Muhamed, Abera Ayalew (2015) ***Moduli Spaces of Topological Solitons.*** **Doctor of Philosophy (PhD) thesis, University of Kent,.**

MODULI SPACES OF TOPOLOGICAL SOLITONS

ABERA AYALEW MUHAMED

A THESIS SUBMITTED TO
THE UNIVERSITY OF KENT AT CANTERBURY
IN THE SUBJECT OF APPLIED MATHEMATICS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY BY RESEARCH

March 2015

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Acknowledgements

It is my pleasure to extend my deepest gratitude and indebtedness to Dr. Steffen Krusch, who has been helpful to me, supervised this thesis irrespective of enormous commitment and for enlightening with his genuine advice and giving open discussion. He has offered valuable guidance and support. I would also like to express my gratitude to Dr. Jim Shank for his guidance and discussions.

I want to thank the University of Kent for giving me the opportunity to conduct the research in an academically stimulating environment. In particular. I thank the School of Mathematics, Statistics and Actuarial Science for providing me with the needed financial support for my tuition and maintenance throughout my stay in Kent.

My greatest thanks to my family for a tremendous amount of love and encouragement . My wife; Ruhy, I thank you for your support, patience and unfailing love. My brother; Assen, I should say really thank you as you always encourage in an indescribable way. I dedicate this thesis to my brother; Assen Ayalew. My family have played a crucial role and inspired me to pursue knowledge to the best of my abilities all through my life. Without their love, support and guidance, I would not have come this far, I thank you family. Above all, thanks to The Most High Almighty God (Allah) for giving me the opportunity, the grace and the strength to successfully complete this thesis.

Declaration

Apart from chapter 1 and 2 that have an expository nature, the candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

Chapter 6 had collaborative work with Alexander Cockburn. However, most of the work, calculation of the formulas and analysis was done before the discussion when he was visiting Kent.

Abstract

This thesis presents a detailed study of phenomena related to topological solitons (in 2-dimensions). Topological solitons are smooth, localised, finite energy solutions in non-linear field theories. The problems are about the moduli spaces of lumps in the projective plane and vortices on compact Riemann surfaces.

Harmonic maps that minimize the Dirichlet energy in their homotopy classes are known as lumps. Lump solutions in real projective space are explicitly given by rational maps subject to a certain symmetry requirement. This has consequences for the behaviour of lumps and their symmetries. An interesting feature is that the moduli space of charge 3 lumps is a 7-dimensional manifold of cohomogeneity one. In this thesis, we discuss the charge 3 moduli space, calculate its metric and find explicit formula for various geometric quantities. We discuss the moment of inertia (or angular integral) of moduli spaces of charge 3 lumps. We also discuss the implications for lump decay. We discuss interesting families of moduli spaces of charge 5 lumps using the symmetry property and Riemann-Hurwitz formula. We discuss the Kähler potential for lumps and find an explicit formula on the 1-dimensional charge 3 lumps.

The metric on the moduli spaces of vortices on compact Riemann surfaces where the fields have zeros of positive multiplicity is evaluated. We calculate the metric, Kähler potential and scalar curvature on the moduli spaces of hyperbolic 3- and some submanifolds of 4-vortices. We construct collinear hyperbolic 3- and 4-vortices and derive explicit formula of their corresponding metrics. We find interesting subspaces in both 3- and 4-vortices on the hyperbolic plane and find an explicit formula for their respective metrics and scalar curvatures.

We first investigate the metric on the totally geodesic submanifold $\Sigma_{n,m}$, $n + m = N$ of the moduli space M_N of hyperbolic N-vortices. In this thesis, we discuss the Kähler potential on $\Sigma_{n,m}$ and an explicit formula shall be derived in three different approaches. The first is using the direct definition of Kähler potential. The second is based on the regularized action in Liouville theory. The third method is applying a scaling argument. All the three methods give the same result. We discuss the geometry of $\Sigma_{n,m}$, in particular when $n = m = 2$ and $m = n - 1$. We evaluate the vortex scattering angle-impact parameter relation and discuss the $\frac{\pi}{2}$ vortex scattering of the space $\Sigma_{2,2}$. Moreover, we study the $\frac{\pi}{n}$ vortex scattering of the space $\Sigma_{n,n-1}$. We also compute the scalar curvature of $\Sigma_{n,m}$.

Finally, we discuss vortices with impurities and calculate explicit metrics in the presence of impurities.

Chapter 1

Introduction

Although the name topological solitons was not mentioned, the idea of topological solitons has its origins as far back as the work of Lord Kelvin to describe the knotted vortex model of atoms [67]. Moreover, a theory in the 1920s by Dirac [12] is another example. Modern topological solitons research began with Abrikosov vortices [1] in superconductors and the Skyrme model [53,54]. Quantum field theory is a theoretical framework for constructing models in theoretical physics (see for instance [49, 70]). This theory became popular in the 1950s following the work of Feynman, Schwinger and Tomonaga. Physicists and mathematicians started to exhibit the classical field equations in their fully non-linear form and they explained some of the solutions as candidates for a particle of the theory.

Mainly, the topological behaviour of the field can be determined by the winding number N of the field [36]. This winding number N is also called the topological charge or degree, and can be considered as the net number of individual particles. A single particle, $N = 1$, is called a topological soliton. The minimal energy field configuration for this case is smooth, classically stable and concentrated in some finite region of space. In this sense, they behave like ordinary particles. They are also stable solutions of systems of partial differential equations. The soliton numbers are stable because they carry a topological charge N. Since the topological charge is a conserved quantity, a single soliton cannot decay, that is the soliton number is conserved due to a topological constraint, such as a winding number [36].

A powerful result regarding the existence of topological solitons is Derrick's theorem [11]. It states that under spatial rescaling if there is no stationary point of the energy, as a function of the scaling, then there are no non-vacuum solutions of the associated Euler-Lagrange equation. In many field theories the variation of the energy functional against certain spatial re-scaling is never zero for any static non-vacuum field configuration and therefore these theories do not have localized solitons.

Bogomolnyi [5] showed for several field theories that the energy E and the topological charge N satisfy the inequality $E \geq \pi|N|$. This bound is known as the Bogomolnyi bound. Equality holds if the fields, which minimize the energy, satisfy the Bogomolyni equations. Their solutions are static and stable, and all solutions of the same charge have the same energy. These field theories are called of Bogomolyni type, and the stable, static minimal energy solutions of the Bogomolyni equations form a moduli space.

The classical low-energy dynamics of monopoles has been studied by Manton [30] using the geodesic approximation. He conjectured that the N-soliton trajectories are approximated by geodesics in the moduli space of static N-solitons with respect to the L^2 metric induced by the kinetic energy functional of the field theory. It was rigorously proven by Stuart [62] that the approximation works. This idea was also extended by Ward [69] who exploited this approach in order to understand the dynamics of N moving lumps by restricting the field dynamics to M_N. The geodesics on the moduli space M_N of lumps, when traversed at slow speed, are the low-energy dynamical solutions of the model, in which N lumps move slowly and interact on Σ. Manton's approximation has been used to study many solitons see e.g. [57, 61, 62].

It is impossible to present all the historical innovation and development regarding topological solitons. An excellent explanation to topological solitons can be found in [36]. Instead, we mainly focus on topological solitons known as lumps and vortices.

The results presented in chapter 3 are related to the $\mathbb{C}P^1$ sigma model. The $\mathbb{C}P^1$-sigma model is a $2+1$-dimensional field theory of Bogomolnyi type. The $O(3)$-sigma model admits static stable finite energy solutions in the plane [28] and is equivalent to the $\mathbb{C}P^1$-sigma model in the classical sense. In this case the Bogomolyni equations, also known as the self-dual equations, are in fact the Cauchy-Riemann equations.

Rational maps of degree N are solutions of the Bogomolny equation of the $\mathbf{O}(3)$ sigma model with topological charge N and energy $2\pi|N|$. Belavin and Polyakov [45] studied the Bogomolyni equations by a change of variables, and they exploited the Lagrangian density of the $\mathbb{C}P^1$ sigma model from the Lagrangian density of the $O(3)$-sigma model. The algebraic topology of rational maps and the construction of harmonic maps between surfaces have been studied by Segal [52] and by Eells and Lemaire [15], respectively. Speight and Sadun [50] showed the moduli space for a compact Riemann surface is geodesically incomplete and so is the metric. The metric on the space of holomorphic maps is given by restricting the kinetic energy term where the moduli are allowed to depend on time. This metric is Kähler [48]. The low energy dynamics of a $\mathbb{C}P^1$ lump on the space-time $S^2 \times \mathbb{R}$ [56] and the geometry of a space of rational maps of degree N [38] have been studied. The Fubini-Study metric γ_{FS} of rational maps of degree one has been studied by Krusch and Speight [25], here identifying a rational map with the projective equivalence classes of its coefficients such that M_1 is an open subset of $\mathbb{C}P^3$ and $\mathbb{C}P^3$ is equipped with the Fubini-Study metric of constant holomorphic sectional curvature 4. In this thesis, we evaluate the Fubini-Study metric on a particular moduli space of charge three rational maps and also calculate the Kähler potential of totally geodesic submanifolds of M_1. Lumps can decay but it has been shown in [27, 69] that the scattering of lumps takes place before lump decay using the geodesic approximation. A head-on collisions between lumps in the $2+1$ dimensional $\mathbb{C}P^1$ model on a flat torus has been studied numerically by Cova and Zakrzewski [10] and analytically by Speight [58]. Rational maps also play an important role in related models. For example, the rational map ansatz gives a good approximations for the symmetries of Skyrme configurations and the Finkelstein-Rubinstein constraint can be calculated directly from this ansatz using homotopy theory [22]. We study the symmetries of rational maps to understand the geometry and dynamics of lumps.

Harmonic maps are solutions of Laplace's equation on Riemannian manifolds. They minimize the energy within their homotopy class and are usually known as lumps. Denote by M_N the moduli space of degree N lumps. M_N is a $2N+1$-dimensional smooth complex Riemannian manifold. There is a natural Riemannian metric on M_N, which is called the L^2 metric. The L^2 metric is undefined when the physical space is plane $\mathbb{R}^2$ because some zero modes are non-normalizable [69] and hence some coordinates on the moduli space are frozen [27]. However, the L^2 metric is well defined on a compact Riemann surface as these non-normalizable zero modes are absent [38]. The main aim of this chapter is to understand the L^2 geometry of the moduli spaces of lumps. The L^2 metric γ can be derived from the restriction of the kinetic energy functional. Note that harmonic maps are extrema of the Dirichlet energy functional. For the $\mathbb{C}P^1$ model, one can have an explicit expression of harmonic maps in terms of rational maps. Rational maps give exact descriptions and show the symmetries of the lumps. A rational map is a function given by the ratio of two polynomials with no common roots. Lumps could be understood in terms of rational maps on the projective plane with their metric and lump decay. Speight [59] studied the L^2 metric on the moduli spaces of degree 1 harmonic maps on both S^2 and $\mathbb{R}P^2$ and obtained an explicit formula. We focus mainly on charge three rational maps between $\mathbb{R}P^2$, acquiring a detailed and careful understanding of their L^2 geometry. The L^2 metric plays an important role in slow lump dynamics just as Samols' [51] metric does for vortices.

The results given in chapter 4, 5 and 6 are related to the moduli spaces of vortices. Vortices are topologically stable time-independent solutions to a set of classical field equations that have finite energy. Vortices in two dimensions are particle-like and in three dimensions form vortex curves [32]. Vortices are the static solutions of the Ginzburg-Landau equations. These equations are the Euler-Lagrange equations for the action functional given by the Ginzburg-Landau Lagrangian [36].

Solving the Bogomolyni equation of Abelian vortices is equivalent to solving the Kazdan-Warner equation [7]. In [4], the relation between Abelian vortices and Riemannian metrics has been formulated with a natural understanding of vortices as degenerate Hermitian metrics which satisfy some curvature equation. No explicit solutions for vortices on flat space are known; however, Witten [71] noted the Bogomolyni equations on the hyperbolic plane $\mathcal{H}$ are integrable and Strachan [60] calculated explicitly the metric on 2-vortex moduli space.

The nature of interactions among vortices are determined by the coupling constant λ [26, 36]. For $\lambda < 1$, vortices attract and for $\lambda > 1$ vortices repel. When the coupling constant λ takes the critical value 1, there are no net static forces between the vortices. This value separates Type I ($\lambda < 1$) and Type II ($\lambda > 1$) superconductivity. For critical coupling $\lambda = 1$, there exist static configurations satisfying the first order Bogomolyni equations. Then the Abelian Higgs model admits static and finite energy Bogomolnyi solutions which minimize the energy in their homotopy classes. The N-vortex solutions in $\mathbb{C}$ can be uniquely characterized by where the Higgs field ϕ vanishes and the multiplicities of the zeros of ϕ are all positive where the sum of multiplicities is the winding number N of the solution [21,64,65]. The N unordered Higgs zero locations in $\mathbb{C}$ are therefore the natural coordinates parameterizing the space of static N-vortex solutions. This space is called the moduli space for N vortices, denoted by M_N. M_N has a natural Kähler structure inherited from the kinetic terms of the Lagrangian.

Manton and Rink [33] postulated that hyperbolic vortices can be constructed geometrically from holomorphic maps between hyperbolic surfaces. We shall consider Ginzburg-Landau vortices moving on the hyperbolic plane and the Bogomolyni equations for static hyperbolic N-vortices. These equations can be reduced to an integrable Liouville's equation on a disc. In [60], Strachan showed an implicit formula for the Riemannian metric γ based on the Higgs field near the vortex center and he evaluated an explicit formula for the moduli space of M_1 and M_2. For $N \geq 3$, the computation of an explicit metric γ is complicated. We study the case $N = 3$ and submanifolds of the case $N = 4$. We mainly compute and study the metric, Kähler potential and scalar curvature of hyperbolic 3- and 4-vortices for the Abelian Higgs model with coupling constant $\lambda = 1$, using the Poincaré disc model. We shall derive a generalized Samols' metric on the moduli space of vortices when the zeros of the Higgs field have any multiplicity. The geometric properties of hyperbolic double vortices will also be studied.

The exact moduli space metrics on some totally geodesic submanifolds $\Sigma_{r,t}$ of the moduli space of static hyperbolic N-vortices [24] for $r > t + 1$ or $r = t + 1$ and the metric of N vortices on regular N-gon [26] were studied. Using a scaling argument and the Polyakov conjecture [44] by considering the work in [63], Manton and Chen [9] calculated the Kähler potential of Abelian Higgs vortices on $\mathbb{R}$ and upper half-plane $\mathcal{H}$ with Samols' metric. We evaluate the Kähler potential for the metric on the symmetric space $\Sigma_{r,t}$ of the moduli space M_N.

The geometry of $\Sigma_{2,0}$ in [60] and $\Sigma_{2,1}$ in [24] were discussed in detail. In [32], the dynamics of vortices can be understood in terms of the geometry of the moduli space M_N of static N-solitons since the moduli space approximation evaluation of the partition function of vortices can be reduced to the computation of the volume of the moduli space. The interactions of critically coupled vortices that do not exert a force on each other when at rest do however affect one another when in motion [57]. The numerical work in [40] implied the scattering of critically coupled vortices is non-trivial. Thatcher and Morgan [66] studied intervortex forces in the numerical simulations that show the scattering of two vortices depends on the phase gradient of the Higgs fields. We shall study the geometry of $\Sigma_{2,2}$, for example, the metric, curvature and the relation between the impact parameter-scattering angle are calculated. There is a $\frac{\pi}{2}$ scattering in this space. Furthermore, we evaluate an explicit formula for the scattering-impact parameter relation and (Matlab plot) show $\frac{\pi}{r}$ scattering for the symmetric space $\Sigma_{r,r-1}$.

The BPS dynamics of vortices in the presence of electric and magnetic impurities was studied by Tong and Wong [68]. Although the presence of electric impurities altered the dynamics accompanied by a connection term, the metric remains invariant. Magnetic impurities altered both the metric and the dynamics of vortices. We shall investigate vortices with impurities and evaluate an explicit metric on moduli spaces of vortices with magnetic impurities.

This thesis is organized as follows. In chapter 2 the concept of homotopy theory and fundamental group are introduced. The main results of the thesis are in the following four chapters. Chapter 3 is the moduli spaces of lumps in a projective space. In chapter 4 the moduli spaces of hyperbolic vortices are studied. Chapter 5 is geometry and dynamics of vortices. In chapter 6 vortices with impurities is presented. A conclusion to this thesis is provided in chapter 7.

Chapter 2

Homotopy in Topological Solitons

2.1 Homotopy Theory

This section investigates the homotopy theory which is very important to study topological solitons. Algebraic topology plays an important role in understanding the classification and stability of solitons in field theory. Homotopy theory studies topological objects up to homotopy equivalence. Homotopy equivalence is a weaker relation than topological equivalence, i.e., homotopy classes of spaces are larger than homeomorphism classes. Homotopy equivalence plays a more important role than homeomorphism, essentially homotopy groups are invariant with respect to homotopy equivalence. The classification of solitons in field theories uses homotopy theory and topological degree. The classification depends on both the domain and target spaces. In the homotopy sense, we can demonstrate for two configuration spaces whether two maps can be continuously deformed into each other or not. This deformation is called homotopy. Time evolution of the system is an example of homotopy in the physics sense. If the field configurations can be continuously deformed into each other, they are equivalent or equal; otherwise, they are different. Topological features in field theories refer to the properties of the fields. The set of possible field configurations are classified according to their topological properties and only the topology of scalar fields is considered here.

Two configurations are said to be topologically equivalent if it is possible to deform one of them continuously into the other. This is an equivalence relation and therefore divides the set of all field configurations of the same topology. Hence, topologically different field configurations are forbidden by an infinite energy barrier from being transformed into one another. The classification of fields based on their topological features is carried out by dividing the field into homotopy classes. It is not guaranteed that static, localized solutions of the field equations are found in all classes. The space of all possible field configurations may be given a non-trivial topology by the condition that some functional of the various fields is finite. In classical field theory this functional is potential energy (or total energy, if non-static).

For a map $\phi : X \to Y$ which is a field at a given time, homotopy theory can be applied to a scalar field theory governed by a Lagrangian. If the field satisfies the Euler-Lagrange equation, then it is continuous in space and time. Thus, its homotopy class is well defined and invariant with respect to time. The homotopy class is a topological conserved quantity. Hence, homotopy theory can be applied to field theories defined in a manifold of finite dimension. Let $\phi : X \to Y$ be a continuous map between two manifolds. Let $x_0 \in X, y_0 \in Y$ be such that $\phi(x_0) = y_0$. Then ϕ is a based map.

Definition 1. *Let $I = [0, 1]$ be an interval. Two continuous maps $f, g : X \to Y$ are homotopic, denoted by $f \simeq g$, if there exists a continuous map $F : I \times X \to Y$ such that $F(0, x) = f(x)$ and $F(1, x) = g(x)$ for all $x \in X$.*

Note that $f \simeq g$ means f and g can be continuously deformed into each other. Homotopy is symmetric because the time interval can be reversed, it is transitive because the unit interval can be adjoined and scaled, and it is reflexive because f is trivially homotopic to itself. Since homotopy is an equivalence relation, maps $f : X \to Y$ can be classified into equivalence classes, denoted by $[f]$. For example, let $c : X \to \mathbb{R}$, $c(x) = c$ be a constant map. Then the class of maps homotopic to the constant map c consists of all real continuous functions $f(x)$, with homotopy given by $F(x,t) = (1-t)f(x) + tc, t \in [0,1]$.

Definition 2. *A map $f : X \to Y$ is a homotopy equivalence if there is a map $g : Y \to X$ such that $fg \simeq 1_Y$ and $gf \simeq 1_X$. We can write $X \simeq Y$ and say that X and Y have the same homotopy type.*

This defines an equivalence relation on the set of topological spaces, and this equivalence is strictly weaker than that of being homeomorphic.

Definition 3. *Let X and Y be topological spaces. Let A be a subspace of X. Let $f, g : X \to Y$ be continuous maps from X to Y such that $f|_A = g|_A$. We say that f and g are homotopic relative to A, denoted by $f \simeq g_{rel} A$, if and only if there exists a homotopy $H : X \times I \to Y$ such that*

$$H(x,0) = f(x), \quad H(x,1) = g(x), \forall x \in X \quad \textit{and} \quad H(a,t) = f(a) = g(a), \forall a \in A.$$

Note that homotopy relative to a subspace of X is also an equivalence relation on the set of all continuous maps between topological spaces X and Y.

2.1.1 Fundamental Group

The fundamental group of a space X is defined so that its elements are homotopy classes of loops in X starting and ending at a fixed base point $x_0 \in X$. That is the homotopy classes of based loops on X.

Definition 4. *In a topological space X, we define a path as a continuous map $\alpha : I \longrightarrow X$ where I is the unit interval $[0,1]$. A path or path class is called a loop, if the initial and terminal points are the same.*

Let $x_0 \in X$. Define an equivalence relation on the set of all loops based at x_0, where two such loops α and β are equivalent if and only if $\alpha \simeq \beta_{rel}\{0,1\}$. This equivalence class is referred to as the based homotopy class of loop α. The set of equivalence classes of loops based at x_0 is denoted by $\pi_1(X, x_0)$. All paths each of which is homotopic to a path α is called a homotopy class of α, denoted by $[\alpha]$. Thus two loops α and β represent the same element of $\pi_1(X, x_0)$ if and only if $\alpha \simeq \beta_{rel}\{0,1\}$. That is, there exists a homotopy $F : I \times I \to X$ between α and β such that

$$F(t,0) = \alpha(t), \quad F(t,1) = \beta(t), \forall t \in I$$

$$F(0,s) = \alpha(0) = \beta(0), \;\; F(1,s) = \alpha(1) = \beta(1).$$

Given two loops α and β such that $\alpha(1) = \beta(0)$, there is a composition $\alpha \cdot \beta$ that traverses first α and then β, defined by the formula

$$\alpha \cdot \beta(s) = \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2}, \\ \beta(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Definition 5. *If $\alpha : I \to X$ is a loop in X, we can define the inverse of α as $\alpha^{-1} : I \to X$ by $\alpha^{-1}(s) = \alpha(1-s)$, it satisfies $\alpha^{-1} \cdot \alpha(s) \simeq \alpha \cdot \alpha^{-1}(s) \simeq \alpha_0(s)$, where α_0 is a constant loop at the base point x_0.*

The product $\alpha \cdot \beta$ induces a product on homotopy classes given by $[\alpha] \cdot [\beta] := [\alpha \cdot \beta]$, where α and β are loops at x_0 [55].

Theorem 1. *Let $x_0 \in X$ and let $\pi_1(X, x_0)$ be the set of all homotopy classes of loops based at the point x_0. Then $\pi_1(X, x_0)$ is a group, the group of multiplication on $\pi_1(X, x_0)$ being defined according to the rule $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$ and $[\alpha]^{-1} = [\alpha^{-1}]$ for all loops α and β based at x_0.*

Proof. One can find the proof of the theorem in [55]. □

Definition 6. *(Fundamental Group) The set of all homotopy classes* $[\alpha]$ *of loops* $\alpha : I \longrightarrow X$ *at the base point* x_0 *is a group with respect to the product* $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$*, which is called the fundamental group of* X *at the base point* x_0*, denoted* $\pi_1(X, x_0)$.

If X is path connected, then for any base points $x_0, y_0 \in X$, the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, y_0)$ are isomorphic. Thus, we can ignore the base point and denote the fundamental group simply as $\pi_1(X)$.

Note that the concept of fundamental group is powerful. For example, the simply connectedness of a set can easily be checked if its fundamental group is known: by definition any connected set is simply connected if and only if its fundamental group is trivial. For example, $\pi_1(S^2) = 0$, because S^2 is a simply connected space. If scalar fields are continuous maps to $\mathbb{R}^d$, their topology becomes trivial. So, there are no topological solitons but imposing some restrictions on fields, a different situation emerges. Consider scalar fields whose energy density approaches rapidly to zero when the distance from the origin ρ goes to ∞. This requirement of the energy is zero at infinity imposes boundary conditions on the fields, crucial for the topological classification which becomes interesting. Denote $\mathcal{V}$ by the vacuum space where the energy vanishes. Thus, whenever $\rho \to \infty$, the field $\phi(\rho) \to \bar{\phi} \in \mathcal{V}$, providing a boundary condition. This shows the existence of non-trivial topology with a localized finite energy solution. Hence, topological solitons may exist.

2.1.2 The homotopy classes of maps on S^n

The n-sphere is the set of points in $\mathbb{R}^{n+1}$ at a unit distance from the origin. That is, $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. Let $\phi : S^n \longrightarrow S^n$ be a field. Identify the n-sphere S^n as $\mathbb{R}^n$ with all points

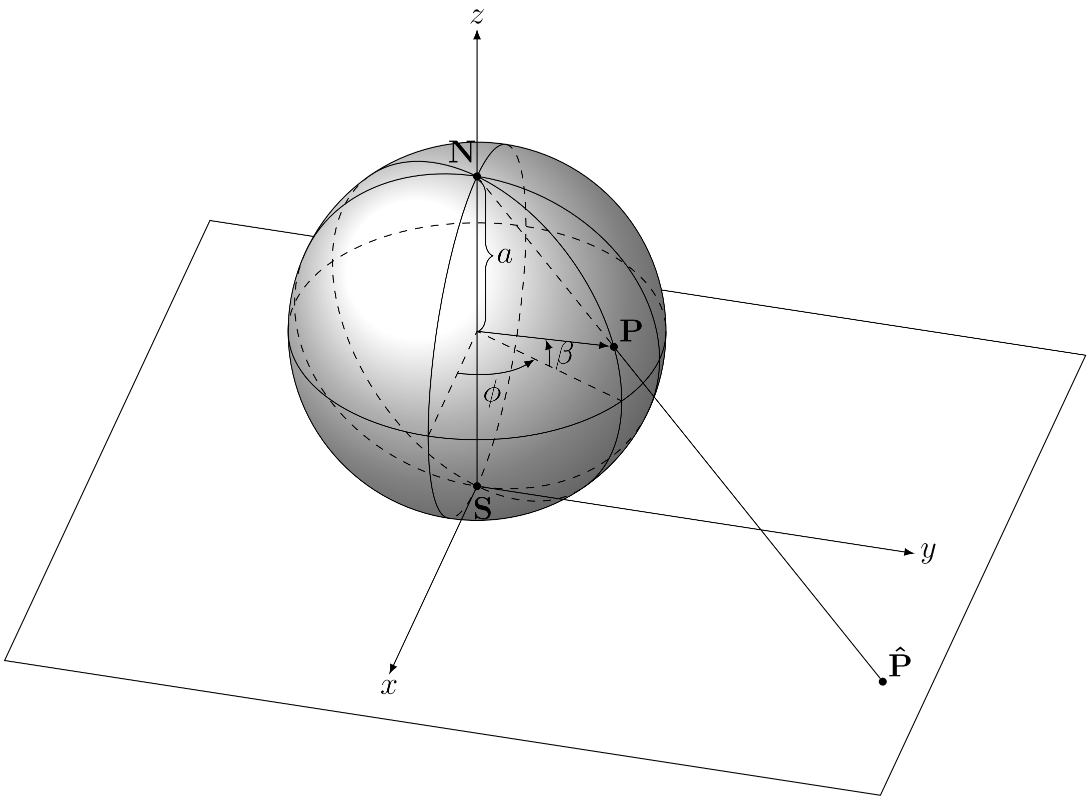


Figure 2.1: Stereographic Projection from North Pole N through P projects to $\hat{P}$.

at infinity identified using stereographic projection from the north pole N as seen in fig.2.1, then the stereographic projection is a one-to-one correspondence between $S^n - \{N\}$ and $\mathbb{R}^n$.

Definition 7. *The set of homotopy classes of based maps $\phi : S^k \longrightarrow S^n$ is the k^{th} homotopy group of S^n and denoted by $\pi_k(S^n)$, where k is a positive integer.*

Example 1. *Consider a map $\phi : S^1 \longrightarrow S^1$ and its image in S^1 is called a loop. We have that $\pi_1(S^1) = \mathbb{Z}$.*

One can check the proof of $\pi_1(S^1) = \mathbb{Z}$ in [36]. The map $\phi : S^1 \to S^1$ can be defined by a continuous function $h(\theta)$ on $[0, 2\pi]$ such that $h(0) = 0$ is the base point and $h(2\pi) = 2\pi k$, $k \in \mathbb{Z}$. Here, $h(\theta)$ is the angle on the codomain. We will see in later section, k is the topological degree or winding number of the map.

Here we will discuss the field configurations characterized by linear and non-linear field theory for the classification of possible topological solitons, see table (2.1). As an example for a linear field theory, let $X = \mathbb{R}^d$ and $Y = \mathbb{R}^D$. Then the corresponding total energy E ginen by

$$E = \int \left(\frac{1}{2} \nabla \phi_l \cdot \nabla \phi_l + U(\phi) \right) d^d x, \quad l = 1, 2, \cdots, D,$$

where $U(\phi)$ is the potential function and ϕ is a D-point of vector. Now consider the vacuum manifold $\mathcal{V} = \{\phi | U(\phi) = 0\}$. One kind of non-trivial topology arises from the properties of the vacuum manifold. To ensure finite energy configurations, there should be a constraint on the field configuration at $\mathcal{V}$. Thus, the boundary values remain fixed and $\phi(\infty) \in \mathcal{V}$. Define a map $\phi^\infty : S^{d-1} \mapsto \mathcal{V}$ which is the asymptotic data of the configuration field ϕ, where S^{d-1} is the sphere at infinity in $\mathbb{R}^d$. Homotopically, no information is lost if only ϕ^∞ is considered, namely, ϕ^∞ and $\tilde{\phi}^\infty$ are homotopic if and only if ϕ and $\tilde{\phi}$ are homotopic. Hence, the homotopy class of ϕ^∞ belongs to the homotopy classes $\pi_{d-1}(\mathcal{V})$ which determines the topological behaviour of the field ϕ. The field may be non-trivial if $\pi_{d-1}(\mathcal{V})$ is non-trivial. Due to triviality, this type of topological solitons do not exist for example if $\mathcal{V} = \mathbb{R}^d$ or a single point. Let $\mathcal{V} = S^{d-1}$. We can see that the field ϕ^∞ is a topologically non-trivial map if $d = 2$ or 3.

In the non-linear case, the homotopy classifications of the topological solitons is carried out by elements of the homotopy group $\pi_d(Y)$, where Y is a closed manifold. Let $\phi : \mathbb{R}^d \to Y$ be a non-linear field. Due to finiteness of the energy, the field tend to a constant value as $r \to \infty$. By the stereographic projection we can extend the field ϕ to a based map $\phi : S^d \to Y$ when the field ϕ has finite energy and the potential minimizes to zero in the vacuum field $\mathcal{V}$.

	Linear		Non-Linear	
	Topological Solitons	Homotopy $\pi_{d-1}(\mathcal{V})$	Topological Solitons	homotopy $\pi_d(Y)$
1	Kinks	$\pi_0(S^0)$	Non-linear kink	$\pi_1(S^1)$
2	Vortices	$\pi_1(S^1)$	Sigma Model Lumps	$\pi_2(S^2)$
3	Monopoles	$\pi_2(S^2)$	Skyrmions	$\pi_3(S^3)$

Table 2.1: The taxonomy of possible topological solitons.

2.2 Topological Degree

Topological degree is useful for calculating the homotopy classes of maps. Let $\phi : X \longrightarrow Y$ be a map between two closed manifolds with same dimension and both X and Y are oriented, and the map ϕ should be differentiable everywhere, with continuous derivatives. Let $\int_Y \Omega = 1$, where Ω is a normalized volume form on Y.

Definition 8. *Let $\phi : X \longrightarrow Y$ be a differentiable map between two oriented closed manifolds such that $\dim X = \dim Y$ and let Ω be a normalized volume form on Y, $\phi^\star(\Omega)$ the pull-back of Ω to X using the map ϕ. Then*

$$\deg \phi = \int_X \phi^\star(\Omega) \tag{2.2.1}$$

is called the topological degree of ϕ, and it is an integer. That is $\deg \phi \in \mathbb{Z}$.

Note that if $\Omega = \beta(y) dy^1 \wedge ... \wedge dy^d$, and ϕ is represented by functions $y(x)$, then

$$\begin{aligned}
\phi^\star(\Omega) &= \beta(y(x)) \frac{\partial y^1}{\partial x^j} dx^j \wedge .... \wedge \frac{\partial y^d}{\partial x^l} dx^l \\
&= \beta(y(x)) \det \left(\frac{\partial y^i}{\partial x^j} \right) dx^1 \wedge ... \wedge dx^d \\
&= \beta(y(x)) J(x) dx^1 \wedge ... \wedge dx^d,
\end{aligned}$$

where $J(x) = \det(\frac{\partial y^j}{\partial x^i})$ is the Jacobian of the map at x.

The topological degree (2.2.1) is a homotopy invariant and independent of the choice of Ω since it cannot be changed under a continuous deformation and the difference of two normalized volume forms on the target space is a $d-$form whose integral is zero.

Example 2. *Let $R : S^2 \longrightarrow S^2$ be a rational map, where S^2 is the complex plane with one point at infinity adjoined, $\mathbb{C} \cup \{\infty\}$. Let z be a coordinate in $\mathbb{C}$ and $\bar{z}$ the complex conjugate of z. A rational map is a function $R(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials in z and they must* *not have common zeros. We will show later that the degree of $R(z)$, N, is defined as*

$$N = \frac{1}{4\pi}\int\left(\frac{(1+|z|^2)}{(1+|R|^2)}|\partial_z R|\right)^2\frac{2idzd\bar{z}}{(1+|z|^2)^2},$$

and that N is equal to the topological degree of the rational map $k_{alg} = \max\{\deg(p), \deg(q)\}$.

2.3 Covering Spaces and Fundamental Groups of $\mathbb{R}P^n$

In this section, we will explain the properties of covering spaces that will help us to study and understand some facts on the moduli spaces of lumps.

Definition 9. *A covering space of X is a pair consisting of space $\tilde{X}$ and a continuous map $p : \tilde{X} \longrightarrow X$ such that the following holds: Each $x \in X$ has an path connected open neighborhood U such that each arc component of $p^{-1}(U)$ is mapped homeomorphically onto U by p. $\tilde{X}$ is called the covering space and X is the base space of the covering projection.*

Proposition 1. *If a group G acts evenly[1] on Y, then the projection $p : Y \to Y/G$ is a covering map. That is, the covering map p exhibits its base space as a quotient space of its covering space.*

Proof. The proof can be found in [16]. □

Example 3. *The group $G = \mathbb{Z}_2$ acts on the S^n with the nontrivial element taking a point to its antipodal point. The quotient space is $\mathbb{R}P^n$, the real projective space, and the quotient mapping $p : S^n \to \mathbb{R}P^n$ is a covering map. Thus, for $n = 2$, (S^2, p) is a covering space of $\mathbb{R}P^2$.*

Let $\varphi : X \to Y$ be a continuous map. Then $\varphi : X \to Y$ induces a homomorphism $\varphi_\star : \pi_1(X) \to \pi_1(Y)$, defined by composing loops $\alpha : I \to X$ with φ, that is $\varphi_\star[\alpha] = [\varphi\alpha]$.

[1] We say that G acts evenly if any point in Y has a neighborhood V such that g.V and h.V are disjoint for any distinct elements g and h in G.

$$\begin{tikzcd}
& S^2 \arrow[d, "\rho"] \\
\mathbb{R}P^2 \arrow[ur, "\bar{\phi}"] \arrow[r, "\phi"'] & \mathbb{R}P^2
\end{tikzcd}$$

Figure 2.2: A homotopy lifting of $\phi : \mathbb{R}P^2 \to \mathbb{R}P^2$.

Theorem 2. *(Homotopy Lifting Lemma) Let $(\tilde{X}, p)$ be a covering space of X and Y a connected and locally arc-wise connected space, $\tilde{x_0} \in \tilde{X}, x_0 = p(\tilde{x_0}), y_0 \in Y$. Given a map $\phi : (Y, y_0) \longrightarrow (X, x_0)$; there exists a map $\tilde{\phi} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x_0})$ such that $p \cdot \tilde{\phi} = \phi$ if and only if $\phi_\star(\pi_1(Y, y_0))$ is contained in $p_\star(\pi_1(\tilde{X}, \tilde{x_0}))$: such a lifting $\tilde{\phi}$, when it exists, is unique.*

Proof. The proof can be found in [37]. □

Example 4. *Let $\phi : \mathbb{R}P^2 \longrightarrow \mathbb{R}P^2$ be a map. Since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$, we can apply example 3 and theorem 2 . Hence, we have that the projection map that is the covering map $p : S^2 \longrightarrow \mathbb{R}P^2$ and ϕ lifts to $\bar{\phi} : \mathbb{R}P^2 \longrightarrow S^2$ if and only if the induced homomorphism $\phi_\star = 0$, as shown in fig. 2.2 .*

Definition 10. *(Automorphism group) For any covering $p : Y \to X$, there is a group $Aut(Y, p)$ of covering transformations:*

$$Aut(Y, p) = \{\varphi : Y \to Y : \varphi \text{ is a homeomorphism and } p \circ \varphi = p\}.$$

This is a group by composition of mappings, and it acts on Y; it is called the automorphism group of the covering.

