On the linearity of order-isomorphisms

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Abstract

A basic problem in the theory of partially ordered vector spaces is to characterise those cones on which every order-isomorphism is linear. We show that this is the case for every Archimedean cone that equals the inf-sup hull of the sum of its engaged extreme rays. This condition is milder than existing ones and is satisfied by, for example, the cone of positive operators in the space of bounded self-adjoint operators on a Hilbert space. We also give a general form of order-isomorphisms on the inf-sup hull of the sum of all extreme rays of the cone, which extends results of Artstein-Avidan and Slomka to infinite dimensional partially ordered vector spaces, and prove the linearity of homogeneous order-isomorphisms in a variety of new settings.

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1 Introduction

A fundamental problem in the study of partially ordered vector spaces is to understand the structure of their order-isomorphisms, i.e., order preserving bijections whose inverses are also order preserving. In particular one would like to characterise those partially ordered vector spaces on which all order-isomorphisms are affine.

Pioneering research on this problem was motivated by special relativity theory where the causal order is considered on the Minkowski spacetime. During the 1950s and 1960s several results were obtained in finite dimensional spaces by Alexandrov and Ovčinnikova [3] and Zeeman [15], who showed that the order-isomorphisms from the

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causal cone onto itself are linear. Later Alexandrov [2] extended his result to order-isomorphisms on finite dimensional ordered vector spaces, where every extreme ray of the cone is engaged, that is to say, each extreme ray of the cone lies in the linear span of the other extreme rays. Rothaus [11] obtained a similar result where the domain of the order-isomorphism could also be the interior of the cone, but he assumes that the cone does not have any isolated extreme rays, which is a stronger assumption than the one used by Alexandrov. In the 1970s Noll and Schäffer made numerous contributions to this area in a series of papers, [9, 10, 12, 13]. Like Alexandrov, they considered the case where the cone is the sum of its engaged extreme rays, but they do not require the partially ordered vector spaces to be finite dimensional. More recently, Artstein-Avidan and Slomka [5] obtained a complete description of the order-isomorphisms between finite dimensional partially ordered vector spaces.

In many natural infinite dimensional settings the results of Noll and Schäffer are not applicable. A case in point is the space $B(H)_{\mathrm{sa}}$ consisting of bounded self-adjoint operators on a Hilbert H, ordered by the cone of positive (semi-definite) operators. Even though the cone $B(H)_{\mathrm{sa}}^+$ contains many engaged extreme rays, namely the rays through the rank-one projections, it does not satisfy the condition of Noll and Schäffer. Even so Molnár [7] showed, by using operator algebra techniques, that every order-isomorphism on $B(H)_{\mathrm{sa}}^+$ is linear. In this paper we obtain a generalisation of [10, Theorem A] by Noll and Schäffer that is sufficiently strong to yield Molnár's result.

Before we outline the main results in the paper, we point out that the domain on which the order-isomorphisms are considered plays a key role. In the paper we will work on so called upper sets, i.e, sets which contain all upper bounds of its elements. Such domains include cones, the interiors of cones, and the whole vector space. It turns out that without this assumption order-isomorphisms can be more complicated. Indeed, Šemrl [14] gave a complete characterisation of the order-isomorphisms on order intervals of $B(H)_{\mathrm{sa}}$, which include maps that are not affine.

Our generalisation of [10, Theorem A] exploits the fact that infima and suprema in a partially ordered vector space are preserved under order-isomorphisms. Instead of the conditions imposed by Noll and Schäffer, we assume that the cone, C, is equal to the inf-sup hull of the positive span of its engaged extreme rays, which is much weaker. In other words, we require that each $x \in C$ can be written as $x = \inf_{\alpha \in A}(\sup_{\beta \in B} x_{\alpha,\beta})$, where each $x_{\alpha,\beta}$ belongs to

$$[0,\infty)_{\mathcal{R}_E} = \{r_1 + \cdots + r_n \colon r_i \in C \text{ is an engaged extreme vector of } C \text{ for all } i\},$$

A and B arbitrary index sets, and for the infimum and supremum we only consider lower bounds and upper bounds in C. The main result can be formulated as follows.

Theorem 1.1. *Suppose $U \subseteq (X, C)$ and $V \subseteq (Y, K)$ are upper sets in Archimedean partially ordered vector spaces, and $f \colon U \to V$ is an order-isomorphism. If (X, C) is directed and C equals the inf-sup hull of $[0,\infty)_{\mathcal{R}_E}$, then f is affine.*

Here $f \colon U \to V$ is affine if it is the restriction of an affine map $F \colon \mathrm{aff}(U) \to Y$.

A key step in our argument is Theorem 3.10, which says that every order-isomorphism f from $[a,\infty) = \{a + x \colon x \in C\}$ onto $[b,\infty) = \{b + y \colon y \in K\}$ is the restriction of an affine map on the affine span of $[a,\infty)_{\mathcal{R}_E} = a + [0,\infty)_{\mathcal{R}_E}$. The proof requires a careful reworking of some of the ideas in [10].

Of course not every order-isomorphism is affine. Simply consider the space $C(K)$, consisting of continuous real functions on a compact Hausdorff space K, and the

map $f \mapsto f^3$. On $C(K)$ Schäffer [12] showed that each order-isomorphism, which is homogeneous (of degree one), is linear. In [13] he strengthened this result to general order unit spaces. In finite dimensional spaces the existence of a disengaged extreme ray in the cone is necessary and sufficient to yield a nonlinear order-isomorphism. This follows from [5, Theorem 1.7] by Artstein-Avidan and Slomka, who showed that any order-isomorphism in a finite dimensional space has a particular diagonal form. In Section 5 we obtain an infinite dimensional analogue of this result. We also give an alternative condition that guarantees that all homogeneous order-isomorphisms are linear, which can be applied in partially ordered vector spaces without an order unit such as $\ell^p(\mathbb{N})$ spaces.

2 Preliminaries

Let X be a real vector space and C be a cone in X, so C is convex, $\lambda C \subseteq C$ for all $\lambda \geq 0$, and $C \cap -C = \{0\}$. The cone C induces a partial order on X by $x \leq_C y$ if $y - x \in C$. The pair (X, C) is called a *partially ordered vector space*. For simplicity we write $\leq$ instead of $\leq_C$ if C is clear from the context, and we write $x < y$ if $x \leq y$ and $x \neq y$.

A partially ordered vector space (X, C) is said to be *Archimedean* if for each $x \in X$ and $y \in C$ with $nx \leq y$ for all $n \geq 1$ we have that $x \leq 0$. A subset G of X is said to be *directed* if for each $x, y \in G$ there exists $z \in G$ such that $x \leq z$ and $y \leq z$. It is well known that X is directed if and only if C is generating, i.e., $X = C - C$. Given $x \leq y$ we define the *order interval* by $[x, y] = \{z \in X : x \leq z \leq y\}$. We denote the cone with apex a by

$$[a, \infty) = \{a + x \colon x \in C\}.$$

Extreme rays of the cone play an important role in this paper. A vector $e \in X \setminus \{0\}$ is called an *extreme vector* if $0 \leq e$, and $0 \leq x \leq e$ implies that $x = \lambda e$ for some $\lambda \geq 0$, or, if $e \leq 0$, and $e \leq x \leq 0$ implies $x = \lambda e$ for some $\lambda \geq 0$. For an element $x \in C$ we define the *ray through* x as $R_x = \{\lambda x : \lambda \geq 0\}$. If $e \in C$ is an extreme vector, R_e is said to be an *extreme ray*. The notion of an extreme ray coincides with the ray being extreme in the convex sense. Indeed, a ray R in C is extreme if, and only if, for any two rays R_1 and R_2 in C satisfying $R = \alpha R_1 + (1 - \alpha) R_2$ for some $\alpha \in (0, 1)$ we have that $R_1 = R_2$, see [4, Lemma 1.43]. Given an extreme ray R we call $z + R$ an *extreme half-line with apex z*. The following elementary property of extremal vectors will be used frequently in the sequel, see [4, Lemma 1.44].

