Geometric aspects of the ODE/IM correspondence

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Abstract

This review describes a link between Lax operators, embedded surfaces and Thermodynamic Bethe Ansatz equations for integrable quantum field theories. This surprising connection between classical and quantum models is undoubtedly one of the most striking discoveries that emerged from the off-critical generalisation of the ODE/IM correspondence, which initially involved only conformal invariant quantum field theories. We will mainly focus of the KdV and sinh-Gordon models. However, various aspects of other interesting systems, such as affine Toda field theories and non-linear sigma models, will be mentioned. We also discuss the implications of these ideas in the AdS/CFT context, involving minimal surfaces and Wilson loops. This work is a follow-up of the ODE/IM review published more than ten years ago by JPA, before the discovery of its off-critical generalisation and the corresponding geometrical interpretation.
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Contents

1 Introduction

There is a deep connection between integrable equations in two dimensions and the embedding of surfaces in higher-dimensional manifolds. The simplest instance of this relation appeared in the works of 19th-century geometers [?, ?] on the description of pseudo-spherical and minimal surfaces sitting in 3-dimensional Euclidean space $\mathbb{R}^3$. The structural equations describing their embedding, the Gauss-Mainardi-Codazzi (GMC) system, are today known as the sine-Gordon and Liouville equations, respectively. More recently, in the works of Lund, Regge, Pohlmeyer and Getmanov [?, ?, ?], a general correspondence has been suggested and subsequently formalised by Sym [?, ?, ?, ?, ?]. These results showed that any integrable field theory, with associated linear problem based on a semi-simple Lie algebra $\mathfrak{g}$, could be put in the form of a GMC system for a surface embedded in a $\dim(\mathfrak{g})$-dimensional space.

The connection between embedded surfaces and integrable models has proven especially fruitful in the context of the AdS/CFT correspondence. In this framework, the semiclassical limit of a string worldsheet theory in an AdS_{n+1} space can be exploited to compute certain observables of conformal field theory (CFT) living on the boundary of that space. The canonical example of this correspondence deals with $\mathrm{AdS}_5 \times \mathrm{S}_5$. In this case, semiclassical worldsheet solutions are used to describe, in the dual CFT, states with large quantum numbers [?], expectation values of Wilson loop operators [?, ?] and universal properties of Maximally Helicity Violating (MHV) gluon scattering amplitudes [?, ?]. The connection with integrable models allows these quantities to be related to certain known universal structures of integrability, such as the Y-system or the corresponding set of Thermodynamic Bethe Ansatz (TBA) equations [?, ?].
Generally speaking, the ODE/IM correspondence, discovered in [?], is instead a link between quantum Integrable Models, studied within the formalism of [?, ?] where analytic properties and functional relations are the main ingredients, and the theory of Ordinary Differential Equations in the complex domain [?, ?]. The relationship is far more general than initially thought, with concrete ramifications in string theory, AdS/CFT, and aspects of the recently-discovered correspondences between supersymmetric gauge theories and integrable models [?, ?, ?, ?, ?, ?, ?, ?, ?]. The ODE/IM correspondence relies on an exact equivalence between spectral determinants associated with certain generalised Sturm-Liouville problems, and the Baxter T and Q functions emerging within the Bethe Ansatz framework. Currently, the link mainly involves the finite volume/temperature Bethe Ansatz equations associated with 2D integrable quantum field theories. However, there are mild hopes that it can be generalised to accommodate also integrable lattice models [?].

The primary purpose of this review is to describe the deep connection existing between the ODE/IM correspondence and the theory of embedded surfaces in higher-dimensional manifolds.

The rest of the article is organised as follows. A brief review on the KdV theory and associated integrals of motion, at both the classical and quantum level, is contained in sections ?? and ??. Section ?? contains a preliminary discussion of the ODE/IM correspondence for the quantum KdV (mKdV/sinh-Gordon) hierarchy, the relevant Schrödinger equation is introduced, and some general facts about the correspondence are described. Section ?? is devoted to a schematic derivation of the Baxter TQ relation from the Schrödinger equation (more details can be found in the original works [?], [?], [?] and in the review [?]). Section ?? describes how the local integrals of motion emerge from the semiclassical quantisation. A short discussion of generalisations to excited states and to models related to higher-rank algebras is contained in section ??.

The problem associated with the off-critical variant of the ODE/IM correspondence, the connection with the sinh-Gordon model (shG) and surfaces embedded in AdS spaces is discussed in section ??. In particular, section ?? contains a general introduction to embedded surfaces in AdS_{n+1}, while in section ?? the specific case of minimal surfaces in AdS_3 is discussed in more detail, together with their relation with Lax equations and the modified sinh-Gordon model (mshG). In section ??, the generalised potential appearing in the modified sinh-Gordon model is interpreted within a Wilson loop type setup while in sections ??–?? the associated linear problem is linked, also with the help of a WKB analysis, to the T- and Y-systems. Starting from the Y-system and the WKB asymptotics, the corresponding Thermodynamic Bethe Ansatz equations are derived in section ?? and the interpretation of the surface area in terms of the free energy is given in section ??. Finally, section ?? contains our conclusions.

2 Classical and quantum KdV, the light-cone shG model, and local integrals of motion

The starting point of the work [?] by Bazhanov, Lukyanov and Zamolodchikov (BLZ) is the Korteweg-de Vries equation[1]

$$u_{,\mathfrak{t}}(x,\mathfrak{t}) + 12\, u_{,x}(x,\mathfrak{t})u(x,\mathfrak{t}) + 2\, u_{,xxx}(x,\mathfrak{t}) = 0\,, \tag{2}$$

on a segment of length $L = 2\pi$ with periodic boundary conditions $u(x+2\pi,\mathfrak{t}) = u(x,\mathfrak{t})$. In the following we will often omit the time dependence of u, since we will mainly work within

[1] In the following, we will denote partial derivatives with subscripts after a comma:

$$F_{,x_1x_2,\dots}(x_1,x_2,\dots) = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\dots F(x_1,x_2,\dots) = \partial_{x_1}\partial_{x_2}\dots F(x_1,x_2,\dots)\ . \tag{1}$$

the Hamiltonian formalism. It is well-known (see, for example, [?]) that from the point of view of integrability, the KdV equation is also deeply connected with the light-cone classical sinh-Gordon model

$$\phi_{,xt}(x,t) + \sinh(\phi(x,t)) = 0\,, \tag{3}$$

since they formally share the same set of local integrals of motion. Note that we have used different font styles for the KdV time parameter t in equation (??) and the sinh-Gordon time t in equation (??). As will become apparent from later considerations, this is to underline the fact that the corresponding Hamiltonians, when considered as part of the same hierarchy of conserved charges for one of the two models, evolve field configurations along different 'generalised time directions'.

2.1 Lax pair and classical conserved charges

The purpose of this section is to derive the expression of the classical integrals of motion for the KdV model through the introduction of a pair of Lax operators which depend on a spectral parameter. We will essentially sketch the derivation presented in the book [?], to which the interested reader is addressed for further details.

First of all, notice that the KdV equation (??) can be written as a Zero Curvature Condition (ZCC)

$$A_{\mathrm{t},x} - A_{x,\mathrm{t}} - [A_x, A_{\mathrm{t}}] = 0\,, \tag{4}$$

for the $\mathfrak{sl}(2)$ connection[2] $A = A_x\, dx + A_{\mathrm{t}}\, d\mathrm{t}$, with components

$$A_x = \begin{pmatrix} 0 & 1 \\ \lambda^2 - u & 0 \end{pmatrix}\,, \qquad A_{\mathrm{t}} = -2\begin{pmatrix} -u_{,x} & 4\,\lambda^2 + 2\,u \\ 4\lambda^4 - 2\lambda^2\,u - u_{,xx} - 2\,u^2 & u_{,x} \end{pmatrix}\,, \tag{5}$$

where λ is the spectral parameter. In turn, equation (??) coincides with the compatibility condition of the following pair of linear systems of (first-order) differential equations:

$$(\mathbb{1}\partial_x - A_x)\begin{pmatrix} \Psi \\ \chi \end{pmatrix} = 0\,, \qquad (\mathbb{1}\partial_{\mathrm{t}} - A_{\mathrm{t}})\begin{pmatrix} \Psi \\ \chi \end{pmatrix} = 0\,. \tag{6}$$

The first equation in (??) gives $\chi = \Psi_{,x}$, together with the Schrödinger-type equation

$$(\mathbf{L} - \lambda^2)\Psi = 0\,, \quad \mathbf{L} = \partial_x^2 + u\,. \tag{7}$$

The second relation in (??) leads instead to the time-evolution equation

$$(\partial_{\mathrm{t}} - \mathbf{M})\Psi = 0\,, \quad \mathbf{M} = -2(\partial_x^3 + 3\,u\,\partial_x + 3\,u_{,x})\,. \tag{8}$$

[2]That is, an $\mathfrak{sl}(2)$-valued one-form.

The compatibility between equations (??) and (??) gives

$$\mathbf{L}_{,\mathrm{t}} - [\mathbf{M}, \mathbf{L}] = 0\,, \tag{9}$$

a constraint which is also equivalent to the original KdV equation (??).

A direct consequence of the zero-curvature condition (??), which involves the arbitrary parameter λ, is the existence of an infinite tower of independent conserved charges. The generator of these quantities is the trace

$$\mathcal{T}(\lambda) = \mathrm{tr}(\mathcal{M}(\lambda))\,, \tag{10}$$

of the so-called monodromy matrix

$$\mathcal{M}(\lambda) = \overleftarrow{\exp}\left(\int_0^{2\pi} dx\, A_x(x,\mathfrak{t},\lambda)\right) = \lim_{\delta x\to 0}(1+\delta x\, A_x(x_n,\mathfrak{t},\lambda))\ldots(1+\delta x\, A_x(x_1,\mathfrak{t},\lambda))\,. \tag{11}$$

In (??), the symbol $\overleftarrow{\exp}$ denotes the path-ordered exponential and $x_1 = 0 < x_2 < \cdots < x_n = 2\pi$.

Since A_x and $A_\mathfrak{t}$ belong to the $sl(2)$ algebra we can introduce the matrices

$$\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{E}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{E}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{12}$$

with $[\mathbf{H}, \mathbf{E}_\pm] = \pm 2\mathbf{E}_\pm$, $[\mathbf{E}_+, \mathbf{E}_-] = \mathbf{H}$ and, expand the connection A_x over the basis $\{\mathbf{H}, \mathbf{E}_-, \mathbf{E}_+\}$ as

$$A_x = A_h\,\mathbf{H} + A_-\,\mathbf{E}_- + A_+\,\mathbf{E}_+\,. \tag{13}$$

Notice that $\mathcal{T}(\lambda)$, defined in (??), is invariant under (periodic) gauge transformations of A_x

$$A_x \to {}^{g}A_x = g^{-1}\,A_x\,g - g^{-1}\,g_{,x}\,. \tag{14}$$

Therefore, we can gauge transform (??) such that ${}^{g}A_- = {}^{g}A_+ = 0$. We first perform the gauge transformation $g_1 = \exp(f_-\,\mathbf{E}_-)$, which leads to

$${}^{g_1}A_x = (A_h + A_+ f_-)\,\mathbf{H} - (f_{-,x} + 2\,A_h\,f_- + A_+\,f_-^2 - A_-)\,\mathbf{E}_- + A_+\,\mathbf{E}_+\,. \tag{15}$$

Setting

$$f_- = \frac{1}{A_+}\,(\nu - \mathcal{A})\;, \quad \mathcal{A} = A_h - \frac{1}{2}\partial_x \ln A_+\,, \tag{16}$$

the vanishing of the coefficient A_- of $\mathbf{E}_-$ in (??) becomes equivalent to the solution of the following Riccati equation:

$$\nu_{,x} + \nu^2 = V\,, \quad V = \mathcal{A}_{,x} + \mathcal{A}^2 + \mathcal{A}_-\,\mathcal{A}_+\,, \tag{17}$$

that, with the standard replacement $\nu(x) = \partial_x \ln y(x)$, can be recast into the Schrödinger-type form

$$\left(\partial_x^2 - V(x,\lambda)\right) y(x) = 0\,. \tag{18}$$

Since the potential in (??) is periodic, $V(x+2\pi,\lambda) = V(x,\lambda)$, we can introduce a pair of independent Bloch solutions $\{y_+, y_-\}$ such that the corresponding Wronskian $W[y_+, y_-] = 1$ and

$$y_\pm(x+2\pi,\lambda) = \exp(\pm\mathcal{P}(\lambda))y_\pm(x,\lambda)\,, \tag{19}$$

where $\mathcal{P}$ is the quasi-momentum:

$$\mathcal{P}(\lambda) = \ln\left(\frac{y_+(2\pi,\lambda)}{y_+(0,\lambda)}\right) = \int_0^{2\pi} dx\, \nu(x,\lambda)\,. \tag{20}$$

However, in (??), the coefficient A_+ is still unfixed and A_h may still depend on the coordinate x. Following [?], we can perform two further independent gauge transformations, g_2 and g_3, without spoiling the $A_- = 0$ constraint. In fact, the combined transformation $g = g_1\, g_2\, g_3$ with

$$g_2 = \exp(f_+\, \mathbf{E}_+)\,, \quad g_3 = \exp(h\,\mathbf{H})\,, \tag{21}$$

and

$$f_+ = A_+\, y_+\, y_-\,, \quad h = \frac{1}{2}\ln\left(A_+\, y_+^2 \exp\left(-2\,\mathcal{P}(\lambda)\frac{x}{2\pi}\right)\right)\,, \tag{22}$$

leads to

$${}^g A_x = \frac{1}{2\pi}\mathcal{P}(\lambda)\,\mathbf{H}\,, \tag{23}$$

giving

$$\mathcal{T}(\lambda) = \mathrm{tr}(\mathcal{M}(\lambda)) = 2\cosh\left(\mathcal{P}(\lambda)\right)\,. \tag{24}$$

For the KdV model under consideration, we have (cf. (??), (??) and (??))

$$A_h = 0\,, \quad A_- = \lambda^2 - u\,, \quad A_+ = 1\,, \tag{25}$$

while the Riccati and the Schrödinger equations are

$$\nu_{,x} + \nu^2 = \lambda^2 - u\,, \quad (\mathbf{L} - \lambda^2)y = 0\,. \tag{26}$$

To find the local conserved charges, we expand ν as series in the spectral parameter around $\lambda^2 = \infty$:

$$\nu = \lambda + \sum_{n=0}^{\infty}(-1)^n\frac{\nu_n}{\lambda^n}\,, \tag{27}$$

and therefore

$$\mathcal{P}(\lambda) = \int_0^{2\pi} dx\,\nu(x) = 2\pi\lambda + \sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda^n}\int_0^{2\pi} dx\,\nu_n(x)\,. \tag{28}$$

Finally, plugging (??) into (??) we find the recursion relation

$$\nu_{n+1} = \frac{1}{2}\left(\nu_{,x} + \sum_{p=0}^{n} \nu_p \nu_{n-p}\right), \quad \nu_0 = 0\,, \quad \nu_1 = \frac{1}{2}u\,. \tag{29}$$

The first few coefficients are

$$\begin{aligned}
\nu_1 &= \frac{1}{2}u\,, \quad \nu_2 = \frac{1}{4}u_{,x}\,, \quad \nu_3 = \frac{1}{8}(u^2 + u_{,xx})\,, \\
\nu_4 &= \frac{1}{2}\nu_{3,x} + \frac{1}{8}u\,u_{,x}\,, \quad \nu_5 = \frac{1}{2}\nu_{4,x} + \frac{1}{32}(u_{,x})^2 + \frac{1}{16}u\,u_{,xx} + \frac{1}{16}u^3\,,
\end{aligned} \tag{30}$$

which correspond, when normalised as in [?] and up to total derivatives, to the following integrals of motion:

$$\begin{aligned}
I_1^{(cl)} = I_1^{[\mathbf{KdV}]} &= \int_0^{2\pi} \frac{dx}{2\pi}\, u(x)\,, \quad I_3^{(cl)} = I_3^{[\mathbf{KdV}]} = \int_0^{2\pi} \frac{dx}{2\pi}\, u^2(x)\,, \\
I_5^{(cl)} = I_5^{[\mathbf{KdV}]} &= \int_0^{2\pi} \frac{dx}{2\pi}\left(u^3(x) - \frac{1}{2}u_{,x}^2(x)\right).
\end{aligned} \tag{31}$$

The relation between the KdV and the modified KdV (mKdV) equations emerges through the Miura transformation

$$u(x,\mathfrak{t}) = -v^2(x,\mathfrak{t}) - v_{,x}(x,\mathfrak{t})\,, \tag{32}$$

which implies

$$u_{,\mathfrak{t}} + 2\,u_{,xxx} + 12\,uu_{,x} = -(2\,v + \partial_x)\left(v_{,\mathfrak{t}} + 2\,v_{,xxx} - 12\,v^2\,v_{,x}\right) = 0\,. \tag{33}$$

Hence a solution $v(x,\mathfrak{t})$ of the mKdV equation

$$v_{,\mathfrak{t}}(x,\mathfrak{t}) + 2\,v_{,xxx}(x,\mathfrak{t}) - 12\,v^2(x,\mathfrak{t})\,v_{,x}(x,\mathfrak{t}) = 0\,, \tag{34}$$

can be mapped into a KdV solution through the Miura transformation (??). A straightforward consequence of this fact is that the quantities $I_n^{(cl)}$ coincide with the integrals of motion $I_n^{[\mathbf{mKdV}]}$ of the mKdV theory

$$I_n^{[\mathbf{mKdV}]}[v] = -I_n^{[\mathbf{KdV}]}[u = -v^2 - v_{,x}]\,, \tag{35}$$

that is

$$I_1^{[\mathbf{mKdV}]} = \int_0^{2\pi} \frac{dx}{2\pi}\, v^2(x)\,, \quad I_3^{[\mathbf{mKdV}]} = -\int_0^{2\pi} \frac{dx}{2\pi}\,\left(v^4(x) + (v_{,x}(x))^2\right)\,, \ldots \tag{36}$$

Furthermore, the sinh-Gordon model (??) also possesses the same set of local charges, provided the formal identification $v(x,\mathfrak{t}) = \phi_{,x}(x,t)/2$ is made at fixed times $\mathfrak{t}$ and t:

$$I_n^{[\mathbf{shG}]}[\phi] = I_n^{[\mathbf{mKdV}]}[v = \tfrac{1}{2}\phi_{,x}]\,. \tag{37}$$

In fact, the sinh-Gordon Lagrangian in light-cone coordinates is

$$\mathcal{L}^{[\mathbf{shG}]} = \frac{1}{2\pi}\left(\phi_{,t}(x,t)\,\phi_{,x}(x,t) - \cosh(\phi(x,t)) + 1\right) , \tag{38}$$

and the conjugated momentum and Hamiltonian are

$$\pi(x,t) = \frac{1}{2\pi}\phi_{,x}(x,t) , \quad H^{[\mathbf{shG}]} = \int_0^{2\pi} \frac{dx}{2\pi}\left(\cosh\phi(x,t) - 1\right) . \tag{39}$$

Then $\{\phi(x,t), \pi(x',t)\} = \delta(x-x')$, and the sinh-Gordon equations of motion can be written as

$$\phi_{,xt}(x,t) = 2\,v_{,t}(x,\mathfrak{t},t) = 2\,\{v(x,\mathfrak{t},t), H^{[\mathbf{shG}]}\} . \tag{40}$$

Notice that in (??), t denotes the sinh-Gordon time, which differs from the KdV (mKdV) time $\mathfrak{t}$ appearing in (??) and (??).[3]

In addition, imposing periodic boundary conditions $\phi(x+2\pi,t) = \phi(x,t)$ and using the equation of motion, it is not difficult to prove that

$$\{I^{(cl)}_{2n+1}[v = \frac{1}{2}\phi_{,x}], H^{[\mathbf{shG}]}\} = 0 , \quad (\forall n \in \mathbb{Z}_{\geq}) . \tag{41}$$

For example:

$$\{I^{(cl)}_1[v = \frac{1}{2}\phi_{,x}], H^{[\mathbf{shG}]}\} = \int_0^{2\pi} \frac{dx}{4\pi}\phi_{,x}\,\phi_{,xt} = -\int_0^{2\pi} \frac{dx}{4\pi}\partial_x \cosh(\phi(x,t)) = 0 . \tag{42}$$

Therefore, and as mentioned in the previous section, the KdV conserved charges $\{I^{(cl)}_n\}$ are also integrals of motion for the sinh-Gordon model (??). We will see later that the off-critical field theory generalisation of the ODE/IM correspondence described in this review is naturally based on the sinh-Gordon perspective of this connection.

2.2 Quantisation of the local conserved charges

It is well known (cf. [?]) that the KdV model admits two equivalent Hamiltonian structures. The first Hamiltonian is

$$H = I^{(cl)}_3 = \int_0^{2\pi} \frac{dx}{2\pi}\, u^2(x) , \tag{43}$$

[3] At least formally, relation (??) can be regarded as a particular instance of the KdV/mKdV hierarchy of equations [?]:

$$v_{,\mathfrak{t}_{2k-1}}(\{\mathfrak{t}_i\}) = \{I^{[\mathbf{mKdV}]}_{2k-1}, v(\{\mathfrak{t}_i\})\} ,$$

where $\{\mathfrak{t}_i\}$, with $i \in 2\mathbb{Z}+1$, is the set of generalised time directions with the identifications $\mathfrak{t}_1 = x$, $\mathfrak{t}_3 = \mathfrak{t}$ and also $\mathfrak{t}_{-1} = t$, i.e. $I^{[\mathbf{mKdV}]}_{-1} = H^{[\mathbf{shG}]}$ (see, for example [?,?]).

with Poisson bracket

$$\frac{1}{2\pi}\{u(x),u(y)\} = 2(u(x)+u(y))\delta_{,x}(x-y)+\delta_{,xxx}(x-y)\,. \tag{44}$$

The second possibility is instead

$$H' = I_5^{(cl)} = \int_0^{2\pi}\frac{dx}{2\pi}\left(u^3(x)-\frac{1}{2}(u_{,x}(x))^2\right)\,, \tag{45}$$

with Poisson bracket

$$\frac{1}{2\pi}\{u(x),u(y)\}' = 2\,\delta_{,x}(x-y)\,. \tag{46}$$

Both options lead to the KdV equation:

$$\partial_t u = \{H,u\} = \{H',u\}' = -12\,u\,u_{,x} - 2\,u_{,xxx}\,. \tag{47}$$

Furthermore, through the change of variables $u(x) = -(\phi_{,x}(x))^2 - \phi_{,xx}(x)$, the first Poisson bracket (??) reduces to

$$\frac{1}{2\pi}\{\phi(x),\phi(y)\} = \frac{1}{2}\epsilon(x-y)\,, \tag{48}$$

with $\epsilon(x) = n$ for $2\pi n < x < 2\pi(n+1)$ and $n \in \mathbb{Z}$. This is the standard Poisson bracket involving a single bosonic field $\phi(x,t)$ with periodic boundary conditions and conjugated momenta $\pi(x,t)$ as in (??).

