

Kent Academic Repository

Full text document (pdf)

Citation for published version

Deaño, Alfredo and Simm, Nicholas J (2020) Characteristic polynomials of complex random matrices and Painlevé transcendents. *International Mathematics Research Notices*. ISSN 1073-7928. (Submitted)

DOI

Link to record in KAR

<https://kar.kent.ac.uk/80235/>

Document Version

Pre-print

Copyright & reuse

Content in the Kent Academic Repository is made available for research purposes. Unless otherwise stated all content is protected by copyright and in the absence of an open licence (eg Creative Commons), permissions for further reuse of content should be sought from the publisher, author or other copyright holder.

Versions of research

The version in the Kent Academic Repository may differ from the final published version.

Users are advised to check <http://kar.kent.ac.uk> for the status of the paper. **Users should always cite the published version of record.**

Enquiries

For any further enquiries regarding the licence status of this document, please contact:

researchsupport@kent.ac.uk

If you believe this document infringes copyright then please contact the KAR admin team with the take-down information provided at <http://kar.kent.ac.uk/contact.html>

Characteristic polynomials of complex random matrices and Painlevé transcendents

Alfredo Deaño and Nick Simm

ABSTRACT. We study expectations of powers and correlation functions for characteristic polynomials of $N \times N$ non-Hermitian random matrices. For the 1-point and 2-point correlation function, we obtain several characterizations in terms of Painlevé transcendents, both at finite- N and asymptotically as $N \rightarrow \infty$. In the asymptotic analysis, two regimes of interest are distinguished: boundary asymptotics where parameters of the correlation function can touch the boundary of the limiting eigenvalue support and bulk asymptotics where they are strictly inside the support. For the complex Ginibre ensemble this involves Painlevé IV at the boundary as $N \rightarrow \infty$. Our approach, together with the results in [47] suggests that this should arise in a much broader class of planar models. For the bulk asymptotics, one of our results can be interpreted as the merging of two ‘planar Fisher-Hartwig singularities’ where Painlevé V arises in the asymptotics. We also discuss the correspondence of our results with a normal matrix model with d -fold rotational symmetries known as the *lemniscate ensemble*, recently studied in [14, 17]. Our approach is flexible enough to apply to non-Gaussian models such as the truncated unitary ensemble or induced Ginibre ensemble; we show that in the former case Painlevé VI arises at finite- N . Scaling near the boundary leads to Painlevé V, in contrast to the Ginibre ensemble.

1. Introduction and results

The purpose of this paper is to study correlation functions of characteristic polynomials, of the form

$$(1.1) \quad R_{\vec{\gamma}}(\vec{z}) := \mathbb{E} \left(\prod_{i=1}^m |\det(G_N - z_i)|^{\gamma_i} \right)$$

where $\vec{\gamma} = (\gamma_1, \dots, \gamma_m)$ are parameters¹ and $\vec{z} \in \mathbb{C}^m$. In the first instance we take G_N to be a standard complex Ginibre random matrix, that is $G_N = \frac{1}{\sqrt{N}} G$ where G is an $N \times N$ matrix of *i.i.d.* standard complex normal random variables, though we shall also discuss other non-Gaussian examples.

Particular attention will be paid to the cases $m = 1$ and $m = 2$, where the average (1.1) turns out to be intimately related to Painlevé transcendents. These are solutions of a distinguished class of non-linear second order differential equations of second degree, labelled Painlevé I - Painlevé VI, see for instance [30, Chapter 32]. The relevance of Painlevé transcendents in random matrix theory arose most famously in the work of Tracy and Widom in connection with the largest eigenvalue of a Hermitian random matrix [60], building on the earlier developments of Jimbo, Miwa, Mōri and Sato [50]. They are now among the most important special functions of mathematical physics, and they appear in connection with reduction of integrable PDEs, models in statistical mechanics, combinatorics and orthogonal polynomials, to name only a few applications, see [62] for

¹We frequently consider the case that $\vec{\gamma} = 2\vec{k}$ where $\vec{k} \in \mathbb{N}^m$, though throughout the article we will discuss the extent to which our results can be extended or interpreted for general values of $\vec{\gamma}$

a recent survey, and the manuscript [32]. The characterization of averages involving Hermitian (or unitary) random matrices in terms of Painlevé transcendents has attracted considerable activity in the last decades. However, the fact that Painlevé transcendents also play an important role in the context of non-Hermitian random matrices appears to be somewhat less appreciated and will be the main focus of this work. For the complex Ginibre ensemble, this has been discussed to some extent in the physics literature: Nishigaki and Kamenev [58] applied methods from supersymmetry to compute (1.1) when $m = 1$ and a special case of the $m = 2$ correlation function. In subsequent work of Kanzieper [51], in the context of replica field theories, it was pointed out that the results in [58] imply a relation to Painlevé IV and Painlevé V transcendents. Here, in addition to studying the general correlations (1.1), we will arrive at these results using what we believe to be a mathematically more transparent approach, circumventing the need for supersymmetry techniques. Another advantage of our approach is its applicability beyond the Gaussian distributed cases.

In the case of Hermitian (or unitary) random matrices, correlation functions of characteristic polynomials of type (1.1) were studied quite intensively over the last two decades. The pioneering work of Keating and Snaith [53] demonstrated their importance in modelling statistical properties of the Riemann zeta function high up on the critical line. For GUE random matrices, higher order correlation functions were studied by Brézin and Hikami [22, 23] in various asymptotic regimes and the importance of determinantal structures was emphasized. More recently there has been a resurgence of interest in characteristic polynomials of random matrices due to their association with log-correlated Gaussian fields [42, 44, 48, 63]. Along these lines, the recently studied question of ‘moments of moments’ [10, 12] (where Painlevé V also arises) is closely related to correlation functions of the type considered here, albeit in the Hermitian context. Our main object (1.1) can be viewed as the multi-point Laplace transform of the *random process* $\phi_N(z) = \log |\det(G_N - z)|$ which is asymptotically Gaussian and logarithmically correlated, see [6, 59, 64]. In the present non-Hermitian setting the understanding of (1.1) remains at an early stage, especially at the level of asymptotic analysis.

Let us now be more precise about the asymptotic regimes under consideration. Under the chosen normalization of the matrices G_N , we have the famous *circular law*: in the limit $N \rightarrow \infty$ the eigenvalues of G_N are uniformly distributed on the unit disc in the complex plane. Our focus will be on large N asymptotics of (1.1) in the microscopic regime, that is where the points $\{z_i\}_{i=1}^m$ have a separation on the order of the mean eigenvalue spacing of G_N of size $1/\sqrt{N}$. This is to be distinguished from the macroscopic or mesoscopic regimes which involve separations at larger scales. The possible microscopic scalings of (1.1) are then further divided into *boundary asymptotics* where the z_i ’s are close to the boundary of the eigenvalue support (close to the unit circle for the matrices G_N) and *bulk asymptotics* where the z_i ’s are strictly inside the support. The boundary asymptotics, although interesting in their own right, have an application to the normal matrix model which we now discuss.

Recall that the normal matrix model is the probability measure on N points $\lambda_1, \dots, \lambda_N$ in the complex plane defined by

$$(1.2) \quad d\mathbb{P}(\lambda_1, \lambda_2, \dots, \lambda_N) = \frac{1}{\mathcal{Z}_N} \prod_{j=1}^N e^{-NV(\lambda_j)} \prod_{1 \leq k < j \leq N} |\lambda_k - \lambda_j|^2 d^2\lambda_1 \dots d^2\lambda_N.$$

where \mathcal{Z}_N is a normalization constant, known as the *partition function*, defined by

$$(1.3) \quad \mathcal{Z}_N = \int_{\mathbb{C}^N} \prod_{j=1}^N e^{-NV(\lambda_j)} \prod_{1 \leq k < j \leq N} |\lambda_k - \lambda_j|^2 d^2\lambda_1 \dots d^2\lambda_N.$$

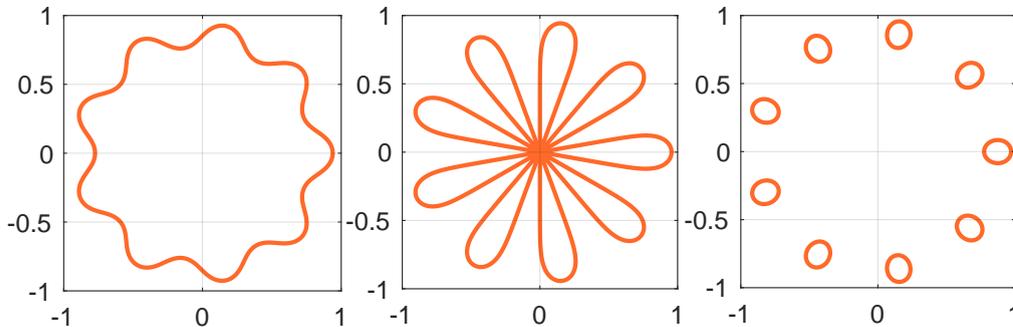


FIGURE 1. The boundary of the droplet for the lemniscate potential (1.4) for $d = 9$ with critical value $t_c = 1/\sqrt{d}$. From left to right the plots show $t < t_c$, $t = t_c$ and $t > t_c$.

Although the matrix G_N is almost surely not normal, its eigenvalue distribution is known to take the form (1.2) with $V(\lambda) = |\lambda|^2$. Therefore, the eigenvalues of the complex Ginibre ensemble can be viewed as the most basic yet non-trivial example of the model (1.2). In general, the function $V(\lambda)$ is called the external potential and subject to suitable growth and regularity conditions, in the limit $N \rightarrow \infty$ the random points $\{\lambda_j\}_{j=1}^N$ are supported on a compact subset of the plane known as the *droplet*, denoted \mathcal{S} . The density of eigenvalues in \mathcal{S} is described by an equilibrium measure coming from weighted potential theory, see [7] for further background. In the electrostatic interpretation of the model (1.2), quantity (1.1) can be viewed as a partition function of the form (1.3) subjected to the insertion of m point charges at locations \vec{z} and strengths $\vec{\gamma}$, see [9, 56, 57].

Recently, there has been interest in studying the model (1.2) corresponding to the class of potentials

$$(1.4) \quad V^{(d)}(\lambda, t) = |\lambda|^{2d} - t(\lambda^d + \bar{\lambda}^d), \quad d \in \mathbb{N}, \quad t \in \mathbb{R},$$

which are invariant under the d -fold symmetry $\lambda \rightarrow \lambda e^{2\pi i/d}$. This model was introduced by Balogh and Merzi [15] who showed that the equilibrium measure is supported on the interior of the domain, $\mathcal{S} = \{\lambda : |\lambda^d - t| \leq t_c\}$ where $t_c = d^{-1/2}$. Due to the lemniscate type shape of \mathcal{S} , the random point configuration corresponding to (1.4) has been referred to as the *lemniscate ensemble* [8], see Figure 1. When t passes through t_c , the topology of the droplet undergoes a transition from being simply connected to consisting of d connected components. The case $t = t_c$ has been singled out as a model with a non-trivial boundary singularity (see [8]) and the effect of this singularity on the partition function asymptotics is of interest (see also Remark 3.7).

In the recent work [17] the orthogonal polynomials for the planar weight $e^{-NV^{(d)}(\lambda, t)}$ have been analysed. Close to the transition $t = t_c$ the asymptotics were expressed in terms of the Hamiltonian of the Painlevé IV system. This followed earlier works [14, 56] on the off-critical cases $t \neq t_c$ which were expressed in terms of classical special functions, see also [13]. A natural question is to look at the partition function associated with (1.4). The following lemma shows its relation to characteristic polynomials of the Ginibre ensemble.

LEMMA 1.1. Let $\mathcal{Z}_N^{(\text{Lem}_d)}(t)$ denote the partition function corresponding to (1.4). Then we have the identity

$$(1.5) \quad \mathcal{Z}_{Nd}^{(\text{Lem}_d)}(t) = e^{(Ntd)^2} \tilde{c}_{N,d} \prod_{\ell=0}^{d-1} \mathbb{E} \left(|\det(G_N - t\sqrt{d})|^{\gamma_\ell} \right)$$

where

$$(1.6) \quad \gamma_\ell = -2 \left(1 - \frac{\ell+1}{d} \right), \quad \ell = 0, \dots, d-1.$$

Here $\tilde{c}_{N,d}$ is an explicit constant that we give later in Section 3.3.

Notice that double scaling the parameter t close to $t_c := \frac{1}{\sqrt{d}}$ in (1.5) corresponds precisely to boundary asymptotics of (1.1) with $m = 1$. In what follows we will present the boundary asymptotics in the Ginibre case; consequences for (1.5) are given in Section 3.3. The best result we are aware of for the average in (1.5) holds strictly away from the boundary and the authors do not comment on the connection with the lemniscate ensemble.

THEOREM 1.2 (Webb and Wong '17 [64]). Fix $\delta > 0$ and consider the closed disc $D_{1-\delta}$ of radius $1 - \delta$. Then provided $\text{Re}(\gamma) > -2$, the following asymptotic formula holds uniformly for $z \in D_{1-\delta}$

$$(1.7) \quad \mathbb{E} (|\det(G_N - z)|^\gamma) = e^{\frac{N\gamma}{2}(|z|^2-1) + \frac{\gamma^2}{8} \log(N)} \frac{(2\pi)^{\frac{\gamma}{4}}}{G(1 + \frac{\gamma}{2})} (1 + o(1)), \quad N \rightarrow \infty,$$

where $G(z)$ is the Barnes G -function. The expansion (1.7) holds uniformly in compact subsets of $\text{Re}(\gamma) > -2$.

We now state our main results.

1.1. Complex Ginibre ensemble: Boundary asymptotics. Our first result extends the asymptotic expansion (1.7) uniformly to the boundary $|z| = 1$ and leads to a new behaviour in the constant term. In order to state our result, recall that a *GUE random matrix* is a Hermitian random matrix H whose density is proportional to $e^{-\text{Tr}(H^2)/2}$ with respect to the Lebesgue measure. We denote by $\lambda_{\max}^{(\text{GUE}_k)}$ the largest eigenvalue of the matrix H , having size $k \times k$.

THEOREM 1.3. Let z belong to the closed disc D_R of radius $R = 1 + L/\sqrt{N}$ for some positive constant $L > 0$. Then for any $k \in \mathbb{N}$, the following asymptotic expansion holds uniformly on D_R ,

$$(1.8) \quad \mathbb{E} \left(|\det(G_N - z)|^{2k} \right) = e^{Nk(|z|^2-1) + \frac{k^2}{2} \log(N)} \frac{(2\pi)^{\frac{k}{2}}}{G(1+k)} F_k(\sqrt{N}(1-|z|^2)) \\ \times (1 + o(1)), \quad N \rightarrow \infty,$$

where $F_k(x)$ is the distribution function of the largest eigenvalue of a $k \times k$ GUE random matrix,

$$(1.9) \quad F_k(x) = \mathbb{P}(\lambda_{\max}^{(\text{GUE}_k)} < x).$$

PROOF. See Section 3.1 □

REMARK 1.4. A result of Tracy and Widom characterizes the above GUE probability in terms of Painlevé transcendents. In particular in [60] we find

$$(1.10) \quad F_k(x) = \exp \left(- \int_x^\infty \sigma_k^{(\text{IV})}(s) ds \right),$$

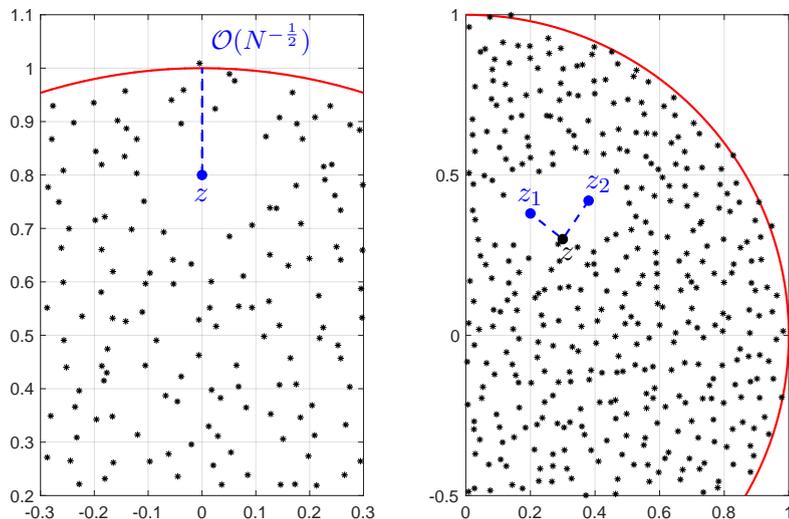


FIGURE 2. Schematic insertion of a charge near the boundary of distance on the order of $N^{-1/2}$ (left), and of two charges z_1 and z_2 of distance on the order of $N^{-1/2}$ from a fixed point z in the bulk of the complex Ginibre ensemble (right). In both cases $N = 1000$, and the unit circle is depicted in red.

where $\sigma_k^{(\text{IV})}(s)$ is the unique solution of the Jimbo-Miwa-Okamoto σ -form of the Painlevé IV equation

$$(1.11) \quad (\sigma'')^2 + 4(\sigma')^2(\sigma' + 2k) - 4(s\sigma' - \sigma)^2 = 0$$

subject to the boundary condition $\sigma_k(s) \sim -2ks - k^2/s$ as $s \rightarrow -\infty$ (see [36, Equation 4.18]).

We expect that the form of expansion (1.8) also holds more generally when $k = \gamma/2$ and γ satisfies the hypothesis of Theorem 1.2, noting that the above Painlevé characterization does not require integrality of k . In the later sections, we will provide some heuristic computations lending support to such an extrapolation to non-integer k . This is based on the fact that for finite N , the left-hand side of (1.8) can be expressed in terms of a Painlevé V transcendent for general $k = \frac{\gamma}{2}$, see Section 3.2. The Painlevé IV on the right-hand side of (1.8) then arises from a rescaling (confluent limit) of this equation. Notice that when $|z| < 1$ is fixed inside the unit disc and N is large, we can replace the GUE probability in (1.9) with 1 and we recover precisely (1.7) with $\gamma = 2k$. On the other hand, if we are close to the boundary with $|z| = 1 - u/\sqrt{N}$ with u fixed, the GUE probability is not negligible and approaches a non-trivial limit involving $F_k(2u)$. When $|z| > 1 + \delta$ a different type of expansion holds, see Appendix A for details. As we already mentioned, Painlevé IV type objects also appeared in the recent work [17], which (taking into account relation (1.5)) is likely to be related with our result here, see Section 3.3. Analogous results on the line are known: the work [18] considers a jump singularity colliding with the spectral edge of GUE random matrices, which they describe in terms of Painlevé II.