Lemma 2.1. *In a partially ordered vector space (X, C) any three extremal vectors in C that generate three distinct extremal rays are linearly independent.*

Another useful basic observation is the following.

Lemma 2.2. *Let (X, C) be Archimedean. If $x, y \in X$ are such that $0 \leq y \leq x$, and for each $0 \leq \lambda \leq 1$ we have that $y \leq \lambda x$ or $\lambda x \leq y$, then there exists a $\mu \geq 0$ such that $y = \mu x$.*

Proof. Let $x, y \in X$ be as in the statement. We may assume without loss of generality that x and y are non-zero. Now define $\mu = \sup\{\lambda \geq 0 \colon \lambda x \leq y\}$. By assumption μ is well-defined and $0 \leq \mu \leq 1$.

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Note that $\mu x \leq y$. Indeed, for $n \geq 1$ we have that $(\mu - 1/n)x \leq y$, so that $n(\mu x - y) \leq x$, which implies that $\mu x \leq y$, as (X, C) is Archimedean.

To show that $y \leq \mu x$ we distinguish two cases: $0 \leq \mu < 1$ and $\mu = 1$. In the case $0 \leq \mu < 1$ we have that $y \leq (\mu + 1/n)x$ for all n sufficiently large. Thus, $n(y - \mu x) \leq x$, which shows that $y \leq \mu x$, as the space is Archimedean. If $\mu = 1$, then $x = y$, since $y \leq x$ by assumption, and $x = \mu x \leq y$ as shown before. $\square$

Given vector spaces X and Y, a map $f \colon X \to Y$ is called *affine* if it is a translation of a linear map, that is, there is $a \in X$ such that $x \mapsto f(x + a) - f(a)$ is linear.

Let (X, C) and (Y, K) be partially ordered vector spaces. A set $U \subseteq X$ is called an *upper set* if $x \in U$ and $y \geq x$ imply $y \in U$. So, X, C and translations thereof are all upper sets in (X, C). Let $U \subseteq X$ be an upper set. A map $f \colon U \to Y$ is called *affine* or *linear* if it is the restriction of an affine map $F \colon \mathrm{aff}(U) \to Y$ or a linear map $F \colon \mathrm{span}(U) \to Y$, respectively. If C is generating then we have $\mathrm{aff}(U) = \mathrm{span}(U) = X$. A map $f \colon U \to Y$ is affine if and only if $f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$ for all $x_1, \ldots, x_n \in U$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\lambda_1 + \cdots + \lambda_n = 1$ such that $\lambda_1 x_1 + \cdots + \lambda_n x_n \in U$. It is a well-known fact that, if the upper set U is convex, then $f \colon U \to Y$ is affine if and only if f is *convex-linear*, that is, for each $x, y \in U$ and $0 \leq \lambda \leq 1$ we have that $f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y)$.

An element u in a partially ordered vector space (X, C) is an *order unit* if for all $x \in X$ there exists a $\lambda \geq 0$ such that $-\lambda u \leq x \leq \lambda u$. If C is generating, then $u \in C$ is an order unit if and only if for every $x \in C$ there exists $\lambda \geq 0$ with $x \leq \lambda u$. If (X, C) is Archimedean and $u \in C$ is an order unit then the formula

$$\|x\|_u := \inf\{\lambda \geq 0 \colon -\lambda u \leq x \leq \lambda u\}$$

defines a norm on X, called the *order unit norm*. A triple (X, C, u), where (X, C) is an Archimedean partially ordered vector space and u is an order unit in (X, C), is called an *order unit space*. In an order unit space we denote the interior of the cone C with respect to the order unit norm by C°. The set C° is an upper set and consists of all order units of (X, C).

3 Linearity of order-isomorphisms

In the sequel (X, C) and (Y, K) will be Archimedean partially ordered vector spaces. Initially we only consider order-isomorphisms $f \colon [a, \infty) \to [b, \infty)$, where $a \in X$ and $b \in Y$. However, the main result, Theorem 1.1, holds for more general domains.

A key role in the analysis of order-isomorphisms is played by extreme half-lines. This idea has been exploited to analyse order-isomorphisms on finite dimensional partially ordered vector spaces [5] as well as in infinite dimensions in [10]. In infinite dimensions, however, the extreme half-lines are not as useful, as there are cones that have none or only very few extreme rays. The following order theoretic characterization of extreme half-lines is due to Noll and Schäffer, see [10, Proposition 1]. For completeness we provide a proof.

Proposition 3.1. *If (X, C) is Archimedean and $x \in X$, then $H \subseteq [x, \infty)$ is an extreme half-line with apex x if and only if H is maximal among subsets $G \subseteq [x, \infty)$ with $x \in G$ that satisfy:*

(P1) G *is directed.*

(P2) For any $y \in G$ *the order interval* $[x, y]$ *is totally ordered.*

(P3) G *contains at least two distinct points.*

Proof. Suppose $H \subseteq X$ is maximal among subsets $G \subseteq [0, \infty)$ that satisfy properties (P1)–(P3). We first argue that H is contained in a half-line. Let $y, w \in H$ be given, so $x \leq y, w$. Due to (P1) there exists a $z \in H$ such that $y, w \leq z$. Since $\leq$ is preserved under addition, (P2) guarantees that the order interval $[0, z - x]$ is totally ordered. Moreover, it contains $y - x$, $w - x$, and $\lambda(z - x)$ for all $0 \leq \lambda \leq 1$. Therefore, by Lemma 2.2 there exist $\alpha, \beta \geq 0$ such that $y - x = \alpha(z - x)$ and $w - x = \beta(z - x)$. This shows that y and w are on the half-line through z with apex x. We conclude that any pair of points in H lie on a half-line with apex x, and hence H is contained in a half-line with apex x. Let R be a ray in C such that $H \subseteq x + R$.

By (P3) there exists an $r \in C \backslash \{0\}$ such that $x + r \in H$ and $x + R = \{x + \lambda r : \lambda \geq 0\}$. Note that $x + R$ satisfies properties (P1) and (P3). We now show that $x + R$ also satisfies (P2). Consider $y = x + \lambda r$ with $\lambda > 0$. Then $[x, y] = [x, x + \lambda r]$ equals the interval $[x, r]$ up to dilation. We know that $[x, x + r]$ is totally ordered, as $x + r \in H$ and H satisfies property (P2). Hence $[x, y]$ is also totally ordered. It now follows from the maximality assumption on H that $H = x + R$.

To see that $x + R$ is an extreme half-line, we note that $[0, r]$ is totally ordered, as $[x, x + r]$ is totally ordered. It follows from Lemma 2.2 that r is an extreme vector.

Conversely, suppose $H = x + R$ is an extreme half-line. Clearly, H satisfies properties (P1)–(P3). Suppose $G \supseteq H$ also satisfies (P1)–(P3) and $y \in G$. Since G is directed, there exists a $z \in G$ with $z \geq y, x + r$. Moreover, $[x, z]$ is totally ordered by (P2) and, hence, $[0, z - x]$ is totally ordered and $y - x, r \in [0, z - x]$. If $y - x \leq r$, then there is a $\mu \geq 0$ such that $y - x = \mu r$, as r is extreme, so that $y = x + \mu r \in H$. Otherwise, we have $r \leq y - x$ and for each $0 \leq \lambda \leq 1$ we have $\lambda(y - x) \in [0, z - x]$, so $r \leq \lambda(y - x)$ or $\lambda(y - x) \leq r$. By Lemma 2.2 it follows that there is a $\sigma \geq 0$ such that $r = \sigma(y - x)$. Then $\sigma \neq 0$ and $y = x + \sigma^{-1} r \in H$. $\square$

We note that property (P3) is only a necessary condition if C does not have any extreme rays and can be dropped otherwise.

As a direct corollary we obtain the following result.

Corollary 3.2. *If* $f : [a, \infty) \to [b, \infty)$ *is an order-isomorphism, then* f *maps an extreme half-line with apex* $x \in [a, \infty)$ *onto an extreme half-line with apex* $f(x) \in [b, \infty)$.

