Cluster algebras and discrete integrability

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March 21, 2019

Abstract

Cluster algebras are a class of commutative algebras whose generators are defined by a recursive process called mutation. We give a brief introduction to cluster algebras, and explain how discrete integrable systems can appear in the context of cluster mutation. In particular, we give examples of birational maps that are integrable in the Liouville sense and arise from cluster algebras with periodicity, as well as examples of discrete Painlevé equations that are derived from Y-systems.

1 Introduction

Cluster algebras are a special class of commutative algebras that were introduced by Fomin and Zelevinsky almost twenty years ago [21], and rapidly became the hottest topic in modern algebra. Rather than being defined a priori by a given set of generators and relations, the generators of a cluster algebra are produced recursively by iteration of a process called mutation. In certain cases, a sequence of mutations in a cluster algebra can correspond to iteration of a birational map, so that a discrete dynamical system is generated. The reason why cluster algebras have attracted so much attention is that cluster mutations and associated discrete dynamical systems or difference equations arise in such a wide variety of contexts, including Teichmuller theory [19, 20], Poisson geometry [29], representation theory [11], and integrable models in statistical mechanics and quantum field theory [14, 33, 66], to name but a few.

The purpose of this review is to give a brief introduction to cluster algebras, and describe certain situations where the associated dynamics is

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completely integrable, in the sense that a discrete version of Liouville's theorem in classical mechanics is valid. Furthermore, within the context of cluster algebras, we will describe a way to detect whether a given discrete system is integrable, based on an associated tropical dynamical system and its connection to the notion of algebraic entropy. Finally, we describe how discrete Painlevé equations can arise in the context of cluster algebras.

2 Cluster algebras: definition and examples

A cluster algebra with coefficients, of rank N, is generated by starting from a seed $(B, \mathbf{x}, \mathbf{y})$ consisting of an *exchange matrix* $B = (b_{ij}) \in \mathrm{Mat}_N(\mathbb{Z})$, an N-tuple of *cluster variables* $\mathbf{x} = (x_1, x_2, \dots, x_N)$, and another N-tuple of *coefficients* $\mathbf{y} = (y_1, y_2, \dots, y_N)$. The exchange matrix is assumed to be skew-symmetrizable, meaning that there is a diagonal matrix D, consisting of positive integers, such that DB is skew-symmetric. For each integer $k \in [1, N]$, there is a mutation μ_k which produces a new seed $(B', \mathbf{x}', \mathbf{y}') = \mu_k(B, \mathbf{x}, \mathbf{y})$. The mutation μ_k consists of three parts: *matrix mutation*, which is applied to B to produce $B' = (b'_{ij}) = \mu_k(B)$, where

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \mathrm{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise}, \end{cases} \tag{1}$$

with $\mathrm{sgn}(a)$ being ± 1 for positive/negative $a \in \mathbb{R}$ and 0 for $a = 0$, and

$$[a]_+ = \max(a, 0);$$

coefficient mutation, defined by $\mathbf{y}' = (y'_j) = \mu_k(\mathbf{y})$ where

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j \left(1 + y_k^{-\mathrm{sgn}(b_{jk})}\right)^{-b_{jk}} & \text{otherwise}; \end{cases} \tag{2}$$

and *cluster mutation*, given by $\mathbf{x}' = (x'_j) = \mu_k(\mathbf{x})$ with the *exchange relation*

$$x'_k = \frac{y_k \prod_{i=1}^N x_i^{[b_{ki}]_+} + \prod_{i=1}^N x_i^{[-b_{ki}]_+}}{(1 + y_k)x_k}, \tag{3}$$

and $x'_j = x_j$ for $j \neq k$.

Given an initial seed, one can apply an arbitrary sequence of mutations, which produces a sequence of seeds. This can be visualized by attaching the initial seed to the root of an N-regular tree $\mathbb{T}_N$ (with N branches attached to each vertex), and then labelling the seeds as $(B_{\mathbf{t}}, \mathbf{x}_{\mathbf{t}}, \mathbf{y}_{\mathbf{t}})$ with "time" $\mathbf{t} \in \mathbb{T}_N$. Note that mutation is an involution, $\mu_k \cdot \mu_k = \mathrm{id}$, but in general two successive mutations do not commute, i.e. typically $\mu_j \cdot \mu_k \neq \mu_k \cdot \mu_j$ for $j \neq k$. Moreover, in general the exponents and coefficients appearing in the exchange relation (3) change at each stage, because the matrix B and the $\mathbf{y}$ variables are altered by each of the previous mutations.

Definition 1. The cluster algebra $\mathcal{A}(B, \mathbf{x}, \mathbf{y})$ is the algebra over $\mathbb{C}(\mathbf{y})$ generated by the cluster variables produced by all possible sequences of mutations applied to the seed $(B, \mathbf{x}, \mathbf{y})$.

We will also consider the case of coefficient-free cluster algebras, for which the $\mathbf{y}$ variables are absent, the seeds are just $(B, \mathbf{x})$, and the cluster mutation is defined by the simpler exchange relation

$$x_k' = \frac{\prod_{i=1}^N x_i^{[b_{ki}]_+} + \prod_{i=1}^N x_i^{[-b_{ki}]_+}}{x_k}. \tag{4}$$

Remark 1. The original definition of a cluster algebra in [21] involves a more general setting in which the coefficients $\mathbf{y}$ are elements of a semifield $\mathbb{P}$, that is, an abelian multiplicative group together with a binary operation $\oplus$ that is commutative, associative and distributive with respect to multiplication. In that setting, with the N-tuple $\mathbf{y} \in \mathbb{P}^N$, the algebra $\mathcal{A}(B, \mathbf{x}, \mathbf{y})$ is defined over $\mathbb{Z}[\mathbb{P}]$, and the addition in the denominator of (3) is given by $\oplus$. The case we consider here corresponds to $\mathbb{P} = \mathbb{P}_{univ}$, the universal semifield, consisting of subtraction-free rational functions in the variables y_j, in which case $\oplus$ becomes ordinary addition in the field of rational functions $\mathbb{C}(\mathbf{y})$. However, starting with the more general setting, we can also consider the case of the trivial semifield with one element, $\mathbb{P} = \{1\}$, which yields the coefficient-free case (4).

In order to illustrate the above definitions, we now present a number of concrete examples. For the sake of simplicity, we concentrate on the coefficient-free case in the rest of this section, and return to the equations with coefficients $\mathbf{y}$ at a later stage.

Example 1. The cluster algebra of type B_2: A particular cluster algebra of rank $N = 2$ is given by taking the exchange matrix

$$B = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}, \tag{5}$$

and the initial cluster $\mathbf{x} = (x_1, x_2)$, to define a seed $(B, \mathbf{x})$. The matrix B is skew-symmetrizable: the diagonal matrix $D = \mathrm{diag}(1, 2)$ is such that

$$DB = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

is skew-symmetric. Applying the mutation μ_1 and using the rule (1) gives a new exchange matrix

$$B' = \mu_1(B) = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} = -B,$$

while the coefficient-free exchange relation (4) gives a new cluster $\mathbf{x}' = (x_1', x_2)$ with

$$x_1' = \frac{x_2^2 + 1}{x_1}.$$

Since mutation acts as an involution, we have $\mu_1(B', \mathbf{x}') = (B, \mathbf{x})$, so nothing new is obtained by applying μ_1 to this new seed. Thus we consider $\mu_2(B', \mathbf{x}') = \mu_2 \cdot \mu_1(B, \mathbf{x})$ instead, which produces $\mu_2(B') = B$ and $\mu_2(\mathbf{x}') = (x_1', x_2')$, where

$$x_2' = \frac{x_1' + 1}{x_2} = \frac{x_1 + x_2^2 + 1}{x_1 x_2}.$$

Once again, a repeat application of the same mutation μ_2 returns to the previous seed, so instead we consider applying μ_1 to obtain $\mu_1 \cdot \mu_2 \cdot \mu_1(B) = -B$ and $\mu_1 \cdot \mu_2 \cdot \mu_1(\mathbf{x}) = (x_1'', x_2')$, with

$$x_1'' = \frac{x_1^2 + 2x_1 + x_2^2 + 1}{x_1 x_2^2}.$$

Repeating this sequence of mutations, it is clear that the exchange matrix just changes by an overall sign at each step. Perhaps more surprising is the fact that after obtaining $(\mu_2 \cdot \mu_1)^2(\mathbf{x}) = (x_1'', x_2'')$, with

$$x_2'' = \frac{x_1 + 1}{x_2},$$

the variable x_1 reappears in the cluster after a further step, i.e. $\mu_1 \cdot (\mu_2 \cdot \mu_1)^2(\mathbf{x}) = (x_1, x_2'')$, and finally $(\mu_2 \cdot \mu_1)^3(\mathbf{x}) = (x_1, x_2) = \mathbf{x}$, so that the initial seed $(B, \mathbf{x})$ is restored after a total of six mutations. Thus the cluster algebra has a finite number of generators in this case, since there are only the six cluster variables $x_1, x_2, x_1', x_2', x_1'', x_2''$. This example is called the cluster

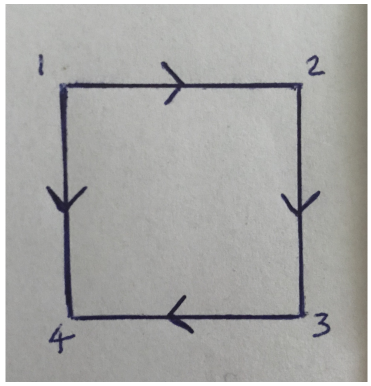

Figure 1: The quiver Q corresponding to the exchange matrix (6).

algebra of type B_2, since the initial matrix B is derived from the Cartan matrix of the B_2 root system, that is

$$C = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix},$$

by replacing the diagonal entries in C with 0, and changing signs of the off-diagonal entries so that b_{ij} and b_{ji} have opposite signs for $i \neq j$.