1.2. Complex Ginibre ensemble: Bulk collision of two singularities. Consider the particular case $m = 2$ of (1.1) and set

$$(1.12) \quad z_1 = z + \frac{u_1}{\sqrt{N}}, \quad z_2 = z + \frac{u_2}{\sqrt{N}},$$

where z is a point in the bulk, *i.e.* $|z| < 1 - \delta$ for some fixed $\delta > 0$. That is, we consider the case of two singularities merging (or colliding) inside the bulk of the spectrum, see Figure 2. The corresponding asymptotics of (1.1) turns out to be related to the LUE (Laguerre Unitary Ensemble) defined as follows: Let $G_{k,p}$ denote a rectangular $k \times p$ matrix of *i.i.d.* independent standard complex normal random variables with $p \geq k$. Then the matrix $W = G_{k,p}G_{k,p}^\dagger$ is said to be distributed according to the LUE with p degrees of freedom, denoted $\text{LUE}_{k,p}$. The names *Wishart matrix* or *sample covariance matrix* are also in common use, see the text [34] for further background. The largest eigenvalue of W will be denoted $\lambda_{\max}^{(\text{LUE}_{k,p})}$.

THEOREM 1.5. *Let $k_1, k_2 \in \mathbb{N}$, where we assume without loss of generality that $k_1 \geq k_2$ and take z_1, z_2 scaled according to (1.12) with u_1 and u_2 fixed. Then we have the following asymptotic expansion,*

$$(1.13) \quad \mathbb{E} \left(|\det(G_N - z_1)|^{2k_1} |\det(G_N - z_2)|^{2k_2} \right) = e^{k_1 N(|z_1|^2 - 1) + k_2 N(|z_2|^2 - 1) + \frac{k_1^2}{2} \log(N) + \frac{k_2^2}{2} \log(N)} \\ \times e^{-2k_1 k_2 \log|z_2 - z_1|} \frac{(2\pi)^{(k_1 + k_2)/2}}{G(1 + k_1)G(1 + k_2)} F_{k_1, k_2}(N|z_2 - z_1|^2) (1 + o(1)), \quad N \rightarrow \infty,$$

where $F_{k_1, k_2}(x)$ is the distribution function of the largest eigenvalue of a $k_2 \times k_2$ LUE matrix with k_1 degrees of freedom:

$$(1.14) \quad F_{k_1, k_2}(x) = \mathbb{P} \left(\lambda_{\max}^{(\text{LUE}_{k_2, k_1})} < x \right).$$

PROOF. See Section 4. □

REMARK 1.6. *As with Theorem 1.3, the probability $F_{k_1, k_2}(x)$ has a characterization in terms of a non-linear second order differential equation. In the same work of Tracy and Widom [60] one finds the result*

$$(1.15) \quad F_{k_1, k_2}(x) = \exp \left(- \int_x^\infty \frac{\sigma_{k_2, k_1 - k_2}^{(V)}(s)}{s} ds \right)$$

where $\sigma_{k, \alpha}^{(V)}(s)$ satisfies the Jimbo-Miwa-Okamoto σ -form of the Painlevé V equation,

$$(1.16) \quad (t\sigma'')^2 - [\sigma - t\sigma' + 2(\sigma')^2 + (2k + \alpha)\sigma']^2 + 4(\sigma')^2(k + \alpha + \sigma')(k + \sigma') = 0.$$

If we consider a limiting case where the separation $|u_2 - u_1| \rightarrow \infty$ we can replace the LUE probability with 1. The remaining terms in the asymptotic expansion (1.13) are consistent with a conjectured formula of Webb and Wong [64] for the *global asymptotics* of (1.1) (defined by a separation between z_2 and z_1 of order $O(1)$). Indeed, the appearance of the term $-2k_1 k_2 \log|z_2 - z_1|$ should be interpreted as the covariance function of the random process $\phi_N(z) := \log|\det(G_N - z)|$. Such *logarithmic correlations* are widely anticipated to appear at global or mesoscopic regimes because of the association of $\phi_N(z)$ with the Gaussian free field [6, 59]. We remark that the particular case of (1.13) with $k_1 = k_2$ and $z = 0$ appeared in the mentioned work of Nishigaki and Kamenev [58] using a nonlinear σ model (saddle point analysis involving commuting and anti-commuting variables), but the terms of order $\log(N)$ appearing in (1.13) appear to be absent in [58]. This could be due to their intended application to replica limits which could render such terms negligible. Finally we remark that a one-dimensional version of this merging of singularities has been investigated in the context of Toeplitz (or Hankel) determinants (with non-integer exponents), and also involves Painlevé V [24, 25].

As with Theorem 1.3, we believe that Theorem 1.5 continues to hold when k_1 and k_2 are general real numbers such that the left-hand side of (1.13) is finite, with equation

(3.17) defining the extrapolation to such non-integral exponents. To give support to this we give an example involving (partially) non-integer exponents, as follows.

THEOREM 1.7. *Let $k_2 \in \mathbb{N}$ and γ be a real number such that $\gamma \geq k_2$. Consider the left-hand side of (1.13) scaled according to (1.12) at the center of the eigenvalue support, $z = 0$, with $u_1 = 0$. Then the expansion on the right-hand side of (1.13) remains true with $k_1 = \gamma$.*

PROOF. See Section 4. □

As we explain in Section 4, Theorem 1.7 is in fact an asymptotic expansion for the $m = 1$ case of (1.1), with G_N replaced by a certain non-Gaussian matrix: the *induced Ginibre ensemble* introduced and studied in [31], see equation (4.34).

1.3. Truncated unitary ensemble and Painlevé transcendents. Let M be a positive integer and consider $U(M)$, the group of $M \times M$ unitary matrices. The CUE (Circular Unitary Ensemble) is $U(M)$ equipped with the normalized Haar measure $d\mu(U)$. A *truncation* of U is simply a sub-block, which by invariance of the Haar measure we can take to be the upper-left $N \times N$ minor of U , that we denote here by T and assume $N < M$. The eigenvalues $\lambda_1, \dots, \lambda_N$ of the sub-unitary matrix T lie strictly inside the unit disc $|\lambda_j| \leq 1$ for all $j = 1, \dots, N$. Their joint probability density function has been computed by Życzkowski and Sommers [66] as

$$(1.17) \quad P(\lambda_1, \dots, \lambda_N) \propto \prod_{j=1}^N (1 - |\lambda_j|^2)^{M-N-1} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^2.$$

Using (1.17) the authors showed that statistics of the eigenvalues depend sensitively on the relative scaling of N and M as $M \rightarrow \infty$. Roughly speaking, unless N is very close to M , the matrix T is expected to have similar spectral characteristics as the Ginibre matrix G_N from the previous sections. However, when $M - N = \kappa = O(1)$, so that only a constant order of rows and columns are truncated, one expects new behaviour. This is known as the *weak non-unitarity limit*, see [45, 54] for further background and applications.

THEOREM 1.8 (Boundary asymptotics). *Let $M - N = \kappa$ be a fixed positive integer and fix $k \in \mathbb{N}$. Consider the boundary scaling*

$$(1.18) \quad |z| = 1 - \frac{u}{N}, \quad u > 0.$$

Then the following asymptotic formula holds

$$(1.19) \quad \mathbb{E} \left(|\det(T - z)|^{2k} \right) = N^{k^2} \frac{G(k + \kappa + 1)}{G(k + 1)G(\kappa + 1)} (2u)^{-k^2 - k\kappa} F_k^{(\kappa)}(2u) (1 + o(1)), \quad N \rightarrow \infty,$$

where $F_k^{(\kappa)}(u)$ is the distribution function of the largest eigenvalue in the LUE with $k + \kappa$ degrees of freedom,

$$(1.20) \quad F_k^{(\kappa)}(x) = \mathbb{P} \left(\lambda_{\max}^{(\text{LUE}_{k, k+\kappa})} < x \right).$$

PROOF. See Section 5. □

REMARK 1.9. *As in Theorems 1.5 and 1.3, the quantity (1.20) is expressible in terms of Painlevé transcendents. In particular (1.20) is given by*

$$(1.21) \quad F_k^{(\kappa)}(x) = \exp \left(- \int_x^\infty \frac{\sigma_{k, \kappa}^{(\text{V})}(s)}{s} ds \right)$$

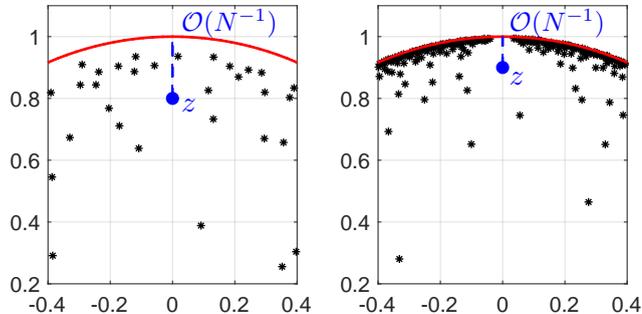


FIGURE 3. Schematic insertion of a charge z near the boundary in the truncated CUE ensemble. On the left $M = 200$ and $N = 180$, and on the right $M = 2000$ and $N = 1980$. The unit circle is depicted in red.

where $\sigma_{k,\kappa}^{(V)}(s)$ satisfies the equation (3.17) with parameters $k_2 = k$ and $\alpha = \kappa$. Via this representation, our expansion (1.19) has a natural interpretation for both non-integer k and non-integer κ .

Compared to Theorem 1.3, the most interesting point of departure concerns the change from Painlevé IV to Painlevé V appearing in the asymptotic expansion. A fundamental difference here is that the boundary forms a *hard edge*: eigenvalues of T are forbidden from leaving the unit disc and thus a different boundary behaviour can be expected. Properties of two-dimensional models of type (1.2) with a hard edge have been studied recently [7] and were shown to deviate from the boundary statistics known for the Ginibre ensemble.

For our final result on the truncated unitary ensemble, we state an analogue of (1.19) at finite N . In order to state the result, let W_1 and W_2 be two independent LUE matrices with p_1 and p_2 degrees of freedom, both of dimension k . The ratio of LUE matrices

$$(1.22) \quad J = W_1(W_1 + W_2)^{-1}$$

has eigenvalues supported on the interval $[0, 1]$. This set of eigenvalues is known as the *Jacobi Unitary Ensemble*, again see [34] for further background. We denote the largest eigenvalue of J by $\lambda_{\max}^{(\text{JUE}_k^{(a,b)})}$ where $a = p_1 - k$ and $b = p_2 - k$ is a standard notation for the parameters of the ensemble.

THEOREM 1.10 (Truncated CUE and Painlevé VI). *Let $k \in \mathbb{N}$. Then we have the exact identity*

$$(1.23) \quad \mathbb{E} \left(|\det(T - z)|^{2k} \right) = \frac{C_{M,N,k}}{(1 - |z|^2)^{k(M-N)+k^2}} \mathbb{P} \left(\lambda_{\max}^{\text{JUE}_k^{(M-N,N)}} < 1 - |z|^2 \right),$$

where

$$(1.24) \quad C_{M,N,k} = \prod_{j=0}^{k-1} \frac{\Gamma(M - N + 1 + j)\Gamma(N + 1 + j)}{\Gamma(M + 1 + j)\Gamma(1 + j)}.$$

PROOF. See Section 5. □

REMARK 1.11. *As with the previous cases involving GUE and LUE, one anticipates a Painlevé representation for (1.23). A third order equation was obtained in the same paper of Tracy and Widom [60], but this was not expressed in second order form and identified in terms of Painlevé transcendents until the later work [1, 19, 38, 46, 65]. These studies*

show, with a variety of approaches, that for $0 \leq x \leq 1$,

$$(1.25) \quad \mathbb{P} \left(\lambda_{\max}^{\text{JUE}_k^{(a,b)}} < x \right) = \exp \left(- \int_x^1 \frac{\sigma_{a,b}^{(\text{VI})}(t) - b_1 b_2 t + \frac{b_1 b_2 + b_3 b_4}{2}}{t(1-t)} dt \right),$$

The function $\sigma_{a,b}$ satisfies the Jimbo-Miwa-Okamoto σ -form of the Painlevé VI equation

$$(1.26) \quad \sigma'(t(1-t)\sigma'')^2 - \left(\sigma'(2\sigma + (1-2t)\sigma') + \prod_{i=1}^4 b_i \right)^2 + \prod_{i=1}^4 (\sigma' - b_i^2) = 0,$$

with parameters

$$(1.27) \quad b_1 = b_2 = k + \frac{a+b}{2}, \quad b_3 = \frac{a+b}{2}, \quad b_4 = \frac{b-a}{2}.$$

This continues to give an interpretation of the right-hand side of (1.23) even when k is not an integer. We will prove in this finite- N setting that the extrapolation to non-integer k in this fashion gives the correct result.

REMARK 1.12. In the limit $\kappa \rightarrow 0$ the matrix T reduces to a Haar distributed unitary matrix whose spectrum lies precisely on the unit circle. In that case the asymptotics of order N^{k^2} is close to the Keating and Snaith result used to conjecture the corresponding moments of the Riemann zeta function [53]. In this purely unitary case $\kappa = 0$ the association with Painlevé VI and the boundary scaling in terms of Painlevé V was discussed in [38]. More recently, Painlevé V arose for boundary asymptotics of derivatives of CUE characteristic polynomials [11].

1.4. Multiple charges. In this section we state our results for the correlation function (1.1) for $m > 2$. At the edge, we found a natural generalization of the GUE type Theorem 1.3 which replaces the probability (1.9) with an object related to non-intersecting Brownian motion.

Recall that the transition density of a standard Brownian motion on \mathbb{R} , leaving from u and arriving at v is

$$(1.28) \quad p_t(u, v) = \frac{1}{\sqrt{2\pi t}} e^{-(u-v)^2/(2t)}.$$

Now consider k non-intersecting Brownian motions on \mathbb{R} , leaving from positions (u_1, \dots, u_k) at time 0 and arriving at (v_1, \dots, v_k) at time 1. The probability that all particles belong to the set E at time t is, up to a normalizing factor, given by the Karlin-McGregor formula (see [52] or [3, Sec. 7.4])

$$(1.29) \quad \mathcal{Z}_t(\vec{u}, \vec{v}, E) = \int_{E^k} \det \left\{ p_t(u_i, x_j) \right\}_{i,j=1}^k \det \left\{ p_{1-t}(x_i, v_j) \right\}_{i,j=1}^k d\vec{x}.$$

Of particular interest here will be the case $E = \mathbb{R}_+$ which corresponds to the probability that the smallest particle is positive at time t .

THEOREM 1.13. Consider the boundary scaling

$$(1.30) \quad x_j = z + \frac{u_j}{\bar{z}\sqrt{N}}, \quad \bar{y}_j = \bar{z} + \frac{\bar{v}_j}{z\sqrt{N}}, \quad j = 1, \dots, k.$$

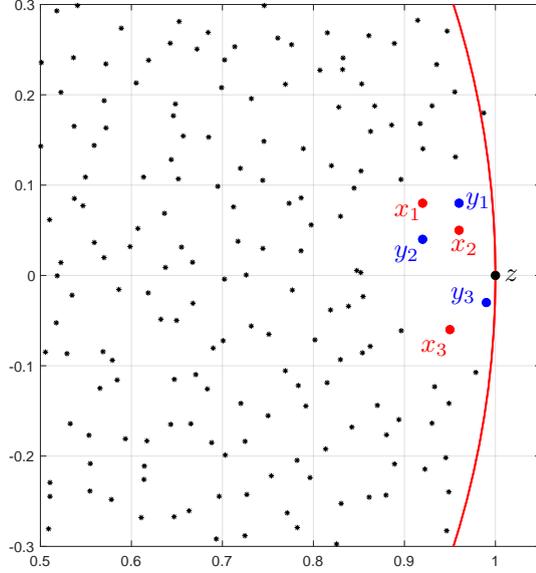


FIGURE 4. Schematic insertion of two families of charges (with $k = 3$) near the point $z = 1$ on the boundary in the complex Ginibre ensemble.

where z is a fixed point on the boundary, $|z| = 1$, with fixed complex vectors $\vec{u}, \vec{v} \in \mathbb{C}^k$, which may contain degeneracies. Then we have the following asymptotic expansion,

$$(1.31) \quad \mathbb{E} \left(\prod_{j=1}^k \det(G_N - x_j) \det(G_N^\dagger - \bar{y}_j) \right) = \exp \left(- \sum_{j=1}^k \left(\sqrt{N}(u_j + \bar{v}_j) - \frac{u_j^2 + \bar{v}_j^2}{2} \right) + \frac{k^2}{2} \log(N) \right) \\ \times \frac{(2\pi)^k}{k!} F_k^{(\text{edge})}(\vec{u}, \vec{v}) (1 + o(1)), \quad N \rightarrow \infty.$$

where

$$(1.32) \quad F_k^{(\text{edge})}(\vec{u}, \vec{v}) = \frac{\mathcal{Z}_{1/2}(\vec{u}, \vec{v}, \mathbb{R}_+)}{\prod_{1 \leq i < j \leq k} (u_j - u_i)(\bar{v}_j - \bar{v}_i)}$$

In the event that the vectors \vec{u} or \vec{v} contain degeneracies, the expansion (1.31) continues to hold by taking appropriate limits in (1.32).