The quantisation of (??) is then achieved by performing the following replacements [?]:

$$\frac{1}{2\pi}\{\,,\,\} \to \frac{\mathbb{i}c}{6\pi}[\,,\,]\,,\quad u(x) \to -\frac{6}{c}\mathbf{T}(x)\,. \tag{49}$$

Expanding

$$\mathbf{T}(x) = \frac{c}{24} + \sum_{n=-\infty}^{\infty}\mathbf{L}_{-n}e^{\mathbb{i}nx}\,, \tag{50}$$

we see, from (??), that the operators $\mathbf{L}_n$ satisfy the Virasoro algebra

$$[\mathbf{L}_n,\mathbf{L}_m] = (n-m)\mathbf{L}_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}\,. \tag{51}$$

Alternatively, performing first a quantum Miura transformation

$$-\beta^2\mathbf{T}(x) = :\hat{\phi}_{,x}(x)^2: + (1-\beta^2)\hat{\phi}_{,xx}(x) + \frac{\beta^2}{24}\,, \tag{52}$$

and expanding the fundamental quantum field $\hat{\phi}(x)$ in plane-wave modes as

$$\hat{\phi}(x) = \mathbb{i}\mathbf{Q} + \mathbb{i}\mathbf{P}\,x + \sum_{n\neq 0}\frac{\mathbf{a}_{-n}}{n}e^{\mathbb{i}nx}\,, \tag{53}$$

we obtain the Heisenberg algebra

$$[\mathbf{Q},\mathbf{P}]=\frac{\mathbb{i}}{2}\beta^2,\quad [\mathbf{a}_n,\mathbf{a}_m]=\frac{n}{2}\beta^2\delta_{n+m,0}\,. \tag{54}$$

The relation between the central charge c appearing in the Virasoro algebra (??) and the parameter β in equation (??) is

$$\beta=\sqrt{\frac{1-c}{24}}-\sqrt{\frac{25-c}{24}}\,. \tag{55}$$

The highest weight (vacuum) vector $|\mathbb{p}\rangle$ over the Heisenberg algebra is defined by

$$\mathbf{P}|\mathbb{p}\rangle=\mathbb{p}|\mathbb{p}\rangle\,,\quad a_n|\mathbb{p}\rangle=0\,,\quad (\forall n>0). \tag{56}$$

In terms of the Virasoro representation, the states $|\mathbb{p}\rangle$ are highest weights with conformal dimensions

$$\Delta=\left(\frac{\mathbb{p}}{\beta}\right)^2+\frac{c-1}{24}\,, \tag{57}$$

$$\mathbf{L}_0|\mathbb{p}\rangle=\Delta|\mathbb{p}\rangle\,,\quad \mathbf{L}_n|\mathbb{p}\rangle=0,\quad (\forall n>0)\,. \tag{58}$$

The quantum charges were first determined in [?] under the replacement of classical fields with the corresponding operators ($\phi\to\hat{\phi}$), and by following the scheme

1. $\mathbf{I}_n=:I_n^{(cl)}:,\quad (n=1,3)$;
2. $\mathbf{I}_n=:I_n^{(cl)}:+\sum_{k=1}^{n}(\beta)^{2k}:I_n^{(k)}:,\quad (n=5,7,\dots)$;
3. The quantum corrections $:I_n^{(k)}:$ do not contain any of the $:I_m^{(cl)}:$ as a part (see [?] for more details.);
4. $[\mathbf{I}_n,\mathbf{I}_m]=0\,,\quad (\forall\, n,m\in 2\,\mathbb{Z}_{\geq}+1)\,.$

The first three non-vanishing local integrals of motion, written in terms of the generators of the Virasoro algebra (??), are:

$$\begin{aligned}\mathbf{I}_1 &= \mathbf{L}_0-\frac{c}{24}\,,\quad \mathbf{I}_3=2\sum_{n=1}^{\infty}\mathbf{L}_{-n}\mathbf{L}_n+\mathbf{L}_0^2-\frac{c+2}{12}\mathbf{L}_0+\frac{c\,(5\,c+22)}{2880}\,,\\ \mathbf{I}_5 &= \sum_{n_1+n_2+n_3=0}:\mathbf{L}_{n_1}\mathbf{L}_{n_2}\mathbf{L}_{n_3}:+\sum_{n=0}^{\infty}\left(\frac{c+11}{6}n^2-1-\frac{c}{4}\right)\mathbf{L}_{-n}\mathbf{L}_n+\frac{3}{2}\sum_{n=0}^{\infty}\mathbf{L}_{1-2n}\mathbf{L}_{2n-1}\\ &- \frac{c+4}{8}\mathbf{L}_0^2+\frac{(c+2)(3\,c+20)}{576}\mathbf{L}_0-\frac{c\,(3\,c+14)(7\,c+68)}{290304}\,.\end{aligned} \tag{59}$$

In equation (??), the normal ordering : : means that the operators $\mathbf{L}_{n_i}$ with larger n_i are placed to the right. The corresponding expectation values $I_n^{vac} = \langle \mathbb{p}|\mathbf{I}_n|\mathbb{p}\rangle$ on the vacuum states are

$$\begin{aligned} I_1^{vac} &= \Delta - \frac{c}{24}\,, \quad I_3^{vac} = \Delta^2 - \frac{c+2}{12}\Delta + \frac{c\,(5\,c+22)}{2880}\,, \\ I_5^{vac} &= \Delta^3 - \frac{c+4}{8}\Delta^2 - \frac{(c+2)(3\,c+20)}{576}\Delta - \frac{c\,(3\,c+14)(7\,c+68)}{290304}\,, \end{aligned} \tag{60}$$

where c and Δ are related to $\mathbb{p}$ and β through equations (??) and (??). An alternative, but more sophisticated, method leading to the same result (??) is described in [?].

3 The ODE/IM correspondence for the quantum KdV-shG hierarchy

The simplest instance of the ODE/IM correspondence involves, on the ODE side, the second order differential equation [?, ?]

$$\Big(-\partial_x^2 + P(x)\Big)\chi(x) = 0 \tag{61}$$

with

$$P(x) = P_0^{\mathbf{[KdV]}}(x, E, l, M) = \left(x^{2M} + \frac{l(l+1)}{x^2} - E\right). \tag{62}$$

The generalised potential P and wavefunction χ depend, therefore, on three extra parameters: the energy or spectral parameter E, the ‘angular-momentum’ l, and the exponent M. For simplicity, throughout this review, M and l will be kept real with $M \geq 0$. However, there are no serious limitations forbidding the extension of both M and l to the complex domain. The range $-1 \leq M \leq 0$ is essentially equivalent, by a simple change of variables, to the $M > 0$ regime [?, ?].[4] We will see that for $M \geq -1$ equation (??) is related, through the ODE/IM correspondence, to the conformal field theory with central charge $c \leq 1$ associated to the quantisation of the KdV-shG theory.[5]

The ODE/IM correspondence is based on the observation that the CFT version of Baxter's TQ equation [?] for the six-vertex model, and the quantum Wronskians introduced in the works by BLZ [?], exactly match the Stokes relations and Wronskians between independent solutions of (??). BLZ introduced a continuum analogue of the lattice transfer matrix $\mathbb{T}$ for the quantum KdV equation, an operator-valued function $\mathbf{T}(\lambda, \mathbb{p})$, together with the Baxter $\mathbf{Q}_{\pm}(\lambda, \mathbb{p})$ operators with $\mathbf{Q}(\lambda, \mathbb{p}) \equiv \mathbf{Q}_{+}(\lambda, \mathbb{p}) = \mathbf{Q}_{-}(\lambda, -\mathbb{p})$, where $\mathbb{p}$ is the quasi-momentum

[4] In fact, with the identification $\beta^{-2} = M + 1$, the equivalence $(-1 \leq M \leq 0) \leftrightarrow (M \geq 0)$ coincides with the $\beta^2 \to \beta^{-2}$, duality in the integrals of motion in the quantum KdV model (see, for example, [?]).

[5] The regime $M < -1$ is also interesting, since it is related to the Liouville field theory [?].

[?]. Both the $\mathbf{Q}$ and $\mathbf{T}$ operators are entire in the spectral parameter λ with

$$[\mathbf{T}(\lambda, \mathbb{p}), \mathbf{Q}_{\pm}(\lambda, \mathbb{p})] = 0 \,. \tag{63}$$

All the descendent CFT states in the Verma module associated to the highest-weight vector $|\mathbb{p}\rangle$ are characterised by the real parameter $\mathbb{p}$. Since $\mathbf{T}$ and $\mathbf{Q}_{\pm}$ commute, we can work directly with their eigenvalues

$$T(\lambda, \mathbb{p}) = \langle \mathbb{p} | \mathbf{T}(\lambda, \mathbb{p}) | \mathbb{p} \rangle \,, \quad Q_{\pm}(\lambda, \mathbb{p}) = \langle \mathbb{p} | \lambda^{\mp \mathbb{p}/\beta^2} \mathbf{Q}_{\pm}(\lambda, \mathbb{p}) | \mathbb{p} \rangle \tag{64}$$

which satisfy the TQ relation [?]

$$T(\lambda, \mathbb{p}) Q_{\pm}(\lambda, \mathbb{p}) = e^{\mp \mathrm{i} 2\pi \mathbb{p}} Q_{\pm}(q^{-1}\lambda, \mathbb{p}) + e^{\pm \mathrm{i} 2\pi \mathbb{p}} Q_{\pm}(q\lambda, \mathbb{p}) \tag{65}$$

with $q = \exp(i\pi\beta^2)$.

It turns out that equation (??) exactly matches a *Stokes relation*, i.e. a connection formula, for particular solutions of the ODE (??). The precise correspondence between the parameters in (??) and those in (??) is:

$$\beta^{-2} = M + 1 \,, \quad \mathbb{p} = \frac{2l+1}{4M+4} \,, \quad \lambda = (2M{+}2)^{-2M/(M+1)} \, \Gamma\left(\frac{M}{M+1}\right)^{-2} E \,. \tag{66}$$

Supplemented with the analytic requirement that both T and Q are entire in λ, (??) leads to the Bethe Ansatz equations. At a zero $\lambda = \lambda_i$ of $Q(\lambda, \mathbb{p}) = Q_{+}(\lambda, \mathbb{p})$, the RHS of (??) vanishes since $T(\lambda_i, \mathbb{p})$ is finite, and hence

$$\frac{Q(q^{-1}\lambda_i, \mathbb{p})}{Q(q\,\lambda_i, \mathbb{p})} = -e^{\mathrm{i}4\pi\mathbb{p}} \,. \tag{67}$$

As a result, the link between (??) and the Baxter relation (??) for the quantum KdV model is more than formal: the resulting T and Q functions emerging from these two – apparently disconnected – setups are exactly the same.

3.1 Derivation of Baxter's TQ relation from the ODE

Consider the ODE (??), where we will henceforth allow x to be complex, living on a suitable cover C of the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ so as to render the equation and its solutions single-valued. A straightforward WKB analysis shows that for large x close to the positive real axis a generic solution has a growing leading asymptotic of the form

$$\chi(x) \sim c_{+} \, P(x)^{-1/4} \exp\left(\int^{x} dx' \, \sqrt{P(x')}\right) \,, \quad (\mathrm{Re}[x] \to +\infty) \,. \tag{68}$$

Even at fixed normalisation c_{+}, this asymptotic does not uniquely characterise the solution, since an exponentially decreasing contribution can always be added to $\chi(x)$ without spoiling the large-x behaviour (??). The exponentially small term can explicitly emerge from the asymptotics only if the nontrivial solution to (??) is carefully chosen such that the coefficient of the exponentially growing term vanishes. In this special situation

$$\chi(x) \sim c_- \, P(x)^{-1/4} \exp\left(- \int^x dx' \, \sqrt{P(x')} \right) , \quad (\mathrm{Re}[x] \to +\infty) . \tag{69}$$

Apart for the arbitrariness of the overall normalisation factor c_-, the asymptotic (??) now uniquely specifies the solution of (??). This was formalised by Sibuya and collaborators in the following statement, which holds not only on the real axis but also in an M-dependent wedge of the complex plane: the ODE (??) has a basic solution $y(x,E,l)$ with the following properties, which fix it uniquely:

1. $y(x,E,l)$ is an entire function of E, and a holomorphic function of $x \in C$, where C is a suitable cover of the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$;

2. the asympotic behaviour of $y(x,E,l)$ for $|x| \to \infty$ with $|\arg(x)| < 3\pi/(2M{+}2)$ is

   $$y \sim \frac{1}{\sqrt{2\mathbb{i}}} \, x^{-\frac{M}{2}} \exp\left(-\frac{x^{M+1}}{M{+}1} \right) , \; y_{,x} \sim -\frac{1}{\sqrt{2\mathbb{i}}} \, x^{\frac{M}{2}} \exp\left(-\frac{x^{M+1}}{M{+}1} \right) , \tag{70}$$

   though there are small modifications in the asymptotics (??) for $M \le 1$ (see, for example, [?]).

To proceed with our analysis, it is necessary to continue x even further into the complex plane, beyond the wedge where Sibuya's initial result applies. We define general rays in the complex plane by setting $x = \varrho e^{\mathbb{i}\vartheta}$ with ϱ and ϑ real. Substituting into the WKB formulas (??) and (??), we detect two possible asymptotic behaviours

$$\chi_\pm \sim P^{-1/4} \exp\left(\pm \frac{1}{M{+}1} e^{\mathbb{i}\vartheta(1+M)} \varrho^{1+M} \right) . \tag{71}$$

For most values of ϑ, one of these solutions will be exponentially growing, or *dominant*, and the other exponentially decaying, or *subdominant*. However, for

$$\mathrm{Re}\left[e^{\mathbb{i}\vartheta(1+M)} \right] = 0 \tag{72}$$

both solutions oscillate, and neither dominates the other. The values

$$\vartheta = \pm \frac{\pi}{2M{+}2} \, , \;\; \pm \frac{3\pi}{2M{+}2} \, , \;\; \pm \frac{5\pi}{2M{+}2} \, , \;\; \dots \, , \tag{73}$$

where this happens, and the two solutions (??) exchange rôles, are called *anti-Stokes lines*.[6] The *Stokes lines* are instead the lines along which χ either grows or shrinks the fastest, and

[6] We are following here the convention used, for example, in [?]. Unfortunately, the lines characterised by the condition (??) are sometimes called instead *Stokes lines*.

Figure 1: Stokes, WKB sectors and convention for the branch cut when $2M \notin \mathbb{Z}_{\geq}$.

in the current case they lie right in the middle, between adjacent anti-Stokes lines, and are characterised by

$$\operatorname{Im}\left[e^{\mathrm{i}\vartheta(1+M)}\right] = 0\,. \tag{74}$$

The wedges between adjacent anti-Stokes lines are called Stokes sectors, and we will label them as

$$\mathcal{S}_k = \left\{ x \in \mathbb{C} : \left| \arg(x) - \frac{2\pi k}{2M+2} \right| < \frac{\pi}{2M+2} \right\}\,. \tag{75}$$

In this notation the region of validity of the asymptotic (??) is the union of wedges

$$\mathcal{S}_{\mathrm{WKB}} = \mathcal{S}_{-1} \cup \overline{\mathcal{S}_0} \cup \mathcal{S}_1 \tag{76}$$

where $\overline{\mathcal{S}_0}$ is the closure of $\mathcal{S}_0$.

Finding the large $|x|$ behaviour of the particular solution $y(x, E, l)$ outside the region (??) is a non-trivial task: the continuation of a limit is not in general the same as the limit of a continuation, and so (??) no longer holds once $\mathcal{S}_{\mathrm{WKB}}$ is left. This issue is related to the so-called *Stokes phenomenon*, wherein the quantities of principal interest are the *Stokes multipliers*, encoding the switching-on of small (subdominant) exponential terms as Stokes lines are crossed [?].

Thus far we have discussed the behaviour of solutions to (??) when $|x|$ is large. Consider now the region $x \simeq 0$. For $M > -1$, the origin corresponds to a regular singularity, and the associated indicial equation shows that a generic solution to (??) behaves as a linear combination of x^{l+1} and x^{-l} as $x \to 0$. This allows a special solution $\psi(x, E, l)$ to be specified by the requirement

$$\psi(x, E, l) \sim x^{l+1} + \mathcal{O}(x^{l+3}) \,. \tag{77}$$

This boundary condition defines $\psi(x, E, l)$ uniquely provided Re$[l] > -3/2$. A second solution can be obtained from $\psi(x, E, l)$ by noting that, since the differential equation (??) is invariant under the analytic continuation $l \to -1-l$, $\psi(x, E, -1-l)$ is also a solution. Near the origin, $\psi(x, E, -1-l) \sim x^{-l} + O(x^{-l+2})$, therefore for generic values of the angular momentum l the two solutions

$$\psi_{+}(x, E) = \psi(x, E, l) \,, \quad \psi_{-}(x, E) = \psi(x, E, -1-l) \,, \tag{78}$$

are linearly independent, i.e. the Wronskian $W[\psi_+, \psi_-]$ is non-vanishing. Some subtleties arise at the isolated points

$$l + \frac{1}{2} = \pm \left(m_1 + (M+1) m_2 \right) \,, \quad (m_1, m_2 \in \mathbb{Z}_{\geq}) \,, \tag{79}$$

where $\{\psi_+, \psi_-\}$ fails to be a basis of solutions [?]. For $2M \in \mathbb{Z}_{\geq}$, this is just the standard resonant phenomenon in the Frobenius method, which predicts that one of the two independent solutions may acquire a logarithmic component, when the two roots of the indicial equation differ by an integer. For the remainder of this review we will steer clear of such points, but see [?] for some further discussion of the issue.

A natural eigenproblem for a Schrödinger equation, the so-called *radial* or *central* problem, is to look for values of E at which there exists a solution that vanishes as $x \to +\infty$, and behaves as x^{l+1} at origin. For Re$[l] > -1/2$, this boundary condition is equivalent to demanding the square integrability of the solution on the half line, and for Re$[l] > 0$ to the requirement that the divergent x^{-l-1} term is absent. For Re$[l] \leq -1/2$, the problem can be defined by analytic continuation.

Addressing the reader to [?] and [?] for more details, we proceed by adopting a trick due to Sibuya [?]. Starting from the uniquely-defined solution $y(x, E, l)$, subdominant in the Stokes sector $\mathcal{S}_0$, we generate a set of functions

$$y_k(x, E, l) = \omega^{k/2} y(\omega^{-k} x, \; \omega^{2k} E, \; l) \;\; , \quad \omega = e^{\frac{2\pi \mathrm{i}}{2M+2}} \;\; , \quad (k \in \mathbb{Z}) \,, \tag{80}$$

all of which solve (??). Notice that the asymptotic expansion

$$y_{\pm 1}(x, E, l) \sim \pm \sqrt{\mathrm{i}} \frac{x^{-M/2}}{\sqrt{2}} \exp\left(\frac{x^{M+1}}{M+1} \right) , \tag{81}$$

is valid in the Stokes sector $\mathcal{S}_0$ containing the real line. Hence, we can compute the Wronskians $W\left[y, y_{\pm 1}\right]$ using the expansions (??) and (??), finding that they are non-zero: $W\left[y, y_{\pm 1}\right] = \pm 1$. As a consequence $\{y, y_{\pm 1}\}$ are bases of the two-dimensional space of solutions to the ODE

(??). More generally, a similar consideration shows that $W\left[y_k, y_{k+1}\right] = 1$ and hence any pair $\{y_k, y_{k+1}\}$ constitutes a basis. In particular, y_{-1} can be written as a linear combination of the basis elements $y = y_0$ and y_1 as $y_{-1} = Cy + \tilde{C}y_1$, or equivalently

$$C(E,l)\, y(x,E,l) = y_{-1}(x,E,l) - \tilde{C}(E,l)\, y_{\,1}(x,E,l)\,, \tag{82}$$

where the connection coefficients $\tilde{C}$ and C are the Stokes multipliers. For the right-hand side of of (??) to match the exponentially decreasing behaviour on the left, we must set $\tilde{C} = -1$ (cf. equation (??)) and so

$$C(E,l)\, y_0(x,E,l) = y_{-1}(x,E,l) + y_1(x,E,l)\,, \tag{83}$$

where the sole non-trivial Stokes multiplier $C(E,l)$ takes, in the chosen normalisations (??) and (??) for $y(x)$ and $y_k(x)$, the simple form:

$$C(E,l) = W[\,y_{-1}\,, y_1]/W[\,y_0, y_1] = W[\,y_{-1}\,, y_1]\,. \tag{84}$$

We now project $y(x,E,l)$ onto another solution, defined by its asymptotics as $x \to 0$. Taking the Wronskian of both sides of (??) with $\psi(x,E,l)$ results in the x-independent equation

$$C(E,l)W[y_0,\psi](E,l) = W[y_{-1},\psi](E,l) + W[y_1,\psi](E,l)\,. \tag{85}$$

To relate the objects on the right-hand side of this equation back to $W[y_0,\psi]$, we first define another set of 'rotated' solutions, by analogy with (??):

$$\psi_k(x,E,l) = \omega^{k/2}\psi(\omega^{-k}x,\ \omega^{2k}E,\ l)\,, \quad (k \in \mathbb{Z})\,. \tag{86}$$

The functions (??) also solve (??) and a consideration of their behaviour as $x \to 0$ shows that

$$\psi_k(x,E,l) = \omega^{-(l+1/2)k}\psi(x,E,l)\,. \tag{87}$$

In addition,

$$W[y_k,\psi_k](E,l) = \omega^k\, W[y(\omega^{-k}x,\omega^{2k}E,l),\psi(\omega^{-k}x,\omega^{2k}E,l)] = W[y,\psi](\omega^{2k}E,l)\,. \tag{88}$$

Combining these results,

$$W[y_k,\psi](E,l) = \omega^{(l+1/2)k}W[y,\psi](\omega^{2k}E,l)\,, \tag{89}$$

and setting

$$D(E,l) = W[y,\psi](E,l)\,, \tag{90}$$

the projected Stokes relation (??) becomes

$$C(E,l)D(E,l) = \omega^{-(l+1/2)}D(\omega^{-2}E,l) + \omega^{(l+1/2)}D(\omega^2E,l)\,. \tag{91}$$

Therefore, as anticipated at the end of section ??, with the identifications $T = C$ and $Q = D$ and (??), the Stokes equation (??) exactly matches the Baxter TQ relation (??) for the quantum KdV theory described in [?]. Finally, the constraint $W[\,y_k\,, y_{k+1}] = 1$, becomes

$$\det \begin{pmatrix} \omega^{-\frac{2l+1}{4}} D_-(\omega^{-1}E) & \omega^{\frac{2l+1}{4}} D_-(\omega E) \\ \omega^{\frac{2l+1}{4}} D_+(\omega^{-1}E) & \omega^{-\frac{2l+1}{4}} D_+(\omega E) \end{pmatrix} = (2l+1)\ , \tag{92}$$

with $D_-(E) = D(E, l)$ and $D_+(E) = D(E, -l-1)$. Equation (??) is known in the literature as *quantum Wronskian* [?], and is a special case of the QQ-systems of [?]. In turn, the QQ-systems are x-independent versions of the ψ-systems of [?].

3.2 All orders semiclassical expansion and the quantum integrals of motion

We first note that with a simple change of variables [?], the Schrödinger equation (??) can be recast into the form

$$\left(-\varepsilon^2 \partial_w^2 + Z(w)\right) y(w) = 0\,, \tag{93}$$

where

$$Z(w) = \frac{1}{4\hat{l}^2}\, w^{1/\hat{l}-2}(w^{M/\hat{l}} - 1)\,, \quad \hat{l} = l + \frac{1}{2}\,, \quad \varepsilon = E^{-(M+1)/2M}\,. \tag{94}$$

A key feature of equations (??) and (??) is that the E-dependence, contained in ε, has been factored out of the transformed potential $Z(w)$. Suppose now that (??) has a solution of the form

$$y(w) = \exp\left(\frac{1}{\varepsilon}\sum_{n=0}^{\infty} \varepsilon^n S_n(w)\right)\,. \tag{95}$$

For equation (??) to be fulfilled order-by-order in ε, the derivatives $S_{n,w}(w)$ must obey the following recursion relation:

$$S_{0,w}(w) = -\sqrt{Z(w)}\ , \quad 2\, S_{0,w}\, S_{n,w} + \sum_{j=1}^{n-1} S_{j,w}\, S_{n-j,w} + S_{n-1,ww} = 0\,, \quad (n \geq 1)\,. \tag{96}$$

The first few terms of the solution are

$$\begin{aligned} S_{1,w} &= -\frac{Z_{,w}}{4Z}\,, \quad S_{2,w} = -\frac{1}{48}\left(\frac{Z_{,ww}}{Z^{3/2}} + 5\,\partial_w\left(\frac{Z_{,w}}{Z^{3/2}}\right)\right)\,, \\ S_{3,w} &= -\frac{Z_{,ww}}{16Z^2} + \frac{5(Z_{,w})^2}{64Z^3} = \partial_w\left(\frac{5(Z_{,w})^2}{64Z^3} - \frac{Z_{,ww}}{16Z^2}\right)\,, \end{aligned} \tag{97}$$

and further terms are very easily obtained using, for example, Mathematica. Keeping only the first two contributions, S_0 and S_1, corresponds to the standard physical optics or WKB approximation. Near the turning points $Z = 0$ the approximation breaks down, and further work is needed to find the connection formulae for the continuation of WKB-like solutions of given order from one region of non-vanishing Z to another (see, for example, section 10.7 of [?]).