There are two significant features of the preceding example, namely the fact that there are only finitely many clusters, and the fact that the cluster variables are all Laurent polynomials (polynomials in x_1, x_2 and their reciprocals) with integer coefficients. The first feature is rare: a cluster algebra is said to be of finite type if there are only finitely many clusters, and it was shown in [23] that all such cluster algebras are generated from seeds corresponding to the finite root systems that appear in the Cartan-Killing classification of finite-dimensional semisimple Lie algebras. The second feature (the Laurent phenomenon) is ubiquitous [22], and follows from the following result, proved in [22].

Proposition 1. *All cluster variables in a coefficient-free cluster algebra $\mathcal{A}(B, \mathbf{x})$ are Laurent polynomials in the variables from the initial cluster, with integer coefficients, i.e. they are elements of the ring of Laurent polynomials, that is $\mathbb{Z}[\mathbf{x}^{\pm 1}] := \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_N^{\pm 1}]$.*

There is an analogous statement in the case that coefficients are included, and in fact it is possible to prove the stronger result that all of the coefficients of the cluster variables have positive integer coefficients, so they belong to $\mathbb{Z}_{>0}[\mathbf{x}^{\pm 1}]$ (see [35, 52], for instance).

Example 2. The cluster algebra of type $\tilde{A}_{1,3}$: As an example of rank $N = 4$, we take the skew-symmetric matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \tag{6}$$

which is obtained from the Cartan matrix of the affine root system $A_3^{(1)}$ [48], namely

$$C = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix},$$

by replacing each of the diagonal entries of C with 0, and making a suitable adjustment of signs for the off-diagonal entries, such that $b_{ij} = -b_{ji}$. Since B is a skew-symmetric integer matrix, it can be associated with a quiver Q without 1- or 2-cycles, that is, a directed graph specified by the rule that b_{ij} is equal to the number of arrows $i \to j$ if it is non-negative, and minus the number of arrows $j \to i$ otherwise (see Fig.1). If the mutation μ_1 is applied, then the new exchange matrix is

$$B' = \mu_1(B) = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix},$$

which corresponds to a new quiver Q' obtained by a cyclic permutation of the vertices of the original Q (see Fig.2), while the initial cluster $\mathbf{x} = (x_1, x_2, x_3, x_4)$ is mutated to $\mu_1(\mathbf{x}) = (x_1', x_2, x_3, x_4)$, where x_1' is defined by the relation

$$x_1 x_1' = x_2 x_4 + 1.$$

Rather than trying to describe the effect of every possible choice of mutation, we consider what happens when μ_1 is followed by μ_2, and once more observe that, at the level of the associated quiver, this just corresponds to applying the same cyclic permutation as before to the vertex labels 1,2,3,4. The new cluster obtained from this is $\mu_2 \cdot \mu_1(\mathbf{x}) = (x_1', x_2', x_3, x_4)$, with x_2' defined by

$$x_2 x_2' = x_3 x_1' + 1,$$

and if μ_3 is applied next, then $\mu_3 \cdot \mu_2 \cdot \mu_1(\mathbf{x}) = (x_1', x_2', x_3', x_4)$, with

$$x_3 x_3' = x_4 x_2' + 1.$$

Continuing in this way, it is not hard to see that the composition $\mu_4 \cdot \mu_3 \cdot \mu_2 \cdot \mu_1$ takes the original B to itself, and applying this sequence of mutations repeatedly in the same order generates a new cluster variable at each step, with the sequence of cluster variables satisfying the nonlinear recurrence relation

$$x_n x_{n+4} = x_{n+1} x_{n+3} + 1 \tag{7}$$

(where we have made the identification $x_1' = x_5$, $x_2' = x_6$, and so on). Regardless of other possible choices of mutations, this particular sequence of mutations alone generates an infinite set of distinct cluster variables, as can be seen by fixing some numerical values for the initial cluster. In fact, as was noted in [42], for any orbit of (7) there is a constant K such that the iterates satisfy the linear recurrence

$$x_{n+6} + x_n = K x_{n+3}. \tag{8}$$

Upon fixing $(x_1, x_2, x_3, x_4) = (1, 1, 1, 1)$, the nonlinear recurrence generates the integer sequence

$$1, 1, 1, 1, 2, 3, 4, 9, 14, 19, 43, 76, \ldots,$$

which also satisfies the linear recurrence (8) with $K = 5$; so the terms grow exponentially with n, and the integers x_n are distinct for $n \geq 4$. This is called the $\tilde{A}_{1,3}$ cluster algebra, because the corresponding quiver is an orientation of the edges of an affine Dynkin diagram of type A with one anticlockwise arrow and three clockwise arrows.

The skew-symmetry of B is preserved under matrix mutation, and for any skew-symmetric integer matrix there is an equivalent operation of *quiver mutation* which acts on the associated quiver Q: to obtain the mutated quiver $\mu_k(Q)$ one should (i) add pq arrows $i \xrightarrow{pq} j$ whenever Q has a path of length two passing through vertex k with p arrows $i \xrightarrow{p} k$ and q arrows $k \xrightarrow{q} j$; (ii) reverse all arrows in Q that go in/out of vertex k; (iii) delete any 2-cycles created in the first step.

Unlike the B_2 cluster algebra, the above example is not of finite type, because there are infinitely many clusters. However, it turns out that it is of finite mutation type, in the sense that there are only a finite number of exchange matrices produced under mutation from the initial B. Cluster

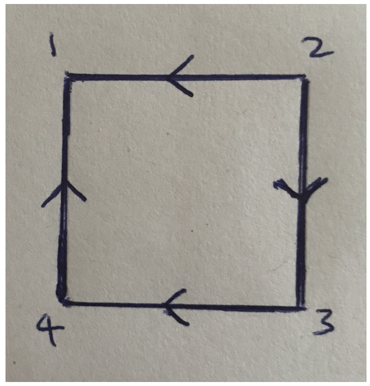

Figure 2: The mutated quiver $Q' = \mu_1(Q)$ obtained by applying μ_1 to (6).

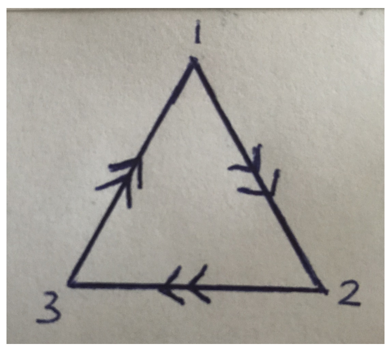

Figure 3: The quiver corresponding to the exchange matrix (9).

algebras of finite mutation type have also been classified [16, 17]: as well as those of finite type, they include cluster algebras associated with triangulated surfaces [19, 20], cluster algebras of rank 2, plus a finite number of exceptional cases.

Example 3. Cluster algebra related to Markoff's equation: For $N = 3$, consider the exchange matrix

$$B = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}, \tag{9}$$

which is associated with the quiver in Fig.3. After any sequence of matrix mutations, one can obtain only B or $-B$, so this is another example of finite mutation type: it is connected to the moduli space of once-punctured tori, and the Markoff equation

$$x^2 + y^2 + z^2 = 3xyz \tag{10}$$

which arises in that context as well as in Diophantine approximation theory [7, 9]. Upon applying μ_1 to the initial cluster (x_1, x_2, x_3), the result is (x_1', x_2, x_3) with

$$x_1 x_1' = x_2^2 + x_3^2,$$

and a subsequent application of μ_2 yields (x_1', x_2', x_3), where

$$x_2 x_2' = x_3^2 + x_1'^2.$$

Repeated application of the mutations $\mu_3 \cdot \mu_2 \cdot \mu_1$ in that order produces a new cluster variable at each step, and upon identifying $x_4 = x_1'$, $x_5 = x_2'$, and so on, the sequence of cluster variables (x_n) is generated by a recurrence of third order, namely

$$x_n x_{n+3} = x_{n+1}^2 + x_{n+2}^2. \tag{11}$$

It can also be shown that on each orbit of (11) there is a constant K such that the nonlinear relation

$$x_{n+3} + x_n = K x_{n+1} x_{n+2}$$

holds for all n, and by using the latter to eliminate x_{n+3} it follows that

$$K = \frac{x_n^2 + x_{n+1}^2 + x_{n+2}^2}{x_n x_{n+1} x_{n+2}} \tag{12}$$

is an invariant for (11), independent of n. In particular, taking the initial values to be $(1, 1, 1)$ gives $K = 3$, and each adjacent triple $(x, y, z) = (x_n, x_{n+1}, x_{n+2})$ in the resulting sequence

$$1, 1, 1, 2, 5, 29, 433, 37666, 48928105, \ldots \tag{13}$$

is an integer solution of Markoff's equation (10). The terms of this sequence have double exponential growth: $\log x_n$ grows exponentially with n.

The next example is generic, in the sense that there are both infinitely many clusters and infinitely many exchange matrices.

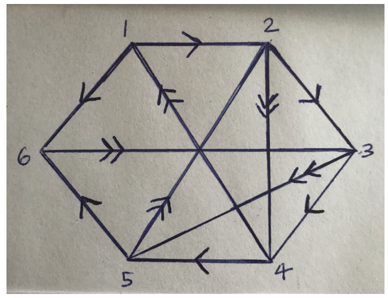

Figure 4: The Somos-6 quiver corresponding to the exchange matrix (14).