PROOF. See Section 3.4. □

To obtain (1.1), we have to merge entries of the vectors \vec{u} and \vec{v} into several distinct families and eventually take $\vec{u} = \vec{v}$ in the ratio (1.32). In the Brownian motion interpretation, the confluence of terminal and initial points is well studied and has appeared several times in the literature. For example, as in [3, Section 7.4] we can consider m families $u_i^{(1)}, \dots, u_i^{(k_i)}$ and $v_i^{(1)}, \dots, v_i^{(k_i)}$ for $i = 1, \dots, m$ each of which are merged to a single point a_i and b_i respectively for each $i = 1, \dots, m$ with $k_1 + \dots + k_m = k$, finally setting $a_i = b_i$. Thus we are considering k non-intersecting Brownian motions with k_i paths starting at a_i and

terminating at b_i . This results in the block structure

$$(1.33) \quad \mathcal{Z}_t(\vec{u}, \vec{v}, E) \sim c_k(\vec{u}, \vec{v}) \int_{(tE)^k} \prod_{j=1}^k e^{-\frac{tx_j^2}{2(1-t)}} \\ \times \det \begin{pmatrix} \left\{ x_j^{i-1} e^{a_1 x_j} \right\}_{i=1, j=1}^{k_1, k} \\ \vdots \\ \left\{ x_j^{i-1} e^{a_m x_j} \right\}_{i=1, j=1}^{k_m, k} \end{pmatrix} \times \det \begin{pmatrix} \left\{ x_j^{i-1} e^{b_1 x_j} \right\}_{i=1, j=1}^{k_1, k} \\ \vdots \\ \left\{ x_j^{i-1} e^{b_m x_j} \right\}_{i=1, j=1}^{k_m, k} \end{pmatrix} d\vec{x}.$$

Although the vectors \vec{u}, \vec{v} should be real for this probabilistic interpretation, the formulae (1.32) and (1.33) continue to make sense for complex values. Precisely this type of block structure has appeared in the context of m -component KP hierarchy and multiple Hermite polynomials [3, 27, 28]. The recent work [57] studied the relation between the planar orthogonal polynomials associated with (1.1) and type II multiple orthogonal polynomials, but only at the finite- N level. It would be interesting to understand if the multiple orthogonality known to be associated with (1.33) is related. In the particular case $m = 1$ and $p_1 = k$, one has k Brownian motions all starting from a_1 : it is straightforward to show that this recovers precisely the GUE probability (1.9). At the level of a differential equation analogous to (1.11), it has been shown in a series of works of Adler, Van Moerbeke and collaborators that (1.33) satisfies a system of m non-linear PDEs where the $\{p_i\}_{i=1}^m$ are parameters and can in principle be interpreted for non-integer values, see [3] and references therein.

Finally we discuss the generalization of Theorem 1.5 to multiple charges colliding in the bulk.

THEOREM 1.14. *Fix $\delta > 0$ and take z inside the unit disc $|z| < 1 - \delta$. Consider points $\{z_i\}_{i=1}^k$ and $\{w_i\}_{i=1}^k$ centered and scaled*

$$(1.34) \quad z_i = z + \frac{u_i}{\sqrt{N}}, \quad \bar{w}_i = \bar{z} + \frac{\bar{v}_i}{\sqrt{N}},$$

for $i = 1, \dots, k$ where the complex vectors \vec{u} and \vec{v} are fixed, but may contain degeneracies. Then we have the following asymptotic expansion

$$(1.35) \quad \mathbb{E} \left(\prod_{i=1}^k \det(G_N - z_i) \det(G_N^\dagger - \bar{w}_i) \right) = \exp \left(Nk(|z|^2 - 1) + \sqrt{N} \sum_{i=1}^k (z\bar{v}_i + \bar{z}u_i) + \frac{k^2}{2} \log(N) \right) \\ \times \frac{(2\pi)^{k/2}}{G(1+k)} F_k^{(\text{bulk})}(\vec{u}, \vec{v}) (1 + o(1)), \quad N \rightarrow \infty,$$

in terms of the following integral over the group $U(k)$ of $k \times k$ unitary matrices,

$$(1.36) \quad F_k^{(\text{bulk})}(\vec{u}, \vec{v}) = \int_{U(k)} e^{\text{Tr}(UAU^\dagger \bar{B})} d\mu(U)$$

where $d\mu(U)$ is the normalized Haar measure on $U(k)$ and

$$(1.37) \quad A = \text{diag}(u_1, \dots, u_k), \quad \bar{B} = \text{diag}(\bar{v}_1, \dots, \bar{v}_k).$$

PROOF. See Section 4. □

The integral appearing in (1.36) is the famous Harish-Chandra Itzykson Zuber integral. As we discuss in Section 4 there is a well-known explicit formula for this integral [49]. However, in order to obtain (1.1) or the result in Theorem 1.5, a rather delicate merging

procedure is required and the explicit formula is not always easy to use in this case. We explain how to overcome this in Section 4.

REMARK 1.15. *A remark about universality: Our proof of the aforementioned results relies essentially only on local asymptotics of the relevant correlation kernel, either close to the boundary or in the bulk (in contrast to the approaches in [51, 58] which exploit the Gaussian structure of the Ginibre ensemble). For the more general model (1.2), the correlations in the bulk and at the edge are now known to be strongly independent of the potential $V(\lambda)$ (universal) up to some restrictions on the regularity of $V(\lambda)$ and the droplet. This has been accomplished at the edge in the recent development [47] and at the bulk in [9]. Indeed, considering Theorem 1.3 for example, based on the results in [47] we expect that the same Painlevé IV object describes the collision of a point charge with the boundary of the droplet for a broad class of normal matrix models. In a different direction, bulk universality for correlations of non-Hermitian characteristic polynomials was recently investigated in [4] where a formula essentially equivalent to (1.35) was derived for a class of independent entry random matrices.*

To summarize, we tabulate below the results described in this section. Each Theorem can be viewed as a certain duality mapping the planar problem under consideration to objects typically encountered in the study of Hermitian or unitary matrices. The last column indicates the associated Painlevé system.

Microscopic behaviour of the correlation function (1.1)				
Planar model	m	Regime	Duality	Painlevé
\mathbb{C} -Ginibre	1	Finite- N	LUE	V
—”—	1	Edge, $N \rightarrow \infty$	GUE	IV
—”—	2	Bulk, $N \rightarrow \infty$	LUE	V
—”—	$m > 2$	Edge, $N \rightarrow \infty$	Non-intersecting paths	
—”—	$m > 2$	Bulk, $N \rightarrow \infty$	Itzykson Zuber integral	
T-CUE	1	Finite- N	JUE	VI
—”—	1	Edge, $N \rightarrow \infty$	LUE	V

TABLE 1. The above table summarises the main results of the present work.

The structure of this paper is as follows. In Section 2 we recall some well known facts about the normal matrix model (1.2) and its relation to determinantal structures and orthogonal polynomials in the complex plane. Then in Section 3 we compute the one point correlation (object (1.1) with $m = 1$) at finite N and discuss its relation to Painlevé V for finite N and Painlevé IV near the boundary, proving Theorem 1.3. These boundary asymptotics are then used to give a conjecture for the critical asymptotics of the partition function (1.3) with potential (1.4). Section 4 is devoted to proving the results about bulk asymptotics and merging of singularities namely Theorems 1.5, 1.7 and 1.14. Finally Section 5 is devoted to studying the 1-point correlation for the truncated unitary ensemble, its relation to Painlevé VI and boundary asymptotics in terms of Painlevé V.

2. Planar orthogonal polynomials and determinantal structures

In this section we set some basic notation for the paper and recall some known facts regarding the normal matrix model and planar orthogonal polynomials. Consider the normal matrix model corresponding to a weight $w(\lambda)$ as defined in (1.2), where $w(\lambda) = e^{-NV(\lambda)} \geq 0$. From the weight $w(\lambda)$ we may construct a unique family of degree k monic polynomials $p_k(\lambda)$, $k = 0, 1, 2, \dots$ satisfying the planar orthogonality

$$(2.1) \quad \int_{\mathbb{C}} p_k(\lambda) \overline{p_j(\bar{\lambda})} w(\lambda) d^2\lambda = \delta_{k,j} h_k$$

and these polynomials can be used to characterize statistical properties of (1.2). A distinguished role is played by a function of two variables known as the *correlation kernel* of the model, defined by

$$(2.2) \quad K_N(\lambda_1, \bar{\lambda}_2) = \sqrt{w(\lambda_1)w(\lambda_2)} \sum_{j=0}^{N-1} \frac{p_j(\lambda_1) \overline{p_j(\bar{\lambda}_2)}}{h_j}.$$

For example, the k -point eigenvalue correlation functions corresponding to (1.2) are given explicitly in terms of (2.2),

$$(2.3) \quad \begin{aligned} R_k(\lambda_1, \dots, \lambda_k) &:= \frac{N!}{(N-k)!} \int_{\mathbb{C}^{N-k}} P(\lambda_1, \dots, \lambda_N) d^2\lambda_{k+1} \dots d^2\lambda_N \\ &= \det \left\{ K_N(\lambda_i, \bar{\lambda}_j) \right\}_{i,j=1}^k, \end{aligned}$$

which expresses the fact that eigenvalues of normal random matrices form a determinantal point process with kernel (2.2). The partition function (1.3) is also expressed in terms of orthogonal polynomials, via the norming constants in (2.1) we have

$$(2.4) \quad \mathcal{Z}_N = N! \det \left\{ \int_{\mathbb{C}} \lambda^i \bar{\lambda}^j w(\lambda) d^2\lambda \right\}_{i,j=0}^{N-1} = N! \prod_{j=0}^{N-1} h_j.$$

A simple but very useful fact we will exploit throughout the paper is that if $w(\lambda)$ is rotationally invariant, i.e. $w(\lambda) = w(|\lambda|)$, then the planar orthogonal polynomials are the monomials, i.e. $p_j(\lambda) = \lambda^j$ for each non-negative integer j .

The kernel (2.2) is also relevant for the computation of our central object of interest (1.1), as implied by the following result. In what follows, and in the rest of the paper, we will use the notation

$$(2.5) \quad \Delta(\vec{x}) := \prod_{1 \leq i < j \leq k} (x_k - x_i) = \det \left\{ x_i^{j-1} \right\}_{i,j=1}^k$$

to denote the Vandermonde determinant in the entries of the vector \vec{x} , whose dimension may vary depending on the context.

THEOREM 2.1 (Akemann and Vernizzi [5]). *Consider the polynomial kernel*

$$(2.6) \quad B_N(x, \bar{y}) := \sum_{j=0}^{N-1} \frac{p_j(x) \overline{p_j(\bar{y})}}{h_j} = K_N(x, \bar{y}) [w(x)w(y)]^{-1/2}$$

and generic complex numbers $\{x_j\}_{j=1}^k$ and $\{y_j\}_{j=1}^k$. The following exact identity holds

$$(2.7) \quad \mathbb{E} \left(\prod_{j=1}^N \prod_{i=1}^k (\lambda_j - x_i) (\bar{\lambda}_j - \bar{y}_i) \right) = \frac{\det \left\{ B_{N+k}(x_i, \bar{y}_j) \right\}_{i,j=1}^k}{\Delta(\vec{x}) \overline{\Delta(\vec{y})}} \prod_{j=0}^{k-1} h_{j+N}$$

where the expectation is taken with respect to (1.2).

The idea behind this result goes back to Brézin and Hikami [22] who developed it in the context of Hermitian random matrices. In order to reproduce averages of the type (1.1), we have to merge some of the points $\{x_j\}_{j=1}^k, \{y_j\}_{j=1}^k$ which is easily done with L'Hôpital's rule and basic facts for differentiating determinants. For example, in the case $m = 1$ of (1.1), we have to merge together all points $x_i \rightarrow x$ and $y_i \rightarrow y$ for all $i = 1, \dots, m$ which gives

$$(2.8) \quad \frac{\det \left\{ A(x_i, y_j) \right\}_{i,j=1}^k}{\Delta(\vec{x})\Delta(\vec{y})} \rightarrow \left(\prod_{j=0}^{k-1} \frac{1}{(j!)^2} \right) \det \left\{ \frac{\partial^{i+j-2} A(x, y)}{\partial x^{i-1} \partial y^{j-1}} \right\}_{i,j=1}^k$$

for any smooth function of two variables $A(x, y)$. This representation also turns out to be useful for studying asymptotics, as it reduces the analysis to that of the two-variable kernel (2.6) and its derivatives. In order to calculate (or identify) the determinant on the right-hand side of (2.8) we will often exploit the following.

LEMMA 2.2 (Andréief identity). *For a domain D and two sets of integrable functions $\{f_j(t)\}_{j=1}^k$ and $\{g_j(t)\}_{j=1}^k$ we have*

$$(2.9) \quad \det \left\{ \int_D f_i(t) g_j(t) dt \right\}_{i,j=1}^k = \frac{1}{k!} \int_{D^k} \det \{f_j(t_i)\}_{i,j=1}^k \det \{g_j(t_i)\}_{i,j=1}^k d\vec{t}$$

3. The Ginibre case: finite- N and boundary asymptotics

In this section we begin by discussing in some detail the object (1.1) with $m = 1$, that is the moments

$$(3.1) \quad R_{2k}(z) = \mathbb{E} \left(|\det(G_N - z)|^{2k} \right),$$

where G_N is a standard complex Ginibre random matrix. Of interest will be the relation to Painlevé transcendents at finite- N and the asymptotics $N \rightarrow \infty$. In particular we will prove Theorem 1.3 and discuss the problem of continuing k off the integers. Then we apply the obtained results to the lemniscate partition function. Finally, we extend the considerations to multiple products of characteristic polynomials and prove Theorem 1.13.

3.1. Duality with the LUE. Recall the definition of the Laguerre Unitary Ensemble from the introduction: form the matrix $W = G_{k,p} G_{k,p}^\dagger$ where $G_{k,p}$ is a $k \times p$ matrix of standard complex Gaussian random variables and we assume $p \geq k$. The eigenvalues of t_1, \dots, t_k of W have a well-known joint probability density function proportional to

$$(3.2) \quad P(t_1, \dots, t_k) \propto \prod_{j=1}^k t_j^\alpha e^{-t_j} \Delta^2(\vec{t}).$$

where $\alpha = p - k$ is referred to as the *parameter* of the LUE and k as the size, see [34] for further details.

PROPOSITION 3.1 ([35, 58]). *The following exact duality identity holds:*

$$(3.3) \quad R_{2k}(z) = \frac{N^{k^2/2}}{\prod_{j=0}^{k-1} j!(j+1)!} \int_{[0, \infty)^k} \prod_{j=1}^k e^{-\sqrt{N}t_j} \left(|z|^2 + \frac{t_j}{\sqrt{N}} \right)^N \Delta^2(\vec{t}) d\vec{t}$$

$$(3.4) \quad = N^{-Nk} e^{Nk|z|^2} \left(\prod_{j=1}^k \frac{\Gamma(j+N)}{\Gamma(j)} \right) \mathbb{P}(\lambda_1 > N|z|^2)$$

where λ_1 is the smallest eigenvalue in the Laguerre Unitary Ensemble with parameter $\alpha = N$ and size k .

To our knowledge the identity (3.3) first appeared in [58] where it was derived with Grassmann integration techniques and was then generalized by Forrester and Rains in [35] using a symmetric functions approach. Similar dualities involving positive definite matrices have appeared in the work [43], see also [41] for a different generalization. We remark that the mentioned approaches appear to be quite specific to the Ginibre setting. In the following we will give a different proof using the general formula (2.7).

PROOF OF PROPOSITION 3.1. Starting with identity (2.7), the weight in the Ginibre ensemble² is $w(\lambda) = e^{-|\lambda|^2}$ and the corresponding planar orthogonal polynomials are $p_j(\lambda) = \lambda^j$. The orthogonality norms follow from an explicit integration as $h_j = \pi j!$ and this gives rise to the usual Ginibre finite- N kernel

$$(3.5) \quad B_{N+k}(x, y) = \frac{1}{\pi} \sum_{j=0}^{N+k-1} \frac{(xy)^j}{j!} = \frac{x^{N+k}}{\pi \Gamma(N+k)} \int_0^\infty e^{-tx} (y+t)^{N+k-1} dt,$$

where, for simplicity (and without loss of generality), we have assumed $x > 0$ and $y > 0$. The Ginibre kernel has been written in this way to facilitate computing its partial derivatives, as in (2.8). Since the multiplicative factors in (3.5) drop out of the determinant, it is enough to calculate $\det\{A(x_i, y_j)\}_{i,j=1}^k$ where

$$(3.6) \quad A(x, y) = \int_0^\infty e^{-tx} (y+t)^{N+k-1} dt.$$

Now we merge the points $x_1, \dots, x_k \rightarrow x$ and $y_1, \dots, y_k \rightarrow y$. Differentiating and applying Lemma 2.2 yields

$$(3.7) \quad \begin{aligned} & \det \left\{ \frac{\partial^{i+j} A(x, y)}{\partial x^i \partial y^j} \right\}_{i,j=0}^{k-1} = c_{N,k} \det \left\{ \int_0^\infty t^{i-1} e^{-tx} (y+t)^{N+k-j} dt \right\}_{i,j=1}^k \\ & = \frac{c_{N,k}}{k!} \int_{[0,\infty)^k} \prod_{j=1}^k e^{-t_j x} (y+t_j)^N \det \left\{ t_i^{j-1} \right\}_{i,j=1}^k \det \left\{ (y+t_i)^{k-j} \right\}_{i,j=1}^k d\vec{t} \\ & = \frac{c_{N,k} (-1)^{\frac{k(k-1)}{2}}}{k!} \int_{[0,\infty)^k} \prod_{j=1}^k e^{-xt_j} (y+t_j)^N \Delta^2(\vec{t}) d\vec{t}, \end{aligned}$$

where

$$(3.8) \quad c_{N,k} = \Gamma(N+k)^k (-1)^{\frac{k(k-1)}{2}} \prod_{j=1}^k \frac{1}{\Gamma(N+k-(j-1))}.$$

Collecting the multiplicative factors in (3.5) and the product of norms in (2.7) gives

$$(3.9) \quad \begin{aligned} & \mathbb{E} \left(\det(G-x)^k \det(G^\dagger - y)^k \right) \\ & = \left(\prod_{j=1}^k \frac{1}{(j-1)! j!} \right) \int_{[0,\infty)^k} \prod_{j=1}^k e^{-t_j} (xy+t_j)^N \Delta^2(\vec{t}) d\vec{t}, \end{aligned}$$

where we made the change of variables $t_j \rightarrow t_j/x$. As an identity between polynomials in x and y , (3.9) now holds for any $x, y \in \mathbb{C}$. Identity (3.3) follows from the rescalings $x \rightarrow \sqrt{N}x$, $y \rightarrow \sqrt{N}y$ and $t_j \rightarrow \sqrt{N}t_j$. \square

²Here it is convenient to begin with the N -independent weight, to obtain (3.3) the factor of N can be easily restored at the end of the calculation.

Now we will prove Theorem 1.3.

