle_{H^1(0,1)^2} \right|^2 dk. \end{aligned}$$

This completes the proof.

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