Proposition 2. *If $p : Y \to X$ is a $G-$covering, and Y is connected, then the canonical homeomorphism $G \to Aut(Y, p)$ is an isomorphism.*

Proof. The proof can be found in [16]. □

Definition 11. *A covering $p : Y \to X$ is called regular if $p_\star(\pi_1(Y, y))$ is a normal subgroup of $\pi_1(X, x)$.*

Proposition 3. *Let Y be a connected, locally arc-wise connected topological space and G be a properly discontinuous group of homeomorphisms of Y. Let $p : Y \longrightarrow Y/G$ denote the natural projection of Y onto its quotient space. Then, (Y, p) is a regular covering space of Y/G and $G = Aut(Y, p)$.*

Proof. The proof can be found in [37]. $\square$

Definition 12. *Let $p : Y \to X$ be a covering and let Y be simply connected. Then the covering space (Y, p) is called the universal covering space of X.*

Example 5. *Let $\rho : S^2 \longrightarrow S^2$ be the antipodal map defined by $z \longmapsto -\frac{1}{\bar{z}}$. As ρ^2 is the identity transformation, a properly discontinuous cyclic group G of order 2 of homeomorphism of S^2 is generated by ρ. Since S^2 is simply connected, (S^2, p) is a universal and regular covering space of S^2/G. Hence $G = \mathbb{Z}_2$ is of order 2 and $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$.*

Chapter 3

Moduli Space of Lumps on a Projective Space

3.1 The $O(3)$-Sigma model on a Riemann surface

A sigma model is a widely studied non-linear scalar field theory. Given a d-dimensional Riemann manifold Σ, then the non-linear sigma model on the spacetime $\mathbb{R} \times \Sigma$ with target space Y is defined by the Lagrangian

$$L = \frac{1}{4} \int_{\Sigma} d\mu \partial_\mu \phi_\delta \partial^\mu \phi_\sigma H^{\delta\sigma}, \tag{3.1.1}$$

where $d\mu$ is the volume form on Σ, g_Σ is the Riemannian metric on Σ, ϕ is a scalar field defined on Σ, $\partial^\mu = g^{\mu\nu}\partial_\nu$ and $g^{\mu\nu}$ are the components of the inverse of the Lorentzian metric

$$g = dt^2 - g_\Sigma$$

on the spacetime $\mathbb{R} \times \Sigma$ and $H_{\delta\sigma}$ is the metric on Y. The $O(3)$-sigma model is a famous example of a non-linear sigma model. In the $O(3)$-sigma model, the field can be parameterized as a three-component unit vector, $\phi = (\phi_1, \phi_2, \phi_3)$ with $\phi \cdot \phi = 1$. That is, the $O(3)$-sigma model can be formulated in terms of fields (ϕ_1, ϕ_2, σ) such that $\sigma = \pm\sqrt{1 - (\phi_1^2 + \phi_2^2)}$ where ϕ_1 and ϕ_2 are locally unconstrained [36]. Note that for the energy to be finite ϕ must be constant at spatial infinity. Take $\phi^\infty = (0,0,1)$. Thus, the target space can be identified with the Riemann sphere S^2. For the $(2+1)$-dimensional Minkowski spacetime $\mathbb{R}^{2,1}$, the energy of a static field configuration is given by

$$E = \frac{1}{4}\int_{\mathbb{R}^2} (\partial_i\phi \cdot \partial_i\phi)\, d^2x, \tag{3.1.2}$$

where $i = 1, 2$. Due to finiteness of the energy, the field ϕ should be constant at spatial infinity, say $\phi^\infty = (0,0,1)$. The Riemann sphere S^2 is the compactification to $\mathbb{R}^2 \cup \{\infty\}$. Let $\phi : S^2 \to S^2$ be a based map. Since $\pi_2(S^2) = \mathbb{Z}$, then the field ϕ is classified by the topological degree. The topological degree N of the map ϕ is given by

$$N = \frac{1}{4\pi}\int_{S^2} \phi \cdot (\partial_1\phi \times \partial_2\phi) d^2x, \tag{3.1.3}$$

where the integrand is the pull-back of the normalized area form on S^2. The energy E and the topological degree N satisfy the the Bogomolny bound $E \geq 2\pi|N|$. This bound can be shown by integrating the inequality

$$\begin{aligned}
&(\partial_i\phi \pm \epsilon_{ij}\phi \times \partial_j\phi)\cdot(\partial_i\phi \pm \epsilon_{ik}\phi \times \partial_k\phi) = 2\left(\partial_i\phi\cdot\partial_i\phi \mp \epsilon_{ij}\phi\cdot(\partial_i\phi \times \partial_j\phi)\right) \geq 0 \\
&\Rightarrow \partial_i\phi\cdot\partial_i\phi \mp 2\phi\cdot(\partial_1\phi \times \partial_2\phi) \geq 0 \\
&\Rightarrow \frac{1}{4}\int \partial_i\phi\cdot\partial_i\phi\, d^2x \geq \frac{1}{2}\left|\int \phi\cdot(\partial_1\phi\times\partial_2\phi)\, d^2x\right| = 2\pi\left|\frac{1}{4\pi}\int \phi\cdot(\partial_1\phi\times\partial_2\phi)\, d^2x\right| \\
&\Rightarrow E \geq 2\pi|N|.
\end{aligned}$$

Equality holds if and only if the field is the solution of the Bogomolny equations

$$(\partial_i\phi \pm \epsilon_{ij}\phi \times \partial_j\phi) = 0.$$

Consider R the stereographic Riemann sphere coordinate image of ϕ on the target space projected from the north pole. The coordinate R is given by $R = \frac{(\phi_1 + i\phi_2)}{(1+\phi_3)}$ and let the local complex coordinate be $z = x^1 + ix^2$ with conjugate $\bar{z} = x^1 - ix^2$. One can then explicitly express ϕ in terms of R as

$$\phi = \left(\frac{R+\bar{R}}{1+|R|^2}, \frac{R-\bar{R}}{i(1+|R|^2)}, \frac{|R|^2-1}{1+|R|^2} \right).$$

Since $R = R(z, \bar{z})$ is the function of z and $\bar{z}$, the Lagrangian (3.1.1) becomes

$$L = \int_{S^2} dS \frac{\partial_\mu R \partial^\mu \bar{R}}{(1+|R|^2)^2} \tag{3.1.4}$$

where $\mu = z, \bar{z}$. This Lagrangian is referred to as that of the $\mathbb{C}P^1$ sigma model. The $\mathbb{C}P^1$ sigma model in $(2+1)$- dimensions is a non-linear field theory possessing topological solitons, called lumps.

The model can be generalised to physical space being the 2 dimensional compact Riemann surface Σ with metric g, that is the spacetime $\mathbb{R} \times \Sigma$. We mainly consider $\Sigma = S^2$ and $\Sigma = \mathbb{R}P^2$, but other examples have been discussed in the literature [58]. The kinetic energy functional induces a natural metric γ on the moduli space of degree N lumps M_N: M_N is a finite dimensional, smooth Riemannian manifold.

For $\Sigma = S^2$, the energy E (3.1.2) and the topological degree (3.1.3) are then simplified and written in terms of R as

$$E = \frac{1}{2} \int_{S^2} \frac{(|\partial_z R|^2 + |\partial_{\bar{z}} R|^2)(1+|z|^2)^2}{(1+|R|^2)^2} \frac{dz d\bar{z}}{(1+|z|^2)^2}, \tag{3.1.5}$$

$$N = \frac{1}{4\pi} \int_{S^2} \frac{(|\partial_z R|^2 - |\partial_{\bar{z}} R|^2)(1+|z|^2)^2}{(1+|R|^2)^2} \frac{dz d\bar{z}}{(1+|z|^2)^2}, \tag{3.1.6}$$

respectively where $\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$. Now one can similarly show that $E \geq 2\pi|N|$, the Bogomolny bound. This is because for $N \geq 0$, we can see that

$$\begin{aligned}
E &= \frac{1}{2}\int_{S^2} \frac{(|\partial_z R|^2 + |\partial_{\bar{z}} R|^2)(1+|z|^2)^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} \\
&= \frac{1}{2}\int_{S^2} \frac{(|\partial_z R|^2 - |\partial_{\bar{z}} R|^2)(1+|z|^2)^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} + \int_{S^2} \frac{(|\partial_{\bar{z}} R|^2)(1+|z|^2)^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} \\
&= 2\pi N + \int_{S^2} \frac{(|\partial_{\bar{z}} R|^2)(1+|z|^2)^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.
\end{aligned}$$

Similarly, for $N \leq 0$, one can also find that

$$\begin{aligned}
E &= \frac{1}{2}\int_{S^2} \frac{(|\partial_z R|^2 + |\partial_{\bar{z}} R|^2)(1+|z|^2)^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} \\
&= \frac{1}{2}\int_{S^2} \frac{(|\partial_{\bar{z}} R|^2 - |\partial_z R|^2)(1+|z|^2)^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} + \int_{S^2} \frac{(|\partial_z R|^2)(1+|z|^2)^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} \\
&= -2\pi N + \int_{S^2} \frac{(|\partial_z R|^2)(1+|z|^2)^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.
\end{aligned}$$

Hence, $E \geq 2\pi|N|$. Equality holds for $N \geq 0$ if and only if the Cauchy-Riemann equation is satisfied,

$$\partial_{\bar{z}} R = 0, \tag{3.1.7}$$

whose solutions are holomorphic function $R(z)$. We can do a similar calculation for $N \leq 0$ to show that

$$\partial_z R = 0, \tag{3.1.8}$$

satisfied by antiholomorphic functions $R(\bar{z})$. In summary, the energy E is minimized to $2\pi|N|$ to each topological class by a solution of the Cauchy-Riemann equation

$$\begin{cases} \partial_{\bar{z}} R = 0 & \text{if } N \geq 0, \\ \partial_z R = 0 & \text{if } N \leq 0. \end{cases}$$

Lump solutions can be explicitly given by rational maps between Riemann spheres S^2. From now onwards, we mainly focus on the geometry and topology of rational maps. A rational map is a function given by the ratio of two polynomials

$$R(z) = \frac{p(z)}{q(z)}, \tag{3.1.9}$$

where p and q must have no common roots. The rational map (3.1.9) is a smooth map from S^2 to S^2.

The space of rational maps of degree N is the space consisting of all holomorphic maps of degree N from the Riemann sphere S^2 to itself, denoted by Rat_N. It is a smooth connected complex manifold of dimension $2N+1$ and its fundamental group, $\pi_1(Rat_N)$, is a cyclic group of order $2N$ for $N \geq 1$ [52]. For example, consider $N = 2$. Then Rat_2 is the space of all quadratic rational maps from the Riemann sphere S^2 to itself which is a smooth complex 5-manifold.

The topological degree N of a rational map is the number of pre-images of a given point C, counted with multiplicity which can be found by solving the equation $R(z) = C$. We have to solve the polynomial equation $p(z) - Cq(z) = 0$. The number of zeros is given by the highest power of z in either $p(z)$ or $q(z)$. The net number of pre-images of C doesn't vary as C changes. Suppose $p(z) - Cq(z)$ has one or more leading power of z missing. Then the missing finite zeros of the polynomial equation $p(z) - Cq(z) = 0$ are considered as at infinity [36]. Thus, the topological degree N of R is $K_{alg} = \max\{\deg p, \deg q\}$. Since R is holomorphic, the energy E of this rational map R is $2\pi|N|$, where N is the topological degree (3.1.6) of R given by

$$\deg R = N = \frac{1}{4\pi}\int_{S^2} \frac{(1+|z|^2)^2|\partial_z R|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}. \tag{3.1.10}$$

Definition 13. *Let $f(z) = a_d z^d + \cdots + a_0$ and $g(z) = b_l z^l + \cdots + b_0$ be two polynomials of degrees d and l, respectively. Their resultant $Res(f,g)$ is the determinant of the $(d+l)\times(d+l)$ Sylvester matrix $Syl(f,g)$ given by*

$$\begin{pmatrix} a_d & a_{d-1} & a_{d-2} & \cdots & 0 & 0 & 0 \\ 0 & a_d & a_{d-1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_0 & 0 \\ 0 & 0 & 0 & \cdots & a_2 & a_1 & a_0 \\ b_l & b_{l-1} & b_{l-2} & \cdots & 0 & 0 & 0 \\ 0 & b_l & b_{l-1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_1 & b_0 & 0 \\ 0 & 0 & 0 & \cdots & b_2 & b_1 & b_0 \end{pmatrix}.$$

Suppose f has d roots $\chi_1, \cdots, \chi_d$ and g has l roots $\eta_1, \cdots, \eta_l$ (not necessary distinct). Then [18]

$$Res(f,g) = a_d^l b_l^d \prod_{i=1}^{d} \prod_{j=1}^{l} (\chi_i - \eta_j). \tag{3.1.11}$$

Note that $Res(f,g) \neq 0$ if and only if f and g have no common roots. The rational map (3.1.9) is a meromorphic function, so it may be infinity or any finite complex value for finite z. Recall that rational maps are given by ratio of two polynomials $R(z) = \frac{p(z)}{q(z)}$, where p and q have no common zeros. This condition can be rewritten as

$$Res(p,q) \neq 0,$$

where $Res(p,q)$ is the determinant of the Sylvester matrix $Syl(p,q)$.

Example 6. *Each map R in the space Rat_1 can be expressed as a ratio*

$$R(z) = \frac{p(z)}{q(z)} = \frac{a_1 z + a_2}{b_1 z + b_2}$$

with degree $\max\{\deg(p), \deg(q)\}$ *equal to* 1*. Hence, we can identify Rat_1 with the Zariski open subset of complex projective 3-space consisting of all points $(a_1 : a_2 : b_1 : b_2) \in \mathbb{C}P^3$ for which the resultant*

$$Res(p,q) = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = a_1 b_2 - a_2 b_1 \neq 0.$$

This suffices that Rat_1 can be identified with a projective equivalence class of $GL(2,\mathbb{C})$. We can in fact choose a unimodular matrix as the representative for each equivalence class. Then the space Rat_1 can be identified with the group $PSL(2,\mathbb{C})$ consisting of all Möbius transformations from the Riemann sphere S^2 to itself.

3.2 The moduli space of harmonic maps between $\mathbb{R}P^2$

Before discussing the moduli space of harmonic maps on the projective plane, let us first discuss on the moduli space of harmonic maps on the Riemann sphere S^2. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a map where $\mathcal{M}$ and $\mathcal{N}$ are Riemann manifolds with metrics g and h, respectively. There is a natural Riemannian metric, called the L^2 metric, which is for each pair of tangent vectors $U, V \in T_\phi M_N$ the inner product

$$\gamma(U,V) = \int_{\mathcal{M}} d\mu_g h_\phi(U,V) \tag{3.2.1}$$

where $d\mu_g$ is the Riemannian volume element on $\mathcal{M}$. When $\dim(\mathcal{M}) = 2$, Speight [59] studied the energy density is localized in lump-like structures distributed over $\mathcal{M}$. For $\mathcal{M} = \mathcal{N} = S^2$, harmonic maps can be parametrized by rational maps. Recall that rational maps minimize energy within their homotopy classes. The homotopy classes of continuous maps $\phi : S^2 \longrightarrow$

S^2, by the Hopf degree theorem [6], are labeled by their topological degree $N \in \mathbb{Z}$. The map ϕ can be distinguished by the general degree N rational map of the form

$$R(z) = \frac{p(z)}{q(z)} = \frac{p_0 + p_1 z + ... + p_N z^N}{q_0 + q_1 z + ... + q_N z^N} \tag{3.2.2}$$

where $p_i, q_i \in \mathbb{C}, i = 0...N$ are constants and p_n and q_n do not both vanish simultaneously and $Res(p,q) \neq 0$. For a non-zero $\mu \in \mathbb{C}$, the points $(\mu p_0, ..., \mu p_N, \mu q_0, ..., \mu q_N) \in \mathbb{C}^{2N+2}$. Since this point determines the same rational map as $(p_0, .., p_N, q_0, ...q_N)$, $M_N \subset \mathbb{C}P^{2N+1}$. Since $Res(p,q) \neq 0$, the inclusion is open. Consider $q_N \neq 0$. We can define complex coordinates $b_\alpha = \frac{p_\alpha}{q_N}, \alpha = 0, ..., N$ and $b_{N+1+\alpha} = \frac{q_\alpha}{q_N}, \alpha = 0, ..., N-1$. Then the map (3.2.2) can be parametrized as

$$R(z) = \frac{b_0 + b_1 z ... + b_N z^N}{b_{N+1} + b_{N+2} z + ... + z^N}. \tag{3.2.3}$$

There is a theorem in [59] states that for $N > 0$, (M_N, γ) is Kähler with respect to the complex structure induced by the open inclusion $M_N \subset \mathbb{C}P^{2N+1}$. Then the metric γ is Kähler in b_α coordinate system and given by

$$\gamma = \gamma_{\alpha\beta} db^\alpha \overline{db^\beta}, \tag{3.2.4}$$

where repeated indices are summed over, and

$$\gamma_{\alpha\beta} = \int_{\mathbb{C}} \frac{dz d\bar{z}}{(1 + |z^2|)^2 (1 + |R|^2)^2} \left(\frac{\partial R}{\partial b^\alpha} \right) \left(\frac{\partial \bar{R}}{\partial b^\beta} \right). \tag{3.2.5}$$

Since γ is hermitian, then $\gamma_{\alpha\beta} = \bar{\gamma}_{\alpha\beta}$ [41]. Then for all $\forall \alpha, \beta, \delta$, we can find that

$$\frac{\partial \gamma_{\alpha\beta}}{\partial b^\delta} \equiv \frac{\partial \gamma_{\delta\beta}}{\partial b^\alpha}, \quad \frac{\partial \gamma_{\alpha\beta}}{\partial \bar{b}^\delta} \equiv \frac{\partial \gamma_{\alpha\beta}}{\partial \bar{b}^\delta}.$$

Homotopy, covering spaces and fundamental groups play important roles in studying the moduli spaces of harmonic maps between projective spaces $\mathbb{R}P^2$ and classify their homotopy classes.

Lemma 1. *The homotopy classes of a continuous map $\phi : \mathbb{R}P^2 \to \mathbb{R}P^2$ contains a harmonic reperesentative if and only if the induced map $\phi_\star : \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^2)$ is injective.*

Proof. This lemma is an immediate consequence of a theorem in [15] and one can find a similar proof in [14]. □

Lemma 1 means that the homotopy classes of a continuous map $\phi : \mathbb{R}P^2 \to \mathbb{R}P^2$ associated to the induced homomorphism $\phi_\star : \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^2)$ belong to two families. This is due to the fact that the fundamental group of the projective space , $\pi_1(\mathbb{R}P^2)$, is $\mathbb{Z}_2$. The first family is the zero morphism family which contains a trivial class and a non-trivial class which is a non-harmonic representative. The second family contains an infinite number of classes belong to an identity family such that the elements are determined by the absolute value of the degree, that takes any odd value. Let $p : S^2 \to \mathbb{R}P^2$ denotes the covering projection. Any harmonic map ϕ in the identity family homotopy class lifts to a rational map $\bar{\phi} : S^2 \to S^2$:

$$\begin{tikzcd} S^2 \arrow[r, "\bar{\phi}"'] \arrow[d, "p"'] & S^2 \arrow[d, "p"] \\ \mathbb{R}P^2 \arrow[r, "\phi"'] & \mathbb{R}P^2 \end{tikzcd}$$

Therefore, the identity family homotopy classes consists of an infinite number of classes. All of the classes comprise harmonic representatives which are given by rational maps (3.2.2) and satisfy

$$R\left(-\frac{1}{\bar{z}}\right) = -\frac{1}{\bar{R}(z)}. \tag{3.2.6}$$

That is, $q_i = (-1)^i \bar{p}_{N-i}$. Denote by $\bar{M}_N$ the moduli space of the absolute value of the degree N harmonic maps between $\mathbb{R}P^2$. Note that $\bar{M}_N \subset M_N$.

Theorem 3. *For an odd $N \geq 1$, $\bar{M}_N$ is a totally geodesic Lagrangian submanifold of M_N.*

Proof. The proof can be found in [59]. □

Note that for $\mu \in \mathbb{C}^\times$, all rational maps of the form (3.2.2) can be written as

$$R(z) = \mu \frac{(z - z_1)...(z - z_N)}{(z - w_1)...(z - w_N)}. \tag{3.2.7}$$

Using (3.2.6), a rational map on the projective plane can then be rewritten as

$$R(z) = \lambda \frac{(z - z_1)...(z - z_N)}{(1 + \bar{z}_1 z)...(1 + \bar{z}_N z)}, \tag{3.2.8}$$

where N is odd and $\lambda \in \mathbb{C}^\times$ with $|\lambda| = 1$.

3.3 The metric on the moduli space of degree 1 rational maps

Let G be the degree preserving group acts isometrically on M_1 and define a map $h : M_1 \longrightarrow M_1$, $R(z) \mapsto \bar{R}(\bar{z})$. Then we can consider $G \cong SO(3) \rtimes SO(3) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2 = \{\mathbb{I}d, h\}$.

Proposition 4. *$M_1 \cong PU(2) \rtimes \mathbb{R}^3$ as a manifold, where $PU(2) \cong SU(2)/\mathbb{Z}_2 \cong SO(3)$ is the subgroup of $PSL(2, \mathbb{C})$ consisting of all rotations of S^2.*

Proof. From example 6, there is a well known matrix representation of Möbius transformation of R as

$$R(z) = \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot z = A \cdot z$$

where $A \in GL(2, \mathbb{C})$. Since A can be taken as unimodular, so is $-A$. Therefore, $SL(2, \mathbb{C})$ is a double cover of the moduli space and the moduli space of degree 1 lumps is $SL(2, \mathbb{C})/\mathbb{Z}_2$.

For all $A \in SL(2,\mathbb{C})$, there are unique $U \in SU(2)$ and H, a positive definite, unimodular, hermitian 2×2 matrix satisfying $A = UH$ [42, 43]. Then the space $SL(2,\mathbb{C})$ is locally a product of S^3 and $\mathbb{R}^3$ (the parameter space of the positive definite, unimodular, hermitian 2×2 matrices), i.e $SL(2,\mathbb{C}) \cong S^3 \rtimes \mathbb{R}^3$. Since $SU(2) \cong S^3$, $SL(2,\mathbb{C}) \cong SU(2) \rtimes \mathbb{R}^3$. Let $\tau_a, a = 1,2,3$ be the standard Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $[U] = \pm U \in PU(2), \tilde{\boldsymbol{\lambda}} \in \mathbb{R}^3$. In [42], it is shown that any $[A] \in PSL(2,\mathbb{C})$ has a unique decomposition

$$[A] = [U](\Lambda \mathbb{I}_2 + \tilde{\boldsymbol{\lambda}} \cdot \tau) \tag{3.3.1}$$

where $\Lambda = \sqrt{1+\lambda^2}$, $\lambda = |\tilde{\boldsymbol{\lambda}}|$ and $\cdot$ denotes the $\mathbb{R}^3$ scalar product. Define a basis $\{t_a = \frac{i}{2}\tau_a,\ a = 1,2,3\}$ for $su(2)$. Let $\sigma_a,\ a = 1,2,3$ be the left invariant on $PU(2)$ associated with the basis $\{t_a\}$. Then $su(2) \cong T_{\mathcal{I}_2}PU(2)$ and the moving coframe is $\{d\lambda_a, \sigma_a\}$ [59]. Thus, as a maifold $M_1 \cong PU(2) \rtimes \mathbb{R}^3$. $\square$

Now let us parameterize the algebra of $SU(2)$ using the generators [3]

$$t_1 = \frac{1}{2}i\tau_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, t_2 = \frac{1}{2}i\tau_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, t_3 = \frac{1}{2}i\tau_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

where $\tau_a, a = 1,2,3$ are the Pauli matrices; the commutators are $[t_a, t_b] = -\epsilon_{abc}t_c$. Let θ, ϕ, ψ be the Euler angles such that $\theta \in [0,\pi), \phi \in [0, 2\pi), \psi \in [0, 4\pi)$ for parameterizing $SU(2)$, then define

$$g(\phi,\theta,\psi) = e^{\phi t_3}e^{\theta t_2}e^{\psi t_3} = \begin{pmatrix} e^{\frac{i}{2}(\psi+\phi)}\cos\frac{1}{2}\theta & e^{\frac{i}{2}(\phi-\psi)}\sin\frac{1}{2}\theta \\ -e^{\frac{i}{2}(\psi-\phi)}\sin\frac{1}{2}\theta & e^{-\frac{i}{2}(\psi+\phi)}\cos\frac{1}{2}\theta \end{pmatrix}. \tag{3.3.2}$$

Then we can define a left-invariant 1-forms η_a on $SU(2)$ via

$$g^{-1}dg = \eta_1 t_1 + \eta_2 t_2 + \eta_3 t_3 \tag{3.3.3}$$

$$= \frac{i}{2}\begin{pmatrix} \eta_3 & \eta_1 - i\eta_2 \\ \eta_1 + i\eta_2 & -\eta_3 \end{pmatrix} \tag{3.3.4}$$

and then the η_i's are computed as

$$\eta_1 = -\sin\psi d\theta + \cos\psi \sin\theta d\phi$$

$$\eta_2 = \cos\psi d\theta + \sin\psi \sin\theta d\theta$$

$$\eta_3 = d\psi + \cos\theta d\phi$$

satisfying $d\eta_i = \frac{1}{2}\epsilon_{ijk}\eta_j \wedge \eta_k$.

Proposition 5. *Let τ be a G invariant symmetric $(0,2)$ tensor on M_1 which is Hermitian and whose J-associated 2-form $\hat{\tau}$(i.e. $\hat{\tau}(X,Y) := \tau(JX,Y)$) is closed. Then there exists a smooth function $F : [0,\infty) \longrightarrow \mathbb{R}$ such that*

$$\tau = F_1 d\lambda \cdot d\lambda + F_2(\lambda \cdot d\lambda)^2 + F_3 \sigma \cdot \sigma + F_4(\lambda \cdot \sigma)^2 + F_5 \lambda \cdot (\sigma \times d\lambda) \tag{3.3.5}$$

where

$$F_1 = F(\lambda), F_2 = \frac{F(\lambda)}{1+\lambda^2} + \frac{F^{'}(\lambda)}{\lambda}, F_3 = (\frac{1+2\lambda^2}{4})F(\lambda), F_4 = (\frac{1+\lambda^2}{4\lambda})F^{'}(\lambda), F_5 = F(\lambda)$$

$F^{'}$ denotes the derivative of F, σ_a are the left one-forms on $PU(2)$ associated with the basis $\{\frac{i}{2}\tau_a, a = 1,2,3\}$, $\times$ denotes the $\mathbb{R}^3$ vector product and juxtaposition of vectors denotes symmetrized tensor product.

Details of the proof of the proposition can be found in [59]. Let Γ be the radial curve, $\Gamma =$

$\{[\mathbb{I}_2], (0,0,\lambda) : \lambda \geq 0\}$, of rational maps $W_\lambda : z \mapsto \mu(\lambda) z, \mu(\lambda) = (\sqrt{1+\lambda^2}+\lambda)^2 = (\Lambda + \lambda)^2$ and we consider the metric as the squared length of the vector $\frac{\partial}{\partial \lambda_1}$ at this curve corresponding to the rational map $W_\lambda(z)$.

Corollary 1. *The L^2 metric on M_1 is*

$$\gamma = F_1 d\lambda \cdot d\lambda + F_2(\lambda \cdot d\lambda)^2 + F_3 \sigma \cdot \sigma + F_4(\lambda \cdot \sigma)^2 + F_5 \lambda \cdot (\sigma \times d\lambda), \tag{3.3.6}$$

where $F_1, ..., F_5$ are functions of λ only and

$$F = 4\pi\mu \frac{[\mu^4 - 4\mu^2 \log \mu - 1]}{(\mu^2-1)^3}, \mu = (\sqrt{1+\lambda^2}+\lambda)^2. \tag{3.3.7}$$

Moreover, the L^2 metric is given by

$$\gamma\left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \lambda_1}\right) = \frac{4\pi\mu^2\left[(\mu^2+1)\log \mu - \mu^2 + 1\right]}{(\mu^2-1)^3}. \tag{3.3.8}$$

For a proof of the first part one can see in [59]. However, to prove the second part we can compute (3.3.8) as it is twice the kinetic energy by computing the L^2 norm of the zero mode $\frac{\partial}{\partial \mu} \in T_{(\mu,0)} M_1$ of the field $w_\lambda(z,t) = (\mu + t)z$:

$$\begin{aligned}
\gamma\left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \lambda_1}\right) &= \int_{\mathbb{C}} \frac{|\dot{w}_\lambda(z,0)|^2}{(1+|w_\lambda(z,0)|^2)^2} \frac{dz d\bar{z}}{(1+|z|^2)^2} \\
&= 4\pi\mu^2 \int_0^\infty \frac{r^3 dr}{(1+\mu^2 r^2)^2 (1+r^2)^2} \\
&= \frac{4\pi\mu^2\left[(\mu^2+1)\log \mu - \mu^2 + 1\right]}{(\mu^2-1)^3}.
\end{aligned}$$

We calculated the metric on the totally geodesic submanifold of M_1 and agrees with the metric done in [59].

3.4 Symmetric lumps in real projective space

In the following, we derive families of symmetric rational maps in real projective space. We start by considering rational maps between Riemann spheres. A rational map $R(z)$ has a discrete symmetry if

$$R(\kappa(z)) = M_\kappa(R(z)) \tag{3.4.1}$$

where κ and M_κ are Möbius transformations. Let $\mathcal{K} \subset SO(3)$ be a group which acts on S^2 as an $SU(2)$ Möbius transformation. The map R is $\mathcal{K}$-symmetric if for each $\kappa \in \mathcal{K}$, there exists an M_κ such that (3.4.1) satisfied. For consistency, the pairs (κ, M_κ) have the same composition rule as in $\mathcal{K}$. Then $R(\kappa_1\kappa_2(z)) = M_{\kappa_1}M_{\kappa_2}(R(z))$, for all $\kappa_1, \kappa_2 \in \mathcal{K}$. The map $\kappa \longrightarrow M_\kappa$ is a homomorphism.

Lemma 2. *Suppose $R(z)$ has symmetry (3.4.1). Then the rational map $\tilde{R}(z) = \hat{M}_1(R(\hat{M}_2(z)))$ has the symmetry*

$$\tilde{R}(z) = \tilde{M}_1(\tilde{R}(\tilde{M}_2(z)))$$

where $\tilde{M}_1(z) = \hat{M}_1(M_\kappa(\hat{M}_1^{-1}(z)))$ and $\tilde{M}_2 = \hat{M}_2^{-1}(\kappa(\hat{M}_2(z)))$, and $\hat{M}_i$, $i = 1, 2$ are Möbius transformations.

The proof of the lemma can be found in [22]. One can choose the symmetry to be around convenient axes using a change of coordinates in domain and target. Define a C_n^k symmetry of a rational map as a rotation in space by $\alpha = 2\pi/n$ followed by a rotation in target space by $\beta = k\alpha$, $-n < k \leq n$. The following lemma classifies which C_n^k symmetries are allowed for a rational map of degree N.

Lemma 3. *A rational map of degree N can have a C_n^k symmetry if and only if $N \equiv 0 \mod n$ or $N \equiv k \mod n$.*

One can find a proof in [23]. Recall that rational map R in real projective space should satisfy the constraint (3.2.6) and it can then be written in the general form

$$R(z) = \mathrm{e}^{i\phi} \frac{\sum\limits_{k=0}^{N} a_k z^k}{\sum\limits_{k=0}^{N} (-1)^k \bar{a}_{N-k} z^k}, \tag{3.4.2}$$

where $a_k \in \mathbb{C}$ and $\phi \in \mathbb{R}$. This severely restricts the number of allowed symmetries. Denote by $\widetilde{Rat_N}$ the degree N rational maps in real projective space. Note that $\widetilde{Rat_N}$ is a submanifold of Rat_N. In fact, we obtain the following lemma.

Lemma 4. *A rational map of degree N satisfying the constraint (3.2.6) has a C_n^k symmetry if and only if $N \equiv k \mod n$. If $n \geq N$ then the rational map has D_∞ symmetry.*

Proof:

Without loss of generality we choose coordinates such that one C_n^k rotation is around the z axis in space and target space and satisfies the boundary condition $R(\infty) = \infty$.

First consider $N \equiv 0 \mod n$. Then $N = nl$ and a C_n^k rational map can be written as

$$R(z) = \frac{r(z^n)}{z^{n-k} s(z^n)},$$

where $r(z)$ has degree l and $s(z)$ has degree less than l. On the other hand, given $r(z)$ the constraint (3.2.6) fixes the coefficients of the denominator. In particular, only coefficients of z^n will be non-zero. Hence, the only compatible solution is $k = 0$.

Consider the case $N \equiv k \mod n$ which includes the $k = 0$ case for $N \equiv 0 \mod n$. Set $k = N \mod n$ and $s = (N-k)/n$, then the rational map is given by

$$R(z) = \frac{\sum\limits_{j=0}^{s} a_j z^{jn+k}}{\sum\limits_{j=0}^{s} b_{s-j} z^{jn}}. \tag{3.4.3}$$

The inversion symmetry (3.2.6) leads to the following two constraints on the coefficients

$$(-1)^{nj}\bar{b}_{s-j} = \lambda a_j \tag{3.4.4}$$

and

$$(-1)^{k+1}a_{s-j}(-1)^{nj} = \lambda\bar{b}_j, \tag{3.4.5}$$

where λ takes account of the fact that numerator and denominator can be multiplied with a common factor. Taking the modulus, we obtain that $|\lambda| = 1$, so that $\bar{\lambda} = 1/\lambda$. By relabelling $j \mapsto s - j$, equation (3.4.5) becomes

$$a_j = \bar{\lambda}(-1)^{k+n(s-j)}\bar{b}_{s-j}. \tag{3.4.6}$$

This is compatible with equation (3.4.4) provided $ns + k \equiv N$ is odd. For $N = n$, we obtain the map

$$R(z) = \lambda\frac{a_1 z^N + a_0}{(-1)^N \bar{a}_0 z^N + \bar{a}_1}. \tag{3.4.7}$$

Performing a Möbius transformation in target space to remove the phase λ this is equivalent to a Möbius transformation of the axial map.

Similarly, for $n > N$, the rational map (3.4.3) reduces to

$$R(z) = \frac{a_0 z^N}{\lambda a_0}$$

since in this case $N = k$ and $s = 0$. This is again a the axial map. □

In the following, we will discuss the case $N = 3$ in more detail. According to Lemma 3 and 4, imposing C_n symmetry with $n \geq 3$, we obtain D_∞ symmetry, given by maps of the form

$$R(z) = \mathrm{e}^{i\phi} \frac{az^3 + b}{-\bar{b}z^3 + \bar{a}}, \tag{3.4.8}$$

where $a, b \in \mathbb{C}$, a and b are not both zero, and $\phi \in \mathbb{R}$. Here the rotation axis in space has been chosen to be the x_3 axis. This choice corresponds to fixing two real parameters. Here, the symmetry is a C_3^0. Consider $a \neq 0$. Then the rational map (3.4.8) can be rewritten as

$$R(z) = e^{i\phi} \frac{a}{\bar{a}} \frac{(z^3 + \frac{b}{a})}{(-\frac{\bar{b}}{\bar{a}} z^3 + 1)} = e^{i\psi} \frac{z^3 + c}{-\bar{c}z^3 + 1},$$

where $e^{\psi} = e^{i\phi} a/\bar{a}$ and $c = b/a$. Hence, the moduli space of the symmetric lumps of (3.4.8) is parametrized by one complex number and a phase which together with the choice of axes gives real dimension 5. The moduli space can also be viewed as the orbit under rotations and iso-rotations of the map

$$R(z) = z^3. \tag{3.4.9}$$

Since rotation and isorotations act independently apart from the axial symmetry around the third axis, the dimension of the moduli space of symmetric lumps of (3.4.9) is again 5.

The only rational maps that are compatible with a C_2 symmetry around some axis are given by

$$R(z) = \mathrm{e}^{i\phi} \frac{az^3 + bz}{\bar{b}z^2 + \bar{a}}. \tag{3.4.10}$$

By Möbius transformations preserving this symmetry, name rotations around the third axis in space and target space, the rational map can be brought into the form

$$R(z) = \frac{z^3 + cz}{cz^2 + 1}, \tag{3.4.11}$$

such that c is real and non-negative. The surprising fact is that this map has D_2 symmetry, since it is also symmetric under

$$R(z) = \frac{1}{R\left(\frac{1}{z}\right)}.$$

Hence, imposing C_2 symmetry automatically gives a family of D_2 symmetric maps. Since rotations and isorotations act independently and cannot change the magnitude of the parameter c, the moduli space of the symmetric lumps of (3.4.11) has real dimension 7. Another way of calculating the dimension of the symmetry orbit of (3.4.11) is as follows. A general rational map can be written as equation (3.2.7) for $N = 3$:

$$R(z) = \mathrm{e}^{i\phi} \frac{(z - z_1)(z - z_2)(z - z_3)}{(1 + \bar{z}_1 z)(1 + \bar{z}_2 z)(1 + \bar{z}_3 z)}. \tag{3.4.12}$$

When we impose symmetry under a π rotation around the z-axes in space followed by an isorotation around the z-axis in target space, one zero has to be equal to zero and the other two map into each other under $z \mapsto -z$. The symmetric rational map is then given by

$$R(z) = \mathrm{e}^{i\phi} \frac{z(z - z_1)(z + z_1)}{(1 + \bar{z}_1 z)(1 - \bar{z}_1 z)}. \tag{3.4.13}$$

Hence, the rational map is parametrized by $\phi \in \mathbb{R}$ and $z_1 \in \mathbb{C}$, that is 3 real parameters we have chosen. Furthermore, we have the rotation axis in space and target space, so again we have that the space is 7-dimensional.