In cases where $Z(w)$ is an entire function of the coordinate w, with just a pair of well-separated simple zeros on the real axis, Dunham [?] found a remarkably simple formulation of the final quantisation condition, valid to all orders in ε:

$$\frac{1}{\mathbb{i}}\oint_{\gamma} dw\,\left(\sum_{n=0}^{\infty}\varepsilon^{n-1}S_{n,w}(w)\right)=2\pi\,k\ ,\quad (k\in\mathbb{Z}_{\geq})\,. \tag{98}$$

In (??), the contour γ encloses the two turning points; it closes because for such a Z all of the functions $S_{n,w}$ derived from (??) are either entire or else have a pair of square root branch points which can be connected by a branch cut along the real axis. Notice that the contour γ can be taken to lie far from the two turning points where the WKB series breaks down and so there is no need to worry about connection formulae. All of the terms $S_{2n+1,w}$, $n\geq 1$, turn out to be total derivatives and can, therefore, be discarded, while the contribution of $\frac{1}{2\mathbb{i}}S_{1,w}=-\frac{1}{8\mathbb{i}}Z_{,w}/Z$ is a simple factor $\pi/2$, when integrated round the two zeros of Z. Dunham's condition then becomes

$$\frac{1}{\mathbb{i}}\oint_{\gamma} dw\,\left(\sum_{n=0}^{\infty}\varepsilon^{2n-1}S_{2n,w}(w)\right)=(2k{+}1)\pi\,,\quad (k\in\mathbb{Z}_{\geq})\,. \tag{99}$$

In the current situation, we are interested in the radial connection problem, where the integration contour runs initially on the segment $w\in(0,1)$:

$$\oint_{\gamma} dw\ S_{2n,w}(w)\to 2\int_0^1 dw\ S_{2n,w}(w)\,. \tag{100}$$

However, for generic values of $\hat{l}$, M and n the integrand in (??) is divergent at $w=0$ and/or at $w=1$. We need, therefore, a consistent regularisation prescription. To this end we replace the integration on the segment $w\in(0,1)$ with an integral over the Pochhammer contour γ_P, represented in figure ??, around the branch points at $w=0$ and $w=1$. To proceed, we first perform a change of variable $z=w^{M/\hat{l}}$,

$$\check{I}_{2n-1}(M,\hat{l})=\frac{2}{\mathbb{i}}\int_0^1 dw\ S_{2n,w}(w)=\frac{2}{\mathbb{i}}\frac{\hat{l}}{M}\int_0^1 dz\ S_{2n,w}\left(z^{\hat{l}/M}\right)\ z^{\hat{l}/M-1}\,. \tag{101}$$

Setting

$$\tilde{S}_{2n}(z)=\frac{2}{\mathbb{i}}\frac{\hat{l}}{M}S_{2n,w}\left(z^{\hat{l}/M}\right)\ z^{\hat{l}/M-1}\,, \tag{102}$$

the monodromies around $z=0$ and $z=1$ are:

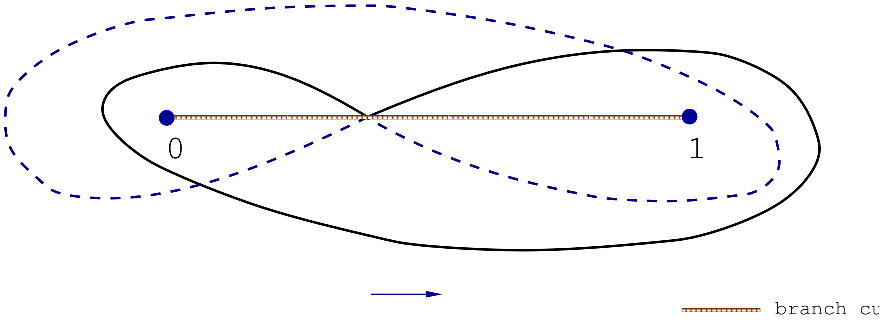


Figure 2: The Pochhammer contour γ_P.

$$\tilde{S}_{2n}(ze^{\mathrm{i}2\pi}) \to e^{\mathrm{i}\frac{\pi}{M}(1-2n)}\tilde{S}_{2n}(z)\,, \quad \tilde{S}_{2n}((z-1)e^{\mathrm{i}2\pi}+1) \to -\tilde{S}_{2n}(z)\,. \tag{103}$$

Therefore, we can replace the integral over the interval $(0,1)$ with an integral over γ_P, provided the extra contribution introduced by integrating over the Pochhammer contour is properly balanced by a normalisation factor. The result is

$$\check{I}_{2n-1}(M,\hat{l}) = \frac{1}{2\left(1-e^{\mathrm{i}\frac{\pi(1-2n)}{M}}\right)} \oint_{\gamma_P} dz\, \tilde{S}_{2n}(z)\,, \tag{104}$$

which is now well defined for generic values of M and λ and can always be written as a finite sum of Euler Beta functions. The explicit outcome is:

$$\check{I}_{2n-1}(M,\hat{l}) = (-1)^n \frac{\sqrt{\pi}\,\Gamma\left(1-\frac{(2n-1)}{2M}\right)}{\Gamma\left(\frac{3}{2}-n-\frac{(2n-1)}{2M}\right)} \frac{(4M+4)^n}{(2n-1)\,n!} I_{2n-1}(M,\hat{l})\,, \tag{105}$$

where $I_{-1}=1$, while the coefficients $I_{2n-1}(M,\hat{l})$, with $n>0$, coincide with the local KdV conserved charges for the vacuum states (??), provided the following identifications are made:

$$c = 1-\frac{6M^2}{M+1}\,, \qquad \Delta = \frac{(2l+1)^2-4M^2}{16(M+1)}\,. \tag{106}$$

The exact link between the all-order WKB coefficients and the integrals of motion (??) is another striking result of the ODE/IM correspondence.

3.3 Simple generalisations

First of all, the link between the ODE (??) and the vacuum states of the quantum KdV model in finite volume $L=2\pi$ can be generalized to accommodate the whole tower of excited states [?] (see also [?]). The basic replacement is to send $P_0^{\mathbf{[KdV]}} \to P_{\mathrm{exc}}^{\mathbf{[KdV]}}$ in (??) with

$$P_{\mathrm{exc}}^{\mathbf{[KdV]}}(x,E,l,M,\{z_k\}) = \left(x^{2M}+\frac{l(l+1)}{x^2}-2\partial_x^2\left(\sum_{k=1}^{K}\ln(x^{2M+2}-z_k)\right)-E\right), \tag{107}$$

where the constants $\{z_k\}$ satisfy the auxiliary Bethe Ansatz type equations:

$$\sum_{\substack{j=1\\j\neq k}}^{K} \frac{z_k(z_k^2+(M+3)(2M+1)z_kz_j+M(2M+1)z_j^2)}{(z_k-z_j)^3} - \frac{Mz_k}{4(M+1)} + \Delta = 0\,. \tag{108}$$

Generalisations of the ODE/IM correspondence for both the vacuum and the excited states involving families of higher-order differential operators were studied in [?, ?, ?, ?, ?, ?, ?, ?].

In the following, instead of describing the setup of [?] or [?, ?, ?, ?, ?, ?, ?, ?, ?] we shall focus on an off-critical variant, which is related to the classical problem of embedded surfaces in AdS$_3$ and also to polygonal Wilson loops [?, ?]. As a preliminary remark, we notice that a natural generalisation of the Sturm-Liouville problem associated with (??)-(??) corresponds to polynomial potentials of the form

$$P_0^{\textbf{[HsG]}}(x,\{x_k\}) = \prod_{k=1}^{2N}(x-x_k)\,, \quad (2N\in\mathbb{Z}_>)\,, \tag{109}$$

where x_1 can be set to zero by shifting x, while the remaining constants x_k $(i=2,\ldots,2N)$ are free parameters. It was argued in [?] that the choice (??), is connected to the Homogeneous sine-Gordon model (hsG) in its CFT limit or equivalently to the $SU(2N)_2/U(1)^{2N-1}$ parafermions [?, ?, ?]. The specific choices of the set x_k which lead to

$$P_0^{\textbf{[Vir]}}(x,m,m') = x^{m-2}(x^{m'-m}-\tilde{E})\,, \tag{110}$$

correspond to the Virasoro minimal models $\mathrm{M}_{m,m'}$. As described in [?], the generalised potential (??) is related to the original instance of the ODE/IM correspondence, discussed in the previous sections, by a simple change of variables.

We shall see in the remaining part of this review that the polynomial potentials (??) appear naturally in the description of Wilson loops in AdS$_3$ with polygonal boundaries.

4 Classical integrable equations and embedded surfaces

In this section we wish to recall the general properties of minimal and constant mean curvature (CMC) surfaces embedded in AdS$_{n+1}$ and explain how a linear differential system arises as a structural constraint on the functions describing the embedding of these surfaces. We will then focus on the simplest non-trivial case of minimal surfaces embedded in AdS$_3$. Here a single field $\tilde{\varphi}$ is present, parametrizing the conformal factor of the metric. This field satisfies the modified sinh-Gordon equation [?, ?, ?, ?], with (anti)-holomorphic potentials A and $\bar{A}$,[7]

[7]As shown in section ??, these functions intuitively measure how 'curved' the surface is, and enter in the definition of the Gauss curvature.

whose singularity structure has profound effects on the shape of the embedded surface. In particular, the presence of an irregular singularity (e.g. when A is a polynomial) corresponds to the presence of a Stokes phenomenon in the linear differential system which then translates into the existence of light-like edges of the surface at the conformal boundary of AdS_3. For A and $\bar{A}$ polynomials of order $2N \in \mathbb{Z}_>$, the embedded surface will sit on a light-like $4(N+1)$-gon on the conformal boundary. Finally, we will explain how to encode the full information of this embedding into a set of finite difference equations, the T-system and the Baxter TQ equation, which can then be converted into non-linear integral equation form.

4.1 Surfaces embedded in AdS_{n+1}

The $(n+1)$-dimensional anti de-Sitter space AdS_{n+1} can be described by a pseudo-spherical restriction of the pseudo-Riemannian flat space $\mathbb{R}^{2,n}$. More precisely, consider $\vec{Y} = (Y^{-1}, Y^0, \cdots, Y^n)^{\mathrm{T}} \in \mathbb{R}^{2,n}$, where the superscript T denotes the operation of matrix transposition; then the condition

$$\vec{Y} \cdot \vec{Y} \equiv -\left(Y^{-1}\right)^2 - \left(Y^0\right)^2 + \sum_{k=1}^{n} \left(Y^k\right)^2 = -\alpha^2 , \qquad (\alpha \in \mathbb{R}) , \tag{111}$$

represents an immersion of AdS_{n+1} with radius α inside $\mathbb{R}^{2,n}$. Here and below we use the dot to denote the scalar product of vectors in $\mathbb{R}^{2,n}$:

$$\vec{Y} \cdot \vec{Y}' = \eta_{AB} Y^A Y'^B , \qquad \eta_{AB} = \mathrm{diag}\left(-1 , -1 , \underbrace{1 , \ldots , 1}_{n} \right) . \tag{112}$$

Concerning the indices we will adopt the convention

$$A, B, C, \ldots = -1, 0, 1, \ldots, n , \qquad \mu, \nu, \ldots = 0, 1 , \tag{113a}$$

$$j, k, l, \ldots = 1, 2, \ldots, n , \qquad a, b, \ldots = 1, 2 . \tag{113b}$$

The AdS_{n+1} space can be parametrised by global coordinates $(\rho, \tau, \theta_1, \ldots, \theta_{n-1})$ as

$$\begin{aligned} Y^{-1} &= \alpha \cosh(\rho) \cos(\tau) , \\ Y^0 &= \alpha \cosh(\rho) \sin(\tau) , \\ Y^j &= \alpha \sinh(\rho) \cos(\theta_{n-j+1}) \prod_{k=1}^{n-j} \sin(\theta_k) , \qquad \theta_n = 0 . \end{aligned} \tag{114}$$

From the last equations we can read the standard AdS metric

$$ds^2 = \alpha^2 \left(-\cosh^2(\rho)\, d\tau^2 + d\rho^2 + \sinh^2(\rho)\, d\Omega_{n-1}^2 \right) , \tag{115}$$

where $d\Omega_{n-1}^2$ is the metric of the unit $(n-1)$-dimensional sphere. The conformal boundary of AdS_{n+1} can be reached by taking the limit $\rho \to \infty$ jointly with a rescaling of the arc-length $ds \to ds/\sinh(\rho)$. The resulting metric is that of a cylinder in $\mathbb{R}^{1,n}$:

$$ds_\partial^2 = \alpha^2 \left(-d\tau^2 + d\Omega_{n-1}^2\right) \ . \tag{116}$$

Let us mention another useful parametrization of the space AdS_{n+1}: the *Poincaré coordinates* $\{\mathsf{r}, \mathsf{t}, \vec{\mathsf{x}}\}$

$$\begin{aligned}
Y^{-1} &= \frac{\alpha^2}{2\mathsf{r}} + \mathsf{r}\frac{\alpha^2 + |\vec{\mathsf{x}}|^2 - \mathsf{t}^2}{2\alpha^2} \ , \\
Y^n &= -\frac{\alpha^2}{2\mathsf{r}} + \mathsf{r}\frac{\alpha^2 - |\vec{\mathsf{x}}|^2 + \mathsf{t}^2}{2\alpha^2} \ , \\
Y^0 &= \frac{\mathsf{r}}{\alpha}\mathsf{t} \ , \qquad Y^j = \frac{\mathsf{r}}{\alpha}\mathsf{x}^j \ , \quad 1 \leq j < n \ .
\end{aligned} \tag{117}$$

In these coordinates the metric reads

$$ds^2 = \frac{\alpha^2}{\mathsf{r}^2}d\mathsf{r}^2 - \frac{\mathsf{r}^2}{\alpha^2}d\mathsf{t}^2 + \frac{\mathsf{r}^2}{\alpha^2}|d\vec{\mathsf{x}}|^2 \ , \tag{118}$$

from which we see that $\mathsf{r} \to \infty$ approaches the boundary $\partial\mathrm{AdS}_{n+1}$. The singularity $\mathsf{r} = 0$ is an apparent one, called *Poincaré-Killing horizon* and shows that the Poincaré coordinates are not global.

Now that we have defined our embedding space, AdS_{n+1}, we move on to the construction of the embedded surface Σ. Here we have a choice to make: we need to decide whether the time-like direction of AdS_{n+1} lies in the tangent space $T\Sigma$, in which case we will have what is known as a time-like surface, or is orthogonal to it which will yield a space-like surface. This choice will dictate the type of reality conditions we need to impose on the parametrisation of Σ. For time-like surfaces we will need to describe the surface with Minkowski coordinates ξ^μ or, equivalently, with light-cone coordinates $(\xi^+ = \xi^0 + \xi^1, \xi^- = \xi^0 - \xi^1) \in \mathbb{R}^2$. On the contrary, space-like surfaces will be parametrised by Euclidean coordinates x^a or, which is the same, complex coordinates $(z = x^1 + \mathbb{i}\, x^2, \bar{z} = x^1 - \mathbb{i}\, x^2) \in \mathbb{C}$. In the following we will concentrate on the latter type of surfaces. The same type of analysis can be carried over with some modifications for time-like surfaces. As is usual when dealing with the Euclidean plane, we will let the coordinates $(z, \bar{z})$ take values in the full two dimensional complex space $\mathbb{C}^2$ while keeping the real slice condition $z^* = \bar{z}$ in the back of our minds, imposing it only when we see fit. Furthermore, we will continue to denote partial derivatives with subscripts after a comma, i.e.:

$$f_{,z}(z,\bar{z}) = \frac{\partial}{\partial z}f(z,\bar{z}) = \partial f(z,\bar{z}) \ , \quad f_{,\bar{z}}(z,\bar{z}) = \frac{\partial}{\partial \bar{z}}f(z,\bar{z}) = \bar{\partial} f(z,\bar{z}) \ . \tag{119}$$

Finally, whenever it is not necessary, we will drop the explicit dependence on the coordinates.

The description of the embedding of Σ in AdS_{n+1} is carried by the embedding function $\vec{Y} : \mathbb{C}^2 \longrightarrow \mathbb{R}^{2,n}$, such that $\vec{Y}(z,\bar{z}) \cdot \vec{Y}(z,\bar{z}) = -\alpha^2$. From it we can immediately construct the tangent space $T_p\Sigma$ at any point $p \in \Sigma$ as the span of the two vectors $\vec{Y}_{,z}$ and $\vec{Y}_{,\bar{z}}$, and compute the metric tensor, also known as *first fundamental form*:

$$\mathrm{I} = ds^2 = g_{zz}\,(dz)^2 + 2\,g_{z\bar{z}}\,dz\,d\bar{z} + g_{\bar{z}\bar{z}}\,(d\bar{z})^2 \ , \qquad g = \begin{pmatrix} \vec{Y}_{,z}\cdot\vec{Y}_{,z} & \vec{Y}_{,z}\cdot\vec{Y}_{,\bar{z}} \\ \vec{Y}_{,z}\cdot\vec{Y}_{,\bar{z}} & \vec{Y}_{,\bar{z}}\cdot\vec{Y}_{,\bar{z}} \end{pmatrix} . \tag{120}$$

It is an established fact [?, ?, ?, ?] that, at least locally, one can choose isothermal coordinates $(z', \bar{z}')$ such that

$$ds^2 = 2\,g'_{z'\bar{z}'}\,dz'\,d\bar{z}' \ . \tag{121}$$

In the following we will fix these coordinates and drop the primes. The requirements $\vec{Y}_{,z}\cdot\vec{Y}_{,z} = \vec{Y}_{,\bar{z}}\cdot\vec{Y}_{,\bar{z}} = 0$ are known as Virasoro constraints and we see that these immediately imply that the (real) vectors $\vec{Y}_{,1} = \vec{Y}_{,z} + \vec{Y}_{,\bar{z}}$ and $\vec{Y}_{,2} = -\mathbb{i}\vec{Y}_{,z} + \mathbb{i}\vec{Y}_{,\bar{z}}$ satisfy the following identities

$$\vec{Y}_{,1}\cdot\vec{Y}_{,1} = \vec{Y}_{,2}\cdot\vec{Y}_{,2} \ , \qquad \vec{Y}_{,1}\cdot\vec{Y}_{,2} = 0 \ . \tag{122}$$

As a consequence, since we already have one independent time-like vector $\vec{Y}$ and in $\mathbb{R}^{2,n}$ there can be at most 2, we conclude that

$$\vec{Y}_{,1}\cdot\vec{Y}_{,1} > 0 \ , \qquad \vec{Y}_{,2}\cdot\vec{Y}_{,2} > 0 \quad \Longrightarrow \quad \vec{Y}_{,z}\cdot\vec{Y}_{,\bar{z}} > 0 \ . \tag{123}$$

Due to the AdS constraint $\vec{Y}\cdot\vec{Y} = -\alpha^2$, we see that the triple $\left(\vec{Y}, \vec{Y}_{,z}, \vec{Y}_{,\bar{z}}\right)$ spans, at any point of Σ, a three-dimensional subspace of AdS_{n+1}. In order to understand the structure of the embedding, we now need to augment the above triple to a full basis of $\mathbb{R}^{2,n}$ and we can do this by introducing the following set of orthonormal real vectors,[8]

$$\left\{\vec{N}_j\right\}_{j=1}^{n-1} \ , \qquad \vec{N}_i\cdot\vec{N}_j = \eta_{ij} \ , \qquad \eta_{ij} = \mathrm{diag}\,(-1\,,1\,,\dots\,,1) \ , \tag{124}$$

spanning, together with $\vec{Y}$, the normal space $(T_p\Sigma)^{\perp}$ at any point $p \in \Sigma$:

$$\vec{N}_i\cdot\vec{Y} = \vec{N}_i\cdot\vec{Y}_{,z} = \vec{N}_i\cdot\vec{Y}_{,\bar{z}} = 0 \ . \tag{125}$$

For each of these vectors there exists a *second fundamental form* II_j, defined as

$$\begin{aligned} \mathrm{II}_j &= (d_j)_{zz}\,(dz)^2 + 2\,(d_j)_{z\bar{z}}\,dz\,d\bar{z} + (d_j)_{\bar{z}\bar{z}}\,(d\bar{z})^2 \ , \\ d_j &= \begin{pmatrix} \vec{Y}_{,zz}\cdot\vec{N}_j & \vec{Y}_{,z\bar{z}}\cdot\vec{N}_j \\ \vec{Y}_{,z\bar{z}}\cdot\vec{N}_j & \vec{Y}_{,\bar{z}\bar{z}}\cdot\vec{N}_j \end{pmatrix} . \end{aligned} \tag{126}$$

[8] To have a basis of $\mathbb{R}^{2,n}$ we need 2 time-like vectors. One, $\vec{Y}$, we already have, the other has to be one of these normals. We choose it to be $\vec{N}_1$.

Note that while in principle we should also have a fundamental form associated to the normal direction $\vec{Y}$,[9] this turns out to be trivial:

$$d_0 = \begin{pmatrix} \vec{Y}_{,zz} \cdot \vec{Y} & \vec{Y}_{,z\bar{z}} \cdot \vec{Y} \\ \vec{Y}_{,z\bar{z}} \cdot \vec{Y} & \vec{Y}_{,\bar{z}\bar{z}} \cdot \vec{Y} \end{pmatrix} = \begin{pmatrix} -\vec{Y}_{,z} \cdot \vec{Y}_{,z} & -\vec{Y}_{,z} \cdot \vec{Y}_{,\bar{z}} \\ -\vec{Y}_{,z} \cdot \vec{Y}_{,\bar{z}} & -\vec{Y}_{,\bar{z}} \cdot \vec{Y}_{,\bar{z}} \end{pmatrix} = -g \ . \tag{127}$$

It is now a good point to simplify the notation by introducing the following functions

$$e^{\tilde{\varphi}} = \vec{Y}_{,z} \cdot \vec{Y}_{\bar{z}} \ , \qquad H_j = e^{-\tilde{\varphi}} \vec{Y}_{,z\bar{z}} \cdot \vec{N}_j \ , \tag{128a}$$

$$A_j = \vec{Y}_{,zz} \cdot \vec{N}_j \ , \qquad \bar{A}_j = \vec{Y}_{,\bar{z}\bar{z}} \cdot \vec{N}_j \ . \tag{128b}$$

The field $\tilde{\varphi} \in \mathbb{R}$ is sometimes called the Pohlmeyer field. From the first and the second fundamental forms one can construct the shape operators

$$w_j = d_j g^{-1} = \begin{pmatrix} H_j & e^{-\tilde{\varphi}} A_j \\ e^{-\tilde{\varphi}} \bar{A}_j & H_j \end{pmatrix} \ , \tag{129}$$

whose invariants compute the total Gauss curvature K and the components H_j of the mean curvature vector $\vec{H}$

$$H_j = \frac{1}{2} \mathrm{tr}\,(w_j) = \frac{\vec{Y}_{,z\bar{z}} \cdot \vec{N}_j}{\vec{Y}_{,z} \cdot \vec{Y}_{,\bar{z}}} = H_j \ , \tag{130a}$$

$$K = \sum_{j=1}^{n-1} \det\,(w_j) = \sum_{j=1}^{n-1} \left(H_j H_j - e^{-2\tilde{\varphi}} A_j \bar{A}_j \right) \ . \tag{130b}$$

Now we have, at any point $p \in \Sigma$, a complete set of orthogonal vectors in $\mathbb{R}^{2,n}$ which we collect as the rows of a matrix $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = \left(\vec{Y} \ , \ \vec{Y}_{,z} \ , \ \vec{Y}_{,\bar{z}} \ , \ \vec{N}_1 \ , \ \cdots \ , \ \vec{N}_{n-1} \right)^{\mathrm{T}} \ . \tag{131}$$

This object is known as the *frame field* or *moving frame* and is anchored on the surface Σ. Consequently, its motion along the surface has to satisfy certain constraints and, since $\boldsymbol{\sigma}$ provides a basis everywhere on Σ, these take the form of a set of linear equations, called the Gauss–Weingarten (GW) system:

$$\boldsymbol{\sigma}_{,z} = \mathcal{U} \boldsymbol{\sigma} \ , \qquad \boldsymbol{\sigma}_{,\bar{z}} = \bar{\mathcal{U}} \boldsymbol{\sigma} \ . \tag{132}$$

Finally, this system immediately implies a consistency condition which, in the geometry literature, is known as the Gauss-Codazzi-Mainardi (GMC) equation

$$\mathcal{U}_{,\bar{z}} - \bar{\mathcal{U}}_{,z} + \left[\mathcal{U}, \bar{\mathcal{U}} \right] = 0 \ . \tag{133}$$

[9] We will identify this direction with the index 0.