Proof. Suppose that R is an extreme ray of C. Then $f(x + R) \subseteq [f(x), \infty)$ and satisfies properties (P1)–(P3), as f is an order-isomorphism. So by Proposition 3.1 we find that $f(x + R) = f(x) + S$, where S is an extreme ray of K. $\square$

Our next step is to show that order-isomorphisms $f : [a, \infty) \to [b, \infty)$ possess an additive property on extreme half-lines, which was proved in [10, Lemma 1]. For the reader's convenience we include the proof.

Lemma 3.3. *Let* R *and* S *be distinct extreme rays of* C *and* $f : [a, \infty) \to [b, \infty)$ *be an order-isomorphism. For each* $x \in [a, \infty)$, $r \in R$ *and* $s \in S$ *we have that*

$$f(x + r + s) - f(x + s) = f(x + r) - f(x). \tag{3.1}$$

Proof. The equality in the statement holds trivially if either r or s equals zero. Assume $r \neq 0$ and $s \neq 0$. Then $R_j = x + js + R$ for $j \in \{0, 1, 2\}$ are three distinct parallel extreme half-lines. Due to Corollary 3.2, their images $f(R_j)$ are extreme half-lines in Y and they are distinct as f is injective. For each $\lambda \geq 0$, the set $x + S + \lambda r$ is an extreme half-line that intersects R_j for each $j \in \{0, 1, 2\}$, so, by Corollary 3.2, $f(x + S + \lambda r)$ is an extreme half-line and

$$f(x + S + \lambda r) \text{ intersects } f(R_j) \text{ for each } j \in \{0, 1, 2\} \text{ and } \lambda \geq 0. \qquad (3.2)$$

We obtain that $f(x + S + \lambda r)$ is not parallel to any of the $f(R_j)$, as R and S are distinct and f is injective.

We aim to show that $f(R_0)$, $f(R_1)$, and $f(R_2)$ are parallel. We do so in two steps. As a first step we show that if two of them are parallel, then all three of them are parallel. Indeed, assume that $f(R_j)$ and $f(R_k)$ are parallel, with $j, k \in \{0, 1, 2\}$, $j \neq k$. Since $f(R_j)$ and $f(R_k)$ are distinct parallel half-lines, it follows from (3.2) that the half-line $f(x + S + \lambda r)$ is in their affine span for every $\lambda \geq 0$. Then the half-line $f(R_i)$ with $i \in \{0, 1, 2\} \setminus \{j, k\}$ is in that affine span, too, as it intersects $f(x + S + \lambda r)$ for two distinct values of λ. Thus, $f(x + S)$, $f(R_i)$, and $f(R_j)$ are three extreme half-lines in the affine plane spanned by $f(R_j)$ and $f(R_k)$. By Lemma 2.1, it follows that at least two of the half-lines $f(x + S)$, $f(R_i)$, and $f(R_j)$ must be parallel, which yields that $f(R_i)$ and $f(R_j)$ must be parallel. Thus, $f(R_i)$, $f(R_j)$, and $f(R_k)$ are parallel.

As a second step we argue by contradiction that at least two of the half-lines $f(R_0)$, $f(R_1)$, and $f(R_2)$ are parallel. For $i \in \{0, 1, 2\}$, take $w_i \in Y$ such that

$$f(R_i) = \{f(x + is) + \lambda w_i \colon \lambda \geq 0\}.$$

Suppose that no two of the three extreme half-lines $f(R_0)$, $f(R_1)$, and $f(R_2)$ are parallel. After translation they correspond to three distinct extremal rays, so that Lemma 2.1 yields that w_0, w_1, and w_2 are linearly independent. Define

$$W_0 = f(x) + \operatorname{span}\{w_0, w_2\},$$
$$W_2 = f(x + 2s) + \operatorname{span}\{w_0, w_2\},$$
$$\ell_1 = \{f(x + s) + \lambda w_1 \colon \lambda \in \mathbb{R}\}.$$
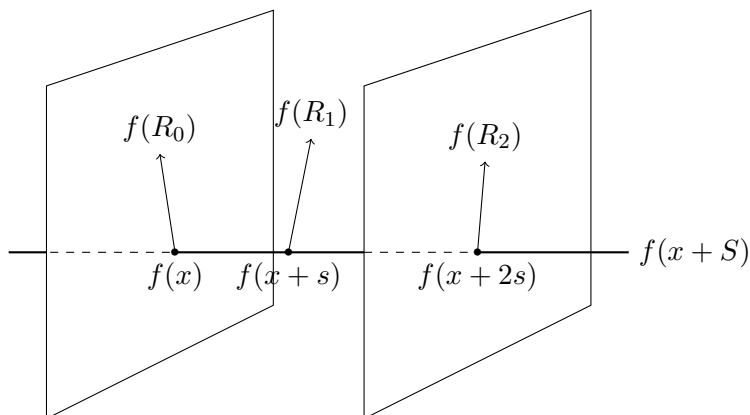
We observe that W_0 and W_2 are parallel and distinct planes. Moreover, $f(R_0) \subseteq W_0$, $f(R_2) \subseteq W_2$ and $f(R_1) \subseteq \ell_1$. The affine span $\operatorname{aff}(W_0, W_2)$ of W_0 and W_2 is three

dimensional and contains ℓ_1. Indeed, for every $z \in f(R_1)$ there is $\lambda \geq 0$ with $z = f(x + s + \lambda r)$, and by (3.2), $\text{aff}(W_0, W_2)$ contains the half-line $f(x + S + \lambda r)$. This shows that $f(R_1) \subseteq \text{aff}(W_0, W_2)$, and hence $\ell_1 \subseteq \text{aff}(W_0, W_2)$. Since w_1 is linearly independent of w_0 and w_2, we conclude that ℓ_1 intersects W_0 and W_2.

We proceed by showing that the half-line $f(R_1)$ intersects W_0 or W_2. Loosely speaking, the point $f(x + s)$ on ℓ_1 lies between W_0 and W_2 and, therefore, the points where ℓ_1 intersects W_0 and W_2 cannot be both at the same side of $f(x+s)$. To make this idea precise, let $v \in Y$ be such that

$$f(x + S) = \{f(x) + \lambda v \colon \lambda \geq 0\}.$$

Observe that $v \in K$, as $f(x + S) \subseteq [f(x), \infty)$. Then

$$\text{aff}(W_0, W_2) = \{f(x + s) + \lambda w_0 + \mu w_2 + \sigma v \colon \lambda, \mu, \sigma \in \mathbb{R}\}.$$

As $f(x + s) + w_1 \in f(R_1) \subseteq \text{aff}(W_0, W_2)$, there are $\lambda, \mu, \sigma \in \mathbb{R}$ such that $w_1 = \lambda w_0 + \mu w_2 + \sigma v$. By linear independence of w_0, w_1 and w_2, we have $\sigma \neq 0$. Consider the case $\sigma < 0$. Then $f(R_1)$ intersects W_0, so there is a $t > 0$ such that $f(x+s+tr) \in W_0$. As $f(x + R) = f(R_0) \subseteq W_0$, it follows that the half-line $f(x + S + tr)$ contains two distinct points of W_0, so that $f(x + S + tr) \subseteq W_0$. Therefore $f(x + 2s + tr) \in W_0 \cap f(R_2) \subseteq W_0 \cap W_2$, which is a contradiction. Otherwise, in case $\sigma > 0$, then $f(R_1)$ intersects W_2, and we similarly arrive at a contradiction. Hence at least two of the half-lines $f(R_0)$, $f(R_1)$, and $f(R_2)$ are parallel, so by the first step all three of them are parallel.

Now we complete the proof. As $f(R_0)$ and $f(R_1)$ are parallel, we have that the vectors $f(x + r) - f(x)$ and $f(x + s + r) - f(x + s)$ have the same direction. By interchanging the roles of R and S we obtain that the vectors $f(x + s) - f(x)$ and $f(x + s + r) - f(x + r)$ have the same direction. Thus, $f(x)$, $f(x + r)$, $f(x + s + r)$, and $f(x + s)$ are the consecutive corners of a parallellogram, which concludes the proof. $\square$

It is interesting to note that the proof of Lemma 3.3 does not work if the domain of the order-isomorphism is bounded. In fact, there exist examples of order-isomorphisms on bounded order intervals for which equation (3.1) does not hold, see for example [14] where order-isomorphisms on order intervals in $B(H)_{\text{sa}}$ are studied.

