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Superimposed Renewal Processes in Reliability*

Shaomin Wu[a]*

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Abstract: This paper reviews the existing literature on the superimposed renewal process, with its foci on probabilistic and statistical properties, statistical inference, and applications in reliability analysis and maintenance policy optimisation. It then proposes future research topics. Copyright © 2019 John Wiley & Sons, Ltd.

1. Introduction

Let $\{N(t), t > 0\}$ be a nonnegative integer-valued stochastic process and the number of occurrences of an event during the time interval $(0, t]$, where the time durations between consecutive occurrences are independent and identically distributed random variables. Then, $\{N(t), t > 0\}$ is a renewal process (see stat05084 for more discussion).

Suppose there are n independent renewal processes $\{N_i(t), t > 0\}$ with $i = 1, 2, ..., n$. The stochastic process $\{\sum_{i=1}^{n} N_i(t), t > 0\}$ is the superposition of the n renewal processes, or a superimposed renewal process (SRP).

For the convenience of expression, we denote $\{X_{i,k}, k = 1, 2, ...\}$ as the inter-renewal times of the i-th renewal process, $F_i(t)$ as the cumulative distribution function (CDF) of $X_{i,k}$, $\{Y_{n,k}, k = 1, 2, ...\}$ as the successive occurrence times between events of the SRP, and $G_{n,k}(t)$ as the CDF of $Y_{n,k}$.

In the language of reliability mathematics, the above notions can be explained as follows. Suppose that a system is composed of n sockets, each of which contains a component. Whenever a component fails, the system fails and the failed component is replaced with a new and identical one. Since the replacement of a component in its corresponding socket can result in the socket as being "good as new" immediately after the replacement, the replacement can be regarded as a renewal. Thus, the counting process that counts the observed cumulative number of failures in socket i, $N_i(t)$, forms a renewal process. $X_{i,k}$ is therefore the lifetime of the component in socket i after the $(k-1)$-th replacement (where $X_{i,k} \geq 0$); $F_i(t)$ is the lifetime distribution of the components in socket i; $Y_{n,k}$ is the lifetime of the system after the $(k-1)$-th failure; and $G_{n,k}(t)$ is the CDF of the life time between the $(k-1)$-th and k-th failures of the system.

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[a]**Kent Business School, University of Kent, Canterbury, Kent CT2 7FS, UK**
***Email: s.m.wu@kent.ac.uk**

Figure 1 illustrates an SRP formed by the failure process of a series system composed of three sockets. The failures are marked by the blue crosses on the three sockets, which are represented by the top three horizontal lines, and the failures of the system, marked by the red crosses on the bottom horizontal line, are formed by the union of all the failures of the three sockets.

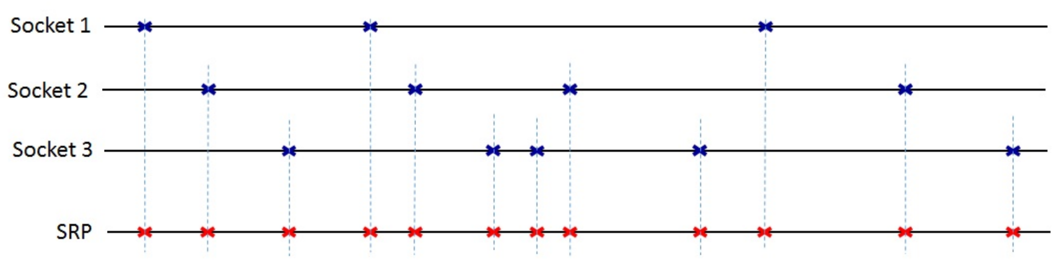

Figure 1. Superimposed renewal process.

It is known in the reliability literature that the mean cumulative number of failures (MCF) is one of the most widely used reliability indices. Denote $M_i(t)$ as the MCF of the i-th socket. Since the occurrences of failures of the components in the n different sockets are independent, the MCF of the system can be estimated by

$$M(t) = \sum_{i=1}^{n} M_i(t), \tag{1}$$

where $M_i(t)$ can be estimated with the renewal function: $M_i(t) = E[N_i(t)] = \sum_{k=1}^{\infty} F_i^{(k)}(t)$ and $F_i^{(k)}(t)$ is the k-fold convolution of $F_i(t)$.