Example 4. A Somos-6 recurrence: A sequence that is generated by a quadratic recurrence relation of the form

$$x_n x_{n+k} = \sum_{j=1}^{\lfloor k/2 \rfloor} \alpha_j \, x_{n+j} x_{n+k-j},$$

where α_j are coefficients, is called a Somos-k sequence (see [22, 28, 40, 62, 63]). A certain class of Somos-6 sequences can be generated by starting from the exchange matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & -2 & 0 & 1 \\ -1 & 0 & 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 & 2 & -2 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 0 & 2 & -2 & -1 & 0 & 1 \\ -1 & 0 & 2 & 0 & -1 & 0 \end{pmatrix}, \tag{14}$$

which corresponds to the quiver in Fig.4. Upon applying cyclic sequences of mutations ordered as $\mu_6 \cdot \mu_5 \cdot \mu_4 \cdot \mu_3 \cdot \mu_2 \cdot \mu_1$, a sequence of cluster variables (x_n) is produced which satisfies the particular Somos-6 recurrence

$$x_n x_{n+6} = x_{n+1} x_{n+5} + x_{n+3}^2. \tag{15}$$

If six 1s are chosen as initial values, then an integer Somos-6 sequence beginning with

$$1, 1, 1, 1, 1, 1, 2, 3, 4, 8, 17, 50, 107, 239, \ldots$$

is produced. For this sequence, $\log x_n$ grows like n^2. However, applying successive mutations other than these cyclic ones generally causes the magnitude of the entries of the exchange matrices to grow - for instance,

$$\mu_5 \cdot \mu_4 \cdot \mu_2(B) = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -3 & -10 & 4 & 0 \\ -1 & 3 & 0 & -1 & 0 & -2 \\ 0 & 10 & 1 & 0 & -3 & 3 \\ 0 & -4 & 0 & 3 & 0 & -1 \\ -1 & 0 & 2 & -3 & 1 & 0 \end{pmatrix};$$

and (e.g. starting with the initial seed evaluated as $(1, 1, 1, 1, 1, 1)$ as before) typically this results in the values of cluster variables showing double exponential growth with the number of steps.

3 Cluster algebras with periodicity

The exchange relation (3) can be regarded as a birational map in $\mathbb{C}^N$. Alternatively, $\mathbf{x} \in (\mathbb{C}^*)^N$ can be viewed as coordinates in a toric chart for some algebraic variety, and a mutation $\mathbf{x} \mapsto \mu_k(\mathbf{x}) = \mathbf{x}'$ as a change of coordinates to another chart. The latter point of view is passive, in the sense that there is some fixed variety and mutation just selects different choices of coordinate charts. Instead of this, we would like to take an active view, regarding each mutation as an iteration in a discrete dynamical system. However, there is a problem with this, because a general sequence of mutations is specified by a "time" $\mathbf{t}$ belonging to the tree $\mathbb{T}_N$, and (except for the case of rank $N = 2$) this cannot naturally be identified with a discrete time belonging to the set of integers $\mathbb{Z}$. Furthermore, there is the additional problem that matrix mutation, as in (1), typically changes the exponents appearing in the exchange relation, so that in general it is not possible to interpret successive mutations as iterations of the same map.

Despite the above comments, it turns out that the most interesting cluster algebras appearing "in nature" have special symmetries, in the sense that they display periodic behaviour with respect to at least some subset of the possible mutations. In fact, all of the examples in the previous section are of this kind. Here we consider a notion of periodicity that was introduced by Fordy and Marsh [25] in the context of skew-symmetric exchange matrices B, which correspond to quivers.

Definition 2. An exchange matrix B is said to be *cluster mutation-periodic* with period m if (for a suitable labelling of indices) $\mu_m \cdot \mu_{m-1} \cdot \ldots \cdot \mu_1(B) = \rho^m(B)$, where ρ is the cyclic permutation

$$\rho : (1, 2, 3, \ldots, N) \mapsto (N, 1, 2, \ldots, N-1).$$

In the context of quiver mutation, the case of cluster mutation-periodicity with period $m = 1$ means that the action of mutation μ_1 on Q is the same as the action of ρ, which is such that the number of arrows $i \to j$ in Q is the same as the number of arrows $\rho^{-1}(i) \to \rho^{-1}(j)$ in $\rho(Q)$. This means that the cluster map $\varphi = \rho^{-1} \cdot \mu_1$ acts as the identity on Q (or equivalently, on B), but in general $\mathbf{x} \mapsto \varphi(\mathbf{x})$ has a non-trivial action on the cluster. Mutation-periodicity with period 1 implies that iterating this map is equivalent to iterating a single recurrence relation.

Example 5. Although it is not skew-symmetric, the exchange matrix (5) in Example 1 is cluster mutation-periodic with period 2, since $\mu_2 \cdot \mu_1(B) = \rho^2(B) = B$ where ρ is the switch $1 \leftrightarrow 2$. Defining the cluster map to be $\varphi = \rho^{-2} \cdot \mu_2 \cdot \mu_1$, the action of φ is periodic with period 3 for any choice of initial cluster, i.e. $\varphi^3 =$id.

Example 6. The exchange matrix (6) in Example 2 is cluster mutation-periodic with period 1. The cluster map $\varphi = \rho^{-1} \cdot \mu_1$ is given by

$$\varphi: \ (x_1, x_2, x_3, x_4) \mapsto \left(x_2, x_3, x_4, \frac{x_2 x_4 + 1}{x_1} \right), \tag{16}$$

whose iterates are equivalent to those of the nonlinear recurrence (7).

Example 7. The exchange matrix (9) in Example 3 is cluster mutation-periodic with period 2. The cluster map $\varphi = \rho^{-2} \cdot \mu_2 \cdot \mu_1$ is given by

$$\varphi: \ (x_1, x_2, x_3) \mapsto \left(x_3, x_1', \frac{x_3^2 + x_1'^2}{x_2} \right), \quad \text{with} \quad x_1' = \frac{x_2^2 + x_3^2}{x_1}. \tag{17}$$

Each iteration of (17) is equivalent to two iterations of the nonlinear recurrence (11). This period 2 example is exceptional because, in general, cluster mutation-periodicity with period $m > 2$ does not give rise to a single recurrence relation (see [25] for more examples).

Example 8. The exchange matrix (14) in Example 4 is cluster mutation-periodic with period 1. The cluster map is given by

$$\varphi: \ (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto \left(x_2, x_3, x_4, x_5, x_6, \frac{x_2 x_6 + x_4^2}{x_1} \right), \tag{18}$$

whose iterates are equivalent to those of the nonlinear recurrence (15).

Remark 2. There is a more general notion of periodicity, due to Nakanishi [57], which extends Definition 2. This yields broad generalizations of Zamolodchikov's Y-systems [66], a set of functional relations, arising from the thermodynamic Bethe ansatz for certain integrable quantum field theories, that were the prototype for the coefficient mutation (2) in a cluster algebra. We shall introduce examples of generalized Y-systems in the sequel.

Fordy and Marsh gave a complete classification of period 1 quivers. Their result can be paraphrased as follows.

Theorem 1. *Let $(a_1,\ldots,a_{N-1})$ an $(N-1)$-tuple of integers that is palindromic, i.e. $a_j = a_{N-j}$ for all $j \in [1, N-1]$. Then the skew-symmetric exchange matrix $B = (b_{ij})$ with entries specified by*

$$b_{1,j+1} = a_j \quad \textit{and} \quad b_{i+1,j+1} = b_{ij} + a_i[-a_j]_+ - a_j[-a_i]_+,$$

for all $i, j \in [1, N-1]$, is cluster mutation-periodic with period 1, and every period 1 skew-symmetric B arises in this way.

The above result says that a period 1 skew-symmetric B matrix is completely determined by the entries in its first row (or equivalently, its first column), and these form a palindrome after removing b_{11}. The entries a_j in the palindrome are precisely the exponents that appear in the exchange relation defining the cluster map φ, whose iterates are equivalent to those of the nonlinear recurrence relation

$$x_n x_{n+N} = \prod_{j:\, a_j>0} x_{n+j}^{a_j} + \prod_{j:\, a_j<0} x_{n+j}^{-a_j}. \tag{19}$$

Thus (19) corresponds to a special sequence of mutations in a particular subclass of cluster algebras. Such a nonlinear recurrence is an example of a generalized T-system, in the terminology of [57].

Next we would like to turn to the question of which recurrences of this special type correspond to discrete integrable systems. We begin our approach to this question in the next section, by considering the notion of algebraic entropy, which gives a measure of the growth of iterates in a discrete dynamical system defined by iteration of rational functions.

4 Algebraic entropy and tropical dynamics

There are various different ways of quantifying the growth, or complexity, of a discrete dynamical system (see [1], for instance). In the context of discrete integrability of birational maps, Bellon and Viallet introduced the concept of algebraic entropy, and proposed that zero algebraic entropy should be a criterion for integrability [4]. For a birational map φ, one can calculate the degree $d_n = \deg \varphi^n$, given by the maximum of the degrees of the components of the map φ^n, and then the algebraic entropy is defined to be

$$\mathcal{E} := \lim_{n\to\infty} \frac{\log d_n}{n}.$$

Typically, the degree d_n grows exponentially with n, so $\mathcal{E} > 0$, but in rare cases there can be subexponential growth, leading to vanishing entropy. In the case of birational maps in two dimensions, the types of degree growth have been fully classified [12], and there are only four possibilities: bounded degrees, linear growth, quadratic growth, or exponential growth; the first three cases, with zero entropy, coincide with the existence of invariant foliations. Thus, at least for maps of the plane, the requirement of zero entropy identifies symplectic maps that are integrable in the sense that they satisfy the conditions needed for a discrete analogue of the Liouville-Arnold theorem to hold [6, 53, 65].