The following observation is a simple consequence of the previous lemma.

Corollary 3.4. *Suppose $r, s \in X$ are extreme vectors with $r \neq \lambda s$ for all $\lambda \in \mathbb{R}$ and $f \colon [a, \infty) \to [b, \infty)$ is an order-isomorphism. If $x \in [a, \infty)$ is such that $x + r + s, x + r, x + s \in [a, \infty)$ then*

$$f(x + r + s) - f(x + r) = f(x + s) - f(x).$$

Proof. We only discuss the proof for the case $r \leq 0$ and $s \leq 0$, and leave the other two remaining cases to the reader, as they are proved in a similar way. By writing $y = x + r + s$, we get

$$f(x+r+s) - f(x+s) = f(y) - f(y-r) = f(y-s) - f(y-r-s) = f(x+r) - f(x)$$

by Lemma 3.3. $\square$

Using this corollary we now show the following lemma.

Lemma 3.5. *Let $f\colon [a,\infty) \to [b,\infty)$ be an order-isomorphism. Suppose $s_1,\ldots,s_n, r \in X$ are extreme vectors such that $r \neq \lambda s_i$ for all $\lambda \in \mathbb{R}$ and $i = 1,\ldots,n$. If $x, x+s_1+\cdots+s_n+r, x+s_1+\cdots+s_n, x+r \in [a,\infty)$, then*

$$f\left(x + r + \sum_{i=1}^{n} s_i\right) - f\left(x + \sum_{i=1}^{n} s_i\right) = f(x+r) - f(x).$$

Proof. By relabelling we may assume that there exists $k \in \{0,\ldots,n\}$ such that $s_i > 0$ for all $i \leq k$ and $s_i < 0$ for all $i > k$. Then $x + r + \sum_{i=1}^{m} s_i \in [a,\infty)$ and $x + \sum_{i=1}^{m} s_i \in [a,\infty)$ for $m = 1,\ldots,n$. By Corollary 3.4 we have

$$f\left(\left(x + \sum_{i=1}^{n-1} s_i\right) + s_n + r\right) - f\left(\left(x + \sum_{i=1}^{n-1} s_i\right) + s_n\right) = f\left(x + \sum_{i=1}^{n-1} s_i + r\right) - f\left(x + \sum_{i=1}^{n-1} s_i\right).$$

Repeating this argument yields the desired conclusion. $\qquad \square$

We can use Lemma 3.5 to get the following identity.

Lemma 3.6. *Let $f\colon [a,\infty) \to [b,\infty)$ be an order-isomorphism. Suppose $x \in [a,\infty)$ and $s_1,\ldots,s_n$ are extreme vectors in X such that $s_i \neq \lambda s_j$ for all $\lambda \in \mathbb{R}$ and $i \neq j$, $x+s_1+\cdots+s_n \in [a,\infty)$, and $x+s_i \in [a,\infty)$ for all $i = 1,\ldots,n$, then*

$$f\left(x + \sum_{i=1}^{n} s_i\right) - f(x) = \sum_{i=1}^{n} \left(f(x+s_i) - f(x)\right).$$

Proof. By relabelling we may assume that there exists $k \in \{0,\ldots,n\}$ such that $s_i > 0$ for all $i \leq k$ and $s_i < 0$ for all $i > k$. Then $x + \sum_{i=1}^{m} s_i \in [a,\infty)$ for $m = 1,\ldots,n$. Using a telescoping sum and Lemma 3.5 we obtain

$$
\begin{aligned}
f\left(x + \sum_{i=1}^{n} s_i\right) - f(x) &= f\left(x + \sum_{i=1}^{n} s_i\right) - f\left(x + \sum_{i=1}^{n-1} s_i\right) + \cdots + f(x+s_1) - f(x) \\
&= \sum_{m=1}^{n} \left(f(x+s_m) - f(x)\right).
\end{aligned}
$$

$\qquad \square$

Let $\mathcal{R}$ denote the collection of all extreme rays in C, and define

$$[a,\infty)_{\mathcal{R}} = \{a + r_1 + \cdots + r_n \in [a,\infty) \colon r_i \in C \text{ is an extreme vector for all } i\}.$$

Lemma 3.7. *Let $f\colon [a,\infty) \to [b,\infty)$ be an order-isomorphism and $x,y \in [a,\infty)_{\mathcal{R}}$. Suppose that $y - x = s_1 + \cdots + s_n$, where $s_i \in X$ is an extreme vector for $i = 1,\ldots,n$. If $r \in X$ is an extreme vector with $r \neq \lambda s_i$ for all $\lambda \in \mathbb{R}$ and $i = 1,\ldots,n$, and $x + r, y + r \in [a,\infty)$, then*

$$f(x+r) - f(x) = f(y+r) - f(y).$$

Proof. Note that

$$
\begin{aligned}
f(y+r) - f(y) &= f(x + (y-x) + r) - f(x + (y-x)) \\
&= f(x + s_1 + \cdots + s_n + r) - f(x + s_1 + \cdots + s_n) \\
&= f(x+r) - f(x)
\end{aligned}
$$

by Lemma 3.5. $\qquad\square$

In the setting of Lemma 3.7, if $r = \lambda s_i$ for some λ and i, and $r \in \mathrm{span}\{s\colon s \in S$ and $S \in \mathcal{R} \setminus \{R\}\}$ where $R = \{\lambda r\colon \lambda \geq 0\}$, then one could replace s_i by a linear combination of extreme vectors not contained in $R \cup -R$ and thus obtain $y - x = s_1' + \cdots + s_m'$ with $r \neq \lambda s_j'$ for all λ and j. Then the conclusion of Lemma 3.7 still holds. This motivates the following definition from [10].

Definition 3.8. Let $\mathcal{S}$ be a collection of rays in a cone C in a vector space X. A ray $R \in \mathcal{S}$ is called *engaged (in $\mathcal{S}$)* whenever

$$
R \subseteq \mathrm{span}(\mathcal{S} \setminus \{R\}) = \mathrm{span}\{s\colon s \in S \text{ and } S \in \mathcal{S}\setminus\{R\}\}
$$

holds, and R is called *disengaged (in $\mathcal{S}$)* otherwise.

It can be shown that an extreme ray of a finite dimensional cone is disengaged (in the set of extreme rays) if and only if the cone equals the Cartesian product of the ray and another subcone. Cones that do not allow such a decomposition are considered in [2].

Recall that $\mathcal{R}$ denotes the collection of all extreme rays of C. We denote the collection of all engaged extreme rays in $\mathcal{R}$ by $\mathcal{R}_E$ and the collection of all disengaged extreme rays in $\mathcal{R}$ by $\mathcal{R}_D$. We remark that being an engaged ray is relative to the collection it is viewed in. Nevertheless, we have that the elements of $\mathcal{R}_E$ are again engaged in $\mathcal{R}_E$. For simplicity we say that an extreme vector $r \in R \cup -R$ is *engaged* if $R \in \mathcal{R}_E$.