PROOF OF THEOREM 1.3. As our starting point we consider the duality formula (3.3). Changing variable $t_j \rightarrow t_j + \sqrt{N}(1 - |z|^2)$ for each $j = 1, \dots, k$ transforms the integral to

$$(3.10) \quad e^{Nk(|z|^2-1)} \int_{(-\sqrt{N}(1-|z|^2), \infty)^k} \prod_{j=1}^k \exp\left(-\sqrt{N}t_j + N \log\left(1 + t_j/\sqrt{N}\right)\right) \Delta^2(\vec{t}) d\vec{t}.$$

Now notice that pointwise, we have the limit

$$(3.11) \quad \exp\left(-\sqrt{N}t_j + N \log\left(1 + t_j/\sqrt{N}\right)\right) \rightarrow e^{-t_j^2/2}, \quad N \rightarrow \infty.$$

The claim is that in fact the following estimate is uniform provided z stays inside a disc of radius $1 + c/\sqrt{N}$ for some constant c ,

$$(3.12) \quad \begin{aligned} & \int_{(-\sqrt{N}(1-|z|^2), \infty)^k} \prod_{j=1}^k \exp\left(-\sqrt{N}t_j + N \log\left(1 + t_j/\sqrt{N}\right)\right) \Delta^2(\vec{t}) d\vec{t} \\ &= \int_{(-\sqrt{N}(1-|z|^2), \infty)^k} \prod_{j=1}^k e^{-t_j^2/2} \Delta^2(\vec{t}) d\vec{t} \times (1 + o(1)), \quad N \rightarrow \infty \\ &= \mathcal{Z}_k^{(\text{GUE})} \mathbb{P}\left(\lambda_{\max}^{(\text{GUE}_k)} < \sqrt{N}(|z|^2 - 1)\right) \times (1 + o(1)), \quad N \rightarrow \infty. \end{aligned}$$

To pass to the final equality we simply recognised the joint density of GUE eigenvalues, up to a normalization factor given explicitly by

$$(3.13) \quad \mathcal{Z}_k^{(\text{GUE})} := \int_{\mathbb{R}^k} \prod_{j=1}^k e^{-t_j^2/2} \Delta^2(\vec{t}) d\vec{t} = (2\pi)^{k/2} \prod_{j=0}^{k-1} (j+1)!.$$

Inserting these results into (3.3) proves Theorem 1.3 after recalling that $G(1+k) = \prod_{j=0}^{k-1} j!$. It remains to verify the uniform asymptotics in the first equality of (3.12). To this end, observe that the magnitude of the resulting error in the *difference* between the two integrals is largest when $z = 0$ (since z only enters through the region of integration, and this is maximized when $z = 0$). The condition $|z| \leq 1 + c/\sqrt{N}$ ensures that the lower limit of integration remains bounded from above and permits the multiplicative error bound in (3.12). Hence the verification of (3.12) reduces to showing that

$$(3.14) \quad \begin{aligned} & \lim_{N \rightarrow \infty} \int_{(-\sqrt{N}, \infty)^k} \prod_{j=1}^k \exp\left(-\sqrt{N}t_j + N \log\left(1 + \frac{t_j}{\sqrt{N}}\right)\right) \Delta^2(\vec{t}) d\vec{t} \\ &= \int_{\mathbb{R}^k} \prod_{j=1}^k e^{-t_j^2/2} \Delta^2(\vec{t}) d\vec{t} \end{aligned}$$

and this can be proved with a dominated convergence argument, for example using the pointwise limit (3.11) and the bound

$$(3.15) \quad -\sqrt{N}t + N \log\left(1 + \frac{t}{\sqrt{N}}\right) = -\int_0^t \frac{s ds}{1 + \frac{s}{\sqrt{N}}} \leq -\frac{t^2}{2(1+|t|)}, \quad t > -\sqrt{N}.$$

□

REMARK 3.2. If $\sqrt{N}(|z|^2 - 1)$ is not bounded from above, this implies that z is too far from the eigenvalue support for our approximation to hold, and we are no longer guaranteed the desired multiplicative error $(1 + o(1))$. In fact the asymptotic expansion in Theorem 1.3 takes a different form if z is far outside the unit disc, see Appendix A. In the case $k = 1$

(3.3) becomes a one-dimensional integral related to a truncated exponential series, where the uniformity of various asymptotic approximations has been studied. In the work [21], a reference to an unpublished approximation of Jet Wimp can be used to give an analogue of Theorem 1.3 (but only for $k = 1$) which is uniform on the entire complex plane.

3.2. Reduction to integrals over $U(N)$ for general γ and Painlevé V. As explained in the introduction, it is interesting to consider extrapolating Theorem 1.3 to non-integer values of k . We will now consider this problem, starting with the second representation (3.4) at finite- N . The work of Tracy and Widom [60] implies the representation

$$(3.16) \quad \mathbb{P}(\lambda_1 > x) = \exp\left(\int_0^x \frac{\sigma_{k,\alpha}(t)}{t} dt\right)$$

where $\sigma_{k,\alpha}(t)$ satisfies the Jimbo-Miwa-Okamoto σ -form of the Painlevé V equation,

$$(3.17) \quad (t\sigma'')^2 - [\sigma - t\sigma' + 2(\sigma')^2 + (2k + \alpha)\sigma']^2 + 4(\sigma')^2(k + \alpha + \sigma')(k + \sigma') = 0$$

with boundary condition

$$(3.18) \quad \sigma_k(t) \sim -t^{N+1} \frac{\Gamma(N+1+k)}{\Gamma(N+2)\Gamma(N+1)\Gamma(k)}, \quad t \rightarrow 0.$$

Similarly, for the largest eigenvalue we have

$$(3.19) \quad \mathbb{P}(\lambda_{\max} < x) = \exp\left(-\int_x^\infty \frac{\sigma_{k,\alpha}(t)}{t} dt\right).$$

As k now plays the role of a *parameter* in the differential equation (3.17), it can be considered for non-integer values. We now consider the left-hand side of (3.3) for non-integer k and show that it coincides with the extrapolation given by (3.17). Via a somewhat different route, this has also been demonstrated in the work of Kanzieper [51] in the context of replica field theories. However, as the association of (3.1) with Painlevé V seems not well known, we will give a different proof below. The proof exploits a Toeplitz determinant identity obtained rather recently in the work [64] where the identification in terms of Painlevé V did not appear. For completeness, we sketch the main ideas in [64] which lead to the following.

THEOREM 3.3. *Let $\gamma > -2$ and let $d\mu(U)$ denote the normalized Haar measure on the group of $N \times N$ unitary matrices. Then we have the identity*

$$(3.20) \quad \begin{aligned} R_\gamma(z) &= R_\gamma(0) \int_{U(N)} \det(U)^{-\gamma/4} |\det(I+U)|^{\gamma/2} \exp(N|z|^2 \text{Tr}(U)) d\mu(U) \\ &= \frac{R_\gamma(0)}{N!(2\pi)^N} \int_{[-\pi, \pi]^N} \prod_{j=1}^N e^{-i\gamma\theta_j/4} |1 + e^{i\theta_j}|^{\gamma/2} e^{N|z|^2 e^{i\theta_j}} |\Delta(e^{i\theta})|^2 d\vec{\theta} \end{aligned}$$

where the constant is

$$(3.21) \quad R_\gamma(0) = N^{-\frac{\gamma N}{2}} \prod_{j=0}^{N-1} \frac{\Gamma(\frac{\gamma}{2} + j + 1)}{\Gamma(j + 1)}.$$

Furthermore, we have the representation in terms of solutions of (3.17),

$$(3.22) \quad R_\gamma(z) = R_\gamma(0) e^{N|z|^2 \gamma/2} \exp\left(\int_0^{N|z|^2} \frac{\sigma_{\gamma/2, N}(t)}{t} dt\right).$$

PROOF. Since $k = \gamma/2$ is no longer an integer, clearly the formula (2.7) cannot be used directly. Instead we apply identity (2.4) and work with the planar polynomials $p_j(\lambda)$ orthogonal with respect to the weight $|\lambda - z|^\gamma e^{-N|\lambda|^2}$ with $\lambda \in \mathbb{C}$. Note that we also cannot use the previous trick to compute $p_j(\lambda)$ because the weight is not radial. The

fundamental idea is to use Green's theorem to reduce the planar orthogonality of $p_j(\lambda)$ to orthogonality on a suitable contour, which can then be deformed to the unit circle. In the present context this idea goes back to [13]. Applying this strategy (see [64, Lemma 2.4] for full details) yields, for $k \leq j$

$$(3.23) \quad \begin{aligned} h_j \delta_{j,k} &= \int_{\mathbb{C}} p_j(\lambda) \bar{z}^k |z - \lambda|^\gamma e^{-N|\lambda|^2} d^2\lambda \\ &= \frac{\pi \Gamma(\frac{\gamma}{2} + k + 1)}{N^{\frac{\gamma}{2} + k + 1}} \oint_{\Sigma} p_j(\lambda) \lambda^{-k - \gamma/2} (\lambda - |z|)^{\gamma/2} e^{-N|z|\lambda} \frac{d\lambda}{2\pi i \lambda} \end{aligned}$$

where the roots are defined using the principal branch. The contour Σ is any closed curve encircling the interval $[0, |z|]$, possibly passing through $|z|$, but no other points of $[0, |z|]$, and oriented counterclockwise; let us take $\Sigma = |z|S^1$ where S^1 is the unit circle. The change of variable $\lambda \rightarrow |z|\lambda$ implies that we get norms on the unit circle of the form

$$(3.24) \quad \begin{aligned} \tilde{h}_j \delta_{j,k} &:= \oint_{\Sigma} p_j(\lambda) \lambda^{-k - \gamma/2} (\lambda - |z|)^{\gamma/2} e^{-N|z|\lambda} \frac{d\lambda}{2\pi i \lambda} \\ &= \oint_{S^1} \tilde{p}_j(\lambda) \lambda^{-k - \gamma/2} |\lambda + 1|^{\gamma/2} e^{N|z|^2\lambda} \frac{d\lambda}{2\pi i \lambda} \end{aligned}$$

where $\tilde{p}_j(\lambda)$ is another family of degree j monic polynomials. To get the second equality in (3.24) we exploited the fact that writing $\lambda = e^{i\theta}$, with $\theta \in [-\pi, \pi)$ we have the identity

$$(3.25) \quad \lambda^{-\gamma/2} (\lambda - 1)^{\gamma/2} = e^{i \operatorname{sgn}(\theta) \gamma \pi / 4} \lambda^{-\gamma/4} |\lambda - 1|^{\gamma/2}, \quad \theta \in (-\pi, \pi)$$

and scaled $\lambda \rightarrow -\lambda$. Now applying (2.4) in reverse we get the expression as a Toeplitz determinant

$$(3.26) \quad \begin{aligned} R_\gamma(z) &= \frac{N! \prod_{j=0}^{N-1} h_j}{\mathcal{Z}_N^{(\operatorname{Gin})}} = N^{-\frac{\gamma N}{2}} \prod_{j=0}^{N-1} \frac{\Gamma(\frac{\gamma}{2} + j + 1)}{\Gamma(j + 1)} \prod_{j=0}^{N-1} \tilde{h}_j \\ &= R_\gamma(0) \det \left\{ \oint_{S^1} \lambda^{j-i-\gamma/4} |\lambda + 1|^{\gamma/2} e^{N|z|^2\lambda} \frac{d\lambda}{2\pi i \lambda} \right\}_{i,j=0}^{N-1}. \end{aligned}$$

Apart from the simple identity (3.25), the determinant (3.26) is of the same form found in [64, Lemma 2.5]. Now (3.20) follows from Heine's identity, expressing the above Toeplitz determinant as an integral over the unitary group. To arrive at (3.22) we simply observe that the quantities on the right-hand side of (3.20) have known associations with Painlevé V, see for example [2, Proposition 4.2] and [37, Eq. (1.46)] where these group integrals appear explicitly and are identified in terms of solutions of the Painlevé V equation. \square

REMARK 3.4. *We make some observations implied by the above calculation. The polynomials appearing in (3.24) (orthogonal on the unit circle) have been studied in relation to Riemann-Hilbert problems in the works [39, 40], which involves Painlevé V. We can therefore make the observation that, as implied by (3.24), up to some explicit constants the results in [39, 40] also apply to the corresponding planar orthogonal polynomials. At the level of asymptotics, the recurrence coefficients for these polynomials were studied in [29] using Riemann-Hilbert techniques related to a Painlevé IV parametrix.*

REMARK 3.5. *In the case $\gamma = 2k$, the Toeplitz determinant (3.26) can be written as the generating function of certain combinatorial objects, associated with the longest increasing subsequence problem [61]. The appearance of such generating functions in the present context of the complex Ginibre ensemble seems to be a new observation. The equivalence of (3.20) and (3.3) for $\gamma = 2k$ implies an interesting duality between averages over the unitary group and over the Laguerre ensemble of positive definite matrices. This duality appeared from a different point of view in [37, Proposition 3.9], which contains further discussion based on hypergeometric functions of matrix argument.*

We will now show that the appearance of Painlevé IV in Theorem 1.3 can be justified for non-integer exponents $k = \gamma/2$, at least to a physical level of rigour. The idea is to take the representation (3.22) and rescale the differential equation (3.17) near the boundary, setting $|z|^2 = 1 - \frac{2u}{\sqrt{N}}$ with u fixed. First notice that the pre-factor give the main terms in the asymptotic expansion of Theorem 1.3:

$$(3.27) \quad R_\gamma(0) = N^{-\frac{\gamma N}{2}} \frac{G(\frac{\gamma}{2} + N + 1)}{G(\frac{\gamma}{2} + 1)G(N + 1)} \sim e^{-\frac{\gamma}{2}N} N^{\frac{\gamma^2}{8}} (2\pi)^{\frac{\gamma}{4}} (1 + o(1)), \quad N \rightarrow \infty.$$

Inserting this into (3.22) and changing variables $t = N(1 - s/\sqrt{N})$, we obtain

$$(3.28) \quad R_\gamma(z) = e^{\frac{N\gamma}{2}(|z|^2 - 1)} \frac{N^{\frac{\gamma^2}{8}} (2\pi)^{\frac{\gamma}{4}}}{G(1 + \frac{\gamma}{2})} \exp\left(\int_{2u}^{\sqrt{N}} \frac{N^{-1/2}\sigma(N(1 - s/\sqrt{N}))}{1 - s/\sqrt{N}} ds\right) (1 + o(1)), \quad N \rightarrow \infty.$$

Now we claim that the function $v(s) = -N^{-1/2}\sigma(N(1 - s/\sqrt{N}))$ has a limit. The equation satisfied by $v(s)$ is

$$(3.29) \quad (\sqrt{N}(1 - s/\sqrt{N})v'')^2 - [-\sqrt{N}v - N(1 - s/\sqrt{N})v' + 2(v')^2 + (\gamma + N)v']^2 + 4(v')^2(N + \gamma/2 + v')(\gamma/2 + v') = 0$$

Equating terms of order N (leading order) in the above equation, we obtain a limiting equation

$$(3.30) \quad (v'')^2 + 4(v')^2(v' + \gamma/2) - (v - sv')^2 = 0$$

which is precisely equation (1.11) of Theorem 1.3 with $k = \gamma/2$.

3.3. Application to the lemniscate ensemble. The partition function of the lemniscate particle system is defined by

$$(3.31) \quad \mathcal{Z}_N^{(\text{Lem}_d)}(t) := \int_{\mathbb{C}^N} \prod_{j=1}^N e^{-N(|\lambda_j|^{2d} - t(\lambda_j^d + \bar{\lambda}_j^d))} |\Delta(\vec{\lambda})|^2 d\vec{\lambda}, \quad d \in \mathbb{N}, \quad t \in \mathbb{R}.$$

In this section we will discuss this partition function for finite N , its relation to Painlevé transcendents and asymptotic expansions in the three regimes $t < t_c$, $t = t_c$ and $t > t_c$ (recall that $t_c := d^{-1/2}$). We begin, for convenience, by repeating Lemma 1.1 from the introduction.

LEMMA 3.6 (Reduction to Ginibre). *We have the identity*

$$(3.32) \quad \mathcal{Z}_{Nd}^{(\text{Lem}_d)}(t) = e^{(Ntd)^2} c_{N,d} (\mathcal{Z}_N^{(\text{Gin})})^d \prod_{\ell=0}^{d-1} \mathbb{E}\left(|\det(G_N - t\sqrt{d})|^{\gamma_\ell}\right)$$

where

$$(3.33) \quad \gamma_\ell = -2 \left(1 - \frac{\ell + 1}{d}\right), \quad \ell = 0, \dots, d-1,$$

and $c_{N,d} = (Nd)! d^{-N(Nd+2d+1)/2} (N!)^{-d}$. The partition function for the Ginibre ensemble (i.e. (1.3) with $V(\lambda) = |\lambda|^2$) is

$$(3.34) \quad \mathcal{Z}_N^{(\text{Gin})} = \pi^N \frac{\prod_{k=1}^N k!}{N^{\frac{N(N+1)}{2}}}.$$

PROOF. We give a quick proof based on ideas of [14]. We will abbreviate throughout

$$(3.35) \quad V^{(d)}(\lambda, t) = |\lambda|^{2d} - t(\lambda^d + \bar{\lambda}^d).$$

If $p_j(\lambda)$ are the monic planar polynomials orthogonal with respect to (3.35), by the symmetry $\lambda \rightarrow \lambda e^{2\pi i/d}$ there must exist a family of monic polynomials $q_j^{(\ell)}(\lambda)$ such that

$$(3.36) \quad p_{jd+\ell}(\lambda) = \lambda^\ell q_j^{(\ell)}(\lambda^d).$$

The change of variables $u = \lambda^d$ comes with a Jacobian $d^2\lambda = d^{-2}|u|^{\frac{2}{d}-2} d^2u$ and we get

$$(3.37) \quad \begin{aligned} h_{jd+\ell} \delta_{j,k} &= \int_{\mathbb{C}} |\lambda|^{2\ell} q_k^{(\ell)}(\lambda^d) \overline{q_j^{(\ell)}(\lambda^d)} e^{-Nd(|\lambda|^{2d} - t(\lambda^d + \bar{\lambda}^d))} d^2\lambda \\ &= d^{-1} e^{Ndt^2} \int_{\mathbb{C}} q_k^{(\ell)}(u+t) \overline{q_j^{(\ell)}(u+t)} |u+t|^{\gamma_\ell} e^{-Nd|u|^2} d^2u \end{aligned}$$

where the γ_ℓ exponents are as in definition (3.33) (note that a factor d^{-1} is absorbed from the d -fold rotation). Scaling $u \rightarrow u/\sqrt{d}$ and using (2.4) twice, we find

$$(3.38) \quad \mathcal{Z}_{Nd}^{(\text{Lem}_d)}(t) = (Nd)! \prod_{\ell=0}^{d-1} \prod_{j=0}^{N-1} h_{jd+\ell} = e^{(Ntd)^2} \frac{c_{N,d}}{(\mathcal{Z}_N^{(\text{Gin})})^d} \prod_{\ell=0}^{d-1} R_{\gamma_\ell}(t\sqrt{d}).$$

where $R_\gamma(z)$ is defined in (3.1), see also (3.20). It is also interesting to consider a second proof going explicitly via the determinant formula in (2.4). We have

$$(3.39) \quad \mathcal{Z}_{Nd}^{(\text{Lem}_d)}(t) = (Nd)! \det \left\{ \int_{\mathbb{C}} \lambda^j \bar{\lambda}^k e^{-NdV^{(d)}(\lambda,t)} d^2\lambda \right\}_{j,k=0}^{Nd-1}.$$