The above equation represents a set of structural conditions for the surface, imposing non-linear constraints on the functions defining the shape and properties of Σ. Its functional form is completely general and appears as a condition for every surface embedded in any space, the details of the particular problem at hand being contained in the form of the matrices $\mathcal{U}$ and $\bar{\mathcal{U}}$. In a more geometrical language, $\mathcal{U}$ and $\bar{\mathcal{U}}$ are the components of a connection one-form $\mathcal{U}dz + \bar{\mathcal{U}}d\bar{z}$ and the GMC equation above is a vanishing condition on the curvature two-form associated to said connection, completely analogous to the ZCC (??) which appeared in the case of the KdV equation. In our case, for a generic surface embedded in AdS_{n+1}, $\mathcal{U}$ and $\bar{\mathcal{U}}$ are $(n+2) \times (n+2)$ matrices, which depend on

- the real Pohlmeyer field $\tilde{\varphi}$,
- the $n-1$ real mean curvatures H_j,
- the $n-1$ complex functions A_j,
- the $\frac{1}{2}n\,(n-1)$ complex functions $B_{ij} = -B_{ji}$, describing the rotation of the normal space $(T\Sigma)^{\perp}$ under motion along the surface:

$$B_{ij} = \vec{N}_{i,z} \cdot \vec{N}_j = -\vec{N}_i \cdot \vec{N}_{j,z} \; . \tag{134}$$

The curvatures H_j and the functions A_j are usually treated as inputs, identifying the type of surface one is dealing with. An interpretation of the functions A_j for the case $n=2$ is presented in section ??. On the other hand, the Pohlmeyer field $\tilde{\varphi}$ and the functions B_{ij} are to be treated as proper dynamical variables.

We will not give the explicit expressions, in the general case, for the matrices $\mathcal{U}$ and $\bar{\mathcal{U}}$ nor for the GMC equation, as the case of interest of this review, presented below, is $n=2$. The reader can easily extract them by derivation from the various constraints amongst the vectors in $\boldsymbol{\sigma}$. We wish however to note that for general n the matrices $\mathcal{U}$ and $\bar{\mathcal{U}}$ entering the GW system (??) can be seen to belong to the affine untwisted Kač-Moody algebra of type B or C. By appropriately redefining the quantities listed above, one can connect this system with the corresponding Toda field theory. Off-critical generalisations of the ODE/IM correspondence associated to higher-rank algebras have been discussed in [?, ?, ?, ?, ?, ?, ?, ?], although without specific analysis of the connection with surface embedding. The case we focus on here, that is $n=2$, is particularly simple as the associated algebra turns out to be $B_1^{(1)} = \mathfrak{so}_3^{(1)} \equiv A_1^{(1)} = \mathfrak{su}_2^{(1)}$.

4.2 Minimal surfaces in AdS_3

While in section ?? the description of embedded surfaces in AdS_{n+1} was reviewed, here we concentrate on the simple case of minimal surfaces embedded in AdS_3.[10] The number of

[10]In three dimensions, a minimal surface is defined by the vanishing of the mean curvature $H \equiv H_1 = 0$.

functions we have to deal with collapses now to two: the real Pohlmeyer field $\tilde{\varphi}$ and the complex function $A_1 = A$. The former will be our unknown function, while we will consider A as a given.

As mentioned in section ??, the structural data of an embedded surface $\Sigma \subset \mathrm{AdS}_3$ is contained in a pair of 4×4 matrices $\mathcal{U}$ and $\bar{\mathcal{U}}$ satisfying the Gauss-Codazzi-Mainardi equation

$$\mathcal{U}_{,\bar{z}} - \bar{\mathcal{U}}_{,z} + [\mathcal{U}, \bar{\mathcal{U}}] = 0 \ . \tag{135}$$

These matrices depend on the complex variables $(z, \bar{z})$ through the Pohlmeyer field $\tilde{\varphi}$, its derivatives and the function A. In the case of a minimal surface in AdS_3 they take the following explicit form

$$\mathcal{U} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \tilde{\varphi}_{,z} & 0 & -A \\ \frac{1}{\alpha^2} e^{\tilde{\varphi}} & 0 & 0 & 0 \\ 0 & 0 & -e^{-\tilde{\varphi}} A & 0 \end{pmatrix} , \qquad \bar{\mathcal{U}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{\alpha^2} e^{\tilde{\varphi}} & 0 & 0 & 0 \\ 0 & 0 & \tilde{\varphi}_{,\bar{z}} & -\bar{A} \\ 0 & -e^{-\tilde{\varphi}} \bar{A} & 0 & 0 \end{pmatrix} , \tag{136}$$

and the GMC equation reduces to the non-linear partial differential equation

$$\tilde{\varphi}_{,z\bar{z}} = \frac{1}{\alpha^2} e^{\tilde{\varphi}} - A \bar{A} e^{-\tilde{\varphi}} \ , \qquad A_{,\bar{z}} = \bar{A}_{,z} = 0 \ . \tag{137}$$

This can be further simplified by introducing the quantities

$$\varphi = \tilde{\varphi} - \ln(2\alpha^2) \ , \qquad P(z) = \frac{1}{2\mathbb{i}\alpha} A(z) \ , \qquad \bar{P}(\bar{z}) = -\frac{1}{2\mathbb{i}\alpha} \bar{A}(\bar{z}) \ , \tag{138}$$

in terms of which the matrices $\mathcal{U}$ and $\bar{\mathcal{U}}$ read

$$\mathcal{U} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \varphi_{,z} & 0 & -2\mathbb{i}\alpha P \\ 2e^{\varphi} & 0 & 0 & 0 \\ 0 & 0 & -\mathbb{i}\frac{e^{-\varphi}}{\alpha} P & 0 \end{pmatrix} , \quad \bar{\mathcal{U}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2e^{\varphi} & 0 & 0 & 0 \\ 0 & 0 & \varphi_{,\bar{z}} & 2\mathbb{i}\alpha \bar{P} \\ 0 & \mathbb{i}\frac{e^{-\varphi}}{\alpha} \bar{P} & 0 & 0 \end{pmatrix} , \tag{139}$$

and the GMC equation takes the form of the so-called *modified sinh-Gordon equation*

$$\frac{1}{2} \varphi_{,z\bar{z}} = e^{\varphi} - P \bar{P} e^{-\varphi} \ . \tag{140}$$

This equation can be written in the form (??) by a shift of the field φ together with a redefinition of the variables $z, \bar{z}$

$$\varphi(z, \bar{z}) \longrightarrow \varphi(w(z), \bar{w}(\bar{z})) + \frac{1}{2} \ln\left(P(z) \bar{P}(\bar{z})\right) \ , \tag{141a}$$

$$w(z) = 2 \int^{z} \sqrt{P(z')} dz' \ , \qquad \bar{w}(\bar{z}) = 2 \int^{\bar{z}} \sqrt{\bar{P}(\bar{z}')} d\bar{z}' \ . \tag{141b}$$

We wish to remark that the above transformation, making (??) into (??), does alter the geometry on which the equation is considered. Moreover, equation (??) is defined on the space $\mathbb{C}^2$, on which we impose the real slice condition $\bar{z} = z^*$; on the other hand, equation (??) is defined on $\mathbb{R}^2$. Hence the two equations are not to be considered equivalent.

Although it is not immediately evident, the above pair (??) can be gauge rotated to a tensor product form:[11]

$$\mathcal{U}' = U_{\rm L} \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes U_{\rm R} \ , \qquad \bar{\mathcal{U}}' = \bar{U}_{\rm L} \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \bar{U}_{\rm R} \ , \tag{142}$$

where

$$\mathcal{U}' = \Gamma^{-1}\mathcal{U}\Gamma - \Gamma^{-1}\Gamma_{,z} \ , \qquad \bar{\mathcal{U}}' = \Gamma^{-1}\mathcal{U}\Gamma - \Gamma^{-1}\Gamma_{,\bar{z}} \ . \tag{143}$$

The explicit expressions for the 2×2 $U_{\rm R}$, $U_{\rm L}$, $\bar{U}_{\rm R}$ and $\bar{U}_{\rm L}$ matrices are as follows:

$$U_{\rm L} = \begin{pmatrix} -\frac{1}{2}\varphi_{,z} & 1 \\ P & \frac{1}{2}\varphi_{,z} \end{pmatrix} , \qquad \bar{U}_L = \begin{pmatrix} 0 & \bar{P}e^{-\varphi} \\ e^{\varphi} & 0 \end{pmatrix} , \tag{144a}$$

$$U_{\rm R} = \begin{pmatrix} -\frac{1}{2}\varphi_{,z} & \mathbb{i} \\ \mathbb{i}P & \frac{1}{2}\varphi_{,z} \end{pmatrix} , \qquad \bar{U}_R = \begin{pmatrix} 0 & -\mathbb{i}\bar{P}e^{-\varphi} \\ -\mathbb{i}e^{\varphi} & 0 \end{pmatrix} , \tag{144b}$$

while the rotation matrix is

$$\Gamma = \begin{pmatrix} 0 & \mathbb{i}\alpha & \alpha & 0 \\ 0 & 0 & 0 & 2\mathbb{i}\alpha \\ 2\alpha e^{\varphi} & 0 & 0 & 0 \\ 0 & -1 & -\mathbb{i} & 0 \end{pmatrix} . \tag{145}$$

One can further rotate both left and right pairs as

$$L_{\rm L} = e^{\frac{1}{4}\varphi\sigma^3} U_{\rm L} e^{-\frac{1}{4}\varphi\sigma^3} - e^{\frac{1}{4}\varphi\sigma^3}\partial e^{-\frac{1}{4}\varphi\sigma^3} \ , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{146}$$

and similarly for the other three matrices, obtaining the more symmetric form

$$L_{\rm L} = \begin{pmatrix} -\frac{1}{4}\varphi_{,z} & e^{\frac{\varphi}{2}} \\ Pe^{-\frac{\varphi}{2}} & \frac{1}{4}\varphi_{,z} \end{pmatrix} , \qquad \bar{L}_{\rm L} = \begin{pmatrix} \frac{1}{4}\varphi_{,\bar{z}} & \bar{P}e^{-\frac{\varphi}{2}} \\ e^{\frac{\varphi}{2}} & -\frac{1}{4}\varphi_{,\bar{z}} \end{pmatrix} , \tag{147a}$$

$$L_{\rm R} = \begin{pmatrix} -\frac{1}{4}\varphi_{,z} & \mathbb{i}e^{\frac{\varphi}{2}} \\ \mathbb{i}Pe^{-\frac{\varphi}{2}} & \frac{1}{4}\varphi_{,z} \end{pmatrix} , \qquad \bar{L}_{\rm R} = \begin{pmatrix} \frac{1}{4}\varphi_{,\bar{z}} & -\mathbb{i}\bar{P}e^{-\frac{\varphi}{2}} \\ -\mathbb{i}e^{\frac{\varphi}{2}} & -\frac{1}{4}\varphi_{,\bar{z}} \end{pmatrix} . \tag{147b}$$

[11] It is an easy exercise to verify that the GMC equations (and thus the structural data of Σ) is invariant under the gauge rotation

$$(\mathcal{U},\bar{\mathcal{U}}) \longrightarrow (\Gamma^{-1}\mathcal{U}\Gamma - \Gamma^{-1}\Gamma_{,z}, \Gamma^{-1}\bar{\mathcal{U}}\Gamma - \Gamma^{-1}\Gamma_{,\bar{z}}) \ ,$$

where Γ is some 4×4 matrix depending on $(z, \bar{z})$.

As a consequence of the above decomposition, the rotated frame $\boldsymbol{\sigma}' = \left(e^{\frac{1}{4}\varphi\sigma^3} \otimes e^{\frac{1}{4}\varphi\sigma^3}\right)\Gamma^{-1}\boldsymbol{\sigma}$ is also decomposed as

$$\boldsymbol{\sigma}' = \boldsymbol{\Psi}\mathbf{M}_0 \ , \qquad \boldsymbol{\Psi} = \boldsymbol{\Psi}_{\mathrm{L}} \otimes \boldsymbol{\Psi}_{\mathrm{R}} \ , \tag{148}$$

where $\mathbf{M}_0$ is a constant 4×4 matrix, while $\boldsymbol{\Psi}_{\mathrm{L}}$ and $\boldsymbol{\Psi}_{\mathrm{R}}$ are solutions to their respective linear problems

$$\boldsymbol{\Psi}_{\mathrm{L},z} = L_{\mathrm{L}}\boldsymbol{\Psi}_{\mathrm{L}} \ , \qquad \boldsymbol{\Psi}_{\mathrm{L},\bar{z}} = \bar{L}_{\mathrm{L}}\boldsymbol{\Psi}_{\mathrm{L}} \ , \tag{149a}$$

$$\boldsymbol{\Psi}_{\mathrm{R},z} = L_{\mathrm{R}}\boldsymbol{\Psi}_{\mathrm{R}} \ , \qquad \boldsymbol{\Psi}_{\mathrm{R},\bar{z}} = \bar{L}_{\mathrm{R}}\boldsymbol{\Psi}_{\mathrm{R}} \ . \tag{149b}$$

Recapitulating, given two solutions of the above systems (??,??), one can reconstruct the corresponding embedding function $\vec{Y}$ for the minimal surface in AdS_3 as

$$\vec{Y} \equiv \vec{e}_1^{\,\mathrm{T}}\boldsymbol{\sigma} = \vec{e}_1^{\,\mathrm{T}}\Gamma\left(e^{-\frac{1}{4}\varphi\sigma^3} \otimes e^{-\frac{1}{4}\varphi\sigma^3}\right)\left(\boldsymbol{\Psi}_{\mathrm{L}} \otimes \boldsymbol{\Psi}_{\mathrm{R}}\right)\mathbf{M}_0 \ , \tag{150}$$

$$\vec{e}_1^{\,\mathrm{T}} = \begin{pmatrix} 1 \ , & 0 \ , & 0 \ , & 0 \end{pmatrix} \ .$$

Let us also mention that the matrix $\mathbf{M}_0$ is not completely general. In fact its form can be almost entirely fixed by considering the orthogonality and normalisation conditions on the scalar products of the basis vectors, which in terms of $\boldsymbol{\sigma}$ can be written as

$$\begin{aligned}\boldsymbol{\sigma}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\boldsymbol{\sigma}^{\mathrm{T}} &= \begin{pmatrix} \vec{Y}\cdot\vec{Y} & \vec{Y}\cdot\vec{Y}_{,z} & \vec{Y}\cdot\vec{Y}_{,\bar{z}} & \vec{Y}\cdot\vec{N} \\ \vec{Y}_{,z}\cdot\vec{Y}_{,z} & \vec{Y}\cdot\vec{Y}_{,z} & \vec{Y}_{,z}\cdot\vec{Y}_{,\bar{z}} & \vec{Y}_{,z}\cdot\vec{N} \\ \vec{Y}_{,\bar{z}}\cdot\vec{Y} & \vec{Y}_{,\bar{z}}\cdot\vec{Y}_{,z} & \vec{Y}_{,\bar{z}}\cdot\vec{Y}_{,\bar{z}} & \vec{Y}_{,\bar{z}}\cdot\vec{N} \\ \vec{N}\cdot\vec{Y} & \vec{N}\cdot\vec{Y}_{,z} & \vec{N}\cdot\vec{Y}_{,\bar{z}} & \vec{N}\cdot\vec{N} \end{pmatrix} \\ &= \begin{pmatrix} -\alpha^2 & 0 & 0 & 0 \\ 0 & 0 & e^{\tilde{\varphi}} & 0 \\ 0 & e^{\tilde{\varphi}} & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ .\end{aligned} \tag{151}$$

One then has

$$\left(\boldsymbol{\Psi}_{\mathrm{L}} \otimes \boldsymbol{\Psi}_{\mathrm{R}}\right)\mathbf{M}_0\left(\sigma^3 \otimes \mathbb{1}_2\right)\mathbf{M}_0^{\mathrm{T}}\left(\boldsymbol{\Psi}_{\mathrm{L}} \otimes \boldsymbol{\Psi}_{\mathrm{R}}\right)^{\mathrm{T}} = \frac{\mathbb{i}}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ , \tag{152}$$

or, equivalently,

$$\mathbf{M}_0\left(\sigma^3 \otimes \mathbb{1}_2\right)\mathbf{M}_0^{\mathrm{T}} = \frac{\mathbb{i}/2}{\det(\boldsymbol{\Psi}_{\mathrm{L}})\,\det(\boldsymbol{\Psi}_{\mathrm{R}})}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ . \tag{153}$$

It is a matter of straightforward computation to verify that the following matrix

$$\mathbf{M}_{\text{spec}} = \frac{1}{2\sqrt{\det(\mathbf{\Psi}_{\mathrm{L}})\,\det(\mathbf{\Psi}_{\mathrm{R}})}} \left(\begin{array}{cccc} 0 & \mathbb{i}b & \mathbb{i}b & 0 \\ -\frac{1}{c} & 0 & 0 & \frac{1}{c} \\ \mathbb{i}c & 0 & 0 & \mathbb{i}c \\ 0 & \frac{1}{b} & -\frac{1}{b} & 0 \end{array} \right) , \tag{154}$$

represents a particular solution to the equation (??). In order to derive the general solution, we can reason as follows. Let $\mathbf{M}$ be a solution to (??) and $\mathbf{R} \in GL(4)$ a generic non-singular matrix. Then we can write $\mathbf{M} = \mathbf{R}\mathbf{M}_{\text{spec}}$. Due to both matrices solving the same equation, the matrix $\mathbf{R}$ has to satisfy the following relation

$$\mathbf{R}\,(\varsigma \otimes \varsigma)\,\mathbf{R}^t = (\varsigma \otimes \varsigma) \ , \qquad \varsigma = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) . \tag{155}$$

Expanding this relation in 2×2 blocks, we obtain the following three equations

$$\begin{aligned} \mathbf{R}_{11}\varsigma\mathbf{R}_{12}^t &= -\left(\mathbf{R}_{11}\varsigma\mathbf{R}_{12}^t\right)^t \ , \\ \mathbf{R}_{21}\varsigma\mathbf{R}_{22}^t &= -\left(\mathbf{R}_{21}\varsigma\mathbf{R}_{22}^t\right)^t \ , \\ \mathbf{R}_{11}\varsigma\mathbf{R}_{22}^t &+ \left(\mathbf{R}_{21}\varsigma\mathbf{R}_{12}^t\right)^t = \varsigma \ , \end{aligned} \tag{156}$$

where, evidently, $\mathbf{R}_{ij}$ are the 2×2 blocks of the matrix $\mathbf{R}$. The first two relations are solved by

$$\mathbf{R}_{11} = a\varsigma\left(\mathbf{R}_{12}^t\right)^{-1}\varsigma^{-1} = \frac{a}{\det\left(\mathbf{R}_{12}\right)}\mathbf{R}_{12} \ , \qquad \mathbf{R}_{21} = a'\varsigma\left(\mathbf{R}_{22}^t\right)^{-1}\varsigma^{-1} = \frac{a'}{\det\left(\mathbf{R}_{22}\right)}\mathbf{R}_{22} \ , \tag{157}$$

where a and a' are some undetermined constants. Plugging the above solutions into the third equation of (??), we have

$$a\varsigma\left[\left(\mathbf{R}_{12}\mathbf{R}_{22}^{-1}\right)^t\right]^{-1} - a'\mathbf{R}_{12}\mathbf{R}_{22}^{-1}\varsigma = \varsigma \ , \tag{158}$$

or, equivalently,

$$\left(a\frac{\det\left(\mathbf{R}_{22}\right)}{\det\left(\mathbf{R}_{12}\right)} - a'\right)\mathbf{R}_{12}\mathbf{R}_{22}^{-1} = \mathbb{1}_2 \ , \tag{159}$$

from which we deduce

$$\mathbf{R}_{22} = a''\mathbf{R}_{12} \quad , \quad aa'' - \frac{a'}{a''} = 1 \ . \tag{160}$$

From these manipulations we conclude that

$$\mathbf{R} = \left(\begin{array}{cc} \frac{a}{\det(\mathbf{R}_{12})} & 1 \\ \frac{b}{c\det(\mathbf{R}_{12})} & c \end{array} \right) \otimes \mathbf{R}_{12} \ . \tag{161}$$

We have found that we can write the general solution to (??) as follows

$$\mathbf{M}_0 = (\mathbf{M}_L \otimes \mathbf{M}_R)\,\mathbf{M}_{\text{mix}} \ , \tag{162}$$

where $\mathbf{M}_L$ and $\mathbf{M}_R$ are $SL(2)$ matrices that rotate, respectively, the solutions Ψ_L and Ψ_R, while $\mathbf{M}_{\text{mix}}$ takes the following form

$$\mathbf{M}_{\text{mix}} = \frac{1}{2\sqrt{\det\left(\boldsymbol{\Psi}_L^M\right)\det\left(\boldsymbol{\Psi}_R^M\right)}} \begin{pmatrix} 0 & \mathbb{i}b & \mathbb{i}b & 0 \\ -\frac{1}{c} & 0 & 0 & \frac{1}{c} \\ \mathbb{i}c & 0 & 0 & \mathbb{i}c \\ 0 & \frac{1}{b} & -\frac{1}{b} & 0 \end{pmatrix} \ , \tag{163}$$

with $\boldsymbol{\Psi}_L^M = \boldsymbol{\Psi}_L \mathbf{M}_L$ and similarly for the right one. We thus see that a generic constant matrix $\mathbf{M}_0$ in (??) is determined by 10 complex parameters, 4 for each $SL(2)$ rotation $\mathbf{M}_{L/R}$ and an additional pair for the matrix $\mathbf{M}_{\text{mix}}$. Note that 10 is the real dimension of the isometry group of the space $\mathbb{R}^{2,2}$, in which AdS_3 is immersed. A further condition on the constant matrix $\mathbf{M}_0$ comes from the reality properties of the basis vectors

$$\boldsymbol{\sigma}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \boldsymbol{\sigma} \ , \tag{164}$$

which implies

$$\left(\boldsymbol{\Psi}_{\text{L}} \otimes \boldsymbol{\Psi}_{\text{R}}\right)^* \mathbf{M}_0^* = \mathbb{i} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \left(\boldsymbol{\Psi}_{\text{L}} \otimes \boldsymbol{\Psi}_{\text{R}}\right) \mathbf{M}_0 \ . \tag{165}$$

and reduces the 10 complex parameter determining $\mathbf{M}_0$ to 10 real ones. Hence our embedded surface determined by (??) is uniquely determined up to isometries of $\mathbb{R}^{2,2}$.