Measuring the degree growth and seeking maps with zero algebraic entropy is a useful tool for identifying discrete integrable systems. (For another approach, based on the growth of heights in orbits defined over $\mathbb{Q}$ or a number field, see [36].) Once such a map has been identified, it leaves open the question of Liouville integrability; this is discussed in the next section. For now, we concentrate on the case of maps arising from cluster algebras, and consider algebraic entropy in that setting.

The advantage of working with cluster maps is that, due to the Laurent property, it is sufficient to consider the growth of degrees of the denominators of the cluster variables in order to determine the algebraic entropy. In particular, in the period 1 case, by Proposition 1 every iterate of (19) can be written in the form

$$x_n = \frac{\mathrm{P}_n(\mathbf{x})}{\mathbf{x}^{\mathbf{d}_n}}, \tag{20}$$

where the polynomial P_n is not divisible by any of the x_j from the initial cluster, and the monomial $\mathbf{x}^{\mathbf{d}_n} = \prod_{j=1}^N x_j^{d_n^{(j)}}$ is specified by the integer vector

$$\mathbf{d}_n = (d_n^{(1)}, \ldots, d_n^{(N)})^T,$$

known as a d-vector. From the fact that cluster variables are subtraction-free rational expressions in $\mathbf{x}$ (or, a fortiori, from the fact that these Laurent polynomials are now known to have positive integer coefficients [35]), it follows that the d-vectors in a cluster algebra satisfy the max-plus tropical analogue of the exchange relations for the corresponding cluster variables [19, 24], where the latter is obtained from (4) by replacing each addition with max, and each multiplication with addition. In the case of (19), this implies the following result.

Proposition 2. *If x_n given by (20) satisfies (19), then the sequence of vectors $\mathbf{d}_n$ satisfies the tropical recurrence relation*

$$\mathbf{d}_n + \mathbf{d}_{n+N} = \max\left(\sum_{j:\, a_j>0} a_j \mathbf{d}_{n+j}, - \sum_{j:\, a_j<0} a_j \mathbf{d}_{n+j} \right). \tag{21}$$

Note that the equality in (21) holds componentwise. For a detailed proof of this result, see [26].

In the period 1 situation, the problem of determining the evolution of d-vectors can be simplified further, upon noting that the first component of $\mathbf{d}_n$ has the initial values

$$d_1^{(1)} = -1, \qquad d_j^{(1)} = 0 \quad \text{for} \quad 2 \leq j \leq N, \tag{22}$$

while each of the other components $d_n^{(k)}$ for $k \in [2, N]$ has the same set of initial values but shifted by $k-1$ steps (so, for instance, $d_1^{(2)} = 0$, $d_2^{(2)} = -1$ and then $d_j^{(2)} = 0$ for $3 \leq j \leq N+1$, since the first division by the variable x_2 appears in x_{N+2}, etc.). The total degree of the monomial $\mathbf{x}^{\mathbf{d}_n}$ is the sum $\sum_{k=1}^{N} d_n^{(k)}$ of the components of the d-vector, and if the components are all non-negative then this coincides with the degree of the denominator of the rational function (20). Unless there is periodicity of d-vectors, corresponding to degrees remaining bounded (which can only happen in finite type cases like Example 1), then all these components are positive for large enough n. Moreover, it is not hard to see that the growth of the degree of the numerators P_n appearing in the Laurent polynomials (20) is controlled by that of the denominators. Thus, to determine the growth of degrees of Laurent polynomials generated by (19), it is sufficient to consider the solution of the scalar version of (21), with initial data given by (22), and the growth of this determines the algebraic entropy.

Example 9. For the recurrence (7) in Example 2, the tropical equation for determining the degrees of d-vectors is given in scalar form by

$$d_n + d_{n+4} = \max(d_{n+1} + d_{n+3}, 0). \tag{23}$$

If we take initial values $d_1 = -1$, $d_2 = d_3 = d_4 = 0$, corresponding to (22), then by induction it follows that $d_n \geq 0$ for all $n \geq 2$, so that the max on the right-hand side of (23) can be replaced by its first entry, to yield the linear recurrence

$$d_n + d_{n+4} = d_{n+1} + d_{n+3} \quad \text{for} \quad n \geq 1.$$

The characteristic polynomial of the latter factorizes as $(\lambda-1)^2(\lambda^2+\lambda+1) = 0$, leading to the solution

$$d_n = n/3 - 1 + \frac{\mathrm{i}}{3\sqrt{3}}(\varepsilon^n - \varepsilon^{-n}), \quad \varepsilon = (-1 + \mathrm{i}\sqrt{3})/2.$$

Thus we have a sequence that grows linearly with n, beginning with

$$-1, 0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, \ldots,$$

where each positive integer appears three times in succession, which corresponds to the degree of the denominator of x_n in each of the variables x_1, x_2, x_3, x_4 separately. Clearly the total degree of the denominator also grows linearly, and the algebraic entropy is $\lim_{n\to\infty}(\log d_n)/n = 0$ in this case.

Example 10. The exchange matrix (9) in Example 3 is period 2 rather than period 1, but we can still calculate the growth of d-vectors in the recurrence (11) by taking its tropical version, namely

$$d_n + d_{n+3} = 2\,\max(d_{n+1}, d_{n+2}),$$

and choosing the initial values $d_1 = -1$, $d_2 = d_3 = 0$, which produces a sequence beginning

$$-1, 0, 0, 1, 2, 4, 7, 12, 20, 33, 54, 88, \ldots.$$

By induction one can show that $d_{n+2} \geq d_{n+1}$ for $n \geq 0$, so in fact the linear recurrence

$$d_{n+3} + d_n = 2d_{n+2}$$

holds for this sequence, with characteristic equation $(\lambda - 1)(\lambda^2 - \lambda - 1) = 0$, and it turns out that the differences

$$F_n = d_{n+3} - d_{n+2}$$

are just the Fibonacci numbers. Hence there is a constant $\mathrm{C} > 0$ such that

$$d_n \sim \mathrm{C} \left(\frac{1+\sqrt{5}}{2} \right)^n ,$$

and the algebraic entropy $\mathcal{E} = \log((1+\sqrt{5})/2)$, which is the same as the limit $\lim_{n\to\infty} (\log \log x_n)/n$ for the sequence (13) - see [39].

Example 11. For the period 1 exchange matrix (14) in Example 4, we consider the recurrence

$$d_n + d_{n+6} = \max(d_{n+1} + d_{n+5}, 2d_{n+3}), \tag{24}$$

which is the max-plus analogue of (15), and take initial data

$$d_1 = -1, \quad d_2 = d_3 = \cdots = d_6 = 0, \tag{25}$$

which generates a degree sequence beginning

$$-1, 0, 0, 0, 0, 0, 1, 1, 1, 2, 2, 3, 3, 3, 5, 5, 6, 7, 7, 9, 9, 10, 12, 12, 14, 15, 16, \ldots. \tag{26}$$

In order to simplify the analysis of (24), we observe that the combination

$$U_n = d_{n+2} - 2d_{n+1} + d_n \tag{27}$$

satisfies a recurrence of fourth order, namely

$$U_{n+4} + 2U_{n+3} + 3U_{n+2} + 2U_{n+1} + U_n = \max(U_{n+3} + 2U_{n+2} + U_{n+1}, 0). \tag{28}$$

(The origin of the substitution (27) will be explained in the next section.) The values in (25) correspond to the initial conditions

$$U_1 = -1, \quad U_2 = U_3 = U_4 = 0$$

for (28), which generate a sequence (U_n) beginning with

$$\begin{aligned} &-1, 0, 0, 0, 1, -1, 0, 1, -1, 1, -1, 0, 2, -2, 1, 0, -1, 2, -2, 1, 1, -2, \\ &2, -1, 0, 1, -2, \ldots, \end{aligned}$$

and further calculation with a computer shows that this sequence does not repeat for the first 40 steps, but then $U_{42} = -1$ and $U_{43} = U_{44} = U_{45} = 0$, so it is periodic with period 41. Thus in terms of the shift operator $\mathcal{S}$, which sends $n \to n + 1$,

$$(\mathcal{S}^{41} - 1)U_n = (\mathcal{S}^{41} - 1)(d_{n+2} - 2d_{n+1} + d_n) = (\mathcal{S}^{41} - 1)(\mathcal{S} - 1)^2 d_n = 0,$$

which is a linear recurrence of order 43 satisfied by the degree sequence (26). Clearly the characteristic polynomial of the latter has $\lambda = 1$ as a triple root, and all other characteristic roots have modulus 1. Therefore, for some constant $\mathrm{C}' > 0$,

$$d_n \sim \mathrm{C}' n^2$$

as $n \to \infty$, which implies that (15) has algebraic entropy $\mathcal{E} = 0$.

The preceding examples indicate that we should regard (7) and (15) as being integrable in some sense, and (11) as non-integrable. According to the relation (8), we know that (7) has at least one conserved quantity K; and it turns out to have three independent conserved quantities [42]. The recurrence (11) also has a conserved quantity, given by (12), but it is possible to show that it can have no other algebraic conserved quantitites, independent of this one. In the next section we will derive two independent conserved quantities for (15), and we will discuss the interpretation of all these examples from the viewpoint of Liouville integrability.