We claim that most of the entries of the above moment matrix are zero due to symmetry. Indeed, because $V^{(d)}(\lambda, t)$ is invariant under the rotation $\lambda \rightarrow \lambda e^{2\pi i/d}$ we see that in a given $d \times d$ sub-block parameterized by $j = j'd + \ell$, $k = k'd + \ell'$, where $\ell = 0, \dots, d-1$ and $\ell' = 0, \dots, d-1$, one has

$$(3.40) \quad \int_{\mathbb{C}} \lambda^i \bar{\lambda}^j e^{-NdV^{(d)}(\lambda,t)} d^2\lambda = e^{2i\pi(j'd - k'd)/d} e^{2i\pi(\ell - \ell')/d} \int_{\mathbb{C}} \lambda^i \bar{\lambda}^j e^{-NdV^{(d)}(\lambda,t)} d^2\lambda$$

By the constraints on ℓ and ℓ' , it is impossible for $(\ell - \ell')/d$ to be an integer unless $\ell = \ell'$. This shows that our moment matrix in (3.39) consists of $d \times d$ diagonal blocks B_{jk} defined such that

$$(3.41) \quad (B_{jk})_{\ell\ell} = \int_{\mathbb{C}} \lambda^{j'd+\ell} \bar{\lambda}^{k'd+\ell} e^{-NdV^{(d)}(\lambda,t)} d^2\lambda$$

Since all these blocks commute one can treat them as scalars and compute the determinant as for an ordinary matrix of scalars, followed by the determinant of the resulting diagonal matrix:

$$(3.42) \quad \begin{aligned} \mathcal{Z}_{Nd}^{(\text{Lem}_d)}(t) &= (Nd)! \prod_{\ell=0}^{d-1} \det \left\{ \int_{\mathbb{C}} |\lambda|^{2\ell} (\lambda^d)^j (\bar{\lambda}^d)^k e^{-NdV^{(d)}(\lambda,t)} d^2\lambda \right\}_{j,k=0}^{N-1} \\ &= (Nd)! \prod_{\ell=0}^{d-1} \det \left\{ d^{-1} \int_{\mathbb{C}} |u|^{\gamma_\ell} u^j \bar{u}^k e^{-Nd|u-t|^2 + Nd|t|^2} d^2u \right\}_{j,k=0}^{N-1} \\ &= \frac{(Nd)! d^{-Nd}}{(N!)^d} \prod_{\ell=0}^{d-1} \int_{\mathbb{C}^N} \prod_{j=1}^N |u_j|^{\gamma_\ell} e^{-Nd|u_j-t|^2 + Nd|t|^2} |\Delta(\vec{u})|^2 d^2\vec{u} \end{aligned}$$

where we made the same change of variable $u = \lambda^d$ as before. Now shifting $u_j \rightarrow u_j + t$ followed by $u_j \rightarrow u_j/\sqrt{d}$ completes the proof. \square

Combining this result with Lemma 3.6 gives the following:

COROLLARY 1. *The partition function of the lemniscate ensemble given by (3.31) has an exact representation in terms of Painlevé V . Specifically, consider the solution $\sigma_{k,\alpha}(t)$ of differential equation (3.17) with boundary condition (3.18). Then*

$$(3.43) \quad \begin{aligned} \mathcal{Z}_{Nd}^{(\text{Lem}_d)}(t) &= e^{(Ntd)^2} c_{N,d} (\mathcal{Z}_N^{(\text{Gin})})^d \prod_{\ell=0}^{d-1} R_{\gamma_\ell}(t\sqrt{d}) \\ &= e^{(Ntd)^2 - Nt^2 \frac{d(d-1)}{2}} c_{N,d} \prod_{\ell=0}^{d-1} \left(a_{N,\gamma_\ell} \exp \left(\int_0^{t^2 d N} \frac{\sigma_{\gamma_\ell/2,N}(s)}{s} ds \right) \right) \end{aligned}$$

where the constants are given in Lemma 3.6 and (3.21).

Combining Lemma 3.6 with Theorem 1.2 gives an asymptotic expansion of the lemniscate partition function in the sub-critical phase.

COROLLARY 2. *In the sub-critical regime $|t| < t_c$ with t fixed, we have*

$$(3.44) \quad \frac{\mathcal{Z}_{Nd}^{(\text{Lem}_d)}(t)}{\mathcal{Z}_{Nd}^{(\text{Lem}_d)}(0)} = e^{N^2|t|^2 d^2 - N|t|^2 \frac{d(d-1)}{2}} (1 + o(1)), \quad N \rightarrow \infty.$$

In the critical phase, based on the previous computations and Theorem 1.3 we can make a conjecture for the expansion at $t = t_c$.

CONJECTURE 1. *In the critical regime $t = t_c + \frac{\tau t_c}{\sqrt{N}}$ with $\tau \in \mathbb{R}$ fixed, we have*

$$(3.45) \quad \frac{\mathcal{Z}_{Nd}^{(\text{Lem}_d)}(t)}{\mathcal{Z}_{Nd}^{(\text{Lem}_d)}(0)} = e^{N^2|t|^2 d^2 - N|t|^2 \frac{d(d-1)}{2}} \prod_{\ell=0}^{d-1} F_{\gamma_\ell/2}(2\tau) (1 + o(1)), \quad N \rightarrow \infty$$

where $F_{\gamma_\ell/2}(x)$ denotes the right-hand side of (1.10) with k replaced with $\gamma_\ell/2$ in (1.11).

It seems likely that the proof of the Conjecture 1 above will require Riemann-Hilbert techniques and is postponed to a future investigation. In the super-critical phase $t > t_c$, by Lemma 3.6 one requires asymptotics in the exterior region outside the circular law spectrum of the Ginibre ensemble, $|z| > 1$. This corresponds to Laplace asymptotics of a random matrix linear statistic with a test function which is smooth on the support of the equilibrium measure - consequently this regime should be the simplest to analyze. Indeed, as we explain in Appendix, the results of [6, 59] suggest that

$$(3.46) \quad \frac{\mathcal{Z}_{Nd}^{(\text{Lem}_d)}(t)}{(\mathcal{Z}_N^{(\text{Lem}_1)}(t\sqrt{d}))^d} = c_{N,d} |t/t_c|^{N(d-1)} |1 - |t/t_c|^{-2}|^{-\kappa_d} (1 + o(1)), \quad N \rightarrow \infty.$$

where

$$(3.47) \quad \kappa_d := \frac{1}{4} \sum_{\ell=0}^{d-1} \gamma_\ell^2 = \frac{d(d-1)(2d-1)}{6d^2}.$$

REMARK 3.7. *Asymptotic expansion of the partition function of planar models has received some attention recently. In the works [16] and [55] a partition function which includes (1.3) as a special case has been studied (the Coulomb and Riesz gases), but their asymptotic expansion only gives terms up to order N , while the expansions here (albeit for specific models) include the constant term.*

REMARK 3.8. *Note that the normalizations $\mathcal{Z}_{Nd}^{(\text{Lem}_d)}(0)$ and $(\mathcal{Z}_N^{(\text{Lem}_1)}(t\sqrt{d}))^d$ appearing in the denominators on the left-hand sides of (3.44), (3.45) and (3.46) involve radial weights and are simple to calculate explicitly.*

3.4. Multiple charges near the boundary.

PROOF OF THEOREM 1.13. Proceeding as in the previous sections, we are going to start with (2.7) under the microscopic scaling

$$(3.48) \quad x_i = z + \frac{u_i}{\bar{z}\sqrt{N}}, \quad \bar{y}_i = \bar{z} + \frac{\bar{v}_i}{z\sqrt{N}}$$

for $i = 1, \dots, k$, where z is a point on the boundary $|z| = 1$. The polynomial part of the finite- N Ginibre kernel is

$$(3.49) \quad B_{N+k}(x, \bar{y}) = \frac{N}{\pi} \sum_{j=0}^{N+k-1} \frac{(x\bar{y}N)^j}{j!}$$

By Lemma 9.4 of [20], under the scaling (3.48) we have, for any fixed $z \in S^1$, the asymptotic

$$(3.50) \quad B_{N+k}(x, \bar{y}) \sim \frac{N}{2\pi} e^{N+\sqrt{N}(u+\bar{v})+u\bar{v}} \operatorname{erfc}\left(\frac{u+\bar{v}}{\sqrt{2}}\right), \quad N \rightarrow \infty$$

uniformly for u and v in compact subsets of \mathbb{C} . By Stirling's formula we obtain

$$(3.51) \quad \prod_{j=0}^{k-1} h_{j+N} \sim \pi^k e^{-Nk} N^{-k/2} (2\pi)^{k/2} (1 + o(1)), \quad N \rightarrow \infty,$$

and $\Delta(\vec{x})\Delta(\vec{y}) = N^{k(k-1)/2} \Delta(\vec{u})\Delta(\vec{v})$. Defining the *error function kernel*

$$(3.52) \quad K_{\operatorname{erf}}(u, \bar{v}) := e^{-\frac{(u-\bar{v})^2}{2}} \operatorname{erfc}\left(\frac{u+\bar{v}}{\sqrt{2}}\right)$$

and inserting the previous results into (2.7), we get the formula

$$(3.53) \quad \mathbb{E} \left(\prod_{i=1}^k \det(G - x_i) \det(G^\dagger - \bar{y}_i) \right) \sim \exp \left(-\sqrt{N} \sum_{i=1}^k (u_i + \bar{v}_i) + \frac{k^2}{2} \log(N) - \sum_{i=1}^k \frac{u_i^2 + \bar{v}_i^2}{2} \right) \\ \times \left(\frac{\pi}{2} \right)^{k/2} \frac{\det \left\{ K_{\operatorname{erf}}(u_i, \bar{v}_j) \right\}_{i,j=1}^k}{\Delta(\vec{u})\Delta(\vec{v})}, \quad N \rightarrow \infty.$$

The relation to non-intersecting paths goes via the integral representation

$$(3.54) \quad K_{\operatorname{erf}}(u, \bar{v}) = 2\sqrt{2\pi} \int_0^\infty p_{1/2}(u, x) p_{1/2}(x, \bar{v}) dx$$

in terms of the Brownian motion transition density

$$(3.55) \quad p_t(u, v) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(u-v)^2}{2t}}.$$

Consequently, by Lemma 2.2 we can immediately identify

$$(3.56) \quad \det \left\{ K_{\operatorname{erf}}(u_i, \bar{v}_j) \right\}_{i,j=1}^k = \frac{2^k (2\pi)^{k/2}}{k!} \mathcal{Z}_{1/2}(\vec{u}, \vec{v}, \mathbb{R}_+).$$

and (1.31) follows. \square

4. Bulk asymptotics: proofs

The purpose of this section will be to prove our results on bulk asymptotics, namely Theorems 1.5 and 1.14 which deal with integer charge exponents and Theorem 1.7 which deals with a particular case of non-integer exponents. The proofs are both based on formula (2.7) but proceed in different ways, one exploits the Harish-Chandra Itzykson Zuber integral while the other exploits the exactly solvable structure of the *induced Ginibre ensemble* studied in [31]. Our proof of Theorem 1.7 can be interpreted as the computation of the one-point correlator for the induced Ginibre ensemble.

PROOF OF THEOREMS 1.5 AND 1.14. We focus to begin with on the proof of Theorem 1.5. Taking (2.7) as our starting point, with the Ginibre weight $w(\lambda) = e^{-N|\lambda|^2}$ the usual formulas apply and we have $p_j(\lambda) = \lambda^j$, $h_j = N^{-j-1}\pi j!$, and the asymptotic formula

$$(4.1) \quad \prod_{j=0}^{k-1} h_{j+N} \sim \pi^k (2\pi)^{k/2} e^{-Nk} N^{-k/2}.$$

The polynomial part of the correlation kernel follows as

$$(4.2) \quad B_{N+k}(x, \bar{y}) = \frac{N}{\pi} \sum_{j=0}^{N+k-1} \frac{(x\bar{y}N)^j}{j!}.$$

The idea of the proof will be to merge the variables in (2.7) in two distinct groups, the first group creating a singularity with strength k_1 and the second with strength k_2 and $k = k_1 + k_2$. It is convenient to first impose the microscopic scaling

$$(4.3) \quad x_j = z + \frac{u_j}{\sqrt{N}}, \quad \bar{y}_j = \bar{z} + \frac{\bar{v}_j}{\sqrt{N}}, \quad j = 1, \dots, k$$

and then perform the merging (still keeping N finite)

$$(4.4) \quad (u_1, \dots, u_{k_1}) \rightarrow (u_1, \dots, u_1), \quad (u_{k_1+1}, \dots, u_{k_1+k_2}) \rightarrow (u_2, \dots, u_2)$$

and

$$(4.5) \quad (\bar{v}_1, \dots, \bar{v}_{k_1}) \rightarrow (\bar{v}_1, \dots, \bar{v}_1), \quad (\bar{v}_{k_1+1}, \dots, \bar{v}_{k_1+k_2}) \rightarrow (\bar{v}_2, \dots, \bar{v}_2).$$

Doing this, we can straightforwardly write down the exact identity

$$(4.6) \quad \begin{aligned} & \mathbb{E} \left(|\det(G_N - z_1)|^{2k_1} |\det(G_N - z_2)|^{2k_2} \right) = e^{Nk|z|^2 + \sqrt{N}k_1(u_1\bar{z} + \bar{u}_2z) + \sqrt{N}k_2(u_2\bar{z} + \bar{u}_2z)} \\ & \times \left(\prod_{j=0}^{k-1} h_{j+N} \right) N^k \pi^{-k} N^{\frac{k(k-1)}{2}} \\ & \times \lim_{\substack{v_2 \rightarrow u_2, v_1 \rightarrow u_1 \\ (\bar{v}_1, \dots, \bar{v}_{k_1}) \rightarrow (\bar{v}_1, \dots, \bar{v}_1)}} \lim_{\substack{(u_{k_1+1}, \dots, u_{k_1+k_2}) \rightarrow (u_2, \dots, u_2) \\ (\bar{v}_{k_1+1}, \dots, \bar{v}_{k_1+k_2}) \rightarrow (\bar{v}_2, \dots, \bar{v}_2)}} \frac{\det \left\{ f_{N+k}(u_i, \bar{v}_j) \right\}_{i,j=1}^k}{\Delta(\vec{u}) \overline{\Delta(\vec{v})}} \end{aligned}$$

where we used that $\Delta(\vec{x}) \overline{\Delta(\vec{y})} = N^{-\frac{k(k-1)}{2}} \Delta(\vec{u}) \overline{\Delta(\vec{v})}$ and defined

$$(4.7) \quad f_{N+k}(u, \bar{v}) = e^{-N|z|^2 - \sqrt{N}(u\bar{z} + \bar{v}z)} \sum_{j=0}^{N+k-1} \frac{\left(N \left(z + \frac{u}{\sqrt{N}} \right) \left(\bar{z} + \frac{\bar{v}}{\sqrt{N}} \right) \right)^j}{j!}, \quad f(u, v) = e^{u\bar{v}}$$

in such a way that $\lim_{N \rightarrow \infty} f_N(u, \bar{v}) = f(u, \bar{v})$ uniformly in compact subsets of u and v (see *e.g.* [20, Lemma 9.2] or [21, Proposition 2]). The degenerate limit in (4.6) can be written as the determinant of a block matrix involving partial derivatives of $f_{N+k}(u, v)$ with respect to the u and v variables. Indeed, all derivatives of $f_N(u, \bar{v})$ to any finite order

converge uniformly to the corresponding derivatives of $f(u, \bar{v})$ as $N \rightarrow \infty$, and thus the various limiting operations can be interchanged. In order to compute the asymptotics as $N \rightarrow \infty$ in (4.6) we may therefore replace the function $f_{N+k}(u, \bar{v})$ with the limiting kernel $f(u, \bar{v}) = e^{u\bar{v}}$. Therefore, the limit $N \rightarrow \infty$ of the ratio of determinants in (4.6) is equal to

$$(4.8) \quad \lim_{\substack{(u_1, \dots, u_{k_1}) \rightarrow (u_1, \dots, u_1), (u_{k_1+1}, \dots, u_{k_1+k_2}) \rightarrow (u_2, \dots, u_2) \\ (\bar{v}_1, \dots, \bar{v}_{k_1}) \rightarrow (\bar{v}_1, \dots, \bar{v}_1), (\bar{v}_{k_1+1}, \dots, \bar{v}_{k_1+k_2}) \rightarrow (\bar{v}_2, \dots, \bar{v}_2)}}} \frac{\det \left\{ e^{u_i \bar{v}_j} \right\}_{i,j=1}^k}{\Delta(\vec{u}) \Delta(\vec{v})}.$$