One may also use the rate of occurrence of failures (ROCOF), $m(t)$, to describe the reliability of a repairable system, as defined below,

$$m(t) = \frac{dM(t)}{dt}. \tag{2}$$

Denote $m_i(t) = \frac{dM_i(t)}{dt}$, then $m(t) = \sum_{i=1}^{n} m_i(t) = \sum_{i=1}^{n} \frac{dM_i(t)}{dt}$.

Examples. In the reliability literature, there are many examples of the SRP. For example, Kallen (2011) considers a steel structure that is protected against corrosion by a coat of paint. Once 3% of the total surface of the coating is damaged, these damaged areas are repaired. The remaining 97% of the surface is left as such. The author then regards that the coated surface is composed of a grid of cells and that the arrival of damages to the coating in each cell follows a renewal process. As a result, the arrivals of damages on the entire surface form an SRP. Other reliability examples can also be found in Krivtsov & Frankstein (2014); Zhang et al. (2017) and Song & Xie (2018).

The literature on the SRP is rich. Cinlar (1972) provides an excellent review of the SRP. This paper aims to give a literature review of the existing research conducted since 1972. The remainder of the paper is structured as follows. Sections 2, 3, and 4 review the probabilistic and statistical properties of the SRP, the statistical inference of the SRP, and the applications of the SRP in reliability engineering, respectively. Section 5 concludes the paper and proposes future research topics.

2. Probabilistic and Statistical Properties

In the literature, the foci on the SRP are on the limiting properties of the probability distribution $G_{n,k}(t)$ and its relevant statistics such as the expected value and the variance.

Cox & Smith (1954) prove that, as n approaches to infinity and $F_i(t)$ are the same with $i = 1, 2, \ldots, n$, the SRP becomes a homogenous Poisson process (HPP). For the case that $F_i(t)$ are different with $i = 1, 2, \ldots, n$, Drenick (1960) shows that when n is infinity and the time is far away from the origin, an SRP behaves like an HPP (Drenick's theorem). He also proved that the distribution of the time to first failure of a series system with $n \to \infty$ is approximately the exponential distribution.

Corrections to the exponential distribution that account for the case with $n < \infty$ are also provided by Drenick (1960) and Cox (1962), respectively. Later, Blumenthal et al. (1973) develop correction terms for the finite age of the system as well (i.e., for the case of $t < \infty$).

If $X_{i,k} > 0$, then a renewal process $\{N_i(t), t > 0\}$ is called an ordinary renewal process. Samuels (1974) shows that if the superposition of two ordinary renewal processes is an ordinary renewal process, then all processes are Poisson. Ferreira (2000) later generalises Samuels's result to the case in which the inter-renewal times of a process may be zero (i.e., $X_{i,k} \geq 0$) and showed that, besides the Poisson processes, there are two pairs of binomial-like processes whose superposition is a renewal process as well. Teresalam & Lehoczky (1991) extend the asymptotic results for ordinary renewal processes to the SRP. In particular, the ordinary renewal functions, renewal equations, and the key renewal theorem are extended to the SRP.

An SRP tends toward (statistical) equilibrium as the time of operation becomes very large. Blumenthal et al. (1984) show that for the SRP at equilibrium, the exponential distribution for times between failures is reasonably well justified. In the transient state, the times between failures follow the exponential distribution as well, but with a time varying scale parameter. The authors state that ignoring this parameter and using a partly aged system to make inferences about the mean life of a fully aged system can lead to very misleading outcomes. Franken (1963); Grigelionis (1963) and Grigelionis (1964) show that the number of events, $\sum_{i=1}^{n} N_i(t)$, in time interval $(0, s)$ has a complex distribution and it can be approximated by the Poisson distribution for the case of n being large, which agrees with the results of (Drenick, 1960; Cox & Smith, 1954). This approximation is inaccurate if n is not large enough(Blumenthal, 1993). If no individual renewal process $\{N_i(t), t > 0\}$ has more than one event in time interval $(0, s)$, Blumenthal (1993) gives a binomial-like expression of the probability of exactly r events for the SRP.