Lemma 3.9. *If $r \in X$ is an extreme vector, then the following assertions hold:*

(i) *$f(x + \lambda r) - f(x)$ is a scalar multiple of $f(x+r) - f(x)$ for every $x \in [a, \infty)$ and $\lambda \in \mathbb{R}$ such that $x + r, x + \lambda r \in [a, \infty)$;*

(ii) *If r is engaged and $x, y, x + r, y + r \in [a, \infty)$ and $y - x \in \mathrm{span}\,\mathcal{R}$, then*

$$
f(x+r) - f(x) = f(y+r) - f(y).
$$

Proof. Assertion (i) follows from Corollary 3.2. Remark that if r is engaged then there exist extreme vectors $s_1, \ldots, s_n$ with $y - x = s_1 + \cdots + s_n$ such that $r \neq \lambda s_i$ for all $\lambda \in \mathbb{R}$ and $i = 1, \ldots, n$. So (ii) follows from Lemma 3.7. $\qquad\square$

The following result is an extension of [10, Theorem A]. Recall that $\mathcal{R}_E$ denotes the collection of engaged extreme rays in $\mathcal{R}$. We define

$$
[a, \infty)_{\mathcal{R}_E} = \{a + r_1 + \cdots + r_n \in [a, \infty)\colon r_i \in C \text{ is an engaged extreme vector for all } i\}.
$$

Theorem 3.10. *If $f\colon [a, \infty) \to [b, \infty)$ is an order-isomorphism, then f is affine on $[a, \infty)_{\mathcal{R}_E}$.*

Proof. Let R be an engaged extreme ray of C and fix $r \in R \backslash \{0\}$. Let $\lambda \in \mathbb{R}$ and take $x \in [a, \infty)_{\mathcal{R}}$ such that $x + \lambda r \geq a$. Then $x, x + r, x + \lambda r \in [a, \infty)$. So, by Lemma 3.9(i), there exists a unique $g_{r,x}(\lambda) \in \mathbb{R}$ such that

$$f(x + \lambda r) - f(x) = g_{r,x}(\lambda)(f(x + r) - f(x)). \tag{3.3}$$

As r is engaged, it follows from Lemma 3.9(ii) that $g_{r,x}(\lambda)$ does not depend on x. Thus there exists a unique function $g_r \colon \mathbb{R} \to \mathbb{R}$ such that for every $\lambda \in \mathbb{R}$ and $x \in [a, \infty)_{\mathcal{R}}$ with $x + \lambda r \geq a$ we have

$$f(x + \lambda r) - f(x) = g_r(\lambda)(f(x + r) - f(x)). \tag{3.4}$$

Clearly, $g_r(1) = 1$ and g_r is a monotone increasing function. For $\lambda, \mu \in \mathbb{R}$ there exists an $x \in [a, \infty)_{\mathcal{R}}$ such that $x + \lambda r \geq a$, $x + \mu r \geq a$, and $x + \lambda r + \mu r \geq a$. Moreover

$$
\begin{aligned}
g_r(\lambda + \mu)(f(x + r) - f(x)) &= f(x + (\lambda + \mu)r) - f(x) \\
&= f(x + \lambda r + \mu r) - f(x + \lambda r) + f(x + \lambda r) - f(x) \\
&= g_r(\mu)(f(x + \lambda r + r) - f(x + \lambda r)) + g_r(\lambda)(f(x + r) - f(x)).
\end{aligned}
$$

Since r is engaged, Lemma 3.9(ii) gives $f(x + \lambda r + r) - f(x + \lambda r) = f(x + r) - f(x)$. Note that $f(x + r) - f(x) \neq 0$, as $r \neq 0$ and f is injective, and hence

$$g_r(\lambda + \mu) = g_r(\lambda) + g_r(\mu).$$

As g_r is monotone increasing, additive, and $g_r(1) = 1$, a result by Darboux (see [1, Theorem 1 in Section 2.1]) yields that $g_r(\lambda) = \lambda$ for all $\lambda \in \mathbb{R}$.

To show that f is affine it suffices to show that f is convex-linear on $[a, \infty)_{\mathcal{R}_E}$. Let $x, y \in [a, \infty)_{\mathcal{R}_E}$ and $0 \leq t \leq 1$. Then $x = a + \sum_{i=1}^{n} \lambda_i r_i$ and $y = a + \sum_{i=1}^{n} \mu_i r_i$ where each $r_i \in C$ is an engaged extreme vector and $r_i \neq \lambda r_j$ for all $\lambda \in \mathbb{R}$ and $i \neq j$. Moreover, $\lambda_i, \mu_i \geq 0$ and $\lambda_i + \mu_i \neq 0$ for all i. Put $s_i = (t\lambda_i + (1 - t)\mu_i)r_i$. As $a + s_i \in [a, \infty)$ for all i, we can apply Lemma 3.6 to get

$$
\begin{aligned}
f(tx + (1 - t)y) - f(a) &= f\left(a + \sum_{i=1}^{n} s_i\right) - f(a) \\
&= \sum_{i=1}^{n}(f(a + s_i) - f(a)) \\
&= \sum_{i=1}^{n}(f(a + (t\lambda_i + (1 - t)\mu_i)r_i) - f(a)) \\
&= \sum_{i=1}^{n}(t\lambda_i + (1 - t)\mu_i)(f(a + r_i) - f(a)) \\
&= t\sum_{i=1}^{n}\lambda_i(f(a + r_i) - f(a)) + (1 - t)\sum_{i=1}^{n}\mu_i(f(a + r_i) - f(a)) \\
&= t\sum_{i=1}^{n}(f(a + \lambda_i r_i) - f(a)) + (1 - t)\sum_{i=1}^{n}(f(a + \mu_i r_i) - f(a)) \\
&= t\left(f\left(a + \sum_{i=1}^{n}\lambda_i r_i\right) - f(a)\right) + (1 - t)\left(f\left(a + \sum_{i=1}^{n}\mu_i r_i\right) - f(a)\right) \\
&= tf(x) + (1 - t)f(y) - f(a), \tag{3.5}
\end{aligned}
$$

where we have used (3.4) and the fact that each r_i is engaged in the forth and sixth equality, and Lemma 3.6 in the seventh one. This completes the proof. $\qquad\square$

Remark 3.11. It is interesting to note that in the proof of Theorem 3.10 we have only used the assumption that r is an engaged extreme vector to show that the map $g_r \colon \mathbb{R} \to \mathbb{R}$ satisfying (3.3) is independent of x and additive. However, if r is a disengaged extreme vector, then (3.3) still holds. In Section 5 we will exploit this observation. Moreover, we remark that it is necessary to work with the positive linear span of engaged extreme vectors, $[a, \infty)_{\mathcal{R}_E}$. Indeed, to apply Lemma 3.6 we need for each i that $a + s_i$ is in the domain of f.

Let us now see how we can use Theorem 3.10 to generalise [10, Theorem A]. Fix $a \in X$. For $V \subseteq [a, \infty)$ and $x \in [a, \infty)$ we say that $x = \inf V$ in $[a, \infty)$ if x is the infimum of V in $[a, \infty)$, that is, $x \in [a, \infty)$ is a lower bound of V and for every other lower bound $z \in [a, \infty)$ of V we have $x \geq z$. Note that we only consider lower bounds in $[a, \infty)$ here. An infimum where all lower bounds in X instead of all lower bounds in $[a, \infty)$ are considered may be different. Similarly, we write $x = \sup V$ in $[a, \infty)$ if x is the least upper bound of V in $[a, \infty)$.

As order-isomorphisms preserve infima and suprema, the following definition will be useful.

Definition 3.12. Given $a \in X$ and $V \subseteq [a, \infty)$ the *inf-sup hull of V in $[a, \infty)$* is the set

$$\{x \in [a, \infty) \colon \text{ there exist } v_{\alpha,\beta} \in V \text{ for } \alpha \in A \text{ and } \beta \in B \text{ such that}$$
$$x = \inf_{\alpha \in A} (\sup_{\beta \in B} v_{\alpha,\beta}) \text{ in } [a, \infty)\},$$

where A and B are arbitrary index sets. The defining condition is understood to include the existence of the infimum and supremum.

Note that the inf-sup hull of a set $D \subseteq [a, \infty)$ in $[a, \infty)$ can be obtained by translating the inf-sup hull of $D - a$ in C over a.