In the same way, one can show

$$(4.9) \quad \mathbb{E} \left(\prod_{i=1}^k \det(G - x_i) \det(G^\dagger - \bar{y}_i) \right) = \exp \left(Nk(|z|^2 - 1) + \sqrt{N} \sum_{i=1}^k (z\bar{v}_i + \bar{z}u_i) + \frac{k^2}{2} \log(N) \right) \\ \times (2\pi)^{k/2} \frac{\det \left\{ e^{u_i \bar{v}_j} \right\}_{i,j=1}^k}{\Delta(\vec{u}) \Delta(\vec{v})} (1 + o(1))$$

where the asymptotics on the right-hand side is defined by taking a limit if some of the points in the vectors \vec{u} or \vec{v} coincide. To conclude the proof of Theorem 1.14 we recognise the right-hand side of (4.9) in terms of the famous Harish-Chandra Itzykson Zuber integral [49], which is the exact formula

$$(4.10) \quad \frac{\det \left\{ e^{u_i \bar{v}_j} \right\}_{i,j=1}^k}{\Delta(\vec{z}) \Delta(\vec{w})} = \frac{1}{G(1+k)} \int_{U(k)} e^{\text{Tr}(UAU^\dagger \bar{B})} d\mu(U),$$

where $d\mu(U)$ is the normalized Haar measure on the group $U(k)$ of $k \times k$ unitary matrices, and $A = \text{diag}(u_1, \dots, u_k)$, $\bar{B} = \text{diag}(\bar{v}_1, \dots, \bar{v}_k)$ are diagonal matrices. The rather intricate merging in (4.6) can now be performed directly by creating degeneracies in the diagonal matrices A and \bar{B} . For Theorem 1.5 we simply put

$$(4.11) \quad A = \text{diag}(\overbrace{u_1, \dots, u_1}^{k_1}, \overbrace{u_2, \dots, u_2}^{k_2}) \\ \bar{B} = \text{diag}(\overbrace{\bar{v}_1, \dots, \bar{v}_1}^{k_1}, \overbrace{\bar{v}_2, \dots, \bar{v}_2}^{k_2})$$

where eventually we will set $u_1 = v_1$ and $u_2 = v_2$. This motivates us to split the unitary matrix U into four sub-blocks. Let us write

$$(4.12) \quad U = \begin{pmatrix} a_{k_1 \times k_1} & b_{k_1 \times k_2} \\ c_{k_2 \times k_1} & d_{k_2 \times k_2} \end{pmatrix}.$$

Then a direct calculation using the unitarity of U shows that

$$(4.13) \quad \text{Tr}(UAU^\dagger \bar{B}) = u_1 \bar{v}_1 k_1 + u_2 \bar{v}_2 k_2 - [(u_2 - u_1)(\bar{v}_2 - \bar{v}_1)] \text{Tr}(cc^\dagger).$$

It turns out that the induced distribution on the product matrix cc^\dagger has been intensively studied and its eigenvalue statistics (in particular its trace) are well known. Assuming that $k_1 \geq k_2$ the joint distribution of eigenvalues of cc^\dagger is (see [26, 33])

$$(4.14) \quad P(t_1, \dots, t_{k_2}) = \frac{1}{C_{k_2}} \prod_{j=1}^{k_2} t_j^{k_1 - k_2} \Delta^2(\vec{t})$$

where C_{k_2} is a normalization constant,

$$(4.15) \quad C_{k_2} = \int_{[0,1]^{k_2}} \prod_{j=1}^{k_2} t_j^{k_1-k_2} \Delta^2(\vec{t}) d\vec{t},$$

which is a particular case of the Jacobi Unitary Ensemble (JUE). Consequently, inserting (4.13) into the Harish-Chandra Itzykson Zuber integral and using (4.14), we find

$$(4.16) \quad \int_{U(k)} e^{N\text{Tr}(UAU^\dagger \bar{B})} d\mu(U) = e^{u_1 \bar{v}_1 k_1 + u_2 \bar{v}_2 k_2} \mathbb{E}(e^{s\text{Tr}(cc^\dagger)}).$$

$$= \frac{e^{u_1 \bar{v}_1 k_1 + u_2 \bar{v}_2 k_2}}{C_{k_2}} \int_{[0,1]^{k_2}} \prod_{j=1}^{k_2} t_j^{k_2-k_1} e^{-st_j} \Delta^2(\vec{t}) d\vec{t},$$

where we defined the variable $s = (u_2 - u_1)(\bar{v}_2 - \bar{v}_1)$. Finally, we need to write this as an expectation over the Laguerre ensemble. For this, we specialize to $u_1 = v_1$ and $u_2 = \bar{v}_2$ so that $s = |u_2 - u_1|^2 = N|z_2 - z_1|^2$. We have

$$(4.17) \quad \frac{1}{C_{k_2}^{(\text{JUE}_{k_1-k_2,0})}} \int_{[0,1]^{k_2}} \prod_{j=1}^{k_2} t_j^{k_1-k_2} e^{-st_j} \Delta^2(\vec{t}) d\vec{t}$$

$$= \frac{C_{k_2}^{(\text{LUE}_{k_1-k_2})}}{C_{k_2}^{(\text{JUE}_{k_1-k_2,0})}} s^{-k_1 k_2} \mathbb{P}_{k_2}(\lambda_{\max}^{(\text{LUE}_{k_1-k_2})} < s)$$

The ratio of normalization constants is

$$(4.18) \quad \frac{C_{k_2}^{(\text{LUE}_{k_1-k_2})}}{C_{k_2}^{(\text{JUE}_{k_1-k_2,0})}} = \prod_{j=0}^{k_2-1} \frac{\Gamma(1+k_1+j)}{\Gamma(1+j)}$$

$$= \frac{G(k_1+k_2+1)}{G(k_1+1)G(k_2+1)},$$

the numerator of which exactly cancels the denominator in (4.10). Therefore, we get the following asymptotic formula, uniformly in compact subsets of u_1 and u_2 ,

$$(4.19) \quad \mathbb{E} \left(|\det(G - z_1)|^{2k_1} |\det(G - z_2)|^{2k_2} \right) = e^{k_1 N(|z_1|^2-1) + k_2 N(|z_2|^2-1) + \frac{k_1^2}{2} \log(N) + \frac{k_2^2}{2} \log(N)}$$

$$\times e^{-2k_1 k_2 \log|z_2-z_1|} \frac{(2\pi)^{(k_1+k_2)/2}}{G(1+k_1)G(1+k_2)} F_k(N|z_2 - z_1|^2)(1 + o(1)), \quad N \rightarrow \infty,$$

where $F_k(s)$ is the distribution function of the largest eigenvalue in a $k_2 \times k_2$ Laguerre ensemble with parameter $k_1 - k_2$:

$$(4.20) \quad F_k(s) = \mathbb{P}_{k_2} \left(\lambda_{\max}^{(\text{LUE}_{k_1-k_2})} < s \right).$$

This concludes the proof of Theorem 1.5. \square

PROOF OF THEOREM 1.7. With the assumptions of Theorem 1.7, the two-point correlator on the left-hand side of (1.13) simplifies to

$$(4.21) \quad \mathbb{E} \left(|\det(G_N)|^{2\gamma_1} |\det(G_N - z_2)|^{2k_2} \right)$$

$$= \frac{1}{Z_N^{(\text{Gin})}} \int_{\mathbb{C}^N} \prod_{j=1}^N |\lambda_j|^{2\gamma_1} |\lambda_j - z_2|^{2k_2} e^{-N|\lambda_j|^2} |\Delta(\vec{\lambda})|^2 d^2 \vec{\lambda}$$

where $z_2 = \frac{u_2}{\sqrt{N}}$ and u_2 is fixed in the microscopic scaling. By the radial symmetry, we can assume without loss of generality that $u_2 > 0$ throughout. The point of this proof, in

contrast to the proof of Theorem 1.5, is that γ_1 is *not constrained to be a positive integer*. We assume that $\gamma_1 \geq k_2 > 0$ and $k_2 \in \mathbb{N}$.

We now proceed by viewing (4.21) as a one-point correlator for the reference weight $w(\lambda) = |\lambda|^{2\gamma_1} e^{-N|\lambda|^2}$ and apply the techniques of the previous sections. The ensemble with this particular weight was introduced under the name *induced Ginibre ensemble*, see [31]. Adapting the proof of Proposition 3.1, we again have a rotationally invariant weight, implying $p_j(\lambda) = \lambda^j$, but now the orthogonality norms are

$$(4.22) \quad h_j = \int_{\mathbb{C}} |\lambda|^{2j+2\gamma_1} e^{-N|\lambda|^2} d^2\lambda = \pi N^{-j-\gamma_1-1} \Gamma(j + \gamma_1 + 1).$$

Under the microscopic scaling, the points in (2.7) take the form

$$(4.23) \quad x_i = \frac{u_i}{\sqrt{N}}, \quad y_i = \frac{v_i}{\sqrt{N}}, \quad i = 1, \dots, k$$

and the product of Vandermonde determinants in the denominator of (2.7) rescale to $\Delta(\vec{x})\Delta(\vec{y}) = N^{-k_2(k_2-1)/2} \Delta(\vec{u})\Delta(\vec{v})$. From the explicit formulae for $p_j(\lambda)$ and h_j , we get the polynomial part of the kernel

$$(4.24) \quad B_N(x, y) = \frac{N^{\gamma_1+1}}{\pi} \sum_{j=0}^{N-1} \frac{(xyN)^j}{\Gamma(j + \gamma_1 + 1)}.$$

Defining the functions

$$(4.25) \quad f_N(u, v) := v^{\gamma_1} \sum_{j=0}^{N-1} \frac{(uv)^j}{\Gamma(j + \gamma_1 + 1)}, \quad f(u, v) := v^{\gamma_1} \sum_{j=0}^{\infty} \frac{(uv)^j}{\Gamma(j + \gamma_1 + 1)}$$

we first observe that (2.7) and (2.8) give the *exact identity*

$$(4.26) \quad \mathbb{E} \left(|\det(G_N)|^{2\gamma_1} |\det(G_N - z_2)|^{k_2} \right) = \frac{\mathcal{Z}_N^{(\text{Ind}_{\gamma_1})}}{\mathcal{Z}_N^{(\text{Gin})}} N^{k_2\gamma_1} \pi^{-k_2} u_2^{-\gamma_1 k_2} N^{k_2(k_2+1)/2} \\ \times \prod_{j=0}^{k_2-1} h_{j+N} \frac{1}{\prod_{j=0}^{k_2-1} (j!)^2} \lim_{u \rightarrow u_2, v \rightarrow u_2} \det \left\{ \frac{\partial^{i+j-2} f_{N+k_2}(u, v)}{\partial u^{i-1} \partial v^{j-1}} \right\}_{i,j=1}^{k_2},$$

where the partition function of the induced Ginibre ensemble is

$$(4.27) \quad \mathcal{Z}_N^{(\text{Ind}_{\gamma_1})} := \int_{\mathbb{C}^N} \prod_{j=1}^N |\lambda_j|^{2\gamma_1} e^{-N|\lambda_j|^2} |\Delta(\vec{\lambda})|^2 d^2\vec{\lambda} = \prod_{j=0}^{N-1} (\pi N^{-j-\gamma_1-1} \Gamma(j + \gamma_1 + 1)).$$

Now the large- N asymptotics of (4.26) can be derived, exploiting the fact that the convergence $f_N(u, v) \rightarrow f(u, v)$ as $N \rightarrow \infty$ is uniform. Furthermore, all mixed partial derivatives of $f_N(u, v)$ converge uniformly to those of $f(u, v)$ and this allows the interchange of the various limiting operations. Consequently, it suffices to compute the determinant in (4.26) with $f_N(u, v)$ replaced with the limiting kernel $f(u, v)$, for which we have the integral representation

$$(4.28) \quad f(u, v) = \frac{1}{\Gamma(\gamma_1)} \int_0^v e^{ut} (v-t)^{\gamma_1-1} dt.$$

The needed partial derivatives of the kernel can be computed using the Leibniz formula for differentiating under the integral, using that the integrand vanishes at the upper end

point. This gives

$$(4.29) \quad \begin{aligned} \frac{\partial^{i+j-2} f(u, v)}{\partial u^{i-1} \partial v^{j-1}} &= \frac{1}{\Gamma(\gamma_1 - (j-1))} \int_0^v t^{i-1} e^{ut} (v-t)^{\gamma_1-1-(j-1)} dt \\ &= \frac{v^{\gamma_1+i-j} (-1)^{i-1} e^{uv}}{\Gamma(\gamma_1 - (j-1))} \int_0^1 t^{\gamma_1-1} e^{-uvt} (t-1)^{i-1} t^{-(j-1)} dt. \end{aligned}$$

Now exploiting Lemma 2.2,

$$(4.30) \quad \begin{aligned} &\det \left\{ \frac{\partial^{i+j-2} f(u, v)}{\partial u^{i-1} \partial v^{j-1}} \right\}_{i,j=1}^{k_2} \\ &= \frac{v^{\gamma_1 k_2} e^{uv k_2}}{k_2!} \prod_{j=0}^{k_2-1} \frac{1}{\Gamma(\gamma_1 - j)} \int_{[0,1]^{k_2}} \prod_{j=1}^{k_2} t_j^{\gamma_1-1} e^{-uvt_j} \det \left\{ (t_i - 1)^{j-1} \right\} \det \left\{ t_i^{-(j-1)} \right\} d\vec{t} \\ &= \frac{v^{\gamma_1 k_2} (uv)^{-\gamma_1 k_2} e^{uv k_2}}{k_2!} \prod_{j=0}^{k_2-1} \frac{1}{\Gamma(\gamma_1 - j)} \int_{[0,uv]^{k_2}} \prod_{j=1}^{k_2} t_j^{\gamma_1-k_2} e^{-t_j} \Delta^2(\vec{t}) d\vec{t}, \end{aligned}$$

where we now spot the joint distribution of eigenvalues of the Laguerre ensemble appearing in (4.30). We have therefore arrived at the determinant identity

$$(4.31) \quad \det \left\{ \frac{\partial^{i+j-2} f(u, v)}{\partial u^{i-1} \partial v^{j-1}} \right\}_{i,j=1}^{k_2} = \frac{v^{\gamma_1 k_2} (uv)^{-\gamma_1 k_2} G(1+k_2) e^{uv k_2}}{k_2!} \mathbb{P}(\lambda_{\max} < uv)$$

where λ_{\max} is the largest eigenvalue in the $k_2 \times k_2$ Laguerre ensemble with parameter $\gamma_1 - k_2$. It remains just to deal with the constants in (4.26). A standard computation using the asymptotic expansion of the Barnes G-function gives,

$$(4.32) \quad \frac{\mathcal{Z}_N^{(\text{Ind}_{\gamma_1})}}{\mathcal{Z}_N^{(\text{Gin})}} = \frac{1}{G(1+\gamma_1)} e^{-\gamma_1 N + \frac{\gamma_1^2}{2} \log(N)} (2\pi)^{\gamma_1/2} (1 + o(1)), \quad N \rightarrow \infty,$$

while Stirling's formula gives

$$(4.33) \quad \prod_{j=0}^{k_2-1} h_{j+N} \sim \pi^{k_2} (2\pi)^{k_2/2} e^{-N k_2} N^{-k_2/2}, \quad N \rightarrow \infty.$$

Now inserting (4.33), (4.32) and (4.31) into (4.26) gives the formula

$$(4.34) \quad \begin{aligned} &\mathbb{E} \left(|\det(G_N)|^{2\gamma_1} |\det(G_N - z_2)|^{2k_2} \right) = e^{u_2^2 k_2 - (\gamma_1 + k_2) N + \frac{\gamma_1^2}{2} \log(N) + \frac{k_2^2}{2} \log(N)} \\ &\times e^{2k_2 \gamma_1 \log(\sqrt{N} u_2)} \frac{(2\pi)^{\frac{\gamma_1 + k_2}{2}}}{G(1+k_2) G(1+\gamma_1)} \mathbb{P}(\lambda_{\max} < u_2^2) (1 + o(1)), \quad N \rightarrow \infty \end{aligned}$$

which is precisely of the form (1.13), with $k_1 = \gamma_1$ and $z = z_1 = 0$. \square

5. Truncated CUE: finite- N and boundary asymptotics

The purpose of this section is to prove analogous results for the one-point function as obtained in Section 1.2. Therefore, we consider the moments

$$(5.1) \quad R_{2k}(z) = \mathbb{E} \left(|\det(T - z)|^{2k} \right)$$

where T is an $N \times N$ truncation of a Haar distributed unitary matrix of size $M \times M$. We begin by showing that (5.1) is expressible in terms of solutions of Painlevé VI via a duality with the Jacobi Unitary Ensemble. Then we will study the boundary asymptotics and show that they are expressed in terms of Painlevé V. This will prove Theorems 1.10 and 1.8. As in Section 1.2, we also investigate the continuation of Theorem 1.10 off the integers,

that is we will prove that setting $k = \gamma/2$ in (1.25) gives the correct interpretation of (5.1). Along the way, we will study planar orthogonal polynomials associated to the weight

$$(5.2) \quad w(\lambda) = |\lambda - z|^\gamma (1 - |\lambda|^2)^{M-N-1}, \quad |\lambda| \leq 1.$$

5.1. Duality and the Jacobi Unitary Ensemble. We begin by deriving the analogue of identity (3.3). Recall the construction of the Jacobi Unitary Ensemble: take two independent k -dimensional LUE matrices (Wishart matrices) W_1 with parameter α and W_2 with parameter β and form the ratio $J = W_1(W_1 + W_2)^{-1}$. The eigenvalues t_1, \dots, t_k are said to belong to the JUE and have the joint probability density function

$$(5.3) \quad P(t_1, \dots, t_k) \propto \prod_{j=1}^k t_j^\alpha (1 - t_j)^\beta \Delta^2(\vec{t}).$$

PROPOSITION 5.1. *Let x and y be any complex numbers and let $k \in \mathbb{N}$. We have*

$$(5.4) \quad \begin{aligned} & \mathbb{E} \left(\det(T - x)^k \det(T^\dagger - y)^k \right) \\ &= \frac{1}{C_{\alpha+N+2,1}^{(\text{JUE}_k)}} \int_{[0,1]^k} \prod_{j=1}^k t_j^{\alpha+1} (1 + (xy - 1)t_j)^N \Delta^2(\vec{t}) d\vec{t} \end{aligned}$$

where

$$(5.5) \quad C_{\alpha+N+2,1}^{(\text{JUE}_k)} := \int_{[0,1]^k} \prod_{j=1}^k t_j^{\alpha+N+1} \Delta^2(\vec{t}) d\vec{t} = \prod_{j=0}^{k-1} \frac{\Gamma(\alpha + N + 2 + j) j! (j+1)!}{\Gamma(\alpha + N + 2 + k + j)}.$$