In [26], a detailed analysis of the behaviour of the tropical recurrences (21) led to the conjecture that the algebraic entropy of (19) should be positive if and only if the following condition holds:

$$\max\left(\sum_{j=1}^{N-1}[a_j]_+, -\sum_{j=1}^{N-1}[-a_j]_+\right) \geq 3. \tag{29}$$

In other words, in order for the cluster map defined by (19) to have a zero entropy, the degree of nonlinearity cannot be too large. The analysis of algebraic entropy for other types of cluster maps has been carried out more recently using methods based on Newton polytopes [27], and using the same methods it is also possible to prove the above conjecture[1]. By enumerating the possible choices of exponents that lie below the bound (29), this leads to a complete proof of a classification result for nonlinear recurrences of the form (19), as stated in [26].

Theorem 2. *A cluster map φ given by a recurrence (19) has algebraic entropy $\mathcal{E} = 0$ if and only if it belongs to one of the following four families:*
(i) For even $N = 2m$, recurrences of the form

$$x_n x_{n+2m} = x_{n+m} + 1. \tag{30}$$

[1]P. Galashin, private communication, 2017

(ii) For $N \geq 2$ and $1 \leq q \leq \lfloor N/2 \rfloor$, recurrences of the form

$$x_n x_{n+N} = x_{n+q} x_{n+N-q} + 1. \tag{31}$$

(iii) For even $N = 2m$ and $1 \leq q \leq m-1$, recurrences of the form

$$x_n x_{n+2m} = x_{n+q} x_{n+2m-q} + x_{n+m}. \tag{32}$$

(iv) For $N \geq 2$ and $1 \leq p < q \leq \lfloor N/2 \rfloor$, recurrences of the form

$$x_n x_{n+N} = x_{n+p} x_{n+N-p} + x_{n+q} x_{n+N-q}. \tag{33}$$

Case (i) is somewhat trivial: the recurrence (30) is equivalent to taking m copies of the Lyness 5-cycle

$$x_n x_{n+2} = x_{n+1} + 1,$$

for which every orbit has period 5, corresponding to the cluster algebra of finite type associated with the root system A_2; so in this case the dynamics is purely periodic and there is no degree growth. Both case (ii), which corresponds to affine quivers of type $\tilde{A}_{q,N-q}$, and case (iii) display linear degree growth, similar to Example 9. Case (iv) consists of Somos-N recurrences, which display quadratic degree growth [56], as in Example 11. Hence only zero, linear, quadratic or exponential growth is displayed by the cluster recurrences (19). Interestingly, these are the only types of growth found in the other families of cluster maps considered in [27]. We do not know if other types of growth are possible; are there cluster maps with cubic degree growth, for instance?

5 Poisson and symplectic structures

So far we have alluded to the concept of integrability, but have skirted around the issue of giving a precise definition of what it means for a map to be integrable. An expected feature of integrability is the ability to find explicit solutions of the equations being considered; the recurrence (7) displays this feature, because all of its iterates satisfy a linear recurrence of the form (8), which can be solved exactly. There are many other criteria that can be imposed: existence of sufficiently many conserved quantities or symmetries, or compatibility of an associated linear system (Lax pair), for instance; and not all of these requirements may be appropriate in different circumstances. It is an unfortunate fact that the definition of an integrable system varies depending on the context, i.e. whether it be autonomous or non-autonomous ordinary differential equations, partial differential equations, difference equations, maps or something else that is being considered. Thus we need to address this problem and clarify the context, in order to specify what integrability means for maps associated with cluster algebras.

There is a precise definition of Liouville integrability in the context of finite-dimensional Hamiltonian mechanics, on a real symplectic manifold M of dimension $2m$, with associated Poisson bracket $\{\, ,\, \}$: given a particular function H, the Hamiltonian flow generated by H is completely integrable, in the sense of Liouville, if there exist m independent functions on M (including the Hamiltonian), say $H_1 = H$, $H_2, \ldots, H_m$, which are in involution with respect to the Poisson bracket, i.e. $\{H_j, H_k\} = 0$ for all j, k. In the context of classical mechanics, this notion of integrability provides everything one could hope for. To begin with, systems satisfying these requirements have (at least) m independent conserved quantities: all of the first integrals $H_1, \ldots, H_m$ are preserved by the time evolution, so each of the trajectories lies on an m-dimensional intersection of level sets for these functions. Furthermore, Liouville proved that the solution of the equations of motion for such systems can be reduced to a finite number of quadratures, so they really are "able to be integrated" as one would expect; and Arnold showed in addition that the flow reduces to quasiperiodic motion on compact m-dimensional level sets, which are diffeomorphic to tori T^m [2], so nowadays the combined result is referred to as the Liouville-Arnold theorem. Another approach to integrability is to require a sufficient number of symmetries, and this is a consequence of the Liouville definition: the Hamilton's equations arising from H have the maximum number of commuting symmetries, namely the flows generated by each of the first integrals H_j.

The notion of Liouville integrability can be extended to symplectic maps in a natural way [6, 53, 65]. However, the requirement of working in even dimensions is too restrictive for our purposes, so instead of a symplectic form we start with a (possibly degenerate) Poisson structure and consider Poisson maps φ, defined in terms of the pullback of functions, given by $\varphi^* F = F \cdot \varphi$.

Definition 3. Given a Poisson bracket $\{\, ,\}$ on a manifold M, a map $\varphi : M \to M$ is called a Poisson map if

$$\{\varphi^* F, \varphi^* G\} = \varphi^* \{F, G\}$$

holds for all functions F, G on M.

(We are being deliberately vague about what sort of Poisson manifold $(M, \{,\})$ is being considered, e.g. a real smooth manifold, or a complex algebraic variety, and what sort of functions, e.g. smooth/analytic/rational, because this may vary according to the context.)

In order to have a suitable notion of integrability for cluster maps, we first require a compatible Poisson structure of some kind. In general, given a difference equation or map, there is no canonical way to find a compatible Poisson bracket. Fortunately, it turns out that for cluster algebras there is often a natural Poisson bracket, of log-canonical type, that is compatible with cluster mutations; and there is always a log-canonical presymplectic form [18, 29, 30, 47].

Example 12. Somos-5 Poisson bracket: The skew-symmetric exchange matrix

$$\begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ -1 & 0 & 2 & 0 & -1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ -1 & 1 & 1 & -1 & 0 \end{pmatrix} \tag{34}$$

is cluster mutation-periodic with period 1. Its associated cluster map is a Somos-5 recurrence, which belongs to family (iv) above, given by (33) with $N = 5$, $p = 1$, $q = 2$. The skew-symmetric matrix $P = (p_{ij})$ given by

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 & 3 \\ -2 & -1 & 0 & 1 & 2 \\ -3 & -2 & -1 & 0 & 1 \\ -4 & -3 & -2 & -1 & 0 \end{pmatrix}$$

defines a Poisson bracket, given in terms of the original cluster variables $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ by

$$\{x_i, x_j\} = p_{ij} x_i x_j. \tag{35}$$

This bracket is called log-canonical because it is just given by the constant matrix P in terms of the logarithmic coordinates $\log x_i$. It is also compatible with the cluster algebra structure, in the sense that it remains log-canonical under the action of any mutation, i.e. writing $\mathbf{x} \mapsto \mu_k(\mathbf{x}) = \mathbf{x}' = (x_i')$, in the new cluster variables it takes the form

$$\{x_i', x_j'\} = p_{ij}' x_i' x_j'$$

for some constant skew-symmetric matrix $P' = (p_{ij}')$. Moreover, under the cluster map $\varphi = \rho^{-1} \cdot \mu_1$ defined by

$$\varphi : \, (x_1, x_2, x_3, x_4, x_5) \mapsto \left(x_2, x_3, x_4, x_5, \frac{x_2 x_5 + x_3 x_4}{x_1} \right), \tag{36}$$

the bracket (35) is preserved, in the sense that for all $i, j \in [1, 5]$ the pullback of the coordinate functions by the map satisfies

$$\{\varphi^* x_i, \varphi^* x_j\} = \varphi^* \{x_i, x_j\}.$$

Hence φ is a Poisson map with respect to this bracket.

Given a Poisson map, we can give a definition of discrete integrability, by adapting a definition from [64], that applies in the continuous case of Hamiltonian flows on Poisson manifolds.

Definition 4. Suppose that the Poisson tensor is of constant rank $2m$ on a dense open subset of a Poisson manifold M of dimension N, and that the algebra of Casimir functions is maximal, i.e. it contains $N - 2m$ independent functions. A Poisson map $\varphi : M \to M$ is said to be completely integrable if it preserves $N - m$ independent functions $F_1, \ldots, F_{N-m}$ which are in involution, including the Casimirs.