Finally, let us also mention that minimal surfaces are naturally related to string theory. The very fact of being minimal implies the possibility of obtaining their defining relations by means of the minimisation of some quantity which, as it turns out, is nothing but the action of a non-linear sigma model

$$\mathscr{A}_{\text{NLSM}} = \int_\Sigma dz\, d\bar{z} \left(\vec{Y}_{,z} \cdot \vec{Y}_{,\bar{z}} + \Lambda\left(\vec{Y}\cdot\vec{Y} + \alpha^2\right)\right) \ , \tag{166}$$

where the Lagrange multiplier Λ imposes the constraint (??), forcing the target space to be AdS_3. The equations of motion

$$\vec{Y}_{,z\bar{z}} = \frac{1}{\alpha^2}\left(\vec{Y}_{,z}\cdot\vec{Y}_{,\bar{z}}\right)\vec{Y} \ , \qquad \vec{Y}_{,z}\cdot\vec{Y}_{,z} = \vec{Y}_{,\bar{z}}\cdot\vec{Y}_{,\bar{z}} = 0 \ , \tag{167}$$

are rather easily connected with (??) [?, ?, ?]. The area $\mathcal{A}$ of the worldsheet is then computed thanks to the metric g as follows

$$\mathcal{A} = \int_\Sigma dz\, d\bar{z} \sqrt{-\det(g)} = \int_\Sigma dz\, d\bar{z} \left(\vec{Y}_{,z} \cdot \vec{Y}_{,\bar{z}} \right) = \int_\Sigma dz\, d\bar{z}\, e^{\tilde{\varphi}} \; . \tag{168}$$

Note that, due to the modified sinh-Gordon equation (??), one has

$$\mathcal{A} = 2\alpha^2 \int_\Sigma dz\, d\bar{z} \left(\varphi_{,z\bar{z}} + P\bar{P}e^{-\varphi} \right) = 2\alpha^2 \int_\Sigma dz\, d\bar{z}\; P\bar{P}e^{-\varphi} + \text{total derivatives} \; , \tag{169}$$

where the total derivative term is a constant independent of the kinematics. This area is divergent and needs to be regularized. As will be explained below, the asymptotic behaviour as $|z| \to \infty$ of the modified sinh-Gordon field is $\varphi \sim \ln |P|$ and one can define a regularized area

$$\mathcal{A}_{\text{reg}} = 2\alpha^2 \int_\Sigma dz\, d\bar{z} \left(P\bar{P}e^{-\varphi} - \left(P\bar{P} \right)^{\frac{1}{2}} \right) \; . \tag{170}$$

4.3 A boundary interpretation of the function P and the Wilson loop

Let us recall that the function P – equivalently A (??) – is related to the Gauss curvature through equation (??). In the current case we have

$$K = -e^{-2\tilde{\varphi}} A\bar{A} = -\frac{1}{\alpha^2} e^{-2\varphi} P\bar{P} \; . \tag{171}$$

Thus, since we wish the surface Σ to be everywhere regular, we must demand for solutions to (??) to compensate for divergences of P. More concretely, we impose that

$$\lim_{(z,\bar{z})\to(z_c,\bar{z}_c)} \frac{1}{|P|} = 0 \quad \Longrightarrow \quad \varphi \underset{(z,\bar{z})\to(z_c,\bar{z}_c)}{\sim} \ln |P| \; . \tag{172}$$

Note that this asymptotic behaviour at the singularities of P is consistent with equation (??). From now on we will assume that the function P is a polynomial of order $2N$, then the only singular point is $|z| \to \infty$. The Gaussian curvature is, therefore, asymptotically a constant

$$K_\infty = \lim_{|z|\to\infty} K = -\frac{1}{\alpha^2} \; , \tag{173}$$

and in this limit the matrices of the linear system (??) become

$$L_{\text{L}} \sim \begin{pmatrix} 0 & z^{N/2}\bar{z}^{N/2} \\ z^{3N/2}\bar{z}^{-N/2} & 0 \end{pmatrix} , \qquad \bar{L}_{\text{L}} \sim \begin{pmatrix} 0 & z^{-N/2}\bar{z}^{3N/2} \\ z^{N/2}\bar{z}^{N/2} & 0 \end{pmatrix} , \tag{174a}$$

$$L_{\text{R}} \sim \begin{pmatrix} 0 & \mathbb{i} z^{N/2}\bar{z}^{N/2} \\ \mathbb{i} z^{3N/2}\bar{z}^{-N/2} & 0 \end{pmatrix} , \qquad \bar{L}_{\text{R}} \sim \begin{pmatrix} 0 & -\mathbb{i} z^{-N/2}\bar{z}^{3N/2} \\ -\mathbb{i} z^{N/2}\bar{z}^{N/2} & 0 \end{pmatrix} . \tag{174b}$$

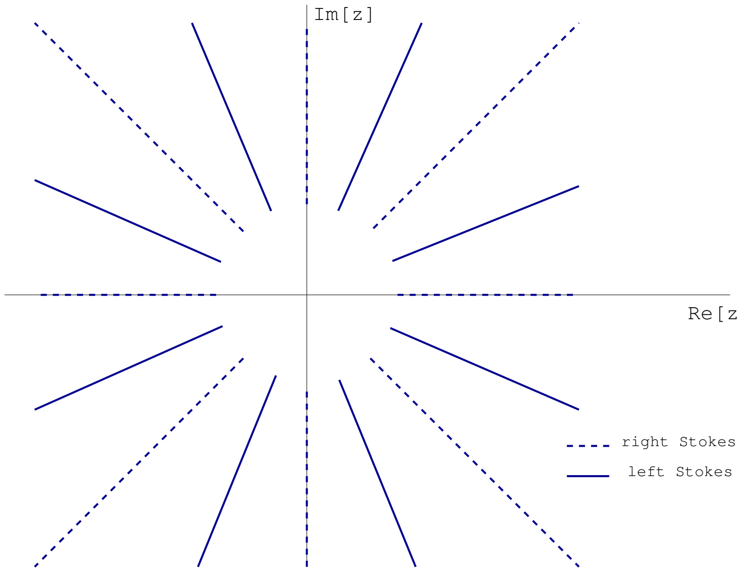


Figure 3: Representation of the Stokes sectors and lines in the complex $(z, \bar{z})$ plane, for the linear system (??), with $P \sim z^{2N}$ and $N = 3$.

In order to study what happens to the boundary of AdS$_3$ we need to jump ahead of ourselves and consider the first order in the WKB expansion of the solutions $\mathbf{\Psi}_\mathrm{L}$ and $\mathbf{\Psi}_\mathrm{R}$. A more detailed analysis of the WKB solutions and the Stokes phenomenon will be given in section ??; here we will just present some facts which will be useful in deriving the boundary of the minimal surface. A simple WKB analysis (cf. section ??) yields

$$\mathbf{\Psi}_\mathrm{L} \propto \begin{pmatrix} e^{\frac{2\,\varrho^{N+1}}{N+1}\cos((N+1)\vartheta)} & -e^{-\mathrm{i}N\vartheta} e^{-\frac{2\,\varrho^{N+1}}{N+1}\cos((N+1)\vartheta)} \\ e^{\mathrm{i}N\vartheta} e^{\frac{2\,\varrho^{N+1}}{N+1}\cos((N+1)\vartheta)} & e^{-\frac{2\,\varrho^{N+1}}{N+1}\cos((N+1)\vartheta)} \end{pmatrix}, \tag{175a}$$

$$\mathbf{\Psi}_\mathrm{R} \propto \begin{pmatrix} e^{\frac{2\,\varrho^{N+1}}{N+1}\sin((N+1)\vartheta)} & e^{-\mathrm{i}N\vartheta} e^{-\frac{2\,\varrho^{N+1}}{N+1}\sin((N+1)\vartheta)} \\ -e^{\mathrm{i}N\vartheta} e^{\frac{2\,\varrho^{N+1}}{N+1}\sin((N+1)\vartheta)} & e^{-\frac{2\,\varrho^{N+1}}{N+1}\sin((N+1)\vartheta)} \end{pmatrix}, \tag{175b}$$

with $z = \varrho e^{\mathrm{i}\vartheta}$ and $\bar{z} = \varrho e^{-\mathrm{i}\vartheta}$. We see that the linear problem displays a Stokes phenomenon at $\varrho \to \infty$, meaning that we can pin down the asymptotic of a specific solution only in certain sectors of the complex plane (see figure ??). These sectors, which we denote by $\mathcal{S}_\mathrm{L}^{(i)}$ and $\mathcal{S}_\mathrm{R}^{(i)}$, are bounded by the anti-Stokes lines which are given by $\cos\left((N+1)\,\vartheta\right) = \mathrm{Re}\left[z^{N+1}\right] = 0$ for the left solution and by $\sin\left((N+1)\,\vartheta\right) = \mathrm{Im}\left[z^{N+1}\right] = 0$ for the right one.

Now, we choose a solution $\mathbf{\Psi}_\mathrm{L}^{(i)} \otimes \mathbf{\Psi}_\mathrm{R}^{(i)}$ having the above asymptotic behaviour in a definite sector of the complex plane, which happens to be the overlap of $\mathcal{S}_\mathrm{L}^{(i)}$ with $\mathcal{S}_\mathrm{R}^{(i)}$. Suppose that we rotate our solution in the complex plane and, at some point, we cross a left anti-Stokes line. Then the asymptotic of our solution will change, since the diverging solution might obscure the presence of a smaller decaying solution. In mathematical terms,

$$\mathbf{\Psi}_{\mathrm{L}}^{(i)} \otimes \mathbf{\Psi}_{\mathrm{R}}^{(i)} = \left(\mathbf{\Psi}_{\mathrm{L}}^{(i+1)} \mathbf{S}\left(\gamma_{\mathrm{L}}^{(i)}\right)\right) \otimes \mathbf{\Psi}_{\mathrm{R}}^{(i)} , \qquad \mathbf{S}(\gamma) = \begin{pmatrix} 0 & -1 \\ 1 & \gamma \end{pmatrix} . \tag{176}$$

A similar jump will happen for the right solution at the right anti-Stokes lines, meaning we have $4(N+1)$ parameters $\left\{\gamma_{\mathrm{L}}^{(i)}, \gamma_{\mathrm{R}}^{(i)}\right\}_{i=1}^{2(N+1)}$, one for each anti-Stokes line.

Now let us consider what happens to the surface embedding function $\vec{Y}$ for $|z| \to \infty$. We will see things more clearly by working in Poincaré coordinates (??):

$$\mathsf{r} = Y_{-1} + Y_2 , \qquad \mathsf{x}^{\pm} = \mathsf{x} \pm \mathsf{t} = \frac{Y_1 \pm Y_0}{Y_{-1} + Y_2} , \tag{177}$$

where we have introduced the light-cone Poincaré coordinates $\mathsf{x}^{\pm}$. Some simple but tedious computation shows that these coordinates have the following expression[12] for our embedding (??)

$$\begin{aligned} \mathsf{r} &= \mathbb{i}\alpha c \frac{\mathbf{\Psi}^M_{L,22}\mathbf{\Psi}^M_{R,11} + \mathbb{i}\mathbf{\Psi}^M_{L,12}\mathbf{\Psi}^M_{R,21}}{\sqrt{\det\left(\mathbf{\Psi}^M_L\right)\det\left(\mathbf{\Psi}^M_R\right)}} , \\ \mathsf{x}^+ &= \frac{b}{c}\frac{\mathbf{\Psi}^M_{L,21}\mathbf{\Psi}^M_{R,11} + \mathbb{i}\mathbf{\Psi}^M_{L,11}\mathbf{\Psi}^M_{R,21}}{\mathbf{\Psi}^M_{L,22}\mathbf{\Psi}^M_{R,11} + \mathbb{i}\mathbf{\Psi}^M_{L,12}\mathbf{\Psi}^M_{R,21}} , \\ \mathsf{x}^- &= \frac{1}{\mathbb{i}b\,c}\frac{\mathbf{\Psi}^M_{L,22}\mathbf{\Psi}^M_{R,12} + \mathbb{i}\mathbf{\Psi}^M_{L,12}\mathbf{\Psi}^M_{R,22}}{\mathbf{\Psi}^M_{L,22}\mathbf{\Psi}^M_{R,11} + \mathbb{i}\mathbf{\Psi}^M_{L,12}\mathbf{\Psi}^M_{R,21}} , \end{aligned} \tag{178}$$

where we used (??) and (??) while $\mathbf{\Psi}^M_{L,ij}$ and $\mathbf{\Psi}^M_{R,ij}$ are the components of the rotated solutions $\mathbf{\Psi}_L M_L$ and $\mathbf{\Psi}_R M_R$, respectively.

Let us suppose we are in a Stokes sector, away from Stokes lines; in the next few expressions, in order to lighten the notation, we will omit the superscript (i) specifying the Stokes sector. Then, as $|z| \to \infty$, the components $\mathbf{\Psi}^M_{L,ij}$ and $\mathbf{\Psi}^M_{R,ij}$ will be naturally expressed by a superposition of a growing and a decaying solution:

$$\mathbf{\Psi}^M_{L,ij} = c^{\text{large}}_{L,j}\psi^{\text{large}}_{L,i} + c^{\text{small}}_{L,j}\psi^{\text{small}}_{L,i} , \tag{179}$$

where the functions $\psi^{\text{large}}_{L/R,i}$ and $\psi^{\text{small}}_{L/R,i}$ are the components of two arbitrary vector solutions to the linear system (??) respectively diverging and decaying[13] as $|z| \to \infty$ in our chosen Stokes

[12]Note that we have not implemented the reality condition (??) in the above expression. When doing so, these embedding functions will be, clearly, real.

[13]In sec. ?? we will define more precisely solutions to the linear problem according to their asymptotic behaviour. There we will refer to them as *dominant* and *subdominant*. For the moment, however, we content ourselves with this intuitive definition as it will be sufficient to gain a qualitative understanding of the asymptotic behaviour of the embedded surface. For this same reason we follow the example of [?] and denote them as *large* and *small*.

sector. We easily verify that

$$c_{L,j}^{\text{large}} = \frac{\det\begin{pmatrix} \boldsymbol{\Psi}_{L,1j}^{M} & \psi_{L,1}^{\text{small}} \\ \boldsymbol{\Psi}_{L,2j}^{M} & \psi_{L,2}^{\text{small}} \end{pmatrix}}{\det\begin{pmatrix} \psi_{L,1}^{\text{large}} & \psi_{L,1}^{\text{small}} \\ \psi_{L,2}^{\text{large}} & \psi_{L,2}^{\text{small}} \end{pmatrix}}\,, \qquad c_{L,j}^{\text{small}} = -\frac{\det\begin{pmatrix} \boldsymbol{\Psi}_{L,1j}^{M} & \psi_{L,1}^{\text{large}} \\ \boldsymbol{\Psi}_{L,2j}^{M} & \psi_{L,2}^{\text{large}} \end{pmatrix}}{\det\begin{pmatrix} \psi_{L,1}^{\text{large}} & \psi_{L,1}^{\text{small}} \\ \psi_{L,2}^{\text{large}} & \psi_{L,2}^{\text{small}} \end{pmatrix}}\,. \tag{180}$$

Equivalent expressions hold for the constants $c_{R,j}^{(\text{large})/(\text{small})}$. Finally plugging (??) into (??), we see that in the limit $|z| \to \infty$, the Poincaré radius diverges[14] $\mathsf{r} \to \infty$ – signalling that we are indeed approaching the boundary ∂AdS_3 – while the light cone coordinates take the following simple form

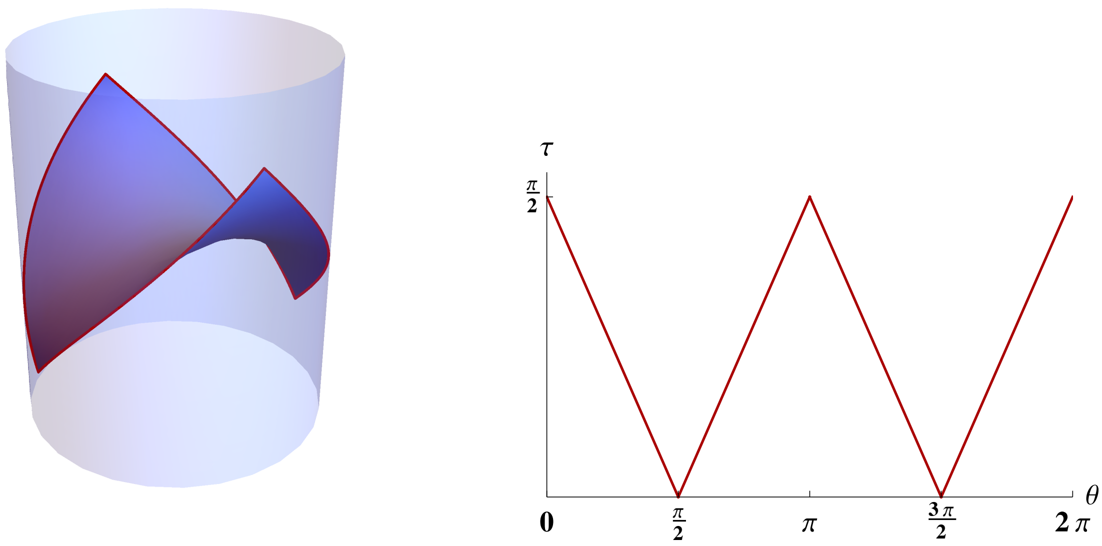
$$\mathsf{x}^{+} = \frac{b}{c}\frac{\det\begin{pmatrix} \boldsymbol{\Psi}_{L,11}^{M} & \psi_{L,1}^{\text{small}} \\ \boldsymbol{\Psi}_{L,21}^{M} & \psi_{L,2}^{\text{small}} \end{pmatrix}}{\det\begin{pmatrix} \boldsymbol{\Psi}_{L,12}^{M} & \psi_{L,1}^{\text{small}} \\ \boldsymbol{\Psi}_{L,22}^{M} & \psi_{L,2}^{\text{small}} \end{pmatrix}}\,, \quad \mathsf{x}^{-} = \frac{1}{\mathbb{i}bc}\frac{\det\begin{pmatrix} \boldsymbol{\Psi}_{R,12}^{M} & \psi_{R,1}^{\text{small}} \\ \boldsymbol{\Psi}_{R,22}^{M} & \psi_{R,2}^{\text{small}} \end{pmatrix}}{\det\begin{pmatrix} \boldsymbol{\Psi}_{R,11}^{M} & \psi_{R,1}^{\text{small}} \\ \boldsymbol{\Psi}_{R,21}^{M} & \psi_{R,2}^{\text{small}} \end{pmatrix}}\,. \tag{181}$$

Note that, while the expressions (??) depend on the choice of normalization for the functions $\psi_{L/R,i}^{\text{large}}$ and $\psi_{L/R,i}^{\text{small}}$, the boundary light-cone coordinates above are independent of it.

Given these results, we can easily see what happens when a Stokes line, say a left one, is crossed. Let us reinstate the explicit index for the sector: $\mathsf{x}_{(i)}^{+}$ and $\mathsf{x}_{(i)}^{-}$ are given by the above expressions, where each of the components of the solutions $\boldsymbol{\Psi}_L^M$, $\boldsymbol{\Psi}_R^M$, ψ_L^{small}, ψ_L^{large} are defined in the overlap of the i-th Stokes sectors $\mathcal{S}_L^{(i)} \cap \mathcal{S}_R^{(i)}$. Looking back at (??), we notice that crossing a left Stokes line, only the light-cone coordinate $\mathsf{x}_{(i)}^{+}$ is influenced, while $\mathsf{x}_{(i)}^{-}$ is the same on both sides of the left Stokes line. In other words, in $\mathcal{S}_L^{(i)} \cap \mathcal{S}_R^{(i)}$ we have light-cone boundary coordinates $\left(\mathsf{x}_{(i)}^{+}, \mathsf{x}_{(i)}^{-}\right)$, while in $\mathcal{S}_L^{(i+1)} \cap \mathcal{S}_R^{(i)}$ they are $\left(\mathsf{x}_{(i+1)}^{+}, \mathsf{x}_{(i)}^{-}\right)$. The same exact reasoning repeats for the crossing of a right Stokes line. Hence we conclude that points on the boundary determined by solutions lying in neighboring Stokes sectors are light-like separated.

Recapitulating, we have seen that the order $2N$ polynomial P defines $4(N+1)$ distinct Stokes sectors on the $(z, \bar{z})$ plane and, consequently, $4(N+1)$ points on the boundary of AdS_3. These are connected by $4(N+1)$ light-like lines, forming a light-like $4(N+1)$-gon on the boundary of AdS_3. In figure ?? we plotted the minimal surface, along with its Wilson loop, for the simplest possible case $P = \bar{P} = 1$, $\alpha = 1$ and $\varphi = 0$. The polygon on the boundary has the interpretation, in the CFT living on ∂AdS_3, as a light-like Wilson loop and, according to the proposal of [?, ?], we can measure its expectation value by computing the area of the minimal surface Σ in AdS_3 having the Wilson loop as its boundary. Moreover, as explained in [?,?], this

[14]Indeed, the numerator of r in (??) is dominated by $\psi_{L,i}^{\text{large}}$ and $\psi_{L,i}^{\text{small}}$, while the denominator is a constant.



(a) (b)

Figure 4: Minimal surface for the case $P = \bar{P} = 1$, $\alpha = 1$ and $\varphi = 0$ in AdS_3 and its Wilson loop. Figure ?? is a representation with $\tanh(\rho)$ as a radius, τ as a vertical direction and θ as an angle where (ρ, τ, θ) are AdS_3 global coordinates (??). The shaded cylinder is the conformal boundary and the red line is the Wilson loop. Figure ?? is a plot of the Wilson loop on the plane (θ, τ) corresponding to the boundary $\tanh \rho = 1$.

same area can be used to compute the gluon scattering amplitude, at leading order in strong coupling, in the boundary theory.

We will now turn to a more in-depth analysis of the solutions to the linear problem (??). As we will see, the presence of the Stokes phenomenon, instead of being a hindrance, will allow us to derive a closed set of functional equations for a collection of functions Y_k[15]. These can then be exploited to reconstruct the solutions $\mathbf{\Psi}_{\mathrm{L}}$ and $\mathbf{\Psi}_{\mathrm{R}}$ and compute the area (??) of the minimal surface.

[15]These functional equations form a closed set only if $P(z)$ lives on a finite cover of $\mathbb{C}$. This can be understood intuitively from the fact that there exists a function Y_k for each generator of the first homology group $\mathrm{H}_1(\mathcal{R}_{\mathrm{WKB}}, \mathbb{Z})$ of the Riemann surface $\mathcal{R}_{\mathrm{WKB}}$ associated to $\sqrt{P}$. If we allow non-rational powers in P, then the first homology group of this Riemann surface will not be finitely generated and we will have to deal with an infinite set of functions Y_k. From a physical point of view, in this case on the boundary of AdS_3 there will be an infinity of light-like lines, never closing themselves into a polygon.