Given two stochastic processes: an SRP in which $F_i(t)$ ($i = 1, 2, ..., n$) are mixed exponential distributions and whose MCF is $M(t)$, and a nonhomogeneous-Poisson-process (NHPP) $\{N^*(t); t \geq 0\}$ with $E[N^*(t)] = M(t)$, Arjas et al. (1991) obtain the relationship between their variances: $\mathrm{Var}[N^*(t)] \leq \mathrm{Var}[M(t)]$.

Barlow & Proschan (1996) give general discussions of SRPs and some limiting results. In general, the SRP will not be a renewal process unless the component renewal processes are homogenous Poisson processes.

Krivtsov & Frankstein (2014) discuss methods of statistical hypothesis testing to distinguish two situations: that all or most of the failures of a multi-socket system with the same type of component in each socket (namely, an SRP with identical $F_i(t)$ for $i = 1, 2, ..., n$) are caused by the failure of one socket because of a system problem (e.g., components are stressed more in one socket, relative to the others) and that the failures of the system may be caused by different sockets. The authors illustrate their method using the failure data of a 4–cylinder petroleum engine with four identical spark plugs.

Simulation Examples. We use Monte Carlo simulation to generate the failure processes of six series systems. The first three systems are composed of 250, 2000, and 20000 sockets, in which components are identical, respectively, and the second three systems are composed of the same numbers of sockets as the first three systems, however, the components in different sockets in a system may be different. The lifetime distributions of the components follows Weibull distributions, i.e., $F_i(t) = 1 - e^{-(\frac{t}{\beta})^{\alpha}}$, where α and β may be different. We then calculate the failure rates within time intervals $(k-1, k]$ $(k = 1, 2, \ldots)$, that is, the number of failures within time interval $(k-1, k]$ divided by the number of components. Figures 2-7 show the failure rate functions of the systems that are composed of $n = 250$, $n = 2,000$, and $n = 20,000$ components, respectively, as indicated by the caption of each figure. That is, in each figure, the X axis shows time and the Y-axis show failure rates.

- All of the components in Figures 2, 3, and 4 have the same shape parameter and scale parameter, $\alpha = 1.5$ and $\beta = 20$;
- Figures 5, 6, and 7 have different shape parameters and scale parameters and are randomly generated with constraints: $\alpha \in [0.5, 4]$ and $\beta \in [12, 60]$.

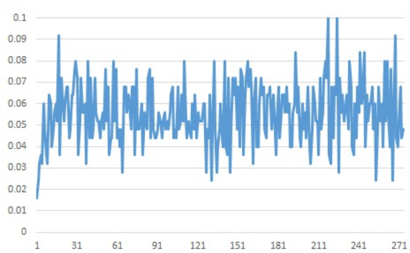

Figure 2. $n = 250$, $\alpha = 1.5$, and $\beta = 20$.

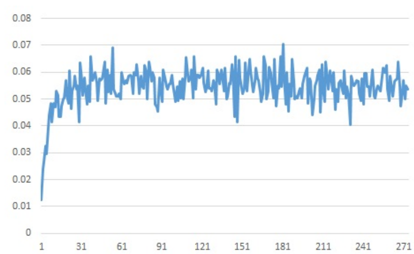

Figure 3. $n = 2000$, $\alpha = 1.5$, and $\beta = 20$.

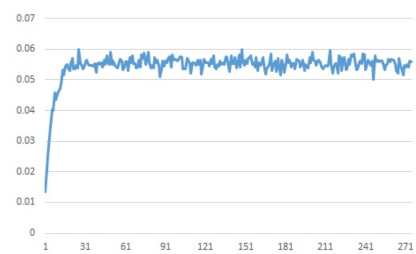

Figure 4. $n = 20000$, $\alpha = 1.5$, and $\beta = 20$.

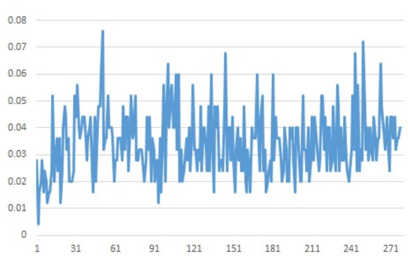

Figure 5. $n = 250$,$\alpha \in [0.5, 4]$, and $\beta \in [12, 60]$.