Example 13. Complete integrability of the $\tilde{A}_{1,3}$ cluster map: Setting $P = B$ with the exchange matrix (6) in Example 2, the bracket

$$\{x_i, x_j\} = b_{ij} x_i x_j$$

is compatible with the cluster algebra structure, and is preserved by the cluster map φ corresponding to (7). At points where all coordinates x_j are non-zero, the Poisson tensor has full rank 4, since B is invertible; so there are no Casimirs. Note that $B^{-1} = -\frac{1}{2}B$, so B is proportional to its own inverse, and the map φ is symplectic, i.e. $\varphi^*\omega = \omega$, where up to overall rescaling the symplectic form is

$$\omega = \sum_{i<j} \frac{b_{ij}}{x_i x_j} \, \mathrm{d}x_i \wedge \mathrm{d}x_j. \tag{37}$$

Now, observe that the recurrence can be rewritten with a 2×2 determinant, as

$$|D_n| = 1, \quad \text{where} \quad D_n = \begin{pmatrix} x_n & x_{n+1} \\ x_{n+3} & x_{n+4} \end{pmatrix},$$

and construct the 3×3 matrix sequence

$$\tilde{D}_n = \begin{pmatrix} x_n & x_{n+1} & x_{n+2} \\ x_{n+3} & x_{n+4} & x_{n+5} \\ x_{n+6} & x_{n+7} & x_{n+8} \end{pmatrix}.$$

Then, by the method of Dodgson condensation [13], the determinant can be expanded as

$$|\tilde{D}_n| = \frac{1}{x_{n+4}} \begin{vmatrix} |D_n| & |D_{n+1}| \\ |D_{n+3}| & |D_{n+4}| \end{vmatrix} = 0.$$

Further calculation shows that the kernel of $\tilde{D}_n$ is spanned by the vector $(1, -J_n, 1)^T$, where J_n is periodic with period 3, so there is a linear relation

$$x_{n+2} - J_n x_{n+1} + x_n = 0, \quad \text{with} \quad J_{n+3} = J_n. \tag{38}$$

Similarly, the linear relation (8), with invariant K (independent of n), corresponds to the fact that the kernel of $\tilde{D}_n^T$ is spanned by $(1, -K, 1)^T$. The J_i can be considered as functions of the phase space coordinates x_1, x_2, x_3, x_4, by writing

$$J_1 = \frac{x_1 + x_3}{x_2},$$

and similarly for J_2, J_3. Computing the Poisson bracket between these functions yields

$$\{J_i, J_{i+1}\} = 2J_iJ_{i+1} - 2,$$

where the indices are read mod 3, so that J_1, J_2, J_3 form a Poisson subalgebra of dimension 3; and

$$K = J_1J_2J_3 - J_1 - J_2 - J_3$$

is a Casimir for this subalgebra, in the sense that $\{J_i, K\} = 0$ for $i = 1, 2, 3$. The map φ preserves any symmetric function of the J_i, so picking the three independent functions

$$F_1 = K, \quad F_2 = J_1J_2 + J_2J_3 + J_3J_1, \quad F_3 = J_1 + J_2 + J_3$$

we have $\varphi^* F_j = F_j$ for all j, but at most two of these can be in involution: $\{F_1, F_2\} = 0 = \{F_1, F_3\}$, but $\{F_2, F_3\} \neq 0$. Thus, choosing just F_1 and F_2, say, the conditions of Definition 4 are satisfied, and the map φ given by (16) is completely integrable.

Remark 3. The fact that cluster variables obtained from affine quivers satisfy linear relations with constant coefficients, such as (8), has been shown in various different ways: for type A in [25, 26], using Dodgson condensation (equivalently, the Desnanot-Jacobi formula); for types A and D in the context of frieze relations [3]; and for all simply-laced types A, D, E in [49], using cluster categories (but see also [11, 61] for another family of quivers made from products of finite and affine Dynkin types A). The fact that there are additional linear relations with periodic coefficients, like (38), was shown for all $\tilde{A}_{p,q}$ quivers in [26], where it was also found that the quantities J_i are coordinates in the dressing chain for Schrödinger operators, and this has recently been extended to affine types D and E [60].

Example 14. Non-existence of a log-canonical bracket for $\tilde{A}_{1,2}$: For the cluster algebra of type $\tilde{A}_{1,2}$, defined by the skew-symmetric exchange matrix

$$B = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

it is easy to verify that there is no bracket of log-canonical form, like (35), that is compatible with cluster mutations. However, iterates of the cluster map, defined by the recurrence

$$x_n x_{n+3} = x_{n+1} x_{n+2} + 1,$$

satisfy the linear relation $x_{n+4} - K x_{n+2} + x_n = 0$, for a first integral K. In fact, setting $u_n = x_n x_{n+1}$ yields a recurrence of second order,

$$u_n u_{n+2} = u_{n+1}(u_{n+1} + 1), \tag{39}$$

and, rewriting K in terms of u_1, u_2, this corresponds to a symplectic map $\hat{\varphi}$ in the (u_1, u_2) plane with symplectic form $\hat{\omega} = \mathrm{d}\log u_1 \wedge \mathrm{d}\log u_2$ and one first integral; so the map $\hat{\varphi}$ is completely integrable.

Example 15. Casimirs for Somos-5: The Poisson tensor for the Somos-5 map (36), defined by (35), has rank 2 on $\mathbb{C}^5 \setminus \{x_i = 0\}$ (away from the coordinate hyperplanes). The kernel of the matrix P is spanned by the vectors

$$\tilde{\mathbf{v}}_1 = (1, -2, 1, 0, 0)^T,\ \tilde{\mathbf{v}}_2 = (0, 1, -2, 1, 0)^T,\ \tilde{\mathbf{v}}_3 = (0, 0, 1, -2, 1)^T, \tag{40}$$

which correspond to three independent Casimir functions

$$F_1 = \mathbf{x}^{\tilde{\mathbf{v}}_1} = \frac{x_1 x_3}{x_2^2}, \quad F_2 = \mathbf{x}^{\tilde{\mathbf{v}}_2} = \frac{x_2 x_4}{x_3^2}, \quad F_3 = \mathbf{x}^{\tilde{\mathbf{v}}_3} = \frac{x_3 x_5}{x_4^2},$$

whose Poisson bracket with any other function G vanishes: $\{F_j, G\} = 0$ for $j = 1, 2, 3$. There are two independent first integrals H_1, H_2, i.e. functions that are preserved by the action of φ, so that $\varphi^* H_i = H_i \cdot \varphi = H_i$ for $i = 1, 2$;

and these are themselves Casimirs because they can be written in terms of the F_j [40]:

$$H_1 = F_1F_2F_3 + \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_1F_2F_3}, \tag{41}$$

$$H_2 = F_1F_2 + F_2F_3 + \frac{1}{F_1F_2} + \frac{1}{F_2F_3} + \frac{1}{F_1F_2^2F_3}. \tag{42}$$

However, the full algebra of Casimirs is not preserved by the map φ, because F_1, F_2, F_3 transform as

$$\varphi^*F_1 = F_2, \quad \varphi^*F_2 = F_3, \quad \varphi^*F_3 = \frac{F_2F_3+1}{F_1F_2^2F_3^2}.$$

Hence the Somos-5 map is not completely integrable with respect to this bracket.

The previous two examples show that if the exchange matrix B is degenerate, then the cluster coordinates may not be the correct ones to use, as either there is no invariant log-canonical bracket in these coordinates, as in the case of $\tilde{A}_{1,2}$, or even if there is such a bracket, a full set of Casimirs is not preserved by the cluster map. (A Poisson map sends Casimirs to other Casimirs, but need not preserve each Casimir individually.) The way out of this quandary, which was already hinted at in Example 14, is to work on a reduced space where the map φ reduces to a symplectic map $\hat{\varphi}$. It turns out that there is a canonical way to do this, based on the presymplectic form ω associated with the cluster algebra, which in general, for any skew-symmetric exchange matrix $B = (b_{ij})$, is given by the formula (37) above.

In the case that B is nondegenerate (which is possible for even N only, as in Example 2), ω is a closed, nondegenerate 2-form, so the cluster map is symplectic, but otherwise ω has a null distribution, generated by vector fields of the form

$$\sum_{j=1}^{N} w_j x_j \frac{\partial}{\partial x_j}, \quad \text{for} \quad \mathbf{w} = (w_j) \in \ker B.$$

These vector fields all commute with other, and can be integrated to yield a commuting set of scaling symmetries: each $\mathbf{w} \in \ker B$ generates a one-parameter scaling group

$$\mathbf{x} \mapsto \tilde{\mathbf{x}} = \lambda^{\mathbf{w}} \cdot \mathbf{x}, \qquad \lambda \in \mathbb{C}^*, \tag{43}$$

where the notation means that each component is scaled so that $\tilde{x}_j = \lambda^{w_j} x_j$. Regarding B as a linear transformation on $\mathbb{Q}^N$, skew-symmetry means that there is an orthogonal direct sum decomposition $\mathbb{Q}^N = \operatorname{im} B \oplus \ker B$. If B has rank $2m$, then an integer basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2m}$ for $\operatorname{im} B$ yields a complete set of rational functions invariant under the symmetries (43), given by the monomials

$$u_j = \mathbf{x}^{\mathbf{v}_j}, \quad j = 1, \ldots, 2m. \tag{44}$$

In the case that B has period 1, it was shown in [26] that by choosing the basis suitably, the rational map $\pi : \mathbf{x} \mapsto \mathbf{u} = (u_j)$ reduces φ to a birational symplectic map $\hat{\varphi}$ in dimension $2m$, with symplectic form $\hat{\omega}$, in the sense that $\hat{\varphi} \cdot \pi = \pi \cdot \varphi$, and $\pi^* \hat{\omega} = \omega$, where

$$\hat{\omega} = \sum_{i<j} \frac{\hat{b}_{ij}}{u_i u_j} \, \mathrm{d}u_i \wedge \mathrm{d}u_j \tag{45}$$

(for a certain skew-symmetric matrix $\hat{B} = (\hat{b}_{ij})$) is also log-canonical. In [44] it was further shown that (up to an overall sign) there is a canonical choice of basis for $\operatorname{im} B \cap \mathbb{Z}^N$ with the property that

$$\varphi^* \mathbf{x}^{\mathbf{v}_j} = \mathbf{x}^{\mathbf{v}_{j+1}}, \quad \text{for} \quad j \in [1, 2m-1].$$

This is called a *palindromic basis*, because the first $N - 2m + 1$ entries of $\mathbf{v}_1$ form a palindrome, with the remaining $2m - 1$ entries being zero, and this palindrome is just shifted along to get the other basis elements; the basis is fixed uniquely if the first entry of $\mathbf{v}_1$ is chosen to be positive. The advantage of a palindromic basis is that the birational map $\hat{\varphi}$ is equivalent to an iteration of a single recurrence relation.