To list all families of degree 3 rational maps on the projective plane, we can use also the Riemann-Hurwitz formula [17]. Suppose that $p \in S^2$ is a ramification point of R so in local coordinates the map R can be represented as $z \mapsto z^{d_p} f(z),\ d_p > 1$, where f is analytic function with $f(0) \neq 0$. The number d_p is called the ramification index at p. By the Riemann-Hurwitz formula, for a degree N rational map $R(z)$ ramified at point p_i in S^2, we have that

$$\chi(S^2) = N\chi(S^2) - \sum_{p_i} (d_{p_i} - 1)\,, \tag{3.4.14}$$

where $\chi(S^2)$ is the Euler charactersic of S^2 and d_{p_i} is the ramification index of $R(z)$ at point p_i. Thus, for $N = 3$, $\sum_{p_i} (d_{p_i} - 1) = 4$ implies there are two possibilities. The first possibility is $d_{p_1} = 3$ which fixes $d_{p_2} = 3$. Hence, we will have the first family which is the symmetry orbit of rational maps z^3 which coincides with the family of maps (3.4.8). The second possibility is when $d_{p_i} = 2, i = 1, 2, 3, 4$. For a rational map R of degree $N > 1$ with critical points w, and near w, R behaves like of $z \mapsto z^n$ near the critical point, for some $n > 1$ [39]. Choose a critical point at $z = 0$ with $R(0) = 0$. Then using Möbius transformations, we can find the family of rational map as

$$R(z) = \frac{z^2(z-a)}{1+az}, \quad a > 0. \tag{3.4.15}$$

Note, the sign of a can be changed by $R(z) \mapsto -R(-z)$. Here rotations and isorotations act independently and cannot change the magnitude of a, hence the dimension of the space is 7 dimensional. This is compatible with writing the rational map as equation (3.4.12). For $N = 3$, under a Möbius transformation in space, $z \mapsto M(z)$, the zeros z_j map to $M^{-1}(z_j)$, and similarly for poles, hence zeros remain opposite poles. The phase is invariant under a Möbius transformation. For a Möbius transformations in target space, the zeros become poles and vice-versa. Suppose that rational function R is of the form (3.4.12). One can show that R has critical points, say c. Therefore for (3.4.12) near c, the behavior of R is like that of $z \longmapsto z^3$ near the origin; thus R is highly contracting near c.

Given the two submanifolds Rat_3^a, one parameter family of rational maps of the form (3.4.15), and Rat_3^c, one parameter family of rational functions of the form (3.4.11), we now define the spaces $\widetilde{Rat_3^a}$ and $\widetilde{Rat_3^c}$ as the symmetry orbit of Rat_3^a and Rat_3^c, respectively, which satisfy the constraint (3.2.6). Furthermore, consider the submanifold $\widetilde{Rat_3^0}$ of $\widetilde{Rat_3}$, where $\widetilde{Rat_3^0}$ is the symmetry orbit of rational maps of z^3. We will show that Rat_3^a has hidden D_2 symmetry, and that $\widetilde{Rat_3^a}$ and $\widetilde{Rat_3^c}$ are identical submanifolds using the angular integral.

3.5 The moduli space $\widetilde{Rat_3}$ on $\mathbb{R}P^2$

In this section we discuss the moduli space of charge $N = 3$ lumps on projective space. We calculate the metric and various geometric quantities. We first discuss maps of the form (3.4.15) and (3.4.11) which possess dihedral symmetries D_2. Then we describe the 7-dimensional symmetry orbit of Rat_3^a in section 3.6.

3.5.1 The moduli space Rat_3^a

For $0 < a < 1$, the rational map $R \in Rat_3^a$ has zeros inside the unit disc at $z = a, z = 0$ and poles at $z = -1/a, z = \infty$, respectively. While for $a > 1$, it has zeros outside the unit disc and poles inside the unit disc. For $a = 0$, the rational map R becomes z^3. Thus, the energy density is symmetric and its metric is equivalent to the metric on the moduli space $\{\xi z^n : \xi \in \mathbb{C}^\times\}$ with $\xi = 1$ and $n = 3$ which was shown in [38]. Furthermore, for $a \to \infty$, it is easy to see that

$$R(z) = \frac{z^2(z-a)}{(1+az)} \longrightarrow -z.$$

Similarly, the energy density is symmetric and the metric γ on Rat_3^a as $a \to \infty$ is equivalent to the metric on $w(z) = -z$ which was shown in [59].

Our next task is to calculate the metric on the moduli space Rat_3^a. We have the ingredient to calculate the metric on Rat_3^a as

$$\gamma = \gamma_{aa} da^2$$

where γ_{aa} is a smooth positive function as

$$\gamma_{aa} = \int_D \frac{|\partial_a R|^2 dz d\bar{z}}{(1+|z|^2)^2(1+|R|^2)^2} = \int_D \frac{|z|^4|z^2+1|^2 dz d\bar{z}}{(1+|z|^2)^2(|1+az|^2+|z|^4|z-a|^2)^2},$$

where $D = \{z \in \mathbb{C} : |z| \leq 1\}$.

Consider the coordinate system, $z = re^{i\theta}$. Then we can rewrite the integrand of γ_{aa} in terms of the coordinates r and θ:

$$\frac{|\partial_a R|^2}{(1+|z|^2)^2(1+|R|^2)^2} = \frac{r^4(1+r^2)^2 - 4r^6\sin^2\theta}{(1+r^2)^4(1-r^2+r^2(r^2+a^2)+2ar(1-r^2)\cos\theta)^2}$$
$$=: \tilde{F}(r,\theta,a).$$

Thus γ_{aa} is rewritten as follows:

$$\gamma_{aa} = \int_0^1 \int_0^{2\pi} \tilde{F}(r,\theta,a) r d\theta dr.$$

In fact, γ_{aa} is a non-negative smooth function since $\tilde{F}(r,\theta,a)$ is non-negative because the integrand of γ_{aa}:

$$\frac{|\partial_a R|^2}{(1+|z|^2)^2(1+|R|^2)^2} \geq 0,$$

which implies $\tilde{F}(r,\theta,a) \geq 0$ and $\gamma_{aa} \geq 0$. Let $\alpha = 1-r^2+r^2(r^2+a^2)$ and $\beta = 2ar(1-r^2)$. Since $r \geq 0$ and $a \geq 0$, it is easily seen that $\alpha > \beta$. Then substituting α and β in $\tilde{F}(r,\theta,a)$, we find that

$$r\tilde{F}(r,\theta,a) = \frac{r^5}{(1+r^2)^2(\alpha+\beta\cos\theta)^2} - \frac{4r^7\sin^2\theta}{(1+r^2)^4(\alpha+\beta\cos\theta)^2}.$$

Applying then the residue theorem, we can find a simplified form of γ_{aa} as follows:

$$\gamma_{aa} = 2\pi\int_0^1 \frac{r^5}{(1+r^2)^2}\left(\frac{\alpha}{(\sqrt{\alpha^2-\beta^2})^3} + \frac{4r^2}{(1+r^2)^2\beta^2}\left(1-\frac{\alpha}{\sqrt{\alpha^2-\beta^2}}\right)\right)dr. \qquad (3.5.1)$$

The function γ_{aa} is an even function and smooth. Note that the integrand of γ_{aa} is rational, so it can be computed explicitly, though the expressions become complicated. Fig.3.1 is the energy density of the moduli space of maps (3.4.15) for different values of a. At $a=0$, the energy density is symetric and as $a\to\infty$, the energy density decrease and form spike; however, at $a=\infty$, it is becomes the energy density of the map $R(z)=z$.

Proposition 6. *Rat_3^a has a submanifold with length $l(Rat_3^a) = \frac{(36-\sqrt{3}\pi)}{81}\pi^2$.*

Proof. Suppose $z = re^{i\theta}$ such that the $\mathbb{C}P^1$ field supports solutions with the $\mathcal{K}$ invariance

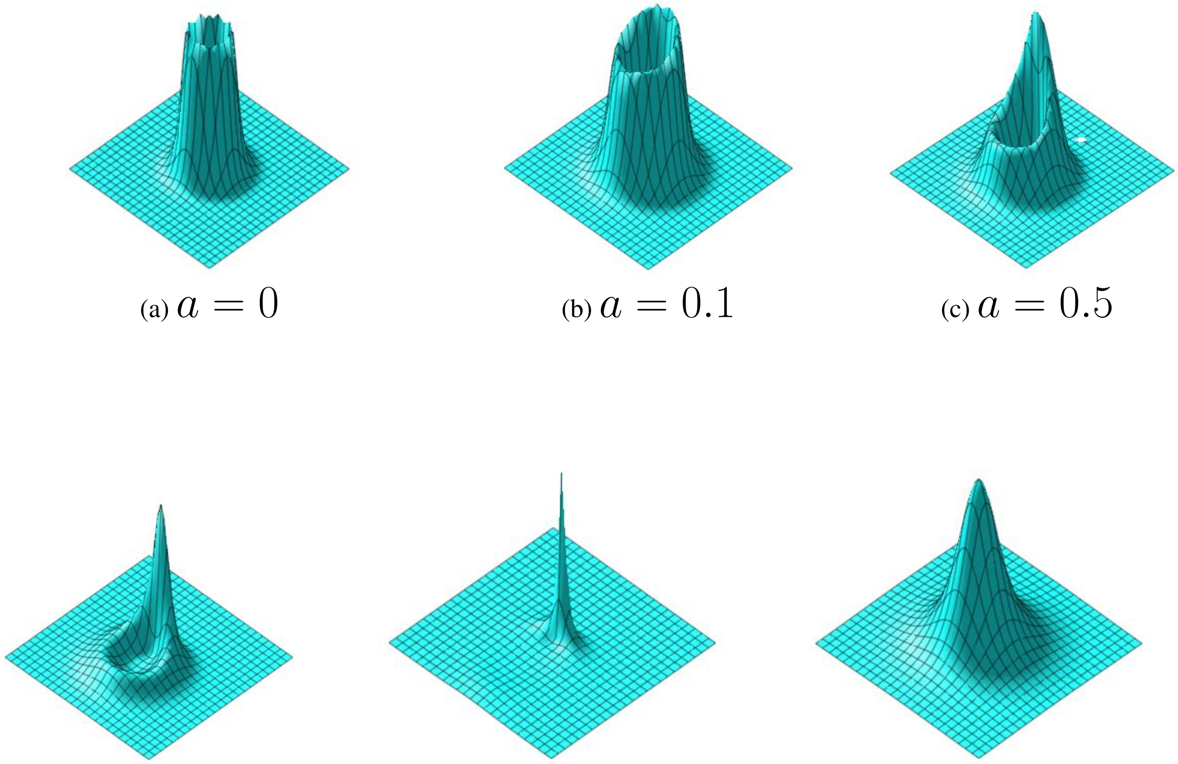
$$R(r,\theta,t) = \frac{r^2e^{i2\theta}(re^{i\theta}-a(t))}{1+a(t)re^{i\theta}},$$

where $a(t)$ is a positive real function. The real function $a(t)$ is differentiable and nonvanishing.. Since the metric on Rat_3^a is $\mathcal{K}_0$ invariant and hermitian, so

$$\gamma = F(\xi)(d\xi^2+\xi^2d\psi^2)$$

for some smooth positive function F. Here to compute $F(\xi)$ from the squared L^2 norm of the zero mode $\frac{\partial}{\partial\xi}$ in the family of $R(z,t) = \frac{z^2(z-(\xi+t))}{1+(\xi+t)z}$:

$$\begin{aligned}F(\xi) &= \int \frac{dzd\bar{z}}{(1+|z|^2)^2}\frac{|\dot{R}(z,0)|^2}{(1+|R|^2)^2} \\ &= 2\pi\int_0^1 \frac{r^5dr}{(1+r^2)^2}\left(\frac{A}{(\sqrt{A^2-B^2})^3} + \frac{4r^2}{(1+r^2)^2B^2}\left(1-\frac{A}{\sqrt{A^2-B^2}}\right)\right),\end{aligned}$$



(a) $a = 0$ (b) $a = 0.1$ (c) $a = 0.5$

(d) $a = 1$ (e) $a = 2$ (f) $a = \infty$

Figure 3.1: These figures are the energy densities of charge three lumps, Rat_3^a.

where $A = 1 - r^2 + r^4 + \xi^2 r^2$ and $B = 2\xi r(1 - r^2)$. Hence, applying Fubini-Tonelli theorem, we can find that

$$
\begin{aligned}
l(Rat_3^a) &= \int_0^{2\pi} d\psi \int_0^{\infty} \xi F(\xi) d\xi \\
&= 2\pi^2 \int_0^1 \frac{(1-r^4)^2 + 2r^4(\ln(r^2) - \ln(1 - r^2 + r^4))}{(1-r^2)^2(1+r^2)^4} r dr \\
&= \frac{(36 - \sqrt{3}\pi)\pi^2}{81}.
\end{aligned}
$$

Therefore the result holds. □

Proposition 7. *The moduli space (Rat_3^a, γ_{aa}) has finite length.*

Proof. Since the integral $\int_0^{\infty} \sqrt{\gamma_{aa}} da < \infty$, it shows automatically the length is finite. Therefore, the boundary of (Rat_3^a, γ_{aa}) at infinity lies at finite distance. □

3.6 The metric on the symmetry orbit of Rat_3^a

Consider the rotation group $SO(3)$ action on the coordinate systems z and R, $R \in Rat_3^a$. Let's suppose first the action $z \mapsto Uz, U \in SO(3) \cong PU(2) \cong SU(2)/\mathbb{Z}_2$. We can expand the left invariant $1-$form $U^{-1}dU$ in terms of a convenient basis of the Lie algebra $t_a = \frac{i}{2}\tau_a, a = 1, 2, 3,$ where τ_a are Pauli matrices as

$$U^{-1}dU = \sigma \cdot \frac{i}{2}\tau_a = \sigma_1 t_1 + \sigma_2 t_2 + \sigma_3 t_3 \tag{3.6.1}$$

where $d\sigma_i = \frac{1}{2}\varepsilon_{ijk}\sigma_j \wedge \sigma_k$. Similarly, for the action $R \mapsto MR, M \in SO(3)$. With similar argument, we have an expression in the Lie algebra basis:

$$M^{-1}dM = \eta \cdot \frac{i}{2}\tau_a = \eta_1 t_1 + \eta_2 t_2 + \eta_3 t_3 \tag{3.6.2}$$

where $d\eta_i = \frac{1}{2}\varepsilon_{ijk}\eta_j \wedge \eta_k$. For example, consider also $\tilde{M} \in SU(2)$ defined by [3]

$$\tilde{M} = \begin{pmatrix} e^{\frac{i}{2}(\psi+\phi)}\cos(\frac{\theta}{2}) & e^{\frac{i}{2}(\psi-\phi)}\sin(\frac{\theta}{2}) \\ -e^{\frac{i}{2}\phi-\psi)}\sin(\frac{\theta}{2}) & e^{-\frac{i}{2}(\psi+\phi)}\cos(\frac{\theta}{2}) \end{pmatrix}.$$

Therefore, we can see that $\tilde{M}^{-1}d\tilde{M} = \eta_1 t_1 + \eta_2 t_2 + \eta_3 t_3$ where the η_i's are computed as

$$\eta_1 = -\sin\psi d\theta + \cos\psi\sin\theta d\phi \tag{3.6.3}$$

$$\eta_2 = \cos\psi d\theta + \sin\psi\sin\theta d\phi \tag{3.6.4}$$

$$\eta_3 = d\psi + \cos\theta d\phi. \tag{3.6.5}$$

Furthermore, let $\tilde{R} \in SU(2)$ and $M \mapsto M\tilde{R}, M \in SO(3)$. Then we can find that $\sigma \mapsto \mathcal{R}\sigma$ and $\eta \mapsto \mathcal{R}\eta$, where $\mathcal{R} \in SO(3)$ with matrix component $R_{ab} = \frac{1}{2}tr(\tau_a\tilde{R}^\dagger\tau_b\tilde{R})$. Hence both σ and η

transform as $3-$vectors under rotations. One can change from the coordinate basis on $SO(3)$, $\{d\alpha, d\beta, d\gamma\}$, to the left invariant 1-forms on $SO(3)$ which can be found the same expressions as before, but with the range of angles automatically appropriate to $SO(3)$, $\alpha \in [0,\pi), \beta \in [0,2\pi), \gamma \in [0,2\pi)$ [3].

The metric should be invariant under spatial rotations. Then by considering da, σ and η, we can construct the most general possible metric as

$$g = A_{ij}(a)\sigma_i\sigma_j + B_i(a)\sigma_i da + C(a)da^2 + D_{ij}(a)\eta_i\eta_j + E_i(a)\eta_i da + F_{ij}(a)\sigma_i\eta_j, \quad (3.6.6)$$

where $i,j = 1,2,3$ and each the component functions depends only on a and is independent of the Euler's angles.

The transformations $\rho : z \mapsto \bar{z}$ and $w : R \mapsto \bar{R}$ are mapping lumps to anti-lumps because each reverses the sign of the topological degree and so both are not an isometry of the moduli space. In fact, the composite transformation $w \circ \rho$ is an isometry. Consider the isometry transformation $U \mapsto \bar{U}$, where $U \in SO(3)$ as a $SU(2)$ Möbius transformation and suppose again $a \mapsto a$. Then

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) \mapsto (-\sigma_1, \sigma_2, -\sigma_3) \quad (3.6.7)$$

$$\eta = (\eta_1, \eta_2, \eta_3) \mapsto (-\eta_1, \eta_2, -\eta_3). \quad (3.6.8)$$

These two isometries remove $B_i(a)$ and $E_i(a)$ for $i = 1,3$ from the general possible metric equation (3.6.6) because for $a \mapsto a$, we have that

$$\sigma \cdot da \mapsto (-\sigma_1 da, \sigma_2 da, -\sigma_3 da),$$

$$\eta \cdot da \mapsto (-\eta_1 da, \eta_2 da, -\eta_3 da).$$

The isometries (3.6.7) and (3.6.8) also result in $A_{12}(a) \equiv A_{21}(a) \equiv A_{23}(a) \equiv A_{32}(a) \equiv 0$ and $D_{12}(a) \equiv D_{21}(a) \equiv D_{23}(a) \equiv D_{32}(a) \equiv F_{12}(a) \equiv F_{21}(a) \equiv F_{13}(a) \equiv F_{31}(a) \equiv F_{23}(a) \equiv F_{32}(a) \equiv 0$.

Our next task should be finding the remaining metric functions of a by taking the appropriate Euler's angle. Firstly, let's consider the parametrization of $SO(3)$ by

$$M(\alpha,\theta,\varphi) = \begin{pmatrix} \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\cos\theta & i\sin\frac{\alpha}{2}(\cos\varphi + i\sin\varphi)\sin\theta \\ i\sin\frac{\alpha}{2}(\cos\varphi - i\sin\varphi)\sin\theta & \cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\cos\theta \end{pmatrix}. \tag{3.6.9}$$

Take first the action $R \mapsto R_\star = e^{i\alpha}R$. That is, we consider $\theta = 0$ in $M(\alpha,\theta,\varphi)$. Then we have a metric of the form $\gamma = \gamma_{\alpha\alpha}(a)d\alpha^2$, where

$$D_{33}(a) = \gamma_{\alpha\alpha}(a) = \int_D \frac{|\partial_\alpha R_\star|^2}{(1+|R_\star|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} = \int_D \frac{|R|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

Now we can also evaluate $D_1(a)$ and $D_2(a)$. Suppose we are taking $\theta = \frac{\pi}{2}$ and $\varphi = 0$. Then with the action $R \mapsto R_\star = MR$, where M is given by the matrix (3.6.9) with $\theta = \frac{\pi}{2}$ and $\varphi = 0$, from $\eta_1 = d\alpha$ we have a metric of the form $\gamma = \gamma_{\alpha\alpha}(a)d\alpha^2$ where

$$D_{11}(a) = \gamma_{\alpha\alpha}(a) = \int_D \frac{|\partial_\alpha R_\star|^2}{(1+|R_\star|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} = \frac{1}{4}\int_D \frac{|1-R^2|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

Similarly, to find out what $D_{22}(a)$ is, let's take $\alpha = \frac{\pi}{2}$ and $\varphi = 0$ in our parametrization of $SO(3)$ in (3.6.9). When we consider the action $R \mapsto R_\star = MR$, then with a similar argument as we saw in evaluating $D_{11}(a)$ and $D_{33}(a)$, we have that the metric is of the form, from $\eta_2 = d\theta$, $\gamma = \gamma_{\theta\theta}(a)d\theta^2$ where

$$D_{22}(a) = \gamma_{\theta\theta}(a) = \int_D \frac{|\partial_\theta R_\star|^2}{(1+|R_\star|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} = \frac{1}{4}\int_D \frac{|1+R^2|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

Similarly, one can also calculate the other metric functions, $D_{13}(a)$ and $D_{31}(a)$:

$$D_{13}(a) = D_{31}(a) = -\frac{1}{2}\int_D \frac{\Re(\bar{R}\,(1-R^2))}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

Note that each corresponding metric functions, $D_{ii}, i, j = 1,2,3$, D_{13} and D_{31} are positive functions.

Secondly, to find the functions $A_{ij}(a), i, j = 1,2,3$, we can follow the same argument as in evaluating the $D_{ij}(a), i, j = 1,2,3$. Suppose we are considering the same parametrization of $SO(3)$ as (3.6.9) and the $SO(3)$ action on z. For instance, let $z \mapsto e^{i\alpha}z$. Then $R \mapsto R_\star = R(e^{i\alpha}z)$. Therefore, we are now able to find $A_{33}(a)$ from the metric of the form $\gamma = \gamma_{\alpha\alpha}(a)d\alpha^2$ where

$$A_{33}(a) = \gamma_{\alpha\alpha}(a) = \int_D \frac{|\partial_\alpha R_\star|^2}{(1+|R_\star|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} = \int_D \frac{|z|^2|\frac{dR}{dz}|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

We can also find the other two functions $A_{11}(a)$ and $A_{22}(a)$ by taking the Euler angles $\theta = \frac{\pi}{2}$ and $\varphi = 0$, and $\alpha = \frac{\pi}{2}$ and $\varphi = 0$, respectively. Considering the above Euler angles and from the metrics of the form $\gamma = \gamma_{\alpha\alpha}(a)d\alpha^2$ and $\gamma = \gamma_{\theta\theta}(a)d\theta^2$, we can find that

$$A_{11}(a) = \gamma_{\alpha\alpha}(a) = \frac{1}{4}\int_D \frac{|1-z^2|^2|\frac{dR}{dz}|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}$$

and

$$A_{22}(a) = \gamma_{\theta\theta}(a) = \frac{1}{4}\int_D \frac{|1+z^2|^2|\frac{dR}{dz}|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

Furthermore, we can calculate the metric functions $A_{13}(a)$ and $A_{31}(a)$ as:

$$A_{13}(a) = A_{31}(a) = -\frac{1}{2}\int_D \frac{\Re(\bar{z}\,(1-z^2))\left|\frac{dR}{dz}\right|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

The corresponding metric functions, $A_{ii}, i, j = 1, 2, 3$,A_{13} and A_{31} are positive functions. Similarly, we can find the following metric functions $B_2(a)$, $E_2(a)$ and $F_{ii}(a), i = 1, 2, 3$ as

$$\begin{aligned}
B_2(a) &= -\frac{1}{2}\int_D \frac{\Re\big((1+z^2)\frac{dR}{dz}\overline{\partial_a R}\big)}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2},\\
E_2(a) &= -\frac{1}{2}\int_D \frac{\Re\big((1+R^2)\overline{\partial_a R}\big)}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2},\\
F_{11}(a) &= \frac{1}{4}\int_D \frac{\Re\big((1-z^2)(1-\bar{R}^2)\frac{dR}{dz}\big)}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2},
\end{aligned}$$

$$\begin{aligned}
F_{22}(a) &= \frac{1}{4}\int_D \frac{\Re\big((1+z^2)(1+\bar{R}^2)\frac{dR}{dz}\big)}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2},\\
F_{33}(a) &= \int_D \frac{\Re\big(z\bar{R}\frac{dR}{dz}\big)}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}\\
F_{13}(a) = F_{31}(a) &= -\frac{1}{2}\int_D \frac{\Re\big((1-z^2)\bar{R}\frac{dR}{dz}\big)}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.
\end{aligned}$$

One can see the plots of these metric functions in fig. 3.2.

Finally, the function $C(a)$ is the same function calculated before which is

$$C(a) = \int_D \frac{|\partial_a R|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

Note that $C(a) = \gamma_{aa}$, where γ_{aa} is a metric function discussed in an ealrier section (3.5) of (3.5.1). Hence, the metric on the 7-dimensional charge three lumps is given by

$$\begin{aligned}
g =& A_{ii}(a)\sigma_i^2 + A_{13}(a)(\sigma_1\sigma_3 + \sigma_3\sigma_1) + C(a)da^2 + D_{ii}(a)\eta_i^2 + D_{13}(a)(\eta_1\eta_3 + \eta_3\eta_1)\\
&+ B_2(a)\sigma_2 da + E_2(a)\eta_2 da + F_{ii}(a)\sigma_i\eta_i, i = 1, 2, 3.
\end{aligned}$$

Proposition 8. *$\widetilde{Rat_3^a}$ is a non-compact totally geodesic Lagrangian submanifold of Rat_3.*

Proof. One can find a similar proof in [56]. We can find that $\widetilde{Rat_3^a}$ is the fixed point set of an isometry of Rat_3. $\widetilde{Rat_3^a}$ is a submanifold of Rat_3 and the Kähler form restrict to 0 on $\widetilde{Rat_3^a}$, and the dimension of Rat_3 is two times the dimension of $\widetilde{Rat_3^a}$. Hence, $\widetilde{Rat_3^a}$ is a totally geodesic Lagrangian submanifold of Rat_3. Let $R \in Rat_3^a$. Its degree drops by 2 as $a \to \infty$, that is an antipodal pair of lump forms, collapses to an infinitely sharp spike and disappears. Thus from this we conclude that Rat_3^a is not compact and the result holds. □

Proposition 9. *The moduli space $\widetilde{Rat_3^a}$ with respect to the general metric g has finite volume, that is, the volume of $(\widetilde{Rat_3^a}, g)$ is finite.*

Proof. The volume on the moduli space Rat_3 is given by

$$Vol(\widetilde{Rat_3^a}) = \int_{SO(3)\times SO(3)\times\mathbb{R}} \sqrt{(|\det(g_{ij})|)}Vol_g(Rat_3^a), \tag{3.6.10}$$

where $Vol_g(\widetilde{Rat_3^a})$ is the volume element on g is given by

$$Vol_g(\widetilde{Rat_3^a}) = \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge da. \tag{3.6.11}$$

Using (3.6.3), (3.6.4) and (3.6.5), we can find that $\eta_1 \wedge \eta_2 \wedge \eta_3 = \sin(\theta) d\phi d\theta d\psi$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 2\pi$. Then we have that

$$Vol(SO(3)) = \int_{SO(3)} \eta_1 \wedge \eta_2 \wedge \eta_3 = 8\pi^2$$

and similarly

$$Vol(SO(3)) = \int_{SO(3)} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = 8\pi^2.$$

The metric functions satisfy the following conditions

$$A_{11} \geq \frac{1}{2}\left(A_{13} + F_{11} + F_{13}\right), \quad A_{22} \geq \frac{1}{2}\left(F_{22} + B_2\right), \quad A_{33} \geq \frac{1}{2}\left(A_{13} + F_{33} + F_{13}\right), \tag{3.6.12}$$

$$D_{11} \geq \frac{1}{2}\left(D_{13} + F_{11} + F_{13}\right), \quad D_{22} \geq \frac{1}{2}\left(F_{22} + E_2\right), \quad D_{33} \geq \frac{1}{2}\left(D_{13} + F_{33} + F_{13}\right), \tag{3.6.13}$$

$$\text{and} \quad C \geq \frac{1}{2}\left(B_2 + E_2\right). \tag{3.6.14}$$

Using Hadamard's inequality [19] and the inequalities (3.6.12) to (3.6.14), the determinant of the matrix (g_{ij}) satisfies

$$
\begin{aligned}
|\det(g_{ij})| &\le \prod_{i=1}^{7}\left(\sum_{j=1}^{7} g_{ij}^2\right)^{\frac{1}{2}} && (3.6.15)\\
&\le \prod_{i=1}^{7}\left(2g_{ii}^2\right)^{\frac{1}{2}} = 8\sqrt{2}\prod_{i=1}^{7} g_{ii} && (3.6.16)\\
&= 8\sqrt{2}A_{11}A_{22}A_{33}D_{11}D_{22}D_{33}C. && (3.6.17)
\end{aligned}
$$

We can also find the following inequalities in the metric functions which are $D_{11} \le \frac{\pi}{6} + \frac{2\pi^2\sqrt{3}}{243}$, $D_{22} \le \frac{\pi}{6} + \frac{2\pi^2\sqrt{3}}{243}$ and $D_{33} \le \frac{\pi}{6}$. We can also see that $A_{11} \le \frac{2\pi}{3}$, $A_{22} \le \frac{2\pi}{3}$ and $A_{33} \le \frac{2\pi}{3} + \frac{8\pi^2\sqrt{3}}{243}$. Thus, $8\sqrt{2}A_{11}A_{22}A_{33}D_{11}D_{22}D_{33} < 1$. The metric function C satisfy the inequality $C \le \frac{4}{(1+a^2)^2}$. Using (3.6.17), $\det(g_{ij}) \le \frac{4}{(1+a^2)^2}$. Note that $\int_0^\infty \frac{2}{1+a^2}da = \pi$ which implies the following integral

$$
\begin{aligned}
Vol(\widetilde{Rat_3^a}) &= 64\pi^4 \int_0^\infty \sqrt{|\det(g_{ij}|}\, da\\
&\le 64\pi^4 \int_0^\infty \frac{2}{1+a^2}\, da = 64\pi^5.
\end{aligned}
$$

This proves the volume of the moduli space of charge three lumps is finite. □

3.6.1 The Moduli Space Rat_3^c

It is easy to check that a rational map $R_c \in Rat_3^c$ has zeros at $z = 0, \pm i\sqrt{c}$ and poles at $z = \pm i\sqrt{\frac{1}{c}}$. The Wronskian of R_c is a polynomial of degree 4 that is $w(z) = cz^4 + (3-c^2)z^2 + c$. For $0 < c < 1$, the zeros of R_c lie inside the unit disc and its poles are found outside the unit disc. For $c = 1$, the poles and zeros of R_c come together and cancel each other, then R_c becomes a rational map of degree one which is z. For $c > 1$, the poles of R_c are found outside the unit disc. Finally, for $c = \infty$, the poles and zeros come together and cancel each other. Then R_c becomes the rational map $\frac{1}{z}$. One can see the energy density of this space in fig.3.3 that shows the energy density is symmetric at $c = 0$. As $c \to 1$ and $c = \infty$, the energy densities dissociate and form spikes. However, for $c = 1$ and $c = \infty$, the energy density becomes the energy density of the rational map $R(z) = z$.

Recall from section 3.6 that the way we have computed the metric functions of the moduli space of Rat_3^a with the isometry transformation of $SO(3)$ as a $SU(2)$ Möbius transformation. Hence the metric should be invariant under rotations and its most general possible metric on $\widetilde{Rat_3^c}$ is given by

$$g_c = \tilde{A}_{ij}(c)\sigma_i\sigma_j + \tilde{B}_i(c)\sigma_i dc + \tilde{C}(c)dc^2 + \tilde{D}_{ij}(c)\eta_i\eta_j + \tilde{E}_i(c)\eta_i dc + \tilde{F}_{ij}(c)\sigma_i\eta_j, \quad (3.6.18)$$

where $i, j = 1, 2, 3$ and each of the component functions depends only on c and is independent of the Euler's angles. Following the same method as earlier in section 3.6, we can remove some metric functions from the metric (3.6.18) and find that

$$\begin{aligned} g_c =& \tilde{A}_{ii}(c)\sigma_i^2 + \tilde{A}_{13}(c)(\sigma_1\sigma_3 + \sigma_3\sigma_1) + \tilde{C}(c)dc^2 + \tilde{D}_{ii}(c)\eta_i^2 + \tilde{D}_{13}(c)(\eta_1\eta_3 + \eta_3\eta_1) \\ &+ \tilde{B}_2(c)\sigma_2 dc + E_2(c)\eta_2 dc + \tilde{F}_{ii}(c)\sigma_i\eta_i, i = 1, 2, 3. \end{aligned}$$

Here take π rotation around the third axis:

$$(\sigma_1, \sigma_2, \sigma_3) \mapsto (-\sigma_1, -\sigma_2, \sigma_3) \text{ and } (\eta_1, \eta_2, \eta_3) \mapsto (-\eta_1, -\eta_2, \eta_3).$$

These two isometries remove $\tilde{A}_{13}, \tilde{A}_{31}, \tilde{D}_{13}, \tilde{D}_{31}, \tilde{F}_{13}, \tilde{F}_{31}, \tilde{B}_2(c)$ and $\tilde{E}_2(c)$ from the general possible metric equation (3.6.18) because we have that

$$\sigma_1\sigma_3 \mapsto -\sigma_1\sigma_3, \ \eta_1\eta_3 \mapsto -\eta_1\eta_3, \ \sigma_2 dc \mapsto -\sigma_2 da \text{ and } \eta_2 dc \mapsto -\eta_2 dc.$$

Therefore, the general possible metric on the moduli space $\widetilde{Rat_3^c}$ is of the form

$$g_c = \tilde{A}_{ii}(c)\sigma_i^2 + \tilde{C}(c)dc^2 + \tilde{D}_{ii}(c)\eta_i^2 + \tilde{F}_{ii}(c)\sigma_i\eta_i, i = 1,2,3, \tag{3.6.19}$$

where $\tilde{A}_{ii}, \tilde{D}_{ii}$ and $\tilde{F}_{ii}$ are the metric functions of $\widetilde{Rat_3^c}$ which have similar structure as the metric functions of $\widetilde{Rat_3^a}$ and in [29].