If $V \subseteq C$ and x and y are in the inf-sup hull of V in C, then $x = \inf_{\alpha \in A}(\sup_{\beta \in B} x_{\alpha,\beta})$ and $y = \inf_{\sigma \in S}(\sup_{\tau \in T} y_{\sigma,\tau})$ in C, with all $x_{\alpha,\beta}$ and $y_{\sigma,\tau}$ in V, and hence for all $\lambda, \mu \geq 0$ we have, in C, that

$$\lambda x + \mu y = \inf_{\alpha \in A} (\sup_{\beta \in B} \lambda x_{\alpha,\beta}) + \inf_{\sigma \in S} (\sup_{\tau \in T} \mu y_{\sigma,\tau}) = \inf_{\alpha \in A} (\sup_{\beta \in B} \lambda x_{\alpha,\beta} + \inf_{\sigma \in S} (\sup_{\tau \in T} \mu y_{\sigma,\tau}))$$

$$= \inf_{\alpha \in A} (\inf_{\sigma \in S} (\sup_{\beta \in B} \lambda x_{\alpha,\beta} + \sup_{\tau \in T} \mu y_{\sigma,\tau})) = \inf_{\alpha \in A} (\inf_{\sigma \in S} (\sup_{\beta \in B} (\sup_{\tau \in T} \lambda x_{\alpha,\beta} + \mu y_{\sigma,\tau})))$$

$$= \inf_{(\alpha,\sigma) \in A \times S} (\sup_{(\beta,\tau) \in B \times T} \lambda x_{\alpha,\beta} + \mu y_{\sigma,\tau}), \qquad (3.6)$$

which shows that $\lambda x + \mu y$ is also in the inf-sup hull of V in C. In particular we see that the inf-sup hull in C of a convex subset of C is again a convex set.

Lemma 3.13. *Let $f \colon [a, \infty) \to [b, \infty)$ be an order-isomorphism and let $D \subseteq [a, \infty)$ be convex. If f is affine on D, then f is affine on the inf-sup hull of D in $[a, \infty)$.*

Proof. We first assume that $a = 0$ and $b = 0$. Suppose $V \subseteq C$ and $v \in C$ are such that $v = \sup(V)$. Then $f(v)$ is an upper bound of $f(V)$ in K. Moreover, if $w \in K$ is

another upper bound of $f(V)$, then $f^{-1}(w) \in C$ is an upper bound of V, since f^{-1} is order preserving. As $v = \sup(V)$ in C we deduce that $v \le f^{-1}(w)$, so that $f(v) \le w$. This implies that $f(v) = \sup(f(V))$ in K. In the same way it can be shown that if $W \subseteq C$ and $w \in C$ are such that $w = \inf(W)$ in C, then $f(w) = \inf(f(W))$ in K. In short, f preserves infima and suprema in the cone.

To complete the proof in this case, it suffices to show that f is convex-linear on the inf-sup hull E of D in C. Indeed, E is a convex set by (3.6). Suppose that $x, y \in E$ and $0 \le t \le 1$. Write $x = \inf_\alpha \sup_\beta x_{\alpha,\beta}$ and $y = \inf_\sigma \sup_\tau y_{\sigma,\tau}$ in C, with $x_{\alpha,\beta}, y_{\sigma,\tau} \in D$ for all α, β, σ and τ. By repeatedly using the fact that f preserves infima and suprema in the cone we get

$$
\begin{aligned}
f(tx + (1-t)y) &= \inf_{\alpha \in A}(\sup_{\beta \in B}(\inf_{\sigma \in S}(\sup_{\tau \in T} f(tx_{\alpha,\beta} + (1-t)y_{\sigma,\tau})))) \\
&= \inf_{\alpha \in A}(\sup_{\beta \in B}(\inf_{\sigma \in S}(\sup_{\tau \in T} tf(x_{\alpha,\beta}) + (1-t)f(y_{\sigma,\tau})))) \\
&= tf(\inf_{\alpha \in A}(\sup_{\beta \in B} x_{\alpha,\beta})) + (1-t)f(\inf_{\sigma \in S}(\sup_{\tau \in T} y_{\sigma,\tau})) = tf(x) + (1-t)f(y),
\end{aligned}
$$

where all the infima and suprema are taken in C or K.

To deal with the general case, consider the translations $S(x) = x + a$, $x \in C$, and $T(y) = y - b$, $y \in [b, \infty)$. Then $S \colon C \to [a, \infty)$ and $T \colon [b, \infty) \to K$ are both affine order-isomorphisms. Hence $\tilde{f} = T \circ f \circ S \colon C \to K$ is an order-isomorphism and therefore it is affine by the first part of the proof. It follows that $f = T^{-1} \circ \tilde{f} \circ S^{-1}$ is affine. $\qquad\square$

Combination of Theorem 3.10 and Lemma 3.13 yields the next conclusion.

Proposition 3.14. *Every order-isomorphism $f \colon [a, \infty) \to [b, \infty)$ is affine on the inf-sup hull of $[a, \infty)_{\mathcal{R}_E}$ in $[a, \infty)$.*

We can now prove our main result Theorem 1.1.

Proof of Theorem 1.1. Let $a \in U$ be given. As C is the inf-sup hull of $[0, \infty)_{\mathcal{R}_E}$ in C, we get that the interval $[a, \infty)$ equals the inf-sup hull of $[a, \infty)_{\mathcal{R}_E}$ in $[a, \infty)$. So it follows from Proposition 3.14 that f is affine on $[a, \infty)$. As X is directed the cone C is generating, and hence $C - C = X$. This implies that there exists a unique affine map $g \colon X \to Y$ such that g restricted to $[a, \infty)$ coincides with f.

In the same way we find that for any $b \in U$ the map f is affine on $[b, \infty)$. Using that C is directed, we know there exists $c \in U$ such that $c \ge a, b$. We remark that the intersection $[a, \infty) \cap [b, \infty)$ contains the interval $[c, \infty)$. Therefore, f and g coincide on $[b, \infty)$ for all $b \in U$. Since $U = \bigcup_{b \in U}[b, \infty)$, we conclude that g coincides with f on U, which completes the proof. $\qquad\square$

Theorem 1.1 is a generalisation of [10, Theorem A] by Noll and Schäffer. It would be interesting to have a complete characterisation of the (infinite dimensional) directed Archimedean partially ordered vector spaces (X, C) for which every order-isomorphism $f \colon C \to C$ is linear. To our knowledge, Theorem 1.1 is the most general result at present. It can, however, not be applied in a variety of settings such as the space $C([0, 1]) \oplus \mathbb{R}$ with cone $\{(f, \alpha) \colon \|f\|_\infty \le \alpha\}$. In this space the cone has exactly two disengaged extreme rays: $\{\lambda(\mathbb{1}, 1) \colon \lambda \ge 0\}$ and $\{\lambda(-\mathbb{1}, 1) \colon \lambda \ge 0\}$, where

$\mathbb{1}(x) = 1$ for all $x \in [0,1]$, but it has no engaged extreme rays. We believe, however, that each order-isomorphism on the cone is linear in this space.

We end this section with a simple observation concerning direct sums. Let (X_1, C_1) and (X_2, C_2) be directed Archimedean partially ordered vector spaces. Then the direct sum $X_1 \oplus X_2$ is a directed Archimedean partially ordered vector space with cone $C_1 \times C_2$. Moreover $(r, s) \in C_1 \times C_2$ is an (engaged) extreme vector if and only if r is an (engaged) extreme vector and $s = 0$, or, s is an (engaged) extreme vector and $r = 0$. It is straightforward to infer that if (X_1, C_1) and (X_2, C_2) satisfy the conditions on (X, C) in Theorem 1.1, then so does $(X_1 \oplus X_2, C_1 \times C_2)$.

4 Self-adjoint operators on a Hilbert space

Let H be a Hilbert space and $B(H)_{\mathrm{sa}}$ be the space of bounded self-adjoint operators on H, ordered by the cone $B(H)_{\mathrm{sa}}^+$ of positive semi-definite operators. In this section we show that $B(H)_{\mathrm{sa}}$ satisfies the conditions of Theorem 1.1.

It is easy to show that the extreme rays of $B(H)_{\mathrm{sa}}^+$ are the rays spanned by rank-one projections. We will denote the collection of all extreme rays of $B(H)_{\mathrm{sa}}^+$ by $\mathcal{R}$. Furthermore, for a closed subspace V of H we denote the orthogonal projection onto by V by P_V, and for $x \in H$ we write $P_x = P_{\mathrm{span}(\{x\})}$.