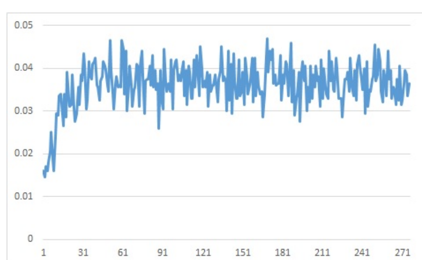

Figure 6. $n = 2000$, $\alpha \in [0.5, 4]$, and $\beta \in [12, 60]$.

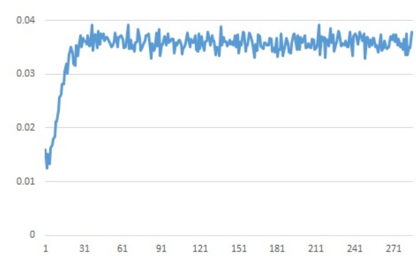

Figure 7. $n = 20000$, $\alpha \in [0.5, 4]$, and $\beta \in [12, 60]$.

From the above six figures, it can be seen that, for systems composed of identical components and for systems composed of different components, the failure rate functions reach an equilibrium state when n becomes large and time becomes large, which agrees with the results of the limiting lifetime distribution $G_{n,k}(t)$ of the SRP for n and k approaching infinite (Cox & Smith, 1954; Drenick, 1960; Cox, 1962). From the six figures, the failure rate functions are increasing in the early period of each figure, which confirms the result of Khinchin's statement that the SRP behaves like a NHPP if the number of sockets in an SRP is large (Khinchin, 1956). Unfortunately, we are not able to confirm the result of Drenick (1960) that the distribution of the time to first failure of a series system with $n \to \infty$ is approximately the exponential distribution, which is because there is only one observation representing the first time to failure and therefore there is no dynamic movement that can be observed in each figure.

In addition to the above interpretation of the figures, we have the following two notes.

- It is noted that the fluctuations may become less severe if we draw the failure rate functions based on a wider time interval, for example, calculate failure rates within time interval $(k-1, k+1]$, where $k = 2k_0 - 1$ and $k_0 = 1, 2, \ldots$. However, if the unit of the scale parameter β is year, for example, $\beta = 20$ reflecting a 20 year design life, then it is reasonable to calculate the failure rate of a system within time interval $(k-1, k+1]$, i.e., on a one-year basis.
- The failure rates are calculated on discrete intervals and should have therefore shown different points on the figures. In the figures, however, the failure rates are linked for the sake of visualisation.

3. Statistical Inference

As discussed in Section 1, an SRP may include the following quantities:

- n: the number of sockets;
- $F_i(t)$: the cumulative probability distribution of the times between successive occurrences;
- $M_i(t)$: the mean cumulative number of failures (MCF).

In the literature, there are publications on estimating the above three quantities, which are reviewed in this section.

3.1. *Estimation of n*

Nayak (1991) considers a superposition of an unknown number of independent HPPs with unknown rates. The author then estimates the number of renewal processes (i.e., the number of sockets) and their rates by observing the failure process of the system for some time.

Dewanji et al. (2012) propose a nonparametric method to estimate the number of the renewal processes in an SRP with $F_i(t)$ being identical distribution function for $i = 1, 2, \ldots, n$.

3.2. *Estimation of Reliability Indices*

The reliability of a series system, for example, socket i, can be characterised by one of the two functions: $F_i(t)$ and $M_i(t)$. Obviously, the quantity $m_i(t)$ discussed in Section 1 (i.e., the rate of occurrence of failures (ROCOF)), can be derived from $M_i(t)$.

3.2.1. *Estimation of $F_i(t)$*

There have been few studies on estimating the component failure-time distribution $F_i(t)$ in an SRP. For applications involving aircraft components, Peixoto (2009) proposes a method to estimate the failure-time distribution $F_i(t)$ by assigning the event times to sockets randomly, and then using simulation to correct for bias. In Peixoto's study, a second layer of simulation is needed to quantify statistical uncertainty (e.g., to compute confidence intervals).