4.4 The associated linear problem, the spectral parameter and the WKB solutions

The left and right pair of matrices (??) are, essentially, the Lax operators for the modified sinh-Gordon model appearing in [?]:

$$\mathcal{L}(\lambda) = \begin{pmatrix} -\frac{1}{4}\varphi_{,z} & \lambda e^{\frac{\varphi}{2}} \\ \lambda P e^{-\frac{\varphi}{2}} & \frac{1}{4}\varphi_{,z} \end{pmatrix} , \qquad \bar{\mathcal{L}}(\lambda) = \begin{pmatrix} \frac{1}{4}\varphi_{,\bar{z}} & \frac{1}{\lambda}\bar{P} e^{-\frac{\varphi}{2}} \\ \frac{1}{\lambda} e^{\frac{\varphi}{2}} & -\frac{1}{4}\varphi_{,\bar{z}} \end{pmatrix} . \tag{182}$$

The only missing element in the pairs (??) is the spectral parameter λ. However we immediately notice that by specialising the value of λ one has

$$L_{\mathrm{L}} = \mathcal{L}(\lambda = 1) , \qquad \bar{L}_{\mathrm{L}} = \bar{\mathcal{L}}(\lambda = 1) , \tag{183a}$$

$$L_{\mathrm{R}} = \mathcal{L}(\lambda = \mathbb{i}) , \qquad \bar{L}_{\mathrm{R}} = \bar{\mathcal{L}}(\lambda = \mathbb{i}) . \tag{183b}$$

The analysis of the Lax pair (??) has been carried out in [?] for the particular case of the function $P(z) = z^{2M} - s^{2M}$. There it was shown that the generalised monodromy data for the linear problem

$$\mathbf{\Phi}_{,z} = \mathcal{L}\,\mathbf{\Phi} , \qquad \mathbf{\Phi}_{,\bar{z}} = \bar{\mathcal{L}}\,\mathbf{\Phi} , \tag{184}$$

is connected with the integrable structures of the quantum sine-Gordon (for $M > 0$) or sinh-Gordon (for $M < -1$) models. As mentioned above, in what follows we will think of $P(z)$ as a polynomial function of order $2N$.[16] For further simplicity, we will concentrate on polynomials having only real roots; hence, from now on we will set

$$P(z) = z^{2N} + \sum_{k=0}^{2N-1} P_k\, z^k = \prod_{k=1}^{2N} (z - z_k) , \qquad (z_k, P_k \in \mathbb{R}) . \tag{185}$$

The first thing we notice about the linear problem (??) is that it possesses a $\mathbb{Z}_2$ symmetry

$$\left(\mathcal{L}(z,\bar{z}|\lambda), \bar{\mathcal{L}}(z,\bar{z}|\lambda)\right) = \left(\sigma^3 \mathcal{L}(z,\bar{z}|-\lambda)\,\sigma^3, \sigma^3 \bar{\mathcal{L}}(z,\bar{z}|-\lambda)\,\sigma^3\right) , \tag{186}$$

which implies that, given a solution $\mathbf{\Phi}(z,\bar{z}|\lambda)$, then $\sigma^3 \mathbf{\Phi}\left(z,\bar{z}|e^{\mathbb{i}\pi}\lambda\right)$ is also a solution. This fact will be useful momentarily, when we discuss the Stokes phenomenon associated with our linear problem. A simple way to study the linear problem (??) is to gauge rotate it by the matrix $\exp\left(\frac{1}{4}\varphi\sigma^3\right)$, so that one obtains

$$\tilde{\mathbf{\Phi}}_{,z} = \tilde{\mathcal{L}}\,\tilde{\mathbf{\Phi}} , \qquad \tilde{\mathbf{\Phi}}_{,\bar{z}} = \tilde{\bar{\mathcal{L}}}\,\tilde{\mathbf{\Phi}} , \qquad \tilde{\mathbf{\Phi}} = e^{-\frac{1}{4}\varphi\sigma^3}\,\mathbf{\Phi} , \tag{187}$$

[16] We might think of considering more general multi-valued potentials, e.g. $P(z) = z^{2N} - s^{2N}$ where $N \notin \frac{1}{2}\mathbb{Z}$ but we still ask that $N \in \mathbb{Q}$. The presence of non-integer powers in the function $P(z)$ would force us to consider the linear problem on an appropriate finite covering of the complex plane. Since the substance of our analysis would not change, we will avoid this complication.

where

$$\tilde{\mathcal{L}} = e^{-\frac{1}{4}\varphi\sigma^3}\,\mathcal{L}\,e^{\frac{1}{4}\varphi\sigma^3} - e^{-\frac{1}{4}\varphi\sigma^3}\partial\left(e^{\frac{1}{4}\varphi\sigma^3}\right) = \begin{pmatrix} -\frac{1}{2}\varphi_{,z} & \lambda \\ \lambda P & \frac{1}{2}\varphi_{,z} \end{pmatrix} \,, \tag{188}$$

and

$$\tilde{\bar{\mathcal{L}}} = e^{-\frac{1}{4}\varphi\sigma^3}\bar{\mathcal{L}}e^{\frac{1}{4}\varphi\sigma^3} - e^{-\frac{1}{4}\varphi\sigma^3}\bar{\partial}e^{\frac{1}{4}\varphi\sigma^3} = \begin{pmatrix} 0 & \frac{1}{\lambda}\bar{P}e^{-\varphi} \\ \frac{1}{\lambda}e^{\varphi} & 0 \end{pmatrix} \,. \tag{189}$$

With this form of the linear problem, it is easier to obtain the WKB expansion.

We start from the following ansatz

$$\tilde{\mathbf{\Phi}} = \frac{1}{\sqrt{S_{,z}}}\begin{pmatrix} 1 & 1 \\ \left(S + \frac{\varphi - \ln(\partial S)}{2\lambda}\right)_{,z} & \left(-S + \frac{\varphi - \ln(\partial S)}{2\lambda}\right)_{,z} \end{pmatrix} \cdot e^{-\lambda S\sigma^3} \,, \tag{190}$$

where S is a function of the variables $(z, \bar{z})$ and of the square of the spectral parameter λ, with asymptotic expansion as $\lambda^2 \to \infty$

$$S = S\left(z, \bar{z}|\lambda^2\right) = \sum_{k=0}^{\infty} \lambda^{-2k} S_k\left(z, \bar{z}\right) \,. \tag{191}$$

The solution $\tilde{\mathbf{\Phi}}$ is normalized in such a way that

$$\det(\tilde{\mathbf{\Phi}}) = -2 \quad \Longrightarrow \quad \det(\mathbf{\Phi}) = -2 \,. \tag{192}$$

The linear system (??) then reduces to a pair of equations for the function S,

$$S_{,z}^2 - \frac{1}{2\lambda^2}\{S, z\} = \frac{\varphi_{,z}^2 - 2\varphi_{,zz}}{4\lambda^2} + P \,, \qquad \{S, z\} = \frac{S_{,zzz}}{S_{,z}} - \frac{3}{2}\left(\frac{S_{,zz}}{S_{,z}}\right)^2 \,, \tag{193a}$$

$$S_{,\bar{z}} - \frac{\bar{P}}{\lambda^2}e^{-\varphi}S_{,z} = 0 \,, \tag{193b}$$

which, as one can easily check, are mutually compatible. Exploiting the series representation (??) we turn this pair of equations into an infinite triangular system for the coefficients S_k, which we then solve by iteration, the first few equations being

$$S_{0,z}^2 = P \,, \qquad S_{0,\bar{z}} = 0 \,, \tag{194a}$$

$$S_{1,z} = \frac{1}{8\sqrt{P}}\left(\frac{P_{,zz}}{P} - \frac{5}{4}\left(\frac{P_{,z}}{P}\right)^2 + \varphi_{,z}^2 - 2\varphi_{,zz}\right) \,, \qquad S_{1,\bar{z}} = e^{-\varphi}\sqrt{P}\bar{P} \,, \tag{194b}$$

$$\cdots \,, \qquad \cdots \,.$$

We thus have expressed the solution to the linear problem (??) as an expansion around $\lambda \to \infty$ as follows:

$$\mathbf{\Phi} = e^{\frac{1}{4}\varphi\sigma^3} \begin{pmatrix} e^{-\lambda S_0 - \frac{1}{4}\ln P + \frac{1}{\lambda}S_1 + \mathcal{O}\left(\lambda^{-2}\right)} & e^{\lambda S_0 - \frac{1}{4}\ln P - \frac{1}{\lambda}S_1 + \mathcal{O}\left(\lambda^{-2}\right)} \\ e^{-\lambda S_0 + \frac{1}{4}\ln P + \frac{1}{\lambda}\left(S_1 + \frac{\varphi_{,z}}{2\sqrt{P}} - \frac{P_{,z}}{4P^{3/2}}\right) + \mathcal{O}\left(\lambda^{-2}\right)} & -e^{\lambda S_0 + \frac{1}{4}\ln P - \frac{1}{\lambda}\left(S_1 + \frac{\varphi_{,z}}{2\sqrt{P}} - \frac{P_{,z}}{4P^{3/2}}\right) + \mathcal{O}\left(\lambda^{-2}\right)} \end{pmatrix}, \tag{195}$$

with

$$S_0 = \int\limits_{z_*} dz\,\sqrt{P}\ , \qquad S_1 = \int\limits_{z_*} \frac{dz}{8\sqrt{P}} \left(\frac{P_{,zz}}{P} - \frac{5}{4}\left(\frac{P_{,z}}{P}\right)^2 + \varphi_{,z}^2 - 2\varphi_{,zz} \right)\ , \tag{196}$$

and z_* some arbitrarily-chosen base point.

A similar analysis for the linear system (??), gauge rotated with the matrix $\exp\left(-\frac{1}{4}\varphi\sigma^3\right)$, yields the small-$\lambda$ behaviour

$$\mathbf{\Phi} = e^{-\frac{1}{4}\varphi\sigma^3} \begin{pmatrix} e^{-\frac{1}{\lambda}\bar{S}_0 + \frac{1}{4}\ln\bar{P} + \lambda\left(\bar{S}_1 + \frac{\varphi_{,\bar{z}}}{2\sqrt{\bar{P}}} - \frac{\bar{P}_{,\bar{z}}}{4\bar{P}^{3/2}}\right) + \mathcal{O}\left(\lambda^2\right)} & -e^{\frac{1}{\lambda}\bar{S}_0 + \frac{1}{4}\ln\bar{P} - \lambda\left(\bar{S}_1 + \frac{\varphi_{,\bar{z}}}{2\sqrt{\bar{P}}} - \frac{\bar{P}_{,\bar{z}}}{4\bar{P}^{3/2}}\right) + \mathcal{O}\left(\lambda^2\right)} \\ e^{-\frac{1}{\lambda}\bar{S}_0 - \frac{1}{4}\ln\bar{P} + \lambda\bar{S}_1 + \mathcal{O}\left(\lambda^2\right)} & e^{\frac{1}{\lambda}\bar{S}_0 - \frac{1}{4}\ln\bar{P} - \lambda\bar{S}_1 + \mathcal{O}\left(\lambda^2\right)} \end{pmatrix}, \tag{197}$$

with

$$\bar{S}_0 = \int\limits_{z_*} d\bar{z}\sqrt{\bar{P}}\ , \qquad \bar{S}_1 = \int\limits_{z_*} \frac{d\bar{z}}{8\sqrt{\bar{P}}} \left(\frac{\bar{P}_{,\bar{z}\bar{z}}}{\bar{P}} - \frac{5}{4}\left(\frac{\bar{P}_{,\bar{z}}}{\bar{P}}\right)^2 + \varphi_{,\bar{z}}^2 - 2\varphi_{,\bar{z}\bar{z}} \right)\ . \tag{198}$$

4.5 WKB geometry, Stokes sectors and subdominant solutions

Now, let us think more carefully about the geometry of what we are doing. By recasting (??) into the system (??) we have moved from an equation defined on $\mathbb{C}^2$ to a system living on the Riemann surface $\mathcal{R}_{\text{WKB}}$ defined by the algebraic equation $\zeta^2 = P(z)$. The quantities S_k appearing in the expansion (??) are line integrals along curves on $\mathcal{R}_{\text{WKB}}$:

$$S_k(z,\bar{z}) = \int\limits_{z_*}^{(z,\bar{z})} s_k\ , \qquad \bar{S}_k = \int\limits_{z_*}^{(z,\bar{z})} \bar{s}_k\ , \tag{199}$$

with s_k and $\bar{s}_k$ being one-forms on $\mathcal{R}_{\text{WKB}}$, e.g.

$$s_0 = \sqrt{P}dz\ , \qquad s_1 = \frac{dz}{8\sqrt{P}} \left(\frac{P_{,zz}}{P} - \frac{5}{4}\left(\frac{P_{,z}}{P}\right)^2 + \varphi_{,z}^2 - 2\varphi_{,zz} \right)\ , \qquad \cdots\ . \tag{200}$$

Figure ?? depicts the first sheet of the Riemann surface in the case of a polynomial $P(z)$ having real roots. In order to define the WKB solutions (??, ??) correctly, on the one hand it is necessary to be careful in the choice of the base point z_* and the integration contour. On the other hand, however, it is possible to pin down the specific solution correctly only in a certain sector of

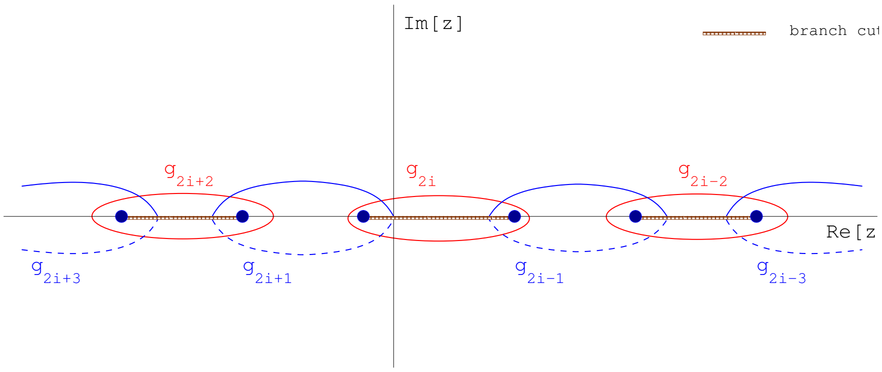


Figure 5: An example of the (first sheet of the) Riemann surface $\mathcal{R}_{\rm WKB}$ for a polynomial $P(z)$ having real roots, and a basis $\{\gamma_i\}$ of cycles on that surface.

the complex plane; this is an example of the Stokes phenomenon and is a direct consequence of the presence of an irregular singularity at $(z, \bar{z}) \to \infty$.

To be more precise, consider the solution (??) at large distances both from the origin and from any critical values of $P(z)$. Then $P(z)$ behaves as $P(z) \sim z^{2N}$ and we can compute the leading behaviour of the coefficients S_0 and S_1 :

$$S_0 \underset{|z|\to\infty}{\sim} \int_{z_*}^{z} dz\, z^N = \frac{z^{N+1} - z_*^{N+1}}{N+1}\,, \qquad S_1 \underset{|z|\to\infty}{\sim} \frac{N}{8}\frac{N+2}{N+1}\left(z^{-N-1} - z_*^{-N-1}\right)\,. \tag{201}$$

Similar expressions hold for $\bar{S}_0$ and $\bar{S}_1$. More generally, as shown in (??), solutions to the modified sinh-Gordon equation (??) behave at leading order in $|z| \to \infty$ as $\varphi \sim 2N \ln |z|$; the only remaining terms in S and $\bar{S}$ are then, respectively, S_0 and $\bar{S}_0$. Hence one finds

$$\mathbf{\Phi} \underset{|z|\to\infty}{\sim} \frac{\bar{z}^{N/4}}{z^{N/4}} \begin{pmatrix} 1 & 1 \\ \frac{z^{N/2}}{\bar{z}^{N/2}} & -\frac{z^{N/2}}{\bar{z}^{N/2}} \end{pmatrix} \cdot e^{-\frac{\lambda z^{N+1} + \frac{1}{\lambda}\bar{z}^{N+1}}{N+1}\sigma^3}\,. \tag{202}$$

Let us denote by $\Phi^{(d)}$ and $\Phi^{(s)}$ the two column vectors comprising the matrix $\mathbf{\Phi}$

$$\mathbf{\Phi} = \begin{pmatrix} \Phi^{(s)} & \Phi^{(d)} \end{pmatrix}\,, \tag{203}$$

so that for large $|z|$ and $|\vartheta| < \frac{\pi}{N+1}$ these vectors behave as

$$\Phi^{(s)} \underset{|z|\to\infty}{\sim} \begin{pmatrix} e^{-\mathbb{i}\frac{N}{4}\vartheta} \\ -e^{\mathbb{i}\frac{N}{4}\vartheta} \end{pmatrix} \exp\left(-\frac{2\,\varrho^{N+1}}{N+1}\cos\left((N+1)\,\vartheta - \mathbb{i}\upsilon\right)\right)\,, \tag{204a}$$

$$\Phi^{(d)} \underset{|z|\to\infty}{\sim} \begin{pmatrix} e^{-\mathbb{i}\frac{N}{4}\vartheta} \\ e^{\mathbb{i}\frac{N}{4}\vartheta} \end{pmatrix} \exp\left(\frac{2\,\varrho^{N+1}}{N+1}\cos\left((N+1)\,\vartheta - \mathbb{i}\upsilon\right)\right)\,, \tag{204b}$$

where $z = \varrho e^{\mathbb{i}\vartheta}$, $\bar{z} = \varrho e^{-\mathbb{i}\vartheta}$ and $\lambda = e^{\upsilon}$. Much as before, we will call $\Phi^{(s)}$ the *subdominant solution* and $\Phi^{(d)}$ the *dominant solution*. It is clear from the above expressions that if we

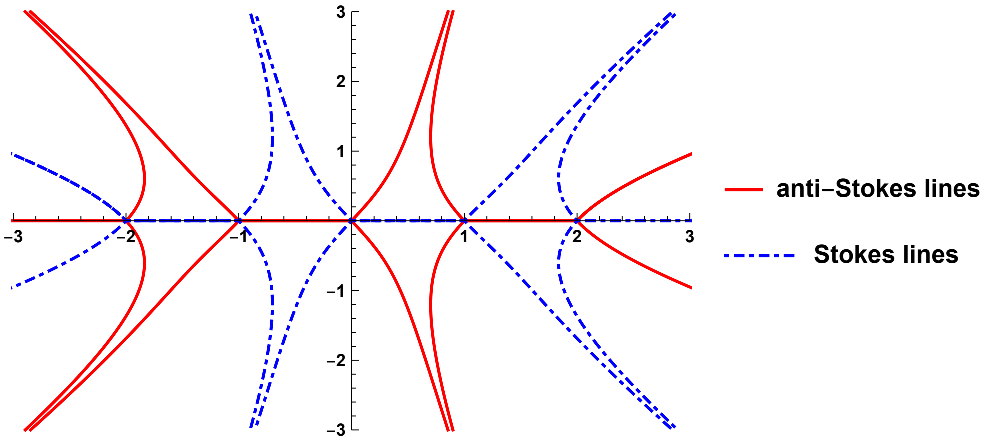


Figure 6: Stokes and anti-Stokes lines for the function $P(z) = z\left(z^2-1\right)\left(z^2-4\right)$, with $\lambda \in \mathbb{R}$. Although not shown here, there are branch cuts connecting -2 with -1, 0 with $+1$ and $+2$ with ∞.

analytically continue from (ϱ, ϑ) to $\left(\varrho, \vartheta + \frac{\pi}{N+1}\right)$ the two asymptotics seem to swap rôles. However, while we can precisely pin down the asymptotic of $\Phi^{(s)}$, since no other term can be added to it without spoiling its asymptotic behaviour, the behaviour (??) might be hiding a contribution coming from a decaying exponential, with a coefficient which in general will change as the Stokes line in the middle of this sector is crossed. Hence when we perform the analytic continuation, we will obtain the following asymptotics, valid for $|\vartheta| < \frac{\pi}{N+1}$ and $\vartheta^{(+)} = \vartheta + \frac{\pi}{N+1}$:

$$\Phi^{(s)}\left(\varrho, \vartheta^{(+)}\right) \underset{|z|\to\infty}{\sim} \begin{pmatrix} e^{-\mathbb{i}\frac{N}{4}\vartheta^{(+)}} \\ -e^{\mathbb{i}\frac{N}{4}\vartheta^{(+)}} \end{pmatrix} \exp\left(\frac{2\,\varrho^{N+1}}{N+1}\cos\left((N+1)\,\vartheta - \mathbb{i}\upsilon\right)\right) = \text{dominant}\,, \tag{205a}$$

$$\Phi^{(d)}\left(\varrho, \vartheta^{(+)}\right) \underset{|z|\to\infty}{\sim} c_+(\lambda) \begin{pmatrix} e^{-\mathbb{i}\frac{N}{4}\vartheta^{(+)}} \\ -e^{\mathbb{i}\frac{N}{4}\vartheta^{(+)}} \end{pmatrix} \exp\left(\frac{2\,\varrho^{N+1}}{N+1}\cos\left((N+1)\,\vartheta - \mathbb{i}\upsilon\right)\right)\,. \tag{205b}$$

Therefore, for ϑ in the sector $|\vartheta| < \frac{\pi}{N+1}$, the continued solution $\Phi^{(d)}\left(\varrho, \vartheta^{(+)}\right)$ is in general dominant but, exceptionally, it will be subdominant at zeros of the coefficient $c_+(\lambda)$. The story is similar to that of section ??, and the preliminary discussion reported there will be formalised in the following sections.

Summarising, we see that the function $P(z)$ partitions the Riemann surface $\mathcal{R}_{\text{WKB}}$ into *Stokes sectors* $\mathcal{S}_j$, bounded by *anti-Stokes lines*, defined by $\operatorname{Re}[\lambda S_0] = 0$. In each of these sectors we can define a matrix solution $\mathbf{\Phi}_j$ composed of a dominant and a subdominant solution

$$\mathbf{\Phi}_j = \begin{pmatrix} \Phi_j^{(s)} & \Phi_j^{(d)} \end{pmatrix}\,. \tag{206}$$

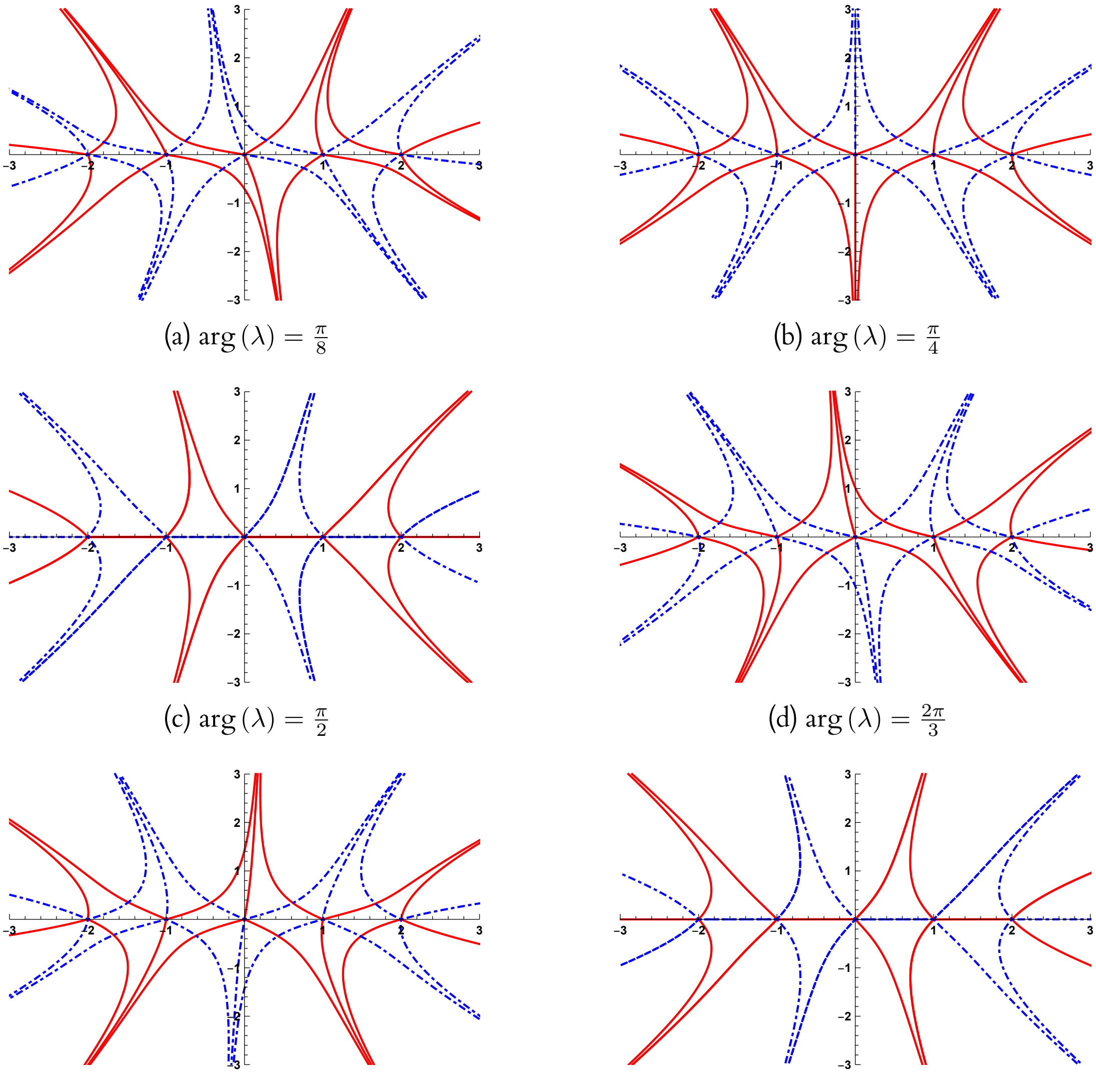
The decay (or growth) of this solution is largest whenever the solution lies on a *Stokes line*, defined by $\operatorname{Im}[\lambda S_0] = 0$. Figure ?? depicts an example of the Stokes and anti-Stokes lines for a particular choice of $P(z)$, while figure ?? is a view of the same picture from very large $|z|$. The

Figure 7: Figure ?? looked at from very large $|z|$. The fine details of the function $P(z)$ disappear and we only see the lines defined by Re $\left[z^{7/2}\right]=0$ and Im $\left[z^{7/2}\right]=0$, that is $\vartheta_k^{\mathrm{aS}}=\pi\frac{2k+1}{7}$ and $\vartheta_k^{\mathrm{S}}=\pi\frac{2k}{7}$ with $k=-3,-2,-1,0,1,2,3$. The Stokes sectors $\mathcal{S}_k$ are labeled by the index k of the angles ϑ_k^S.

definition of Stokes and anti-Stokes lines depends on the phase of the spectral parameter λ and, as displayed in figure ??, a counter-clockwise rotation of λ rotates the sectors in a clockwise direction. When $\arg(\lambda)=\pi$, one returns to the same situation as for $\arg(\lambda)=0$, but with the sectors exchanged in a clockwise fashion. Consequently, exploiting the $\mathbb{Z}_2$ symmetry (??), we can define the solutions $\mathbf{\Phi}_j$ as

$$\mathbf{\Phi}_j\left(z,\bar{z}|\lambda\right)=\left(\sigma^3\right)^j\mathbf{\Phi}\left(z,\bar{z}|e^{j\mathrm{i}\pi}\lambda\right)\ , \tag{207}$$

where $\mathbf{\Phi}$, our starting solution, is defined in what we choose to be the 0-th sector $\mathcal{S}_0$. In what follows we will label the sectors according to the index k of the $\vartheta_k^{\mathrm{S}}=\pi\frac{k}{N+1}$ solution of the Stokes line equation Im $\left[z^{N+1}\right]$ for large $|z|$. Hence the sector $\mathcal{S}_0$ will be for $\lambda\in\mathbb{R}$ the one containing the positive real line at large enough $|z|$. See figure ?? for an example.