Proposition 10. *Rat_3^c has two non-overlapping submanifolds of which one is for $0 < c_1 < 1$ with length $l_1(Rat_3^{c_1}) = \frac{5\pi^2}{12}(1 + 2\ln(2))$ and and the second one is for $1 < c_2 < \infty$ with length $l_2(Rat_3^{c_2}) = \frac{5\pi^2}{12}\left(\frac{1}{3} - 2\ln(2) + \frac{4\pi\sqrt{3}}{27}\right)$.*

Proof. One can refer to proposition 6 for the method of proof. □

Proposition 11. *The moduli space (Rat_c, γ_{cc}) has finite length, where $\gamma_{cc}(c) = \tilde{C}(c)$, $c \in [0.1)$.*

Proof. Since the integral $\int_0^1 \sqrt{\gamma_{cc}}dc < \infty$, it shows the length is finite. Therefore, the boundary of (Rat_3^c, γ_{cc}) at infinity lies at finite distance. □

Secondly, we consider rational functions of the form $R(z) = z^3$ and its symmetry orbit denoted by $\widetilde{Rat_3^0}$. The energy density is symmetric and its metric is equivalent to the metric on the moduli space $\{\xi z^n : \xi \in \mathbb{C}^\times\}$ with $\xi = 1$ and $n = 3$ which was shown in [38]. Note that $\widetilde{Rat_3^0}$ is a totally geodesic submanifold of $\widetilde{Rat_3}$. The general metric on $\widetilde{Rat_3^0}$ is given by

$$g_0 = f_i\sigma_i^2 + h_i\eta_i^2 + F_{33}\sigma_3\eta_3, \tag{3.6.20}$$

$$= f_1\sigma_1^2 + f_2\sigma_2^2 + f_3(\sigma_3 + 3\eta_3)^2 + h_1\eta_1^2 + h_2\eta_2^2 \quad i = 1,2,3, \tag{3.6.21}$$

where

$$f_1 = f_2 = \frac{2\pi^2\sqrt{3}}{27}, \quad h_1 = h_2 = \frac{\pi}{6} + \frac{2\pi^2\sqrt{3}}{243},$$
$$f_3 = \frac{3\pi}{2} - \frac{4\pi^2\sqrt{3}}{27}, \quad h_3 = \frac{\pi}{6} - \frac{4\pi^2\sqrt{3}}{243}, \quad F_{33} = \frac{\pi}{2} - \frac{4\sqrt{3}\pi^2}{81}.$$

Proposition 12. *$\widetilde{Rat_3^0}$ has a finite volume with*
$Vol(\widetilde{Rat_3^0}) = \frac{\pi^{13/2}}{19683}\left(4\,\pi\,\sqrt{3}+81\right)\sqrt{324-32\,\sqrt{3}}.$

Proof. Here to avoid over-counting, we divide the volume element by two since the space has an additional C_2 symmetry. As earlier in proposition 3, we have that $Vol(SO(3)) = \int_{SO(3)} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = 8\pi^2$. From (3.6.3) and (3.6.3), $\eta_1 \wedge \eta_2 = \sin(\theta) d\phi d\theta$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, then the integral is $\int_{SO(3)/SO(2)} \eta_1 \wedge \eta_2 = 4\pi$. The volume is given by

$$Vol(\widetilde{Rat_3^0}) = \int \sqrt{|f_1 f_2 f_3 h_1 h_2|}\; \eta_1 \wedge \eta_2 \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3,$$

which can be evaluated as

$$Vol(\widetilde{Rat_3^0}) = 32\pi^3\sqrt{|f_1 f_2 f_3 h_1 h_2|} = \frac{4\pi^{13/2}}{19683}\left(4\,\pi\,\sqrt{3}+81\right)\sqrt{324-32\,\sqrt{3}}.$$

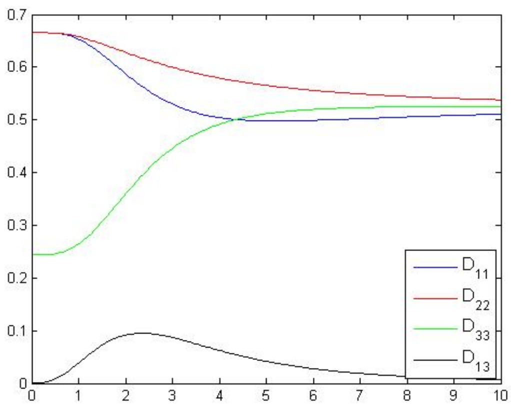
□

Proposition 13. *The moduli space $(\widetilde{Rat_3^0}, \tilde{\gamma}_{cc})$ has finite length.*

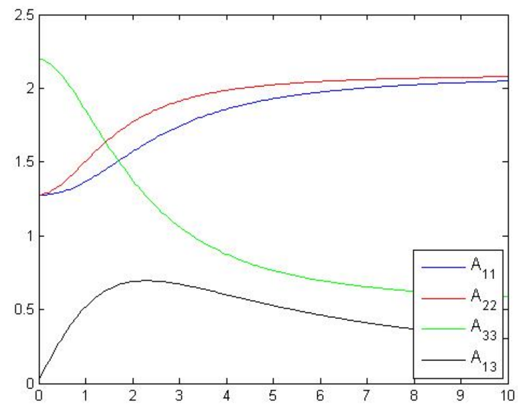
Proof. The length can be computed as follows:

$$l = \int_0^\infty \sqrt{\tilde{\gamma}_{cc}}dc = 2\sqrt{\pi\left(\frac{1}{6}+\frac{2}{2443}\,\pi\,\sqrt{3}\right)}\int_0^\infty \frac{1}{1+c^2}dc = \pi^{\frac{3}{2}}\left(\frac{1}{6}+\frac{2\sqrt(3)\pi}{243}\right)^{\frac{1}{2}}.$$

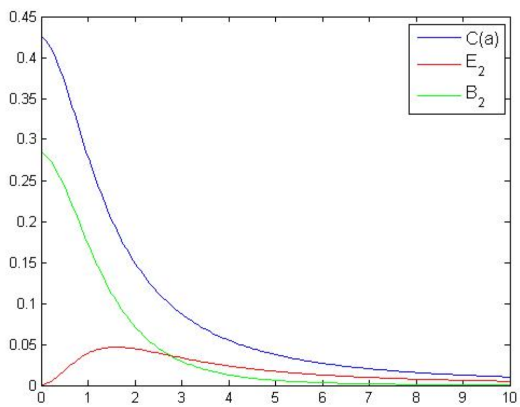
□



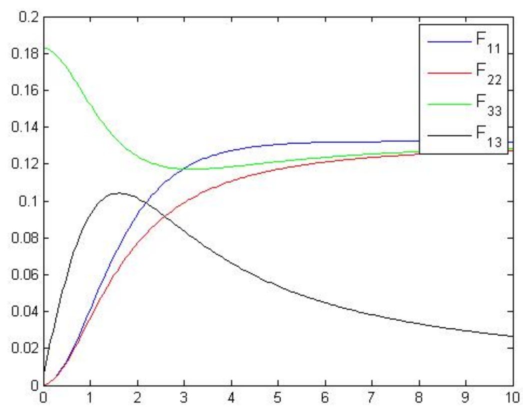
(a)



(b)

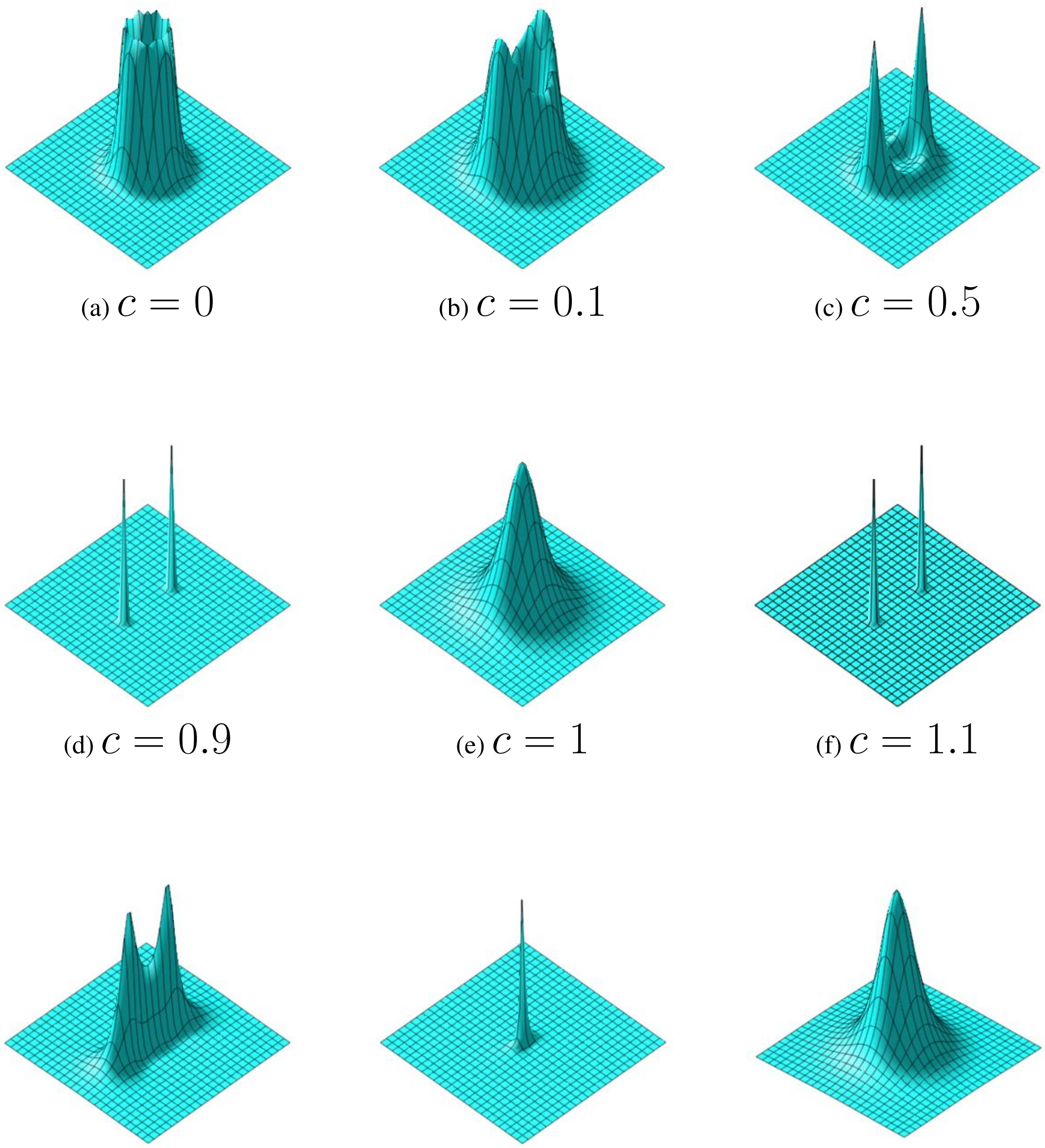


(c)



(d)

Figure 3.2: (a) The metric functions $D_{11}(a)$, $D_{22}(a)$, $D_{33}(a)$ and $D_{13}(a) = D_{31}(a)$ (blue, red, green and black, respectively), (b) The metric functions $A_{11}(a)$ (blue), $A_{22}(a)$ (red), $A_{33}(a)$ (green) and $A_{13}(a)$ (black), (c) The metric function $C(a)$ (blue) which is also the plot of the metric function γ_{aa} that discussed earlier, $E_2(a)$ (green) and $B_2(a)$ (red) show the metric functions are finite and positive definite. Also one can observe that as a increases, all the metric functions approach to zero and (d) The metric functions $F_{11}(a), F_{22}(a)$, $F_{33}(a)$ and $F_{13}(a)$ (blue, red, green and black, respectively).



(a) $c = 0$ (b) $c = 0.1$ (c) $c = 0.5$

(d) $c = 0.9$ (e) $c = 1$ (f) $c = 1.1$

(g) $c = 2$ (h) $c = 10$ (i) $c = \infty$

Figure 3.3: These figures are the energy densities of charge three lumps, Rat_3^c.

3.7 Lump decay

Since a zero and a corresponding pole are opposite each other, a single lump cannot decay. This leads to interesting lump decay channels. Consider first the moduli space Rat_3^a: Following the zeros and poles as for $a \in [0, \infty]$, we start with the axial map with three zeros at the origin 0 and three poles at ∞. Then one zero moves from 0 to ∞ along the positive x axis while one pole moves from ∞ to 0 along the negative x axis. The zero cancels with a pole at ∞ while the pole cancels with a zero at 0. Secondly, take the space Rat_3^c : For $c = 0$ we have the axial maps with three zeros at the origin and three poles at ∞. For $0 < c < 1$ one zero remains fixed while one zero moves up and one down along the imaginary axis. Also, one pole remains fixed at ∞ while two poles travel towards 0 along the positive and negative imaginary axis (y-axis) respectively. For $c = 1$ two poles and two zeros cancel. For $1 < c < \infty$ the poles move towards 0 while the poles move towards ∞ where they cancel. Due to the extra symmetry, lump decay is more complicated on $\mathbb{R}P^2$. Finally, let us consider the moduli space R_∞. The symmetry of the axial map prevents lump decay. For $N = 1$ the space of allowed rational maps is $SO(3)$ and therefore lumps cannot decay. For $N = 3$ it's the 5 dimensional symmetry orbit of z^3. For $c = 0$, we have the axial maps with three zeros at the origin and three poles at ∞. For $0 < c < \infty$, three zeros move from 0 to ∞ along the positive x-axis and the poles come from ∞ to 0 along the negative x -axis. No zeros cancel poles and vice-versa. That is the zeros move from 0 to ∞ and become poles while poles come from ∞ to 0 along the negative x-axis and their final destiny is to become zeros.

3.8 Angular integral

The angular integral of a degree N rational map R is given by

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1+|z|^2}{1+|R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{dz d\bar{z}}{(1+|z|^2)^2}. \tag{3.8.1}$$

Minimizing $\mathcal{I}$ is to minimize the energy configuration for a fixed N. It can be shown that $\mathcal{I} \geq N^2$ [36]. This quantity $\mathcal{I}$ is invariant under rotation in space and target space. Therefore, it distinguishes between maps not related by the symmetry group. Our next task is to calculate $\mathcal{I}$ for rational maps belonging to $\tilde{Rat}_3$. For $R_c \in Rat_3^c$, $R_c(z) \to z^3$ as $c \to 0$ and $R_c(z) \to \frac{1}{z}$ as $c \to \infty$. We can therefore see that R_a and R_c are related by a Möbius transformation for $c \in [0, 1)$ and $a \in [0, \infty)$. In fact, as $iR_c(z) = R_c(iz)$, $c \to -c$, hence we can take $c \geq 0$. We can calculate the angular integral for $a = c = 0$ and denote it by $\mathcal{I}_0$. Then $\mathcal{I}_0 = \frac{81+16\sqrt{3}\pi}{9}$. Fig.3.4 shows the angular integral diverges at $c = 1$ for R_c case. We can compute the $\mathcal{I}_{crit}$ analytically by first calculating the point where the derivative of the angular integral with respect to c vanishes but it is unlikely that there is a simple explicit expression due to the complicated structure of the integrand of $\mathcal{I}$. We calculated I_{crit} numerically and surprisingly, we found that $I_{crit} = I_0$. Here the critical value is at $c = 3$. Hence, there are two families of rational maps. The angular integral $\mathcal{I}$ as a function of a for R_a in (3.4.15) is also displayed in Fig. 3.4. This indicates that the maps $R_a \in \widetilde{Rat}_3^a$ and $R_c \in \widetilde{Rat}_3^c$ with the same value of $\mathcal{I}$ are related to each other via Möbius transformations. In fact, this can be checked using Maple by explicitly computing the relevant Möbius transformations. Hence the moduli space of lumps of (3.4.11) and (3.4.15) parametrize the same space, and therefore (3.4.15) has a hidden D_2 symmetry. So, $\widetilde{Rat}_3^a$ and $\widetilde{Rat}_3^c$ are different parametrizations of $\widetilde{Rat}_3$.

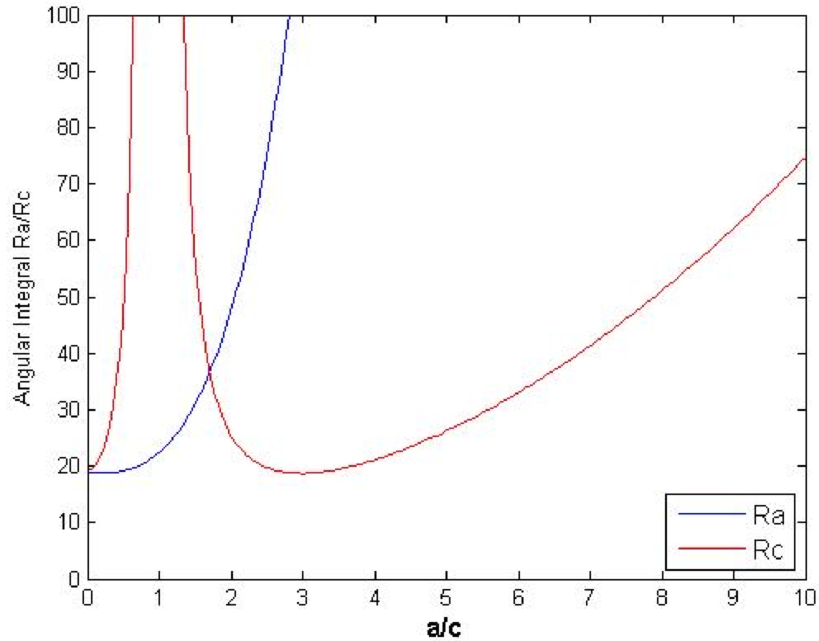


Figure 3.4: Angular integral plot of R_a and R_c

3.9 $N = 5$

The C_n^k symmetry discussed in section 3.4 suggests there are four families of moduli spaces of degree 5 rational maps. In addition, the Riemann-Hurwitz formula indicates that there are five families of moduli spaces of degree 5 rational maps. Let's first consider the former one. Lemma 3 and 4 show a D_∞ symmetry when a C_n symmetry is imposed with $n \geq 5$ which is given by rational maps of the form

$$R(z) = e^{i\psi} \frac{az^5 + b}{-\bar{b}z^5 + \bar{a}}, \tag{3.9.1}$$

where a and b do not both vanish simultaneously and complex, and $\psi \in \mathbb{R}$. Refer to section 3.4 to see that with a similar argument, the symmetry is C_5^0 and its moduli space having dimension 5 can be viewed as the symmetry of z^5.

The rational map which is grouped to the C_2^1 symmetry family is given by

$$R(z) = e^{i\psi}\frac{z(a_0 + a_1 z^2 + a_2 z^4)}{\bar{a}_2 - \bar{a}_1 z^2 + \bar{a}_0 z^4}. \tag{3.9.2}$$

An interesting example of this kind of rational map which satisfies the D_2 symmetry is given by

$$R(z) = \frac{z(1 + iaz^2 + bz^4)}{b + iaz^2 + z^4},\ a, b \in \mathbb{R}. \tag{3.9.3}$$

Surprisingly, when $b = 1$, lump decay can be observed as four zeros cancel with four poles at 0 and four poles cancel with four zeros at ∞. Furthermore, when $a = 0$, $R(z)$ has the symmetry of a square. If $b = -5$, then there is octahedral symmetry. The other family of maps that can be observed from the C_n^k symmetry is the C_3^2 case. In this symmetry family, the rational map is given by

$$R(z) = e^{i\psi}\frac{z^2(a_0 + a_1 z^3)}{\bar{a}_1 - \bar{a}_0 z^3}. \tag{3.9.4}$$

By Möbius transformations preserving this symmetry, namely rotations around the third axis in space and target space, the rational map can be brought into the form

$$R(z) = \frac{z^2(z^3 + a)}{1 - az^3},\ a \in [0, \infty). \tag{3.9.5}$$

The remaining family of maps satisfying the C_n^k symmetry is the C_4^1 which the rational map is given by

$$R(z) = \frac{z(z^4 + a)}{1 + az^4},\ a \in [0, 1) \cup (1, \infty). \tag{3.9.6}$$

Proposition 14. *For N where $n + k = N$, n is even and $n > k$, the moduli spaces of rational maps of degree N of the C_n^k symmetry is 7-dimensional and satisfies the D_n symmetry.*

Proof. The rational map that satisfies the symmetry C_n^k is given by

$$R(z) = e^{i\psi}\frac{z^k(a_1 z^n + a_0)}{\bar{a}_0 z^n + \bar{a}_1}. \tag{3.9.7}$$

By Möbius transformation, the rational map can be brought into the form

$$R(z) = \frac{z^k(z^n + a)}{1 + az^n},\ a \in [0, \infty). \tag{3.9.8}$$

The map (3.9.8) satisfies the condition

$$R\left(\frac{1}{z}\right) = \frac{1}{R(z)}. \tag{3.9.9}$$

Hence, the moduli space of the rational map has D_n symmetry. □

Note that the C_4^1 family of maps is a subset of the C_2^1 family of map.

To see another family of degree 5 rational maps on the projective plane, we use the Riemann-Hurwitz formula [17]. Thus, for $N = 5$, $\sum_{p_i}(d_{p_i} - 1) = 8$ implies there are five possibilities. Here, the index $d_{p_1} = 5$ is the first possibility which fixes $d_{p_2} = 5$. Then the first family is the symmetry of rational map z^5 which coincides with the family of the map (3.9.1). The second possibility of the index d_{p_i} is when $d_{p_1} = d_{p_2} = 4$ which fix $d_{p_3} = d_{p_4} = 2$. In this case, the family of rational maps is given by

$$R(z) = \frac{z^4(z + a)}{1 - az}. \tag{3.9.10}$$

The third possibility of the index d_{p_i} is when $d_{p_1} = d_{p_2} = 3$ and $d_{p_i} = 2$, $i = 3, 4, 5, 6$ which results in the third family of rational maps given by ($b \neq 0$)

$$R(z) = \frac{z^3(z^2 + az + b)}{(1 - \bar{a}z + bz^2)}, \tag{3.9.11}$$

where $|a|^2(b^2+1)+(b^2-1)^2+b(a^2+\bar{a}^2)\neq 0$. The fourth possibility of the index d_{p_i} is when $d_{p_i}=3, i=1,2,3,4$. Thus, the fourth family of rational maps is given by ($b\neq 0$)

$$R(z)=\frac{z^3(z^2+az+b)}{(1-az+bz^2)}, \tag{3.9.12}$$

where $a^2(b+1)^2+(b^2-1)^2\neq 0$.

Conjecture 1. *The family of the map (3.9.12) is a* 9*-dimensional space with* C_2 *symmetry.*

Finally, the remaining possibility of index d_{p_i} is when $d_{p_i}=2,\ i=1,...,8$. Hence, the family of map is given by ($c\neq 0$)

$$R(z)=\frac{z^2(z^3+az^2+bz+c)}{1-\bar{a}z+\bar{b}z^2-cz^3}, \tag{3.9.13}$$

where the parameter $a,\ b$ and c are coefficients such that the determinant of the Sylvester matrix $Syl(\tilde{p},\tilde{q})$ is never zero, where $\tilde{p}=z^3+az^2+bz+c$ and $\tilde{q}=1-\bar{a}z+\bar{b}z^2-cz^3$, and furthermore the Wronskian of (3.9.13) has to have 7 simple roots.

3.10 The Kähler potential and Fubini-Study metric of $\mathbb{R}P^2$ lumps

In this section, we will compute the Kähler potential of the moduli space of Rat_3^a and its Fubini-Study metric. In the example, we shall evaluate the Kähler potential for the totally geodesic submanifold of lumps.

Definition 14. *A Kähler manifold is a Hermitian manifold* (M,g) *whose Kähler form* ω *is closed:* $d\omega=0$*. The metric* g *is called the Kähler metric of* M*.*

The metric on the moduli space of holomorphic maps defined by the sigma model kinetic energy is Kähler whenever both the domain and target manifolds are Kähler [48].

Theorem 4. *Let M be a complex manifold with Riemannian metric g. Then the following are equivalent*

- *g is a Kähler metric.*
- *$d\omega = 0$.*
- *For each point $P \in M$, there is a smooth real function $\mathcal{K}$ in a neighborhood of P such that $\omega = i\partial\bar{\partial}\mathcal{K}$.*

$\mathcal{K}$ is called the Kähler potential.

Proof. The first and second conditions are directly from the definition. One can see the equivalence of the second and the third conditions in [41]. □

$\mathcal{K}$ is unique up to Kähler transformations, $\mathcal{K}(z, \bar{z}) \mapsto \mathcal{K}(z, \bar{z}) + f(z) + f(\bar{z})$ for any holomorphic function f and cannot be a globally defined smooth function on the boundary of compact spaces. Because suppose now that $\mathcal{K}$ is a globally defined. Then ω would be an exact form and so is $vol = \frac{\omega^m}{m!}$. If vol were exact, its integral over M would vanish [13], which is a contradiction with the assumption that the metric is non-degenerate. Therefore, $\mathcal{K}$ cannot be defined globally. For a rational map $R(z) = \frac{p(z)}{q(z)}$, the geodesics in Rat_3^a endowed with the Kähler metric $\gamma_{aa} = \partial_a^2 \mathcal{K}$ where

$$\mathcal{K} = \int_D \log(|p(z)|^2 + |q(z)|^2) \frac{dz d\bar{z}}{(1 + |z|^2)^2},$$

computes the Kähler potential from the static solution configurations $R = R(z, a)$. The complex projective space $\mathbb{C}P^N$ is a Kähler manifold which means that Kähler metrics naturally occur on complex projective varieties. For example there is a standard Fubini-Study metric γ_{FS} on $\mathbb{C}P^N$ which is also positive definite and if $X \subset \mathbb{C}P^N$ is a complex submanifold, the restriction of γ_{FS} to X is Kähler. Therefore, the Fubini-Study metric on $\mathbb{C}P^3$ gives a Kähler metric on M_1 given by the open inclusion $M_1 \subset \mathbb{C}P^3$. Let $(z_0, ..., z_N) \in \mathbb{C}^{N+1}$ be a homogeneous coordinate on $\mathbb{C}P^N$. The Fubini-Study metric γ_{FS} on $\mathbb{C}P^N$ is given by

$$\gamma_{FS} = i\partial\bar{\partial}\log(|z_0|^2 + ... + |z_N|^2).$$

Define coordinate charts by (U_0, ϕ) such that $U_0 = \{[z_0, z_1, ..., z_N] \in \mathbb{C}P^N, z_0 \neq 0\}$. Let $w = (w_1, ..., w_N) \in \mathbb{C}^N, w_i = \frac{z_i}{z_0}$ a section of ϕ over U_0 and hence for $w \in \mathbb{C}^N$, the Fubini-Study metric γ_{FS}

$$\gamma_{FS} = i\left(\frac{\sum_{i=1}^N dw_i d\bar{w}_i}{(1 + \sum_{i=1}^N |w_i|^2)} - \frac{\sum_{i=1}^N \bar{w}_i dw_i \wedge \sum_{k=1}^N w_k d\bar{w}_k}{(1 + \sum_{i=1}^N |w_i|^2)^2}\right).$$

Let $w(z) = \frac{a_1 z + a_2}{a_3 z + a_4}$ with $a_4 \neq 0$ be a rational map of degree 1. Define the inhomogeneous coordinates as $b_i = \frac{a_i}{a_4}, i = 1, 2, 3$. Then using the coordinates $b_i, i = 1, 2, 3$, we can express the Fubini-Study metric as:

$$\gamma_{FS} = \frac{(1 + \sum |b_a|^2)(\sum db_a d\bar{b}_b) - (\sum \bar{b}_a db_a)(\sum b_b d\bar{b}_b)}{(1 + \sum |b_a|^2)^2}. \tag{3.10.1}$$

Using (3.10.1), one can find that the Fubini-Study metric on Rat_3^a as

$$\gamma_{FS} = \frac{da^2}{(1 + a^2)^2}.$$

Example (The Kähler potential for charge one lump)

In this example, we will compute the Kähler potential on the moduli space M_1 where the field is given by $w_\lambda(z) = \mu z$, where $\mu = \frac{\sqrt{1+\lambda^2}+\lambda}{\sqrt{1+\lambda^2}-\lambda}$. Since we can take $p(z) = (\sqrt{1 + \lambda^2} + \lambda)z$ and

$q(z) = \sqrt{1+\lambda^2} - \lambda$, then the Kähler potential is

$$\mathcal{K} = \frac{1}{2}\int_{\mathbb{C}} \left(4\log(1+\mu^2|z|^2) - \log(\mu)\right) \frac{4dxdy}{(1+|z|^2)^2}$$
$$= 16\pi \int_0^\infty \left(\log(1+\mu^2 r^2) - \log(\mu)\right) \frac{rdr}{(1+r^2)^2} = 4\pi\left(\frac{2\mu^2\log(\mu)}{\mu^2-1} - \log(\mu)\right).$$

Note that using this Kähler potential, we can evaluate the L^2 metric on M_1 as

$$\gamma\left(\frac{\partial}{\partial\lambda_1}, \frac{\partial}{\partial\lambda_1}\right) = \frac{4\pi\mu^2\left[(\mu^2+1)\log\mu - \mu^2 + 1\right]}{(\mu^2-1)^3}.$$

Note that this metric is the same metric that we calculated in the previous section.

3.11 Summary of chapter

In this chapter, the geometry and topology of moduli spaces of charge three lumps was studied. The L^2 metric on these moduli spaces was explicitly derived by an isometry transformation of $SO(3)$ as a $SU(2)$ Möbius transformations of space and target space. The moduli spaces have finite length and volume. The zeros and poles of rational maps was considered to study lump decay. The energy densities of lumps for the parameters $a = 0, a = \infty$ and $c = 0, c = 1, c = \infty$ are symmetric. The minimal value of the moment of inertia, or angular integrals, of moduli spaces of charge three lumps were explicitly evaluated. The angular integrals, particularly their plots, played an important role in identifying the family of rational maps. We applied the Riemann-Huwrtiz formula to find possible families of moduli spaces of charge 5 lumps. The Fubini-Study metric on a 1-dimensional moduli space of charge 3 lumps and the corresponding Kähler potential were explicitly computed.

Chapter 4

Vortices on the Hyperbolic Plane

The geometry of the moduli spaces M_N of static N-solitons plays an important role in the study of the dynamics of topological solitons of a field theory of Bogomlyni type: the low energy soliton scattering, the thermodynamics of soliton gases and the quantum mechanics of solitons [36]. The L^2 metric γ on M_N can be taken as the restriction of the kinetic energy of the field. In this chapter we will study the moduli space of hyperbolic vortices and compute the explicit formula for the metric γ on hyperbolic 3- and 4-vortices. The Abelian Higgs model [20] is a field theory which consists of a complex scalar field ϕ coupled to a $U(1)$ connection. The Abelian Higgs vortices are topological solitons in two dimensional space minimizing the energy functional. We will consider Ginzburg-Landau vortices moving on the hyperbolic plane and the Bogomolyni equations for static hyperbolic N-vortices can be reduced to an integrable Liouville equation on a disc.

Let $\mathbb{R}^{1,2}$ be the space-time with coordinates $(x_0 = t, x_1, x_2)$. The $2+1$-dimensional Abelian Higgs model is governed by the action

$$S(\mathcal{A}, \phi) = \int_0^{t_0} (T(\mathcal{A}, \phi) - V(\mathcal{A}, \phi)) dx_0, \tag{4.0.1}$$

where $T(\mathcal{A}, \phi)$ is the kinetic energy, $V(\mathcal{A}, \phi)$ is the potential energy, $\mathcal{A}$ is a $U(1)$-connection on

$\mathbb{R}^{1,2}$ given by the 1-form

$$\mathcal{A} = A_0 dx_0 + A_1 dx_1 + A_2 dx_2,$$

and ϕ is a Higgs field given by a smooth complex function $\phi : \mathbb{R}^{1,2} \longrightarrow \mathbb{C}$. The Lagrangian density at critical coupling constant, $\lambda = 1$, is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\overline{D_\mu\phi}D^\mu\phi - \frac{1}{8}(1-|\phi|^2)^2, \mu = 0,1,2, \tag{4.0.2}$$

where ϕ is a complex scalar field on $\mathbb{R}^{1,2}$ coupled to a $U(1)$ gauge potential $\mathcal{A}$, $D_\mu\phi = \partial_\mu\phi - iA_\mu\phi$ is the covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the gauge field strength. Note that $\mathbb{R}^{1,2}$ is equipped with the Minkowski metric of signature $(+,-,-)$. The terms in the right side of (4.0.2) are then

$$\overline{D_\mu\phi}D^\mu\phi = D_0\phi\overline{D_0\phi} - D_1\phi\overline{D_1\phi} - D_2\phi\overline{D_2\phi} \tag{4.0.3}$$

$$F_{\mu\nu}F^{\mu\nu} = -2(E_1^2 + E_2^2 - F_{12}^2), \tag{4.0.4}$$

where $E_i = F_{0i}$, $i = 1,2$ is the electric components of the field while F_{12} is the magnetic component of the field. The Lagrangian (4.0.2) is also invariant under the following gauge transformations

$$\phi \mapsto e^{i\alpha(t,x)}\phi, \;\; A_\mu \mapsto A_\mu + \partial_\mu\alpha(t,x),$$

where $\alpha(t,x)$ is arbitrary function on $\mathbb{R}^{1,2}$. The field equations are

$$D^\mu D_\mu\phi - \frac{1}{2}(1-|\phi|^2)\phi = 0 \tag{4.0.5}$$

$$\partial_\nu F^{\nu\mu} + \frac{i}{2}(\bar{\phi}D^\mu\phi - \phi\overline{D^\mu\phi}) = 0, \tag{4.0.6}$$

where $D^{\mu} = \eta^{\mu\nu} D_{\nu}$, with $\eta^{\mu\nu}$ being the inverse component of the Minkowski metric η on $\mathbb{R}^{1,2}$. To ensure finiteness of the total energy, we need to put some constraint on the field configuration. The constraint can be explicitly expressed as $|\phi| \to 1$, $D_{\mu}\phi \to 0$ as $|x| \to \infty$. Thus, ϕ takes its value in the vacuum field. Vortices are known to exist in the $(2+1)$-dimensional Abelian Higgs model. They are the static field configurations minimizing the Ginzburg-Landau energy functional.

In sections 4.1 and 4.2, we will give a review of the metric on the moduli space of compact Riemann surfaces and hyperbolic vortices, respectively and in section 4.3, the exact formula for the metric and other geometric correspondences for 3-hyperbolic vortices will be derived. Subsections 4.3.1 and 4.3.3 present the metric on some subspaces of hyperbolic 3-vortices. In subsection 4.3.4, we will study double vortices on the hyperbolic plane and exploit its metric, scalar curvature and Kähler potential. Subsection 4.3.6 discusses the metric and scalar curvature of $m+1$-vortices where $m \geq 2$. Then subsection 4.3.8 will present three collinear hyperbolic vortices. In section 4.4, we study the metric on 4-hyperbolic vortices. Mostly, the metric, Kähler potential and scalar curvature on the moduli space of hyperbolic 3 and submanifolds of 4-vortices will be computed.

4.1 The metric on M_N (with multiplicities)

In this section, we give a brief computation of a metric on a compact Riemann surface when the Abelian Higgs field ϕ has zeros of positive multiplicity m. The metric on M_N is the L^2 metric derived from the restriction of the kinetic energy to M_N. Following Strachan [60], a general formula for the L^2 metric on the moduli space was exploited by Samols where the vortex positions are distinct. To compute the metric on M_N where the vortex positions have multiplicity m, remember that we follow the computation steps and methods of the procedure in the literature [36]. Let Σ be a Riemann surface with the local complex coordinate $z = x_1 + ix_2$ and its complex conjugate $\bar{z} = x_1 - ix_2$. The metric on $\mathbb{R} \times \Sigma$ with conformal factor Ω is

$$ds^2 = dx_0^2 - \Omega(z, \bar{z})dzd\bar{z} = dx_0^2 - \Omega(x_1, x_2)(dx_1^2 + dx_2^2). \tag{4.1.1}$$

The static Ginzberg-Landau (GL) potential energy functional at the critical coupling is

$$V = \frac{1}{2}\int_{\Sigma}\left(\Omega^{-1}F_{12}^2 + \overline{D_i\phi}D_i\phi + \frac{\Omega}{4}(1 - \phi\bar{\phi})^2\right)dx_1dx_2, \tag{4.1.2}$$

where ϕ is a complex valued scalar field on Σ and $F_{12} = \partial_1 A_2 - \partial_2 A_1$ is the curvature. One can rewrite the Ginzberg-Landau (GL) energy by completing the square and we have that

$$V = \frac{1}{2}\int_{\Sigma}\left(\Omega^{-1}\left(F_{12} - \frac{\Omega}{2}(1 - \phi\bar{\phi})\right)^2 + (D_1 + iD_2)\phi\overline{(D_1 + iD_2)\phi} + F_{12}\right)dx_1dx_2. \tag{4.1.3}$$

Vortices, which minimize the energy functional, satisfy the first order Bogomolnyi equations

$$(D_1 + iD_2)\phi = 0 \tag{4.1.4}$$

$$F_{12} - \frac{\Omega}{2}(1 - |\phi|^2) = 0, \tag{4.1.5}$$

where $|\phi|^2 = \phi\bar{\phi}$. The first Chern number classifies the solutions into topologically stable sectors

$$N = \frac{1}{2\pi}\int_{\Sigma}F_{12}, \tag{4.1.6}$$

where N is an integer. The energy satisfies the Bogomolnyi bound

$$V \geq \pi |N|. \tag{4.1.7}$$

Here our aim is to calculate the metric on M_N where the scalar field has zeros of multiplicity m. Suppose $\{\phi(t), A_i(t)\}$ be a family of N-vortex solutions of the Bogomolnyi equations with vortex locations $\{z_r(t)\}$ of multiplicity m_r. The vortices are moving slowly and the time derivative of the Bogomolnyi equations is held by the time derivatives of the field configurations, $\{\partial_0\phi, \partial_0 A\}$. Take $A_0 = 0$, known as the temporal gauge. The Gauss law constraint for the gauge field $E_i = F_{i0} = \partial_i A_0 - \partial_0 A_i$ reads

$$\partial_i E_i + \frac{i}{2}\left(\bar{\phi}\partial_0\phi - \phi\partial_0\bar{\phi}\right) = 0. \tag{4.1.8}$$

The kinetic energy functional is

$$T = \frac{1}{2}\int_\Sigma \left(\partial_0 A_i \partial_0 A_i + \Omega|\partial_0\phi|^2\right) dx_1 dx_2. \tag{4.1.9}$$

Let $\phi = e^{\frac{1}{2}h + i\chi}$. Note that h is a gauge invariant quantity and finite except at the zeros of ϕ. That is, h has logarithmic singularities, and becomes infinitely negative. Using the Bogomolnyi equations we can eliminate χ because from the first Bogomolnyi equation we have that $A_1 = \frac{1}{2}\partial_2 h + i\partial_1\chi$ and $A_2 = -\frac{1}{2}\partial_1 h + i\partial_2\chi$. Then the second Bogomolnyi equation will give (by including a delta-function source for h)

$$\nabla^2 h + \Omega - \Omega e^h = 4\pi\sum_{r=1}^{N} m_r\delta(z - z_r), \tag{4.1.10}$$

where $\nabla^2 = 4\partial_{\bar{z}}\partial_z$ is the standard Laplacian and the equation is valid only away from the zeros of ϕ. As ϕ has zeros with multiplicity m_r, $|\phi| \sim |z - z_r|^{m_r}$. Thus, $h \sim 2m_r \log|z - z_r|$.