Theorem 4.1. *If H is a Hilbert space, with $\dim H \geq 2$, and $U, W \subseteq B(H)_{\mathrm{sa}}$ are upper sets, then every order-isomorphism $f \colon U \to W$ is affine.*

Proof. We verify that $B(H)_{\mathrm{sa}}$ satisfies the conditions of Theorem 1.1. Evidently, $B(H)_{\mathrm{sa}}$ is directed and Archimedean. We first show that all extreme rays of $B(H)_{\mathrm{sa}}^+$ are engaged. So, suppose $P \in \mathcal{R}$. Then there exists an $x \in H$ such that $P = P_x$. As $\dim H \geq 2$ we can find non-zero $y, z \in H$ such that y and z are orthogonal and x, y, z lie in a two-dimensional subspace V. Then $P_V = P_y + P_z$, so that

$$P_x = P_V - (I - P_x)P_V = P_y + P_z - P_{\{x\}^\perp}P_V = P_y + P_z - P_w,$$

where $w \in \{x\}^\perp \cap (V \backslash \{0\})$. We conclude that P_x can be written as a linear combination of rank-one projections different from P_x and, hence, the ray spanned by P_x is engaged in $\mathcal{R}$.

It follows from [6, Corollary 3] that for each $0 \leq A \leq I$ we have

$$A = \sup\{\lambda P_x \colon \lambda \geq 0 \text{ and } x \in H \text{ such that } \lambda P_x \leq A\}.$$

Note that for each $B \in B(H)_{\mathrm{sa}}^+$ there exists $\mu > 0$ such that $0 \leq \mu B \leq I$, and hence the inf-sup hull (or indeed the sup hull) of the engaged extreme rays of $B(H)_{\mathrm{sa}}^+$ equals $B(H)_{\mathrm{sa}}^+$. □

We remark that Theorem 4.1 was first proved, using different arguments, by Molnár [7] for order-isomorphisms from $B(H)_{\mathrm{sa}}$ onto itself, and in this situation one cannot apply [10, Theorem A]. It is also interesting to note that in Theorem 4.1 we only require U and V to be upper sets in $B(H)_{\mathrm{sa}}$. In fact, this can be exploited to recover another result by Molnar [8], which says that there exists no order-isomorphism from $B(H)_{\mathrm{sa}}$ onto the interior of $B(H)_{\mathrm{sa}}^+$, as such an order-isomorphism would need to be affine by Theorem 4.1. There are potentially other interesting upper sets, such as the set $\{A \in B(H)_{\mathrm{sa}} \colon \langle Ax, x \rangle + \langle Ay, y \rangle \geq \alpha\}$ for $x, y \in H$ and $\alpha \in \mathbb{R}$, to which Theorem 4.1 can be applied.

5 Order-isomorphisms in related problems

In this section we proceed the discussion of Section 3 and relate to results by Artstein-Avidan and Slomka and Schäffer in settings somewhat different than in Theorem 1.1. We obtain three results. First, we present a "diagonalization formula" for order-isomorphisms between cones, see (5.1) below. Second, we apply the results of Section 3 to positively homogeneous order-isomorphisms between cones and obtain that they must be linear if one of the cones equals the inf-sup hull of the positive span of its extreme rays. Third, we consider separable complete order unit spaces where in one of them the inf-sup hull of the positive linear span of the engaged extreme rays is big enough to intersect the interior of the cone. In that case we derive from Theorem 1.1 that every order-isomorphism between upper sets must be affine.

We begin with the following infinite dimensional analogue of a result by Artstein-Avidan and Slomka [5, Theorem 1.7].

Proposition 5.1. *Let (X, C) and (Y, K) be Archimedean partially ordered vector spaces and suppose that $f \colon C \to K$ is an order-isomorphism. Let $(v_\alpha)_{\alpha \in A}$ be a collection of linearly independent extreme vectors in C. Then there exist corresponding monotone increasing bijections $g_\alpha \colon [0, \infty) \to [0, \infty)$, for $\alpha \in A$, such that for all $\lambda_1, \ldots, \lambda_n \geq 0$ and $\alpha_1, \ldots, \alpha_n \in A$ we have*

$$f\left(\sum_{i=1}^{n} \lambda_i v_{\alpha_i}\right) = \sum_{i=1}^{n} g_{\alpha_i}(\lambda_i) f(v_{\alpha_i}). \tag{5.1}$$

Proof. Note that $f(0) = 0$. Let $r \in C$ be an extreme vector. According to Corollary 3.2, f maps the extreme ray through r bijectively onto the extreme ray through $f(r)$. Hence there exists a nonnegative scalar $g_r(\lambda)$ such that $f(\lambda r) = g_r(\lambda) f(r)$, for all $\lambda \geq 0$. Moreover, the function $g_r \colon [0, \infty) \to [0, \infty)$ is a monotone increasing bijection. Equation (5.1) now follows from Lemma 3.6. $\square$

In [5, Theorem 1.7], also the finite dimensional cases $f \colon X \to X$ and $f \colon C^\circ \to C^\circ$ are considered. In the situation of Proposition 5.1, if f is an order-isomorphism from X to Y and $f(0) = 0$, then one can easily verify that the maps g_r are actually defined on $\mathbb{R}$ and that (5.1) holds for all $\lambda \in \mathbb{R}$. The infinite dimensional version of the case where $f \colon C^\circ \to K^\circ$ is not so strong. Indeed, if (X, C) and (Y, K) are infinite dimensional order unit spaces, then one can adapt the proof of Proposition 5.1 to show that for each order-isomorphism $f \colon C^\circ \to K^\circ$ and each collection $(v_\alpha)_{\alpha \in A}$ of linearly independent extreme vectors of C, there are linearly independent extreme vectors $(w_\alpha)_{\alpha \in A}$ of K and monotone increasing bijections $g_\alpha \colon [0, \infty) \to [0, \infty)$, $\alpha \in A$, such that for all $\lambda_1, \ldots, \lambda_n \geq 0$ and $\alpha_1, \ldots, \alpha_n \in A$ we have (5.1) where $f(v_{\alpha_i})$ is replaced by w_{a_i}, provided that $\sum_{i=1}^{n} \lambda_i v_{\alpha_i} \in C^\circ$. However, in general infinite dimensional order unit spaces most elements of the interior of the cone cannot be written as a positive linear combination of finitely many positive extreme vectors and, thus, the use of this result is limited.

Let us next consider positively homogeneous order-isomorphisms. If $U \subseteq X$ and $V \subseteq Y$ are such that $\lambda u \in U$ and $\lambda v \in V$ for every $u \in U$, $v \in V$, and $\lambda > 0$, then a map $f \colon U \to V$ is called *positively homogeneous* if $f(\lambda u) = \lambda f(u)$ for every $u \in U$ and $\lambda > 0$. If U and V are generating Archimedean cones, then this condition implies

that $f(0) = 0$, which yields the more common definition that includes $\lambda = 0$. The definition given here also applies to maps on interiors of cones.

In [13, Theorem B], Schäffer provides the next result.

Theorem 5.2 (Schäffer). *Let (X, C, u) and (Y, K, v) be order unit spaces. Then every positively homogeneous order-isomorphism $f: C^\circ \to K^\circ$ is linear.*

The results of Section 3 yield the following alternative statement, in which the requirement of an order unit is replaced by a condition involving extreme rays.