(a) $\arg(\lambda) = \frac{\pi}{8}$ (b) $\arg(\lambda) = \frac{\pi}{4}$

(c) $\arg(\lambda) = \frac{\pi}{2}$ (d) $\arg(\lambda) = \frac{2\pi}{3}$

(e) $\arg(\lambda) = \frac{4\pi}{5}$ (f) $\arg(\lambda) = \pi$

Figure 8: Plots of Stokes and anti-Stokes lines for the polynomial function $P(z) = z\left(z^2-1\right)\left(z^2-4\right)$ and various phases of the spectral parameter λ. One sees that a counter-clockwise rotation of λ corresponds to a clockwise rotation of the sectors. For $\arg(\lambda) = \pi$, the picture looks the same as figure ??, but the sectors have been exchanged in a clockwise fashion.

4.6 The connection matrices, the T-system and the Hirota equation

We can now make the relations (??) more precise as follows:[17]

$$\Phi^{(s)}_{j-1}(z,\bar{z}|\lambda) = \Phi^{(d)}_{j}(z,\bar{z}|\lambda) \tag{208a}$$
$$\Phi^{(d)}_{j-1}(z,\bar{z}|\lambda) = -\Phi^{(s)}_{j}(z,\bar{z}|\lambda) + T\left(e^{j\mathrm{i}\pi}\lambda\right)\Phi^{(d)}_{j}(z,\bar{z}|\lambda)\ , \tag{208b}$$

[17]The -1 sign in the second equality is necessary to have $\det(\mathbf{\Phi}_{j-1}) = \det(\mathbf{\Phi}_j) = -2$.

or, in matrix notation

$$\mathbf{\Phi}_{j-1}(z,\bar{z}|\lambda)=\mathbf{\Phi}_j(z,\bar{z}|\lambda)\,\mathbf{T}\left(e^{j\mathbb{i}\pi}\lambda\right)\ ,\qquad \mathbf{T}(\lambda)=\begin{pmatrix}0 & -1\\ 1 & T(\lambda)\end{pmatrix}\ . \tag{209}$$

It is immediate to see that

$$T(\lambda)=\frac{1}{\det(\mathbf{\Phi}_0)}\det\begin{pmatrix}\Phi_0^{(s)} & \Phi_{-1}^{(d)}\end{pmatrix}=-\frac{1}{2}\det\begin{pmatrix}\Phi_0^{(s)} & \Phi_{-2}^{(s)}\end{pmatrix}\ , \tag{210}$$

where we have used (??) and the fact that $\Phi_j^{(d)}=\Phi_{j-1}^{(s)}$. We can generalize this construction, introducing the *lateral connection matrices* $\mathbf{T}_k(\lambda)$ which, as the name suggests, relate solutions living in (next)^k-neighbouring Stokes sectors:

$$\mathbf{\Phi}_j(z,\bar{z}|\lambda)=\mathbf{\Phi}_{j+k}(z,\bar{z}|\lambda)\,\mathbf{T}_k\left(\lambda e^{\left(j+\frac{k+1}{2}\right)\mathbb{i}\pi}\right)\ . \tag{211}$$

The form of these matrices is constrained by noticing that they need to satisfy the following consistency relation

$$\mathbf{T}_k(\lambda)=\mathbf{T}_{k-j}\left(e^{\frac{j}{2}\mathbb{i}\pi}\lambda\right)\mathbf{T}_j\left(e^{\frac{j-k}{2}\mathbb{i}\pi}\lambda\right)\ , \tag{212}$$

which implies that we can parametrise the lateral connection matrices as follows

$$\mathbf{T}_k(\lambda)=\begin{pmatrix}-T_{k-2}(\lambda) & -T_{k-1}\left(e^{-\frac{1}{2}\mathbb{i}\pi}\lambda\right)\\ T_{k-1}\left(e^{\frac{1}{2}\mathbb{i}\pi}\lambda\right) & T_k(\lambda)\end{pmatrix}\ . \tag{213}$$

Each function $T_k(\lambda)$, which we call a Stokes multiplier or *lateral connection coefficient*, can be computed as a determinant of subdominant solutions defined in distinct Stokes sectors:

$$T_{2k-1}(\lambda)=\frac{1}{2}\det\begin{pmatrix}\Phi_{-k-1}^{(s)} & \Phi_{k-1}^{(s)}\end{pmatrix}\ , \tag{214a}$$

$$T_{2k}\left(\lambda e^{\frac{1}{2}\mathbb{i}\pi}\right)=\frac{1}{2}\det\begin{pmatrix}\Phi_{-k-1}^{(s)} & \Phi_{k}^{(s)}\end{pmatrix}\ . \tag{214b}$$

One must clearly have $\mathbf{T}_0(\lambda)=\mathbf{1}$, implying that

$$T_{-2}(\lambda)=-1\ ,\qquad T_{-1}(\lambda)=0\ ,\qquad T_0(\lambda)=1\ , \tag{215}$$

which agree with the determinant expressions (??).

The relation (??) can be used to extract a series of additional constraints on the functions $T_k(\lambda)$. First of all one has the unimodularity condition

$$\det(\mathbf{T}_k(\lambda))=1\ , \tag{216}$$

to which we will return momentarily. Another obvious relation is the following

$$\mathbf{T}_0\left(e^{-\frac{j}{2}\mathbb{i}\pi}\lambda\right)=\mathbf{1}=\mathbf{T}_{-j}(\lambda)\,\mathbf{T}_j(\lambda)\quad\Longrightarrow\quad T_{-k-1}(\lambda)=-T_{k-1}(\lambda)\ . \tag{217}$$

We also require that a rotation of $2(N+1)$ Stokes sectors brings us back to the same solution (modulo a ± 1 factor), from which we deduce that

$$\mathbf{T}_{j+2N+2}(\lambda) = \pm \mathbf{T}_j\left(e^{(N+1)\mathrm{i}\pi}\lambda\right) \quad \Longrightarrow \quad T_{2N+1}(\lambda) = 0 \; . \tag{218}$$

Finally, we obtain a recursive definition for $T_k(\lambda)$ by looking at the components of (??)

$$T_k(\lambda) = T_j\left(e^{\frac{j-k}{2}\mathrm{i}\pi}\lambda\right) T_{k-j}\left(e^{\frac{j}{2}\mathrm{i}\pi}\lambda\right) - T_{j-1}\left(e^{\frac{j-k-1}{2}\mathrm{i}\pi}\lambda\right) T_{k-j-1}\left(e^{\frac{j+1}{2}\mathrm{i}\pi}\lambda\right) \; , \tag{219}$$

which is called the T-system. An equivalent, more elegant, form is obtained by the simple unimodularity requirement mentioned above

$$\det\left(\mathbf{T}_{k+1}(\lambda)\right) = 1 \quad \Longrightarrow \quad T_k\left(e^{\frac{1}{2}\mathrm{i}\pi}\lambda\right) T_k\left(e^{-\frac{1}{2}\mathrm{i}\pi}\lambda\right) = 1 + T_{k+1}(\lambda)\, T_{k-1}(\lambda) \; . \tag{220}$$

This equation needs to be supported by the boundary conditions found above, $T_0(\lambda) = 1$ and $T_{2N+1}(\lambda) = 0$, and is known in the literature as *Hirota bilinear equation* [?, ?, ?]. One can check that the T-system is obtained by iteration from the Hirota equation.

There are various manipulations one can perform on the Hirota equation. For example, one can formally solve it by parametrizing the functions $T_k(\lambda)$ by a pair of Q functions $\{Q_a(\lambda)\}_{a=1,2}$ as follows

$$T_k(\lambda) = \det\begin{pmatrix} Q_1\left(e^{\frac{k+1}{2}\mathrm{i}\pi}\lambda\right) & Q_1\left(e^{-\frac{k+1}{2}\mathrm{i}\pi}\lambda\right) \\ Q_2\left(e^{\frac{k+1}{2}\mathrm{i}\pi}\lambda\right) & Q_2\left(e^{-\frac{k+1}{2}\mathrm{i}\pi}\lambda\right) \end{pmatrix} \; . \tag{221}$$

Then it is easy to see that the Hirota equation is equivalent to the following one

$$\det\begin{pmatrix} Q_1\left(e^{\frac{1}{2}\mathrm{i}\pi}\lambda\right) & Q_1\left(e^{-\frac{1}{2}\mathrm{i}\pi}\lambda\right) \\ Q_2\left(e^{\frac{1}{2}\mathrm{i}\pi}\lambda\right) & Q_2\left(e^{-\frac{1}{2}\mathrm{i}\pi}\lambda\right) \end{pmatrix} = 1 \; , \tag{222}$$

which, in the literature, is known as a *quantum Wronskian* [?, ?]. The relation (??) is the off-critical version of the constraint (??), obtained within the quantum KdV context. From (??) and (??) we obtain Baxter's TQ equation

$$T_1(\lambda)\, Q_a(\lambda) = Q_a\left(e^{\mathrm{i}\pi}\lambda\right) + Q_a\left(e^{-\mathrm{i}\pi}\lambda\right) \; , \qquad (a = 1, 2) \; , \tag{223}$$

by simply expanding the trivial identity

$$\det\begin{pmatrix} Q_1\left(e^{\mathrm{i}\pi}\lambda\right) & Q_1(\lambda) & Q_1\left(e^{-\mathrm{i}\pi}\lambda\right) \\ Q_2\left(e^{\mathrm{i}\pi}\lambda\right) & Q_2(\lambda) & Q_2\left(e^{-\mathrm{i}\pi}\lambda\right) \\ Q_a\left(e^{\mathrm{i}\pi}\lambda\right) & Q_a(\lambda) & Q_a\left(e^{-\mathrm{i}\pi}\lambda\right) \end{pmatrix} = 0 \; , \qquad (a = 1, 2) \; . \tag{224}$$

The matrix

$$\mathbf{Q}(\lambda) = \begin{pmatrix} Q_1\left(e^{\mathbb{i}\pi}\lambda\right) & Q_1(\lambda) \\ Q_2\left(e^{\mathbb{i}\pi}\lambda\right) & Q_2(\lambda) \end{pmatrix} \tag{225}$$

has a geometrical interpretation: it is the *central connection matrix* of the central problem for our linear system. In other words, it relates the solutions $\mathbf{\Phi}_j$ to another fundamental solution $\mathbf{\Xi}$, defined via local analysis at a point where no Stokes phenomenon is present[18]. Then $\mathbf{\Xi}$ is insensitive to the rotation of λ by integer multiple of $\mathbb{i}\pi$ and one has the relation

$$\mathbf{\Phi}_j(z,\bar{z}|\lambda) = \mathbf{\Xi}(z,\bar{z}|\lambda)\,\mathbf{Q}\left(e^{j\mathbb{i}\pi}\lambda\right) \ . \tag{226}$$

Playing with this relation and (??), we obtain the following identity

$$\mathbf{Q}(\lambda) = \mathbf{Q}\left(e^{k\mathbb{i}\pi}\lambda\right)\mathbf{T}_k\left(e^{\frac{k+1}{2}\mathbb{i}\pi}\lambda\right) \ , \tag{227}$$

from which it is possible to derive both the Baxter TQ equation (??) (by simply setting $k = 1$) and the parametrization (??) of the functions T_k (by Cramer's rule). The QQ-system (??) corresponds to the unimodularity requirement $\det(\mathbf{Q}(\lambda)) = 1$.

Although Q-functions are interesting objects, we find it more convenient to introduce a new set of functions: the Y-functions. These are defined as follows

$$Y_k(\lambda) = T_{k-1}(\lambda)\,T_{k+1}(\lambda) \ , \quad (k = 1,\ldots,2N-1) \ , \tag{228}$$

or, in a more invariant form, and using the fact that $\det\begin{pmatrix} \Phi_k^{(s)} & \Phi_{k+1}^{(s)} \end{pmatrix} = -\det\mathbf{\Phi}_0$,

$$Y_{2k}(\lambda) = \frac{\det\begin{pmatrix} \Phi_{-k-2}^{(s)} & \Phi_k^{(s)} \end{pmatrix}\det\begin{pmatrix} \Phi_{-k-1}^{(s)} & \Phi_{k-1}^{(s)} \end{pmatrix}}{\det\begin{pmatrix} \Phi_{-k-1}^{(s)} & \Phi_{-k-2}^{(s)} \end{pmatrix}\det\begin{pmatrix} \Phi_k^{(s)} & \Phi_{k-1}^{(s)} \end{pmatrix}} \ , \tag{229a}$$

$$Y_{2k+1}\left(\lambda e^{\frac{1}{2}\mathbb{i}\pi}\right) = \frac{\det\begin{pmatrix} \Phi_{-k-2}^{(s)} & \Phi_{k+1}^{(s)} \end{pmatrix}\det\begin{pmatrix} \Phi_{-k-1}^{(s)} & \Phi_k^{(s)} \end{pmatrix}}{\det\begin{pmatrix} \Phi_{-k-1}^{(s)} & \Phi_{-k-2}^{(s)} \end{pmatrix}\det\begin{pmatrix} \Phi_{k+1}^{(s)} & \Phi_k^{(s)} \end{pmatrix}} \ . \tag{229b}$$

In term of the functions Y, the Hirota equation (??) becomes

$$Y_k\left(\lambda e^{\frac{1}{2}\mathbb{i}\pi}\right)Y_k\left(\lambda e^{-\frac{1}{2}\mathbb{i}\pi}\right) = \left(1 + Y_{k+1}(\lambda)\right)\left(1 + Y_{k-1}(\lambda)\right) \ . \tag{230}$$

This set of equations is known in the literature as a *Y-system*; see for example [?, ?, ?, ?].

[18]In the first incarnations of the ODE/IM correspondence [?, ?] this point was the origin $z = 0$, which represents a regular singularity of the differential equation. Consequently the solution obtained by local analysis around $z = 0$ does not exhibit any Stokes phenomena. The term "central" also descends from these first examples, in which the eigenvalue problem associated to the central connection matrix concerned functions with behaviour defined at $z = 0$ and $|z| \to \infty$. In our case the linear system possesses no singularity at finite z, however we can still define an eigenvalue problem for functions with given behaviour as $|z| \to \infty$ and at an arbitrary point z which, being regular, will not give rise to a Stokes phenomenon. We stick to the tradition and call such an eigenvalue problem "central".

4.7 Properties of the Y-functions and the TBA equation

Although the rewriting (??) of the Hirota equation does not seem to change the situation much, it actually allows us to derive an integral equation for the logarithm of the Y functions. Let us briefly review how this is done.

Using the definition (??) of the WKB solution, we easily see that

$$Y_{2k}\left(\lambda\right) = \exp\left(-\lambda\oint\limits_{\gamma_{2k}} s\right)\,,\qquad Y_{2k+1}\left(\lambda e^{\frac{1}{2}\mathbb{i}\pi}\right) = \exp\left(-\lambda\oint\limits_{\gamma_{2k+1}} s\right)\,,\tag{231}$$

where $s = \sum_{k=0}^{\infty}\lambda^{-2k}s_k$ and the one-forms s_k were introduced in (??). The γ_k are closed contours, elements of a basis of the first homology group $\mathrm{H}_1\left(\mathcal{R}_{\mathrm{WKB}},\mathbb{Z}\right)$. Since our branch cuts can all be taken to lie on the real axis (remember, we chose the polynomial $P\left(z\right)$ to only have real roots), we can arrange them as shown in figure ??. It is evident that the $Y_k\left(\lambda\right)$ functions are analytic in λ with essential singularities sitting at $\lambda = 0$ and $\lambda = \infty$. In particular, a perturbative analysis of the WKB solutions tells us that

$$\ln Y_{2k} = -\lambda\oint\limits_{\gamma_{2k}} dz\,\sqrt{P} + \mathcal{O}\left(\lambda^{-1}\right)\,,\qquad \ln Y_{2k+1} = \mathbb{i}\lambda\oint\limits_{\gamma_{2k+1}} dz\,\sqrt{P} + \mathcal{O}\left(\lambda^{-1}\right)\,.\tag{232}$$

A similar result holds for the expansion around $\lambda = 0$, with $\sqrt{P}dz$ replaced by $\sqrt{\bar{P}}d\bar{z}$. Hence we find that the Y functions have the following asymptotic for large $|\upsilon|$, with $\upsilon = \ln\lambda$,

$$\ln Y_k\left(\upsilon\right) \underset{|\upsilon|\to\infty}{\sim} -m_k\cosh(\upsilon)\qquad \begin{cases} m_{2k} = 2\oint_{\gamma_{2k}} dz\,\sqrt{P} = 2\oint_{\gamma_{2k}} d\bar{z}\,\sqrt{\bar{P}} \\[2ex] m_{2k+1} = -\mathbb{i}\,2\oint_{\gamma_{2k+1}} dz\,\sqrt{P} = -\mathbb{i}\,2\oint_{\gamma_{2k+1}} d\bar{z}\,\sqrt{\bar{P}} \end{cases}.\tag{233}$$

Note that this behaviour is valid for $\mathrm{Im}\left[\upsilon\right] \in \left(-\pi,\pi\right)$, since beyond this range, the WKB approximation we have used may no longer be reliable.[19] The quantities m_k can be shown to be real when all the zeroes of $P\left(z\right)$ are real.[20]

[19] Actually the WKB approximation can be shown to be valid in the range $\mathrm{Im}\left[\upsilon\right] \in \left(-\frac{3}{2}\pi,\frac{3}{2}\pi\right)$.

[20] Consider a polynomial with $2N$ roots

$$P\left(z\right) = \left(z-z_1\right)\left(z-z_2\right)\cdots\left(z-z_{2N}\right)\,,$$

and suppose that $z_1, z_2 \in \mathbb{R}$. We wish to compute the integral

$$I = \oint\limits_{\gamma_{1,2}} dz\sqrt{P\left(z\right)}\,,$$

where $\gamma_{1,2}$ is a cycle encircling in a counter-clockwise sense the cut running from z_1 to z_2. Moreover, without

Now, from the properties just mentioned, we deduce that the auxiliary function

$$y_k(\upsilon) = \ln\left(Y_k(\upsilon)\, e^{m_k \cosh(\upsilon)}\right) \ , \tag{234}$$

is analytic in the strip $\mathsf{S}_\upsilon = |\mathrm{Im}\,[\upsilon]\,| < \frac{\pi}{2}$ and decays at large $|\mathrm{Re}\,[\upsilon]\,|$ therein. Moreover it obeys the logarithmic form of (??)

$$y_k\left(\upsilon + \frac{1}{2}\mathbb{i}\pi\right) + y_k\left(\upsilon - \frac{1}{2}\mathbb{i}\pi\right) = \ln\left(1 + Y_{k+1}(\upsilon)\right) + \ln\left(1 + Y_{k-1}(\upsilon)\right) \ . \tag{235}$$

This form is very useful, because the operator effecting the shift in the right-hand side above is inverse to the convolution kernel $\mathcal{K}(\upsilon) = \frac{1}{2\pi\cosh(\upsilon)}$. In mathematical terms

$$\begin{aligned} &[\mathcal{K} * y_k]\left(\upsilon + \frac{1}{2}\mathbb{i}\pi\right) + [\mathcal{K} * y_k]\left(\upsilon - \frac{1}{2}\mathbb{i}\pi\right) \\ &= \int_{\mathbb{R}} \frac{d\upsilon'}{2\pi} \frac{y_k\left(\upsilon' + \frac{1}{2}\mathbb{i}\pi\right) + y_k\left(\upsilon' - \frac{1}{2}\mathbb{i}\pi\right)}{\cosh(\upsilon - \upsilon')} = \oint_{\partial\mathsf{S}_\upsilon} \frac{d\upsilon'}{2\pi\mathbb{i}} \frac{y_k(\upsilon')}{\sinh(\upsilon - \upsilon')} = y_k(\upsilon) \ , \end{aligned} \tag{236}$$

where $\partial\mathsf{S}_\upsilon$ is the boundary of the strip $\mathsf{S}_\upsilon = |\mathrm{Im}\,[\upsilon]\,| < \frac{\pi}{2}$ and we used, in turn, that y_k decays in S_υ for $\mathrm{Re}\,[\upsilon] \to \pm\infty$, and that it has no singularities in S_υ. Thus we have arrived at the following integral TBA-like equation [?]

$$\varepsilon_k(\upsilon) = m_k \cosh(\upsilon) - \int_{\mathbb{R}} \frac{d\upsilon'}{2\pi} \frac{\ln\left(1 + e^{-\varepsilon_{k-1}(\upsilon')}\right) + \ln\left(1 + e^{-\varepsilon_{k+1}(\upsilon')}\right)}{\cosh(\upsilon - \upsilon')} \ , \tag{237}$$

where we introduced the *pseudo-energies* (borrowing the language of the TBA)

$$Y_k(\upsilon) = e^{-\varepsilon_k(\upsilon)} \ . \tag{238}$$

If we were to choose a polynomial $P(z)$ with complex roots, then everything that has been said and shown above will essentially remain the same, with the exception of the assertion $m_k \in \mathbb{R}$. What will now happen is that the 'masses' m_k will be complex numbers and the TBA equation (??) will need to be adjusted to the following, more general, form

$$\varepsilon_k(\upsilon) = \frac{m_k}{2}e^{\upsilon} + \frac{m_k^*}{2}e^{-\upsilon} - \int_{\mathbb{R}} \frac{d\upsilon'}{2\pi} \frac{\ln\left(1 + e^{-\varepsilon_{k-1}(\upsilon')}\right) + \ln\left(1 + e^{-\varepsilon_{k+1}(\upsilon')}\right)}{\cosh(\upsilon - \upsilon')} \ . \tag{239}$$

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loss of generality, suppose $z_1 = 0$, $z_2 > 0$ and $z_j \notin [0, z_1]$, $\forall j = 3, \ldots, 2N$. Then our integral becomes

$$I = -2\int_0^{z_2} dz\, \sqrt{z(z - z_2)\cdots(z - z_{2N})}\ ,$$

since the integrals on infinitesimal circles around $z = 1$ and $z = z_2$ vanish. The integral I is explicitely a real number, as long as $z_j \in \mathbb{R}\,,\ \forall j = 3, \ldots 2N$.