Define $\eta = \partial_0 \log\phi$. The kinetic energy (4.1.9) reduces to

$$T = -i\sum_{r=1}^{N}\int_{C_r} \bar{\eta}\partial_{\bar{z}}\eta d\bar{z}, \tag{4.1.11}$$

where C_r is the boundary of the disc D_r centered at z_r, of radius ϵ [36]. The expansion of $h = \log|\phi|^2$ around the point z_r has the form

$$\begin{aligned} h =& 2m_r \log|z - z_r| + a_r + \frac{1}{2}\bar{b}_r(z - z_r) + \frac{1}{2}b_r(\bar{z} - \bar{z}_r) + \bar{c}_r(z - z_r)^2 \\ &- \frac{\Omega(z_r)}{4}(z - z_r)(\bar{z} - \bar{z}_r) + c_r(\bar{z} - \bar{z}_r)^2 + O(|z - z_r|^3), \end{aligned}$$

which is a convergent Taylor series with a_r is real and b_r measures the extent to which contours of h close to z_r differ from circles centered at z_r. To avoid logarithmic singularities of h, let $h_{reg} = h - 2m_r \log|z - z_r|$ be the regularized function. Then

$$b_r = 2\frac{\partial h_{reg}}{\partial \bar{z}}\bigg|_{z=z_r}.$$

We next construct $\bar{\eta}$ and $\partial_{\bar{z}}\eta$ on the boundary C_r and differentiate h with respect to the vortex position z_s. We will have that

$$\begin{aligned} \frac{\partial h}{\partial z_s} =& \frac{-m_r\delta_{rs}}{z - z_r} + \frac{\partial a_r}{\partial z_s} + \frac{1}{2}\frac{\partial \bar{b}_r}{\partial z_s}(z - z_r) - \frac{1}{2}\bar{b}_r\delta_{rs} + \frac{1}{2}\frac{\partial b_r}{\partial z_s}(\bar{z} - \bar{z}_r) - 2\bar{c}_r(z - z_r)\delta_{rs} \\ &- \frac{\Omega(z_r)}{4}(\bar{z} - \bar{z}_r)\delta_{rs} + O(|z - z_r|^3). \end{aligned}$$

One can see that $\frac{\partial h}{\partial z_r}$ has a pole at z_r with residue $-m_r$ that results in

$$(\nabla^2 - e^h)\frac{\partial h}{\partial z_r} = -4\pi m_r \partial_z \delta^2(z - z_r). \tag{4.1.12}$$

Note that, in [36], the term η is given by $\eta = \sum_{r=1}^{N} \dot{z}_r \frac{\partial h}{\partial z_s}$. The derivative of $\frac{\partial h}{\partial z_s}$ with respect to $\bar{z}$ is

$$\partial_{\bar{z}}\left(\frac{\partial h}{\partial z_s}\right) = \frac{1}{2}\frac{\partial b_r}{\partial z_s} + \frac{\Omega(z_r)\delta_{rs}}{4} + O(|z - z_r|),$$

which implies the derivative of η with respect to $\bar{z}$ is

$$\partial_{\bar{z}}\eta = \sum_{r=1}^{N} \dot{z}_s \left(\frac{1}{2}\frac{\partial b_r}{\partial z_s} + \frac{\Omega(z_r)\delta_{rs}}{4}\right) + O(|z - z_r|).$$

Then using the residue theorem, we can compute the integral as

$$\int_{C_r} \bar{\eta}\partial_{\bar{z}}\eta d\bar{z} = m_r \pi i \dot{\bar{z}}_r \sum_{s=1}^{N} \dot{z}_s \left(\frac{\partial b_r}{\partial z_s} + \frac{\Omega(z_s)}{2}\delta_{rs}\right).$$

Summing over all C_s, one can find that the kinetic energy as

$$T = \frac{1}{2}\pi \sum_{r,s} m_r \left(\Omega(z_r)\delta_{rs} + 2\frac{\partial b_s}{\partial z_r}\right) \dot{z}_r \dot{\bar{z}}_s. \tag{4.1.13}$$

Therefore, the metric on M_N where the vortex positions have multiplicities m_r is

$$ds^2 = \pi \sum_{r,s} m_r \left(\Omega(z_r)\delta_{rs} + 2\frac{\partial b_s}{\partial z_r}\right) dz_r d\bar{z}_s. \tag{4.1.14}$$

From the hermitian property of the metric, one can see the following relation

$$\frac{\partial b_s}{\partial z_r} = \frac{\partial \bar{b}_r}{\partial \bar{z}_s}. \tag{4.1.15}$$

The translational invariance of the the entire system results in $\sum_{r=1}^{N} b_r = \sum_{r=1}^{N} \bar{b}_r = 0$ and the rotational invariance produces $\sum_{r=1}^{N} b_r \bar{z}_r = \sum_{r=1}^{N} \bar{b}_r z_r$ [46, 47]. Moreover, the symmetric property which can be found in [47] is

$$\frac{\partial b_r}{\partial \bar{z}_s} = \frac{\partial b_s}{\partial \bar{z}_r}. \tag{4.1.16}$$

The symmetry property implies that the metric on M_N is Kähler [48]. The associated Kähler 2-form, which is closed, is given by

$$\omega = \frac{i\pi}{2} \sum_{r,s} m_r \left(\Omega(z_s)\delta_{rs} + 2\frac{\partial b_r}{\partial z_s} \right) dz_s \wedge d\bar{z}_r. \tag{4.1.17}$$

Because of the symmetry of the second partial derivatives and (4.1.16), one can verify that

$$\begin{aligned}
d\omega &= i\pi \sum_{r,s,t} m_s \left(\frac{\partial^2 b_s}{\partial \bar{z}_t \partial z_r} d\bar{z}_t \wedge dz_r \wedge d\bar{z}_s + \frac{\partial^2 b_s}{\partial z_t \partial z_r} dz_t \wedge dz_r \wedge d\bar{z}_s \right) \\
&= i\pi \sum_{r,s,t} m_s \left(\frac{\partial^2 \bar{b}_r}{\partial \bar{z}_t \partial \bar{z}_s} d\bar{z}_t \wedge dz_r \wedge d\bar{z}_s + \frac{\partial^2 b_s}{\partial z_t \partial z_r} dz_t \wedge dz_r \wedge d\bar{z}_s \right) \\
&= 0.
\end{aligned}$$

Note that $m_s \neq 0$, because by definition m_s is positive integer.

4.2 The metric on hyperbolic vortices

The upper half-plane is given by $\mathcal{H} = \{z = x + iy \in \mathbb{C} : Im(z) > 0\}$. A metric of constant negative curvature on the upper half-plane $\mathcal{H}$ is

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{4.2.1}$$

This space is also called the hyperbolic plane. The hyperbolic plane can be represented by the disc model. The hyperbolic plane $\mathcal{H}$ is isometric to the Poincaré disc model, $D = \{z \in \mathbb{C} : |z| < 1\}$, since we can find an isometry between D and $\mathcal{H}$ using the Cayley-transform

$$c : D \longrightarrow \mathcal{H}, \;\; z \mapsto \frac{i+z}{1+iz}.$$

Note that there is no radial symmetry in the upper half-plane, so for our purpose we will consider later on the Poincaré disc model. Let Σ be a Riemann surface of local complex coordinate z, $\bar{z}$ its complex conjugate, with metric

$$ds^2 = \Omega(z, \bar{z}) dz d\bar{z},$$

where Ω is the conformal factor. The Gauss curvature of the half-plane model is negative. So, the hyperbolic plane has negative Gauss curvature. Note that the Gauss curvature is given by $\kappa = -\frac{2}{\Omega}\partial_z \partial_{\bar{z}} \log(\Omega)$. The metric on the hyperbolic plane with curvature $-\frac{1}{2}$ is

$$ds^2 = \Omega(z, \bar{z}) dz d\bar{z}, \;\; \Omega(z, \bar{z}) = \frac{8}{(1-|z|^2)^2}. \tag{4.2.2}$$

Our next goal is to study the dynamics of the moduli spaces of hyperbolic vortices. The critically coupled Ginzburg-Landau vortices on $\mathcal{H}$ are minimals of the potential energy (4.1.2) but the integral is over $\mathcal{H}$ and the scalar field ϕ is $\phi : \mathcal{H} \longrightarrow \mathbb{C}$. The boundary of the domain is a circle at infinity and there should be a constraint at infinity to ensure finiteness of the energy because for static field, the total energy is finite. This static energy is the Ginzburg-Landau potential energy. Thus the field ϕ satisfies the boundary condition $|\phi| \to 1$ for $|z| \to 1$. The Ginzburg-Landau energy V on $\mathcal{H}$ is minimized by homotopy classes of solutions of the Bogomolyni equations (4.1.4) and (4.1.5). We call these solutions Bogomolnyi vortices or hyperbolic N-vortices. The zeros of ϕ are the vortex centers where ϕ vanishes. These zeros are interpreted as individual vortex positions and the number of vortices N counted with multiplicity is the first Chern number (4.1.6).

Recall from section 4.1 that setting $\phi = e^{\frac{1}{2}h+i\chi}$, where h is gauge invariant quantity and finite except at the zeros of ϕ, one can find the following equation

$$4\frac{\partial^2 h}{\partial z \partial \bar{z}} + \Omega - \Omega e^h = 0, \tag{4.2.3}$$

where this equation is valid only away from the vortex positions. Setting $h = 2g + 2\log\frac{1}{2}(1 - |z|^2)$ the equation for h becomes Liouville's equation [71],

$$4\frac{\partial^2 g}{\partial z \partial \bar{z}} - e^{2g} = 0. \tag{4.2.4}$$

One can solve this equation exactly such that the solution is given by

$$g = \log\left(\frac{2|\frac{df}{dz}|}{1-|f|^2}\right), \tag{4.2.5}$$

where $f(z)$ is an arbitrary, complex analytic function. Using g, we can reconstruct ϕ as

$$|\phi|^2 = \left|e^{g+\log\frac{1}{2}(1-|z|^2)+i\chi}\right|^2 = \frac{(1-|z|^2)^2}{(1-|f|^2)^2}\left|\frac{df}{dz}\right|^2.$$

Then with a simple choice of phase, the scalar field ϕ can be seen as

$$\phi = \frac{1-|z|^2}{1-|f|^2}\frac{df}{dz}. \tag{4.2.6}$$

Note that the scalar field ϕ in (4.2.6) vanishes at the critical points of f satisfying the boundary condition $|\phi| \to 1$ as $|z| \to 1$. The zeros of ϕ are the vortex positions of the moduli space. These are also the critical points of the function f. The boundary condition $|\phi| \to 1$ for $|z| \to 1$ and this ϕ nonsingular inside the disc, $|z| < 1$ results in $|f| \to 1$ on the boundary and $|f| < 1$ inside the unit disc. Hence these conditions are satisfied by choosing a Blaschke function of the form

$$f(z) = z \prod_{i=1}^{N} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right), \tag{4.2.7}$$

where $a_1, ..., a_N \in \mathbb{C}, |a_i| < 1$. Taubes [21] showed that the zeros of ϕ if they exist, can be specified by N unordered points $\{z_1, z_2, ..., z_N\}$. Then the hyperbolic N-vortices are in one-to-one correspondence with degree N polynomials. Let $\{z_1, z_2, ..., z_N\}$ be the zeros of the field ϕ and define a polynomial of degree N with all of whose roots lie in the open disc $|z| < 1$,

$$\begin{aligned} P(z) &= \prod_{r=1}^{N} (z - z_r), |z_r| < 1 \\ &= z^N + w_1 z^{N-1} + ... + w_N. \end{aligned}$$

One can see that there is a one-to-one correspondence between an ordered set of arbitrary complex numbers coefficients $\{w_1, ..., w_N\}$. The set of unordered points $\{z_1, z_2, ..., z_N\}$ determine the coefficients of P as each w_i can be explicitly written in terms of the z_j:

$$w_1 = -(z_1 + ... + z_N), ..., w_N = (-1)^N z_1 z_2 ... z_N.$$

Conversely, the coefficients determine $P(z)$ and hence N-vortices. The function f is a rational map having exactly $2N$ critical points, counted with multiplicity. This can be seen as follows. The function f satisfies $f \circ i = i \circ f$ where $i : z \mapsto \bar{z}^{-1}$. If z is the critical point of f, so is $i(z)$. Therefore, f has N critical points inside the unit disc and N critical points outside the unit disc because f has no critical points on the boundary $|z| = 1$.

The kinetic energy (4.1.9) induces a natural Riemannian (Kähler) metric γ on M_N. Samols [51, 60] derived the metric and the associated Kähler form on the $N-$vortex moduli space by taking distinct N zeros of the Higgs field ϕ. Let the positions of N-vortices be allowed to move along trajectories $z_i(t), i = 1, ..., k$. There are k-vortices z_i with multiplicity m_i and $\sum_{i=1}^{k} m_i = N$. We can then write $h = \log|\phi|^2$ as

$$h = \log|\phi|^2 = 2\log(1 - z\bar{z}) + 2\log|\partial_z f| - 2\log(1 - f\bar{f}).$$

Now suppose that the zeros of ϕ are say z_r of multiplicity m_r, then near the moving zero z_r

$$\partial_z f = (z - z_r)^{m_r}\tilde{f}, \tag{4.2.8}$$

where $\tilde{f}$ is an analytic function. Then one can see that

$$h = 2\log(1 - z\bar{z}) + 2m_r \log|z - z_r| + \log\tilde{f} + \log\bar{\tilde{f}} - 2\log(1 - f\bar{f}).$$

The gauge invariant quantity $h = \log|\phi|^2$ has the series expansion around the zero z_r of ϕ (as earlier in section 4.1)

$$h = 2m_r \log|z - z_r| + a_r + \frac{1}{2}\bar{b}_r(z - z_r) + \frac{1}{2}b_r(\bar{z} - \bar{z}_r) + c_r(z - z_r)^2 + ..., \tag{4.2.9}$$

where $a_r, b_r, \bar{b}_r$ are all functions of the separations between vortex position z_r and all other vortex positions $z_s, s \neq r$. Samols [51] showed that b_r and $\bar{b}_r$ play a central role in the formula for the metric on the moduli space. In order to avoid the logarithmic singularities of h in (4.2.9) near $z = z_r$, we define the regularized version of h as

$$h_{reg} = \log|\phi|^2 - m_r \log(z - z_r) - m_r \log(\bar{z} - \bar{z}_r)$$
$$= 2\log(1 - z\bar{z}) + \log \tilde{f} + \log \bar{\tilde{f}} - 2\log(1 - f\bar{f}).$$

Then the coefficient b_r can be computed as

$$b_r = 2\frac{\partial h_{reg}}{\partial \bar{z}}\bigg|_{z=z_r} = \frac{-4z_r}{1 - |z_r|^2} + 2B_r, \tag{4.2.10}$$

where $B_r = \left(\frac{\partial_{\bar{z}} \bar{\tilde{f}}}{\bar{\tilde{f}}}\right)\bigg|_{z=z_r}$. Samols' argument [51] can be generalized and the kinetic energy (4.1.9) of trajectories in M_N becomes

$$T = 2\pi \sum_{r,s=1}^{k} m_r \frac{\partial B_r}{\partial z_s} \dot{z}_r \dot{\bar{z}}_s, \tag{4.2.11}$$

where $B_r = \left(\frac{\partial_{\bar{z}} \bar{\tilde{f}}}{\bar{\tilde{f}}}\right)\bigg|_{z=z_r}$. The reality property of the kinetic energy shows that

$$\frac{\partial \bar{B}_s}{\partial \bar{z}_r} = \frac{\partial B_r}{\partial z_s}. \tag{4.2.12}$$

Then the Hermitian metric with respect to the canonical complex structure on M_N induced by T is

$$\gamma = 4\pi \sum_{r,s=1}^{k} m_r \frac{\partial B_r}{\partial z_s} dz_r d\bar{z}_s \tag{4.2.13}$$

with Kähler form

$$\omega = 2i\pi \sum_{r,s=1}^{k} m_r \frac{\partial B_r}{\partial z_r} dz_r \wedge d\bar{z}_s. \tag{4.2.14}$$

Example 7. *In this example, we shall see how the metric on the moduli space M_m, where m is* *a multiplicity of a single critical point, is calculated. Consider*

$$f(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^{m+1}, |a| < 1.$$

Then f has a single critical point of multiplicity m at a, since $\frac{df}{dz} = \frac{(z-a)^m(m+1)(1+|a|^2)}{(1-\bar{a}z)^{m+2}}$. The critical point at $z = a$ with multiplicity m is the vortex position. The expansion of h near $z = a$ is given by

$$h = (m+1)\log|z-a| + a_r + \frac{1}{2}\bar{b}_r(z-a) + \frac{1}{2}b_r(\bar{z}-\bar{a}) + ... \tag{4.2.15}$$

with

$$b_a = \frac{-4a}{1-|a|^2} + \frac{2(m+2)a}{1-|a|^2}.$$

So, from this we calculate that

$$\frac{\partial b_a}{\partial a} = \frac{2m}{(1-|a|^2)^2}.$$

From our earlier result, the metric on M_m is given by

$$ds^2 = m\pi\left(\Omega(a) + 2\frac{\partial b_a}{\partial a}\right)dad\bar{a} = \pi m(m+2)\frac{4dad\bar{a}}{(1-|a|^2)^2}.$$

We will see in the next two sections the Kähler potential and the scalar curvature are calculated as

$$\mathcal{K} = -4(m^2+2m)\pi\log(1-|a|^2) \quad \textit{and} \quad \kappa = \frac{-1}{\pi m(m+2)}$$

respectively. Here, the metric, the Kähler potential and the scalar curvature of the moduli space depend on the multiplicity of vortex positions.

4.3 The metric on hyperbolic 3-vortices

In this section, we will study the moduli space of hyperbolic 3-vortices and calculate the non-explicit metric on the moduli space. We shall evaluate an explicit metric for some specific cases and their respective scalar curvature properties and Kähler potentials. The moduli space will then consist of three families based on the zeros of the field ϕ. Recall that with a simple choice of phase, the scalar field ϕ is given by

$$\phi = \frac{1 - z\bar{z}}{1 - f\bar{f}}\frac{df}{dz},$$

where f is a holomorphic map from the hyperbolic plane $\mathcal{H}$ into itself such that $|f| < 1$ with boundary behavior $|f(z)| \to 1$ as $|z| \to 1$ and f is the Blaschke function

$$f(z) = z\prod_{i=1}^{N}\frac{z - a_i}{1 - \bar{a}_i z}, |a_i| < 1.$$

The moduli spaces of hyperbolic 3-vortices shall be studied as follows. Consider the submanifold of the moduli space that is given by

$$\tilde{M_3} = \left\{ f(z) = \frac{z(z^3 + az^2 + bz + c)}{(cz^3 + bz^2 + az + 1)} \, : \, a, b, c \in \mathbb{R} \right\}, \tag{4.3.1}$$

such that the roots of f say z_i are in the unit disc (i.e $|z_i| < 1$). Equivalently,

$$\tilde{M_3} = \tilde{M}_{31} \cup \tilde{M}_{32},$$

where

$$\tilde{M}_{31} = \left\{ f(z) = \frac{z(z - z_1)(z - z_2)(z - \bar{z}_2)}{(1 - z_1 z)(1 - z_2 z)(1 - \bar{z}_2 z)} \, : \, z_1 \in \mathbb{R}, z_2 \in \mathbb{C} \right\} \tag{4.3.2}$$

and

$$\tilde{M}_{32} = \left\{ f(z) = \frac{z(z - z_1)(z - z_2)(z - z_3)}{(1 - z_1 z)(1 - z_2 z)(1 - z_3 z)} \, : \, z_1, z_2, z_3 \in \mathbb{R} \right\}, \tag{4.3.3}$$

where $|z_1| < 1$, $|z_2| < 1$ and $|z_3| < 1$. The function f in (4.3.1), (4.3.2) or (4.3.3) is a Blaschke function and its zeros satisfy the C_2 symmetry, $z \mapsto \bar{z}$. The space $\tilde{M}_3$ is a subspace of M_3 with dimension $\dim_{\mathbb{R}}(\tilde{M}_3) = 3$. The parameters $\{a, b, c\}$ in (4.3.1), and $\{z_1, z_2, \bar{z}_2\}$ in (4.3.2) or $\{z_1, z_2, z_3\}$ in (4.3.3) satisfy the relations

$$a = -(z_1 + z_2 + \bar{z}_2),\ b = |z_2|^2 + z_1(z_2 + \bar{z}_2) \text{ and } c = -z_1|z_2|^2 \tag{4.3.4}$$

or

$$a = -(z_1 + z_2 + z_3),\ b = z_1z_2 + z_1z_3 + z_2z_3 \text{ and } c = -z_1z_2z_3. \tag{4.3.5}$$

We know that the vortex positions are the critical points of the function f. Hence, it is a simple matter that $\frac{df}{dz}$ vanishes when the following polynomials of degree 6 which is the numerator of $\frac{df}{dz}$ vanishes:

$$P(z) = cz^6 + 2bz^5 + (3a + ab - bc)z^4 + (4 + 2a^2 - 2c^2)z^3 + (3a + ab - 2bc)z^2 + 2bz + c. \tag{4.3.6}$$

The zeros of ϕ (positions of the vortices) are the zeros of the polynomial (4.3.6) and these vortices lie inside the unit disc $|z| < 1$. As f has no critical points on $|z| = 1$, f has exactly 3 critical points inside the unit disk and 3 outside. The denominator of this rational map is uniquely determined by its numerator, so on $\tilde{M}_3 \subset \mathbb{C}^3$ with real dimension three and with this kind of f, there are three vortices inside the unit disk $|z| < 1$. The polynomial $P(z)$ has then the form

$$P(z) = c(z - \alpha)\left(z - \frac{1}{\bar{\alpha}}\right)(z - \beta)\left(z - \frac{1}{\bar{\beta}}\right)(z - \gamma)\left(z - \frac{1}{\bar{\gamma}}\right), \tag{4.3.7}$$

such that $|\alpha| < 1$, $|\beta| < 1$ and $|\gamma| < 1$. Since the coefficients of the polynomial $P(z)$ (4.3.6) are real, if Z is a zero of $P(z)$, so is $\bar{Z}$ [2]. Using (4.3.6) and (4.3.7), we find $\alpha = \bar{\alpha}$ implies α

is real, but β and γ can be real or complex. Generally, the vortex positions depend on the zeros of the function f. If z_2 is complex or real, the vortex position β can be either complex or real. Let us first see when β is complex. Then γ should be the conjugate of β. Thus, we can rewrite the polynomial (4.3.7) as

$$P(z) = c(z-\alpha)\left(z-\frac{1}{\alpha}\right)(z-\beta)\left(z-\frac{1}{\bar{\beta}}\right)(z-\bar{\beta})\left(z-\frac{1}{\beta}\right). \tag{4.3.8}$$

Hence, α, β and $\bar{\beta}$ are the zeros of ϕ and implied that the zeros of the rational function f and the vortex positions have similar structure. That is, one of the vortex positions is at the real axis and the other two positions are conjugate to each other. The second case is when both β and γ are real. We can find two sub-cases of which one is when β and γ are equal and otherwise. The former sub-case shows the existence of double vortices while the later one implies the existence of collinear vortices along the real axis. From the series expansion of $h = \log|\phi|^2$ around the zeros of ϕ, $z=\alpha$, $z=\beta$ and $z=\bar{\beta}$, and since $|\phi|^2$ is explicitly known on the hyperbolic space, hence the dependence of $b_r, r = \alpha, \beta, \bar{\beta}$ on the vortex positions α, β and $\bar{\beta}$ can be determined explicitly too. We assumed α to be complex but then it is chosen to be real. In fact, the properties of b_r can be calculated and have the following general forms: for $\partial_z f = (z-\alpha)\tilde{f}$,

$$\begin{aligned} b_\alpha &= \frac{-4\alpha}{1-|\alpha|^2} + 2\left(\frac{\partial_{\bar{z}}\bar{\tilde{f}}}{\bar{\tilde{f}}}\right)\Bigg|_{z=\alpha} && (4.3.9) \\ &= \frac{-6\alpha}{1-|\alpha|^2} + 4Re\left(\frac{1}{\alpha-\beta} + \frac{\beta}{\alpha\beta-1}\right) - \frac{4(a+2b\alpha+3c\alpha^2)}{1+a\alpha+b\alpha^2+c\alpha^3} && (4.3.10) \end{aligned}$$

and also for $\partial_z f = (z-\beta)\tilde{f}$

$$b_\beta = \bar{b}_{\bar{\beta}} = \frac{-4\beta}{1-|\beta|^2} + 2\left(\frac{\partial_{\bar{z}}\bar{\tilde{f}}}{\bar{\tilde{f}}}\right)\bigg|_{z=\beta} \tag{4.3.11}$$

$$= \frac{-6\beta}{1-|\beta|^2} + \frac{2}{\bar{\beta}-\alpha} + \frac{2}{\bar{\beta}-\beta} + \frac{2\alpha}{\alpha\bar{\beta}-1} + \frac{2\bar{\beta}}{\bar{\beta}^2-1} - \frac{4(a+2b\bar{\beta}+3c\bar{\beta}^2)}{1+a\bar{\beta}+b\bar{\beta}^2+c\bar{\beta}^3}. \tag{4.3.12}$$

Using Samols metric and the Kähler property, we can find that the metric on $\tilde{M}_3$ must have the form

$$ds^2 = \pi\sum_{r,s}\left(\Omega(z_r)\delta_{rs} + 2\frac{\partial b_{z_r}}{\partial z_s}\right)dz_r d\bar{z}_s$$

$$= F_1(\alpha,\beta,\bar{\beta})d\alpha d\bar{\alpha} + F_2(\alpha,\beta,\bar{\beta})d\alpha d\bar{\beta} + F_3(\alpha,\beta,\bar{\beta})d\alpha d\beta + F_4(\alpha,\beta,\bar{\beta})d\beta d\bar{\beta},$$

for some metric functions $F_i(\alpha,\beta,\bar{\beta}), i = 1,...,4$. From the b_α and $b_\beta = \bar{b}_{\bar{\beta}}$, we can compute the metric functions $F_i(\alpha,\beta,\bar{\beta}), i = 1,...,4$. Since we know that the metric is given by

$$ds^2 = \pi\left(\Omega(\alpha) + 2\frac{\partial b_\alpha}{\partial\alpha}\right)d\alpha d\bar{\alpha} + 2\pi\left(\Omega(\beta) + 2\frac{\partial b_\beta}{\partial\beta}\right)d\beta d\bar{\beta} \tag{4.3.13}$$

$$+ 2\pi\left(\frac{\partial b_\alpha}{\partial\beta} + \frac{\partial b_{\bar{\beta}}}{\partial\alpha}\right)d\alpha d\bar{\beta} + 2\pi\left(\frac{\partial b_\alpha}{\partial\bar{\beta}} + \frac{\partial b_\beta}{\partial\alpha}\right)d\alpha d\beta \tag{4.3.14}$$

$$= F_1(\alpha,\beta,\bar{\beta})d\alpha d\bar{\alpha} + F_2(\alpha,\beta,\bar{\beta})d\alpha d\bar{\beta} + F_3(\alpha,\beta,\bar{\beta})d\alpha d\beta + F_4(\alpha,\beta,\bar{\beta})d\beta d\bar{\beta}, \tag{4.3.15}$$

where

$$F_1(\alpha,\beta,\bar{\beta}) = \pi\left(\Omega(\alpha) + 2\frac{\partial b_\alpha}{\partial\alpha}\right), \tag{4.3.16}$$

$$F_2(\alpha,\beta,\bar{\beta}) = F_3(\alpha,\beta,\bar{\beta}) = 2\pi\left(\frac{\partial b_\alpha}{\partial\beta} + \frac{\partial b_{\bar{\beta}}}{\partial\alpha}\right), \tag{4.3.17}$$

$$F_4(\alpha,\beta,\bar{\beta}) = 2\pi\left(\Omega(\beta) + 2\frac{\partial b_\beta}{\partial\beta}\right). \tag{4.3.18}$$

The metric (4.3.15) can be evaluated when $\alpha,\ \beta,\ \gamma \in \mathbb{R}$ and has the same kind of structure.

Next we shall study the dynamics on moduli spaces of these hyperbolic 3-vortices in two major cases, and the relationship between the vortex positions and the zeros of the holomorphic function f. Furthermore, we are interested in seeing the metric on this space. Suppose that the vortex positions are vertices of an isosceles triangle. We simply call this kind of vortices isosceles vortices. For example see fig.4.1 that shows the positions of three vortices.

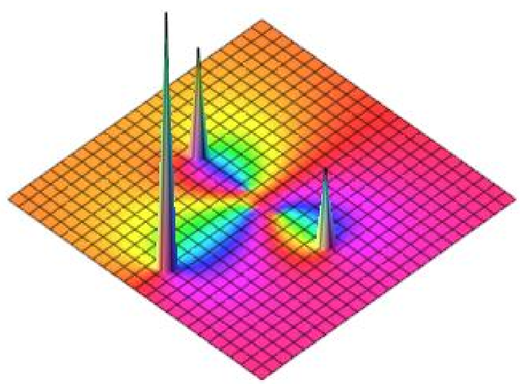


Figure 4.1: The plot of hyperbolic 3- vortices where the positions are at $\frac{i}{2}$, $-\frac{i}{2}$ and $-\frac{1}{2}$.

Expanding (4.3.8) and equating its coefficients with the coefficients of (4.3.6), one can find that

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\bar{\beta}}+\alpha+\beta+\bar{\beta}=\frac{-2b}{c} \tag{4.3.19}$$

$$|\beta|^2+\alpha(\beta+\bar{\beta})+\frac{1}{|\beta|^2}+\frac{1}{\alpha}\left(\frac{1}{\beta}+\frac{1}{\bar{\beta}}\right)+\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\bar{\beta}}\right)(\alpha+\beta+\bar{\beta})=\frac{3a+ab-bc}{c} \tag{4.3.20}$$

$$\frac{1}{\alpha|\beta|^2}+\alpha|\beta|^2+\frac{(\alpha+\beta+\bar{\beta})^2}{\alpha|\beta|^2}+\frac{(|\beta|^2+\alpha(\beta+\bar{\beta}))^2}{\alpha|\beta|^2}=\frac{-(4+2a^2-2c^2)}{c}. \tag{4.3.21}$$

We use these equations for deriving the metric on the moduli space of our hyperbolic 3-vortices though they are complicated to solve explicitly. We calculate the metric on the following submanifolds of M_3.

1. When all zeros of a Blaschke function f in (4.3.2) z_1, z_2 and $\bar{z}_2$ are not at the origin (that is, when $z_1|z_2|^2 \neq 0$). In this case, we first consider the zeros of f at z_2 and $\bar{z}_2$ are fixed and $z_2 + \bar{z}_2 = 0$. We construct vortices whose vortex positions found at the vertices of isosceles triangle and discuss this submanifold in subsection 4.3.1. Similarly, we consider the zeros of f at z_1, z_2 and $\bar{z}_2$ are vertices of an equilateral triangle. We construct vortices whose vortex positions found at the vertices of equilateral triangle and discuss this submanifold in subsection 4.3.2.

2. When a Blaschke function f in (4.3.2) has fixed zero(s) at the origin (that is, when $z_1|z_2|^2 = 0$).

   Firstly, we consider $z_1 = 0, \;\; z_2 \neq 0$ with $z_2 + \bar{z}_2 \neq 0$. We construct two families of vortices. The first family is when the critical points of f are simple. We discuss this submanifold in subsection 4.3.3. The vortex positions are found at the vertices of an isosceles triangle. The second family is when f has a double vortex and we discuss this in subsection 4.3.4. The motivation is to see the fixed vortex at the origin as a defect and study its geometric properties when the two vortices move. Similarly, if we consider $z_1 = 0, \;\; z_2 \neq 0$ with $z_2 + \bar{z}_2 = 0$, the vortex positions found at $ib, \; 0$ and $-ib$, $b \in \mathbb{R}$. The metric on this submanifold was done in [24].

   Secondly, we consider $z_1 \neq 0, \;\; z_2 = \bar{z}_2 = 0$. We discuss this submanifold in subsection 4.3.6 and generalised for $m \geq 2$ fixed vortices at the origin. In this subsection 4.3.6, our motivation to study and understand the metric and scalar curvature relative to the number of defects at the origin.

3. When a Blaschke function f in (4.3.3) has zeros at $z_1 = 0$ (fixed) and $z_2 = z_3$. We discuss this submanifold in subsection 4.3.8. The motivation of studying this submanifold is to calculate the metric when the position of one vortex depends on the other.

4.3.1 Metric on hyperbolic 3-vortices, $c \neq 0$

In this section we discuss the metric on hyperbolic 3-vortices when $c \neq 0$ by taking a specific case. Consider the zeros of the function f where z_1, z_2 and $\bar{z}_2$ are placed on the vertices of a triangle such that $z_1 \neq 0$ and $z_2 + \bar{z}_2 = 0$ implies that z_2 is purely imaginary say, $z_2 = iy, y > 0$. Suppose that z_2 is fixed. Using the coefficients of $P(z)$ in (4.3.6) and (4.3.8), we can find a simplified form of equations, naively

$$\begin{aligned} z_1 &= \frac{2\alpha|\beta|^2}{(1+\alpha^2)|\beta|^2 + \alpha(\beta+\bar{\beta})(1+|\beta|^2)} \\ y^2 &= \frac{1}{2}(-\Gamma + \sqrt{\Gamma^2 + 12}), \end{aligned}$$

where

$$\Gamma = |\beta|^2 + \frac{1}{|\beta|^2} + \left(\alpha + \frac{1}{\alpha}\right)(\beta+\bar{\beta})\frac{(1+|\beta|^2)}{|\beta|^2} + \frac{(\beta+\bar{\beta})^2}{|\beta|^2}.$$

Hence we can find the metric on the moduli space of the hyperbolic 3-vortices. From equation (4.3.4), we have a simplified form of the parameters $a = -z_1, b = y^2$ and $c = -z_1 y^2$. Hence we are now in the position of computing the coefficients b_α, b_β and $b_{\bar{\beta}}$ explicitly. For the coefficients $b_r, r = \alpha, \beta, \bar{\beta}$, the computation is similar to (4.3.10) and (4.3.12). Suppose the vortex position at the real axis is moving and the other two vortices are fixed. Here we want to see what the metric looks like if two vortices are fixed at either different positions or at the origin. Consider the vortex positions at $z = \alpha, z = \beta$ and $z = \bar{\beta}$. Let $|\beta| = r$. Suppose that β is fixed. Furthermore, one can see that either analytically or using Maple $|\alpha| < 1$ for $|z_1| < 1$ and $r < 1$ for $y < 1$. Now since α can be written explicitly in terms of z_1 and since also β is fixed, we have that

$$\alpha = \alpha(z_1)$$

and the metric on this moduli space has the form

$$ds^2 = F(\alpha)d\alpha^2.$$

Recall that the two vortices at the positions β and $\bar{\beta}$ are fixed and the other one vortex at α is moving at a constant speed. Now following the same argument and procedure done earlier, one can find the coefficient b_α as

$$b_\alpha = \begin{cases} \frac{1}{\alpha}\left(\frac{-6\alpha^2}{1-\alpha^2} + \frac{4\alpha^2}{\alpha^2+r^2} + \frac{4r^2\alpha^2}{1+\alpha^2 r^2} + \frac{8\alpha^2(1-y^2+2y^2\alpha^2)}{(1-\alpha^2)(1+\alpha^2 y^2)}\right), \beta \neq 0 \\ \frac{1}{\alpha}\left(\frac{-6\alpha^2}{1-\alpha^2} + 4 + \frac{4\nu\alpha^2}{1-\alpha^2}\right), \beta = 0, \end{cases}$$

where

$$\nu = \frac{3(1+\alpha^2) - \sqrt{9(1-\alpha^2)^2 + 4\alpha^2}}{4\alpha^2}.$$

Hence the metric is

$$ds^2 = \pi\left(\Omega(\alpha) + \frac{1}{\alpha}\frac{d(\alpha b_\alpha)}{d\alpha}\right)d\alpha^2$$
$$= \begin{cases} \left(\frac{12\pi}{(1-\alpha^2)^2} + g(\alpha)\right)d\alpha^2, \beta \neq 0 \\ \frac{4\pi}{(1-\alpha^2)^2}\left(1 + \frac{2(1+\alpha^2)}{\sqrt{9(1-\alpha^2)^2+4\alpha^2}}\right)d\alpha^2, \beta = 0, \end{cases}$$

where

$$g(\alpha) = \frac{8\pi r^2}{(\alpha^2+r^2)^2} + \frac{8\pi r^2}{(1+\alpha^2 r^2)^2} - \frac{16\pi y^2}{(1+\alpha^2 y^2)^2}.$$

Note that when the fixed vortices are at the origin (that is, β=0), the result agrees with [24] as expected. When $\beta \neq 0$, the metric is more general than the result in [24]. Before discussing the case $c = 0$, let us see the following that will give as the same metric as studied by Krusch and Speight [24].