Definition 5. Given a cluster mutation-periodic skew-symmetric exchange matrix B with period 1, of rank $2m$, and the symplectic coordinates $(u_j) \in \mathbb{C}^{2m}$ defined by (44) with a palindromic basis, the *U-system* is the recurrence corresponding to the reduced cluster map $\hat{\varphi}$, which, for some rational function $\mathcal{F}$, has the form

$$u_n u_{n+2m} = \mathcal{F}(u_{n+1}, \ldots, u_{n+2m-1}). \tag{46}$$

We have already seen an example of a U-system, namely the reduced recurrence (39) for $\tilde{A}_{1,2}$. An integrable U-system corresponds to the canonical version of integrability for maps: the U-system is equivalent to a symplectic map in dimension $2m$, so m independent first integrals in involution are needed for complete integrability.

Example 16. Complete integrability of the Somos-5 U-system: With B given by (34), a palindromic basis for $\mathrm{im}\, B$ is written using (40) as

$$\mathbf{v}_1 = \tilde{\mathbf{v}}_1 + \tilde{\mathbf{v}}_2 = (1, -1, -1, 1, 0)^T, \quad \mathbf{v}_2 = \tilde{\mathbf{v}}_2 + \tilde{\mathbf{v}}_3 = (0, 1, -1, -1, 1)^T,$$

so the reduced coordinates are

$$u_1 = \frac{x_1 x_4}{x_2 x_3} = F_1 F_2, \quad u_2 = \frac{x_2 x_5}{x_3 x_4} = F_2 F_3,$$

and $\omega = \sum_{i<j} b_{ij} \mathrm{d}\log x_i \wedge \mathrm{d}\log x_j$ reduces to the symplectic form

$$\hat{\omega} = \frac{\mathrm{d}u_1 \wedge \mathrm{d}u_2}{u_1 u_2}$$

in these coordinates. The cluster map (36) reduces to an iteration of the U-system

$$u_n u_{n+2} = \frac{u_{n+1} + 1}{u_{n+1}}, \tag{47}$$

and although H_1 does not survive this reduction, the first integral H_2 can be rewritten in terms of u_1, u_2, to yield the function

$$H = u_1 + u_2 + \frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_1 u_2},$$

so the U-system corresponds to a completely integrable symplectic map in two dimensions. The generic level sets of H are cubic curves of genus 1, and this is an example of a symmetric QRT map (see [38] and references).

The general Somos-6 recurrence, with constant coefficients α, β, γ, has the form

$$x_n x_{n+6} = \alpha x_{n+1} x_{n+5} + \beta x_{n+2} x_{n+4} + \gamma x_{n+3}^2, \tag{48}$$

which has the Laurent property [22], but cannot come from a cluster algebra when $\alpha\beta\gamma \neq 0$, due to there being too many terms on the right-hand side. In fact, it appears in the more general setting of mutations in LP algebras, which allow exchange relations with more terms [51]. Being quadratic relations, Somos recurrences are reminiscent of Hirota bilinear equations for tau functions in soliton theory, and indeed, the general Somos-6 recurrence is a reduction of Miwa's equation [10], which is the bilinear discrete BKP equation, also known as the cube recurrence in algebraic combinatorics. Here we conclude our discussion of Example 4, by setting $\beta = 0$, to obtain a bilinear equation with a total of three terms, which can be obtained as a reduction of the discrete Hirota equation (bilinear discrete KP, or octahedron recurrence), that is

$$T_1T_{-1} = T_2T_{-2} + T_3T_{-3}, \tag{49}$$

where the tau function $T = T(m_1, m_2, m_3)$ and the subscript $\pm j$ denotes a shift in the jth independent variable, so e.g. $T_{\pm 1} = T(m_1 \pm 1, m_2, m_3)$, and so on. The advantage of making a reduction from this equation with more independent variables is that it has a Lax pair, which reduces to a Lax pair for the Somos recurrence, and there is an associated spectral curve, whose coefficients provide first integrals.

Example 17. A Somos-6 U-system: Setting $\beta = 0$ in (48) produces

$$x_nx_{n+6} = \alpha x_{n+1}x_{n+5} + \gamma x_{n+3}^2. \tag{50}$$

This differs from (15) and (18) by the inclusion of coefficients α, γ, which can be achieved by augmenting the cluster algebra with frozen variables that appear in the exchange relations but do not themselves mutate (see [25] and references, for instance), and does not change other features such as Poisson brackets or the (pre)symplectic forms. Upon applying the method in [45], we can obtain (48) as a plane wave reduction of (49), by setting

$$T(m_1, m_2, m_3) = a_1^{m_1^2}a_2^{m_2^2}a_3^{m_3^2}\,x_n, \qquad n = m_0 + 3m_1 + 2m_3,$$

with m_0 arbitary, and taking $\alpha = a_3^2/a_1^2$, $\gamma = a_2^2/a_1^2$. Under this reduction, the linear system whose compatibility gives the discrete KP equation becomes

$$\begin{array}{rcl} Y_n\psi_{n+3} + \alpha\zeta\psi_{n+2} & = & \xi\psi_n, \\ \psi_{n+3} - X_n\psi_{n+1} & = & \zeta\psi_n, \end{array} \tag{51}$$

where ψ_n is a wave function, ζ, ξ are spectral parameters, and

$$X_n = \frac{x_{n+2}x_{n+3}}{x_{n+4}x_{n+1}}, \quad Y_n = \frac{x_{n+4}x_n}{x_{n+3}x_{n+1}}.$$

The equation (50) is the compatibility condition for these two linear equations for ψ_n (to be precise, the parameter γ arises as an integration constant). This is more conveniently seen by writing the second linear equation in matrix form, with a vector $\Psi_n = (\psi_n, \psi_{n+1}, \psi_{n+2})^T$, as

$$\Psi_{n+1} = \mathbf{M}_n\psi_n, \quad \mathbf{M}_n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta & X_n & 0 \end{pmatrix}, \tag{52}$$

and then using the second linear equation in (51) to reformulate the first one as an eigenvalue problem with Ψ_n as the eigenvector, that is

$$\mathbf{L}_n\Psi_n = \xi\Psi_n, \quad \mathbf{L}_n = \begin{pmatrix} \zeta Y_n & u_n & \zeta\alpha \\ \zeta^2\alpha & \zeta(Y_{n+1}+\alpha X_n) & u_{n+1} \\ \zeta u_{n+2} & \zeta^2\alpha + u_{n+1}^{-1} & \zeta(Y_{n+2}+\alpha X_{n+1}) \end{pmatrix}. \tag{53}$$

In the above expression for the Lax matrix $\mathbf{L}_n$, we have introduced the quantities

$$u_n = X_nY_n = \frac{x_nx_{n+2}}{x_{n+1}^2},$$

which for $n = 1,2,3,4$ give a set of symplectic coordinates obtained from the palindromic basis $\mathbf{v}_1 = (1,-2,1,0,0,0)^T$, $\mathbf{v}_2 = (0,1,-2,1,0,0)^T$, $\mathbf{v}_3 = (0,0,1,-2,1,0)^T$, $\mathbf{v}_4 = (0,0,0,1,-2,1)^T$ for $\operatorname{im} B$, with B as in (14), and satisfy the U-system

$$u_nu_{n+4} = \frac{\alpha u_{n+1}u_{n+2}^2u_{n+3}+\gamma}{u_{n+1}^2u_{n+2}^3u_{n+3}^2} \tag{54}$$

(which should be compared with the tropical formulae (27) and (28) above), correponding to the reduced cluster map $\hat{\varphi}$. The symplectic form $\hat{\omega}$, such that $\hat{\varphi}^*\hat{\omega} = \hat{\omega}$, is

$$\hat{\omega} = \sum_{i<j}\hat{b}_{ij}\mathrm{d}\log u_i \wedge \mathrm{d}\log u_j, \quad \hat{B} = (\hat{b}_{ij}) = \begin{pmatrix} 0 & 1 & 2 & 1 \\ -1 & 0 & 2 & 2 \\ -2 & -2 & 0 & 1 \\ -1 & -2 & -1 & 0 \end{pmatrix},$$

so the associated nondegenerate Poisson bracket for these coordinates is given by $\{u_i,u_j\} = \hat{p}_{ij}u_iu_j$ with $(\hat{p}_{ij}) = \hat{B}^{-1}$. The compatibility condition of the matrix system given by (52) and (53) is the discrete Lax equation

$$\mathbf{L}_{n+1}\mathbf{M}_n = \mathbf{M}_n\mathbf{L}_n,$$

which is equivalent to the U-system (54). So this is an isospectral evolution, and the spectral curve

$$\det(\mathbf{L}_n - \xi\mathbf{1}) = -\xi^3 + H_1\xi^2 + (1-H_2\zeta^2)\xi + \alpha^3\zeta^5 + \gamma\zeta^3 = 0 \tag{55}$$

is independent of n, with the non-trivial coefficients being H_1, given by

$$(u_n+u_{n+3})u_{n+1}u_{n+2}+\alpha\left(\frac{1}{u_nu_{n+1}}+\frac{1}{u_{n+1}u_{n+2}}+\frac{1}{u_{n+2}u_{n+3}}\right)+\frac{\gamma}{u_nu_{n+1}^2u_{n+2}^2u_{n+3}},$$