PROOF OF PROPOSITION 5.1, THEOREM 1.10 AND THEOREM 1.8. Compared with the Ginibre case in Proposition 3.1, the main differences here are that we now work with the weight $w(\lambda) = (1 - |\lambda|^2)^\alpha$ supported on the disc $\lambda \in \mathbb{D}$ (we will abbreviate $\alpha := M - N - 1$ in what follows). As before the corresponding orthogonal polynomials are $p_j(\lambda) = \lambda^j$ but now the normalizations are

$$(5.6) \quad h_j := \int_{\mathbb{D}} |\lambda|^{2j} (1 - |\lambda|^2)^\alpha d^2\lambda = \pi \frac{j! \Gamma(\alpha + 1)}{\Gamma(j + \alpha + 2)}$$

and the polynomial part of the kernel follows as (see [53]),

$$(5.7) \quad \begin{aligned} B_{N+k}(x, y) &= \frac{1}{\Gamma(\alpha + 1)} \sum_{j=0}^{N+k-1} \frac{\Gamma(j + \alpha + 2) (xy)^j}{\pi j! \Gamma(\alpha + 1)} \\ &= x^{N+k} \frac{\Gamma(\alpha + N + k + 2)}{\pi \Gamma(\alpha + 1) (N + k - 1)!} \int_0^\infty (tx + 1)^{-(N+k)-\alpha-2} (t + y)^{N+k-1} dt \end{aligned}$$

where, for now (and without loss of generality) we will assume that x and y are positive reals. The integral representation (5.7) follows *e.g.* from Euler's integral representation for the hypergeometric function [30, 15.6.1]. Its advantage here is that we have separated the x and y variables in the integrand. Indeed, considering the function

$$(5.8) \quad A_{N+k}^{(\alpha)}(x, y) := \int_0^\infty (tx + 1)^{-N-k-\alpha-2} (t + y)^{N+k-1} dt$$

we can directly calculate the derivatives required on the right-hand side of (2.8),

$$(5.9) \quad \frac{\partial^{i+j-2} A(x, y)}{\partial x^{i-1} \partial y^{j-1}} = d_{i,j,N,k,\alpha} \int_0^\infty t^{i-1} (tx + 1)^{-N-k-\alpha-2-(i-1)} (t + y)^{N+k-1-(j-1)} dt$$

where

$$(5.10) \quad d_{i,j,N,k,\alpha} := \frac{\Gamma(N + k + \alpha + i + 1) \Gamma(N + k) (-1)^{i-1}}{\Gamma(N + k + \alpha + 2) \Gamma(N + k - (j - 1))}$$

By Lemma 2.2 we get

$$\begin{aligned}
 & \det \left\{ \frac{\partial^{i+j-2} A_{N+k}^{(\alpha)}(x, y)}{\partial x^{i-1} \partial y^{j-1}} \right\}_{i,j=1}^k \\
 (5.11) \quad &= e_{N,k,\alpha} \int_{[0,\infty)^k} \det \left\{ t_i^{j-1} (t_i x + 1)^{-N-k-i-\alpha-1} \right\}_{i,j=1}^k \det \left\{ (t_i + y)^{N+k-j} \right\}_{i,j=1}^k d\vec{t} \\
 &= (-1)^{\frac{k(k-1)}{2}} e_{N,k,\alpha} \int_{[0,\infty)^k} \prod_{j=1}^k (t_j x + 1)^{-N-\alpha-1-2k} (t_j + y)^N \Delta^2(\vec{t}) d\vec{t}
 \end{aligned}$$

where

$$(5.12) \quad e_{N,k,\alpha} := \frac{(-1)^{\frac{k(k-1)}{2}}}{k!} \left(\frac{\Gamma(N+k)}{\Gamma(N+k+\alpha+2)} \right)^k \prod_{j=1}^k \left(\frac{\Gamma(N+k+\alpha+j+1)}{\Gamma(N+j)} \right).$$

To obtain the second line in (5.11) we made use of the determinant identities

$$\begin{aligned}
 & \det \left\{ (t_i + y)^{k-j} \right\}_{i,j=1}^k = (-1)^{\frac{k(k-1)}{2}} \Delta(\vec{t}), \\
 (5.13) \quad & \det \left\{ (t_i^{j-1} (t_i x + 1)^{-k-i}) \right\}_{i,j=1}^k = \prod_{j=1}^k (t_j x + 1)^{-2k} \Delta(\vec{t}).
 \end{aligned}$$

After sending $t_j \rightarrow t_j/x$ for each $j = 1, \dots, k$ in (5.11) and some further standard manipulations, we arrive at the identity

$$\begin{aligned}
 & \det \left\{ \frac{\partial^{i+j-2} A_{N+k}^{(\alpha)}(x, y)}{\partial x^{i-1} \partial y^{j-1}} \right\}_{i,j=1}^k \\
 (5.14) \quad &= (-1)^{\frac{k(k-1)}{2}} e_{N,k,\alpha} x^{-Nk-k^2} \int_{[0,1]^k} \prod_{j=1}^k t_j^{\alpha+1} (1 + (xy-1)t_j)^N \Delta^2(\vec{t}) d\vec{t}
 \end{aligned}$$

Putting everything together, we have

$$\begin{aligned}
 & \mathbb{E} \left(\det(T - x)^k \det(T^\dagger - \bar{y})^k \right) \\
 (5.15) \quad &= \frac{\prod_{j=0}^{k-1} h_{j+N}}{\prod_{j=0}^{k-1} (j!)^2} x^{k(N+k)} \left(\frac{\Gamma(\alpha + N + 2)}{\pi \Gamma(\alpha + 1)(N-1)!} \right)^k \det \left\{ \frac{\partial^{i+j-2} A_{N+k}^{(\alpha)}(x, y)}{\partial x^{i-1} \partial y^{j-1}} \right\}_{i,j=1}^k \\
 &= \frac{1}{C_{\alpha+N+2,1}^{(\text{JUE})}} \int_{[0,1]^k} \prod_{j=1}^k t_j^{\alpha+1} (1 + (xy-1)t_j)^N \Delta^2(\vec{t}) d\vec{t},
 \end{aligned}$$

which completes the proof of Proposition 5.1. To obtain the last line above we inserted (5.14) and identified the pre-factors as

$$(5.16) \quad \frac{\prod_{j=0}^{k-1} h_{j+N}}{\prod_{j=0}^{k-1} (j!)^2} \left(\frac{\Gamma(\alpha + N + 2)}{\pi \Gamma(\alpha + 1)(N-1)!} \right)^k (-1)^{\frac{k(k-1)}{2}} e_{N,k,\alpha} = \frac{1}{C_{\alpha+N+2,1}^{(\text{JUE})}}.$$

This proves Proposition 5.1. Initially we assumed $x > 0$ and $y > 0$, but since (5.15) is an identity between polynomials in x and y , it holds on the whole complex plane. Now Theorem 1.10 follows by setting $x = \bar{y} = z$ and the change of variables $t_j \rightarrow t_j/(1 - |z|^2)$ which expresses the right-hand side of (5.4) as a gap probability in the Jacobi Unitary

Ensemble. For the boundary limit, note that if $|z| = 1 - \frac{u}{N}$ and $\kappa = M - N = \alpha + 1$ is fixed, we have the limit

$$\begin{aligned}
(5.17) \quad & \lim_{N \rightarrow \infty} \int_{[0,1]^k} \prod_{j=1}^k t_j^\kappa \left(1 + \left(-\frac{2u}{N} + \frac{u^2}{N^2} \right) t_j \right)^N \Delta^2(\vec{t}) d\vec{t} \\
&= \int_{[0,1]^k} \prod_{j=1}^k t_j^\kappa e^{-2ut_j} \Delta^2(\vec{t}) d\vec{t} \\
&= \frac{G(2+k)G(\kappa+k+1)}{G(\kappa+1)} (2u)^{-k^2-k\kappa} F_k^{(\kappa)}(2u)
\end{aligned}$$

identified as a gap probability in the Laguerre Unitary Ensemble. To complete the proof of Theorem 1.8 note that the constant in Proposition 5.1 has the asymptotics

$$(5.18) \quad C_{\alpha+N+2,1}^{(\text{JUE})} = G(1+k)G(2+k)N^{-k^2} (1 + o(1)), \quad N \rightarrow \infty.$$

□

REMARK 5.2. *In the limiting case $\kappa \rightarrow 0$ and $x = y = 1$, the right-hand side of (5.4) becomes completely explicit and re-derives the exact formula for moments of the characteristic polynomial of a Haar distributed unitary matrix due to Keating and Snaith [53].*

5.2. Painlevé VI to Painlevé V scaling heuristics. Considering the boundary scaling $|z|^2 = 1 - \frac{2u}{N}$, we consider scaling the identity (1.23) in the weak non-unitarity limit, so that $M - N = \kappa$ is fixed. A change of variable $t \rightarrow t/N$ in (1.23) gives the characterization

$$(5.19) \quad \mathbb{P} \left(\lambda_{\max}^{\text{JUE}_k^{(\kappa,N)}} < \frac{2u}{N} \right) = \exp \left(- \int_{2u}^N \frac{\sigma_{\kappa,N}^{(\text{VI})} \left(\frac{t}{N} \right) - b_1 b_2 \frac{t}{N} + \frac{b_1 b_2 + b_3 b_4}{2}}{t(1 - \frac{t}{N})} dt \right)$$

Defining the centered and scaled function

$$(5.20) \quad v(t) = \sigma_{\kappa,N}^{(\text{VI})} \left(\frac{t}{N} \right) - b_1 b_2 \frac{t}{N} + \frac{b_1 b_2 + b_3 b_4}{2}$$

it is easily shown that v satisfies the equation

$$\begin{aligned}
(5.21) \quad & N^3 \left(v' + \frac{b_1 b_2}{N} \right) t^2 \left(1 - \frac{t}{N} \right)^2 (v'')^2 + \\
& - (N(N-2t)(v')^2 + (2vN + (b_1 b_2 - b_3 b_4)N - 2b_1 b_2 t)v' + 2vb_1 b_2)^2 \\
& + N^4 \prod_{i=1}^4 \left(v' + \frac{b_1 b_2 - b_i^2}{N} \right) = 0.
\end{aligned}$$

Now dividing both sides of (5.21) by N^4 and computing the limit $N \rightarrow \infty$, we arrive at the limiting equation

$$(5.22) \quad (tv'')^2 - (2(v')^2 - tv' + (2k + \kappa)v' + 2v)^2 + 4(v')^2(v' + k)(v' + \kappa + k).$$

which is precisely the σ -form of Painlevé V of (3.17) with parameter $\alpha = \kappa$. Thus

$$(5.23) \quad \lim_{N \rightarrow \infty} \mathbb{P} \left(\lambda_{\max}^{\text{JUE}_k^{(\kappa,N)}} < \frac{2u}{N} \right) = \exp \left(- \int_{2u}^{\infty} \frac{v(t)}{t} dt \right)$$

Clearly these scaling heuristics are consistent with Theorem 1.8.

5.3. Planar orthogonal polynomials and reduction to integrals over $U(N)$. We now consider the quantity (5.1) with $k = \gamma/2$ and γ a general real parameter up to the integrability constraint $\gamma > -2$. As with the Ginibre case, this average over non-Hermitian matrices can be reduced to an integral over the unitary group.

THEOREM 5.3. *Let $\gamma > -2$ and let $d\mu(U)$ denote the normalized Haar measure on the group of $N \times N$ unitary matrices. Then we have the identity*

$$(5.24) \quad \begin{aligned} R_\gamma(z) &= a_{N,\gamma,\kappa}^{(\text{tCUE})} \int_{U(N)} \det(U)^{-\frac{\gamma}{4}} |\det(I+U)|^{\frac{\gamma}{2}} \det(I+|z|^2U)^{\kappa+\frac{\gamma}{2}} d\mu(U) \\ &= \frac{a_{N,\gamma,\kappa}^{(\text{tCUE})}}{N!(2\pi)^N} \int_{[-\pi,\pi]^N} \prod_{j=1}^N e^{-\frac{i\gamma\theta_j}{4}} |1 + e^{i\theta_j}|^{\frac{\gamma}{2}} (1 + |z|^2 e^{i\theta_j})^{\kappa+\frac{\gamma}{2}} |\Delta(e^{i\theta})|^2 d\vec{\theta} \end{aligned}$$

where the constant is

$$(5.25) \quad a_{N,\gamma,\kappa}^{(\text{tCUE})} := \prod_{j=0}^{N-1} \frac{\Gamma(\frac{\gamma}{2} + j + 1)\Gamma(j + \kappa + 1)}{\Gamma(j + 1)\Gamma(\frac{\gamma}{2} + j + \kappa + 1)}.$$

Furthermore, we have the exact representation in terms of the Jimbo-Miwa-Okamoto σ -form of Painlevé VI:

$$(5.26) \quad R_\gamma(z) = R_\gamma(1) \exp\left(\int_{|z|^2}^1 \frac{h(t) - e_2't + \frac{1}{2}e_2}{t(1-t)} dt\right).$$

where h satisfies the equation

$$(5.27) \quad h'(t(1-t)h'')^2 + (h'(2h - (2t-1)) + \tilde{b}_1\tilde{b}_2\tilde{b}_3\tilde{b}_4)^2 = \prod_{j=1}^4 (h' + \tilde{b}_j^2)$$

with parameters

$$(5.28) \quad \begin{aligned} \tilde{b}_1 &= \frac{\kappa + N}{2}, & \tilde{b}_2 &= \frac{\kappa + \gamma + N}{2} \\ \tilde{b}_3 &= \frac{-\kappa + N}{2}, & \tilde{b}_4 &= -\frac{N + \gamma + \kappa}{2}. \end{aligned}$$

and e_2 (respectively e_2') is the elementary symmetric polynomial of degree 2 in $(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4)$ (respectively in $(\tilde{b}_1, \tilde{b}_3, \tilde{b}_4)$).

PROOF. By rotational symmetry the function $R_\gamma(z)$ only depends on $|z|$, so we set $z = x > 0$ without loss of generality. We will again start from orthogonal polynomials in the complex plane and apply the Green's theorem trick to reduce the planar orthogonality to contour orthogonality over the boundary of the unit disc. The proof is very similar to that of Theorem 3.3 so we only indicate the main points. Let $p_j(\lambda)$ denote the monic degree j polynomials satisfying the planar orthogonality ($k \leq j$)

$$(5.29) \quad h_j \delta_{j,k} = \int_{\mathbb{D}} P_j(\lambda) \bar{\lambda}^k |\lambda - x|^\gamma (1 - |\lambda|^2)^{\kappa-1} d^2\lambda.$$

Proceeding as in the Ginibre case [64], the analogue of identity (3.23) is

$$(5.30) \quad \begin{aligned} h_j \delta_{k,j} &:= \int_{\mathbb{D}} P_j(\lambda) \bar{\lambda}^k |\lambda - x|^\gamma (1 - |\lambda|^2)^{\kappa-1} d^2\lambda \\ &= \pi \frac{\Gamma(\frac{\gamma}{2} + j + 1)\Gamma(\kappa)}{\Gamma(\frac{\gamma}{2} + j + \kappa + 1)} \oint_{S^1} P_j(\lambda) \lambda^{-k} |\lambda - x|^\gamma (1 - x\lambda)^\kappa \frac{d\lambda}{2\pi i \lambda}. \end{aligned}$$

Again, the same reasoning that led to (3.26) gives, in this context, the Toeplitz determinant expression

$$(5.31) \quad R_\gamma(x) = \frac{N!}{\mathcal{Z}_N^{(\text{tCUE})}} \prod_{j=0}^{N-1} h_j = \frac{N!}{\mathcal{Z}_N^{(\text{tCUE})}} \prod_{j=0}^{N-1} \left(\pi \frac{\Gamma(\frac{\gamma}{2} + j + 1)\Gamma(\kappa)}{\Gamma(\frac{\gamma}{2} + j + \kappa + 1)} \right) \\ \times \det \left\{ \oint_{S^1} \lambda^{j-i} |\lambda - x|^\gamma (1 - x\lambda)^\kappa \frac{d\lambda}{2\pi i \lambda} \right\}_{i,j=0}^{N-1}.$$

Making appropriate changes of variables and deformations of the contour, the integrals inside the determinant can be written in the equivalent form

$$(5.32) \quad \oint_{S^1} \lambda^{j-i} |\lambda - x|^\gamma (1 - x\lambda)^\kappa \frac{d\lambda}{2\pi i \lambda} = (-x)^{j-i} \oint_{S^1} \lambda^{j-i} \lambda^{-\frac{\gamma}{4}} |1 + \lambda|^{\frac{\gamma}{2}} (1 + x^2\lambda)^{\kappa + \frac{\gamma}{2}} \frac{d\lambda}{2\pi i \lambda}$$

where we used identity (3.25). Now (5.24) follows from the same considerations given in the proof of Theorem 3.3. The association of the group integrals in (5.24) with Painlevé VI has been studied by Forrester and Witte, for example see Equations (1.6), (1.21), (1.34) and Proposition 13 in [38], from which identity (5.26) follows. \square

REMARK 5.4. *As with the Ginibre case, the identity (5.30) implies that the planar orthogonal polynomials are expressible in terms of certain polynomials orthogonal on the unit circle. These polynomials have been characterized in terms of fundamental objects associated with the Painlevé VI system, see [40]. Discrete N -recurrences for the group integrals in (5.26) were derived in [39]. As before, identities (5.24) and (5.4) (with $x = y = |z|$) imply a duality between averages over the CUE and the JUE. This has been discussed in the context of generalized hypergeometric functions in Section 3.3 of [38]. Its appearance here is a new manifestation of the duality arising from computing an average of truncated unitary matrices in two different ways. Interestingly, averages of this type have been shown to arise in last passage percolation and the Ising model, see Section 5.3 of [38].*

REMARK 5.5. *The constant $R_\gamma(1)$ in (5.26) can be computed explicitly from (5.24) using the identity (3.25) to reinstate the absolute value. This leads to the Morris integral evaluation,*

$$(5.33) \quad R_\gamma(1) = \frac{a_{N,\gamma,\kappa}^{(\text{tCUE})}}{N!(2\pi)^N} \int_{[-\pi,\pi]^N} \prod_{j=1}^N e^{\frac{i\kappa\theta_j}{2}} |1 + e^{i\theta}|^{\gamma+\kappa} |\Delta(e^{i\theta})|^2 d\vec{\theta} \\ = a_{N,\gamma,\kappa}^{(\text{tCUE})} \prod_{j=0}^{N-1} \frac{\Gamma(1 + \kappa + \gamma + j)\Gamma(1 + j)}{\Gamma(1 + \kappa + \frac{\gamma}{2} + j)\Gamma(1 + \frac{\gamma}{2} + j)} \\ = \prod_{j=0}^{N-1} \frac{\Gamma(\kappa + j + 1)\Gamma(\kappa + \gamma + j + 1)}{\Gamma(\kappa + \frac{\gamma}{2} + j + 1)^2}.$$

Writing this in terms of Barnes- G functions and inserting their asymptotic expansion gives

$$(5.34) \quad R_\gamma(1) = N^{\frac{\gamma^2}{4}} \frac{G(\kappa + \frac{\gamma}{2} + 1)^2}{G(\kappa + 1)G(\kappa + \gamma + 1)} (1 + o(1)), \quad N \rightarrow \infty$$

which is consistent with the limit $u \rightarrow 0$ of (1.19) with $k = \frac{\gamma}{2}$.