4.3.2 Vortices of zero centre of mass coordinate

In this section we will consider hyperbolic 3-vortices with their center of mass coordinate to be zero, i.e. $\alpha + \beta + \bar{\beta} = 0$ and then study the geometric quantities of the corresponding moduli space. In fact, using $\alpha + \beta + \bar{\beta} = 0$, equations (4.3.19) to (4.3.21) results in the following two simplified equations

$$\frac{|\beta|^2 - \alpha^2}{\alpha|\beta|^2} = \frac{-2b}{c}, \quad |\beta|^2 - \alpha^2 = \frac{3a + ab - bc}{c}$$

and

$$\frac{1}{\alpha|\beta|^2} + \alpha|\beta|^2 + \frac{(|\beta|^2-\alpha^2)^2}{\alpha|\beta|^2} = \frac{-4-2a^2+2c^2}{c}.$$

These equations also give the following cubic polynomial equation

$$\alpha^3 + p\alpha + q = 0,$$

where

$$p = \frac{3a + ab - bc}{c} \quad \text{and} \quad q = \frac{pc}{2b} = \frac{3a + ab - bc}{2b}.$$

From the facts about roots of cubic polynomials of real coefficient, the above cubic equation has at least one real root. In particular, suppose the above polynomial has one real root and the other two are complex. Since $|\beta|^2 = p + \alpha^2$ and $\beta + \bar{\beta} = -\alpha$, then $\beta = \frac{-\alpha}{2} + i\sqrt{p + \frac{3}{4}\alpha^2}$. In fact, either using Cardano formula or with direct computation using the above three equations, we can find that $a = 0$ and $b = 0$. Hence, we are now able to write c explicitly in terms of α, β and $\bar{\beta}$. That is,

$$2c^2 - \left(\frac{1}{\alpha^3} + \alpha^3\right) c - 4 = 0,$$

which is equivalently

$$c = \frac{(1+\alpha^6) - \sqrt{(1+\alpha^6)^2 + 32\alpha^6}}{4\alpha^3} = \frac{(1+\alpha^6) - \sqrt{(1-\alpha^6)^2 + 36\alpha^6}}{4\alpha^3}.$$

Note that the vortex positions $\alpha, \beta, \bar{\beta}$ are placed on an equilateral triangle. Hence, since these vortices are placed on the regular polygons (equilateral triangle), they will rotate at constant speed about their center off mass coordinate (centroid). For the zeros of ϕ (i.e. for the vortex positions α, β and $\bar{\beta}$), one can find that the coefficients b_r as:

$$b_\alpha = \frac{1}{\alpha}\left(\frac{-4\alpha^2}{1-\alpha^2} + 2 - \frac{6\alpha^6}{1-\alpha^6} - \frac{12c\alpha^3}{(1+c\alpha^3)}\right)$$

and

$$b_\beta = \bar{b}_{\bar{\beta}} = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) b_\alpha.$$

Therefore, the metric on this moduli space is of the form

$$ds^2 = G(\alpha)d\alpha^2,$$

where

$$\begin{aligned} G(\alpha) &= 3\pi\Omega(\alpha) + 3\pi\frac{b_\alpha}{\alpha} + 3\pi\left(\frac{\partial b_\alpha}{\partial \alpha}\right) \\ &= \frac{108\pi\alpha^4}{(1-\alpha^6)^2}\left(1 + \frac{6(1+\alpha^6)}{\sqrt{(1-\alpha^6)^2 + 36\alpha^6}}\right). \end{aligned}$$

Note that this metric agrees with the metric in [24]. For $\alpha \to 1$, the term $1 + \frac{6(1+\alpha^6)}{\sqrt{(1-\alpha^6)^2+36\alpha^6}}$ approaches 3 and one can observe that the moving vortices α, β and $\bar{\beta}$ are very far apart that results in the induced metric approaches to the product metric on $(M_1)^3$. We can also show that the metric is asymptotic to

$$ds^2_\infty = \frac{36\pi}{(1-\alpha^2)^2}d\alpha^2.$$

Because the metric on the moduli space $M_1 = \{f(z) = \frac{(z-a)^2}{(1-\bar{a}z)^2}\}$ is $ds^2 = \frac{12\pi}{(1-|a|^2)^2}dad\bar{a}$, then the product metric on $(M_1)^3$ is given by as follows:

$$ds^2_{prod} = \sum_{r=1}^{3} \frac{12\pi}{(1-|z_r|^2)^2}dz_r d\bar{z}_r$$

and the metric

$$ds^2 = G(\alpha)d^2\alpha = \left(\frac{36\pi}{(1-\alpha^2)^2} + O(1-\alpha^2)\right)d\alpha^2.$$

Note that this moduli space has a uniformly constant negative curvature, $-\frac{1}{9\pi}$.

4.3.3 Metric on hyperbolic 3-vortices, $c = 0$

Here we are interested in studying the moduli space of hyperbolic 3-vortices when $c = 0$. We will then study first the dynamics of hyperbolic 3-vortices where $z_1 = 0$ and $z_2 \neq 0$ and then find interesting double hyperbolic 3-vortices. Here, we find that from (4.3.4), the following simplified equations as

$$\frac{a(b+3)}{2b} = A\left(1 + \frac{1}{B}\right) \tag{4.3.22}$$

$$\frac{a^2+2}{b} = B + \frac{1+A^2}{B}, \tag{4.3.23}$$

where a and b are the coefficients (4.3.4) of the polynomial (4.3.6) and $A = -(\beta+\bar{\beta})$, $B = |\beta|^2$, where β and $\bar{\beta}$ are the vortex positions. Then we can solve these two equations simultaneously in terms of β as a function of a and b or vise-versa. Suppose the vortex position at the origin is fixed. From equation (4.3.12), we find the coefficient b_β as

$$b_\beta = \bar{b}_{\bar{\beta}} = \frac{-6\beta}{1-|\beta|^2} + \frac{2}{\bar{\beta}} + \frac{2}{\bar{\beta}-\beta} + \frac{2\bar{\beta}}{\bar{\beta}^2-1} - \frac{8b\bar{\beta}}{1+b\bar{\beta}^2}. \tag{4.3.24}$$

From (4.3.22) and (4.3.23), $b(\beta, \bar{\beta})$ satisfies the equation

$$(p_1 b + p_2)(b+3)^2 + p_3 b^2 = 0, \tag{4.3.25}$$

where $p_1 = B^2(BA^2 + A^2 + 1)$, $p_2 = -2A^2B^2$ and $p_3 = -4A^4(B+1)^2$. The metric on this space is calculated as

$$ds^2 = -8\pi\left(\frac{1}{(1-|\beta|^2)^2} + \frac{2}{|\beta|}\frac{d}{d|\beta|}\left(\frac{b\bar{\beta}^2}{1+b\bar{\beta}^2}\right)\right)d\beta d\bar{\beta}, \tag{4.3.26}$$

where b is the solution of (4.3.25).

4.3.4 Hyperbolic double vortices

In this subsection, we will first show how to parametrize double vortex positions and derive the metric on the moduli space. We discuss and evaluate its corresponding Kähler potential and scalar curvature. In order to derive the metric, we again follow the same procedure of finding the coefficients of the Taylor series expansion of the gauge invariant quantity h. Recall from section 4.3 that the zeros β and γ of the polynomial (4.3.6) can be real. When $\beta = \gamma$, hyperbolic double vortices may exist. Here we shall take $c = 0$. Since $c = z_1|z_2|^2$, $z_1 = 0$ or $z_2 = 0$. Let us first take $z_1 = 0$ and $z_2 \neq 0$ such that $z_2 + \bar{z}_2 \neq 0$. One can see that there exists a double critical point say a of f. That is $\partial_z f$ can be rewritten as

$$\partial_z f = \frac{(z-a)^2(z-\frac{1}{a})^2}{(1-z_2z)^2(1-\bar{z}_2z)^2}.$$

That is, the function f has a Taylor series expansion about $z = a$ of the form

$$f(z) = B_0^a + B_3^a(z-a)^3 + B_4^a(z-a)^4 + \cdots , \tag{4.3.27}$$

where B_0^a and $B_i^a, i = 3, 4, \cdots$ are the coefficients of $(z-a)^0$ and $(z-a)^i$, respectively. Let $z_2 = x + iy$. Then by equating both the equations of $\partial_z f$, we can find

$$x = \frac{3(1+a^2) - \sqrt{9(1-a^2)^2 + 4a^2}}{4a}$$

and

$$y = \frac{\sqrt{2}}{4}\sqrt{\frac{-9-2a^2-2a^4-2a^6-9a^8+3(1+a^2+a^4+a^6)\sqrt{9(1-a^2)^2+4a^2}}{a^2(1+4a^2+a^4)}}.$$

In fact, we have already derived a general metric on the moduli space of vortices on the hyperbolic plane where the zeros of the scalar Higgs field are of multiplicity m. Hence, as a is a double zero for $\partial_z f$, the metric on $\tilde{M}_3$ is calculated as:

$$ds^2 = \frac{16\pi}{3}\partial_a\left(\frac{B_4^a}{B_3^a}\right)da^2, \tag{4.3.28}$$

where B_3^a and B_4^a are the coefficients of $(z-a)^3$ and $(z-a)^4$ of the Taylor series of f about $z=a$, respectively. Hence the metric is

$$ds^2 = \frac{4\pi}{(1-a^2)^2}\left(3+\frac{1+a^2}{\sqrt{9(1-a^2)^2+4a^2}}\right)da^2. \tag{4.3.29}$$

The metric on the moduli spaces of the double 3-vortices on the hyperbolic plane is invariant under $a \mapsto \alpha = e^{i\gamma}a$.Then the metric on the moduli space of double vortices is given by

$$ds^2 = \frac{4\pi}{(1-|\alpha|^2)^2}\left(3+\frac{1+|\alpha|^2}{\sqrt{9(1-|\alpha|^2)^2+4|\alpha|^2}}\right)d\alpha d\bar{\alpha}. \tag{4.3.30}$$

In this case, the double hyperbolic 3-vortices with metric (4.3.30) is a 2-dimensional a submanifold of $\tilde{M}_3$. When $|\alpha| \to 1$, the metric is approximated by the metric on the moduli space of two coincident vortices. Thus, when the hyperbolic double vortices move away from the origin together, then their moduli space is similar to two coincident vortices.

Using (4.3.30), the Kähler potential for the moduli space of the double hyperbolic vortices is

$$\mathcal{K} = -12\pi\log(1-|\alpha|^2) - 6\pi\tanh^{-1}\left(\frac{\sqrt{2}}{8}(-7+9|\alpha|^2)\right) + \mathcal{K}_0,$$

where

$$\mathcal{K}_0 = 2\pi \left[3\tanh^{-1}\left(\frac{-9+7|\alpha|^2}{3\sqrt{9(1-|\alpha|^2)^2+4|\alpha|^2}}\right) + 2\tanh^{-1}\left(\frac{1+|\alpha|^2}{\sqrt{9(1-|\alpha|^2)^2+4|\alpha|^2}}\right)\right].$$

The curvature of this space can be evaluated since we have already calculated the metric. The moduli space of this double hyperbolic vortices has a uniformly negative curvature. The curvature κ for this metric is calculated as

$$\kappa = \frac{A + B\sqrt{9\,|a|^4 - 14\,|a|^2 + 9}}{\left(9\,|a|^4 - 14\,|a|^2 + 9\right)^{3/2}\pi\left(1 + |a|^2 + 3\sqrt{9\,|a|^4 - 14\,|a|^2 + 9}\right)^3}, \tag{4.3.31}$$

where A and B are

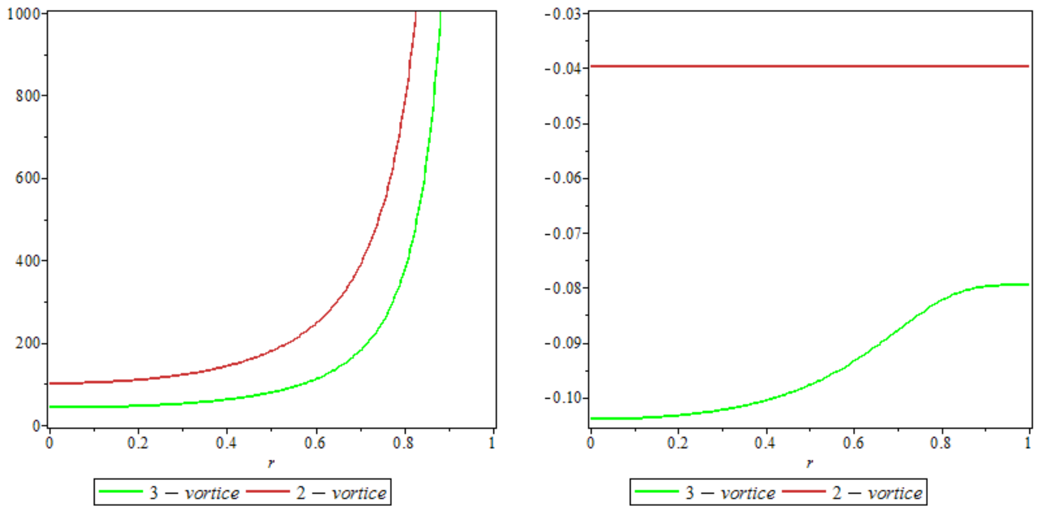
$$A = -6714\,|a|^{12} + 30996\,|a|^{10} - 67686\,|a|^8 + 86168\,|a|^6 - 67686\,|a|^4 + 30996\,|a|^2 - 6714,$$

$$B = -6\,\left(|a|^2 + 1\right)\left(117\,|a|^8 - 404\,|a|^6 + 590\,|a|^4 - 404\,|a|^2 + 117\right).$$

In fact, one can see the result of the plots of the curvature with respect to the parameter $|a|$ in fig. 4.2b (green) and it approaches to $-\frac{1}{8\pi}$ as $|\alpha| \to 1$. Furthermore, one can compare the double hyperbolic 3-vortices with a double hyperbolic 2-vortices at one point. As shown in fig. 4.2, both have uniformly negative curvature and their metric functions are strictly increasing. We also notice that in both metric functions and curvatures, the corresponding geometric quantity of the double hyperbolic 3-vortices is always less than a double 2-vortices case.

4.3.5 Hyperbolic 3-vortices along the y-axis

Let us consider vortices along the y-axis. Here the three vortices are placed along the y-axis where one is fixed at the origin and the other two vortices are placed at the imaginary axis of



(a) Metric functions

(b) Curvatures

Figure 4.2: Plot of curvature and metric functions for double hyperbolic 2- and 3-vortices.

which one is the conjugate of the other. Firstly, consider $z_1 = 0$ and $z_2 \neq 0$ where $z_2 + \bar{z}_2 = 0$. Hence, f has a critical point if and only if the polynomial

$$P(z) = 2z(bz^4 + 2z^2 + b) \tag{4.3.32}$$

has roots in the unit disk. The fourth degree polynomial $q(z) = bz^4 + 2z^2 + b$ is a factor of the polynomial (4.3.32) and it has purely imaginary roots that guarantees the polynomial (4.3.32) too, say β and $\bar{\beta}$, $\frac{1}{\beta}$, $\frac{1}{\bar{\beta}}$. Since $\partial_z f$ vanishes at $z = 0$, we can take simply $\alpha = 0$ and since β is purely imaginary, $\beta + \bar{\beta} = 0$. The coefficient b_i, $i = \alpha, \beta, \bar{\beta}$ of the Taylor series expansion of the gauge invariant quantity h are

$$b_\alpha = -4b \qquad \text{and} \qquad b_\beta = \frac{2}{\bar{\beta}} - \frac{6\beta}{1 - |\beta|^2} - \frac{4b\bar{\beta}}{1 + b\bar{\beta}^2}.$$

The metric agrees with [24] as expected, the scalar curvature and Kähler potential on this particular moduli space are

$$ds^2 = \frac{96\pi|\beta|^2}{(1-|\beta|^4)^2} d\beta d\bar{\beta}, \quad \kappa = -\frac{1}{6\pi} \quad \text{and} \quad \mathcal{K} = -24\pi \log(1-|\beta|^4), \tag{4.3.33}$$

respectively. We notice here that the moduli space has uniformly constant negative curvature.

4.3.6 The metric on hyperbolic $(m+1)$-vortices, $m \geq 2$

In this subsection, we will first evaluate the metric on the moduli space of hyperbolic 3-vortices where one vortex is at the real axis and the other two vortices are fixed at the origin. Then we will generalize for $m+1$-vortices on the hyperbolic plane. Consider $z_2 = 0$ and $z_1 \neq 0$. In other words, all the three vortices are placed on the x-axis. The vortex postions are the zeros of the polynomial

$$P(z) = z^2(3az^2 + (4+a^2)z + 3a)$$

in the unit disc. The vortex positions are at $\beta = 0$ of multiplicity 2 and α of single multiplicity:

$$\alpha = \frac{-(2+a^2) + \sqrt{a^4 - 5a^2 + 4}}{3a}.$$

The derivative of f is

$$\partial_z f = (z-\alpha)\tilde{f}, \text{ where } \tilde{f} = \frac{z^2(\alpha z - 1)}{\alpha(1+az)^2}.$$

The coefficients of the Taylor series expansion of $\log|\phi|^2$, that is, b_α is given by

$$\begin{aligned} b_\alpha =& \frac{-4\alpha}{1-\alpha^2} + 2\frac{\partial_{\bar{z}}\bar{\tilde{f}}}{\bar{\tilde{f}}}|_{z=\alpha} \\ =& \frac{-6\alpha}{1-\alpha^2} + \frac{4}{\alpha} - \frac{4a}{1+a\alpha}. \end{aligned}$$

Suppose that the two vortices (i.e vortex positions) at $z = \beta$ and $z = \bar{\beta}$ are fixed at the origin and the other vortex at $z = \alpha$ moves. Then the metric on the moduli space of these vortices is

$$ds^2 = \frac{4\pi}{(1-\alpha^2)^2}\left(1 + \frac{2(1+\alpha^2)}{\sqrt{9(1-\alpha^2)^2 + 4\alpha^2}}\right) d\alpha^2.$$

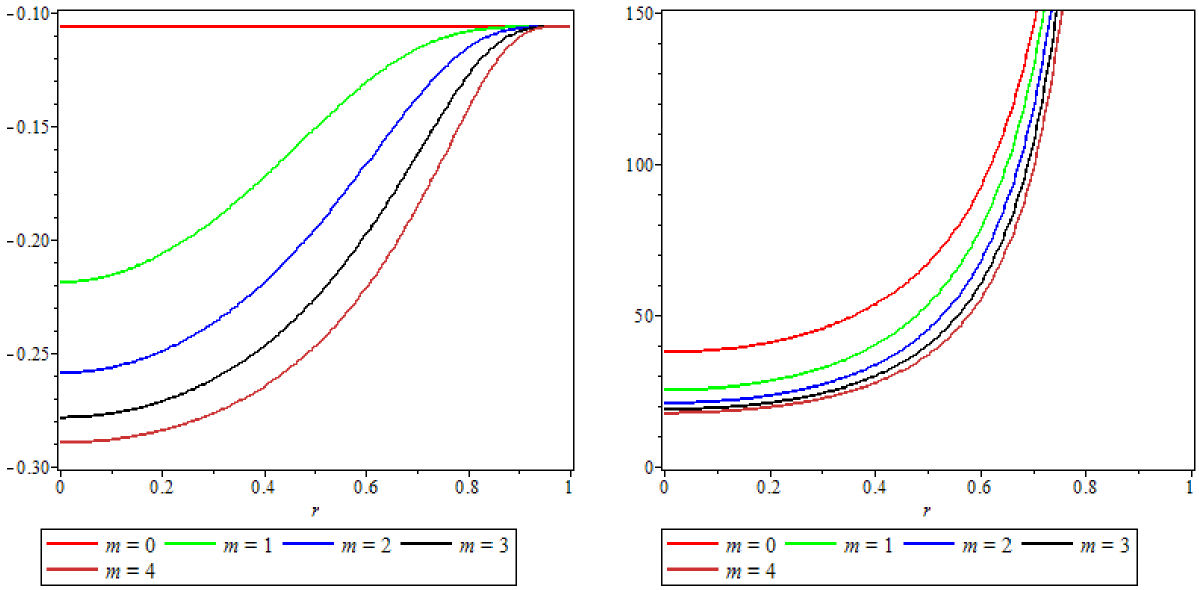
In general, suppose that m vortices are fixed at the origin (i.e. the vortex position at $z = \beta = 0$ of multiplicity m) and the other vortex position at $z = \alpha$ is rotating. Then one can find that the metric on this moduli space is given by

$$ds^2 = \frac{4\pi}{(1-|\alpha|^2)^2}\left(1 + \frac{2(1+|\alpha|^2)}{\sqrt{(m+1)^2(1-|\alpha|^2)^2 + 4|\alpha|^2}}\right) d\alpha d\bar{\alpha}. \qquad (4.3.34)$$

Fig.4.3 shows the geometric properties of the metric (4.3.34). Fig 4.3 shows that for all m, the metric functions of the respective moduli space having same structure and as m increases the metric functions decreases. For all values of m, the metric functions are bounded below by $\frac{4\pi}{(1-|\alpha|^2)^2}$ and above by $\frac{12\pi}{(1-|\alpha|^2)^2}$ and the curvature κ of the moduli space is uniformly negative and as m goes to ∞, the scalar curvature $\kappa \to -\frac{1}{3\pi}$.

4.3.7 Comparison of hyperbolic 2- and 3-vortices

In this subsection we will discuss the comparison between hyperbolic 2- and 3-vortices based on their curvature and metric functions. We can see some geometric properties of hyperbolic 2-

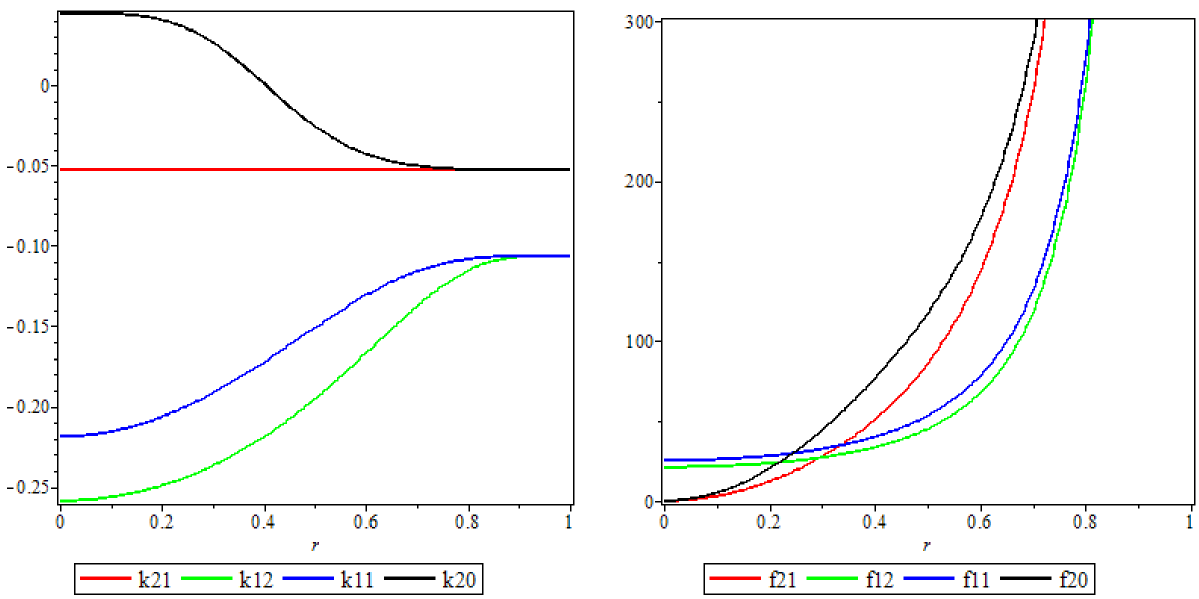


(a) Curvatures when m=0,1,2,3,4.

(b) Metric functions when m=0,1,2,3,4.

Figure 4.3: Plot of scalar curvatures and metric functions for hyperbolic $(m+1)$-vorices, $m = 0, 1, 2, 3, 4$, when m-vortices fixed at the origin.

and 3-vortices. Fig. 4.4 shows the plot of curvature and metric functions of hyperbolic 2- and 3- vortices where the vortex positions placed at the origin and real axis. Fig.4.4a and 4.4b refer the plot of scalar curvature and metric functions of moduli spaces hyperbolic 2- and 3- vortices, respectively. The black curve refers when 2-vortices placed at opposite positions and there are no other vortices in between (that is, the vortex positions say are -α and α). The moduli space (black) has positive scalar curvature near $|\alpha| = 0$ and negative near $|\alpha| = 1$. The red curve represents when 2-vortices placed at opposite positions and one vortex found in between (that is, the vortex positions say are -α, 0, and α). The moduli space (red) has constant scalar curvature which is $-\frac{1}{6\pi}$. As shown in the figure, scalar curvature of the moduli spaces (black and red) approach to $-\frac{1}{6\pi}$ and their metric functions have similar behaviour. The blue curve refers when one vortex placed at a position α and one vortex position at the origin. The moduli space (blue) has uniformly negative scalar curvature. The green curve refers when one vortex placed at a position α and two vortices at the origin. The moduli space (green) has uniformly negative scalar curvature. As shown in the figure, scalar curvature of the moduli spaces (blue and green) approach to $-\frac{1}{3\pi}$ and their metric functions have similar behaviour. In general, one can see from the figure that the geometric quantities (scalar curvature and metric functions) of the 3-vortices is less than the 2-vortices. Another comparison is also see in 4.2.



(a) Curvatures.

(b) Metric functions.

Figure 4.4: Plot of curvature and metric functions for hyperbolic 2 and 3-vortices. kij represents scalar curvature and fij represent the metric functions.

4.3.8 Three collinear vortices

In this subsection, we will discuss collinear hyperbolic 3-vortices. Their vortex positions of these three collinear hyperbolic 3-vortices can be exploited from the Blaschke's function f where f is given by (4.3.35). We will then derive the metric (4.3.36). Consider the space of functions of the form

$$M_{cr} = \{f : f(z) = \frac{z^2(z-a)^2}{(1-\bar{a}z)^2}, a \in \mathbb{C}, |a| < 1\}. \tag{4.3.35}$$

One can see that f has critical points in the unit disc: $z = 0, z = a, z = \frac{1-\sqrt{1-|a|^2}}{\bar{a}} := c$ of which all are in a straight line. Suppose the vortex position at $z = 0$ is fixed and the other two vortices at $z = a$ and $z = c$ move. One of the surprising properties of these vortices is that the two vortices positioned at a and c move in the same direction since when a increases (or decreases) so does c and vice-versa. We can compute the metric as usual using the coefficients of $h = \log |\phi|^2$. The coefficients b_a are given by

$$b_a = \frac{6}{\bar{a}} + \frac{2a}{1-|a|^2} \text{ and } b_c = \frac{2c}{1-|c|^2} = \frac{a}{\sqrt{1-|a|^2}}.$$

Let $a = re^{i\psi}$ and $c = Re^{i\psi}$. Then the metric on this moduli space is

$$ds^2 = f_1(r)d^2r + f_2(r)d^2\psi, \tag{4.3.36}$$

where $f_2(r) = \left(r^2 + R^2 + rR\left(4 + \frac{6}{1-r^2} + \frac{6-r^2}{(1-r^2)^{\frac{3}{2}}}\right)\right) f_1(r)$, $R = \frac{1-\sqrt{1-r^2}}{r}$ and

$$f_1(r) = \frac{12\pi}{(1-r^2)^2} + \frac{12\pi(R'(r))^2}{(1-R^2)^2} + 2\pi R'(r)\left(4 + \frac{6}{1-r^2} + \frac{6-r^2}{(1-r^2)^{\frac{3}{2}}}\right).$$

Now for $a << 1$ and for $a \to 1$, we can see $f_1(r)$ and $f_2(r)$ as

$$f_1(r) \sim \frac{32\pi}{(1-r^2)^2} \quad \text{and} \quad f_2(r) \sim 8r^2 f_1(r).$$

This is in other words, the two moving vortices approach each other. Here, all the three vortices lie in a straight line in such a way that one is fixed at the origin and the other two of them of which one is dependent on the other, move together towards to the origin and recede away from the origin in the same direction simultaneously.

4.4 The metric on 4-vortices for hyperbolic plane

In this section, we will study the moduli space of hyperbolic 4-vortices where the zeros of the Blaschke function satisfying the C_2 symmetries: $z \mapsto -z$ and $z \mapsto \bar{z}$. Consider the space of functions where their zero sets satisfy these two C_2 symmetries given by

$$\tilde{M_4} = \left\{ f : f(z) = z \prod_{i=1}^{4} \left(\frac{z - Z_i}{1 - \bar{Z}_i z} \right) \right\} \tag{4.4.1}$$

where $Z_i \in \mathbb{C}$ or $Z_i \in \mathbb{R}$ such that $|Z_i| < 1$. Thus $\tilde{M_4}$ can be rewritten as the form:

$$\tilde{M_4} = \left\{ f : f(z) = \frac{z(z^4 + Az^2 + B)}{(1 + Az^2 + Bz^4)}, A, B \in \mathbb{R} \right\}, \tag{4.4.2}$$

where $A = -(Z_1^2 + \bar{Z_1}^2)$ and $B = |Z_1|^4$ or $A = -2Z_1^2$ and $B = Z_1^4$. The space $\tilde{M_4}$ is a submanifold of M_4 and $\dim_{\mathbb{R}}(\tilde{M_4}) = 2$. If Z_1 is real, the vortex positions are on the real axis forming four collinear vortices. The metric will be calculated in the next section when $a = b \in \mathbb{R}$. However, if Z_1 is complex, the critical points of f can be either complex or real. So, we can find two families of vortex positions. In fact, the numerator of the derivative $\frac{df}{dz}$ which is of the form

$$P(z) = Bz^8 + (3 - B)Az^6 + (A^2 - 3B^2 + 5)z^4 + (3 - B)Az^2 + B \tag{4.4.3}$$

implies the zeros of the scalar field (that is the vortex positions) are of the form $\pm\alpha$ and $\pm\bar{\alpha}$, where α is a complex number. Thus, the vortex positions fall into two families and show that the first family is when α is real which leads to the construction of double hyperbolic 4-vortices. The second family of vortex postions is when α is complex. Interestingly, the vortex positions satisfy the C_2 symmetry : $z \to \bar{z}$ and $z \to -z$. Hence, the polynomial (4.4.3) can be rewritten as

$$P(z) = B(z^2-\alpha^2)(z^2-\bar{\alpha}^2)\left(z^2-\frac{1}{\alpha^2}\right)\left(z^2-\frac{1}{\bar{\alpha}^2}\right). \tag{4.4.4}$$

The function f is invariant under the action $Z_1 \mapsto \bar{Z}_1$ from which it follows that Z_1 can be seen as the product of a function of α and a positive function ν, ν is a function of $|\alpha|$ only, such that $Z_1 = \alpha\nu(|\alpha|)$. For example, if $\Im(Z_1) = \Re(Z_1)$, the vortex positions are at the vertices of a square with centre at the origin. Therefore, using the two equations (4.4.3) and (4.4.4), we can find ν as

$$\nu^4 = \frac{-(1+|\alpha|^8)+\sqrt{(1-|\alpha|^8)^2+64|\alpha|^8}}{6|\alpha|^8}. \tag{4.4.5}$$

The metric on the moduli space then is given by

$$ds^2 = \frac{256\pi\alpha^6}{(1-\alpha^8)^2}\left(1+\frac{8(1+\alpha^8)}{\sqrt{(1-\alpha^8)^2+64\alpha^8}}\right)d\alpha^2.$$

One can see that the metric agrees with the metric in [24]. Hyperbolic double 4-vortices can be noticed either on the real or imaginary axis. Let us take the vortex positions α be real. Then the double hyperbolic 4-vortices exist and will discuss in a later subsection 4.4.17.

4.4.1 Hyperbolic four collinear vortices

Consider the space of functions of the form

$$\tilde{M}_{4c} = \left\{ f(z) = \frac{z(z^2-z_1^2)^2}{(1-\bar{z}_1^2z^2)^2}, z_1 \in \mathbb{C} \right\}. \tag{4.4.6}$$

Note that the zeros of the function satisfies the symmetry $z \mapsto -z$. Take $z_1 = ia$ purely imaginary. Then the space (4.4.6) can be rewritten as

$$\tilde{M}_{4c} = \left\{ f(z) = \frac{z(z^2+a^2)^2}{(1+a^2z^2)^2}, a \in \mathbb{R} \right\}. \tag{4.4.7}$$

The partial derivative of f with respect to z is then

$$\partial_z f = \frac{(z^2+a^2)(z^2+c^2)(c^2z^2+1)}{c^2(1+a^2z^2)^3},$$

where

$$c^2 = \frac{5-3a^4-\sqrt{9a^8-34a^4+25}}{2a^2}$$

satisfying $|c| < 1$. Hence, $\pm ia$ and $\pm ic$ are the four simple zeros of ϕ (that is, the vortex positions) inside the unit disc. These vortices have zero centroid, which remains fixed as these vortices rotates. The coefficients b_a and b_c are

$$b_a = -b_{-a} = \frac{1}{\bar{a}} \left(\frac{-4|a|^2}{1-|a|^2} + \frac{6(1+|a|^4)}{1-|a|^4} \right) \tag{4.4.8}$$

and

$$b_c = -b_{-c} = \frac{1}{\bar{c}} \left(\frac{-4|c|^2}{1-|c|^2} + 1 + \frac{4\bar{c}^2}{\bar{c}^2-\bar{a}^2} - \frac{4|c|^4}{1-|c|^4} + \frac{12a^2\bar{c}^2}{1-a^2\bar{c}^2} \right), \tag{4.4.9}$$

respectively. One can in fact easily find the following partial derivatives which help us to calculate the metric on our particular moduli spaces of four vortices

$$\frac{\partial b_a}{\partial a} = \frac{2}{(1-|a|^2)^2} - \frac{6}{(1+|a|^2)^2} \tag{4.4.10}$$

and

$$\frac{\partial b_c}{\partial c} = \frac{-4(1+4|c|^2+|c|^4)}{(1-|c|^4)^2} + \frac{16|c|^2}{(1-|c|^4)^2}\left(1+\frac{4(1+|c|^4)}{\sqrt{1+62|c|^4+|c|^8}}\right). \tag{4.4.11}$$

At this stage we have all the ingredients to set up the metric and it is

$$\begin{aligned}
ds^2 &= 2\pi\left(\Omega(a)+4\frac{\partial b_a}{\partial a}\right)da^2 + 2\pi\left(\Omega(c)+4\frac{\partial b_c}{\partial c}\right)c'(a)^2 da^2 + 8\pi\left(\frac{\partial b_a}{\partial c}+\frac{\partial b_c}{\partial a}\right)c'(a)da^2 \\
&= 16\pi\left(\frac{2}{(1-a^2)^2}-\frac{3}{(1+a^2)^2}\right)da^2 + \frac{16\pi(1+c^4)}{(1-c^4)^2}\left(-1+\frac{6c^2}{\sqrt{(1-c^4)^2+64c^4}}\right)dc^2 \\
&+ \frac{32\pi c^2}{(1-c^4)^2}\left(1+\frac{4(1+c^4)}{\sqrt{(1-c^4)^2+64c^4}}\right)dc^2.
\end{aligned}$$

Note that the function c^2 is a monotonic function. Now when the vortex position at $z = a$ moves towards the origin, so does the vortex position at $z = c$. Hence, the four vortices move to the origin simultaneously. Also, when the vortex position at $z = a$ moves far from the origin, the other three vortices also move far apart. That is, the four vortices moves far apart from the origin simultaneously. Furthermore, as a tends to 1, so does c. Hence, the vortices are far apart. Note that an interesting thing happened in this moduli space such that as the vortices move far from the origin, there exists a double vortex. Then the metric is approximately

$$ds^2 = \frac{64\pi}{(1-a^2)^2}da^2.$$

Hence, the moduli space has approximate curvature $-\frac{1}{3\pi}$.