Theorem 5.3. *Let (X, C) and (Y, K) be Archimedean partially ordered vector spaces such that (X, C) is directed and C equals the inf-sup hull of $[0, \infty)_{\mathcal{R}}$ in C. Then every positively homogeneous order-isomorphism $f: C \to K$ is linear.*

Proof. We first show that f is additive on $[0, \infty)_{\mathcal{R}}$. Let $s_1, \ldots, s_n$ be extreme vectors in C. It suffices to show that $f\left(\sum_{i=1}^n s_i\right) = \sum_{i=1}^n f(s_i)$. In order to apply Lemma 3.6, we combine terms of s_i that lie on the same ray. Indeed, for $j = 1, \ldots, m$, let $I_j \subseteq \{1, \ldots, n\}$ be disjoint with $\bigcup_{j=1}^m I_j = \{1, \ldots, n\}$ such that for every $i, k \in \{1, \ldots, n\}$ we have $s_i = \lambda s_k$ for some $\lambda \geq 0$ if and only if there exists $j \in \{1, \ldots, m\}$ with $i, k \in I_j$. Denote $r_j = \sum_{i \in I_j} s_i$ and for every $i \in I_j$ let λ_i be such that $s_i = \lambda_i r_j$. Then $\sum_{i \in I_j} \lambda_i = 1$ for $j = 1, \ldots, m$. With the aid of Lemma 3.6 and the positive homogeneity of f we obtain

$$f\left(\sum_{i=1}^n s_i\right) = f\left(\sum_{j=1}^m r_j\right) = \sum_{j=1}^m f(r_j) = \sum_{j=1}^m \sum_{i \in I_j} \lambda_i f(r_j)$$

$$= \sum_{j=1}^m \sum_{i \in I_j} f(\lambda_i r_j) = \sum_{i=1}^n f(s_i).$$

As f is positively homogeneous, it follows that f is linear on $[0, \infty)_{\mathcal{R}}$. Due to Lemma 3.13 we obtain that f is linear on the inf-sup hull of $[0, \infty)_{\mathcal{R}}$ in C, which equals C. $\square$

If in Theorem 5.3 f is an order-isomorphism from X to Y and f is homogeneous instead of only positively homogeneous, then it can be shown along similar lines that f is affine.

It is useful to compare Theorem 5.2 and Theorem 5.3 and identify the differences. Let (X, C, u) and (Y, K, v) be order unit spaces. Suppose that $f: C \to K$ is a positively homogeneous order-isomorphism. Then straightforward verification yields $f(C^\circ) = K^\circ$. Hence it follows by Theorem 5.2 that f is linear on C°. As C is the inf hull in C of the convex set C°, it follows from Lemma 3.13 that f is linear on C. Thus, any homogeneous order-isomorphism between cones of order unit spaces is linear. Theorem 5.3 provides a condition, alternative to having an order unit, that yields the same conclusion. For example, the space $\ell^p(\mathbb{N})$ for $1 \leq p \leq \infty$ with coordinate-wise order satisfies the conditions of Theorem 5.3 but fails to have an order unit. Hence Schäffer's Theorem 5.2 does not imply our Theorem 5.3.

Our third interest in this section is an intermediate result by Schäffer, which has a milder homogeneity condition than Theorem 5.2. In [13, Corollary A1] Schäffer shows for order unit spaces (X, C, u) and (Y, K, v), where either $(X, \|.\|_u)$ or $(Y, \|.\|_v)$ is separable and complete, that any order-isomorphism $f: C^\circ \to K^\circ$ is linear, provided

there exists a $w \in C^\circ$ such that $f(\lambda w) = \lambda f(w)$ for all $\lambda \geq 0$. Compared to [13, Theorem B], the positively homogeneous condition of f is weakened to only being positively homogeneous on a ray through the interior of the cone, at the cost of one of the order unit spaces being separable and complete. In conjunction with Theorem 1.1 this yields the following.

Theorem 5.4. *Let (X, C, u) and (Y, K, v) be order unit spaces, and $U \subseteq X$ and $V \subseteq Y$ be upper sets. Suppose that the inf-sup hull of $[0, \infty)_{\mathcal{R}_E}$ in C has a non-empty intersection with C°, and that either $(X, \|.\|_u)$ or $(Y, \|.\|_v)$ is separable and complete. Then every order-isomorphism $f \colon U \to V$ is affine.*

Proof. Firstly, we consider the case $U = C^\circ$ and $V = K^\circ$. Let C_E denote the inf-sup hull in C of the positive linear span of the engaged extreme rays of C. By assumption there exists $x \in C_E \cap C^\circ$. We recall that an order unit space is directed and Archimedean. Hence, Proposition 3.14 says that f is affine on $C_E \cap C^\circ$. As f is an order-isomorphism mapping C° onto K°, it is straightforward to infer that f is in fact linear on $C_E \cap C^\circ$. In particular, $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$. Now [13, Corollary A1] yields that f is linear on C°.

Next we consider the case $U = C$ and $V = K$. Just as in the previous paragraph, there exists an $x \in C^\circ$ such that $f(\lambda x) = \lambda x$ for all $\lambda \geq 0$. We infer that $f(C^\circ) = K^\circ$. Indeed, let $y \in K$. As $x \in C^\circ$ there exists $\lambda \geq 0$ such that $\lambda x \geq f^{-1}(y)$. This yields that $\lambda f(x) = f(\lambda x) \geq y$. Therefore, $f(x)$ is an order unit in (Y, K) and hence $f(x) \in K^\circ$. Now let $y \in C^\circ$. Then there exists $m > 0$ such that $mx \leq y$. We get $mf(x) = f(mx) \leq f(y)$. In particular, $f(y)$ is an order unit and we conclude that $f(y) \in K^\circ$. Hence $f(C^\circ) \subseteq K^\circ$. We remark that for all $\lambda \geq 0$ we have $f^{-1}(\lambda f(x)) = \lambda x = \lambda f^{-1}(f(x))$, in other words f^{-1} is positively homogeneous along the ray through $f(x)$. Therefore, the previous steps applied to f^{-1} instead of f yield the converse inclusion $K^\circ \subseteq f(C^\circ)$. By the first part of the proof we obtain that f is linear on C°. Since C is the inf hull of the convex set C°, it follows from Lemma 3.13 that f is linear on C.

Suppose $a \in X$ and $b \in Y$ are such that $U = [a, \infty)$ and $V = [b, \infty)$. The order-isomorphism $\hat{f}$ defined by $\hat{f}(c) = f(c + a) - b$ maps C to K. By the previously considered case $\hat{f}$ is linear, and hence f is affine.

The general case where $U \subseteq X$ and $V \subseteq Y$ are upper sets follows by arguments similar to those made in the proof of Theorem 1.1. Indeed, for every $a \in U$, f is an order-isomorphism from $[a, \infty)$ to $[f(a), \infty)$, so that f is affine on $[a, \infty)$ by the previous case. Then $f|_{[a,\infty)}$ extends to a unique affine map $F \colon X \to Y$, which is independent of $a \in U$, as (X, C) is directed. $\qquad \square$

To conclude the paper we provide an example to which Theorem 5.4 applies, but not Theorem 1.1. Consider the order unit space (X, C, u) consisting of the real vector space $X = C([0, 1] \cup [2, 3]) \oplus \mathbb{R}$, the Archimedean cone

$$C = \{(f, \lambda) \colon \|f\|_\infty \leq \lambda\}$$

and the order unit $u = (0, 1) \in C$. Then $(X, \|.\|_u)$ is complete and separable. The unit ball

$$B = \{f \in C([0, 1] \cup [2, 3]) \colon \|f\|_\infty \leq 1\}$$

has four extreme points: $\pm\mathbb{1}_{[0,1]}$ and $\pm\mathbb{1}_{[2,3]}$, where $\mathbb{1}_{[0,1]}$ and $\mathbb{1}_{[2,3]}$ denote the indicator functions of $[0,1]$ and $[2,3]$, respectively. Therefore, C has four extreme rays, namely the rays through $(\pm\mathbb{1}_{[0,1]}, 1)$ and $(\pm\mathbb{1}_{[2,3]}, 1)$. As

$$(\mathbb{1}_{[0,1]}, 1) + (-\mathbb{1}_{[0,1]}, 1) = 2u = (\mathbb{1}_{[2,3]}, 1) + (-\mathbb{1}_{[2,3]}, 1),$$

all four extreme rays are engaged, and u which lies in C° is contained in the positive linear span of the engaged extreme rays. We conclude that the order unit space (X, C, u) satisfies the conditions of Theorem 5.4. However, the inf-sup hull in C of the sum of the engaged extreme rays consists only of elements of the form $(\lambda\mathbb{1}_{[0,1]} + \mu\mathbb{1}_{[2,3]}, \nu)$, with $\lambda, \mu \geq 0$ and $|\lambda|, |\mu| \leq \nu$, and hence (X, C) does not satisfy the conditions of Theorem 1.1.

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