Note that as long as $\left|\arg(m_k) - \arg(m_{k+1})\right| < \pi/2\,,\ \forall k$, the above equation is perfectly well defined. However, as soon as we go beyond this regime, it is necessary to pick out the appropriate pole contribution from the kernel.[21] Although the integral equation changes form, the functions Y turn out to be continuous; this phenomenon is known as wall-crossing and has been discussed in [?, ?].

We have arrived at an integral equation whose only inputs are the 'masses' m_k, i.e. the integrals of the WKB one-form s_0 along the basis cycles of $\mathrm{H}_1(\mathcal{R}_{\mathrm{WKB}}, \mathbb{Z})$, and whose outputs are some functions ε_k of the spectral parameter λ. As we will now explain, the knowledge of these functions will allow us to compute the regularized area (??) of the minimal surface in AdS_3, the boundary of which is a polygonal light-like Wilson loop determined by the function $P(z)$, as explained in section ??.

4.8 The area as the free energy

Now we wish to show that the regularized area is really the *Free Energy* associated to the TBA equation (??) – or, more generally, (??). In order to do so we will take a route which might appear to be slightly convoluted, so bear with us. First of all, consider the expression (??) for the regularized area

$$\mathcal{A}_{\text{reg}} = 2\alpha^2 \int_{\Sigma} dz\, d\bar{z} \left(P\bar{P}e^{-\varphi} - \sqrt{P\bar{P}} \right) . \tag{240}$$

We notice that it is possible to write this in terms of the one-forms s_0 and $\bar{s}_0$ (??) and a one-form u

$$s_0 = \sqrt{P}dz \;, \qquad \bar{s}_0 = \sqrt{\bar{P}}d\bar{z} \;, \qquad u = u_z dz + u_{\bar{z}} d\bar{z} \;, \tag{241}$$

as

$$\mathcal{A}_{\text{reg}} = 2\,\alpha^2 \int_{\mathcal{R}_{\mathrm{WKB}}} \left(s_0 \wedge u - s_0 \wedge \bar{s}_0 \right) \;, \tag{242}$$

where, in order to reproduce (??), we are forced to fix the anti-holomorphic part of u as

$$u_{\bar{z}} = \sqrt{P}\bar{P}e^{-\varphi} \;. \tag{243}$$

It is evident that both s_0 and $\bar{s}_0$ are exact, since their components are, respectively, holomorphic and anti-holomorphic. In general the form u is not exact, but it can be made so by precisely choosing the z component u_z, which does not contribute to the integral (??). One easily verifies that the following choice

$$u = \left(\frac{\varphi_{,z}^2 - 2\varphi_{,zz}}{8\sqrt{P}} + f(z) \right) dz + \sqrt{P}\bar{P}e^{-\varphi} d\bar{z} \;, \tag{244}$$

[21] In fact, the equations (??) can be rewritten in the form (??), by shifting $\upsilon \to \upsilon - \arg(m_k)$. These equations will involve kernels $1/\cosh\left(\upsilon - \upsilon' - \mathbb{i}\arg(m_k) + \mathbb{i}\arg(m_{k\pm1})\right)$, which present singularities on the real υ'-line whenever $\left|\arg(m_k) - \arg(m_{k+1})\right| = (2n+1)\,\pi/2\,,\ n \in \mathbb{Z}_{\geq}$.

where $f(z)$ is an arbitrary function of z, fits the bill since

$$du = \frac{e^{\varphi}}{2\sqrt{P}} \frac{\partial}{\partial z} \left(P\bar{P}e^{-2\varphi} + \frac{1}{2}\varphi_{,z\bar{z}} e^{-\varphi} \right) dz \wedge d\bar{z} = 0 \ , \tag{245}$$

due to the modified sinh-Gordon equation (??). We still have the freedom to choose the function $f(z)$ at will, and in the following we take

$$f(z) = \frac{1}{8\sqrt{P}} \left(\frac{P_{,zz}}{P} - \frac{5}{4} \left(\frac{P_{,z}}{P} \right)^2 \right) \ , \tag{246}$$

so that we can express the form u in terms of s_1 (??) as

$$u = s_1 + \sqrt{P}\bar{P}e^{-\varphi} d\bar{z} \ . \tag{247}$$

We are then able to rewrite the regularized area as an integral (??) over the Riemann surface $\mathcal{R}_{\mathrm{WKB}}$ of the external product of two exact one-forms: s_0 and $u - \bar{s}_0$. Why would we want to do this? The answer comes from the following neat property of integration on Riemann surfaces:

Theorem. *[?] Consider a Riemann surface Σ_g of genus g and let $\{a_i, b_i\}_{i=1}^{g}$ be a standard basis of cycles, i.e. a standard basis of $H_1(\Sigma_g, \mathbb{Z})$. Take two exact one-forms ω and ω' and define*

$$\alpha_i = \oint_{a_i} \omega \ , \qquad \beta_i = \oint_{b_i} \omega \ ,$$

$$\alpha'_i = \oint_{a_i} \omega' \ , \qquad \beta'_i = \oint_{b_i} \omega' \ .$$

Then the integral of the two-form $\omega \wedge \omega'$ over the Riemann surface can be decomposed as

$$\int_{\Sigma_g} \omega \wedge \omega' = \sum_{i=1}^{g} \left(\alpha_i \beta'_i - \beta_i \alpha'_i \right) \ . \tag{248}$$

Thanks to this result we can write the expression (??) for the area as

$$\mathcal{A}_{\mathrm{reg}} = 2\alpha^2 \sum_{i,j} w_{i,j} \left(\oint_{\gamma_i} s_0 \right) \left(\oint_{\gamma_j} s_1 - \hat{\bar{s}}_0 \right) \ , \tag{249}$$

where

$$\hat{\bar{s}}_0 = \sqrt{P}\bar{P}e^{-\varphi} d\bar{z} - \bar{s}_0 = \sqrt{\bar{P}} \left(\sqrt{P\bar{P}}e^{-\varphi} - 1 \right) d\bar{z} \ , \tag{250}$$

$\{\gamma_i\}$ is a basis of $\mathrm{H}_1\left(\mathcal{R}_{\mathrm{WKB}}, \mathbb{Z}\right)$ and $w_{i,j}$ are the intersection numbers of these cycles.[22]

Now we need to identify the contour integrals in (??). To this end, let us introduce the functions $\hat{\varepsilon}_k$ defined as

$$\hat{\varepsilon}_{2k}\left(\upsilon\right) = \varepsilon_{2k}\left(\upsilon\right) \ , \qquad \hat{\varepsilon}_{2k+1}\left(\upsilon\right) = \varepsilon_{2k+1}\left(\upsilon + \mathbb{i}\frac{\pi}{2}\right) \ . \tag{251}$$

We can describe their large λ behaviour in two equivalent ways:

- using the expression (??) in terms of WKB integrals

$$\hat{\varepsilon}_k = \lambda \oint_{\gamma_k} s_0 + \frac{1}{\lambda}\oint_{\gamma_k} s_1 + \mathcal{O}\left(\lambda^{-2}\right) \ , \tag{252}$$

- using the TBA equation (??)

$$\hat{\varepsilon}_k = \lambda \oint_{\gamma_k} s_0 + \frac{1}{\lambda}\left(\oint_{\gamma_k} \bar{s}_0 - \frac{1}{\pi}\int_{-\infty}^{\infty} dv' e^{v'} \sum_j w^{k,j} \ln\left(1 + e^{-\hat{\varepsilon}_j(v')}\right)\right) + \mathcal{O}\left(\lambda^{-2}\right) \ , \tag{253}$$

  where we have used the definition (??) of the dimensionless mass parameters m_k and their complex conjugates m_k^*.

  In the case in which the parameters m_k satisfy $\left|\arg\left(m_k\right) - \arg\left(m_{k+1}\right)\right| < \pi/2$, $w^{j,k}$ has the simple expression $w^{j,k} = \delta^{j+1,k} + \delta^{j-1,k}$, and if $2N \in 2\,\mathbb{Z}_{\geq} + 1$ it is invertible with inverse given by the cycle intersection number $w_{i,j}$ introduced above.

Since the above two large-λ expansions must agree term by term, we find the *exact* expression for the integral of the 1-form s_1 on the contours γ_k:

$$\oint_{\gamma_k} s_1 = \oint_{\gamma_k} \bar{s}_0 - \frac{1}{\pi}\int_{-\infty}^{\infty} dv' e^{v'} \sum_j w^{k,j} \ln\left(1 + e^{-\hat{\varepsilon}_j(v')}\right) \ . \tag{254}$$

The expression for the area (??) then takes the following form:

$$\mathcal{A}_{\mathrm{reg}} = 2\frac{\alpha^2}{\pi}\sum_{i,j} w_{i,j} Z_i \left(\int_{-\infty}^{\infty} dv' e^{v'} \sum_j w^{j,k} \ln\left(1 + e^{-\varepsilon_k(v')}\right)\right) \ , \tag{255a}$$

$$Z_i = -\oint_{\gamma_i} s_0 \ . \tag{255b}$$

[22]The cycles γ_i depicted in figure ?? do form a basis but not a normalized one. Hence the need to insert the intersection numbers.

The exact same reasoning as above can be repeated for the small λ limit; this yields

$$\mathcal{A}_{\text{reg}} = 2\frac{\alpha^2}{\pi}\sum_{i,j} w_{i,j}\bar{Z}_i \left(\int_{-\infty}^{\infty} dv' e^{-v'} \sum_j w^{j,k} \ln\left(1+e^{-\varepsilon_k(v')}\right)\right) . \tag{256}$$

Finally, as these two expressions must give the same result,[23] we can take their mean value to find

$$\mathcal{A}_{\text{reg}} = \frac{\alpha^2}{\pi}\sum_i |m_i| \int_{\mathbb{R}} dv \cosh(v) \ln\left(1+e^{-\varepsilon_i\left(v-\mathrm{i}\arg(m_i)\right)}\right) , \tag{257}$$

which coincides with the free energy expression for the TBA equation (??). Note that we made the implicit assumptions that $\sum_j w_{i,j}w^{j,k} = \delta_i^k$, which is true only if $2N \in 2\,\mathbb{Z}_{\geq} + 1$, and $\left|\arg(m_k) - \arg(m_{k+1})\right| < \pi/2$. If instead we have $N \in \mathbb{Z}_{\geq}$ with the constraint on the phases of the masses still in place, the area keeps the form (??), though acquiring an extra term as studied in detail in [?]. On the other hand, if this constraint is relaxed and we cross a wall, new cycles enter the game and one needs to track their contributions with care. However by adapting the derivation we followed it is possible to show that an expression of the form (??) continues to hold. See [?], appendix B, for more details.

4.9 The IM side of ODE/IM correspondence and the conformal limit

We conclude this excursion in the realm of minimal surfaces by briefly making contact with the IM side of the ODE/IM correspondence. In fact what we have done so far in this section pertains to the ODE part of the correspondence: we investigated the classical linear problem (??) and showed how its monodromy data can be used to compute the area of a minimal surface in AdS_3 sitting on a light-like polygonal loop on the boundary ∂AdS_3. Through some non-trivial manipulations of the monodromy data, we arrived at the expression (??) in terms of a set of auxiliary functions $\varepsilon_k(v)$ which satisfy the system of non-linear integral equations (??). As mentioned above, these equations have the flavour of TBA equations for quantum integrable field theories and, as a matter of fact, have appeared earlier in the literature as the equations describing the finite-size ground state spectrum of the $SU(2N)_2/U(1)^{2N-1}$ Homogeneous sine-Gordon model[24] [?, ?, ?, ?, ?, ?, ?]. Hence we conclude that the linear system (??) works as a bridge, connecting the geometry of minimal surfaces in AdS_3 – and, consequently, the properties of light-like Wilson loops in ∂AdS_3 – to the properties of the quantum $SU(2N)_2/U(1)^{2N-1}$ HsG model in finite-size geometry.

[23]This statement is equivalent to the requirement that the total momentum of the TBA vanishes identically, or, in other words, that the *pseudo-energies* ε_k are even functions of v.

[24]Actually, the equations (??) are associated to a particular instance of the $SU(2N)_2/U(1)^{2N-1}$ HsG model, in which the so-called *resonance parameters* are chosen to vanish, see [?, ?].

It is known [?,?] that the CFT limit of the $G_k/U(1)^{r_G}$, with G a compact simple Lie group, r_G the rank of the group G and k its level, is described by the parafermionic $G_k/U(1)^{r_G}$ coset CFT with central charge

$$c = \frac{k-1}{k+h_G} r_G\, h_G \ , \tag{258}$$

where h_G is the Coxeter number of the group G. In the case considered in this section, that is $G = SU(2N)$, one has $r_G = 2N-1$ and $h_G = 2N$ and choosing $k=2$ one obtains the central charge

$$c = N\frac{2N-1}{N+1} \ . \tag{259}$$

As mentioned in section ??, the integrable structure of these CFTs is conjectured to be described by a Sturm-Liouville problem for (??) with the particular choice (??) of the potential $P(x)$. In order to verify this fact, we need to perform the *conformal limit* on the linear system (??). We thus first pick a generic point $(z_0, \bar{z}_0)$, such that $P(z_0) = p_0 \neq 0, \infty$ and $\bar{P}(\bar{z}_0) = \bar{p}_0 \neq 0, \infty$. Without loss of generality we will suppose that $(z_0, \bar{z}_0) = (0,0)$. As the point $(0,0)$ need to be generic, we require the Gauss curvature (??) to be a finite constant at that point

$$e^{-2\varphi} P\bar{P} \underset{(z,\bar{z})\to(0,0)}{\sim} \mathcal{O}\left(z^0, \bar{z}^0\right) \ , \tag{260}$$

which means that the sinh-Gordon field φ will have the following simple, regular behaviour

$$\varphi(z,\bar{z}) \underset{(z,\bar{z})\to(0,0)}{\sim} \frac{1}{2}\ln\left(P_0\bar{P}_0\right) + \sum_{k=1}^{\infty}\left(\varphi_k z^k + \bar{\varphi}_k \bar{z}^k\right) \ . \tag{261}$$

The coefficients φ_k and $\bar{\varphi}_k$ are fixed by inserting the above ansatz into the modified sinh-Gordon equation (??); their explicit form is of no relevance, but we list here the first few

$$\varphi_1 = \frac{P_1}{2P_0} \ , \quad \varphi_2 = \frac{P_2}{2P_0} - \frac{P_1^2}{4P_0^2} \ , \quad \varphi_3 = \frac{P_3}{2P_0} - \frac{P_1 P_2}{2P_0^2} + \frac{P_1^3}{6P_0^3} \ , \tag{262a}$$

$$\varphi_4 = \frac{P_4}{2P_0} - \frac{P_2^2 + 2P_1P_3}{4P_0^2} + \frac{P_1^2 P_2}{2P_0^3} - \frac{P_1^4}{8P_0^4} \ , \tag{262b}$$

with

$$P(z) = P_0 + \sum_{k=1}^{2N} P_k z^k = \prod_{k=1}^{2N}(z - z_k) \ . \tag{263}$$

Similar expressions hold for $\bar{\varphi}_k$ and $\bar{p}(\bar{z})$. We see that when taking the *light-cone* limit $\bar{z} \to 0$, the field assumes the following form

$$\varphi(z,\bar{z}) \underset{\bar{z}\to 0}{\sim} \frac{1}{2}\ln\left(P_0\bar{P}_0\right) + \sum_{k=1}^{\infty}\varphi_k z^k \ . \tag{264}$$

Let us look back at the linear system (??)

$$\Phi_{,z} = \mathcal{L}\Phi \ , \qquad \Phi_{,\bar{z}} = \bar{\mathcal{L}}\Phi \ , \tag{265}$$

with

$$\mathcal{L}(\lambda) = \begin{pmatrix} -\frac{1}{4}\varphi_{,z} & \lambda e^{\frac{\varphi}{2}} \\ \lambda P e^{-\frac{\varphi}{2}} & \frac{1}{4}\varphi_{,z} \end{pmatrix} , \qquad \bar{\mathcal{L}}(\lambda) = \begin{pmatrix} \frac{1}{4}\varphi_{,\bar{z}} & \frac{1}{\lambda}\bar{P} e^{-\frac{\varphi}{2}} \\ \frac{1}{\lambda} e^{\frac{\varphi}{2}} & -\frac{1}{4}\varphi_{,\bar{z}} \end{pmatrix} . \tag{266}$$

We now consider the unknown Φ as a vector, i.e. an arbitrary column of a generic matrix solution of (??), which we can parametrise in the two following ways

$$\Phi = \begin{pmatrix} \lambda e^{\frac{\varphi}{4}}\chi \\ e^{-3\frac{\varphi}{4}}\partial\left(e^{\frac{\varphi}{2}}\chi\right) \end{pmatrix} = \begin{pmatrix} e^{-3\frac{\varphi}{4}}\bar{\partial}\left(e^{\frac{\varphi}{2}}\bar{\chi}\right) \\ \frac{1}{\lambda}e^{\frac{\varphi}{4}}\bar{\chi} \end{pmatrix} . \tag{267}$$

One then easily checks that the linear problem reduces to the following pair of second order differential equations

$$\chi_{,zz}(z,\bar{z}) + \left(\frac{1}{2}v(z,\bar{z}) - \lambda^2 P(z)\right)\chi(z,\bar{z}) \ = \ 0 \ , \tag{268}$$

$$\bar{\chi}_{,\bar{z}\bar{z}}(z,\bar{z}) + \left(\frac{1}{2}\bar{v}(z,\bar{z}) - \frac{1}{\lambda^2}\bar{P}(\bar{z})\right)\bar{\chi}(z,\bar{z}) \ = \ 0 \ , \tag{269}$$

where

$$v(z,\bar{z}) = \varphi_{,zz}(z,\bar{z}) - \frac{1}{2}\varphi_{,z}(z,\bar{z})^2 \ , \qquad \bar{v}(z,\bar{z}) = \varphi_{,\bar{z}\bar{z}}(z,\bar{z}) - \frac{1}{2}\varphi_{,\bar{z}}(z,\bar{z})^2 \ , \tag{270}$$

are the *Miura transforms* of the field φ.

Now we will consider the conformal limit in the form of a double limit: we first take the light cone limit $\bar{z} \to 0$, which will 'freeze' the anti-holomorphic dependence, and subsequently consider the regime $z \sim 0$. In order to consistently perform this last limit, we first rescale all the quantities in play by the appropriate power of λ as follows

$$z = \lambda^{-\frac{1}{N+1}}x \ , \qquad \bar{z} = \lambda^{\frac{1}{N+1}}\bar{x} \ , \tag{271}$$

and scale the zeroes z_k of the potential $P(z)$ as $z \to 0$ so that

$$P(z) = \prod_{k=1}^{2N}(z - z_k) = \lambda^{-\frac{2N}{N+1}}\prod_{k=1}^{2N}(x - x_k) = \lambda^{-\frac{2N}{N+1}}P(x) \ , \tag{272}$$

then consider the limit $\lambda \to \infty$. Let us first concentrate on what happens to equation (??) when we send $\bar{z} \to 0$. The Miura transform v becomes

$$v(z,\bar{z}) = \mathcal{O}\left(z^0\right) = \lambda^{\frac{2}{N+1}}\mathcal{O}\left(\lambda^{-\frac{2}{N+1}}\right) \ , \tag{273}$$

while the differential equation itself now reads

$$\chi_{,xx}\left(x,\bar{x}\right)-\left(\mathcal{O}\left(\lambda^{-\frac{2N}{N+1}}\right)+P\left(x\right)\right)\chi\left(x,\bar{x}\right)=0\;. \tag{274}$$

Then we take the limit $\lambda\to\infty$ while keeping the scaling variables x and x_k finite, so that we arrive at the following equation

$$\chi_{,xx}\left(x\right)=P\left(x\right)\chi\left(x\right)\;, \tag{275}$$

which is clearly holomorphic in form and the reason why we dropped the $\bar{x}$ dependence of ϕ.

What is the fate of the equation (??)? Let us look at what happens to the potential $\bar{P}$ in the light-cone limit

$$\bar{P}\left(\bar{z}\right)=\prod_{k=1}^{2N}\left(\bar{z}-z_k\right)\underset{\bar{z}\to 0}{\sim}\prod_{k=1}^{2N}z_k=\lambda^{-\frac{2N}{N+1}}\prod_{k=1}^{2N}x_k=\lambda^{-\frac{2N}{N+1}}X_N\;. \tag{276}$$

On the other hand, in the light-cone limit we have $\bar{v}\to 0$. Consequently the equation (??) reduces to

$$\bar{\chi}_{,\bar{x}\bar{x}}\left(x,\bar{x}\right)-\lambda^{-\frac{4N}{N+1}}X_N\bar{\chi}\left(x,\bar{x}\right)=0\;, \tag{277}$$

which in the limit $\lambda\to\infty$ becomes

$$\bar{\chi}_{,\bar{x}\bar{x}}\left(x,\bar{x}\right)=0\;. \tag{278}$$

We easily check that this equation is consistent with the relation imposed by the two parametrizations (??) of the vector Φ, since considering the identity

$$\chi=\frac{1}{\lambda}e^{-\varphi}\bar{\partial}\left(e^{\frac{\varphi}{2}}\bar{\chi}\right)\;, \tag{279}$$

and taking a derivative with respect to $\bar{z}$, we obtain

$$\chi_{,\bar{z}}=\frac{e^{-\frac{\varphi}{2}}}{\lambda}\left(\bar{\chi}_{,\bar{z}\bar{z}}\left(z,\bar{z}\right)+\frac{1}{2}\bar{v}\left(z,\bar{z}\right)\bar{\chi}\left(z,\bar{z}\right)\right)\;\longrightarrow\;0\;. \tag{280}$$

This proves that in the double scaling limit, the function ϕ is indeed holomorphic. Hence, as expected, we have recovered the ODE (??) with a potential

$$P\left(x\right)=\prod_{k=1}^{2N}\left(x-x_k\right)\;,\quad\left(2N\in\mathbb{Z}_{>}\right), \tag{281}$$

of the same form as (??).

5 Conclusions

The discovery of a connection between the theory of ordinary differential equations and 2D quantum field theories was a completely unexpected surprise for the integrable model community. It has allowed the investigation of problems in pure mathematics, in statistical mechanics and condensed matter physics, strings and supersymmetric gauge theories. However, most of the mathematical structures and connections that have emerged over the past 20 years in the ODE/IM context have only been superficially explored. Among the many mysterious facts concerning the ODE /IM correspondence, perhaps one of the most fascinating is that it provides a compelling alternative way to quantise classical integrable systems. In this respect, it will be essential to put more effort toward the implementation of this novel quantisation scheme in the context of non-linear sigma models, as initiated in [?].

The ODE/IM correspondence might also provide a way to extend fundamental concepts related to the renormalisation group to the Hamiltonian picture [?] and to implement the quantisation of effective quantum field theories.

Concerning the last topic, the so-called $T\bar{T}$-perturbation, where $T\bar{T}$ is the composite operator defined as the determinant of the stress-energy tensor [?], is known to be integrable at both classical and quantum level [?, ?, ?, ?, ?]. On the classical side, deformed EoMs and Lax operators coincide with the undeformed quantities up to a field-dependent local change of the space-time coordinates [?, ?, ?]. The effect of this deformation on the finite-size quantum TBA spectrum is also well understood; however, what is still missing are the ODE/IM steps connecting the classical to the quantum TBA answer. For instance, it would interesting to know the fate of the polygonal Wilson loops, in particular of the area/ free-energy equivalence described in this review, under the $T\bar{T}$ perturbation or the Lorentz-breaking generalisations studied in [?, ?, ?].

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