4.4.2 Double hyperbolic 4-vortices

Consider the space of functions

$$M_4^\star = \left\{ f(z) = \frac{z(z^2-a^2)(z^2-b^2)}{(1-\bar{a}^2z^2)(1-\bar{b}^2z^2)}, a, b \in \mathbb{C} \right\}. \tag{4.4.12}$$

Let f have double critical points at $\pm\alpha$, where α is the vortex position and $|\alpha| < 1$. f has 4 critical points z_r at the end point of line segment with multiplicity 2: $z_r = (-1)^{r-1}\alpha$. Then we can relate the vortex position $z_1 = \alpha$ to the complex parameters a and b. Note that f is unchanged under both $a \mapsto -a$ and $b \mapsto -b$ which means a and b can be rewritten as

$$a = \alpha\nu_1(|\alpha|) \quad \text{and} \quad b = \alpha\nu_2(|\alpha|),$$

where ν_1 and ν_2 are positive real functions of $|\alpha|$ only. It follows that $ab = \nu_1\nu_2\alpha^2$. We can compute ν_1 and ν_1 as follows: from the space (4.4.12), one can find the derivative of f with respect to z as

$$\partial_z f = \frac{(\bar{c}z^8 + \bar{d}z^6 + ez^4 + dz^2 + c)}{(1 - \bar{a}^2z^2)^2(1 - \bar{b}^2z^2)^2}, \tag{4.4.13}$$

where

$$c = a^2b^2$$

$$d = b^2(|a|^4 - 3) + a^2(|b|^4 - 3)$$

$$e = 5 - 3|a|^4|b|^4 + |a|^4 + |b|^4 + \bar{a}^2b^2 + a^2\bar{b}^2.$$

For $n > 0$, the equation $p(z) = a_0 + a_1z + ... + a_nz^n = 0$ has at least one root. We can factor $p(z)$ as

$$p(z) = (z - \alpha_1).....(z - \alpha_n)$$

where $\alpha_1, ..., \alpha_n$ are not necessary distinct. Let α be a double critical point of f. Using (4.4.13), one expects that the degree 8 polynomial equation, $q(z) = \bar{c}z^8 + \bar{d}z^6 + ez^4 + dz^2 + c = 0$, should be factored and therefore, this results in a double zero at $z = \alpha$ and $z = -\alpha$. Since $\frac{1}{\bar{\alpha}}$

and $\frac{-1}{\bar{\alpha}}$ are also critical points of f, one can factor $q(z)$ as

$$q(z) = \bar{c}(z^2 - \alpha^2)^2(z^2 - \frac{1}{\bar{\alpha}^2})^2 = \bar{c}(z^4 - (\alpha^2 + \frac{1}{\bar{\alpha}^2})z^2 + \frac{\alpha^2}{\bar{\alpha}^2})^2. \tag{4.4.14}$$

It follows from the coefficients (4.4.14) of $q(z)$, we can find the following equations:

$$\frac{(ab)^2}{(\bar{a}\bar{b})^2} = \frac{\alpha^4}{\bar{\alpha}^4} \tag{4.4.15}$$

$$b^2(|a|^4 - 3) + a^2(|b|^4 - 3) = -2\alpha^2(|\alpha|^4 + 1)\nu_1^2\nu_2^2 \tag{4.4.16}$$

$$5 - 3|a|^4|b|^4|a|^4 + |b|^4 + a^2\bar{b}^2 + b^2\bar{a}^2 = (2|\alpha|^4 + (|\alpha|^4 + 1)^2)\nu_1^2\nu_2^2 \tag{4.4.17}$$

$$\bar{b}^2(|a|^4 - 3) + \bar{a}^2(|b|^4 - 3) = -2\bar{\alpha}^2(|\alpha|^4 + 1)\nu_1^2\nu_2^2. \tag{4.4.18}$$

Also, from (4.4.17), we have that

$$5 - 3(|\alpha|^4\nu_1^2\nu_2^2)^2 + |\alpha|^4(\nu_1^2 + \nu_2^2)^2 = (|\alpha|^8 + 4|\alpha|^4 + 1)\nu_1^2\nu_2^2 \tag{4.4.19}$$

and similarly from (4.4.16), one can find that

$$(|\alpha|^4\nu_1^2\nu_2^2 - 3)(\nu_1^2 + \nu_2^2) = -2(|\alpha|^4 + 1)\nu_1^2\nu_2^2. \tag{4.4.20}$$

We can rewrite equation (4.4.20), as follows

$$\nu_1^2\nu_2^2 = \frac{3\nu}{|\alpha|^4(\nu + 2) + 2}, \tag{4.4.21}$$

where $\nu = \nu_1^2 + \nu_2^2$. Now using (4.4.19) and (4.4.21), a quartic equation can be constructed as

$$A\nu^4 + B\nu^3 + C\nu^3 + D\nu + E = 0, \tag{4.4.22}$$

where A, B, C, D are all functions of $|\alpha|$ where

$$A = 20(1 + |\alpha|^4)^2,\ B = -6\left(|\alpha|^{12} - \frac{10}{6}|\alpha|^8 - \frac{10}{6}|\alpha|^4 - 1\right),$$
$$C = |\alpha|^4\left(|\alpha|^8 - 26|\alpha|^4 + 1\right),\ D = 4|\alpha|^4(1 + |\alpha|^4),\ E = |\alpha|^{12}.$$

The quartic equation (4.4.22) can be solved (using for example Maple) in terms of ν which gives four real roots of which two of them are negative and $\nu = \frac{1+|\alpha|^4}{|\alpha|^4}$ is one of the real root, though we do not use it to calculate the metric. The other real root ν is

$$\nu = -\frac{1}{3|\alpha|^4}\left(5(1 + |\alpha|^4) - Y - \frac{7(|\alpha|^8 + 14|\alpha|^4 + 1)}{Y}\right),$$

where $Y = (10(1 + |\alpha|^4)(|\alpha|^8 - 34|\alpha|^4 + 1) + Z)^{\frac{1}{3}}$, where
$Z = 3i\sqrt{27\,|\alpha|^{24} + 2334\,|\alpha|^{20} + 1157\,|\alpha|^{16} + 83556\,|\alpha|^{12} + 11157\,|\alpha|^8 + 2334\,|\alpha|^4 + 27}.$

We can compute the coefficient b_α such that

$$\bar{\alpha}b_\alpha = -\bar{\alpha}b_{-\alpha} = \frac{-4|\alpha|^2}{1 - |\alpha|^2} + \frac{8|\alpha|^4}{1 - |\alpha|^4} + 8|\alpha|^4\nu\left(\frac{|\alpha|^4\nu - 4|\alpha|^4 + 2}{|\alpha|^8\nu(1 - \nu) - |\alpha|^4(\nu - 2) + 2}\right) := G(|\alpha|).$$

Hence, the metric on this space is

$$ds^2 = 2\pi\left(\frac{8}{(1 - |\alpha|^2)^2} + \frac{1}{|\alpha|}\frac{dG(|\alpha|)}{d|\alpha|}\right)d\alpha d\bar{\alpha}. \tag{4.4.23}$$

Here, the second term of the metric in the parentheses is a bit long but using Maple we can plot the metric function of (4.4.23). One can see the plots of the metric function of this moduli space (see fig.4.5) and scalar curvature plot (see fig. 4.6). Note that the term a in the figure is $|\alpha|$.

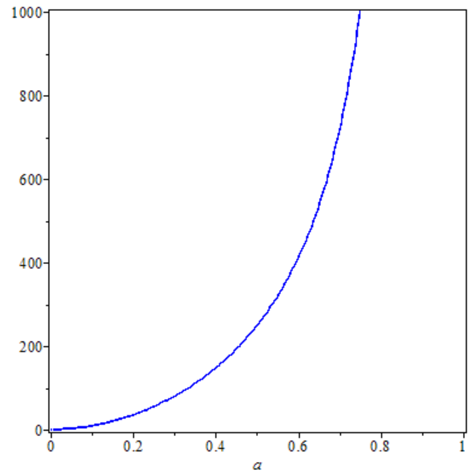


Figure 4.5: The plot of metric functions of double 4-vortices. Note that $a = |\alpha|$.

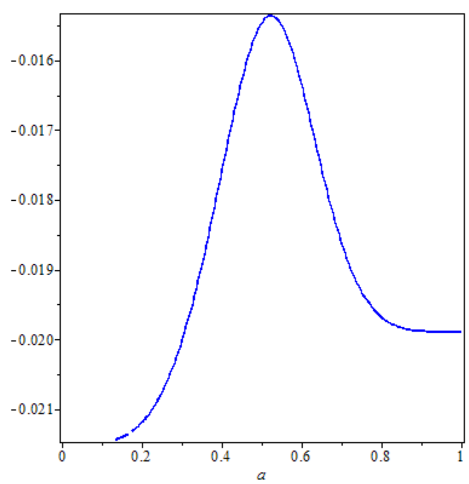


Figure 4.6: The plot of scalar curvature of double 4-vortices. Note that $a = |\alpha|$.

4.5 Summary of chapter

In this chapter, the metric on the moduli spaces of vortices with positive multiplicity, the general and implicit metric on the moduli space of hyperbolic 3- and 4-vortices where at least one of the vortex positions is not at the origin (that is, when $c \neq 0$ or $c = 0$) were evaluated. It has been shown the coefficient b_r of the Taylor series expansion of the gauge invariant quantity h is independent of the multiplicity. The metric, Kähler potential and curvature of moduli space of hyperbolic 3-vortices where the vortex positions have zero centre of mass coordinate have been calculated and agree with the metric in [24]. The double hyperbolic 3 and 4-vortices were shown and their corresponding metrics and curvatures were computed. Interestingly, numerical results showed that their respective curvatures are uniformly negative and increase as the modulus of the vortex position increases to 1 and approaches to $-\frac{1}{3\pi}$ and $-\frac{1}{4\pi}$, respectively. In addition, the Kähler potential for the moduli space of double hyperbolic 3-vortices is computed. We have studied also collinear hyperbolic 3- and 4-vortices and have calculated their metrics and curvatures. Similarly, their corresponding curvature is uniformly negative as a numerical plot suggests.

Chapter 5

Geometry and Dynamics of Vortices

In two dimensions, vortices are particle-like objects. The kinetic energy functional determines the Riemannian metric and its geodesics on the moduli space of vortices [31, 34, 35]. Manton studied the adiabatic trajectories, called the geodesics of the Riemannian metric, such that any dynamic solution of this kind can be obtained as perturbation of some geodesic trajectories. The moduli space approximation [30] is a good approach to study the dynamics of solitons at low energy when most of the degree of freedom are frozen and the solitonic dynamics can be studied by the dynamics in a reduced finite dimensional moduli space. The potential of N-vortices is at the absolute minimum on the moduli space M_N. Moreover, the kinetic energy term of the reduced dynamics dominates because for a small velocity, vortices are unable to move away from M_N. The geodesic motion on M_N is the dynamics projected on M_N.

In this chapter, section 5.1 presents the Kähler potential of a totally geodesic submanifold $\Sigma_{r,t}$ of M_N using the direct definition of the Kähler potential and metric. Section 5.2 gives the Kähler potential of $\Sigma_{r,t}$ using the regularized action of the Liouville theory and a scaling argument. In section 5.3, we discuss the geometry of $\Sigma_{r,t}$.

5.1 The Kählerp potential for some totally geodesic submanifolds of the moduli space M_N

Let r and t be positive integers such that $r+t=N$. Denote by $\Sigma_{r,t}$ be the space of C_r-invariant vortex configurations with r single vortices at the vertices of a regular polygon and t coincident vortices at the polygon's centre. The exact moduli space metrics on some totally geodesic submanifolds of the moduli space of static hyperbolic N-vortices for $r > t+1$ or $r = t+1$ and the metric of N vortices on regular N-gon were studied in [24] and [26], respectively. To calculate the Kähler potential for totally geodesic submanifolds, we first recalculate the metric on this space and generalize the metric on $\Sigma_{r,t}$ for $r \neq t+1$ or $r = t+1$. Recall that points in M_N are in one-to-one correspondence with monic degree N polynomials with roots only in the unit disk. Consider the moduli space such that we identify a N-vortex with the monic polynomial defined by the numerator of the rational map $f(z)$:

$$\Sigma_{r,t} = \{f : f(z) = \frac{z^{t+1}(z^r + \alpha_r)}{(1-\bar{\alpha_r} z^r)}, \alpha_r \in \mathbb{C},\ r+t = N, r, t \geq 0\}. \tag{5.1.1}$$

Assume that $\alpha_r = -a^r,\ a \in \mathbb{C}$. Note that z is a critical point of f if and only if λz is a critical point since $f(\lambda z) \equiv \lambda^{t+1} f(z)$, $\lambda = e^{\frac{2\pi i}{r}} \in U(1)$. The numerator of $\frac{df}{dz}$ vanishes if and only if the polynomial

$$p(z) = z^t\left((t+1)\bar{a}^r z^{2r} - (t+1+r+(t+1-r)|a|^{2r}))z^r + (t+1)a^r\right)$$

vanishes. So, we can find that f has $2r$ critical points z_i such that r of them say $z_i = \lambda^{i-1}\beta$ inside the unit disc at the vertices of some regular r-gon which are the the vortices where the Higgs field vanishes, r critical points outside the unit disc which can be disregarded and the other t critical points should be coincident at 0. It follows $\Sigma_{r,t}$ consists of vortex configuration with r vortices located at the vertices of a regular r-gon centered at 0 and t vortices coincident at $z = 0$. Therefore, using the critical points $z_i = \lambda^{i-1}\beta$ of f, we can find the metric on $\Sigma_{r,t}$. Let us consider $\beta = z_1 \in \mathbb{C}$ which can be related to the complex parameter a. Then $a = \beta\nu(|\beta|)$ where ν is a positive function of $|\beta|$ only. Now choose $|a| < 1$ and $|\beta| < 1$. Then we find that

$$\nu^r = \begin{cases} \frac{(t+1)(1+|\beta|^{2t}) - \sqrt{(t+1)^2(1-|\beta|^{2r})^2 + 4r^2|\beta|^{2r}}}{2|\beta|^{2r}(t+1-r)}, & r \neq t+1, \\ \\ \frac{2}{1+|\beta|^{2r}}, & r = t+1. \end{cases}$$

To avoid logarithmic singularities of h near β, define the regularized form of $h = \log|\phi|^2$ as

$$h_{reg} = \log|\phi|^2 - \log(z-\beta) - \log(\bar{z}-\bar{\beta}). \tag{5.1.2}$$

Since β is a simple zero, then we have that

$$b_1 = 2\frac{\partial h_{reg}}{\partial \bar{z}}|_{z=\beta}, \tag{5.1.3}$$

then we can find that

$$b_1 = \frac{1}{\bar{\beta}}\left(2t + r - 1 - \frac{2r\beta^r\bar{\beta}^r}{1-\beta^r\bar{\beta}^r} + \frac{4r\nu^r\beta^r\bar{\beta}^r}{1-\beta^r\bar{\beta}^r} - \frac{4\beta\bar{\beta}}{1-\beta\bar{\beta}}\right). \tag{5.1.4}$$

Using rotational symmetry, we can determine the other coefficients, $b_i, i \geq 2$, in terms of b_1. So, using rotational invariance and the Kähler form on M_N, the metric on the moduli space $\sum_{r,t}$ is

$$\gamma = F(|\beta|)d\beta d\bar{\beta}, \tag{5.1.5}$$

where $F(|\beta|)$ is the conformal factor [24]:

$$F(|\beta|) = \pi r \left(\Omega(|\beta|) + \frac{1}{|\beta|} \frac{d}{d|\beta|} (\bar{\beta} b_1) \right). \tag{5.1.6}$$

By substituting the above b_1 into $F(|\beta|)$, the metric on the space $\sum_{r,t}$ is

$$\gamma = \begin{cases} \frac{4\pi r^3 |\beta|^{2r-2}}{(1-|\beta|^{2r})^2} \left(1 + \frac{2r(1+|\beta|^{2r})}{\sqrt{(t+1)^2(1-|\beta|^{2r})^2 + 4r^2|\beta|^{2r}}} \right) d\beta d\bar{\beta}, & r \neq t+1, \\ \frac{12\pi r^3 |\beta|^{2r-2}}{(1-|\beta|^{2r})^2} d\beta d\bar{\beta}, & r = t+1. \end{cases} \tag{5.1.7}$$

The Kähler potential $\mathcal{K}$ and the Kähler metric $\gamma_{\alpha\beta}$ on a complex manifold is given by

$$\gamma_{\alpha\beta} = \partial_\alpha \partial_\beta \mathcal{K}. \tag{5.1.8}$$

Now we are calculated an explicitly expression of the Kähler potential for $\Sigma_{r,t}$ when $r = t+1$ and for $\Sigma_{r,t}$ when $r \neq t+1$. Thus, the Kähler potential for $\Sigma_{r,t}$ is given by

$$\mathcal{K} = \begin{cases} -4\pi r \log(1 - |\beta|^{2r}) + \mathcal{K}_0, & r \neq t+1, \\ -12\pi r \log(1 - |\beta|^{2r}), & r = t+1, \end{cases} \tag{5.1.9}$$

with

$$\begin{aligned} \mathcal{K}_0 &= 4r\pi \int \frac{\sqrt{(t+1)^2(1-|\beta|^{2r})^2 + 4r^2|\beta|^{2r}}}{(1-|\beta|^{2r})} \frac{d\beta}{\beta} \\ &= 8r\pi \tanh^{-1}\left(\frac{r(1+|\beta|^{2r})}{k_0} \right) - 4\pi(t+1) \ln \left(\frac{1}{k_0} - (t+1)(1-|\beta|^{2r}) \right) \\ &- 4\pi \tanh^{-1} \left(\frac{(t+1)(1-|\beta|^{2r})}{k_0} \right) + C, \end{aligned}$$

where $k_0 = \sqrt{(t+1)^2(1-|\beta|^{2r})^2 + 4r^2|\beta|^{2r}}$ and C is a constant. For example, for $t = 0$, $\mathcal{K}_0$ is

$$\begin{aligned}\mathcal{K}_0 =& 4\pi\left(-\ln(2r^2 - 1 + |\beta|^{2r} + \sqrt{(1-|\beta|^{2r})^2 + 4r^2|\beta|^{2r}})\right)\\ &- 4\pi\tanh^{-1}\left(\frac{1+(2r^2-1)|\beta|^{2r}}{\sqrt{(1-|\beta|^{2r})^2+4r^2|\beta|^{2r}}}\right)\\ &+ 8r\pi\tanh^{-1}\left(\frac{r(1+|\beta|^{2r})}{\sqrt{(1-|\beta|^{2r})^2+4r^2|\beta|^{2r}}}\right).\end{aligned}$$

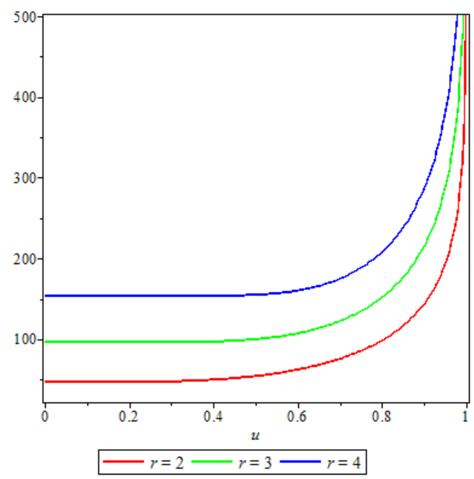


Figure 5.1: The Kähler potential for $\Sigma_{2,0}, \Sigma_{3,0}$ and $\Sigma_{4,0}$ such that the red, the green and the blue correspond, respectively. Note that $u = |\beta|$.

Notice that one can see that the metric γ is given by

$$\gamma = \left(\frac{12r\pi}{(1-|\beta|^2)^2} - rt(t+2)\pi + O(1-|\beta|^2)\right) d\beta d\bar{\beta},$$

and it means that the Kähler potential $\mathcal{K}$ for $\sum_{r,t}$ can be calculated as

$$\mathcal{K} = -12r\pi\log(1-|\beta|^2) - r(t^2+2t)\pi + r(t^2+2t)\pi(1-|\beta|^2) + \mathcal{K}_0(1-|\beta|^2),$$

where $\partial_\beta\partial_{\bar{\beta}}\mathcal{K}_0(1-|\beta|^2) = O(1-|\beta|^2)$. Fig. 5.1 implies the Kähler potential on $\Sigma_{2,0}, \Sigma_{3,0}$ and

$\Sigma_{4,0}$ shows these Kähler potential is unbounded above and as r increases and so do the Kähler potentials.

5.2 The Liouville field in the hyperbolic vortices

In this section, we will first study the natural existence of the Liouville field in the hyperbolic vortices and then the Kähler potential of vortices in the Abelian Higgs vortices following the work of Chen and Manton [9]. Witten [71] and Strachan [60] exploited the fact that a Liouville field naturally exists in hyperbolic vortices. This interesting fact distinguishes the hyperbolic vortices from the flat vortices. The Liouville field then can be used to understand the N-vortices instead of the Taylor series expansion of h. It was known that the explicit metric on the moduli space of the hyperbolic vortices can be computed up to two vortices. However, for some $N \geq 3$, we can evaluate an explicit metric on the submanifolds of moduli space of hyperbolic N-vortices. For example, the metric on totally geodesic submanifold as we discussed in section 5.1, and also see chapter 4 on hyperbolic 3- and 4-vortices.

Let Σ be a Riemann surface with Riemannian metric g. The metric g can be expressed in terms of a conformal factor Ω as

$$g = \Omega dz d\bar{z}. \tag{5.2.1}$$

Set $\phi = e^{\frac{1}{2}h + i\chi}$ as earlier. Bradlow and Garcia-Prada [8] showed the vortices exist with arbitrary locations for the N-vortex field equation on M defined by

$$\nabla^2 h - \Omega(e^h - 1) = 0. \tag{5.2.2}$$

Recall that for hyperbolic plane $\mathcal{H}$ with scalar curvature $-\frac{1}{2}$, the metric g is given by (4.2.2).

Let $g = e^{\sigma} dz d\bar{z}$ where σ satisfies Liouville's equation $4\partial_z \partial_{\bar{z}} \sigma = e^{\sigma}$. Suppose a conformal transformation on a hyperbolic plane $\mathcal{H}$ as $g \mapsto \hat{g} = e^h g = e^{h+\sigma} dz d\bar{z}$. Let $\psi = h + \sigma$. Suppose that $\hat{g}$ has scalar curvature $-\frac{1}{2}$. The conformal factor e^{ψ} satisfies the Liouville equation $4\partial_z \partial_{\bar{z}} \psi = e^{\psi}$. The Taylor series expansion of ψ is the Taylor series expansion of $h + \sigma$. Refer (4.2.9) for the Taylor series expansion of h about z_i. Hence, the Taylor series expansion of ψ is given by

$$\psi = \log |z - z_i|^2 + A_i + \frac{1}{2} \bar{B}_i (z - z_i) + \frac{1}{2} B_i (\bar{z} - \bar{z}_i) + \cdots \tag{5.2.3}$$

where $A_i = a_i + \log \left(\frac{8}{(1 - z_i \bar{z}_i)^2} \right)$, $B_i = b_i + \frac{4z_i}{1 - z_i \bar{z}_i}$, and a_i and b_i are the coefficients in (4.2.9). The unregularised action for ψ is given by

$$s_{\psi} = \frac{i}{2\pi} \int_{\hat{D}} dz \wedge d\bar{z} \left(2\partial_z \psi \partial_{\bar{z}} \psi + e^{\psi} \right), \tag{5.2.4}$$

where $\hat{D} = \{ D - \cup_{i=1}^{N} |z - z_i| < \epsilon \}$, D is the unit disc. The integrals of e^{ψ} and ψ are finite when $\epsilon \to 0$. However, there is a singularity on the integral of the component $\partial_z \partial_{\bar{z}} \psi$. The boundary conditions results in the variation of s_{ψ} and h as

$$\delta s_{\psi} = \frac{i}{2\pi} \int_{\hat{D}} dz \wedge d\bar{z} \left(2\partial_z (\delta \psi \partial_{\bar{z}} \psi) + 2\partial_{\bar{z}} (\delta \psi \partial_z \psi) \right), \tag{5.2.5}$$

and

$$\delta \psi = \delta A_i + \frac{1}{2} \delta \bar{B}_i (z - z_i) + \frac{1}{2} \delta B_i (\bar{z} - \bar{z}_i) + ... \tag{5.2.6}$$

respectively. Substituting $\delta \psi$ into δs_{ψ}, one can see that $\delta s_{\psi} = -4 \sum_{i=1}^{N} \delta A_i$ [31] . Thus, the regularized action for h is given by [9]

$$S = \lim_{\epsilon \to 0} \left\{ \frac{i}{2\pi} \int_{\hat{D}} dz \wedge d\bar{z} \left(2\partial_z \psi \partial_{\bar{z}} \psi + e^{\psi} \right) + 4 \sum_{i=1}^{N} A_i + 4N \log \epsilon \right\}. \tag{5.2.7}$$

Note that the term $4\sum_{i=1}^{N} A_i$ and $4N\log\epsilon$ cancel unnecessary variational quantity and give a well defined variational derivative and we can find that

$$
\begin{aligned}
\frac{\partial S}{\partial z_i} &= \lim_{\epsilon\to 0}\left\{\frac{i}{2\pi}\int_{\hat{D}} dz\wedge d\bar{z}\frac{\partial}{\partial z_i}\left[2\partial_z\psi\partial_{\bar{z}}\psi + e^{\psi}\right]\right\} \\
&- \lim_{\epsilon\to 0}\left\{\frac{i}{2\pi}\oint_{\partial\hat{D}} d\bar{z}\frac{\partial}{\partial z_i}\left[2\partial_z\psi\partial_{\bar{z}}\psi + e^{\psi}\right]\right\} + 4\sum_{i=1}^{N}\frac{\partial A_i}{\partial z_i} \\
&= \lim_{\epsilon\to 0}\left\{\frac{i}{2\pi}\oint_{\hat{D}} dz\wedge d\bar{z}\left[2\partial_z(\frac{\partial\psi}{\partial z_i}\partial_{\bar{z}}\psi) + 2\partial_{\bar{z}}(\frac{\partial\psi}{\partial z_i}\partial_z\psi)\right]\right\} \\
&- \lim_{\epsilon\to 0}\left\{\frac{i}{2\pi}\oint_{\partial\hat{D}} d\bar{z}\frac{\partial}{\partial z_i}\left(2\partial_z\psi\partial_{\bar{z}}\psi + e^{\psi}\right)\right\} + 4\sum_{i=1}^{N}\frac{\partial a_i}{\partial z_i} \\
&= 2\bar{B}_i,
\end{aligned}
$$

where B_i is the coefficient of the Taylor series expansion of ψ. Due to the reality of the action $\frac{\partial S}{\partial \bar{z}_i} = 2B_i$. The action S is then calculated as

$$
S = 2\int B_i d\bar{z}_i = 2\int \bar{B}_i dz_i. \tag{5.2.8}
$$

Hence, $2\frac{\partial B_i}{\partial z_s} = \frac{\partial^2 S}{\partial z_s\partial\bar{z}_i}$ gives the Samols metric having the form

$$
ds^2 = \pi\sum_{i,s}\left(\Omega(z_i)\delta_{is} + \frac{\partial^2 S}{\partial z_s\partial\bar{z}_i}\right)dz_i d\bar{z}_s. \tag{5.2.9}
$$

Therefore, the Kähler potential is given by

$$
\mathcal{K} = \mathcal{K}_\Sigma + S + C, \tag{5.2.10}
$$

where $\mathcal{K}_\Sigma$ and C are the Kähler potential on Σ and constant, respectively. Here our aim is to calculate the Kähler potential on the totally geodesic submanifold $\Sigma_{r,t}$. From (5.1.4), the coefficient B_β of Taylor series expansion of ψ can be given as

$$B_\beta = \frac{4\beta}{1-\beta\bar{\beta}} + \frac{1}{\bar{\beta}}\left(2t + r - 1 - \frac{2r\beta^r\bar{\beta}^r}{1-\beta^r\bar{\beta}^r} + \frac{4r\nu^r\beta^r\bar{\beta}^r}{1-\beta^r\bar{\beta}^r} - \frac{4\beta\bar{\beta}}{1-\beta\bar{\beta}}\right). \tag{5.2.11}$$

Then using (5.2.8) to (5.2.11), the Kähler potential can be calculated as

$$\mathcal{K} = -4\pi r\log(1-|\beta|^{2r}) + 4\pi r\int\frac{\sqrt{(t+1)^2(1-|\beta|^{2r})^2 + 4r^2|\beta|^{2r}}d\beta}{(1-|\beta|^{2r})\beta} + C, \tag{5.2.12}$$

where C is a constant and the integral $\int\frac{\sqrt{(t+1)^2(1-|\beta|^{2r})^2+4r^2|\beta|^{2r}}d\beta}{(1-|\beta|^{2r})\beta}$ is the same as (5.1.9).

5.2.1 The Kähler potential using a scaling argument

We can also evaluate the Kähler potential of the totally geodesic submanifold $\Sigma_{r,t}$ of M_N using scaling argument. Following similar argument of derivation of (4.1.15) and (4.1.16), we can find the following two equations

$$\frac{\partial B_s}{\partial z_i} = \frac{\partial \bar{B}_i}{\partial \bar{z}_s} \text{ and } \frac{\partial B_i}{\partial \bar{z}_s} = \frac{\partial B_s}{\partial \bar{z}_i}, \tag{5.2.13}$$

where B_i is the coefficient of the Taylor series expansion of $\psi = h + \sigma$. For a collective coordinates $\{z_i = \beta\alpha^{i-1}, i = 1, ..., r, \lambda = e^{2i\pi/r}; \bar{z}_i\}$, equations (5.2.13) are satisfied if there exists a real function f [9] such that

$$\frac{\partial f}{\partial \bar{z}_i} = 2B_i \text{ and } \frac{\partial f}{\partial z_i} = 2\bar{B}_i. \tag{5.2.14}$$

Let $B = \sum_{i=1}^{r} \bar{z}_i B_i = r B_\beta$, where B_β is the value given by (5.2.11). Then one can find the following linear combination of (5.2.14) as

$$\sum_{i=1}^{r} \left(z_i \frac{\partial f}{\partial z_i} + \bar{z}_i \frac{\partial f}{\partial \bar{z}_i} \right) = 2(B_i + \bar{B}_i) = 4B = 4r\bar{\beta}B_\beta. \tag{5.2.15}$$

Let $z_i = \eta_i e^{i\alpha_i}$, where η_i and α_i are the distance and angle of the i^{th}-vortex away from the origin. Then one can find that

$$\sum_{i=1}^{r} \eta_i \frac{\partial f}{\partial \eta_i} = 4B = 4r\bar{\beta}B_\beta. \tag{5.2.16}$$

Let all the value η_i are parametrized by dimensionless parameter τ as $\eta_i \equiv \eta_i(\tau)$ and $z_i \equiv z_i(\tau) \equiv \eta_i e^{i\alpha_i}$. Let z_i be proportional to the scaling parameter τ. Then $\tau \frac{dz_i}{d\tau} = z_i$. Therefore, we can find that

$$\tau \frac{df}{d\tau} = \sum_{i=1}^{r} \left(z_i \frac{\partial f}{\partial z_i} + \bar{z}_i \frac{\partial f}{\partial \bar{z}_i} \right) = 4B = 4r\bar{\beta}B_\beta. \tag{5.2.17}$$

Hence, we can find the real function f as

$$f = 4r \int B_\beta d\bar{\beta} = 4r \int \bar{B}_\beta d\beta.$$

Therefore, the entire Kähler potential is given by

$$\begin{aligned}\mathcal{K} &= 8r \log(1 - \beta\bar{\beta}) + f \\ &= -4\pi r \log(1 - |\beta|^{2r}) + 4\pi r \int \frac{\sqrt{(t+1)^2(1-|\beta|^{2r})^2 + 4r^2|\beta|^{2r}}d\beta}{(t+1-r)(1-|\beta|^{2r})\beta} + C,\end{aligned}$$

where C is a constant. Note that, we have found the same formula for all the three approaches of evaluating the Kähler potential for the totally geodesic submanifold $\Sigma_{r,t}$ of M_N.

5.3 The geometry of $\sum_{r,t}$

The geometry of $\sum_{r,r-1}$ and, in particular the geodesic flow on $\sum_{2,1}$, was discussed in detail in [24], using that $\Sigma_{r,r-1}$ has continuously negative curvature, $-\frac{1}{3\pi r}$. Here we will see the case for $\sum_{2,2}$, we compare the relation between the impact parameter and the scattering angle. Recall that the geodesic trajectories of the hyperbolic plane intersects the boundary of the disc at right angle. Now also recall that the isometry from D to $\mathcal{H}$ by the Cayley-transform which takes geodesics to geodesics and it is the restriction of $\mathcal{H}$ a Möbius transformation S^2 in to S^2 which takes circles and lines to circles and lines, preserves angles and maps the real axis to the unit circle in $\mathbb{C}$. Recall that the metric on $\sum_{2,2}$ is given by

$$ds^2 = g(|\beta|^2)d\beta d\bar{\beta} = \frac{32\pi|\beta|^2}{(1-|\beta|^4)^2}\left(1 + \frac{4(1+|\beta|^4)}{\sqrt{9(1-|\beta|^4)^2 + 16|\beta|^4}}\right)d\beta d\bar{\beta},$$

where β is a vortex positions as we discussed earlier. Now let $\beta = \tilde{r}e^{i\theta}$. Thus, the metric can be rewritten as $ds^2 = g(\tilde{r}^2)(d\tilde{r}^2 + \tilde{r}^2 d\theta^2)$ where

$$g(\tilde{r}^2) = \frac{32\pi\tilde{r}^2}{(1-\tilde{r}^4)^2}\left(1 + \frac{4(1+\tilde{r}^4)}{\sqrt{9(1-\tilde{r}^4)^2 + 16\tilde{r}^4}}\right).$$

Then the equations of motion are

$$\begin{cases} \ddot{\tilde{r}} - \frac{g'}{2g}\dot{\tilde{r}}^2 - \frac{h^2}{g^2\tilde{r}^3}(1 + \frac{g'}{2g}\tilde{r}) = 0 \\ \\ \dot{\theta} = \frac{h}{g\tilde{r}^2}, \quad \text{h is constant.} \end{cases}$$

Note that for $\tilde{r} \to 1$, the metric is approximated by

$$ds^2 = \frac{96\pi|\beta|^2}{(1-|\beta|^4)^2} d\beta d\bar{\beta}, \tag{5.3.1}$$

and this metric is isometric to the metric on $\sum_{2,1}$. So, a geodesic intersects the boundary at right angles say at $e^{i\psi}, 0 \leq \psi \leq \frac{\pi}{2}$. For $\lambda \in U(1)$, the radial curves $\chi(\tilde{t}) = \tilde{t}\lambda, |\tilde{t}| < 1$ are geodesic which corresponds to the 4-vortex motion in which two vortices remains fixed at the origin and the other two moves towards to each other and scatter along the trajectories $z(\tilde{t}) = \pm\chi(\tilde{t})^{\frac{1}{2}}$. Note that let $\sum_{2,2}$ be the symmetric configurations such that the vortices are equidistant from the origin and separated from one another by relative angles of π. Configurations in $\sum_{2,2}$ are symmetric under rotations, so any movement starting in $\sum_{2,2}$ with initial velocity along $\sum_{2,2}$ will remain in $\sum_{2,2}$. So, we can consider the motion with in $\sum_{2,2}$.

The geometry of vortex scattering on $\mathcal{H}$ discussed in [24] such that the vortex trajectories are seen as the geodesic arcs. Krusch and Speight [24] give the definition of the scattering angle and impact parameter on the hyperbolic plane as in fig.5.2 such that the dashed curves are the vortex trajectories on the hyperbolic plane when the hyperbolic vortices do not affect each other. The solid curves are vortex trajectories if the vortices affect each other. The geodesic in $\mathcal{H}$ is a circular arc with radius $\frac{1}{2}\tan(\psi)$. Here, refer to fig.5.2, b is the impact parameter and θ_s is the scattering angle such that $\xi + \theta_s = -\frac{\psi}{2}$. Using trigonometric properties, one can calculate ξ as

$$\xi = -\tan^{-1}\left(\frac{1}{2}\tan(\psi)\right). \tag{5.3.2}$$

Suppose we have geodesic motion in $\sum_{2,2}$ which passes through the origin. As b decreases to zero, one passes to the other side of the origin in $\sum_{2,2}$ along a straight line in $\sum_{2,2}$, and emerges with b increasing and ψ changed by half its range. From trigonometric property, the impact parameter b can be calculated (it was derived by Steffen Krusch and Martin Speight [24])

$$b = 4\sqrt{2}\tanh^{-1}\left(\sqrt{1 + \left(\frac{1}{2}\tan(\psi)\right)^2} - \frac{1}{2}\tan(\psi)\right) \tag{5.3.3}$$

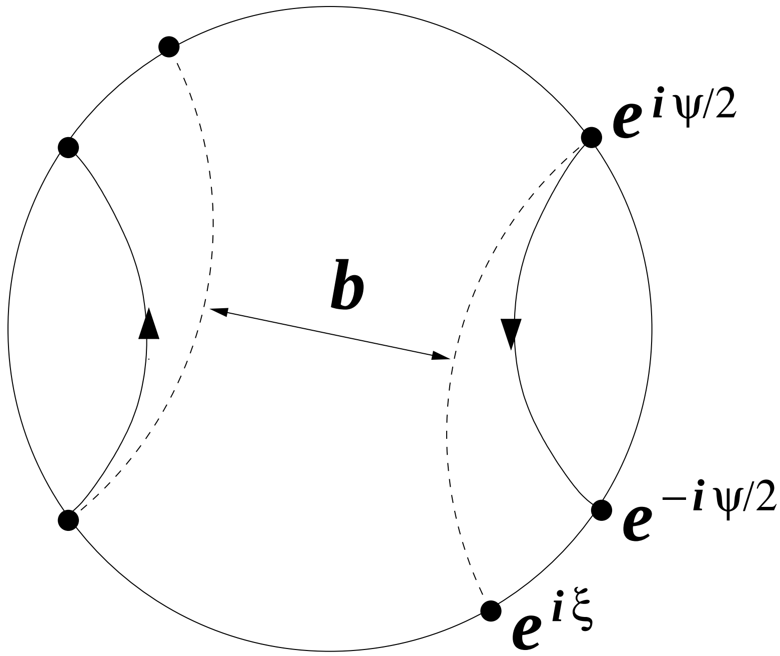


Figure 5.2: Plots of the scattering angle and the impact parameter on the hyperbolic plane taken from [24].