and

$$
\begin{aligned}
H_2 &= \alpha\left(\frac{u_n u_{n+1}}{u_{n+3}}+\frac{u_{n+2}u_{n+3}}{u_n}\right)+\gamma\left(\frac{1}{u_n u_{n+1}u_{n+2}}+\frac{1}{u_{n+1}u_{n+2}u_{n+3}}\right)\\
&\quad +\alpha^2\left(\frac{1}{u_n u_{n+1}^2 u_{n+2}}+\frac{1}{u_{n+1}u_{n+2}^2 u_{n+3}}\right)+\frac{\alpha\gamma}{u_n u_{n+1}^3 u_{n+2}^3 u_{n+3}},
\end{aligned}
$$

which provide two independent first integrals. It can be verified directly that $\{H_1, H_2\} = 0$, which shows that each iteration of (54) corresponds to a completely integrable symplectic map $\hat{\varphi}$ (in different coordinates, the involutivity of these quantities was also shown in [43]). The trigonal spectral curve (55) has genus 4, and admits the involution $(\zeta, \xi) \mapsto (-\zeta, -\xi)$, giving a quotient curve of genus 2, with a Prym variety that is isomorphic to the Jacobian of a second genus 2 curve, analogous to the situation for the general Somos-6 map in [15]. However, in this case there is a more direct way to find the second genus 2 curve, as the hyperelliptic spectral curve of a 2×2 Lax pair obtained by deriving (50) as a reduction of a discrete time Toda equation on a 5-point lattice [45, 46]. For explicit analytic formulae for the solutions in terms of genus 2 sigma functions, see [43, 15].

6 Discrete Painlevé equations from coefficient mutation

The continuous Painlevé equations are a special set of non-autonomous ordinary differential equations of second order that are characterized by the absence of movable critical points in their solutions, which is known as the Painlevé property. Discrete Painlevé equations are a particular class of ordinary difference equations which, like their continuous counterparts, are non-autonomous (meaning that the independent variable appears explicitly); in many cases, they appeared from the search for an appropriate discrete analogue of the Painlevé property [34]. The resulting notion of singularity confinement turned out to be much weaker than the Painlevé property for differential equations, and is not sufficient for integrability, although it is a very useful tool when used judiciously in tandem with other techniques for identifying integrable maps or discrete Painlevé equations [55]. In fact, singularity confinement seems to be very closely related to the Laurent property [41], and it is interesting to speculate whether all discrete integrable systems are related to a system with the Laurent property by introducing a tau function or some other lift of the coordinates [8, 37, 54].

Recently there have been various studies that show how certain discrete Painlevé equations and their higher order analogues can arise from mutation of coefficients in cluster algebras. Here we concentrate on the methods used in [44], but for other related approaches see the work of Okubo [58, 59] and that of Bershtein et al. [5].

A *Y-system* is a set of difference equations arising as relations between coefficients appearing from a sequence of mutations in a cluster algebra with periodicity. The original Y-systems were obtained by Zamolodchikov as a set of functional equations in certain quantum field theories associated with simply-laced affine Lie algebras [66], yet they arise from cluster algebras of finite type obtained from the corresponding finite-dimensional root systems, and display purely periodic dynamics. Generalized Y-systems were defined by Nakanishi [57] starting from a general notion of periodicity in a cluster algebra, and typically display complicated dynamical behaviour.

Here we concentrate on the case of cluster mutation-periodic quivers with period 1, for which the Y-system can be written as a single scalar difference equation, given by

$$y_n y_{n+N} = \frac{\prod_{j=1}^{N-1}(1+y_{n+j})^{[a_j]_+}}{\prod_{j=1}^{N-1}(1+y_{n+j}^{-1})^{[-a_j]_+}}, \tag{56}$$

where, as in Theorem 1, $a_j = b_{1,j+1}$ are the components of the palindromic $(N-1)$-tuple that determines the exchange matrix. (Here we assume that the first non-zero component a_j is positive; there is no loss of generality in doing so, due to the freedom to replace $B \to -B$, but some signs are reversed compared with [44] and [57].) In this context, the coefficient-free recurrence (19) that defines the cluster map is referred to as the *T-system*. It was first observed in [24] that there is a relation between the evolution of coefficients $\mathbf{y}$ under mutations (2) in a cluster algebra, and the evolution of cluster variables $\mathbf{x}$ due to the associated coefficient-free cluster mutations given by (4), which can be summarized by the slogan that "the T-system provides a solution of the Y-system." In the case at hand, the precise statement is that making the subsitution

$$y_n = \prod_{j=1}^{N-1} x_{n+j}^{a_j} \tag{57}$$

in (56) provides a solution of the Y-system whenever x_n satisfies the coefficient-free T-system (19).

Although the equations (19) and (56) are both of order N, there can be a discrepancy between the solutions of the T-system and the Y-system, in the sense that the general solution of the former does not yield the general solution of the latter. This discrepancy is determined by the following result.

Proposition 3. *Let x_n satisfy the modified T-system*

$$x_n x_{n+N} = \mathcal{Z}_n \left(\prod_{j=1}^{N-1} x_{n+j}^{[a_j]_+} + \prod_{j=1}^{N-1} x_{n+j}^{[-a_j]_+} \right). \tag{58}$$

Then the substitution (57) yields a solution of the Y-system (56) if and only if $\mathcal{Z}_n$ satisfies the Z-system

$$\prod_{j=1}^{N-1} \mathcal{Z}_{n+j}^{a_j} = 1. \tag{59}$$

Each iteration of the modified T-system (58) with non-autonomous coefficients evolving according to (59) preserves the presymplectic form given by (37) in terms of the entries of the exchange matrix B, and if B is degenerate we can use a palindromic basis for $\mathrm{im}\, B$ to reduce this to a non-autonomous recurrence in lower dimension that preserves the symplectic form (45).

Definition 6. The pair of equations (58) and (59) is called the *T_z-system.* The *U_z-system* associated with (58) is given by (59) together with

$$u_n u_{n+2m} = \mathcal{Z}_n \, \mathcal{F}(u_{n+1}, \dots, u_{n+2m-1}),$$

where the rational function $\mathcal{F}$ is the same as in (46).

We conclude this section with a couple of examples.

Example 18. Somos-5 Y-system and q-Painlevé II: The Y-system associated with the exchange matrix (34) is

$$y_n y_{n+5} = \frac{(1 + y_{n+1})(1 + y_{n+4})}{(1 + y_{n+2}^{-1})(1 + y_{n+3}^{-1})},$$

and (noting that on the right-hand side of the substitution (57) there is the freedom to shift $n \to n+1$) the general solution of this can be written as

$$y_n = \frac{x_n x_{n+3}}{x_{n+1} x_{n+2}},$$

where x_n satisfies the non-autonomous Somos-5 relation

$$x_n x_{n+5} = \mathcal{Z}_n (x_{n+1} x_{n+4} + x_{n+2} x_{n+3}), \quad \text{with} \quad \frac{\mathcal{Z}_n \mathcal{Z}_{n+3}}{\mathcal{Z}_{n+1} \mathcal{Z}_{n+2}} = 1.$$

Equivalently, we can identify $y_n = u_n$ and solve the third order Z-system for $\mathcal{Z}_n$ to write the U_z-system as a non-autonomous version of the QRT map (47), that is

$$u_n u_{n+2} = \mathcal{Z}_n(1 + u_n^{-1}), \quad \text{with} \quad \mathcal{Z}_n = \beta_n \mathfrak{q}^n, \ \beta_{n+2} = \beta_n.$$

The latter is equivalent to a q-Painlevé II equation identified in [50], having a continuum limit to the Painlevé II differential equation

$$\frac{d^2u}{dz^2} = 2u^3 + zu + \alpha.$$

Example 19. A q-Somos-6 relation: The Y-system corresponding to (14) is

$$y_n y_{n+6} = \frac{(1 + y_{n+1})(1 + y_{n+5})}{(1 + y_{n+3}^{-1})^2}.$$

Its general solution can be written as

$$y_n = \frac{x_n x_{n+4}}{y_{n+2}^2},$$

where x_n satisfies a q-Somos-6 relation given by

$$x_n x_{n+6} = \mathcal{Z}_n(x_{n+1}x_{n+5} + x_{n+3}^2), \quad \text{with} \quad \mathcal{Z}_n = \alpha_\pm \mathfrak{q}_\pm^n, \tag{60}$$

with the solution of the fourth order Z-system

$$\frac{\mathcal{Z}_n \mathcal{Z}_{n+4}}{\mathcal{Z}_{n+2}^2} = 1$$

being given in terms of quantities $\alpha_\pm$ and $\mathfrak{q}_\pm$ that alternate with the parity of n. Alternatively, one can write

$$y_n = u_n u_{n+1}^2 u_{n+2}$$

with u_n satisfying a non-autonomous version of (54), that is

$$u_n u_{n+4} = \frac{\mathcal{Z}_n(u_{n+1}u_{n+2}^2u_{n+3} + 1)}{u_{n+1}^2 u_{n+2}^3 u_{n+3}^2},$$

with $\mathcal{Z}_n$ as in (60). The latter should be regarded as a fourth order analogue of a discrete Painlevé equation.

7 Conclusions

We have just scratched the surface in this brief introduction to cluster algebras and discrete integrability. Among other important examples that we have not described here, we would like to mention pentagram maps [31] and cluster integrable systems related to dimer models [14, 33]. A slightly different viewpoint, with some different choices of topics, can be found in the review [32].

Acknowledgements

The work of all three authors is supported by EPSRC Fellowship EP/M004333/1. ANWH is grateful to the School of Mathematics & Statistics, UNSW for hospitality and additional support under the Distinguished Researcher Visitor Scheme.

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