Appendix A. Relation to central limit theorems and Gaussian free field

In this appendix we will discuss the asymptotic expansion in Theorem 1.3 when z is in the exterior region $|z| > 1 + \delta$ where δ is fixed. As with the proof of Theorem 1.3, we take

the duality 3.1 as our starting point. In particular, rescaling equation (3.3) via $t_j \rightarrow t_j \sqrt{N}$ for each $j = 1, \dots, k$ gives the identity

$$(A.1) \quad \mathbb{E} \left(|\det(G_N - z)|^{2k} \right) = \frac{N^{k^2}}{\prod_{j=0}^{k-1} (j!(j+1)!)} I_N^{(2k)}(z)$$

where

$$(A.2) \quad I_N^{(2k)}(z) := \int_{[0, \infty)^k} \prod_{j=1}^k e^{-N\phi(t_j)} \Delta^2(\vec{t}) d\vec{t}$$

and

$$(A.3) \quad \phi(t) = t - \log(|z|^2 + t)$$

For $|z| > 1$ fixed, the function $\phi(t)$ achieves its minimum at $t = 0$, the endpoint of integration. We have,

$$(A.4) \quad \phi(t) = -2 \log |z| + t(1 - |z|^{-2}) + O(t^2), \quad t \rightarrow 0.$$

Therefore by classical saddle point arguments we get the approximation

$$(A.5) \quad \begin{aligned} I_N^{(2k)}(z) &\sim |z|^{2Nk} \int_{[0, \infty)^k} \prod_{j=1}^k e^{-Nt_j(1-|z|^{-2})} \Delta^2(\vec{t}) d\vec{t} \\ &= |z|^{2Nk} N^{-k^2} (1 - |z|^{-2})^{-k^2} \prod_{j=0}^{k-1} j!(j+1)! \end{aligned}$$

where we used the Selberg integral type formula

$$(A.6) \quad \int_{[0, \infty)^k} \prod_{j=1}^N e^{-t_j} \Delta^2(\vec{t}) d\vec{t} = \prod_{j=0}^{k-1} j!(j+1)!.$$

Now inserting (A.5) into (A.1) we obtain the asymptotic expansion

$$(A.7) \quad \mathbb{E} \left(|\det(G_N - z)|^{2k} \right) = e^{2Nk \log |z| - k^2 \log(1-|z|^{-2})} (1 + o(1)), \quad N \rightarrow \infty.$$

The expansion (A.7) could also be arrived at from the global fluctuation of smooth linear statistics studied in [6, 59]. In this sense, we have the interpretation

$$(A.8) \quad \mathbb{E} (|\det(G_N - z)|^\gamma) = \mathbb{E}(e^{\gamma L_N[f_z]})$$

where

$$(A.9) \quad L_N[f_z] := \log |\det(G - z)| = \sum_{j=1}^N f_z(\lambda_j), \quad f_z(\lambda) := \log |\lambda - z|,$$

and if $|z| > 1$, $f_z(\lambda)$ is a smooth function of λ on the eigenvalue support. In fact consider a general *linear statistic*

$$(A.10) \quad L[f] := \sum_{j=1}^N f(\lambda_j)$$

where λ_j are the N eigenvalues of the complex Ginibre random matrix G_N . Then for sufficiently smooth f , it holds that

$$(A.11) \quad \mathbb{E}(e^{\gamma L_N[f]}) = e^{N\gamma m_1(f) + \frac{\gamma^2}{2} m_2(f)} (1 + o(1)), \quad N \rightarrow \infty.$$

where the asymptotic mean is

$$(A.12) \quad m_1 = \frac{1}{\pi} \int_{\mathbb{D}} f(\lambda) d^2 \lambda$$

and the asymptotic variance is given by

$$(A.13) \quad m_2 = \frac{1}{4\pi} \int_{\mathbb{D}} |\nabla f(\lambda)|^2 d^2\lambda.$$

The gradient squared term in (A.13) is associated with Gaussian free field fluctuations. In the particular case of $f_z(\lambda) = \log|\lambda - z|$, $|z| > 1$, the mean and variance can be computed explicitly from these formulae as $m_1 = \log|z|$ and $m_2 = -\frac{1}{2} \log(1 - |z|^{-2})$. Setting $k = \gamma/2$ then shows the agreement between (A.7) and (A.11). This immediately gives the supercritical asymptotic expansion of the partition function given in (3.46). On the other hand when $|z| \leq 1$ or z is too close to the boundary, $f(\lambda)$ lacks smoothness on the support of the eigenvalues and the asymptotic expansion (A.11) takes a more complicated form, as Theorems 1.2 and 1.3 demonstrate.

Acknowledgements

The authors would like to thank Gernot Akemann, Yan V. Fyodorov, Arno B. J. Kuijlaars and Seung-Yeop Lee for helpful discussions on the results of this work. We thank Peter J. Forrester for comments and for pointing out reference [46] to us. A. D. gratefully acknowledges financial support from EPSRC, First Grant project ‘‘Painlevé equations: analytical properties and numerical computation’’, reference EP/P026532/1. N. S. gratefully acknowledges support of the Royal Society University Research Fellowship ‘‘Random matrix theory and log-correlated Gaussian fields’’.

References

- [1] M. Adler and P. van Moerbeke. Hermitian, symmetric and symplectic random ensembles: PDEs for the distribution of the spectrum. *Ann. of Math. (2)*, 153(1):149–189, 2001.
- [2] M. Adler and P. van Moerbeke. Integrals over classical groups, random permutations, Toda and Toeplitz lattices. *Comm. Pure Appl. Math.*, 54(2):153–205, 2001.
- [3] M. Adler, P. van Moerbeke, and P. Vanhaecke. Moment matrices and multi-component KP, with applications to random matrix theory. *Comm. Math. Phys.*, 286(1):1–38, 2009.
- [4] Ie. Afanasiev. On the Correlation Functions of the Characteristic Polynomials of Non-Hermitian Random Matrices with Independent Entries. *J. Stat. Phys.*, 2019.
- [5] G. Akemann and G. Vernizzi. Characteristic polynomials of complex random matrix models. *Nuclear Phys. B*, 660(3):532–556, 2003.
- [6] Y. Ameur, H. Hedenmalm, and N. Makarov. Fluctuations of eigenvalues of random normal matrices. *Duke Math. J.*, 159(1):31–81, 2011.
- [7] Y. Ameur, N-G. Kang, and N. Makarov. Rescaling Ward Identities in the Random Normal Matrix Model. *Constr. Approx.*, 50(1):63–127, 2019.
- [8] Y. Ameur, N-G. Kang, N. Makarov, and A. Wennman. Scaling limits of random normal matrix processes at singular points. eprint = [arXiv:1510.08723](https://arxiv.org/abs/1510.08723), 2015.
- [9] Y. Ameur, N-G. Kang, and S-M. Seo. The random normal matrix model: insertion of a point charge. eprint = [arXiv:1804.08587](https://arxiv.org/abs/1804.08587), 2018.
- [10] A. Assiotis and J. Keating. Moments of moments of characteristic polynomials of random unitary matrices and lattice point counts. eprint = [arXiv:1905.06072](https://arxiv.org/abs/1905.06072), 2019.
- [11] E. C. Bailey, S. Bettin, G. Blower, B. Conrey, A. Prokhorov, M. O. Rubinstein, and N. C. Snaith. Mixed moments of characteristic polynomials of random unitary matrices. eprint = [arXiv:1901.07479](https://arxiv.org/abs/1901.07479), 2019.
- [12] E. C. Bailey and J. P. Keating. On the Moments of the Moments of the Characteristic Polynomials of Random Unitary Matrices. *Comm. Math. Phys.*, 2019.
- [13] F. Balogh, M. Bertola, S-Y. Lee, and K. D. McLaughlin. Strong asymptotics of the orthogonal polynomials with respect to a measure supported on the plane. *Comm. Pure Appl. Math.*, 68(1):112–172, 2015.
- [14] F. Balogh, T. Grava, and D. Merzi. Orthogonal polynomials for a class of measures with discrete rotational symmetries in the complex plane. *Constr. Approx.*, 46(1):109–169, 2017.
- [15] F. Balogh and D. Merzi. Equilibrium measures for a class of potentials with discrete rotational symmetries. *Constr. Approx.*, 42(3):399–424, 2015.
- [16] R. Bauerschmidt, P. Bourgade, M. Nikula, A., and H-T. Yau. The two-dimensional Coulomb plasma: quasi-free approximation and central limit theorem. eprint = [arXiv:1609.08582](https://arxiv.org/abs/1609.08582), 2016.

- [17] M. Bertola, J. G. Elias Rebelo, and T. Grava. Painlevé IV critical asymptotics for orthogonal polynomials in the complex plane. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 14:Paper No. 091, 34, 2018.
- [18] A. Bogatskiy, T. Claeys, and A. Its. Hankel determinant and orthogonal polynomials for a Gaussian weight with a discontinuity at the edge. *Comm. Math. Phys.*, 347(1):127–162, 2016.
- [19] A. Borodin and P. Deift. Fredholm determinants, Jimbo-Miwa-Ueno τ -functions, and representation theory. *Comm. Pure Appl. Math.*, 55(9):1160–1230, 2002.
- [20] A. Borodin and C. D. Sinclair. The Ginibre ensemble of real random matrices and its scaling limits. *Comm. Math. Phys.*, 291(1):177–224, 2009.
- [21] R. Boyer and W. M. Y. Goh. On the zero attractor of the Euler polynomials. *Adv. in Appl. Math.*, 38(1):97–132, 2007.
- [22] E. Brézin and S. Hikami. Characteristic polynomials of random matrices. *Comm. Math. Phys.*, 214(1):111–135, 2000.
- [23] E. Brézin and S. Hikami. Characteristic polynomials of random matrices at edge singularities. *Phys. Rev. E (3)*, 62(3, part A):3558–3567, 2000.
- [24] T. Claeys and B. Fahs. Random matrices with merging singularities and the Painlevé V equation. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 12:Paper No. 031, 44, 2016.
- [25] T. Claeys and I. Krasovsky. Toeplitz determinants with merging singularities. *Duke Math. J.*, 164(15):2897–2987, 2015.
- [26] Benoît Collins. Product of random projections, Jacobi ensembles and universality problems arising from free probability. *Probab. Theory Related Fields*, 133(3):315–344, 2005.
- [27] E. Daems and A. B. J. Kuijlaars. A Christoffel-Darboux formula for multiple orthogonal polynomials. *J. Approx. Theory*, 130(2):190–202, 2004.
- [28] E. Daems and A. B. J. Kuijlaars. Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions. *J. Approx. Theory*, 146(1):91–114, 2007.
- [29] D. Dai and A. B. J. Kuijlaars. Painlevé IV asymptotics for orthogonal polynomials with respect to a modified Laguerre weight. *Stud. Appl. Math.*, 122(1):29–83, 2009.
- [30] *NIST Digital Library of Mathematical Functions*. <http://dlmf.nist.gov/>, Release 1.0.20 of 2018-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.
- [31] J. Fischmann, W. Bruzda, B. A. Khoruzhenko, H.-J. Sommers, and K. Życzkowski. Induced Ginibre ensemble of random matrices and quantum operations. *J. Phys. A*, 45(7):075203, 31, 2012.
- [32] A. S. Fokas, A. R. Its, A. A. Kapaev, and V. Y. Novokshenov. *Painlevé transcendents. The Riemann-Hilbert approach*, volume 128 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.
- [33] P. J. Forrester. Quantum conductance problems and the Jacobi ensemble. *J. Phys. A*, 39(22):6861–6870, 2006.
- [34] P. J. Forrester. *Log-gases and random matrices*, volume 34 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2010.
- [35] P. J. Forrester and E. M. Rains. Matrix averages relating to Ginibre ensembles. *J. Phys. A*, 42(38):385205, 13, 2009.
- [36] P. J. Forrester and N. S. Witte. Application of the τ -function theory of Painlevé equations to random matrices: PIV, PII and the GUE. *Comm. Math. Phys.*, 219(2):357–398, 2001.
- [37] P. J. Forrester and N. S. Witte. Application of the τ -function theory of Painlevé equations to random matrices: P_V, P_{III}, the LUE, JUE, and CUE. *Comm. Pure Appl. Math.*, 55(6):679–727, 2002.
- [38] P. J. Forrester and N. S. Witte. Application of the τ -function theory of Painlevé equations to random matrices: P_{VI}, the JUE, CyUE, cJUE and scaled limits. *Nagoya Math. J.*, 174:29–114, 2004.
- [39] P. J. Forrester and N. S. Witte. Discrete Painlevé equations, orthogonal polynomials on the unit circle, and N -recurrences for averages over $U(N)$ —P_{III'} and P_V τ -functions. *Int. Math. Res. Not.*, (4):160–183, 2004.
- [40] P. J. Forrester and N. S. Witte. Bi-orthogonal polynomials on the unit circle, regular semi-classical weights and integrable systems. *Constr. Approx.*, 24(2):201–237, 2006.
- [41] Y. V. Fyodorov, J. Grela, and E. Strahov. On characteristic polynomials for a generalized chiral random matrix ensemble with a source. *J. Phys. A*, 51(13):134003, 30, 2018.
- [42] Y. V. Fyodorov and J. P. Keating. Freezing transitions and extreme values: random matrix theory, and disordered landscapes. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 372(2007):20120503, 32, 2014.
- [43] Y. V. Fyodorov and B. A. Khoruzhenko. On absolute moments of characteristic polynomials of a certain class of complex random matrices. *Comm. Math. Phys.*, 273(3):561–599, 2007.
- [44] Y. V. Fyodorov, B. A. Khoruzhenko, and N. J. Simm. Fractional Brownian motion with Hurst index $H = 0$ and the Gaussian unitary ensemble. *Ann. Probab.*, 44(4):2980–3031, 2016.

- [45] Y. V. Fyodorov and H.-J. Sommers. Random matrices close to Hermitian or unitary: overview of methods and results. *J. Phys. A: Math. Gen.*, 36(12):3303–3347, 2003.
- [46] L. Haine and J-P. Semengue. The Jacobi polynomial ensemble and the Painlevé VI equation. *J. Math. Phys.*, 40(4):2117–2134, 1999.
- [47] H. Hedenmalm and A. Wennman. Planar orthogonal polynomials and boundary universality in the random normal matrix model. eprint = [arXiv:1710.06493](https://arxiv.org/abs/1710.06493), 2017.
- [48] C. P. Hughes, J. P. Keating, and N. O’Connell. On the characteristic polynomial of a random unitary matrix. *Comm. Math. Phys.*, 220(2):429–451, 2001.
- [49] C. Itzykson and J. B. Zuber. The planar approximation. II. *J. Math. Phys.*, 21(3):411–421, 1980.
- [50] M. Jimbo, T. Miwa, Y. Mōri, and M. Sato. Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent. *Phys. D*, 1(1):80–158, 1980.
- [51] E. Kanzieper. Exact replica treatment of non-Hermitian complex random matrices. *Frontiers in Field Theory*, pages 23–51, 2005.
- [52] S. Karlin and J. McGregor. Coincidence probabilities. *Pacific J. Math.*, 9:1141–1164, 1959.
- [53] J. P. Keating and N. C. Snaith. Random matrix theory and $\zeta(1/2 + it)$. *Comm. Math. Phys.*, 214(1):57–89, 2000.
- [54] B. A. Khoruzhenko and H.-J. Sommers. Non-Hermitian ensembles. In *The Oxford handbook of random matrix theory*, pages 376–397. Oxford Univ. Press, Oxford, 2011.
- [55] T. Leblé and S. Serfaty. Large deviation principle for empirical fields of log and Riesz gases. *Invent. Math.*, 210(3):645–757, 2017.
- [56] S.-Y. Lee and M. Yang. Discontinuity in the asymptotic behavior of planar orthogonal polynomials under a perturbation of the Gaussian weight. *Comm. Math. Phys.*, 355(1):303–338, 2017.
- [57] S.-Y. Lee and M. Yang. Planar orthogonal polynomials as Type II multiple orthogonal polynomials. *J. Phys. A: Math. Gen.*, 52(27), 2019.
- [58] S. M. Nishigaki and A. Kamenev. Replica treatment of non-Hermitian disordered Hamiltonians. *J. Phys. A: Math. Gen.*, 35(21), 2002.
- [59] B. Rider and B. Virág. The noise in the circular law and the Gaussian free field. *Int. Math. Res. Not. IMRN*, (2):Art. ID rnm006, 33, 2007.
- [60] C. A. Tracy and H. Widom. Fredholm determinants, differential equations and matrix models. *Comm. Math. Phys.*, 163(1):33–72, 1994.
- [61] C. A. Tracy and H. Widom. On the distributions of the lengths of the longest monotone subsequences in random words. *Probab. Theory Related Fields*, 119(3):350–380, 2001.
- [62] W. Van Assche. Orthogonal and multiple orthogonal polynomials, random matrices, and Painlevé equations. eprint = [arXiv:1904.07518](https://arxiv.org/abs/1904.07518), 2019.
- [63] C. Webb. The characteristic polynomial of a random unitary matrix and Gaussian multiplicative chaos—the L^2 -phase. *Electron. J. Probab.*, 20:no. 104, 21, 2015.
- [64] C. Webb and M. D. Wong. On the moments of the characteristic polynomial of a Ginibre random matrix. *J. London Math. Soc.*, 118(5):1017–1056, 2017.
- [65] N. S. Witte, P. J. Forrester, and C. M. Cosgrove. Gap probabilities for edge intervals in finite Gaussian and Jacobi unitary matrix ensembles. *Nonlinearity*, 13(5):1439–1464, 2000.
- [66] K. Życzkowski and H.-J. Sommers. Truncations of random unitary matrices. *J. Phys. A*, 33(10):2045–2057, 2000.

(A. Deaño)

SCHOOL OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE. UNIVERSITY OF KENT, CANTERBURY CT2 7NF, UNITED KINGDOM. A.Deano-Cabrera@kent.ac.uk

(N. Simm)

MATHEMATICS DEPARTMENT. UNIVERSITY OF SUSSEX, FALMER CAMPUS, BRIGHTON, BN1 9RH, UNITED KINGDOM. n.j.simm@sussex.ac.uk