There has been some other work on estimating component lifetime distributions $F_i(t)$. Trindade & Haugh (1980) propose a non-parametric estimator of the lifetime distribution, based on the deconvolution of the renewal equation. Baxter (1994) derives a non-parametric estimator of the lifetime distribution function in a discretised manner and investigates the performance of the method proposed based on the renewal function deconvolution. Tortorella (1996) proposes an estimation procedure by building a pooled discrete renewal process model and estimating the component reliability based on a maximum likelihood-like method.

Kallen (2011) proposes an asymptotic inter-failure time distribution method to estimate the lifetime distribution $F_i(t)$, which requires the system to have run for a long time (i.e., t is large) before the asymptotic approximation becomes valid.

In practice, it often happens that times between failures are known and the causes of failure observations are obtained. Such data are referred to as *masked times between failure data*. For example, Usher (1987) notes, '... when large computer systems fail, analysis is often performed such that a small subset of components, perhaps a circuit board, is identified as the cause of failure. In an attempt to repair the system as quickly as possible, the entire subset of components is replaced and the exact failing component may not be investigated further'.

Based on the consideration that the likelihood for a single SRP is the sum of the likelihoods for all possible data configurations that could have led to the observed SRP, Zhang et al. (2017) estimate the lifetime distribution of a component from the aggregated event data for a fleet with multiple identical systems, namely, a set of identical SRP's. Here, by the aggregated data, the authors mean "masked times between failure data". The procedure works well when the number of events for each SRP is relatively small and the number of systems is sufficiently large, but has limitation for SRPs with a large number of events.

3.2.2. Estimation of $M(t)$

Mukhopadhyay & Samuel (2010) propose a Bayesian approach to estimating the expected number of system repairs (i.e, $M(t)$), the system failure rate, and the conditional intensity function by sampling from posterior distributions of the Weibull parameters.

Assuming the underlying lifetime distribution of the components to be identically mixed-exponential and each failure is masked with the same probability p for all components, where p is unknown and can be estimated from field data, Hansen & Thyregod (1990) suggest a least squares technique based method for an SRP to approximate the mean cumulative number of failures in a repairable system and compare its properties with the properties of the maximum likelihood method by means of simulation studies. Their work indicates that for a given type of data, the approximate method seems to provide efficient and almost unbiased estimates. Moreover, to find the parameter estimates, the method only requires a fraction of the time needed for the corresponding maximum likelihood estimation.

4. Approximation and Applications of the SRP

In the reliability literature, the effectiveness of maintenance is classified into the following three categories.

- AGAN (as good as new): If a new, identical item is used to replace a failed item, then the reliability of the new item equals to that of the failed item. That is, the effectiveness of the maintenance (precisely, replacement in this case) is AGAN.
- ABAO (as bad as old): If a maintenance restores a failed item to the status immediately before the item failed, we say the maintenance is an ABAO one or the maintenance is a *minimal* maintenance.
- imperfect: an imperfect maintenance restores the maintained item to a status between AGAN and ABAO.

The renewal process is widely used for modelling the failure process of a socket with the AGAN maintenance effectiveness and the NHPP is for modelling the ABAO. The maintenance effectiveness of an event occurrence in an SRP, on the other hand, is a typical example of the imperfect maintenance. For an SRP with different $F_i(t)$, the effectiveness of replacing failed components in different sockets may be different on the system and can be regarded as a random variable. This makes the SRP different from most existing repair models, including the virtual age process (Kijima & Sumita, 1986), the geometric process (Lam, 1988), and the doubly geometric process (Wu, 2018), in all of which the maintenance effectiveness is deterministic.

4.1. Approximation of the SRP

Due to the complexity of the SRP, to develop approaches to approximating the SRP offers an alternative solution, which has received much attention. Below we simply review approximation methods for solving reliability problems.

Hansen & Thyregod (1990) state that if only masked time between failure data are available, the most common approach is to approximate the SRP model by a NHPP with the same MCF. This approximation is particularly reasonable in the case where either data are sparse, or where the number of sockets is large.