Using the approximate metric (5.3.1) and the definition of θ_s and b in (5.3.2) and (5.3.3), one can find the scattering angle-impact parameter relation of $\Sigma_{2,2}$ as

$$\theta_s(b) = \left(2\tan^{-1}\left(\frac{1-\tilde{k}^2}{2\tilde{k}}\right) - \tan^{-1}\left(\frac{1-\tilde{k}^2}{\tilde{k}}\right)\right) \tag{5.3.4}$$

where $\tilde{k} = \tanh(\frac{b}{4\sqrt{2}})$ and it holds for large impact parameters. For the scattering angle θ_s and the impact parameter b associated with the ψ geodesic, one has initially $\theta_s = 0$ and b decreases which represents a head-on collision. Thus, θ_s decreases to to zero as b increases. Fig.5.3 is the plot of the relation between the impact parameter-scattering angle relation of the moduli space $\Sigma_{2,2}$ that shows $\frac{\pi}{2}$ scattering and fig. 5.6c (blue) also shows the scalar curvature plot of $\Sigma_{2,2}$ for $r = t = 2$. The motion of these vortices in the geodesic approximation shows that for head-on collisions, the scattering is by angle $\frac{\pi}{2}$ [51, 60]. In general, for a metric given by

$$ds^2 = g(\tilde{r})(d\tilde{r}^2 + \tilde{r}^2 d\theta^2), \tag{5.3.5}$$

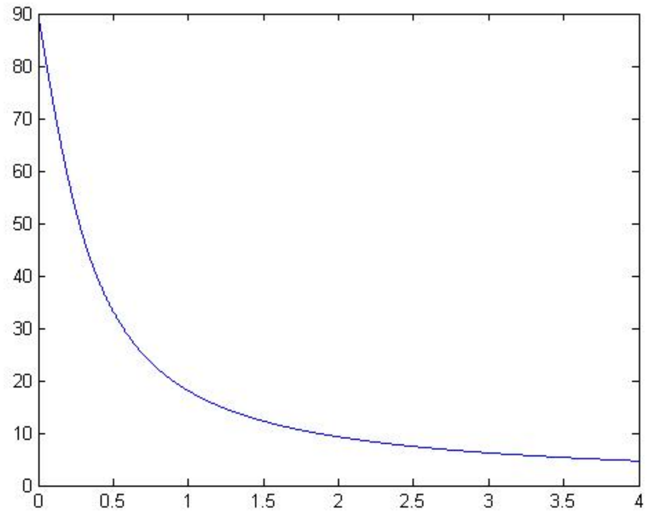


Figure 5.3: The relation between the impact parameter and the scattering angle of $\Sigma_{2,2}$.

the geodesic motions conserve energy E and angular momentum h

$$E = \frac{1}{2} g(\tilde{r}) \left(\dot{\tilde{r}}^2 + \tilde{r}^2 \dot{\theta}^2 \right) \tag{5.3.6}$$

$$h = g(\tilde{r}) \tilde{r}^2 \dot{\theta}^2, \tag{5.3.7}$$

where $g(\tilde{r})$ is the metric function (5.3.5), respectively. Then from (5.3.6) and (5.3.7), one can find that

$$\dot{\tilde{r}}^2 = \frac{2E}{g(\tilde{r})} - \tilde{r}^2 \dot{\theta}^2 \tag{5.3.8}$$

$$\dot{\theta}^2 = \frac{h}{\tilde{r}^2 g(\tilde{r})}. \tag{5.3.9}$$

Using (5.3.8) and (5.3.9), we can find that

$$\frac{d\theta}{d\tilde{r}} = \frac{h}{\sqrt{g(\tilde{r})}\tilde{r}^2} \frac{1}{\sqrt{2E - \frac{h^2}{g(\tilde{r})\tilde{r}^2}}} = \frac{b}{\tilde{r}\sqrt{g(\tilde{r})\tilde{r}^2 - b^2}}, \tag{5.3.10}$$

where $b = \frac{h}{2E}$. We therefore integrate (5.3.10), and find that the relation between the scattering angle and the impact parameter as

$$\theta_s = \int_{\rho_0}^{1} \frac{2bd\tilde{r}}{\tilde{r}\sqrt{g(\tilde{r})\tilde{r}^2 - b^2}}, \tag{5.3.11}$$

where ρ_0 is a turning point satisfying $\tilde{r}_0\sqrt{g(\tilde{r}_0)} = b$. This formula agrees with the scattering angle-impact parameter relation of vortices in flat spaces in [40]. Fig.5.4 and 5.5 are the plot of the relation between the impact parameter-scattering angle relation of the moduli space $\Sigma_{2,2}$ and $\Sigma_{r,r-1}$ for $r = 2, 3, 4, 5$ that shows $\frac{\pi}{2}$ and $\frac{\pi}{r}$ scattering, respectively. That is, $r = 2, 3, 4$ and 5 implies 2-vortices placed opposite position and one vortex fixed at the origin, 3-vortices at the verices of an equilateral triangle and 2-vortices fixed at the origin, 4-vortices placed at the verices of a square and 3-vortices fixed at the origin, 5-vortices placed at regular pentagon and 4-vortices fixed at the origin, respectively.

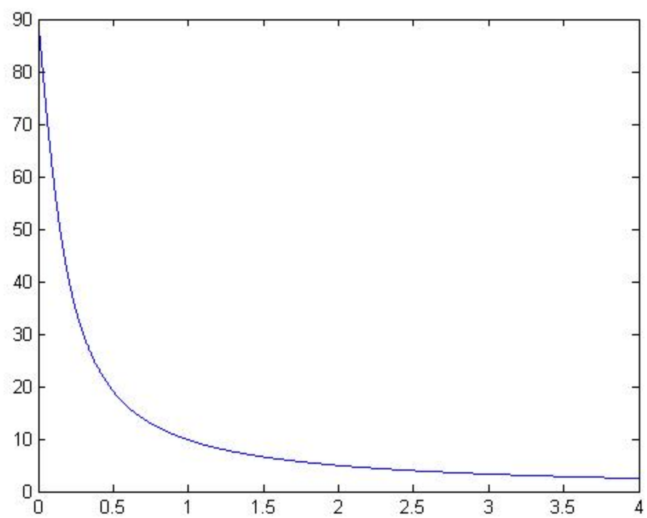


Figure 5.4: Plot of the scattering angle against the impact parameter of $\Sigma_{2,2}$ using the formula (5.3.11).

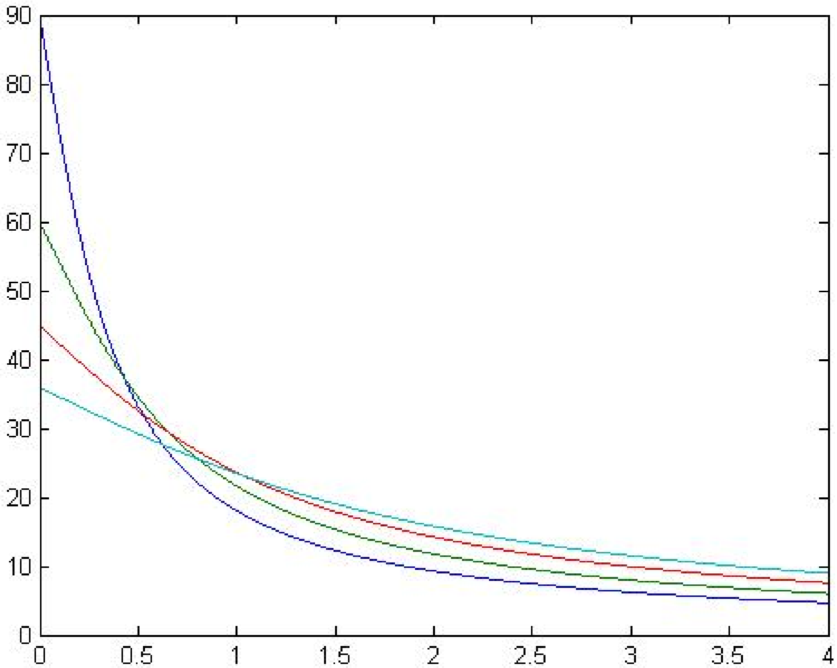
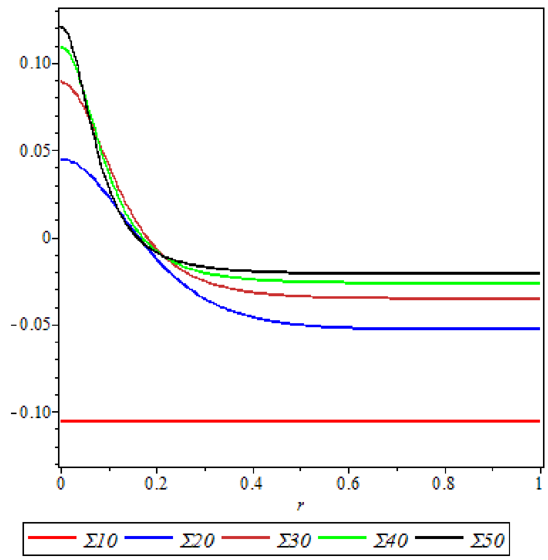


Figure 5.5: Plots of the scattering angle against the impact parameter that show $\frac{\pi}{r}$ scattering where $r = 2, 3, 4, 5$.

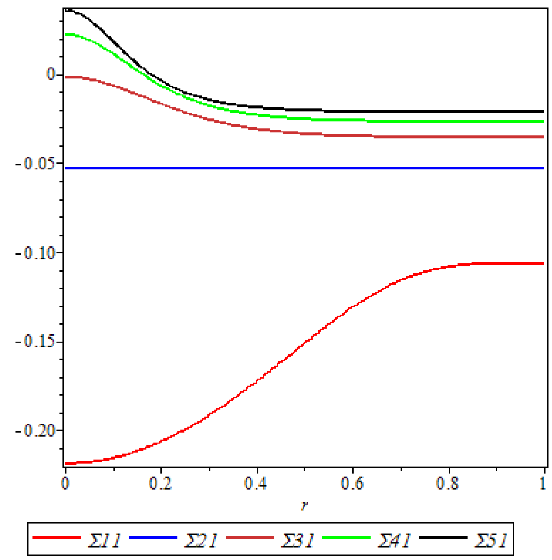
Moreover, the curvature of $\sum_{r,t}$ is given by

$$\kappa = -\frac{1}{2|\alpha|G(|\alpha|)}\frac{d}{d|\alpha|}\left(\frac{|\alpha|}{G(|\alpha|)}\frac{dG(|\alpha|)}{d|\alpha|}\right),$$

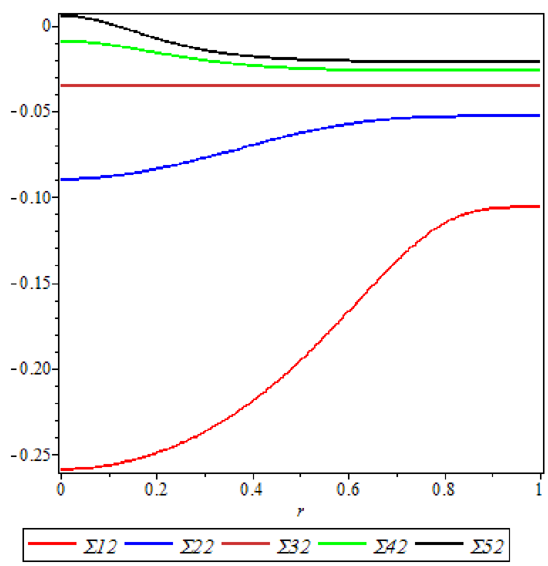
where $G(|\alpha|)$ is the profile function of the metric $(5.3.8)$. The curvature κ is negative uniformly for $t \geq r$, however, for $t \leq r$, the curvature can have a positive value. As $r \to t+1$, this curvature is negative uniformly which is $-\frac{1}{3r\pi}$. For example, $\sum_{2,2}$ has uniformly negative curvature. Fig. 5.6 shows plots of curvature for different values of t and r and here all the horizontal lines are the graphs of the curvature when $r = t+1$. In fig. 5.6a, for $t = 0$ and $r = 2, 3, 4, 5$, one can see that the scalar curvature can have a positive value. Similarly, one can find a positive scalar curvature submanifold in fig. 5.6b for $t = 1$ and $t = 3, 4, 5$. Generally, from fig.5.6, we see that there are submanifolds that have positive and negative scalar curvature and some have uniformly negative curvature.



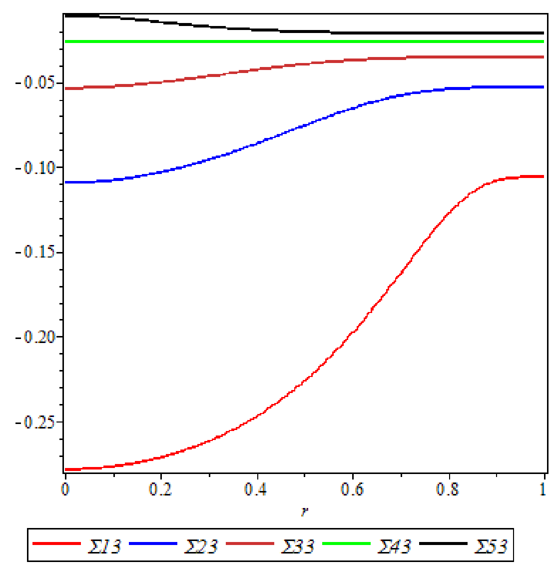
(a) t=0, r=1,2,3,4,5



(b) t=1, r=1,2,3,4,5



(c) t=2, r=1,2,3,4,5



(d) t=3, r=1,2,3,4,5

Figure 5.6: The plots of the scalar curvature of some totally geodesic submanifold of M_N. Note that $\tilde{r} = |\alpha|$.

5.4 Summary of chapter

In this chapter, the Kähler potential for $\Sigma_{r,t}$ of M_N was computed using three methods. These methods are the scaling argument, the regularized action and using direct definition of Kähler potential of a Kähler metric. We found identical results in all cases. The scattering angle of hyperbolic vortices on $\Sigma_{r,r-1}$ is $\frac{\pi}{r}$. In a head-on collision, the scattering angle of vortices on the space $\Sigma_{2,2}$ is $\frac{\pi}{2}$ as for $\Sigma_{2,0}$ and $\Sigma_{2,1}$. We noticed also that the curvature of $\Sigma_{r,t}$ of M_N depends on r and t.

Chapter 6

Vortices with Impurities

The BPS dynamics of vortices with the presence of electric and magnetic impurities was studied by Tong and Wong [68]. The dynamics on the moduli space in the presence of impurities can be deformed. The presence of magnetic impurities changes the metric on the moduli space; however, in the presence of electric impurities the metric is kept invariant where as there is an additional connection term.

We discuss the critically coupled vortices in the presence of impurities. In section 6.1, we investigate vortices with electric impurities and evaluate the explicit metric on the totally geodesic submanifold $\Sigma_{1,t}$ of hyperbolic $t+1$-vortices. In section 6.2, we study vortices with magnetic impurities. We evaluate the explicit metric on the totally geodesic submanifold $\Sigma_{r,t}$ of manifold M_N of hyperbolic N-vortices with impurities.

6.1 Vortices with electric impurities

In this section, we discuss the dynamics of vortices in the presence of electric impurities in the theory by adding impurities in the theory.

The action of the impurity free vortices is given by

$$S = \int d^3x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi \overline{D^\mu \phi} - \frac{1}{2}(1 - |\phi|^2)^2 \right], \tag{6.1.1}$$

where ϕ is the scalar field, $D_\mu \phi = \partial_\mu \phi - i A_\mu \phi$ is the covariant derivative and A_μ is the gauge potential. From (6.1.1), we can find the Bogomolyni equations (4.1.4) and (4.1.5). The general solution to these Bogomolyni equations with charge N has $2N$ parameters [64] that can be written as $(A_i; \phi)$, where $A_i = A_i(z; z^a), \phi = \phi(z; z^a)$, z is a coordinate of the domain and $z^a,\ a = 1, \cdots, 2N$ are the collective coordinates. The slow moving BPS soliton dynamics is determined by the metric on the moduli space of static solitons [30]. Then $(\partial_0 \phi, \partial_0 A_i)$ satisfies the linearized equations given by [68]

$$\delta_a A_i = \frac{\partial A_i}{\partial z^a} + \partial_i \alpha_a, \;\; \delta_a \phi = \frac{\partial \phi}{\partial z^a} + i \alpha_a \phi, \tag{6.1.2}$$

where $\alpha_a(z; z^a)$ are functions which are taken to ensure Gauss' law (4.1.8) holds. Then we consider $\alpha_a(z; z^a)$ to be a $U(1)$ connection over the moduli space.

Let $\rho(z)$ be a static electric charge density that will be added to the action of dynamics of vortices by adding a source term $\rho(z) A_0$ for the gauge field. When we associate the action with an electric impurity parameter ρ, the action with electric impurity is going to be

$$S_{im} = \int d^3x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi \overline{D^\mu \phi} - \frac{1}{2}(1 + |\phi|^2)^2 - \rho A_0 \right]. \tag{6.1.3}$$

The action for vortex dynamics has the form (detailed explanation can be found in [68])

$$S = \int dt \left[\frac{1}{2} \gamma_{ab} \dot{z}^a \dot{\bar{z}}^b + A_a(z) \dot{z}^a \right], \tag{6.1.4}$$

where γ_{ab} is the metric on the moduli space and A_a is a connection given by

$$A_a(z) = \int d^2x \rho(z) \alpha_a(z, z^a). \tag{6.1.5}$$

We can see the following example. Consider the moduli space M_{t+1} of $t+1$-vortices on the hyperbolic plane in such a way that one vortex position at α is moving in the presence of a delta function electric impurity $\rho(z) = \epsilon\delta(z)$ where $\epsilon \ll 1$ and dimensionless, and there are t coincident fixed vortices at the origin. The dynamics of the vortices of this moduli space ($A_0 = 0$, $A = A_1 dx^1 + A_2 dx^2$) is

$$S_{t+1} = \int d\tilde{t} \left[\frac{1}{2}\gamma_{\alpha\alpha}\dot{\alpha}\dot{\bar{\alpha}} + \epsilon A(\alpha)\dot{\alpha}\right], \text{ where } A = -i\partial_{\bar{z}} \log\left(\frac{1-|z|^2}{1-|f|^2}\right), \tag{6.1.6}$$

and f is a complex analytic function given by $f(z) = \frac{z^{t+1}(z-\alpha)}{1-\bar{\alpha}z}$ and $\gamma_{\alpha\alpha}$ is the metric function on the moduli space M_{t+1} given by

$$\gamma_{\alpha\alpha} = \frac{4\pi}{(1-|\alpha|^2)^2}\left(1 + \frac{4(1+|\alpha|^2)}{\sqrt{(t+1)^2(1-|\alpha|^2)^2 + 4|\alpha|^2}}\right). \tag{6.1.7}$$

Thus, the action can be simplified as

$$S_{t+1} = \int dt \left[\frac{1}{2}\gamma_{\alpha\alpha}\dot{\alpha}\dot{\bar{\alpha}} + \epsilon\frac{2\dot{\alpha}}{1-|\alpha|^2}\right]. \tag{6.1.8}$$

The geodesic equations depend on the impurity parameter ϵ. In general, the presence of the electric impurity changes the trajectories of the geodesic motions. Now let $\alpha = \tilde{r}e^{i\theta}$. Then the geodesic equations become

$$\begin{cases} \ddot{\tilde{r}} - \frac{\gamma'}{2\gamma}\dot{\tilde{r}}^2 - \frac{h^2}{\gamma^2 \tilde{r}^3}(1 + \frac{\gamma'}{2\gamma}\tilde{r}) = 0 \\ \dot{\theta} = \frac{h}{\gamma\, \tilde{r}^2},\ h \text{ constant}, \end{cases} \tag{6.1.9}$$

where $\gamma = \frac{1}{2}\left(\gamma_{\alpha\alpha}(\tilde{r}) + \frac{4\epsilon}{1-\tilde{r}^2}\right)$. Then, one can see that in the presence of electric impurities, θ depends also on ϵ. This may alter the impact parameter-scattering angle relation, for example, one can see the approximated scattering angle-impact parameter relation on the space $\Sigma_{2,1}$ as shown in fig.6.1.

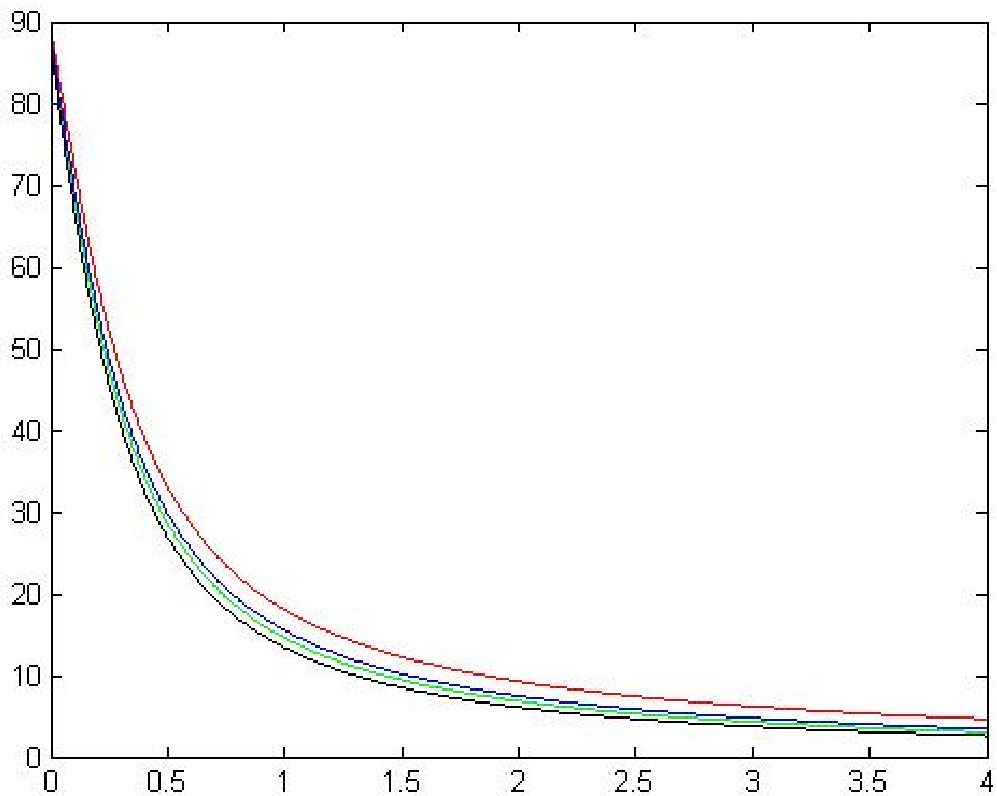


Figure 6.1: Plots of scattering angle-impact parameter relation in the presence of electric impurity $\epsilon \to 0$ and $\epsilon = 0.175, 0.25, 0.5$, (red, blue, green and black), respectively.

6.2 Vortices with magnetic impurities

In this section, we discuss vortices on compact Riemann surfaces Σ by doping with magnetic impurities. The dynamics of vortices in the presence of magnetic impurities are not suplimented by any connection term like the electric case. They change the static solutions and the corresponding moduli space metric. Suppose the action of the vortices associated with a magnetic impurity of parameter η. Then the action [68] becomes

$$S_{im} = \int d^3x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi \overline{D^\mu \phi} - \frac{1}{2}(1 + \eta - |\phi|^2)^2 - \eta F_{12} \right]. \tag{6.2.1}$$

Using same procedure as we did before in the case of doping with an electric impurity, imposing a gauge fixing condition $A_0 = 0$, the Ginzburg-Landau potential energy functional is given by

$$V_{im} = \frac{1}{2} \int_\Sigma \left(\Omega^{-1} \tilde{F}_{12}^2 + \overline{D_i \phi} D_i \phi + \frac{\Omega}{4}(1 + \eta - \phi\bar{\phi})^2 + \eta \tilde{F}_{12} \right) dx_1 dx_2. \tag{6.2.2}$$

By completing the square, the Bogomolyni equations become

$$(D_1 + iD_2)\phi = 0 \quad \text{ and } \quad \tilde{F}_{12} = \frac{\Omega}{2}(1 - |\phi|^2) + \frac{\Omega}{2}\eta. \tag{6.2.3}$$

Note that the Bogomolnyi equations depends on the impurity parameter η. As usual, we are interested in the solutions of these equations. The first Chern number is given by

$$\tilde{N} = \frac{1}{2\pi} \int_\Sigma F_{12} = \frac{1}{4\pi} \int_\Sigma \Omega(1 - |\phi|^2) + \frac{1}{4\pi} \int_\Sigma \Omega\eta = N + \frac{1}{4\pi} \int_\Sigma \Omega\eta. \tag{6.2.4}$$

As $\tilde{N} \in \mathbb{Z}$, we can choose $\eta = 4\pi\Omega^{-1}\epsilon\delta(z)$, where the parameter $\epsilon \in \mathbb{R}$ is the strength of the defect and $z = x_1 + ix_2$. From to (4.1.5), $F_{12} = \frac{\Omega}{2}(1 - |\phi|^2)$ is the impurity free Bogomolyni equation on Σ. As the square of the delta function is not defined, the Bogomolyni equations (6.2.5) imply that the gauge potential A_μ must be divergent in some way near the origin. To ensure the energy is finite, $D_\mu\phi$ should be finite at the origin. Thus, the field $|\phi|$ should have a vortex position at the origin. The Bogomolnyi equations (6.2.5) become

$$(D_1 + iD_2)\phi = 0 \quad \text{and} \quad \tilde{F}_{12} = \frac{\Omega}{2}(1 - |\phi|^2) + 2\pi\epsilon\delta(z). \tag{6.2.5}$$

For a singular gauge transformation $\phi \mapsto \frac{|z|}{z}\phi$, one can find that

$$A_{\bar{z}} \to A_{\bar{z}} - \frac{i}{2\bar{z}} \tag{6.2.6}$$

$$\tilde{F}_{12} \to \tilde{F}_{12} - 2\pi\delta(z). \tag{6.2.7}$$

For finite $\epsilon \in \mathbb{N}$, the Bogomolyni equations become impurity free and they become

$$D_{\bar{z}}\phi = 0 \quad \text{and} \quad \tilde{F}_{12} = \frac{\Omega}{2}(1 - |\phi|^2). \tag{6.2.8}$$

Hence, the solutions of the equations with impurity are same the solutions of the impurity free equations for a finite integer ϵ. The metric on the moduli space of vortices in the presence of impurities with $\epsilon \in \mathbb{N}$ is the restriction of the usual impurity-free vortex metric to the submanifold of solutions where a certain number of vortices are fixed. Then the moduli space metric is the metric on a moduli space of solutions to the impurity free equations[1].

For example, let us consider a totally geodesic submanifold $\sum_{r,t}$ of a moduli space M_N, $r+t = N$. We generalize the metric on $\sum_{r,t}$ to magnetic impurities. The derivation goes through as in section 5.1 of the impurity-free case. For $0 < \epsilon < 1$, the ansatz $g(z) = z^\epsilon f(z)$, where f is a

[1] This step was first pointed out by Alex Cockburn for the case $\sum_{1,0}$

Blaschke function in (5.1.1), solves the equation (6.2.5). Hence the function $g(z)$ given by

$$g(z) = \frac{z^{t+\epsilon+1}(z^r - a^r)}{1 + \bar{a}^r z^r}, \ r + t = N, \ a \in \mathbb{C}. \tag{6.2.9}$$

Refer to chapter 5 section 5.1, the critical points of g given by $z_r = \lambda^{r-1}\beta, \ \lambda \in U(1)$, and a can be given by

$$a = \beta\nu(|\beta|), \tag{6.2.10}$$

where ν is a positive function of $|\beta|$ such that

$$\nu^r = \frac{(t+\epsilon+1)(1+|\beta|^{2r}) - \sqrt{(t+\epsilon+1)^2(1-|\beta|^{2r})^2 + 4r^2|\beta|^{2r}}}{2|\beta|^{2r}(t+\epsilon+1-r)}.$$

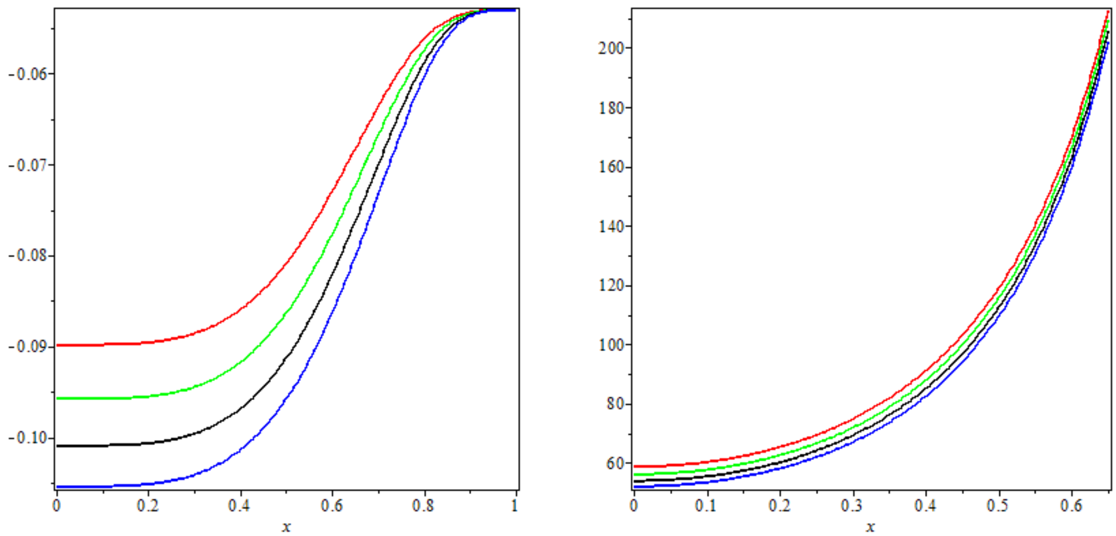
The coefficient b_1 also computed as

$$b_1 = \frac{1}{\bar{\beta}}\left(2(t+\epsilon) + r - 1 - \frac{2r\beta^r\bar{\beta}^r}{1-\beta^r\bar{\beta}^r} + \frac{4r\nu^r\beta^r\bar{\beta}^r}{1-\beta^r\bar{\beta}^r} - \frac{4\beta\bar{\beta}}{1-\beta\bar{\beta}}\right). \tag{6.2.11}$$

Using the Kähler property and the rotational invariance on M_N, the metric on $\Sigma_{n,m}$ with impurity given by (5.1.5). Then, we can find the metric on this space as

$$\begin{aligned}
\gamma &= \pi r\left(\Omega(|\beta|) + \frac{1}{|\beta|}\frac{d}{d|\beta|}(\bar{\beta}b_1)\right)d\beta d\bar{\beta} && (6.2.12)\\
&= \frac{4\pi r^3|\beta|^{2r-2}}{(1-|\beta|^{2r})^2}\left(1 + \frac{2r(1+|\beta|^{2r})}{\sqrt{(t+\epsilon+1)^2(1-|\beta|^{2r})^2 + 4r^2|\beta|^{2r}}}\right)d\beta d\bar{\beta}. && (6.2.13)
\end{aligned}$$

The presence of the magnetic impurity alters the trajectories of the geodesics. Adding magnetic impurities vary the dynamics of the moduli space of the vortices. The presence of impurities changes the scalar curvature of the spaces $\Sigma_{r,t}$. For example, for $\epsilon = 0.25, 0.5, 0.75$ and without impurity, fig. 6.2 shows the scalar curvature and metric function of $\Sigma_{2,2}$ changes in the presence of magnetic impurities. As the impurity increases, both the scalar curvature and metric functions decrease as we see from the plot.



(a) Scalar curvature with impurtity

(b) Metric function with impurity

Figure 6.2: Plots of scalar curvature and the metric function of $\Sigma_{2,2}$ with $\epsilon = 0.25, 0.5, 0.75$, (green, black and blue), respectively and with out impurity (red). Note that $x = |\beta|$.

Chapter 7

Conclusion

This thesis mainly focused on the moduli spaces of lumps in real projective space and vortices on Riemann surfaces. It begins with homotopy theory. Homotopy classes play an important role to ensure stability of vortices. We also investigated covering spaces and homotopy lifting and studied the relation between harmonic maps between Riemann spheres. Rational maps on a projective space associated with antipodal map were studied. We generalized the $\mathbb{C}P^1$ sigma model to physical space being the 2-dimensional compact Riemann surface Σ with metric g and mainly considered $\Sigma = S^2$ and $\Sigma = \mathbb{R}P^2$. The moduli spaces of lumps in a projective plane using rational maps of degree n subject to symmetry requirement was studied. We studied symmetric lumps in a projective plane and then mainly focused on rational maps of degree 3. We then derived families of symmetric rational maps and showed that the moduli space $\widetilde{Rat_3^a}$ consists of two orbits of the symmetry group, namely a 5-dimensional orbit of the axial map and a 7-dimensional orbit of dihedral symmetry generated by the map (3.5.1) or (3.6.1). We followed Martin Speight [59] who derived the metric on the moduli spaces of charge 1 lumps. We were able to evaluate the metric on moduli spaces of charge 3 lumps using $SU(2)$ Möbius transformation for both orbits of maps and explicitly evaluated some volumes of a submanifold of the orbits of maps. We showed that the volume for both these orbits of maps is finite. Both the dihedral and axial orbits of maps have finite length along the parameter c. We showed lump decay by considering zeros and poles of a rational map. In particular, in the orbit of maps (3.5.1) the zero cancel with the pole at ∞ and the pole cancels with the zero at 0 while in the dihedral orbits of maps (3.6.1) one zero fixed at the origin and two poles cancel with two zeros. Due to the symmetry (3.2.6), lump decay is more complicated on $\mathbb{R}P^2$. In the axial orbit of maps, three zeros cancel with three poles. The moduli space of maps (3.5.1) in the real projective space is a non-compact space since its degree decays by 2, collapses to an infinitely sharp spike and disappears.We showed the moduli spaces (3.5.1) and (3.6.1) have equal minimum value of the angular integral $\mathcal{I}$ at $a = c = 0$. Moreover, (3.6.1) has also the same minimum value at $c = 3$. When $N = 5$, we have moduli space of charge 5 lumps. In this spaces, we only understand some of the spaces, for example, the axial map z^5. It is an interesting problem which we suggest for further studies. For the moduli spaces of lumps in the projective plane possible future work would be to study the moduli spaces of charge 5 lumps in the projective plane and evaluate their corresponding geometric quantities.

In chapter 4, we have calculated the general (and for some cases an explicit) metric on the moduli spaces of hyperbolic 3-vortices where at least one of the vortex positions is not at the origin. Depending on the parameters a, b and c, we saw two families of these hyperbolic 3-vortices of which one is for $c \neq 0$ and the other for $c = 0$. For each family, we noticed also that three families can be constructed from the roots of (4.3.7). The metric, Kähler potential and scalar curvature for the moduli spaces of hyperbolic 3-vortices where vortex positions with zero center of mass coordinate was calculated. Results were compared and as expected agree with the metric in [24]. We constructed an interesting subspace which is the double hyperbolic 3-vortices and their corresponding metric and scalar curvature were computed. Interestingly, numerical results show that its scalar curvature is uniformly negative and increases as the modulus of the vortex position increase to 1 and approaches to $-\frac{1}{3\pi}$. In addition, the Kähler potential for the moduli space of double hyperbolic 3-vortices was computed. We also studied collinear hyperbolic 3-vortices and calculated their metric and scalar curvature which is uniformly negative as numerical plot suggests. Furthermore, we calculated the implicit metric for hyperbolic 4-vortices where the zeros of the rational function satisfy the symmetry $z \mapsto -z$ and $z \mapsto \bar{z}$. We found an interesting subspace of collinear hyperbolic 4-vortices and evaluated explicitly its metric. We constructed double 4-vortices on the hyperbolic plane. The corresponding metric was calculated explicitly and it was shown that this subspace has uniformly negative scalar curvature. The Kähler potential and metric on the moduli space of hyperbolic vortices are directly proportional. For the more difficult case of general hyperbolic 4-vortices, the calculation is rather challenging and still an open problem.

One of the motivations behind this work is that lumps on the projective plane are in many ways similar to vortices on compact Riemann surfaces. Both lumps and vortices attain a Bogomolnyi type topological lower bound on energy within their homotopy classes. Then they satisfy the first order self-dual equations (that is, the Cauchy-Riemann equations in case of lumps and Bogomolnyi equations in vortices). The solution of these equations is highly non-trivial and difficult to find an explicit formula for the metric in both lumps and vortices. A way of understanding the underlying connection between lumps and vortices is in terms of rational maps.

We have studied lumps on the projective plane and vortices on some compact Riemann surfaces using the concepts of rational maps. In the case of lumps, the rational map R commutes with the antipodal map $\rho : z \mapsto -\frac{1}{\bar{z}}$ while our rational map f in the moduli space of vortices commutes with the involution $z \mapsto \frac{1}{\bar{z}}$. The computation of the metric of the moduli space of vortices is far more difficult than the computation for lumps because in the case of vortices, the derivative of the rational function is involved. This is because the vortex positions are the critical points of the rational function. An other difference is the moduli space M_N of lumps is geodesically incomplete [50] and the geodesic approximation predicts that lumps may decay and form singularities in finite time.

The next piece of work focused on the moduli space of vortices on compact Riemann surfaces. We showed that the Samols metric can be generalized from vortices of single multiplicity to arbitrary vortices of positive multiplicity and explicitly calculated the metric on the moduli spaces of vortices on compact Riemann surface with positive multiplicity. Following Speight and Krusch [24], we were able to compute the metric on the totally geodesic submanifold $\Sigma_{r,t}$ of M_N, $r + t = N$ for all t and r. The Kähler potential of the symmetric space $\Sigma_{r,t}$ was then evaluated in three ways. The first one is just following the definition of Kähler potential of a Kähler metric while the second one is following Chen and Manton's method of the regularized action of the Taylor series expansion of the gauge invariant quantity $h = \log|\phi|^2$. The geometry of $\Sigma_{2,2}$ and particularly the impact parameter-scattering angle relation where two fixed are vortices placed at the origin and two vortices are placed at opposite positions was studied. These vortices form a right angle vortex scattering. We also studied the impact parameter-scattering angle relation of the space $\Sigma_{r,t}$ where there is $\frac{\pi}{r}$ scattering. For some vortices, the curvature κ of the space $\Sigma_{r,t}$ is uniformly negative for $r \geq t$, however, for $t \leq r$, the curvature can have a positive value. When the scalar curvature is negative, the metric, Kähler potential and scalar curvature are directly proportional. In this chapter a further extension of this work would be to study the vortex scattering on $\Sigma_{2,t}$, $t \geq 3$ and compare the result to $\Sigma_{2,t}$, $0 \leq t \leq 2$.

In chapter 6, vortice and impurities have been studied and investigated. We investigated vortices in the presence of electric and magnetic impurities that was proposed by Tong and Wong [68]. The Bogomolyni equations in the presence of impurities were investigated and analyzed. Therefore, the dynamics is altered by the presence of electric impurities. Electric impurities keep the metric invariant but introduce a connection term. However, the presence of magnetic impurities does change the metric on the moduli space. The dynamics of moduli space of the symmetric space $\Sigma_{r,t}$ of vortices in the presence of electric and magnetic impurities in the theory by doping an impurity on the action of the dynamics of vortices has investigated. We evaluated the action and corresponding metric on the moduli space $\Sigma_{1,t}$ in the presence of electric impurities. In addition, we computed the explicit metric on the symmetric space $\Sigma_{r,t}$ with magnetic impurities. In the presence impurities, one way of taking the work further would be to determine and study the dynamics of hyperbolic N-vortices and evaluate their geometric properties.

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