Wu & Scarf (2017) and Wu (2019) assume that the components are heterogeneous in the series system and the causes of the system failures are unknown, or masked. They then propose methods to approximate the SRP for the cases when there are only masked time between failure data available.

Wu & Scarf (2017) propose two models, Model I and Model II, to approximate the failure process of a series system composed of different components. Model I regards the failure process of a series system equivalent to that of a virtual system consisting of two sockets into each of which a virtual component is inserted. Whenever the system fails, replacement occurs at socket 1 and no maintenance is conducted on the virtual component in socket 2. Model II regards the failure process of a series system equivalent to that of a virtual system consisting of a socket and a subsystem. The socket contains one virtual component and the subsystem contains a number of sockets into each of which there is an inserted virtual component. Broadly speaking, whenever the series system fails, replacement occurs at socket 1 and the virtual subsystem is imperfectly repaired.

By taking the work of Wu & Scarf (2017) forward, Wu (2019) proposes other two models. The first model simply regards the failure process of a real-world system equivalent to that of a virtual system composed of a number of different virtual components and the second assumes that those virtual components are identical. In either model, it is assumed: if the real system fails, the oldest virtual component is replaced. According to the comparison between the two models and the nine other existing models on artificially generated data, with respect to the model performance metrics Akaike's information criterion (AIC), corrected Akaike's information criterion (AICc) and Bayesian information criterion (BIC), the second model proposed in his paper outperforms the ten other models. It also outperforms the nine other existing models on fifteen real-world datasets.

To overcome the problem of the data scarcity and that of the unknown failure modes, Wu's method essentially builds a time series model on the failure intensity functions after system failures (Wu, 2019).

4.2. Applications of the SRP in Reliability

Barnett & Kenward (1996, 1998) use the SRP in scheduling inspection policies for alarm security systems. They obtain some theoretical properties of the relevant SRP model for the case of a Poisson alarm process both when the inter-inspection interval is constant and when it takes the form of another independent HPP. These properties are used to motivate the choice of inference procedures for examining the basic nature of the underlying processes. Kim & Kuo (2011) compares two burn-in options: component burn-in and system burn-in, where one of the systems considered is a series system whose failures form an SRP.

5. Conclusions and Future Research

This paper reviewed the existing research on the superimposed renewal process. Most publications are concentrated on its probabilistic properties and statistical inference.

As above-discussed, one can safely regard the maintenance effectiveness of the replacements in the SRP is imperfect on the entire system. In the literature, much research on modelling imperfect maintenance for single-component systems has been published. The SRP is a good model for modelling the failure processes of multi-component systems and deserves more study in the reliability literature. Since real-world systems are normally composed of a limited number of components, simply applying those limiting probabilistic properties may lead to biased results. As such, the following research topics may be of interest.

- How can maintenance policies for a system modelled by the SRP be developed to minimise relevant cost due to failures.
- Sensitivity analysis should be performed on the approaches to approximating an SRP and associated risks may then be analysed.
- Further to the work of Wu & Scarf (2017); Wu (2019), better methods to approximate the failure process of a system with a small number of sockets should be developed.

Further Readings

As above-discussed, the renewal process and the nonhomogeneous Poisson process are the two most commonly used stochastic processes in the reliability literature. In addition to the superposition of renewal processes, there are some other research on the superposition of other stochastic process in the reliability literature.

Zhao & Xie (1994) use the EM algorithm to solve the problem of maximum likelihood estimation for the superposition of nonhomogeneous Poisson process models (see, stat04548, for more discussion on Poisson processes). This result makes it very easy to obtain maximum likelihood estimates of parameters. Weckman et al. (2001) estimate the parameters of the superimposed NHPP process for a series system composed of identical components and illustrate the method with an example of a jet engine application.

Lam (1993) studies the superposition of Markov renewal processes (SMRP) with countable state spaces. The author defines the SMRP equations associated with the superposed process. The solutions of the superposition-Markov renewal equations are derived and the asymptotic behaviours of these solutions are studied.

On publications on the super-imposed renewal process related to the other processes, the reader is referred to Stadje (2012) and Alamilla et al. (2015), for example.

Other relevant and interesting readings include stat04170, stat04174, stat04177, stat04178, stat04179, stat04181, stat04182, stat04183.

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